$$e^{i\theta} = \cos(\theta) + i\,\mathrm{sen}(\theta)$$

$$\hookrightarrow e^{i\pi} = \underbrace{\cos(\pi)}_{\to -1} + i\,\underbrace{\mathrm{sen}(\pi)}_{\to 0}$$

$$\hookrightarrow e^{i\pi} = -1 \to e^{i\pi} + 1 = 0$$

$$z \cdot z^{*} = |z|^2 \text{ or } |z| = \sqrt{z \cdot z^{*}}$$

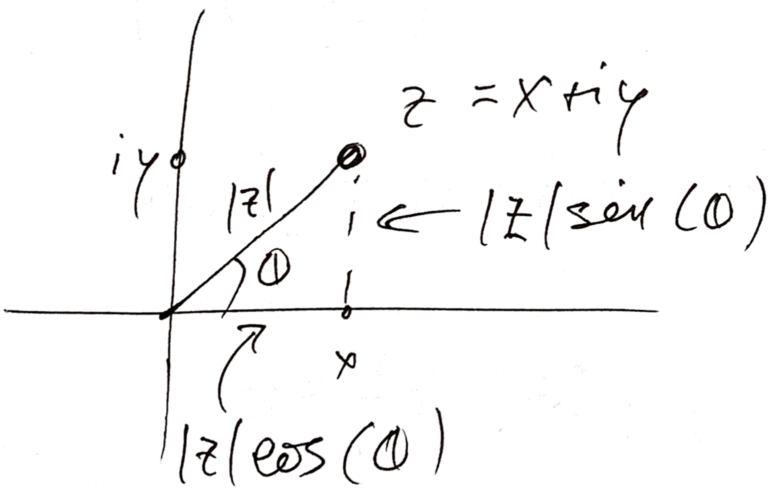
$i = \sqrt{-1}$

$i^2 = \left((-1)^{1/2}\right)^2 = -1$

$i^3 = -i$

$i^4 = 1$; $i^{-1} = -i$

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{(i)^{*}}{(i)^{*}} = \frac{-i}{i \cdot (-i)} = -i$$

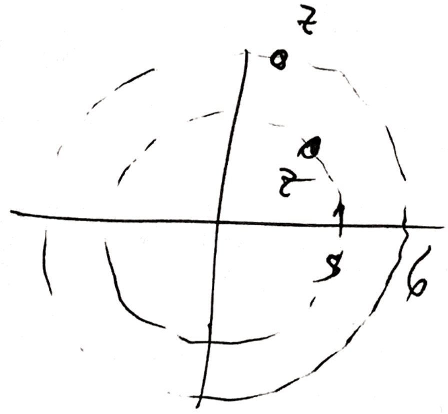


$$z = x + iy = |z|\cos(\theta) + i|z|\,\mathrm{sen}(\theta)$$

$$= |z|\left(\cos(\theta) + i\,\mathrm{sen}(\theta)\right)$$

$$= |z|\,e^{i\theta}$$

$|z| = 5 < |\tilde{z}| = 6$



Complex E-vals:

for $\frac{d\vec{x}}{dt} = A\vec{x}$

we know $\lambda_1, \lambda_2 = \frac{1}{2}\left(tr(A) \pm \left((tr(A))^2 - 4\det(A)\right)^{1/2}\right)$

$$= \frac{1}{2}tr(A) \pm \left(\left(\frac{tr(A)}{2}\right)^2 - \det(A)\right)^{1/2}$$

So, if $\left(\frac{tr(A)}{2}\right)^2 < \det(A) \to$

$\lambda_1, \lambda_2 = \mu \pm i\omega$ where

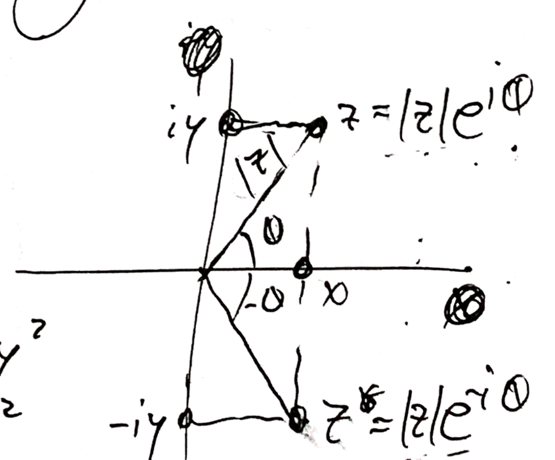
$$\mu = \frac{1}{2}tr(A) \; ; \; \omega = \left(\left|\left(\frac{tr(A)}{2}\right)^2 - \det(A)\right|\right)^{1/2}$$

note, since $\lambda_1 = \mu + i\omega$ we say $\lambda_2 = \lambda_1^*$ "Conjugate"

$\underline{\lambda_2 = \mu - i\omega}$

in general, for $z = x+iy$, $z^* = x-iy$

$$z \cdot z^* = (x+iy)(x-iy) = x^2+y^2 = |z|^2$$



So, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\underline{a_{ij} \in \mathbb{R}}$

↳ $a_{ij}^* = a_{ij} \rightarrow$ <u>no imag. part.</u>

↳ $A^* = A$

So if $A\vec{v} = \lambda \underline{\vec{v}} \Rightarrow (A\vec{v})^* = (\lambda\vec{v})^*$

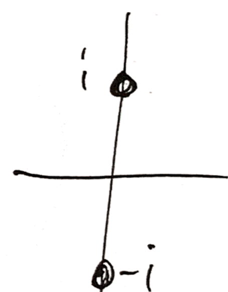
$$\Rightarrow A^*\vec{v}^* = \lambda^*\vec{v}^*$$

$$\Rightarrow A\vec{v}^* = \lambda^*\vec{v}^*$$

So if A is real & $\lambda \in \mathbb{C}$: $A = (\vec{v} \mid \vec{v}^*) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix} (\vec{v} \mid \vec{v}^*)^{-1}$

Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \lambda^2 + 1 = 0$

↳ $\underline{\lambda = \pm i}$



$\lambda = i$

$\underline{(A - \lambda \underset{\sim}{I})\vec{v} = 0} \rightarrow \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right)$

$(-i \cdot (-i) = -1)$

$i^2 = -1$

$\boxed{\begin{array}{l} i = \sqrt{-1} \quad \frac{1}{i} = -i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{array}}$

$\xrightarrow{R_2 - iR_1} \left(\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

↳ $-iv_1 - v_2 = 0 \rightarrow v_2 = -iv_1$

↳ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -iv_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -i \end{pmatrix}$

↳ $\vec{v} = \underline{\begin{pmatrix} 1 \\ -i \end{pmatrix}} \Rightarrow \lambda^* = (i)^* = -i$

$$\vec{v}^* = \begin{pmatrix} 1 \\ -i \end{pmatrix}^* = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

So $A = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$

note: $\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = \frac{1}{i-(-i)} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$

$$= \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{pmatrix}$$

note: $\frac{1}{i} = \frac{1}{i} \frac{(-i)}{(-i)} = \frac{-i}{i \cdot (-i)} = \frac{-i}{-i^2} = -i.$

So: $A = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \frac{1}{2} & i/2 \\ \frac{1}{2} & -i/2 \end{pmatrix}$

So now, if we used this for an ODE:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{x} \; ; \; \vec{x}(0) = \vec{x}_0 \in \mathbb{R}^2$$

Then, from $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{pmatrix}$

So $\vec{x}(t) = c_1\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{it} + c_2\begin{pmatrix} 1 \\ i \end{pmatrix}e^{-it}$ (General Soln.)

So... how do we see this in the phase plane?

well, if $\vec{x}(0) = \vec{x}_0$, we also have:

$$\boxed{\vec{x}(t) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix}\vec{x}_0}$$

or $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$

Now suppose $\vec{x}_0 \in \mathbb{R}^2$ ... and so we grind

$$\vec{x}(t) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{pmatrix}\vec{x}_0$$

$$= \begin{pmatrix} e^{it} & e^{-it} \\ -ie^{it} & ie^{-it} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix}\vec{x}_0$$

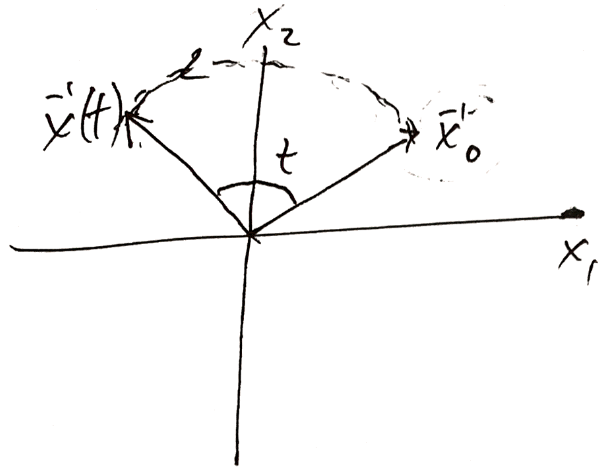
$$= \begin{pmatrix} \frac{1}{2}(e^{it}+e^{-it}) & \frac{i}{2}(e^{it}-e^{-it}) \\ -\frac{i}{2}(e^{it}-e^{-it}) & \frac{1}{2}(e^{it}+e^{-it}) \end{pmatrix}\vec{x}_0$$

So $e^{\pm it} = \cos(t) \pm i\sin(t)$

$$\boxed{\frac{1}{2}(e^{it}+e^{-it}) = \cos(t)\ ;\ \frac{1}{2i}(e^{it}-e^{-it}) = \sin(t)}$$

$$\vec{x}(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\vec{x}_0 .$$

Lct

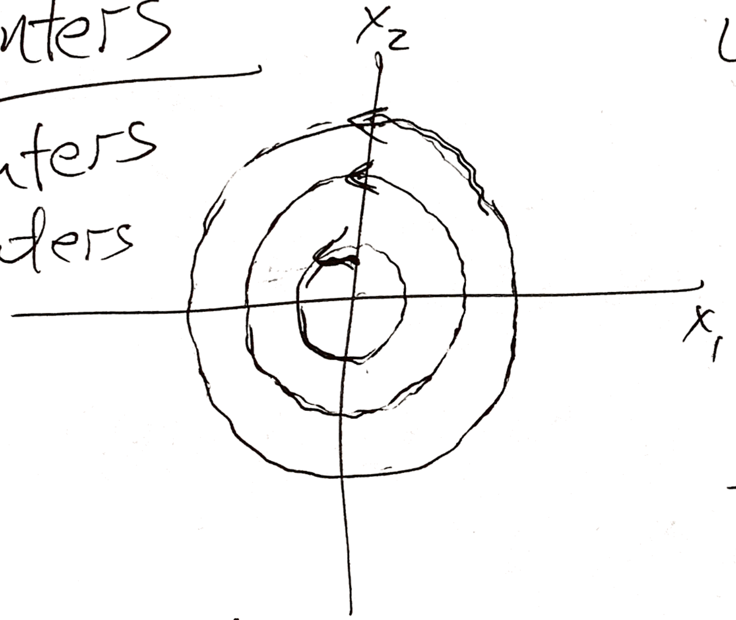


or in the phase plane:

"s"centers

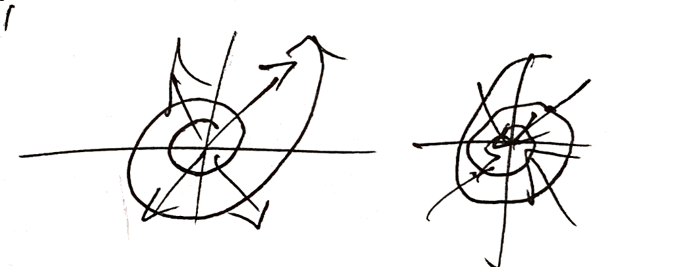
"s"enters

scenters

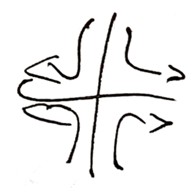


→ $\lambda = \pm i$

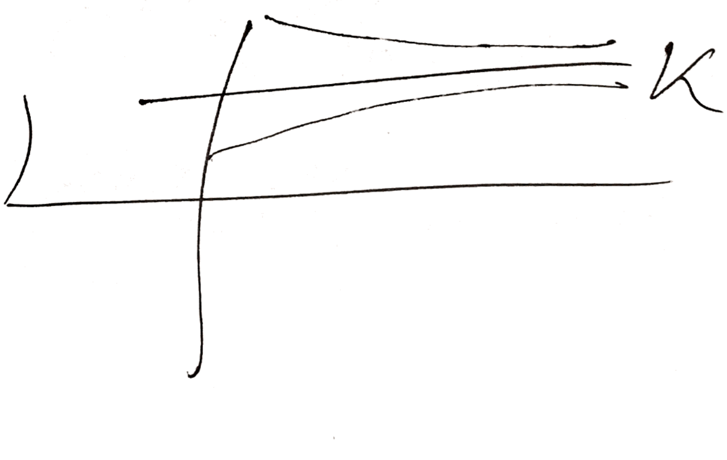
→ periodic oscillation



→ $\frac{dy}{dt} + p(t)\,y = \cos(t)$



$$\frac{dN}{dt} = rN(K-N)$$



So, with all that done, in general for

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0 \in \mathbb{R}^2$$

where $\left(\frac{tr(A)}{2}\right)^2 < det(A) \longrightarrow$

$$\lambda, \lambda^* = \mu \pm i\omega, \quad \mu = \frac{1}{2}tr(A); \; \omega = \left(\left|\left(\frac{tr(A)}{2}\right)^2 - det(A)\right|\right)^{\frac{1}{2}}$$

we always have

$$A = (\vec{v} \,|\, \vec{v}^*) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix} (\vec{v} \,|\, \vec{v}^*)^{-1}$$

$$\hookrightarrow \vec{x}(t) = (\vec{v} \,|\, \vec{v}^*) \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda^* t} \end{pmatrix} (\vec{v} \,|\, \vec{v}^*)^{-1} \vec{x}_0.$$

So

$$e^{\lambda t} = e^{(\mu + i\omega)t} = e^{\mu t} e^{i\omega t} = e^{\mu t}(\cos(\omega t) + i\sin(\omega t))$$

$$e^{\lambda^* t} = e^{(\mu - i\omega)t} = e^{\mu t} e^{-i\omega t} = e^{\mu t}(\cos(\omega t) - i\sin(\omega t))$$

in general, we see now:

$$\vec{x}(t) = e^{\mu t}\left(\vec{v} \,|\, \vec{v}^*\right)\left(\cos(\omega t)\,\underline{I} + i\sin(\omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\left(\vec{v}\,|\,\vec{v}^*\right)^{-1}\vec{x}_0$$

$$= e^{\mu t}\left(\cos(\omega t)\,I + i\sin(\omega t)\left(\vec{v}\,|\,\vec{v}^*\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\vec{v}\,|\,\vec{v}^*\right)^{-1}\right)\vec{x}_0$$

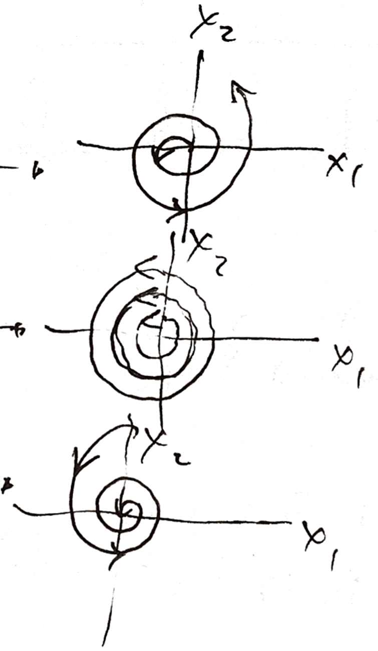
must be real... but to be fair, it's messy.

oscillation terms

$\mu > 0$: growth

$\mu = 0$: neutral

$\mu < 0$: decay



Repeated Evals:

for $\frac{d\vec{x}}{dt} = A\vec{x}$; $\vec{x}(0) = \vec{x}_0$

if $\left(\frac{tr(A)}{2}\right)^2 = det(A) \rightarrow \lambda = \frac{1}{2}tr(A)$.

then either $A = cI$ or

$$A = (\vec{v}|\vec{\omega})\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}(\vec{v}|\vec{\omega})^{-1}$$

called the Jordan Normal Form.

now $A\vec{v} = \lambda\vec{v}$. But what is $\vec{\omega}$?

$\hookrightarrow A(\vec{v}|\vec{\omega}) = (\vec{v}|\vec{\omega})\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$

$\hookrightarrow (A\vec{v}|A\vec{\omega}) = (\lambda\vec{v}|\vec{v} + \lambda\vec{\omega})$

$\hookrightarrow \boxed{A\vec{v} = \lambda\vec{v}\ ;\ (A-\lambda)\vec{\omega} = \vec{v}}$

let $A = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$ $\to$ $tr(A) = 6$; $det(A) = 9$ (14)

so $\frac{(tr(A))^2}{4} = \frac{36}{4} = 9 = det(A)$

$\to$ $\lambda = 3$

$(A - 3I)\vec{v} = 0 \to \left(\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

so $v_1 + v_2 = 0 \to \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Now to find $\vec{w}$, we have to solve

$(A - 3I)\vec{w} = \vec{v}$

$\to \left(\begin{array}{cc|c} 2 & 2 & 1 \\ -2 & -2 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right)$

$$\omega_1 + \omega_2 = 1/2 \longrightarrow \omega_2 = -\omega_1 + 1/2$$

$$\text{or } \vec{\omega} = \begin{pmatrix} \omega_1 \\ -\omega_1 + 1/2 \end{pmatrix} = \underbrace{\omega_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\substack{= \omega_1 \vec{v} \\ \text{homogeneous} \\ \text{solution}}} + \underbrace{\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}}_{\substack{\vec{\omega}_p \\ \text{particular}}}$$

we only use the particular: $\vec{\omega}_p = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$$\leadsto A = \begin{pmatrix} 1 & 0 \\ -1 & 1/2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1/2 \end{pmatrix}^{-1}$$

Now what about dynamics?

$$\text{If } A = (\vec{v} | \vec{\omega}) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\vec{v} | \vec{\omega})^{-1}$$

$$\leadsto \frac{d\vec{x}}{dt} = A\vec{x} \rightarrow \frac{d\vec{x}}{dt} = (\vec{v} | \vec{\omega}) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\vec{v} | \vec{\omega})^{-1} \vec{x}$$

letting $\vec{y}' = (\vec{v}'|\vec{w}')^{-1}\vec{x}'$ $\longrightarrow$

$$\frac{d\vec{y}'}{dt} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{y}' \quad ; \quad \vec{y}'(0) = (\vec{v}'|\vec{w}')^{-1}\vec{x}_0 .$$

$\leadsto \dfrac{dy_1}{dt} = \lambda y_1 + y_2 \; ; \; \dfrac{dy_2}{dt} = \lambda y_2 \rightarrow \dfrac{dy_2}{y_2} = \lambda dt$

Clearly, we get $y_2 = y_2(0) e^{\lambda t}$

$\leadsto \dfrac{dy_1}{dt} = \lambda y_1 + y_{2,0} e^{\lambda t}$

$\leadsto \dfrac{d}{dt}\left(y_1 e^{-\lambda t}\right) = y_{2,0}$ $\rightarrow$

$\leadsto y_1 e^{-\lambda t} - y_{1,0} = y_{2,0} t$

or $y_1(t) = y_{1,0} e^{\lambda t} + y_{2,0} t e^{\lambda t}$

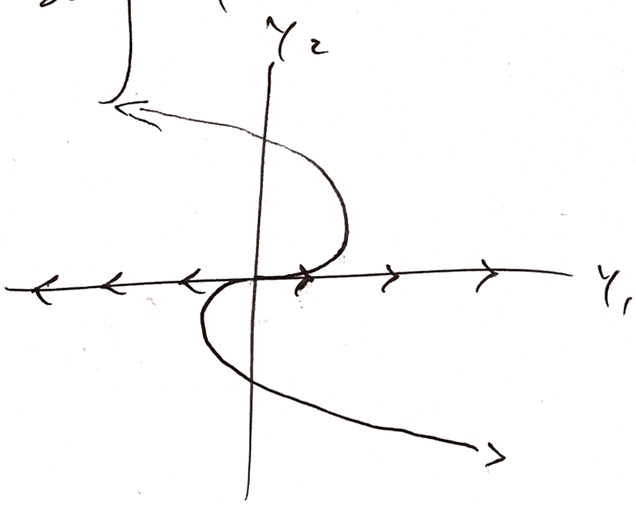
So $\vec{y}(t) = \begin{pmatrix} y_{1,0} e^{\lambda t} + y_{2,0} t e^{\lambda t} \\ y_{2,0} e^{\lambda t} \end{pmatrix}$

$$= y_{1,0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} + y_{2,0} \begin{pmatrix} t \\ 1 \end{pmatrix} e^{\lambda t}$$

So $\vec{x}(t) = (\vec{v} | \vec{w}) \, \vec{y}(t)$

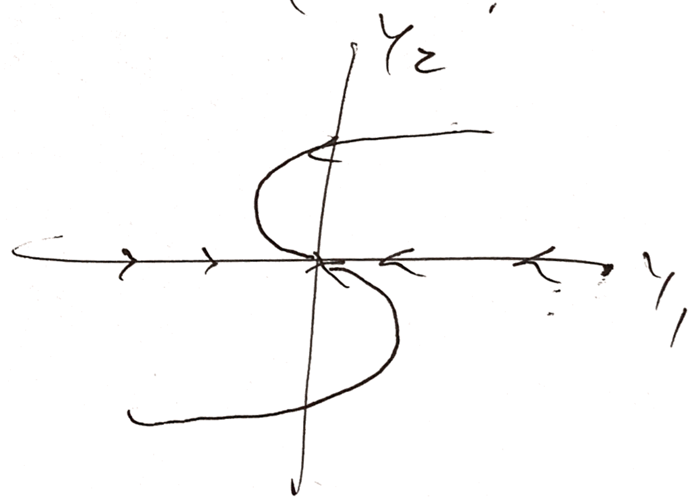
$$= y_{1,0} \vec{v} e^{\lambda t} + y_{2,0} (t\vec{v} + \vec{w}) e^{\lambda t}$$

Say $\lambda > 0$:



y-space

$\lambda < 0$:



y-space.