$$x_p(t) = \int_0^t f(s)\sin(t-s)\,ds$$

$$f(s) = \begin{cases} As, & 0 \le s \le \pi \\ A(2\pi - s), & \pi \le s \le 2\pi \\ 0, & 2\pi < s \end{cases}$$

Note: $x_p(0) = 0$

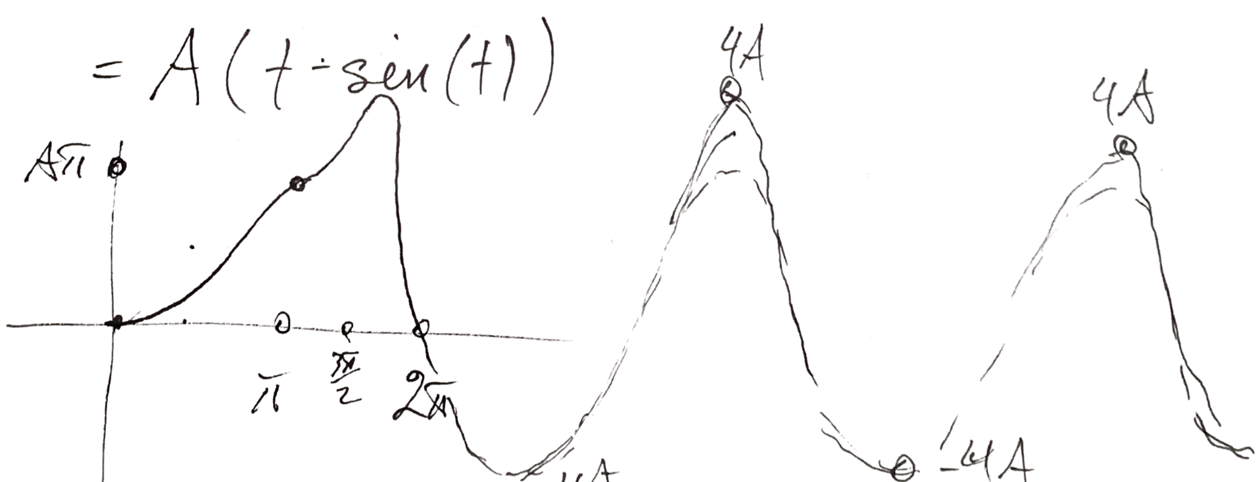
$$\frac{dx_p}{dt} = \int_0^t f(s)\cos(t-s)\,ds \to \left.\frac{dx_p}{dt}\right|_{t=0} = 0.$$

So, for $0 \le t \le \pi$:

$$\to x_p(t) = \int_0^t As\sin(t-s)\,ds$$

$$= A\left\{ s\cos(t-s)\Big|_{s=0}^{s=t} + \sin(t-s)\Big|_{s=0}^{s=t} \right\}$$

$$= A(t - \sin(t))$$



For $\pi < t \le 2\pi$: $x_p(t) = A\int_0^{\pi} s\sin(t-s)\,ds + A\int_{\pi}^{t}(2\pi-s)\sin(t-s)\,ds$ (2)

$$x_p(t) = A\left\{ s\cos(t-s)\Big|_{s=0}^{s=\pi} + \sin(t-s)\Big|_{s=0}^{s=\pi} \right\}$$

$$+A\left\{ (2\pi-s)\cos(t-s)\Big|_{s=\pi}^{s=t} - \sin(t-s)\Big|_{s=\pi}^{s=t} \right\}$$

$$= A\left\{ \pi\cos(t-\pi) + \sin(t-\pi) - \sin(t) \right\}$$

$$+ A\left\{ (2\pi - t) - \pi\cos(t-\pi) + \sin(t-\pi) \right\}$$

$$= A\left\{ 2\pi - t + 2\sin(t-\pi) - \sin(t) \right\}$$

$$= A\left\{ 2\pi - t - 3\sin(t) \right\}.$$

For $t > 2\pi$: $x_p(t) = A\int_0^{\pi} s\sin(t-s)\,ds + A\int_{\pi}^{2\pi}(2\pi-s)\sin(t-s)\,ds$

$$x_p(t) = A\left\{ \pi\cos(t-\pi) + \sin(t-\pi) - \sin(t) \right\}$$

$$+ A\left\{ -\pi\cos(t-\pi) - \sin(t-2\pi) + \sin(t-\pi) \right\}$$

$$= -4A\sin(t)$$

$$x_p(t) = \frac{1}{\sigma^2-\omega^2}\left(\cos(\omega t) - \cos(\sigma t)\right)$$

let $\alpha = \frac{1}{2}(\omega+\sigma)$; $\beta = \frac{1}{2}(\omega-\sigma)$

$$\omega = \alpha+\beta; \quad \sigma = \alpha-\beta$$

$$\cos((\alpha+\beta)t) - \cos((\alpha-\beta)t)$$

$$= \cos(\alpha t)\cos(\beta t) - \sin(\alpha t)\sin(\beta t)$$

$$- \left(\cos(\alpha t)\cos(\beta t) + \sin(\alpha t)\sin(\beta t)\right)$$

$$= \frac{2}{\omega^2-\sigma^2}\sin\left(\frac{1}{2}(\omega+\sigma)t\right)\sin\left(\frac{1}{2}(\omega-\sigma)t\right)$$

$$= \underbrace{\frac{\sin\left(\frac{(\omega-\sigma)}{2}t\right)}{\left(\frac{\omega-\sigma}{2}\right)}}_{\text{"slow"}} \underbrace{\frac{\sin\left(\frac{1}{2}(\omega+\sigma)t\right)}{\omega+\sigma}}_{\text{fast}}$$

For $f(t)$, we define $F(s)=\mathcal{L}\{f\}$ to be, for $s>0$

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt=\lim_{N\to\infty}\int_0^N e^{-st}f(t)\,dt$$

a) Find $\mathcal{L}\{1\}$

$$\hookrightarrow \mathcal{L}\{1\}=\int_0^\infty e^{-st}\,dt=-\frac{1}{s}e^{-st}\Big|_0^\infty=\frac{1}{s}$$

b) Find $\mathcal{L}\{t\}$

$$\hookrightarrow \mathcal{L}\{t\}=\int_0^\infty te^{-st}\,dt=-\frac{1}{s}te^{-st}\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$=-\frac{1}{s^2}e^{-st}\Big|_0^\infty=\frac{1}{s^2}$$

c) Find $\mathcal{L}\{t^n\}$ using $\mathcal{L}\{t^n\}=\int_0^\infty t^n e^{-st}\,dt=\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$

$$=\frac{n}{s}\mathcal{L}\{t^{n-1}\}.$$

$\to \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

True for $n = 0, 1$

$\to \mathcal{L}\{t^{n+1}\} = \frac{(n+1)}{s} \mathcal{L}\{t^n\} = \frac{(n+1)}{s} \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$.

$\to \boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$ $\to$ so polynomials get flipped.

d) Find $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$

$= \frac{1}{s-a}, \; s > a$

$\to$ Exponentials turn into rational functions.

THM: Let $\mathcal{L}\{f_1\} = F_1(s)$ for $s > a_1$

$\mathcal{L}\{f_2\} = F_2(s)$ for $s > a_2$

Then $\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha \mathcal{L}\{f_1\} + \beta \mathcal{L}\{f_2\}$ for $s > \max\{a_1, a_2\}$

pf|

So, if $\mathcal{L}\{\alpha f_1 + \beta f_2\} = \lim_{N \to \infty} \int_0^N e^{-st}(\alpha f_1 + \beta f_2)\,dt$

we see:

$$\int_0^N e^{-st}(\alpha f_1 + \beta f_2)\,dt = \alpha \int_0^N e^{-st} f_1\,dt + \beta \int_0^N e^{-st} f_2\,dt$$

if $s > \max\{a_1, a_2\} \to \lim_{N \to \infty}$ exists for both integrals

and so we see

$$\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha \mathcal{L}\{f_1\} + \beta \mathcal{L}\{f_2\}.$$

So to find $\mathcal{L}\{\cos(\omega t)\}$ we use:

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$\Rightarrow$ $$\mathcal{L}\{\cos(\omega t)\} = \frac{1}{2}\left(\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}\right)$$

$$\mathcal{L}\{e^{i\omega t}\} = \int_0^\infty e^{i\omega t} e^{-st}\,dt = \int_0^\infty e^{(i\omega - s)t}\,dt$$

$$= \frac{1}{i\omega - s} e^{(i\omega - s)t}\Big|_0^\infty$$

$$= \frac{1}{s - i\omega}, \quad s > 0.$$

$\Rightarrow$ $$\mathcal{L}\{\cos(\omega t)\} = \frac{1}{2}\left\{\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{(s-i\omega)(s+i\omega)}\right\} = \frac{s}{s^2+\omega^2}$$

Find $\mathcal{L}\{\sin(\omega t)\}$ using $\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$$= \frac{1}{2i}\left\{\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right\} = \frac{\omega}{s^2+\omega^2}; \quad s>0$$

(5)

Finally, find $\mathcal{L}\{f(t)\}$; $f(t) = \begin{cases} e^{2t} & 0 \le t < 1 \\ 4 & t \ge 1 \end{cases}$

$$F(s) = \int_0^1 e^{-st} e^{2t}\,dt + 4\int_1^\infty e^{-st}\,dt$$

$$= \frac{1}{2-s} e^{(2-s)t}\Big|_0^1 - \frac{4}{s} e^{-st}\Big|_1^\infty$$

$$= \frac{1}{2-s}\left(e^{(2-s)} - 1\right) + \frac{4}{s} e^{-s}, \quad s>0$$