# Probability

↳ makes extensive use of set operations.

Set : collection of objects which are elements of the set.

If $S$ is a set & $x$ is an element of set then we write $x \in S$

if $x$ is not an element $x \notin S$

$\phi \rightarrow$ Empty set

## Set operations

Union $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

Intersection $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

Complement. $\bar{A} = \{x : x \notin A\}$

Diff $A - B = \{x : x \in A \text{ but } x \notin B\}$

A collection of sets is said to be a partition of a set $S$ if sets in the collection are disjoint & their union is $S$.

$S \cup T = T \cup S$ $\qquad A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\qquad A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$

$(A^c)^c = A$ , $A \cap A^c = \phi$ , $A \cup \Omega = \Omega$ , $A \cap \Omega = A$

**Probability** : how likely sthg is to happen
→ func defined on events

Sample space $S$ : set of all possible outcomes

Probabilistic Model : mathematical description of some uncertain situation

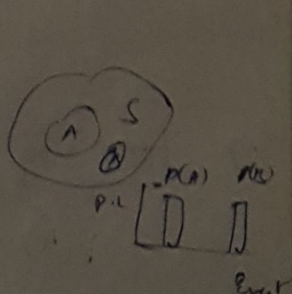
Elements of $\mathcal{J}$ : 1) Sample Space

2) Probabilistic law

[ assigns prob to each event of $S$ ]

↳ Event : Subsets of $S$, representing a particular outcome or set of outcomes

$P(A) = \frac{n(A)}{n(S)}$



→ Diff elements of sample space shd be mutually exclusive so that when experiment is carried out, there is a unique outcome

## Axioms

1) All prob are non-negative $P(A) \geq 0$
2) Prob of sample space is 1 $P(S) = 1$
3) For any 2 disjoint events $A$ & $B$ $P(A \cup B) = P(A) + P(B)$

$$P(A_1 \cup A_2 \cup A_3 \ldots) = P(A_1) + P(A_2) \ldots$$

Any other prob relationships can be derived from the axioms.

## Theorems on prob. spaces

1.) Empty set has prob 0. $\boxed{P(\phi) = 0}$

Proof: ~~A~~

$A = A \cup \phi$

$P(A) = P(A \cup \phi)$ $A$ & $\phi$ are disjoint

$P(A) = P(A) + P(\phi) \Rightarrow P(\phi) = 0$

2.) Complement rule. $\boxed{P(A^c) = 1 - P(A)}$

$S = A \cup A^c$

$P(S) = P(A \cup A^c)$

axiom 3 — $P(A) + P(A^c) = P(S)$

$= 1$ {axiom 2}

$P(A^c) = 1 - P(A)$

3.) For any event $A$, $\boxed{0 \leq P(A) \leq 1}$

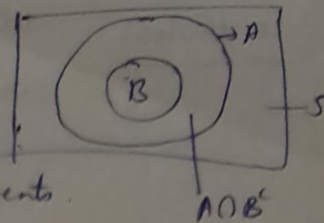
$P(A^c) = 1 - P(A)$

[Axiom 1] $P(A^c) \geq 0 \Rightarrow 1 - P(A) \geq 0$

$\Rightarrow P(A) \leq 1$

4.) If $B \subseteq A$ then ii, $P(B) \leq P(A)$

i, $P(A \cap B') = P(A) - P(B)$




Proof: i, $B$ & $A \cap B^c$ are mutually exclusive events.

$$A = B \cup (A \cap B^c)$$

$$P(A) = P[B \cup (A \cap B^c)]$$

$$P(A) = P(B) + P(A \cap B^c)$$

$$P(A \cap B^c) = P(A) - P(B)$$

ii, From prev. $P(A \cap B^c) \geq 0$ [Axiom 1]

$$P(A) - P(B) \geq 0$$

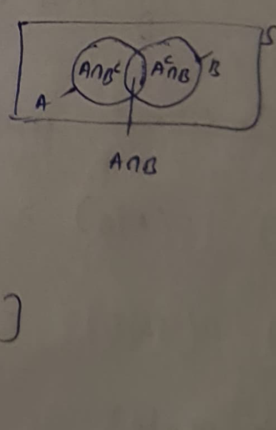
$$P(A) \geq P(B)$$

Theorem 5: If $A$ & $B$ are 2 events & not disjoint then

Law of Add^n of Prob

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$A^c \cap B = B - (A \cap B) \quad -①$$




$$P(A \cup B) = P[A \cup (A^c \cap B)]$$

$$= P(A) + P(A^c \cap B)$$

& [From ①]

$$= P(A) + P(B) - P(A \cap B)$$

Sample space < Discrete, Continuous

(1) Discrete: when $S$ contains finite no of outcomes.

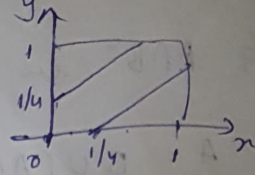
Discrete prob law: prob of any event (collection of outcomes) is the sum of the prob of individual outcomes that make up the event.

- If all outcomes equally likely then $P(A) = \frac{n(A)}{n(S)}$

Eg: Rolling a pair of dice

2.) Continuous: When S contains infinite no of possible outcomes

$1 - 3/4 \cdot 3/4 = \frac{7}{16}$




Conditional probabilities: finding prob of event A, gvn that event B has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

If A & B are mutually exclusive events then $P(B/A) = 0$

$$P(A \cap B) = P\left(\frac{A}{B}\right) P(B) \quad \text{[Multiplication Rule of Prob]}$$

prob of both events occuring simultaneously if A & B are dependent events

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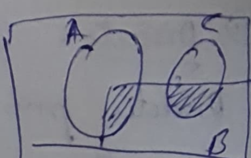
Conditional prob share properties of ordinary prob

1.) $P(A|B) \geq 0$ Assuming $P(B) > 0$ $\left[\frac{P(A \cap B) \to \geq 0}{P(B) \to \geq 0} \text{ overall } \geq 0\right]$

2.) $P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

$P(B|B) = \frac{P(B \cap B)}{P(B)} = 1$

3.) If $A \cap C = \phi$ $P(A \cup C|B) = P(A|B) + P(C|B)$

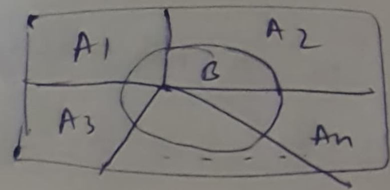



$$P(A \cup C|B) = \frac{P((A \cup C) \cap B)}{P(B)}$$

$$= \frac{P((A \cap B) \cup (C \cap B))}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)}$$

$$= P(A|B) + P(C|B)$$

# Total Law of Probability




Let $A_1, \dots A_n$ be disjoint ~~events~~ events that form partition of the sample space & assume $P(A_i) > 0$

$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i)$$

$$S = A_1 \cup A_2 \cup \dots A_n$$

$$B = B \cap S$$

$$= B \cap (A_1 \cup A_2 \cup \dots A_n)$$

$$P(B) = P[\underbrace{(B \cap A_1) \cup (B \cap A_2)}_{} \cup \dots \cup (B \cap A_n)]$$

these all are disjoint sets

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \dots P(B \cap A_n)$$ [Axiom 3]

$$= P(A_1) P(B|A_1) + P(A_2) P(B|A_2) + \dots P(A_n) P(B|A_n)$$

↳ By def of cond^n prob [Multiplication rule of prob]

$$\therefore P(B) = P(A_1) P(B|A_1) + P(A_2) P(B|A_2) \dots + P(A_n) P(B|A_n)$$

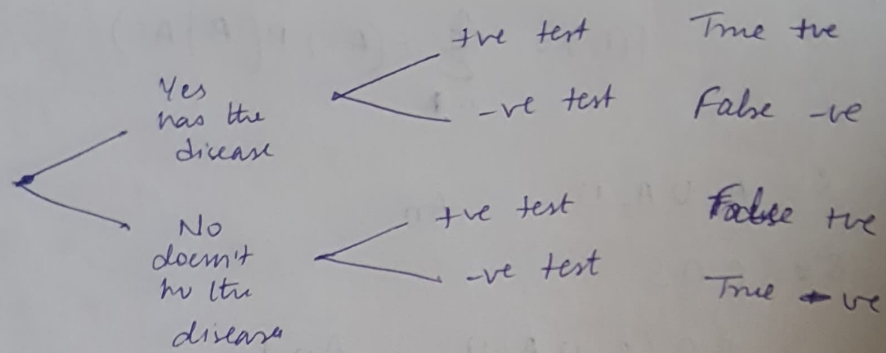
$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i)$$

## Bayes Thm

Let $A_1, A_2 \ldots A_n$ be n mutually exclusive event with $P(A_i) \neq 0$

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\,P(B|A_i)}$$

Example




So given particular test +ve, prob that an individual has disease

$$P(\text{has disease} \mid \text{+ve test}) = \frac{\text{True +ve}}{\text{True +ve} + \text{False +ve}}$$

We're going to ~~back~~ start at the end branches & backtrack to find the beginning.

[If we already know condⁿ prob, we use Bayes thm to find reverse prob]

Proof: According to condⁿ prob,

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \quad -① $$

Using multiplication rule of prob

$$P(A_i \cap B) = P(A_i)\,P(B|A_i) \quad -②$$

Using Total prob thm

$$P(B) = \sum_{k=1}^{n} P(A_k)\,P\left(\frac{B}{A_k}\right) \quad -③$$

Put (2), (3) in (1)

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{\sum_{k=1}^{n} P(A_k)\,P(B|A_k)}$$