AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 5

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Linear Equations of Relative Motion (HCW)

- If the motion of the deputy w.r.t. chief is small as compared with the orbit radius, (4.74-4.76) can be linearized about the origin of the chief-fixed frame
- The linearized equations of relative motion are called the Hill-Clohessy-Wiltshire equations (HCW) and were developed by CW in the early 1960s to analyze spacecraft rendezvous
- We expand the right-hand side of (4.74-4.76) into a Taylor series about the origin and retain first order terms only

$$-\frac{\mu(a_0+x)}{\left[(a_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \approx n_0^2(2x-a_0)$$

$$-\frac{\mu y}{\left[(a_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \approx -n_0^2 y$$

$$-\frac{\mu z}{\left[(a_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \approx -n_0^2 z$$

$$\boxed{\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= d_x + u_x \\ \ddot{y} + 2n\dot{x} &= d_y + u_y \\ \ddot{z} + n^2 z &= d_z + u_z \end{aligned}} \quad \begin{matrix} (5.7) \\ (5.9) \end{matrix}$$

Nonhomogeneous form with perturbations and control acceleration

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State-Space Representation of HCW

- It is convenient to write the linear differential equations in state-space form
- Choosing the state vector

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \qquad (5.13)$$

$$\mathbf{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \qquad (5.14)$$

- The eigenvalues of A are $\{\pm nj, \pm nj, 0, 0\}$ so periodic and secular modes are expected to appear in the solution
- The solution of (5.13) is straightforward according to the standard procedure to compute the transition matrix

$$\mathbf{x}(t) = \mathrm{e}^{A(t-t_0)}\mathbf{x}(t_0) \qquad (5.15)$$

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Solution of HCW (1)

- The resulting HCW transition matrix is (t_o = 0)

$$e^{At} = \begin{bmatrix} 4-3\,c_{nt} & 0 & 0 & \frac{s_{nt}}{n} & \frac{2}{n}-\frac{2c_{nt}}{n} & 0 \\ -6\,nt+6\,s_{nt} & 1 & 0 & -\frac{2}{n}+\frac{2c_{nt}}{n} & \frac{4s_{nt}}{n}-3\,t & 0 \\ 0 & 0 & c_{nt} & 0 & 0 & \frac{s_{nt}}{n} \\ 3\,ns_{nt} & 0 & 0 & c_{nt} & 2\,s_{nt} & 0 \\ -6\,n+6\,nc_{nt} & 0 & 0 & -2\,s_{nt} & -3+4\,c_{nt} & 0 \\ 0 & 0 & -ns_{nt} & 0 & 0 & c_{nt} \end{bmatrix} \quad (5.16)$$

- which provides the solutions to the relative position (and velocity) components

$$x(t) = \left[4x(0)+\frac{2\dot{y}(0)}{n}\right] + \frac{\dot{x}(0)}{n}\sin(nt) - \left[3x(0)+\frac{2\dot{y}(0)}{n}\right]\cos(nt) \quad (5.17)$$

$$y(t) = -[6nx(0)+3\dot{y}(0)]t + \left[y(0)-\frac{2\dot{x}(0)}{n}\right] + \left[6x(0)+\frac{4\dot{y}(0)}{n}\right]\sin(nt) + \frac{2\dot{x}(0)}{n}\cos(nt) \quad (5.22)$$

$$z(t) = \frac{\dot{z}(0)}{n}\sin(nt) + z(0)\cos(nt)$$

Drift linear with time: in-plane motion is unstable

Out-of-plane motion decoupled from in-plane motion

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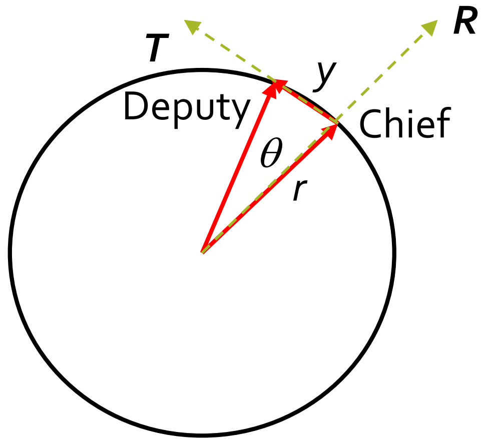
Solution of HCW (2)

- From a dynamics systems perspective, the HCW linear system (5.13) has multiple equilibria $\{x = 0, y = \text{const.}, z = 0\}$ while the original nonlinear system has a continuum of equilibria (4.77)
- This might look as a contradiction but two considerations can be made:
  1. The new equilibria apply in the vicinity of the chief-fixed rotating frame
  2. In the linear system, the coordinate *y* (along-track displacement) can be written as $y = a\delta\theta$ where $\delta\theta$ is a *curvilinear coordinate* along the circumference of the reference orbit
- The following condition needs to be applied to stabilize the along-track motion

$$\dot{y}(0) = -2nx(0) \qquad (5.23)$$



- These initial conditions yield a bounded relative motion to *first order*, i.e. only for HCW. There exists initial conditions which violate (5.23) but satisfy the energy matching condition for (4.74-4.76)

Solution of HCW (3)

- If (5.23) holds, the HCW's solution can be written in magnitude-phase form

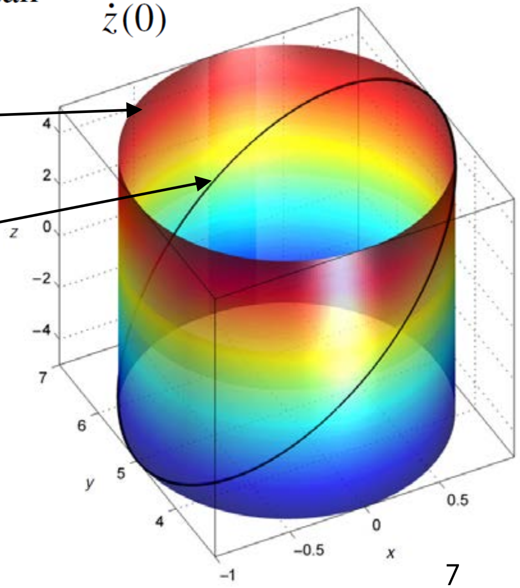
$$(5.24)\quad(5.31)\qquad \boxed{\begin{aligned} x(t) &= \rho_x \sin(nt + \alpha_x) \\ y(t) &= \rho_y + 2\rho_x \cos(nt + \alpha_x) \\ z(t) &= \rho_z \sin(nt + \alpha_z) \end{aligned}}$$

$$\left\{\begin{aligned} \rho_x &= \frac{\sqrt{\dot{x}^2(0) + x^2(0)n^2}}{n} \\ \rho_y &= [y(0) - 2\dot{x}(0)/n] \\ \rho_z &= \frac{\sqrt{\dot{z}^2(0) + z^2(0)n^2}}{n} \\ \alpha_x &= \tan^{-1}\frac{nx(0)}{\dot{x}(0)} \\ \alpha_z &= \tan^{-1}\frac{nz(0)}{\dot{z}(0)} \end{aligned}\right.$$

- These equations constitute a parametric representation of an *elliptic cylinder*

Phase matching
$\alpha_x = \alpha_z$

3D ellipse centered at
$(0, \rho_y, 0)$



Solution of HCW (4)

- RT or *xy*-projection is a 2:1 ellipse
  - Semimajor axis $2\rho_x$
  - Semiminor axis ρ_x
  - Eccentricity $\sqrt{1-\rho_x^2/(4\rho_x^2)} = \sqrt{3}/2$
  - Line of apsides along *y* axis
  - Center of motion at $[0,\ \rho_y]$

- RN or *xz*-projection is an oval

$$\alpha_z = \pi/2 + \alpha_x$$ ellipse

$$\alpha_z = \pi/2 + \alpha_x, \qquad \rho_x = \rho_z$$ circle

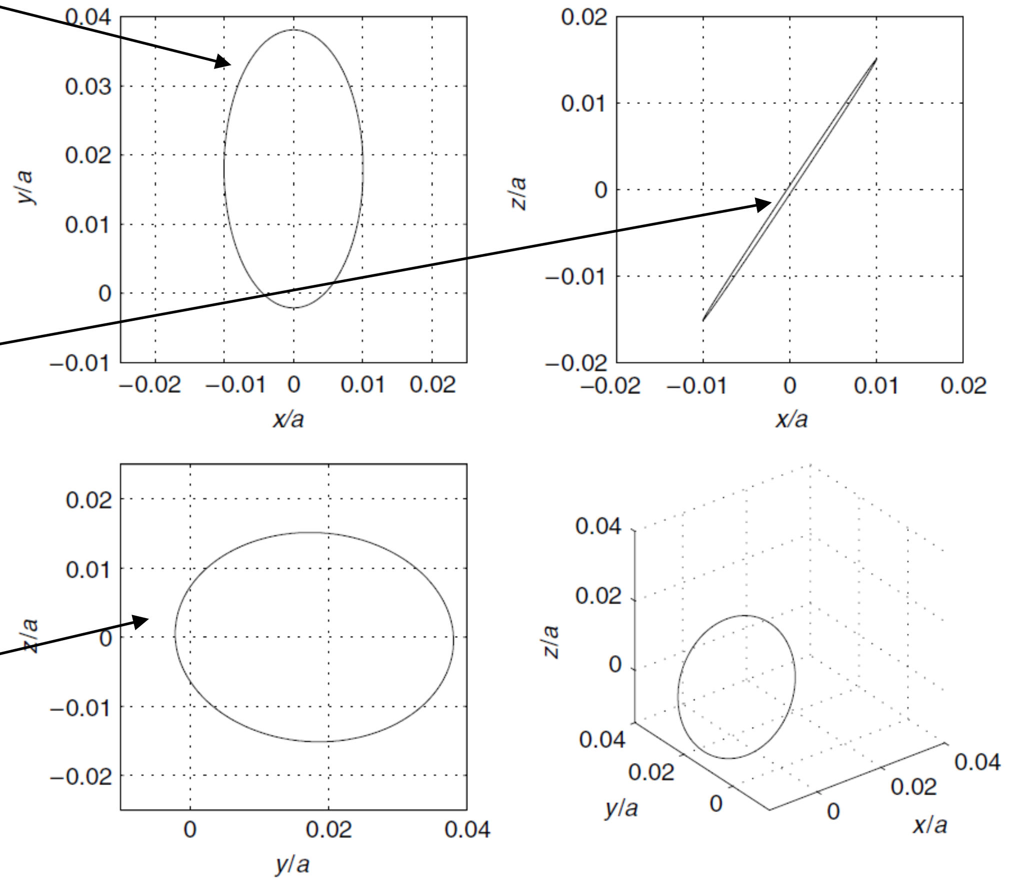
$$\alpha_x = \alpha_z$$ linear

- TN or *yz*-projection is an oval

$$\alpha_x = \alpha_z$$ ellipse

$$\alpha_x = \alpha_z, \qquad \rho_z = 2\rho_x$$ circle

$$\alpha_z = \pi/2 + \alpha_x$$ linear



Solution of HCW (Example, 1)

Example 5.1. *Two satellites are launched aboard a Delta launcher. The satellites are to fly in a close formation. An initial velocity impulse injects the satellites into a circular LEO of* 600 km. *An additional velocity impulse applied by the deputy de-attaches it from the chief so that the relative position components are*

$$x(0) = 69.78 \text{ km}, \quad y(0) = 139.56 \text{ km}, \quad z(0) = 104.67 \text{ km}$$

and the relative velocity components are

$$\dot{x}(0) = 7.5579 \text{ m/s}, \quad \dot{y}(0) = -151.116 \text{ m/s}, \quad \dot{z}(0) = 15.116 \text{ m/s}$$

Determine the geometry of the relative orbit. What initial conditions should be chosen instead of the given initial conditions to yield circular xz- and yz-projections?

- From input data

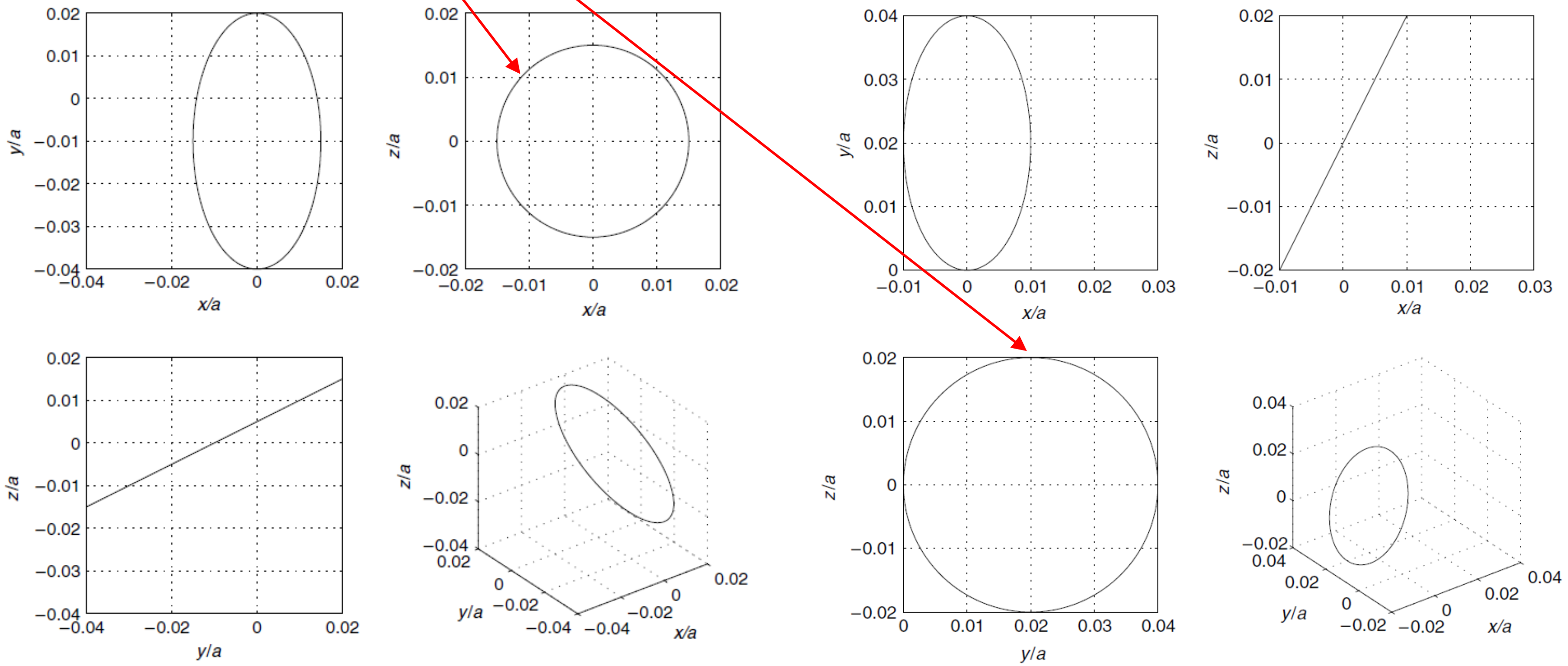
$$a = 600 + 6378 = 6978 \text{ km}$$
$$n = \sqrt{3.986 \cdot 10^5 / 6978^3} \text{ rad/s}$$

- Normalized coordinates

$$\bar{x}(0) = x(0)/a = 0.01, \ \bar{y}(0) = y(0)/a = 0.02$$
$$\bar{z}(0) = z(0)/a = 0.015$$
$$\bar{x}'(0) = 0.001, \ \bar{y}'(0) = -0.002, \ \bar{z}'(0) = 0.002$$

Solution of HCW (Example, 2)

- Solution of HCW provides relative motion depicted in slide 8
- Circular RN projection is given by $\bar{x}(0) = 0, \bar{x}'(0) = \bar{z}(0) = 0.015, \bar{z}'(0) = 0.$
- Circular TN projection is given by $\bar{z}(0) = 2\bar{x}(0) = 0.02, \bar{x}'(0) = \bar{z}'(0) = 0.$



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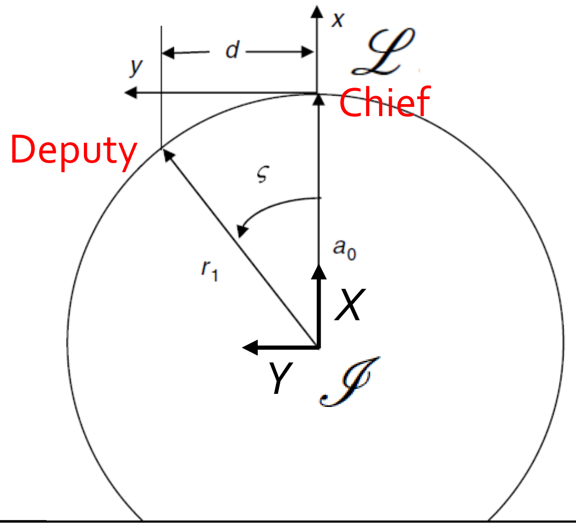
Cartesian vs. Curvilinear Coordinates

- The HCW equations have been derived under the following assumptions
    1. Primary as point mass (or spherically symmetric)
    2. Chief orbit is circular
    3. Distance between deputy and chief is small compared with radius (a_0)
- According to 3. only linear terms in differential gravity are retained
- We reduce the effect of the neglected nonlinear terms by a change of variables
- In fact, the linearization affects the initial conditions, e.g.

Initial conditions of deputy in RTN frame

$$
\begin{aligned}
&x(0) = a_0\left(\cos\varsigma - 1\right), \quad \dot{x}(0) = 0 \\
&y(0) = a_0 \sin\varsigma, \quad \dot{y}(0) = 0 \\
&z(0) = 0, \quad \dot{z}(0) = 0
\end{aligned}
\tag{5.71}
$$



Along-track Drift Caused by HCW Initialization

- Substitution of initial conditions (5.71) in HCW solution (5.18) provides

Radial offset

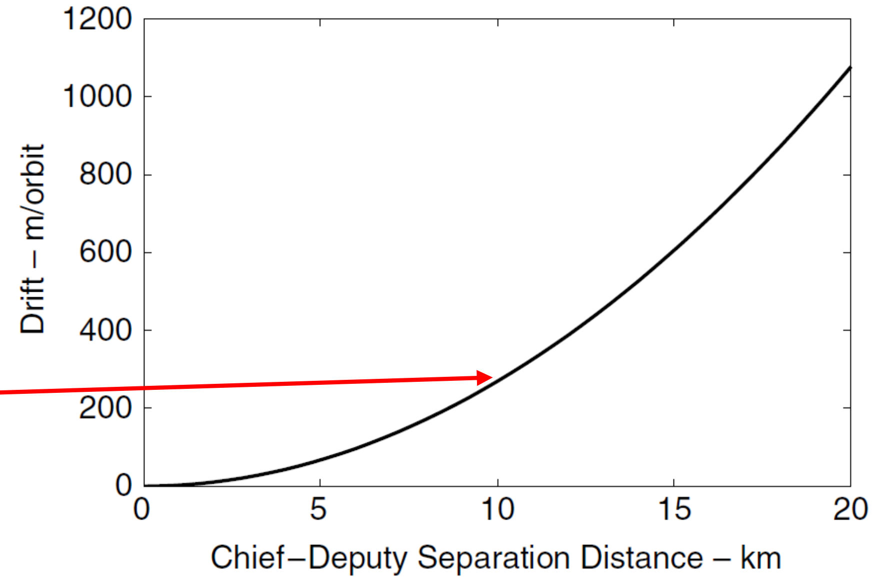
$$\left[4x(0) + \frac{2\dot{y}(0)}{n}\right] = 4a_0(\cos\varsigma - 1) \approx -2a_0\varsigma^2 \qquad (5.72)$$

Along-track drift after 1 orbit

$$-[6nx(0) + 3\dot{y}(0)]t = 12\pi a_0(1 - \cos\varsigma) \approx 6\pi a_0\varsigma^2 \qquad (5.73)$$

Drift/orbit caused by the HCW when using Cartesian reference frame for initialization:

269 m/orbit for 10 km separation at a_o = 7000 km



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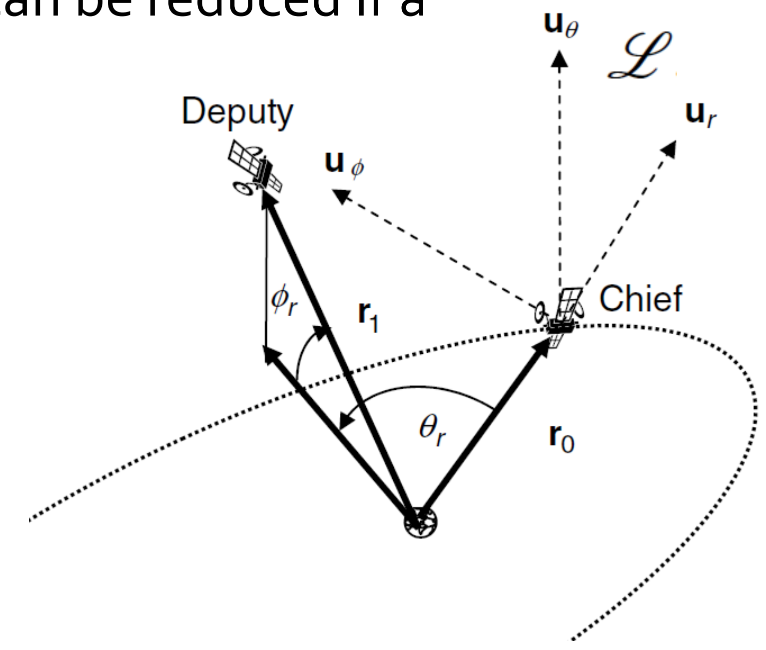
Compensation of Nonlinear Effects in HCW

- If the Cartesian rotating coordinate system is used to compute the initial deputy and chief states, the HCW solution provides an erroneous drift even if the satellites are on the same orbit

- This nonlinear effect is due to the orbit curvature and can be reduced if a curvilinear coordinate system is used instead

$$(\delta r, \theta_r, \phi_r) \longleftrightarrow (x, y, z)$$



- The equations of relative motion can be re-derived using curvilinear coordinates:

Kinematics for deputy

$$\mathbf{v}_1 = \delta\dot{r}\hat{\mathbf{r}}_1 + (a_0 + \delta r)\,\boldsymbol{\omega}_1 \times \hat{\mathbf{r}}_1$$

$$\ddot{\mathbf{r}}_1 = \delta\ddot{r}\hat{\mathbf{r}}_1 + 2\delta\dot{r}\boldsymbol{\omega}_1 \times \hat{\mathbf{r}}_1 + r_1\dot{\boldsymbol{\omega}}_1 \times \hat{\mathbf{r}}_1 + r_1\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \hat{\mathbf{r}}_1)$$

$$\ddot{\mathbf{r}}_1 = \left(\delta\ddot{r}_1 - 2n_0a_0\dot{\theta}_r - n_0^2\delta r - n_0^2a_0\right)\hat{\mathbf{u}}_r + \left(a_0\ddot{\theta}_r + 2a_0n_0\delta\dot{r} - a_0n_0^2\theta_r\right)\hat{\mathbf{u}}_\theta - a_0\ddot{\phi}_r\hat{\mathbf{u}}_\phi \qquad (5.79) \quad (5.85)$$

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HCW Equations using Curvilinear Coordinates

- The gravitational acceleration can be expressed in curvilinear coordinates, expanded in a Taylor series, and linearized in $(\delta r, \theta_r, \phi_r)$ to provide

Dynamics for deputy

$$-\frac{\mu}{r_1^2}\hat{\mathbf{r}}_1 = -\frac{\mu}{(a_0+\delta r)^2}\hat{\mathbf{r}}_1 = -n_0^2\left(a_0\hat{\mathbf{u}}_r - 2\delta r\hat{\mathbf{u}}_r - a_0\theta_r\hat{\mathbf{u}}_\theta - a_0\phi_r\hat{\mathbf{u}}_\phi\right) \qquad (5.87)$$

- Equating (5.85) and (5.87) gives the equations of relative motion in curvilinear coordinates

$$\boxed{\begin{aligned} &\delta\ddot{r} - 2a_0 n_0\dot{\theta}_r - 3n_0^2\delta r = 0 \\ &a_0\ddot{\theta}_r - 2n_0\delta\dot{r} = 0 \\ &\ddot{\phi}_r + n_0^2\phi_r = 0 \end{aligned}} \qquad (5.88)$$

- Note that (5.88) is identical to the HCW equations where (x, y, z) are replaced by $(\delta r, a_0\theta_r, a_0\phi_r)$, i.e. the same mathematical solution applies

Linearized Equations for Eccentric Orbits (1)

- The HCW equations loose accuracy when the eccentricity of the chief spacecraft is not zero
- It is possible to derive linearized equations of relative motion valid about chief orbits of arbitrary eccentricity
- This allows the usage of tools from linear dynamics systems based on the state transition matrix (as shown with HCW)
- We can normalize the general equations of relative motion (4.14-4.16) using r_0 and the true anomaly f as independent variable

Derivative w.r.t. f

$$\bar{x} = x/r_0,\ \bar{y} = y/r_0,\ \bar{z} = z/r_0 \qquad \frac{d(\cdot)}{dt} = (\cdot)'\dot{f}$$

- Additional non-dimensional potential functions are introduced for simplicity

$$\mathcal{W} = \frac{1}{1 + e_0 \cos f_0}\left[\frac{1}{2}(\bar{x}^2 + \bar{y}^2 - e_0\bar{z}^2 \cos f_0) - \mathcal{U}\right] \qquad \mathcal{U} = -\frac{1}{\left[(1+\bar{x})^2 + \bar{y}^2 + \bar{z}^2\right]^{\frac{1}{2}}} + 1 - \bar{x}$$

(4.20) (4.26)

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Linearized Equations for Eccentric Orbits (2)

- This enables to write the general nondimensional equations of relative motion as

$$\boxed{\begin{aligned} \bar{x}'' - 2\bar{y}' &= \frac{\partial \mathcal{W}}{\partial \bar{x}} \\ \bar{y}'' + 2\bar{x}' &= \frac{\partial \mathcal{W}}{\partial \bar{y}} \\ \bar{z}'' &= \frac{\partial \mathcal{W}}{\partial \bar{z}} \end{aligned}}$$

(4.27)

(4.28)

(4.29)

- For short normalized separations, Lawden (orbit transfers), DeVries (particles), Tschauner-Hempel (rendezvous), independently derived

$$\underbrace{k = 1 + e\cos f}_{\text{Only difference w.r.t. HCW}}$$

$$\boxed{\begin{aligned} \bar{x}'' &= \frac{3}{k}\bar{x} + 2\bar{y}' \\ \bar{y}'' &= -2\bar{x}' \\ \bar{z}'' &= -\bar{z}' \end{aligned}}$$

In- and out-of-plane relative motion decoupled by linearization

(5.101a)

(5.101b)

(5.101c)

Solution of Tschauner-Hempel Equations

- The most popular solution of TH is given by the Yamanaka-Ankersen (YA) equations and the respective state transition matrix (simple to program)

- The out-of-plane motion (5.101c) is a simple harmonic oscillator as from HCW

- The along-track motion (5.101b) can be integrated once to produce $\bar{y}'$ which is then substituted into the radial motion (5.101a) to give

$$\bar{x}'' + \left[4 - \frac{3}{k}\right]\bar{x} = 2d \qquad (5.102)$$

Integration constant

- Solving for $\bar{x}$ enables to obtain $\bar{y}$

$$\boxed{\begin{aligned} \bar{x} &= c_1 k \sin f + c_2 k \cos f + c_3(2 - 3ekI \sin f) \\ \bar{y} &= c_4 + c_1(1 + 1/k)\cos f - c_2(1 + 1/k)\sin f - 3c_3k^2 I \\ \bar{z} &= c_5 \cos f + c_6 \sin f \end{aligned}} \qquad \begin{aligned} &(5.124a) \\ &(5.124b) \\ &(5.124c) \end{aligned}$$

- where $c_{1:6}$ are integration constants and $I = \int_{f(0)}^{f} \frac{1}{k(f)^2}\mathrm{d}f = \frac{\mu^2}{h^3}(t - t_0)$

Yamanaka-Ankersen's State Transition Matrix

- The described procedure can be used to obtain the YA's state transition matrix by eliminating the integration constants
- The state vector is defined as $\bar{\mathbf{x}} = [\bar{x}, \bar{x}', y, \bar{y}', z, \bar{z}']^T$ and the complete solution is given by

$$\bar{\mathbf{x}}(t) = \phi(f)\phi^{-1}(f(0))\bar{\mathbf{x}}(t_0) \tag{5.125}$$

$$\phi(f) = \begin{bmatrix} s & c & 2 - 3esI & 0 & 0 & 0 \\ s' & c' & -3e\left(s'I + \frac{s}{k^2}\right) & 0 & 0 & 0 \\ c\left(1 + \frac{1}{k}\right) & -s\left(1 + \frac{1}{k}\right) & -3k^2 I & 1 & 0 & 0 \\ -2s & e - 2c & -3(1 - 2esI) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos f & \sin f \\ 0 & 0 & 0 & 0 & -\sin f & \cos f \end{bmatrix} \tag{5.126}$$

- where

$$c = k \cos f$$
$$s = k \sin f$$

Initial Conditions to Prevent Secular Drift (1)

- The HCW and TH equations and their associated State Transition Matrices can be used to determine initial conditions to prevent secular drift
- Even if they represent approximate linearized conditions, they provide good estimates for the desired initial conditions
- So far we have encountered two conditions
  - From energy matching: identical semi-major axis
  - From HCW: $\dot{y} + 2nx = 0 \quad (5.143)$
- (5.143) is an approximation of the zero differential semi-major axis condition, with effects of eccentricity and nonlinearity neglected
- An equivalent condition derived from TH should give an improvement since eccentricity effects are taken into account
- This can be done by setting $c_3 = 0$ in YA (5.124b) and solving for initial relative position and velocity at an arbitrary $f(0)$

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