#4) Show solution $\vec{x}$ $\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{g}(t)$; $\vec{x}(0) = \vec{x}_0$

can be written as

$\vec{x} = \vec{u} + \vec{v}$

II Put together

I Solve these two problems.

where: $\frac{d\vec{u}}{dt} = A\vec{u}$; $\vec{u}(0) = \vec{x}_0$

$\frac{d\vec{v}}{dt} = A(t)\vec{v} + \vec{g}(t)$; $\vec{v}(0) = 0$

Step I: Solve subproblems i.e. $\vec{u}$ & $\vec{v}$ problems

Step II: Add $\vec{u}$ & $\vec{v}$ to find solution to original problem.

$$t^2 y'' + 4t y' + y = e^{-t}; \quad y(0) = y_0; \; y'(0) = y_1$$

$\to$ $y(t) = t^\lambda = e^{\lambda \ln t}$ so let $\underline{x = \ln t}$

$$\lambda(\lambda - 1) + 4\lambda + 1 = 0$$

$$\lambda^2 + 3\lambda + 1 = 0$$

$$\to \lambda = \frac{1}{2}\left\{-3 \pm (9-4)^{1/2}\right\}$$

$$\lambda_{+/-} = \frac{1}{2}\left\{-3 \pm \sqrt{5}\right\}$$

$\to$ $\underline{y(1) = y_0 \; ; \; y'(1) = y_1}$

$$\underline{y(t) = c_1 t^{\lambda_+} + c_2 t^{\lambda_-}}$$

$\to$ $y(1) = c_1 + c_2 = y_0$

$y'(1) = \lambda_+ c_1 + \lambda_- c_2 = y_1$

$$\to \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

Ex: Solve

$$y'' + y = e^{-t}\cos(2t) \; ; \; y(0) = 2, \; y'(0) = 1$$

$\boxed{\text{pg } 39}$

↳ $\mathcal{L}\{y'\} = sY - y(0)$

$\mathcal{L}\{y''\} = s^2 Y - s y(0) - y'(0)$

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \; ; \; \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$$

↳ $(s^2+1)Y - 2s - 1 = \dfrac{(s+1)}{(s+1)^2+4}$

↳ $Y = \dfrac{(2s+1)}{s^2+1} + \dfrac{(s+1)}{(s+1)^2+4} \dfrac{1}{1+s^2}$

fine, corresponds to Y_h

needs a little more work, corresponds to Y_p.

So:

$$\frac{(s+1)}{\left((s+1)^2+4\right)(s^2+1)} = \frac{as+b}{s^2+1} + \boxed{\frac{c(s+1)+2d}{(s+1)^2+2^2}}$$

$$\hookrightarrow \boxed{(s+1) = (as+b)\left((s+1)^2+4\right) + \left(c(2s+1)+2d\right)(s^2+1)}$$

$$\hookrightarrow \begin{aligned} 5b+c+2d &= 1 &&: s^0 \\ 5a+2b+c &= 1 &&: s^1 \\ 2a+b+c+2d &= 0 &&: s^2 \\ a+c &= 0 &&: s^3 \end{aligned}$$

$$\hookrightarrow a = -c$$

$$\hookrightarrow \begin{aligned} 5b+c+2d &= 1 \\ 2b-4c &= 1 \\ b-c+2d &= 0 \end{aligned} \longrightarrow \boxed{\left(\begin{array}{ccc|c} 5 & 1 & 2 & 1 \\ 2 & -4 & 0 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right)}$$

$\hookrightarrow$ hoor fun... zu pg. 322

In contrast, once we know for Ansatz

$$y''+y=e^{-t}\cos(2t) \qquad y=e^{\lambda t} \rightarrow \lambda^2+1=0$$

$\hookrightarrow$ Homogeneous: $y_1=\cos(t)$, $y_2=\sin(t)$

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}$$

$\hookrightarrow W=y_1y_2'-y_1'y_2=1$ $\qquad \lambda=\mp i$

$\hookrightarrow y_h=c_1\cos(t)+c_2\sin(t)$ $\qquad \hookrightarrow y=c_1e^{it}+c_2e^{-it}$

$=c_1(\cos(t)+i\sin(t))$

$\hookrightarrow c_1=y(0)=2;\ c_2=y'(0)=1$ $\qquad =c_1\cos(t)+c_2\sin(t)$

Scalar form of:

$\hookrightarrow$ Variation of Parameters

$$y(t)=2\cos(t)+\sin(t)-\cos(t)\int_0^t \sin(s)e^{-s}\cos(2s)\,ds$$

$$+\sin(t)\int_0^t \cos(s)\cos(2s)e^{-s}\,ds$$

$$\sin(s)\cos(2s)=\frac{1}{2}\left(\sin(s+2s)+\sin(s-2s)\right)$$

$$\cos(s)\cos(2s)=\frac{1}{2}\left(\cos(s-2s)+\cos(s+2s)\right)\ \ldots$$

Likewise, we can generalize to

$$a y'' + b y' + c y = f(t), \quad y(0) = y_0, \; y'(0) = y_1$$

$$\hookrightarrow a(s^2 \bar{Y} - s y_0 - y_1) + b(s\bar{Y} - y_0) + c\bar{Y} = F(s)$$

$$\hookrightarrow (as^2 + bs + c)\bar{Y} - a s y_0 - a y_1 - b y_0 = F(s)$$

$$\hookrightarrow \bar{Y} = \underbrace{\frac{(as+b)y_0 + a y_1}{as^2 + bs + c}}_{\text{Homogeneous}} + \underbrace{\frac{F(s)}{as^2 + bs + c}}_{\text{Particular}}$$

Or... using variation of parameters (scalar form):

$$y = e^{\lambda t} \hookrightarrow a\lambda^2 + b\lambda + c = 0$$

$$\hookrightarrow \lambda_{+/-} = \frac{1}{2a}\left(-b \pm (b^2 - 4ac)^{1/2}\right)$$

$$\rightarrow y_1 = e^{\lambda_+ t}; \; y_2 = e^{\lambda_- t} \quad (b^2 - 4ac \neq 0)$$

↳ $W = y_1 y_2' - y_1' y_2 = (\lambda_- - \lambda_+) e^{(\lambda_+ + \lambda_-)t}$

↳ Initial Conditions: $y_1(t) = e^{\lambda_+ t}$; $y_2(t) = e^{\lambda_- t}$

↳ $c_1 y_1(0) + c_2 y_2(0) = c_1 + c_2 = y_0$

$c_1 y_1'(0) + c_2 y_2'(0) = c_1 \lambda_+ + c_2 \lambda_- = y_1$

↳ $\begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

↳ $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

$$y_p(t) = \frac{1}{\lambda_- - \lambda_+} \left\{ -e^{\lambda_+ t} \int_0^t e^{-\lambda_- s} f(s)\,ds + e^{\lambda_- t} \int_0^t e^{-\lambda_+ s} f(s)\,ds \right\}$$

I'll take my chances here. Thanks.

And for systems:

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{g}(t) \; ; \; x(0) = \vec{x}_0$$

$$\Rightarrow s\vec{\bar{X}} - \vec{x}_0 = A\vec{\bar{X}} + \vec{G}(s)$$

$$\Rightarrow (sI - A)\vec{\bar{X}} = \vec{x}_0 + \vec{G}(s)$$

$$\Rightarrow \vec{\bar{X}} = (sI - A)^{-1}\vec{x}_0 + (sI - A)^{-1}\vec{G}(s)$$

becomes below
by eval

or:

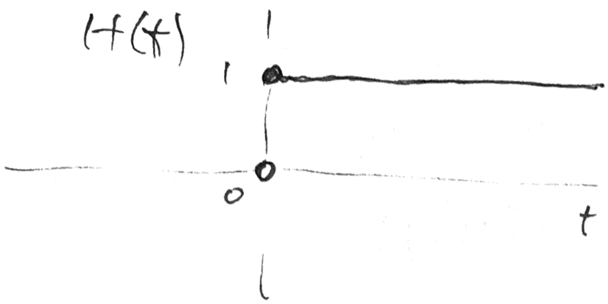
find $A = V\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1} \left(\text{or } = V\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1} \ldots \right)$

$$\Rightarrow \vec{x}(t) = V\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1}\vec{x}_0 + V\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-\lambda_1 s} & 0 \\ 0 & e^{-\lambda_2 s} \end{pmatrix} V^{-1}\vec{g}(s)\,ds$$

5.5 Discontinuous forcings

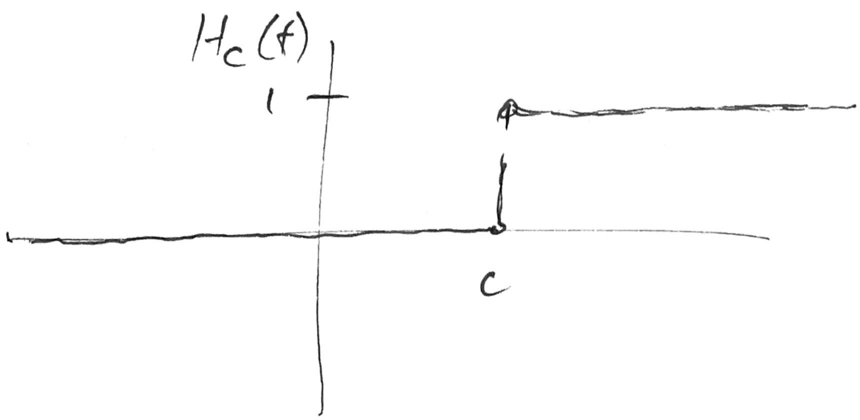
Heaviside Function:

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



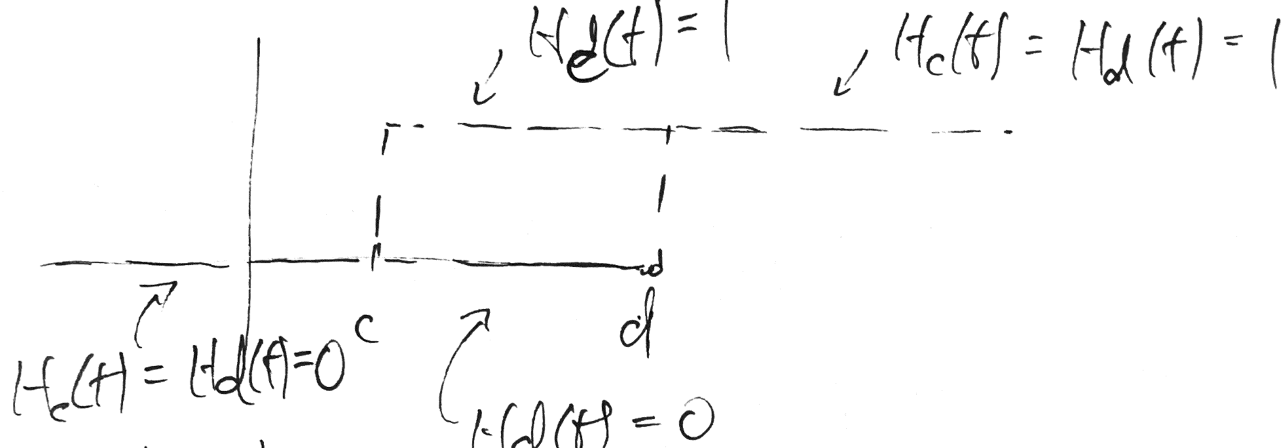
$$H_c(t) = H(t-c) = \begin{cases} 0 & t-c < 0 \\ 1 & t-c \geq 0 \end{cases}$$

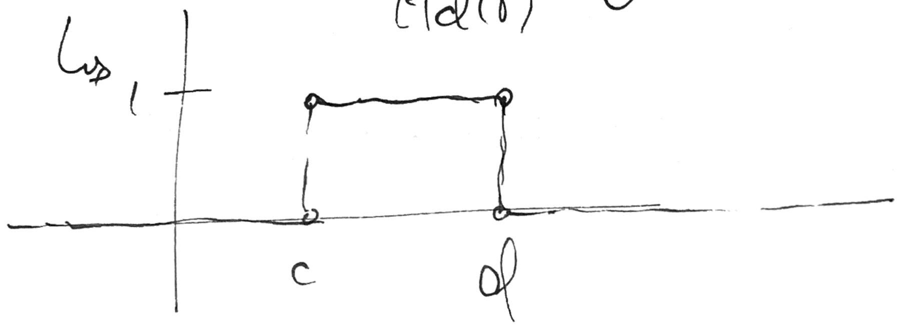
$$= \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



So let $d > c$

↳ $H_c(t) - H_d(t)$



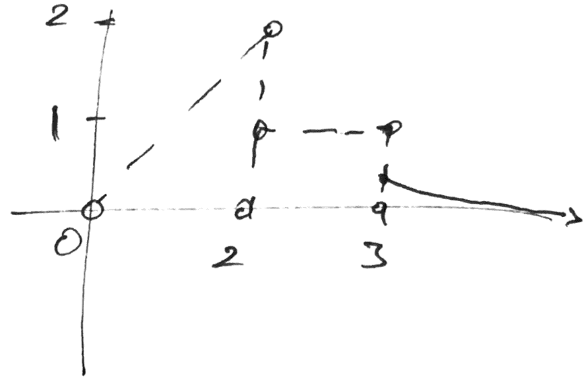
↳ 

Let: $f(t) = \begin{cases} 0 & t < 2 \\ 3 & 2 < t < 4 \\ 0 & t > 4 \end{cases}$

↳ $f(t) = 3\left(H_2(t) - H_4(t)\right)$

$= 3 H_{(2,4)}(t)$

Let $f(t) = \begin{cases} t & 0 < t < 2 \\ 1 & 2 \le t \le 3 \\ e^{-2t} & 3 \le t \end{cases}$



So $f(t) = t(H_{(0,2)}(t)) + H_{[2,3)}(t) + e^{-2t} H_{[3,\infty)}(t)$

$H_0(t) = 1$ for $t \geq 0$

$$= t(1 - H_2(t)) + (H_2(t) - H_3(t)) + e^{-2t} H_3(t)$$

$$= t + (1-t) H_2(t) + (e^{-2t} - 1) H_3(t)$$

Now:

$$\mathcal{L}\{H_c(t)\} = \int_0^\infty e^{-st} H(t-c)\,dt = \int_0^c + \int_c^\infty$$

$$= \int_c^\infty e^{-st}\,dt$$

$$= -\frac{1}{s} e^{-st} \Big|_c^\infty = \frac{1}{s} e^{-sc}; \quad s > 0.$$

Shifted Functions:

Say $g(t) \equiv \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$

or $g(t) = H_c(t)\, f(t-c)$

Los $\mathcal{L}\{H_c(t) f(t-c)\} = \int_0^\infty e^{-st} H_c(t) f(t-c)\,dt$

$$= \int_0^c e^{-st} H_c(t) f(t-c)\,dt + \int_c^\infty e^{-st} H_c(t) f(t-c)\,dt$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt$$

$$\tilde{t} = t - c \rightarrow t = \tilde{t} + c$$

$$= e^{-sc} \int_0^\infty e^{-s\tilde{t}} f(\tilde{t})\,d\tilde{t}$$

$$= e^{-sc} F(s).$$

So now for

$$f(t) = \begin{cases} t & 0 < t < 2 \\ 1 & 2 \le t < 3 \\ e^{-2t} & 3 \le t \end{cases}$$

$\rightarrow$ $f(t) = t + (1-t)H_2(t) + (e^{-2t} - 1)H_3(t)$

$= t + (1-(t-2+2))H_2(t) + (e^{-2(t-3+3)} - 1)H_3(t)$

$= t - (1+(t-2))H_2(t) + (e^{-6}e^{-2(t-3)} - 1)H_3(t)$

$\rightarrow$ $\mathcal{L}\{f\} = \frac{1}{s^2} - \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-2s} + \left(\frac{e^{-6}}{s+2} - \frac{1}{s}\right)e^{-3s}.$