

<sub>1</sub> Physics-Infused Reduced-Order Modeling for Analysis of  
<sub>2</sub> Ablating Hypersonic Thermal Protection Systems

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### Abstract

<sub>6</sub> This work presents a *physics-infused reduced-order modeling* (PIROM) framework  
<sub>7</sub> towards the design, analysis, and optimization of non-decomposing ablating hyper-  
<sub>8</sub> sonic thermal protection systems (TPS). It is demonstrated via the modeling of trans-  
<sub>9</sub>ient thermo-ablative behavior of non-decomposing multi-layered hypersonic TPS. The  
<sub>10</sub> PIROM architecture integrates a reduced-physics backbone, based on the lumped-  
<sub>11</sub> capacitance model (LCM), with data-driven correction dynamics formulated via a  
<sub>12</sub> coarse-graining approach rooted in the Mori-Zwanzig formalism. The LCM is coupled  
<sub>13</sub> to a surface velocity model (SVM) to capture the recession of the ablating TPS as a  
<sub>14</sub> function of the surface temperature. While the LCM and SVM capture the dominant  
<sub>15</sub> physics of the ablating TPS response, the correction terms compensate for residual  
<sub>16</sub> dynamics arising from higher-order non-linear interactions and heterogeneities across  
<sub>17</sub> material layers. The PIROM consistently achieves errors below 1% for a wide range of  
<sub>18</sub> extrapolative settings of design parameters involving time-and-space varying boundary  
<sub>19</sub> conditions and SVM models, and improves by  $x\%$  over the LCM alone. Moreover, the  
<sub>20</sub> PIROM delivers online evaluations that are two orders of magnitude faster than the  
<sub>21</sub> full-order model (FOM). These results demonstrate that PIRO effectively reconciles  
<sub>22</sub> the trade-offs between accuracy, generalizability, and efficiency, providing a promising  
<sub>23</sub> framework for optimizing multi-physical dynamical systems, such as TPS under diverse  
<sub>24</sub> operating conditions.

<sub>25</sub>

## 1 Introduction

<sub>26</sub> At hypersonic speeds, aerospace vehicles experience extreme aero-thermo-dynamic environ-  
<sub>27</sub> ments that require specialized thermal protection systems (TPS) to shield internal sub-  
<sub>28</sub> structures, electronics, and possibly crew members from the intense aerodynamic heating.

29 The TPS is often composed of ablating materials – a high-temperature capable fibrous  
30 material injected with a resin that fills the pore network and strengthens the composite  
31 [Amar2016](#). The TPS design promotes the exchange of mass through thermal and  
32 chemical reactions (i.e., pyrolysis), effectively mitigating heat transfer to the sub-structures.

33 As a result, accurate prediction for the ablating TPS response under extreme hyper-  
34 sonic heating becomes fundamental to ensuring survivability, performance, and safety of  
35 hypersonic vehicles. Not only is it necessary to assess the performance of the thermal man-  
36 agement systems, but also the shape changes of the vehicle’s outer surface induced by the  
37 ablating material, and its impact on the aerodynamics, structural integrity, and controlla-  
38 bility. Nonetheless, high-fidelity simulations of ablating TPS remains a formidable challenge  
39 both theoretically and computationally.

40 Unfortunately, high-fidelity simulations of ablating TPS remains a formidable challenge  
41 both theoretically and computationally.

42 On the theoretical side, the thermo-chemical reactions, coupled with the irregular pore  
43 network structure, translate into simplifying assumptions to reduce non-linearities, and make  
44 the resulting equations more amenable for engineering application and design analysis [x](#).  
45 For instance, one of the most notable codes is the one-dimensional [CMA](#) code that was  
46 developed by Aerotherm Corporation in the 1960s [Howard2015](#). Despite its practical use  
47 in...

48 Another example is the CHarring Ablator Response (CHAR) ablation code, which ignores  
49 elemental decompositions of the pyrolyzing gases, assumes the gases to be a mixture of perfect  
50 gases in thermal equilibrium, and assumes no reaction or condensation with the porous  
51 network [1].

52 [theoretically:](#)

53 [computationally:](#)

## 54 2 Modeling of Ablating Thermal Protection Systems

55 This section presents the problem of modeling a non-decomposing ablating TPS subjected to  
56 extreme hypersonic heating. Two different but mathematically connected solution strategies  
57 are provided: (1) a high-fidelity full-order model (FOM) based on a finite element method  
58 (FEM), and (2) a low-fidelity reduced-physics model (RPM) based on a lumped capacitance  
59 model (LCM) and a one-dimensional surface velocity model (SVM). The FOM is compu-  
60 tationally expensive but provides the highest fidelity, while the RPM is computationally  
61 efficeint but has low predictive fidelity; both models are amenable to high-dimensional de-  
62 sign variables. The RPM is used in the subsequent sections for deriving the PIROM.

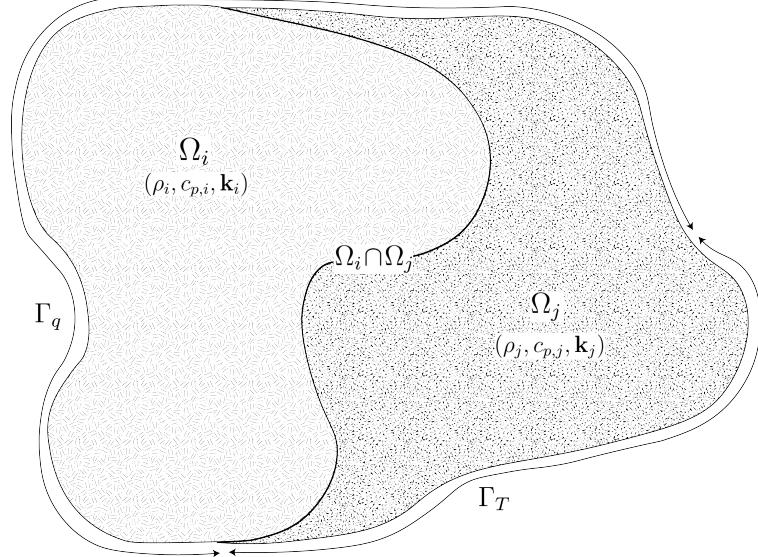


Figure 1: General domain  $\Omega$  with prescribed Neumann and Dirichlet boundary conditions on  $\Gamma_q$  and  $\Gamma_T$ . Mesh displacement  $w(x, t)$  occurs on the  $\Gamma_q$  boundary.

## 63 2.1 Governing Equations

64 The multi-physics for a non-decomposing ablating TPS involves the *energy equation* which  
 65 models the transient heat conduction inside the TPS, and the *pseudo-elasticity equation*,  
 66 which models the mesh motion due to surface recession. The governing PDEs for the ablating  
 67 TPS are summarized in this section.

### 68 2.1.1 Energy Equation

69 Consider a generic domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ , illustrated in Fig. 1. Let  $\partial\Omega = \Gamma_q \cup \Gamma_T$  and  
 70  $\Gamma_q \cap \Gamma_T = \emptyset$ , where a Neumann  $q_b(x, t)$  boundary condition is prescribed on the  $\Gamma_q$  boundary,  
 71 and represents the surface exposed to the hypersonic boundary layer. The Dirichlet  $T_b(x, t)$   
 72 boundary condition is prescribed on the boundary  $\Gamma_T$ . The TPS is divided into  $N$  non-  
 73 overlapping components  $\{\Omega_i\}_{i=1}^N$ , as illustrated in Fig. 1 for  $N = 2$ . The  $i$ -th component  $\Omega_i$   
 74 is associated with material properties  $(\rho_i, c_{p,i}, \mathbf{k}_i)$ , that are assumed to be continuous within  
 75 one component, and can be discontinuous across two neighboring components.

76 The transient heat conduction is described by the energy equation,

$$\rho c_p \left( \frac{\partial T}{\partial t} + \tilde{\mathbf{v}}(x, t) \cdot \nabla T \right) - \nabla \cdot (\mathbf{k} \nabla T) = 0, \quad x \in \Omega \quad (1a)$$

$$-\mathbf{k} \nabla T \cdot \mathbf{n} = q_b(x, t), \quad x \in \Gamma_q \quad (1b)$$

$$T(x, t) = T_b(x, t), \quad x \in \Gamma_T \quad (1c)$$

$$T(x, 0) = T_0(x), \quad x \in \Omega \quad (1d)$$

77 where the density  $\rho$  is constant, while the heat capacity  $c_p$  and thermal conductivity  $\mathbf{k} \in \mathbb{R}^{d \times d}$   
78 may depend on temperature. In the order they appear, the terms in eq. (1a) include, the  
79 unsteady energy storage, heat conduction, temperature advection due to mesh motion, and  
80 source terms due to boundary conditions. The boundary conditions for the energy equation  
81 includes Neumann eq. (1b) and Dirichlet eq. (1c) on  $\Gamma_T$ .

82 An Abirtrary Lagrangian-Eulerian (ALE) description is used to account for mesh motion  
83 due to surface recession, where  $\tilde{\mathbf{v}}(x, t)$  is the relative velocity of the material with respect to  
84 the mesh,

$$\tilde{\mathbf{v}}(x, t) = \mathbf{v}_s(x, t) - \mathbf{v}_m(x, t) \quad (2)$$

85 where  $\mathbf{v}_s(x, t)$  and  $\mathbf{v}_m(x, t)$  are the physical material velocity and mesh velocity, respectively.  
86 In this work, the physical material velocity is assumed to be zero, i.e.,  $\mathbf{v}_s(x, t) = \mathbf{0}$ , and thus  
87 the relative velocity is simply the negative of the mesh velocity,  $\tilde{\mathbf{v}}(x, t) = -\mathbf{v}_m(x, t)$ .

### 88 2.1.2 Pseudo-Elasticity Equation

89 The mesh motion is described by the steady-state pseudo-elasticity equation without body  
90 forces,

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{w}) = \mathbf{0}, \quad \forall t \in \mathcal{T}, \quad \mathbf{x} \in \Omega \quad (3a)$$

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{w}_q(\mathbf{x}, t), \quad \forall t \in \mathcal{T}, \quad \mathbf{x} \in \Gamma_q \quad (3b)$$

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{0}, \quad \forall t \in \mathcal{T}, \quad \mathbf{x} \notin \Gamma_q \quad (3c)$$

$$\mathbf{w}(\mathbf{x}, 0) = \mathbf{0}, \quad \forall \mathbf{x} \in \Omega \quad (3d)$$

91 where the stress tensor  $\boldsymbol{\sigma}$  is related to the strain tensor  $\boldsymbol{\epsilon}(\mathbf{w})$  through Hooke's law,

$$\boldsymbol{\sigma}(\mathbf{w}) = \mathbb{D} : \boldsymbol{\epsilon}(\mathbf{w})$$

92 where  $\mathbb{D}$  is the fourth-order positive definite elasticity tensor, and “ $:$ ” is the double con-  
93 traction of the full-order tensor  $\mathbb{D}$  with the second-order tensor  $\boldsymbol{\epsilon}$ . The elasticity tensor

94 ordinarily possess a number of symmetries, effectively reducing the number of components  
 95 that describe it [2]. The symmetric strain tensor  $\epsilon$  measures the deformation of the mesh  
 96 due to displacements  $\mathbf{w}(x, t)$ , and is defined as,

$$\epsilon(\mathbf{w}) = \frac{1}{2} (\nabla \mathbf{w} + \nabla \mathbf{w}^\top)$$

97 The “material” properties for the mesh are chosen to tailor the mesh deformation, and need  
 98 not represent the actual material being modeled [1].

99 For the pseudo-elasticity equations, the boundary conditions include prescribed displace-  
 100 ments  $\mathbf{w}_q(x, t)$  on the heated boundary  $\Gamma_q$  in eq. (3b), and zero displacements on the unheated  
 101 boundaries in eq. (3c). The initial condition for the mesh displacements is zero in eq. (3d).  
 102 Particularly, the surface velocity due to the ablating material is a function of the surface  
 103 temperature  $T_q(x, t)$  for  $x \in \Gamma_q$  on the heated boundary. For the  $i$ -th material component,  
 104 the mesh velocity on the heated boundary is imposed based on the following relation,

$$\hat{\mathbf{n}} \cdot \mathbf{v}_m(x, t) = f(T_q(x, t)), \quad x \in \Gamma_q \quad (4)$$

105 where  $\hat{\mathbf{n}}$  is the unit normal vector on the heated boundary  $\Gamma_q$ , and  $f$  is a function obtained  
 106 from tabulated data for the material, commonly referred to as a B’ table [1]. The B’ table  
 107 provides a model for the recession velocity as a function of the surface temperature, and is  
 108 pre-computed based on high-fidelity simulations of the ablation process for a one-dimensional  
 109 slab of the material, and is independent of the TPS geometry. Provided the surface velocity,  
 110 the boundary condition in eq. (5) for the mesh displacements are computed by integrating  
 111 the surface velocity over time,

$$\mathbf{w}_q(x, t) = \int_0^t \mathbf{v}_m(x, \tau) d\tau \quad (5)$$

## 112 2.2 Full-Order Model: Finite-Element Method

113 To obtain the full-order numerical solution, the *energy equation* is spatially discretized using  
 114 variational principles of the Discontinuous Galerkin (DG) method [3]. Note that the choice  
 115 of DG approach is mainly for theoretical convenience, and is exclusively performed on the  
 116 energy equation, as it is the surface temperature that drives the ablation process. The  
 117 equivalence between DG and FEM is noted upon their convergence. For the *pseudo-elasticity*  
 118 *equation* standard FEM is used to compute the mesh displacements based on the surface  
 119 temperature provided by the DG solution of the energy equation [2].

120 Consider a conforming mesh partition domain, where each element belongs to one and

<sup>121</sup> only one component. Denote the collection of all  $M$  elements as  $\{E_i\}_{i=1}^M$ . In an element  $E_i$ ,  
<sup>122</sup> its shared boundaries with another element  $E_j$ , Neumann BC, and Dirichlet BC are denoted  
<sup>123</sup> as  $e_{ij}$ ,  $e_{iq}$ , and  $e_{iT}$ , respectively. Lastly,  $|e|$  denotes the length ( $n_d = 2$ ) or area ( $n_d = 3$ ) of a  
<sup>124</sup> component boundary  $e$ .

<sup>125</sup> For the  $i$ -th element, use a set of  $P$  trial functions, such as polynomials, to represent the  
<sup>126</sup> temperature distribution,

$$T_i(x, t) = \sum_{l=1}^P \phi_l^i(x) u_l^i(t) \equiv \boldsymbol{\phi}_i^\top(x) \mathbf{u}_i(t), \quad i = 1, 2, \dots, M \quad (6)$$

<sup>127</sup> Without loss of generality, the trial functions are assumed to be orthogonal, so that,

$$\int_{E_i} \phi_l^i(x) \phi_k^i(x) d\Omega = 0, \quad l \neq k$$

<sup>128</sup> where  $\delta_{lk}$  is the Kronecker delta function. Furthermore, for simplicity, choose  $\phi_1^i = 1$ . Thus,  
<sup>129</sup> by orthogonality,

$$\int_{E_i} \phi_1^i(x) d\Omega = |E_i|, \quad \int_{E_i} \phi_k^i(x) d\Omega = 0, \quad k = 2, 3, \dots, P$$

<sup>130</sup> Under the choice of basis functions,  $u_1^i$  is simply the average temperature of element  $E_i$ ,  
<sup>131</sup> denoted as  $\bar{u}_i$ .

<sup>132</sup> By standard variational processes, e.g., [3], the element-wise governing equation is de-  
<sup>133</sup> noted as,

$$\mathbf{A}_i \dot{\mathbf{u}}_i = (\mathbf{B}_i + \mathbf{C}_i) \mathbf{u}_i + \sum_{j \in \mathcal{N}_i \cup \{T_b\}} (\mathbf{B}_{ij}^i \mathbf{u}_i + \mathbf{B}_{ij}^j \mathbf{u}_j) + \mathbf{f}_i(t), \quad \text{for } i = 1, 2, \dots, M \quad (7)$$

<sup>134</sup> which is collected as the following ODE for the all the elements in the mesh,

$$\mathbf{A}(\mathbf{u}) \dot{\mathbf{u}} = [\mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u})] \mathbf{u} + \mathbf{f}(t) \quad (8)$$

<sup>135</sup> where  $\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M]^\top \in \mathbb{R}^{MP}$  includes all the DG variables,  $\mathbf{f} \in \mathbb{R}^{MP}$  is the exter-  
<sup>136</sup> nal forcing, and the system matrices  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are the matrices due to heat capacity,  
<sup>137</sup> heat conduction, and temperature advection due to mesh motion, respectively. A detailed  
<sup>138</sup> derivation of eqs. (7) and (8) and their matrices is provided in Appendix A.

<sup>139</sup> **2.3 Reduced-Physics Model**

<sup>140</sup> The RPM for predicting the response of the ablating TPS consists of two components: (1)  
<sup>141</sup> the *lumped-capacitance model* (LCM), and (2) the *surface velocity model* (SVM). The LCM is  
<sup>142</sup> described as a first-order system of ODEs for predicting the average temperatures inside the  
<sup>143</sup> components of the TPS, and provides a low-fidelity (under estimate) for the component's  
<sup>144</sup> surface temperature. The SVM provides a relation between the surface temperature and  
<sup>145</sup> the surface recession velocity based on pre-computed B' tables for the material, enabling the  
<sup>146</sup> computation of one-dimensional surface displacements. The LCM and SVM are combined to  
<sup>147</sup> define the RPM, providing low-fidelity estimates for the temperatures and surface recession  
<sup>148</sup> of the ablating TPS.

<sup>149</sup> **2.3.1 Lumped Capacitance Model**

<sup>150</sup> A general form of the LCM is provided in this section; details regarding the derivation for  
<sup>151</sup> the four-component TPS in Fig. 2 are provided in Appendix A. The LCM is a classical  
<sup>152</sup> physics-based low-order model for predicting the temporal variation of average temperature  
<sup>153</sup> in multiple interconnected components [4]. The LCM is derived at the component level from  
<sup>154</sup> a point of view of energy conservation, and leads to the following system of ODEs for the  
<sup>155</sup> average temperatures on the components,

$$\bar{\mathbf{A}}\dot{\bar{\mathbf{u}}} = \bar{\mathbf{B}}\bar{\mathbf{u}} + \bar{\mathbf{f}}(t) \quad (9)$$

<sup>156</sup> Where the states and inputs,

$$\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N]^\top \in \mathbb{R}^N, \quad \bar{\mathbf{f}} = [\bar{f}_1, \bar{f}_2, \dots, \bar{f}_N]^\top \in \mathbb{R}^N \quad (10)$$

<sup>157</sup> include the average temperatures  $\bar{\mathbf{u}}$  and spatially-integrated inputs  $\bar{\mathbf{f}}$  for the  $N$  components.  
<sup>158</sup> For  $i, j = 1, 2, \dots, N$  the  $(i, j)$ -th elements of the  $\bar{\mathbf{A}} \in \mathbb{R}^{N \times N}$ ,  $\bar{\mathbf{B}} \in \mathbb{R}^{N \times N}$ , and  $\bar{\mathbf{f}} \in \mathbb{R}^N$   
<sup>159</sup> matrices are given by,

$$\bar{A}_i = \begin{cases} \int_{\Omega_i} \rho c_p d\Omega_i, & i = j \\ 0, & i \neq j \end{cases}, \quad \bar{B}_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \bar{B}_{ij}^i, & i = j \\ \bar{B}_{ij}^{(j)}, & i \neq j \end{cases}, \quad (11a)$$

$$\mathbf{f}_i = \begin{cases} |e_{iq}| \bar{q}_i + \frac{|e_{iT}|}{R_i} \bar{T}_i, & i = j \\ 0, & i \neq j \end{cases} \quad (11b)$$

160 where,

$$\bar{q}_i = \frac{1}{|e_{iq}|} \int_{e_{iq}} q_b d e_{iq}, \quad \bar{T}_i = \frac{1}{|e_{iT}|} \int_{e_{iT}} T_b d e_{iT}, \quad \bar{B}_{ij}^i = -\frac{|e_{ij}|}{R_{ij}}, \quad \bar{B}_{ij}^j = \frac{|e_{ij}|}{R_{ij}} \quad (12)$$

161 where  $R_{ij}$  is the equivalent thermal resistance between two neighboring components  $\Omega_i$  and  
162  $\Omega_j$ , and  $R_i$  is the thermal resistance between component  $\Omega_i$  and the Dirichlet boundary.  
163 The thermal resistances are computed based on the geometry and material properties of the  
164 components; details regarding their computation are provided in Appendix A.

### 165 2.3.2 Surface Velocity Model

166 The displacement is assumed to be *one-dimensional* on the heated boundary  $\Gamma_q$ , i.e., the  
167 surface recedes only in the direction of the applied load, and occurs only for  $\tilde{N} \leq N$  compo-  
168 nents. For example, in Fig. 2, the surface displacement on the heated boundary occurs only  
169 in the negative  $y$ -direction for the three components exposed to the hypersonic boundary  
170 layer; the fourth component is the substrate and does not ablate. Displacements along the  
171  $x$  direction is small relative to displacements in the  $y$  direction, and are thus neglected.

172 For the  $i$ -th component, the SVM considered in this work takes the form,

$$\dot{\mathbf{w}} = \boldsymbol{\Xi} \bar{\mathbf{u}} - \tilde{\mathbf{f}} \quad (13)$$

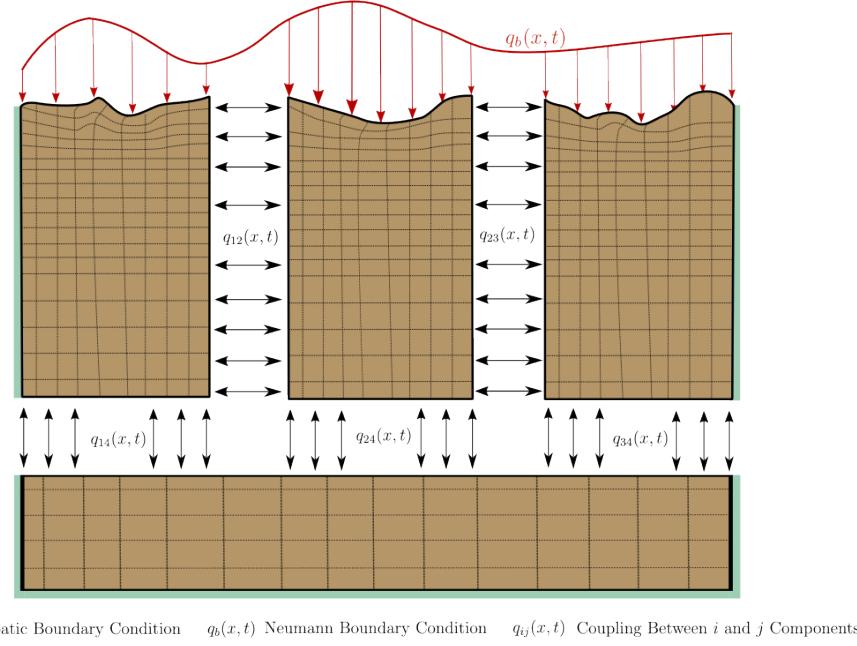
173 where  $\boldsymbol{\Xi} = \text{diag}(\xi_1, \dots, \xi_{\tilde{N}})$  and  $\tilde{\mathbf{f}} = (\xi_1 \bar{u}_{0,1}, \dots, \xi_{\tilde{N}} \bar{u}_{0,\tilde{N}})^\top$ . The constants  $\xi_i$  are small  
174 material-dependent constants, determined from the B' table, and  $\bar{u}_{0,i}$  is the constant initial  
175 temperature of the component. The SVM provides a relation between the surface tempera-  
176 ture and the surface recession velocity based on pre-computed B' tables for the material.

### 177 2.3.3 Coupled Reduced-Physics Model

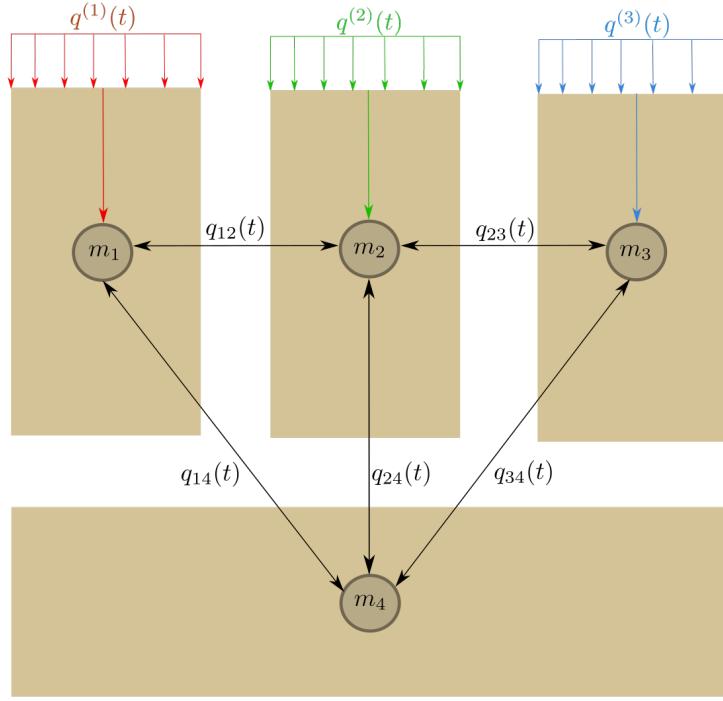
178 The LCM and SVM are combined to define the RPM for predicting the thermo-ablative  
179 response of the TPS under hypersonic boundary layers. Specifically, the RPM is defined as  
180 the LCM as in eq. (9), where the *geometry-dependent matrices*  $\bar{\mathbf{A}}$ ,  $\bar{\mathbf{B}}$ , and  $\tilde{\mathbf{f}}$  are updated  
181 at each time step based on the current surface displacements  $\mathbf{w}$  provided by the SVM. The  
182 RPM is formally stated as,

$$\bar{\mathbf{A}}(\mathbf{s}) \dot{\mathbf{s}} = \bar{\mathbf{B}}(\mathbf{s}) \mathbf{s} + \mathcal{F}(t) \quad (14)$$

183 where the state  $\mathbf{s} = [\bar{\mathbf{u}}, \mathbf{w}]^\top \in \mathbb{R}^{N+\tilde{N}}$  includes the average temperatures for  $N$  components  
184 and the one-dimensional surface displacements for the  $\tilde{N}$  ablating components. The matrices



(a) TPS Decomposition



(b) Lumped Mass Representation

Figure 2: Partition of the TPS into three ablating and one non-ablating components with the corresponding lumped-mass representation.

185 are given as,

$$\mathcal{A} = \begin{bmatrix} \bar{\mathbf{A}}(\mathbf{w}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathcal{B}(\mathbf{s}) = \begin{bmatrix} \bar{\mathbf{B}}(\mathbf{w}) & \mathbf{0} \\ \Xi & \mathbf{0} \end{bmatrix}, \quad \mathcal{F}(t) = \begin{bmatrix} \bar{\mathbf{f}}(t) \\ -\tilde{\mathbf{f}} \end{bmatrix} \quad (15)$$

186 In the matrices  $\bar{\mathbf{A}}$  and  $\bar{\mathbf{B}}$ , the surface displacements  $\mathbf{w}$  are used to define the dimensions for  
187 the  $\Omega_i$  component used in eqs. (11b) and (12), thus effectively coupling the LCM and SVM.

## 188 2.4 Summary of Modeling Approaches

189 The FOM (i.e., DG-FEM) and RPM (i.e., LCM with SVM) are two different but mathemati-  
190 cally connected solution strategies. Specifically, the LCM in eq. (9) not only resembles the  
191 functional form of the DG model in eq. (8), but can be viewed as a special case of the latter,  
192 where the mesh partition is extremely coarse, and the trial and test functions are piece-wise  
193 constants. This removes all spatial variations within each component, and neglects advection  
194 effects due to mesh motion.

195 For example, consider the case where each component  $\Omega_i$  is treated as one single element,  
196 and each element employs one constant basis function  $\phi_i = 1$ . The element-wise DG model  
197 in eq. (7) simplifies into a scalar ODE,

$$\mathbf{A}^{(i)} = \bar{A}_i, \quad \mathbf{C}^{(i)} = 0, \quad \mathbf{B}_{ij}^{(i)} = -\sigma|e_{ij}|, \quad \mathbf{B}_{ij}^{(j)} = \sigma|e_{ij}|, \quad \mathbf{f}_i = |e_{iq}|\bar{q}_i + \sigma|e_{iT}|\bar{T}_i \quad (16)$$

198 Clearly, the LCM is a coarse zeroth-order DG model with the inverse of thermal resistance  
199 chosen as the element-wise penalty factors. Or conversely, the DG model is a refined version  
200 of LCM via  $hp$ -adaptation.

201 The FOM and RPM represent two extremes in the modeling fidelity and computational  
202 cost spectrum. On one hand, the FOM is the most accurate but computationally expensive  
203 to evaluate due to the fine mesh discretizations for both the temperature and displacement  
204 fields, leading to possibly millions of state variables. On the other hand, the RPM considers  
205 only the average temperature of the material from which one-dimensional surface displace-  
206 ments are computed. This considerably reduces the computational cost, but sacrifices local  
207 temperature information that are critical to properly capture higher-order effects due to  
208 mesh motion and thermal gradients within each component. Thus, neither the FOM nor  
209 the RPM is an universal approach for real-world analysis, design, and optimization tasks for  
210 ablating TPS, where thousands of high-fidelity model evaluations may be necessary. This  
211 issue motivates the development of the PIROM, which can achieve the fidelity of FOM at  
212 a computational cost close to the RPM, while maintaining the generalizability to model

213 parameters.

## 214 3 Physics-Infused Reduced-Order Modeling

215 The formulation of PIROM for ablating TPS starts by connecting the FOM, i.e., the DG-  
216 FEM, and the RPM, i.e., the LCM, via a coarse-graining procedure. This procedure pin-  
217 points the missing dynamics in the LCM when compared to DG-FEM. Subsequently, the  
218 Mori-Zwanzig (MZ) formalism is employed to determine the model form for the missing dy-  
219 namics in PIROM. Lastly, the data-driven identification of the missing dynamics in PIROM  
220 is presented.

### 221 3.1 Deriving the Reduced-Physics Model via Coarse-Graining

222 The subsequent coarse-graining formulation is performed on the DG-FEM in eq. (8) to derive  
223 the LCM in eq. (9). This process constraints the trial function space of a full-order DG model  
224 to a subset of piece-wise constants, so that the variables  $\mathbf{u}$ , matrices  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , and forcing  
225 vector  $\mathbf{f}$  are all approximated using a single state associated to the average temperature.  
226 Note that the coarse-graining is exclusively performed on the thermal dynamics, as it is  
227 the surface temperature that drives the one-dimensional recession via the SVM. Hence, the  
228 coarse-graining of the mesh dynamics is not included in the following procedure.

#### 229 3.1.1 Coarse-Graining of States

230 Consider a DG model as in eq. (8) for  $M$  elements and an LCM as in eq. (9) for  $N$  components;  
231 clearly  $M \gg N$ . Let  $\mathcal{V}_j = \{i | E_i \in \Omega_j\}$  be the indices of the elements belonging to the  $j$ -th  
232 component, so  $E_i \in \Omega_j$  for all  $i \in \mathcal{V}_j$ . The number of elements in the  $j$ -th component is  $|\mathcal{V}_j|$ .  
233 The average temperature on  $\Omega_j$  is,

$$\bar{u}_j = \frac{1}{|\Omega_j|} \sum_{i \in \mathcal{V}_j} \int_{E^{(i)}} \boldsymbol{\phi}^{(i)}(x)^T \mathbf{u}^{(i)} d\Omega = \frac{1}{|\Omega_j|} \sum_{i \in \mathcal{V}_j} |E_i| \boldsymbol{\varphi}_i^{j\top} \mathbf{u}^{(i)}, \quad j = 1, 2, \dots, N \quad (17)$$

234 where  $|\Omega_j|$  and  $|E_i|$  denote the area ( $d = 2$ ) or volume ( $d = 3$ ) of component  $j$  and element  
235  $i$ , respectively. The orthogonal basis functions are defined as  $\boldsymbol{\varphi}_i^{j\top} = [1, 0, \dots, 0]^\top \in \mathbb{R}^P$ .

236 Conversely, given the average temperatures of the  $N$  components,  $\bar{\mathbf{u}}$ , the states of an  
237 arbitrary element  $E_i$  is written as,

$$\mathbf{u}^{(i)} = \sum_{k=1}^N \boldsymbol{\varphi}_i^k \bar{u}_k + \delta \mathbf{u}^{(i)}, \quad i = 1, 2, \dots, M \quad (18)$$

<sup>238</sup> where  $\varphi_i^k = 0$  if  $i \notin \mathcal{V}_k$ , and  $\delta\mathbf{u}^{(i)}$  represents the deviation from the average temperature and  
<sup>239</sup> satisfies the orthogonality condition  $\varphi_i^{k\top} \delta\mathbf{u}^{(i)} = 0$  for all  $k$ .

<sup>240</sup> Equations eqs. (17) and (18) are combined and written in matrix form as,

$$\bar{\mathbf{u}} = \Phi^+ \mathbf{u}, \quad \mathbf{u} = \Phi \mathbf{u} + \delta\mathbf{u} \quad (19)$$

<sup>241</sup> where  $\Phi \in \mathbb{R}^{MP \times N}$  is a matrix of  $M \times N$  blocks, with the  $(i, j)$ -th block as  $\varphi_i^j$ ,  $\Phi^+ \in \mathbb{R}^{N \times MP}$   
<sup>242</sup> is the left inverse of  $\Phi$ , with the  $(i, j)$ -th block as  $\varphi_i^{j+} = \frac{|E_i|}{|\Omega_j|} \varphi_i^{j\top}$ , and  $\delta\mathbf{u}$  is the collection of  
<sup>243</sup> deviations. By their definitions,  $\Phi^+ \Phi = \mathbf{I}$  and  $\Phi^+ \delta\mathbf{u} = \mathbf{0}$ .

### <sup>244</sup> 3.1.2 Coarse-Graining of Dynamics

<sup>245</sup> Next, consider a function of states in the form of  $\mathbf{M}(\mathbf{u})\mathbf{g}(\mathbf{u})$ , where  $\mathbf{g} : \mathbb{R}^{MP} \rightarrow \mathbb{R}^{MP}$   
<sup>246</sup> is a vector-valued function, and  $\mathbf{M} : \mathbb{R}^{MP} \rightarrow \mathbb{R}^{p \times MP}$  is a matrix-valued function with an  
<sup>247</sup> arbitrary dimension  $p$ . Define the projection matrix  $\mathbf{P} = \Phi \Phi^+$  and the projection operator  
<sup>248</sup>  $\mathcal{P}$  as,

$$\begin{aligned} \mathcal{P} [\mathbf{M}(\mathbf{u})\mathbf{g}(\mathbf{u})] &= \mathbf{M}(\mathbf{P}\mathbf{u})\mathbf{g}(\mathbf{P}\mathbf{u}) \\ &= \mathbf{M}(\Phi\bar{\mathbf{u}})\mathbf{g}(\Phi\bar{\mathbf{u}}) \end{aligned} \quad (20)$$

<sup>249</sup> so that the resulting function depends only on the average temperatures  $\bar{\mathbf{u}}$ . Correspondingly,  
<sup>250</sup> the residual operator  $\mathcal{Q} = \mathcal{I} - \mathcal{P}$ , and  $\mathcal{Q}[\mathbf{M}(\mathbf{u})\mathbf{g}(\mathbf{u})] = \mathbf{M}(\mathbf{u})\mathbf{g}(\mathbf{u}) - \mathbf{M}(\Phi\bar{\mathbf{u}})\mathbf{g}(\Phi\bar{\mathbf{u}})$ . When  
<sup>251</sup> the function is not separable, the projection operator is simply defined as  $\mathcal{P}[\mathbf{g}(\mathbf{u})] = \mathbf{g}(\mathbf{P}\mathbf{u})$ .

<sup>252</sup> Subsequently, the operators defined above are applied to coarse-grain the dynamics. First,  
<sup>253</sup> write the DG-FEM in eq. (8) as,

$$\dot{\mathbf{u}} = \mathbf{A}(\mathbf{u})^{-1} \mathbf{B}(\mathbf{u}) \mathbf{u} + \mathbf{A}(\mathbf{u})^{-1} \mathbf{C}(\mathbf{u}) \mathbf{u} + \mathbf{A}(\mathbf{u})^{-1} \mathbf{f}(t) \quad (21)$$

<sup>254</sup> and multiply both sides by  $\Phi^+$  to obtain,

$$\Phi^+ \dot{\mathbf{u}} = \Phi^+ (\Phi \dot{\bar{\mathbf{u}}} + \delta \dot{\mathbf{u}}) = \dot{\bar{\mathbf{u}}} = \Phi^+ \mathbf{r}(\mathbf{u}, t) \quad (22)$$

<sup>255</sup> Apply the projection operator  $\mathcal{P}$  and the residual operator  $\mathcal{Q}$  to the right-hand side to obtain,

$$\dot{\bar{\mathbf{u}}} = \mathcal{P} [\Phi^+ \mathbf{r}(\mathbf{u}, t)] + \mathcal{Q} [\Phi^+ \mathbf{r}(\mathbf{u}, t)] \equiv \mathbf{r}^{(1)}(\mathbf{u}, t) + \mathbf{r}^{(2)}(\mathbf{u}, t) \quad (23)$$

<sup>256</sup> where  $\mathbf{r}^{(1)}(\mathbf{u}, t)$  is resolved dynamics that depends on  $\bar{\mathbf{u}}$  only, and  $\mathbf{r}^{(2)}(\mathbf{u}, t)$  is the un-resolved  
<sup>257</sup> or residual dynamics. Detailed derivations and analysis of  $\mathbf{r}^{(1)}(\mathbf{u}, t)$  and  $\mathbf{r}^{(2)}(\mathbf{u}, t)$  can be

258 found in the Appendix.

259 It follows from Ref. [8] that the resolved dynamics is exactly the LCM, where the ad-  
260 vection term reduces to zero, i.e.,  $\bar{\mathbf{C}}(\bar{\mathbf{u}}) = \mathbf{0}$  as shown in the Appendix. Using the notation  
261 from eq. (9), it follows that,

$$\begin{aligned}\mathbf{r}^{(1)}(\mathbf{u}, t) &= \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{B}}(\bar{\mathbf{u}}) \bar{\mathbf{u}} + \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{C}}(\bar{\mathbf{u}}) \bar{\mathbf{u}} + \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{f}}(\bar{\mathbf{u}}) \\ &= \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{B}}(\bar{\mathbf{u}}) \bar{\mathbf{u}} + \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{f}}(t)\end{aligned}\quad (24)$$

262 where the following relations hold,

$$\bar{\mathbf{A}}(\bar{\mathbf{u}}) = \mathbf{W} (\Phi^+ \mathbf{A} (\Phi \bar{\mathbf{u}})^{-1} \Phi)^{-1} \quad \bar{\mathbf{C}}(\bar{\mathbf{u}}) = \mathbf{0} \quad (25a)$$

$$\bar{\mathbf{B}}(\bar{\mathbf{u}}) = \mathbf{W} \Phi^+ \mathbf{B} (\Phi \bar{\mathbf{u}}) \Phi \quad \bar{\mathbf{f}}(t) = \mathbf{W} \Phi^+ \mathbf{f} \quad (25b)$$

263 where  $\mathbf{W} \in \mathbb{R}^{N \times N}$  is a diagonal matrix with the  $i$ -th element as  $[\mathbf{W}]_{ii} = |\mathcal{V}_k|$  if  $i \in \mathcal{V}_k$ .

264 The examination of the second residual term  $\mathbf{r}^{(2)}(\mathbf{u}, t)$  in eq. (23) is shown in the Appendix,  
265 and demonstrates that the physical sources of missing dynamics in the LCM include: the  
266 approximation of non-uniform temperature within each component as a constant, and the  
267 elimination of the advection term due to coarse-graining. In sum, the above results not  
268 only show that the LCM is a result of coarse-graining of the full-order DG-FEM, but also  
269 reveal the discrepancies between the LCM and the DG-FEM. These discrepancies propagate  
270 into the SVM, which as a result of the averaging in the LCM formulation, under-predicts  
271 the surface recession rates. In the subsequent section, the discrepancies in the LCM are  
272 corrected to formulate the PIROM.

### 273 3.2 Physics-Infusion Via Mori-Zwanzig Formalism

274 The Mori-Zwanzig (MZ) formalism is an operator-projection technique used to derive ROMs  
275 for high-dimensional dynamical systems, especially in statistical mechanics and fluid dynam-  
276 ics [5, 6, 7]. It provides an exact reformulation of a high-dimensional Markovian dynamical  
277 system, into a low-dimensional observable non-Markovian dynamical system. The proposed  
278 ROM is subsequently developed based on the approximation to the non-Markovian term in  
279 the observable dynamics. Particularly, eq. (23) shows that the DG-FEM dynamics can be  
280 decomposed into the resolved dynamics  $\mathbf{r}^{(1)}(\mathbf{u}, t)$  and the orthogonal dynamics  $\mathbf{r}^{(2)}(\mathbf{u}, t)$ , in  
281 the sense of  $\mathcal{P}\mathbf{r}^{(2)} = 0$ . In this case, the MZ formalism can be invoked to express the dynamics

282  $\bar{\mathbf{u}}$  in terms of  $\bar{\mathbf{u}}$  alone as the projected Generalized Langevin Equation (GLE) [5, 6, 7],

$$\dot{\bar{\mathbf{u}}}(t) = \mathbf{r}^{(1)}(\bar{\mathbf{u}}, t) + \int_0^t \tilde{\kappa}(t, s, \bar{\mathbf{u}}) ds \quad (26)$$

283 where the first and second terms are referred to as the Markovian and non-Markovian terms,  
284 respectively. The non-Markovian term accounts for the effects of past un-resolved states on  
285 the current resolved states via a memory kernel  $\tilde{\kappa}(t, s, \bar{\mathbf{u}})$ , which in practice is computationally  
286 expensive to evaluate.

287 **3.2.1 Markovian Reformulation**

288 This section details the formal derivation of the PIROM for the thermal dynamics, based on  
289 approximations to the memory kernel. Specifically, the kernel  $\tilde{\kappa}$  is examined via a leading-  
290 order expansion, based on prior work [9]; this can be viewed as an analog of zeroth-order  
291 holding in linear system theory with a sufficiently small time step. In this case, the memory  
292 kernel is approximated as,

$$\tilde{\kappa}(t, s, \bar{\mathbf{u}}) \approx \mathbf{r}^{(1)}(\bar{\mathbf{u}}, t) \cdot \nabla_{\bar{\mathbf{u}}} \mathbf{r}^{(2)}(\Phi \bar{\mathbf{u}}, t) \quad (27)$$

293 Note that the terms in  $\mathbf{r}^{(1)}$  have a common factor  $\bar{\mathbf{A}}^{-1}$ ; this motivates the following heuristic  
294 modification of the model form in eq. (26),

$$\dot{\bar{\mathbf{u}}} = \mathbf{r}^{(1)}(\bar{\mathbf{u}}, t) + \bar{\mathbf{A}}^{-1}(\bar{\mathbf{u}}) \int_0^t \kappa(t, s, \bar{\mathbf{u}}) ds \quad (28a)$$

$$\bar{\mathbf{A}}(\bar{\mathbf{u}}) \dot{\bar{\mathbf{u}}} = \bar{\mathbf{B}}(\bar{\mathbf{u}}) \bar{\mathbf{u}} + \bar{\mathbf{f}}(t) + \int_0^t \kappa(t, s, \bar{\mathbf{u}}) ds \quad (28b)$$

295 where the original kernel  $\tilde{\kappa}$  is effectively normalized by  $\bar{\mathbf{A}}^{-1}$ . Intuitively, such choice of kernel  
296 reduces its dependency on the averaged material properties, and simplifies the subsequent  
297 design of model form.

298 Subsequently, the hidden states are introduced to “Markovianize” the system eq. (26).  
299 In this manner, eq. (28b) is converted into a pure state-space model, with the functional  
300 form of the LCM retained; since LCM is a physics-based model, then it encodes the phys-  
301 ical information and retains explicit parametric dependence of the problem. Consider the  
302 representation of the kernel as a finite sum of simpler functions, e.g., exponentials,

$$\kappa(t, s, \bar{\mathbf{u}}) = \sum_{j=1}^m \mathcal{K}_j(t, s, \bar{\mathbf{u}}) [\mathbf{p}_j + \mathbf{d}_j(\bar{\mathbf{u}})] \phi_j(s, \bar{\mathbf{u}}) \quad (29)$$

303 where,

$$\mathcal{K}_j(t, s, \bar{\mathbf{u}}) = e^{-\int_s^t (\lambda_j + e_j(\bar{\mathbf{u}})) d\tau}, \quad \phi_j(s, \bar{\mathbf{u}}) = \mathbf{q}_j^\top \bar{\mathbf{u}}(s) + \mathbf{g}_j(\bar{\mathbf{u}})^\top \bar{\mathbf{u}}(s) + \mathbf{r}_j^\top \bar{\mathbf{f}}(s) \quad (30)$$

304 with suitable coefficients  $\mathbf{p}_j, \mathbf{d}_j, \mathbf{q}_j, \mathbf{g}_j, \mathbf{r}_j \in \mathbb{R}^N$  and decay rates  $\lambda_j, e_j(\bar{\mathbf{u}}) > 0$ , that need to  
305 be identified from data.

306 Define the hidden states as,

$$\beta_j(t) = \int_0^t \mathcal{K}_j(s, \bar{\mathbf{u}}) \phi_j(s, \bar{\mathbf{u}}) ds \quad (31)$$

307 then through its differentiation with respect to time,

$$\dot{\beta}_j(t) = -[\lambda_j + e_j(\bar{\mathbf{u}})] \beta_j(t) + \mathbf{q}_j^\top \bar{\mathbf{u}}(t) + \mathbf{g}_j(\bar{\mathbf{u}})^\top \bar{\mathbf{u}}(t) + \mathbf{r}_j^\top \bar{\mathbf{f}}(t) \quad (32)$$

308 and the memory term becomes,

$$\int_0^t \boldsymbol{\kappa}(t, s, \bar{\mathbf{u}}) ds = \sum_{j=1}^m [\mathbf{p}_j + \mathbf{d}_j(\bar{\mathbf{u}})] \beta_j(t) \quad (33)$$

309 Then, eq. (28b) is recast as the extended Markovian system,

$$\bar{\mathbf{A}}(\bar{\mathbf{u}}) \dot{\bar{\mathbf{u}}} = \bar{\mathbf{B}}(\bar{\mathbf{u}}) \bar{\mathbf{u}} + [\mathbf{P} + \mathbf{D}(\bar{\mathbf{u}})] \boldsymbol{\beta} + \bar{\mathbf{f}}(t) \quad (34a)$$

$$\dot{\boldsymbol{\beta}} = [\mathbf{Q} + \mathbf{G}(\bar{\mathbf{u}})] \bar{\mathbf{u}} + [\mathbf{E}(\bar{\mathbf{u}}) - \boldsymbol{\Lambda}] \boldsymbol{\beta} + \mathbf{R} \bar{\mathbf{f}}(t) \quad (34b)$$

310 where the data-driven operators associated to the hidden dynamics are collected as,

$$\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^{m \times m}, \quad \mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m] \in \mathbb{R}^{N \times m} \quad (35a)$$

$$\mathbf{D}(\bar{\mathbf{u}}) = [\mathbf{d}_1(\bar{\mathbf{u}}), \mathbf{d}_2(\bar{\mathbf{u}}), \dots, \mathbf{d}_m(\bar{\mathbf{u}})] \in \mathbb{R}^{N \times m}, \quad \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m] \in \mathbb{R}^{m \times N} \quad (35b)$$

$$\mathbf{G}(\bar{\mathbf{u}}) = [\mathbf{g}_1(\bar{\mathbf{u}}), \mathbf{g}_2(\bar{\mathbf{u}}), \dots, \mathbf{g}_m(\bar{\mathbf{u}})] \in \mathbb{R}^{m \times N}, \quad \mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m] \in \mathbb{R}^{m \times N} \quad (35c)$$

$$\mathbf{E}(\bar{\mathbf{u}}) = \text{diag}(e_1(\bar{\mathbf{u}}), e_2(\bar{\mathbf{u}}), \dots, e_m(\bar{\mathbf{u}})) \in \mathbb{R}^{m \times m} \quad (35d)$$

311 Note that since the hidden states  $\boldsymbol{\beta}$  serve as the memory, their initial conditions are set to  
312 zero, i.e.,  $\boldsymbol{\beta}(t_0) = \mathbf{0}$ , no memory at the beginning. The physics-infused model in eq. (34b)  
313 retains the structure of the LCM, while the hidden states account for missing physics through  
314 corrections to the stiffness and advection matrices, as well as the forcing term.

315 **3.2.2 Coupling with Surface Velocity Model**

316 Provided the physics-infused formulation in eq. (34b), the last step in the formulation is to  
317 couple it with the SVM to account for the ablative surface recession. To this end, consider  
318 the observable of interest as the surface temperature, denoted as  $\mathbf{z} \in \mathbb{R}^{n_z}$ , where  $n_z$  is the  
319 number of monitored surface locations, and collect the resolved and hidden states into one  
320 state vector  $\mathbf{y} = [\bar{\mathbf{u}}, \boldsymbol{\beta}]^T \in \mathbb{R}^{n_y}$  with  $n_y = N + m$ . Define the data-driven observable mapping  
321 operator  $\mathbf{M} \in \mathbb{R}^{n_z \times n_y}$  and define the PIROM's observable as,

$$\mathbf{z} = \mathbf{My} \quad (36)$$

322 The PIROM is coupled to the SVM in eq. (13) by computing the surface recession velocity  
323 based on the PIROM's surface temperature prediction  $\mathbf{z}$ . Specifically, the surface recession  
324 velocity is updated as,

$$\dot{\mathbf{w}} = \mathbf{E}\mathbf{z} - \tilde{\mathbf{f}} \quad (37)$$

325 Thus, the PIROM is formally stated as,

$$\tilde{\mathbf{A}}\dot{\mathbf{y}} = [\tilde{\mathbf{B}} + \tilde{\mathbf{C}}] \mathbf{y} + \mathbf{H}\bar{\mathbf{f}}(t) \quad (38a)$$

$$\mathbf{z} = \mathbf{My} \quad (38b)$$

326 where,

$$\tilde{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}(\bar{\mathbf{u}}) & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \in \mathbb{R}^{n_y \times n_y}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}(\bar{\mathbf{u}}) & \mathbf{P} \\ \mathbf{Q} & -\Lambda \end{bmatrix} \in \mathbb{R}^{n_y \times n_y}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \mathbf{D}(\bar{\mathbf{u}}) \\ \mathbf{G}(\bar{\mathbf{u}}) & \mathbf{E}(\bar{\mathbf{u}}) \end{bmatrix} \in \mathbb{R}^{n_y \times n_y}, \quad (39a)$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{I} \\ \mathbf{R} \end{bmatrix} \in \mathbb{R}^{n_y \times N}, \quad \mathbf{M} \in \mathbb{R}^{n_z \times n_y} \quad (39b)$$

327 In eq. (38), the terms  $\bar{\mathbf{A}}$ ,  $\bar{\mathbf{B}}$ , and  $\bar{\mathbf{f}}$  are the LCM terms. The collection of matrices,

$$\boldsymbol{\Theta} = \{\mathbf{P}, \mathbf{D}(\bar{\mathbf{u}}), \mathbf{G}(\bar{\mathbf{u}}), \Lambda, \mathbf{E}(\bar{\mathbf{u}}), \mathbf{Q}, \mathbf{G}(\bar{\mathbf{u}}), \mathbf{R}, \mathbf{M}\}, \in \mathbb{R}^{n_\theta} \quad (40)$$

328 are learnable parameters to capture the memory effects. Particularly, the matrices  $\mathbf{P}, \Lambda, \mathbf{Q}, \mathbf{R}$   
329 are constants that need to be identified from data, and account for the effects of coarse-  
330 graining on the stiffness and forcing matrices. The matrices  $\mathbf{D}(\bar{\mathbf{u}}), \mathbf{E}(\bar{\mathbf{u}}), \mathbf{G}(\bar{\mathbf{u}})$  are state-  
331 dependent matrices, and account for the effects of coarse-graining on the advection matrix.  
332 Leveraging the DG-FEM formula for the advection matrix in eq. (49c) in the Appendix, and  
333 noting that the mesh displacements are functions of the ablating velocity as in eq. (4), the

<sup>334</sup> state-dependent matrices for the  $i$ -th component are written as,

$$\mathbf{D}^{(i)}(\bar{\mathbf{u}}) \approx f^{(i)}(\bar{u}^{(i)})\mathbf{D}^{(i)}, \quad \mathbf{E}^{(i)}(\bar{\mathbf{u}}) \approx f^{(i)}(\bar{u}^{(i)})\mathbf{E}^{(i)}, \quad \mathbf{G}^{(i)}(\bar{\mathbf{u}}) \approx f^{(i)}(\bar{u}^{(i)})\mathbf{G}^{(i)} \quad (41)$$

<sup>335</sup> where  $f^{(i)}(\bar{u}^{(i)})$  is the surface recession velocity function in ?? for the  $i$ -th component,  
<sup>336</sup> and  $\mathbf{D}^{(i)}, \mathbf{E}^{(i)}, \mathbf{G}^{(i)}$  are constant matrices to be identified from data. In eq. (38),  $\mathbf{M}$  is a  
<sup>337</sup> fully-populated matrix that extracts the observables, i.e., the surface temperatures, from  
<sup>338</sup> the PIROM states  $\mathbf{y}$ . The PIROM incorporates explicit information on the temperature-  
<sup>339</sup> dependent material properties through the LCM matrices, as well as the surface recession  
<sup>340</sup> velocity function through eq. (41). The next step is focused on identifying the unknown  
<sup>341</sup> parameters  $\Theta$  characterizing the hidden dynamics.

### <sup>342</sup> 3.3 Learning the Hidden Dynamics from Data

<sup>343</sup> The learning of the PIROM is achieved through a neural-ODE like approach [Chen2018](#).  
<sup>344</sup> For ease of presentation, consider the following compact form of the PIROM in eq. (38),

$$\mathcal{F}(\dot{\mathbf{y}}, \mathbf{y}; \boldsymbol{\xi}, \Theta) = \mathbf{0} \quad (42)$$

<sup>345</sup> where  $\boldsymbol{\xi}$  defines the parametrization of the problem, i.e., operating conditions, such as the  
<sup>346</sup> BC's, as well as the material properties. Consider a dataset of  $N_s$  high-fidelity trajectories  
<sup>347</sup> of observables over a time interval  $[t_0, t_f]$ ,

$$\mathcal{D} = \left\{ \left( t_k, \mathbf{z}_{\text{HF}}^{(l)}(t_k), \boldsymbol{\xi}^{(l)} \right) \right\}_{l=1}^{N_s}, \quad k = 0, 1, \dots, K \quad (43)$$

<sup>348</sup> The learning problem is formulated as the following differentially-constrained problem,

$$\min_{\Theta} \quad \frac{1}{N_s} \sum_{l=1}^{N_s} \frac{1}{K} \sum_{k=0}^K \left\| \mathbf{z}_{\text{HF}}^{(l)}(t_k) - \mathbf{M}\mathbf{y}^{(l)}(t_k) \right\|_2^2 \quad (44a)$$

$$\text{s.t.} \quad \mathcal{F} \left( \dot{\mathbf{y}}^{(l)}, \mathbf{y}^{(l)}; \boldsymbol{\xi}^{(l)}, \Theta \right) = \mathbf{0}, \quad t \in [t_0, t_f], \quad l = 1, 2, \dots, N_s \quad (44b)$$

$$\mathbf{y}^{(l)}(t_0) = \mathbf{y}_0(\boldsymbol{\xi}^{(l)}), \quad l = 1, 2, \dots, N_s \quad (44c)$$

## <sup>349</sup> 4 Application to Thermal Protection Systems

<sup>350</sup> In this section,

<sub>351</sub> **5 Conclusions**

352 **A Technical Details**

353 This appendix presents the technical details of the PIROM framework applied to the TPS  
354 ablation problem. The first section provides the mathematical details for the definition of  
355 the DG-FEM. The second section follows the projection procedures from Ref. x, and demon-  
356 strates the effects of coarse-graining on the advection matrix. The third section presents the  
357 derivation of the LCM model from an energy-conservation perspective.

358 **A.1 Full-Order Model**

359 To obtain the full-order numerical solution, the governing equation is spatially discretized  
360 using variational principles of Discontinuous Galerkin (DG) to result in a high-dimensional  
361 system of ordinary differential equations (ODEs). The DG-FEM model is written in an  
362 element-wise form, which is beneficial for subsequent derivations of the lower-order models.  
363 Note that the choice of DG approach here is mainly for theoretical convenience in the sub-  
364 sequent coarse-graining formulation. In the numerical results, the full-order TPS ablation  
365 simulations is computed using standard FEM instead, and the equivalence between DG and  
366 standard FEM is noted upon their convergence.

367 **A.1.1 Domain Discretization**

368 Consider a conforming mesh partition of the domain, as shown in Fig. DOMAIN, where each  
369 element belongs to one and only one component. Denote the collection of all  $M$  elements  
370 as  $\{E_i\}_{i=1}^M$ . To ease the description of the DG model, a graph structure is employed. The  
371 elements are treated as vertices, the set of which is denoted  $\mathcal{V} = \{m\}_{m=1}^M$ . Two neighboring  
372 elements,  $E_i$  and  $E_j$ , are connected by an edge  $(i, j)$ , and the shared boundary between them  
373 is denoted  $e_{ij}$ . The collection of all edges are denoted  $\mathcal{E}$ , and  $\mathcal{G}$  is referred to as a graph.  
374 In the graph, the edges are unidirected, meaning if  $(i, j) \in \mathcal{E}$  then  $(j, i) \in \mathcal{E}$ . Furthermore,  
375 denote the neighbors of the  $i$ -th element as  $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$ . Lastly, for the ease of  
376 notation, introduce two special indices:  $T$  for the boundary of an element that overlaps with  
377 the Dirichlet boundary condition, and similarly  $q$  for the Neumann boundary condition.

<sup>378</sup> **A.1.2 Weak Form of Discontinuous Galerkin Method**

<sup>379</sup> Choosing appropriate basis functions  $\phi_k$  and  $\phi_l$  and using the Interior Penalty Galerkin  
<sup>380</sup> (IPG) scheme [3], the variational bilinear form for eq. (1a) is,

$$\sum_{i=1}^M a_{\epsilon,i}(\phi_k, \phi_l) = \sum_{i=1}^M L_i(\phi_k) \quad (45)$$

<sup>381</sup> where  $\epsilon$  is an user-specified parameter and,

$$a_{\epsilon,i}(\phi_k, \phi_l) = \int_{E^{(i)}} \left( \rho c_p \phi_k \frac{\partial \phi_l}{\partial t} + \nabla \phi_k \cdot (\mathbf{k} \nabla \phi_l) - \rho c_p \phi_k \mathbf{v} \cdot \nabla \phi_l \right) dE^{(i)} \quad (46a)$$

$$\begin{aligned} &= - \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \int_{e_{ij}} \{ \mathbf{k} \nabla \phi_k \cdot n \} [\phi_l] de_{ij} + \epsilon \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \int_{e_{ij}} \{ \mathbf{k} \nabla \phi_l \cdot n \} [\phi_k] de_{ij} \\ &\quad + \sigma \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \int_{e_{ij}} [\phi_k] [\phi_l] de_{ij} \end{aligned} \quad (46b)$$

$$L_i(v) = \epsilon \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \int_{e_{ij}} (\mathbf{k} \nabla \phi_l \cdot n) T_b de_{ij} + \int_{e_{iq}} \phi_k q_b de_{iq} + \sigma \int_{e_{iT}} \phi_k T_b de_{iT} \quad (46c)$$

<sup>382</sup> In the bi-linear form above, the notations  $[]$  and  $\{\}$  are respectively the jumps and averages  
<sup>383</sup> at the boundary  $e_{ij}$  share by two elements  $E_i$  and  $E_j$ ,

$$[u] = u|_{E_i} - u|_{E_j}, \quad \{u\} = \frac{1}{2} \left( u|_{E_i} + u|_{E_j} \right), \quad \text{for } x \in e_{ij} = E_i \cap E_j$$

<sup>384</sup> Furthermore, in the bi-linear form, the terms associated with  $\sigma$  are introduced to enforce  
<sup>385</sup> the Dirichlet boundary conditions;  $\sigma$  is a penalty factor whose value can depend on the size  
<sup>386</sup> of an element. Depending on the choice of  $\epsilon$ , the bi-linear form corresponds to symmetric  
<sup>387</sup> IPG ( $\epsilon = -1$ ), non-symmetric IPG ( $\epsilon = 1$ ), and incomplete IPG ( $\epsilon = 0$ ). All these schemes  
<sup>388</sup> are consistent with the original PDE and have similar convergence rate with respect to mesh  
<sup>389</sup> size. In the following derivations, the case  $\epsilon = 0$  is chosen for the sake of simplicity.

<sup>390</sup> **A.1.3 Discontinuous Galerkin Model**

<sup>391</sup> Next, the DG-based model is written in an element-wise form. For the  $i$ -th element, use a  
<sup>392</sup> set of  $P$  trial functions to represent the temperature as in eq. (6). Without loss of generality,  
<sup>393</sup> the trial functions are assumed to be orthogonal, so that  $\int_{E^{(i)}} \phi_k^{(i)}(x) \phi_l^{(i)}(x) dx = |E^{(i)}| \delta_{kl}$ ,  
<sup>394</sup> where  $|E^{(i)}|$  is the area ( $n_d = 2$ ) or volume ( $n_d = 3$ ) of the  $i$ -th element, and  $\delta_{kl}$  is the  
<sup>395</sup> Kronecker delta.

<sup>396</sup> Using test functions same as trial functions, the dynamics  $\mathbf{u}^{(i)}$  is obtained by evaluating

<sup>397</sup> the element-wise bi-linear forms,

$$a_{\epsilon,i}(\phi_k^{(i)}, T^{(i)}) = L_i(\phi_k^{(i)}), \quad k = 1, 2, \dots, P \quad (47)$$

<sup>398</sup> The above procedure yields,

$$\mathbf{A}^{(i)} \dot{\mathbf{u}}^{(i)} = (\mathbf{B}^{(i)} + \mathbf{C}^{(i)}(t)) \mathbf{u}^{(i)} + \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \left( \mathbf{B}_{ij}^{(i)} \mathbf{u}^{(i)} + \mathbf{B}_{ij}^{(j)} \mathbf{u}^{(j)} \right) + \mathbf{f}^{(i)}(t) \quad (48)$$

<sup>399</sup> where for  $k, l = 1, 2, \dots, P$ ,

$$[\mathbf{A}^{(i)}]_{kl} = \int_{E^{(i)}} \rho c_p \phi_k^{(i)} \phi_l^{(i)} dE^{(i)} \quad (49a)$$

$$[\mathbf{B}^{(i)}]_{kl} = - \int_{E^{(i)}} (\nabla \phi_k^{(i)}) \cdot (\mathbf{k} \nabla \phi_l^{(i)}) dE^{(i)} \quad (49b)$$

$$[\mathbf{C}^{(i)}]_{kl} = \int_{E^{(i)}} \rho c_p \phi_k^{(i)} \mathbf{v}^{(i)} \cdot \nabla \phi_l^{(i)} dE^{(i)} \quad (49c)$$

$$[\mathbf{B}_{ij}^{(i)}] = \int_{e_{ij}} \left\{ \mathbf{k} \nabla \phi_k^{(i)} \cdot \hat{n} \right\} \phi_l^{(i)} - \sigma [\phi_k^{(i)}] \phi_l^{(i)} de_{ij} \quad (49d)$$

$$[\mathbf{B}_{ij}^{(j)}] = \int_{e_{ij}} - \left\{ \mathbf{k} \nabla \phi_k^{(i)} \cdot \hat{n} \right\} \phi_l^{(j)} + \sigma [\phi_k^{(i)}] \phi_l^{(j)} de_{ij} \quad (49e)$$

$$[\mathbf{f}^{(i)}]_k = \int_{e_{iq}} \phi_k^{(i)} q_b de_{iq} + \sigma \int_{e_{iT}} \phi_k^{(i)} de_{iT} \quad (49f)$$

<sup>400</sup> The matrices  $\mathbf{A}^{(i)} \in \mathbb{R}^{P \times P}$ ,  $\mathbf{B}^{(i)} \in \mathbb{R}^{P \times P}$ , and  $\mathbf{C}^{(i)} \in \mathbb{R}^{P \times P}$  are respectively the capacitance,  
<sup>401</sup> conductivity, and advection matrices for element  $i$ . These matrices depend on  $\rho$ ,  $c_p$ ,  $\mathbf{k}$ , and  
<sup>402</sup>  $\mathbf{v}$ , and hence can be non-linear functions of  $\mathbf{u}^{(i)}$ . Since the trial functions are orthogonal, if  
<sup>403</sup>  $\rho c_p$  is constant within an element,  $\mathbf{A}^{(i)}$  is diagonal; otherwise,  $\mathbf{A}_i$  is symmetric and positive  
<sup>404</sup> definite as  $\rho c_p > 0$ .

<sup>405</sup> For compactness, the element-wise model in eq. (48) is also written in matrix form,

$$\mathbf{A}(\dot{\mathbf{u}}) = [\mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u})] \mathbf{u} + \mathbf{f}(t) \quad (50)$$

<sup>406</sup> where  $\mathbf{u} = [\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(M)}]^T \in \mathbb{R}^{MP}$  includes all DG variables,  $\mathbf{f} = [\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \dots, \mathbf{f}^{(M)}]^T \in \mathbb{R}^{MP}$ ,  $\mathbf{A}$  and  $\mathbf{C}$  are matrices of  $M$  diagonal blocks whose  $i$ -th blocks are  $\mathbf{A}^{(i)}$  and  $\mathbf{C}^{(i)}$ , and  
<sup>407</sup>  $\mathbf{B}$  is a matrix of  $M \times M$  blocks whose  $(i, j)$ -th block is,  
<sup>408</sup>

$$\mathbf{B}_{ij} = \begin{cases} \mathbf{B}^{(i)} + \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \mathbf{B}_{ij}^{(i)}, & i = j \\ \mathbf{B}_{ij}^{(j)}, & i \neq j \end{cases} \quad (51)$$

<sup>409</sup> The dependency of  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  on  $\mathbf{u}$  is explicitly noted in eq. (50), which is the source of  
<sup>410</sup> non-linearity in the current TPS problem. Moreover, the mesh velocity  $\mathbf{v}$  varies with space  
<sup>411</sup> and time, and thus the advection matrix  $\mathbf{C}$  varies with time as a function of  $q_b$ .

## <sup>412</sup> A.2 Coarse-Graining of Dynamics

<sup>413</sup> The LCM is obtained by coarse-graining the full-order DG-FEM. This coarse-graining proce-  
<sup>414</sup> dure produces resolved  $\mathbf{r}^{(1)}(\mathbf{u}, t)$  and residual  $\mathbf{r}^{(2)}(\mathbf{u}, t)$  dynamics as in eq. (23). This section  
<sup>415</sup> presents the detail derivations and magnitude analysis for the resolved and residual dynam-  
<sup>416</sup> ics.

### <sup>417</sup> A.2.1 Resolved Dynamics

<sup>418</sup> Using eq. (20), the resolved dynamics is computed as follows,

$$\mathbf{r}^{(1)}(\mathbf{u}, t) = \mathcal{P} [\Phi^+ \mathbf{A}(\mathbf{u})^{-1} (\mathbf{B}(\mathbf{u})\mathbf{u} + \mathbf{C}(\mathbf{u})\mathbf{u} + \mathbf{f}(t))] \quad (52a)$$

$$\begin{aligned} &= \Phi^+ \mathbf{A}(\mathbf{Pu})^{-1} \mathbf{PB}(\mathbf{Pu}) \mathbf{Pu} + \Phi^+ \mathbf{A}(\mathbf{Pu})^{-1} \mathbf{PC}(\mathbf{Pu}) \mathbf{Pu} \\ &\quad + \Phi^+ \mathbf{A}(\mathbf{Pu})^{-1} \mathbf{Pf}(t, \mathbf{Pu}) \end{aligned} \quad (52b)$$

$$\begin{aligned} &= \underbrace{\Phi^+ \mathbf{A}(\Phi \bar{\mathbf{u}})^{-1} \Phi}_{\#1} \underbrace{\Phi^+ \mathbf{B}(\Phi \bar{\mathbf{u}}) \Phi \bar{\mathbf{u}}}_{\#2} + \Phi^+ \mathbf{A}(\Phi \bar{\mathbf{u}})^{-1} \Phi \underbrace{\Phi^+ \mathbf{C}(\Phi \bar{\mathbf{u}}) \Phi \bar{\mathbf{u}}}_{\#3} \\ &\quad + \Phi^+ \mathbf{A}(\Phi \bar{\mathbf{u}})^{-1} \Phi \underbrace{\Phi^+ \mathbf{f}(t, \Phi \bar{\mathbf{u}})}_{\#4} \end{aligned} \quad (52c)$$

<sup>419</sup> Detailed derivations for the #1, #2, and #4 terms can be found in Ref. [x](#). The effects of  
<sup>420</sup> coarse-graining on the advection term #3 are analyzed next.

<sup>421</sup> **Term #3** The  $\mathbf{C}(\mathbf{u}) \in \mathbb{R}^{MP \times MP}$  matrix contains  $M$  diagonal of size  $P \times P$ , since the  
<sup>422</sup> basis functions are defined locally on each element. Therefore,  $[\mathbf{C}(\mathbf{u})]_{ij} = \mathbf{0}$  for all  $i \neq j$  with  
<sup>423</sup>  $i, j = 1, 2, \dots, M$ . It follows that for  $k, l = 1, 2, \dots, N$ ,

$$[\Phi^+ \mathbf{C}(t, \Phi \bar{\mathbf{u}}) \Phi]_{kl} = \sum_{i=1}^M \sum_{j=1}^M \boldsymbol{\varphi}_i^{k+} [\mathbf{C}(t, \Phi \bar{\mathbf{u}})]_{ij} \boldsymbol{\varphi}_j^l \quad (53a)$$

$$= \sum_{i=1}^M \boldsymbol{\varphi}_i^{k+} [\mathbf{C}(t, \Phi \bar{\mathbf{u}})]_{ii} \boldsymbol{\varphi}_i^l \quad (53b)$$

$$= \sum_{i \in \mathcal{V}_k} \boldsymbol{\varphi}_i^{k+} [\mathbf{C}(t, \Phi \bar{\mathbf{u}})]_{ii} \boldsymbol{\varphi}_i^l \quad (53c)$$

<sup>424</sup> where in the second row, the fact that  $[\mathbf{C}(\mathbf{u})]_{ij} = 0$  for all  $i \neq j$  is used, and in the last row,  
<sup>425</sup> the fact that  $\varphi_i^{k+} = 0$  for all  $i \notin \mathcal{V}_k$  is used. Now, considering that  $[\mathbf{C}(\mathbf{u})]_{ii}$  has a  $(1, 1)$ -th  
<sup>426</sup> zero element, i.e.,  $[C_{11}(t, \Phi\bar{\mathbf{u}})]_{ii} = 0$ , and that if  $k \neq l$  then  $i \notin \mathcal{V}_l$  and thus  $\varphi_i^l = \mathbf{0}$ , it follows  
<sup>427</sup> that for some index  $i \in \mathcal{V}_k$ ,

$$\varphi_i^{k+} [\mathbf{C}(t, \Phi\bar{\mathbf{u}})]_{ii} \varphi_i^l = \varphi_i^{k+} [\mathbf{C}(t, \Phi\bar{\mathbf{u}})]_{ii} \varphi_i^k = \frac{|E_i|}{|\Omega_k|} [C_{11}(t, \Phi\bar{\mathbf{u}})]_{ii} = 0 \quad (54)$$

<sup>428</sup> The matrix  $[\Phi^+ \mathbf{C}(t, \Phi\bar{\mathbf{u}}) \Phi]_{kl} = 0$  for all  $k, l = 1, 2, \dots, N$ , and thus,

$$\bar{\mathbf{C}}(t, \bar{\mathbf{u}}) = \Phi^+ \mathbf{C}(t, \Phi\bar{\mathbf{u}}) \Phi = \mathbf{0} \quad (55)$$

<sup>429</sup> as indicated by the LCM in eq. (9).

### <sup>430</sup> A.2.2 Magnitude Analysis for Residual Dynamics

<sup>431</sup> Next, the magnitude of the residual dynamics  $\mathbf{r}^{(2)}(\mathbf{u}, t)$  is analyzed to pinpoint the missing  
<sup>432</sup> physics in the LCM. By definition,

$$\mathbf{r}^{(2)}(\mathbf{u}, t) = \dot{\bar{\mathbf{u}}} - \mathbf{r}^{(1)}(\bar{\mathbf{u}}, t) \quad (56a)$$

$$= \Phi^+ \mathbf{r}(\mathbf{u}, t) - \mathbf{r}^{(1)}(\mathbf{u}, t) \quad (56b)$$

$$\begin{aligned} &= \underbrace{\Phi^+ \mathbf{A}(\mathbf{u})^{-1} \mathbf{B}(\mathbf{u}) \mathbf{u} - \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{B}}(\bar{\mathbf{u}}) \bar{\mathbf{u}}}_{\#1} + \underbrace{\Phi^+ \mathbf{A}(\mathbf{u})^{-1} \mathbf{C}(\mathbf{u}) \mathbf{u} - \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{C}}(t, \bar{\mathbf{u}}) \bar{\mathbf{u}}}_{\#2} \\ &\quad + \underbrace{\Phi^+ \mathbf{A}(\mathbf{u})^{-1} \mathbf{f}(t) - \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{f}}(t)}_{\#3} \end{aligned} \quad (56c)$$

<sup>433</sup> The magnitude analysis for terms  $\#1$  and  $\#3$  can be found in Ref. [x](#). The analysis for term  
<sup>434</sup>  $\#2$  is presented next. Let  $\mathbf{D}(\bar{\mathbf{u}}) = \mathbf{A}(\Phi\bar{\mathbf{u}})^{-1} \mathbf{P} \mathbf{C}(t, \Phi\bar{\mathbf{u}})$ , then,

$$\Phi^+ \mathbf{A}(\mathbf{u})^{-1} \mathbf{C}(\mathbf{u}) \mathbf{u} - \bar{\mathbf{A}}(\bar{\mathbf{u}})^{-1} \bar{\mathbf{C}}(t, \bar{\mathbf{u}}) \bar{\mathbf{u}} \quad (57a)$$

$$= \Phi^+ \mathbf{A}(\mathbf{u})^{-1} \mathbf{C}(\mathbf{u}) \mathbf{u} - \Phi^+ \mathbf{A}(\Phi\bar{\mathbf{u}})^{-1} \mathbf{P} \mathbf{C}(t, \Phi\bar{\mathbf{u}}) \Phi \Phi^+ \mathbf{u} \quad (57b)$$

$$= \Phi^+ \mathbf{A}^{-1}(\mathbf{u}) \mathbf{C}(\mathbf{u}) \mathbf{u} - \Phi^+ \mathbf{D}(\bar{\mathbf{u}}) \Phi \Phi^+ \mathbf{u} \quad (57c)$$

$$(57d)$$

<sup>435</sup> where  $\mathbf{P} = \Phi\Phi^+$ . Thus,

$$\|\Phi^+\mathbf{A}(\mathbf{u})^{-1}\mathbf{C}(\mathbf{u})\mathbf{u} - \bar{\mathbf{A}}^{-1}(\bar{\mathbf{u}})\bar{\mathbf{C}}(t, \bar{\mathbf{u}})\bar{\mathbf{u}}\| \quad (58a)$$

$$\leq \|\Phi^+\mathbf{A}^{-1}(\mathbf{u})\mathbf{C}(\mathbf{u})\mathbf{u} - \Phi^+\mathbf{D}(\bar{\mathbf{u}})\mathbf{u}\| + \|\Phi^+\mathbf{D}(\bar{\mathbf{u}})\mathbf{u} - \Phi^+\mathbf{D}(\bar{\mathbf{u}})\Phi\Phi^+\mathbf{u}\| \quad (58b)$$

$$\leq \|\Phi^+\| \underbrace{\|\mathbf{A}^{-1}(\mathbf{u})\mathbf{C}(\mathbf{u})\mathbf{u} - \mathbf{D}(\bar{\mathbf{u}})\mathbf{u}\|}_{\#1} + \|\Phi^+\mathbf{D}(\bar{\mathbf{u}})\| \underbrace{\|\mathbf{u} - \Phi\Phi^+\mathbf{u}\|}_{\#2} \quad (58c)$$

<sup>436</sup> where term  $\#2$  is due to the appriximation of non-uniform temperaeture as constants, and  
<sup>437</sup> term  $\#1$  is the error in the advection dynamics due to coarse-graining.

### <sup>438</sup> A.3 Lumped Capacitance Model

<sup>439</sup> The following assumptions are employed: (1) the temperature in component  $(i)$  is described  
<sup>440</sup> by a scalar time-varying average temperature  $\bar{u}^{(i)}$ , (2) between neighboring components  $(i)$   
<sup>441</sup> and  $(j)$  the heat flux is approximated as,

$$q_{ij} = \frac{\bar{u}^{(j)} - \bar{u}^{(i)}}{R_{ij}} \quad (59)$$

<sup>442</sup> where  $R_{ij}$  is the thermal resistance. Empirically, for a component of isotropic heat conduc-  
<sup>443</sup> tivity  $k$ , length  $\ell$ , and cross-section area  $A$ , the thermal resistance is  $R = \ell/kA$ . Between  
<sup>444</sup> components  $i$  and  $j$ , define  $R_{ij} = R_i + R_j$ . In addition, the heat flux due to Dirichlet  
<sup>445</sup> boundary condition is computed as  $q_{iT} = (T_b - \bar{u}^{(i)})/R_i$ .

<sup>446</sup> At component  $i$ , the dynamics of LCM are given by,

$$\int_{E^{(i)}} \rho c_p \dot{\bar{u}}^{(i)} dE^{(i)} = \left( \sum_{j \in \mathcal{N}_i} \int_{e_{ij}} \frac{\bar{u}^{(j)} - \bar{u}^{(i)}}{R_{ij}} de_{ij} \right) + \int_{e_{iq}} q_b de_{iq} + \int_{e_{iT}} \frac{T_b - \bar{u}^{(i)}}{R_i} de_{iT} \quad (60a)$$

$$\bar{A}^{(i)} \dot{\bar{u}}^{(i)} = \left( \sum_{j \in \mathcal{N}_i} \frac{|e_{ij}|}{R_{ij}} (\bar{u}^{(j)} - \bar{u}^{(i)}) \right) + |e_{iq}| \bar{q}^{(i)} + \frac{|e_{iT}|}{R_i} (\bar{T}^{(i)} - \bar{u}^{(i)}) \quad (60b)$$

$$= \sum_{j \in \mathcal{N}_i} \left( -\frac{|e_{ij}|}{R_{ij}} \bar{u}^{(i)} + \frac{|e_{ij}|}{R_{ij}} \bar{u}^{(j)} \right) + \left( -\frac{|e_{iT}|}{R_i} \bar{u}^{(i)} \right) + \left( |e_{iq}| \bar{q}^{(i)} + \frac{|e_{iT}|}{R_i} \bar{T}^{(i)} \right) \quad (60c)$$

$$= \sum_{j \in \mathcal{N}_i \cup \{T_b\}} \left( \bar{B}_{ij}^{(i)} \bar{u}^{(i)} + \bar{B}_{ij}^{(j)} \bar{u}^{(j)} \right) + \bar{f}^{(i)} \quad (60d)$$

<sup>447</sup> where in eq. (60b)  $|e|$  denotes the length ( $d = 2$ ) or area ( $d = 3$ ) of a component boundary  
<sup>448</sup>  $e$ . The  $\bar{A}^{(i)}$ ,  $\bar{B}_{ij}^{(i)}$ , and  $\bar{B}_{ij}^{(j)}$  quantities are provided in eq. (12).

<sup>449</sup> The lumped-mass representation for the four-component TPS is shown in Fig. 2. Let  $v_i$

represent the area of the  $i$ -th element,  $\overline{\rho c_p}_i$ , the heat capacity evaluated using the average temperature  $\bar{u}^{(i)}$ , and  $1/R_{ij} = 1/R_i(\bar{u}^{(i)}) + 1/R_j(\bar{u}^{(j)})$  the equivalent thermal resistance between elements  $i$  and  $j$ . Leveraging the formulas from eqs. (11b) and (12), the LCM matrices are given by,

$$\bar{\mathbf{A}} = \begin{bmatrix} \overline{\rho c_p}_1 v_1 & 0 & 0 & 0 \\ 0 & \overline{\rho c_p}_2 v_2 & 0 & 0 \\ 0 & 0 & \overline{\rho c_p}_3 v_3 & 0 \\ 0 & 0 & 0 & \overline{\rho c_p}_4 v_4 \end{bmatrix}, \quad (61a)$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \frac{1}{R_{12}} + \frac{1}{R_{14}} & -\frac{1}{R_{12}} & 0 & -\frac{1}{R_{14}} \\ -\frac{1}{R_{12}} & \frac{1}{R_{12}} + \frac{1}{R_{24}} + \frac{1}{R_{23}} & -\frac{1}{R_{23}} & -\frac{1}{R_{24}} \\ 0 & -\frac{1}{R_{32}} & \frac{1}{R_{32}} + \frac{1}{R_{34}} & -\frac{1}{R_{34}} \\ -\frac{1}{R_{14}} & -\frac{1}{R_{24}} & -\frac{1}{R_{34}} & \frac{1}{R_{14}} + \frac{1}{R_{24}} + \frac{1}{R_{34}} \end{bmatrix}, \quad \bar{\mathbf{f}} = \begin{bmatrix} \bar{q}^{(1)} \\ \bar{q}^{(2)} \\ \bar{q}^{(3)} \\ 0 \end{bmatrix} \quad (61b)$$

## References

- [1] Adam J. Amar, A. Brandon Oliver, Benjamin S. Kirk, Giovanni Salazar, and Justin Drob. Overview of the charring ablator response (char) code. In AIAA, editor, *AIAA Aviation Forum*, 2016.
- [2] Guy L. Bergel, Frank B. Beckwith, Gabriel J. de Frias, Kevin M. Manktelow, Mark T. Merewether, Scott T. Miller, Krishen J. Parmar, Timothy S. Shelton, Jesse T. Thomas, Jeremy T. Trageser, Benjamin T. Treweek, Michael V. Veilleux, and Ellen B. Wagman. *Sierra/SolidMechanics 5.6 User's Manual*, 2022.
- [3] Gary Cohen and Sébastien Pernet. *Finite Element and Discontinuous Galerkin Methods for Transient Wave Equations*. Springer Dordrecht, Le Chesnay and Toulouse France, 2018.
- [4] Frank P. Incropera, David P. DeWitt, Theodore L. Bergman, and Adrienne S. Lavine. *Fundamentals of Heat and Mass Transfer*. Wiley, Hoboken, NJ, 7th edition, 2011.
- [5] Eric Parish and Karthik Duraisamy. Non-markovian closure models for large eddy simulations using the mori-zwanzig formalism. *Phys. Rev. Fluids*, 2:014604, Jan 2017.
- [6] Eric Parish and Karthik Duraisamy. A unified framework for multiscale modeling using the mori-zwanzig formalism and the variational multiscale method. *ArXiv*, 12 2017.
- [7] Eric Parish and Karthik Duraisamy. *Coarse-Graining Turbulence Using the Mori-Zwanzig Formalism*. Cambridge University Press, Cambridge, 2025.

- <sup>473</sup> [8] Carlos A. Vargas Venegas, Daning Huang, Patrick Blonigan, and John Tencer. Physics-  
<sup>474</sup> infused reduced-order modeling for analysis of multi-layered hypersonic thermal protec-  
<sup>475</sup> tion systems. *ArXiv*, 2025.
- <sup>476</sup> [9] Yin Yu, John Harlim, Daning Huang, and Yan Li. Learning coarse-grained dynamics on  
<sup>477</sup> graph. *arXiv preprint arXiv:2405.09324*, 2024.