Definitions:

~~Define the parity of an integer n is~~

~~p(n)~~

The quotient $\dfrac{\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}$ will be written as B for brevity.

Let $p(m) = \begin{cases} 0 \text{ if } m \text{ is even} \\ 1 \text{ if } m \text{ is odd} \end{cases}$

Note that $p(p(n)+p(m)) = p(n+m)$.

Let 2^{ℓ} be the largest power of 2 which divides an integer n. Define $f(n)=\ell$.

~~We claim that the answer is even positive integers.~~

~~Let $\frac{\operatorname{lcm}(x,y)}{y}=a$, and $\frac{\operatorname{lcm}(z,y)}{y}=c$.~~

~~WLOG the following four lemmas are stated regarding x and a, but apply equally to z and c.~~

Note that $f(m/n) = f(m) - f(n)$.
If and only if $f(n) = 0$, n is odd and $p(n) = 1$.

We claim that the answer is even positive integers.

Let $\frac{\operatorname{lcm}(x,y)}{y} = a$ and $\frac{\operatorname{lcm}(y,z)}{y} = c$.
WLOG the following three lemmas are stated regarding x and a, but apply equally to z and c.

Lemma 1. If $p(x) = 1$ and $p(y) = 0$, $p(a) = 1$.
Note that, since $f(x) = 0$, $f(\operatorname{lcm}(x,y)) = f(y)$.
Thus $f(a) = f(\operatorname{lcm}(x,y)) - f(y) = f(y) - f(y) = 0$.
Since $f(a) = 0$, $p(a) = 1$. $\square$

Lemma 2. If $p(x) = 0$ and $p(y) = 1$, $p(a) = 0$.
Note that, since $f(y) = 0$, $f(\operatorname{lcm}(x,y)) = f(x)$.
Thus $f(a) = f(x) - f(y) = f(x)$. Since $p(x) = 0$, $f(x)$ is non-zero, thus $f(a)$ is non-zero and $p(a) = 0$. $\square$

~~Lemma 1. If $p(x)=1$ and $p(y)=0$, $p(a)=1$.~~

~~Proof. Note that, since $f(x)=0$, $f(\operatorname{lcm}(x,y))=f(y)$,~~

Lemma 3. If $p(x)=p(y)=1$, $p(a)=1$.

Note that a is always an integer since $\operatorname{lcm}(x,y)$ is a multiple of y. Since $f(x)=f(y)=0$, $f(a)=f(\operatorname{lcm}(x,y))-f(y)=f(\operatorname{lcm}(x,y))=0$, and $p(a)=1$. $\square$

~~Lemma 4~~.

We will first show that B cannot be odd. Consider the following 3 cases:

1) x and z are both odd; $p(x)=p(z)=1$:

- If y is even, then, by Lemma 1, $p(a)=p(c)=1$. Thus, (since $B=y\cdot\frac{a+c}{\operatorname{lcm}(x,z)}$), $p(a+c)=0$ and $p(\operatorname{lcm}(x,z))=1$. Since y is even, B cannot be odd in this case.

- If y is odd, then, by Lemma 3, $p(a)=p(c)=1$. Thus $a+c$ is even, and $\operatorname{lcm}(x,z)$ is odd, so B is even.

2) $p(x)=1$, $p(z)=0$; WLOG this covers the case where $p(x)=0$ and $p(z)=1$.

- y is even — thus by Lemma 1, $p(a)=1$.

Observe that if $f(z) > f(y)$, $p(c)=0$; if $f(z) \le f(y)$, $p(c)=1$.

Let us consider both cases:

- $f(z) > f(y)$: $p(c)=0$, so $p(a+c)=1$.
  $B=(a+c)\cdot\frac{y}{\operatorname{lcm}(x,z)}$; $f(\operatorname{lcm}(x,z)) > f(y)$, ~~[illegible]~~ and since $a+c$ is odd, B is ~~[illegible]~~ not an integer.
- $f(z) \le f(y)$: $p(c)=1$, so $p(a+c)=0$.
  $B=(a+c)\cdot\frac{y}{\operatorname{lcm}(x,z)}$; since $f(z) \le f(y)$,
  $f\left(\frac{y}{\operatorname{lcm}(x,z)}\right) = f(y) - f(\operatorname{lcm}(x,z)) \overset{*}{=} f(y) - f(z) \ge 0.$
  Thus $\frac{y}{\operatorname{lcm}(x,z)}$ is 1, even, or a fraction with an odd denominator (when simplified; since $f(y) \ge f(z)$), and since $a+c$ is even, B in this case cannot be an odd integer.

* since x is odd

- y is odd — By lemma 3, (case 2 cont.)

$p(a)=1$, and by lemma 2, $p(c)=0$.

$B = \frac{y(a+c)}{\operatorname{lcm}(x,z)}$; y is odd, $a+c$ is odd, and $\operatorname{lcm}(x,z)$ is even, so ~~B cannot B~~ B is fractional and is not an odd positive integer.

3) $p(x)=p(z)=0$; x and z are even.

- y is even;

~~o $f(x) = f(z)$~~

o $f(x)=f(z)$

– $f(x)>f(y)$: $p(a)=p(c)=0$.

$f(a) = f(x)-f(y) = f(z)-f(y) = f(c)$

$p\left(\frac{f(a)}{2^{f(a)}}\right) = 1 = p\left(\frac{f(c)}{2^{f(c)}}\right) \Rightarrow f(a+c) > f(a)$

Thus $\frac{a+c}{\operatorname{lcm}(x,z)}$ is even and B cannot be odd.

– $f(x) \leq f(y)$: $f(y/\operatorname{lcm}(x,z)) \geq 0$; B cannot be odd.

o WLOG $f(x) > f(z)$

– $f(x) > f(y)$: $f(\operatorname{lcm}(x,z)) = f(x)$;

$f\left(\frac{y}{\operatorname{lcm}(x,z)}\right) = f(y)-f(x)$ As seen in the last case; $f(a+b) > f(y)-f(x)$, so B cannot be odd.

– $f(x) \le f(y)$: $\frac{y}{\operatorname{lcm}(x,z)}$ is consequently even, and since $a+c$ is an integer, B cannot be odd.

• y is odd:

WLOG, let $f(x) \ge f(z)$. Note that $f(a) = f(x)$, and $f(z) = f(c)$.

~~$f(a+c) = f(x)$ unless $f(x) = f(z)$~~

~~since $f\left(2^{f(x)}\left(\frac{a}{2^{f(x)}} + \frac{c}{2^{f(x)}}\right)\right) = f(x)$, unless $f(a) = f(c)$~~

~~in which case $f(a+c) \ge f(x)+1$.~~

If $\frac{\operatorname{lcm}(x,z)}{2^{f(x)}} = y$, then $f(\operatorname{lcm}(x,y)) = f(\operatorname{lcm}(y,z))$; then $f(y(a+c)) = f(x)+1$, in which case B is even. Otherwise, odd terms will remain in the denominator of B, and B is a fraction. Regardless, B cannot be odd.

The casework is complete; it has now been been demonstrated that B cannot be an odd positive integer. $\square$

We now prove that any even positive integer can be written in the form

$$\frac{\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}.$$

For ~~an~~ positive even integer $2n$ (for positive integer n) it may be expressed with:

$$(x,y,z)=(1,n,1)$$

in which case B is:

$$\frac{\operatorname{lcm}(1,n)+\operatorname{lcm}(1,n)}{\operatorname{lcm}(1,1)}=2n.$$

As a result, all positive even integers and only positive even integers can be written in the form

$$\frac{\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}.$$

Q.E.D.