Problem Set #2, Set 2

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Exercise 1

Take any $w, w' \in \mathscr{C}, x \in \mathbb{R}$.

$$\begin{aligned}
|Uw(x) - Uw'(x)| &\leq \rho \sup_U \left| \int \{w[F(x,u,z)] - w'[F(x,u,z)]\}\Phi(dz) \right| \\
&\leq \rho \sup_U \int |w[F(x,u,z)] - w'[F(x,u,z)]|\Phi(dz)| \\
&\leq \rho \sup_U \int \|w - w'\|\Phi(dz) \\
&= \rho\|w - w'\|
\end{aligned}$$

Taking the sup over x finishes the proof $\implies U$ is a contraction mapping. Banach's Fixed Point Theorem implies that U has one and only one fixed point w^*. There exists a policy function $\sigma \in \Sigma$ satisfying $Uw^* = U_\sigma w^*$. For this policy we have $w^* = Uw^* = U_\sigma w^*$. But, v_σ is the only fixed point of U_σ, so $w^* = v_\sigma \implies w^* \leq v^*$, since $v_\sigma \leq v^*, \forall \sigma \in \Sigma$. v_σ is thus the unique fixed point of U in $\mathscr{C}$.

Exercise 2

See `exercises2.ipny` for solutions.