The Phase Plane: $\frac{d\vec{x}}{dt} = A\vec{x}$

for 2×2 A, let $(\operatorname{tr}(A))^2 - 4\det(A) > 0$

then $\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \left((\operatorname{tr}(A))^2 - 4\det(A)\right)^{1/2}\right)$

so now $\lambda_{1,2} \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$

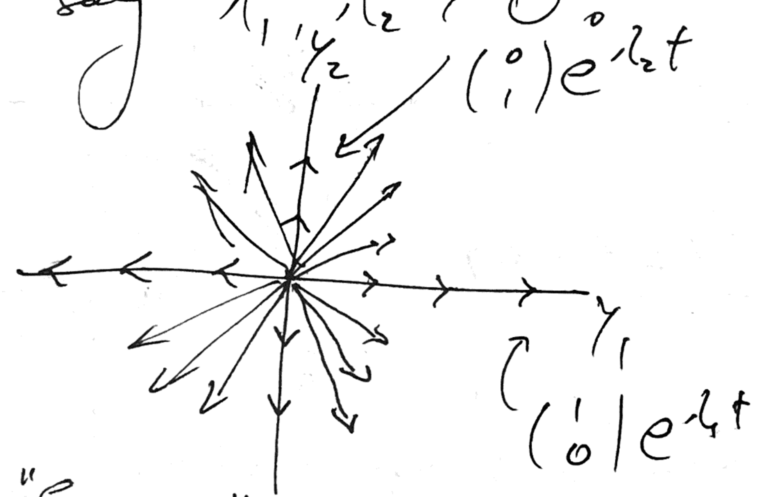
then $A = V\Lambda V^{-1}$, $V = (\vec{v}_1 | \vec{v}_2)$, $A\vec{v}_j = \lambda_j \vec{v}_j$, $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

using $\vec{y} = V^{-1}\vec{x}$ $\longrightarrow$

$$\frac{d\vec{y}}{dt} = \Lambda \vec{y} \longrightarrow \frac{dy_j}{dt} = \lambda_j y_j \longrightarrow y_j(t) = c_j e^{\lambda_j t}$$

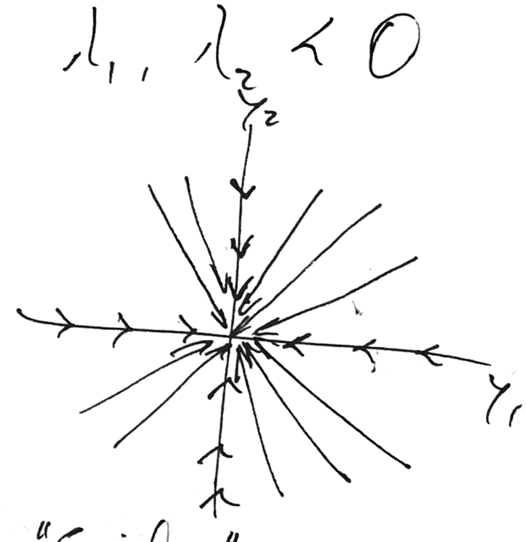
then $\vec{y} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 t}$

So say $\lambda_1, \lambda_2 > 0$:



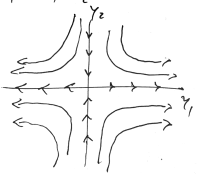
"Source"

$\lambda_1, \lambda_2 < 0$

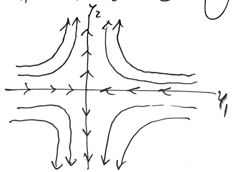


"Sink"

$\lambda_1 > 0,\ \lambda_2 < 0$



$\lambda_1 < 0,\ \lambda_2 > 0$



"Saddle"

So now, what happens when we go back to $\vec{x}$?

Again: $\vec{y} = V^{-1}\vec{x}$ or $\vec{x} = V\vec{y} = (\vec{v}_1 | \vec{v}_2)$

So: $\vec{y} = c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix}0\\1\end{pmatrix} e^{\lambda_2 t}$

So $\vec{x} = V\left(c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix}0\\1\end{pmatrix} e^{\lambda_2 t}\right)$

$= c_1 e^{\lambda_1 t} V \begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{\lambda_2 t} V \begin{pmatrix}0\\1\end{pmatrix}$

$= c_1 e^{\lambda_1 t} \underline{\vec{v}_1} + c_2 e^{\lambda_2 t} \underline{\vec{v}_2}$

So now we follow $\vec{v}_1$ & $\vec{v}_2$.

Example: let $\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \vec{x}$ (3)

If we want to plot the phase plane we

- Find e-vals, e-vecs

$\hookrightarrow \lambda^2 - tr(A)\lambda + det(A) = 0$

$\hookrightarrow \lambda^2 - (-3+3)\lambda + (-9-16) = 0$

$\hookrightarrow \lambda^2 - 25 = 0 \longrightarrow \lambda = \pm 5$

$\lambda_1 = -5$: $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$\hookrightarrow \left(\begin{array}{cc|c} 2 & 4 & 0 \\ 4 & 8 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\hookrightarrow v_1 + 2v_2 = 0$ or $v_1 = -2v_2$.

$\rightarrow \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda_2 = 5$: $(A - \lambda_2 I)\vec{v}_2 = 0$

$\hookrightarrow \left(\begin{array}{cc|c} -8 & 4 & 0 \\ 4 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow -2v_1 + v_2 = 0$

or $v_2 = 2v_1$

$\hookrightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

So again: $A = V \Lambda V^{-1}$

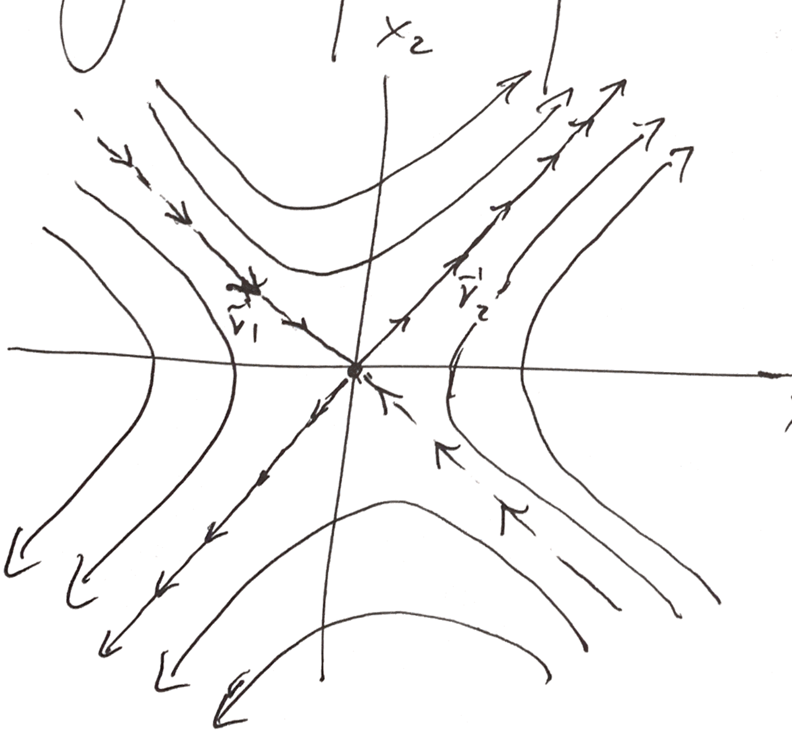
$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}.$$

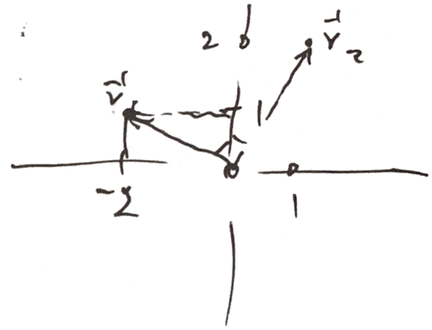
and $\vec{x} = c_1 \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{= \vec{v}_1} e^{\underline{-5t}} + c_2 \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{= \vec{v}_2} e^{\underline{5t}}$

decay

growth

now as for the phase plane:





note $\vec{v}_1 \cdot \vec{v}_2 = 0$

So $\perp$.