Math Problem Set #3

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4.2

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

D is upper triangular, so the eigenvalues are its diagonal entries $\implies$ D has no eigenvalues. Algebraic and geometric multiplicities are both zero.

4.4

(i) For non-zero $x \in V, \lambda \in \mathbb{F}$,

$$\begin{aligned} Ax &= \lambda x \\ (Ax)^H &= (\lambda x)^H \\ x^H A^H &= x^H \lambda^H \end{aligned}$$

Post-multiplying by x,

$$x^H A^H x = x^H \lambda^H x$$

Since A is Hermitian, $A^H = A$,

$$\begin{aligned} \implies x^H A x &= x^H \lambda^H x \\ x^H \lambda x &= \lambda^H x^H x \\ \lambda x^H x &= \lambda^H x^H x \\ \implies \lambda &= \lambda^H \end{aligned}$$

Thus, $\lambda \in \mathbb{R}$.

(ii) $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \bar{\lambda}\|x\|^2 = \langle x, A^H x\rangle = -\langle x, Ax\rangle = -\lambda\|x\|^2 \implies \bar{\lambda} = -\lambda \implies \lambda$ is imaginary.

4.6

Take an upper triangular matrix $A \in M_n(\mathbb{F})$,

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

With arbitrary entries above the diagonal, and zeros below. Take any $\lambda \in \mathbb{F}$. Then,

$$\lambda I - A = \begin{bmatrix} \lambda - \lambda_1 & & & \\ & \lambda - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda - \lambda_n \end{bmatrix}$$

λ is an eigenvalue of A if and only if $\lambda I - A$ is not invertible. Equivalently, for upper triangular matrices, A is invertible if and only if all the diagonal entries are nonzero. Thus, λ is an eigenvalue if and only if it equals one of the numbers $\lambda_1, \lambda_2 \cdots \lambda_n$. Thus, $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigenvalues of A.

4.8

(i) $S = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ forms a basis if it's linearly independent, and if every vector $v \in V$ can be expressed as a linear combination of the elements of S.

(ii)

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii) The kernel of D is one of them.

4.13

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \\ p(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 0.8 & -0.4 \\ -0.2 & \lambda - 0.6 \end{vmatrix} \\ \implies \sigma(A) &= \{1, 0.4\} \end{aligned}$$

The eigenvectors corresponding to eigenvalues 1 and 0.4 are $\begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$, respectively. Thus, the transition matrix is:

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

4.15

Let set $S = \{x_1, x_2, \cdots, x_n\}$ form an eigenbasis of $A \in M_n(\mathbb{F})$. Because A is semisimple, it is diagonalizable. Let $D = P^{-1}AP$ be the diagonal representation of A. Thus,

$$\begin{aligned} f(A) &= a_0 I + a_1 A + \cdots + a_n A^n \\ &= a_0 I + a_1 P^{-1} D P + \cdots + a_n P^{-1} D^n P \end{aligned}$$

The eigenvalues of D are the entries along its diagonal. For $i = 1, \cdots, n$,

$$\begin{aligned} \lambda_i &= a_0 + a_1 \lambda_i + \cdots + a_n \lambda_i^n \\ &= f(\lambda_i) \end{aligned}$$

as desired.

4.16
(i)

$$\begin{aligned}
A &= \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} B \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
\implies A^n &= \begin{bmatrix} 1.5 & -1 \\ -0.5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

As $n \to \infty, 0.4 \to 0$.

$$\begin{aligned}
\implies \lim_{n\to\infty} A^n &= \begin{bmatrix} 1.5 & -1 \\ -0.5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1.5 & 0 \\ -0.5 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -1.5 \\ -1 & 0.5 \end{bmatrix} \\
\implies \| \lim_{n\to\infty} A^n \|_1 &= 4
\end{aligned}$$

(ii)

$$\begin{aligned}
\| \lim_{n\to\infty} A^n \|_\infty &= 4.5 \\
\| \lim_{n\to\infty} A^n \|_F &= \sqrt{tr(A^T A)} \\
&= \sqrt{12.5}
\end{aligned}$$

Yes, it's changing, but I think I messed something up. **TODO: FIX THIS**

(iii) Let matrix $B = 3I + 5A + A^3$, with A as defined above.

$$\begin{aligned}
\sigma(A) &= \{1, 0.4\} \\
\sigma(B) &= \{3 + 5(1) + (1)^3, 3 + 5(0.4) + (0.4)^3\} \\
&= \{9, 5.064\}
\end{aligned}$$

4.18

$$Ax = \lambda x$$

Left-multiplying,

$$\begin{aligned} x^T A x &= x^T \lambda x \\ x^T A x &= \lambda x^T x \\ \implies x^T A &= \lambda x^T \end{aligned}$$

TODO: actually solve this.