Oscillations:

In many cases we can write our dynamical system in the form

$\dot{\theta} = f(\theta)$ where  i.e. we oscillate on the circle.

So, for this to happen, we need

$$f(\theta + 2\pi) = f(\theta) \text{ i.e. } f \text{ is a } 2\pi \text{ periodic function.}$$

Note though, $\theta = \theta(t)$, and so θ has its own "period", say T so that

$\theta(t+T) = \theta(t) + 2\pi$; so θ is not itself actually periodic.

So $f(\theta(t+T)) = f(\theta(t) + 2\pi) = f(\theta(t))$

To compute the period, we note that

$$\frac{d\theta}{dt} = f(\theta) \longrightarrow \frac{d\theta}{f(\theta)} = dt \longrightarrow \int_0^{2\pi} \frac{d\theta}{f(\theta)} = \int_0^T dt = T$$

Thus:

$$\boxed{T = \int_0^{2\pi} \frac{d\theta}{f(\theta)}}$$

Nice Example:

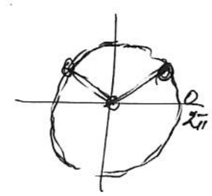
$$\dot{\theta} = \omega \longrightarrow \theta(t) = \omega t + \theta_0$$

$$T = \int_0^{2\pi} \frac{d\theta}{\omega} = \frac{2\pi}{\omega}$$

Less Nice Example:

$$\dot{\theta} = \omega - a\sin(\theta), \quad \omega > 0, \ a \geq 0$$

So before doing any thing else:

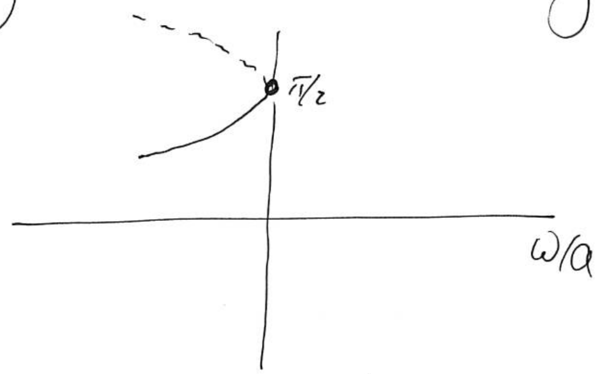


FP's: $\omega - a\sin(\theta) = 0 \longrightarrow \sin(\theta) = \omega/a$

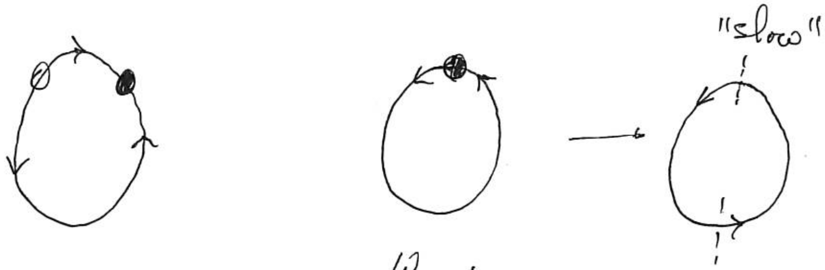
$\hookrightarrow \left|\frac{\omega}{a}\right| < 1 \rightarrow \theta_* = \begin{cases} \sin^{-1}\left(\frac{\omega}{a}\right) \\ \pi - \sin^{-1}\left(\frac{\omega}{a}\right) \end{cases}$

Stb: $\partial_\theta f = -a\cos(\theta_*) = \begin{cases} -a\left(1 - \omega^2/a^2\right)^{1/2} \\ a\left(1 - \omega^2/a^2\right)^{1/2} \end{cases}$

So, we get a Saddle-Node Bifurcation



$\omega/a = 1$



$\frac{\omega}{a} = 1$

$\frac{\omega}{a} < 1$

Too slow, and we come to a halt.

$\frac{\omega}{a} > 1$

$\theta = \pi/2$, $\omega - a\sin(\theta) = \omega - a \to$ slowest

$\theta = 3\pi/2$, $\omega - a\sin(\theta) = \omega + a \to$ fastest

Further, as $\omega/a \gtrsim 1$,

we can make the "slow" portion take arbitrarily long.

So, honestly, Strogatz should have first said

$$\tau = t/t_s$$

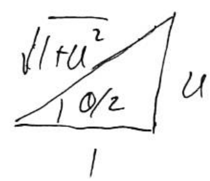
↳ $$\frac{1}{t_s}\frac{d\theta}{d\tau} = \omega - a\sin(\theta)$$

↳ $$\frac{d\theta}{d\tau} = t_s\omega - t_s a\sin(\theta)$$

If we let $t_s a = 1 \longrightarrow \dfrac{d\theta}{d\tau} = \tilde{\omega} - \sin(\theta)$, $\tilde{\omega} = \dfrac{\omega}{a}$.

↳ $$\tilde{T} = \int_0^{2\pi} \frac{d\theta}{\tilde{\omega} - \sin(\theta)} = \int_{-\pi}^{\pi} \frac{d\theta}{\tilde{\omega} + \sin(\theta)}$$

↳ $u = \tan\left(\frac{\theta}{2}\right)$



↳ $$\sin(\theta/2) = \frac{u}{\sqrt{1+u^2}}; \quad \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+u^2}}$$

↳ $$\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2) = \frac{2u}{1+u^2}$$

$$du = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)d\theta \quad \text{or} \quad d\theta = \frac{2}{(1+u^2)}\, du$$

and thus

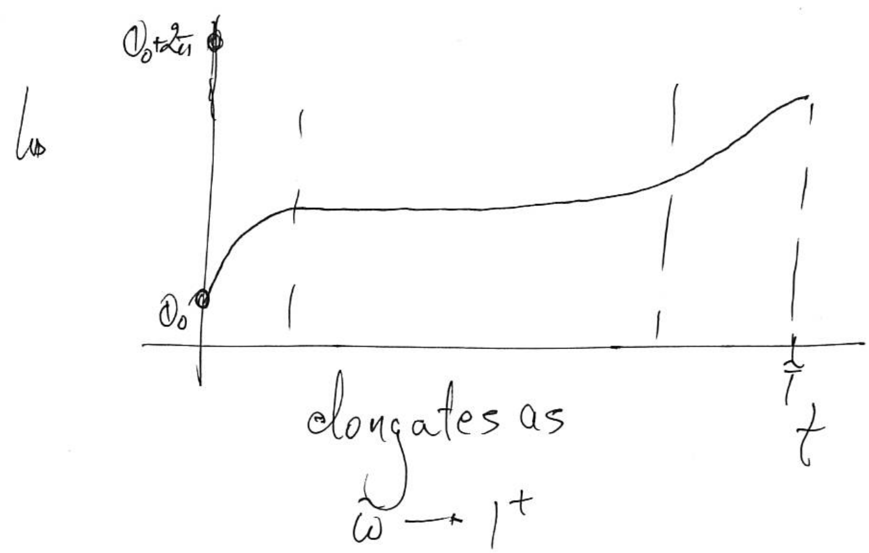
$$\tilde{T} = \int_{-\infty}^{\infty} \frac{2\,du}{(1+u^2)\left(\tilde{\omega} + 2u/(1+u^2)\right)} = \cancel{4\int_{0}^{\infty} \frac{du}{\tilde{\omega}(1+u^2)+2u}}$$

$$= \int_{-\infty}^{\infty} \frac{2\,du}{\tilde{\omega}(1+u^2)+2u}$$

$$= \frac{2}{\sqrt{\tilde{\omega}}} \int_{-\infty}^{\infty} \frac{du}{u^2 + \tilde{\omega} - \frac{1}{\tilde{\omega}}} = \frac{2\pi}{\sqrt{\tilde{\omega}^2 - 1}}$$

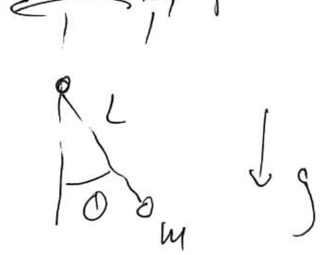
So for $\tilde{\omega} \sim 1$:

$$\tilde{T} \simeq \frac{2\pi}{(1+\tilde{\omega})^{1/2}(-1+\tilde{\omega})^{1/2}} \sim \frac{\pi\sqrt{2}}{(-1+\tilde{\omega})^{1/2}} \text{ for } \tilde{\omega} \sim 1^+$$



Examples:

The Overdamped Pendulum:



damping

external torque

$\hookrightarrow \quad mL^2\ddot{\Theta} + b\dot{\Theta} + mgL\sin(\Theta) = \Gamma$

$\hookrightarrow \quad \frac{L}{g}\ddot{\Theta} + \frac{b}{mgL}\dot{\Theta} + \sin(\Theta) = \tilde{\Gamma}$

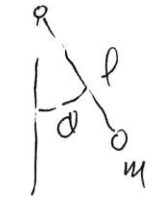
$\hookrightarrow \quad \frac{L}{gT^2}\ddot{\Theta} + \frac{b}{mgLT}\dot{\Theta} + \sin(\Theta) = \tilde{\Gamma} \quad (\tau = t/T)$

$\hookrightarrow \quad T = \frac{b}{mgL} \longrightarrow \frac{L}{gT^2} = mgL\left(\frac{mL^2}{b^2}\right) = \varepsilon \ll 1$

$\hookrightarrow \quad \dot{\Theta} \approx \tilde{\Gamma} - \sin(\Theta)$

So again, for $\tilde{T} > 1$, we get periodic motion, and for $\tilde{T} < 1$, the torque is not enough to overcome damping.

Systems of Equations:

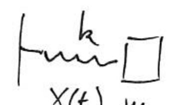


$T = \frac{1}{2} m (\ell \dot{\theta})^2$; $V = mg\ell(1-\cos(\theta))$

$\Rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$

So if $\theta \sim 0 \Rightarrow \sin(\theta) \approx \theta$

$\Rightarrow \quad \ddot{\theta} + \omega^2 \theta = 0$ where $\omega = \sqrt{\frac{g}{\ell}}$.

or 

$\rightarrow \quad \ddot{x} + \frac{k}{m} x = 0 \quad \rightarrow \quad \omega = \sqrt{\frac{k}{m}}$

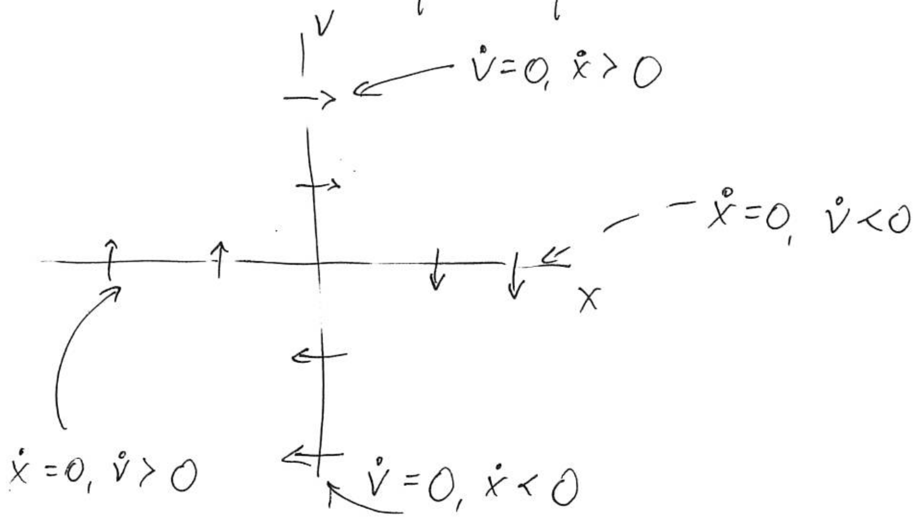
$\Rightarrow \quad \ddot{x} + \omega^2 x = 0$

So, to make this 1st order, we let $v = \dot{x}$

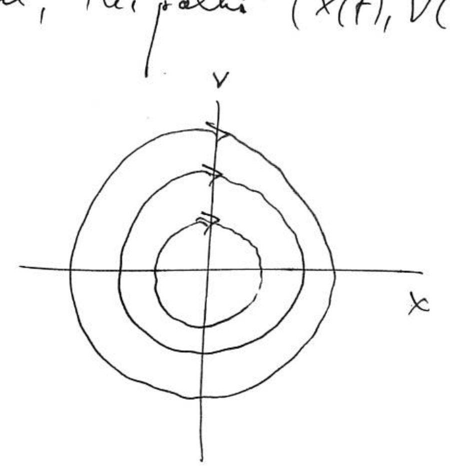
$\Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x \end{cases}$

$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$

We track motion in the phase-plane:



As we will see, the paths $(x(t), v(t))$ are closed circles



So, we in general want to be able to understand how to solve

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} \quad ; \quad \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

or

$$\frac{d}{dt}\vec{x} = A\vec{x}, \quad \vec{x} \in \mathbb{R}^2, \quad \vec{x}(0) = \vec{x}_0$$

To do this, we are going to "diagonalize" A i.e.

1. Find eigenvectors / eigenvalues $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$

   note: we can run into trouble when $\lambda_1 = \lambda_2$, will get to that.

↳ Let $V = \left(\vec{v}_1 \middle| \vec{v}_2 \right)$

↳ $AV = \left(A\vec{v}_1 \middle| A\vec{v}_2 \right) = \left(\lambda_1\vec{v}_1 \middle| \lambda_2\vec{v}_2 \right)$

$$= \left(\vec{v}_1 \middle| \vec{v}_2 \right) \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

↳ $AV = V\Lambda$; $\Lambda = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

↳ $A = V \Lambda V^{-1}$

2. in $\dot{\vec{x}} = A\vec{x} = V\Lambda V^{-1}\vec{x}$

↳ $V^{-1}\dot{\vec{x}} = \Lambda V^{-1}\vec{x}$

↳ $\frac{d}{dt}\left(V^{-1}\vec{x} \right) = \Lambda \left(V^{-1}\vec{x} \right)$

Set $\vec{y} = V^{-1}\vec{x}$

$\hookrightarrow \dfrac{d\vec{y}}{dt} = \Lambda \vec{y} \rightarrow \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}$

$\hookrightarrow \dot{y}_1 = \lambda_1 y_1 \; ; \; \dot{y}_2 = \lambda_2 y_2$

So diagonalization reduces the problem to two very simple ODE's.

$\hookrightarrow y_1(t) = y_{1,0}\, e^{\lambda_1 t} \; ; \; y_2(t) = y_{2,0}\, e^{\lambda_2 t}$

(3) So $\vec{y}(t) = y_{1,0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_1 t} + y_{2,0}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 t}$

To go back to $\vec{x}(t)$ we note that

$\vec{y}_0 = V^{-1}\vec{x}_0$

$\vec{x}(t) = V\vec{y}(t) = V\left\{ y_{1,0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_1 t} + y_{2,0}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 t} \right\}$

$V\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\vec{v}_1 | \vec{v}_2\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{v}_1 \; ; \; \left(\vec{v}_1 | \vec{v}_2\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{v}_2$

(5)

↳

$$\vec{x}(t) = y_{1,0}\vec{v}_1 e^{\lambda_1 t} + y_{2,0}\vec{v}_2 e^{\lambda_2 t}, \quad \vec{y}_0 = V^{-1}\vec{x}_0$$

↳ So all dynamics determined by eigenvectors, eigenvalues.

Tr / Det plane:

$$(A-\lambda)\vec{v} = 0 \rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\rightarrow \lambda^2 - tr(A)\lambda + det(A) = 0$$

$$tr(A) = a+d; \quad det(A) = ad-bc$$

$$\rightarrow \lambda^{\pm} = \frac{1}{2}\left\{ tr(A) \pm \left[(tr(A))^2 - 4 det(A) \right]^{1/2} \right\}$$

$$= \frac{1}{2}\left\{ \tau \pm \left[\tau^2 - 4\Delta \right]^{1/2} \right\}$$

$\Delta < 0$ / saddles: (16)

$$\Delta < 0 \rightarrow \tau^2 - 4\Delta > \tau^2 \rightarrow \underbrace{\left(\tau + (\tau^2-4\Delta)^{1/2}\right)}_{-}\underbrace{\left(\tau - (\tau^2-4\Delta)^{1/2}\right)}_{+} < 0$$

i.e. opposite signs

$\hookrightarrow$ i.e. $\vec{x}(t) = \gamma_{1,0}\vec{v}_1 e^{\lambda_1 t} + \gamma_{2,0}\vec{v}_2 e^{\lambda_2 t}$

$\hookrightarrow$ one direction represents growth, the other decay

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$$

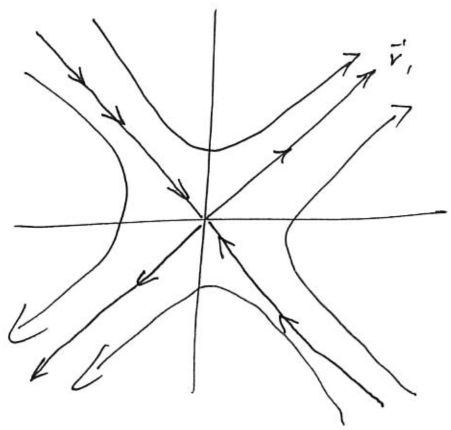
$\hookrightarrow \tau = -1; \ \Delta = -6$

$\hookrightarrow \lambda = \frac{1}{2}\left[1 \pm 5 \right] = 3, -2$

$$\lambda = 3 \rightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda = -2 \rightarrow \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \vec{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

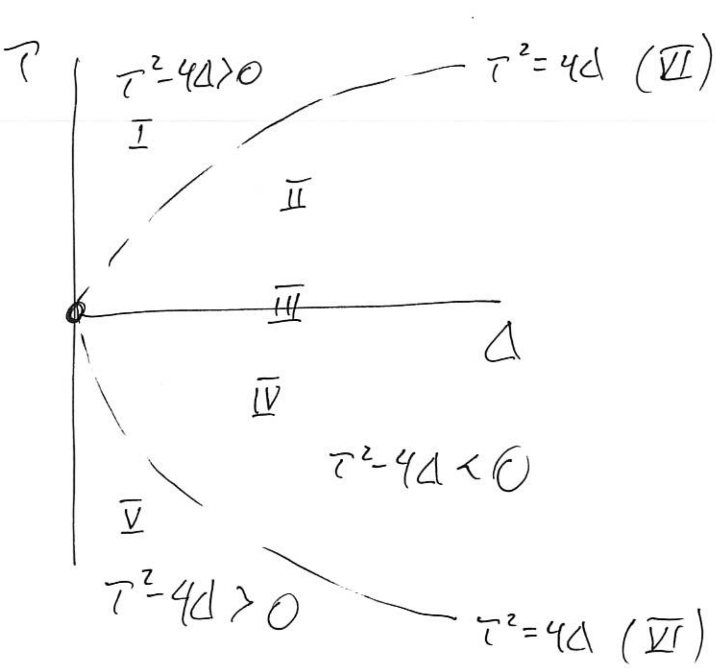
L_0



i.e. we have a "saddle"

$\Delta > 0$: $\tau^2 - 4\Delta > 0$, $\tau^2 - 4\Delta = 0$, $\tau^2 - 4\Delta < 0$

"saddle"



I : $\tau^2 - 4\Delta > 0$, $\tau > 0$

II : $\tau^2 - 4\Delta < 0$, $\tau > 0$

III : $\tau^2 - 4\Delta < 0$, $\tau = 0$

IV : $\tau^2 - 4\Delta < 0$, $\tau < 0$

V : $\tau^2 - 4\Delta > 0$, $\tau < 0$

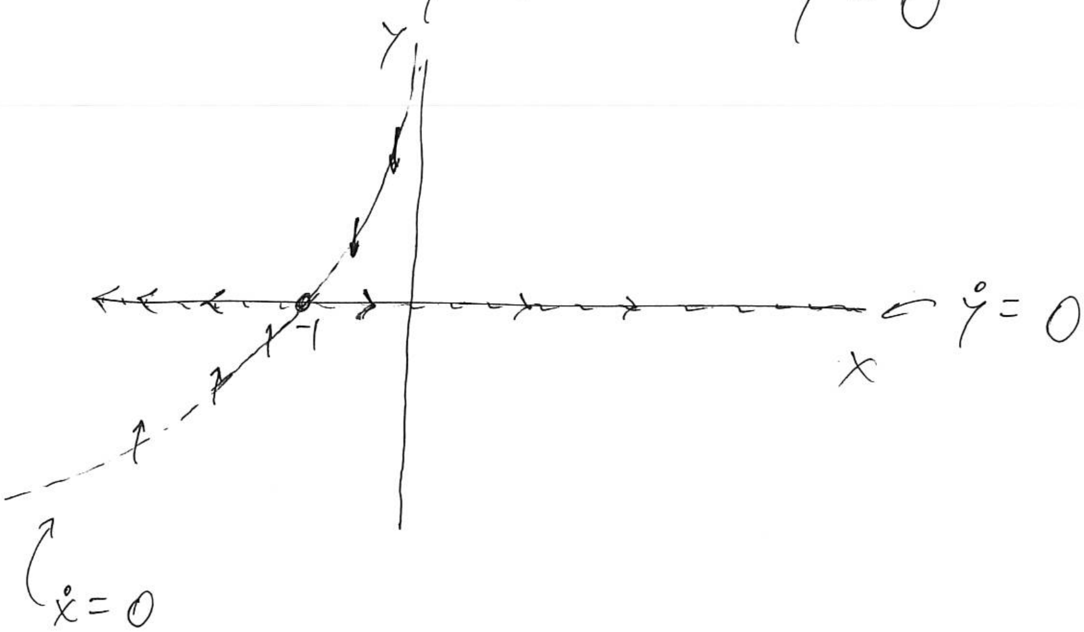
Moving on: Phase-Planes for 2-D nonlinear systems

$$\dot{x} = x + e^{-y}$$

$$\dot{y} = -y$$

↳ Nullclines: $\dot{x} = 0 \rightarrow x = -e^{-y}$

$\dot{y} = 0 \rightarrow y = 0$



So, from this we get one, the appearance of a fixed point at $(-1, 0)$ which is clearly the intersection of null-clines. But we also get the appearance of what very much looks like a saddle.

To wit, for a non-linear system

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

Let (x_*, y_*) be a fixed point i.e. $f(x_*, y_*) = g(x_*, y_*) = 0$

↳ $f(x,y) = f(x_*, y_*) + f_x(x_*, y_*)(x - x_*) + f_y(x_*, y_*)(y - y_*) + \cdots$

$g(x,y) = g(x_*, y_*) + g_x(x_*, y_*)(x - x_*) + g_y(x_*, y_*)(y - y_*) + \cdots$

↳ if $\dot{x} = \frac{d}{dt}(x - x_*)$; $\dot{y} = \frac{d}{dt}(y - y_*)$

↳ $$\frac{d}{dt}\begin{pmatrix} x - x_* \\ y - y_* \end{pmatrix} \approx \begin{pmatrix} f_x(x_*, y_*) & f_y(x_*, y_*) \\ g_x(x_*, y_*) & g_y(x_*, y_*) \end{pmatrix} \begin{pmatrix} x - x_* \\ y - y_* \end{pmatrix}$$

↳ so by finding the Jacobian of our system, we can hopefully linearize the problem and determine the local dynamics around the fixed point.

So for $f(x,y) = x + e^{-y}$; $g(x,y) = -y$

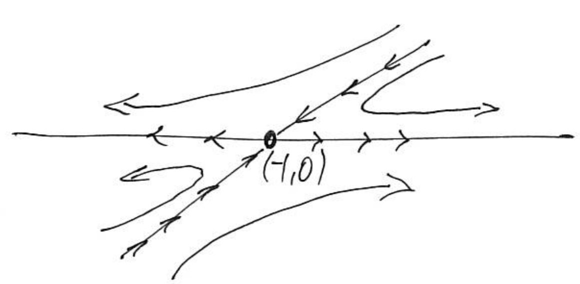
↳ $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1 & -e^{-y} \\ 0 & -1 \end{pmatrix}$

↳ @ $(-1,0) \to \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \to \tau = 0 ; \Delta = -1$

↳ $\lambda = \frac{1}{2}\{0 \pm \{0+4\}\}^{1/2} = \pm 1$

↳ $\lambda = 1 \to \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

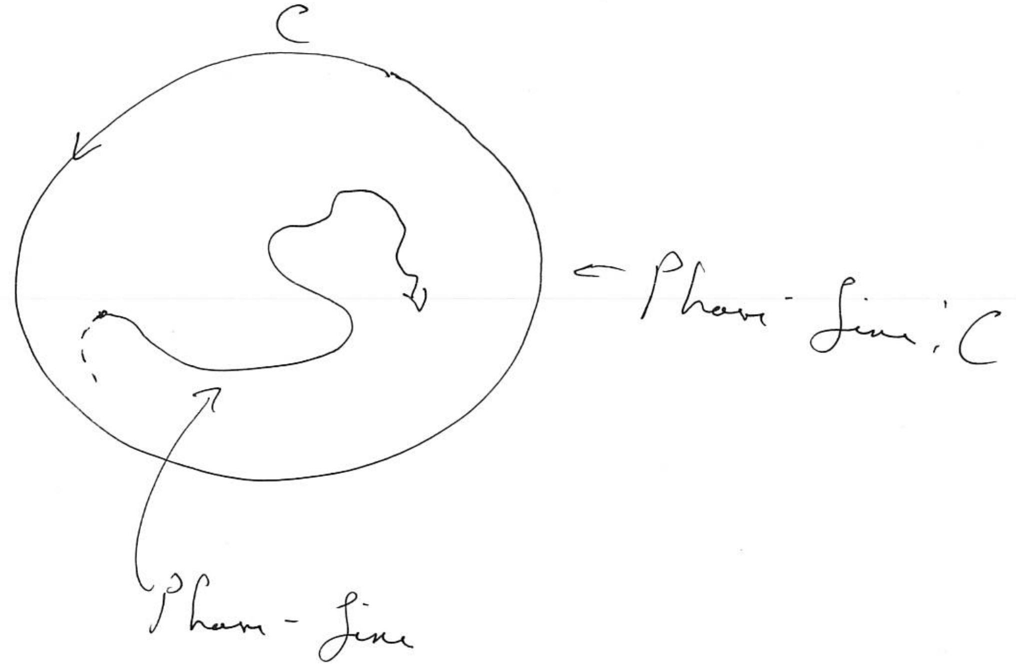
$\lambda = -1 \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



Rules of the Phase-Plane: For any autonomous system, existence and uniqueness of solutions imply:

$$\boxed{\text{Phase Lines Never Cross}}$$

So:



If there is no fixed point within C, then the interior curve must, in the limit, approach C. (Poincaré-Bendixson THM)

Hyperbolic vs Non-Hyperbolic Fixed Points:

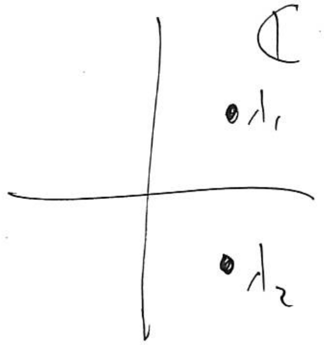
For $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, let $f(x_*,y_*) = g(x_*,y_*) = 0$.

↳ So, when we find the associated evals of the Jacobian via $\lambda = \frac{1}{2}\left(\tau \pm (\tau^2 - 4\Delta)^{1/2}\right)$

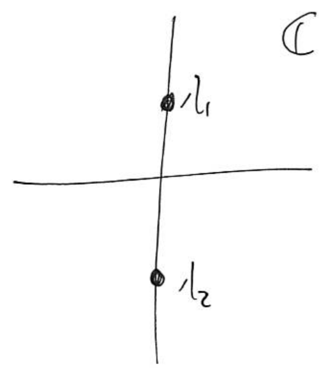
↳ we say the fixed point is

hyperbolic if: $\text{Re}\{\lambda\} \neq 0$

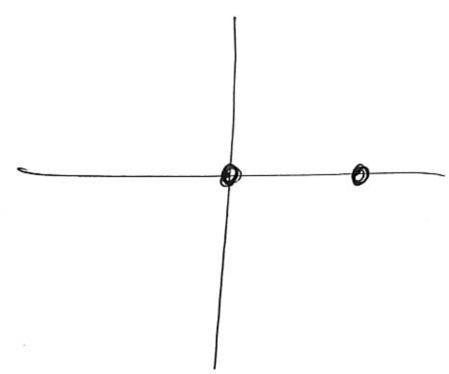
non-hyperbolic if: $\text{Re}\{\lambda\} = 0$



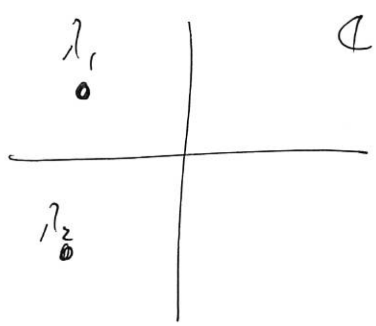
hyperbolic



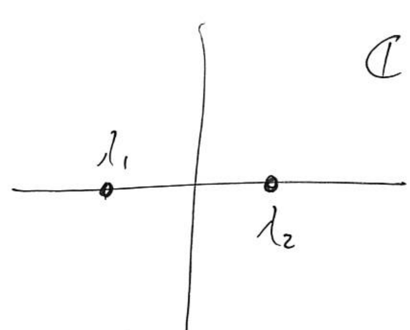
non-hyperbolic



non-hyperbolic



hyperbolic



hyperbolic

Hyperbolicity means what happens in the linearization happens in a local neighborhood of the fixed point.

i.e. in $\begin{cases} \dot{x} = x + e^{-y} \\ \dot{y} = -y \end{cases}$ $\longrightarrow$ $(-1,0)$ is a linearized saddle

so the non-linear flow behaves like a saddle.

Pernicious Example:

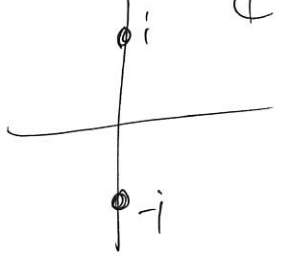
$$\dot{x} = -y + ax(x^2+y^2)$$

$$\dot{y} = x + ay(x^2+y^2)$$

Clearly $(0,0)$ is a fixed point

Jacobian: $\begin{pmatrix} 3ax^2+ay^2 & -1+2axy \\ 1+2axy & ax^2+3ay^2 \end{pmatrix}$

@ $(0,0) \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \lambda^2+1=0 \longrightarrow \lambda = \pm i$

$\Rightarrow$ 

not hyperbolic !

To analyze this, go to polar coordinates:

$$x = r\cos(\theta) \quad , \quad y = r\sin(\theta)$$

$$\dot{x} = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)$$

$$\dot{y} = \dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)$$

$$\Rightarrow \quad \dot{r} = \dot{x}\cos(\theta) + \dot{y}\sin(\theta) = ar^3$$

$$r\dot{\theta} = \dot{y}\cos(\theta) - \dot{x}\sin(\theta) = r$$

$$\Rightarrow \quad \dot{r} = ar^3 \; ; \quad \dot{\theta} = 1$$

$$\Rightarrow \cancel{\int \frac{dr}{r^3} = \int a\,dt \longrightarrow -\frac{1}{2}r^{-2} = at + C}$$