Gradescope Assignment: Due 2/24/21
0 pts for no work
2 pts for attempt
4 pts for full answer

By

$$A\mathbf{x} = \mathbf{0},$$

we mean the matrix problem

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which can be written in the augmented form

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \end{array} \right)$$

or component-by component so that

$$a_{11}x_1 + a_{12}x_2 = 0,\ a_{21}x_1 + a_{22}x_2 = 0.$$

We define the determinant of A, $\det(A)$, to be

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

1. (Short) Show that if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution (i.e. x_1 or x_2 is not zero), then $\det(A) = 0$. Your solution should address the two cases:
   (a) $x_1 \neq 0$, so that $a_{11} = -a_{12}x_2/x_1$ and $a_{21} = -a_{22}x_2/x_1$
   (b) $x_2 \neq 0$, so that $a_{12} = -a_{11}x_1/x_2$ and $a_{22} = -a_{21}x_1/x_2$
2. (Short(ish)) Show that if $\det(A) = 0$ then $A\mathbf{x} = 0$ has an infinite number of nontrivial solutions. You should address the following cases:
   (a) $a_{11} \neq 0$, which leads to the row-reduced echelon matrix problem

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & 0 \end{array} \right)$$

   (b) $a_{21} \neq 0$, which leads to the row-reduced matrix problem

$$\left(\begin{array}{cc|c} 0 & \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{21}} & 0 \\ a_{21} & a_{22} & 0 \end{array} \right)$$

   (c) $a_{21} = 0$ and $a_{11} = 0$, which leads to the augmented matrix problem

$$\left(\begin{array}{cc|c} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \end{array} \right)$$