Last Time:

- Static Attitude Estimation Algorithms
- Attitude Error and random Sampling

Today:

- Kalman Filter

Notes on Expectations:

$$E[f(x)] = \int_{-\infty}^{\infty} f(x)\, p(x)\, dx$$

- The value of a function weighted by the probability distribution
- Gives the value of $f(x)$ you would expect to see averaged over many random samples
- Expectation is a linear opperator

$$E[x+y] = E[x] + E[y], \quad E[\alpha x] = \underbrace{\alpha}_{\text{scalar}} E[x]$$

- Common examples:

$$\mu = E[x], \quad P = E[(x-\mu)(x-\mu)^T]$$

Notes on Stochastic Linear Systems:

- Discrete-Time linear time-varying (LTV) system with additive noise:

$$x_{k+1} = A_k x_k + B_k u_k + \underbrace{w_k}_{\text{process noise}}$$

$$y_k = C_k x_k + \underbrace{v_k}_{\text{measurement noise}}$$

$$w_k \sim \underbrace{N(0, W)}_{\text{Normal distribution with } \mu = 0 \text{ and covariance } W} \quad , \quad v_k \sim N(0, V)$$

Notes on Multivariate Gaussians:

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(P)}} \exp\left(-\tfrac{1}{2}(x-\mu)^T P^{-1}(x-\mu)\right)$$

mean: $\mu = E[x] \in \mathbb{R}^n$ covariance: $P = E[(x-\mu)(x-\mu)^T] \in S^n_{++}$

What is a Kalman Filter

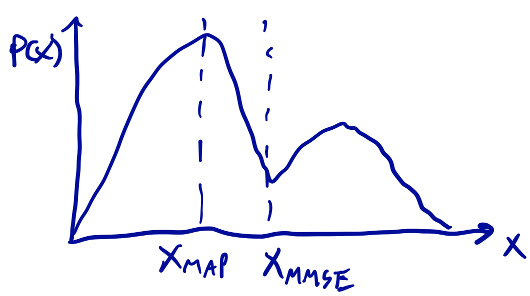
- The optimal linear state estimator for a linear dynamical system.

What are we tryinging to Optimize?

- Minimum mean squared error (MMSE) estimate:

$$\underset{\bar{x}}{\operatorname{argmin}}\ E[(x-\bar{x})^T(x-\bar{x})] = E[\operatorname{Tr}((x-\bar{x})(x-\bar{x})^T)]$$
$$= \operatorname{Tr}[P]$$

- Also known as "least squares" or "minimum variance"



- Maximum a-posteriori (MAP) estimate

$$\underset{x}{\operatorname{argmax}}\ \underbrace{p(x|y)}_{\text{"Probability of } x \text{ conditioned on } y\text{"}}$$

- For a Gaussian, $\bar{x}_{MAP} = \bar{x}_{MMSE} = \mu$

$\Rightarrow$ We can derive the Kalman filter using either criterion

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Kalman Filter as Recursive Linear MMSE Estimator:

- Assume we have an estimate of the state that includes measurement information up to t_k:

$$\bar{x}_{k|k} = E[x_k \mid y_{1\ldots k}]$$

- Assume we know the state error covariance:

$$P_{k|k} = E[(x_k - \bar{x}_{k|k})(x_k - \bar{x}_{k|k})^T]$$

- We want to update $\bar{x}$ and P to include a new measurement at time t_{k+1}

* The Kalman Filter can be broken into two steps:

1) Prediction Step:

$$\begin{aligned}
\bar{x}_{k+1|k} &= E[Ax_k + Bu_k + w_k \mid y_{1\ldots k}] \\
&= AE[x_k \mid y_{1\ldots k}] + Bu_k + E[w_k \mid y_{1\ldots k}]^{\nearrow 0} \\
&= \boxed{A\bar{x}_{k|k} + Bu_k}
\end{aligned}$$

$$\begin{aligned}
P_{k+1|k} &= E[(x_{k+1} - \bar{x}_{k+1|k})(x_{k+1} - \bar{x}_{k+1|k})^T] \\
&= E[(Ax_k + \cancel{Bu_k} + w_k - A\bar{x}_{k|k} - \cancel{Bu_k})(\cdots)^T] \\
&= A\underbrace{E[(x - \bar{x}_{k|k})(x - \bar{x}_{k|k})^T]}_{P_{k|k}}A^T + \underbrace{E[w_k w_k^T]}_{W_k}
\end{aligned}$$

* Noise w_k is uncorrelated with the state x_k:

$$E[x_k w_k^T] = E[w_k x_k^T] = 0$$

$$\boxed{P_{k+1|k} = A P_{k|k} A^T + W}$$

2) Measurement Update

- Define "innovation"

$$z_{k+1} = y_{k+1} - C\bar{x}_{k+1|k} = \underbrace{C x_{k+1} + v_{k+1}}_{y_{k+1}} - C\bar{x}_{k+1|k}$$

- Innovation Covariance

$$S_{k+1} = E[z_{k+1} z_{k+1}^T] = E[(C(x_{k+1} - \bar{x}_{k+1|k}) + v_{k+1})(\cdots)^T]$$

* Noise v_{k+1} and state x_{k+1} are uncorrelated

$$\Rightarrow S_{k+1} = C E[(x_{k+1} - \bar{x}_{k+1|k})(x_{k+1} - \bar{x}_{k+1|k})^T] C^T + E[v_{k+1} v_{k+1}^T]$$

$$= \boxed{C P_{k+1|k} C^T + V}$$

- Innovation is the thing we feed back into the filter to correct our state estimate

- State Update:

$$\bar{x}_{k+1|k+1} = \bar{x}_{k+1|k} + \underbrace{L_{k+1}}_{\text{"Kalman Gain"}} z_{k+1}$$

- Covariance Update:

$$P_{k+1|k+1} = E[(x_{k+1} - \bar{x}_{k+1|k+1})(x_{k+1} - \bar{x}_{k+1|k+1})^T]$$

$$= E[(x_{k+1} - \bar{x}_{k+1|k} - L(C x_{k+1} + v_{k+1} - C\bar{x}_{k+1|k}))(\cdots)^T]$$

$$= (I - L_{k+1}C) E[(x_{k+1} - \bar{x}_{k+1|k})(\cdots)^T](I - L_{k+1}C)^T$$

$$+ L_{k+1} E[v_{k+1} v_{k+1}^T] L_{k+1}^T$$

* Noise v_{k+1} and state x_{k+1} are uncorrelated

$$P_{k+1|k+1} = \underbrace{(I - L_{k+1}C)P_{k+1|k}(I - L_{k+1}C)^T + L_{k+1}VL_{k+1}^T}$$

"Joseph Form" - you may see other versions

- Kalman Gain

$$\text{MMSE} \Rightarrow \text{minimize } E[(x_{k+1} - \bar{x}_{k+1|k+1})^T(x_{k+1} - \bar{x}_{k+1|k+1})]$$

$$= Tr[P_{k+1|k+1}]$$

- $P_{k+1|k+1}$ is a quadratic function of L_{k+1}

$$\Rightarrow \text{set } \frac{\partial Tr[P_{k+1|k+1}]}{\partial L_{k+1}} = 0 \quad \text{and solve for } L_{k+1}$$

$$\Rightarrow -2P_{k+1|k}C^T + 2L_{k+1}CP_{k+1|k}C^T + 2L_{k+1}V = 0$$

$$\Rightarrow -P_{k+1|k}C^T + L_{k+1}S_{k+1} = 0$$

$$\Rightarrow \boxed{L_{k+1} = P_{k+1|k}C^TS_{k+1}^{-1}}$$

KF Algorithm:

1) Start with $\bar{x}_{0|0}$, $P_{0|0}$, W, V

2) Predict: $\bar{x}_{k+1|k} = A\bar{x}_{k|k} + Bu_k$

$$P_{k+1|k} = AP_{k|k}A^T + W$$

3) Calculate innovation + covariance:

$$z_{k+1} = y_{k+1} - C\bar{x}_{k+1|k}, \quad S_{k+1} = CP_{k+1|k}C^T + V$$

4) Calculate Kalman Gain: $L_{k+1} = P_{k+1|k}C^T S_{k+1}^{-1}$

5) Update:

$$\bar{x}_{k+1|k+1} = \bar{x}_{k+1|k} + L_{k+1}z_{k+1}$$

$$P_{k+1|k+1} = (I - L_{k+1}C)P_{k+1|k}(I - L_{k+1}C)^T + L_{k+1}VL_{k+1}^T$$

6) Go to 2