ODE's:

- Problem - let A be 2×2, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = (\vec{a}_1 | \vec{a}_2)$

  - Show $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2$.

- Problem - let A be 2×2, B be 2×2 and $B = (\vec{b}_1 | \vec{b}_2)$

  - Show $AB = (A\vec{b}_1 | A\vec{b}_2)$.

  Hint: $B = (\vec{b}_1 | \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) + (\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} | \vec{b}_2)$

  ↳ $AB = A(\vec{b}_1 | \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) + A(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} | \vec{b}_2)$

- Let $A\vec{v} = \lambda\vec{v}$; $A\vec{w} = \gamma\vec{w}$, A 2×2, $\lambda \neq \gamma$.

  We say $\vec{v}$ & $\vec{w}$ are l. ind. if the only solution to $c_1\vec{v} + c_2\vec{w} = 0$ is $c_1 = c_2 = 0$.

  ↳ equivalent to only solution to

  $$(\vec{v}|\vec{w})\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ is trivial solution.

Problem: Let $\vec{v}, \vec{w}$ be as above. Show they are l.i.

pf| By contradiction, let $\vec{v} = \alpha\vec{w}$, $\alpha \neq 0$. (2)

$\hookrightarrow A\vec{v} = \lambda\vec{v}$

$\hookrightarrow A(\alpha\vec{w}) = \alpha\lambda\vec{w}$

$\hookrightarrow \alpha A\vec{w} = \alpha\lambda\vec{w}$

$\hookrightarrow \alpha\eta\vec{w} = \alpha\lambda\vec{w}$

$\hookrightarrow (\eta - \lambda)\vec{w} = 0$ so $\vec{w} \neq 0 \Rightarrow \eta = \lambda$, but this is contradiction $\boxed{\not{}}$

So, going back to $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \to \lambda = 3, 1$, $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thm | guarantees e-vects are l.i. Though, if we look @ $\underline{V} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, l.i. means $\underline{V}\vec{c} = 0$ only has trivial

$\hookrightarrow \det(\underline{V}) \neq 0$

so $\det(\underline{V}) = -1 - 1 = -2 \neq 0 \to \underline{V}^{-1}$ exists.

so let $A\vec{v} = \lambda\vec{v}$, $A\vec{\omega} = \eta\vec{\omega}$, $\lambda \neq \eta$

↳ $A(\vec{v}|\vec{\omega}) = (A\vec{v}|A\vec{\omega}) = (\lambda\vec{v}|\eta\vec{\omega}) = (\vec{v}|\vec{\omega})\begin{pmatrix}\lambda & \\ & \eta\end{pmatrix}$

↳ $A = (\vec{v}|\vec{\omega})\begin{pmatrix}\lambda & \\ & \eta\end{pmatrix}(\vec{v}|\vec{\omega})^{-1}$

So $A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \to (\vec{v}|\vec{\omega}) = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \to (\vec{v}|\vec{\omega})^{-1} = -\frac{1}{2}\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

↳ $A = \cancel{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3 & \\ & 1\end{pmatrix}\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}$ ... okay cute, but so what?

$= \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}$

↳ $\dfrac{dx}{dt} = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}x$, $x(0) = x_0$

↳ $\dfrac{dx}{dt} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}x$

↳ $\underline{V}^{-1}\dfrac{dx}{dt} = \underline{\Lambda}\,\underline{V}^{-1}x$

↳ $\dfrac{d}{dt}(\underline{V}^{-1}x) = \underline{\Lambda}(\underline{V}^{-1}x)$

so, we let ~~$\underline{V}^{-1}x$~~ $y(t) = \underline{V}^{-1}x(t)$ ← NO TIME!!!

$$\Rightarrow \frac{d}{dt}y = \Lambda y = \begin{pmatrix}\lambda & 0\\ 0 & q\end{pmatrix} y = \begin{pmatrix}\lambda & 0\\ 0 & q\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}\lambda y_1\\ q y_2\end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}\lambda y_1\\ q y_2\end{pmatrix}$$

$$\Rightarrow \frac{dy_1}{dt} = \lambda y_1 \; ; \; \frac{dy_2}{dt} = q y_2$$

$$\Rightarrow y_1(t) = y_1(0)e^{\lambda t} \; ; \; y_2(t) = y_2(0)e^{qt}$$

$$\Rightarrow \begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix} = \begin{pmatrix}e^{\lambda t}y_1(0)\\ e^{qt}y_2(0)\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{qt}\end{pmatrix}\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}$$

$$\Rightarrow \underline{V}^{-1}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{qt}\end{pmatrix}\underline{V}^{-1}\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \underline{V}\begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{qt}\end{pmatrix}\underline{V}^{-1}\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}.$$

So, seriously, that's it:

$$\frac{d}{dt}x = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x \; ; \quad x(0) = x_0$$

So $x(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{pmatrix} \underbrace{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} x_0}_{c}$.

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{t} \end{pmatrix} = \underline{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}}$$