Scallop and Neuro-Symbolic Programming

Lecture 2.5 - Weighted Model Counting and Probabilistic Reasoning

Weighted Model Counting

- The Weighted Model Counting problem
    - Model: Solution to a propositional boolean formula
    - Model Counting: Number of solutions to a propositional boolean formula
        - #SAT: #P-Complete Problem
    - Weighted Model Counting: “number of solutions” with weights assigned to each boolean variable

Model and Model Counting

$$(a \wedge \bar{b}) \vee (b \wedge c)$$

Boolean Formula

Model and Model Counting

$$(a \wedge \bar{b}) \vee (b \wedge c)$$

Boolean Formula

a	b	c	Result
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Truth Table

Model and Model Counting

$$(a \wedge \bar{b}) \vee (b \wedge c)$$

Boolean Formula

a	b	c	Result
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

$a = \mathrm{T}, b = \mathrm{T}, c = \mathrm{T}$

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Truth Table

Model and Model Counting

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Truth Table

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Number of Models

Weight (Probability) with Variables

$$Pr(a) = 0.4, Pr(b) = 0.9, Pr(c) = 0.1$$

a	b	c	Result
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
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Truth Table

"Weighted Number" of Models

Weight of Each Model

$$Pr(a) = 0.4, Pr(b) = 0.9, Pr(c) = 0.1$$

a	b	c	Result
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
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F	T	F	F
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a = T, b = T, c = T

a = T, b = F, c = T

a = T, b = F, c = F

a = F, b = T, c = F

Pr(a) Pr(b) Pr(c) = 0.4 x 0.9 x 0.1 = **0.036**

Pr(a) (1 – Pr(b)) Pr(c) = 0.4 x 0.1 x 0.1 = **0.004**

Pr(a) (1 – Pr(b)) (1 – Pr(c)) = 0.4 x 0.1 x 0.9 = **0.036**

(1 – Pr(a)) Pr(b) (1 – Pr(c)) = 0.6 x 0.9 x 0.9 = **0.486**

Truth Table

"Weighted Number" of Models

Weight of Each Model

a	b	c	Result
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
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a = T, b = T, c = T

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Truth Table

$$Pr(a) = 0.4, Pr(b) = 0.9, Pr(c) = 0.1$$

Weight of each model

Pr(a) Pr(b) Pr(c) = 0.4 x 0.9 x 0.1 = **0.036**

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“Weighted Number” of Models

Weighted Model Counting

$$Pr(a) = 0.4, Pr(b) = 0.9, Pr(c) = 0.1$$

a	b	c	Result
T	T	T	T
T	T	F	F
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a = T, b = T, c = T

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Weight of each model

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(1 – Pr(a)) Pr(b) (1 – Pr(c)) = 0.6 x 0.9 x 0.9 = **0.486**

Pr($(a \wedge \bar{b}) \vee (b \wedge c)$) = **0.562**

Truth Table

"Weighted Number" of Models

How to do Weighted Model Counting?

- Just enumerating the full truth table…
- [Find a SAT Solver to count the models]
- Use Binary Decision Diagrams (BDD) and its extensions (OBDD, ROBDD, etc.)
- Use Sentential Decision Diagrams (SDD)

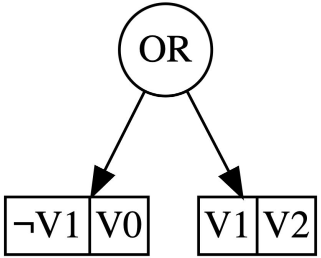
How to do Weighted Model Counting?

- Just enumerating the full truth table...
- Find a SAT Solver to count the models
- Use Binary Decision Diagrams (BDD) and its extensions (OBDD, ROBDD, etc.)
- Use Sentential Decision Diagrams (SDD)

The weighted model counter currently used in Scallop

Sentential Decision Diagram (SDD)

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2) \longrightarrow$$



$$\longrightarrow \mathrm{Pr}(V_0)(1 - \mathrm{Pr}(V_1)) + \mathrm{Pr}(V_1)\mathrm{Pr}(V_2)$$

Boolean Formula

One of the SDDs

Formula for Weighted Model Counting

There could be multiple SDDs

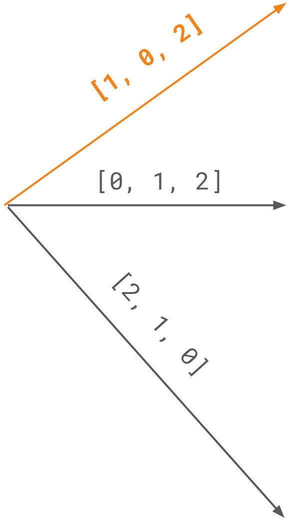
$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula

There could be multiple SDDs

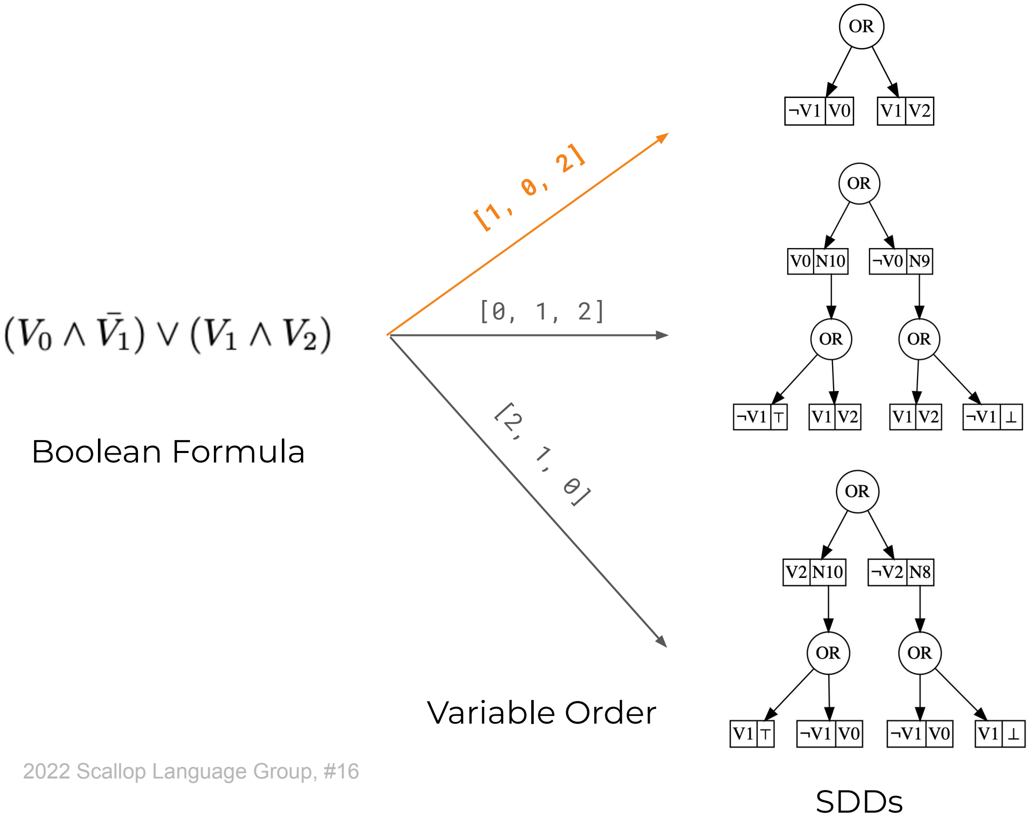
$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula



Variable Order

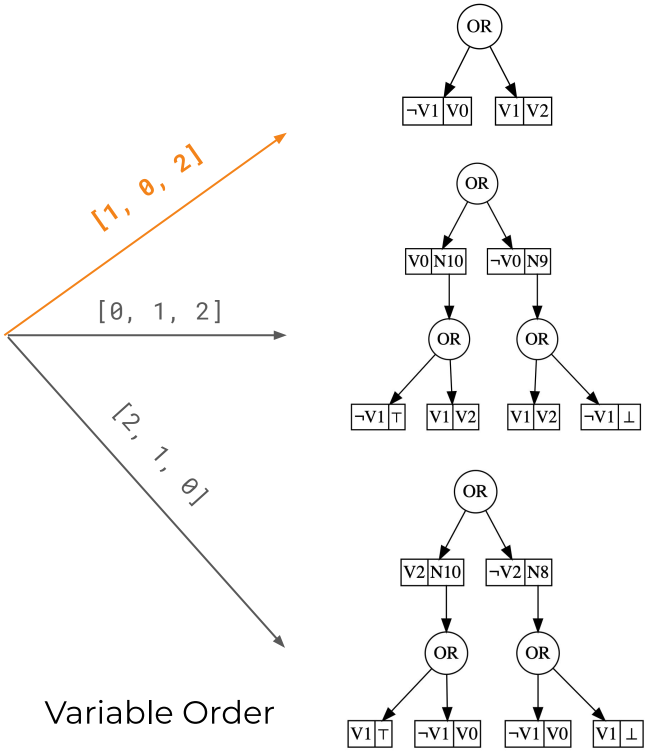
There could be multiple SDDs



There could be multiple SDDs

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula



Variable Order

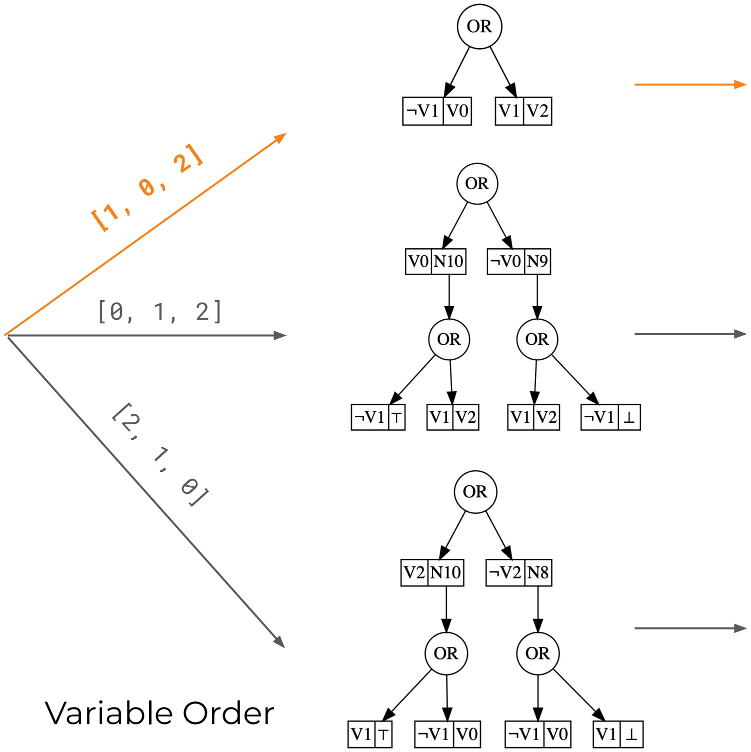
SDDs

There could be multiple SDDs

Let $p_i = \Pr(V_i)$:

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula



$$p_0(1-p_1)+p_1p_2$$

$$p_0((1-p_1)+p_1p_2)+(1-p_0)p_1p_2$$

$$p_2(p_1+(1-p_1)p_0)+(1-p_2)(1-p_1)p_0$$

Variable Order

SDDs

Formula for Weighted Model Counting

Differentiable Weighted Model Counting

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula

$$p_0(1 - p_1) + p_1 p_2$$

Formula for Weighted Model Counting

Differentiable Weighted Model Counting

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula

$$p_0(1 - p_1) + p_1 p_2$$

Formula for Weighted Model Counting

→

The gradient has size 3 since there are 3 input boolean variables

$$\hat{p}_0 = (p_0, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top)$$

$$\hat{p}_1 = (p_1, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top)$$

$$\hat{p}_2 = (p_2, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top)$$

Augmenting simple probabilities into dual numbers!

Differentiable Weighted Model Counting

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula

$$p_0(1 - p_1) + p_1 p_2$$

Formula for Weighted Model Counting

$\longrightarrow$

$$\hat{p}_0 = (p_0, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top)$$
$$\hat{p}_1 = (p_1, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top)$$
$$\hat{p}_2 = (p_2, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top)$$

Augmenting simple probabilities into Dual Numbers!

$$\hat{0} = (0, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top)$$
$$\hat{1} = (1, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top)$$
$$\hat{p} + \hat{q} = (p, \nabla_p) + (q, \nabla_q) = (p + q, \nabla_p + \nabla_q)$$
$$\hat{p} \times \hat{q} = (p, \nabla_p) \times (q, \nabla_q) = (pq, q\nabla_p + p\nabla_q)$$

Operations for Dual Numbers

Differentiable Weighted Model Counting

$$(V_0 \wedge \bar{V_1}) \vee (V_1 \wedge V_2)$$

Boolean Formula

$$p_0(1 - p_1) + p_1 p_2$$

Formula for Weighted Model Counting

$\longrightarrow$

$$\hat{p}_0 = (p_0, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top)$$

$$\hat{p}_1 = (p_1, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top)$$

$$\hat{p}_2 = (p_2, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top)$$

Augmenting simple probabilities into Dual Numbers!

$\longrightarrow$

$$\hat{p}_0(\hat{1} - \hat{p}_1) + \hat{p}_1 \hat{p}_2$$

Formula for Differentiable Weighted Model Counting