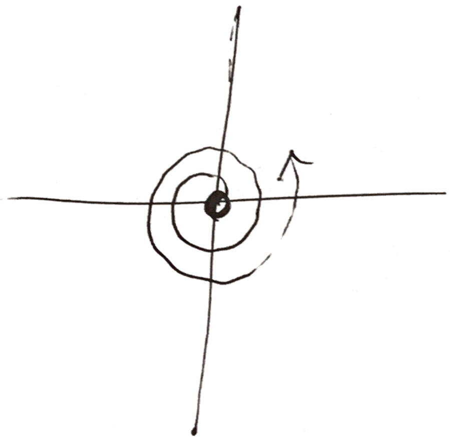
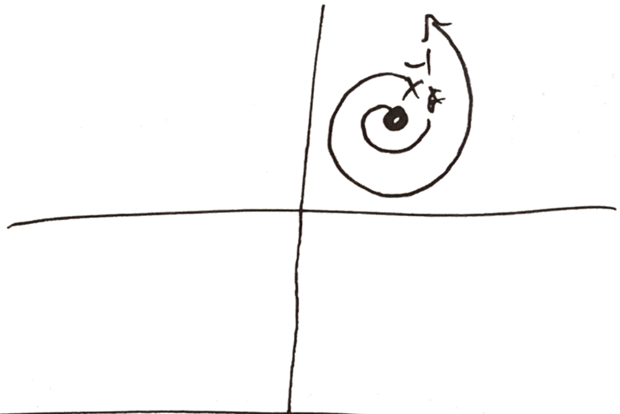
$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \text{vs.} \qquad \frac{d\vec{x}}{dt} = A\vec{x} + \underline{\vec{b}}$$

external forcing

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\vec{x}_* = \begin{pmatrix}0\\0\end{pmatrix} \qquad\qquad \vec{x}_* = -A^{-1}\vec{b}$$





Non-Autonomy:

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}(t)$$

Suppose $A = V\Lambda V^{-1}$

TEMPTATION: $A\vec{x} = -\underline{\vec{b}(t)} \to \vec{x}_*(t) = -A^{-1}\vec{b}(t)$

↳ DEAD END

↳ Irony (Wikipedia)

$$\frac{d\vec{x}}{dt} = V\Lambda V^{-1}\vec{x} + \vec{b}(t)$$

$$\Rightarrow V^{-1}\frac{d\vec{x}}{dt} = \Lambda V^{-1}\vec{x} + V^{-1}\vec{b}(t)$$

$$\Rightarrow \frac{d\vec{y}}{dt} = \Lambda\vec{y} + \vec{c}(t),\quad \vec{y} = V^{-1}\vec{x},\ \vec{c} = V^{-1}\vec{b}$$

$$\Rightarrow \frac{dy_1}{dt} = \lambda_1 y_1 + c_1(t);\quad \frac{dy_2}{dt} = \lambda_2 y_2 + c_2(t)$$

$$\Rightarrow \frac{d}{dt}\left(y_1 e^{-\lambda_1 t}\right) = c_1(t)e^{-\lambda_1 t};\quad \frac{d}{dt}\left(y_2 e^{-\lambda_2 t}\right) = c_2(t)e^{-\lambda_2 t}$$

$$\Rightarrow y_1(t)e^{-\lambda_1 t} - y_{1,0} = \int_0^t c_1(s)e^{-\lambda_1 s}\,ds$$

$$y_2(t)e^{-\lambda_2 t} - y_{2,0} = \int_0^t c_2(s)e^{-\lambda_2 s}\,ds$$

$$y_1(t) = y_{1,0}\, e^{-\lambda_1 t} + \int_0^t c_1(s)\, e^{-\lambda_1 (t-s)}\, ds$$

$$y_2(t) = y_{2,0}\, e^{-\lambda_2 t} + \int_0^t c_2(s)\, e^{-\lambda_2 (t-s)}\, ds$$

$$\Rightarrow \quad \vec{y} = \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \begin{pmatrix} y_{1,0} \\ y_{2,0} \end{pmatrix}$$

$$+ \underbrace{\int_0^t \begin{pmatrix} e^{-\lambda_1 (t-s)} & 0 \\ 0 & e^{-\lambda_2 (t-s)} \end{pmatrix} \begin{pmatrix} c_1(s) \\ c_2(s) \end{pmatrix} ds}_{\text{convolutions}}$$

$$\vec{y} = V^{-1}\vec{x} \rightarrow \vec{x} = V\vec{y}\,; \quad \vec{c} = V^{-1}\vec{b}(t) \rightarrow \vec{b} = V\vec{c}$$

$$\Rightarrow \quad \vec{x} = \underbrace{V \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} V^{-1} \vec{x}_0}_{\vec{x}_h(t)} + \underbrace{V \int_0^t \begin{pmatrix} e^{-\lambda_1 (t-s)} & 0 \\ 0 & e^{-\lambda_2 (t-s)} \end{pmatrix} V^{-1} \vec{b}(s)\, ds}_{\vec{x}_p(t)}$$

i.e. if $\vec{b}(t) = 0$

$$\frac{d\vec{x}}{dt} = A\vec{x}\,; \quad \vec{x}(0) = \vec{x}_0$$

$$\frac{d\vec{x}}{dt} - A\vec{x} = \vec{b}(t)$$

$$L\vec{x} = \vec{b}$$

So, where last we left off, we had solved

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}; \quad \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}(t)$$

for $\vec{x}(0) = \vec{x}_0$.

In all cases, using $A = V \Lambda V^{-1}$ or $V \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1}$ gives us exact solutions.

But now, what of

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}; \quad \vec{x}(0) = \vec{x}_0 \quad ?$$

And here things get a lot more complicated... there is no general solution strategy. But we can develop theory to give us some insights.

Generally, we try to find solutions through trickery.

End Goal: Find two linearly independent solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$.

Again L.I. means: $c_1\vec{x}_1(t)+c_2\vec{x}_2(t)=0$ iff $c_1=c_2$ $\forall t$.

↳ we note that if:

$$\left(\frac{d}{dt}-A\right)\vec{x}_1=0, \quad \left(\frac{d}{dt}-A\right)\vec{x}_2=0$$

then $\left(\frac{d}{dt}-A\right)(c_1\vec{x}_1+c_2\vec{x}_2)$

$$=c_1\left(\frac{d}{dt}-A\right)\vec{x}_1+c_2\left(\frac{d}{dt}-A\right)\vec{x}_2$$

$$=c_1\cdot 0+c_2\cdot 0=0.$$

End Goal: Find two linearly independent solutions to $\frac{d\vec{x}}{dt} = A(t)\vec{x}$, say $\vec{x}_1(t), \vec{x}_2(t)$.

Again L.I. means:

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = 0 \text{ iff } c_1 = c_2 = 0.$$

Another way to encode this is using the Wronskian where

$$W(t) = \det\left(\left(\vec{x}_1(t) \,|\, \vec{x}_2(t)\right)\right)$$

So if $W(t) \neq 0 \Rightarrow \vec{x}_1(t)$ and $\vec{x}_2(t)$ are l.i.

Abel's THM:

Let $\frac{d\vec{x}_1}{dt} = A(t)\vec{x}_1$; $\frac{d\vec{x}_2}{dt} = A(t)\vec{x}_2$.

$$\hookrightarrow \frac{dW}{dt} = \frac{d}{dt}\left(x_{11}x_{22} - x_{12}x_{21}\right)$$

$$\frac{d}{dt}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} ; \quad \frac{d}{dt}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_{11} + a_{12}x_{12} \\ a_{21}x_{11} + a_{22}x_{12} \end{pmatrix} ; \qquad = \begin{pmatrix} a_{11}x_{21} + a_{12}x_{22} \\ a_{21}x_{21} + a_{22}x_{22} \end{pmatrix}$$

So $$\frac{dW}{dt} = \frac{dx_{11}}{dt}x_{22} + x_{11}\frac{dx_{22}}{dt} - \frac{dx_{12}}{dt}x_{21} - x_{12}\frac{dx_{21}}{dt}$$

$$= (a_{11}x_{11} + a_{12}x_{12})x_{22} + x_{11}(a_{21}x_{21} + a_{22}x_{22})$$

$$- (a_{21}x_{11} + a_{22}x_{12})x_{21} - (a_{11}x_{21} + a_{12}x_{22})x_{12}$$

$$= (a_{11} + a_{22})(x_{11}x_{22} - x_{12}x_{21})$$ (after the dust settles)

$$\boxed{\frac{dW}{dt} = tr(A)\, W(t)}$$

↳ $$W(t) = W(t_0)\, e^{\int_{t_0}^{t} tr(A(s))\,ds}$$

Example: Euler Equations

$$ax^2y'' + bxy' + cy = 0$$

Note, letting $u = y'$ and dividing by ax^2, we get:

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0$$

$$\Rightarrow u' + \frac{b}{ax}u + \frac{c}{ax^2}y = 0$$

$$\Rightarrow \frac{d}{dx}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{ax^2} & -\frac{b}{ax} \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$$

To solve this, we guess... let $y(x) = x^\lambda$

$$\Rightarrow y' = \lambda x^{\lambda-1} \; ; \; y'' = \lambda(\lambda-1)x^{\lambda-2}$$

$$\Rightarrow ax^2y'' + bxy' + cy = (a\lambda(\lambda-1) + b\lambda + c)x^\lambda = 0$$

$$\Rightarrow a\lambda(\lambda-1) + b\lambda + c = 0$$

$$\Rightarrow a\lambda^2 + (b-a)\lambda + c = 0.$$

So $\lambda_{+/-} = \frac{1}{2}\left((b-a) \pm \left((b-a)^2 - 4ac\right)^{1/2}\right)$

Thus our candidate solutions are

$$y_1(x) = x^{\frac{1}{2}(b-a+((b-a)^2-4ac)^{1/2})} = x^{\lambda_+}$$

and $y_2(x) = x^{\frac{1}{2}(b-a-((b-a)^2-4ac)^{1/2})} = x^{\lambda_-}$

are these even linearly independent?

Well, certainly not if $(b-a)^2 = 4ac$, so
$(\lambda_+ = \lambda_-)$

1) Suppose $(b-a)^2 \neq 4ac$. $(\lambda_+ \neq \lambda_-)$

So $A(x) = \begin{pmatrix} 0 & 1 \\ -\frac{c}{ax^2} & -\frac{b}{ax} \end{pmatrix}$

$\Rightarrow \operatorname{tr}(A(x)) = -\frac{b}{ax}$

$\Rightarrow W(x) = W(x_0)e^{\int_{x_0}^{x} -\frac{b}{as}\,ds} = W(x_0)e^{-\frac{b}{a}\ln\left|\frac{x}{x_0}\right|}$

or $W(x) = W(x_0)\left|\frac{x_0}{x}\right|^{b/a}$

So, now we need to make only one computation at a reasonable point and then consider cases.

I'd choose $x_0 = 1$ $\leadsto$ $1^{\lambda_+}=1,\ 1^{\lambda_-}=1.$

and $y_1'(x) = \lambda_+ x^{\lambda_+-1}$; $y_2'(x) = \lambda_- x^{\lambda_--1}$

so $y_1'(1) = \lambda_+$; $y_2'(1) = \lambda_-$

so $W(1) = \det\left(\begin{pmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1)\end{pmatrix}\right)$

$= \det\begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_-\end{pmatrix} = \lambda_- - \lambda_+$

so $W(x) = -((b-a)^2 - 4ac)^{1/2}\,|x|^{-b/a}$

if $(b-a)^2 \neq 4ac$, then $W(x)=0$ if $-b/a>0$ and $x=0$,

otherwise <u>l.i.</u>