AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 7

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Schaub

Numerical Simulations (1)

- Do equations (14.122), (14.127), and (14.131) for arbitrary, small, and near-zero eccentricity indeed predict the spacecraft formation geometry?
- Let the chief orbit elements and the orbit element differences be given by

Orbit elements	Value	Units
a	7555	km
e	0.03 or 0.13	
i	48.0	deg
Ω	20.0	deg
ω	10.0	deg
M_0	0.0	deg

Orbit elements	Value	Units
δa	0	km
δe	0.00095316	
δ_i	0.0060	deg
$\delta\Omega$	0.100	deg
$\delta\omega$	0.100	deg
δM_0	-0.100	deg

- The ratio $\rho/r \approx 0.003 << e$, thus we expect poor performance for small and near-zero eccentricity assumptions
- The relative motion is specified to be bounded (identical semi-major axis)

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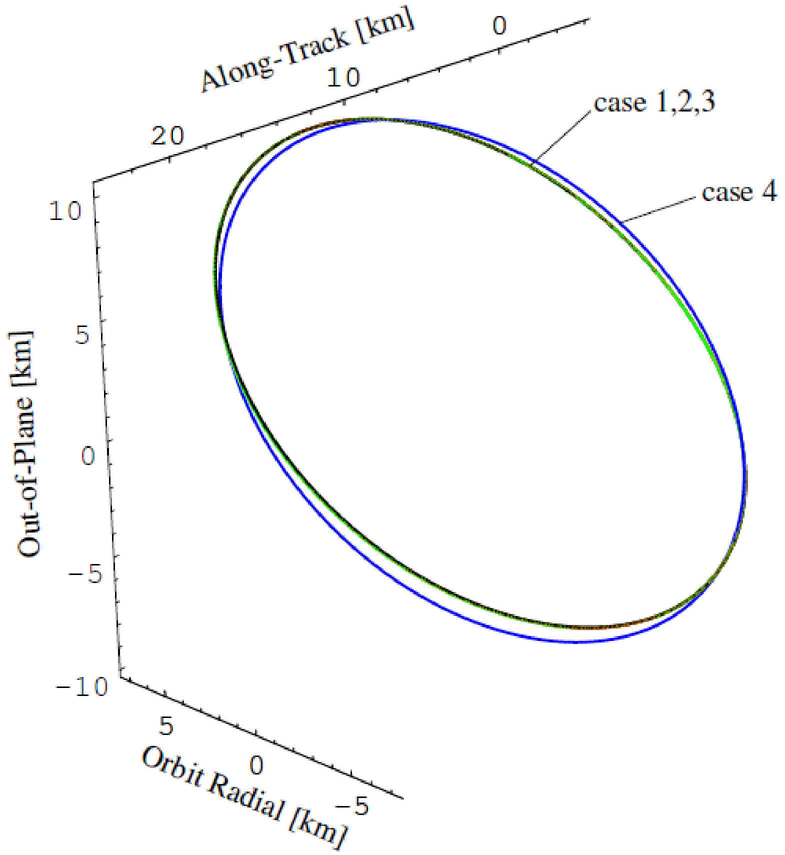
Numerical Simulations (2)

1) True nonlinear equations (numerical integration)

2) Linear solution for arbitrary *e* (analytical)

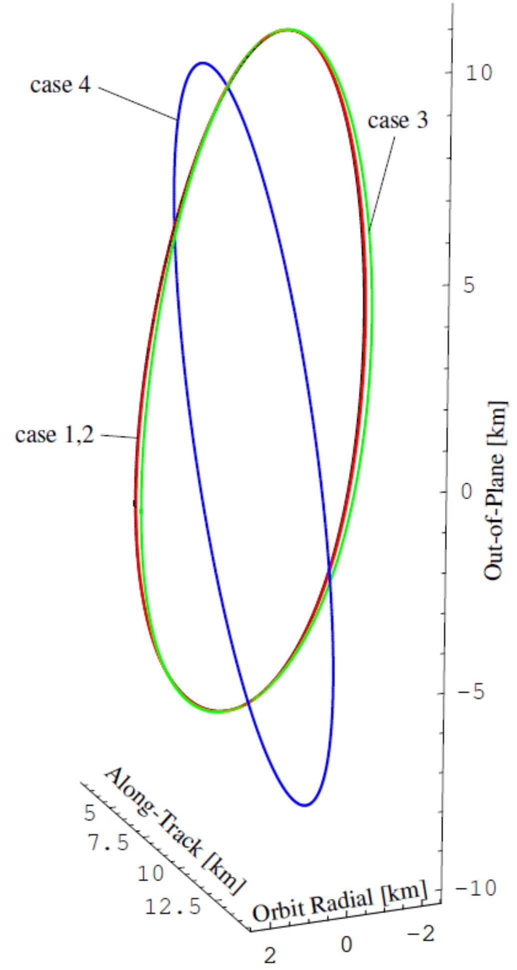
3) Linear solution for small *e* (analytical)

4) Linear solution for near-zero *e* (analytical)



(i) Relative Orbits in Hill Frame for $e =$ 0.03

Increase eccentricity



(ii) Relative Orbits in Hill Frame for e $= 0.13$

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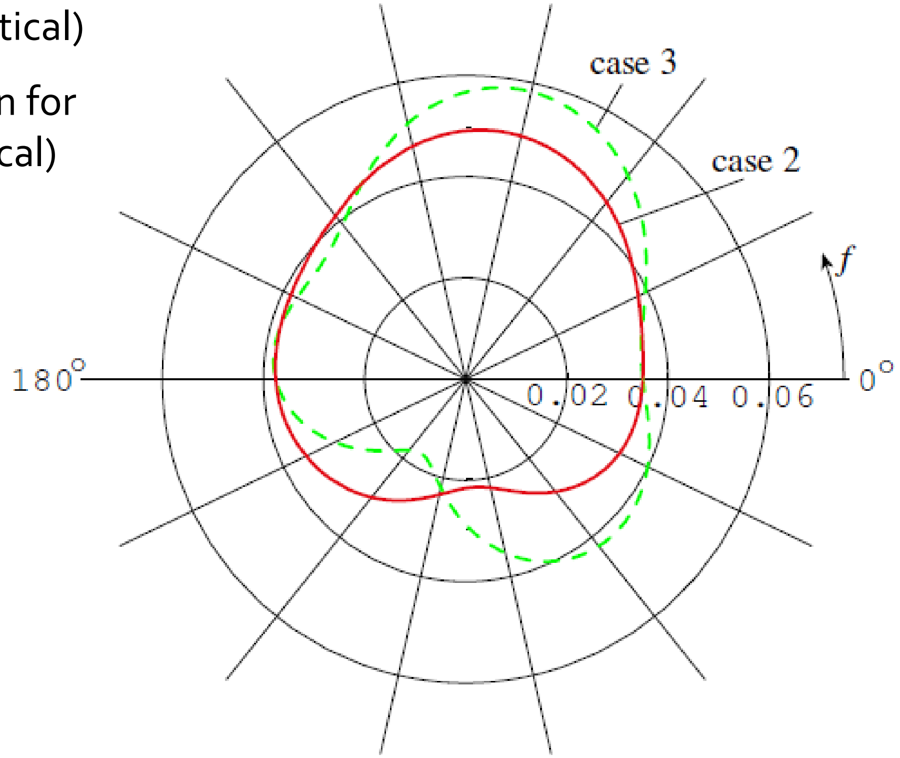
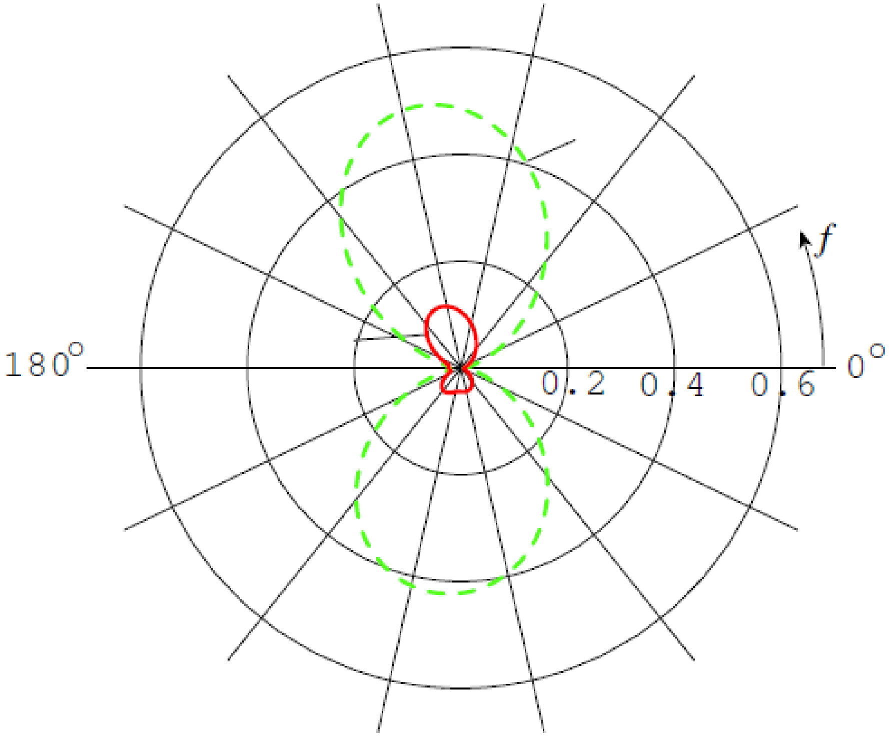
Numerical Simulations (3)

2) Linear solution for arbitrary *e* (analytical)

3) Linear solution for small *e* (analytical)



(iii) RMS Relative Orbit Error in kilometers vs. Chief True Anomaly Angle f for $e = 0.03$



(iv) RMS Relative Orbit Error in kilometers vs. Chief True Anomaly Angle f for $e = 0.13$

D'Amico

Relative Eccentricity and Inclination Vectors (1)

- Usage of nearly non-singular orbital elements lead to an alternative representation of (14.131)

$$(2.1) \qquad \boldsymbol{\alpha} = \begin{pmatrix} a \\ u \\ e_x \\ e_y \\ i \\ \Omega \end{pmatrix} = \begin{pmatrix} a \\ \omega + M \\ e \cos\omega \\ e \sin\omega \\ i \\ \Omega \end{pmatrix}$$

Mean argument of latitude

u

Eccentricity vector

$\boldsymbol{e} = (e_x, e_y)^T$

- The orbit element differences are replaced by so called relative orbit elements

$$(2.2) \quad \delta\boldsymbol{\alpha} = \begin{pmatrix} \delta a \\ \delta\lambda \\ \delta e_x \\ \delta e_y \\ \delta i_x \\ \delta i_y \end{pmatrix} = \begin{pmatrix} (a_d - a)/a \\ (u_d - u) + (\Omega_d - \Omega)\cos i \\ e_{x_d} - e_x \\ e_{y_d} - e_y \\ i_d - i \\ (\Omega_d - \Omega)\sin i \end{pmatrix}$$

Relative semi-major axis

Relative mean longitude

Relative eccentricity vector

Relative inclination vector

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Relative Eccentricity and Inclination Vectors (2)

- Cartesian or polar representations for relative ecc./incl. vectors can be used

$$\delta \boldsymbol{e} = \begin{pmatrix} \delta e_x \\ \delta e_y \end{pmatrix} = \delta e \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \quad (2.3) \qquad \delta \boldsymbol{i} = \begin{pmatrix} \delta i_x \\ \delta i_y \end{pmatrix} = \delta i \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \quad (2.4)$$

- The amplitudes of these vectors (δ) shall not be confused with the orbit element differences of Schaub (Δ) !!
- The phases of these vectors are called relative perigee and ascending node
- Reformulating the linear mapping for near-zero eccentricity provides

$$\begin{array}{lcllll} \delta r_r/a & \approx & \delta a & & -\delta e_x \cos u & -\delta e_y \sin u \\ \delta r_t/a & \approx & -\frac{3}{2}\delta a u & +\delta\lambda & +2\delta e_x \sin u & -2\delta e_y \cos u \\ \delta r_n/a & \approx & & & +\delta i_x \sin u & -\delta i_y \cos u \end{array} \quad (2.17)$$

- or in amplitude/phase form

Compare with HCW !!

$$\boxed{\begin{array}{lclll} \delta r_r/a & = & \delta a & & -\delta e \cos(u-\varphi) \\ \delta r_t/a & = & \delta\lambda & -\frac{3}{2}\delta a u & +2\delta e \sin(u-\varphi) \\ \delta r_n/a & = & & & +\delta i \sin(u-\vartheta) \end{array}} \quad (2.18)$$

Revised Insight into Relative Orbit Geometry

Arithmetic difference (Schaub)

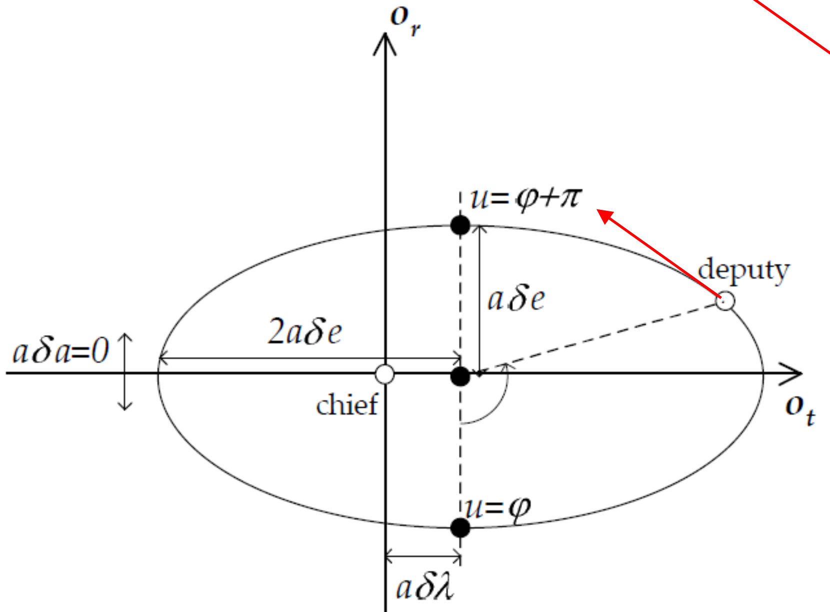
Relative orbit element (D'Amico)

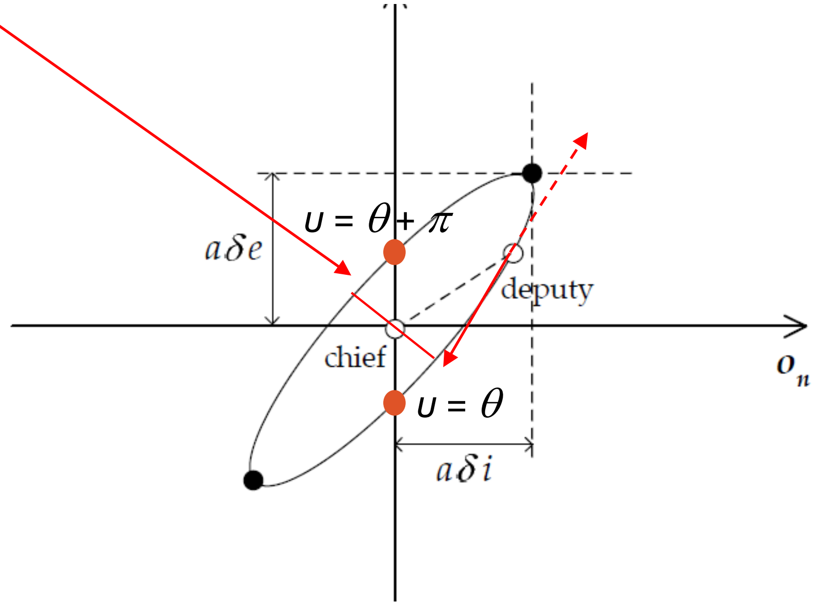
- Bounded relative motion
- Centered relative motion
- Minimum separation in RN

$$\delta a = 0 \tag{2.19}$$

$$\delta\lambda = 0 \Leftrightarrow \Delta u = -\Delta\Omega \cos i$$

$$\delta r_{nr}^{\min} = \frac{\sqrt{2}a\,|\delta\boldsymbol{e}\cdot\delta\boldsymbol{i}|}{\left(\delta e^2 + \delta i^2 + |\delta\boldsymbol{e}+\delta\boldsymbol{i}|\cdot|\delta\boldsymbol{e}-\delta\boldsymbol{i}|\right)^{1/2}} \tag{2.22}$$



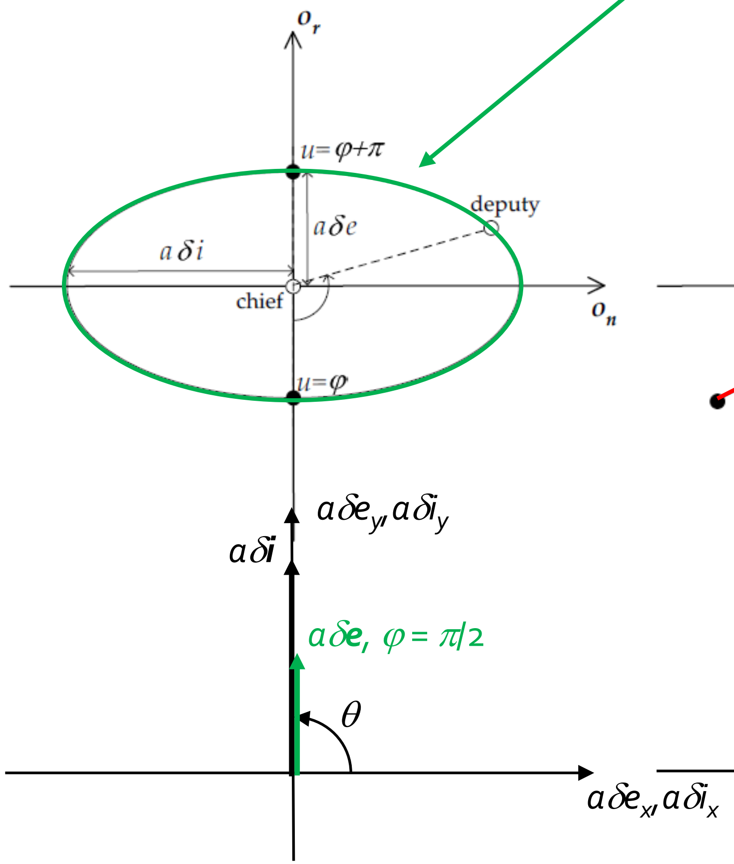


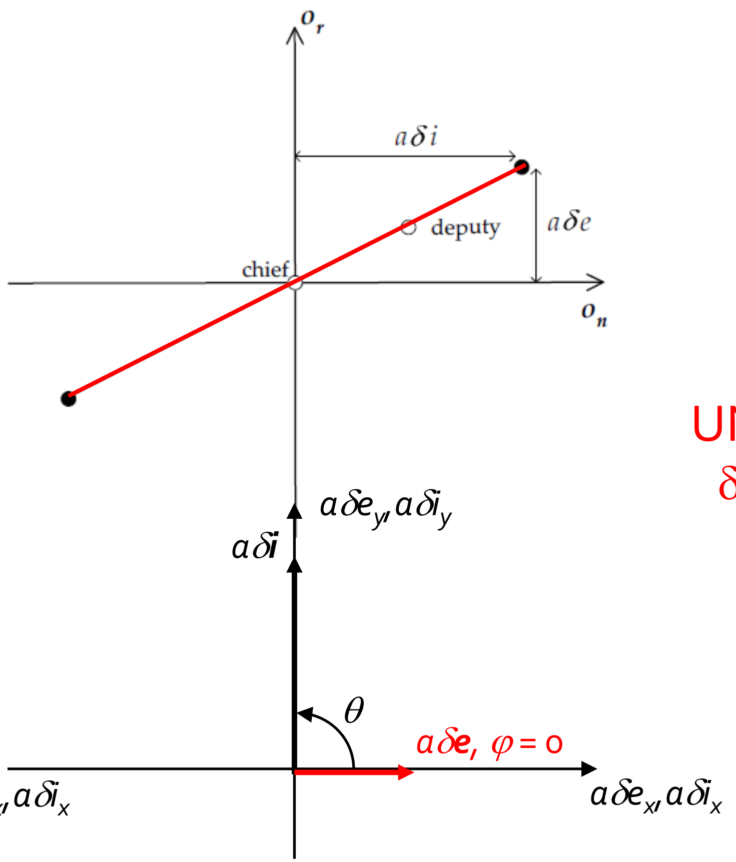
Relative Eccentricity/Inclination Vector Separation

$$\delta r_{nr}^{\min}(u=\varphi, \varphi+k\pi) = a\min\{\delta e, \delta i\}$$

SAFE

$\delta \boldsymbol{e} \,//\, \delta \boldsymbol{i}$



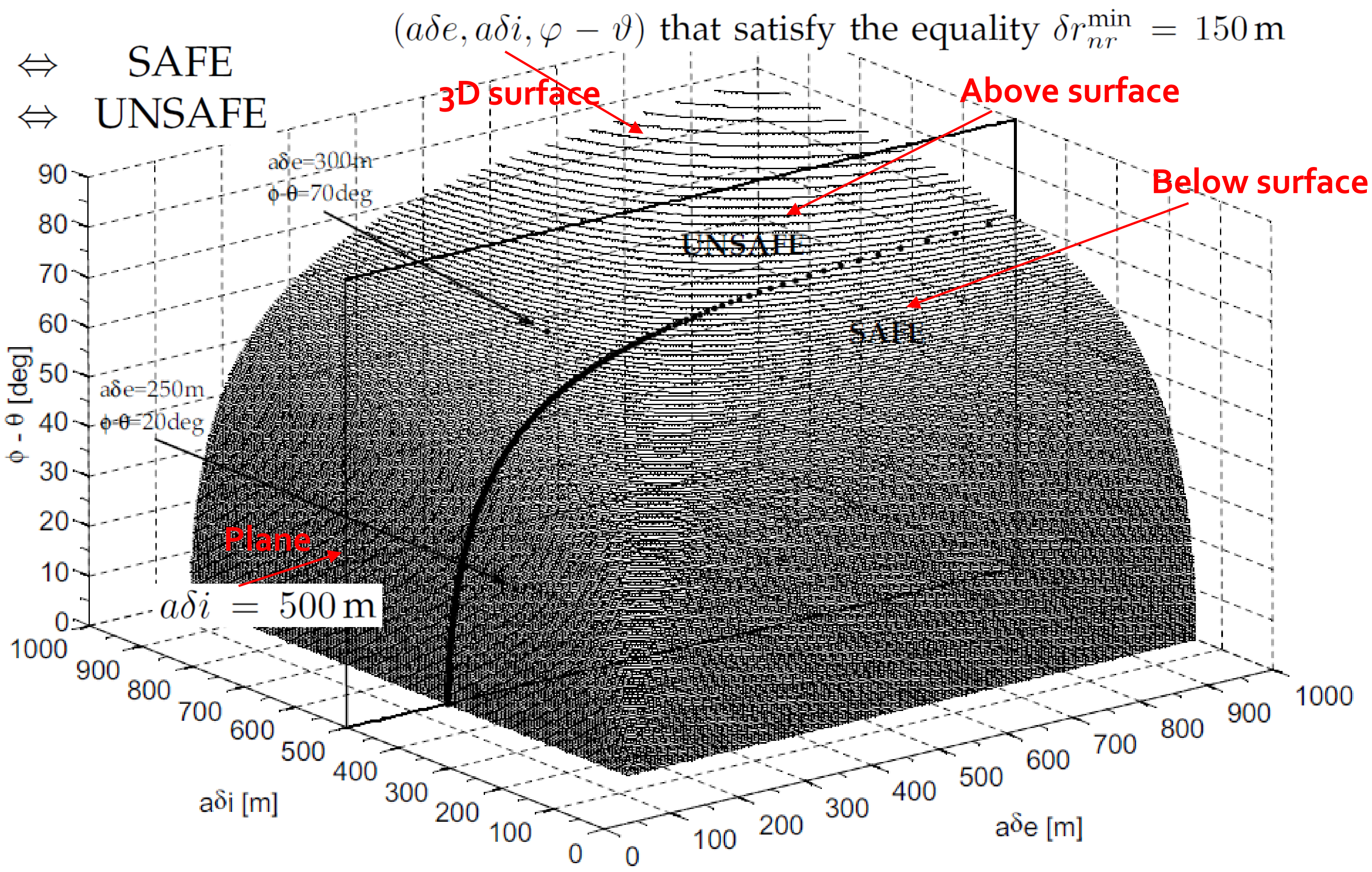


UNSAFE

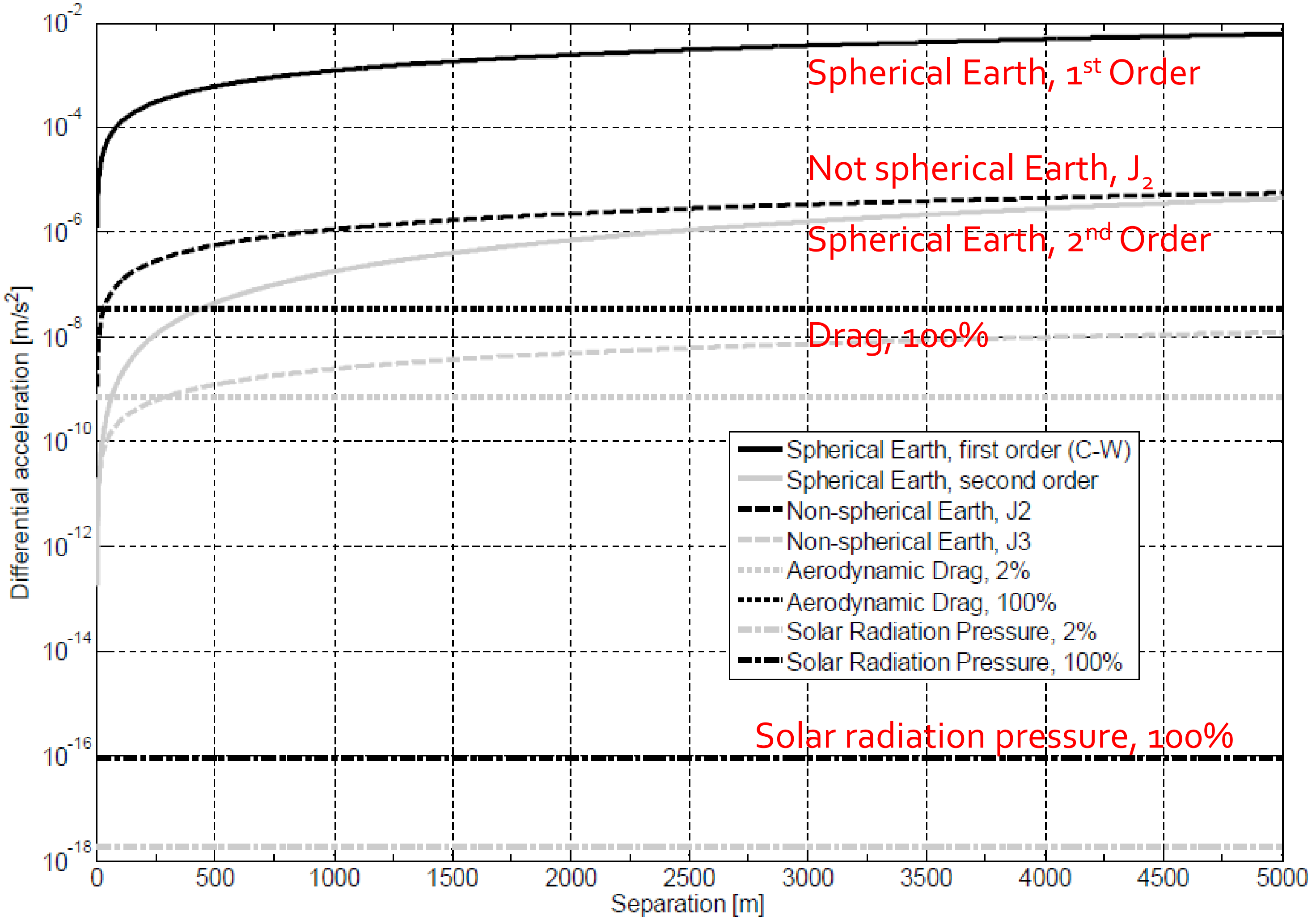
$\delta \boldsymbol{e} \perp \delta \boldsymbol{i}$

Collision-Free Formation Flying Configurations

$$\delta r_{nr}^{\min} \geq d_{\min} \quad \Leftrightarrow \quad \text{SAFE}$$
$$\delta r_{nr}^{\min} < d_{\min} \quad \Leftrightarrow \quad \text{UNSAFE}$$



Perturbed Relative Motion (LEO <1500km)



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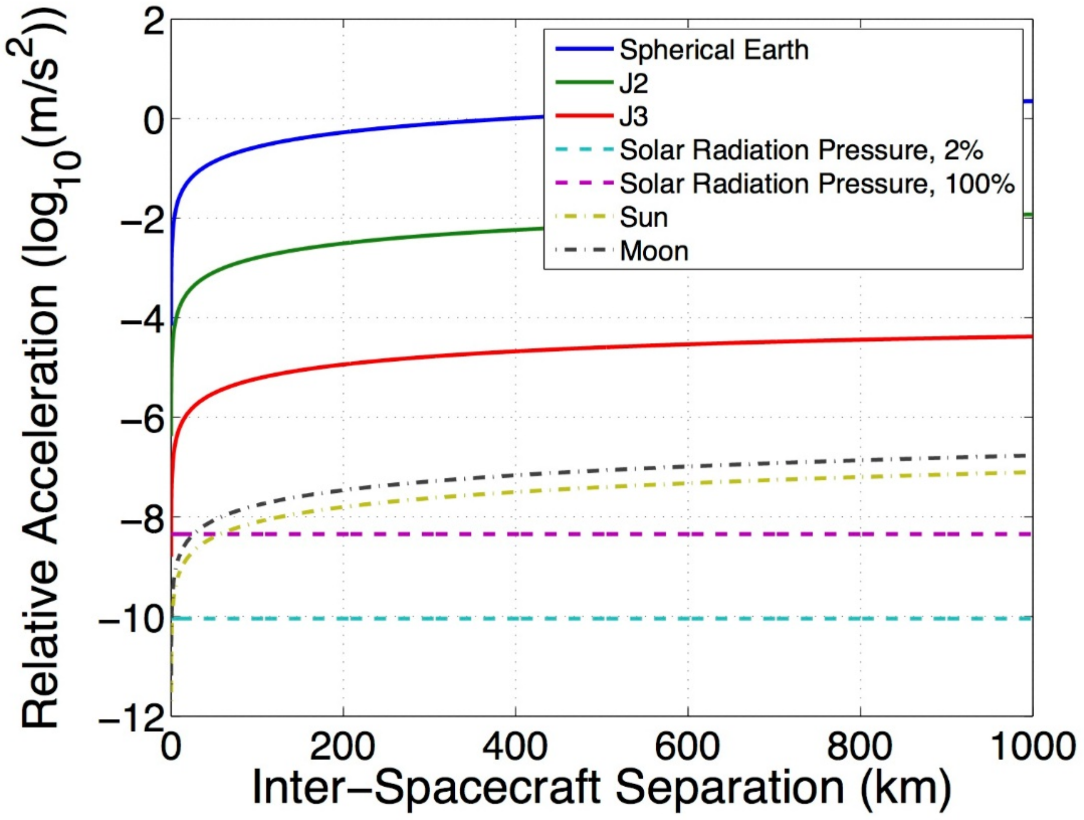
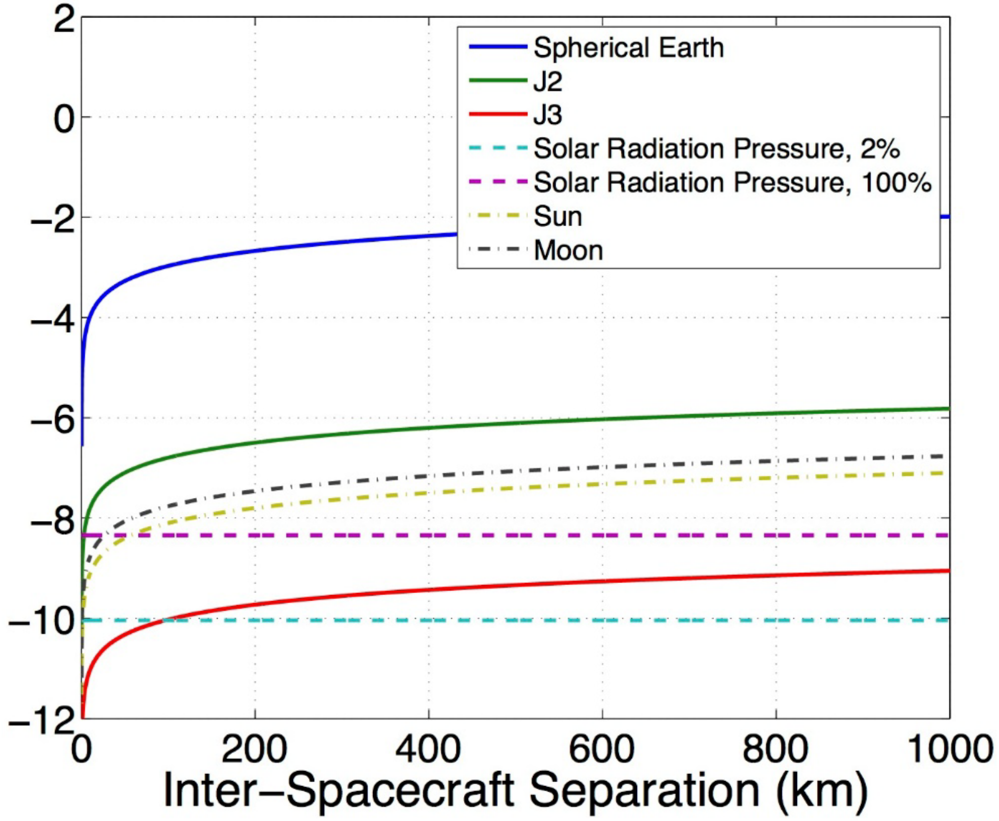
Perturbed Relative Motion at Different Altitudes

LEO Perturbations

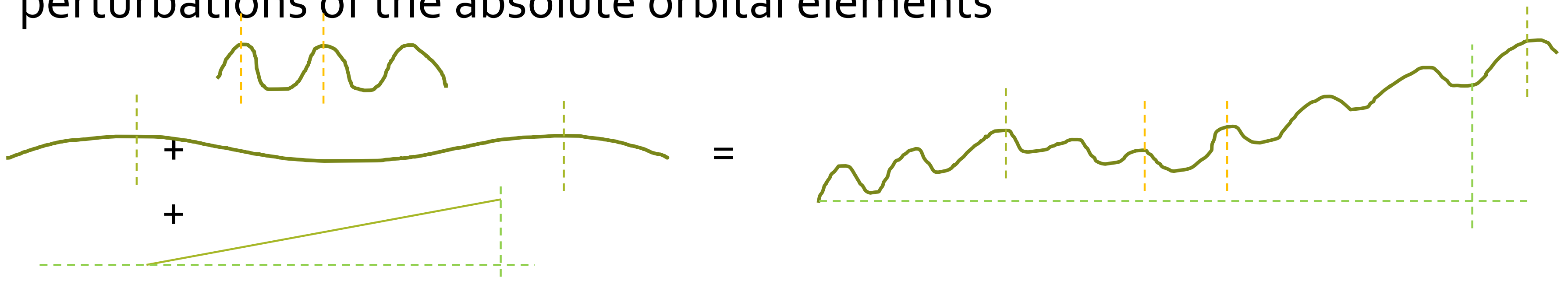


GEO Perturbations



Earth's Oblateness J_2 Effects (Absolute, 1)

- The Earth's equatorial bulge causes short-, long-period and secular perturbations of the absolute orbital elements



- The variation of parameters method (see Brouwer and Lyddane [1959-1963]) provides the analytical tool to capture these effects for the absolute orbit

$$\frac{d}{dt}\begin{pmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ M \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3\gamma n \cos i \\ \frac{3}{2}\gamma n(5\cos^2 i - 1) \\ \frac{3}{2}\gamma\eta n(3\cos^2 i - 1) \end{pmatrix} \qquad (2.26)$$

Secular variations of Keplerian orbital elements caused by J_2

$$\gamma = \frac{J_2}{2}\left(\frac{R_E}{a}\right)^2 \frac{1}{\eta^4} \qquad (2.25)$$

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Earth's Oblateness J_2 Effects (Relative, 2)

- We can substitute the long-period and secular effects into our definition of ROE and neglect 2nd order effects (in *e* and *δ*) as done previously to obtain

Secular variations of relative orbital elements caused by J_2

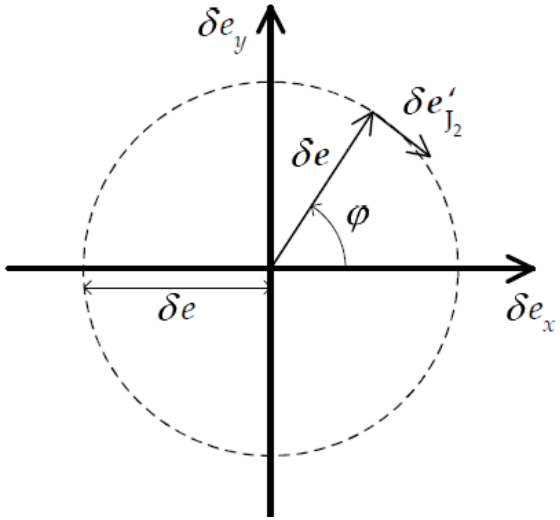
$$\delta\dot{\boldsymbol{\alpha}} = \begin{pmatrix} 0 \\ -\frac{21}{2}\gamma n \sin(2i)\delta i_x \\ -\frac{3}{2}\gamma n(5\cos^2 i - 1)\delta e_y \\ \frac{3}{2}\gamma n(5\cos^2 i - 1)\delta e_x \\ 0 \\ 3\gamma n \sin^2 i \delta i_x \end{pmatrix} \quad (2.28)$$

- Using the mean argument of latitude *u* as independent variable, after integration over u-u_0, we obtain

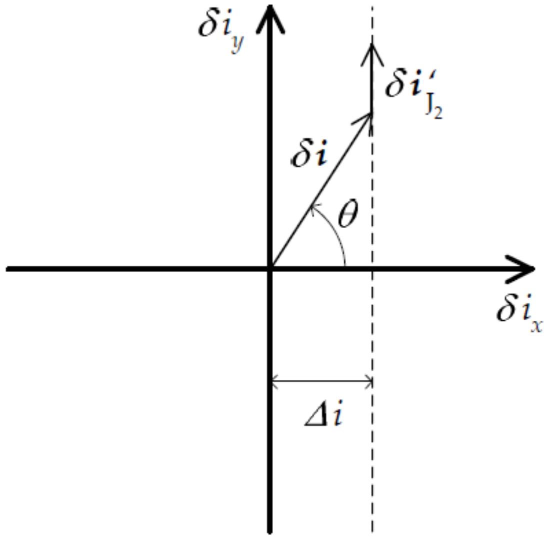
$$\delta\boldsymbol{\alpha}(t) = \begin{pmatrix} \delta a \\ \delta\lambda - \frac{21}{2}(\gamma\sin(2i)\delta i_x + \frac{1}{7}\delta a)(u(t) - u_0) \\ \delta e \cos(\varphi + \varphi'(u(t) - u_0)) \\ \delta e \sin(\varphi + \varphi'(u(t) - u_0)) \\ \delta i_x \\ \delta i_y + 3\gamma\sin^2 i \delta i_x(u(t) - u_0) \end{pmatrix} \quad (2.29)$$

$$\varphi' = \frac{d\varphi}{du} = \frac{3}{2}\gamma(5\cos^2 i - 1) \quad (2.30)$$

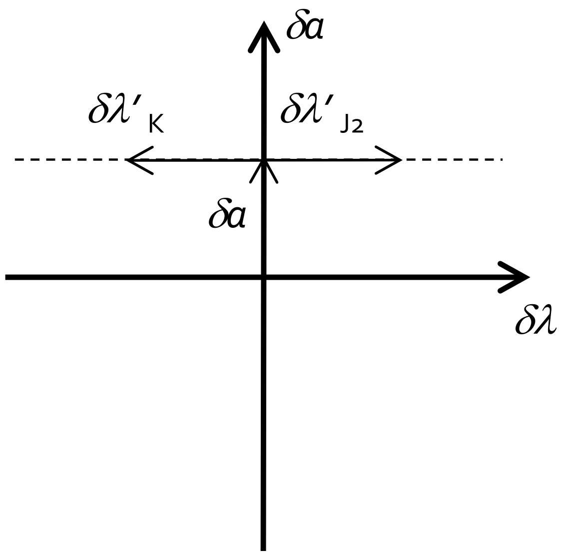
Earth's Oblateness J_2 Effects (Relative, 3)



$$\varphi' = \frac{d\varphi}{du} = \frac{3}{2}\gamma(5\cos^2 i - 1)$$



$$\delta i'_{J2} = 3\gamma\sin^2 i\,\delta i_x$$



$$\delta\lambda' = -\frac{21}{2}\left(\gamma\sin(2i)\delta i_x + \frac{1}{7}\delta a\right)$$

Clockwise for sun-synchronous orbits with period of about 100-200 days or about 1000 times the orbital period. A critical inclination exists!

Proportional to Δi and J_2. Note that $\sin(2i)$ is negative for sun-synchronous orbits and closed relative orbits are given by

$$\Delta a \approx 1000\Delta i$$

Differential Drag Effects (1)

GVE

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\left(\frac{A}{m}\right) C_d \rho \frac{v^3}{an^2}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\left(\frac{A}{m}\right) C_d \rho (e + \cos f) v$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\left(\frac{A}{m}\right) C_d \rho \frac{\sin f}{e} v$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n + \frac{b}{ae}\left(\frac{A}{m}\right) C_d \rho \left(1 + e^2 \frac{r}{p}\right) \sin f v$$

- The interaction of the upper atmosphere with the satellite's surface produces the dominant non-conservative disturbance for LEO spacecraft

$$|\ddot{r}_t| = \frac{1}{2}\rho v^2 C_{\mathrm{D}} \frac{A}{m}$$

Along-track acceleration

$$B = C_{\mathrm{D}} \frac{A}{m}$$

Ballistic coefficient

- If we neglect density variations over distances of less than a few kilometers, the relative along-track acceleration for two formation-flying spacecraft is driven by the differential ballistic coefficient ΔB

$$\delta r_t = \frac{1}{2} a \Delta \ddot{u} (t - t_0)^2 = \frac{3}{4n^2} \Delta B \rho v^2 (u(t) - u_0)^2$$

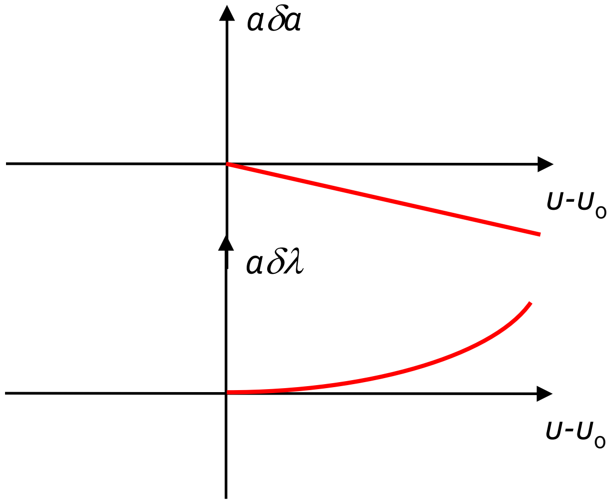
- The first-order relative motion model can be extended to include this accumulated along-track offset, either using Cartesian or ROE parameters

Differential Drag Effects (2)

Only due to drag

$$a\delta\lambda(t) = \frac{3}{8n^2}\Delta B\rho v^2\left(u(t) - u_0\right)^2$$

$$a\delta a(t) = -\frac{1}{2n^2}\Delta B\rho v^2\left(u(t) - u_0\right)$$



- Impact of differential drag can be minimized by employing identically designed spacecraft. The ballistic coefficients can be matched to roughly 1% at launch.
- Mass variations during lifetime can cause an additional difference of 1%
- Considering typical atmospheric density values in LEO, differential accelerations of $<10^1$ nm/s^2 are encountered which require negligible delta-vs
- This conclusion is no longer valid during safe modes (10^2 nm/s^2) or for non-cooperative spacecraft where differential drag can match absolute drag

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New State Transition Matrix based on ROEs

$$\begin{pmatrix} \delta \dot{a} \\ \delta \boldsymbol{\alpha} \end{pmatrix}_t = \boldsymbol{\Phi}(t - t_0) \begin{pmatrix} \delta \dot{a} \\ \delta \boldsymbol{\alpha} \end{pmatrix}_{t_0}$$

$$\begin{array}{cc} & \begin{array}{ccccccc} \text{DRAG} & \text{KEPLER} & & & \text{SECULAR J2} & & \end{array} \\ \boldsymbol{\Phi}_{F,0} = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ (t_F - t_0) & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\nu}{2}(t_F - t_0)^2 & \nu(t_F - t_0) & 1 & 0 & 0 & \mu(t_F - t_0) & 0 \\ 0 & 0 & 0 & 1 & -\dot{\varphi}(t_F - t_0) & 0 & 0 \\ 0 & 0 & 0 & \dot{\varphi}(t_F - t_0) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda(t_F - t_0) & 1 \end{bmatrix} \end{array}$$

$$\nu = -\frac{3}{2}n \qquad \dot{\varphi} = \frac{3}{2}n\gamma(5\cos^2 i - 1) \qquad \mu = -\frac{21}{2}n\gamma \sin 2i \qquad \lambda = 3n\gamma \sin^2 i$$

Maneuver Planning in Near-Circular Orbit

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2a^2 v}{\mu} a_v \\
\frac{de}{dt} &= \frac{1}{v}\left(\frac{r}{a}\sin f\, a_n + 2(e+\cos f) a_v\right) \\
\frac{di}{dt} &= \frac{r\cos\theta}{h} a_h \\
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{h\sin i} a_h \\
\frac{d\omega}{dt} &= \frac{1}{ev}\left(-\left(2e+\frac{r}{a}\right)\cos f\, a_n + 2\sin f\, a_v\right) - \frac{r\sin\theta\cos i}{h\sin i} a_h \\
\frac{dM}{dt} &= n + \frac{b}{aev}\left(\frac{r}{a}\cos f\, a_n - 2\left(1+e^2\frac{r}{p}\right)\sin f\, a_v\right)
\end{aligned}
$$

GVE

- A relative orbit control system is necessary either to maintain the nominal formation geometry over the mission lifetime (i.e., formation keeping) or to acquire new formation geometries (i.e., formation reconfiguration)
- The inversion of the solution of the HCW equations expressed in terms of ROE provides the ideal framework to design closed-form deterministic impulsive maneuvering schemes

Current orbit location $u \geq u_M$

Maneuver location

Desired variation of ROE

$$
\begin{aligned}
a\delta a &\approx && +2\delta v_t/n && \\
a\delta\lambda &\approx -2\delta v_r/n && -3(u-u_M)\delta v_t/n && \\
a\delta e_x &\approx +\delta v_r \sin u_M/n && +2\delta v_t \cos u_M/n && \\
a\delta e_y &\approx -\delta v_r \cos u_M/n && +2\delta v_t \sin u_M/n && \\
a\delta i_x &\approx && && +\delta v_n \cos u_M/n \\
a\delta i_y &\approx && && +\delta v_n \sin u_M/n
\end{aligned}
\qquad (2.38)
$$

Instantaneous variation of velocity

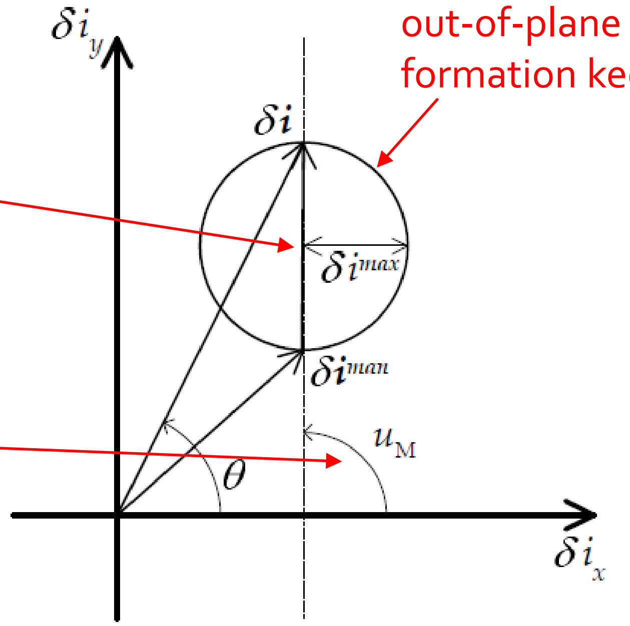
Maneuver Planning: Out-Of-Plane

- The problem consists of 2 unknowns δv_n, u_M and 2 equations and can be solved through a single- or double-impulse

Magnitude of desired correction vector

$$\begin{aligned} \delta v_n &= na \left\| \delta \boldsymbol{i}^{\mathrm{man}} - \delta \boldsymbol{i} \right\| \\ u_{\mathrm{M}} &= \arctan \left[(\delta i_y^{\mathrm{man}} - \delta i_y)/(\delta i_x^{\mathrm{man}} - \delta i_x) \right] \end{aligned} \quad (2.64)$$

Phase of desired correction vector



Control window for out-of-plane formation keeping

Maneuver Planning: In-Plane (1)

- The problem consists of 3 unknowns δv_r, δv_t, u_M and 4 equations (over-determined) and can be solved exactly only through a double-impulse scheme which doubles the number of unknowns (under-determined)

Note: drift of $\delta\lambda$ caused by δa after first maneuver is not taken into account here (becomes guidance problem)

$$
\begin{aligned}
\delta v_{t_1} &= \frac{na}{4}\left[\delta a + \delta e\cos(u_{\mathrm{M}_1} - \xi)\right] & -\frac{na}{4}\chi\left[\frac{\delta\lambda}{2} + \delta e\sin(u_{\mathrm{M}_1} - \xi)\right] \\
\delta v_{r_1} &= \frac{na}{2}\left[-\frac{\delta\lambda}{2} + \delta e\sin(u_{\mathrm{M}_1} - \xi)\right] & -\frac{na}{2}\chi\left[\delta a - \delta e\cos(u_{\mathrm{M}_1} - \xi)\right] \\
\delta v_{t_2} &= \frac{na}{4}\left[\delta a - \delta e\cos(u_{\mathrm{M}_1} - \xi)\right] & +\frac{na}{4}\chi\left[\frac{\delta\lambda}{2} + \delta e\sin(u_{\mathrm{M}_1} - \xi)\right] \\
\delta v_{r_2} &= \frac{na}{2}\left[-\frac{\delta\lambda}{2} - \delta e\sin(u_{\mathrm{M}_1} - \xi)\right] & +\frac{na}{2}\chi\left[\delta a - \delta e\cos(u_{\mathrm{M}_1} - \xi)\right]
\end{aligned}
\qquad (2.42)
$$

First maneuver location

Second maneuver location

$$\chi = \frac{\sin(\Delta u_{\mathrm{M}})}{\cos(\Delta u_{\mathrm{M}}) - 1}$$

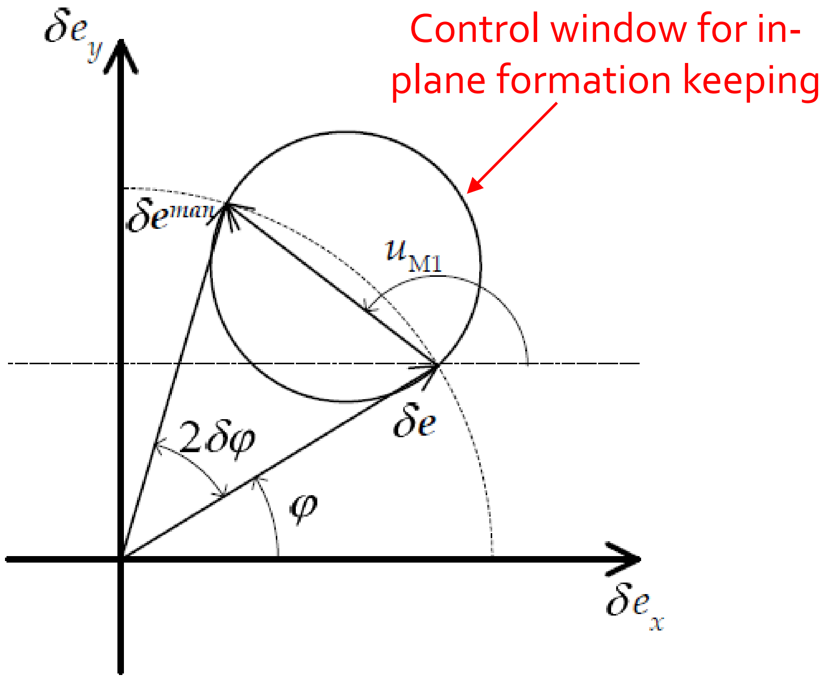
$$\xi = \arctan(\delta e_y / \delta e_x)$$

$$u_{\mathrm{M}_2} - u_{\mathrm{M}_1} \in]0, 2\pi[$$

Maneuver Planning: In-Plane (2)

- The most simple double-impulse scheme with $u_{M1} = \xi$ and $u_{M2} = u_{M1} + \pi$ turns out to be the minimum cost (total delta-v) solution for formation keeping

$$
\begin{aligned}
\delta v_{t_1} &= \frac{na}{4}\left[(\delta a^{\mathrm{man}} - \delta a) + \|\delta \boldsymbol{e}^{\mathrm{man}} - \delta \boldsymbol{e}\|\right] \\
\delta v_{t_2} &= \frac{na}{4}\left[(\delta a^{\mathrm{man}} - \delta a) - \|\delta \boldsymbol{e}^{\mathrm{man}} - \delta \boldsymbol{e}\|\right] \\
u_{\mathrm{M}_1} &= \arctan\left[(\delta e_y^{\mathrm{man}} - \delta e_y)/(\delta e_x^{\mathrm{man}} - \delta e_x)\right]
\end{aligned}
$$



Note: $\delta\lambda$ is controlled through δa^{man} achieved after maneuver pair

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