AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 4

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Equations of Motion with J_2 (1)

- The equations of motion of a satellite under the influence of gravitational and thrust perturbations can be written in the inertial frame as

$$\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}} \mathcal{V} + \mathbf{u} \tag{4.89}$$

Position vector

Total gravitational potential

Thrust acceleration

$$\begin{aligned} \mathcal{V} &= \mathcal{U} - \mathcal{R} \\ &= -\frac{\mu}{r}\left\{1 - \frac{J_2}{2}\frac{R_e{}^2}{r^2}\left[\frac{3}{r^2}(\mathbf{r}\cdot\hat{\mathbf{K}})^2 - 1\right]\right\} \end{aligned} \tag{4.90}$$

Unit vector along polar axis of inertial frame

$$\hat{\mathbf{K}} = \sin\theta_0 \sin i_0 \hat{\mathbf{i}} + \cos\theta_0 \sin i_0 \hat{\mathbf{j}} + \cos i_0 \hat{\mathbf{k}} \tag{4.91}$$

Expressed in RTN frame

Equations of Motion with J_2 (2)

- Derivation of (4.90) and substitution in (4.89) provides the equations of absolute J_2-perturbed motion

Position components in inertial frame

$$\ddot{X} = -\frac{\mu X}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left(5\frac{Z^2}{r^2} - 1\right)\right] \tag{4.93a}$$

$$\ddot{Y} = -\frac{\mu Y}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left(5\frac{Z^2}{r^2} - 1\right)\right] \tag{4.93b}$$

$$\ddot{Z} = -\frac{\mu Z}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left(5\frac{Z^2}{r^2} - 3\right)\right] \tag{4.93c}$$

- Note that the polar component of the angular momentum is now conserved
- The same procedure has been used to include conveniently effects up to J_6
- A nonlinear simulation for multiple satellites is carried out by numerically integrating copies of (4.93), one for each satellite

Relative Motion in Rotating Frame (1)

- The resulting relative displacement and velocity vectors, expressed in the inertial frame, $\mathscr{I}$ are defined as

Deputy

$$\delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0 \qquad (4.96a)$$

$$\delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_0 \qquad (4.96b)$$

Chief

- It is often convenient to express these vectors in the chief-centered RTN frame, $\mathscr{L}$ according to the following transformation (valid for eccentric orbits)

$$x = \frac{\delta\mathbf{r}^T\mathbf{r}_0}{r_0}$$

$$y = \frac{\delta\mathbf{r}^T(\mathbf{h}_0\times\mathbf{r}_0)}{\|\mathbf{h}_0\times\mathbf{r}_0\|}$$

$$z = \frac{\delta\mathbf{r}^T\mathbf{h}_0}{h_0}$$

Differentiation

$$\dot{x} = \frac{\delta\mathbf{v}^T\mathbf{r}_0 + \delta\mathbf{r}^T\mathbf{v}_0}{r_0} - \frac{(\delta\mathbf{r}^T\mathbf{r}_0)(\delta\mathbf{r}_0^T\mathbf{v}_0)}{r_0^3}$$

$$\dot{y} = \frac{\delta\mathbf{v}^T(\mathbf{h}_0\times\mathbf{r}_0) + \delta\mathbf{r}^T(\dot{\mathbf{h}}_0\times\mathbf{r}_0 + \mathbf{h}_0\times\mathbf{v}_0)}{\|\mathbf{h}_0\times\mathbf{r}_0\|} - \frac{\delta\mathbf{r}^T(\mathbf{h}_0\times\mathbf{r}_0)(\mathbf{h}_0\times\mathbf{r}_0)^T(\dot{\mathbf{h}}_0\times\mathbf{r}_0 + \mathbf{h}_0\times\mathbf{v}_0)}{\|\mathbf{h}_0\times\mathbf{r}_0\|^3} \qquad (4.97a)\ (4.98c)$$

$$\dot{z} = \frac{\delta\mathbf{v}^T\mathbf{h}_0 + \delta\mathbf{r}^T\dot{\mathbf{h}}_0}{h_0} - \frac{\delta\mathbf{r}^T\mathbf{h}_0(\mathbf{h}_0^T\dot{\mathbf{h}}_0)}{h_0^3}$$

Relative Motion in Rotating Frame (2)

- These expressions are equivalent to a transformation of coordinates from $\mathscr{L}$ to $\mathscr{I}$ given by

$$T_{\mathscr{L}}^{\mathscr{I}} = \left[\hat{\mathbf{r}}_0 \;\; (\hat{\mathbf{h}}_0 \times \mathbf{r}_0) \;\; \hat{\mathbf{h}}_0\right]$$

- with its inverse

$$T_{\mathscr{I}}^{\mathscr{L}}(\Omega, i, \theta) = \begin{bmatrix} c_\Omega c_\omega - s_\Omega s_\theta c_i & s_\Omega c_\theta + c_\Omega s_\theta c_i & s_\theta s_i \\ -c_\Omega s_\theta - s_\Omega c_\theta c_i & -s_\Omega s_\theta + c_\Omega c_\theta c_i & c_\theta s_i \\ s_\Omega s_i & -c_\Omega s_i & c_i \end{bmatrix} \quad \theta = \omega + f$$

- These expressions allow the computation of the relative motion from numerical simulations
- The direct derivation of the equations of motion in the rotating frame is treated in the following

Initial Conditions and Mean Orbital Elements (1)

- It is often convenient to specify the initial conditions of the satellite orbits for numerical simulations through mean orbital elements
- This is because the formation design/control is typically done through mean orbital elements in order to set the desired secular and long-period properties
- The resulting initial mean orbital elements can be transformed into the respective osculating elements via Brower theory (Appendix E) or other procedures for nonsingular and equinoctial elements (e.g., Gim-Alfriend)
- The osculating elements can then be transformed into the inertial position and velocity coordinates using the principles of orbital mechanics

Example 3.1 (*Mean-to-osculating transformation*). *Let the given mean elements be:*

$$\begin{aligned} \bar{a} &= 7100 \text{ km}, \quad \bar{\theta} = 0 \text{ rad}, \quad \bar{i} = 70^\circ \\ \bar{q}_1 &= 0.05, \quad \bar{q}_2 = 0.05, \quad \bar{\Omega} = 45^\circ \end{aligned} \tag{3.24}$$

Obtain the osculating elements.

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Initial Conditions and Mean Orbital Elements (2)

- The first order Brower transformation results in the following osculating elements

$$
\begin{aligned}
&a = 7109.31795 \text{ km}, \quad \theta = 0.00005 \text{ rad}, \quad i = 1.22196 \text{ rad} \\
&q_1 = 0.05063, \quad q_2 = 0.05003, \quad \Omega = 0.78547 \text{ rad}
\end{aligned}
\tag{3.25}
$$

- Note that there is approximately a 10 km difference between mean and osculating semi-major axis values
- The first-order inverse transformation can be obtained by simply replacing J_2 by $-J_2$ in the Brower transformation and treating the osculating elements as the inputs and the mean elements as the outputs
- The following mean elements are back-computed

Minor differences w.r.t. Example 3.1

$$
\begin{aligned}
&\bar{a} = 7099.996055 \text{ km}, \quad \bar{\theta} = 0.000008 \text{ rad}, \quad \bar{i} = 1.221731 \text{ rad} \\
&\bar{q}_1 = 0.0500006, \quad \bar{q}_2 = 0.04999994, \quad \bar{\Omega} = 0.7853984 \text{ rad}
\end{aligned}
\tag{3.26}
$$

- An iterative procedure can be devised to solve for the osculating from the mean elements through the usage of the Jacobian, $D = \partial \boldsymbol{œ} / \partial \overline{\boldsymbol{œ}}$ (Appendix F)

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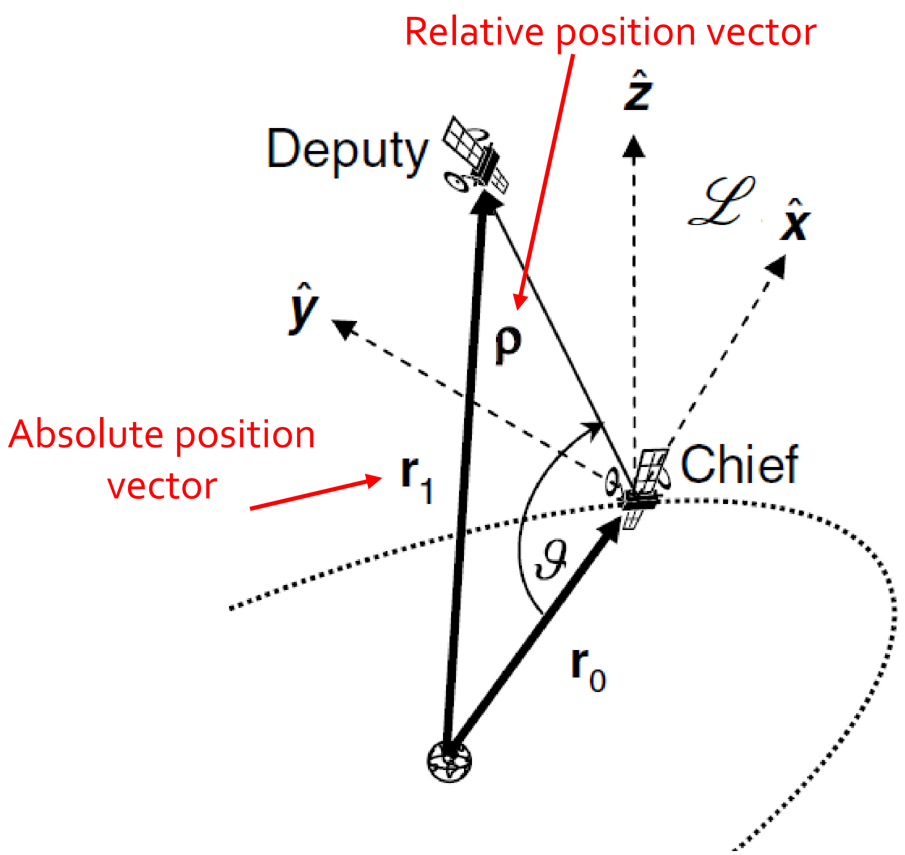
Nonlinear Equations of Relative Motion (1)

- We wish to develop the equations of relative motion under the setup of the two-body problem
- The relative motion is first described in the inertial frame and then transformed into a chief-fixed, RTN or LVLH rotating frame
- Position of deputy relative to chief

$$\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_0$$

$$\ddot{\mathbf{r}}_1 = -\frac{\mu}{r_1^3}\mathbf{r}_1 \qquad \ddot{\mathbf{r}}_0 = -\frac{\mu}{r_0^3}\mathbf{r}_0$$

$$\ddot{\boldsymbol{\rho}} = -\frac{\mu(\mathbf{r}_0 + \boldsymbol{\rho})}{\|\mathbf{r}_0 + \boldsymbol{\rho}\|^3} + \frac{\mu}{r_0^3}\mathbf{r}_0 \qquad (4.4) - (4.9)$$



Nonlinear Equations of Relative Motion (2)

- The relative accelerations w.r.t. the inertial and rotating frames are linked by the theorem of Coriolis

Angular velocity vector of rotating relative to inertial frame

$$\ddot{\boldsymbol{\rho}} = \frac{d^{2\mathscr{L}}\boldsymbol{\rho}}{dt^2} + 2\,{}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \frac{d^{\mathscr{L}}\boldsymbol{\rho}}{dt} + \frac{d^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}}}{dt} \times \boldsymbol{\rho} + {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times ({}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \boldsymbol{\rho}) \tag{4.10}$$

- We can substitute the following expressions in Eq. (4.9)-(4.10) to derive the general equations of relative motion

$$\begin{aligned} {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} &= [0,\, 0,\, \dot{\theta}_0]^T \\ \mathbf{r}_0 &= [r_0,\, 0,\, 0]^T \\ [\boldsymbol{\rho}]_{\mathscr{L}} &= [x,\, y,\, z]^T \end{aligned}$$

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{\theta}_0\dot{y} - \ddot{\theta}_0 y - \dot{\theta}_0^2 x &= -\frac{\mu(r_0+x)}{\left[(r_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} + \frac{\mu}{r_0^2} \\ \ddot{y} + 2\dot{\theta}_0\dot{x} + \ddot{\theta}_0 x - \dot{\theta}_0^2 y &= -\frac{\mu y}{\left[(r_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \\ \ddot{z} &= -\frac{\mu z}{\left[(r_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \end{aligned}}$$

(4.14)

(4.16)

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Nonlinear Equations of Relative Motion (3)

- The equations of relative motion represent a 10-dimensional system (6 relative + 4 absolute position and velocity) of nonlinear differential equations
- For non-constant angular velocity $\ddot{\theta}_0 \neq 0$, these equations admit a single relative equilibrium at $x = y = z = 0$ meaning that the deputy will appear stationary in the chief frame if and only if their positions coincide
- In the presence of external differential perturbations ***d***, and differential control forces ***u***, the equations of the relative motion become

$$\ddot{x} - 2\dot{\theta}_0\dot{y} - \ddot{\theta}_0 y - \dot{\theta}_0^2 x = -\frac{\mu(r_0 + x)}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + \frac{\mu}{r_0^2} + d_x + u_x \qquad (4.17)$$

$$\ddot{y} + 2\dot{\theta}_0\dot{x} + \ddot{\theta}_0 x - \dot{\theta}_0^2 y = -\frac{\mu y}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + d_y + u_y \qquad (4.19)$$

$$\ddot{z} = -\frac{\mu z}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + d_z + u_z$$

Commensurability

- For unperturbed Keplerian motion, a rather simple method exists for generating bounded relative motion between any two spacecraft
- The underlying methodology is based on the concept of orbital-period *commensurability*
  - Two non-zero real numbers c and d are commensurable if and only if c/d is a rational number
  - There exists a real number g, and integers m and n, such that $c=mg$ and $d=ng$
- Two elliptic orbits with orbital periods T_1 and T_2 are said to be $m{:}n$ commensurable if

Definition of Commensurability

$$\frac{T_1}{T_2} = \frac{m}{n} \tag{4.1}$$

Integer numbers

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Energy Matching Condition (1)

- Based on our previous lectures on Keplerian orbits, orbit commensurability translates to a relationship between orbital energies and semi-major axes

$$\frac{T_1}{T_2} = \left(\frac{\mathcal{E}_2}{\mathcal{E}_1}\right)^{3/2} = \left(\frac{a_1}{a_2}\right)^{3/2} \quad (4.2) \qquad \Rightarrow \qquad \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{a_1}{a_2} = \left(\frac{m}{n}\right)^{2/3} \quad (4.3)$$

- Commensurability constraints the ratio between orbital energies and between semi-major axes (not necessarily rational)
- This represents an energy matching condition. $m = n = 1$ is an interesting case for formation-flying, where energies and semi-major axes match
- When formulated in the rotating reference-orbit-fixed frame, orbital commensurability gives a single, simple, algebraic constraint on initial conditions which guarantees bounded relative orbits
- This approach can also be used for initialization and formation-keeping

Energy Matching Condition (2)

- Do the equations of relative motion provide bounded solutions? Intuition:
  - If the orbits are elliptic, the separation can not growth unboundedly
  - If the periods are not commensurate, periodicity won't be exhibited
- The relative motion btw. non-commensurable elliptic orbits is said to be *quasi-periodic*, and may appear to be "locally" unbounded
- In distributed space systems, 1:1 commensurability is the interesting case, whereas other commensurability ratios are of interest in interplanetary travel and orbital transfers
- To implement the energy matching condition for finding periodic relative orbits, we use the theorem of Coriolis for the deputy velocity

$$\mathbf{v}_1 = \frac{d^{\mathscr{L}}}{dt}\boldsymbol{\rho} + \frac{d^{\mathscr{L}}}{dt}\mathbf{r}_0 + {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \mathbf{r}_0 + {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \boldsymbol{\rho} = \begin{bmatrix} \dot{x} - \dot{\theta}_0 y + \dot{r}_0 \\ \dot{y} + \dot{\theta}_0(x + r_0) \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

(4.30) (4.31)

Energy Matching Condition (3)

- The total specific energy of the deputy and chief spacecraft comprise the kinetic and potential energies

$$\mathcal{E}_1 = \frac{1}{2}v_1^2 - \frac{\mu}{r_1} = \frac{1}{2}\{(\dot{x} - \dot{\theta}_0 y + \dot{r}_0)^2 + [\dot{y} + \dot{\theta}_0(x + r_0)]^2 + \dot{z}^2\} \quad (4.34)$$

$$(4.35)$$

$$- \frac{\mu}{\sqrt{(r_0 + x)^2 + y^2 + z^2}} = \mathcal{E}_0 = -\frac{\mu}{2a_0} \quad (4.36)$$

- In order to design a 1:1 bounded formation, we require the following constraint on the initial conditions of the equations of relative motion

$$\boxed{\begin{aligned}&\frac{1}{2}\{[\dot{x}(0) - \dot{\theta}_0(0)y(0) + \dot{r}_0(0)]^2 + \{\dot{y}(0) + \dot{\theta}_0(0)[x(0) + r_0(0)]\}^2 + \dot{z}(0)^2\}\\ &\quad - \frac{\mu}{\sqrt{[r_0(0) + x(0)]^2 + y^2(0) + z^2(0)}} = -\frac{\mu}{2a_0} \qquad (4.37)\end{aligned}}$$

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Energy Matching Condition (4)

- Most often the energy matching constraint (4.37) is normalized, with the distances being measured in units of a_0 and angular velocities in units of n_0
- Normalized quantities are denoted by $(\bar{\cdot})$ and differentiation w.r.t. normalized time by $(\cdot)'$

Example 4.1. *Consider a chief spacecraft on an elliptic orbit. Normalize positions by* a_0 *and angular velocities by* $\sqrt{\mu/a_0^3}$ *so that* $a_0 = \mu = 1$. *Using these normalized units, let*

$$\begin{aligned} \bar{y}(0) = 0, \bar{z}(0) = 0.1, \bar{x}'(0) = 0.02 \\ \bar{y}'(0) = 0.02, \bar{z}'(0) = 0, f_0(0) = 0, e_0 = 0.1 \end{aligned} \tag{4.38}$$

Find $\bar{x}(0)$ *that guarantees a 1:1 bounded relative motion.*

Energy Matching Condition (5)

- From the solution of the Keplerian two-body problem

$$r_0 = \|\mathbf{r}_0\| = \frac{a_0(1-e_0^2)}{(1+e_0\cos f_0)} \quad \Rightarrow \quad \bar{r}_0(0) = \frac{1-e_0^2}{1+e_0\cos f_0(0)} = \frac{1-0.1^2}{1+0.1} = 0.9$$

$$\dot{r}_0 = e_0 \sin f_0 \sqrt{\frac{\mu}{a_0(1-e_0^2)}} \quad \Rightarrow \quad \bar{r}_0'(0) = e_0 \sin f_0(0) \sqrt{\frac{1}{(1-e_0^2)}} = 0$$

$$\dot{f} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1+e\cos f)^2 \quad \Rightarrow \quad \theta_0'(0) = \sqrt{\frac{1}{(1-e_0^2)^3}}(1+e_0\cos f_0(0))^2 = 1.22838$$

- Upon substitution into (4.37), we obtain a 6th order equation for $x(0)$

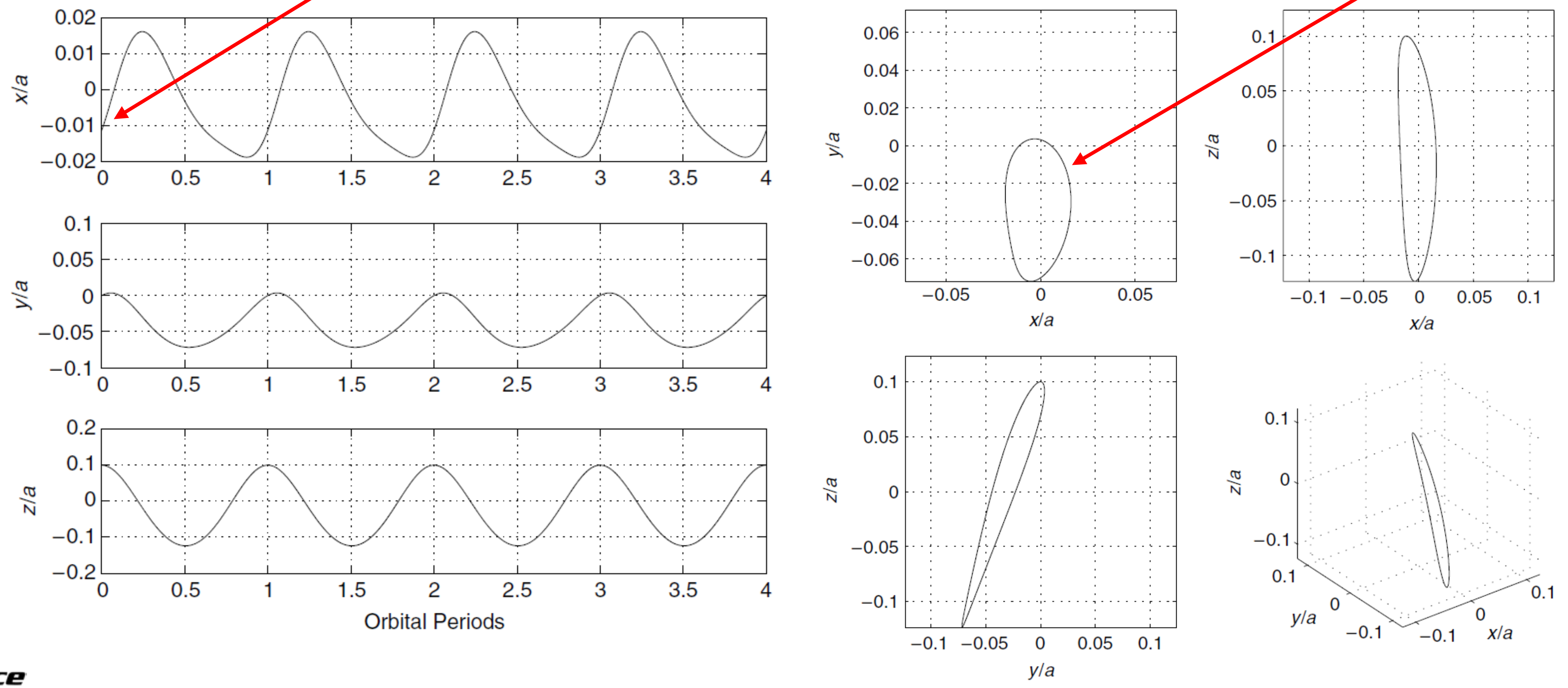
$$2.2768\bar{x}^6(0) + 12.4431\bar{x}^5(0) + 31.3762\bar{x}^4(0) + 45.46062\bar{x}^3(0) + 39.5905\bar{x}^2(0) + 19.5344\bar{x}(0) + 0.2151 = 0$$

Energy Matching Condition (6)

- The energy matching condition provides two real solutions for this example which guarantee 1:1 bounded motion

$$\bar{x}_1(0) = -0.01127, \bar{x}_2(0) = -1.8059$$

Numerical integration of (4.17-4.19)



Circular Chief Orbit (1)

- Previously, we derived the general nonlinear equations of relative motion for arbitrary chief orbits (elliptic)
- A simpler form of the equations is found in practical cases which are typically characterized by a circular chief orbit
- Substituting $\dot{\theta}_0 = n_0 = \text{const.}, \ddot{\theta}_0 = 0$ and $r_0 = a_0 = \text{const}$ in the equations of relative motion yields

$$\boxed{\begin{aligned} \ddot{x} - 2n_0\dot{y} - n_0^2 x &= -\frac{\mu(a_0 + x)}{\left[(a_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + \frac{\mu}{a_0^2} && (4.74) \\ \ddot{y} + 2n_0\dot{x} - n_0^2 y &= -\frac{\mu y}{\left[(a_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} && (4.75) \\ \ddot{z} &= -\frac{\mu z}{\left[(a_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} && (4.76) \end{aligned}}$$

Circular Chief Orbit (2)

- The new equations admit an equilibrium continuum

$$\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$$

- given by

$$z = 0, (x + a_0)^2 + y^2 = a_0^2 \tag{4.77}$$

- This equation conforms to the energy matching condition and defines a circle that coincides with the chief's orbit
- The center of the circle has coordinates $x = -a_0, y = 0$ which are the coordinates of the primary in the RTN frame
- Physical interpretation: the deputy appears stationary in a chief-fixed frame if the deputy is co-located on the same circular orbit of the chief
- From a dynamical systems perspective, we expect that small perturbations near equilibria generate periodic orbits about the equilibria
- The resulting periodic motion is called *libration*

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Linear Equations of Relative Motion (HCW)

- If the motion of the deputy w.r.t. chief is small as compared with the orbit radius, (4.74-4.76) can be linearized about the origin of the chief-fixed frame
- The linearized equations of relative motion are called the Hill-Clohessy-Wiltshire equations (HCW) and were developed by CW in the early 1960s to analyze spacecraft rendezvous
- We expand the right-hand side of (4.74-4.76) into a Taylor series about the origin and retain first order terms only

$$-\frac{\mu(a_0+x)}{\left[(a_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \approx n_0^2(2x-a_0)$$
$$-\frac{\mu y}{\left[(a_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \approx -n_0^2 y$$
$$-\frac{\mu z}{\left[(a_0+x)^2+y^2+z^2\right]^{\frac{3}{2}}} \approx -n_0^2 z$$

$$\boxed{\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= d_x + u_x \\ \ddot{y} + 2n\dot{x} &= d_y + u_y \\ \ddot{z} + n^2 z &= d_z + u_z \end{aligned}} \qquad \begin{aligned}(5.7)\\(5.9)\end{aligned}$$

Nonhomogeneous form with perturbations and control acceleration

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