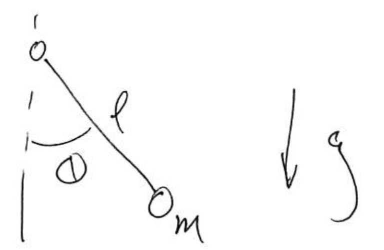
The Pendulum:



$\leadsto \ddot{\theta} + \frac{g}{l}\sin(\theta) = 0$

$\leadsto$ after rescaling $\ddot{\theta} + \sin(\theta) = 0$

$\leadsto \dot{\theta}\ddot{\theta} + \dot{\theta}\sin(\theta) = 0$

$\leadsto \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2\right) - \frac{d}{dt}\cos(\theta) = 0$

$\leadsto \frac{1}{2}\dot{\theta}^2 - \cos(\theta) = E$

Now this is important since if we let $v = \dot{\theta}$

$\leadsto \begin{pmatrix}\dot{\theta} \\ \dot{v}\end{pmatrix} = \begin{pmatrix} v \\ -\sin(\theta)\end{pmatrix} \to$ fp's at $v=0,\ \theta = n\pi$

↳ $J = \begin{pmatrix} 0 & 1 \\ -\cos(\theta) & 0 \end{pmatrix} \longrightarrow J\big|_{(n\pi,0)} = \begin{pmatrix} 0 & 1 \\ -(-1)^n & 0 \end{pmatrix}$

↳ $\lambda^2 + (-1)^n = 0 \longrightarrow \lambda = \begin{cases} \pm i & n \text{ even} \\ \pm 1 & n \text{ odd} \end{cases}$

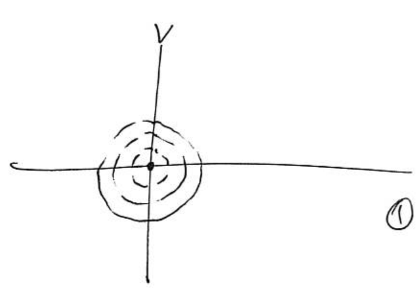
So how do we study dynamics around the points $(2k\pi, 0)$?

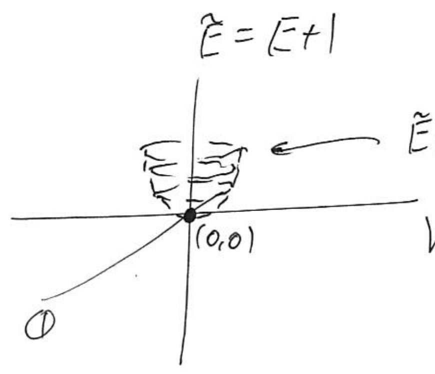
So the equation: $E = \frac{1}{2}v^2 - \cos(\theta)$ is a constraint

i.e. once we fix E, v and θ must satisfy the above equation.

So, for $\theta \sim 0$, $\cos(\theta) = 1 - \frac{\theta^2}{2} + O(\theta^4)$

↳ $E+1 \sim \frac{1}{2}v^2 + \frac{1}{2}\theta^2$, so for $E > -1$, we get "circles" of increasing radius.

↳ 



$\longleftarrow$ $\tilde{E}(\theta, v)$ is a paraboloid.

This also motivates rewriting our constraint as

$$E+1-1 = \frac{1}{2}v^2 - \cos(\theta)$$

$\hookrightarrow$ $$\tilde{E} = \frac{1}{2}v^2 + (1-\cos(\theta))$$

$$= \frac{1}{2}v^2 + 2\sin^2\left(\frac{\theta}{2}\right)$$ and we see a bit more of an equation for a circle.

But to really nail this down, we must use

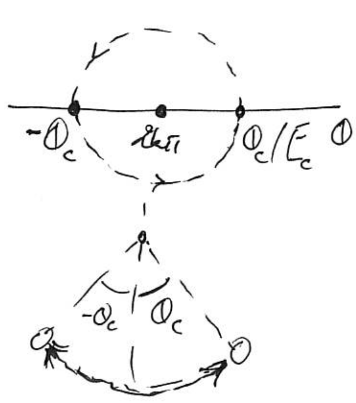
$$\dot{\theta}^2 = 2(E+\cos(\theta))$$

$\hookrightarrow$ $$\frac{d\theta}{\sqrt{2(E+\cos(\theta))}} = dt \quad \text{or} \quad t = \int_{\theta_0}^{\theta(t)} \frac{d\omega}{\sqrt{2(E+\cos(\omega))}}$$

$\hookrightarrow$ Requires the theory of elliptic integrals.

Note, if we know that around the fixed points $(2k\pi, 0)$ we have closed orbits

$\hookrightarrow$



$$\rightarrow \quad T = 2\int_{-\theta_c}^{\theta_c} \frac{d\omega}{\sqrt{2(E_c+\cos(\omega))}}$$

$$= 4\int_{0}^{\theta_c} \frac{d\omega}{\sqrt{2(E_c+\cos(\omega))}}.$$

So, from the integral

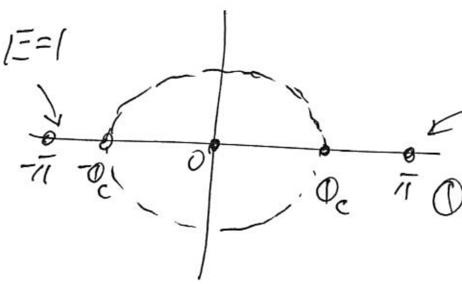
$$t = \int_{\theta_0}^{\theta(t)} \frac{d\omega}{\sqrt{2(E+\cos(\omega))}} \quad \text{and} \quad E = \frac{1}{2}v^2 - \cos(\theta)$$

So @ $\theta = 0$, $v = 0$, $E = -1$

@ $\theta = \pm\pi$, $v = 0$, $E = 1$

and for $-1 < E < 1 \rightarrow E + \cos(\omega)$ has a root for $\omega \in [-\pi, \pi]$

but, if we fix $E = E_c = -\cos(\theta_c)$

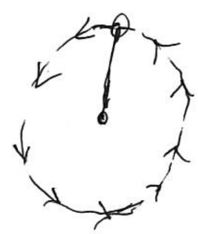


So $E_c + \cos(\omega) \geq 0$ for $\omega \in [-\theta_c, \theta_c]$

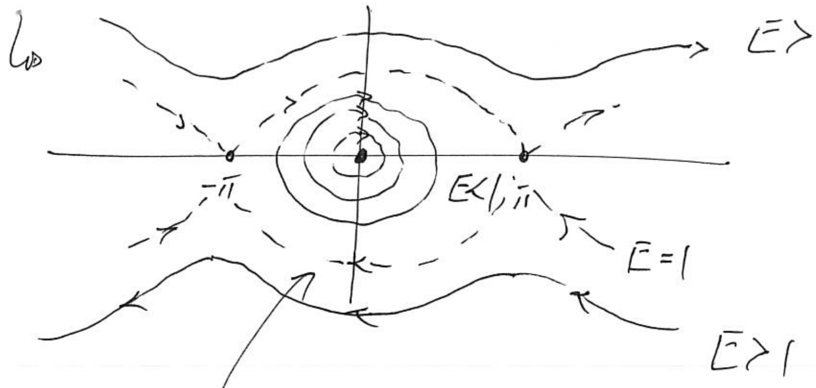
at $E = 1$, we are at $\pm\pi$ or moving in between them,

takes an infinite amount of time.

and for $E > 1$, we just go round and round



↳



Heteroclinic Orbits

More generally, we can study conservative systems:

$$F = ma = m\ddot{x}$$

and if the system is conservative $\rightarrow F = -\dfrac{dV}{dx}$

↳ $\ddot{x} = -\dfrac{1}{m}\dfrac{dV}{dx} \rightarrow \ddot{x}\dot{x} = -\dfrac{1}{m}\dfrac{dV}{dx}\dot{x}$

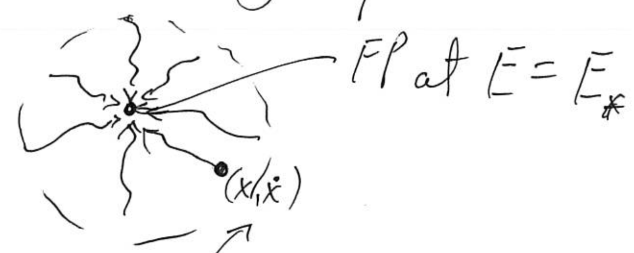
$$\rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{m}V\right) = 0$$

$$\rightarrow \underbrace{\frac{1}{2}\dot{x}^2}_{\text{Kinetic Energy}} + \underbrace{\frac{1}{m}V}_{\text{Potential Energy}} = E \longleftarrow \text{Total Energy}$$

So all motion happens along fixed E contours.

THM: Conservative systems cannot have attracting fixed points.

Pf| Suppose we did:



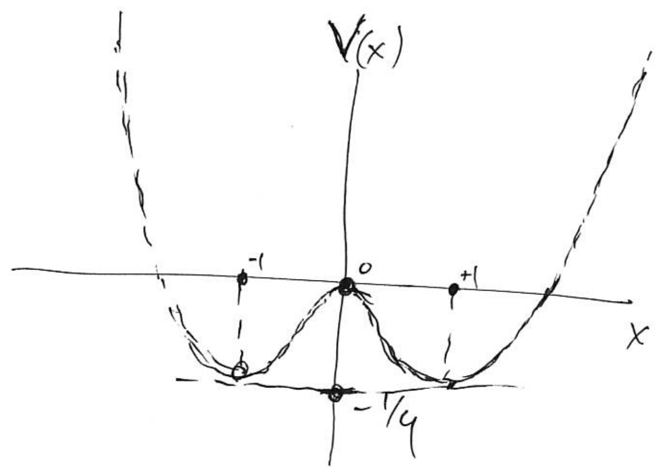
no matter which path I follow, all paths inevitably must be at the same energy E_*, which is clearly nonsensical.

So in conservative systems, we can have saddles and centers, and various connections. From the pendulum we saw heteroclinic connections (between them)

Suppose we choose $V = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, $m = 1$.

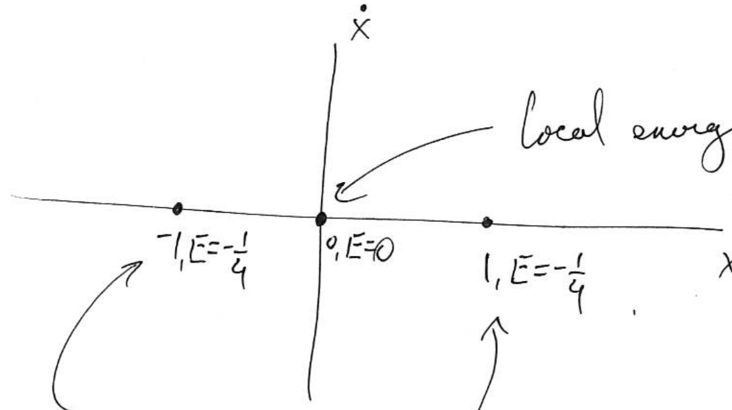
↳ $V'(x) = -x + x^3 = x(x^2 - 1)$

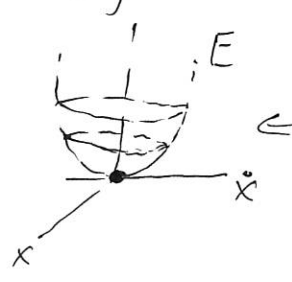
$V''(x) = -1 + 3x^2$

↳ 

"Double-well potential".

↳ $E = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$

↳ 

↳ 

← local minimum around an isolated fixed point ⇒ center!

So if we had instead done the usual thing: $\dot{x} = v$

↳ $\dot{x} = v$

$\dot{v} = x - x^3$

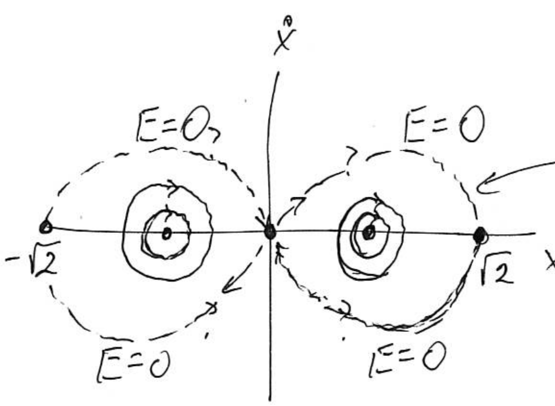
↳ fp's: $(0,0)$, $(\pm 1, 0)$

stab: $J = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$

↳ $(0,0)$; $\lambda = \pm 1$

$(\pm 1, 0)$: $\lambda = \pm i\sqrt{2}$

↳ because of energy considerations, we know this really does mean center.

↳ 

Homoclinic Orbit.

to address ?: $(0,0) \to E=0 \to$ let $\dot{x}=0, E=0 \to x^2\left(\frac{1}{4}x^2 - \frac{1}{2}\right) = 0$

$\to x = \pm\sqrt{2}$