Take Me Out to (Analyze) the Ballgame

Visualization and Analysis Techniques for Big Spatial Data

Chris Comiskey

Oregon State University

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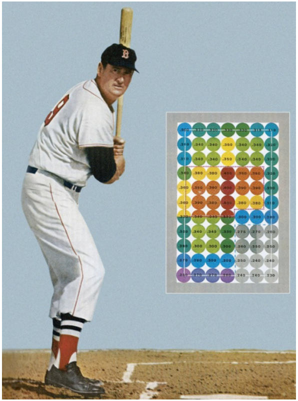
Baseball, Baseball, Baseball

- Chris, rookie year

- Chris, Boston Red Sox

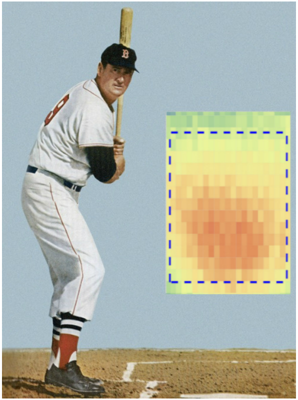
Hitting Analytics

- “The Science of Hitting”$_{1970}$



- Iconic breakthrough, hitter
- No data

- The Statistics of Hitting

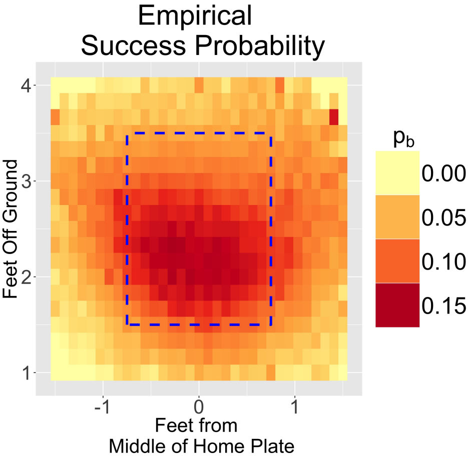


- PITCHf/x data, R, heat maps
- SGLMMs, Stan, PPMs, INLA

Outline

1. Variable-Resolution Heat Maps
2. Interactive Heat Map Confidence Intervals
3. Approaches to Big Data Spatial Mixed Models for Baseball Data
   - Computational Optimization in Stan
   - Predictive Process Models
   - Integrated Nested Laplace Approximation

Empirical Success Probability Heat Map



Empirical Success Probability Heat Map

- Swings: $i = 1, 2, \ldots, N$
- Bernoulli(π_i) trials:

$$S_i = \begin{cases} 1; & \text{swing success} \\ 0; & \text{swing failure} \end{cases}$$

- Location (x_i, y_i)
- Grid boxes $G_1, G_2, \ldots, G_B$
- Box b totals:

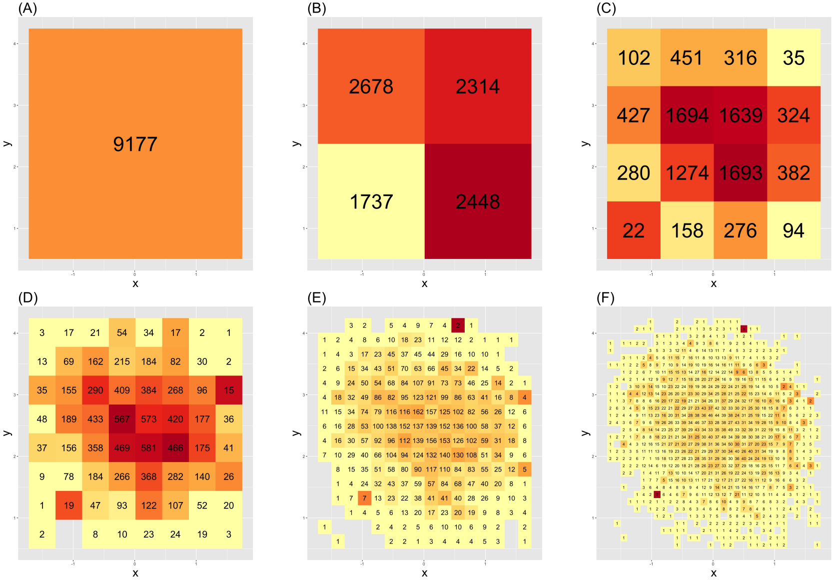
$$N_b = \sum_{i=1}^{N} I_{(x_i, y_i) \in G_b}$$

- Box b empirical success:

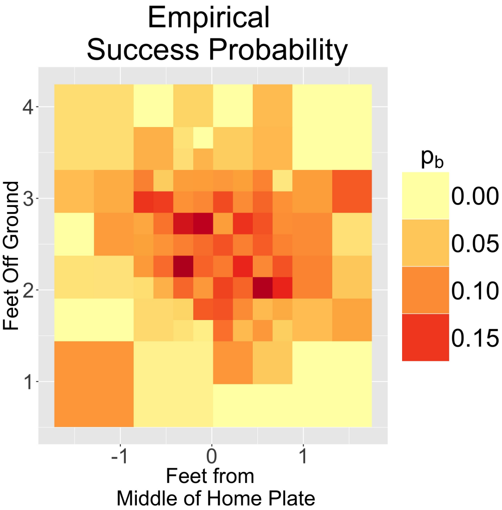
$$p_b = \frac{1}{N_b} \sum_{i=1}^{N} S_i I_{(x_i, y_i) \in G_b}$$

Heat Map Resolution Selection

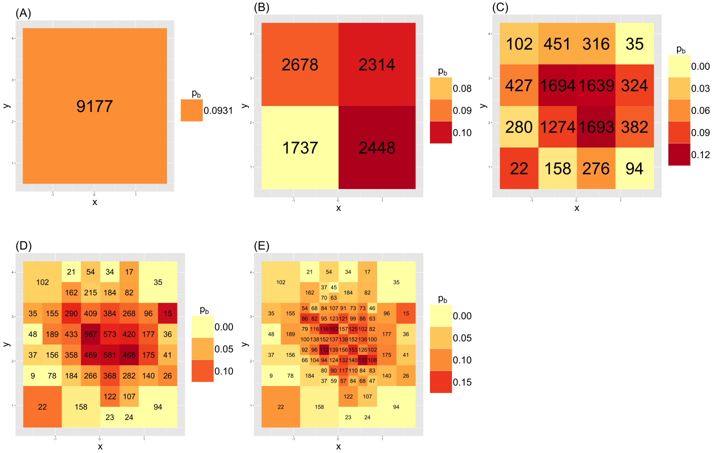
Jhonny Peralta



A Thing of Beauty



- VR algorithm
  - Stopping rule
  - Subdivision method



Variable-Resolution Heat Maps

Combine Resolutions

- Sample size stopping rule
- Combine resolutions
- Resolution conveys varying data abundance
- Improvements:
  - **.gif** option
  - Alternate stopping rule types
  - Alternate subdivision method

varyres(...) in **varyres**

varyres {varyres} R Documentation

A variable-resolution heat map generator

Description

This function creates variable resolution heat maps accroding to a stopping rule

Usage

```
varyres(dataset, cutoff, fun = mean, max = 6)
```

Arguments

`dataset`	data frame with spatial data: x-coordinates (x), y coordinates (y), and Bernoulli responses at those locations (res)
`cutoff`	Box subdivisions cease when a box sample size drops below the cutoff
`fun`	Function to apply to responses in each box
`max`	The maximum number of subdivision iterations the algorithm will perform

Value

A list containing a data frame for each iteration of the subdivision algorithm; and a vector of the number of boxes eligible for subdivision at each iteration.

Examples

```
data(hitter)
data <- varyres(hitter, mean, cutoff = 200, max = 6)
mapit(data[[4]])
```

Outline

Generalized Linear Model

$$Y_i | \mathbf{X}_i(\mathbf{s}_i) \overset{ind}{\sim} \text{Bernoulli}(\pi_i)$$

$$\text{logit}(\pi_i | \boldsymbol{s}_i) = \mathbf{X}_i(\mathbf{s}_i)\beta$$

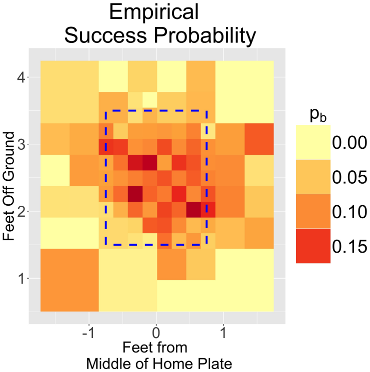
- Swings: $i = 1, 2, \ldots, N$
- Pitch location: $\mathbf{s}_i = (x_i, y_i)$
- Biomechanical covariates: $\mathbf{X}_i(\mathbf{s}_i)$
- Fit to Jhonny Peralta data

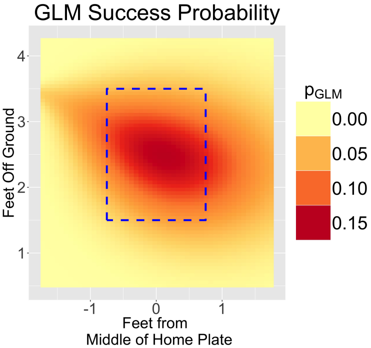
Logistic Regression Model

Jhonny Peralta

$$Y_i | \mathbf{X}_i(\mathbf{s}_i) \overset{ind}{\sim} \text{Bernoulli}(\pi_i)$$

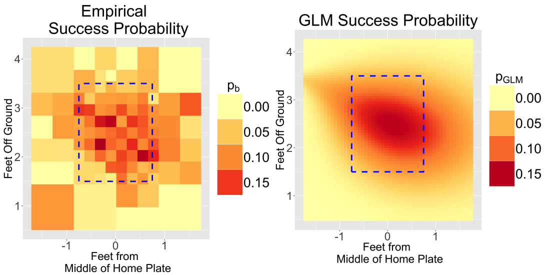
$$\text{logit}(\pi_i | \boldsymbol{s}_i) = \mathbf{X}_i(\mathbf{s}_i)\beta,$$





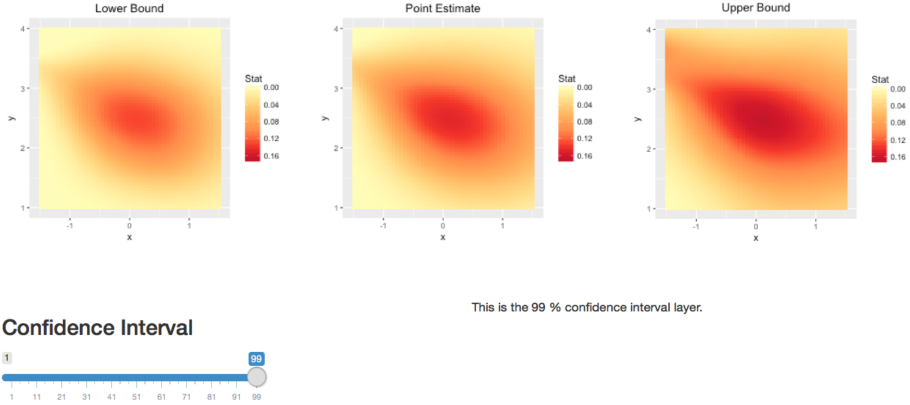
Logistic Regression Model

Jhonny Peralta



- Hosmer-Lemeshow (logistic regression) GOF test
  - H_0: Well fit
    H_A: Lack of fit
  - p-value = 0.8217
- Confidence intervals

Interactive Confidence Intervals



mapapp: An R Package

mapapp: An R Package

- Challenge: Where does user stop and package begin?
- Future improvement

```
all_in_one <- get_CI(model, x, y, levels)
shinyHMCI(all_in_one)
```

1. `get_CI(...)` — creates proper data structure
2. `shinyHMCI(all_in_one)` — creates Shiny app

...and **.gif** option.

Outline

Spatial Generalized Linear Mixed Model (SGLMM)

$$Y_i | \mathbf{X}_i(\mathbf{s}_i) \overset{ind}{\sim} \text{Bernoulli}(\pi_i)$$
$$\text{logit}(\pi_i | \boldsymbol{s}_i) = \mathbf{X}_i(\mathbf{s}_i)\beta + w(\boldsymbol{s}_i)$$

- $w(\boldsymbol{s}_i)$ — spatial random effect, location $\boldsymbol{s}_i$.
- $\boldsymbol{w}(\boldsymbol{s}) = (w(\boldsymbol{s}_1), w(\boldsymbol{s}_2), \ldots, w(\boldsymbol{s}_N))$ — vector
- $\boldsymbol{w}(\boldsymbol{s})$ - Gaussian Random Field (GRF)

Gaussian Random Field: $\mathbf{w(s)}$

$$\boldsymbol{w}(\boldsymbol{s})|\boldsymbol{\theta} \sim \mathrm{MVN}(\mathbf{0}, \Sigma(\boldsymbol{\theta}))$$

$$\Sigma(\phi, \sigma^2)_{i,k} = \sigma^2 \exp(-||\boldsymbol{s}_i - \boldsymbol{s}_k||/\phi)$$

- Spatial exponential covariance
  - $||\boldsymbol{s}_i - \boldsymbol{s}_k||$ - Euclidean distance
  - σ^2 - scale parameter
  - ϕ - range parameter.
- Notice: $\Sigma(\boldsymbol{\theta})$ — $n \times n$

Computational Cost: “Big N” Problem

- Peralta: $n = 9177$
- $\boldsymbol{w}(\boldsymbol{s})$: 9177×9177 cov. matrix
- MCMC iterations require: Σ^{-1}, determinant of Σ
- $\mathcal{O}(n^3)$ rate of increase:

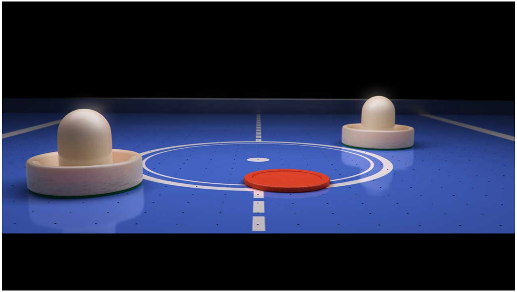
$$t(n) \leq M \cdot n^3$$

$$\text{as } n \to \infty$$

Outline

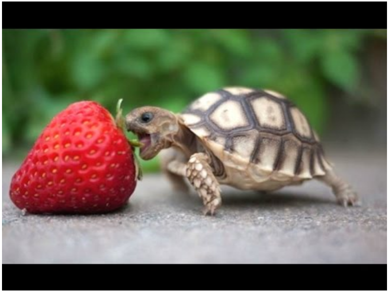
Hamiltonian Monte Carlo (HMC)

- Stan uses Hamiltonian proposal mechanism
  1. Disk on surface, with position $(\beta, \boldsymbol{\theta})$ and momentum
  2. Randomly sample momentum (auxiliary)
  3. Calculate new position (parameters)
  4. That's your Metropolis proposal.



Stan Computational Optimization

- ϕ, $\boldsymbol{\beta}$: informative/proper priors $\rightarrow$ identifiability/cost/convergence
- QR factorization of X, Cholesky decomposition of $\Sigma(\boldsymbol{\theta})$
- Matrices & vectors faster than loops & scalars
- n = 25: 40 seconds $\rightarrow$ 3 seconds
- n = 2000 — overnight, 350 draws

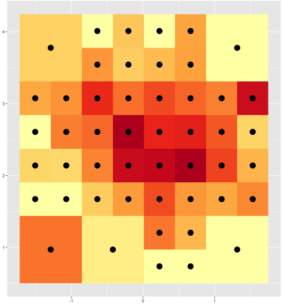


Outline

Predictive Process Models (PPMs)

[Banerjee et al., 2008]

- Knots: $\boldsymbol{S}^* = \{\boldsymbol{s}_1^*, \ldots, \boldsymbol{s}_m^*\}$
  - $m << n$



$$\text{logit}(\pi_i|\boldsymbol{s}_i) = \boldsymbol{X}_i(\boldsymbol{s}_i)\beta + \widetilde{w}(\boldsymbol{s}_i)$$

$$\widetilde{\boldsymbol{w}}(\boldsymbol{s}) \sim \text{MVN}\{0, \widetilde{\Sigma}(\boldsymbol{\theta})\}$$

$$\widetilde{\Sigma}(\boldsymbol{\theta})_{i,j} = \boldsymbol{\sigma}^{*T}(\boldsymbol{s}_i;\boldsymbol{\theta})\cdot\boldsymbol{\Sigma}^{*-1}(\boldsymbol{\theta})\cdot\boldsymbol{\sigma}^*(\boldsymbol{s}_j;\boldsymbol{\theta})$$

- $\boldsymbol{\sigma}^*(\boldsymbol{s}_i;\boldsymbol{\theta})$ = $\text{Cov}(\boldsymbol{s}_i, \boldsymbol{S}^*)$
- $\boldsymbol{\Sigma}^*(\boldsymbol{\theta}) = \text{Var}(\boldsymbol{S}^*)$

PPM Results

- Implement in **spBayes** [Finley et al., 2013].
- MCMC chains did not converge — trace-plots:

  - n = 1000, knots = 97, 10K samples, $\approx$ 6.7 mins
  - n = 1000, knots = 49, 30K samples, $\approx$ 7 mins
  - n = 3000, knots = 49, 80K samples, $\approx$ 54 mins
- Speed issue for extending MCMC chains

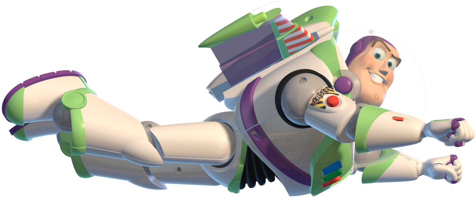
Outline

Integrated Nested Laplace Approximation (INLA)

[Rue et al., 2009]

$$\text{logit}(\pi_i) = \boldsymbol{X}_i(\boldsymbol{s}_i)\beta + w(\boldsymbol{s}_i)$$
$$\boldsymbol{w}(\boldsymbol{s})|\boldsymbol{\theta} \sim \text{MVN}(\boldsymbol{0}, \Sigma(\boldsymbol{\theta}))$$

- Assume Matérn covariance
- Two parts (continuous domain)
  - Part 1: Stochastic Partial Differential Equation (SPDE)
  - Part 2: Integrated Nested Laplace Approximation (INLA)



INLA: Step 1

- Parameter vector: $\boldsymbol{\rho} = (\beta^\mathsf{T}, \widetilde{\boldsymbol{w}}^\mathsf{T})^\mathsf{T}$
  - $\boldsymbol{Q}$: precision matrix of $\boldsymbol{\rho}$
- Hyperparameter vector: $\boldsymbol{\theta} = (\kappa, \sigma)$

1. Gaussian approximation:

$$p(\boldsymbol{\rho}|\boldsymbol{\theta}, \boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\rho}, \boldsymbol{\theta})p(\boldsymbol{\rho}|\boldsymbol{\theta})p(\boldsymbol{\theta})$$

$$p(\boldsymbol{\rho}|\boldsymbol{\theta}, \boldsymbol{y}) \propto \exp\left(-\frac{1}{2}\boldsymbol{\rho}^T\boldsymbol{Q}\boldsymbol{\rho} + \sum_i \log p(y_i|\boldsymbol{\rho}, \boldsymbol{\theta})\right)$$

$$p_G(\boldsymbol{\rho}|\boldsymbol{\theta}, \boldsymbol{y}) \propto \exp\left(-\frac{1}{2}(\boldsymbol{\rho} - \boldsymbol{\mu})^T(\boldsymbol{Q} + \mathrm{diag}(\boldsymbol{c}))(\boldsymbol{\rho} - \boldsymbol{\mu})\right)$$

   - Gaussian kernel: $\boldsymbol{c}$ and $\boldsymbol{\mu}$ depend on second order Taylor expansions of $f(\boldsymbol{\rho}) = \sum_i \log p(y_i|\boldsymbol{\rho}, \boldsymbol{\theta})$

INLA: Step 2

- Fact: $P(A, B) = P(A|B)P(B) = P(B|A)P(A)$

$$p(\boldsymbol{y}|\boldsymbol{\theta}) = \frac{p(\boldsymbol{y}|\boldsymbol{\rho}, \boldsymbol{\theta})p(\boldsymbol{\rho}|\boldsymbol{\theta})}{p(\boldsymbol{\rho}|\boldsymbol{y}, \boldsymbol{\theta})}$$

- Bayes proportionality:

$$\begin{aligned} p(\boldsymbol{\theta}|\boldsymbol{y}) &\propto p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}) \\ &\propto \frac{p(\boldsymbol{y}|\boldsymbol{\rho}, \boldsymbol{\theta})p(\boldsymbol{\rho}|\boldsymbol{\theta})}{p(\boldsymbol{\rho}|\boldsymbol{y}, \boldsymbol{\theta})} \cdot p(\boldsymbol{\theta}) \end{aligned}$$

2. For a given $\boldsymbol{\theta}$, let $\boldsymbol{\rho}_0 = \text{argmax}_{\rho} p(\boldsymbol{\rho}|\boldsymbol{y}, \boldsymbol{\theta})$. Then,

$$\tilde{p}(\boldsymbol{\theta}|\boldsymbol{y}) \propto \frac{p(\boldsymbol{y}|\boldsymbol{\rho}_0, \boldsymbol{\theta})p(\boldsymbol{\rho}_0|\boldsymbol{\theta})}{p_G(\boldsymbol{\rho}_0|\boldsymbol{y}, \boldsymbol{\theta})} \cdot p(\boldsymbol{\theta})$$

INLA: Step 2

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INLA: Step 3 & Step 4

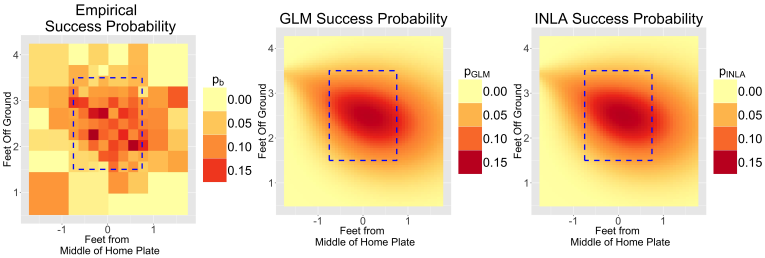
3. Numerical integration:

$$p(\rho_j|\boldsymbol{y}) \approx \int p_{\mathrm{G}}(\rho_j|\boldsymbol{\theta}, \boldsymbol{y})\tilde{p}(\boldsymbol{\theta}|\boldsymbol{y})d\boldsymbol{\theta}$$

4. Numerical integration:

$$p(\theta_k|\boldsymbol{y}) \approx \int \tilde{p}(\boldsymbol{\theta}|\boldsymbol{y})d\boldsymbol{\theta}_{-k}$$

INLA Model Fit: 34 seconds



INLA		
ρ_i	$\hat{\rho}_i$	SE
κ	3.23	$\pm$ 1 SE: (1.35, 7.54)
σ	0.11	$\pm$ 1 SE: (0.05, 0.26)

- $\hat{\kappa}$ — long range correlation
- $\text{SE}(w(\boldsymbol{s})) = 0.11$.
- $p_i = 0.15 \rightarrow p_i \pm 1 \cdot \text{SE} = (0.137, 0.165)$
- $p_i = 0.15 \rightarrow p_i \pm 2 \cdot \text{SE} = (0.125, 0.181)$

Summary

- Resolution selection? Variable-resolution heat maps and **varyres**
- Heat map confidence intervals? Interactive HMCIs and **mapapp**
- Fitting big data SGLMMs to baseball data?
  - Stan - inadequate
  - PPM - Slow and did not converge
  - INLA - Fast, successful(?); to be continued...

Thanks for listening.

Sudipto Banerjee, Alan E Gelfand, Andrew O Finley, and Huiyan Sang. Gaussian predictive process models for large spatial data sets. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 70(4):825–848, 2008.

Andrew O Finley, Sudipto Banerjee, and Alan E Gelfand. spbayes for large univariate and multivariate point-referenced spatio-temporal data models. *arXiv preprint arXiv:1310.8192*, 2013.

Håvard Rue, Sara Martino, and Nicolas Chopin. Approximate bayesian inference for latent gaussian models by using integrated nested laplace approximations. *Journal of the royal statistical society: Series b (statistical methodology)*, 71(2):319–392, 2009.

Finn Lindgren, Håvard Rue, and Johan Lindström. An explicit link between gaussian fields and gaussian markov random fields: the stochastic partial differential equation approach. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 73(4): 423–498, 2011.

Moving Forward

- Improve packages
- Submit packages to CRAN
- Learn a LOT more about INLA results, estimates, standard errors
- Biomechanists at OSU
- Compare GLM and SGLMM with scoring rules
- Exit velocity as response variable
- Independence assumption, pitch sequences
- Randomized pitch selection

If I knew then...

- Skip UNC
- Start research $\rightarrow$ start packages
- Use courses to practice writing, get feedback, etc.
- Emphasize literature review
- Exit velocity as response???

Glad I did know...

- “Write and show as you go.”
- Growth mindset

Responsibilities

- Reproducibility.
- Acknowledge uncertainty.
- Honesty
  - Expertise
  - Data
  - Methods
  - Results
- Warren Buffet anecdote

Important Theorems

- Law of Large Numbers - “Guarantees stable, long term results for random events.”
- Central Limit Theorem
- Bayes Theorem
- Taylor’s Theorem
- Ito’s Lemma - stochastic calculus, mathematical finance, Black-Scholes, options pricing!

GLM Confidence Intervals

Asymptotically...

- $\mathrm{E}(\hat{\beta}) = \beta$
- $\mathrm{Var}(\beta) = (\boldsymbol{X}^{\mathsf{T}} \widehat{\boldsymbol{V}} \boldsymbol{X})^{-1}$
  - $\widehat{\boldsymbol{V}}_{i,i} = p_i(1 - p_i)$

Variable-Resolution Algorithm

Pseudo-code

* Keep track of three containers: History, Current, Iteration

```
function(dataset,
         fun = mean, # function to apply to box data
         cutoff,     # stopping rule threshold
         max ){      # maximum number of iterations

HISTORY_CONTAINER # store iterative results

CURRENT_CONTAINER: box centers, statistic,
  box observations, box heights/widths,
  box vert/horiz, upper/lower bounds

Add CURRENT_CONTAINER to HISTORY_CONTAINER
```

Variable-Resolution Algorithm

Pseudo-code

```
WHILE(exists N_b > cutoff and ITERATION < max) {
  ITERATION CONTAINER <- list()

    FOR(All BOX_b in CURRENT CONTAINER){
      IF(BOX_b count > cutoff){

          Retrieve BOX_b data, subdivide/calculate
            BOX_b sub-box info, add to
            ITERATION CONTAINER
      }
    }
    Update CURRENT_CONTAINER: remove obsolete boxes,
      add ITERATION CONTAINER boxes
    Add CURRENT CONTAINER to HISTORY CONTAINER
  }
```

HMC, Longer Version

- $H(q(t), p(t)) = U(q(t)) + K(p(t))$
- Total energy of a system
- t = time
- q = position (variables of interest), U = potential energy
- p = momentum (auxiliary), K = kinetic energy
- $H(q, p) = -\log f_q(q) + p^T \boldsymbol{M}^{-1} p/2$
- Tidy partial derivatives

Stan Optimization

- QR factorization of X

$$\begin{aligned} \boldsymbol{X} &= \boldsymbol{QR} \\ \boldsymbol{X}\beta &= \boldsymbol{QR}\beta \rightarrow \text{Let } \boldsymbol{\theta} = \boldsymbol{R}\beta \\ \boldsymbol{X}\beta &= \boldsymbol{Q}\boldsymbol{\theta} \\ \beta &= \boldsymbol{R}^{-1}\boldsymbol{\theta} \end{aligned}$$

- Cholesky decomposition of $\Sigma(\boldsymbol{\theta})$

```
L = cholesky_decompose(Sigma);
Z ~ normal(0, 1);
Z_mod = L * Z;
Y ~ bernoulli_logit(Q*theta + Z_mod);
```

Stan Model Fitting

- N = 1000: 7 hours 45 mins for 1500 draws
  - Note: GLM (w/o spatial effect): 6 seconds.
- N = 2000 overnight, 350 draws

Predictive Process Models (PPMs)

[Banerjee et al., 2008]

- $\text{Cov}\left(w(\boldsymbol{s}_i), w(\boldsymbol{s}_k)\right) = \Sigma(\boldsymbol{s}_i, \boldsymbol{s}_k; \boldsymbol{\theta}) = \left[\Sigma(\boldsymbol{s}_i, \boldsymbol{s}_k; \boldsymbol{\theta})\right]_{i,k=1}^{n}$
- Knots: $\boldsymbol{S}^* = \{\boldsymbol{s}_1^*, \ldots, \boldsymbol{s}_m^*\}$
  - $m << n$
- Knot random effects: $\boldsymbol{w}^* = \left[w(\boldsymbol{s}_i^*)\right]_{i=1}^{m}$
- Knot GRF: $\boldsymbol{w}^*|\boldsymbol{\theta} \sim \text{MVN}\{\boldsymbol{0}, \Sigma^*(\boldsymbol{\theta})\}$ — m-dim
  - Knot covariance: $\Sigma^*(\boldsymbol{\theta}) = \left[\Sigma(\boldsymbol{s}_i^*, \boldsymbol{s}_k^*)\right]_{i,k=1}^{m}$
- $\boldsymbol{\sigma}(\boldsymbol{s}_0; \boldsymbol{\theta}) = \left[\Sigma(\boldsymbol{s}_0, \boldsymbol{s}_k^*; \boldsymbol{\theta})\right]_{k=1}^{m}$ — $m \times 1$ vector

PPM Kriging

- $\boldsymbol{\sigma}(\boldsymbol{s}_0;\boldsymbol{\theta}) = \left[\Sigma(\boldsymbol{s}_0, \boldsymbol{s}_k^*;\boldsymbol{\theta})\right]_{k=1}^{m}$ — $m \times 1$ vector
  - $\text{Cov}(w(\boldsymbol{s}_0), \text{ knots})$
- $\widetilde{w}(\boldsymbol{s}_0)$ - interpolated random effect

$$\begin{aligned}\widetilde{w}(\boldsymbol{s}_0) &= E[w(\boldsymbol{s}_0)|\boldsymbol{w}^*] \\ &= \boldsymbol{\sigma}^T(\boldsymbol{s}_0;\boldsymbol{\theta}) \cdot \Sigma^{*-1}(\boldsymbol{\theta}) \cdot \boldsymbol{w}^*\end{aligned}$$

- Spatially varying linear combination
- Minimizes squared error loss function (linear predictors, GRFs)

Now replace $w(\boldsymbol{s}_i)$ with $\widetilde{w}(\boldsymbol{s}_i)$.

The Predictive Process: $\widetilde{w}(\boldsymbol{s})$

Another Gaussian Random Field

$$\widetilde{\boldsymbol{w}}(\boldsymbol{s}) \sim \text{MVN}\{0, \widetilde{\Sigma}(\cdot)\}$$
$$\widetilde{\Sigma}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{\theta}) = \boldsymbol{\sigma}^T(\boldsymbol{s}; \boldsymbol{\theta}) \cdot \boldsymbol{\Sigma}^{*-1}(\boldsymbol{\theta}) \cdot \boldsymbol{\sigma}(\boldsymbol{s}'; \boldsymbol{\theta})$$

Predictive process model:

$$\text{logit}(\pi_i | \boldsymbol{s}_i) = \boldsymbol{X}_i(\boldsymbol{s}_i)\beta + \widetilde{w}(\boldsymbol{s}_i)$$

- Note dimension of covariance matrix.
- Implement in **spBayes** [Finley et al., 2013].
- Convergence problems

Part 1: Stochastic Partial Differential Equation (SPDE)

[Lindgren et al., 2011]

- Goal: Represent Matérn GRF as GMRF
- SPDE:

$$(\kappa^2 - \Delta)\boldsymbol{w}(\boldsymbol{s}) = \boldsymbol{W}(\boldsymbol{s})$$

- Finite Element Method: project SPDE onto basis representation
- Piecewise linear basis representation

$$\widetilde{w}(\boldsymbol{s}) = \sum_k \psi_k(\boldsymbol{s})\omega_k$$

  - Domain triangulation
  - $\psi_k(\boldsymbol{s})$ — deterministic basis functions
  - ω_k — weights, explicit with sparse precision matrix

Part 1: Stochastic Partial Differential Equation (SPDE)

[Lindgren et al., 2011]

- For $\langle f(\boldsymbol{u}), g(\boldsymbol{u})\rangle = \int f(\boldsymbol{u})g(\boldsymbol{u})d\boldsymbol{u}$:

$$\left[\left\langle \phi_l, (\kappa^2 - \Delta)\widetilde{\boldsymbol{w}} \right\rangle\right]_l \stackrel{D}{=} [\langle \phi_l, \boldsymbol{W}\rangle]_l$$

- Galerkin solution, $\boldsymbol{\phi} = \boldsymbol{\psi}$:

$$\left[\left\langle \psi_l, (\kappa^2 - \Delta)\left(\sum_{k=1}^{n_v} \psi_k \omega_k\right)\right\rangle\right]_l \stackrel{D}{=} [\langle \psi_l, \boldsymbol{W}\rangle]_l$$

$$\left(\kappa^2 \boldsymbol{C} + \boldsymbol{G}\right)\boldsymbol{\omega} \stackrel{D}{=} \mathrm{N}(\boldsymbol{0}, \boldsymbol{C})$$

where $\boldsymbol{C} = \langle \psi_l, \psi_k\rangle$ and $\boldsymbol{G} = \langle \psi_l, -\Delta\psi_k\rangle$

- $\boldsymbol{\omega} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{Q}^{-1})$ for

$$\boldsymbol{Q} = \left(\kappa^2 \widetilde{\boldsymbol{C}} + \boldsymbol{G}\right)^T \boldsymbol{C}^{-1} \left(\kappa^2 \widetilde{\boldsymbol{C}} + \boldsymbol{G}\right).$$

Estimates

GLM			
Covariate	β_i	MLE	SE
1	β_0	-4.08	0.70
r	β_1	1.19	0.51
θ	β_2	-1.93	1.90
$r * \theta$	β_3	-1.64	0.70
r^2	β_4	-0.32	0.09
θ^2	β_5	-3.92	1.10
$r^2 * \theta^2$	β_6	-0.46	0.21

INLA			
Covariate	ρ_i	$\hat{\rho}_i$	SE
N/A	κ	3.23	$\pm$ 1 SE: (1.35, 7.54)
N/A	σ	0.11	$\pm$ 1 SE: (0.05, 0.26)
1	β_0	-4.14	0.76
r	β_1	1.25	0.55
θ	β_2	-1.90	1.96
$r * \theta$	β_3	-1.70	0.93
r^2	β_4	-0.33	0.10
θ^2	β_5	-3.93	1.14
$r^2 * \theta^2$	β_6	-0.48	0.22

- $\hat{\kappa}$ — long range correlation
- $\text{SE}(w(\boldsymbol{s})) = 0.11.$
- $\rightarrow 0.15 \pm 1 \cdot \text{SE} = (0.137, 0.165)$
- $\rightarrow 0.15 \pm 2 \cdot \text{SE} = (0.125, 0.181)$