

On $(\text{mod } n)$ Spirals

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August 7, 2014

Goal of Talk

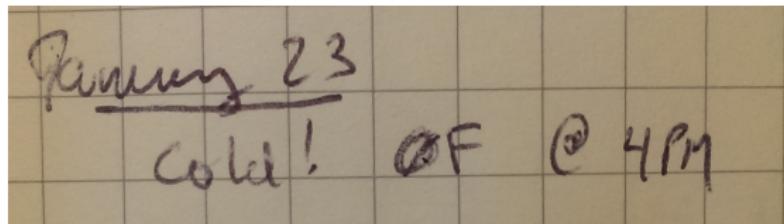
A means to teach students how to think mathematically.

- ▶ Show how one can impose definitions to patterns you see.
- ▶ Observations→Conjecture→Proof
- ▶ Learn to generalize: $\text{dim 2} \rightarrow \text{dim } d$

Aimed at early high school students, but can be useful to all ages/levels.

Motivation for Play

- ▶ Polar Vortex!



- ▶ In bed with flu for a week or so
- ▶ What to do?

MATH!... or at least play with numbers

- ▶ Hackathon!

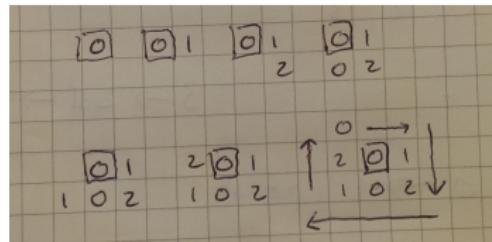
Orderly Doodling

3	6	7	0	1	2	3	4	5	6	7	0	1
2	5	0	1	2	3	4	5	6	7	0	1	2
1	4	7	2	3	4	5	6	7	0	1	2	3
0	3	6	1	4	5	6	7	0	1	2	3	4
7	2	5	0	3	6	7	0	1	2	3	4	5
6	1	4	7	2	5	0	1	2	3	4	5	6
5	0	3	6	1	4	3	2	3	4	5	6	7
4	7	2	5	0	3	6	5	4	5	6	7	0
3	6	1	4	3	2	1	0	7	6	7	0	1
2	5	0	3	6	5	4	3	2	1	0	1	2
1	4	3	2	1	0	3	6	5	4	3	2	3
0	7	6	5	4	3	2	1	0	7	6	5	4

2	2	3	4	5	6	7	8	0	1	2	3	4
1	7	0	1	2	3	4	5	6	7	8	0	5
0	0	8	6	7	8	0	1	2	3	4	1	6
8	8	7	5	2	3	4	5	6	7	5	2	7
7	7	6	4	1	6	7	8	0	8	6	3	8
6	6	5	3	0	5	0	1	1	0	7	4	0
5	5	4	2	8	4	3	2	2	1	8	5	1
4	4	3	1	7	6	5	4	3	2	0	6	2
3	3	2	0	8	9	6	5	4	3	1	7	3
2	2	1	0	8	7	6	5	4	3	2	8	4
1	1	0	8	7	6	5	4	3	2	1	0	5
0	8	7	6	5	4	3	2	1	0	8	7	6

What is the process?

Select an integer $n \geq 2$, say $n = 3$, noting $\mathbb{Z}_3 = \{0, 1, 2\}$



$$\boxed{0} \rightarrow \boxed{0} \quad 1 \rightarrow \boxed{0} \quad \frac{1}{2} \rightarrow \boxed{0} \quad \frac{1}{2} \rightarrow$$

$$\begin{matrix} & & & & 0 & . & . & . \\ 1 & \boxed{0} & \frac{1}{2} & \rightarrow & 2 & \boxed{0} & \frac{1}{2} & \rightarrow & 2 & \boxed{0} & \frac{1}{2} & \dots \\ & 0 & 2 & & 1 & 0 & 2 & & 1 & 0 & 2 & \end{matrix}$$

Formally,

$$l_j^* = j - 1 \pmod{n} \text{ at site } l_j.$$

Patterns Seemed Interesting

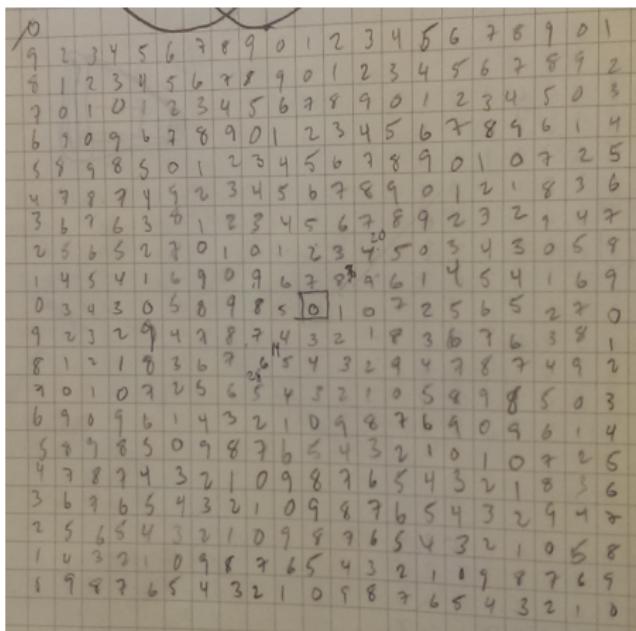
0	1	2	3	4	5	0	1	2	3
5	0	1	2	3	4	5	0	1	4
4	5	2	3	4	5	0	y	2	5
3	4	1	0	2	3	2	3	0	
2	3	0	5	0	4	3	4	1	
1	2	5	4	3	2	5	4	8	2
0	1	7	3	2	1	0	9	0	3
5	0	5	4	3	2	1	0	1	4
4	3	2	1	0	5	4	8	2	5
3	2	1	0	5	4	3	2	1	0

0	1	2	0	1	2	6	1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
1	2	2	0	1	2	0	1	2	6	1	2	0	1	2
0	1	9	0	1	2	0	1	2	0	1	2	0	2	0
2	0	0	2	0	1	2	0	1	2	0	1	1	0	1
1	2	0	1	2	2	0	1	2	0	1	2	2	1	2
0	1	1	0	1	1	0	1	2	0	2	0	0	2	0
2	0	0	2	0	0	2	0	2	0	1	1	0	1	1
1	2	2	1	2	2	1	0	2	2	1	2	2	1	2
0	1	1	0	1	1	0	2	1	0	2	0	0	2	0
2	0	0	2	0	2	1	0	2	1	0	1	1	0	1
1	2	2	1	0	2	1	0	2	1	0	2	2	1	2
0	1	1	0	2	1	0	2	1	0	2	1	0	2	0
2	0	2	1	0	2	1	0	2	1	0	2	1	0	1
1	0	2	1	0	2	1	0	2	1	0	2	1	0	1
0	2	1	0	2	1	0	2	1	0	2	1	0	2	0

...want larger spirals!

Desire to Automate

- ▶ Want bigger spirals:
tedious by hand
- ▶ Write a program!



Programs Call for Definitions

- ▶ *Complete Spiral*
 - ▶ Forms a square and
 - ▶ Last value assigned is $I_i^* = n - 1$
- ▶ “Ond” is spiral in Swahili
 - ▶ First complete spiral achieved is Ond_n^1
 - ▶ Subsequent are Ond_n^k , $k = 2, 3, \dots$
- ▶ *Iteration Count* is number of times \mathbb{Z}_n used in the k^{th} square

6 — 7 — 8 — 9 — 0 — 1 — 2 — 3 — 4 — 5 — 6 — 7 — 8
 |
 5 0 — 1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 0 — 1
 |
 4 9 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 0 — 1 2
 |
 3 8 1 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 2 3
 |
 2 7 0 1 0 — 1 — 2 — 3 — 4 — 5 0 3 4
 |
 1 6 9 0 9 6 — 7 — 8 — 9 6 1 4 5
 |
 0 5 8 9 8 5 0 — 1 0 7 2 5 6
 |
 9 4 7 8 7 4 — 3 — 2 1 8 3 6 7
 |
 8 3 6 7 6 — 5 — 4 — 3 — 2 9 4 7 8
 |
 7 2 5 6 — 5 — 4 — 3 — 2 — 1 — 0 5 8 9
 |
 6 1 4 — 3 — 2 — 1 — 0 — 9 — 8 — 7 — 6 9 0
 |
 5 0 — 9 — 8 — 7 — 6 — 5 — 4 — 3 — 2 — 1 — 0 1
 |
 4 — 3 — 2 — 1 — 0 — 9 — 8 — 7 — 6 — 5 — 4 — 3 — 2

Still unwieldy

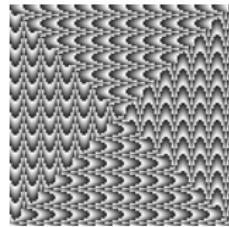
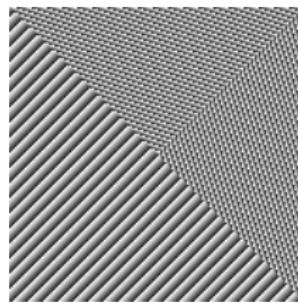
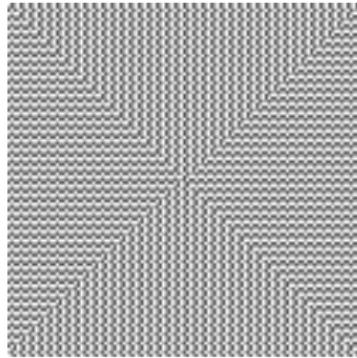
Visualize the patterns

Realize generated Ond_n^k as grayscale graphical images

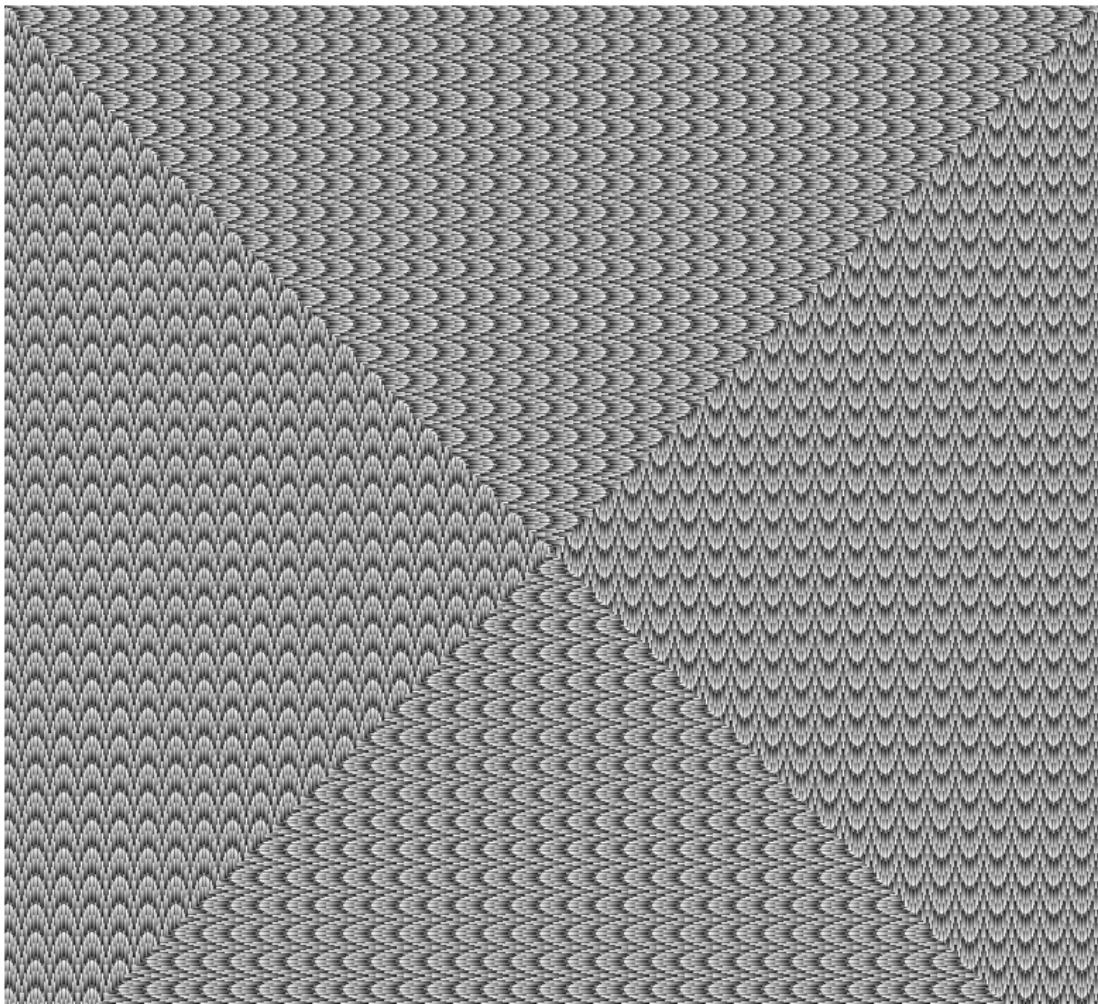
The map $f : \mathbb{Z}_n \rightarrow \mathbb{Z}_{256}$ is defined by a scaled floor function,

$$f(j) = \alpha j, \quad \text{where} \quad \alpha = \left\lfloor \frac{255}{n} \right\rfloor$$

Square Lattice



Ond_3^{39} Ond_8^{39} Ond_9^{39} respectively



Automation to Conjecture

- ▶ Initial image creation guessed at size of generated spiral
- ▶ Is there some formula for *One side length* and *iterations*?
- ▶ Can I use (n, k, \dots) to determine this?

Generate Data and Observe

Output *iteration* and *side length* data when generating *Ond* images

n	k	side	iter	p-factors
3	4	12	48	3
3	5	15	75	3
3	6	18	108	3
4	4	8	16	2 2
4	5	10	25	2 2
4	6	12	36	2 2
5	4	20	80	5
5	5	25	125	5
5	6	30	180	5
6	4	24	96	2 3
6	5	30	150	2 3
6	6	36	216	2 3
7	4	28	112	7
7	5	35	175	7
7	6	42	252	7
8	4	16	32	2 2 2
8	5	20	50	2 2 2
8	6	24	72	2 2 2
9	4	12	16	3 3
9	5	15	25	3 3
9	6	18	36	3 3

- ▶ Included prime factors of n to output to add information

Generate Data and Observe

Output *iteration* and *side length* data when generating *Ond* images

n	k	side	iter	p-factors
3	4	12	48	3
3	5	15	75	3
3	6	18	108	3
4	4	8	16	2 2
4	5	10	25	2 2
4	6	12	36	2 2
5	4	20	80	5
5	5	25	125	5
5	6	30	180	5
6	4	24	96	2 3
6	5	30	150	2 3
6	6	36	216	2 3
7	4	28	112	7
7	5	35	175	7
7	6	42	252	7
8	4	16	32	2 2 2
8	5	20	50	2 2 2
8	6	24	72	2 2 2
9	4	12	16	3 3
9	5	15	25	3 3
9	6	18	36	3 3

- Side lengths: $Ond_3^4 > Ond_4^4 < Ond_5^4 \implies$ non-linearity

Theorem

Let s denote the greatest square divisor of n . The complete spiral Ond_n^k has the following structure:

- (i) If λ is the length of the sides of Ond_n^k , then

$$\lambda = \frac{kn}{\sqrt{s}}. \quad (1)$$

- (ii) If ξ is the iteration count of Ond_n^k , then

$$\xi = \frac{k^2 n}{s}. \quad (2)$$

Greatest Perfect Square Divisor of n

Write down Ond_n^1 , determine side length, solve $\lambda = \frac{n}{\sqrt{s}}$ for s .

Proof of the Conjecture.

- ▶ $n \geq 2$ is fixed
- ▶ λ complete spiral side length, ξ iteration count
- ▶ Spiral is square $\implies \xi n = \lambda^2 \implies n|\lambda^2$.
- ▶ $p|n$ with p prime $\implies p|\lambda^2 \implies p|\lambda$
- ▶ s is greatest square divisor of $n \implies n = q_1^2 q_2$
 - ▶ q_2 is square-free
 - ▶ $s = q_1^2$.
- ▶ $n|\lambda^2 \implies q_1^2|\lambda^2 \implies q_1|\lambda$ and $\implies q_2|(\frac{\lambda}{q_1})^2$
- ▶ q_2 is square-free $\implies q_2|(\frac{\lambda}{q_1})$ and $\lambda = kq_1q_2$ for $k \in \mathbb{Z}_+$.
- ▶ $n|\lambda^2$ for any such λ , thus $\lambda = kq_1q_2$ is the side of the k -th complete spiral Ond_n^k .
- ▶ $\lambda = \frac{kn}{\sqrt{s}}$ follows since $\sqrt{s} = q_1$
- ▶ $\xi = \frac{k^2 n}{s}$ follows since $\xi = \frac{\lambda^2}{n}$.

Generalize!

- ▶ Generalize in dimension: increase d
- ▶ Generalize in shape
- ▶ Think geometrically: filling an area
 - ▶ more abstractly looking at when $n|f(d)$

Generalize in Dimension

- ▶ Generalize to d -dim hypercube by looking at $n|\lambda^d$
 - ▶ Difficult to visualize

Theorem

Let $n|\lambda^d$ for $d = 2, 3, \dots$. Let $n = qm^d$ where $m \in \mathbb{N}$ and q is d -power free. Let q be defined in terms of its prime factors, p_j , as

$$q = \prod_{j=1}^r p_j^{e_j} \quad \text{with each } e_j < d.$$

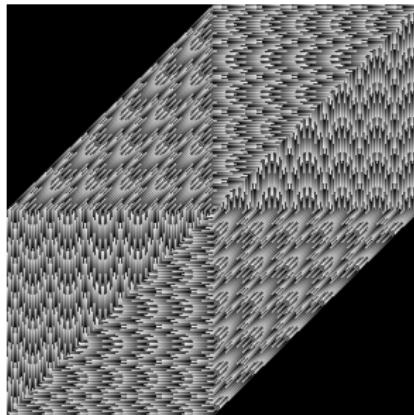
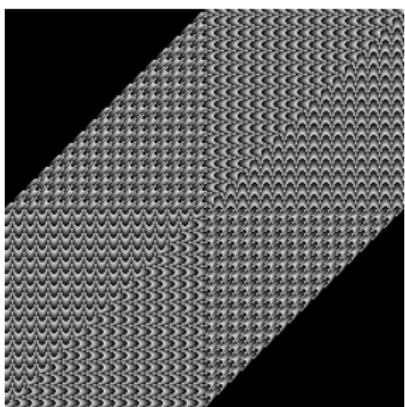
then

$$\lambda = km \prod p_j \quad k = 1, 2, 3, \dots$$

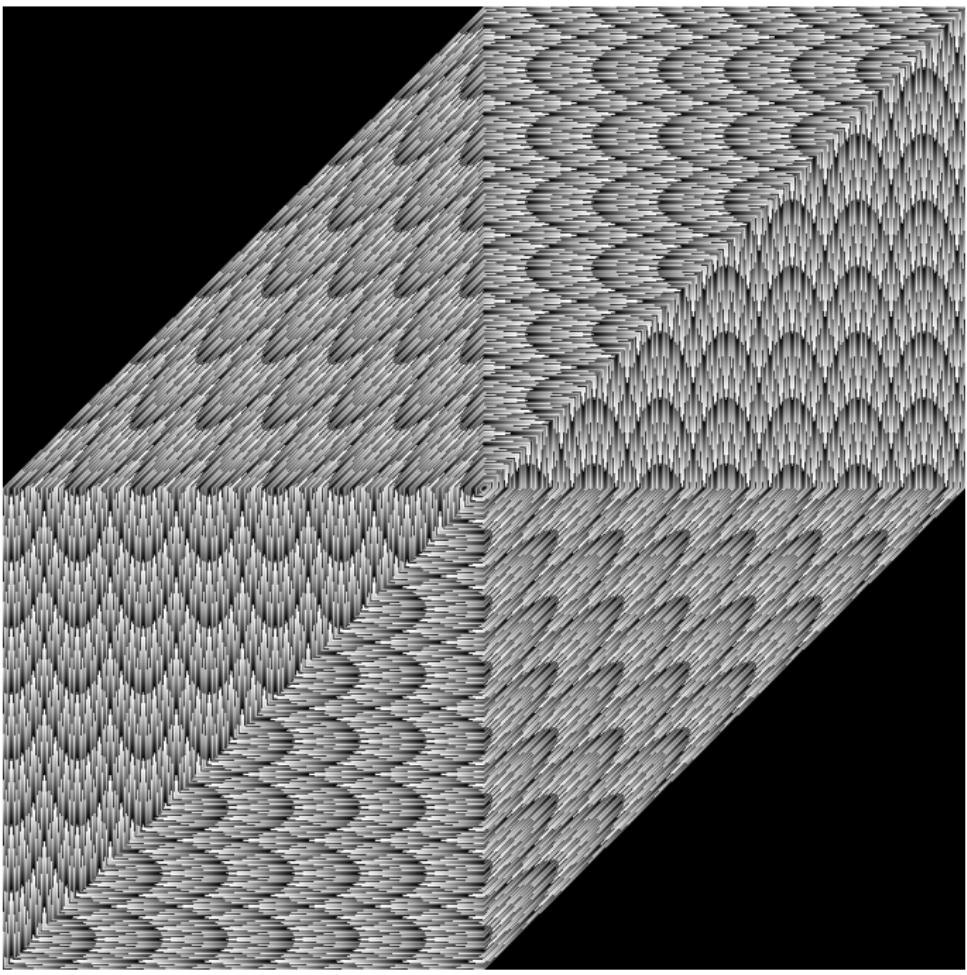
- ▶ Note that m^d is the greatest d -power divisor of n
- ▶ Proof is similar to $d = 2$ proof.

Other Shapes: Centered Hexagons

- ▶ Centered hexagonal numbers are $CH_d = 3d(d + 1) + 1$
- ▶ Investigate $n|CH_d$
- ▶ Not all n will satisfy!
 - ▶ $n = 2, \dots, 6$ don't work but $n = 7, 13, 19, 31, 37, \dots$ will

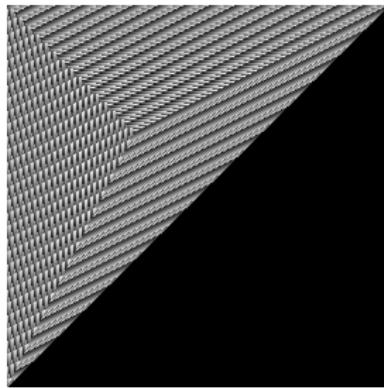
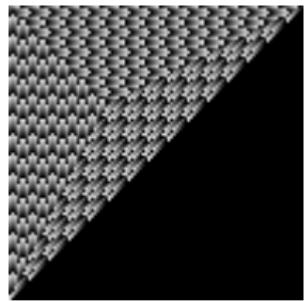


$HexOnd_7^{34}$ and $HexOnd_{37}^{12}$



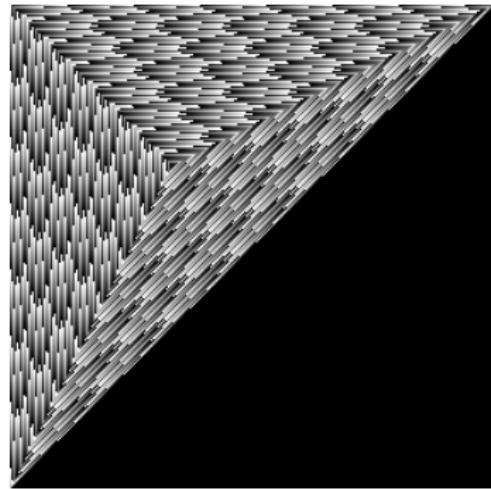
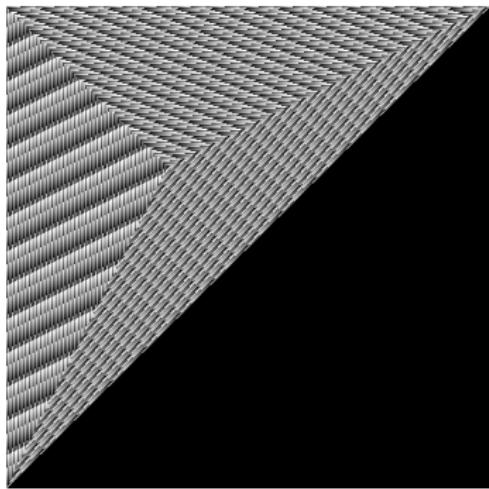
Other Shapes: Triangles

- ▶ Triangle numbers are $T_d = \frac{d(d+1)}{2}$
- ▶ Investigate $n|T_d$



$TriOnd_7^{29}$ and $TriOnd_{27}^{29}$

$TriOnd_{36}^{29}$ and $TriOnd_{39}^{29}$



Note the bias by using right triangle

So what is all this?

- ▶ Show how one can impose definitions to patterns you see.
- ▶ Observations→Conjecture→Proof
- ▶ Learn to generalize

A means to teach how to think mathematically!

Things to Explore

- ▶ Other 2 dimensional lattice shapes
- ▶ Non-hypercube shapes in > 2 dimensions
- ▶ Addition of *Ond*
- ▶ Random *Ond*: Select n_0 from some distribution, cycle through \mathbb{Z}_{n_0} once, choose n_1 , cycle through \mathbb{Z}_{n_1} , ...

Thank You!

We invite you to please take a paper if you are interested in this topic. For code and visualizations, please go to:

<https://github.com/cwcomplex/modNspirals>

- ▶ Robin Young of UMASS-Amherst
- ▶ Veracode, Inc – Hackathon!
- ▶ Jared Carlson of Veracode

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