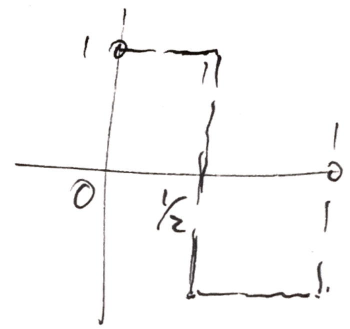
Periodic Functions: $f(t)$ is periodic of period T if

$$f(t+T) = f(t) \quad \forall t \ (\geq 0 \text{ in our case})$$

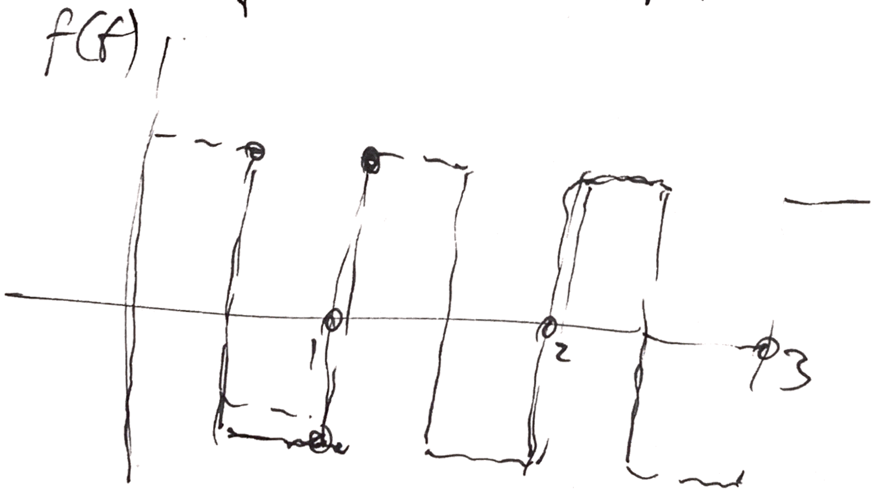
Ex $\cos(t+2\pi) = \cos(t)$; $\sin(t+2\pi) = \sin(t)$

Say $\tilde{f}(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ -1 & \frac{1}{2} < t \leq 1 \end{cases}$ $\rightarrow$



Let $f(t) = \sum_{j=0}^{\infty} \tilde{f}(t-j)\, H_{(j,j+1)}(t)$

Ex



In general then we can think of any periodic function this way:

if $f(t) = f(t+T)$, let $\tilde{f}(t) = f(t)\, H_{(0,T)}(t)$

so $f(t) = \sum_{j=0}^{\infty} \tilde{f}(t - jT)\, H_{(jT,(j+1)T)}(t)$

That said, when we want to find

$\mathcal{L}\{f(t)\}$ for $f(t+T) = f(t)$, I suggest you do:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = \sum_{j=0}^{\infty} \int_{jT}^{(j+1)T} e^{-st} f(t)\, dt$$

$\tilde{t} = t - jT$ or $t = \tilde{t} + jT$

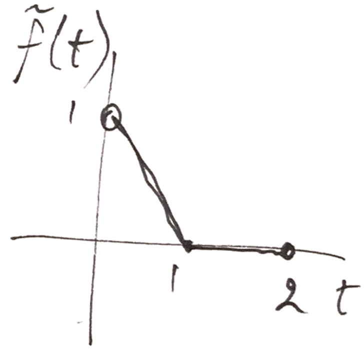
$$= \sum_{j=0}^{\infty} e^{-sjT} \int_0^{T} e^{-s\tilde{t}} f(\tilde{t} + jT)\, d\tilde{t}$$

but $f(\tilde{t} + jT) = f(\tilde{t})$

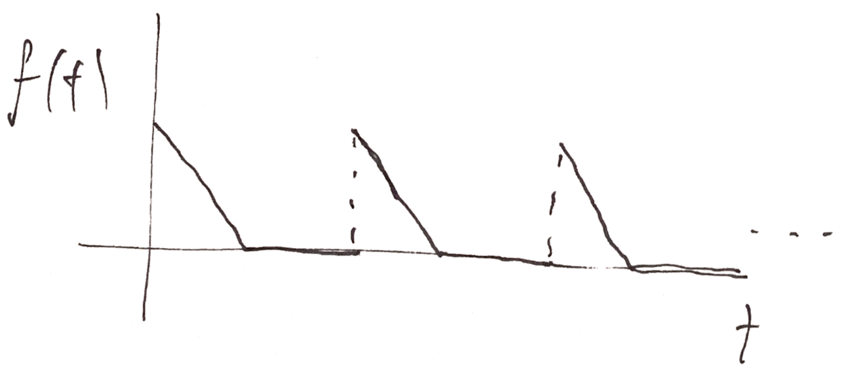
↳ $\mathcal{L}\{f\} = \left(\int_0^T e^{-s\tilde{t}} f(\tilde{t})\, d\tilde{t} \right) \sum_{j=0}^{\infty} \left(e^{-sT}\right)^j$

$$= \left(\frac{1}{1-e^{-sT}} \right) \cdot \int_0^T e^{-st} f(t)\, dt$$

Ex: $\tilde{f}(t) = \begin{cases} 1-t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$



↳ $f(t) = \sum_{j=0}^{\infty} \tilde{f}(t-2j)\, H_{(2j,\,2(j+1))}(t)$



But $\mathcal{L}\{f\} = \dfrac{1}{1-e^{-2s}} \displaystyle\int_0^2 e^{-st} \tilde{f}(t)\,dt$

$$= \frac{1}{1-e^{-2s}} \int_0^1 e^{-st} (1-t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left\{ -\frac{1}{s}(1-t)e^{-st}\Big|_0^1 - \frac{1}{s}\int_0^1 e^{-st}\,dt \right.$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right) \right\}$$