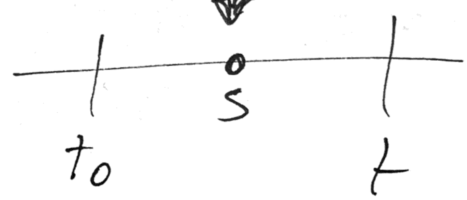
A Note on Integration:

$$\frac{du}{dt} = g(t), \quad u(t_0) = u_0$$

↳ $du = g(t)\,dt$

↳ $$\int_{u_0}^{u(t)} d\tilde{u} = \int_{t_0}^{t} g(s)\,ds$$



or: $t_0 \leq s \leq t$

Integration Variable must be different from bounds.

↳ $$u(t) - u_0 = \int_{t_0}^{t} g(s)\,ds$$; Initial-Value taken into account.

↳ $$u(t_0) - u_0 = \int_{t_0}^{t_0} g(s)\,ds = 0$$

$$\frac{d\vec{x}}{dt} = A\vec{x} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \quad ; \quad \vec{x}(0) = \vec{x}_0$$

$$A = V \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1}$$

$$\vec{y} = V^{-1}\vec{x}$$

$$\hookrightarrow \boxed{\frac{d\vec{y}}{dt} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{y} + \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}} \quad ; \quad \vec{c} = V^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Details

$$\hookrightarrow \frac{d\vec{x}}{dt} = V\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1}\vec{x} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\hookrightarrow V^{-1}\frac{d\vec{x}}{dt} = \underbrace{V^{-1}V}_{=I}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1}x + V^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{d}{dt}\left(V^{-1}\vec{x}\right) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1}x + V^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\hookrightarrow \frac{d}{dt} y_1 = \lambda y_1 + y_2 + c_1$$

$$\frac{d}{dt} y_2 = \lambda y_2 + c_2 \leftarrow$$ y_2 equ is ind. of y_1

$$\hookrightarrow \frac{d}{dt}\left(y_2 e^{-\lambda t}\right) = c_2(t) e^{-\lambda t}$$

$$\hookrightarrow y_2(t) e^{-\lambda t} - y_2(0) = \int_0^t c_2(s) e^{-\lambda s}\, ds$$

$$y_2(t) = y_2(0) e^{\lambda t} + \int_0^t c_2(s) e^{\lambda(t-s)}\, ds$$

$$\boxed{y_2(t) = y_2(0)e^{-\lambda t} + \int_0^t c_2(s)e^{-\lambda(t-s)}\,ds}$$

$$\frac{dy_1}{dt} = \lambda y_1 + y_2(t) + c_1(t)$$

$$\frac{d}{dt}\left(y_1 e^{-\lambda t}\right) = \left(y_2(t) + c_1(t)\right)e^{-\lambda t}$$

$$= y_2(0) + \int_0^t c_2(s)e^{-\lambda s}\,ds + c_1(t)e^{-\lambda t}$$

$$y_1(t)e^{-\lambda t} - y_1(0) = y_2(0)t + \int_0^t\int_0^u c_2(s)e^{-\lambda s}\,ds\,du$$

$$+ \int_0^t c_1(s)e^{-\lambda s}\,ds$$

$$y_1(t) = y_1(0)e^{-\lambda t} + y_2(0)te^{\lambda t} + \int_0^t c_1(s)e^{-\lambda(t-s)}\,ds$$

$$+ \int_0^t\int_0^u c_2(s)e^{-\lambda(t-s)}\,ds\,du$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

"Like with like"

$$+ \int_0^t \begin{pmatrix} c_1(s) \\ c_2(s) \end{pmatrix} e^{\lambda(t-s)} ds$$

$$+ \int_0^t \int_0^u \begin{pmatrix} c_2(s) \\ 0 \end{pmatrix} e^{\lambda(t-s)} ds\,du$$

$$\vec{x} = V\vec{y}, \quad \vec{y} = V^{-1}\vec{x}; \quad \vec{c} = V^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\vec{x} = V\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} V^{-1}\vec{x}_0 + \int_0^t \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} e^{\lambda(t-s)} ds$$

$$+ V \int_0^t \int_0^u \begin{pmatrix} c_2(s) \\ 0 \end{pmatrix} e^{\lambda(t-s)} ds\,du$$

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$$\begin{pmatrix} c_2(s)\\ 0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}c_1(s)\\ c_2(s)\end{pmatrix}$$

$$= \begin{pmatrix}0&1\\0&0\end{pmatrix} V^{-1}\begin{pmatrix}f_1(s)\\ f_2(s)\end{pmatrix}$$

$$\vec{x} = e^{\lambda t} V \begin{pmatrix}1&t\\0&1\end{pmatrix} V^{-1}\vec{x}_0 + \int_0^t \begin{pmatrix}f_1(s)\\ f_2(s)\end{pmatrix} e^{\lambda(t-s)}\,ds$$

$$+ \int_0^t\int_0^u V\begin{pmatrix}0&1\\0&0\end{pmatrix}V^{-1}\begin{pmatrix}f_1(s)\\ f_2(s)\end{pmatrix} e^{\lambda(t-s)}\,ds\,du$$

$$\cancel{= e^{\lambda t}(}$$

$$= e^{\lambda t} V\left(I + t\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)V^{-1}\vec{x}_0$$

$$= e^{\lambda t}\left(I + tV\begin{pmatrix}0&1\\0&0\end{pmatrix}V^{-1}\right)\vec{x}_0$$