Variation of Parameters (4.7)

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$$\frac{d\vec{x}}{dt} = P(t)\vec{x}(t) + \vec{g}(t) \; ; \quad \cancel{x}(0) = x_0$$

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Step I: Solve homogeneous problem

$$\Rightarrow \boxed{\frac{d\vec{x}}{dt} = P(t)\vec{x}(t) \; ; \quad x(0) = x_0}$$

note, from Abel's THM: $W(t) = W(0)e^{\int_0^t tr(P)ds}$

we can hopefully get a sense of where we expect linearly independent solutions.

Assuming we can find region of t around $t=0$ such that $W(t) \neq 0 \Rightarrow$

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t), \quad W(t) = \det\left(\vec{x}_1(t) \,\middle|\, \vec{x}_2(t)\right)$$

or we have $\bar{\bar{X}}_h(t) = \left(\vec{x}_1(t) \,|\, \vec{x}_2(t)\right)$

so that $\vec{x}(t) = \bar{\bar{X}}_h(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \bar{\bar{X}}_h(t)\vec{c}$

↳ $\vec{x}_0 = \bar{\bar{X}}_h(0)\vec{c} \rightarrow \vec{c} = \underline{\bar{\bar{X}}_h^{-1}(0)\vec{x}_0}$

so $\boxed{\vec{x}_h(t) = \bar{\bar{X}}_h(t)\bar{\bar{X}}_h^{-1}(0)\vec{x}_0}$

Now, how to build $\vec{x}_p(t)$? $\longrightarrow$ $\underline{\vec{x}_p(0) = \vec{0}}$

We suppose:

$$\vec{x}_p(t) = u(t)\vec{x}_1(t) + v(t)\vec{x}_2(t)$$

$$= \bar{\bar{X}}_h(t) \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \bar{\bar{X}}_h(t)\vec{u}(t)$$

↳ $\dfrac{d}{dt}\vec{x}_p = \left(\dfrac{d}{dt}\bar{\bar{X}}_h\right)\vec{u} + \bar{\bar{X}}_h\left(\dfrac{d\vec{u}}{dt}\right)$

So show:

$$\frac{d}{dt}\bar{X}_h = P\bar{X}_h \;; \quad \bar{X}_h(t) = \left(\vec{x}_1(t) \,\middle|\, \vec{x}_2(t)\right)$$

pf: $\quad \frac{d\vec{x}_1}{dt} = P\vec{x}_1 \;; \quad \frac{d\vec{x}_2}{dt} = P\vec{x}_2$

$$\hookrightarrow \frac{d}{dt}\left(\vec{x}_1 \,|\, \vec{x}_2\right) = \left(P\vec{x}_1 \,|\, P\vec{x}_2\right)$$

$$= P\left(\vec{x}_1 \,|\, \vec{x}_2\right)$$

or $\frac{d}{dt}\bar{X}_h = P\bar{X}_h$. $\boxed{}$

So then $\vec{x}_p = \bar{X}_h \vec{u}$

$$\frac{d}{dt}\vec{x}_p = P\bar{X}_h\vec{u} + \bar{X}_h(t)\frac{d\vec{u}}{dt}$$

Showed this

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and we want:

$$\frac{d}{dt}\vec{x}_p(t) = P\vec{x}_p + \vec{g}(t) \;; \quad \vec{x}_p(0) = \vec{0}$$

Want this

So pulling our results together, we get: (4)

$$P\vec{x}_p + \bar{X}_h \frac{d\vec{u}}{dt} = P\vec{x}_p + \vec{g}(t)$$

So $\dfrac{d\vec{u}}{dt} = \bar{X}_h^{-1}(t)\,\vec{g}(t)$, $\quad W(t) = \det \bar{X}_h(t)$

$W(t) \neq 0 \Leftrightarrow \bar{X}_h^{-1}(t)$ exists.

or $\vec{u}(t) = \displaystyle\int_0^t \bar{X}_h^{-1}(s)\,\vec{g}(s)\,ds$

and $\vec{x}_p(t) = \bar{X}_h(t) \displaystyle\int_0^t \bar{X}_h^{-1}(s)\,\vec{g}(s)\,ds$

If we let $\bar{X}_h(t) = \left(\vec{x}_1(t) \,\middle|\, \vec{x}_2(t)\right) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$; $\vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

Show

$$\vec{x}_p(t) = \bar{X}_h(t) \int_0^t \frac{1}{W(s)} \begin{pmatrix} x_{22}(s)g_1(s) - x_{12}(s)g_2(s) \\ -x_{21}(s)g_1(s) + x_{11}(s)g_2(s) \end{pmatrix} ds$$

So to see this, we note:

(5)

$$\bar{X}_u^{-1}(s) = \frac{1}{\det \bar{X}_u} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} = \frac{1}{W(s)} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$$

so $\bar{X}_u^{-1}(s) = \frac{1}{W(s)} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

$$= \frac{1}{W(s)} \begin{pmatrix} x_{22} g_1 - x_{12} g_2 \\ -x_{21} g_1 + x_{11} g_2 \end{pmatrix}$$

Suppose we ~~started~~ w/

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = f(t)$$

then let $y = \frac{dx}{dt}$ →

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

so now $\vec{g} = \begin{pmatrix} 0 \\ f \end{pmatrix} \to g_1 = 0,\ g_2 = f$

and $X_h = \begin{pmatrix} x_1 & x_2 \\ dx_1/dt & dx_2/dt \end{pmatrix}$

to find the 1^{st} component of $\vec{x}_p(t)$ from our formula.