or

$$\mathcal{L}\{L[f]\} = (as^2+bs+c)F(s) + (-as-b)f(0) - af'(0).$$

So say we look at:

$$y'' + 2y' + 5y = e^{-t}\ ; \quad y(0) = 1,\ y'(0) = -3$$

$\Rightarrow a = 1,\ b = 2,\ c = 5$

$$\mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st}e^{-t}\,dt = \frac{1}{s+1}$$

$\Rightarrow$ if we let $Y(s) = \mathcal{L}\{y\}$:

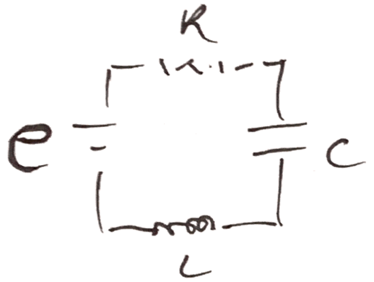
$$(s^2+2s+5)Y - (s+2)y(0) - y'(0) = \frac{1}{s+1}$$

$$\Rightarrow (s^2+2s+5)Y - (s+2) + 3 = \frac{1}{s+1}$$

$$\Rightarrow (s^2+2s+5)Y = \frac{1}{s+1} + s - 1 = \frac{s^2}{s+1}$$

$$\Rightarrow \boxed{Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}}$$

LRC - Circuit



$\longrightarrow \quad i = \frac{dq}{dt}$

$$L\frac{di}{dt} + Ri + \frac{1}{C} q = e$$

w/ $q(0) = q_0$; $i(0) = i_0 = q'(0)$

Lap $\mathcal{L}\{i\} = I(s) = sQ(s) - q_0$ where $Q(s) = \mathcal{L}\{q\}$

and $L\left(sI(s) - i_0\right) + RI + \frac{1}{C} Q = E(s)$

Lap $sQ(s) - I(s) = q_0$

$\frac{1}{C} Q(s) + (sL + R) I = E(s) + i_0 L$

Lap
$$\begin{pmatrix} s & -1 \\ \frac{1}{C} & sL+R \end{pmatrix} \begin{pmatrix} Q(s) \\ I(s) \end{pmatrix} = \begin{pmatrix} q_0 \\ E(s) + i_0 L \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} Q(s) \\ I(s) \end{pmatrix} = \frac{1}{s(sL+R)+\frac{1}{C}} \begin{pmatrix} sL+R & 1 \\ -1/C & s \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 \\ E(s)+i_0 L \end{pmatrix}$$

or

$$Q(s) = \frac{1}{s(sL+R)+\frac{1}{C}} \left((sL+R)\mathcal{E}_0 + E(s) + i_0 L \right)$$

$$I(s) = \frac{1}{s(sL+R)+\frac{1}{C}} \left(-\frac{1}{C}\mathcal{E}_0 + s\left(E(s)+i_0 L\right) \right)$$

<u>End of 5.2</u>

So, going back to our example

$$y'' + 2y' + 5y = e^{-t}, \quad y(0) = 1, \quad y'(0) = -3$$

leads to $Y(s) = \dfrac{s^2}{(s+1)(s^2+2s+5)}$; $Y(s) = \mathcal{L}\{y\}$

So, how do we undo it?

So one of the big results here is $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$

or $e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}$

so we leverage that $s+1$ in the denominator so that

$$Y(s) = \frac{(s+1-1)^2}{(s+1)((s+1-1)^2 + 2(s+1-1)+5)}$$

$$= \frac{(s+1)^2 - 2(s+1) + 1}{(s+1)((s+1)^2 - 2(s+1) + 1 + 2(s+1) - 2 + 5)}$$

$$= \frac{(s+1)^2 - 2(s+1) + 1}{(s+1)((s+1)^2 + 4)} = \mathcal{L}\{e^{-t}\tilde{y}(t)\}$$

let $\tilde{Y}(\tilde{s}) = \mathcal{L}\{\tilde{y}(t)\} = \dfrac{\tilde{s}^2 - 2\tilde{s} + 1}{\tilde{s}(\tilde{s}^2 + 4)}$

So, we carry with this as

$$\frac{\tilde{s}^2 - 2\tilde{s} + 1}{\tilde{s}(\tilde{s}^2+4)} = \frac{A}{\tilde{s}} + \frac{(B\tilde{s}+C)}{\tilde{s}^2+4}$$

↳ $\tilde{s}^2 - 2\tilde{s} + 1 = A(\tilde{s}^2+4) + \tilde{s}(B\tilde{s}+C)$

$= (A+B)\tilde{s}^2 + C\tilde{s} + 4A$

↳ $A+B=1$; $C=-2$; $4A=1 \rightarrow A=\frac{1}{4} \rightarrow B=\frac{3}{4}$

↳ $\tilde{Y}(\tilde{s}) = \frac{1}{4\tilde{s}} + \frac{3}{4}\frac{\tilde{s}}{\tilde{s}^2+4} - \frac{2}{\tilde{s}^2+4}$; we find $\mathcal{L}^{-1}$ of each

↳ $\tilde{y}(t) = \frac{1}{4} + \frac{3}{4}\cos(2t) - \sin(2t)$

↳ $y(t) = e^{-t}\left(\frac{1}{4} + \frac{3}{4}\cos(2t) - \sin(2t)\right)$

Here we used:

$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$; $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$; $\mathcal{L}\{1\} = \frac{1}{s}$.

So now we need techniques for breaking up functions to help us use tables to find $\mathcal{L}^{-1}$ i.e. if we have

$$F(s) = \mathcal{L}\{f\}$$

$\hookrightarrow$ find $f(t) = \mathcal{L}^{-1}\{F\}$.

So if we get something literally off the table, we can get a quick answer.

Ex: $F(s) = \dfrac{4}{s^2+16} = \dfrac{4}{s^2+4^2} \to f(t) = \sin(4t)$

$$F(s) = \frac{6}{(s+2)^4} \Rightarrow F(s) = \tilde{F}(s+2)$$

where $\tilde{F}(s) = \dfrac{6}{s^4} = \mathcal{L}\{t^3\}$

$\hookrightarrow$ $F(s) = \mathcal{L}\{e^{-2t}t^3\}$

or $f(t) = e^{-2t}t^3$.

And since $\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 F_1(s) + c_2 F_2(s)$

$$\Rightarrow c_1 f_1 + c_2 f_2 = \mathcal{L}^{-1}(c_1 F_1(s) + c_2 F_2(s))$$
$$= c_1 \mathcal{L}^{-1}\{F_1\} + c_2 \mathcal{L}^{-1}\{F_2\}$$

Ex: $\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^4} + \frac{3}{s^2+16} + \frac{5(s+1)}{s^2+2s+5}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{5(s+1)}{(s+1)^2+4}\right\}$$

$$= \frac{2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-2t}t^3 + \frac{3}{4}\sin(4t) + 5e^{-t}\cos(2t)$$

and that's all adorable, but it doesn't help us avoid partial fractions.

Ex (Easy): $\dfrac{(s-2)}{s^2+4s-5} = \dfrac{(s-2)}{(s+5)(s-1)}$

$$= \frac{a_1}{s-1} + \frac{a_2}{s+5}$$

$\hookrightarrow$ $s-2 = a_1(s+5) + a_2(s-1)$

$$= (a_1+a_2)s + 5a_1 - a_2$$

$\hookrightarrow$ $a_1 + a_2 = 1$

$5a_1 - a_2 = -2$ $\longrightarrow \begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{-6}\begin{pmatrix} -1 & -1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= -\frac{1}{6}\begin{pmatrix} 1 \\ -7 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 7/6 \end{pmatrix}$$

$\hookrightarrow$ $F(s) = -\frac{1}{6}\left(\frac{1}{s-1}\right) + \frac{7}{6}\left(\frac{1}{s+5}\right)$

$\hookrightarrow$ $f(t) = -\frac{1}{6}e^{t} + \frac{7}{6}e^{-5t}$.

Ex:

$$\hookrightarrow \frac{s^2+20s+31}{(s+2)^2(s-3)} = \frac{a_1}{s-3} + \frac{a_2}{(s+2)} + \frac{a_3}{(s+2)^2}$$

$$s^2+20s+31 = a_1(s+2)^2 + a_2(s-3)(s+2) + a_3(s-3)$$

So we can go a little faster by checking

$$s=3: \quad 25a_1 = 100 \longrightarrow a_1 = 4$$

$$s=-2: \quad -5a_3 = -5 \longrightarrow a_3 = 1$$

$$s=0: \quad 4a_1 - 6a_2 - 3a_3 = 31$$

$$\hookrightarrow -6a_2 = 31-13 = 18 \longrightarrow a_2 = -3.$$

but note, the gold standard is expand and match i.e.

$$s^2+20s+31 = a_1(s^2+4s+4) + a_2(s^2-s-6) + a_3(s-3)$$

$$= (a_1+a_2)s^2 + (4a_1 - a_2 + a_3)s + (4a_1 - 6a_2 - 3a_3)$$

$$\hookrightarrow \quad a_1 + a_2 = 1$$

$$4a_1 - a_2 + a_3 = 20$$

$$4a_1 - 6a_2 - 3a_3 = 31$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 4 & -1 & 1 \\ 4 & -6 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 20 \\ 31 \end{pmatrix}$$