Constraint Satisfaction Problems

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Outline

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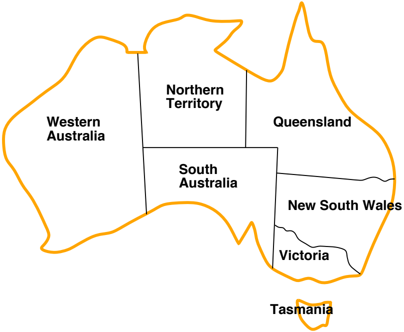
1. Constraint Satisfaction
   - Defining Problems
   - Problem Types
2. Backtracking Search
   - CSPs as Search
   - Search Heuristics
3. Search Alternatives
   - Local Search
   - Tree Search

Example: Map Coloring

Problem

Color all countries with red, green or blue, and with no 2 adjacent countries the same color



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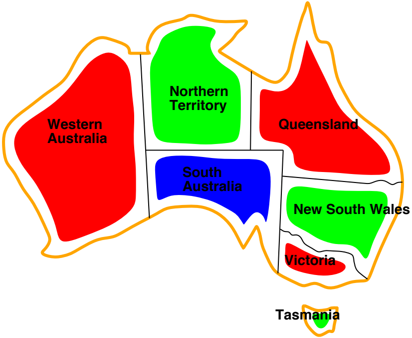

Constraint Satisfaction Problem

Terms

Variables $X_1, X_2, \ldots, X_n$

Domains Allowable values for each variable

Constraints Allowable combinations of variables

States

Assignment of values to variables, $\{X_i = v_i, X_j = v_j, \ldots\}$

Types of States

Consistent No constraints violated

Solution No constraints violated, all variables assigned

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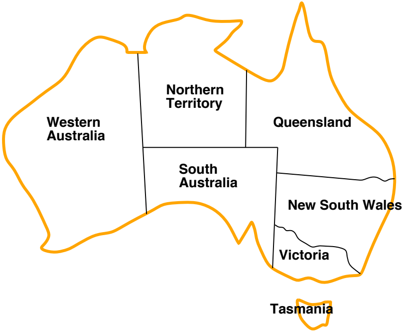

Variables WA, NT, Q, NSW, V, SA, T

Domains $D_i = \{\texttt{red}, \texttt{green}, \texttt{blue}\}$

Constraints $\texttt{WA} \neq \texttt{NT}, \texttt{WA} \neq \texttt{SA}, \texttt{NT} \neq \texttt{SA}, \ldots$

Solution $\left\{ \begin{array}{l} \texttt{WA} = \texttt{red}, \texttt{NT} = \texttt{green}, \texttt{Q} = \texttt{red}, \\ \texttt{NSW} = \texttt{green}, \texttt{V} = \texttt{red}, \texttt{SA} = \texttt{blue}, \\ \texttt{T} = \texttt{green} \end{array} \right\}$

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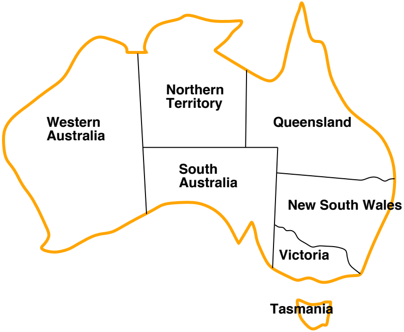

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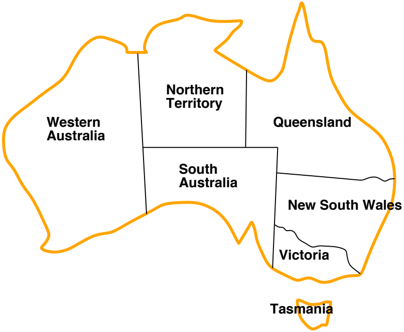

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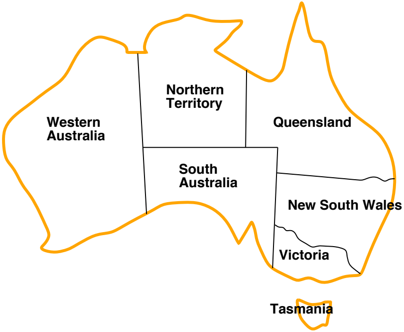

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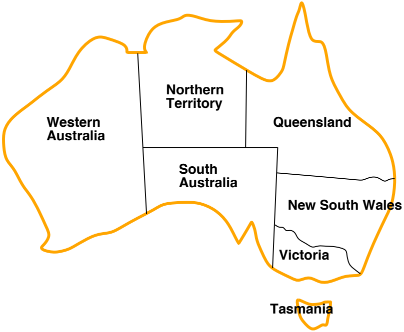

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Example: Cryptarithmetic

$$
\begin{array}{cccc}
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+ & T & W & O \\
\hline
F & O & U & R
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$$

Variables $T, W, O, F, U, R, C_1, C_2, C_3$

Domains $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints

$$
\begin{aligned}
O + O &= R + 10 \cdot C_1 \\
C_1 + W + W &= U + 10 \cdot C_2 \\
C_2 + T + T &= O + 10 \cdot C_3 \\
C_3 &= F
\end{aligned}
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Solution $\left\{ \begin{array}{l} T = 7, W = 3, O = 4, F = 1, U = 6, \\ R = 8, C_1 = 0, C_2 = 0, C_3 = 1 \end{array} \right\}$

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Example: Meeting Scheduling

Problem Description

- 2 hour meetings: Jim & Tammy
- 1 hour meetings: Jim & Martha, Martha & Tammy
- Meetings start on the hour 9:00am to 5:00pm
- Busy: Jim 12-2, Martha 11-1, Tammy 10-11 and 2-3

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Variables $\mathtt{J\&T}$, $\mathtt{J\&M}$, $\mathtt{M\&T}$

Domains $D_{\mathtt{J\&T}} = \{\mathtt{9\text{-}11}, \mathtt{10\text{-}12}, \ldots\}$

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Constraints $\mathtt{J\&T} \cap \mathtt{J\&M} = \emptyset$, $\mathtt{J\&M} \cap \mathtt{M\&T} = \emptyset$,

$\mathtt{12\text{-}2} \cap \mathtt{J\&T} = \emptyset$, $\mathtt{12\text{-}2} \cap \mathtt{J\&M} = \emptyset, \ldots$

Solution $\{\mathtt{J\&T} = \mathtt{3\text{-}5}, \mathtt{M\&T} = \mathtt{1\text{-}2}, \mathtt{J\&M} = \mathtt{10\text{-}11}\}$

Domain Types

Finite Domains

- Examples: map coloring, 8-queens
- Constraints described by enumeration, e.g. $(\mathtt{WA}, \mathtt{NT}) \in \{(\mathtt{red}, \mathtt{green}), (\mathtt{red}, \mathtt{blue}), \ldots\}$

Infinite but Countable Domains

- Examples: scheduling jobs by day or hour
- Need constraint language, e.g. $\mathtt{Start}_{\mathtt{Job1}} + 5 \leq \mathtt{Start}_{\mathtt{Job3}}$

Continuous Domains

- Examples: scheduling jobs by any fraction of time
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Unary Constraints

- Restrict the value of a single variable
- Can be eliminated by preprocessing domains

Binary Constraints

- Relate two variables, e.g. $\mathtt{SA} \neq \mathtt{NSW}$

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- Cost of assignments, e.g. `red` is better than `green`

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CSPs as Search

Formulation

States	Full or partial assignments
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Actions	Assign value to variable, obeying constraints
Goal	Assignment is complete

Benefits

- Same for all CSPs
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Given n variables, d values in domains:

	Root	Level 1	Level 2	...
Branches	nd	$(n-1)d$	$(n-2)d$	...

Leaves: $n! \cdot d^n$ Total possible assignments: d^n

Note

Variable assignments are commutative, e.g.

- `WA = red` then `NT = green`
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Backtracking Search Code

```python
def csp_search(csp, heuristic, assignment=None):
    if assignment is None:
        assignment = {}
    # if assignment is complete, return it
    if len(assignment) == len(csp.variables):
        return assignment
    # select an unassigned variable and order the values
    variable = heuristic.select_variable(csp, assignment)
    # try assigning each value to the variable
    for value in heuristic.order_values(csp, assignment, variable):
        assignment[variable] = value
        # for consistent assignments, recursively check if
        # it is possible to assign the remaining variables
        if csp.is_consistent(assignment):
            result = csp_search(csp, heuristic, assignment)
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        del assignment[variable]
    # all assignments failed
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        # for consistent assignments, recursively check if
        # it is possible to assign the remaining variables
        if csp.is_consistent(assignment):
            result = csp_search(csp, heuristic, assignment)
            if result is not None:
                return result
        del assignment[variable]
    # all assignments failed
    return None
```

Backtracking Example

Variables WA, NT, Q, SA, NSW, V, T

Domains $D_i = \{\texttt{red}, \texttt{green}, \texttt{blue}\}$

Constraints $\texttt{SA} \neq \texttt{WA}$, $\texttt{SA} \neq \texttt{NT}$, $\texttt{SA} \neq \texttt{Q}$, $\texttt{SA} \neq \texttt{NSW}$, $\texttt{SA} \neq \texttt{V}$, $\texttt{WA} \neq \texttt{NT}$, $\texttt{NT} \neq \texttt{Q}$, $\texttt{Q} \neq \texttt{NSW}$, $\texttt{NSW} \neq \texttt{V}$

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$\{\}$

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WA $\neq$ NT, NT $\neq$ Q, Q $\neq$ NSW, NSW $\neq$ V

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{WA=red}

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{ WA = red, NT = green, Q = blue, SA = ???}

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Backtracking Example

Variables WA, NT, Q, SA, NSW, V, T

Domains $D_i = \{\texttt{red}, \texttt{green}, \texttt{blue}\}$

Constraints SA ≠ WA, SA ≠ NT, SA ≠ Q, SA ≠ NSW, SA ≠ V, WA ≠ NT, NT ≠ Q, Q ≠ NSW, NSW ≠ V

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Backtracking Heuristics

Problem

- Basic depth first search is still too inefficient.
- E.g. Can only solve n-queens for $n \approx 25$

How can we be smarter?

- Which variable should be assigned next?
- In what order should its values be tried?
- Can we detect inevitable failure early?

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Minimum Remaining Values

Idea

Select the variable with the fewest legal values

Also Known As

Most Constrained Variable

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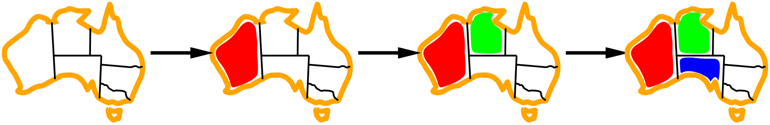

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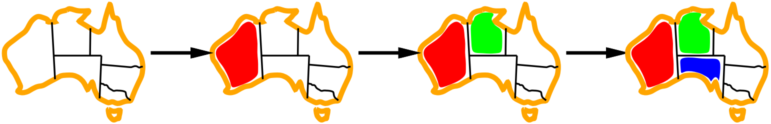

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- But what to do when MRV produces ties?
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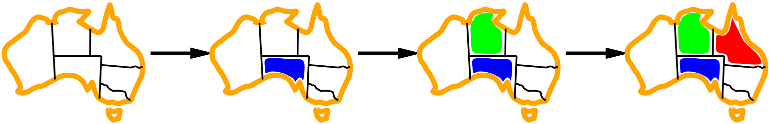

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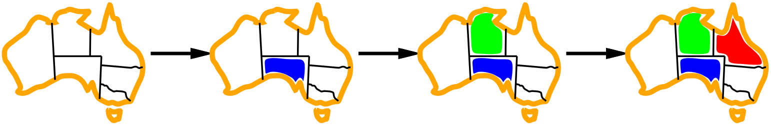

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Least Constraining Value

Idea

Select the variable that rules out the smallest number of values for the remaining variables

	Minimum Remaining Values
+	Degree Heuristic
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≈	1000 queens

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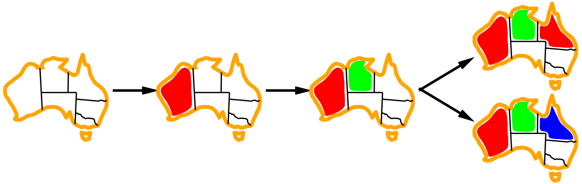

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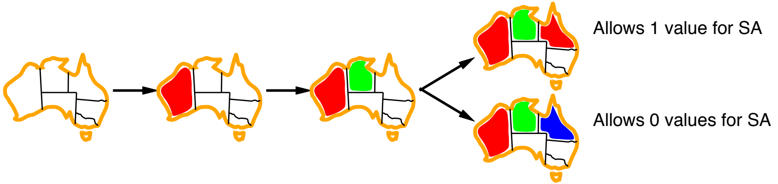

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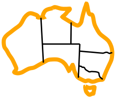
Forward Checking

Idea

- Keep track of remaining legal values for all variables
- Stop search when any variable has no legal values



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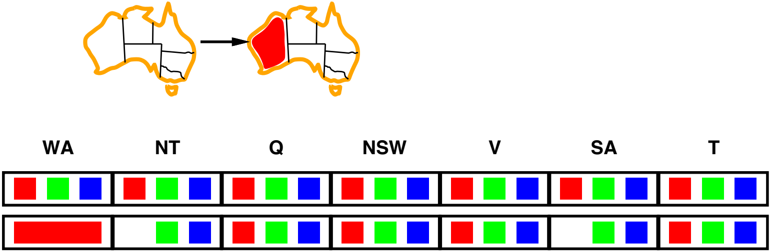

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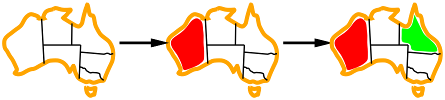

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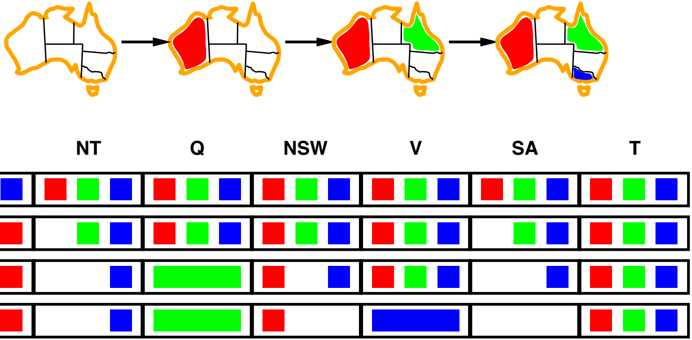

Outline

1. Constraint Satisfaction
   - Defining Problems
   - Problem Types
2. Backtracking Search
   - CSPs as Search
   - Search Heuristics
3. Search Alternatives
   - Local Search
   - Tree Search

CSPs as Local Search

Formulation

States	Complete assignments
Initial	Any complete assignment
Actions	Change value of one variable
Goal	Consistent assignment

Benefits

- Minimal memory consumption
- Emprically very effective

CSPs as Local Search

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Idea

- Pick a variable with constraint violations
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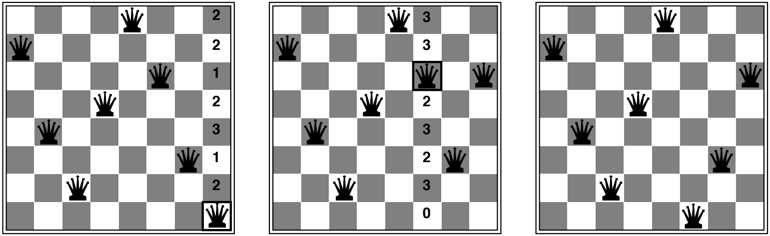

Min-Conflicts Code

```python
def min_conflicts(csp, max_steps):
    # start with an initial complete assignment
    assignment = {}
    for variable in csp.variables:
        assignment[variable] = random.choice(variable.domain)
    # adjust one variable each time through the loop
    for i in range(max_steps):
        # return the assignment when it is consistent
        if csp.is_consistent(assignment):
            return assignment
        # otherwise, select a random conflicted variable
        var = random.choice(csp.get_conflicts(assignment))
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        counts = {}
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Almost constant time for arbitrary n with high probability

Other kinds of CSPs

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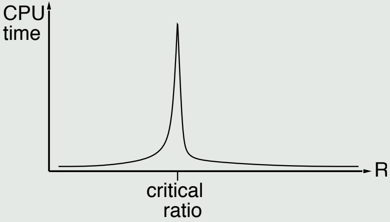
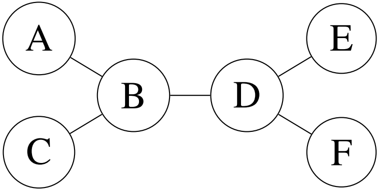


Tree-Structured CSPs

CSPs as graphs:

- Nodes = Variables
- Edges = Constraints



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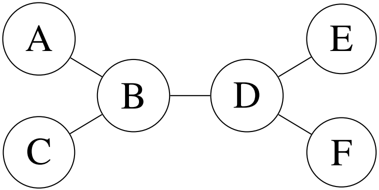
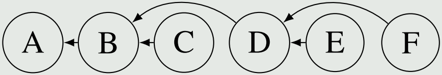


Solver

1. Choose root variable, list parents before children, e.g.



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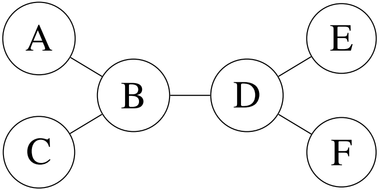

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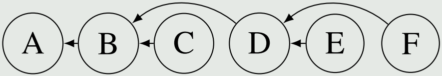

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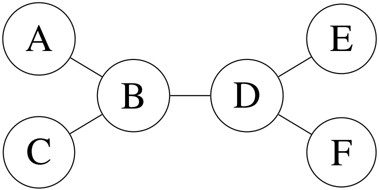

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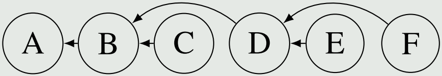

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Schedule jobs A, B, C, D, E for 9, 10, 11 or 12

- B must be before 11:00
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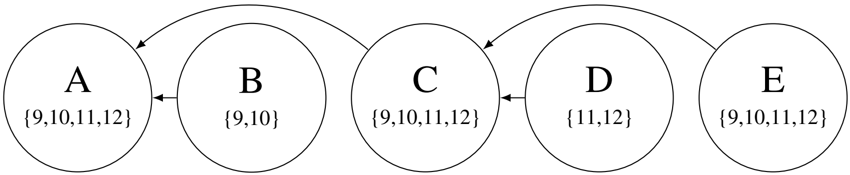

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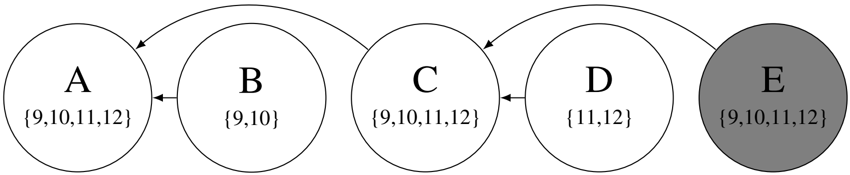

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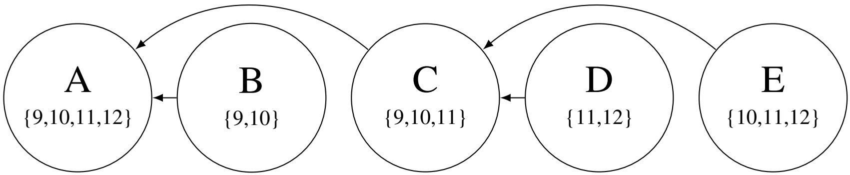

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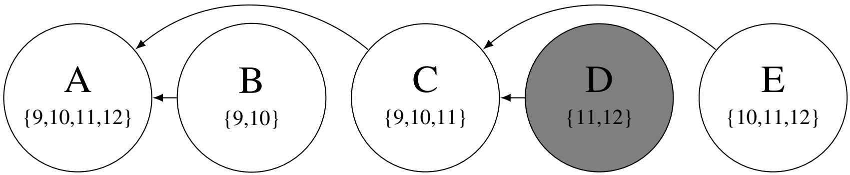

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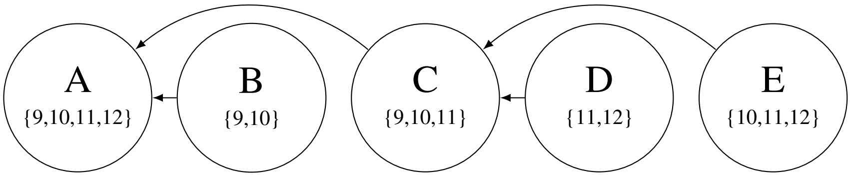

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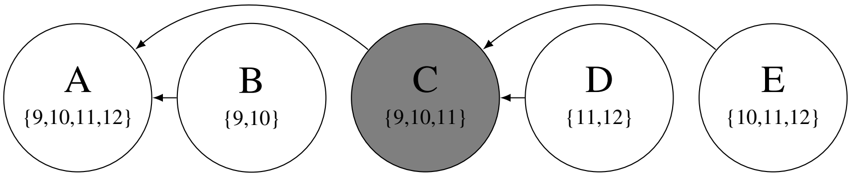

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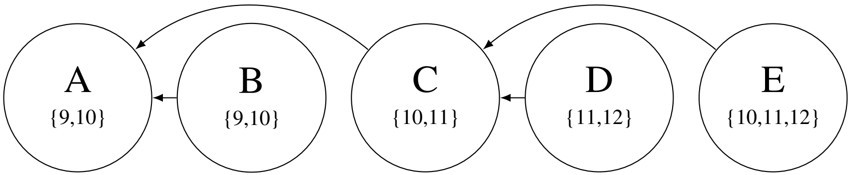

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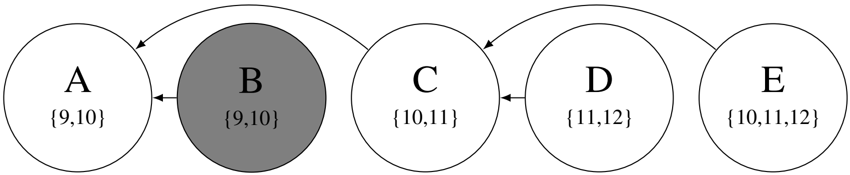

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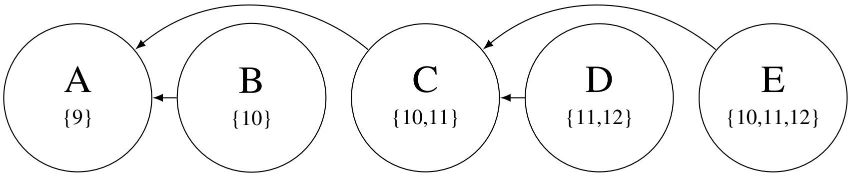

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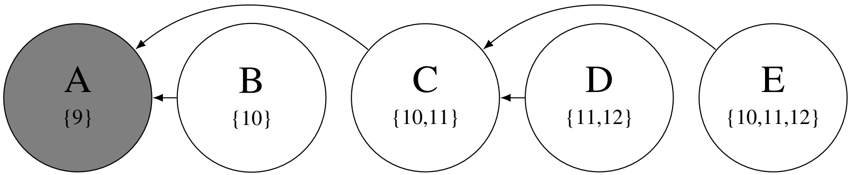

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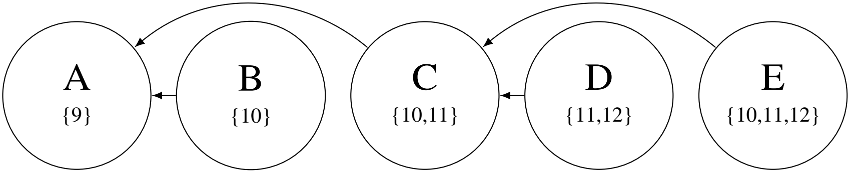

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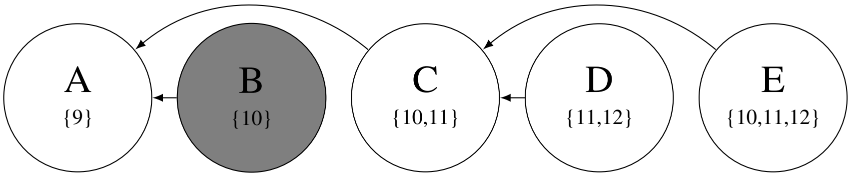

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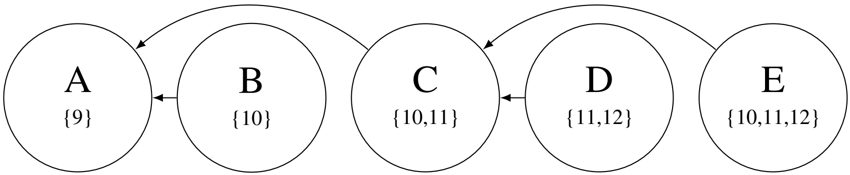

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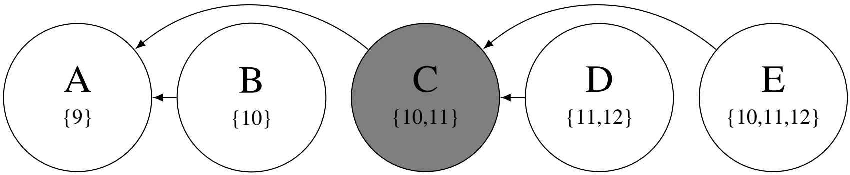

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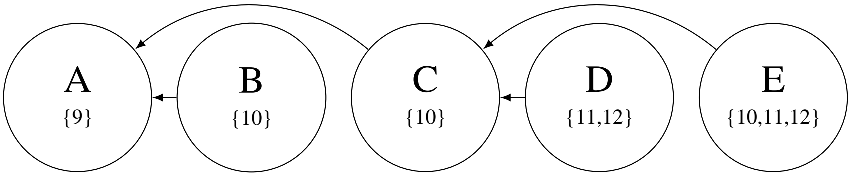

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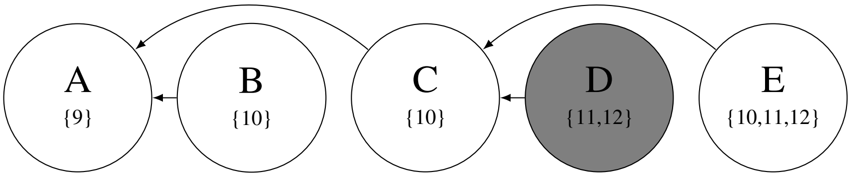

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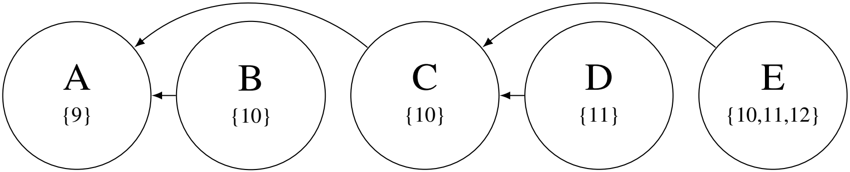

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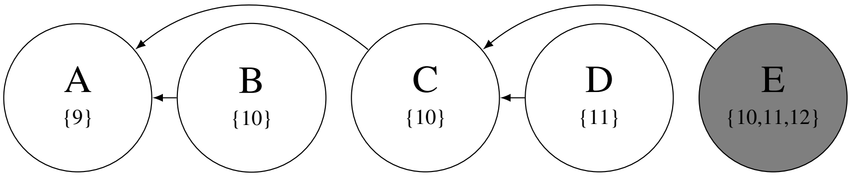

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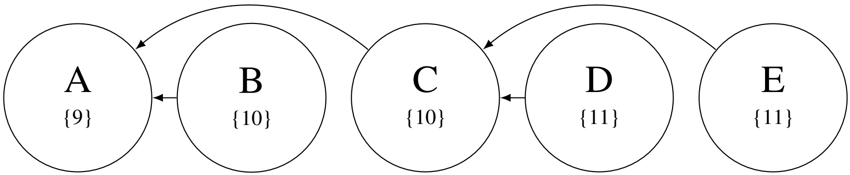
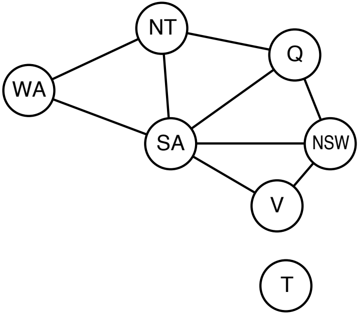


Subproblems and Decomposition

Example

Tasmania, mainland: separate components



If each subproblem has c of the n total variables

- Num. of subproblems: n/c
- Time per subproblem: d^c
- Total work: $O(nd^c/c)$

$n = 80, d = 2, c = 20$

At 10 million nodes/sec:

- $2^{80} = 4$ billion years
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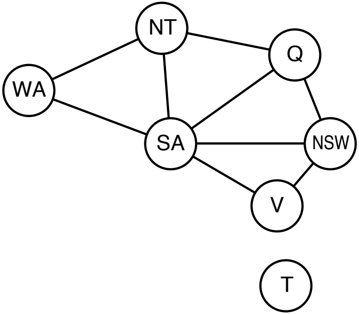

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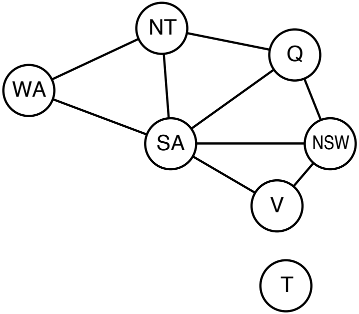

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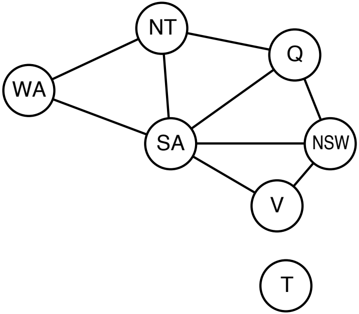

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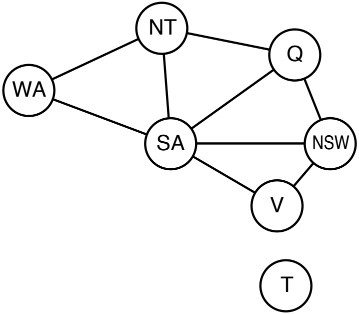

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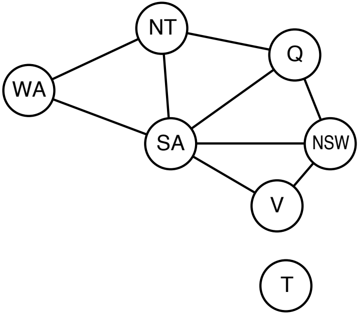

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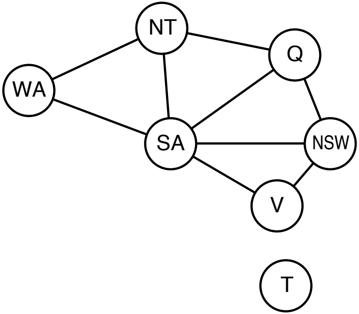

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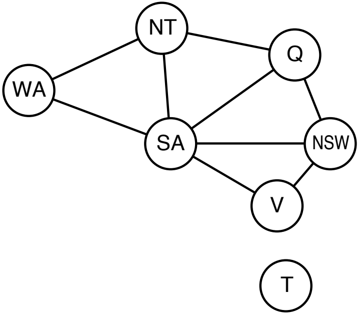

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Tree Decomposition

Convert constraint graphs to trees by assigning variables:

Properties

- Complexity: $O(d^c \cdot (n-c)d^2)$, given cutset size c
- Finding the smallest cycle cutset is NP-hard, but some efficient approximations exist

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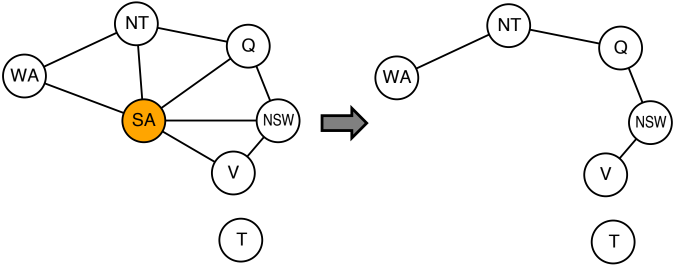

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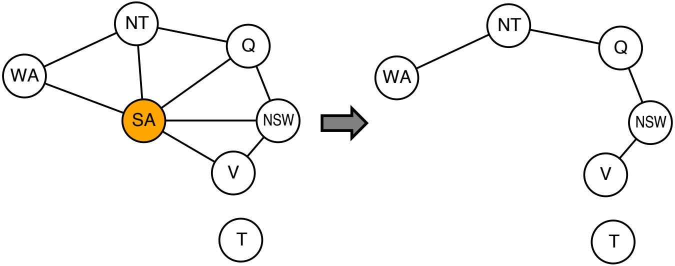

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Key Points

Constraint Satisfaction Problems

- States are assignment of values to variables
- Goals are assignments with no constraint violations

- Backtracking
  - Depth-first search, one variable assigned per node
  - Can be made effective with a number of heuristics
- Min-Conflicts
  - One value changed to reduce violations per iteration
  - Usually effective in practice
- Tree-Structured Search
  - Can be solved in linear time
  - Graphs can sometimes be decomposed into trees