AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 6

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Table of Contents

Initial Conditions to Prevent Secular Drift (1)

- The HCW and TH equations and their associated State Transition Matrices can be used to determine initial conditions to prevent secular drift
- Even if they represent approximate linearized conditions, they provide good estimates for the desired initial conditions
- So far we have encountered two conditions
  - From energy matching: identical semi-major axis
  - From HCW: $\dot{y} + 2nx = 0 \quad (5.143)$
- (5.143) is an approximation of the zero differential semi-major axis condition, with effects of eccentricity and nonlinearity neglected
- An equivalent condition derived from TH should give an improvement since eccentricity effects are taken into account
- This can be done by setting $c_3 = 0$ in YA (5.124b) and solving for initial relative position and velocity at an arbitrary $f(0)$

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Initial Conditions to Prevent Secular Drift (2)

- Normalized condition for no drift

$$k^2(f(0))\bar{y}'(f(0)) + ek(f(0)) \sin f(0)\bar{x}'(f(0)) + \left[2 + 3e \cos f(0) + e^2\right]\bar{x}(f(0)) = 0 \tag{5.146}$$

- Unscaled condition for no drift

$$k(f(0))y'(f(0)) + e \sin f(0) \left[x'(f(0)) - y(f(0))\right] + [2 + e \cos f(0)]\, x(f(0)) = 0 \tag{5.147}$$

- Condition with time as independent variable

$$k(f(0)) \left[\dot{y}(t_0) + \dot{f}(0)x(t_0)\right] + e \sin f(0) \left[\dot{x}(t_0) - \dot{f}(0)y(t_0)\right] + \dot{f}(0)x(t_0) = 0 \tag{5.148}$$

$$\dot{f}(0) = \sqrt{\frac{\mu}{p^3}}k(f(0))^2 \tag{5.149}$$

Linear boundedness condition is a function of the true anomaly

Can be satisfied in several ways since it involves radial and along-track velocities

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Periodic Solutions of TH Equations

- From (5.124) with $c_3 = 0$, the periodic solutions of TH can be written in amplitude/phase form

$$x = \rho_x \sin(f + \alpha_x) \tag{5.158a}$$

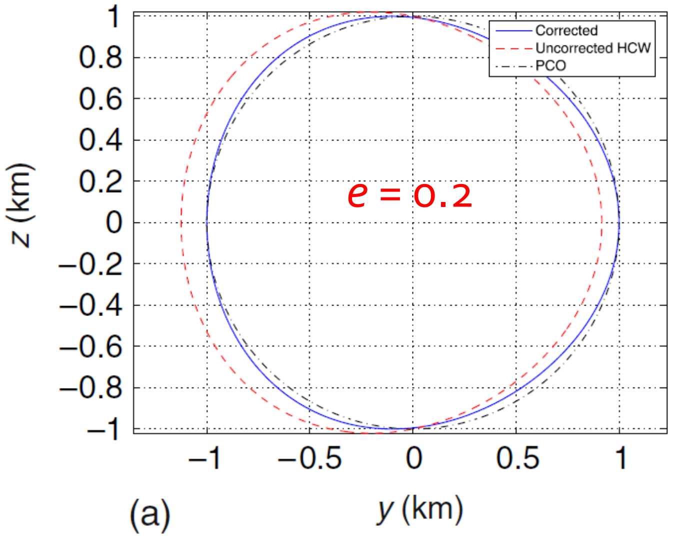
$$y = 2\rho_x \cos(f + \alpha_x)\frac{(1 + (e/2)\cos f)}{(1 + e\cos f)} + \frac{\rho_y}{(1 + e\cos f)} \tag{5.158b}$$

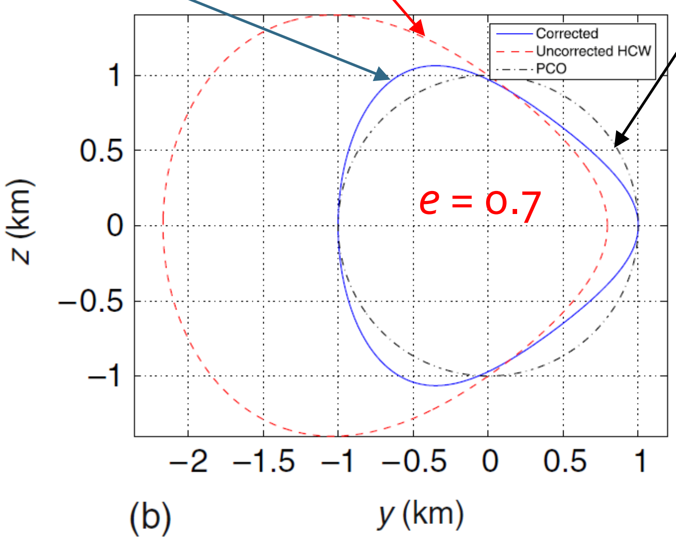
$$z = \rho_z \frac{\sin(f + \alpha_z)}{(1 + e\cos f)} \tag{5.158c}$$

Two main differences if compared with HCW

- These are affected by an along-track bias which can be removed by imposing

$$\rho_y = e\rho_x \cos\alpha_x \tag{5.163}$$





Note: from here equations from Schaub

Orbit Element Difference Description (1)

- Although Cartesian coordinates are common, the differential equations of relative motion must be solved in order to obtain the relative orbit geometry
- The relative orbit is determined through the chief orbit motion and the relative orbit initial conditions

$$\boldsymbol{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)^T \tag{14.37}$$

- To determine where the deputy satellite would be at time t, the differential equations need to be integrated forward from $\boldsymbol{X}(t_0)$ to $\boldsymbol{X}(t)$
- The six initial conditions form six invariant quantities, but they are not convenient to determine the instantaneous geometry of the relative motion
- Instead of using the six invariants (14.37), we propose to use an orbit element difference vector

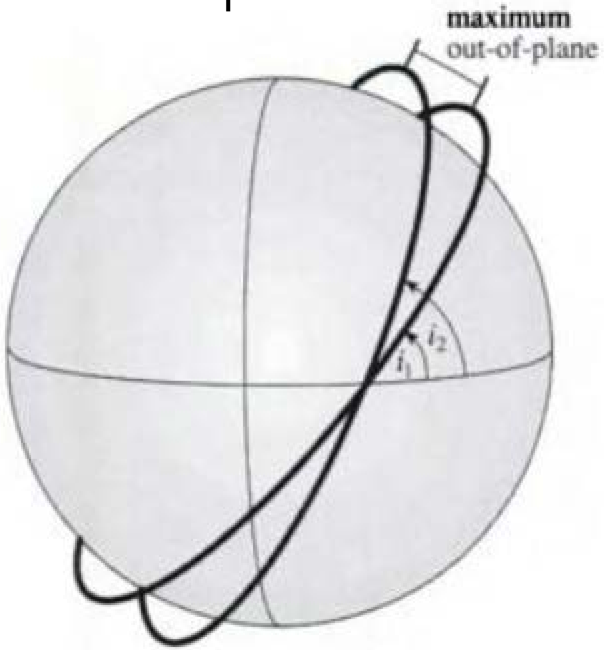
Any orbit element sets could be used!

$$\delta\boldsymbol{e} = \boldsymbol{e}_d - \boldsymbol{e}_c = (\delta a, \delta e, \delta i, \delta\Omega, \delta\omega, \delta M_0)^T \tag{14.40}$$

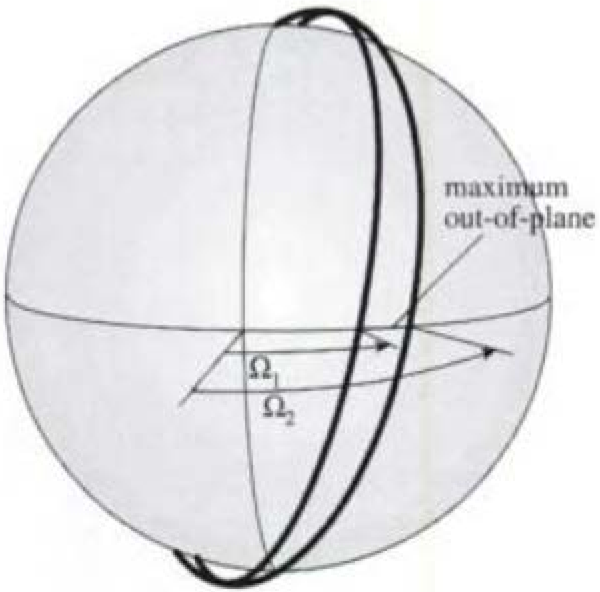
- Given $\boldsymbol{e}_c$ and $\delta\boldsymbol{e}$, the deputy satellite position can be computed at any t by solving the Kepler's equation (without solving differential equations)

Orbit Element Difference Description (2)

- Using orbit element differences, it is possible to make statements regarding the relative orbit geometry, for example
  - Difference in *i* specifies max cross-track motion at extreme latitudes
  - Difference in Ω specifies max cross-track motion at equator



(a) Inclination angle difference



(b) Ascending node difference

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Mapping Cartesian/Orbit Element Differences (1)

- We could use the nonlinear mapping between position&velocity and absolute orbit elements from previous lectures to map between $\boldsymbol{X}$ and $\delta\boldsymbol{e}$
- However, if the relative orbit is small compared with the chief orbit radius, it is possible to obtain a direct linear mapping of this form

$$\boldsymbol{X} = [A(\boldsymbol{e}_c)]\delta\boldsymbol{e} \tag{14.41}$$

- To this end we use three coordinate systems: *C* (RTN at Chief), *D* (RTN at Deputy), and *N* (Inertial at central body). Then

$$[CN] = [CN(\Omega_c, i_c, \theta_c)]$$

- is the direction cosine matrix mapping vectors from *N* to *C*
- The deputy spacecraft inertial position vector can be written in chief and deputy RTN frames as

Inertial orbit radius

$$^{\mathcal{C}}\boldsymbol{r}_d = {^{\mathcal{C}}}(r_c + x, y, z)^T \tag{14.45}$$

$$^{\mathcal{D}}\boldsymbol{r}_d = {^{\mathcal{D}}}(r_d, 0, 0)^T \tag{14.46}$$

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Mapping Cartesian/Orbit Element Differences (2)

- The deputy position vector can be mapped from *D* (14.46) to *C* (14.45) by composing rotation matrices

$$ {}^{\mathcal{C}}\boldsymbol{r}_d = [CN][ND]^{\mathcal{D}}\boldsymbol{r}_d \tag{14.47} $$

- Linearizing about the chief orbit provides

Small angles
No subscript for chief

$$ [ND] \approx [NC] + [\delta NC] \tag{14.48} $$

$$ r_d \approx r + \delta r \tag{14.49} $$

- Equation (14.47) is expanded by dropping second order terms to provide

$$ {}^{\mathcal{C}}\boldsymbol{r}_d = \begin{pmatrix} r + \delta r \\ 0 \\ 0 \end{pmatrix} + r[CN]\begin{pmatrix} \delta NC_{11} \\ \delta NC_{21} \\ \delta NC_{31} \end{pmatrix} \tag{14.51} $$

$$ \delta NC_{11} = NC_{12}\ \delta\theta - NC_{21}\ \delta\Omega + NC_{31} \sin\Omega\ \delta i \tag{14.52} $$

$$ \delta NC_{21} = NC_{22}\ \delta\theta + NC_{11}\ \delta\Omega - NC_{31} \cos\Omega\ \delta i \tag{14.53} $$

$$ \delta NC_{31} = NC_{32}\ \delta\theta + \sin\theta \cos i\ \delta i \tag{14.54} $$

Mapping Cartesian/Orbit Element Differences (3)

- Substitution of (14.52-14.54) into (14.51) and differentiation w.r.t time provides the complete linear mapping between ***X*** and δ***e***
- It is not convenient to use $\delta\theta$ or δf to describe the anomaly difference between formation-flying satellites in eccentric orbits
- Typically δM is used instead, because it remains constant for unperturbed Keplerian motion, even if the chief orbit is elliptic, with $\delta a = 0$
- Differences in true anomaly are written in terms of differences in mean anomaly and differences in eccentricity through linearization of Kepler's eq.

$$\delta f = \frac{(1 + e\cos f)^2}{\eta^3}\delta M + \frac{\sin f}{\eta^2}(2 + e\cos f)\delta e \qquad (14.113)$$

- After substitution in the expressions derived from (14.52-14.54), this provides the desired linear mapping in dimensional form. Normalization by the chief orbit radius and trigonometric identities provide an equivalent to the YA solution in amplitude/phase form

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Mapping Cartesian/Orbit Element Differences (4)

- Linearized relative orbit motion about reference orbit of arbitrary eccentricity

$$u(f) \approx \frac{\delta a}{a} - \frac{e\delta e}{2\eta^2} + \frac{\delta_u}{\eta^2}\left(\cos(f - f_u) + \frac{e}{2}\cos(2f - f_u)\right) \quad (14.122a)$$

$$v(f) \approx \left(\left(1 + \frac{e^2}{2}\right)\frac{\delta M}{\eta^3} + \delta\omega + \cos i\,\delta\Omega\right) - \frac{\delta_u}{\eta^2}\left(2\sin(f - f_u) + \frac{e}{2}\sin(2f - f_u)\right) \quad (14.122b)$$

$$w(f) \approx \delta_w \cos(\theta - \theta_w) \quad (14.122c)$$

- *u*, *v*, and *w* are non-dimensional coordinates (angular for *x*,*y*,*z* << *r*)

Phase angles (not necessarily small)

$$f_u = \tan^{-1}\left(\frac{e\delta M}{-\eta\delta e}\right) \quad (14.120a)$$

$$f_v = \tan^{-1}\left(\frac{\eta\delta e}{e\delta M}\right) = f_u - \frac{\pi}{2} \quad (14.120b)$$

$$\theta_w = \tan^{-1}\left(\frac{\delta i}{-\sin i\ \delta\Omega}\right) \quad (14.120c)$$

Amplitudes (small)

$$\delta_u = \sqrt{\frac{e^2\delta M^2}{\eta^2} + \delta e^2} \quad (14.121a)$$

$$\delta_w = \sqrt{\delta i^2 + \sin^2 i\,\delta\Omega^2} \quad (14.121b)$$

Linear Relative Orbit Motion for Eccentric Orbits

$$\boxed{\begin{aligned} u(f) &\approx \frac{\delta a}{a} - \frac{e\delta e}{2\eta^2} + \frac{\delta_u}{\eta^2}\left(\cos(f - f_u) + \frac{e}{2}\cos(2f - f_u)\right) && (14.122a) \\ v(f) &\approx \left(\left(1 + \frac{e^2}{2}\right)\frac{\delta M}{\eta^3} + \delta\omega + \cos i\delta\Omega\right) \\ &\quad - \frac{\delta_u}{\eta^2}\left(2\sin(f - f_u) + \frac{e}{2}\sin(2f - f_u)\right) && (14.122b) \\ w(f) &\approx \delta_w \cos(\theta - \theta_w) && (14.122c) \end{aligned}}$$

- Difference in ω does not appear in $u(f)$
- Only M and e differences cause periodicity in $u(f)$
- Offsets in v(f) are caused by all orbit element except e difference
- Out-of-plane oscillations are governed by differences in i and Ω
- Although explicit secular terms do not appear in (14.122), they are hidden in δM which grows for non-zero differences in a

Hybrid formulation with f and M as independent variables

Chief

$$M(t) = M_0 + \sqrt{\frac{\mu}{a^3}}(t - t_0)$$

First variation of Kepler equation

$$\delta M = \delta M_0 - \frac{3}{2}\frac{\delta a}{a}(M - M_0) \qquad (14.97)$$

- (14.122) is more accurate than the solution of the TH equations because δa is not approximated by the linearization process, thus bounded orbits can be more accurately designed (no need for extra corrections)
- Note that cross-track separation is extreme at $\theta = \theta_w$ as expected

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Chief Orbits with Small Eccentricity

- Since ***r*** is time dependent for an elliptic chief orbit, the points of maximum angular separation btw. satellites may not correspond to maximum distance
- This difficulty vanishes with small eccentricities of the chief orbit: i.e., $e > \rho/r$ but $e^2 < \rho/r << 1$. In this case we can linearize and drop higher order terms

$$\eta^2 \approx 1 \qquad r = \frac{a\eta^2}{1 + e\cos f} \approx a(1 - e\cos f) \tag{14.125}$$

- The linearized dimensional relative motion can be derived analytically as

$$\boxed{\begin{aligned} x(f) &\approx \delta a + a\delta_x \cos(f - f_x) && (14.127\text{a}) \\ y(f) &\approx a\left(\frac{\delta M}{\eta} + \delta\omega + \cos i\,\delta\Omega\right) \\ &\quad - a\delta_y \sin(f - f_y) - \frac{ae}{2}\sin(2f)\delta e && (14.127\text{b}) \\ z(f) &\approx a\delta_z \cos(\theta - \theta_z) - \frac{ae}{2}\delta_z \cos(2f - f_z) \\ &\quad - \frac{ae}{2}(\sin\omega\,\delta i - \cos\omega \sin i\,\delta\Omega) && (14.127\text{c}) \end{aligned}}$$

Phase angles and amplitudes are now given by Eqs. (14.128-14.129)

Near-Circular Chief Orbits

- For near-circular chief orbits, $e < \rho/r \ll 1$, terms containing the eccentricity can be dropped, thus $r \rightarrow a$, $\eta \rightarrow 1$, $f_x, f_y \rightarrow 0$, $f \rightarrow M = nt$

- The relative orbit motion becomes

$$\boxed{\begin{aligned} x(f) &\approx \delta a - a\cos f\,\delta e \\ y(f) &\approx a(\delta\omega + \delta M + \cos i\,\delta\Omega) + 2a\sin f\,\delta e \\ z(f) &\approx a\sqrt{\delta i^2 + \sin^2 i\,\delta\Omega^2}\cos(\theta - \theta_z) \end{aligned}}$$

(14.131a)
(14.131b)
(14.131c)

- which can be compared with the well known solution of the HCW equations

$$\boxed{\begin{aligned} x(t) &= A_0\cos(nt + \alpha) \\ y(t) &= -2A_0\sin(nt + \alpha) + y_{\text{off}} \\ z(t) &= B_0\cos(nt + \beta) \end{aligned}}$$

(14.130a)
(14.130b)
(14.130c)

- to show that a direct relationship exists between HCW integration constants and orbit element differences

- Here bounded relative motion is assumed with $\delta a = 0$

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