For $f(t)$, we define $F(s) = \mathcal{L}\{f\}$ to be, for $s>0$ (1)

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt = \lim_{N\to\infty} \int_0^{N} e^{-st} f(t)\,dt$$

a) Find $\mathcal{L}\{1\}$

$$= -\frac{1}{s}\Big(\underbrace{\lim_{t\to\infty} e^{-st}}_{\to 0,\ s>0} - 1\Big)$$

$$\hookrightarrow \mathcal{L}\{1\} = \int_0^{\infty} e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s}.$$

$$f(t)=1 \xrightarrow{\mathcal{L}} F(s) = \frac{1}{s}$$ } $\mathcal{L}$ goes from "t"-space to "s"-space

b) Find $\mathcal{L}\{t\}$

$$\hookrightarrow \mathcal{L}\{t\} = \int_0^{\infty} t e^{-st}\,dt = -\frac{1}{s} t e^{-st}\Big|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}\,dt$$

$$= -\frac{1}{s^2} e^{-st}\Big|_0^{\infty} = \frac{1}{s^2}.$$

c) Find $\mathcal{L}\{t^n\}$ using $\mathcal{L}\{t^n\} = \int_0^{\infty} t^n e^{-st}\,dt = \frac{n}{s}\int_0^{\infty} t^{n-1} e^{-st}\,dt$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}}$$

$\hookrightarrow \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$?

True for $n = \underline{0, 1}$ $\quad \mathcal{L}\{t^0\} = \frac{0!}{s^{0+1}}$; $\mathcal{L}\{t\} = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$

$\hookrightarrow \mathcal{L}\{t^{n+1}\} = \frac{(n+1)}{s} \mathcal{L}\{t^n\} = \frac{(n+1)}{s} \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$. $\underline{s > 0}$

$\hookrightarrow \boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$ → so polynomials get flipped.

d) Find $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$

$\underline{a \in \mathbb{R}}$ $\qquad = \frac{1}{s-a}$, $\boxed{s > a}$

$\hookrightarrow$ Exponentials turn into rational functions.

THM: Let $\mathcal{L}\{f_1\} = F_1(s)$ for $s > a_1$

$\mathcal{L}\{f_2\} = F_2(s)$ for $s > a_2$

Then $\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha\mathcal{L}\{f_1\} + \beta\mathcal{L}\{f_2\}$ for $\underline{s > \max\{a_1, a_2\}}$

pf| So, if $\mathcal{L}\{\alpha f_1 + \beta f_2\} = \lim\limits_{N\to\infty} \int_0^N e^{-st}(\alpha f_1 + \beta f_2)\,dt$

we see:

$$\int_0^N e^{-st}(\alpha f_1 + \beta f_2)\,dt = \alpha\int_0^N e^{-st} f_1\,dt + \beta\int_0^N e^{-st} f_2\,dt$$

if $s > \max\{a_1, a_2\} \to \lim\limits_{N\to\infty}$ exists for both integrals

and so we see

$$\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha\mathcal{L}\{f_1\} + \beta\mathcal{L}\{f_2\}.$$

$\boxtimes$

So to find $\mathcal{L}\{\cos(\omega t)\}$ we use:

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right), \quad \omega>0.$$

$$\Rightarrow \mathcal{L}\{\cos(\omega t)\} = \frac{1}{2}\left(\mathcal{L}\{e^{i\omega t}\}+\mathcal{L}\{e^{-i\omega t}\}\right)$$

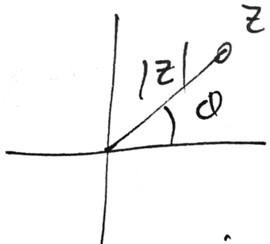
$$\mathcal{L}\{e^{i\omega t}\} = \int_0^\infty e^{i\omega t}e^{-st}\,dt = \int_0^\infty e^{(i\omega-s)t}\,dt$$

$$= \frac{1}{i\omega-s}e^{(i\omega-s)t}\Big|_0^\infty$$

$$= \frac{1}{s-i\omega}, \quad \underline{s>0}.$$

$$|e^{i\theta}| = 1$$

$$|z| = \sqrt{z_r^2+z_i^2}$$



$$\underline{z=|z|e^{i\theta}}$$

$$\Rightarrow \mathcal{L}\{\cos(\omega t)\} = \frac{1}{2}\left\{\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{(s-i\omega)(s+i\omega)}\right\} = \frac{s}{s^2+\omega^2}$$

Find $\mathcal{L}\{\sin(\omega t)\}$ using $\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$$= \frac{1}{2i}\left\{\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right\} = \frac{\omega}{s^2+\omega^2}; \quad s > 0$$

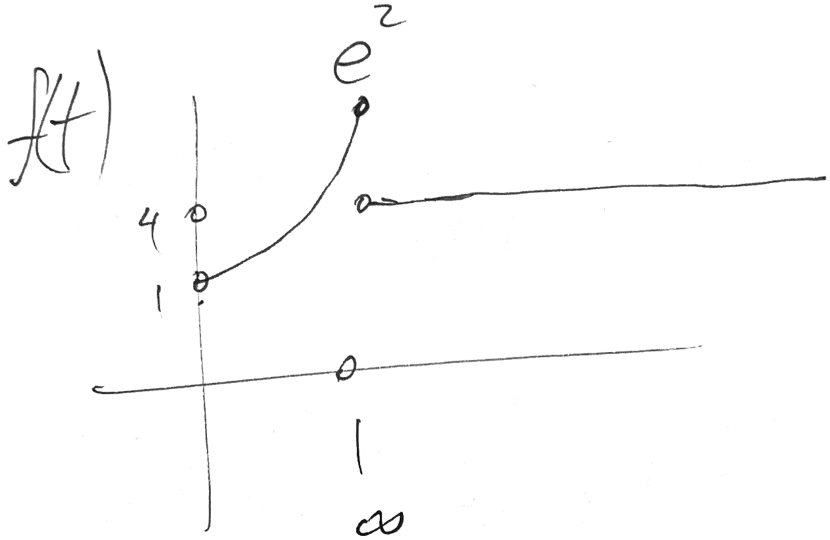
5

Finally, find $\mathcal{L}\{f(t)\}$; $f(t) = \begin{cases} e^{2t} & \underline{0 \leq t < 1} \\ 4 & \underline{t \geq 1} \end{cases}$

$$F(s) = \int_0^1 e^{-st} e^{2t}\,dt + 4\int_1^\infty e^{-st}\,dt$$

$$= \frac{1}{2-s} e^{(2-s)t}\Big|_0^1 - \frac{4}{s} e^{-st}\Big|_1^\infty$$

$$= \frac{1}{2-s}\left(e^{(2-s)} - 1\right) + \frac{4}{s} e^{-s}, \quad s > 0$$



$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^1 f(t)e^{-st}\,dt + \int_1^\infty f(t)e^{-st}\,dt$$

$$= \int_0^1 e^{2t}e^{-st}\,dt + 4\int_1^\infty e^{-st}\,dt$$

$$= \frac{1}{2-s}e^{(2-s)t}\Big|_{t=0}^{t=1} - \frac{4}{s}e^{-st}\Big|_{t=1}^{t=\infty}$$

I LEAVE "s" ALONE.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a, \; a\in\mathbb{R}$$

$$\mathcal{L}\{\cos(\omega t)\} = \frac{\omega}{s^2+\omega^2}, \quad \omega\in\mathbb{R}$$

$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$\mathcal{L}\{e^{(\mu+i\omega)t}\} = \int_0^\infty e^{(\mu+i\omega-s)t}\,dt$$

$$= \frac{1}{\mu+i\omega-s}\,\underline{e^{(\mu+i\omega-s)t}}\Big|_0^{\infty}$$

$$e^{(\mu+i\omega-s)t} = \underbrace{e^{i\omega t}}\,\boxed{e^{(\mu-s)t}} \quad \mu-s<0$$

$$\hookrightarrow \underline{|e^{i\omega t}| = 1} \Rightarrow |\cos(\omega t) + i\sin(\omega t)| = 1.$$

$$\mathcal{L}\{e^{(\mu+i\omega)t}\} = \frac{1}{s-(\mu+i\omega)}, \quad \boxed{s>\mu}$$

Beat: Suppose $f(t) = e^{t^2}$

Ls You want $\int_0^{\infty} e^{t^2} e^{-st}\,dt \ldots$

What is rate of decay as $t \to \infty$?

L'Hopital Analysis: let $s>0$:

$$\lim_{t\to\infty} e^{t^2} e^{-st} = \lim_{t\to\infty} \frac{e^{t^2}}{e^{st}}$$

$$e^{t^2 - st} = e^{t(t-s)} \qquad = \lim_{t\to\infty} \frac{2t\,e^{t^2}}{s\,e^{st}} = \infty.$$

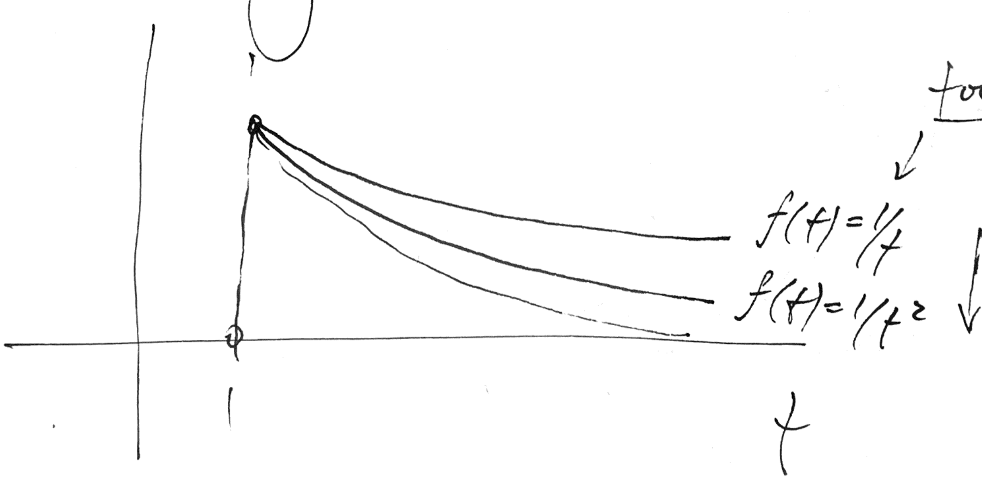
$\int_M^\infty f(t)\,dt$ as a subject of study:

Quick review:

$$\int_1^\infty t^{-p}\,dt = \frac{1}{-p+1} t^{-p+1}\Big|_1^\infty = \begin{cases} \infty & p \le 1 \\ \frac{1}{-1+p} & p > 1. \end{cases}$$

What does this tell us?

We need a fast enough rate of decay in order to ensure the existence of the integral!



as p goes up, so does rate of decay

Show $\int_M^\infty t^k e^{-at}\,dt < \infty$ for $a>0$, $k \in \mathbb{N} = \{1, 2, \dots\}$

So of course $\lim_{t\to\infty} t^k e^{-at} = \lim_{t\to\infty} \frac{t^k}{e^{at}} = \lim_{t\to\infty} \frac{k!}{a^k e^{at}} = 0$.

by L'Hopital.

and by integration-by-parts, we have

$$\int_M^\infty t^k e^{-at}\,dt = -\frac{1}{a} t^k e^{-at}\Big|_{t=M}^{\infty} + \frac{k}{a}\int_M^\infty t^{k-1} e^{-at}\,dt$$

$$= \frac{1}{a} M^k e^{-aM} + \frac{k}{a}\int_M^\infty t^{k-1} e^{-at}\,dt$$

$\hookrightarrow$ repeat.

Moral of Story: Even the smallest amount of exponential decay always beats polynomial growth.

Piecewise Continuous: 5.1 → 5.2

For $f:[0,\infty)\to\mathbb{R}$; $\mathcal{L}\{f\}=\int_0^\infty f(t)e^{-st}\,dt$

if $\exists\,\{t_j\}_{j=0}^{n}$: on $[0,t_0),[t_0,t_1),\dots,[t_{n-1},t_n),[t_n,\infty)$

# of discons are finite

f is continuous such that

$\lim_{t\to t_j^{+/-}} f(t)$ exists

discons are always jumps

Then we say f is piecewise cont.

in other words, we only have a finite number of jumps in f.

So $\mathcal{L}\{f\} = \int_0^{t_0} f(t)e^{-st}\,dt + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} f(t)e^{-st}\,dt$

$+\int_{t_n}^{\infty} f(t)e^{-st}\,dt$

But now we need a far more general tool:

Suppose $\int_M^\infty g(t)\,dt < \infty$ for $g(t) \geq 0$, continuous on $[M,\infty)$

Then if f is continuous on $[M,\infty)$ and $|f(t)| \leq g(t)$

so $\int_M^\infty f(t)\,dt$ exists or $\lim_{\cancel{X} \to \infty} \int_M^{\cancel{X}} f(t)\,dt$ exists.

~~"Proof": Let $G(x) = \int_M^x g(t)\,dt, \quad x \geq M$~~

~~By FTC: $G'(x) = g(x) \geq 0$~~

~~So $G(x)$ is increasing and~~

~~$\lim_{x\to\infty} G(x) = \int_M^\infty g(t)\,dt$~~

~~Likewise, we can define $\tilde{G}(x) = \int_x^\infty g(t)\,dt$~~

~~$\tilde{G}'(x) = -g(x) \leq 0, \quad \lim_{x\to\infty} \tilde{G}(x) = 0$~~

"proof":

we assume: $\int_M^\infty g(t)\,dt < \infty$, $g(t) \geq 0$

$0 \leq |f(t)| \leq g(t)$ for $t \in [M, \infty)$

want to know:

does $\lim_{x\to\infty} \int_M^x f(t)\,dt$ exist?

a) Does $\lim_{x\to\infty} \int_M^x |f(t)|\,dt$ exist?

$\hookrightarrow \int_M^x |f(t)|\,dt \leq \int_M^x g(t)\,dt \xrightarrow{x\to\infty} G < \infty$

$\hookrightarrow \frac{d}{dx} \int_M^x |f(t)|\,dt = |f(x)| \geq 0.$

If we can find $K>0,\ a>0,\ M>0$:

$|f(t)| \le K e^{at}$ for $t \geq M$ we say f is of exponential order.

(e^{t^2})

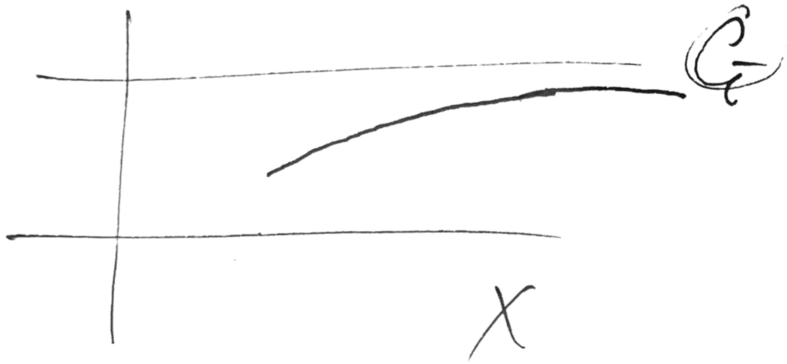
See an important trick!

$$\text{So} \quad \left| \int_M^\infty f(t) e^{-st}\,dt \right| \le \int_M^\infty |f(t)| e^{-st}\,dt$$

$$\le K \int_M^\infty e^{-(s-a)t}\,dt$$

$$\le \frac{K}{s-a} e^{-(s-a)M}, \quad \underline{s > a.}$$

We see for piecewise continuous functions, only real problem is $\int_{t_N}^\infty f(t) e^{-st}\,dt$.



Monotonic Convergence THMs

$\Rightarrow \lim_{X \to \infty} \int_M^X |f(t)|\,dt$ exists.

$\Rightarrow -|f(t)| \leq f(t) \leq |f(t)|$

$\Rightarrow 0 \leq f(t) + |f(t)| \leq 2|f(t)|$

$\Rightarrow \int_M^\infty (f(t) + |f(t)|)\,dt$ exists

$\Rightarrow \int_M^\infty f(t)\,dt$ exists.

$$0 \leq f(t) + |f(t)| \leq 2|f(t)|$$

$$\leadsto 0 \leq \int_M^x f(t)\,dt + \int_M^x |f(t)|\,dt \leq 2 \int_M^x |f(t)|\,dt$$

Properties of the Laplace Transform:

If $F(s)=\mathcal{L}\{f(t)\}$ for $s>a$, and $c\in\mathbb{R}$ $\rightarrow$

$$\mathcal{L}\{e^{ct}f(t)\}=F(s-c), \quad s>a+c$$

pf|

$$\mathcal{L}\{e^{ct}f(t)\}=\int_0^\infty e^{ct}e^{-st}f(t)\,dt$$

$$=\int_0^\infty e^{(c-s)t}f(t)\,dt$$

$$=\int_0^\infty e^{-(s-c)t}f(t)\,dt, \qquad F(s)=\int_0^\infty e^{-st}f(t)\,dt$$

$$=F(s-c)$$

If $F(s) = \mathcal{L}\{f(t)\}$ for $s > a$, and $f(t)$ is piecewise continuous on $[0, \infty)$, then we see

$$\mathcal{L}\{f'\} = sF(s) - f(0).$$

pf| $$\mathcal{L}\{f'\} = \int_0^\infty f'(t)e^{-st}\,dt$$

$$= \int_0^{t_0} f'(t)e^{-st}\,dt + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f'(t)e^{-st}\,dt$$

$$+ \int_{t_n}^\infty f'(t)e^{-st}\,dt$$

So we see, using integration-by parts that

$$\int_a^b f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_a^b + s\int_a^b f(t)e^{-st}\,dt$$

Thus:

$$\mathcal{L}\{f'\} = f(t)e^{-st}\Big|_{t=0}^{t_0} + \sum_{j=0}^{n-1} f(t)e^{-st}\Big|_{t=t_j}^{t_{j+1}} + f(t)e^{-st}\Big|_{t=t_n}^{\infty}$$

$$+\, s\left\{\int_0^{t_0} f(t)e^{-st}\,dt + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} f(t)e^{-st}\,dt + \int_{t_n}^{\infty} f(t)e^{-st}\,dt\right\}.$$

So: $f(t)e^{-st}\Big|_{t=t_j}^{t_{j+1}} = f(t_{j+1})e^{-st_{j+1}} - f(t_j)e^{-st_j}$

$$= \tilde{f}_{j+1} - \tilde{f}_j$$

So

$$\mathcal{L}\{f'\} = \underline{\tilde{f}_0} - f(0) + \underbrace{\sum_{j=0}^{n-1}(\tilde{f}_{j+1} - \tilde{f}_j) - \underline{\tilde{f}_n}}_{=\,0 \text{ because of telescoping series.}} + s\mathcal{L}\{f\}$$

$$= sF(s) - f(0)$$

And now, essentially via ... induction, we have

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0)$$
$$= s(s\mathcal{L}\{f\} - f(0)) - f'(0)$$
$$= s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

Note, we also write this as:

$$\mathcal{L}\{\frac{d}{dt} f(t)\} = s\mathcal{L}\{f\} - f(0)$$

$\Rightarrow$ $\mathcal{L}\{b\frac{d}{dt} f\} = b\mathcal{L}\{\frac{df}{dt}\} = b(s\mathcal{L}\{f\} - f(0))$

$\Rightarrow$ we can start talking about:

$$\mathcal{L}\{a\frac{d^2f}{dt^2} + b\frac{df}{dt} + cf\} = \mathcal{L}\{L[f]\}.$$

$\Rightarrow$ $a(s^2F(s) - sf(0) - f'(0)) + b(sF(s) - f(0)) + cF(s).$

or

$$\mathcal{L}\{L[f]\} = (as^2+bs+c)F(s) + (-as-b)f(0) - af'(0).$$