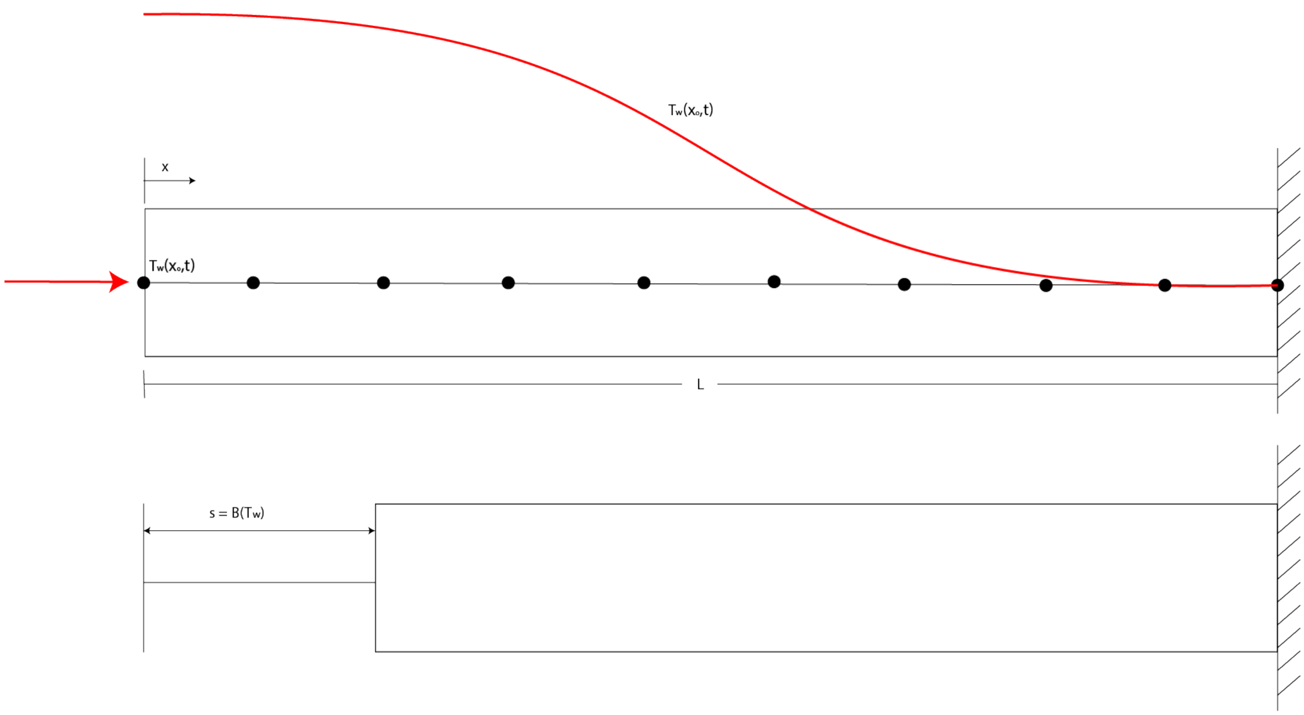
1 Reduced-Physics Model

This section describes the derivation of the RPM model for the ablating TPS surface. The model is composed of two components: (1) a thermal solver to capture the one-dimensional distribution of temperature across the ablating material, and (2) an elasticity solver to capture the one-dimensional mesh motion as a function of the surface temperature. The interface between the thermal solver and the moving mesh is achieved using a B' table, which determines the surface recession velocity as a function of the wall temperature.

1.1 Computational Domain

Consider a 2 mm Titanium slab as shown in Fig. 1.1. The geometry, material properties, and boundary conditions are summarized in Table x. The left surface is exposed to the hypersonic flow (Neumann boundary condition), while the right surface is perfectly insulated (adiabatic boundary condition).



Material	Density, [kg/m^3]	Thermal Conductivity, W/mK	Specific Heat, [J/kgK]
Tungstenn	x	x	x

1.2 Governing Equations

The governing equations for a non-decomposing ablator involves the energy equation with a temperature advection term to account for the moving boundary. An Arbitrary Lagrangian-Eulerian description (ALE) is adopted to incorporate the effects of mesh motion into the energy equation. The ALE approach assumes the computational mesh moves with a velocity $\mathbf{v}(x,t)$ that is different to the material velocity $\mathbf{w}(x,t)$. These effects are taken into

moves the mesh independently of the material movement,

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \left(\mathbf{w}(x,t) - \mathbf{v}(x,t)\right) \cdot \nabla T - \nabla \cdot (\mathbf{k} \nabla T) = 0,\ x \in \Omega \tag{1a}$$

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = 0 \tag{1b}$$

$$-\mathbf{k}\nabla T \cdot \mathbf{n} = q_b(x,t),\ x \in \Gamma_q \tag{1c}$$

$$T(x,0) = T_0(x),\ x \in \Omega \tag{1d}$$

The following simplifications are introduced in the FOM x to aid in the derivation of the RPM,

1. The material properties are independent of temperature.
2. The domain is one dimensional.
3. The spatial discretization is coarse-grained.
4. ...

With these

With the inclusion of the physical assumptions, the FOM in x reduces to the following one-dimensional energy equation with the temperature advection term for the moving mesh,

$$\rho c_p \left(\frac{\partial T}{\partial t} - v(x,t) \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) = 0 \tag{2a}$$

$$\frac{\partial}{\partial x}\left(E \frac{\partial u}{\partial x} \right) = 0 \tag{2b}$$

$$k \frac{\partial T}{\partial x}\bigg|_{x=0} = q(t) \tag{2c}$$

$$k \frac{\partial T}{\partial x}\bigg|_{x=\ell} = 0 \tag{2d}$$

$$u(0,t) = v(t) * (t - t_0) \tag{2e}$$

$$u(\ell,t) = 0 \tag{2f}$$

1.3 Numerical Solution

A numerical solution based on the FEM method is adopted for the governing PDEs.

1.3.1 Elasticity Solver

Assuming the Young's modulus is constant, the PDE simplifies to,

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{3}$$

which has the analytical solution,

$$u(x,t) = a(t)x + b(t) \tag{4}$$

Using the boundary conditions leads to,

$$u(x,t) = u(0,t) * \left(1 - \frac{x}{\ell}\right) \tag{5}$$

The mesh velocity is the time derivative of the displacement,

$$v(x,t) = \frac{\partial u(x,t)}{\partial t} = v(t)\left(1 - \frac{x}{\ell}\right) \tag{6}$$

1.3.2 Thermal Solver

Let $\phi_i^{(e)}(x)$ with $i = 1, 2$ be two linear shape defined over the element $e_i = [x_i, x_{i+1}]$ with length $h_e = x_{i+1} - x_i$,

$$\phi_1^{(e)}(x) = \begin{cases} \frac{x_{i+1}-x}{h_e}, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases}, \quad \phi_2^{(e)}(x) = \begin{cases} \frac{x-x_i}{h_e}, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Letting,

$$x(\xi) = \frac{1-\xi}{2} x_i + \frac{1+\xi}{2} x_{i+1}$$

for $\xi \in [-1, 1]$,

$$\hat{\phi}_1^{(e)}(\xi) = \frac{1-\xi}{2}, \quad \hat{\phi}_2^{(e)}(\xi) = \frac{1+\xi}{2} \tag{8}$$

Multiply through by the test function,

$$\int_\Omega \left[\rho c_p \frac{\partial T}{\partial t} - \rho c_p v(x,t) \frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) \right] \phi_i(x) dx = 0 \tag{9}$$

. . .

$$\int_\Omega \rho c_p \frac{\partial T}{\partial t} \phi_i(x) dx - \int_\Omega \rho c_p v(x,t) \frac{\partial T}{\partial x} \phi_i(x) dx + \int_\Omega k \frac{\partial T}{\partial x} \frac{\partial \phi_i(x)}{\partial x} dx = k \frac{\partial T}{\partial x} \phi_i(x) \bigg|_{\partial\Omega} \tag{10}$$

Perform the finite-element approximation,

$$T(x,t) \approx \sum_j T_j(t) \phi_j(x) \tag{11}$$

and define the matrix elements,

$$M_{ij} = \int_\Omega \rho c_p \phi_i(x) \phi_j(x) dx \tag{12}$$

$$C_{ij}(t) = \int_\Omega \rho c_p v(x,t) \frac{\partial \phi_j}{\partial x} \phi_i(x) dx \tag{13}$$

$$K_{ij} = \int_\Omega k \frac{\partial \phi_i(x)}{\partial x} \frac{\partial \phi_j(x)}{\partial x} dx \tag{14}$$

$$f_i(t) = k \frac{\partial T}{\partial x} \phi_i(x) \bigg|_{\partial\Omega} \tag{15}$$

The time-dependent finite-dimensional ODE system for nodal temperatures $\mathbf{T}(t)$, including the ALE-induced advection effect from mesh motion, is given as,

$$\mathbf{M} \frac{d\mathbf{T}}{dt} + (\mathbf{K} - \mathbf{C}(t)) \mathbf{T} = \mathbf{f}(t) \tag{16}$$

The element-level expressions for the mass, stiffness, advection, and forcing vectors are given as,

$$M_{mn}^{(e)} = \int_{x_i}^{x_{i+1}} \rho c_p \phi_m(x) \phi_n(x) dx = \rho c_p \frac{h_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{17}$$

$$K_{mn}^{(e)} = \int_{x_i}^{x_{i+1}} k \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_n}{\partial x} dx = \frac{k}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{18}$$

$$C_{mn}^{(e)}(t) = \int_{x_i}^{x_{i+1}} \rho c_p v(x,t) \frac{\partial \phi_n(x)}{\partial x} \phi_m(x) dx \tag{19}$$

$$f_1^{(e)}(t) = (q(t), 0)^T \tag{20}$$

1.4 Coarse Graining

$$\mathbf{A}(\mathbf{u})\dot{\mathbf{u}} = (\mathbf{B}(\mathbf{u}) - \mathbf{C}(\mathbf{u}, t))\,\mathbf{u} + \mathbf{f}(t) \tag{21}$$

so that,

$$\dot{\mathbf{u}} = \mathbf{r}(\mathbf{u}, t) = \mathbf{A}(\mathbf{u})^{-1}\left[(\mathbf{B}(u) - \mathbf{C}(\mathbf{u}, t))\,\mathbf{u} + \mathbf{f}(t)\right] \tag{22}$$

The resolved dynamics,

$$\begin{align}
\mathbf{r}^{(1)}(\mathbf{u}, t) &= \mathcal{P}\left[\mathbf{\Phi}^{+}\mathbf{r}(\mathbf{u}, t)\right] \tag{23} \\
&= \mathcal{P}\left[\mathbf{\Phi}^{+}\mathbf{A}^{-1}(\mathbf{u})\mathbf{B}(\mathbf{u})\mathbf{u} - \mathbf{\Phi}^{+}\mathbf{A}^{-1}(\mathbf{u})\mathbf{C}(\mathbf{u}, t)\mathbf{u} + \mathbf{\Phi}^{+}\mathbf{A}^{-1}(\mathbf{u})\mathbf{f}(t)\right] \tag{24}
\end{align}$$

1.5 Numerical Simulation Results