Fractal Attractors:

The Middle Third Cantor Set

$$\begin{tikzcd}[column sep=tiny]
 & & & [0,1] \arrow[dll] \arrow[drr] & & & \\
 & [0,\frac{1}{3}] \arrow[dl] \arrow[dr] & & & & [\frac{2}{3},1] \arrow[dl] \arrow[dr] & \\
[0,\frac{1}{9}] & & [\frac{2}{9},\frac{1}{3}] & & [\frac{2}{3},\frac{7}{9}] & & [\frac{8}{9},1]
\end{tikzcd}$$

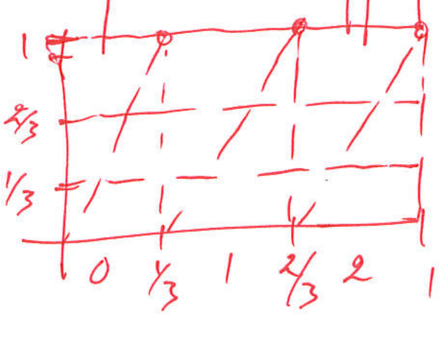
$\swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow$

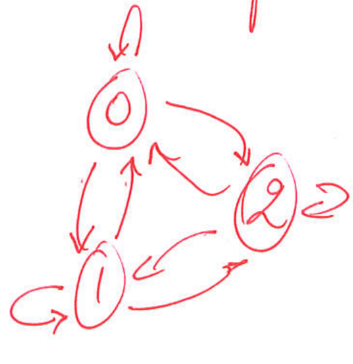
and so forth.

At this point, we should think in base-3

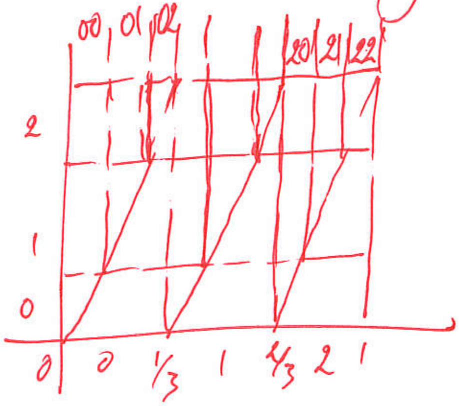
i.e. $x \in [0,1]$, let $x = \sum_{j=1}^{\infty} \frac{b_j}{3^j}$; $b_j = 0,1,2$

Equivalently, we can apply the $3x \bmod 1$ map for x





So after one iteration of $3x \bmod 1$ and comparing to middle 3^{rd} Cantor ...



↳ $\underline{0}$ $\underline{2}$

$\underline{00}$ $\underline{02}$ $\underline{20}$ $\underline{22}$

any itinerary in the middle 3^{rd} Cantor set cannot have a <u>one</u>,
but we can get any sequence of 0 & 2

↳ Middle 3^{rd} follows transition graph to $\circlearrowleft 0 \rightleftarrows 2 \circlearrowright$

↳ Same as binary sequences

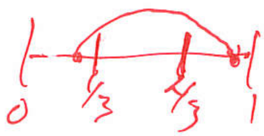
↳ <u>Uncountable</u>.

Cantor Set as Attractor:

relation to current location.

Heads: Go $2/3$ of the way to 0

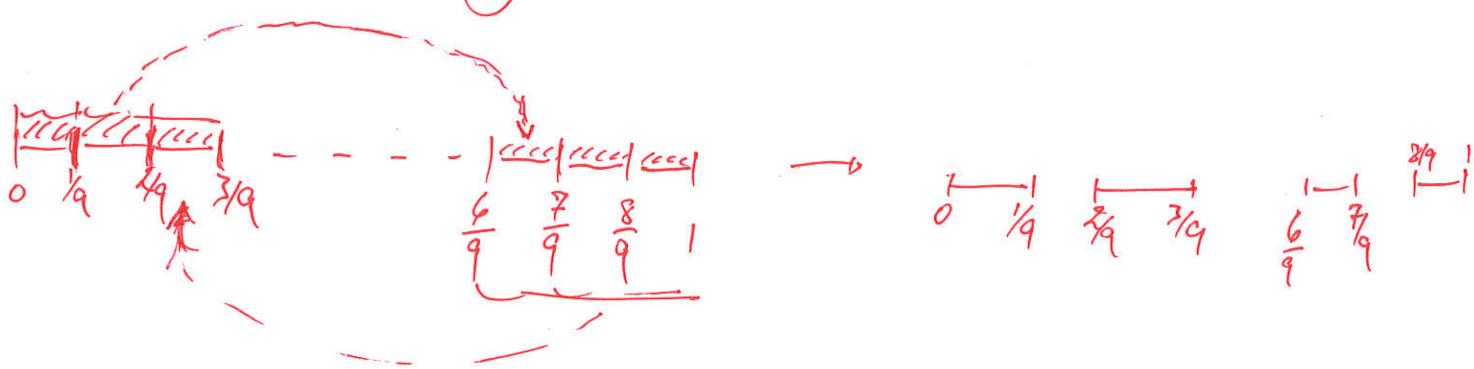
Tails: Go $2/3$ of the way to 1



If we do this for every point in $[0,1]$

$\hookrightarrow [0,1] \Longrightarrow [0, 1/3] \cup [2/3, 1]$

Go back and do it again



Repeat, and clearly we get middle 3rd Cantor, so we clearly see how a funny, "fractal", like set is an attractor for a repeated map.

This is equivalent to doing the following

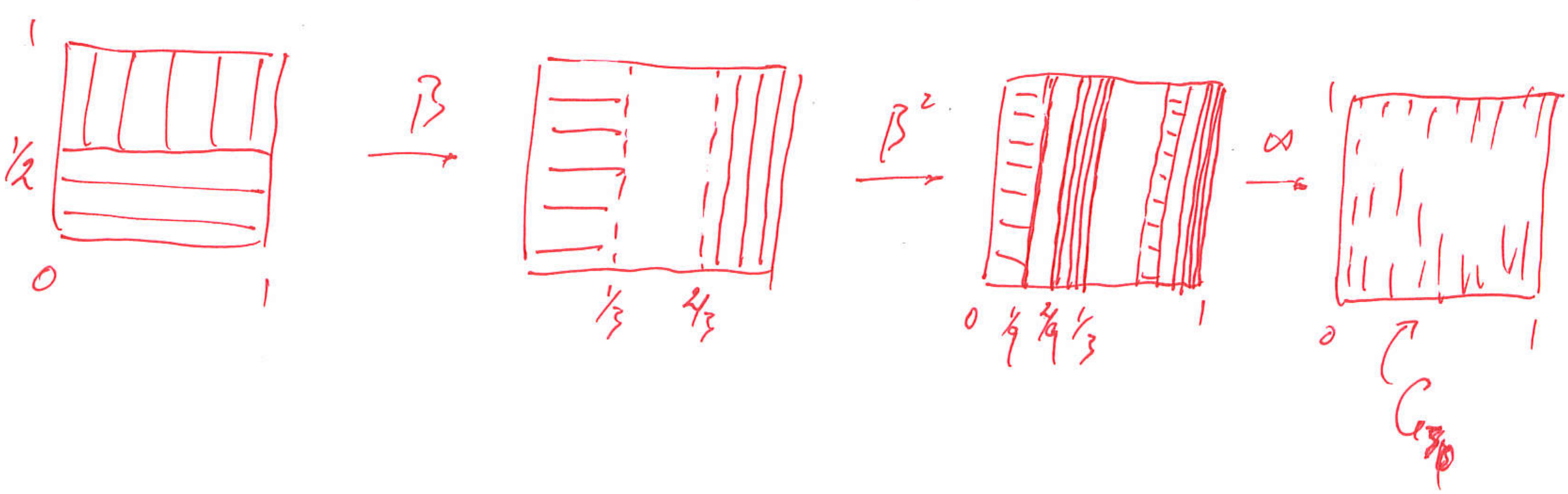
$$F(x) = \begin{cases} f_1(x) = \frac{x}{3} & p = \frac{1}{2} \\ f_2(x) = (2+x)/3 & p = \frac{1}{2} \end{cases}$$

↳ $C_3 = \lim_{n \to \infty} F^n(x) \quad \forall x \in [0,1]$.

We don't need randomness though to get "fractal" attractors

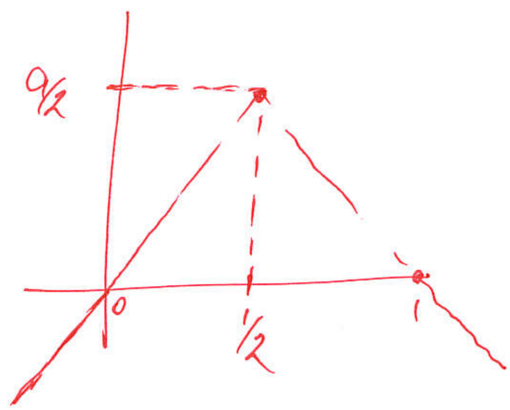
Skinny Baker Maps:

$$B(x,y) = \begin{cases} \left(\frac{1}{3}x, 2y\right) & 0 \le y \le \frac{1}{2} \\ \left(\frac{1}{3}x + \frac{2}{3}, 2y-1\right) & \frac{1}{2} < y \le 1 \end{cases}$$



In this deterministic vein, let us look at

$$T_a(x) = \begin{cases} ax & x \le \frac{1}{2} \\ a(1-x) & x > \frac{1}{2} \end{cases}$$



$0<a<1$: Not only stable, but globally attracting fixed point at $x=0$.

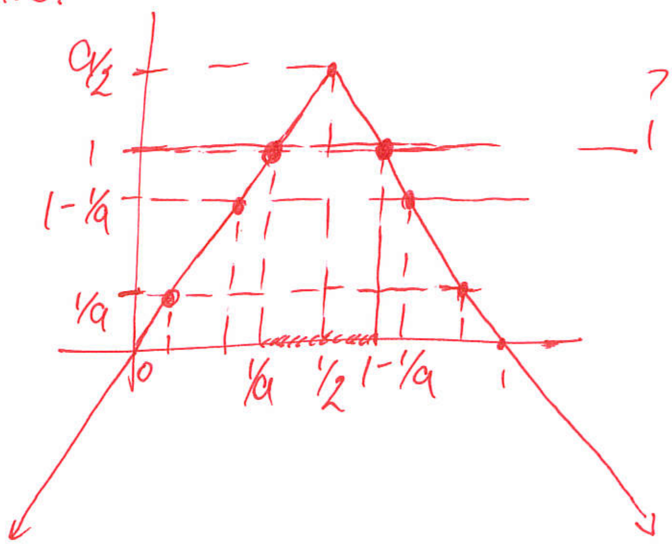
Note : if $x<0 \to T_a(x) < 0$

if $x>1 \to T_a(x) < 0$

So $1 \le a \le 2$, $[0,1] \to [0, \frac{a}{2}]$, while everyone else maps to $-\infty$ in the limit.

But what happens when $a>2$?

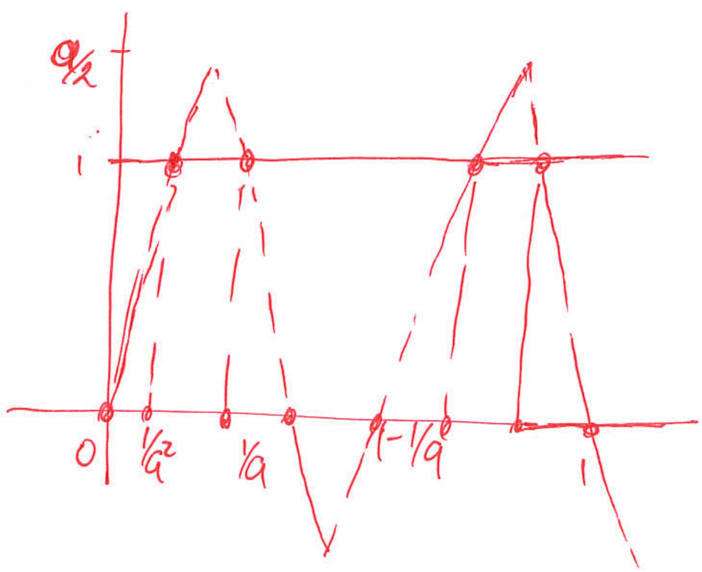
$a>2$:



The points $x_0 \in (\frac{1}{a}, 1-\frac{1}{a})$ map to $(1, \frac{a}{2})$

↳ so they escape to $(-\infty)$ in the limit.

↳ $T_a^2(x)$



Let $N_a = \{x \in \mathbb{R} : \lim_{n\to\infty} T_a^n(x) = -\infty\}$

↳ $N_3^c = C_3$.

Fatou's Theorem:

Let $P(z)$ be some polynomial map. Any attracting cycle "attracts" at least one critical point.

So, for $P_c(z) = z^2 + c \to P_c'(z) = 2z = 0 \to z = 0$.

Thus for $P_{-i}(z) = z^2 - i$

$\to P_{-i}(0) = -i$, $P_{-i}^2(0) = -1 - i \Rightarrow$ repellant?

$\to$ okay the book has me here...

Mandelbrot Set:

$$M = \{c : 0 \notin B_\infty \text{ for } P_c(z) = z^2 + c\}$$

Julia Set:

$J \equiv \partial B_\infty$ i.e. boundary of the basin of ∞.

Fractals in $\mathbb{C}$:

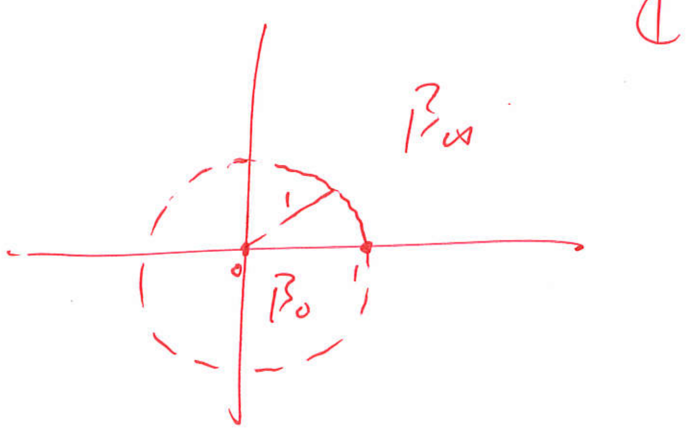
Look at $P_c(z) = z^2 + c$, $z \in \mathbb{C}$, $c \in \mathbb{C}$

$z = z_r + i z_i$, $|z| = (z_r^2 + z_i^2)^{1/2}$

Let $c = 0$: $P_c(z) = z \longrightarrow z = 0, 1$.

$z = 0$ is stable with basin of attraction $|z| < 1$

$|z| = 1$ is invariant under $P_0(z)$, and everything else goes to "∞"

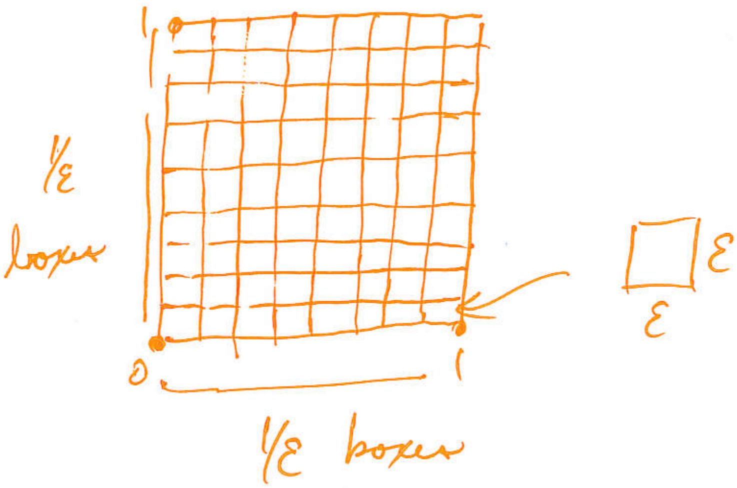


Now, what about more general fixed points for general c:

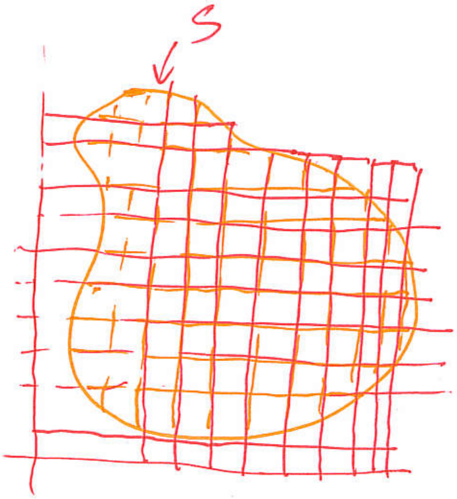
$z^2 + c = z$?

How to Measure the Size a Fractal Occupies

①



↳ N boxes $\simeq \frac{1}{\varepsilon^2}$ ***



→ $N(\varepsilon) = \frac{C}{\varepsilon^2}$, $N(\varepsilon)$ = # of grid boxes which S intersects.

and in general for d-dimensional objects

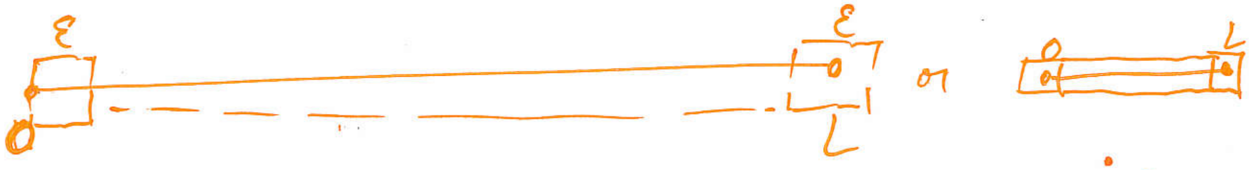
$$N(\varepsilon) = \frac{C}{\varepsilon^d}$$

↳ $\ln N(\varepsilon) = \ln C - d \ln \varepsilon$

(2)

↳ $d \simeq \dfrac{\ln C - \ln N(\varepsilon)}{\ln \varepsilon}$ or we can define

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln \varepsilon^{-1}} \quad \text{(Box Counting Dimension)}$$



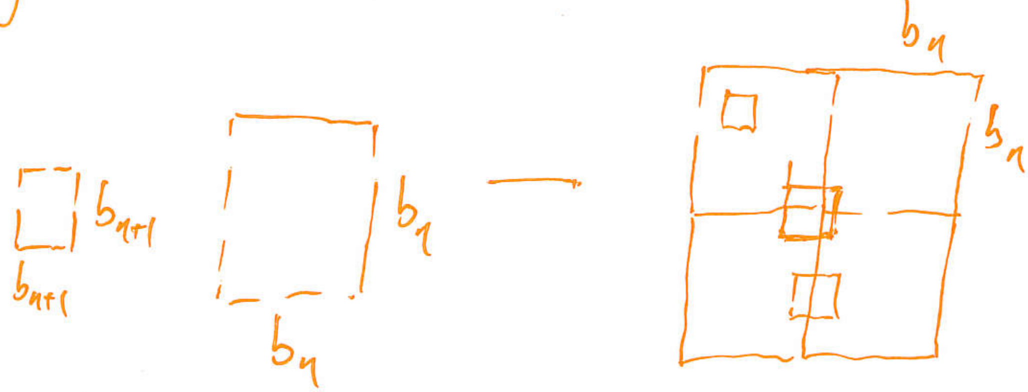
↳ $\varepsilon \cdot N(\varepsilon) \simeq L$ or $N(\varepsilon) \simeq L/\varepsilon$

i.e. we can shift the boxes about a bit.

so $d = \lim_{\varepsilon \to 0} \dfrac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = 1.$

Now we specify sequences $b_n \to 0$, $b_{n+1} < b_n$, $\lim_{n \to \infty} \dfrac{\ln b_{n+1}}{\ln b_n} = 1.$

Suppose for $0 < \varepsilon < 1$ $\exists n$: $b_{n+1} \leq \varepsilon < b_n$



Let some set S be covered by $N(b_{n+1})$ boxes

↳ for $b_{n+1} \leq \varepsilon < b_n \rightarrow N(\varepsilon) \leq 4N(b_{n+1})$

↳ $\dfrac{N(b_n)}{4} \leq N(\varepsilon) \leq 4N(b_{n+1})$

↳ in $\mathbb{R}^m \rightarrow \dfrac{N(b_n)}{2^m} \leq N(\varepsilon) \leq 2^m N(b_{n+1})$

↳ if $b_{n+1} \leq \varepsilon \rightarrow \dfrac{1}{\ln(1/b_{n+1})} \leq \dfrac{1}{\ln(1/\varepsilon)}$

↳ $\dfrac{\ln(2^{-m}N(b_n))}{\ln(1/b_{n+1})} \leq \dfrac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}$

$\varepsilon < b_n \rightarrow \dfrac{1}{\ln(1/\varepsilon)} < \dfrac{1}{\ln(1/b_n)}$

↳ $\dfrac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)} < \dfrac{\ln(2^m N(b_{n+1}))}{\ln(1/b_n)}$

multiply by $\frac{\ln(1/b_n)}{\ln(1/b_n)}$ on the left, $\frac{\ln(1/b_{n+1})}{\ln(1/b_{n+1})}$ on the right (4)

$\Rightarrow$ $d = \lim_{n\to\infty} \frac{\ln N(b_n)}{\ln(1/b_n)}$.

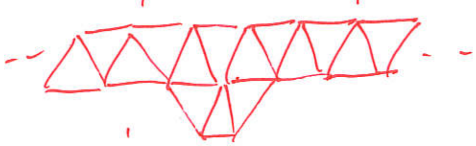
Note, we can also use $N_0(\varepsilon) = \{$ smallest # of boxes of size ε that intersect S $\}$

$\Rightarrow$ $N_0(\varepsilon) \le N(\varepsilon)$

but $N(\varepsilon) \le 2^m N_0(\varepsilon)$

$\Rightarrow$ $N_0(\varepsilon) \le N(\varepsilon) \le 2^m N_0(\varepsilon)$ $\rightarrow$ we can use either.

Can use other shapes: triangles, parallelograms, "spheres" (5)



Note analysis from before must change, but in principle should still hold.

Box counting dimension of $\mathcal{C}_3$:

0 ———————— 1 $n = 0$ (C_0)

0 — $1/3$ $2/3$ — 1 $n = 1$ (C_1)

0 — $1/9$ $2/9$ — $1/3$ $2/3$ — $7/9$ $8/9$ — 1 $n = 2$ (C_2)

⋮

"∞"

so at stage n, we have 2^n intervals of length $\frac{1}{3^n}$

since $\mathcal{C}_3 = \bigcap_{n=0}^{\infty} C_n \rightarrow \mathcal{C}_3 \subset C_n$

so we can set $N_0(b_n) = 2^n$; $1/b_n = 3^n$

$\Rightarrow$ $d = \lim_{n \to \infty} \frac{\ln N_0(b_n)}{\ln 1/b_n} = \frac{\ln 2}{\ln 3}$.

An Introduction to Measure

in $\mathbb{R}$, we say a set $U \subset \mathbb{R}$ is open if

$\forall x \in U \ \exists \varepsilon > 0 : N_\varepsilon(x) \subset U$ where $N_\varepsilon(x) = \{y \in \mathbb{R} \mid |x-y| < \varepsilon\}$.

if U is open, U^c is closed where $U^c = \mathbb{R} \setminus U$.

↳ open (a,b) :  ; $(a,b)^c = (-\infty, a] \cup [b, \infty)$

closed $[a,b]$:  ; $[a,b]^c = (-\infty, a) \cup (b, \infty)$

neither $[a,b)$ or $(a,b]$

let $\{U_j\}_{j=1}^\infty$ be a collection of open sets

↳ $\tilde{U} = \bigcup_{j=1}^\infty U_j$ is open, $\bar{U}_n = \bigcap_{j=1}^n U_j$ is open

but $\bar{U} = \bigcap_{j=1}^\infty U_j$ can be anything in principle.

likewise if C_j are closed

↳ $\tilde{C} = \bigcap_{j=1}^{\infty} C_j$ is also closed, $\bar{C}_n = \bigcup_{j=1}^{n} C_j$

but $\bar{C} = \bigcup_{j=1}^{\infty} C_j$ could in principle be anything.

this gets worse (or better...)

Call $\bar{U}$ a G_δ set, $\bar{C}$ a F_σ set.

$$G_\delta^c \to F_\sigma \; ; \; F_\sigma^c \to G_\delta .$$

↳ Can look at

$F_{\sigma\delta}$ = Sets that are countable intersections of F_σ sets

$G_{\delta\sigma}$ = Countable unions of G_δ sets

↳ $F_{\sigma\delta\sigma}$; $F_{\sigma\delta\sigma\delta}$; ....

$G_{\delta\sigma}$; $G_{\delta\sigma\delta}$; $G_{\delta\sigma\delta\sigma}$; ...

So what do we want out of (Lebesgue) measure?

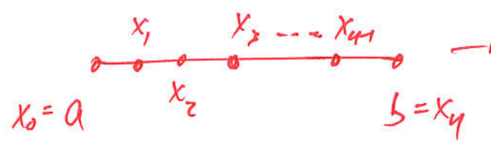
i) let $I = (a,b), [a,b), (a,b], [a,b]$

↳ $m(I) = b-a$.

ii) for $\{A_j\}_{j=1}^{\infty}$, $A_j \cap A_k = \phi$

↳ $m\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m(A_j)$

think about Riemann integration:

$\int_a^b f(x)\,dx \rightarrow$ 

$\rightarrow [a,b] = \bigcup_{j=0}^{n-1} [x_j, x_{j+1}) \cup \{b\}$.

iii) if $m(E)$ is defined, and $E+y = \{x = e+y, e \in E\}$

↳ $m(E+y) = m(E)$.

Given set $\mathfrak{X}$, let $\mathcal{Q}$ be a collection of subsets of $\mathfrak{X}$.

$\mathcal{Q}$ is an algebra | if $A \in \mathcal{Q} \Rightarrow A^c \in \mathcal{Q}$

| if $A, B \in \mathcal{Q} \Rightarrow A \cup B \in \mathcal{Q}$

$\hookrightarrow$ implies <u>finite</u> unions / intersections $\in \mathcal{Q}$.

Now let $\mathcal{C}$ be collection of subsets of $\mathfrak{X}$

$\exists\ \tilde{\mathcal{Q}}$ that is the smallest algebra which contains $\mathcal{C}$.

pf | Let $\{\mathcal{Q}_\gamma\}_{\gamma \in S}$ be the collection of all algebras which contain $\mathcal{C}$.

Let $\tilde{\mathcal{Q}} = \bigcap_\gamma \mathcal{Q}_\gamma$.

Clearly $\mathcal{C} \subset \tilde{\mathcal{Q}}$.

Let $A \in \tilde{\mathcal{Q}} \to A \in \mathcal{Q}_\gamma\ \forall \gamma \to A^c \in \mathcal{Q}_\gamma\ \forall \gamma \to A^c \in \mathcal{Q}$

$$A, B \in \tilde{\mathcal{A}} \to \underset{\forall \gamma}{A, B \in \mathcal{A}_\gamma} \to \underset{\forall \gamma}{A \cup B \in \mathcal{A}_\gamma} \to A \cup B \in \tilde{\mathcal{A}}. \qquad \boxtimes$$

An algebra is a σ-algebra if every countable union of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

i.e. $\{A_j\}_{j=1}^{\infty} \in \mathcal{A} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ $\left(m\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m(A_j) \right)$.

↳ again, if we have $\mathcal{G} \subset \mathfrak{X} \to \exists$ smallest σ-algebra that contains $\mathcal{G}$

in $\mathbb{R}$, let $\mathcal{G} = \{\mathcal{O}_\gamma\}_\gamma$, $\mathcal{O}_\gamma$ open

Let $\mathcal{B}_\sigma$ be the smallest σ-algebra contaiing $\mathcal{G}$

We call $\mathcal{B}_\sigma$ the Borel σ-algebra.

↳ $\mathcal{G}_\delta, \mathcal{F}_\sigma, \mathcal{G}_{\delta\sigma}, \mathcal{F}_{\sigma\delta}, \dots \in \mathcal{B}_\sigma$

↳ every other types of sets!

Outer Measure:

$A \subset \mathbb{R}$, $A \subset \bigcup_{j=1}^{\infty} I_j$, I_j open interval

let $m^*(A) = \inf_{A \subset (\cup I_j)} \sum_{j=1}^{\infty} \ell(I_j)$, $\ell(I_j) = |I_j|$ or length.

THM: $m^*(I) = \ell(I)$ if I is an interval (includes (a,∞)).

THM: $\{A_n\}_{n=1}^{\infty}$, $A_n \subset \mathbb{R}$ $\longrightarrow$ $m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} m^*(A_j)$

Note: not "=" !

To get "=", we need to work on measurable sets E:

$\forall A \subset \mathbb{R}$: $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

additive over disjoint sets.

$m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c)$ from before.

↳ need to show $m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c)$

THM: if $m^*(E) = 0 \to E$ is measurable.

pf| $A \cap E \subset E \to m^*(A \cap E) \le m^*(E) = 0 \to m^*(A \cap E) = 0$.

$A \cap E^c \subset A \to m^*(A) \ge m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c)$

↳ $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ $\boxtimes$

THM: E_1, E_2 measurable $\to E_1 \cup E_2$ measurable.

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap \underbrace{E_1^c \cap E_2^c}_{=(E_1 \cup E_2)^c})$$

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$$

↳ $m^*(A \cap (E_1 \cup E_2)) \le m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$

↳ $m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap E_1^c \cap E_2^c) \le m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap E_1^c \cap E_2^c)$

$\le m^*(A \cap E_1) + m^*(A \cap E_1^c) = m^*(A)$. $\boxtimes$