# Multiple Scales:

van der Pol Equation:

$$\ddot{x} + \underline{\mu}(\underline{x^2-1})\dot{x} + x = 0 \quad \text{with } \mu \gg 1$$

↳ Strong Nonlinear Damping

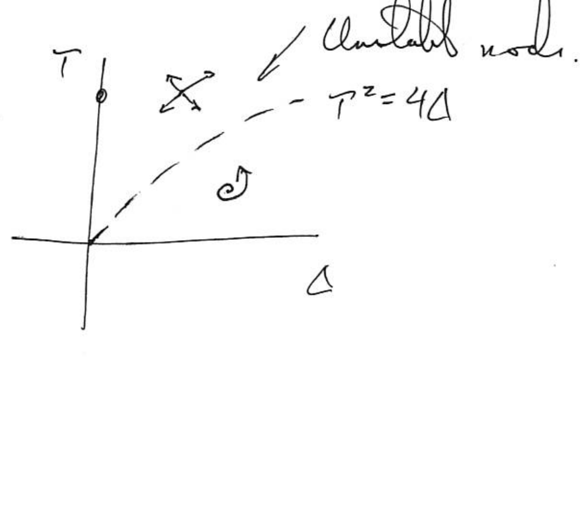
Let $y = \dot{x}$

↳ $\dot{y} = -x - \mu(x^2-1)y$

↳ Only f.p. @ $(0,0)$

$$J = \begin{bmatrix} 0 & 1 \\ -1-2\mu xy & -\mu(x^2-1) \end{bmatrix} \longrightarrow J\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

$\longrightarrow T = \mu;\ \Delta = 1$

$\longrightarrow$ 

But instead of that, we note

$$\ddot{x} + \mu(x^2-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \mu\left(\frac{x^3}{3} - x\right)\right)$$

Set $F(x) = \frac{1}{3}x^3 - x$

So van der Pol becomes:

$$\frac{d}{dt}\left(\dot{x} + \mu F\right) + x = 0$$

So let: $\omega = \dot{x} + \mu F$ — note $(0,0)$ is still the only fixed point.

$$\dot{\omega} = -x$$

$\mu \gg 1 \Rightarrow \varepsilon = \frac{1}{\mu} \ll 1 \Rightarrow$

$\varepsilon\dot{x} = \varepsilon\omega - F$ $\quad$ $\underline{\text{let}}\ \varepsilon \to 0$ $\qquad$ $F(x) = \frac{1}{3}x^3 - x = 0$

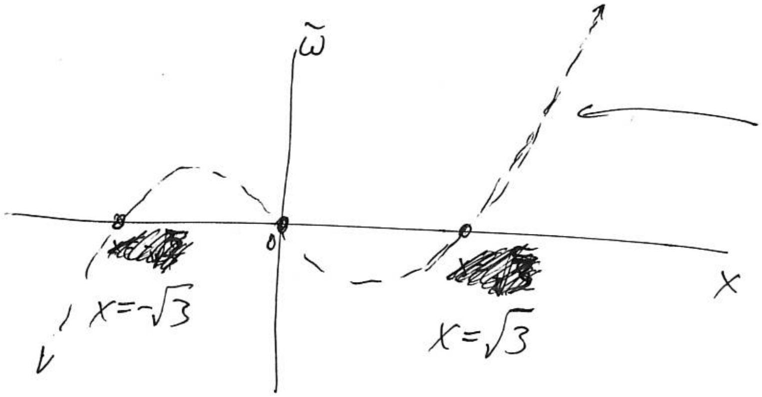
$\dot{\omega} = -x$ $\qquad\qquad\qquad\qquad$ $\dot{\omega} = x$

This is a bit more elegant and powerful if we let $\tilde{\omega} = \varepsilon\omega$

Then $\varepsilon \dot{x} = \tilde{\omega} - F(x)$

$\dot{\tilde{\omega}} = -\varepsilon x$

Then $\varepsilon = 0 \rightarrow \dot{\tilde{\omega}} = 0, \quad \tilde{\omega} = F(x) = \frac{1}{3}x^3 - x$




so for small $\varepsilon$, we expect solutions to be close to this curve. Note, this nullcline is not a solution, but solutions should be close to it for small $\varepsilon$.

However, if we introduce the new, <u>fast</u> time

$$\tau = t/\varepsilon$$

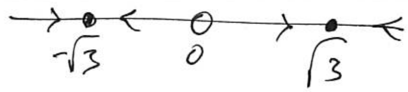
Then $x_\tau = \tilde{\omega} - F(x)$

$\tilde{\omega}_\tau = -\varepsilon^2 x$

If we put this together, we have

$$x_\tau = -F(x) - \varepsilon^2 \int^\tau x(s)\,ds$$

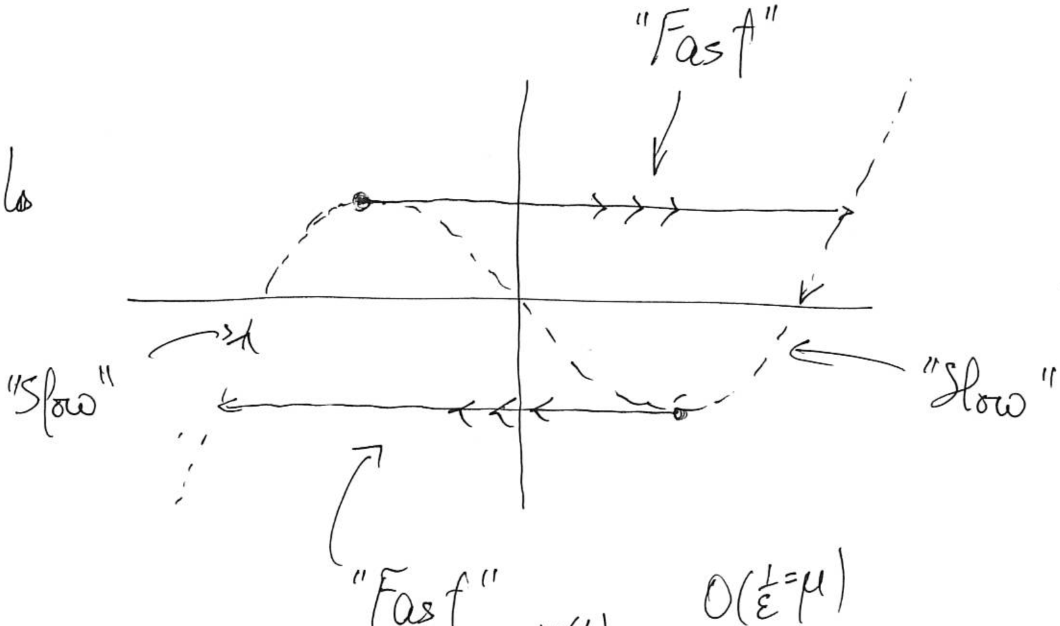
$\Rightarrow$ $x_\tau \simeq -F(x)$ $\longrightarrow$




$\Rightarrow$ $$x(\tau) \simeq \pm \left[ \frac{3}{1 + \left(\frac{3}{x_0^2} - 1\right) e^{-2\tau}} \right]^{1/2}$$

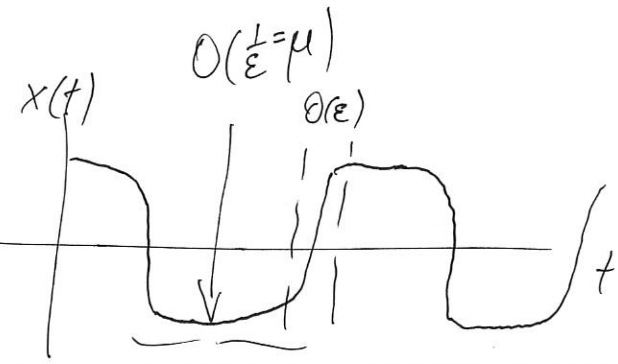
note: $e^{-2\tau} = e^{-2t/\varepsilon}$ decays exponentially quickly

$\Rightarrow$




Limit cycle is constructed from slow/fast "manifolds".

$\Rightarrow$



Formalizing this, we now look at a family of weak non-linear oscillators (5)

$$\ddot{x} + \varepsilon h(x,\dot{x}) + x = 0$$

Regular Perturbation Theory:

$$x = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

Les $\ddot{x}_0 + x_0 = 0$

$$\ddot{x}_1 + x_1 = -h(x_0, \dot{x}_0) \quad \text{and so forth.}$$

Les $x_0(t) = A\cos(t) + B\sin(t)$

or

$$x_0(t) = \tilde{A}e^{it} + \tilde{B}e^{-it} = (\tilde{A}+\tilde{B})\cos(t) + i(\tilde{A}-\tilde{B})\sin(t)$$

let $\tilde{B} = \tilde{A}^* \Rightarrow A = 2\,\mathrm{Re}\,\tilde{A}\,;\; B = -2\,\mathrm{Im}\,\tilde{A}.$

So at the next order, we can find $x_1$ using <u>variation of parameters</u>

↳ Wronskian $W(t) = \begin{vmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{vmatrix} = -2i = \frac{2}{i}$

↳ $x_1(t) = \tilde{A}e^{it} + \tilde{A}^* e^{-it} + \frac{ie^{it}}{2}\int_0^t e^{-is} h(x_0, \dot{x}_0)\,ds - \frac{ie^{-it}}{2}\int_0^t e^{is} h(x_0, \dot{x}_0)\,ds$

So, okay what's the problem?

Suppose $h(x_0, \dot{x}_0) = h_0(e^{it}, e^{-it}) = \underbrace{\sum_{m=-\infty}^{\infty} \hat{h}_m e^{imt}}_{\text{Fourier series}}$

if $h(x_0, \dot{x}_0)$ is real $\longrightarrow$ $\hat{h}_{-m} = \hat{h}_m^*$

(7)

so $e^{it}\int_0^t e^{-is} h(x_0, \dot{x}_0)\,ds = \sum_m \hat{\eta}_m e^{it}\int_0^t e^{i(m-1)s}\,ds$

So we see that if $m = 1$

so we get out a term: $t\hat{\eta}_1 e^{it}$

this grows in $t$!

i.e. we have resonance.

so $x(t) = x_0(t) + \varepsilon t \hat{\eta}_1 e^{it} + \dots$

only remains $O(\varepsilon)$ for $t \sim O(1)$

Thus our approximation breaks down as $t$ gets at all large.

How to remedy this? We introduce a new time scale $\tau = \varepsilon t$. (8)

$\hookrightarrow$ $x = x_0(t,\tau) + \varepsilon x_1(t,\tau) + \cdots$

$\hookrightarrow$ $\dfrac{d}{dt} = \partial_t + \varepsilon \partial_\tau \rightarrow \dfrac{d^2}{dt^2} = \partial_t^2 + 2\varepsilon \partial_{t\tau} + \varepsilon^2 \partial_\tau^2$

$\hookrightarrow$ $\partial_t^2 x_0 + x_0 = 0$

$$\partial_t^2 x_1 + x_1 = -h(x_0, \partial_t x_0) - \underline{2\partial_{t\tau}^2 x_0}$$

Acts as a correction forcing.

$\hookrightarrow$ $x_0(t,\tau) = \underline{A(\tau)e^{it} + A^*(\tau)e^{-it}}$

what were integration constants now depend on the slow time.

↳ and now we have:

$$\eta(x_0, \partial_t x_0) = \sum_{m=-\infty}^{\infty} \hat{\eta}_m(\tau)\, e^{imt} \longrightarrow \hat{\eta}_1(\tau),\ \hat{\eta}_{-1}(\tau) \text{ are offending members.}$$

and again $\hat{\eta}_{-1} = \hat{\eta}_1^*$

↳

$$\eta(x_0, \partial_t x_0) + 2\partial_{t\tau}^2 x_0 = \left(\hat{\eta}_1(\tau) + 2i\partial_\tau A\right) e^{it}$$

$$+ \left(\hat{\eta}_{-1} - 2i\partial_\tau A^*\right) e^{-it} + \underbrace{\ldots}_{\text{stuff we're not worried about}}$$

↳ if we set: $\partial_\tau A = \frac{i}{2}\hat{\eta}_1(\tau)$ then we have removed the offending members.

So, for relatively simple nonlinearities, this is a straightforward process: (10)

van der Pol Oscillator:

$$h(x_0, \dot{x}_0) = (x_0^2 - 1)\dot{x}_0$$

$\hookrightarrow$ if we let $x_0 = Ae^{it} + A^* e^{-it}$

$\hookrightarrow$ $h(x_0, \dot{x}_0) = i(Ae^{it} - A^* e^{-it})(A^2 e^{2it} + 2|A|^2 + (A^*)^2 e^{-2it} - 1)$

$\hookrightarrow$ $h(x_0, \dot{x}_0) = i\left((2|A|^2 - 1)Ae^{it} - |A|^2 A e^{it} + c.c. + \mathcal{O}(e^{3it}, e^{-3it})\right)$

$\hookrightarrow$ $\hat{h}_1(T) = i(|A|^2 - 1)A$

$\hookrightarrow$ from multiple scales analysis we have:

$$\partial_T A = -\frac{1}{2}(|A|^2 - 1)A$$

to solve this let $A = re^{i\theta}$ where $r = |A|$:

$\hookrightarrow$ $\partial_T A = (r_T + ir\theta_T)e^{i\theta}$

$\hookrightarrow$ $r_T + ir\theta_T = -\frac{1}{2}(r^2 - 1)r$

↳ $ir\theta_\tau = 0 \longrightarrow \theta_\tau = 0 \Rightarrow \theta = \theta_0$

$r_\tau = \frac{1}{2}(1-r^2)r$

↳ $\ln\left|\frac{r}{r_0}\right| - \frac{1}{2}\ln\left|\frac{1+r}{1+r_0}\right| - \frac{1}{2}\ln\left|\frac{1-r}{1-r_0}\right| = \frac{\tau}{2}$

↳ $\ln\left|\frac{r}{r_0}\cdot\left(\frac{1-r^2}{1-r_0^2}\right)^{-1/2}\right| = \tau/2$

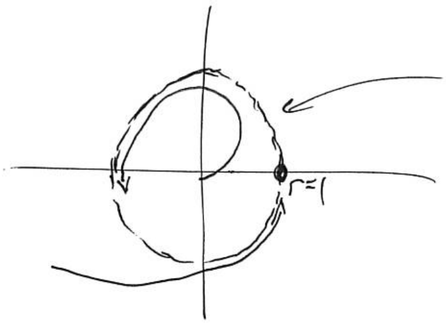
↳ $r(1-r^2)^{-1/2} = r_0((1-r_0^2))^{1/2}e^{\tau/2}$

↳ $$r^2 = \frac{r_0^2((1-r_0^2))e^{\tau}}{1+r_0^2((1-r_0^2))e^{\tau}} \longrightarrow r(\tau) = \frac{r_0((1-r_0^2))^{1/2}e^{\tau/2}}{(1+r_0^2((1-r_0^2))e^{\tau})^{1/2}}$$

↳ as $\tau \to \infty$; $r(\tau) \to 1$;

$$x(t) \sim r(\tau)e^{it+i\theta_0} + r(\tau)e^{-it-i\theta_0} + O(\varepsilon)$$

$$\sim 2r(\tau)\cos(t+\theta_0) + O(\varepsilon)$$




So we have for leading order a limit cycle.

now if $h(x,\dot{x})$ had been more complicated, to find $\hat{h}_1$, we can use orthogonality relations.

$\to$ $x_0 = A e^{it} + A^* e^{-it}$

$\to$ $h(x_0,\dot{x}_0) = \sum_{m=-\infty}^{\infty} \hat{h}_m e^{imt}$

$\to$ $\frac{1}{2\pi}\int_0^{2\pi} h(x_0,\dot{x}_0) e^{-int}\,dt = \sum_{m=-\infty}^{\infty} \hat{h}_m \left\{ \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt \right\}$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}$$

$\to$ $\hat{h}_n = \frac{1}{2\pi}\int_0^{2\pi} h(x_0,\dot{x}_0) e^{-int}\,dt$

So, in more complicated situations, we can hopefully get somewhere with

$$\hat{h}_1 = \frac{1}{2\pi}\int_0^{2\pi} h(x_0(t), \dot{x}_0(t))\, e^{-it}\, dt$$

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Note, at this point, we've essentially turned multiple scales into "averaging" and vice versa, so, yup, that's nice.

The Duffing equation:

$$\ddot{x} + x + \varepsilon x^3 = 0$$

$\Rightarrow$ $h(x_0, \dot{x}_0) = x_0^3 = (Ae^{it} + A^* e^{-it})^3 = 3|A|^2 A e^{it} + \dots$

$\Rightarrow$ $\hat{h}_1(T) = 3|A|^2 A$

$\Rightarrow$ $\partial_T A = \frac{3i}{2}|A|^2 A \longrightarrow A = re^{i\theta}$

$$r_T + ir\theta_T = \frac{3i}{2} r^3 \longrightarrow r_T = 0, \quad \theta_T = \frac{3}{2} r^2$$

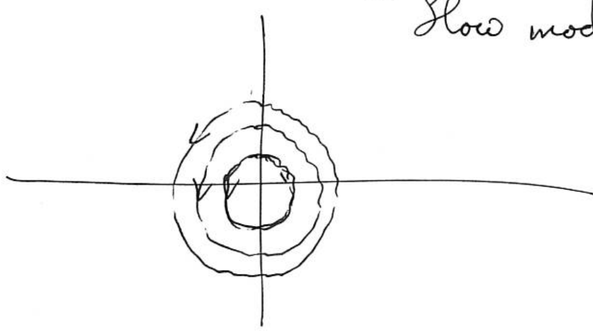
$\Rightarrow \quad r = r_0, \quad \theta = \frac{3}{2} r_0^2 \tau + \theta_0$

$$\Rightarrow \quad x(t) = r_0 e^{i\left(t + \theta_0 + \frac{3}{2}\varepsilon r_0^2 t\right)} + r_0 e^{-i\left(t + \theta_0 + \frac{3}{2}\varepsilon r_0^2 t\right)} + O(\varepsilon)$$

$$= 2r_0 \cos\left(t + \underline{\tfrac{3}{2}\varepsilon r_0^2 t} + \theta_0\right) + O(\varepsilon)$$

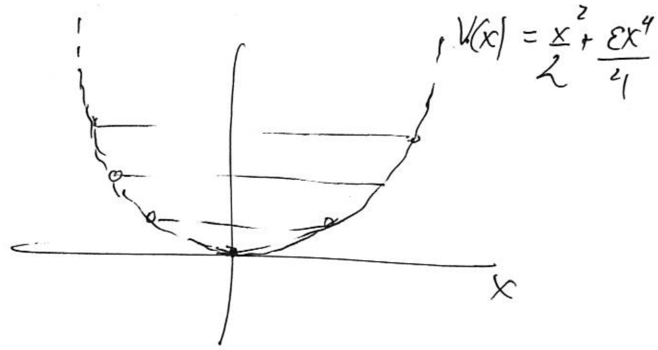
Slow modulation of period.

$\Rightarrow$



Note: Duffing equation may be rewritten as:

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\varepsilon x^4}{4} = E \longrightarrow$$




$\Rightarrow$ so of course we expect periodic orbits. This also helps us appreciate the validity of the method.