Oscillatory Forcing in Underdamped Systems

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

See 4.6 in text

let $b^2 - 4ac < 0$ $\qquad (4.6) \rightarrow \underline{y = \dot{x}}$

↳ $\lambda_1 = -\mu + i\omega$, $\lambda_2 = \lambda_1^* = -\mu - i\omega$

So

$$\begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} = \frac{i}{2\omega}\left\{ \left(\int_0^t f(s) e^{(-\mu - i\omega)(t-s)} ds \right) \begin{pmatrix} 1 \\ -\mu - i\omega \end{pmatrix} - \left(\int_0^t f(s) e^{(-\mu + i\omega)(t-s)} ds \right) \begin{pmatrix} 1 \\ -\mu + i\omega \end{pmatrix} \right\}$$

So, really can just focus on $x_p(t)$ ...

$$\cancel{\int_0^t f(s) e^{(-\mu + i\omega)(t-s)} ds}$$

$\Rightarrow$

$$x_p(t) = \frac{i}{2\omega}\left\{ \int_0^t f(s)e^{(-\mu-i\omega)(t-s)}ds - \int_0^t f(s)e^{(-\mu+i\omega)(t-s)}ds \right\}$$

$$= \frac{i}{2\omega}\int_0^t f(s)e^{-\mu(t-s)}\left(e^{-i\omega(t-s)} - e^{i\omega(t-s)}\right)ds$$

Using: $\boxed{\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)}$ ☆!☆!☆!

$\frac{1}{i} = -i$

$\Rightarrow$

$$x_p(t) = \frac{1}{\omega}\int_0^t f(s)e^{-\mu(t-s)}\sin(\omega(t-s))\,ds$$

Resistance / Friction

oscillatory part of system

so if we look at $f(s) = \cos(\upsilon s)$

$$\Rightarrow \cos(\upsilon s)\sin(\omega(t-s)) = \frac{1}{2}\left\{ \sin(\omega(t-s)+\upsilon s) + \sin(\omega(t-s)-\upsilon s) \right\}.$$

$$= \frac{1}{2}\left\{ \sin(\omega t + (\upsilon-\omega)s) + \sin(\omega t - (\omega+\upsilon)s) \right\}.$$

$$f(s) = \cos(\sigma s);$$

$$x_p(t) = \frac{1}{2\omega}\left\{ \int_0^t e^{-\mu(t-s)} \sin(\omega t + (\sigma - \omega)s)\, ds + \int_0^t e^{-\mu(t-s)} \sin(\omega t - (\omega + \sigma)s)\, ds \right\}.$$

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2 paths $\boxed{4.6}$

1: Integration by Parts (Horribly onerous)

2: Use $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$$x_p(t) = \frac{1}{2\omega}\cdot\frac{1}{2i}\left\{ \int_0^t e^{-\mu(t-s)} \left(e^{i(\omega t + (\sigma-\omega)s)} - c.c.\right) + \int_0^t e^{-\mu(t-s)} \left(e^{i(\omega t - (\sigma+\omega)s)} - c.c.\right) \right\}.$$

$$x_p(t) = \frac{1}{2\omega}\left(\frac{\mu}{\mu^2+(\upsilon-\omega)^2}\left(\sin(\underbrace{(\omega+(\upsilon-\omega))t}_{=\upsilon t}) - \sin(\omega t)e^{-\mu t}\right)\right.$$

$$+\frac{(\upsilon-\omega)}{\mu^2+(\upsilon-\omega)^2}\left(-\cos(\underbrace{(\omega+(\upsilon-\omega))t}_{=\upsilon t}) + \cos(\omega t)e^{-\mu t}\right)$$

$$+\frac{\mu}{\mu^2+(\upsilon+\omega)^2}\left(\sin(\underbrace{(\omega-(\omega+\upsilon))t}_{=-\upsilon t}) - \sin(\omega t)e^{-\mu t}\right)$$

$$\left.-\frac{(\omega+\upsilon)}{\mu^2+(\omega+\upsilon)^2}\left(-\cos(\underbrace{(\omega-(\omega+\upsilon))t}_{=-\upsilon t}) + \cos(\omega t)e^{-\mu t}\right)\right)$$

$$\cancel{e^{i\upsilon t} + e^{(-\mu+i\omega)t} = \cos(\upsilon t) - e^{-\mu t}\cos(\omega t) - i e^{-\mu t}\sin(\omega t)}$$

at $t \to \infty$:

$$x_p(t) = \frac{1}{2\omega}\left\{ \frac{\mu}{\mu^2+(\sigma-\omega)^2} \sin(\sigma t) - \frac{(\sigma-\omega)}{\mu^2+(\sigma-\omega)^2} \cos(\sigma t) \right.$$

$$\left. + \frac{(\omega+\sigma)}{\mu^2+(\omega+\sigma)^2} \cos(\sigma t) - \frac{\mu}{\mu^2+(\sigma+\omega^2)} \sin(\sigma t) \right\}$$

$$= \frac{1}{2\omega}\left\{ -\sqrt{\mu^2+(\sigma-\omega)^2} \cos(\sigma t + \varphi_-) \right.$$

$$\left. + \sqrt{\mu^2+(\omega+\sigma)^2} \cos(\sigma t + \varphi_+) \right\}.$$

$$\cos(\varphi_-) = \frac{\sigma-\omega}{\sqrt{\mu^2+(\sigma-\omega)^2}}; \quad \sin(\varphi_-) = \frac{\mu}{\sqrt{\mu^2+(\sigma-\omega)^2}}$$

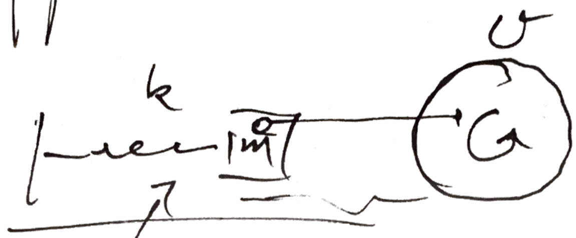
etc...

$$\cos(\varphi_+) = \frac{(\omega+\sigma)}{\sqrt{\mu^2+(\sigma+\omega)^2}}; \quad \sin(\varphi_+) = \frac{\mu}{\sqrt{\mu^2+(\sigma+\omega)^2}}$$

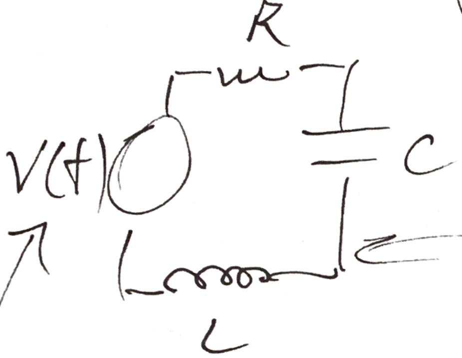
What happens at $\nu \to \omega$?

$\Downarrow$

Lo



natural freq. response is $\underline{\omega}$



natural freq. response is ω

driven w/ ν

Driving system to resonance

Let $\sigma = \omega$:

$$x_p(t) = \frac{1}{2\omega}\left(\frac{1}{\mu}\sin(\omega t) + \frac{2\omega}{\mu^2+4\omega^2}\cos(\omega t) - \frac{\mu}{\mu^2+4\omega^2}\sin(\omega t)\right)$$

$$= \frac{1}{2\omega}\left(\frac{1}{\mu}\sin(\omega t) + \sqrt{\mu^2+4\omega^2}\cos(\omega t + \varphi_+)\right)$$

if $0 < \mu \ll 1 \Rightarrow \frac{1}{\mu} \gg 1$