# Math Problem Set #3

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## 4.2

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$D$ is upper triangular, so the eigenvalues are its diagonal entries $\implies$ $D$ has no eigenvalues. Algebraic and geometric multiplicities are both zero.

## 4.4

(i) For non-zero $x \in V, \lambda \in \mathbb{F}$,

$$\begin{aligned} Ax &= \lambda x \\ (Ax)^H &= (\lambda x)^H \\ x^H A^H &= x^H \lambda^H \end{aligned}$$

Post-multiplying by $x$,

$$x^H A^H x = x^H \lambda^H x$$

Since A is Hermitian, $A^H = A$,

$$\begin{aligned} \implies x^H A x &= x^H \lambda^H x \\ x^H \lambda x &= \lambda^H x^H x \\ \lambda x^H x &= \lambda^H x^H x \\ \implies \lambda &= \lambda^H \end{aligned}$$

Thus, $\lambda \in \mathbb{R}$.

(ii) $\langle Ax, x \rangle = \langle \lambda x, x \rangle = \bar{\lambda}\|x\|^2 = \langle x, A^H x \rangle = -\langle x, Ax \rangle = -\lambda\|x\|^2 \implies \bar{\lambda} = -\lambda \implies \lambda$ is imaginary.

## 4.6

Take an upper triangular matrix $A \in M_n(\mathbb{F})$,

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

With arbitrary entries above the diagonal, and zeros below. Take any $\lambda \in \mathbb{F}$. Then,

$$\lambda I - A = \begin{bmatrix} \lambda - \lambda_1 & & & \\ & \lambda - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda - \lambda_n \end{bmatrix}$$

$\lambda$ is an eigenvalue of $A$ if and only if $\lambda I - A$ is not invertible. Equivalently, for upper triangular matrices, $A$ is invertible if and only if all the diagonal entries are nonzero. Thus, $\lambda$ is an eigenvalue if and only if it equals one of the numbers $\lambda_1, \lambda_2 \cdots \lambda_n$. Thus, $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigenvalues of $A$.

**4.8**

(i) $S = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ forms a basis if it's linearly independent, and if every vector $v \in V$ can be expressed as a linear combination of the elements of $S$.

(ii)

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii) The kernel of $D$ is one of them.

**4.13**

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \\ p(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 0.8 & -0.4 \\ -0.2 & \lambda - 0.6 \end{vmatrix} \\ \implies \sigma(A) &= \{1, 0.4\} \end{aligned}$$

The eigenvectors corresponding to eigenvalues 1 and 0.4 are $\begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$, respectively. Thus, the transition matrix is:

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

**4.15**

Let set $S = \{x_1, x_2, \cdots, x_n\}$ form an eigenbasis of $A \in M_n(\mathbb{F})$. Because $A$ is semisimple, it is diagonalizable. Let $D = P^{-1}AP$ be the diagonal representation of $A$. Thus,

$$\begin{aligned} f(A) &= a_0 I + a_1 A + \cdots + a_n A^n \\ &= a_0 I + a_1 P^{-1} D P + \cdots + a_n P^{-1} D^n P \end{aligned}$$

The eigenvalues of $D$ are the entries along its diagonal. For $i = 1, \cdots, n$,

$$\begin{aligned} \lambda_i &= a_0 + a_1 \lambda_i + \cdots + a_n \lambda_i^n \\ &= f(\lambda_i) \end{aligned}$$

as desired.

**4.16**
(i)

$$\begin{aligned}
A &= \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} B \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
\implies A^n &= \begin{bmatrix} 1.5 & -1 \\ -0.5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

As $n \to \infty, 0.4 \to 0.$

$$\begin{aligned}
\implies \lim_{n\to\infty} A^n &= \begin{bmatrix} 1.5 & -1 \\ -0.5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1.5 & 0 \\ -0.5 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -1.5 \\ -1 & 0.5 \end{bmatrix} \\
\implies \| \lim_{n\to\infty} A^n \|_1 &= 4
\end{aligned}$$

(ii)

$$\begin{aligned}
\| \lim_{n\to\infty} A^n \|_\infty &= 4.5 \\
\| \lim_{n\to\infty} A^n \|_F &= \sqrt{tr(A^T A)} \\
&= \sqrt{12.5}
\end{aligned}$$

(iii) Let matrix $B = 3I + 5A + A^3$, with $A$ as defined above.

$$\begin{aligned}
\sigma(A) &= \{1, 0.4\} \\
\sigma(B) &= \{3 + 5(1) + (1)^3, 3 + 5(0.4) + (0.4)^3\} \\
&= \{9, 5.064\}
\end{aligned}$$

**4.18**

$$Ax = \lambda x$$

Left-multiplying,

$$\begin{aligned} x^T Ax &= x^T \lambda x \\ x^T Ax &= \lambda x^T x \\ \implies x^T A &= \lambda x^T \end{aligned}$$

**4.20**

$$\begin{aligned} B &= U^H AU \\ B^H &= (U^H AU)^H \\ &= U^H AU^{HH} \\ &= U^H AU \\ &= B \end{aligned}$$

$\implies$ $B$ is Hermitian.

**4.24**
(i)

$$\begin{aligned} p(x) = \frac{\langle x, Ax\rangle}{\|x\|^2} &= \frac{\langle Ax, x\rangle}{\|x\|^2} \\ \implies \frac{\lambda\langle x, x\rangle}{\|x\|^2} &= \frac{\bar{\lambda}\langle x, x\rangle}{\|x\|^2} \\ \implies \lambda &= \bar{\lambda} \\ \implies \lambda &\in \mathbb{R} \end{aligned}$$

(ii)

$$\begin{aligned} p(x) = \frac{\langle x, Ax\rangle}{\|x\|^2} &= \frac{\langle -Ax, x\rangle}{\|x\|^2} \\ \implies \frac{\lambda\langle x, x\rangle}{\|x\|^2} &= \frac{-\bar{\lambda}\langle x, x\rangle}{\|x\|^2} \\ \implies -\lambda &= \bar{\lambda} \\ \implies \lambda &\in \mathbb{C} \end{aligned}$$

**4.25**

Any $x_j, j = 1, \cdots, n$ can be written as $x_1 x_1^H x_j + \cdots + x_n x_n^H x_j = (x_1 x_1^H + \cdots + x_n x_n^H) x_j \implies x_1 x_1^H + \cdots + x_n x_n^H = I_n$.

**4.27**

Positive definite implies that $x^T A x > 0$ for all $x \in \mathbb{R}^n$. Let $e_1, \cdots, e_n$ be the standard basis. For positive definite matrix $A, e_i^T A e_i > 0, i = 1, \cdots, n$. For some basis vector $e_i, e_i^T A e_i = a_{ii}$, that is, the $i^{th}$ entry along the diagonal of A. Thus, the diagonal entries of $A$ are all strictly greater than zero.

**4.28**

**4.31**

(i)

$$\begin{aligned}
\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} &= \sup_{x \neq 0} \frac{\|U \Sigma V' x\|_2}{\|x\|_2} \\
&= \sup_{x \neq 0} \frac{\|\Sigma V' x\|_2}{\|x\|_2} \\
&= \sup_{y \neq 0} \frac{\|\Sigma y\|_2}{\|V y\|_2} \\
&= \sup_{y \neq 0} \frac{(\sum_{i=1}^r \sigma_i^2 |y_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^r |y_i|^2)^{\frac{1}{2}}} \\
&\leq \sigma_1
\end{aligned}$$

For $y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$, $\|\Sigma y\|_2 = \sigma_1$, and the supremum is attained.

(ii)

$$\begin{aligned}
\|A\|_2 = \sup_{x \neq 0} \frac{\|A^{-1} x\|_2}{\|x\|_2} &= \sup_{x \neq 0} \frac{\|(U \Sigma V')^{-1} x\|_2}{\|x\|_2} \\
&= \sup_{x \neq 0} \frac{\|V \Sigma U^T x\|_2}{\|x\|_2} \\
&= \sup_{y \neq 0} \frac{\|\Sigma^{-1} y\|_2}{\|V y\|_2} \\
&= \sup_{y \neq 0} \frac{(\sum_{i=1}^r \frac{1}{\sigma_i^2} |y_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^r |y_i|^2)^{\frac{1}{2}}} \\
&\implies \|A^{-1}\|_2 = \sigma_n^{-1}
\end{aligned}$$

(iii)

(iv)

$$
\begin{aligned}
\|UAV\|_2 &= \sup \frac{\|UAVx\|_2}{\|x\|_2} \\
&= \sup \frac{\langle UAVx, x\rangle^{\frac{1}{2}}}{\|x\|_2} \\
&= \sup \frac{\langle Ax, x\rangle^{\frac{1}{2}}}{\|x\|_2} \\
&= \sup \frac{\|Ax\|_2}{\|x\|_2} \\
&= \|A\|_2
\end{aligned}
$$

**4.32**

$$
\begin{aligned}
\|UAV\|_F &= \sqrt{Tr[(UAV)(UAV)^H]} \\
&= \sqrt{Tr(UAVV^H A^H U^H)} \\
&= \sqrt{Tr(UAA^H U^H)} \\
&= \sqrt{Tr(AA^H UU^H)} \\
&= \sqrt{Tr(AA^H)} \\
&= \sqrt{Tr(A^2)} \\
&= \|A\|_F
\end{aligned}
$$

**4.33**

**4.36**
The matrix

$$
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

Has $\det(A) = -1$, eigenvalues $\{i, -i\}$, and singular values $\{1, 1\}$.

**4.38**
(i)

$$
\begin{aligned}
AA^\dagger &= U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H \\
&= I \\
&\implies AA^\dagger A = A
\end{aligned}
$$

(ii) $AA^\dagger = I \implies A^\dagger AA^\dagger = A^\dagger$.
(iii)

$$\begin{aligned}
(AA^\dagger)^H &= A^{\dagger H}A^H \\
&= (V_1\Sigma_1^{-1}U_1^H)^H(U_1\Sigma_1V_1^H)^H \\
&= U_1E_1^{-1}V_1^HV_1\Sigma_1U_1^H \\
&= AA^\dagger
\end{aligned}$$

(iv)

$$\begin{aligned}
(A^\dagger A)^H &= A^HA^{\dagger H} \\
&= (U_1\Sigma_1V_1^H)^H(V_1\Sigma_1^-1U_1^H)^H \\
&= V_1\Sigma_1U_1^HU_1\Sigma_1^{-1}V^H \\
&= A^\dagger A
\end{aligned}$$