So, compared to mass-spring:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F(t)$$

Resist. Friction "Hooke's" Law.

vs.

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV_{in}}{dt}$$

Motivates studying

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t),$$

$a, c > 0, \; b \geq 0$

let $y = \frac{dx}{dt}$

$$\to \; a\frac{dy}{dt} + by + cx = f(t)$$

$$\to \; \frac{dy}{dt} + \frac{b}{a}y + \frac{c}{a}x = \frac{1}{a}f(t)$$

$$\to \; \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t)/a \end{pmatrix}.$$

Homogeneous problem:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So $tr(A) = -\frac{b}{a}$; $det(A) = \frac{c}{a}$

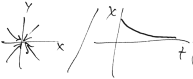
So $\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$ or $\lambda = \frac{1}{2}\left(-\frac{b}{a} \pm \left(\frac{b^2}{a^2} - \frac{4c}{a}\right)^{1/2}\right)$

$$= \frac{1}{2}\left(-\frac{b}{a} \pm \frac{1}{|a|}(b^2 - 4ca)^{1/2}\right)$$

$$= \frac{1}{2a}\left(-b \pm (b^2 - 4ca)^{1/2}\right)$$

So, we end up w/ three regions:

I) Over Damped - $b^2 - 4ca > 0$

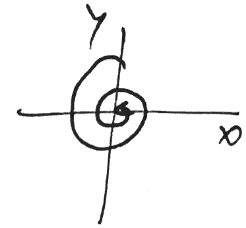
So $\lambda_{+/-} < 0$ → sink 

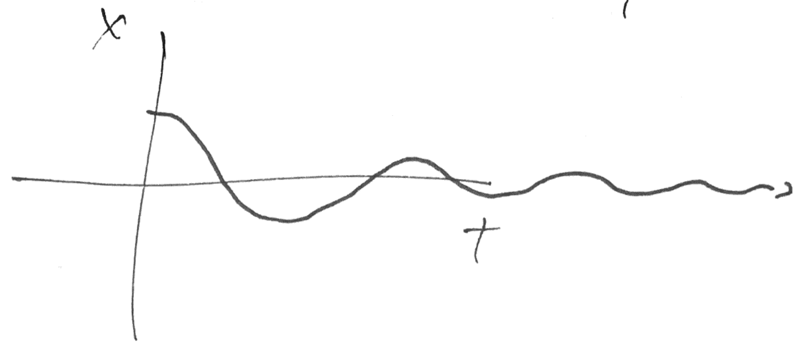
II) Critically Damped - $b^2 - 4ca = 0$ → $\lambda_+ = \lambda_- = -b/2a$

→ degenerate sink. 

III) Under-Damped - $b^2 - 4ca < 0$

↳ $\lambda_+ = \lambda_-^* \in \mathbb{C}$, → decaying spiral





For $b^2 - 4ca > 0$ →

$$(A - \lambda_{+/-} I)\vec{v} = 0 \to \left(\begin{array}{cc|c} -\lambda_{+/-} & 1 & 0 \\ -c/a & -b/a & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} -\lambda_{+/-} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\to y = \lambda_{+/-} x$$

$$\text{or } \vec{v}_{+/-} = \begin{pmatrix} 1 \\ \lambda_{+/-} \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$b^2 - 4ac < 0$

$\hookrightarrow \lambda_{+/-} = \frac{1}{2a}\left(-b \pm i\,(4ca - b^2)^{1/2}\right)$

$= -\mu \pm i\omega, \quad \mu = b/2a \,; \quad \omega = (4ca - b^2)^{1/2}/2a$

$\hookrightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^* \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda^* t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^* \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Finally if $b^2 = 4ac \hookrightarrow b = 2\sqrt{ac}$

$\hookrightarrow A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -2\sqrt{\frac{c}{a}} \end{pmatrix} ; \quad \lambda = -\frac{b}{2a} = -\sqrt{\frac{c}{a}}$

$(A - \lambda I)\vec{v} = 0 \rightarrow \left(\begin{array}{cc|c} \sqrt{\frac{c}{a}} & 1 & 0 \\ -\frac{c}{a} & -\sqrt{\frac{c}{a}} & 0 \end{array}\right)$

$\rightarrow \sqrt{\frac{c}{a}}\, x + y = 0 \rightarrow \vec{v} = \begin{pmatrix} 1 \\ -\sqrt{\frac{c}{a}} \end{pmatrix}$

$(A - \lambda I)\vec{w} = \vec{v} \rightarrow \left(\begin{array}{cc|c} \sqrt{\frac{c}{a}} & 1 & 1 \\ -\frac{c}{a} & -\sqrt{\frac{c}{a}} & -\sqrt{\frac{c}{a}} \end{array}\right) \rightarrow \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$