AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 2

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- Fundamental astrodynamics
- Coordinate systems
- Keplerian 2-body problem
- Solutions of orbital differential equations

Terminology

- Orbital plane
  - Plane which contains position and velocity vectors of orbiting body about primary body under Newton's inverse-square law of gravitation only
- Ecliptic plane
  - Plane which contains the mean orbit of the Earth around the Sun
- Periapsis and Apoapsis
  - Points on the orbit of an orbiting body closest and farthest to the primary
- Orbital angular momentum vector ($\boldsymbol{h}$)
  - Cross product between position and linear momentum vectors
- Vernal equinox (symbol Υ)
  - Date when Sun crosses the celestial equator moving northward. Night and day have almost equal length. First day of Spring. Reference line for inertial measurements.

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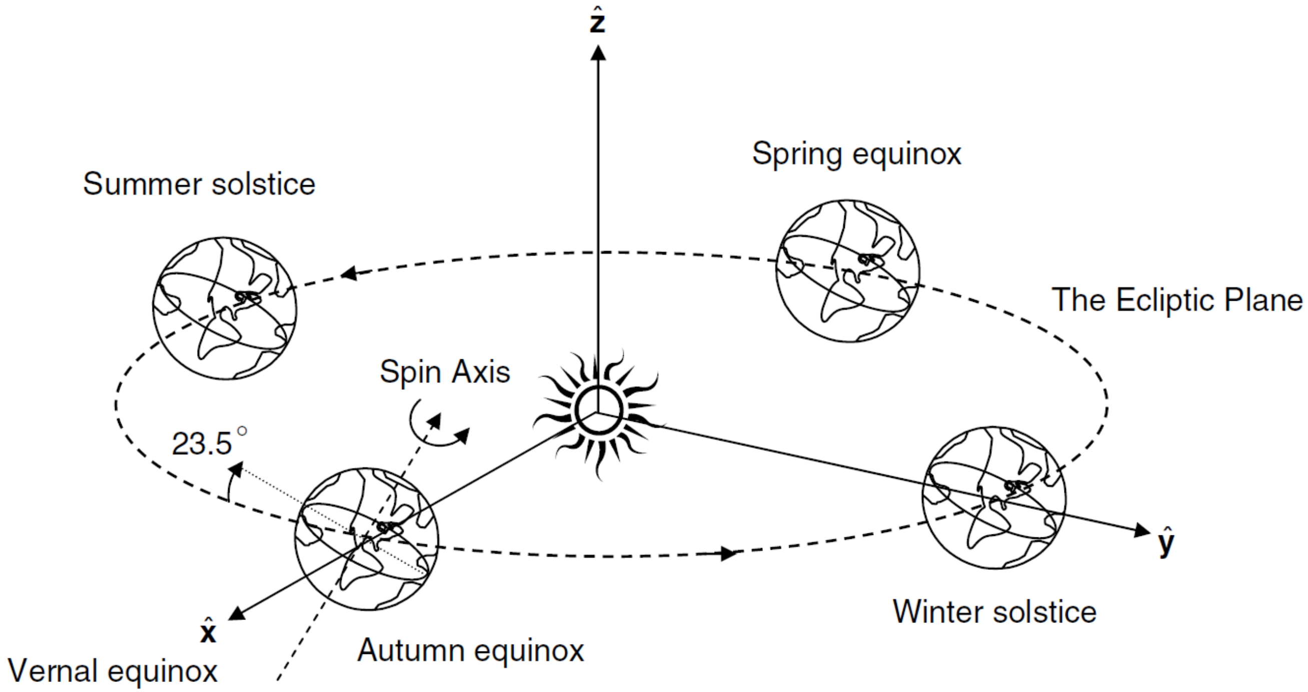
Coordinate Systems

- Problems that involve rates of change of physical quantities require the definition of a reference frame, giving rise to a coordinate system
- Rates are always referred to a coordinate system
- Most coordinate systems are Cartesian, rectangular, dextral (CRD) and are built on a fundamental plane (first two axis) and its normal
  - Inertial $\mathscr{I}$
    - Heliocentric
    - Geocentric (Earth-Centered Inertial)
  - Perifocal $\mathscr{P}$
  - Earth-Centered Earth-Fixed $\mathscr{F}$
  - Local-Vertical Local-Horizontal (LVLH) $\mathscr{L}$
    or Radial, Along-, Cross-track (RTN)
  - Polar Rotating $\mathscr{R}$

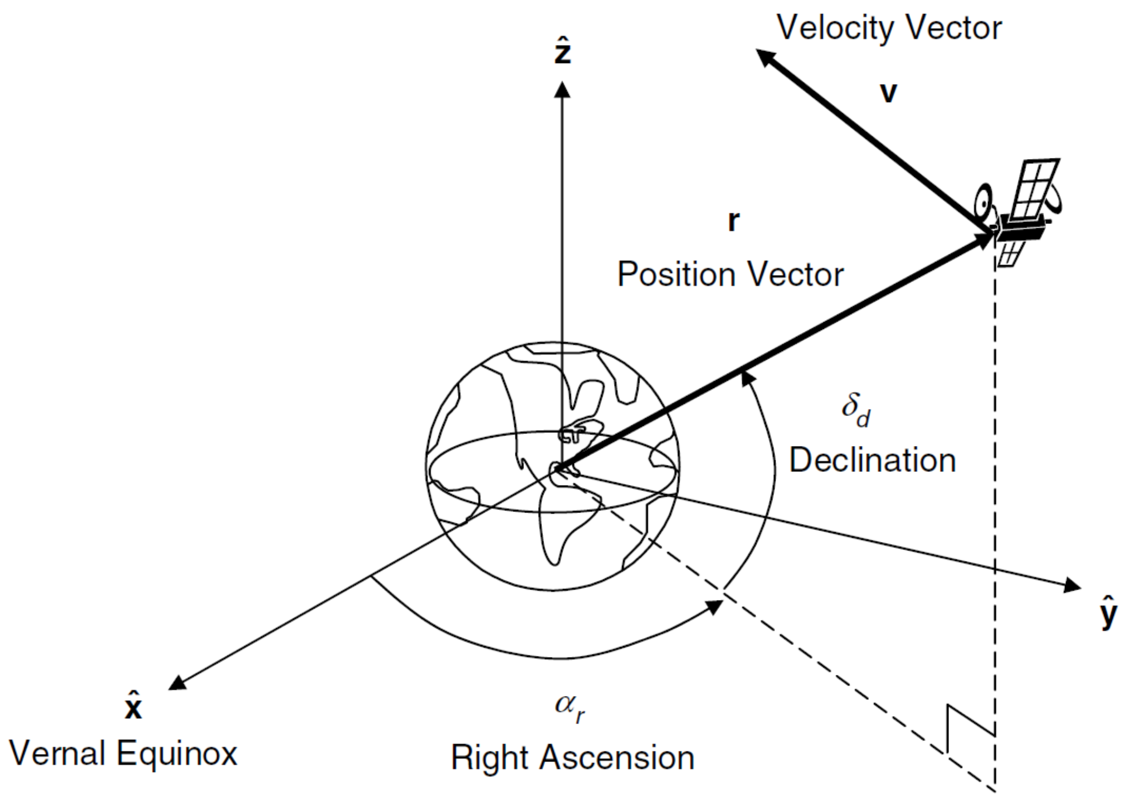
Inertial Heliocentric Coordinate System

- Origin
  - Sun
- Fundamental plane
  - Ecliptic plane
- Axes
  - x → Vernal Equinox
  - z → Normal to ecliptic
  - y → Complete triad
- Usage/Notes
  - Interplanetary not-Earth bounded missions
  - Rarely used in this course



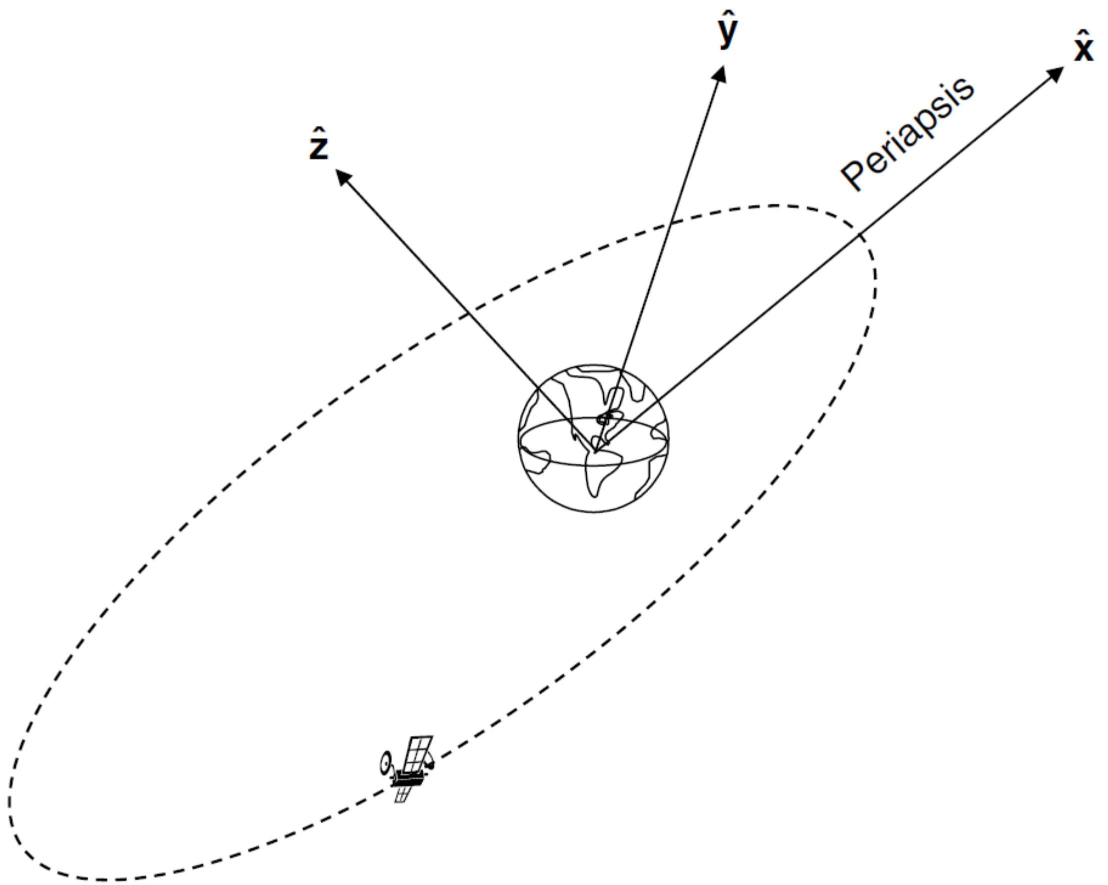
Inertial Geocentric Coordinate System (ECI)

- Origin
  - Earth
- Fundamental plane
  - Equator
- Axes
  - x → Vernal Equinox
  - z → Normal to equator (North pole)
  - y → Complete triad
- Usage/Notes
  - Earth bounded missions
  - Often used in this course
  - Common realization: Earth Mean Equator and Equinox J2000 (EME2000)



Perifocal Coordinate System

- Origin
  - Primary
- Fundamental plane
  - Orbital plane (instantaneous)
- Axes
  - x → Periapsis
  - z → Angular momentum vector
  - y → Complete triad
- Usage/Notes
  - Orbital mechanics
  - Often used in this course
  - Non-inertial in the presence of perturbations (instantaneous quantities)



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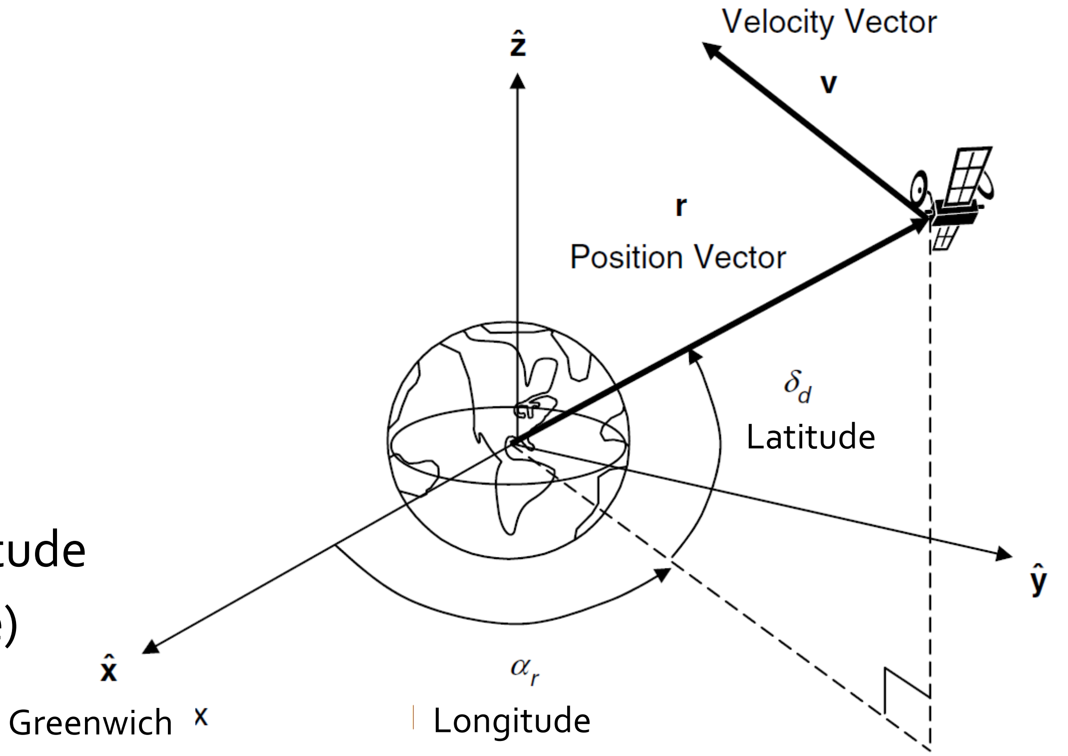
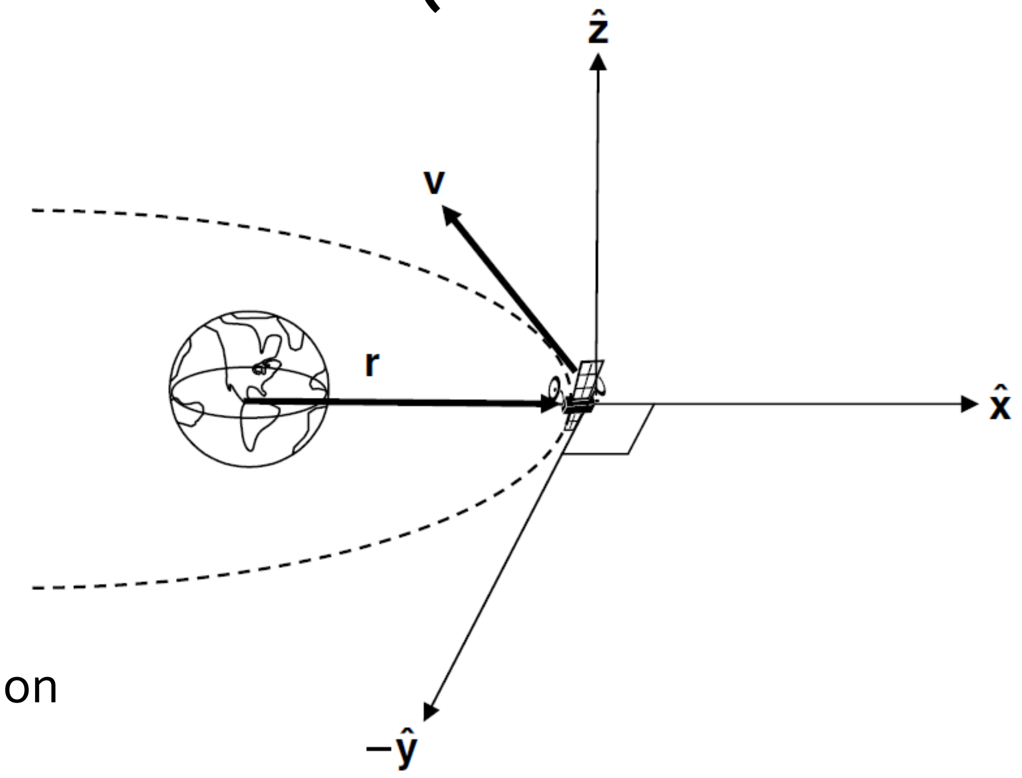
Earth-Centered Earth-Fixed (ECEF)

- Origin
  - Earth
- Fundamental plane
  - Equator
- Axes
  - x → Zero geocentric latitude/longitude
  - z → Normal to equator (North pole)
  - y → Complete triad
- Usage/Notes
  - Rotates with Earth
  - Navigation, mission operations
  - Common realization: World Geodetic System 1984 (WGS84)



Local-Vertical Local-Horizontal (LVLH or RTN)

- Origin
  - Spacecraft (real or virtual)
- Fundamental plane
  - Orbital plane (instantaneous)
- Axes
  - x (R) → Position vector, outwards
  - z (N) → Angular momentum vector
  - y (T) → Complete triad, direction of motion
- Usage/Notes
  - Rotates with spacecraft/orbit
  - Relative navigation, dynamics, control
  - Often used in this course



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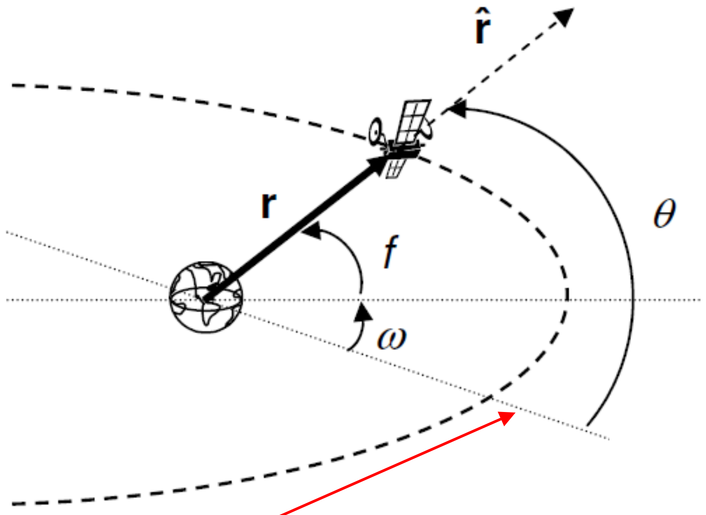
Polar Rotating Coordinate System

- Origin
  - Primary
- Fundamental plane
  - Orbital plane (instantaneous)
- Polar coordinates
  - r → Position vector, outwards
  - θ → Counterclockwise from reference line (here line of nodes)
- Usage/Notes
  - Rotates with spacecraft/orbit
  - Relative dynamics
  - Phase linked to orbit phasing, e.g. true argument of latitude $\theta = f + \omega$



Restricted Two-Body Problem

- Assumptions
  - No forces (external, internal) except Newtonian inverse-squared gravity
  - Gravitating bodies are spherical
  - Primary's mass is much larger than orbiting body's mass
- The two-body equations of motion (fundamental orbital differential equation)

Gravitational parameter of primary

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = \mathbf{0} \qquad (2.1)$$

- can be expressed in polar rotating coordinates $\mathscr{R}$

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} \qquad (2.5)$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \qquad (2.6)$$

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Conservation Laws

- Specific angular momentum is constant (magnitude and direction)
  - From (2.6)
$$\frac{\mathrm{d}}{\mathrm{d}t}\overbrace{(r^2\dot{\theta})}^{h} = r(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (2.7)$$
  - From definition
$$\dot{\mathbf{h}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{r} = \mathbf{0} \qquad (2.10)$$
- Specific mechanical energy is constant
  - From (2.5)
$$\frac{d}{dr}\left(\frac{\dot{r}^2}{2}\right) = \left(\frac{h^2}{r^3} - \frac{\mu}{r^2}\right) \qquad (2.12)$$
  - Integrating both sides in *r* and solving for the constant of integration

Total specific energy

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \frac{h^2}{2r^2} - \frac{\mu}{r} = \underbrace{\frac{\dot{r}^2}{2} + \frac{(r\dot{\theta}^2)}{2}}_{\text{Kinetic}} \quad \text{and} \quad \underbrace{-\frac{\mu}{r}}_{\text{Potential Energy}} = \text{const.} \qquad (2.13)$$

Polar Solution of Equations of Motion (1)

- Popular vis-viva (living force) form of specific mechanical energy

Magnitude of inertial velocity

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} \qquad (2.14)$$

- Utilizing both constants of motion, we can write the solution of (2.5-2.6)

$$\dot{r} = \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right) - \frac{h^2}{r^2}} \qquad (2.15)$$

$$\dot{\theta} = \frac{h}{r^2} \qquad (2.16)$$

- It is often more convenient to use orbit phasing θ instead of time as independent variable, to this end we divide (2.15) by (2.16)

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r^2\sqrt{2\left(\mathcal{E} + \mu/r\right) - h^2/r^2}}{h} \qquad (2.17)$$

Polar Solution of Equations of Motion (2)

- Eq. (2.17) can be directly integrated with initial condition $\theta_0 = \omega$ (Arg. Perigee)

$$\theta = \int \frac{h dr}{r^2\sqrt{2(\mathscr{E} + \mu/r) - h^2/r^2}} + \omega = \cos^{-1}\frac{1/r - \mu/h^2}{\sqrt{2\mathscr{E}/h + \mu^2/h^4}} + \omega \qquad (2.18)$$

- Solving for r yields Keplerian orbits

$$\boxed{r = \frac{h^2/\mu}{1 + \sqrt{1 + 2\mathcal{E}h^2/\mu^2}\cos(\theta - \omega)} = \frac{p}{1 + e\cos f}} \qquad (2.19)\,(2.20)$$

- which is the equation of a conic section in polar coordinates (conic equation)

$$p = h^2/\mu$$

Semilatus rectum or semi-parameter

(2.21)

$$e = \sqrt{1 + 2\mathcal{E}h^2/\mu^2}$$

Eccentricity

(2.22)

$$f = \theta - \omega$$

True anomaly

(2.23)

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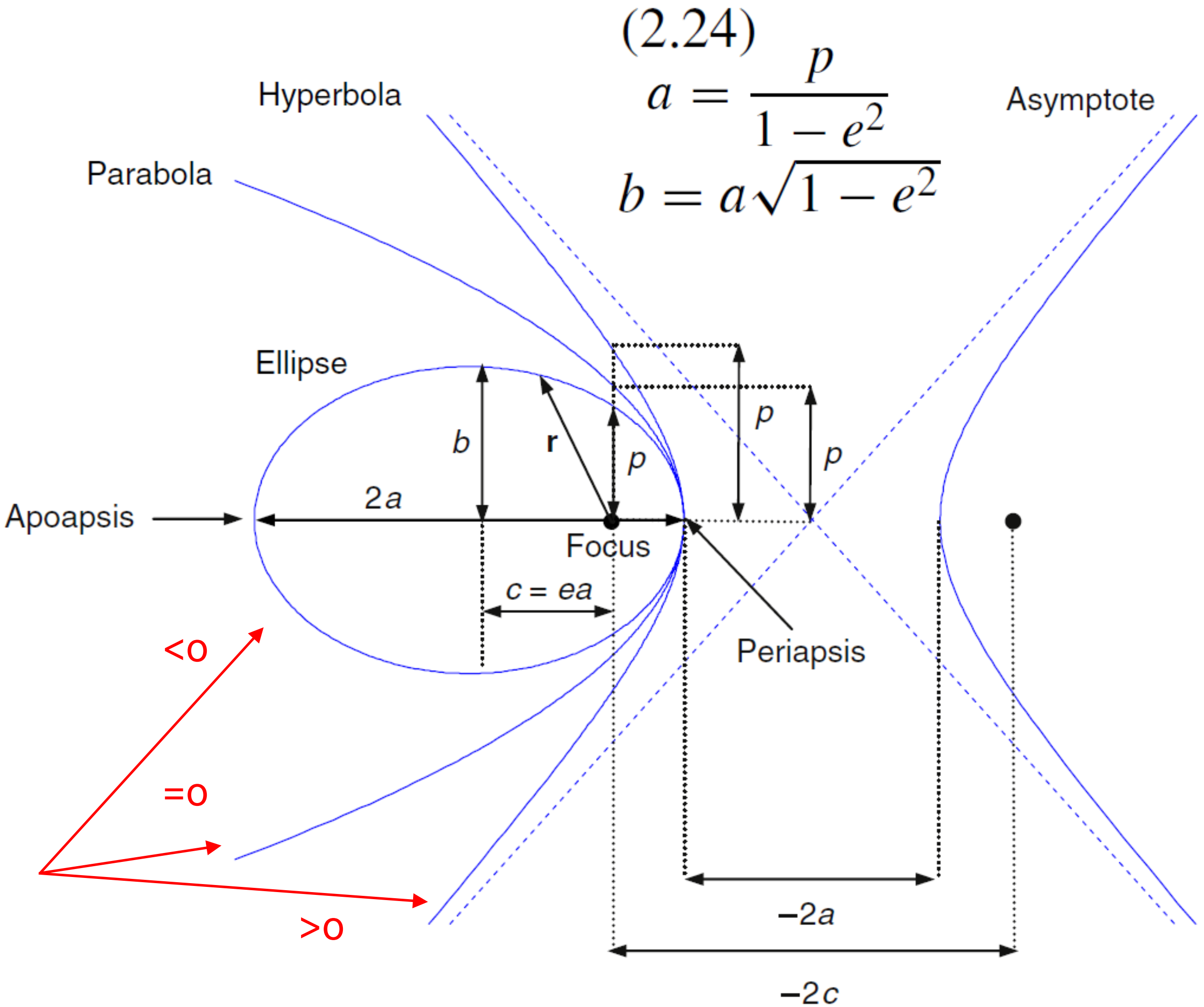
Keplerian Orbits: Ellipse, Parabola, Hyperbola

- Ellipse
  - Constant sum of distances from foci
  - $a > 0$
  - $0 < e < 1$
  - *Circle*: $e = 0$
  - *Parabola*: $e = 1$, $a = \infty$
- Hyperbola
  - Constant difference of distances from foci
  - $a < 0$
  - $e > 1$
- Energy from (2.13)

$$\mathcal{E} = -\frac{\mu}{2a}$$

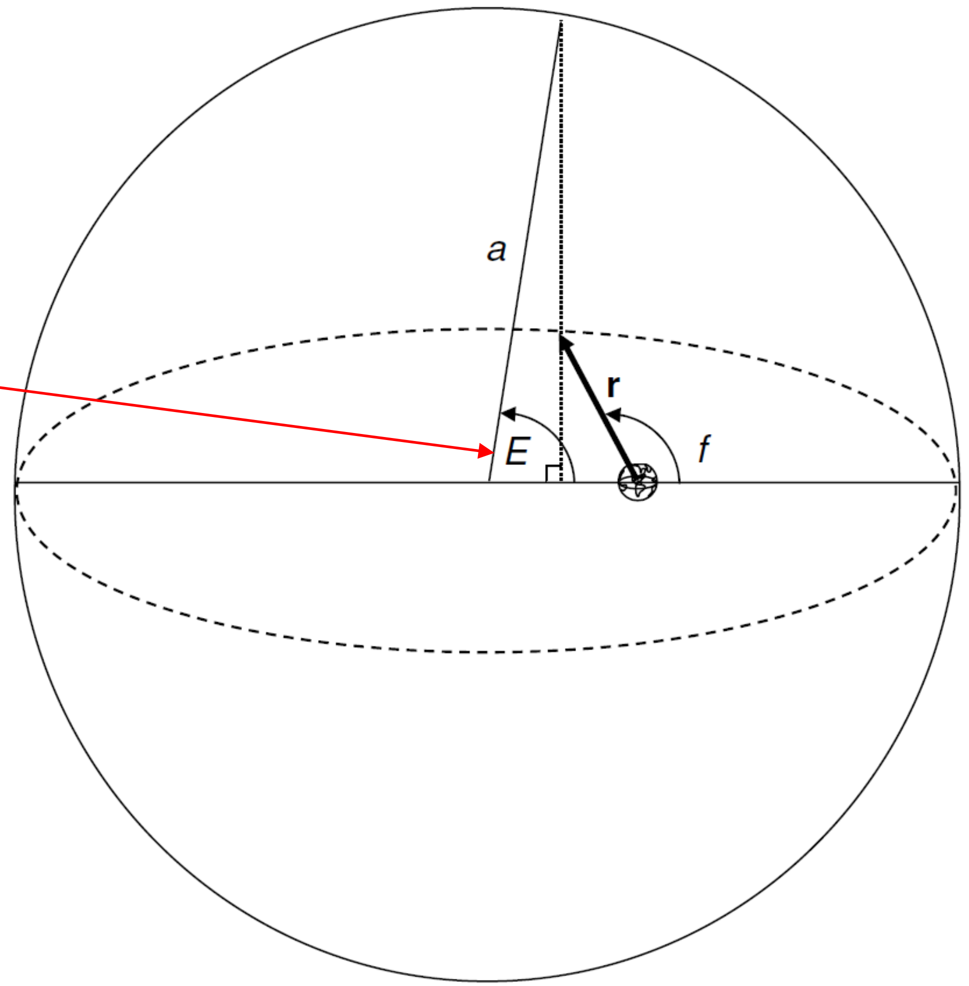


Time Solution of Equations of Motion

- Eq. (2.20) provides position as a function of true anomaly *f*
- We need to express the position as a function of time *t*
- Since relationship of the form $f = f(t)$ does not exist, the eccentric anomaly *E* is introduced such that $f = f(E)$ has a closed form (trigonometric formula)
- *E* can be computed as a function of time through the Kepler's equation

$$\boxed{M = M_0 + n(t - t_0) = E - e \sin E} \quad (2.26)$$

- where *n* is the mean motion $n = \sqrt{\mu / a^3}$
- *M* is the mean anomaly (mathematical construct)



Constants of Motion for Two-Body Problem

- From the fundamental orbital differential equation (2.1), we expect 6 integration constants, we have found 5 constants so far
  - M_0 : Mean anomaly at epoch
  - $\mathcal{E}$: Total specific energy
  - $\boldsymbol{h}$: Three components of the angular momentum vector
- Three constants can be found by recalling that motion in a conservative field yields a constant vector called the *Laplace-Runge-Lenz vector*. In the Keplerian two-body problem, this vector is the *eccentricity vector*

$$(2.28) \quad \mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}$$

$$\mathbf{r} \cdot \mathbf{e} = r\, e \cos f$$
$$e = \|\mathbf{e}\|$$

- The eccentricity unit vector points to the periapsis and is identical to the x-axis of the perifocal frame

Angular Velocity and Orbital Period

- Differentiating the true anomaly w.r.t. time provides the angular velocity along a Keplerian orbit (f and θ differ by the constant ω)

$$(2.29) \quad \dot{f} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1 + e\cos f)^2 \qquad (2.30) \quad \dot{f} = \dot{\theta}$$

- Letting the radius-vector sweep an element area dA in dt

$$\left.\begin{array}{r} dA = r^2 df/2 \\ h = r^2\dot{f} \longrightarrow dt = r^2/h\,df \end{array}\right\} \longrightarrow dt = 2dA/h$$

- We can calculate the orbital period T for an elliptic orbit

$$\int_0^T dt = \frac{2}{h}\int_0^{\pi ab} dA \longrightarrow T = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{\pi\mu}{\sqrt{2(-\mathcal{E})^3}}$$

Known as function of a

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Solution in Inertial Coordinates (1)

- We have a solution to the orbital differential equations in polar coordinates
- In order to solve them in inertial coordinates, it is customary to express position and velocity vectors in the perifocal coordinate system

$$[\mathbf{r}]_{\mathscr{P}} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} = \begin{bmatrix} a(\cos E - e) \\ b\sin E \\ 0 \end{bmatrix} \qquad (2.34)\ (2.35)$$

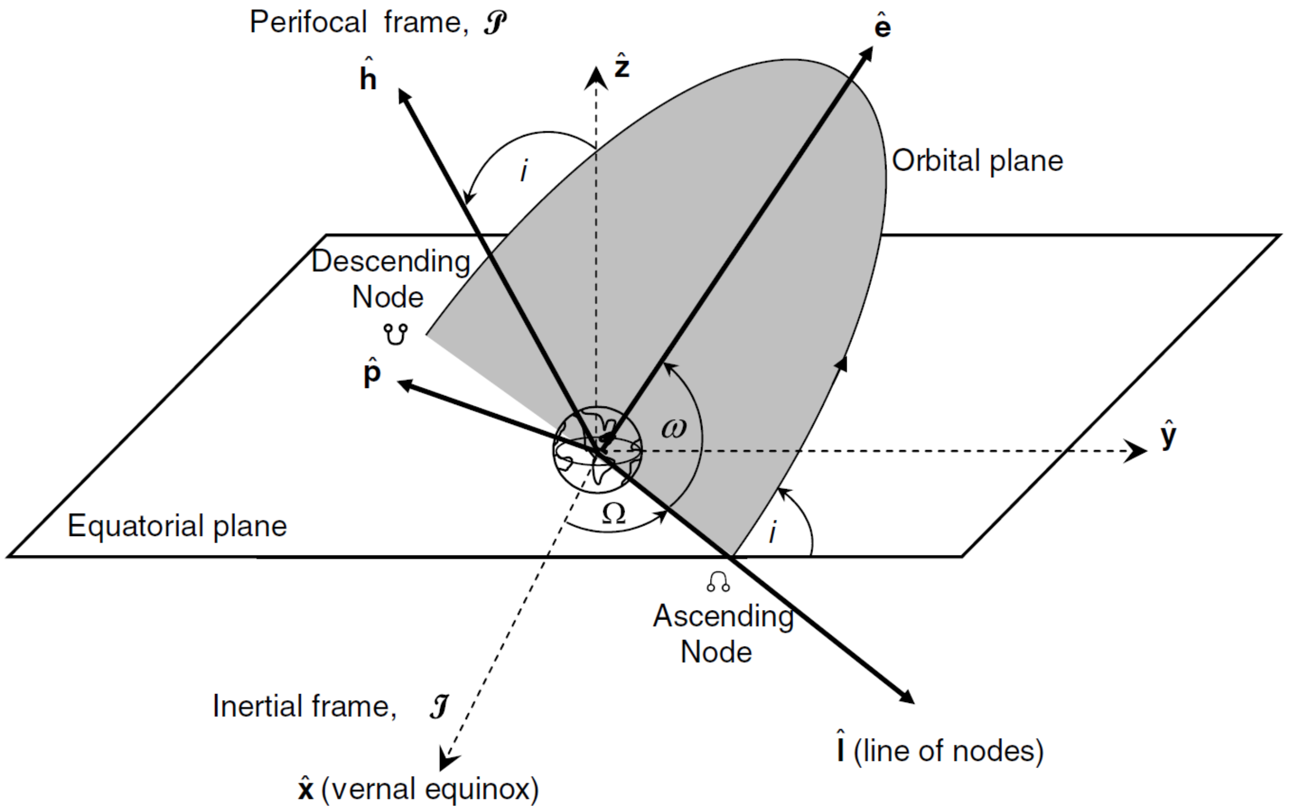
$$[\dot{\mathbf{r}}]_{\mathscr{P}} = \sqrt{\frac{\mu}{a(1-e^2)}} \begin{bmatrix} -\sin f \\ e + \cos f \\ 0 \end{bmatrix} \qquad (2.37)$$

- Inertial position and velocity vectors are obtained through a rotation matrix from perifocal to inertial coordinates based on the classical orbital elements

Coordinate transformation

$$T_{\mathscr{P}}^{\mathscr{I}}(\omega, i, \Omega) = T_3(\Omega, \hat{\mathbf{z}}) T_2(i, \hat{\mathbf{l}}) T_1(\omega, \hat{\mathbf{h}}) \qquad (2.38)$$

Solution in Inertial Coordinates (2)



- $T_1(\omega, \hat{\mathbf{h}})$, a rotation about $\hat{\mathbf{h}}$ by $0 \le \omega \le 2\pi$, mapping $\hat{\mathbf{e}}$ onto $\hat{\mathbf{l}}$.
- $T_2(i, \hat{\mathbf{l}})$, a rotation about $\hat{\mathbf{l}}$ by $0 \le i \le \pi$, mapping $\hat{\mathbf{h}}$ onto $\hat{\mathbf{z}}$.
- $T_3(\Omega, \hat{\mathbf{z}})$, a rotation about $\hat{\mathbf{z}}$ by $0 \le \Omega \le 2\pi$, mapping $\hat{\mathbf{l}}$ onto $\hat{\mathbf{x}}$.

Solution in Inertial Coordinates (3)

- Evaluating the direction cosine matrix from the Euler angles we obtain

No subscript

$$\mathbf{r} = T^{\mathscr{I}}_{\mathscr{P}}(\omega, i, \Omega)\mathbf{r}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{r}(a, e, i, \Omega, \omega, M_0, t)$$
$$= \frac{a(1-e^2)}{1+e\cos f}\begin{bmatrix} \mathrm{c}_{f+\omega}\mathrm{c}_\Omega - \mathrm{c}_i\mathrm{s}_{f+\omega}\mathrm{s}_\Omega \\ \mathrm{c}_i\mathrm{c}_\Omega\mathrm{s}_{f+\omega} + \mathrm{c}_{f+\omega}\mathrm{s}_\Omega \\ \mathrm{s}_i\mathrm{s}_{f+\omega} \end{bmatrix} \tag{2.40}$$

$$\mathbf{v} = \dot{\mathbf{r}} = T^{\mathscr{I}}_{\mathscr{P}}(\omega, i, \Omega)\dot{\mathbf{r}}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{v}(a, e, i, \Omega, \omega, M_0, t)$$
$$= \sqrt{\frac{\mu}{a(1-e^2)}}\begin{bmatrix} -\mathrm{c}_\Omega\mathrm{s}_{f+\omega} - \mathrm{s}_\Omega\mathrm{c}_i\mathrm{c}_{f+\omega} - e(\mathrm{c}_\Omega\mathrm{s}_\omega + \mathrm{s}_\Omega\mathrm{c}_\omega\mathrm{c}_i) \\ \mathrm{c}_\Omega\mathrm{c}_i\mathrm{c}_{f+\omega} - \mathrm{s}_\Omega\mathrm{s}_{f+\omega} - e(\mathrm{s}_\Omega\mathrm{s}_\omega - \mathrm{c}_\Omega\mathrm{c}_\omega\mathrm{c}_i) \\ \mathrm{s}_i(\mathrm{c}_{f+\omega} + e\mathrm{c}_\omega) \end{bmatrix} \tag{2.41}$$

- inertial position and velocity which depend upon time and the classical orbital elements

$$\text{œ} = \{a, e, i, \Omega, \omega, M_0\} \tag{2.42}$$

Nonsingular Orbital Elements

- The Euler angles Ω, ω and phasing f may become degenerate in some cases
  - Circular orbit: ω and f are undefined (no unique line of apsides)
  - Equatorial orbit: Ω is undefined (no unique line of nodes)

- Position and velocity are always well-defined. Hence alternative orbital elements are used to alleviate deficiencies (nonsingular orbital elements)
  - Eccentricity vector, ***e*** or ***q***, and mean argument of latitude, λ

$$q_1 = e\cos\omega, \quad q_2 = e\sin\omega, \quad \lambda = \omega + M \leftarrow \text{or } M_0 \qquad (2.43)$$

  - Equinoctial elements

$$\left\{a,\, e\sin(\omega+\Omega),\, e\cos(\omega+\Omega),\, \tan\frac{i}{2}\sin\Omega,\, \tan\frac{i}{2}\cos\Omega,\, \omega+\Omega+M\right\} \qquad (2.44)$$

  - Using quaternion (Euler parameters, β_i) instead of Euler angles

$$\sum_{i=0}^{3}\beta_i^2 = 1 \qquad \text{œ} = \left\{a,\, \sqrt{1-e^2},\, \beta_1,\, \beta_2,\, \beta_3,\, M_0\right\} \qquad (2.45)\,(2.46)$$

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