ON (MOD n) SPIRALS

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1. Introduction

This note is intended to introduce the process of constructing (mod n) spirals and the idea of a complete spiral. The construction of a (mod n) spiral is similar in nature to Ulam's Spiral [1]. The note also introduces a theorem related to patterns seen regarding the lengths of sides, iteration counts, and ending corners of these objects, and provides a proof. This process also provides a deterministic process for discovering the greatest square divisor of integers $n \geq 2$. Grayscale visualizations of these spirals have been generated in order to further illuminate understanding and interest in these objects. Generalizations to higher dimensions and spirals of other shapes are given. The author's are unaware of other work or literature on this topic, so have no references other than the couple related to code; in part this is why the note has been written and offered to share as it is a candidate for use in classroom learning.

2. Spiral Construction

We describe the construction of a (mod n) spiral and introduce the notation we will use to analyze these. For a fixed integer $n \geq 2$, we will be working with the additive group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, ..., n-1\}$. Let L be a square lattice, which we can take to be $\mathbb{Z}^2$. Denoting the origin by $l_1 = (0, 0)$, we build the spiral by enumerating the lattice sites and assigning numbers from $\mathbb{Z}_n$ in turn. We spiral in a clockwise direction starting in the direction of the positive x-axis, so that $l_2 = (1, 0)$, $l_3 = (1, -1)$, and the next four lattice points are $\{(0, -1), (-1, -1), (-1, 0), (-1, 1)\}$, respectively. In general, once l_j has been assigned, and having chosen a 'direction' by moving from l_{j-1}, the next site l_{j+1} is the site to the right if this is not yet accounted for, or the site ahead otherwise.

$$\boxed{0} \rightarrow \begin{array}{cc} \boxed{0} & 1 \end{array} \rightarrow \begin{array}{cc} \boxed{0} & 1 \\ & 2 \end{array} \rightarrow \begin{array}{cc} \boxed{0} & 1 \\ 0 & 2 \end{array} \rightarrow \begin{array}{ccc} & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array} \rightarrow \begin{array}{ccc} 2 & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array} \rightarrow \begin{array}{ccccc} & 0 & \cdot & \cdot & \cdot \\ & 2 & \boxed{0} & 1 & \cdot \\ & 1 & 0 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Figure 1. Building a (mod 3) spiral

Having enumerated the lattice as above, we now assign values from $\mathbb{Z}_n$ as follows: to the origin we assign $l_1^* = 0$, and we then count in $\mathbb{Z}_n$: $l_2^* = 1$, $l_3^* = 2$, etc., and cycling back to

0 after n steps, so that $l^*_{n+1} = 0$. Continuing in this way, we ‘count’ all lattice sites (mod n). In general, this yields

$$l^*_j = j - 1 \pmod{n} \quad \text{at site} \quad l_j.$$

The first author's initial interest in generating spirals of increasing size for various $\mathbb{Z}_n$ was to investigate some of the patterns seen in figure 3. In an attempt to describe the patterns, he introduced the following definition:

Definition 1. *A* complete spiral *occurs when in the spiral construction, the partially completed spiral forms a square, and the last (mod n) value assigned is $l^*_i = n - 1$, so all of $\mathbb{Z}_n$ has been used an integer number of times.*

We use the following notation: denote the first complete spiral achieved by Ond^1_n; subsequent complete spirals are denoted Ond^k_n, $k = 2, 3, \ldots$. The number of times $\mathbb{Z}_n$ is used in order to complete the k^{th} square is the iteration count. The word "ond" is "spiral" in Swahili.

Figure 4 shows Ond^k_n for $k = 1, 2$ and $n = 1, 2$. We see that Ond^1_3 is a 3×3 square and has iteration count 3, and Ond^2_3 is a 6×6 square with 12 iterations. Similarly, Ond^1_4 is 2×2 with a single iteration, and Ond^2_4 is 4×4 with 4 iterations. The first element, 0, of each spiral in the figures is in a box and last element of the spiral marked with a dot.

The author generated several spirals with pencil and paper in a methodical manner to create sets of complete spirals for various values of n and k. While tedious and not exactly related to the initial goal of looking at diagonal patterns, it helped to realize there seemed to be patterns found in the construction of the spirals. Specifically in the sizes of the complete spirals, iteration counts, and where the last lattice point rested; this led the author to investigate what the patterns were.

2.1. **On the side lengths and iterate counts of Ond^k_n.** In order to investigate these patterns, the author determined that more data was needed and, due to the tedium of pencil and paper, a program should be written to generate complete spirals [2]. This allowed the author to generate a larger number of complete spirals and collect data on lengths, iterations, and ending points.

In looking at the initial data, for small choices of n, k, a few patterns in tuples of (lengths, iterations) were found, including (kn, k^2n), $(\frac{kn}{2}, \frac{k^2n}{4})$, and $(\sqrt{n}k, k^2)$. To establish the pattern, the prime factorization of n for each complete spiral Ond^k_n with the lengths and iteration data was generated for analysis. From this data, the author determined there was a relation involved with finding the greatest square divisor of n, which led to the following observations.

Theorem 2.1. *Let s denote the greatest square divisor of n. The complete spiral Ond^k_n has the following structure:*

(i) If λ is the length of the sides of Ond^k_n, then

$$\lambda = \frac{kn}{\sqrt{s}}. \tag{1}$$

(ii) If ξ is the iteration count of Ond_n^k, then

$$\xi = \frac{k^2 n}{s}. \tag{2}$$

(iii) If $l_{max} \in L$ is the last lattice point in the complete spiral Ond_n^k, then l_{max} is either the top-right corner or bottom-left corner of the square. If both n and k are odd, then l_{max} will be the top-right corner of Ond_n^k. In all other cases, l_{max} will be the bottom-left corner.

One will note that in the case where n is square-free, then $s = 1$ and (1) and (2) reduce to $\lambda = kn$ and $\xi = k^2 n$, respectively. These can be considered an upper bound for all cases. One might also consider these to be the least robust in the case of length and iteration counts.

A most interesting aspect to this process of complete spiral construction is the connection to the greatest square divisor of some integer n. To see this more clearly, choose some n and construct, by hand, the spiral Ond_n^1. At this point, you know λ and ξ, so pick one, substitute $k = 1$ and solve for s. So from a constructivist approach we find the greatest square divisor.

In the process of investigating sizes and iteration counts of Ond_n^k, the first author implemented a method to map the generated spirals to grayscale images. This works best for positive integers, n, less than 256. The map $f : \mathbb{Z}_n \to \mathbb{Z}_{256}$ is defined by a scaled floor function,

$$f(j) = \alpha j, \quad \text{where} \quad \alpha = \left\lfloor \frac{255}{n} \right\rfloor$$

The function f thus maps $\mathbb{Z}_n$ to a set of brightness values which are used to generate grayscale images.

An example of this is Ond_{31}^{39} seen in figure 5. One can see a complex pattern emerging. Further, if one zooms in and out on the image, various different patterns emerge that seem to play tricks on the eye. The image may also be view at [3] and is the file *N=31_k=39-grey.png* which makes it easier for testing the zoom in/out. Further, at this location, one may find the generated images for Ond_n^k for $n = 2, 3, \ldots, 31$ and $k = 1, 2, \ldots, 50$. Another visualization example is provided by Figure 6 and is the spiral Ond_{10}^{39}.

Proof of the Theorem. Recall that $n \geq 2$ is fixed. Let λ be the length of the side of a complete spiral, and let ξ be the corresponding iteration count. Since the spiral is square, we must have $\xi n = \lambda^2$, and so $n | \lambda^2$. Now any prime which divides n must also divide λ^2, and must thus divide λ.

Since s is the greatest square divisor of n, we can uniquely write $n = q_1^2 q_2$ where q_2 is square-free, with $s = q_1^2$. It follows that $q_1^2 | \lambda^2$, so $q_1 | \lambda$ and $q_2 | (\frac{\lambda}{q_1})^2$. Since q_2 is square-free, $q_2 | (\frac{\lambda}{q_1})$ and we have $\lambda = k q_1 q_2$ for some integer k. Clearly $n | \lambda^2$ for any such λ, and we conclude that $\lambda = k q_1 q_2$ is the side of the k-th complete spiral Ond_n^k.

Equation (1) now follows since $\sqrt{s} = q_1$, and (2) follows since $\xi = \frac{\lambda^2}{n}$. Finally, (iii) follows since $l_{max} = \lambda^2$ is even unless k, q_1 and q_2 are all odd, and by construction l_{max}

represents the top-right corner of the square if it is odd, and the bottom-left corner if it is even. □

3. Generalizations

The second author proposed investigating the idea how of *Ond* might work for cubes or higher dimensional squares, since the lattice is the basis for the spiral. In imagining dimension $d = 3$ and $d = 4$, finding the route in the lattice to properly spiral can be quite difficult, let alone an algorithm for any d.

In terms of planar spirals described above, we are interested in finding squares, thus the areal requirement $n|\lambda^2$. For the case of cubic spirals, we have the volume requirement $n|\lambda^3$ and $n|\lambda^4$ for the $4d$ hypercube. Thus we think about the Ond_n for any dimension d as having this same geometric requirement.

Theorem 3.1. *Let $n|\lambda^d$ for $d = 2, 3, \ldots$. Let $n = qm^d$ where $m \in \mathbb{N}$ and q is d-power free. Let q be defined in terms of it's prime factors, p_j, as*

$$q = \prod_{j=1}^{r} p_j^{e_j} \quad \textit{with each} \quad e_j < d.$$

then

$$\lambda = km \prod p_j \quad k = 1, 2, 3, \ldots$$

Proof. Since we have that $n|\lambda^d$, then it follows that

$$\begin{aligned} & qm^d|\lambda^d \\ \Longrightarrow\ & m^d|\lambda^d \\ \Longrightarrow\ & m|\lambda \end{aligned}$$

which implies that writing $\frac{\lambda}{m}$ makes sense. Thus we have that $q|(\frac{\lambda}{m})^d$. Recall the definition of q and it's clear that $p_j|q$. Then because $q|(\frac{\lambda}{m})^d$, it follows that $p_j|(\frac{\lambda}{m})^d$. Since p_j is clearly d-power free, $p_j|\frac{\lambda}{m}$. Since the p_j's are distinct, it follows that $p_1 \ldots p_r|\frac{\lambda}{m}$ and thus $\lambda = km \prod p_j$ for some $k = 1, 2, \ldots$. □

4. Other Planar Shapes

We can expand away from investigating the square spirals to other planar shapes defined on a square lattice. While the spiraling process may vary, the area requirement of achieving *Ond* remains the same.

4.1. Triangle Numbers.

In order to investigate triangle *Ond*, we make use of the triangle numbers [5] to determine the area requirement. It is well known that the triangle numbers are of the form:

$$Area(\triangle d) = T_d = \frac{d(d+1)}{2}$$

and grow as shown in Figure 2. Thus we look at the requirement $n|\frac{d(d+1)}{2}$. Due to the factors d and $d+1$, it is reasonable to suspect that if $n|T_d$ then either $n|T_{d-1}$ or $n|T_{d+1}$. However, it is not always the case that n will divide consecutive triangles.

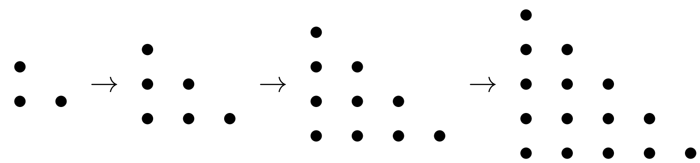

FIGURE 2. Triangles on Lattice $d = 2, 3, 4, 5$

To more clearly see this, we generate the values

$$\{T_d\}_{d=2}^{21} = \{3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231\}$$

and observe when $n|T_d$, for some fixed $n \geq 2$. Here we briefly look at It is clear that only the even valued T_d are Ond_2 achievable, which means that $T_3, T_4, T_7, T_8, T_{11}, T_{12}, T_{15}, T_{16}, T_{19}, T_{20}$ are such examples. From this limited sample, one sees they do appear to obey the consecutive triangle idea. From a similar analysis, one will see that this consecutive triangle notion holds for $n = 3, 4, 5, 7, 8, 9, 11, 12, 13, 16$.

However, the spiral Ond_6 is achieved by $T_3, T_8, T_{11}, T_{12}, T_{15}, T_{20}, T_{23}, T_{24}, \ldots$. In this case, one does see the consecutive triangles, such as in $d = 11, 12$. However, one also sees also other cases in which there are lone triangles. It seems to be that given a consecutive pair that achieve Ond_6, T_j and T_{j+1}, then so will the triangles $T_{j+4}, T_{j+9}, T_{j+12}, T_{j+13}, \ldots$.

The case of Ond_{10} is another case in which there are lone triangles among the various consecutive triangles that achieve this spiral. It seems in this case, given a consecutive pair that achieves the spiral, T_j, T_{j+1}, then so will the following triangles $T_{j+5}, T_{j+16}, T_{j+20}, T_{j+21}, \ldots$. The authors calculated that the cases $n = 14, 15$ also contain lone triangles among consecutive, however, have not characterized the pattern in general.

4.2. **Hexagonal Numbers.** In a similar manner to the triangle case, we can make use of hexagonals on a lattice and the hexagonal numbers [6]. Similar to the triangle numbers, we can consider these numbers as notions of area for achieving in Ond. The hexagonal numbers are defined as

$$H_d = d(2d-1)$$

Thus we are interested in the requirement $n|H_d$. It is necessarily the case that $2d-1$ is odd valued for all d of concern. Thus, if d is even, then H_d is even, and if d is odd, then H_d is odd. It is also the case that if $n|d$, then Ond_n can be achieved by H_d. Thus, if $n|d$, then $H_{kd} = kd(2kd-1)$ for $k = 1, 2, \ldots$ are all Ond_n achievable. Note that this is a subset of possible hexagons that achieve Ond_n. Clearly, it is also the case that if $n|2d-1$, then H_d achieves Ond_n.

From numerical calculations, one sees some patterns in the d values for Ond_n achievable hexagons. For example, in looking at Ond_{11}, the hexagons $H_6, H_{11}, H_{17}, H_{22}, H_{28}, \ldots$. The pattern is $d = 6 \xrightarrow{+5} 11 \xrightarrow{+6} 17 \xrightarrow{+5} 22 \xrightarrow{+6} \ldots$. Similar alternating patterns are found for other Ond_n including Ond_{10} which has $d = 8 \xrightarrow{+2} 10 \xrightarrow{+8} 18 \xrightarrow{+2} 20 \xrightarrow{+8} 28 \xrightarrow{+2} \ldots$ for the hexagons $H_8, H_{10}, H_{18}, H_{20}, H_{28}, \ldots$. It is interesting to note that the sum of consecutive differences is the value of n in both the $n = 10, 11$ cases.

However, not all cases have this feature as in $n = 15, 16$. For $n = 15$, the pattern is over more than two values as in $H_{18}, H_{20}, H_{23}, H_{30}, H_{33}$ which follows $d = 18 \xrightarrow{+2} 20 \xrightarrow{+3} 23 \xrightarrow{+7} 30 \xrightarrow{+3} 33 \xrightarrow{+2} \ldots$. The pattern repeats and the sum over the d differences is 15, which is the value of n, obviously ignoring the last +2. In the case of $n = 16$, difference between d values is constant 16.

There are similar shape numbers that are defined on the planar lattice that can be explored. These include the hex numbers [7], heptagonal numbers [8], and other polygonal numbers [9].

5. Summary

This note presented the process of constructing (mod n) spirals in a square lattice and proof regarding some notions of patterns found in this construction. Further, we have presented a generalization of the notion of achieving a spiral Ond to hypercubes of dimension d by conforming to a volume geometric requirement. Lastly, we began to explore other $d = 2$ shapes in the $\mathbb{Z}^2$ lattice, pushing the notion of the geometric requirement to other forms.

Acknowledgments: The first author recognizes Veracode Inc and Jared Carlson of Veracode.

References

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[2] A. Reiter, https://github.com/cwcomplex/modNspirals
[3] A. Reiter, https://github.com/cwcomplex/modNspirals/tree/master/somegrey
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[5] Wolfram Mathworld, "Triangle Numbers", http://mathworld.wolfram.com/TriangularNumber.html
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[7] Wolfram Mathworld, "Hex Numbers", http://mathworld.wolfram.com/HexNumber.html
[8] Wolfram Mathworld, "Heptagonal Numbers", http://mathworld.wolfram.com/HeptagonalNumber.html
[9] Wolfram Mathworld, "Polygonal Numbers", http://mathworld.wolfram.com/PolygonalNumber.html

6. Appendix A

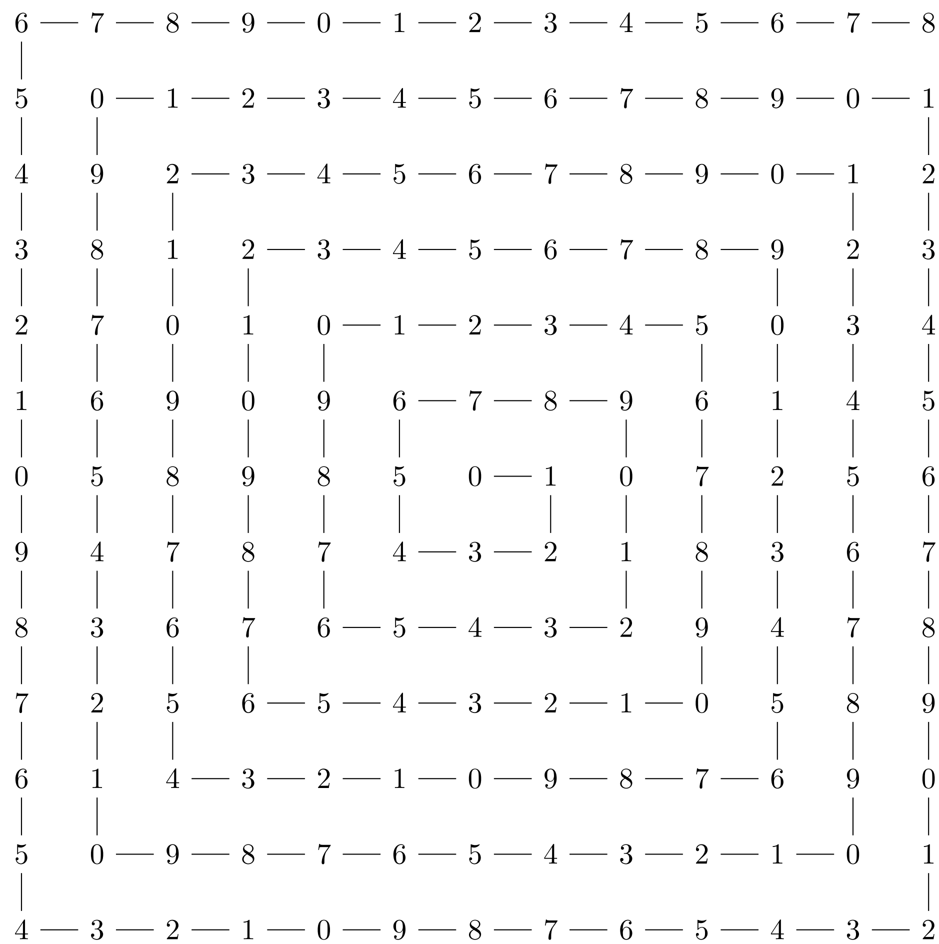

Figure 3. A (mod 10) spiral with some indication of path

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$$\begin{array}{ccc} 0 & 1 & \dot{2} \\ 2 & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array}$$

(A)

$$\begin{array}{cccccc} 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 2 & \boxed{0} & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 1 & 0 & 2 \\ \dot{2} & 1 & 0 & 2 & 1 & 0 \end{array}$$

(B)

$$\begin{array}{cc} \boxed{0} & 1 \\ \dot{3} & 2 \end{array}$$

(C)

$$\begin{array}{cccc} 2 & 3 & 0 & 1 \\ 1 & \boxed{0} & 1 & 2 \\ 0 & 3 & 2 & 3 \\ \dot{3} & 2 & 1 & 0 \end{array}$$

(D)

FIGURE 4. (a) Ond_3^1, (b) Ond_3^2, (c) Ond_4^1, (d) Ond_4^2

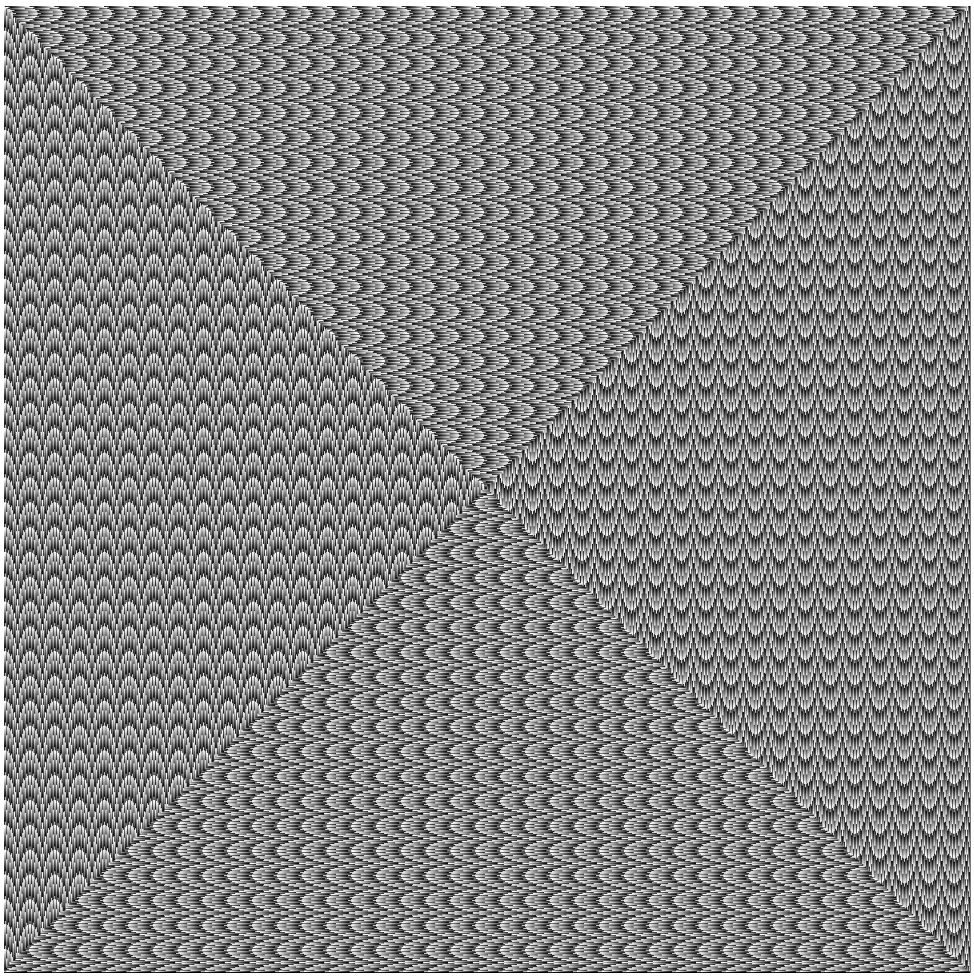

FIGURE 5. Visualization of Ond_{31}^{39}

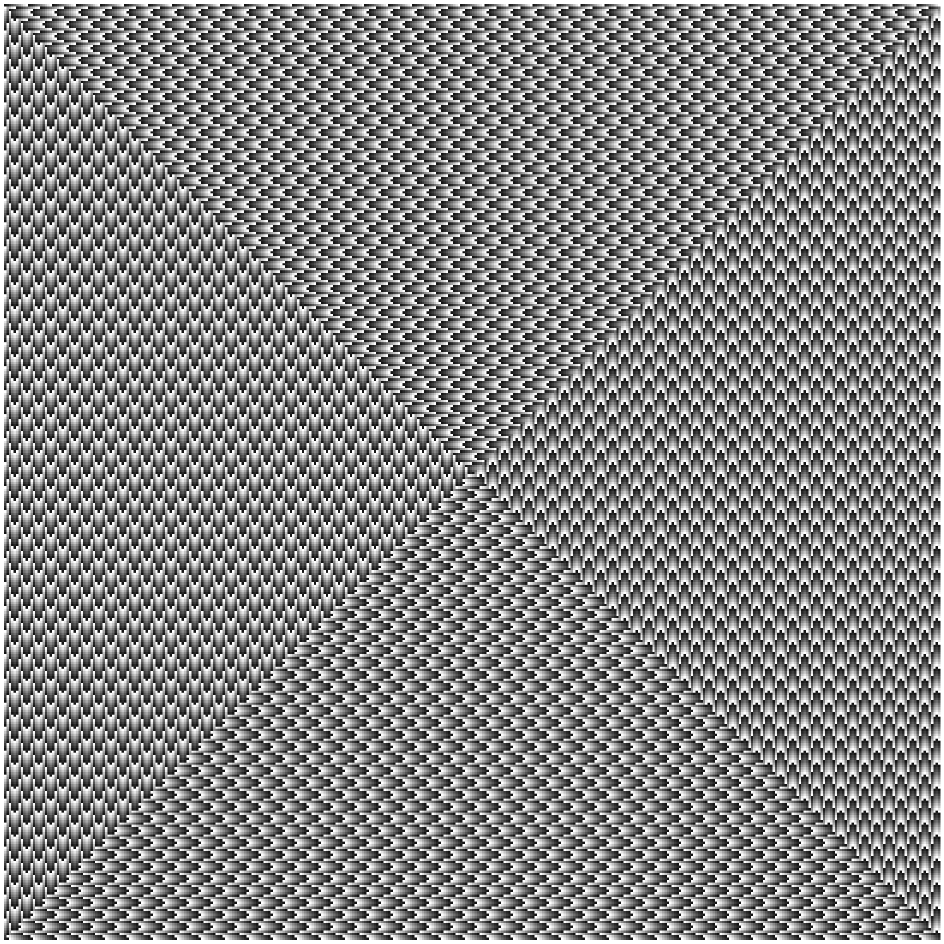

FIGURE 6. Visualization of Ond_{10}^{39}