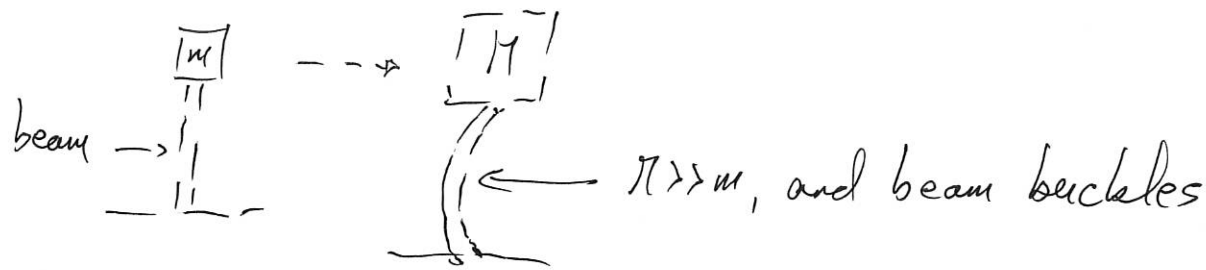
Bifurcations:

Suppose $\dot{x} = f(x, r)$, where r is some parameter.

$\rightarrow \quad x = x(t; r)$

How do the properties of $x(t;r)$ change as we vary r?



A prototypical example: Saddle-Node Bifurcations

$$\dot{x} = r + x^2$$

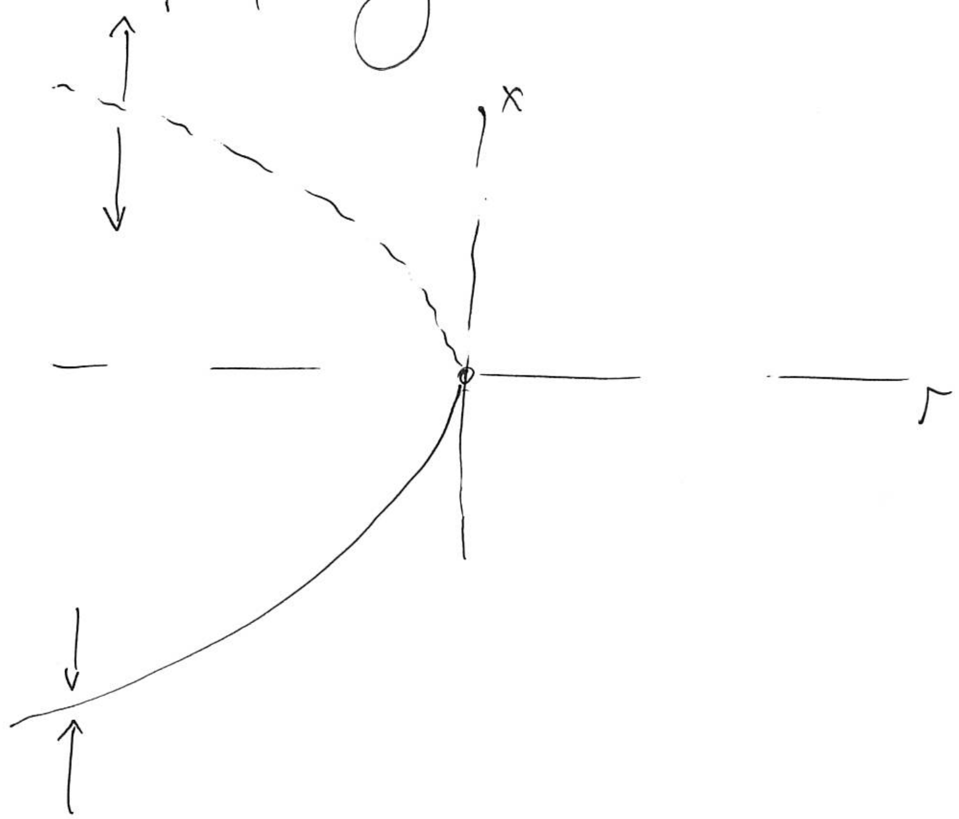
$\rightarrow \quad r + x_*^2 = 0 \quad \longrightarrow \quad x_* = \pm\sqrt{-r}$

So, if $r<0$, we have fixed points. If $r>0$, we do not.

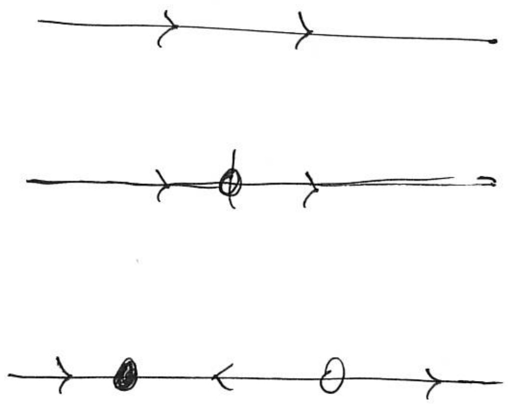
Likewise: $f_x(x_*, r) = 2x_* = \pm 2\sqrt{-r}$

So if $r<0$, $\sqrt{-r}$ is unstable, $-\sqrt{-r}$ is stable

Thus, to capture this information, we make a bifurcation diagram:



If we look at phase plots:



$r > 0$

$r = 0$, "Node"

$r < 0$, "Saddle"

Thus "Saddle-Node" Bifurcation.

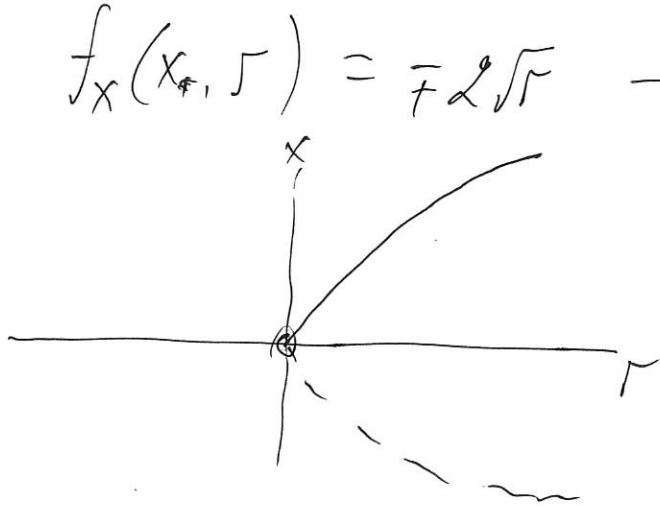
Examine a "Blue-Sky Bifurcation":

$$\dot{x} = r - x^2$$

$\hookrightarrow$ $x_* = \pm\sqrt{r}$ $\rightarrow$ $r > 0$, two fixed points

$r < 0$, none

$f_x(x_*, r) = \mp 2\sqrt{r}$ $\rightarrow$ $+\sqrt{r}$ stable, $-\sqrt{r}$ unstable



Strogatz then says:

Show $\dot{x} = r - x - e^{-x}$ exhibits a saddle-node bifurcation.

Okay, he says something about a graph, blah, blah, blah...

↳ $r - x - e^{-x} = 0$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

for $x \sim 0$:

$$r - x - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = 0$$

↳ $r - 1 - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \cdots = 0$

↳ $r = 1 \rightarrow x_* = 0$. Further, if $r \gtrsim 1 \rightarrow$

$$x_*(r) = \pm(2(r-1))^{1/2} + (r-1)x_1 + (r-1)^{3/2}x_2 + \cdots$$

Or if you like, let $r-1=\varepsilon$

↳ $\varepsilon - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots = 0$

↳ let $x = \pm\sqrt{2\varepsilon} + \varepsilon x_1 + \varepsilon^{3/2} x_2 + \cdots$

$$f_x(x_*(r), r) = -1 + e^{-x_*}$$

$$= -1 + \left(1 - x_* + \frac{x_*^2}{2} \cdots\right)$$

$$= -x_* + \frac{x_*^2}{2} - \cdots$$

Small Smaller

↳ So $-x_*$ determines stability for $r \sim 1$

Normal Forms:

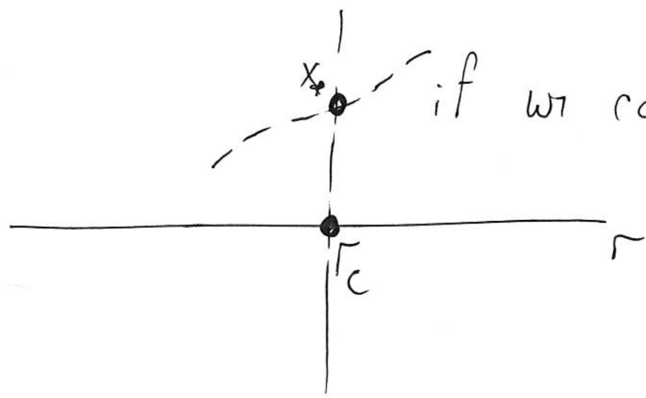
in $\dot{x} = f(x; r)$

Suppose we can show

$$f(x_*; r_c) = 0$$

What makes this a bifurcation?

Well:



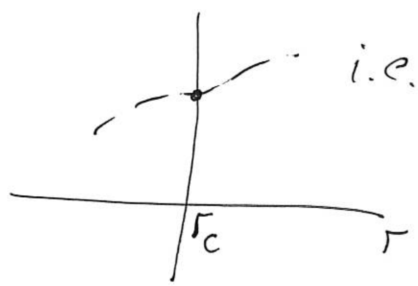
if we can write $x_* = x_*(r)$ around r_c, it is not a bifurcation.

So:

$$f(x; r) = f(x_*; r_c) + f_x(x_*; r_c)(x - x_*) + f_r(x_*; r_c)(r - r_c) + \frac{1}{2} f_{xx}(x - x_*)^2 + f_{xr}(x - x_*)(r - r_c) + \frac{1}{2} f_{rr}(r - r_c)^2 + \cdots = 0$$

So $f(x_*;r_c)=0$, and if $f_x(x_*;r_c)\neq 0$

↳ $$x = x_* - \frac{f_r(x_*;r_c)}{f_x(x_*;r_c)}(r-r_c) + \cdots$$

↳  i.e. no bifurcation!

Requirement 1 for a Bifurcation at (x_*,r_c):

$$\partial_x f(x_*,r_c)=0$$

↳ $\dot{x}=r+x^2 \rightarrow x_*=0,\ r=0,\ f_x(0,0)=0$!

↳ if $f_{xx}(x_*,r_c)\neq 0$:

$$f_{xx}(x-x_*)^2 + 2f_{xr}(r-r_c)(x-x_*) + f_{rr}(r-r_c)^2 + 2f_r(r-r_c) = 0$$

Now we have that:

$$(x-x_*) = \frac{1}{2f_{xx}}\left\{-2f_{xr}(r-r_c) \pm \left(4f_{xr}^2(r-r_c)^2 - 4f_{xx}\left(2f_r(r-r_c)+f_{rr}(r-r_c)^2\right)\right)^{1/2}\right\}$$

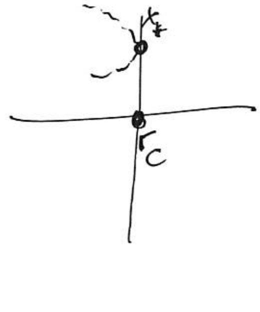
$$= -\frac{f_{xr}}{f_{xx}}(r-r_c) \pm \frac{1}{f_{xx}}\left\{-2f_{xx}f_r(r-r_c) - H(r-r_c)^2\right\}^{1/2}$$

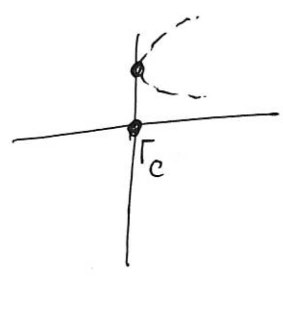
where $H = f_{xx}f_{rr} - f_{xr}^2 = \begin{vmatrix} f_{xx} & f_{xr} \\ f_{xr} & f_{rr} \end{vmatrix}$

So, if we also have $f_r \neq 0$

$$\Rightarrow \quad (x-x_*) \sim \pm\frac{1}{f_{xx}}\sqrt{-2f_{xx}f_r(r-r_c)} + O(r-r_c)$$

$\Rightarrow \quad f_{xx}f_r > 0$ $\qquad\qquad$ $f_{xx}f_r < 0$





But this is hideous to look at.

Let $\tilde{r} = r - r_c$, $u = x - x_*$

↳ $\dot{u} \approx 2f_r \tilde{r} + f_{rr}\tilde{r}^2 + 2f_{xr}\tilde{r}u + f_{xx}u^2$

↳ $\dot{u} \approx a + bu + cu^2$,

$$a = 2f_r\tilde{r} + f_{rr}\tilde{r}^2$$

$$b = 2f_{xr}\tilde{r}$$

$$c = f_{xx}$$ (suppose $c>0$ for sake of argument)

↳ $\dot{u} = a - \frac{b^2}{4c} + \left(\sqrt{c}\,u + \frac{b}{2\sqrt{c}}\right)^2$

↳ $\dot{\tilde{u}} = \tilde{a} + c\tilde{u}^2$, $\tilde{u} = u + \frac{b}{2c}$

↳ $\tilde{u} = \gamma\omega$

↳ $\dot{\omega} = \frac{\tilde{a}}{\gamma} + c\gamma\omega^2 \to \boxed{\dot{\omega} = \bar{a} + \omega^2}$ Normal Form

Now we see how to keep playing this game:

if $f_{xx} \neq 0$ but $f_r = 0$

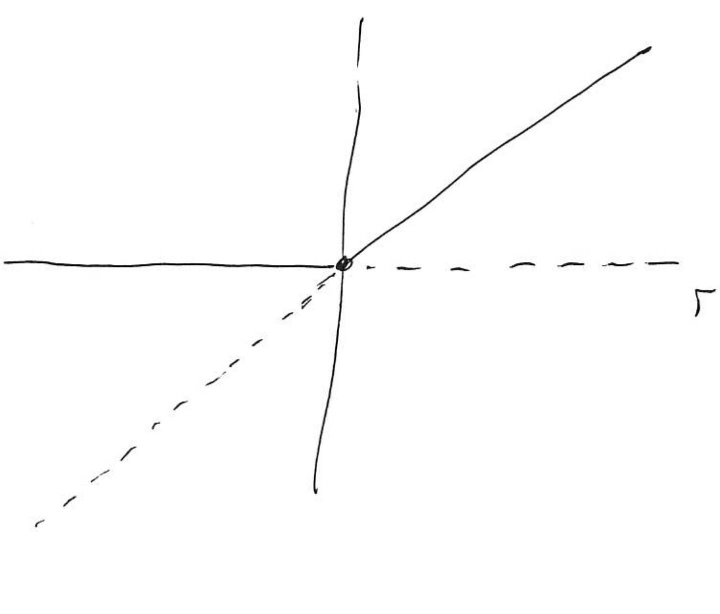
$\Rightarrow$ $$(x - x_*) = -\frac{f_{xr}}{f_{xx}}(r - r_c) \pm \frac{(-H)^{1/2}}{f_{xx}}|r - r_c|$$

$\Rightarrow$ This gives rise to the "transcritical" bifurcation:

$$\dot{x} = rx - x^2$$

$\Rightarrow$ $x_* = 0, r$

$$f_x(x_*, r) = r - 2x_* = \begin{cases} r \\ -r \end{cases}$$



Example:

$$\dot{x} = r \ln x + x - 1$$

So $f(1, r) = 0$

$f_x(x, r) = \frac{r}{x} + 1$

↳ $f_x(1, r) = r + 1 = 0 \rightarrow r_c = -1$ is bifurcation point

$f_r(x, r) = \ln x$

↳ $f_r(1, r) = 0$

$f_{xx}(x, r) = -r/x^2 \rightarrow f_{xx}(1, -1) = 1 \neq 0$

↳ transcritical bifurcation at $(1, -1)$.

Super critical Pitchfork Bifurcation:

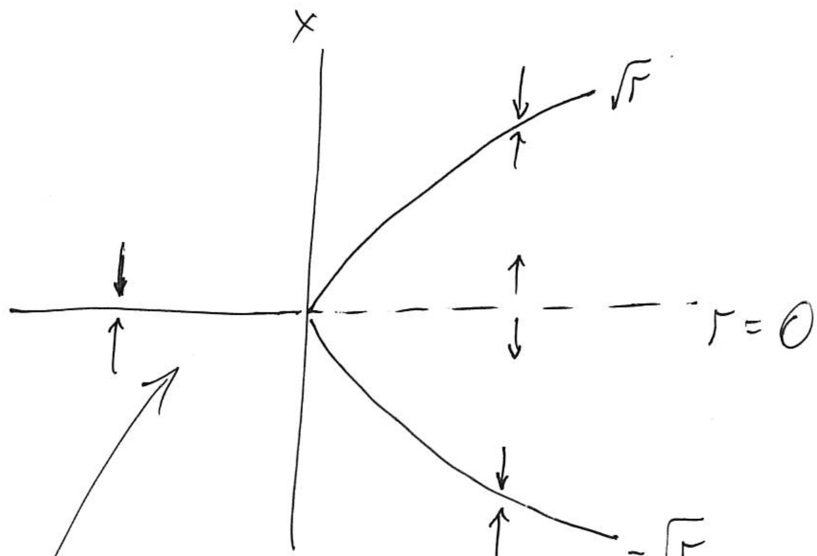
$\dot{x} = rx - x^3$ (i.e. $f_{xx} = 0$ so we must go to higher terms)

$f(x,r) = 0$

$\hookrightarrow x_* = 0, \; x_* = \pm\sqrt{r}$

$$f_x(x_*, r) = r - 3x_*^2 = \begin{cases} r \\ -2r \\ -2r \end{cases}$$

$\hookrightarrow$



$\underbrace{\text{1-fp becomes 3}}$

Note:

$$\dot{x} = rx - x^3 = f(x,r)$$

↳ $f(-x,r) = -f(x,r)$

↳ $\frac{d}{dt}(-x) = -\dot{x} = -f(x,r) = f(-x,r)$

↳ $x(t)$ is a solution $\Rightarrow$ $-x(t)$ is also a soln.

↳ $f(-x,r) = -f(x,r)$ → generally implies a pitchfork bifurcation.

Ex: $\dot{x} = -x + \beta \tanh(x)$

↳ $f(-x,\beta) = x + \beta\tanh(-x) = x - \beta\tanh(x) = -f(x,\beta)$.