Singular Perturbations:

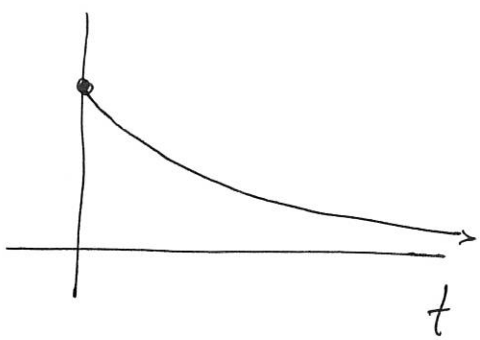
$$\varepsilon \ddot{x} + \dot{x} + x = 0$$

$$x(0) = 1, \quad \dot{x}(0) = 0$$

if we try $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

$\hookrightarrow$ $\dot{x}_0 + x_0 = 0 \rightarrow x_0(t) = \tilde{x}_0 e^{-t} \rightarrow x_0(0) = 1$ but $\underline{\dot{x}_0(0) \neq 0.}$

$\hookrightarrow$



So what to do?

Let $\tau = t/\delta \rightarrow \frac{d}{dt} = \frac{1}{\delta}\frac{d}{d\tau}$

$\hookrightarrow$ $\frac{\varepsilon}{\delta^2}\frac{d^2 x}{d\tau^2} + \frac{1}{\delta}\frac{dx}{d\tau} + x = 0$

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$$\rightarrow \frac{d^2x}{dT^2} + \frac{\delta}{\varepsilon}\frac{dx}{dT} + \frac{\delta^2}{\varepsilon}x = 0$$

If we choose $\delta = \varepsilon \longrightarrow$ "Width of the Boundary Layer"

$$\rightarrow \frac{d^2x}{dT^2} + \frac{dx}{dT} + \varepsilon x = 0$$

Let $x = \bar{X}_0(T) + \varepsilon \bar{X}_1(T) + \varepsilon^2 \bar{X}_2(T) + \cdots$

$$\rightarrow \frac{d^2\bar{X}_0}{dT^2} + \frac{d\bar{X}_0}{dT} = 0$$

Save for Later

Skip for Now!

$$\rightarrow \frac{d\bar{X}_0}{dT} + \bar{X}_0 = C_0 \rightarrow \frac{d}{dT}(\bar{X}_0 e^T) = C_0 e^T$$

$$\rightarrow \bar{X}_0(T) = C_0 + D_0 e^{-T}$$

$$\bar{X}_0(0) = C_0 + D_0 = 1$$

$$\left.\frac{d\bar{X}_0}{dT}\right|_{T=0} = D_0 = 0 \longrightarrow \begin{array}{l} D_0 = 0 \\ C_0 = 1 \end{array}$$

$$\lambda = \frac{1}{2\varepsilon}\left\{ -1 \pm (1-4\varepsilon)^{1/2} \right\}$$

$\left\{ \begin{array}{l} \text{Solve} \\ \varepsilon \ddot{x} + \dot{x} + x = 0 \\ \quad \underline{\text{Directly}} \end{array} \right.$

①

$$x(t) = A e^{\lambda_+ t} + \beta e^{\lambda_- t}$$

$$A + \beta = 1, \quad A\lambda_+ + \beta\lambda_- = 0$$

$$\Leftrightarrow A\lambda_+ + (1-A)\lambda_- = 0$$

$$\Leftrightarrow A(\lambda_+ - \lambda_-) = -\lambda_-$$

$$\Leftrightarrow \left\{ \begin{array}{l} A = \dfrac{\lambda_-}{\lambda_- - \lambda_+} \\ \\ \beta = 1 - \dfrac{\lambda_-}{\lambda_- - \lambda_+} = \dfrac{\lambda_+}{\lambda_- - \lambda_+} \end{array} \right.$$

$$x(t) = \frac{1}{\lambda_- - \lambda_+}\left\{ \lambda_- e^{\lambda_+ t} - \lambda_+ e^{+\lambda_- t} \right\}$$

$$\lambda_- - \lambda_+ = -\frac{1}{\varepsilon}(1-4\varepsilon)^{1/2} = -\frac{1}{\varepsilon}\left(1 - 2\varepsilon + O(\varepsilon^2)\right) = -\frac{1}{\varepsilon} + 2 + O(\varepsilon)$$

$$\lambda_+ = \frac{1}{2\varepsilon}\left\{-1+\left(1-2\varepsilon+O(\varepsilon^2)\right)\right\} = -1+O(\varepsilon)$$

$$\lambda_- = \frac{1}{2\varepsilon}\left\{-1-\left(1-2\varepsilon+O(\varepsilon^2)\right)\right\} = -\frac{1}{\varepsilon}+1+O(\varepsilon)$$

↳ $$x(t) \sim \frac{1}{2-\frac{1}{\varepsilon}+O(\varepsilon)}\left\{\left(1-\frac{1}{\varepsilon}\right)e^{-t} - \left(-1+O(\varepsilon)\right)e^{-t/\varepsilon}\right\}$$

$$\sim \frac{-\varepsilon}{1-2\varepsilon+O(\varepsilon^2)}\left\{\left(1-\frac{1}{\varepsilon}+O(\varepsilon)\right)e^{-t} - \left(-1+O(\varepsilon)\right)e^{-t/\varepsilon}\right\}$$

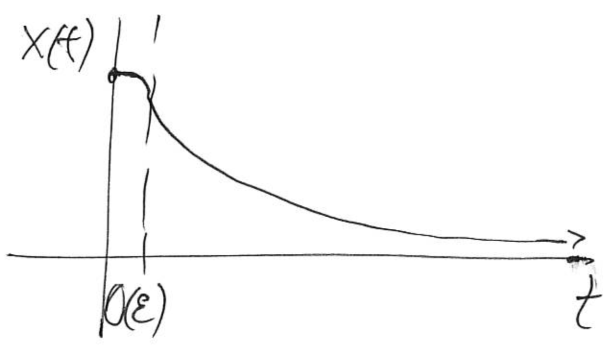
$$\sim -\varepsilon\left(1+2\varepsilon+O(\varepsilon^2)\right)\left(\left(1-\frac{1}{\varepsilon}\right)e^{-t} - \left(-1+O(\varepsilon)\right)e^{-t/\varepsilon}\right)$$

$$\sim e^{-t} + \varepsilon\left(2e^{-t}-e^{-t}-e^{-t/\varepsilon}\right) + O(\varepsilon^2)$$

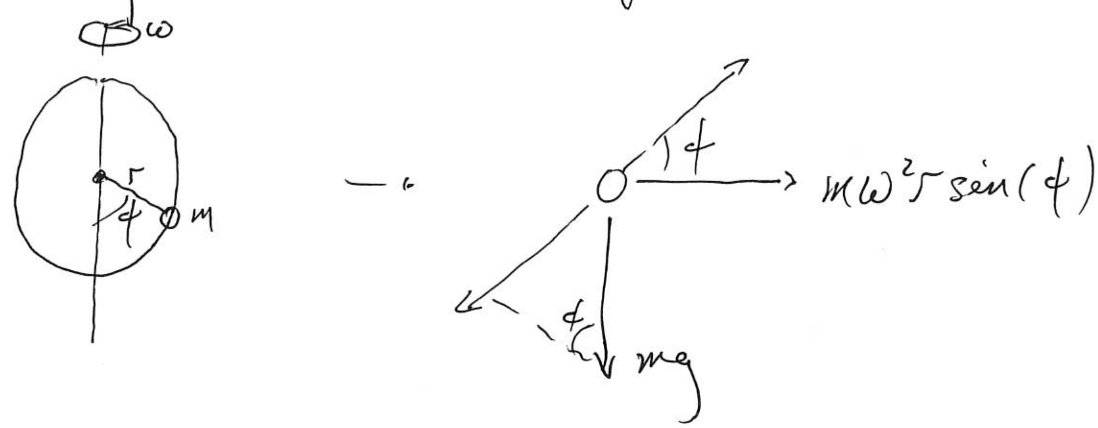
$$\sim e^{-t} + \varepsilon\left(e^{-t}-e^{-t/\varepsilon}\right) + O(\varepsilon^2)$$

↳ $x(0) = 1 + O(\varepsilon^2)$

$\dot{x}(0) = O(\varepsilon)$



Overdamped Bead on a Hoop:



So

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi)$$

So note, ϕ is an angle, and thus it really doesn't have units.

Thus, we let $\tau = t/T$

So $\frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}$, $\frac{d^2}{dt^2} = \frac{1}{T^2}\frac{d^2}{d\tau^2}$

Likewise, we treat mg as the characteristic magnitude of a force.

Thus, after scaling and dividing, we get

$$\frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi)$$

So, let $\gamma = r\omega^2/g$, and now we ask, how to choose T?

Choice 1: $b/mgT = 1$ or $T = b/mg \rightarrow b = F\cdot T = m\frac{L}{T^2}T = m\frac{L}{T}$

↳ Note, this emphasizes the role of friction in the problem.

↳ $$\frac{r}{g}\frac{1}{T^2} = \frac{r}{g}\left(\frac{mg}{b}\right)^2 = rg\left(\frac{m}{b}\right)^2 = \frac{rm}{b^2}mg$$

$$E \sim mv^2 \Rightarrow \frac{m}{b^2} \sim \frac{1}{E} \Rightarrow (rmg)\left(\frac{m}{b^2}\right)$$

is an energy balance.

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Or $\frac{r}{g}\frac{1}{T^2} = (rmg)/(b^2/m)$ ← characteristic work done by friction

characteristic work done by gravity

Now further suppose that

$$rmg \ll b^2/m$$ i.e. friction does most of the work

↳ we then introduce the non-dimensional parameter ε

$$\varepsilon = (rmg)/(b^2/m), \quad \underline{0 < \varepsilon \ll 1}$$

↳ $$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma\sin(\phi)\cos(\phi)$$

Again, we have two initial conditions

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = \phi_1$$

So, if we ignore ε term, ~~we~~ we cannot satisfy both initial conditions.

But let's see what happens when we do anyway

$$\frac{d\phi}{d\tau} = \sin(\phi)(\gamma\cos(\phi) - 1) = f(\phi; \gamma)$$

↳ FP's: $\sin(\phi) = 0, \quad \cos(\phi) = 1/\gamma$

↳ $\phi^* = 0, \pi$ ↳ if $\gamma > 1 \rightarrow \phi^* = \pm\cos^{-1}(1/\gamma)$

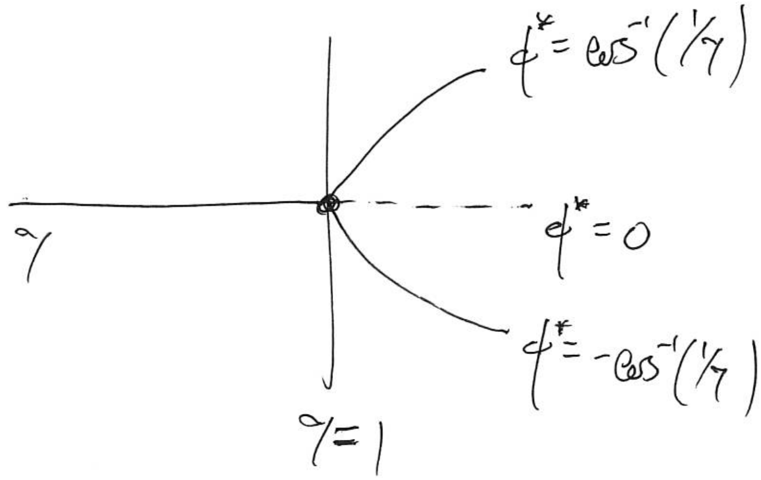
St: $\partial_\phi f = \cos(\phi)(\gamma\cos(\phi) - 1) - \gamma\sin^2(\phi)$

$$= \gamma\cos(2\phi) - \cos(\phi)$$

$$\partial_\phi f\big|_{\phi^*=0} = \gamma - 1 \qquad \partial_\phi f\big|_{\phi^*=\pm\cos^{-1}\left(\frac{1}{\gamma}\right)} = -\gamma\left(1-\frac{1}{\gamma^2}\right)$$

$$\partial_\phi f\big|_{\phi^*=\pi} = \gamma + 1$$

Always unstable

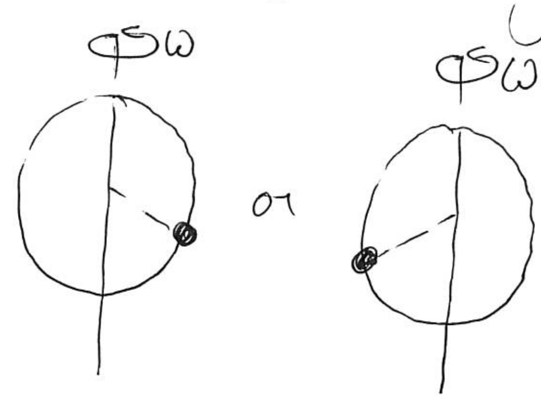


; Supercritical Pitchfork Bifurcation

So if $\gamma = r\omega^2/g \rightarrow$ for $\gamma < 1$ or $\omega^2 < g/r$

↳  bead slides down.

But when $\eta > 1$ or $\omega^2 > g/r$, centrifugal force can hold us aloft $\rightarrow$



The particular perturbation decides which of the two states we fall into. We call this a <u>symmetry broken state</u>.

But what of our angular acceleration?

Let $\bar{t} = \tau/\delta$

$$\hookrightarrow \frac{\varepsilon}{\delta^2}\frac{d^2\phi}{d\bar{t}^2} = -\frac{1}{\delta}\frac{d\phi}{d\bar{t}} - \sin(\phi) + \eta\sin(\phi)\cos(\phi)$$

$$\to \frac{d^2\phi}{d\bar{t}^2} = -\frac{\delta}{\varepsilon}\frac{d\phi}{d\bar{t}} + \frac{\delta^2}{\varepsilon}\sin(\phi)(\gamma\cos(\phi)-1)$$

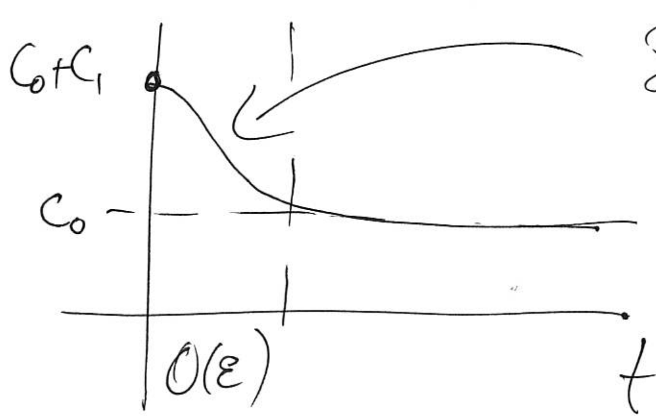
$\to$ if we choose $\delta = \varepsilon$

$$\to \frac{d^2\phi}{d\bar{t}^2} = -\frac{d\phi}{d\bar{t}} + \varepsilon\sin(\phi)(\gamma\cos(\phi)-1)$$

$$\to \frac{d^2\phi}{d\bar{t}^2} + \frac{d\phi}{d\bar{t}} = 0$$

$$\to \phi(\bar{t}) = C_0 + C_1 e^{-\bar{t}}$$

or $\phi(t) = C_0 + C_1 e^{-t/\varepsilon}$



So there is a fast transient in this problem.