So again, we have for $f(t+T)=f(t)$ that

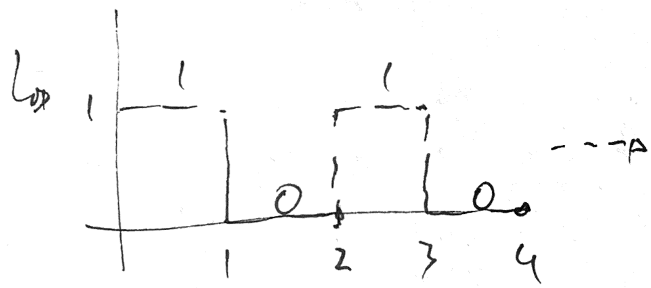
$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \to \frac{1}{1-e^{-sT}} = \sum_{j=0}^{\infty} e^{-sTj}$$

So, if we define $\tilde{f}(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & 1 \le t < 2 \end{cases}$

$\to \mathcal{L}\{f(t-c)H_c(t)\} = e^{-sc}F(s)$

$$\to f(t) = \sum_{j=0}^{\infty} \tilde{f}(t-2j)\, H_{(2j,2(j+1))}(t)$$

$\to$



$$\to \mathcal{L}\{f\} = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2s}}\,\frac{1-e^{-s}}{s} = \frac{1}{(1+e^{-s})\cancel{(1-e^{-s})}}\,\frac{\cancel{(1-e^{-s})}}{s}$$

$$= \frac{1}{s(1+e^{-s})}$$

So in $y'' + \pi^2 y = f(t)$; $y(0)=0,\ y'(0)=0$

$\omega_0 = \pi$ $\quad T = 2 \Rightarrow \omega = \frac{2\pi}{T} = \pi$

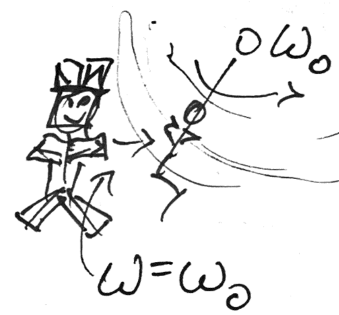
$\Rightarrow (s^2+\pi^2)\,Y = \frac{1}{s(1+e^{-s})}$; $\quad \boxed{\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j \ ; \ |x|<1}$

$$\Rightarrow Y = \frac{1}{s(s^2+\pi^2)}\,\frac{1}{(1+e^{-s})}$$

pattern

$f \to \boxed{L} \to y$

$$= \frac{1}{\pi^2}\left\{\frac{1}{s} - \frac{s}{s^2+\pi^2}\right\}\frac{1}{(1+e^{-s})}$$



now $\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+\pi^2}\right\} = 1-\cos(\pi t)$

using $\frac{1}{1+e^{-s}} = \sum_{k=0}^{\infty} (-1)^k e^{-ks}$; $\quad s>0$

and then $\mathcal{L}\{f(t-c)H_c(t)\} = e^{-sc}F(s)$

$$\Rightarrow Y = \frac{1}{\pi^2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+\pi^2}\right)e^{-ks}$$

$$\Rightarrow y(t) = \frac{1}{\pi^2}\sum_{k=0}^{\infty}(-1)^k\big(1-\cos(\pi(t-k))\big)H_k(t)$$

$$\cos(\pi(t-k)) = \cos(\pi t)\cos(\pi k) = (-1)^k \cos(\pi t)$$

$$\rightarrow \quad y(t) = \frac{1}{\pi^2}\sum_{k=0}^{\infty}\left(\left((-1)^k - \cos(\pi t)\right)\right) H_k(t)$$

So let $t = 2m$ (i.e. t is even)

$$\rightarrow \quad \cos(2\pi m) = 1; \quad H_k(2m) = \begin{cases} 0 & 2m < k \\ 1 & 2m \geq k \end{cases}$$

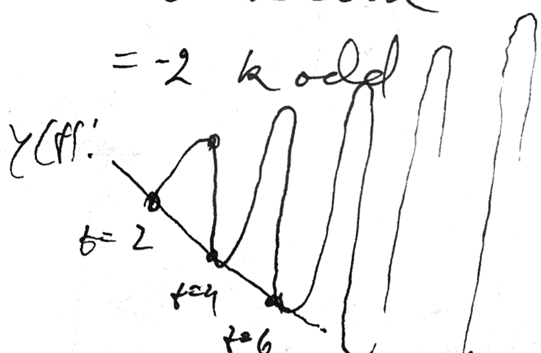
$$\rightarrow \quad y(2m) = \frac{1}{\pi^2}\sum_{k=0}^{2m}\left((-1)^k - 1\right)$$

0 1 .... 2m-1 2m

$= 0$ k even

$= -2$ k odd

$$\rightarrow \; = -\frac{2m}{\pi^2}$$



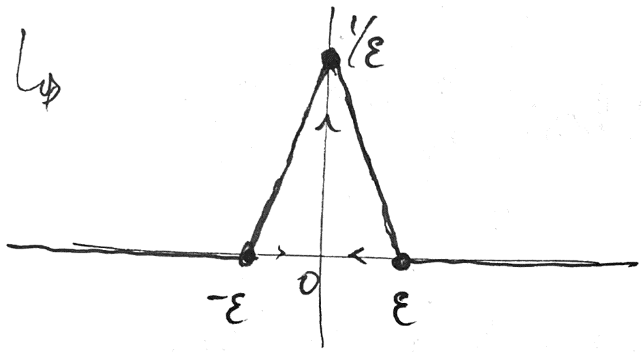
and for $t = 2m+1 \rightarrow \cos(\pi(2m+1)) = -1$

$$H_k(2m+1) = \begin{cases} 0 & 2m+1 < k \\ 1 & 2m+1 \geq k \end{cases}$$

$$y(2m+1) = \frac{1}{\pi^2}\sum_{k=0}^{2m+1}\left((-1)^k + 1\right) = \frac{2(m+1)}{\pi^2}$$

$ay'' + by' + cy = g_\varepsilon(t-t_0)$ where, letting $t_0 > 0$, we have

$$g_\varepsilon(t) = \begin{cases} 0 & |t| > \varepsilon \\ (t+\varepsilon)/\varepsilon^2 & -\varepsilon \le t < 0 \\ (-t+\varepsilon)/\varepsilon^2 & 0 \le t \le \varepsilon \end{cases} \qquad (5.7)$$

↳ 

So as $\varepsilon \to 0^+$, we get a function with a narrowing support but an ever growing height.

$g_\varepsilon(t) \to \delta(t)$

Dirac Delta "function"

↳ $\mathcal{L}\{g_\varepsilon(t-t_0)\} = \int_0 e^{-st} g_\varepsilon(t-t_0)\,dt \; ; \; t_0 > 0$

$\tilde{t} = t - t_0$

$$= e^{-st_0} \int_{-t_0}^{\infty} e^{-st} g_\varepsilon(t)\,dt$$

Need $0 < \varepsilon < t_0$.

So

$$\mathcal{L}\{g_\varepsilon(t-t_0)\} = e^{-st_0}\left(\int_{-\varepsilon}^{0} e^{-st}\frac{(t+\varepsilon)}{\varepsilon^2}\,dt + \int_{0}^{\varepsilon} e^{-st}\frac{(-t+\varepsilon)}{\varepsilon^2}\,dt\right)$$

$$= e^{-st_0}\left(\frac{1}{\varepsilon^2}\int_{-\varepsilon}^{0} e^{-st}t\,dt - \frac{1}{\varepsilon^2}\int_{0}^{\varepsilon} e^{-st}t\,dt + \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} e^{-st}\,dt\right)$$

$$= e^{-st_0}\left\{-\frac{2}{\varepsilon^2}\int_{0}^{\varepsilon}\cosh(st)\,t\,dt + \frac{1}{\varepsilon}\left\{-\frac{1}{s}e^{-st}\Big|_{-\varepsilon}^{\varepsilon}\right\}\right\}$$

$$= e^{-st_0}\left\{-\frac{2}{\varepsilon^2}\left(\frac{\sinh(st)t}{s}\Big|_{0}^{\varepsilon} - \frac{\cosh(st)}{s^2}\Big|_{0}^{\varepsilon}\right) + \frac{2}{\varepsilon s}\sinh(\varepsilon s)\right\}$$

$$= e^{-st_0}\left\{-\frac{2}{\varepsilon^2}\left(\frac{\varepsilon\sinh(\varepsilon s)}{s} - \frac{(\cosh(s\varepsilon)-1)}{s^2}\right) + \frac{2\sinh(\varepsilon s)}{\varepsilon s}\right\}$$

$$= e^{-st_0}\left(\frac{2(\cosh(\varepsilon s)-1)}{\varepsilon^2 s^2}\right)$$

L'Hopital then gives us

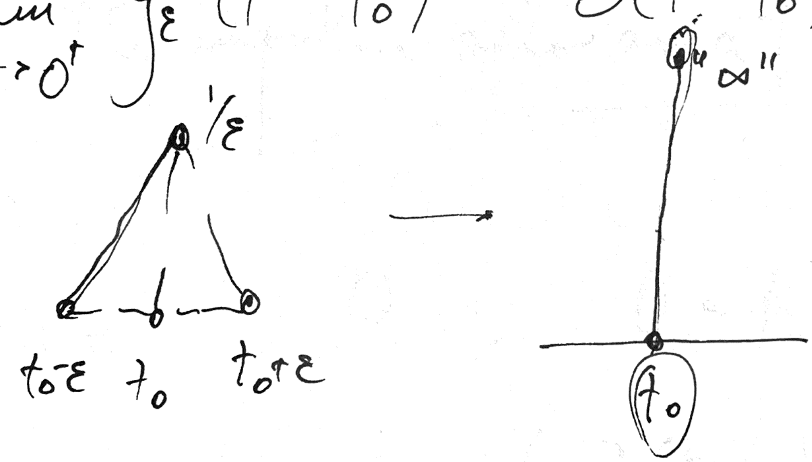
$$\lim_{\varepsilon\to 0^+}\mathcal{L}\{g_\varepsilon(t-t_0)\} = e^{-st_0}$$

Based off this, we define the Dirac δ-function (6)

where $$\int_{-\infty}^{\infty} \delta(t-t_0)\, f(t)\, dt = f(t_0)$$

$$\text{Lo } \mathcal{L}\{\delta(t-t_0)\} = c\int_0^{\infty} e^{-st}\,\delta(t-t_0)\,dt = ce^{-st_0},\ t_0>0.$$

Lo $\lim_{\varepsilon\to 0^+} g_\varepsilon(t-t_0)$ "=" $\delta(t-t_0)$



Point Mass

• m

Point Charges

• q

So for $ay'' + by' + cy = \delta(t-t_0)$; $y(0) = y'(0) = 0$

Lo $(as^2+bs+c)\,\bar{Y} = e^{-st_0}$

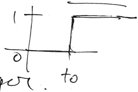
or $$\bar{Y}(s) = \frac{e^{-st_0}}{as^2+bs+c} = \frac{e^{-st_0}}{a\left(s^2+\frac{b}{a}s+\frac{c}{a}\right)}$$

7

so if $s^2 + \frac{b}{a}s + \frac{c}{a} = \frac{1}{(s-\lambda_1)(s-\lambda_2)}$

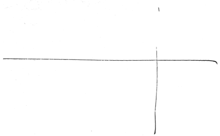
$$= \frac{1}{\lambda_1-\lambda_2}\left(\frac{1}{s-\lambda_1} - \frac{1}{s-\lambda_2}\right)$$

So $y(t) = \frac{1}{a(\lambda_1-\lambda_2)}\left[e^{\lambda_1(t-t_0)} - e^{\lambda_2(t-t_0)}\right] H_{t_0}(t)$



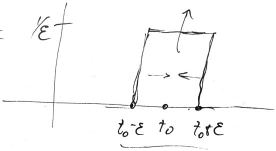
So the impulse acts as a trigger.

$$y(t) = \begin{cases} 0, & 0 < t < t_0 \\ \frac{1}{a(\lambda_1-\lambda_2)}\left(e^{\lambda_1(t-t_0)} - e^{\lambda_2(t-t_0)}\right), & t \geq t_0 \end{cases}$$



$$\tilde{g}_\varepsilon(t-t_0) = \frac{1}{\varepsilon} H_{(t_0-\varepsilon,\, t_0+\varepsilon)}(t)$$

$=$



$$\mathcal{L}\{\tilde{g}_\varepsilon(t-t_0)\} = \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0+\varepsilon} e^{-st}\,dt$$

$$= \frac{1}{\varepsilon}\left(-\frac{1}{s} e^{-st}\Big|_{t_0-\varepsilon}^{t_0+\varepsilon}\right)$$

$$= \frac{1}{s\varepsilon}\left(e^{-st_0}e^{\varepsilon s} - e^{-st_0}e^{-\varepsilon s}\right)$$

$$= \frac{1}{\varepsilon s}\left(e^{-(t_0-\varepsilon)s} - e^{-(t_0+\varepsilon)s}\right)$$

$$a(s-\lambda_1)(s-\lambda_2)\,\bar{Y} = \frac{1}{\varepsilon s}\left(e^{-(t_0-\varepsilon)s} - e^{-(t_0+\varepsilon)s}\right)$$

$$\text{Lo } \bar{Y} = \frac{1}{\varepsilon a}\left(\frac{c_1}{s} + \frac{c_2}{s-\lambda_1} + \frac{c_3}{s-\lambda_2}\right)\left(e^{-(t_0-\varepsilon)s} - e^{-(t_0+\varepsilon)s}\right)$$

$$= \frac{1}{\varepsilon a}\Big(c_1\left(H_{(t_0-\varepsilon)}(t) - H_{(t_0+\varepsilon)}(t)\right)$$

$$+ c_2\left(e^{\lambda_1(t-(t_0-\varepsilon))}H_{(t_0-\varepsilon)}(t) - e^{\lambda_1(t-(t_0+\varepsilon))}H_{(t_0+\varepsilon)}(t)\right) \ldots$$