Last Time

- MEKF Implementation

Today:

- Star tracker in MEKF
- Environmental Disturbance Torques
- Gravity Gradient

MEKF with Star Tracker

- Star trackers compare images of the sky to a star catalog

- They comput a quaternion by solving Wahba's problem and return a 3x3 covariance matrix V_{ST}.

- Because this is a static estimate we can do better by running a filter that includes dynamics and past measurements

- Star trackers can also have issues at high slew rates, so running a filter with other sensors is a good idea.

Star Tracker Measurement Model:

$$y = g(x)$$

- Attitude is already part of the state, so measurement function is trivial.

- Problem: We don't want to subtract quaternions in the innovation calculation.

- Solution: Use quaternion/matrix multiplication and axis-angle error vector.

$$
y_{k+1} = \begin{bmatrix} {}_B q_{ST} \\ {}^B r_1 \\ \vdots \\ {}^B r_N \end{bmatrix} \implies z_{k+1} = \begin{bmatrix} \log(Q^T_{k+1|k} \, Q_{ST}) \\ {}^B r_1 - Q^T_{k+1|k} \, {}^N r_1 \\ \vdots \\ {}^B r_N - Q^T_{k+1|k} \, {}^N r_N \end{bmatrix}
$$

$$
C_{k+1} = \left[\begin{array}{c|c} I_{3\times3} & 0_{3\times3} \\ {}^B \hat{r}^1_{k+1|k} & 0_{3\times3} \\ \vdots & \vdots \\ {}^B \hat{r}^N_{k+1|k} & 0_{3\times3} \end{array}\right]
$$

- The rest of the filter stays the same

Environmental Torques:

* Drag:

  - Atmospheric density scales roughly exponentially

$$\rho(h) = \rho_0 e^{\left(\frac{-(h-h_0)}{H}\right)}$$

"scale height"

  - More accurate models account for temperature variations, seasons, etc.

  - This typically matters for Earth orbits < 500 km altitude

* Solar Radiation Pressure

  - Scales like $1/r^2$, r = distance from sun

  - $$p_{photon} = \frac{E}{c} \Rightarrow F = \frac{\dot{E}}{c} \Rightarrow P = \frac{F}{A} = \alpha \frac{W}{c}$$

momentum, force, pressure, area

solar flux watts/meter²

α varies from 1–2 depending on reflectivity

  - $W \approx 1360\ W/m^2$ at Earth

$$P = \frac{1360}{3\times10^8} \approx 4.5\times10^{-6} \text{ pascals}$$

  - This matters for Earth orbits > 600 km altitude (once it is bigger than drag)

* Magnetic

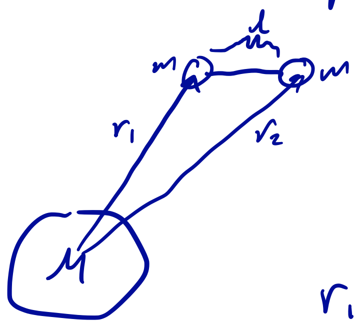
  - Planetary magnetic fields exert torques on spacecraft that have a magnetic moment (due to permanent magnetism or current flow).

  - Scales like $1/r^3$, r = distance from planet

  - Can be used for actuation (magnetic torque coils) from LEO → GEO

* Gravity Gradient

- Gravity varies like $1/r^2$, causing different forces on different ends of a spacecraft:



dumbell satellite

$$F_1 = \frac{\mu m}{r_1^2}, \quad F_2 = \frac{\mu m}{r_2^2}$$

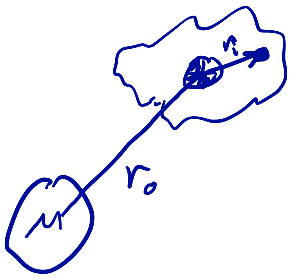
$$r_1 < r_2 \Rightarrow F_1 > F_2$$

- Tends to align long axis of satellite along radial direction

- Can be used to stabilize Earth-pointing spacecraft

* Gravity Gradient Torque:

- General rigid body satellite:



$$m = \sum_{i=1}^{N} m_i$$

$$\underbrace{r_o}_{\sim 10^6 m} \gg \underbrace{r_i}_{\sim 1 m}$$

$$F_{grav} = \sum_{i=1}^{N} \frac{-\mu m_i}{\|r_o + r_i\|^3}(r_o + r_i)$$

* Since $r_o \gg r_i$, we can Taylor expand F_{grav} about r_o. The 1st order approximation will be good to ~6 decimal places, so its essentially exact for our purposes.

$$F(r) \approx \underbrace{F(r_o)}_{\text{Point mass force at COM}} + \underbrace{\sum_{i=1}^{N} \frac{\partial F}{\partial r}\bigg|_{r_o} r_i}_{\text{small adjustment due to finite mass distribution}}$$

$$F(r) = \frac{-\mu m}{(r^T r)^{3/2}} r \Rightarrow \boxed{\frac{\partial F}{\partial r} = \frac{-\mu m}{(r^T r)^{3/2}} I + \frac{3\mu m}{(r^T r)^{5/2}} r r^T}$$

"Gravity gradient (Jacobian)"

$$\tau = \sum_{i=1}^{N} r_i \times F_i = \sum_{i=1}^{N} r_i \times \left(\frac{-\mu m_i}{(r_o^T r_o)^{3/2}} I + \frac{3\mu m_i}{(r_o^T r_o)^{5/2}} r_o r_o^T\right) r_i$$

$$= \frac{3\mu}{(r_o^T r_o)^{5/2}} \sum_{i=1}^{N} r_i \times m_i r_o r_o^T r_i = \frac{3\mu}{(r_o^T r_o)^{5/2}} \sum_{i=1}^{N} r_o \times (-m_i r_i r_i^T) r_o$$

* Remember inertia matrix $J = \sum_i m_i[(r_i^T r_i)I - r_i r_i^T]$

* $(r_i^T r_i) I$ term will cancel in the expression for torque due to cross product.

$$\Rightarrow \tau = \frac{3\mu}{(r_0^T r_0)^{5/2}} \sum_{i=1}^{N} r_0 \times \underbrace{\left(m_i \left[(r_i^T r_i) I - r_i r_i^T \right]\right)}_{J} r_0$$

$$\Rightarrow \boxed{\tau = \frac{3\mu}{(r_0^T r_0)^{5/2}} r_0 \times J r_0}$$

"Gravity Gradient Torque"

- Magnitude scales like $1/r^3$
- Torque is zero whenever radial vector is parallel to a principal axis
- It is usually sufficient to ignore higher-order gravity terms (e.g. J_2) in calculating gravity gradient torque.