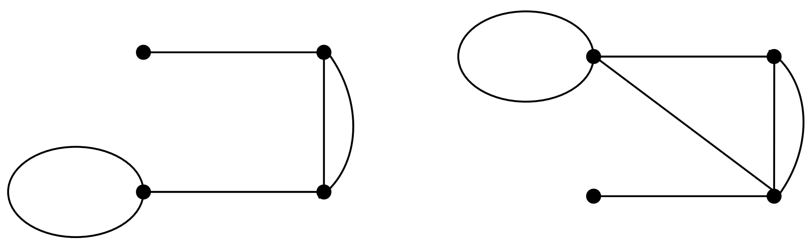
## 1.7. Isomorphic Graphs

**Example:**

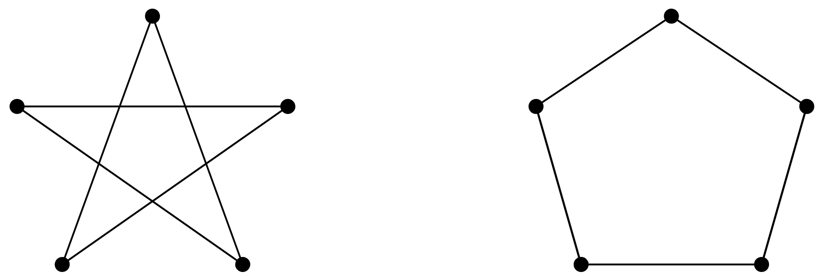
Consider the following graphs, are they the isomorphic, i.e. the "same"?

◆

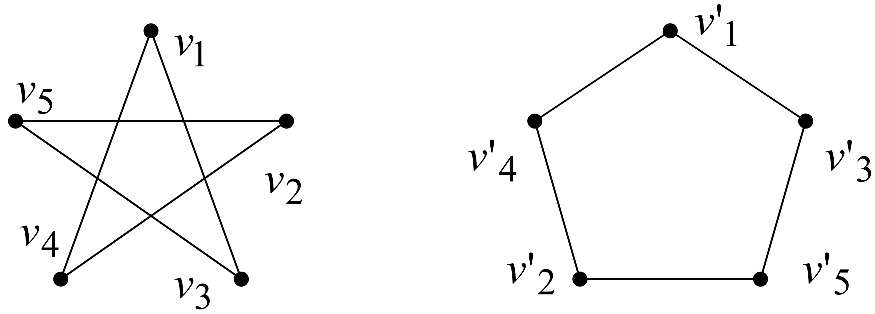


No. The left-hand graph has 5 edges; the right hand graph has 6 edges.

◆



Firstly, label the graphs. It “looks” true, so check all the things we know:




$G$ $G'$

Number of vertices: both 5.

Number of edges: both 5.

Degrees of corresponding vertices: all degree 2.

Connectedness: Each is fully connected.

Number of connected components: Both 1.

Pairs of connected vertices: All correspond.

Number of loops: 0.

Number of parallel edges: 0.

Everything is equal and so the graphs are isomorphic.

More formally:

$G = \{V, E\}$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

$$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$$
$$= \{e_1, e_2, e_3, e_4, e_5\}$$

$G' = \{V', E'\}$ where $V' = \{v_1', v_2', v_3', v_4', v_5'\}$ and

$$E' = \{(v_1', v_2'), (v_2', v_3'), (v_3', v_4'), (v_4', v_5'), (v_5', v_1')\}$$
$$= \{e_1', e_2', e_3', e_4', e_5'\}.$$

Construct 2 functions: $f : V \to V'$ and $g : E \to E'$

| $f : V \to V'$ | | $g : E \to E'$ | |
|---|---|---|---|
| $V$ | $V'$ | $E$ | $E'$ |
| $v_1$ | $v_1'$ | $e_1$ | $e_1'$ |
| $v_2$ | $v_2'$ | $e_2$ | $e_2'$ |
| $v_3$ | $v_3'$ | $e_3$ | $e_3'$ |
| $v_4$ | $v_4'$ | $e_4$ | $e_4'$ |
| $v_5$ | $v_5'$ | $e_5$ | $e_5'$ |

### 1.7.1. Definition

Let $G=\{V,E\}$ and $G'=\{V',E'\}$ be graphs. $G$ and $G'$ are said to be isomorphic if there exist a pair of functions $f:V\to V'$ and $g:E\to E'$ such that *f* associates each element in *V* with exactly one element in $V'$ and vice versa; *g* associates each element in *E* with exactly one element in $E'$ and vice versa, and for each $v\in V$, and each $e\in E$, if *v* is an endpoint of the edge *e*, then $f(v)$ is an endpoint of the edge $g(e)$.

Notes:

* To prove two graphs are isomorphic you must give a formula (picture) for the functions *f* and *g*.

* If two graphs are isomorphic, they must have:

  - the same number of vertices
  - the same number of edges
  - the same degrees for corresponding vertices
  - the same number of connected components
  - the same number of loops

- the same number of parallel edges.

* Further,

  - both graphs are connected or both graphs are not connected, and

  - pairs of connected vertices must have the corresponding pair of vertices connected.

* In general, it is easier to prove two graphs are not isomorphic by proving that one of the above properties fails.