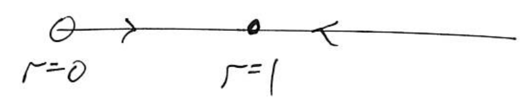
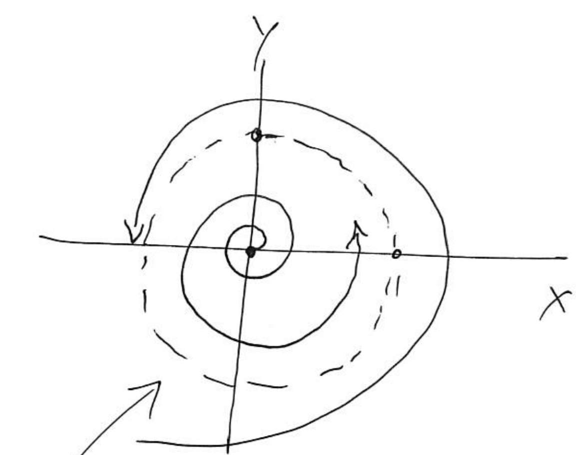
Limit Cycles:

$$\dot{r} = r(1-r^2) \;;\quad \dot{\theta} = 1$$

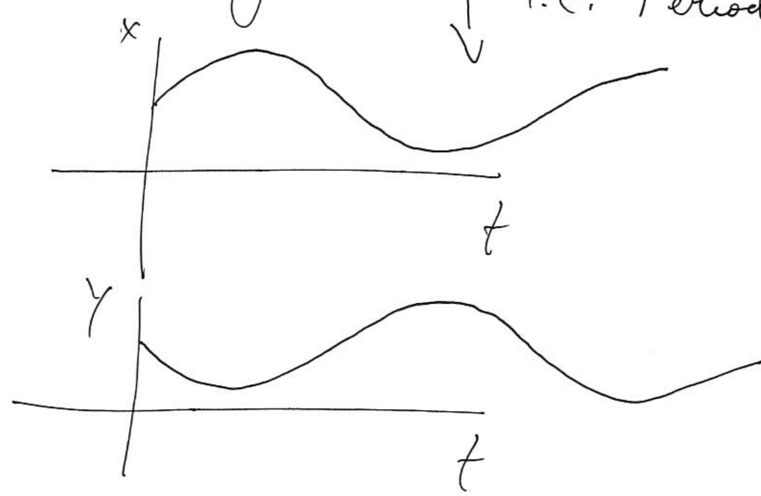
↳ $r = 0, 1 \rightarrow \partial_r f = 1 - 3r^2 \rightarrow \partial_r f|_0 > 0,\ \partial_r f|_{r=1} < 0$

↳ 

↳ 

$x = r\cos(\theta)$
$y = r\sin(\theta)$

$r = 1$: Limit Cycle → i.e. Periodic / Oscillatory Motion w/o External Forcing!

↳ 

Limit Cycles in Nature:

- Beating Heart
- Chemical Reactions
- Hormone Levels
- Temperature / Weather Patterns

and on and on.

Limit Cycles are thus an enormous issue, and a central subject of study so it is important to have some criterion for establishing their existence.

For:

$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

How can I know I have a limit cycle? → Global, not local question → linearization really won't help.

The Poincaré - Bendixson THM:

For $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$

(*) Suppose we have a closed, bounded region R such that

- R does not contain any fixed points.
- There is a trajectory contained within R

i.e. some $(x(t), y(t))$: $(x(t), y(t)) \in R \ \forall t$.

Then: A limit cycle exists in R.

Example (slight variant from the one in the book)

$\dot{r} = r(1-r^2) + \mu r^2 \cos\theta$

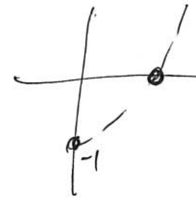
$\dot{\theta} = 1$

So, for $\mu = 0$, we know we have a limit cycle. Can we show one persists for small μ?

So, if $\dot{r} = f(r,\theta) = r(1 - r^2 + \mu r \cos\theta)$

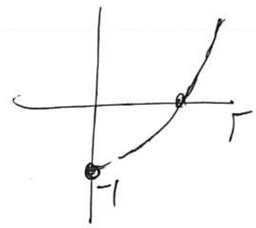
So for $\mu > 0$, $r > 0$, $\frac{f}{r} < 1 - r^2 + \mu r$; $\frac{f}{r} > 1 - r^2 - \mu r$

Find: $1-r^2+\mu r < 0$ or $r^2-\mu r-1 \geq 0$



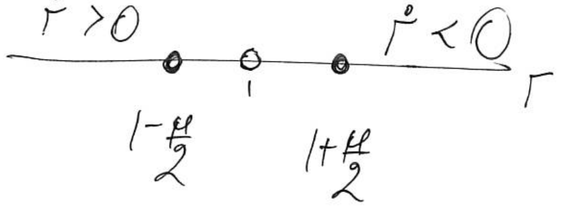
$\Rightarrow r \geq \frac{1}{2}\{\mu + (\mu^2+4)^{1/2}\} \simeq 1+\mu/2$

$1-r^2-\mu r > 0$ or $r^2+\mu r-1 \leq 0$

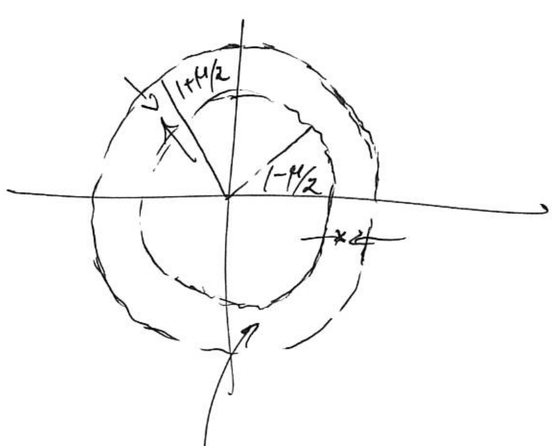


$\Rightarrow r \leq \frac{1}{2}\{-\mu + (\mu^2+4)^{1/2}\} \simeq 1-\frac{\mu}{2}$

$\Rightarrow$



$\Rightarrow$



So we see every orbit is trapped within this annulus.
There are no fixed points, so we are done i.e.
There must be a limit cycle somewhere in the annulus.

Glycolysis: the break down of sugar in living beings.

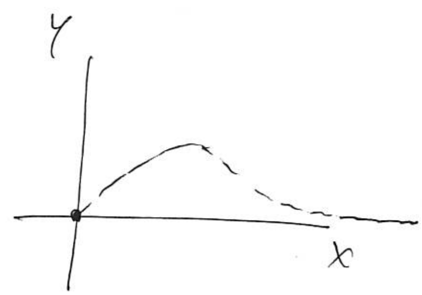
$$\dot{x} = -x + ay + x^2 y, \quad a, b > 0$$
$$\dot{y} = b - ay - x^2 y$$

→ $x \equiv$ concentration of ADP

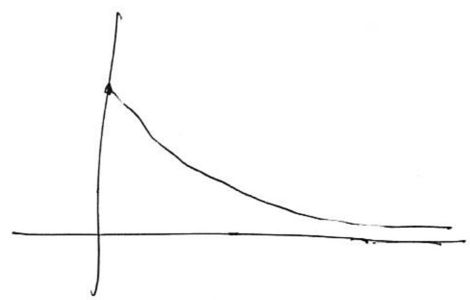
$y \equiv$ concentration of F6P

1 Null-clines:

$\dot{x} = 0$: $y = x/(a+x^2)$ →



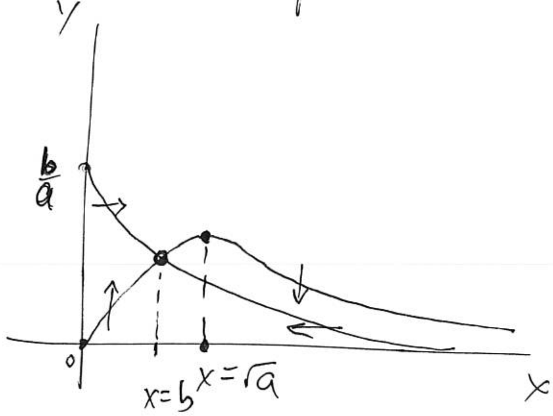
$\dot{y} = 0$: $y = b/(a+x^2)$ →



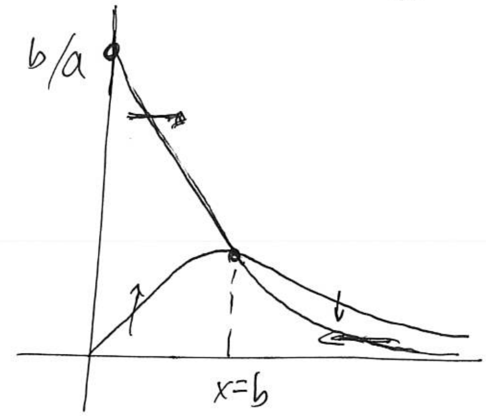
And then Strogatz gets a little glib for my taste...

→ Fixed point: $\dfrac{b}{a+x^2} = \dfrac{x}{(a+x^2)} \longrightarrow x = b.$

Thus, we get the picture, noting that for $y = x/(a+x^2)$, $\frac{dy}{dx} = \frac{a-x^2}{(a+x^2)^2} \rightarrow x = \sqrt{a}$



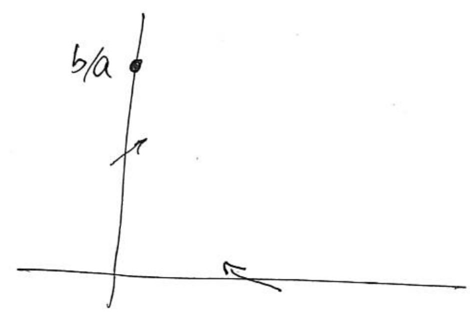
or



$b < \sqrt{a}$ $\qquad\qquad$ $b > \sqrt{a}$

Now, fix $y = 0 \longrightarrow \dot{x} = -x, \; \dot{y} = b \longrightarrow$

$x = 0 \longrightarrow \dot{x} = ay, \; \dot{y} = b - ay$

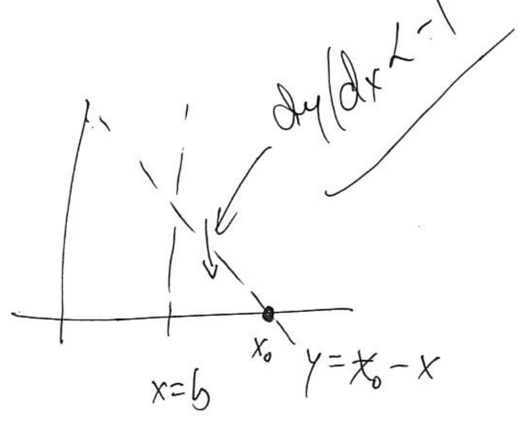


Now note:

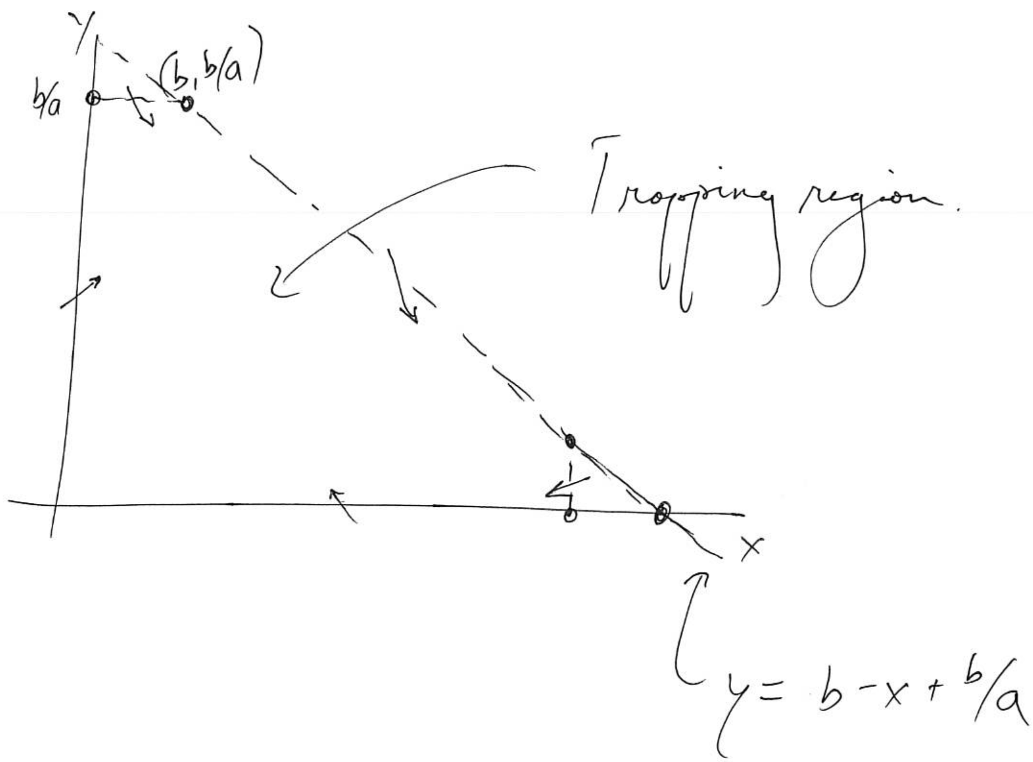
$\dot{x} + \dot{y} = b - x \longrightarrow \dot{x} + \dot{y} < 0$ if $x > b$

$\longrightarrow \frac{dy}{dt} < -\frac{dx}{dt}$

so $\frac{dy}{dx} < -1 \longrightarrow$



↳ thus roughly we get:



But we have a fixed point within our region, so we have more work to do.

↳ $J = \begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{bmatrix}$

↳ $\tau = -1 - a + 2xy - x^2$

$\Delta = a + x^2 > 0$

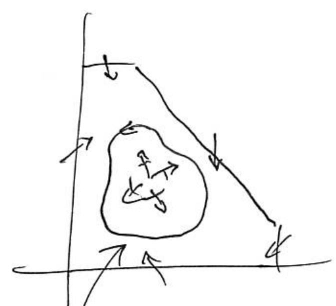
↳ so in principle, sign of τ matters the most.

So, at $(b, b/(a+b^2))$

→ $T = \dfrac{b^2-a}{b^2+a} - (a+b^2)$

→ $T=0$: $b^2 = \frac{1}{2}\left(1-2a \pm (1-8a)^{1/2}\right)$

→ if $T>0$, we have an unstable fixed point

→ 

limit cycle within.

$T<0$, we have a stable spiral/sink.

→