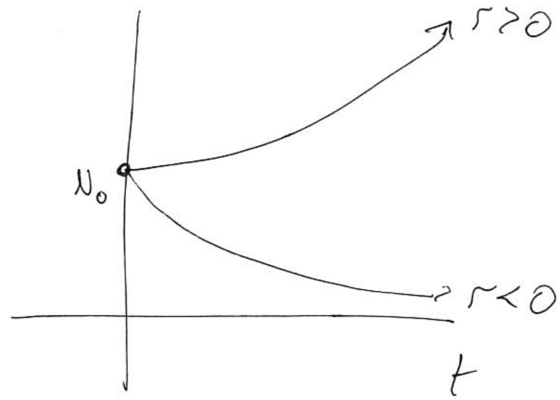
Population Growth:

$N(t) \equiv$ # of members of a species

↳ $\frac{dN}{dt} = rN$ ← Growth/Death Rate

↳ $N(t) = N_0 e^{rt}$

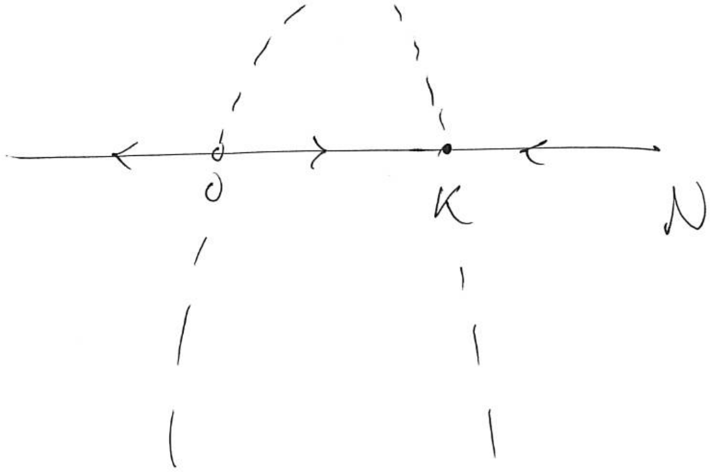


Okay, clearly not so realistic. Thus we move to the Logistic Equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Carrying Capacity

$$rN\left(1-\frac{N}{K}\right)=0 \rightarrow N_*=0, K$$



So if $N(0)=N_0>0 \Rightarrow N(t)\rightarrow K$

But what can rescaling tell us?

$$N=N_s\tilde{N}; \quad \tau=t/t_s$$

$$\frac{dN}{dt}=rN\left(1-\frac{N}{K}\right) \rightarrow \frac{N_s}{t_s}\frac{d\tilde{N}}{d\tau}=rN_s\tilde{N}\left(1-\frac{N_s}{K}\tilde{N}\right)$$

$$\rightarrow \frac{d\tilde{N}}{d\tau}=t_s r\tilde{N}\left(1-\frac{N_s}{K}\tilde{N}\right)$$

$$\rightarrow t_s r=1; \; N_s/K=1.$$

So $t_s = 1/r$; $N_s = K$, and that makes sense. (3)

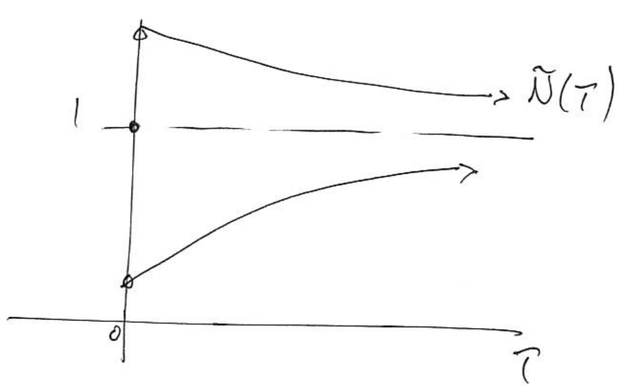
$$\frac{d\tilde{N}}{d\tau} = \tilde{N}(1-\tilde{N}) \rightarrow \frac{d\tilde{N}}{\tilde{N}(1-\tilde{N})} = d\tau$$

$$\rightarrow \left\{\frac{1}{\tilde{N}} + \frac{1}{1-\tilde{N}}\right\} d\tilde{N} = d\tau$$

$$\rightarrow \ln\left|\frac{\tilde{N}}{1-\tilde{N}}\right| = \tau + C$$

$$\rightarrow \frac{\tilde{N}}{1-\tilde{N}} = Ce^{\tau}$$

$$\rightarrow \tilde{N}(\tau) = \frac{Ce^{\tau}}{1+Ce^{\tau}} \rightarrow N(t) = \ldots$$



Linearized Analysis:

$$\dot{x} = f(x)$$

$$\text{w/ } f(x_*) = 0$$

Taylor Series:

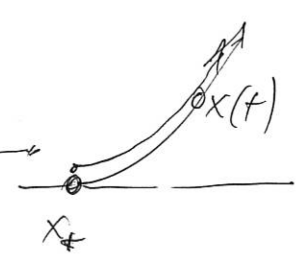
$$f(x) = f(x_*) + f'(x_*)(x - x_*) + O((x-x_*)^2)$$

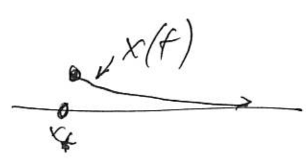
$$= f'(x_*)(x - x_*) + O((x-x_*)^2)$$

$\hookrightarrow \dot{x} \sim f'(x_*)(x - x_*)$

$\hookrightarrow \frac{d}{dt}(x - x_*) \sim f'(x_*)(x - x_*)$

$\hookrightarrow \dot{u} \sim f'(x_*)\, u$ where $u = x - x_*$

$\hookrightarrow f'(x_*) > 0 \rightarrow u(t) \sim e^{f'(x_*)t} \rightarrow$ 

$f'(x_*) < 0 \rightarrow u(t) \sim e^{f'(x_*)t} \rightarrow$ 

i.e. $f'(x_*) > 0 \rightarrow$ small deviations to x_* grow $\rightarrow$

x_* is repellant / unstable.

$f'(x_*) < 0 \rightarrow$ small deviations to x_* grow $\rightarrow$

x_* is attractive / stable.

Example: $\dot{x} = \sin(x)$

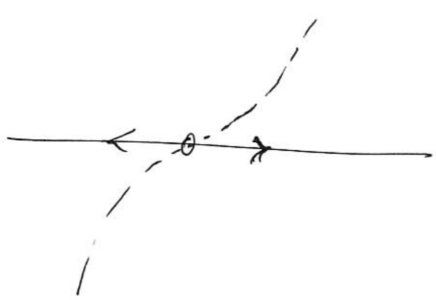
$x_* = n\pi$

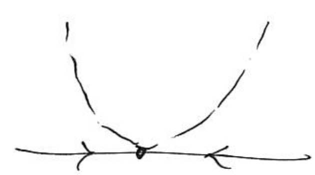
$f'(x_*) = \cos(n\pi) = (-1)^n$

$\hookrightarrow$ n even $\rightarrow f'(x_*) = 1 > 0 \rightarrow$ unstable

n odd $\rightarrow f'(x_*) = -1 < 0 \rightarrow$ stable

$f'(x_*) = 0$? $\dot{x} = -x^3$ vs. $\dot{x} = x^2$





Moral of Story: More Work