On (mod n) Spirals

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Goal of Talk

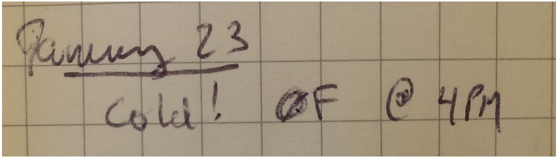
A means to teach students how to think mathematically.

- Show how one can impose definitions to patterns you see.
- Observations$\rightarrow$Conjecture$\rightarrow$Proof
- Learn to generalize: dim $2 \rightarrow$dim d

Aimed at early high school students, but can be useful to all ages/levels.

Motivation for Play

- Polar Vortex!

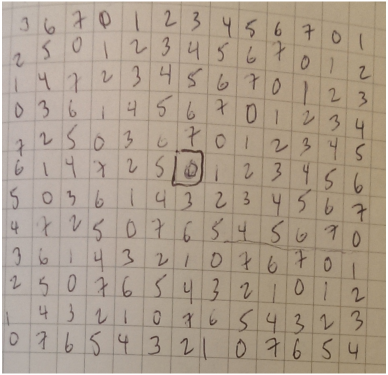


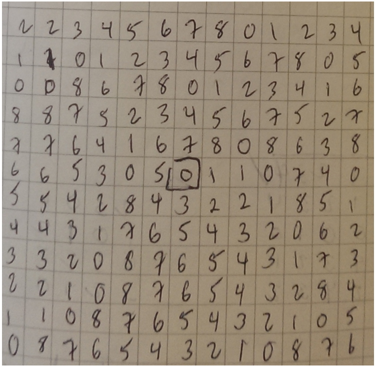
- In bed with flu for a week or so
- What to do?

MATH!... or at least play with numbers

- Hackathon!

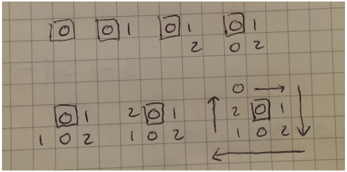
Orderly Doodling





What is the process?

Select an integer $n \geq 2$, say $n = 3$, noting $\mathbb{Z}_3 = \{0, 1, 2\}$



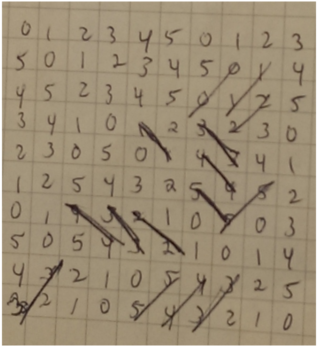
$$\boxed{0} \;\to\; \begin{array}{cc} \boxed{0} & 1 \end{array} \;\to\; \begin{array}{cc} \boxed{0} & 1 \\ & 2 \end{array} \;\to\; \begin{array}{cc} \boxed{0} & 1 \\ 0 & 2 \end{array} \;\to$$

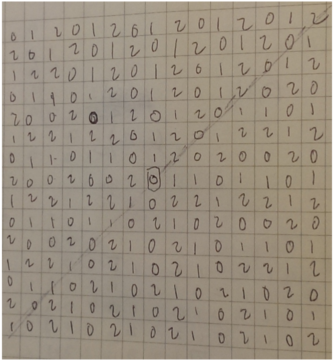
$$\begin{array}{ccc} & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array} \;\to\; \begin{array}{ccc} 2 & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array} \;\to\; \begin{array}{ccccc} & 0 & \cdot & \cdot & \cdot \\ & 2 & \boxed{0} & 1 & \cdot \\ & 1 & 0 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Formally,

$$l_j^* = j - 1 \pmod{n} \quad \text{at site} \quad l_j .$$

Patterns Seemed Interesting

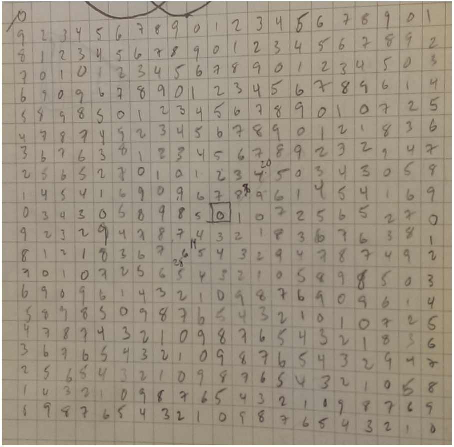




...want larger spirals!

Desire to Automate

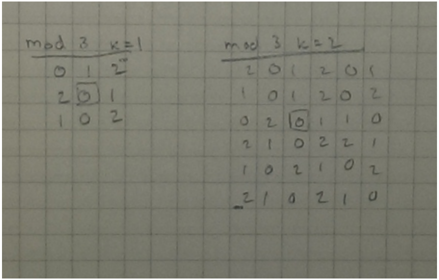
- Want bigger spirals: tedious by hand
- Write a program!



Programs Call for Definitions

- *Complete Spiral*
  - Forms a square and
  - Last value assigned is $l_i^* = n - 1$
- "Ond" is spiral in Swahili
  - First complete spiral achieved is Ond_n^1
  - Subsequent are Ond_n^k, $k = 2, 3, \ldots$.
- *Iteration Count* is number of times $\mathbb{Z}_n$ used in the k^{th} square

Ond examples

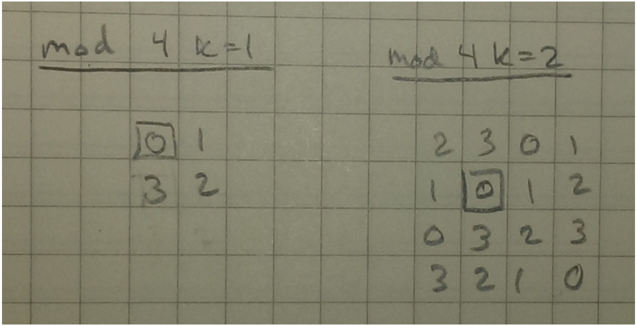


mod 3 k=1

0	1	2
2	$\boxed{0}$	1
1	0	2

mod 3 k=2

2	0	1	2	0	1
1	0	1	2	0	2
0	2	$\boxed{0}$	1	1	0
2	1	0	2	2	1
1	0	2	1	0	2
2	1	0	2	1	0

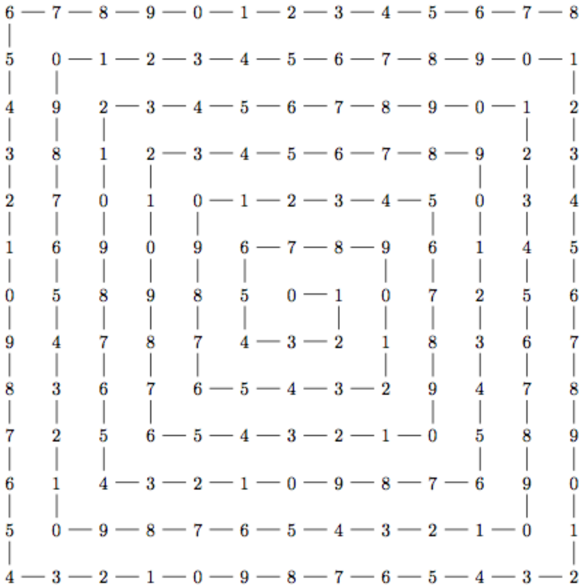


mod 4 k=1

$\boxed{0}$	1
3	2

mod 4 k=2

2	3	0	1
1	$\boxed{0}$	1	2
0	3	2	3
3	2	1	0



Still unwieldy

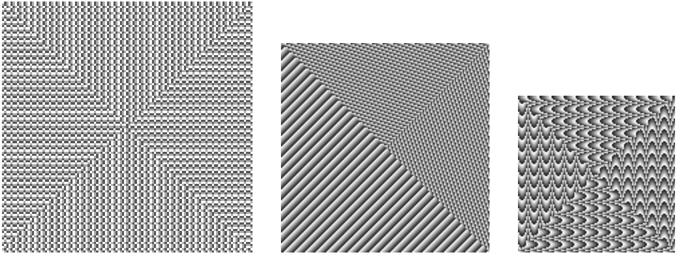
Visualize the patterns

Realize generated Ond_n^k as grayscale graphical images

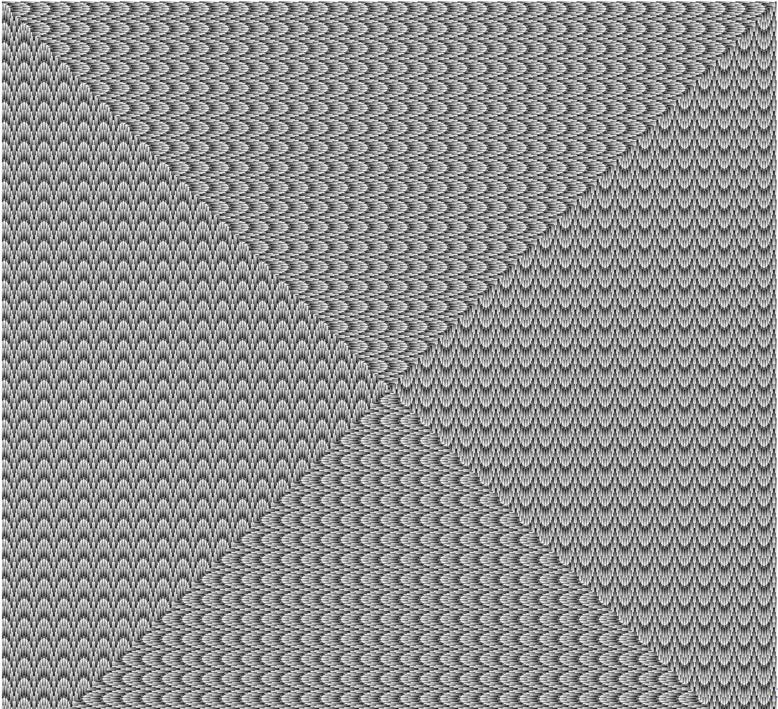
The map $f : \mathbb{Z}_n \to \mathbb{Z}_{256}$ is defined by a scaled floor function,

$$f(j) = \alpha j, \quad \text{where} \quad \alpha = \left\lfloor \frac{255}{n} \right\rfloor$$

Square Lattice



Ond_3^{39} Ond_8^{39} Ond_9^{39} respectively



Automation to Conjecture

- Initial image creation guessed at size of generated spiral
- Is there some formula for *Ond side length* and *iterations*?
- Can I use $(n, k, \ldots)$ to determine this?

Generate Data and Observe

Output *iteration* and *side length* data when generating *Ond* images

n	k	side	iter	p-factors
3	4	12	48	3
3	5	15	75	3
3	6	18	108	3
4	4	8	16	2 2
4	5	10	25	2 2
4	6	12	36	2 2
5	4	20	80	5
5	5	25	125	5
5	6	30	180	5
6	4	24	96	2 3
6	5	30	150	2 3
6	6	36	216	2 3
7	4	28	112	7
7	5	35	175	7
7	6	42	252	7
8	4	16	32	2 2 2
8	5	20	50	2 2 2
8	6	24	72	2 2 2
9	4	12	16	3 3
9	5	15	25	3 3
9	6	18	36	3 3

- Included prime factors of n to output to add information

Generate Data and Observe

Output *iteration* and *side length* data when generating *Ond* images

n	k	side	iter	p-factors
3	4	12	48	3
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4	4	8	16	2 2
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4	6	12	36	2 2
5	4	20	80	5
5	5	25	125	5
5	6	30	180	5
6	4	24	96	2 3
6	5	30	150	2 3
6	6	36	216	2 3
7	4	28	112	7
7	5	35	175	7
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8	4	16	32	2 2 2
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9	4	12	16	3 3
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- Side lengths: $Ond_3^4 > Ond_4^4 < Ond_5^4 \implies$ non-linearity

Theorem

Let s denote the greatest square divisor of n. The complete spiral Ond_n^k has the following structure:

(i) *If λ is the length of the sides of Ond_n^k, then*

$$\lambda = \frac{kn}{\sqrt{s}}. \tag{1}$$

(ii) *If ξ is the iteration count of Ond_n^k, then*

$$\xi = \frac{k^2 n}{s}. \tag{2}$$

Greatest Perfect Square Divisor of n

Write down Ond_n^1, determine side length, solve $\lambda = \frac{n}{\sqrt{s}}$ for s.

Proof of the Conjecture.

- $n \geq 2$ is fixed
- λ complete spiral side length, ξ iteration count
- Spiral is square $\implies \xi n = \lambda^2 \implies n|\lambda^2$.
- $p|n$ with p prime $\implies p|\lambda^2 \implies p|\lambda$
- s is greatest square divisor of $n \implies n = q_1^2 q_2$
  - q_2 is square-free
  - $s = q_1^2$.
- $n|\lambda^2 \implies q_1^2|\lambda^2 \implies q_1|\lambda$ and $\implies q_2|(\frac{\lambda}{q_1})^2$
- q_2 is square-free $\implies q_2|(\frac{\lambda}{q_1})$ and $\lambda = kq_1q_2$ for $k \in \mathbb{Z}_+$.
- $n|\lambda^2$ for any such λ, thus $\lambda = kq_1q_2$ is the side of the k-th complete spiral Ond_n^k.
- $\lambda = \frac{kn}{\sqrt{s}}$ follows since $\sqrt{s} = q_1$
- $\xi = \frac{k^2 n}{s}$ follows since $\xi = \frac{\lambda^2}{n}$.

$\square$

Generalize!

- Generalize in dimension: increase d
- Generalize in shape
- Think of geometrically: filling an area
  - more abstractly looking at when $n|f(d)$

Generalize in Dimension

- Generalize to d-dim hypercube by looking at $n|\lambda^d$
  - Difficult to visualize

Theorem

Let $n|\lambda^d$ for $d = 2, 3, \ldots$. Let $n = qm^d$ where $m \in \mathbb{N}$ and q is d-power free. Let q be defined in terms of it's prime factors, p_j, as

$$q = \prod_{j=1}^{r} p_j^{e_j} \quad \textit{with each} \quad e_j < d.$$

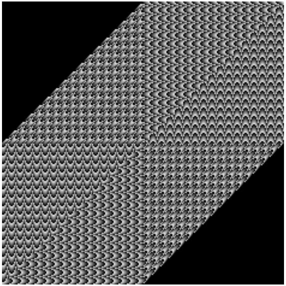
then

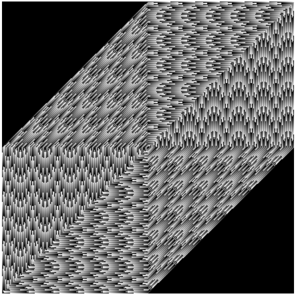
$$\lambda = km \prod p_j \quad k = 1, 2, 3, \ldots$$

- Note that m^d is the greatest d-power divisor of n
- Proof is similar to $d = 2$ proof.

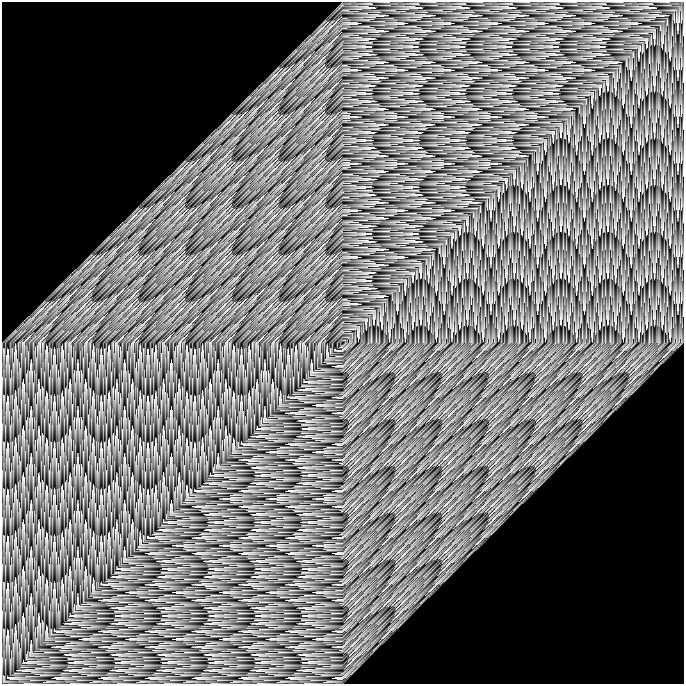
Other Shapes: Centered Hexagons

- Centered hexagonal numbers are $CH_d = 3d(d+1)+1$
- Investigate $n | CH_d$
- Not all n will satisfy!
    - $n = 2, \ldots, 6$ don't work but $n = 7, 13, 19, 31, 37, \ldots$ will



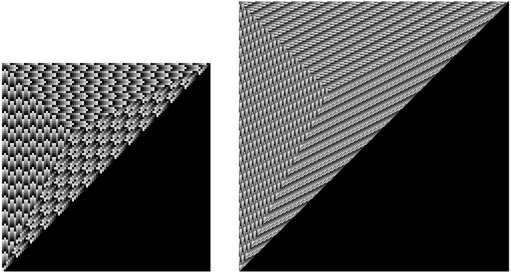


$HexOnd_7^{34}$ and $HexOnd_{37}^{12}$



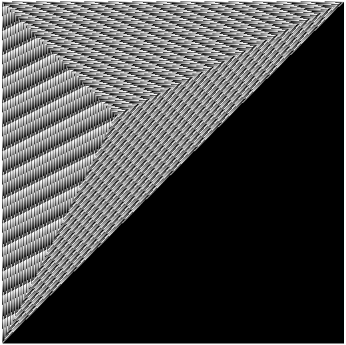
Other Shapes: Triangles

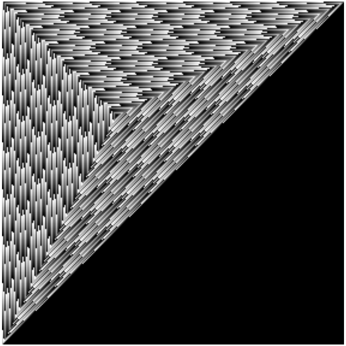
- Triangle numbers are $T_d = \frac{d(d+1)}{2}$
- Investigate $n | T_d$



$TriOnd_7^{29}$ and $TriOnd_{27}^{29}$

$TriOnd^{29}_{36}$ and $TriOnd^{29}_{39}$





Note the bias by using right triangle

So what is all this?

- Show how one can impose definitions to patterns you see.
- Observations$\rightarrow$Conjecture$\rightarrow$Proof
- Learn to generalize

A means to teach how to think mathematically!

Things to Explore

- Other 2 dimensional lattice shapes
- Non-hypercube shapes in > 2 dimensions
- Addition of *Ond*
- Random *Ond*: Select n_0 from some distribution, cycle through $\mathbb{Z}_{n_0}$ once, choose n_1, cycle through $\mathbb{Z}_{n_1}$, ...

Thank You!

We invite you to please take a paper if you are interested in this topic. For code and visualizations, please go to:

https://github.com/cwcomplex/modNspirals

- Robin Young of UMASS-Amherst
- Veracode, Inc – Hackathon!
- Jared Carlson of Veracode

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