# Notes on Thermal Ablation Modeling

Patrick Blonigan

August 25, 2023

## 1 Partial Differential Equation

From Howard and Blackwell's paper, the governing equations for a non-decomposing ablation are the time-dependent energy conservation equation,

$$\frac{\partial \rho_s e_s}{\partial t} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial T}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\rho_s h_s v_{m_i}\right) - \dot{Q} = 0, \tag{1}$$

and the steady state mesh deformation equation,

$$\frac{\partial}{\partial x_j}\left(D_{ijkl}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right) = 0, \tag{2}$$

where $T$ is temperature, $u$ is mesh displacement, $\rho_s$ is the solid density, $e_s \equiv e_{0_s} + \int_{T_0}^{T} c_s(T)T$ is the solid internal energy, $c_s(T)$ is the solid specific heat capacity, $k_{ij} \equiv k(T)$ is the solid thermal conductivity, $h_s \equiv h_s(T)$ is the solid enthalpy, $v_{m_i} \equiv \frac{du_i}{dt}$ is the velocity of a mesh node, $\dot{Q}$ is a volumetric heating term, and $D_{ijkl}$ is the constitutive tensor.

The aero-heating boundary condition for eq. (1) is derived from an energy and mass balance at the ablating surface. The energy balance is,

$$\dot{q}_{aero} = \rho_e u_e C_H (h_r - h_w(T_w)) - \sigma\epsilon(T_w^4 - T_\infty^4) \tag{3}$$

where $T_w$ is wall temperature, $\dot{m}_s'' \equiv \rho_s v_{m_i}$ is mass flux at the wall, $\sigma$ is the Stefan-Boltzmann constant, $\epsilon$ is radiative emissivity of the material, $T_\infty$ is far field temperature, and $\rho_e, u_e, C_H, h_r$ are quantities obtained from aerodynamic models or look-up tables. The third term corresponds to energy gained due to the solid burning, while the fourth term corresponds to energy being transported away from from the wall by the material burning away.

The mass balance is

$$\dot{m}_s'' = \rho_s v_{m_i,w} = \rho_e u_e C_m B_c'(T_w), \tag{4}$$

where $\rho_e u_e C_m$ is obtained from a CFD simulation or a look-up table, and $B_c'$ is a quantity obtained from a look-up table as a function of temperature.

The recession boundary condition imposes nodal displacements to match the nodal velocities at the wall, $v_{m_i,w}$. Since the mesh deformation equation is steady-state, the boundary condition depends on the time step size used for temporal discretization, $\Delta t$,

$$u_i(t+\Delta t) = u_i(t) + \Delta t \hat{n}_i \frac{\dot{m}''_s}{\rho_s} \tag{5}$$

where $\hat{n}_i$ is the unit vector in direction $i$.

# 2 Discrete Ordinary Differential Equation

The above PDEs are discretized with a control-volume finite element method and backward euler timestepping, resulting in a discrete ODE and a linear system for the energy and elasticity equations, respectively.

For a finite element mesh with $N$ nodes, the discrete energy equations are,

$$\rho_s \frac{\boldsymbol{e}(\boldsymbol{T}^i) - \boldsymbol{e}(\boldsymbol{T}^{i-1})}{\Delta t^i} - \boldsymbol{K}(\boldsymbol{T}^i, \boldsymbol{u}^i)\boldsymbol{T}^i - \boldsymbol{C}(\boldsymbol{T}^i, \boldsymbol{u}^i)\frac{\boldsymbol{u}^i - \boldsymbol{u}^{i-1}}{\Delta t^i} - \dot{\boldsymbol{Q}}^i(\boldsymbol{T}^i) - \boldsymbol{b}(\boldsymbol{T}^i, \boldsymbol{u}^i) = 0, \tag{6}$$

where $\boldsymbol{T}^i \in \mathbb{R}^N$ is a vector of nodal temperatures at time step $i$, $\boldsymbol{u}^i \in \mathbb{R}^N$ is a vector of nodal displacements, $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ is the thermal stiffness matrix corresponding to the second term in Eq. (1), $\boldsymbol{C} \in \mathbb{R}^{N\times N}$ is the energy transport term corresponding the third term in Eq. (1), $\dot{\boldsymbol{Q}}^i \in \mathbb{R}^N$ is a vector of heat sources at each node, and $\boldsymbol{b}_T \in \mathbb{R}^N$ is the vector of boundary condition contributions to the energy equation at each node.

For a finite element mesh with $N$ global nodes and $N_w$ wall nodes, the discrete static elasticity equation is

$$\boldsymbol{E}\boldsymbol{u}^i - \boldsymbol{I}_u^T\left(\boldsymbol{I}_u\boldsymbol{u}^{i-1} + \Delta t^i \boldsymbol{v}(\boldsymbol{I}_T\boldsymbol{T}^i)\right) = 0, \tag{7}$$

where $\boldsymbol{E} \in \mathbb{R}^{3N\times 3N}$ is the elasticity operator, $\boldsymbol{I}_u \in \mathbb{R}^{3N_w\times 3N}$ is a matrix that selects all three displacement components on each wall node, $\boldsymbol{I}_T \in \mathbb{R}^{N_w\times N}$ is a matrix that selects temperatures on each wall node, and $\boldsymbol{v} \in \mathbb{R}^{3N_w}$ is a vector of node velocities at the wall computed from the mass balance Eq. (4).

These equations are solved together using a Newton-Raphson Method at each time step, usually with an adaptive time-stepper.