Approaches to Effective Sample Size Estimation for Trend Detection in Time Series

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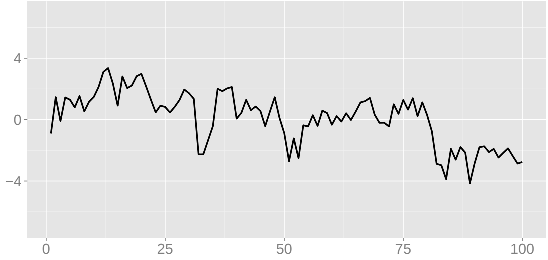
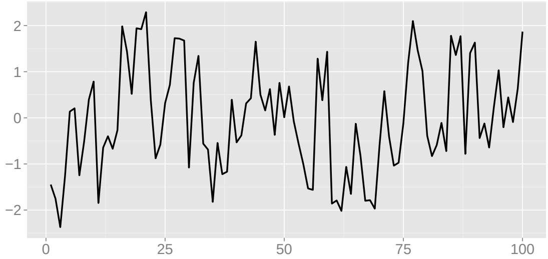
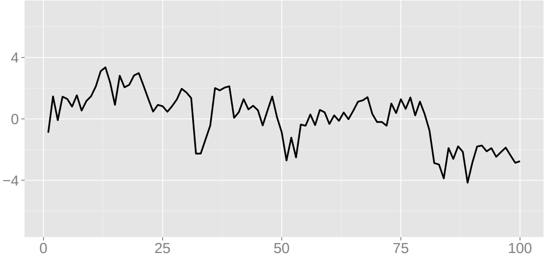
Which AR(1) time series has trend?

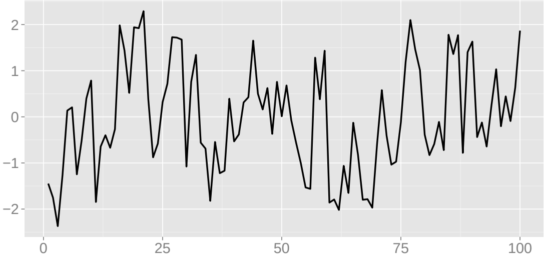




Introduction: Trend Detection

Trick question... neither!





Autocorrelation Autocorrelation Autocorrelation

- Autocorrelation complicates trend detection
  - Trend = change in the mean
- Effective Sample Size (ESS)
- $ESS_{\mu} = \frac{n^2}{\mathbf{1}'\Sigma_{\rho}\mathbf{1}} = n\left(\frac{n}{\mathbf{1}'\Sigma_{\rho}\mathbf{1}}\right)$
  - [Bayley and Hammersley(1946)]
  - For *iid* observations: $\mathrm{Var}(\bar{x}) = \frac{\sigma^2}{n}$
  - For positively correlated observations: $\mathrm{Var}(\bar{x}) > \frac{\sigma^2}{n}$
- Use ESS to modify trend detection tests
  - ESS for the *mean*

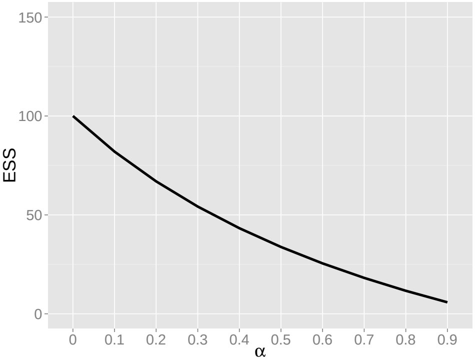
A series of observations $X_1, X_2, \ldots, X_{100}$, where

$$X_t = \alpha X_{t-1} + Z_t.$$

- $0 \leq \alpha < 1$
  - Stationary
- $Z_t \overset{iid}{\sim} N(0, 1)$
- $X_t \sim N\left(0, \frac{1}{1-\alpha^2}\right)$
- $X_t | X_{t-1} \sim N\left(\alpha X_{t-1}, 1\right)$

Effective Sample Size

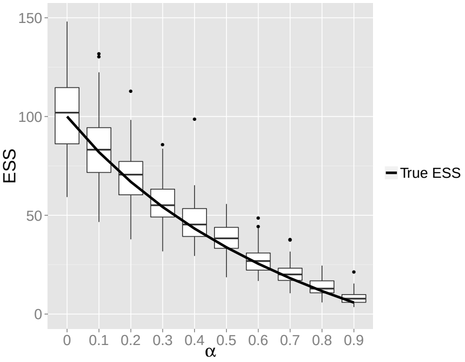
- Attempt to quantify information content
- As autocorrelation increases, ESS decreases.



Can we estimate ESS effectively?

Effective Sample Size: MLE Estimation

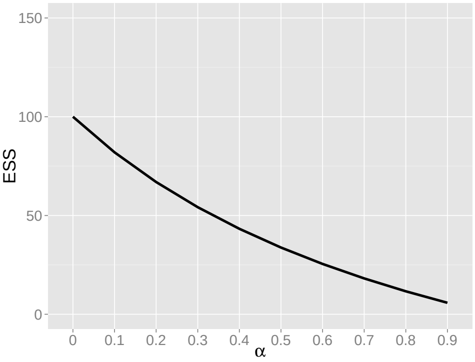
- For every correlation parameter α, 100 simulated AR(1) series, and 100 estimates $\hat{\alpha}_{MLE}$



[Thiébaux and Zwiers(1984)]

Effective Sample Size: Fisher Information

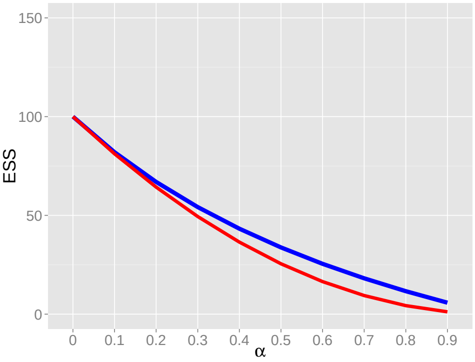
- Back to drawing board...
- Could Fisher Information serve as a proxy?



Let's take a look.

Effective Sample Size: Fisher Information

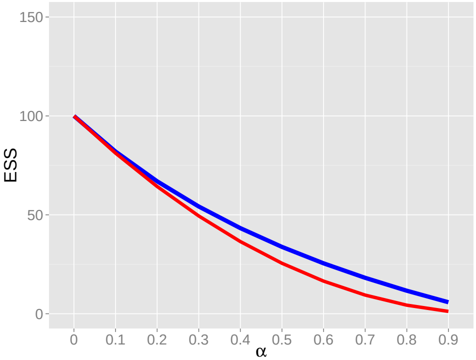
- Back to drawing board...
- Could **Fisher Information** serve as a proxy for **ESS**?



Sure could.

Effective Sample Size: Fisher Information

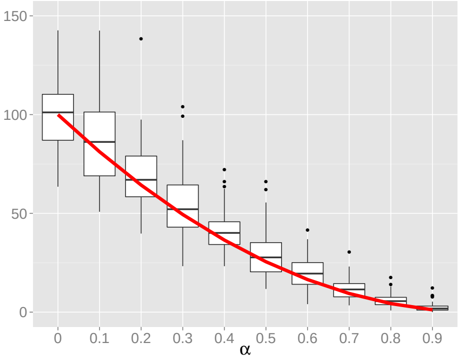
- Is estimation any easier?
- More simulations.



Drumroll...

Effective Sample Size: Fisher Information Estimation

- For every α, 100 simulated AR(1) series
- 100 MLE estimates of FI.



Results virtually identical. Why?

AR(1) Correlation Parameter: α

- $X_t = \alpha X_{t-1} + Z_t$
- The asymptotic distribution of $\hat{\alpha}_{MLE}$ for an AR(1) series of length n is:

$$\hat{\alpha}_n \xrightarrow{d} N\left(\alpha, \frac{1-\alpha^2}{n}\right).$$

- Too large for practical use.
- Back to the drawing board...

Bayesian Paradigm

- We **have** "Prior Information"
  - No shortage of hydrologic data
- Thomas Bayes to the rescue!
- The Bayesian posterior distribution for α given $\mathbf{X}$ is:

$$f(\alpha|\mathbf{X}) \propto f(\mathbf{X}|\alpha)f(\alpha)$$

- $f(\alpha)$ can incorporate prior information

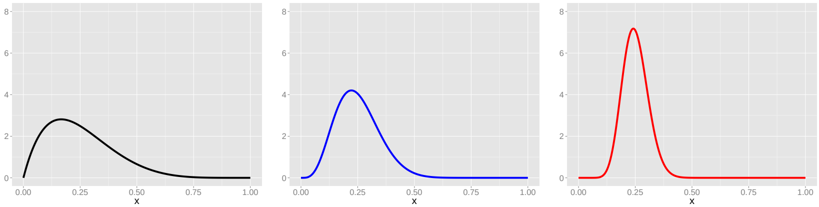
Beta Priors on α

- Beta family is natural choice for prior on α, with $[0, 1)$ support.
- For $\alpha \sim Beta(\alpha_1, \beta_1)$, we have

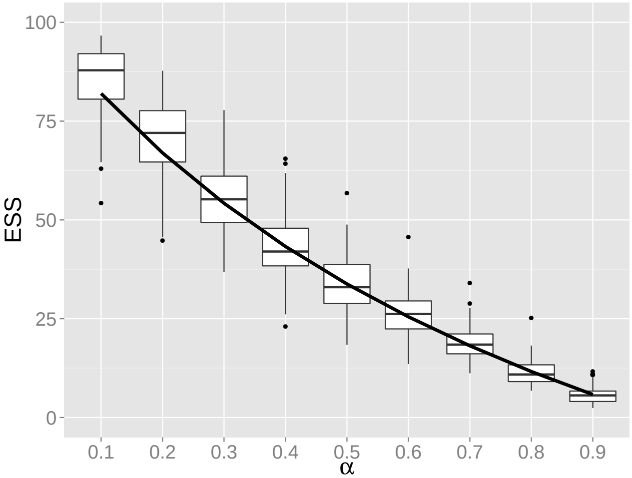
$$E[\alpha] = \frac{\alpha_1}{(\alpha_1 + \beta_1)}$$

$$Var[\alpha] = \frac{\alpha_1 \beta_1}{(\alpha_1 + \beta_1)^2 (\alpha_1 + \beta_1 + 1)}$$

- Suppose data indicates $\alpha = 0.25$
- Three Beta priors—same mean, different variance



How much better do we do?



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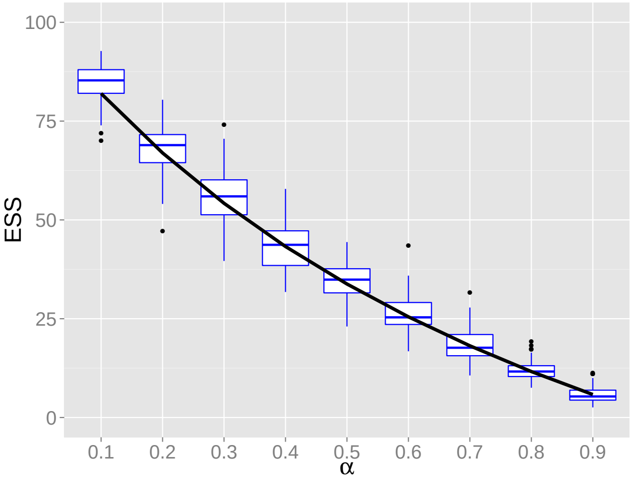

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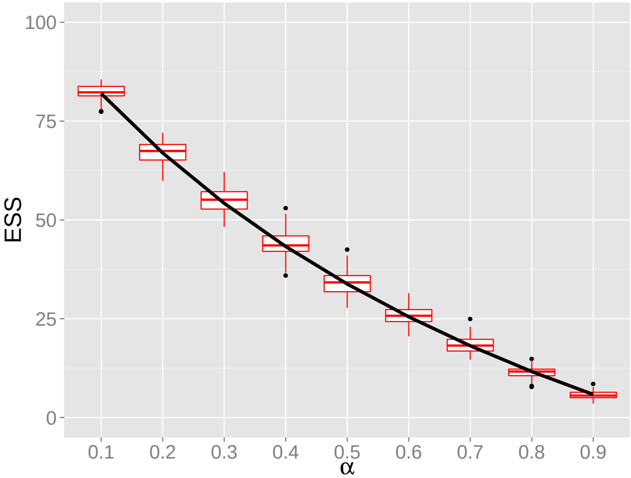
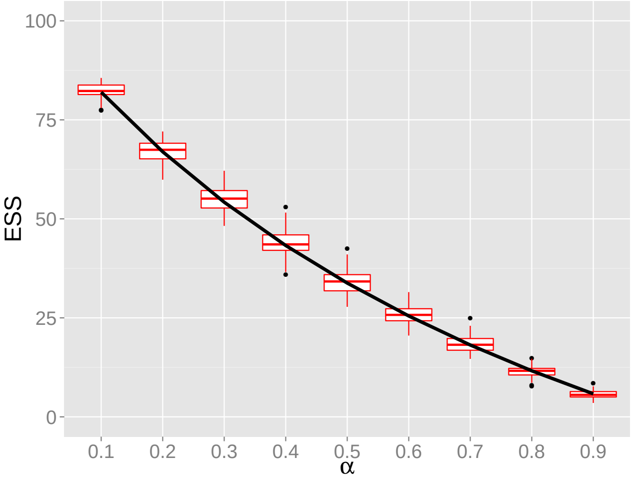


Victory! ...Victory? Not exactly.

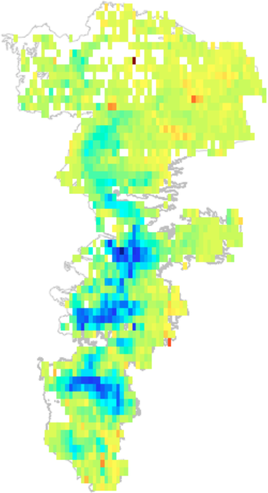


Bayesian Estimation of ESS – The Catch

There is no such thing as a free lunch.

- Beta priors are shifting to have mean equal to true mean
- Of course we do better
- Key point: we do a **lot** better with a **little** help

- Streams exist in networks
- Networks are spatially and temporally correlated
- Key point: we can use this information



G.V. Bayley and J.M. Hammersley.
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H. J. Thiébaux and F. W. Zwiers.
The interpretation and estimation of effective sample size.
Journal of Climate and Applied Meteorology, 23(5): 800–811, 2014/10/31 1984.