AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 3

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- Non-Keplerian motion and perturbations
- Variation of parameters
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Variation of Parameters

- Solutions of the fundamental orbital differential equations have been obtained for nominal, undisturbed, Keplerian motion
- When perturbations act on the body, the motion is no longer Keplerian
- In order to solve for the resulting non-Keplerian motion, Euler and Lagrange have developed the *variation-of-parameters* (VOP) procedure
- VOP is a general method to solve nonlinear differential equations:
  - Take the homogeneous solution of the differential equation
  - Take the integration constants of the "unperturbed" motion
  - Express these constants as a function of time
  - Introduce ad-hoc constraint to simplify equations
  - Solve for new, now time-dependent parameters

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Newman's Example (1)

- Second order differential equation with initial conditions

$$\ddot{x} + x = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \tag{2.59}$$

- Homogeneous solution with integration constants

$$x = s \sin t + c \cos t \tag{2.60}$$

- VOP seeks solution of the form

Now time dependent

$$x = s(t) \sin t + c(t) \cos t \tag{2.61}$$

- Differentiating two times yields fourth order system due to change of variables

$$(x, \dot{x}) \mapsto (s, \dot{s}, c, \dot{c})$$

- Constraint is introduced to remove excess freedom

$$\dot{s}(t) \sin t + \dot{c}(t) \cos t = \Xi(t) \longrightarrow \text{Arbitrary} \tag{2.64}$$

- After double differentiation of (2.61), this allows re-writing system in the form

$$\dot{s}(t) = f(t)\cos(t) - \frac{\mathrm{d}}{\mathrm{d}t}(\Xi \cos t) \tag{2.69}$$

$$\dot{c}(t) = -f(t)\sin t + \frac{\mathrm{d}}{\mathrm{d}t}(\Xi \sin t) \tag{2.70}$$

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Newman’s Example (2)

- Integration of (2.69-2.70) yields

$$s(t) = \int_0^t f(\tau)\cos\tau\, d\tau - \Xi\cos t + s(0) \qquad (2.71)$$

$$c(t) = \int_0^t f(\tau)\sin\tau\, d\tau + \Xi\sin t + c(0) \qquad (2.72)$$

- which after substitution into (2.61) provides the desired solution

$$x = -\cos t\int_0^t f(\tau)\sin\tau\, d\tau + \sin t\int_0^t f(\tau)\cos\tau\, d\tau + s(0)\sin t + c(0)\cos t \qquad (2.73)$$

- In contrast to the new state variables, the solution in terms of the original *x* is invariant to a particular selection of Ξ
- This phenomenon is called *symmetry* and the analogy with orbital elements is straightforward

Lagrange and Gauss Variational Equations

- When subject to a perturbing specific force, ***d***, the fundamental differential equations become

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = \mathbf{d} \qquad (2.74)$$

3 Eq./3 Unknowns

- Applying the VOP formalism, the integration constants, here the orbital elements, become functions of time, thus yielding

$$\mathbf{r} = \mathbf{f}[\text{œ}(t), t] \qquad (2.75)$$

6 Parameters

- Taking the time derivative provides

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{f}}{\partial \text{œ}} \dot{\text{œ}} \qquad (2.76)$$

- In the presence of a conservative, position-only dependent perturbing potential, VOP leads to the **Lagrange's planetary equations (LPE)**
- In the presence of an arbitrary perturbing acceleration or control input in the RTN frame, VOP leads to the **Gauss variational equations (GVE)**

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Lagrange's Planetary Equations (1)

- Differentiating (2.76) and substituting in (2.74) provides three extra degrees of freedom due to the change of state variables (or 3 Eq./6 Unknowns)

$$(\mathbf{r}, \dot{\mathbf{r}}) \mapsto (\text{œ}, \dot{\text{œ}})$$

- Since the resulting system is underdetermined, three extra conditions can be imposed, Lagrange chose

User defined velocity

(2.77) Osculating constraint $\longrightarrow \dfrac{\partial \mathbf{f}}{\partial \text{œ}} \dot{\text{œ}} = \mathbf{0}$ More general $\longrightarrow \dfrac{\partial \mathbf{f}}{\partial \text{œ}} \dot{\text{œ}} = \mathbf{v}_g(\text{œ}, \dot{\text{œ}}, t)$

- Physically this postulates that the trajectory in the inertial configuration space is always tangential to an "instantaneous" ellipse (or hyperbola) defined by the "instantaneous" values of the time-varying orbital elements

$$\text{œ}(t)$$

- The instantaneous orbit is called *osculating orbit* and the orbital elements which satisfy the Lagrange constraint are *osculating orbital elements*

Lagrange's Planetary Equations (2)

- Differentiating (2.76) and substituting in (2.74), considering the osculating constraint and the Keplerian solution, yields

$$\frac{\partial \mathbf{v}}{\partial \boldsymbol{\text{œ}}}\dot{\boldsymbol{\text{œ}}} = \mathbf{d} \qquad (2.85)$$

- which gives 6 differential equations in the osculating orbital elements together with (2.77). These equations can be written in compact form using the Lagrange matrix

$$(2.91) \qquad \dot{\boldsymbol{\text{œ}}} = \mathfrak{L}^{-1}\left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\text{œ}}}\right]^T \mathbf{d}$$

Perturbing potential

$$\mathbf{d} = \nabla_{\mathbf{r}}\mathcal{R} = \frac{\partial \mathcal{R}}{\partial \mathbf{r}} \qquad (2.95)$$

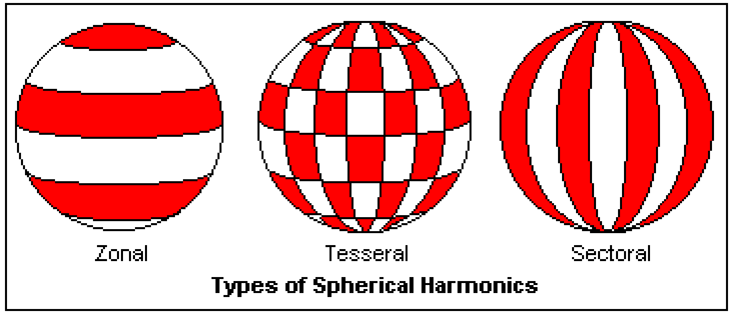
Conservative

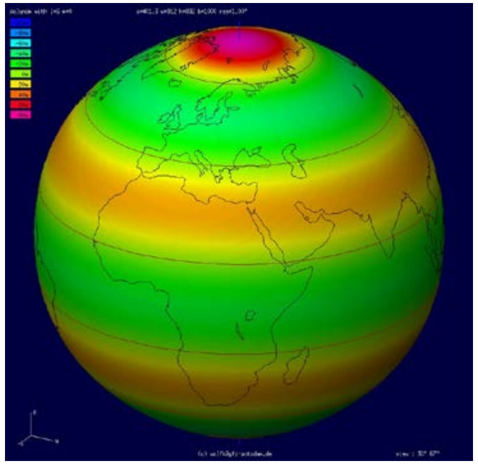
Position-dependent

$$(2.97) \qquad \boxed{\dot{\boldsymbol{\text{œ}}} = \mathfrak{L}^{-1}\frac{\partial \mathcal{R}}{\partial \boldsymbol{\text{œ}}}}$$

Perturbing Potential for Zonal Harmonics

- In a mass distribution sense, primary is not spherical, nor symmetric about equator, nor axis-symmetric
- Irregular mass distribution can be modelled through a geo-potential expansion in spherical harmonics (zonal, tesseral, sectoral)
- Dominant **zonal harmonics** account for axially-symmetric non-spherical primary





$$\mathcal{R} = -\frac{\mu}{r}\sum_{k=2}^{\infty} J_k \left(\frac{R_e}{r}\right)^k P_k(\cos\phi) \qquad (2.98)$$

Colatitude

$$\cos\phi = \sin i \, \sin(f+\omega) \qquad (2.99)$$

Legendre polynomials

$$P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}\left[(x^2-1)^k\right] \qquad (2.101)$$

Zonal coefficients

$J_2 = 1082.63 \times 10^{-6}$ "Bulge"

$J_3 = -2.52 \times 10^{-6}$ "Pear"

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Averaging Theory (1)

- Substitution of (2.98-2.99) in (2.97) and derivation provide the instantaneous rates of the orbital elements caused by zonal harmonics
- The dominant J_2 perturbation causes short-period oscillations, long-period oscillations, and secular drift
- Averaging theory can be used to simplify the equations of motion and derive analytical expressions for the secular drift caused e.g. by J_2

$$\dot{\mathbf{x}} = \epsilon \mathbf{F}(\mathbf{x}, t) \Rightarrow \bar{\mathbf{F}} = \langle \mathbf{F} \rangle \triangleq \frac{1}{T} \int_t^{t+T} \mathbf{F}(\mathbf{x}, \tau) d\tau \Rightarrow \dot{\bar{\mathbf{x}}} = \epsilon \bar{\mathbf{F}}(\bar{\mathbf{x}})$$

Original differential equations (non-autonomous but *T*-periodic)

Averaging operator

Averaged differential equations (autonomous)

- Averaging theorem (Theorem 2.1) guarantees that the two systems coincide to first-order as ε tends to zero, or $\mathbf{x} = \bar{\mathbf{x}} + \mathcal{O}(\epsilon)$

Averaging Theory (2)

- Application of averaging to the Lagrange planetary equations provides the following expression for the perturbing potential due to J_2

$$\bar{\mathcal{R}} = \langle \mathcal{R} \rangle = \frac{\bar{n}^2 J_2 R_e^2}{4(1-\bar{e}^2)^{\frac{3}{2}}}(3\cos^2\bar{i} - 1) \qquad (2.114)$$

- and after substitution the linear differential equations for the *mean classical orbital elements*

(2.115a)
(2.115f)

$$\frac{d\bar{a}}{dt} = 0 \qquad \frac{d\bar{\Omega}}{dt} = -\frac{3}{2}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\cos\bar{i}$$

$$\frac{d\bar{e}}{dt} = 0 \qquad \frac{d\bar{\omega}}{dt} = \frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}(5\cos^2\bar{i} - 1)$$

$$\frac{d\bar{i}}{dt} = 0 \qquad \frac{d\bar{M}_0}{dt} = \frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\bar{\eta}(3\cos^2\bar{i} - 1)$$

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Gauss' Variational Equations (1)

- GVE are obtained by using the chain rule

$$\dot{\text{\oe}} = \frac{\partial \text{\oe}}{\partial t} + \frac{\partial \text{\oe}}{\partial \mathbf{r}}\left(\frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \mathbf{r}}{\partial \text{\oe}}\dot{\text{\oe}}\right) + \frac{\partial \text{\oe}}{\partial \mathbf{v}}\left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \text{\oe}}\dot{\text{\oe}}\right) \qquad (2.102)$$

- by using osculating constraint (2.77) and unperturbed solution

$$\dot{\text{\oe}} = \frac{\partial \text{\oe}}{\partial \mathbf{v}}\mathbf{d} \qquad (2.104)$$

- and by expressing all variables in an RTN frame fixed to the satellite

$$\mathbf{r} = r\hat{\mathbf{R}}, \quad \mathbf{v} = \frac{\partial r}{\partial t}\hat{\mathbf{R}} + r\frac{\partial f}{\partial t}\hat{\mathbf{S}} \qquad \mathbf{d} = d_r\hat{\mathbf{R}} + d_\theta\hat{\mathbf{S}} + d_h\hat{\mathbf{W}} \qquad (2.106)$$

Radial Along-track Cross-track

Gauss' Variational Equations (2)

- The resulting GVE are given by
- GVE provides the time variation of the orbital elements when subject to a generic perturbation
- If the perturbation is a control thrust, GVE shows the effect on the orbital elements
- In this course we'll consider mainly zonal harmonics, atmospheric drag, and control inputs

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= 2\,\frac{d_r a^2 e\,\sin f}{h} + 2\,\frac{d_\theta a^2 p}{hr} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{d_r\, p \sin f}{h} + \frac{d_\theta\,[(p+r)\cos f + re]}{h} \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{d_h r \cos(f+\omega)}{h} \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{d_h r \sin(f+\omega)}{h \sin i} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{d_r\, p\cos f}{he} + \frac{d_\theta\,(p+r)\sin f}{he} \\
&\quad - \frac{d_h r \sin(f+\omega)\cos i}{h \sin i} \\
\frac{\mathrm{d}M_0}{\mathrm{d}t} &= d_r\left[\frac{\left(-2\,e + \cos f + e\,\cos^2 f\right)\left(1-e^2\right)}{e\,(1+e\,\cos f)\,na}\right] \\
&\quad + d_\theta\left[\frac{\left(e^2-1\right)(e\,\cos f + 2)\sin f}{e\,(1+e\,\cos f)\,na}\right]
\end{aligned}
$$

(2.107a) (2.107f)

Often replaced by *M* or *f*

FROM SCHAUB pp 594-597

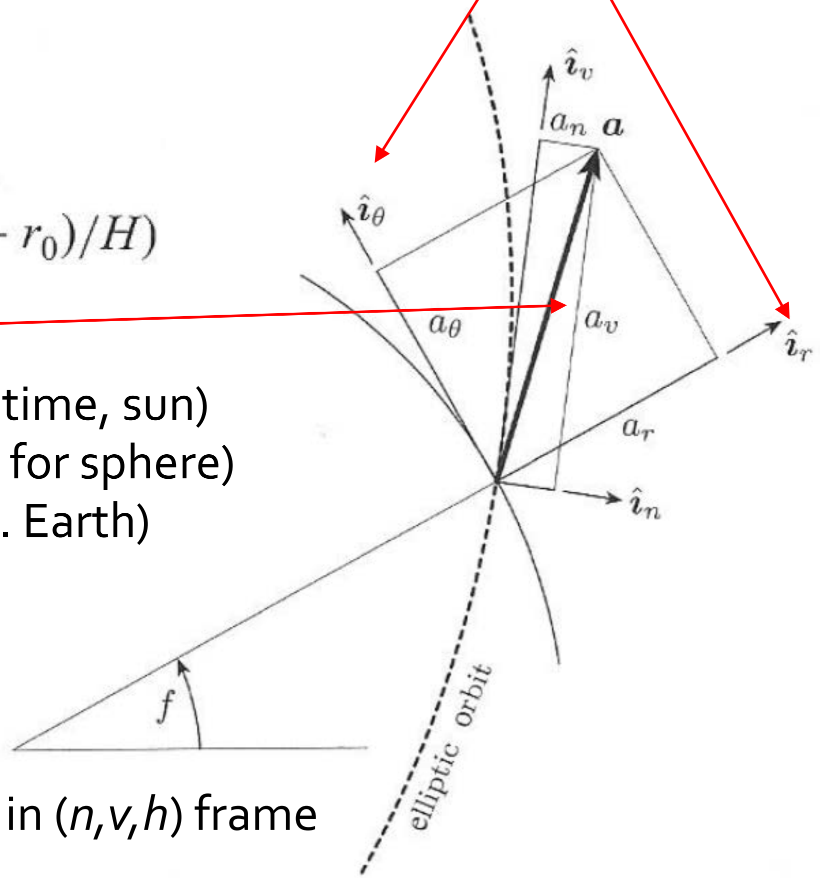
Atmospheric Drag (1)

GVE refer to RTN (r, θ, h)

- Molecules in atmosphere hitting object cause an aerodynamic force

Ballistic coefficient

$$\boldsymbol{a}_D = \underbrace{-\overbrace{\left(\frac{A}{m}\right)C_d}^{\text{Ballistic coefficient}}\rho\frac{v^2}{2}}_{a_v}\hat{\boldsymbol{\imath}}_v \qquad \rho(r) = \rho_0 \exp(-(r - r_0)/H)$$



- ρ = Atmospheric density (height, location, time, sun)
- C_D = Drag coefficient (2.2 for flat plate, 2.0 for sphere)
- $\boldsymbol{v}$ = Velocity relative to atmosphere (≈ w.r.t. Earth)
- A = Cross-section area (size, attitude)
- m = Mass of object

- Atmospheric drag is more easily modelled in (n,v,h) frame

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Atmospheric Drag (2)

- It is convenient to write the GVE with the acceleration vector expressed in the new rotating frame (aligned with velocity, *v*, rather than position, *r*)

$$\boldsymbol{a} = a_r\hat{\boldsymbol{\imath}}_r + a_\theta\hat{\boldsymbol{\imath}}_\theta + a_h\hat{\boldsymbol{\imath}}_h = a_n\hat{\boldsymbol{\imath}}_n + a_v\hat{\boldsymbol{\imath}}_v + a_h\hat{\boldsymbol{\imath}}_h$$

$$\hat{\boldsymbol{\imath}}_n = \frac{h}{pv}\left(\frac{p}{r}\hat{\boldsymbol{\imath}}_r - e\sin f\,\hat{\boldsymbol{\imath}}_\theta\right) \qquad \hat{\boldsymbol{\imath}}_v = \frac{\boldsymbol{v}}{v} = \frac{h}{pv}\left(e\sin f\,\hat{\boldsymbol{\imath}}_r + \frac{p}{r}\hat{\boldsymbol{\imath}}_\theta\right)$$

- This provides the following transformations between the rotating frames

$$\begin{pmatrix} a_r \\ a_\theta \end{pmatrix} = \frac{h}{pv}\begin{bmatrix} p/r & e\sin f \\ -e\sin f & p/r \end{bmatrix}\begin{pmatrix} a_n \\ a_v \end{pmatrix} = \frac{1}{\sqrt{1+e^2+2e\cos f}}\begin{bmatrix} 1+e\cos f & e\sin f \\ -e\sin f & 1+e\cos f \end{bmatrix}\begin{pmatrix} a_n \\ a_v \end{pmatrix}$$

$$\begin{pmatrix} a_n \\ a_v \end{pmatrix} = \frac{h}{pv}\begin{bmatrix} p/r & -e\sin f \\ e\sin f & p/r \end{bmatrix}\begin{pmatrix} a_r \\ a_\theta \end{pmatrix} = \frac{1}{\sqrt{1+e^2+2e\cos f}}\begin{bmatrix} 1+e\cos f & -e\sin f \\ e\sin f & 1+e\cos f \end{bmatrix}\begin{pmatrix} a_r \\ a_\theta \end{pmatrix}$$

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Atmospheric Drag (3)

- The new GVE can be used to incorporate atmospheric drag as follows

$$\frac{da}{dt} = \frac{2a^2 v}{\mu} a_v$$

$$\frac{de}{dt} = \frac{1}{v}\left(\frac{r}{a}\sin f\, a_n + 2(e+\cos f)a_v\right)$$

$$\frac{di}{dt} = \frac{r\cos\theta}{h} a_h$$

$$\frac{d\Omega}{dt} = \frac{r\sin\theta}{h\sin i} a_h$$

$$\frac{d\omega}{dt} = \frac{1}{ev}\left(-\left(2e+\frac{r}{a}\right)\cos f\, a_n + 2\sin f\, a_v\right) - \frac{r\sin\theta\cos i}{h\sin i} a_h$$

$$\frac{dM}{dt} = n + \frac{b}{aev}\left(\frac{r}{a}\cos f\, a_n - 2\left(1+e^2\frac{r}{p}\right)\sin f\, a_v\right)$$

$$\frac{da}{dt} = -\left(\frac{A}{m}\right)C_d\rho\frac{v^3}{an^2}$$

< 0

$$\frac{de}{dt} = -\left(\frac{A}{m}\right)C_d\rho(e+\cos f)v$$

< 0

$$\frac{di}{dt} = 0$$

Secular

$$\frac{d\Omega}{dt} = 0$$

Periodic

$$\frac{d\omega}{dt} = -\left(\frac{A}{m}\right)C_d\rho\frac{\sin f}{e}v$$

$$\frac{dM}{dt} = n + \frac{b}{ae}\left(\frac{A}{m}\right)C_d\rho\left(1+e^2\frac{r}{p}\right)\sin f\, v$$

Orbit Maneuvers (1)

- The GVE can be used to derive analytical control laws for impulsive thrusts, i.e. maneuvers that can be modeled as instantaneous variations of velocity
- For small maneuvers we can integrate over the impulse and obtain a modified version of GVE where the derivatives w.r.t. time become a net variation of the argument after the impulse

(2.107a) $\frac{d}{dt} \longrightarrow \Delta$ (2.107f) $d \longrightarrow \Delta v$

- Simple example is given by corrections of inclination and ascending node

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{d_h r \cos(f+\omega)}{h} \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{d_h r \sin(f+\omega)}{h \sin i}$$

Integration over impulse

$$\Delta i = \frac{r\cos\theta}{h}\Delta v_h \qquad \Delta\Omega = \frac{r\sin\theta}{h\sin i}\Delta v_h$$

Solve for maneuver location and size

$$\theta_c = \arctan\frac{\Delta\Omega \sin i}{\Delta i} \qquad \Delta v_h = \frac{h}{r}\sqrt{\Delta i^2 + \Delta\Omega^2 \sin i^2}$$

Orbit Maneuvers (2)

Example 2.2. *Determine the impulse required to suppress the nodal precession accumulated over one orbit period for a mean circular orbit with* $\bar{a} = 7100$ km *and* $\bar{i} = 70°$.

$$\frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}t} = -\frac{3}{2} J_2 \left(\frac{R_e}{\bar{p}}\right)^2 \bar{n} \cos \bar{i}$$

$$R_e = 6378.1363 \text{ km}$$

$$\mu = 3.98604415 \times 10^5 \text{ km}^3/\text{s}^2$$

$$\Delta\Omega = -0.00282 \text{ rad}$$

$$\Delta v_h = \left| \frac{\Delta\Omega \sqrt{\mu/\bar{a}} \sin \bar{i}}{\sin \theta} \right| = 19.829 \text{ m/s}$$

Example 2.3. *Determine the impulse required to suppress the differential nodal precession accumulated over one orbit period for a mean circular orbit with* $\bar{a} = 7100$ km *and* $\bar{i} = 70°$ *due to a differential inclination* $\delta i = 1/7100$ rad.

$$\delta i \rightarrow \delta\Omega = 3\pi J_2 (R_e/\bar{a})^2 (\sin \bar{i})\delta i \rightarrow \Delta v_h = 7.6732 \times 10^{-3} \text{ m/s}$$

over 1 orbit

41 m/s over 1 year

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