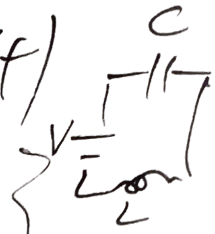
$$x_p(t) = \int_0^t f(s)\sin(t-s)\,ds$$

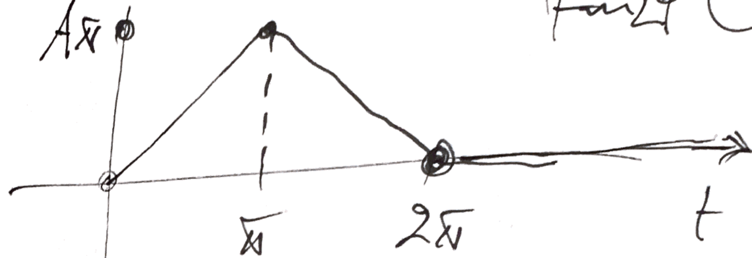
$\cancel{y''+y = f''(t)}$

$$x'' + x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$



$$f(s) = \begin{cases} As, & 0 \le s \le \pi \\ A(2\pi - s), & \pi \le s \le 2\pi \\ 0, & 2\pi < s \end{cases}$$

↳



Note: $x_p(0) = 0$

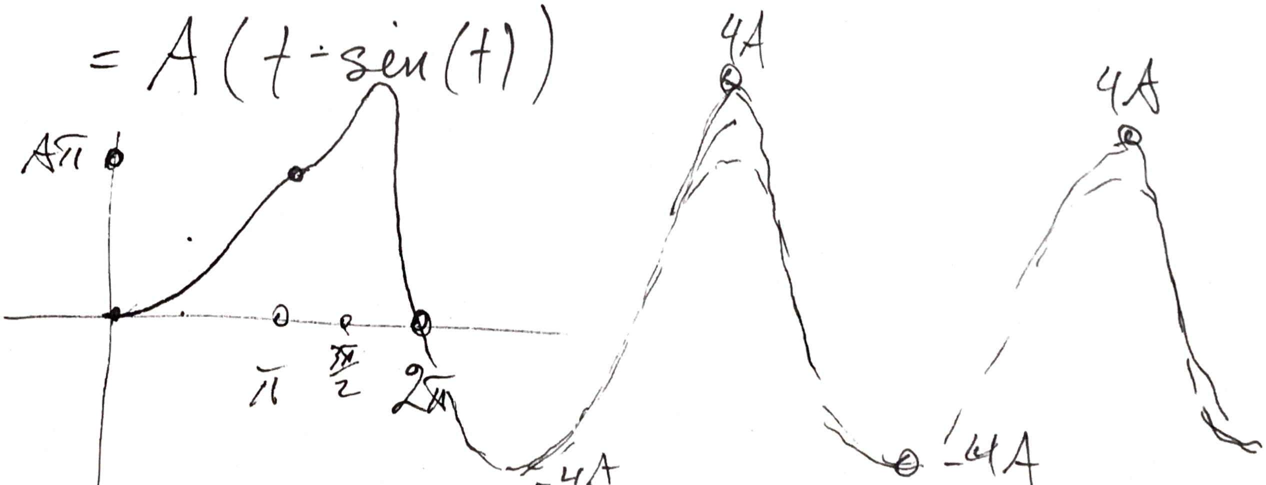
$$\frac{dx_p}{dt} = \int_0^t f(s)\cos(t-s)\,ds \rightarrow \left.\frac{dx_p}{dt}\right|_{t=0} = 0.$$

So, for $0 \le t \le \pi$:

↳ $x_p(t) = \int_0^t As\sin(t-s)\,ds$

$$= A\left\{ s\cos(t-s)\Big|_{s=0}^{s=t} + \sin(t-s)\Big|_{s=0}^{s=t} \right\}$$

$$= A(t - \sin(t))$$



For $\pi < t \leq 2\pi$: $x_p(t) = A\int_0^{\pi} s\sin(t-s)\,ds + A\int_{\pi}^{t}(2\pi-s)\sin(t-s)\,ds$ (2)

$$x_p(t) = A\left\{ s\cos(t-s)\Big|_{s=0}^{s=\pi} + \sin(t-s)\Big|_{s=0}^{s=\pi} \right\}$$

$$+A\left\{ (2\pi-s)\cos(t-s)\Big|_{s=\pi}^{s=t} - \sin(t-s)\Big|_{s=\pi}^{s=t} \right\}$$

$$= A\left\{ \pi\cos(t-\pi) + \sin(t-\pi) - \sin(t) \right\}$$

$$+ A\left\{ (2\pi - t) - \pi\cos(t-\pi) + \sin(t-\pi) \right\}$$

$$= A\left\{ 2\pi - t + 2\sin(t-\pi) - \sin(t) \right\}$$

$$= A\left\{ 2\pi - t - 3\sin(t) \right\}.$$

For $t > 2\pi$: $x_p(t) = A\int_0^{\pi} s\sin(t-s)\,ds + A\int_{\pi}^{2\pi}(2\pi-s)\sin(t-s)\,ds$

$$x_p(t) = A\left\{ \pi\cos(t-\pi) + \sin(t-\pi) - \sin(t) \right\}$$

$$+A\left\{ -\pi\cos(t-\pi) - \sin(t-2\pi) + \sin(t-\pi) \right\}$$

$$= -4A\sin(t)$$

$$x_p(t) = \frac{1}{\upsilon^2-\omega^2}\left(\cos(\omega t) - \cos(\upsilon t)\right)$$

let $\alpha = \frac{1}{2}(\omega+\upsilon)$; $\beta = \frac{1}{2}(\omega-\upsilon)$

$$\omega = \alpha+\beta; \quad \upsilon = \alpha-\beta$$

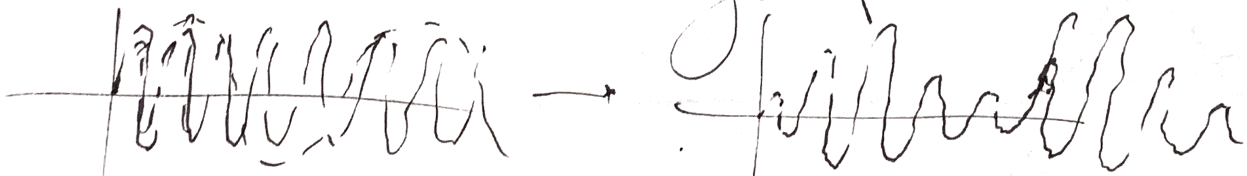
$$\cos((\alpha+\beta)t) - \cos((\alpha-\beta)t)$$

$$= \cos(\alpha t)\cos(\beta t) - \sin(\alpha t)\sin(\beta t)$$

$$- \left(\cos(\alpha t)\cos(\beta t) + \sin(\alpha t)\sin(\beta t)\right)$$

$$= \frac{2}{\omega^2-\upsilon^2}\sin\left(\tfrac{1}{2}(\omega+\upsilon)t\right)\sin\left(\tfrac{1}{2}(\omega-\upsilon)t\right)$$

$$= \underbrace{\frac{\sin\left(\frac{(\omega-\upsilon)}{2}t\right)}{\left(\frac{\omega-\upsilon}{2}\right)}}_{\text{"slow"}} \quad \underbrace{\frac{\sin\left(\frac{1}{2}(\omega+\upsilon)t\right)}{\omega+\upsilon}}_{\text{fast}}$$



↳ $L(y) = y'' + y = g$

$y(0) = y_0, \; y'(0) = y_1$

↳ Homogeneous:

$L(u) = 0$

$u(0) = y_0, \; u'(0) = y_1$

↳

$u(t) = c_1 \cos(t) + c_2 \sin(t)$

$u(0) = c_1 = y_0$

$u'(0) = c_2 = y_1$

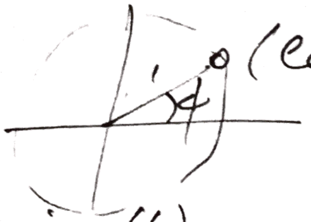
Inhomogeneous:

$L(v) = g$

$v(0) = 0, \; v'(0) = 0$

$v(t) = \int_0^t \sin(t-s) g(s) ds$

↳ $\cos^2(\phi) + \sin^2(\phi) = 1$



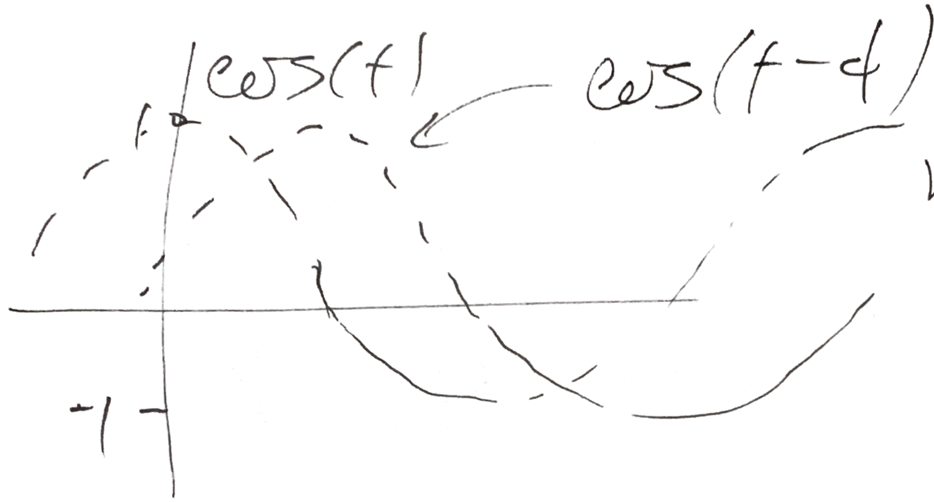
$(\cos(\phi), \sin(\phi))$

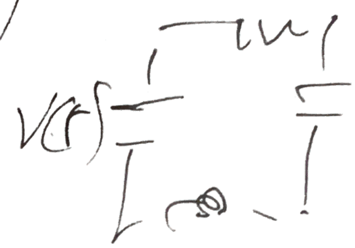
$$y(t) = y_0 \cos(t) + y_1 \sin(t) + \int_0^t \sin(t-s) g(s)\, ds.$$

$$= \sqrt{y_0^2 + y_1^2} \left(\frac{y_0}{\sqrt{y_0^2 + y_1^2}} \cos(t) + \frac{y_1}{\sqrt{y_0^2 + y_1^2}} \sin(t) \right) + \ldots$$

↳ $\cos(\phi) = \dfrac{y_0}{\sqrt{y_0^2 + y_1^2}}$; $\sin(\phi) = \dfrac{y_1}{\sqrt{y_0^2 + y_1^2}}$; $\tan(\phi) = \dfrac{y_1}{y_0}$.

$$= \sqrt{y_0^2+y_1^2}\,\cos(t-\phi) + \int_0^t \text{sen}(t-s)\,g(s)\,ds$$





$$g \rightarrow \boxed{\mathcal{L}} \rightarrow Y$$

$\cos(\omega t)$, $\cos(2\omega t)$

#33

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(\tau)g(\tau)}{W(\tau)}\,d\tau + y_2(t)\int_{t_0}^{t} \frac{y_1(\tau)g(\tau)}{W(\tau)}\,d\tau$$

$$Y'(t) = -y_1'(t)\int_{t_0}^{t} \frac{y_2(\tau)g(\tau)}{W(\tau)}\,d\tau + \frac{-y_1(t)y_2(t)g(t)}{W(t)}$$

$$+ y_2'(t)\int_{t_0}^{t} \frac{y_1(\tau)g(\tau)}{W(\tau)}\,d\tau + \frac{y_2(t)y_1(t)g(t)}{W(t)}$$

$$= -y_1'(t)\int_{t_0}^{t} \frac{y_2(\tau)g(\tau)}{W(\tau)}\,d\tau + y_2'(t)\int_{t_0}^{t} \frac{y_1(\tau)g(\tau)}{W(\tau)}\,d\tau$$

$$\hookrightarrow \underline{Y'(t_0) = 0}$$

(1)

#34

$y'' + y = 0$

↳ $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$

↳ $\frac{d}{dt}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$ $u = y'$

↳ $\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$

↳ if $\lambda = +i \rightarrow -iy + u = 0$

$u = iy$

↳ $\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\begin{pmatrix} y \\ u \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$$

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$

↳ $y = c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\sin(t))$

$= (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t)$

$\hookrightarrow$ $y_1(t) = \cos(t)$

$y_2(t) = \text{sen}(t)$

$\hookrightarrow$ $\dfrac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}$

$y_1(s)y_2'(s) - y_1'(s)y_2(s) = \cos^2(s) + \text{sen}^2(s) = 1.$

$\hookrightarrow$ $= \cos(s)\,\text{sen}(t) - \cos(t)\,\text{sen}(s)$

$= \text{sen}(t-s)$

$\hookrightarrow$ $y(t) = \displaystyle\int_{t_0}^{t} \text{sen}(t-s)\,g(s)\,ds.$

$\boxed{\#39}$ ③

$L(y) = g(t)$

$y(t_0) = y_0$ & $y'(t_0) = y_1$

$y = u + v$; let u & v be such that:

$\hookrightarrow L(u) = 0$, $L(v) = g$

$u(t_0) = y_0$; $v(t_0) = 0$

$u'(t_0) = \cancel{0}\, y_1$ $v'(t_0) = 0$

Let: $y = u + v$

$\hookrightarrow L(y) = L(u+v) = L(u) + L(v) = 0 + g = g$

$y(t_0) = u(t_0) + v(t_0) = y_0 + 0 = y_0$

$y'(t_0) = u'(t_0) + v'(t_0) = y_1 + 0 = y_1$
