So back to the undecoupled case:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^* \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda^* t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^* \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}; \quad \lambda = -\mu + i\omega$$

or if we just want the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \lambda \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} 1 \\ \lambda^* \end{pmatrix} e^{\lambda^* t}$$

Though: $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^* \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

now in real variables:

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -\mu \end{pmatrix} + i \begin{pmatrix} 0 \\ \omega \end{pmatrix}; \quad e^{\lambda t} = e^{-\mu t} \cos(\omega t) + i e^{-\mu t} \sin(\omega t)$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tilde{c}_1 e^{-\mu t} \left(\cos(\omega t) \begin{pmatrix} 1 \\ -\mu \end{pmatrix} - \sin(\omega t) \begin{pmatrix} 0 \\ \omega \end{pmatrix} \right)$$

$$+ \tilde{c}_2 e^{-\mu t} \left(\cos(\omega t) \begin{pmatrix} 0 \\ \omega \end{pmatrix} + \sin(\omega t) \begin{pmatrix} 1 \\ -\mu \end{pmatrix} \right)$$

so if we drop y $\left(= \frac{dx}{dt}\right)$ then

$$x(t) = e^{-\mu t}\left(\tilde{c}_1 \cos(\omega t) + \tilde{c}_2 \sin(\omega t)\right)$$

$$= \sqrt{\tilde{c}_1^{\,2} + \tilde{c}_2^{\,2}}\, e^{-\mu t}\left(\frac{\tilde{c}_1}{\sqrt{\tilde{c}_1^{\,2}+\tilde{c}_2^{\,2}}} \cos(\omega t) + \frac{\tilde{c}_2}{\sqrt{\tilde{c}_1^{\,2}+\tilde{c}_2^{\,2}}} \sin(\omega t)\right)$$

let $\cos(\varphi) = \dfrac{\tilde{c}_1}{\sqrt{\tilde{c}_1^{\,2}+\tilde{c}_2^{\,2}}}$; $\sin(\varphi) = \dfrac{\tilde{c}_2}{\sqrt{\tilde{c}_1^{\,2}+\tilde{c}_2^{\,2}}}$

using $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

damping/attenuation

so $x(t) = \sqrt{\tilde{c}_1^{\,2}+\tilde{c}_2^{\,2}}\, e^{-\mu t} \cos(\omega t - \varphi)$

$$\tan(\varphi) = \tilde{c}_2 / \tilde{c}_1$$

Phase shift

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frequency of oscillation

Now on to forcing:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f/a \end{pmatrix}$$

$$\text{w/ } \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

of course, 1st solve homogeneous problem:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}; \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

so long as $b^2 - 4ac \neq 0$

$$\Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The solution to the forced problem is then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\underbrace{+ \int_0^t \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 (t-s)} & 0 \\ 0 & e^{\lambda_2 (t-s)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{f(s)}{a} \end{pmatrix} ds}_{= \begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} \text{ or "particular solution"}}$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{f(s)}{a} \end{pmatrix} = \frac{f(s)}{a(\lambda_2 - \lambda_1)} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 (t-s)} & 0 \\ 0 & e^{\lambda_2 (t-s)} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} -e^{\lambda_1 (t-s)} \\ e^{\lambda_2 (t-s)} \end{pmatrix} = \begin{pmatrix} -e^{\lambda_1 (t-s)} + e^{\lambda_2 (t-s)} \\ -\lambda_1 e^{\lambda_1 (t-s)} + \lambda_2 e^{\lambda_2 (t-s)} \end{pmatrix}$$

So

$$\begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} = \frac{1}{a(\lambda_2-\lambda_1)} \int_0^t \begin{pmatrix} f(s)\left(e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}\right) \\ f(s)\left(\lambda_2 e^{\lambda_2(t-s)} - \lambda_1 e^{\lambda_1(t-s)}\right) \end{pmatrix} ds$$

$$= \frac{1}{a(\lambda_2-\lambda_1)} \left\{ \left(\int_0^t f(s) e^{\lambda_2(t-s)} ds\right) \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} - \left(\int_0^t f(s) e^{\lambda_1(t-s)} ds\right) \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \right\}$$

So only two things to compute:

$$\int_0^t f(s) e^{\lambda_2(t-s)} ds \quad \text{and} \quad \int_0^t f(s) e^{\lambda_1(t-s)} ds.$$

① overdamped: $b^2 - 4ac > 0$

let $f(s) = \cos(\sigma s)$, $\lambda_1, \lambda_2 < 0$.

$\hookrightarrow$ $$\int_0^t \cos(\sigma s) e^{\lambda(t-s)} ds = e^{\lambda t} \int_0^t \cos(\sigma s) e^{-\lambda s} ds.$$

$$\int_0^t \cos(\sigma s) e^{-\lambda s} ds = -\frac{1}{\lambda} \cos(\sigma s) e^{-\lambda s} \Big|_0^t - \frac{\sigma}{\lambda} \int_0^t \sin(\sigma s) e^{-\lambda s} ds$$

$$= -\frac{1}{\lambda} \cos(\sigma s) e^{-\lambda s} \Big|_0^t - \frac{\sigma}{\lambda} \left\{ -\frac{1}{\lambda} \sin(\sigma s) e^{-\lambda s} \Big|_0^t + \frac{\sigma}{\lambda} \int_0^t \cos(\sigma s) e^{-\lambda s} ds \right\}.$$

$\hookrightarrow$

$$\left(1 + \frac{\sigma^2}{\lambda^2}\right) \int_0^t \cos(\sigma s) e^{-\lambda s} ds = \frac{1}{\lambda} \left(1 - \cos(\sigma t) e^{-\lambda t}\right) + \frac{\sigma}{\lambda^2} \sin(\sigma t) e^{-\lambda t}$$

Lo

$$e^{\lambda t}\int_0^t \cos(\sigma s)e^{-\lambda s}\,ds = \frac{\lambda}{\lambda^2+\sigma^2}e^{\lambda t}$$

decaying transient since $\lambda < 0$

$$+\frac{\sigma}{\lambda^2+\sigma^2}\sin(\sigma t) - \frac{\lambda}{\lambda^2+\sigma^2}\cos(\sigma t)$$

sustained oscillation

$$= \frac{\lambda}{\lambda^2+\sigma^2}e^{\lambda t} - \sqrt{\lambda^2+\sigma^2}\cos(\sigma t + \tilde{\varphi}_f)$$

$$\tan(\tilde{\varphi}_f) = \frac{\sigma}{\lambda}.$$

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forcing frequency drives system but natural frequency of circuit introduces amplitude modulation and phase lag.