ON (MOD N) SPIRALS

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1. Introduction

This note is intended to introduce the process of constructing (mod n) spirals and the idea of a complete spiral. It also introduces a theorem related to patterns seen regarding the lengths of sides, iteration counts, and ending corners of these objects, and provides a proof. This process also provides a deterministic process for discovering the greatest square divisor of integers $n \geq 2$. Grayscale visualizations of these spirals have been generated in order to elucidate further understanding of them. A theorem generalizing aspects of the spirals to cubic and hyper cubic dimensions d, with proof, is also provided to help gain insight into higher dimensions. Lastly, we include some investigation of non-square shapes and their spiral patterns. While not inspired by Ulam's Spiral [1], the construction of a (mod n) spiral is similar in nature. The author's are unaware of other work or literature on this topic, so have no references other the couple related to code; in part this is why the note has been written.

2. Spiral Construction

We describe the construction of a (mod n) spiral and introduce the notation we will use to analyze these. For a fixed integer $n \geq 2$, we will be working with the additive group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, ..., n-1\}$. Let L be a square lattice, which we can take to be $\mathbb{Z}^2$. Denoting the origin by $l_1 = (0, 0)$, we build the spiral by enumerating the lattice sites and assigning numbers from $\mathbb{Z}_n$ in turn. We spiral in a clockwise direction starting in the direction of the x-axis, so that $l_2 = (1, 0)$, $l_3 = (1, -1)$, and the next four lattice points are $\{(0, -1), (-1, -1), (-1, 0), (-1, 1)\}$, respectively. In general, once l_j has been assigned, and having chosen a 'direction' by moving from l_{j-1}, the next site l_{j+1} is the site to the right if this is not yet accounted for, or the site ahead otherwise.

$$\boxed{0} \rightarrow \begin{array}{cc} \boxed{0} & 1 \end{array} \rightarrow \begin{array}{cc} \boxed{0} & 1 \\ & 2 \end{array} \rightarrow \begin{array}{cc} \boxed{0} & 1 \\ 0 & 2 \end{array} \rightarrow \begin{array}{ccc} & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array} \rightarrow \begin{array}{ccc} 2 & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array} \rightarrow \begin{array}{ccccc} & 0 & \cdot & \cdot & \cdot \\ & 2 & \boxed{0} & 1 & \cdot \\ & 1 & 0 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Figure 1. Building a (mod 3) spiral

Having enumerated the lattice as above, we now assign values from $\mathbb{Z}_n$ as follows: to the origin we assign $l_1^* = 0$, and we then count in $\mathbb{Z}_n$: $l_2^* = 1$, $l_3^* = 2$, etc., and cycling back to 0 after n steps, so that $l_{n+1}^* = 0$. Continuing in this way, we 'count' all lattice sites (mod n). In general, this yields

$$l_j^* = j - 1 \pmod{n} \quad \text{at site} \quad l_j.$$

The first author's initial interest in generating spirals of increasing size for various $\mathbb{Z}_n$ was to investigate some of the patterns seen in figure 3. In an attempt to describe the patterns, he introduced the following definition:

Definition 1. *A* complete spiral *occurs when in the spiral construction, the partially completed spiral forms a square, and the last (mod n) value assigned is $l_i^* = n - 1$, so all of $\mathbb{Z}_n$ has been used an integer number of times.*

We use the following notation: denote the first complete spiral achieved by Ond_n^1; subsequent complete spirals are denoted Ond_n^k, $k = 2, 3, \ldots$. The number of times $\mathbb{Z}_n$ is used in order to complete the k^{th} square is the iteration count. The word "ond" is "spiral" in Swahili.

Figure 4 shows Ond_n^k for $k = 1, 2$ and $n = 1, 2$. We see that Ond_3^1 is a 3×3 square and has iteration count 3, and Ond_3^2 is a 6×6 square with 12 iterations. Similarly, Ond_4^1 is 2×2 with a single iteration, and Ond_4^2 is 4×4 with 4 iterations. The first element, 0, of each spiral in the figures is in a box and last element of the spiral marked with a dot.

The author generated several spirals with pencil and paper in a methodical manner to create sets of complete spirals for various values of n and k. While tedious and not exactly related to the initial goal of looking at diagonal patterns, it helped to realize there seemed to be patterns found in the construction of the spirals. Specifically in the sizes of the complete spirals, iteration counts, and where the last lattice point rested; this led the author to investigate what the patterns were.

2.1. **On the side lengths and iterate counts of Ond_n^k.** In order to investigate these patterns, the author determined that more data was needed and, due to the tedium of pencil and paper, a program should be written to generate complete spirals [2]. This allowed the author to generate a larger number of complete spirals and collect data on lengths, iterations, and ending points.

In looking at the initial data, for small choices of n, k, a few patterns in tuples of (lengths, iterations) were found, including $(kn, k^2 n)$, $(\frac{kn}{2}, \frac{k^2 n}{4})$, and $(\sqrt{n}k, k^2)$. To establish the pattern, the prime factorization of n for each complete spiral Ond_n^k with the lengths and iteration data was generated for analysis. From this data, the author determined there was a relation involved with finding the greatest square divisor of n, which led to the following observations.

Theorem 2.1. *Let s denote the greatest square divisor of n. The complete spiral Ond_n^k has the following structure:*

(i) If λ *is the length of the sides of* Ond_n^k*, then*

$$\lambda = \frac{kn}{\sqrt{s}}. \tag{1}$$

(ii) If ξ *is the iteration count of* Ond_n^k*, then*

$$\xi = \frac{k^2 n}{s}. \tag{2}$$

(iii) If $l_{max} \in L$ *is the last lattice point in the complete spiral* Ond_n^k*, then* l_{max} *is either the top-right corner or bottom-left corner of the square. If both* n *and* k *are odd, then* l_{max} *will be the top-right corner of* Ond_n^k*. In all other cases,* l_{max} *will be the bottom-left corner.*

One will note that in the case where n is square-free, then $s = 1$ and (1) and (2) reduce to $\lambda = kn$ and $\xi = k^2 n$, respectively. These can be considered an upper bound for all cases. One might also consider these to be the least robust in the case of length and iteration counts.

A most interesting aspect to this process of complete spiral construction is the connection to the greatest square divisor of some integer n. To see this more clearly, choose some n and construct, by hand, the spiral Ond_n^1. At this point, you know λ and ξ, so pick one, substitute $k = 1$ and solve for s. So from a constructivist approach we find the greatest square divisor.

In the process of investigating sizes and iteration counts of Ond_n^k, the first author implemented a method to map the generated spirals to grayscale images. This works best for positive integers, n, less than 256. The map $f : \mathbb{Z}_n \to \mathbb{Z}_{256}$ is defined by a scaled floor function,

$$f(j) = \alpha j, \quad \text{where} \quad \alpha = \left\lfloor \frac{255}{n} \right\rfloor$$

The function f thus maps $\mathbb{Z}_n$ to a set of brightness values which are used to generate grayscale images.

An example of this is Ond_{31}^{39} seen in figure 5. One can see a complex pattern emerging. Further, if one zooms in and out on the image, various different patterns emerge that seem to play tricks on the eye. The image may also be view at [3] and is the file *N=31_k=39-grey.png* which makes it easier for testing the zoom in/out. Further, at this location, one may find the generated images for Ond_n^k for $n = 2, 3, \ldots, 31$ and $k = 1, 2, \ldots, 50$.

Proof of the Theorem. Recall that $n \geq 2$ is fixed. Let λ be the length of the side of a complete spiral, and let ξ be the corresponding iteration count. Since the spiral is square, we must have $\xi n = \lambda^2$, and so $n | \lambda^2$. Now any prime which divides n must also divide λ^2, and must thus divide λ.

Since s is the greatest square divisor of n, we can uniquely write $n = q_1^2 q_2$ where q_2 is square-free, with $s = q_1^2$. It follows that $q_1^2 | \lambda^2$, so $q_1 | \lambda$ and $q_2 | (\frac{\lambda}{q_1})^2$. Since q_2 is square-free,

$q_2|(\frac{\lambda}{q_1})$ and we have $\lambda = kq_1q_2$ for some integer k. Clearly $n|\lambda^2$ for any such λ, and we conclude that $\lambda = kq_1q_2$ is the side of the k-th complete spiral Ond_n^k.

Equation (1) now follows since $\sqrt{s} = q_1$, and (2) follows since $\xi = \frac{\lambda^2}{n}$. Finally, (iii) follows since $l_{max} = \lambda^2$ is even unless k, q_1 and q_2 are all odd, and by construction l_{max} represents the top-right corner of the square if it is odd, and the bottom-left corner if it is even. □

3. Generalizations

We can further investigate the idea of *Ond* by looking at how spiraling might work for cubes or higher dimensional squares, since the lattice is the basis for the spiral. In imagining dimension $d = 3$ and $d = 4$, finding the route in the lattice to properly spiral can be quite difficult, let alone an algorithm for any d.

In terms of planar spirals described above, we are interested in finding squares, thus the areal requirement $n|\lambda^2$. For the case of cubic spirals, we have the volume requirement $n|\lambda^3$ and $n|\lambda^4$ for the $4d$ hypercube. Thus we think about the *Ond* for any dimension d as having this same geometric requirement.

Theorem 3.1. *Let $n|\lambda^d$ for $d = 2, 3, \ldots$. Let $n = qm^d$ where $m \in \mathbb{N}$ and q is d-power free. Let q be defined in terms of it's prime factors, p_j, as*

$$q = \prod p_j^{e_j} \quad \textit{with each} \quad e_j < d.$$

then

$$\lambda = km \prod p_j \quad k = 1, 2, 3, \ldots$$

Proof. We have that

$$\begin{aligned} & n|\lambda^d \\ \Longrightarrow\ & qm^d|\lambda^d \\ \Longrightarrow\ & m^d|\lambda^d \\ \Longrightarrow\ & m|\lambda \end{aligned}$$

which means that writing $\frac{\lambda}{m}$ makes sense. Thus we have that $q|(\frac{\lambda}{m})^d$. Recall the definition of q and we see

$$\begin{aligned} & p_j^{e_j}|q|(\frac{\lambda}{m})^d \\ \Longrightarrow\ & p_j|q|(\frac{\lambda}{m})^d \\ \Longrightarrow\ & p_j|(\frac{\lambda}{m})^d \\ \Longrightarrow\ & p_j|\frac{\lambda}{m} \\ \Longrightarrow\ & p_1 \ldots p_r|\frac{\lambda}{m} \end{aligned}$$

since p_j are distinct. Thus, we can write $\lambda = km \prod p_j$ for some k. □

4. Triangle and Hexagonal Numbers

We can expand from the square spirals to other planar shapes defined on a square lattice and investigate these. While the spiraling process can vary, one the area requirement of achieving Ond remains the same.

4.1. Triangle Numbers. In order to investigate triangle Ond, we make use of the triangle numbers [5] to determine the area requirement, integral area in this case. It is well known [5] that

$$Area(\triangle d) = T_d = \frac{d(d+1)}{2}$$

Thus we look at the requirement $n|\frac{d(d+1)}{2}$, which tells us that either $n|d$ or $n|(d+1)$. In the case of $n|d$, then not only $n|T_d$ but also $n|T_{d-1}$ since $T_{d-1} = \frac{(d-1)d}{2}$. Further, if it is that $n|(d+1)$, then also $n|T_{d+1}$ since $T_{d+1} = \frac{(d+1)(d+2)}{2}$. One can see that once Ond is achieved for some n by filling some T_d, then it is necessarily the case that T_{d-1} or T_{d+1} are also Ond for n due to the inclusion of successive d factors.

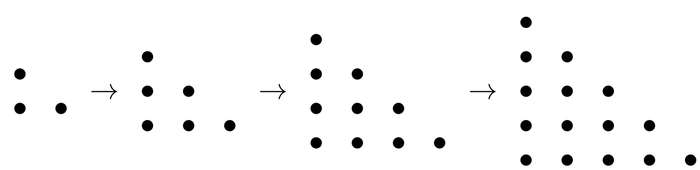

Figure 2. Triangles on Lattice starting with $d = 2$

If we take the first $\{T_d\}_{d=2,\ldots,10}$, we have the set { 3, 6, 10, 15, 21, 28, 36 45, 55 }. We see that Ond_2 can be achieved by T_3 since $d+1 = 4$ and T_4 since $d = 4$. Similarly we see that T_7 and T_8 are also Ond_2 because of $d+1 = 8$ and $d = 8$, respectively. One can clearly see the pairing nature that is described above in action here. If we look at Ond_4, we find the triangles T_7 and T_8 work and in looking at Ond_5 we see T_4, T_5 and T_9, T_{10} work; further demonstrating the pairing nature.

4.2. Hexagonal Numbers. In a similar manner to the triangle case, we can make use of hexagonals on a lattice and their hexagonal numbers [6]. Similar to the triangle numbers, we can consider these notions of area for achieving in Ond. The hexagonal numbers are defined as

$$H_d = d(2d-1)$$

Thus we are interested in the requirement $n|H_d$ which implies either $n|d$ or $n|(2d-1)$. If we look at the case for Ond_2, it is clear that the $2|H_d$ requirement forces that $2|d$ since $2d-1$ is odd for all d; thus we can assume that Ond_2 is achieved for every even d, or every other H_d.

For $d = 2, \ldots, 10$, we have the H_j values 6, 15, 28, 45, 66, 91, 120, 153, 190. If we take $\mathbb{Z}_2$ we have the first 3 Ond to be the case H_2, H_4, H_6.

Lastly, characterizing the visualizations and interpreting their relations to the process would be a neat endeavor.

Acknowledgments: The first author recognizes Veracode Inc and Jared Carlson of Veracode.

References

[1] Wolfram Mathworld, “Prime Spirals”, `http://mathworld.wolfram.com/PrimeSpiral.html`
[2] A. Reiter, `https://github.com/cwcomplex/modNspirals`
[3] A. Reiter, `https://github.com/cwcomplex/modNspirals/tree/master/somegrey`
[4] A. Reiter, `https://github.com/cwcomplex/modNspirals/blob/master/squareoff.py`
[5] Wolfram Mathworld, “Triangle Numbers”, `http://mathworld.wolfram.com/TriangularNumber.html`
[6] Wolfram Mathworld, “Hexagonal Numbers”, `http://mathworld.wolfram.com/HexagonalNumber.html`

5. Appendix A

6	7	8	9	0	1	2	3	4	5	6	7	8
5	0	1	2	3	4	5	6	7	8	9	0	1
4	9	2	3	4	5	6	7	8	9	0	1	2
3	8	1	2	3	4	5	6	7	8	9	2	3
2	7	0	1	0	1	2	3	4	5	0	3	4
1	6	9	0	9	6	7	8	9	6	1	4	5
0	5	8	9	8	5	0	1	0	7	2	5	6
9	4	7	8	7	4	3	2	1	8	3	6	7
8	3	6	7	6	5	4	3	2	9	4	7	8
7	2	5	6	5	4	3	2	1	0	5	8	9
6	1	4	3	2	1	0	9	8	7	6	9	0
5	0	9	8	7	6	5	4	3	2	1	0	1
4	3	2	1	0	9	8	7	6	5	4	3	2

Figure 3. A (mod 10) spiral with some indication of path

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$$\begin{array}{ccc} 0 & 1 & \dot{2} \\ 2 & \boxed{0} & 1 \\ 1 & 0 & 2 \end{array}$$

(A)

$$\begin{array}{cccccc} 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 2 & \boxed{0} & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 1 & 0 & 2 \\ \dot{2} & 1 & 0 & 2 & 1 & 0 \end{array}$$

(B)

$$\begin{array}{cc} \boxed{0} & 1 \\ \dot{3} & 2 \end{array}$$

(C)

$$\begin{array}{cccc} 2 & 3 & 0 & 1 \\ 1 & \boxed{0} & 1 & 2 \\ 0 & 3 & 2 & 3 \\ \dot{3} & 2 & 1 & 0 \end{array}$$

(D)

FIGURE 4. (a) Ond_3^1, (b) Ond_3^2, (c) Ond_4^1, (d) Ond_4^2

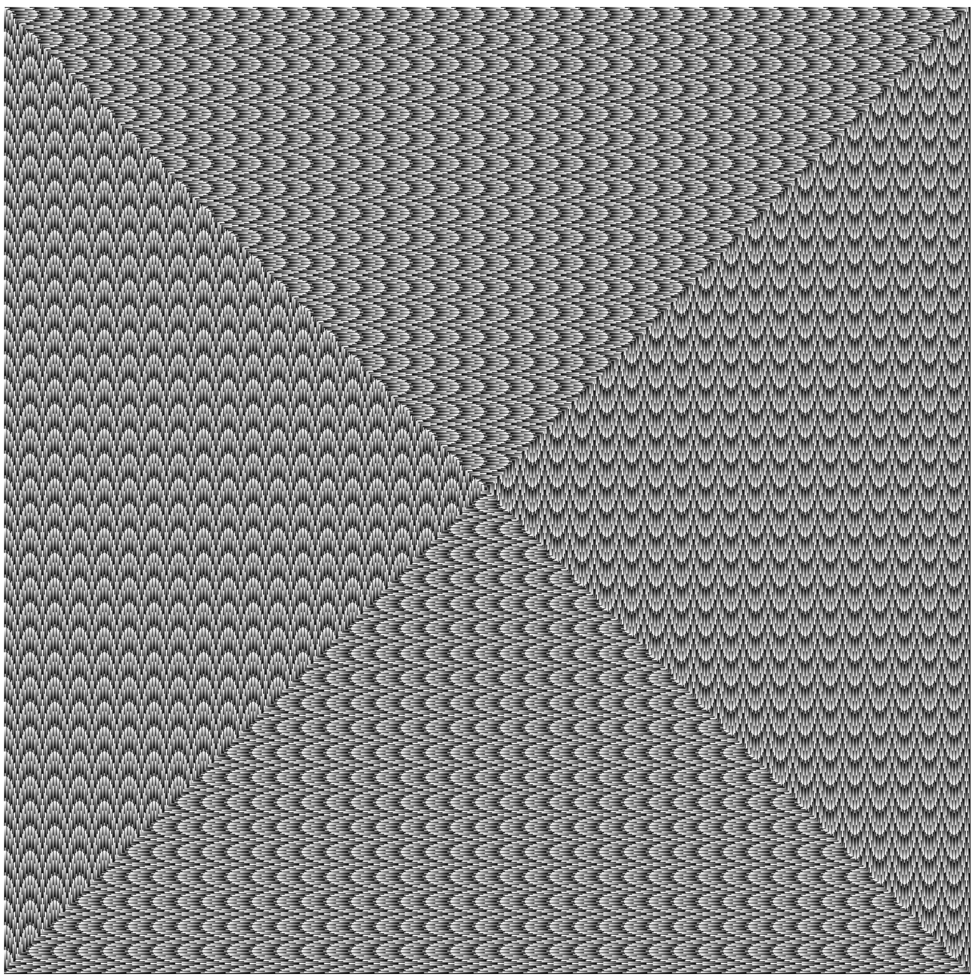

FIGURE 5. Visualization of Ond_{31}^{39}

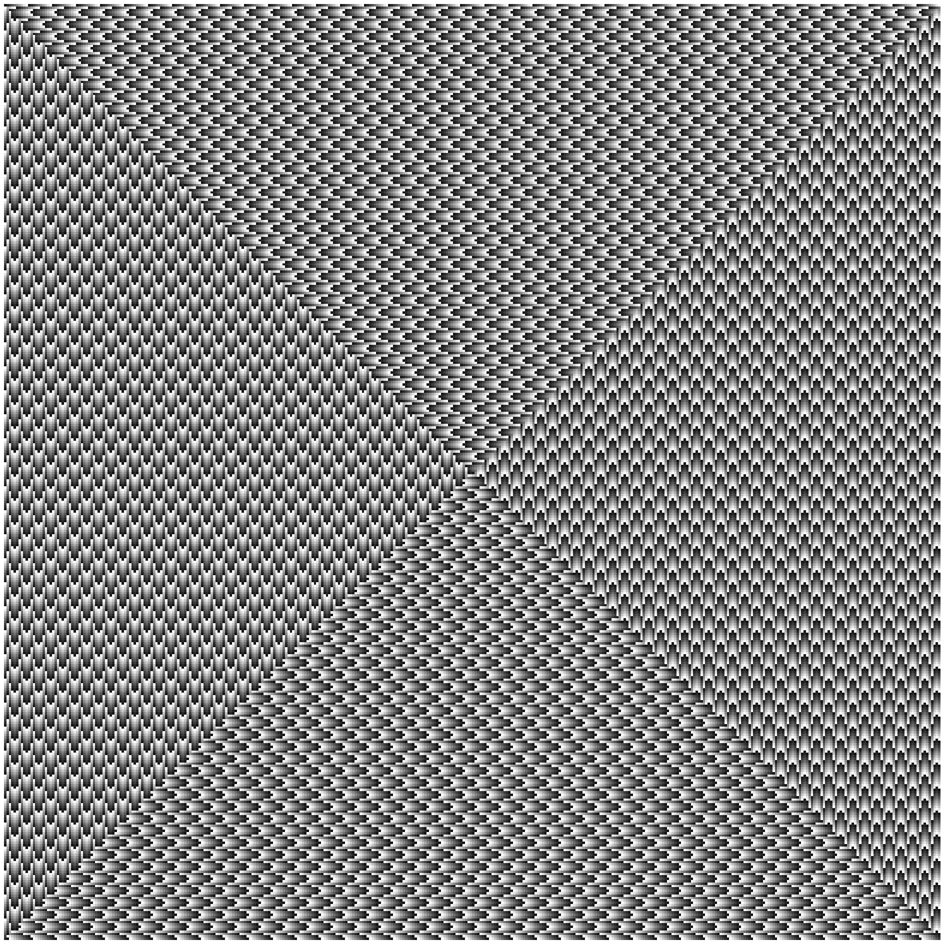

FIGURE 6. Visualization of Ond_{10}^{39}