AA 279 D – SPACECRAFT FORMATION-FLYING AND RENDEZVOUS: LECTURE 10

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Relative Navigation Problem

- Given
  - a coarse a priori relative orbit
  - any type and number of relative orbit related measurements
- Find
  - the relative position and velocity of a spacecraft as a function of time
- By
  - minimizing statistical difference between modelled and actual measurements

Note: for the majority of practical applications there is no need for initial relative orbit determination, i.e. for finding six state parameters from six measurements

Statistical Orbit Determination

- A method is required to deal with multiple solved-for parameters to obtain accurate estimates
- Most state estimation methods use the partial derivatives of the observables with respect to various solved-for parameters (state) to correct an a-priori estimate
- Two basic methods exist to do so
  - *Sequential estimation*: a new estimate of the state vector is obtained after each observation
  - *Batch estimation*: all observation are processed and combined to obtain a single update state vector
- The methods can be combined and show advantages and disadvantages. In general sequential estimation is more sensitive to individual data points and converges more quickly at the cost of stability
- Deterministic methods always provide a solution, require a very rough estimate (if at all), are easy to interpret, but are not able to account for uncertainties and include more state parameters. Not addressed here.

Sequential Estimation for Space

- Extended Kalman Filter (EKF)
  - Optimal state estimate in the sense of minimum variance of state error
  - Support of near-real time processing through sequential estimation
  - Virtually no on-board storage of measurements
  - Linearization done about the newly estimated trajectory
  - Higher computational effort because of re-initialization of differential equations
- Unscented Kalman Filter (UKF)
  - Uses finite number of sigma points to propagate probability accurately
  - Computational complexity similar to EKF
  - Lower technology readiness level and flight heritage for space
  - Advantages are present only for highly non-linear dynamics

Alfriend

Ingredients for State Estimation

- State vector, ***x***
  - *n*-dimensional, all variables necessary for state determination, e.g. sensor biases, misalignments, force parameters
  - State parameters may be constant during the processing interval or time-varying:

$$\dot{\mathbf{x}}(t_k) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t_k) + \mathbf{w}(t_k) \qquad Q = E[\mathbf{w}(t)\ \mathbf{w}(t)^T]$$

Process noise

- Observation vector, ***y***
  - *m*-dimensional, all sensor measurements, e.g. direct sensor readouts such as event times or processed observations

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, t) + \mathbf{v}_k \qquad R_k = E[\mathbf{v}_k\ \mathbf{v}_k^T]$$

Measurement noise

- Modeled observation vector, $\hat{\boldsymbol{y}}$
  - *m*-dimensional, predicted values of the observation vector, ***y***, based on estimated values of the state, ***x***
  - The observation model is typically based on the hardware/physical model of the sensor which is providing the measurements

- Dynamics and measurement model are blended to obtain an optimal estimate $\hat{\mathbf{x}}_k$ and its variance or uncertainty P_k

$$\tilde{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k \qquad P_k = \mathrm{E}[\tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_k^T] = \mathrm{E}[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T]$$

Formal state error

Uncertainty and Assumptions

- Random variables are assumed white with normal distribution (Gaussian), mean and variance (std^2) defined through the expectation operator

Scalar

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \mu_X$$

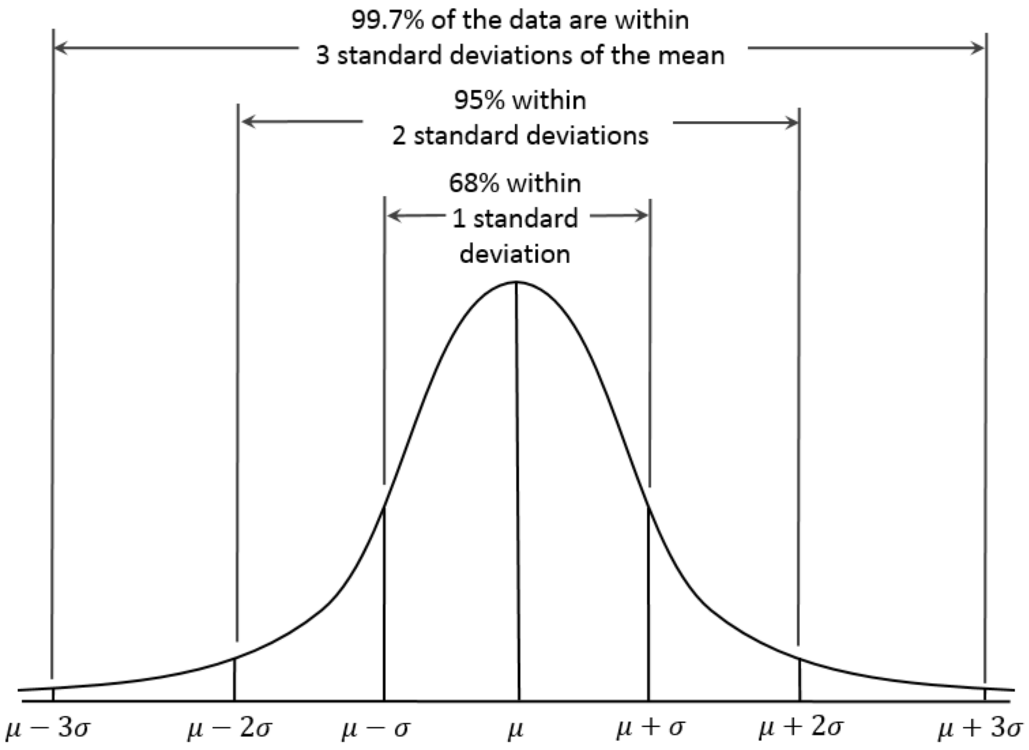
$$V(X) = E[(X - \mu)^2] = E(X^2) - \mu^2 = \sigma_x^2$$

$$SD(X) = \sigma = \sqrt{\sigma^2}$$

2D Vector

$$E(\xi) = E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$V(\xi) = \begin{bmatrix} E(x_1^2) - \mu_1^2 & E(x_1 x_2) - \mu_1\mu_2 \\ E(x_2 x_1) - \mu_2\mu_1 & E(x_2^2) - \mu_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$$ Symmetric positive



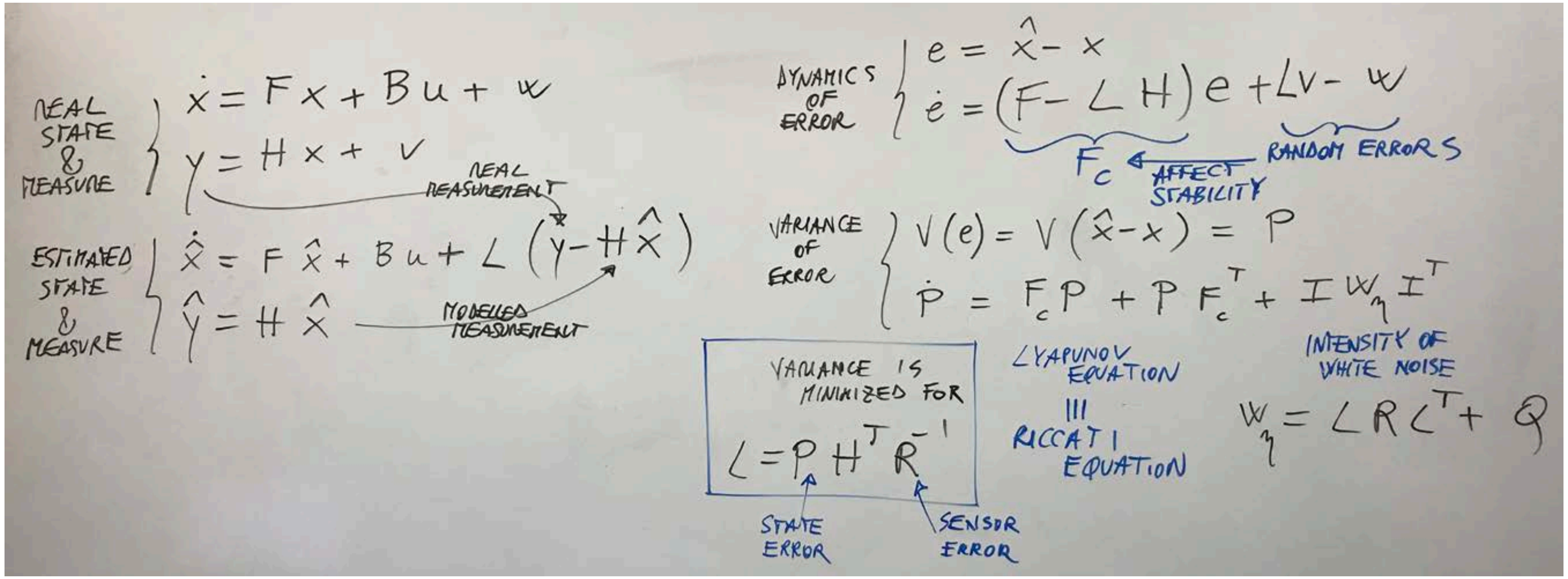
- Statistical properties: sensor noise independent of state and uncorrelated in time, process noise independent of state and measurements

(Extended) Kalman Filter

- Linearization of state and measurement models at each call

$$F_k = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k} \qquad H_k = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k^-} \qquad \Phi_k = e^{F_k \Delta t} \qquad Q_k = \int_0^{\Delta t} e^{F_k \tau} Q (e^{F_k \tau})^T \mathrm{d}\tau$$

- The state estimation error can be expressed as a linear dynamics system where the variance of the state error can be minimized similar to LQR control



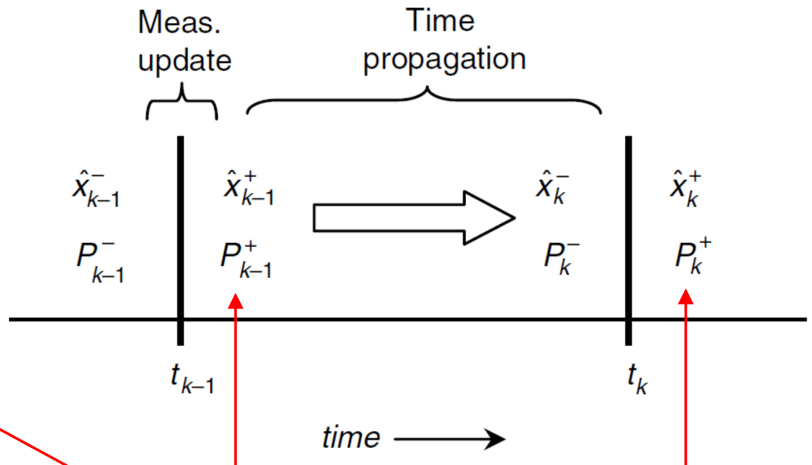
(Extended) Kalman Filter Algorithm

- First Step (always possible)
  - Propagate state and error covariance forward in time

This can be done through numerical integration

$$\hat{\mathbf{x}}_k^- = \Phi_{k-1}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{w}_{k-1} + \mathbf{u}_{k-1}$$

$$P_k^- = \Phi_{k-1} P_{k-1}^+ \Phi_{k-1}^T + Q_{k-1}$$



Input

Output

- Second Step (iff measurement available)
  - Compute gain matrix from uncertainty and partial derivatives

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$

  - Update state and covariance estimates

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- - K_k(\underbrace{\mathbf{y}_k - \mathbf{h}_k(\hat{\mathbf{x}}_k^-)}_{\text{Residual, } \boldsymbol{z}_k})$$

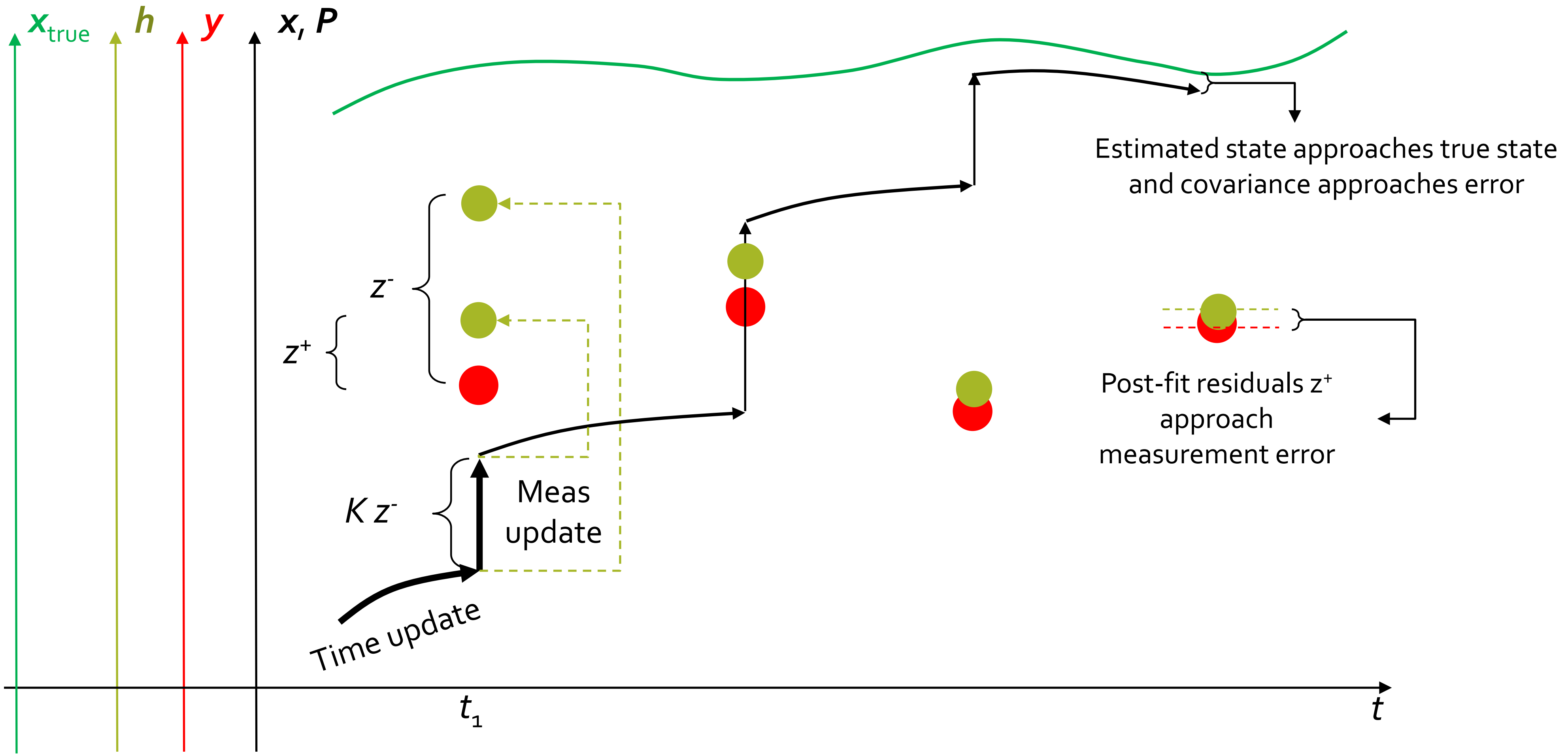
Actual Modelled

Residual, $\boldsymbol{z}_k$

$$P_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T$$

Joseph form (numerically stable)

(Extended) Kalman Filter Procedure



Kinematic vs Reduced-Dynamics Approach

- Reduced-Dynamics estimation
  - A dynamics model is used to model the effect of forces on state parameters
  - Tuning is needed to trade trust in dynamics model (P_o and Q_o) vs trust in measurements (R_o)
  - In the presence of large dynamics model uncertainties
    - Augment state estimation with force model parameters
    - Augment state estimation with so-called empirical accelerations
  - Precise orbit determination relies on the most accurate dynamics and measurements models

- Kinematic estimation
  - A kinematic model for the state parameters replaces the dynamics model
  - Not reliance on dynamics model facilitates including poorly known control input
  - More robustness in closed-loop is expected at the cost of reduced navigation accuracy

$$\dot{x}(t) = Ax(t) + D\tilde{v}(t)$$

$$x(k+1) = Fx(k) + v(k)$$

Continuous vs discrete model with sampling T

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F = e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix} \tilde{q}$$

Constant velocity

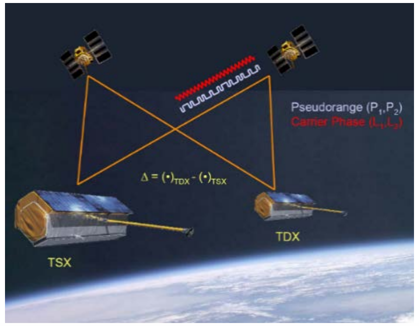
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

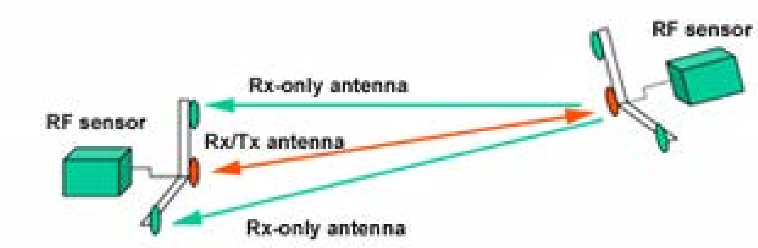
$$F = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} \frac{1}{20}T^5 & \frac{1}{8}T^4 & \frac{1}{6}T^3 \\ \frac{1}{8}T^4 & \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{6}T^3 & \frac{1}{2}T^2 & T \end{bmatrix} \tilde{q}$$

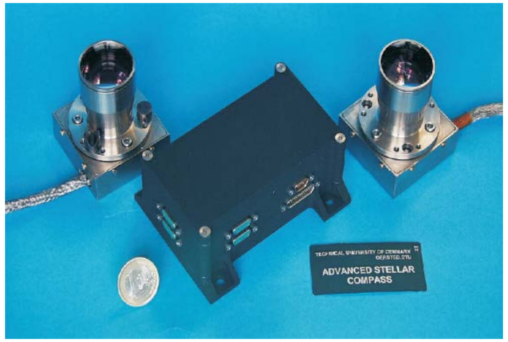
Constant acceleration

Relative Navigation Sensors

- GPS/GNSS
  - Cooperative, rely on GNSS constellation
  - Phase difference = Projected baseline + bias
  - Ambiguity resolution
- Radio Frequency
  - Cooperative, self-contained
  - Pseudolites provide GPS like signals
  - Applicable for deep space navigation
- Vision-Based
  - (Non-)Cooperative, optical/infrared
  - High dynamics range: far- to short-range
  - Support angles-only to full pose estimation

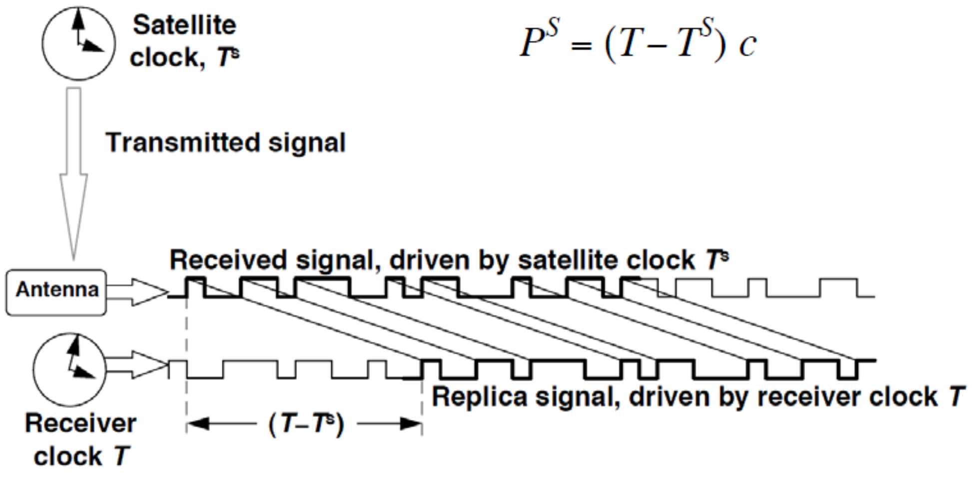




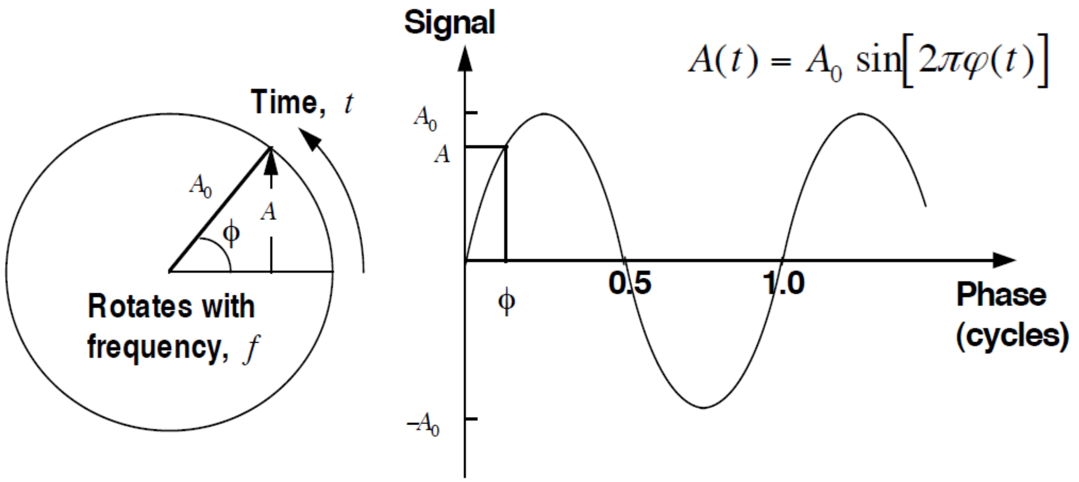


GPS/GNSS Measurement Visuals

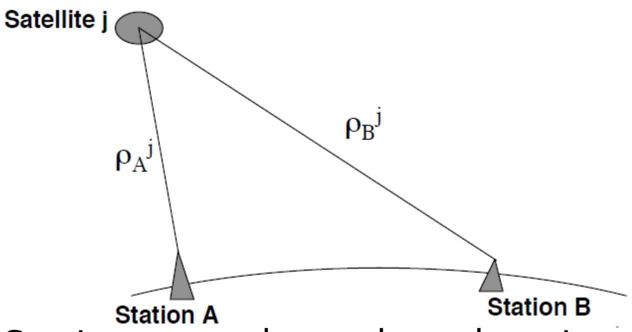
Pseudorange (noise: dm)



Carrier-phase (noise: mm)

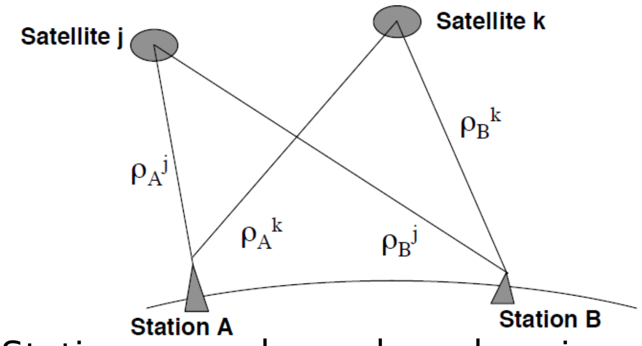


Single differencing (eliminates GNSS clock bias)



Stations record pseudo and carrier-phase measurements to the same satellites

Double differencing (eliminates receiver clock bias)



Stations record pseudo and carrier-phase measurements to the same satellites

GPS/GNSS Measurement Model

- Elementary observables (single-frequency)

Pseudorange

Carrier-phase

Ionosphere path delay

$$\rho_{\mathrm{PR}} = \rho + c(\delta t - \delta t_{\mathrm{GPS}}) + I + S_{\mathrm{PR}} + \epsilon_{\mathrm{PR}}$$

$$\rho_{\mathrm{CP}} = \rho + c(\delta t - \delta t_{\mathrm{GPS}}) - I + \lambda_1 N_{\mathrm{CP}} + S_{\mathrm{CP}} + \epsilon_{\mathrm{CP}}$$

$$I = I_0 L(E) \qquad L(E) = \frac{2.037}{\sqrt{\sin E^2 + 0.076 + \sin E}}$$

$$\epsilon_{\mathrm{CP}} \approx \epsilon_{\mathrm{PR}}/1000$$

- Linear combinations (single-frequency)

GRAPHIC

$$\begin{aligned}\rho_{\mathrm{GR}} &= (\rho_{\mathrm{PR}} + \rho_{\mathrm{CP}})/2 = \\ &= \rho + c(\delta t - \delta t_{\mathrm{GPS}}) + N + S_{\mathrm{GR}} + \epsilon_{\mathrm{GR}}\end{aligned}$$

$$\epsilon_{\mathrm{GR}} \approx \epsilon_{\mathrm{PR}}/2$$

Receivers

Single- and double-differences

$$\begin{aligned}\rho_{\mathrm{SDCP}} = \Delta\rho_{\mathrm{CP}} &= \rho^{\mathrm{M}}_{\mathrm{CP}} - \rho^{\mathrm{T}}_{\mathrm{CP}} = \\ &= \Delta\rho + c\Delta\delta t + 2\Delta N - I_0\Delta L + \Delta S_{\mathrm{CP}} + \Delta\epsilon_{\mathrm{CP}}\end{aligned}$$

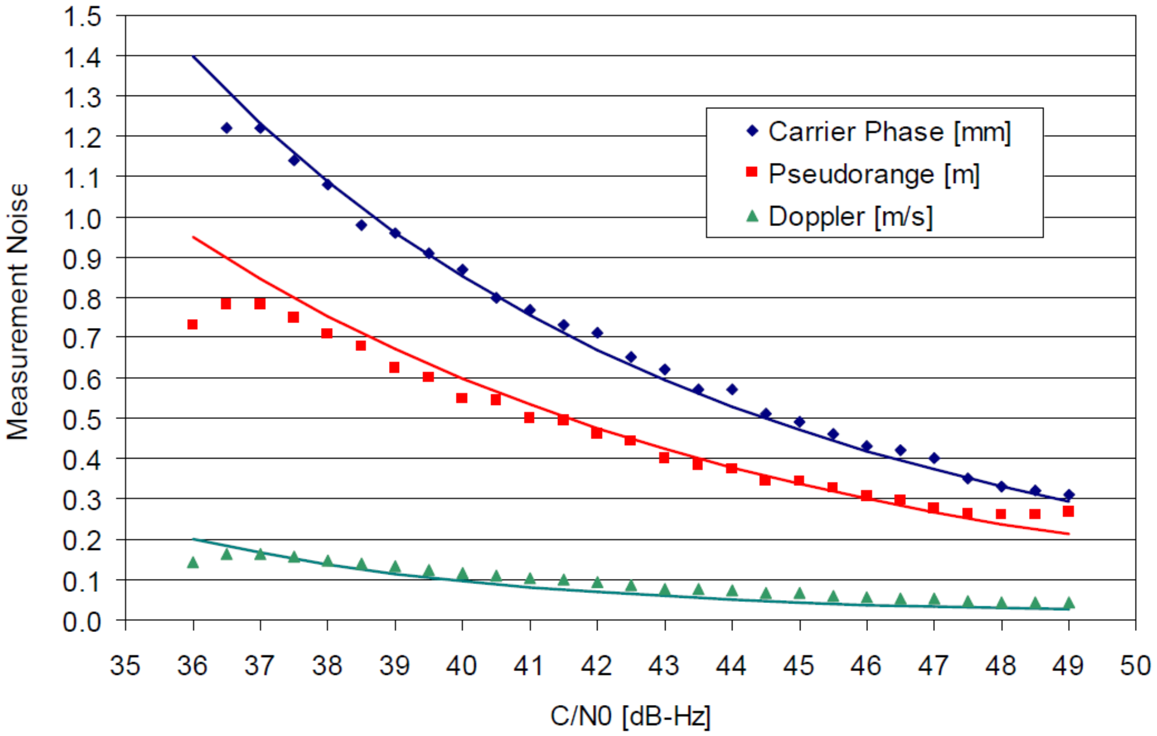
$$\begin{aligned}\rho_{\mathrm{DDCP}} = \Delta^k_j\rho_{\mathrm{SDCP}} &= \rho^{\mathrm{k}}_{\mathrm{SDCP}} - \rho^{\mathrm{j}}_{\mathrm{SDCP}} = \\ &= \Delta^k_j\Delta\rho + \Delta^k_j\Delta N_{\mathrm{C}} \cancel{- I_0\Delta^k_j\Delta L + \Delta^k_j\Delta S_{\mathrm{CP}} + \Delta^k_j\Delta\epsilon_{\mathrm{CP}}}\end{aligned}$$

GNSS satellites

- Since all measurements are assumed unbiased, the estimation state must be augmented with bias parameters
- All measurements must be related to the state through a measurement model

GPS/GNSS Measurement Noise Model

DLR's Phoenix-S GPS receiver



$$b_{\mathrm{dll}} = 1/12\,\mathrm{Hz} \qquad b_{\mathrm{pll}} = 8\,\mathrm{Hz}$$

From tracking loop characteristics

$$\sigma_{\mathrm{PR}} = \frac{c}{1.023 \cdot 10^6} \sqrt{\frac{b_{\mathrm{dll}}}{\mathrm{SNR}}}$$

$$\sigma_{\mathrm{CP}} = \frac{\lambda_1}{2\pi} \sqrt{\frac{b_{\mathrm{pll}}}{\mathrm{SNR}}}$$

$$\mathrm{SNR} = 10^{\frac{C/N_0}{10}}$$

From measurements model

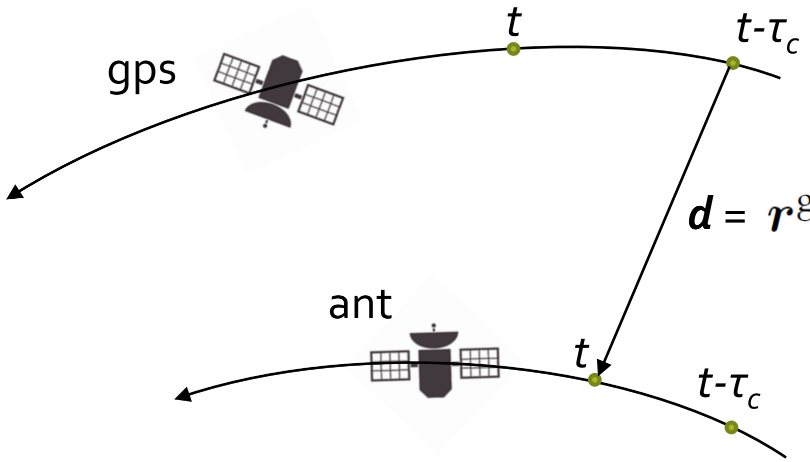
$$\sigma_{\mathrm{GR}} \approx \sigma_{\mathrm{PR}}/2$$

$$\sigma_{\mathrm{SDCP}} \approx \sqrt{2}\sigma_{\mathrm{CP}}$$

$$H_k = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k^-}$$

GPS/GNSS Measurement Partials

- The geometric range is the distance between phase centers of the receiving and transmitting antennas at time of receipt and transmission respectively



$$\boldsymbol{d} = \boldsymbol{r}^{\text{gps}}(t-\tau_{\text{c}}) - \boldsymbol{r}^{\text{ant}}(t)$$

Light time needs to be solved iteratively using

$c\,\tau_{\text{c}} = d$

Time of departure

Time of arrival

Line-of-sight mapped into ECI

$$\frac{\partial \rho(t)}{\partial \boldsymbol{r}} = -\boldsymbol{R}^{\text{ecef}}\boldsymbol{o}_{\text{s}} = -\boldsymbol{R}^{\text{ecef}}\frac{\boldsymbol{r}^{\text{gps}}(t-\tau_{\text{c}}) - \boldsymbol{r}^{\text{ant}}(t)}{\|\boldsymbol{r}^{\text{gps}}(t-\tau_{\text{c}}) - \boldsymbol{r}^{\text{ant}}(t)\|}$$

Info needed on-board:

GPS satellite position and clock offset

Attitude and antenna offset from CoM

Integer Ambiguity Resolution

- Integer ambiguity resolution translates carrier-phase measurements into pseudoranges with millimeter noise level (our ultimate goal)
- We can compare code and phase measurements and round to the nearest integer

$$\rho_{\mathrm{DDCP}} = \Delta_j^k \rho_{\mathrm{SDCP}} = \rho^{\mathrm{k}}_{\mathrm{SDCP}} - \rho^{\mathrm{j}}_{\mathrm{SDCP}} = \\ = \Delta_j^k \Delta\rho + \Delta_j^k \Delta N_{\mathrm{C}} - \cancel{I_0 \Delta_j^k \Delta L + \Delta_j^k \Delta S_{\mathrm{CP}} + \Delta_j^k \Delta\epsilon_{\mathrm{CP}}}$$

GNSS satellites

$$\Longrightarrow \quad \Delta\Delta_{jk}N_c = round[\rho_{DDCP} - \rho_{DDPR}]$$

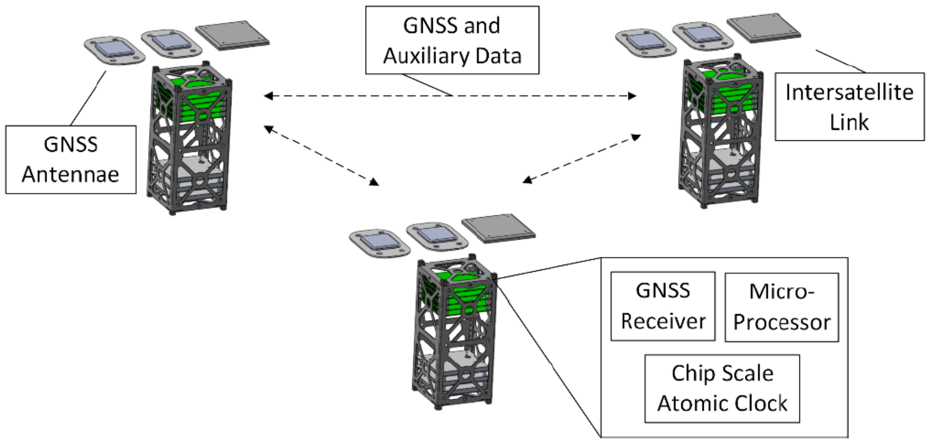
- In practice more advanced methods are required, especially for large baselines, high solar activity, and orbit/attitude maneuvers

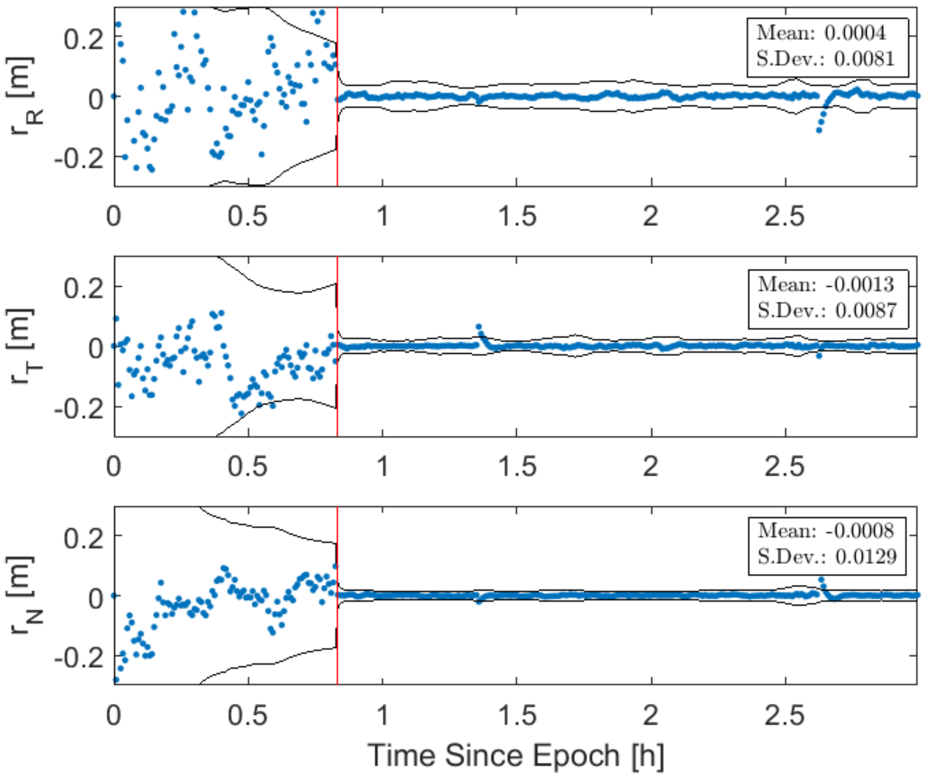
Real-time

Wide- and narrow-lane combinations ⟷ Linear mixed real/integer adjustment problem

Off-line

DiGiTaL





Distributed multi-GNSS Timing/Localization (DiGiTaL) system can achieve centimeter relative positioning accuracy on CubeSat avionics using Integer Ambiguity Resolution

rendezvous Space

State Representation for Navigation

- Different sets of estimation parameters have been considered in relative orbit determination by several authors, what are the key guidelines?
- Fundamentally, the filter may comprise the following types of parameters:
  - spacecraft motion described by position/velocity or orbit elements
  - dynamics model parameters such as atmospheric/solar radiation pressure coefficients, maneuver delta-v, and empirical accelerations
  - measurement model parameters such as receivers' clock-offsets, carrier-phase ambiguities, ionospheric path delays
- Most of these parameters can either be handled as absolute or relative quantities

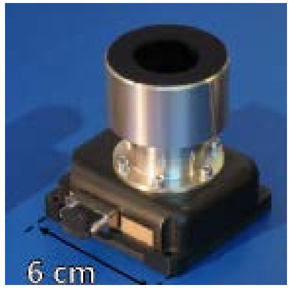
$$\Delta\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$$

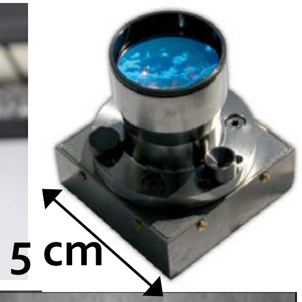
$$\begin{aligned}\boldsymbol{P}_{\text{rel}} &= \mathrm{E}\left[\boldsymbol{e}(\Delta\boldsymbol{x})\boldsymbol{e}(\Delta\boldsymbol{x})^T\right] \\ &= \mathrm{E}\left[\boldsymbol{e}(\boldsymbol{x}_1)\boldsymbol{e}(\boldsymbol{x}_1)^T\right] + \mathrm{E}\left[\boldsymbol{e}(\boldsymbol{x}_2)\boldsymbol{e}(\boldsymbol{x}_2)^T\right] - \mathrm{E}\left[\boldsymbol{e}(\boldsymbol{x}_1)\boldsymbol{e}(\boldsymbol{x}_2)^T\right] - \mathrm{E}\left[\boldsymbol{e}(\boldsymbol{x}_2)\boldsymbol{e}(\boldsymbol{x}_1)^T\right] \\ &= \boldsymbol{P}_1 + \boldsymbol{P}_2 - \boldsymbol{P}_{12} - \boldsymbol{P}_{12}^T\end{aligned}$$

Combination of absolute states

Resulting covariance shows that the relative navigation error is always larger/equal than the absolute navigation error if the cross-covariance between state estimates vanishes or is neglected

Vision-Based Navigation





Angles-only navigation at "far-range"

Pose estimation at "short-range"

Optical relative navigation is characterized by high dynamics-range, low-cost, simplicity

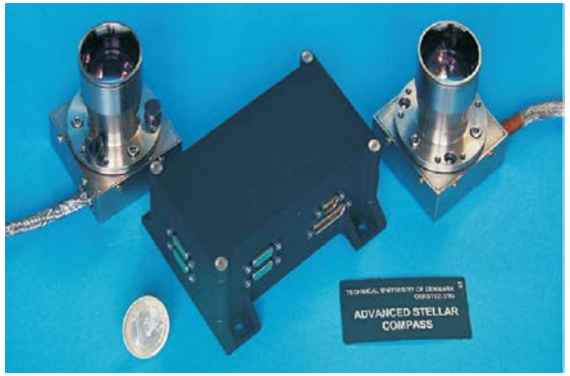
rendezvous Space

Camera Model (Ideal)

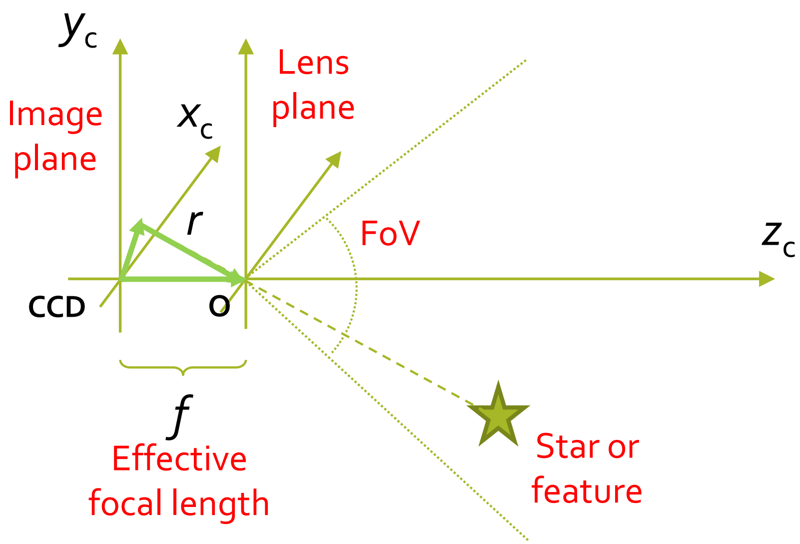
- Detector is activated though impingement of photons over time, e.g. CCD or CMOS
- Corresponding matrix of illuminated pixels after exposure is delivered as image
- The pinhole model is the most popular camera model based on perspective equations

$$\vec{\hat{r}} = \begin{bmatrix} \frac{x_c}{r} & \frac{y_c}{r} & \frac{f}{r} \end{bmatrix} ; r = \sqrt{x_c^2 + y_c^2 + f^2}$$

Units of length



Camera head units (CCD) and digital processing unit of star sensor used on PRISMA



Camera Model (Distortions)

- Optical sensors must be calibrated to remove the following typical errors
  - Origin of lens plane and image plane are not aligned

$$\begin{cases} x_c'' = x_{CCD} - x_o \\ y_c'' = y_{CCD} - y_o \end{cases}$$

Components of intersection of optical axis with the CCD plane

  - Pixels have different width and height

$$\begin{cases} x_c' = x_c'' \\ y_c' = y_c'' \dfrac{dy}{dx} \\ d' = \sqrt{x_c'^2 + y_c'^2} \end{cases}$$

CCD pixel width and height

Intrinsic parameters

  - The optic chain causes distortions (function of distance from center)

$$\begin{cases} x_c = (1 + kd'^2)x_c' \\ y_c = (1 + kd'^2)y_c' \end{cases}$$

Lens distortion coefficient

rendezvous Space

Vision Measurements Partials

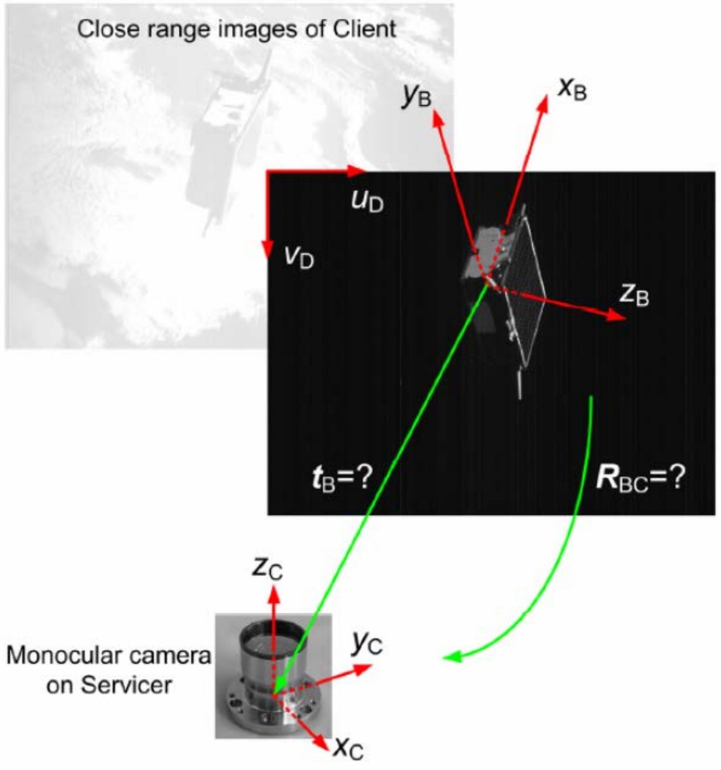
- Each image point (measurement, $\boldsymbol{\rho}$) can be expressed as a function of the unknown pose (state, ***R*** and ***t***) according to the following 3D-2D true perspective projection equations

$$\boldsymbol{r}_{\mathrm{C}} = \left(x_{\mathrm{C}}, y_{\mathrm{C}}, z_{\mathrm{C}}\right)^{t} = \boldsymbol{R}_{\mathrm{BC}}\left(\boldsymbol{p}_{\mathrm{B}} - \boldsymbol{t}_{\mathrm{B}}\right)$$

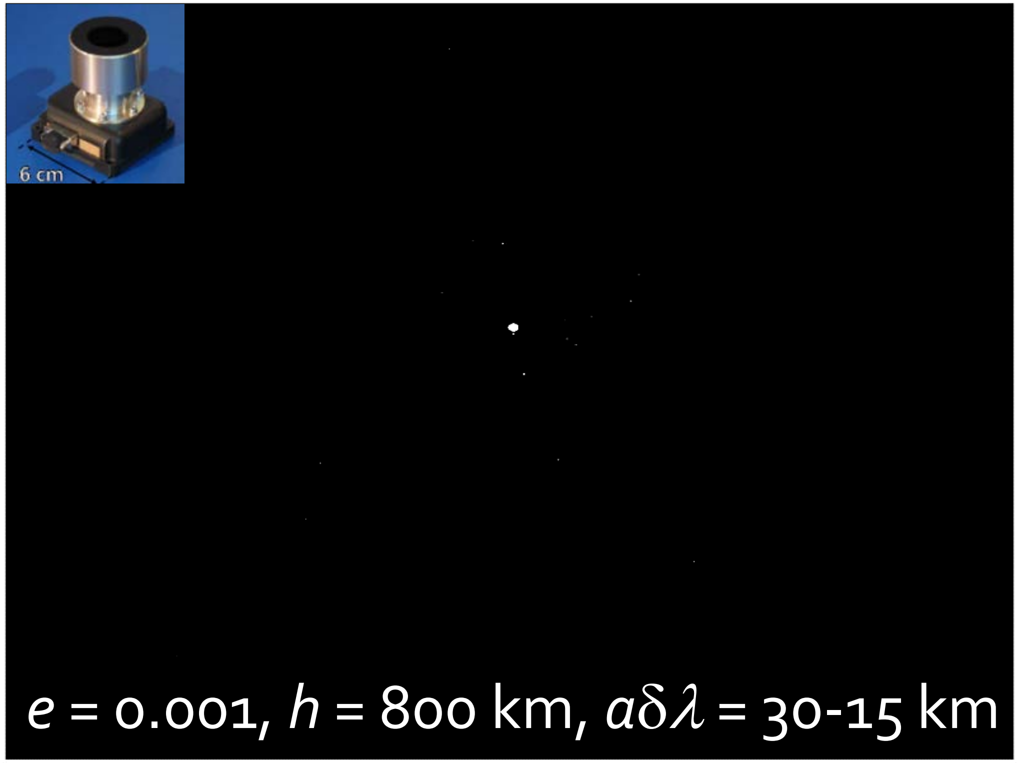
$$\boldsymbol{\rho}_{\mathrm{D}} = \left(u_{\mathrm{D}}, v_{\mathrm{D}}\right) = \left(\frac{x_{\mathrm{C}}}{z_{\mathrm{C}}}\frac{f}{du}, \frac{y_{\mathrm{C}}}{z_{\mathrm{C}}}\frac{f}{dv}\right)$$

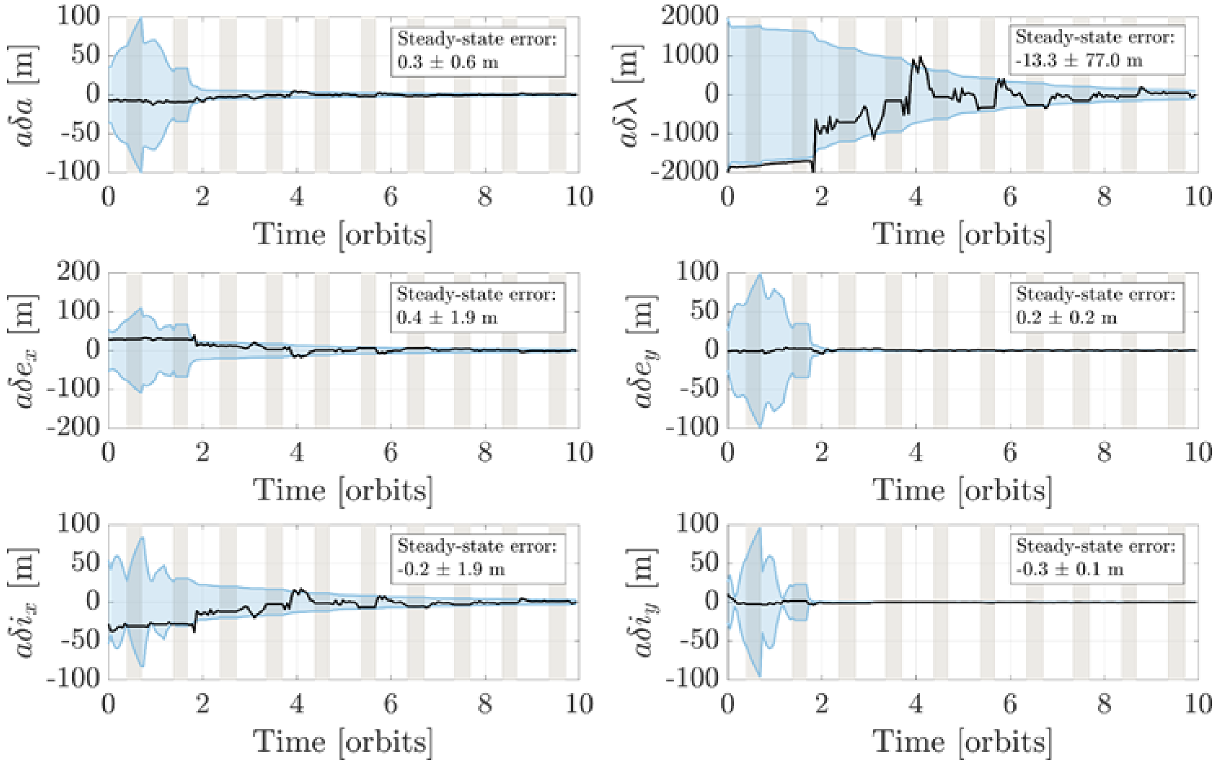
- Assuming known correspondence between features tracked in image and features available in 3D model

$$\frac{\partial \boldsymbol{\rho}_{\mathrm{D}}}{\partial \boldsymbol{r}_{\mathrm{C}}} = \begin{pmatrix} f_u/z & 0 & -f_u x/z^2 \\ 0 & f_v/z & -f_v y/z^2 \end{pmatrix}; \quad \frac{\partial \boldsymbol{r}_{\mathrm{C}}}{\partial \boldsymbol{t}_{\mathrm{B}}} = -\boldsymbol{R}_{\mathrm{BC}}; \quad \frac{\partial \boldsymbol{r}_{\mathrm{C}}}{\partial \boldsymbol{\varphi}_{\mathrm{BC}}} = \frac{\partial \boldsymbol{R}_{\mathrm{BC}}}{\partial \boldsymbol{\varphi}_{\mathrm{BC}}}$$



StarFOX





An UKF can achieve high accuracy in the presence of eclipses, without maneuvers or a-priori info of target using linear ROE dynamics and non-linear measurement model

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