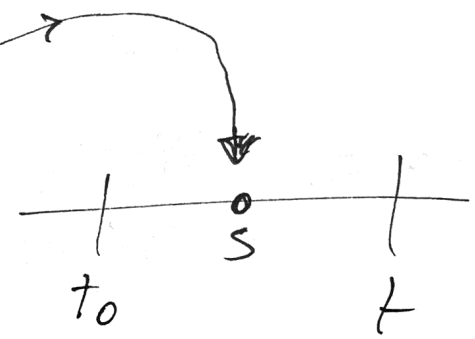


A Note on Integration.

(1)

$$\frac{du}{dt} = g(t), \quad u(t_0) = u_0$$

$$\hookrightarrow du = g(t) dt$$

$$\hookrightarrow \int_{u_0}^{u(t)} d\tilde{u} = \int_{t_0}^t g(s) ds$$


The diagram shows a horizontal line representing the real number line. There are three tick marks labeled t_0 , s , and t from left to right. A curved arrow starts at t_0 and points to t , passing above the line. Another curved arrow starts at t and points back to t_0 , passing below the line. A small circle is drawn around the point s on the line, with an arrow pointing down to it from the word 'or' in the text below.

$$\text{or: } t_0 \leq s \leq t$$

Integration

Variable must be different from bounds.

$$\hookrightarrow u(t) - u_0 = \int_{t_0}^t g(s) ds \quad \left. \begin{array}{l} \text{Initial-Value} \\ \text{taken into account.} \end{array} \right\}$$

$$\hookrightarrow u(t_0) - u_0 = \int_{t_0}^{t_0} g(s) ds = 0$$

$$\frac{d\vec{x}}{dt} = A\vec{x} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}; \quad \vec{x}(0) = \vec{x}_0$$

$$A = V \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} V^{-1}$$

$$\vec{y} = V^{-1}\vec{x}$$

$$L_1 \left[\frac{d\vec{y}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \right]; \quad \vec{c} = V^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Details

$$L_2 \quad \frac{d\vec{x}}{dt} = V \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} V^{-1}\vec{x} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$L_3 \quad V^{-1} \frac{d\vec{x}}{dt} = \underset{=I}{V^{-1}V} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} V^{-1}\vec{x} + V^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{d}{dt}(V^{-1}\vec{x}) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} V^{-1}\vec{x} + V^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(2)

$$\hookrightarrow \frac{d}{dt} y_1 = 1y_1 + y_2 + c_1$$

$$\frac{d}{dt} y_2 = 1y_2 + c_2 \quad \leftarrow y_2 \text{ eqn is ind. of } y_1$$

$$\hookrightarrow \frac{d}{dt} (y_2 e^{-\lambda t}) = c_2(t) e^{-\lambda t}$$

$$\hookrightarrow y_2(t) e^{-\lambda t} - y_2(0) = \int_0^t c_2(s) e^{-\lambda s} ds$$

$$y_2(t) = y_2(0) e^{-\lambda t} + \int_0^t c_2(s) e^{-\lambda(t-s)} ds$$

$$\boxed{y_2(t) = \underline{y_2(0)} e^{-1t} + \int_0^t c_2(s) e^{-1(t-s)} ds} \quad (3)$$

$$\frac{dy_1}{dt} = 1 y_1 + \downarrow y_2(t) + q(t)$$

$$\begin{aligned} \hookrightarrow \frac{d}{dt} (y_1 e^{-1t}) &= (y_2(t) + q(t)) e^{-1t} \\ &= y_2(0) + \int_0^t c_2(s) e^{-1s} ds + q(t) e^{-1t} \end{aligned}$$



$$\begin{aligned} y_1(t) e^{-1t} - y_1(0) &= y_2(0) t + \int_0^t \int_0^u c_2(s) e^{-1s} ds du \\ &\quad + \int_0^t q(s) e^{-1s} ds \end{aligned}$$

$$\begin{aligned} \hookrightarrow y_1(t) &= y_1(0) e^{-1t} + y_2(0) t e^{-1t} + \int_0^t q(s) e^{-1(t-s)} ds \\ &\quad + \int_0^t \int_0^u c_2(s) e^{-1(t-s)} ds du \end{aligned}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} c_1(s) \\ c_2(s) \end{pmatrix} e^{\lambda(t-s)} ds$$

"Like (4) with like"

$$+ \int_0^t \int_0^u \begin{pmatrix} c_2(s) \\ 0 \end{pmatrix} e^{-\lambda(t-s)} ds du$$

$$\bar{x} = V \bar{y}, \quad \bar{y} = \underline{V^{-1} \bar{x}}; \quad \bar{c} = \underline{V^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}$$

$$\bar{x} = V \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} V^{-1} \bar{x}_0 + \int_0^t \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} e^{-\lambda(t-s)} ds$$

$$+ V \int_0^t \int_0^u \begin{pmatrix} c_2(s) \\ 0 \end{pmatrix} e^{-\lambda(t-s)} ds du$$

$$\begin{pmatrix} C_2(s) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V^{-1} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix}$$

$$\begin{aligned} \bar{x}' &= e^{\lambda t} V \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} V^{-1} \bar{x}_0 + \int_0^t \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} e^{-\lambda(t-s)} ds \\ &\quad + \int_0^t \int_0^u V \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V^{-1} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} e^{-\lambda(t-s)} ds du \end{aligned}$$

$$= e^{\lambda t}$$

$$= e^{\lambda t} V \left(I + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) V^{-1} \bar{x}_0$$

$$= e^{\lambda t} \left(I + t V \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V^{-1} \right) \bar{x}_0$$