Another repeated eval:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$$

$\hookrightarrow$ $tr(A) = 4$, $det(A) = 4$

$\hookrightarrow$ $\left(\frac{tr(A)}{2}\right)^2 = 4 = det(A)$

So $\lambda = \frac{tr(A)}{2} = 2$

$\hookrightarrow$ $(A-2I)\vec{v} = 0 \rightarrow \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

So $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Now, as for $\vec{w}$: $(A-2I)\vec{w} = \vec{v} \rightarrow \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right)$

$\hookrightarrow \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$

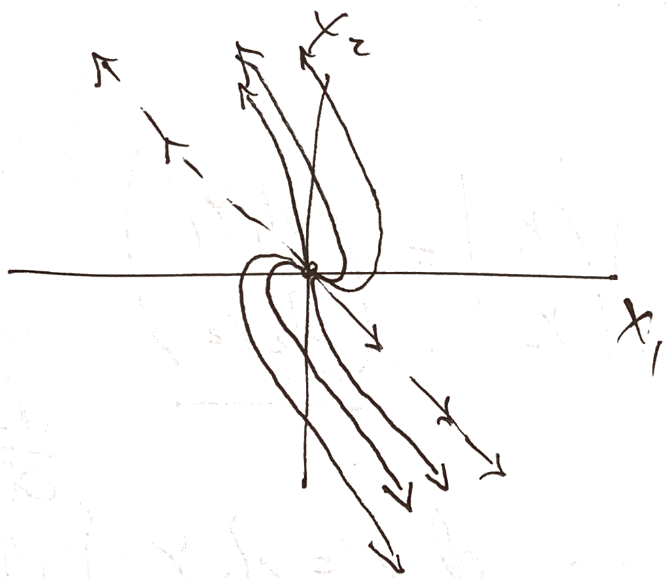
$\hookrightarrow$ $-\omega_1 - \omega_2 = 1$ $\hookleftarrow$ $\vec{\omega} = \begin{pmatrix} \omega_1 \\ -\omega_1 - 1 \end{pmatrix} = \omega_1 \vec{v} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

↳ $\vec{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

↳ $A = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}^{-1}$

and we likewise have

$$\vec{x}(t) = c_1 \vec{v} e^{2t} + c_2 (t\vec{v} + \vec{w}) e^{2t} \; ; \; \underline{\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

↳ 

Beyond $\frac{d\vec{x}}{dt} = A\vec{x}$:

Constant Coefficient, Inhomogeneous :

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

Two ways to look at this

i) let $A = V\Lambda V^{-1}$ or $V\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}V^{-1}$

$\hookrightarrow \frac{d\vec{x}}{dt} = V\Lambda V^{-1}\vec{x} + \vec{b}$

$\hookrightarrow \frac{d\vec{y}}{dt} = \Lambda\vec{y} + V^{-1}\vec{b}$; $\vec{y} = V^{-1}\vec{x}$.

$= \Lambda\vec{y} + \vec{c}$; $\vec{c} = V^{-1}\vec{b}$

$\hookrightarrow \frac{dy_1}{dt} = \lambda_1 y_1 + c_1$; $\frac{dy_2}{dt} = \lambda_2 y_2 + c_2$

$\hookrightarrow \frac{d}{dt}(y_1 e^{-\lambda_1 t}) = c_1 e^{-\lambda_1 t}$; $\frac{d}{dt}(y_2 e^{-\lambda_2 t}) = c_2 e^{-\lambda_2 t}$

ii) focus on equilibrium: $\frac{d\vec{x}_*}{dt} = 0$

$\hookrightarrow A\vec{x}_* + \vec{b} = 0 \rightarrow A\vec{x}_* = -\vec{b}$, so if A^{-1} exists

$\hookrightarrow \vec{x}_* = -A^{-1}\vec{b}$

So now let $\vec{x} = \vec{x}_* + \vec{y}$

$$\hookrightarrow \frac{d\vec{x}}{dt} = \frac{d\vec{y}}{dt}$$

and $A\vec{x} + \vec{b} = A(\vec{x}_* + \vec{y}) + \vec{b} = A\vec{x}_* + \vec{b} + A\vec{y}$

$= A\vec{y}$

$$\hookrightarrow \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b} \Rightarrow \frac{d\vec{y}}{dt} = A\vec{y}$$

$\hookrightarrow$ if $A = V\Lambda V^{-1} \longrightarrow$

$$\vec{x}(t) = \vec{x}_* + c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

Example:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

↳ find $\vec{x}_*$:

$$\left(\begin{array}{cc|c} -3 & 4 & -1 \\ 4 & 3 & -1 \end{array}\right)$$

$$R_2 + \tfrac{4}{3}R_1 \rightarrow \left(\begin{array}{cc|c} -3 & 4 & -1 \\ 0 & \frac{25}{3} & -7/3 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -3 & 4 & -1 \\ 0 & 1 & -7/25 \end{array}\right)$$

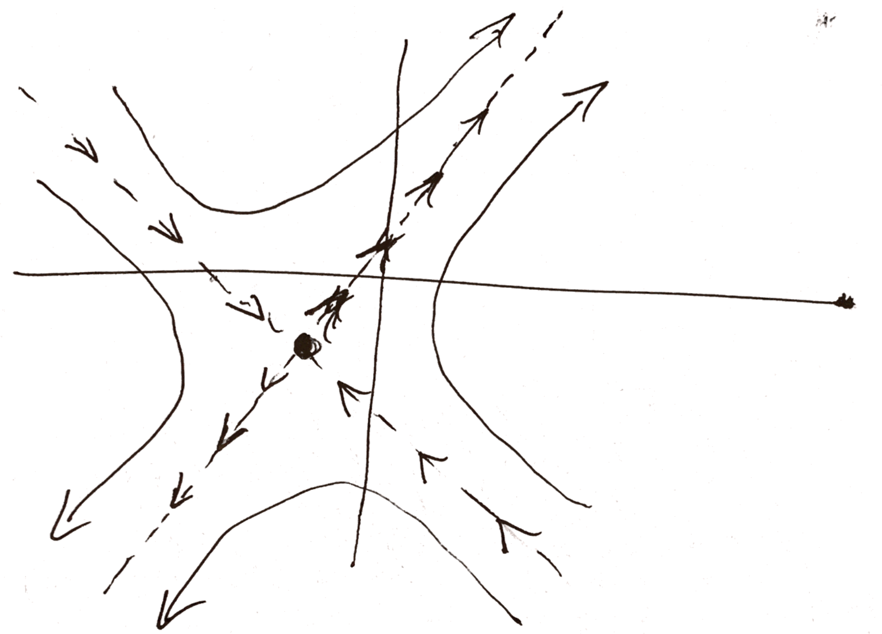
$$R_1 - 4R_2 \rightarrow \left(\begin{array}{cc|c} -3 & 0 & 3/25 \\ 0 & 1 & -7/25 \end{array}\right)$$

$$\hookrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1/25 \\ 0 & 1 & -7/25 \end{array}\right)$$

$$\hookrightarrow \vec{x}_* = \begin{pmatrix} -1/25 \\ -7/25 \end{pmatrix}$$

Now: $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$

↳ $\vec{x}(t) = \begin{pmatrix} -1/25 \\ -7/25 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$

↳ 

So why did we talk about the first approach at all?

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}(t)$$

↳ if $A = V \Lambda V^{-1} \rightarrow \vec{c}(t) = V^{-1} \vec{b}(t)$

↳ $\frac{d}{dt}\left(y_1 e^{-\lambda_1 t}\right) = c_1(t) e^{-\lambda_1 t}$; $\frac{d}{dt}\left(y_2 e^{-\lambda_2 t}\right) = c_2(t) e^{-\lambda_2 t}$