Problem Set #1

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Problem 1

3.6
By independence,

$$\sum_{i\in I} P(A\cap B_i) = P(A)[\sum_{i\in I} P(B_i)]$$

By additivity,

$$P(A)[\sum_{i\in I} P(B_i)] = P(A)P(\bigcup_{i\in I} B_i)$$

Since $\bigcup_{i\in I} B_i = \Omega$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$, $P(\bigcup_{i\in I} B_i) = 1 \implies P(A)P(\bigcup_{i\in I} B_i) = P(A)$, as desired.

3.8

$$\begin{aligned} 1 - \prod_{k=1}^{n}(1 - P(E_k)) &= 1 - \prod_{k=1}^{n} P(E_k^c) \\ &= 1 - P(\bigcap_{k=1}^{n} E_k^c) \\ &= P((\bigcap_{k=1}^{n} E_k^c)^c) \\ &= P(\bigcup_{k=1}^{n} E_k) \end{aligned}$$

Where line (2) follows from independence, and line (4) by DeMorgan's Laws.

3.11
Assume that the DNA test is perfectly accurate in identifying someone who was at the crime scene, i.e. $P(pos|guilty) = 1$. From the description, we have:

$$P(guilty) = \frac{1}{250m}$$
$$P(pos) = \frac{1}{3m}$$

We want the probability that an individual was at the crime scene, given their DNA test was positive, that is $P(guilty|positive)$. Bayes' Rule gives:

$$\begin{aligned} P(guilty|pos) &= \frac{P(pos|guilty)P(guilty)}{P(pos)} \\ &\approx 1.2\% \end{aligned}$$

3.11

First, some setup. Define:

D1 = Morty opens door 1
D2 = Morty opens door 2
D3 = Morty opens door 3
C1 = Car behind door 1
C2 = Car behind door 2
C3 = Car behind door 3

Assume you choose door 1. Then,

$$\begin{aligned}
&P(D3|C1) = P(D2|C1) = \frac{1}{2} \\
&P(D3|C3) = 0 \\
&P(D3|C2) = 1
\end{aligned}$$

And,

$$\begin{aligned}
P(C1|D3) &= \frac{P(D3|C1)P(C1)}{P(D3)} \\
&= \frac{1}{3} \\
P(C2|D3) &= \frac{P(D3|C2)P(C2)}{P(D3)} \\
&= \frac{2}{3}
\end{aligned}$$

For the 10-door case, if Monty opens 8 doors, the contestant has a $\frac{9}{10}$ chance of winning if they switch, and a $\frac{1}{10}$ chance if they do not.