Reversible Systems:

Sometimes symmetry isn't obvious as with $\ddot{x} = \frac{-1}{m}\frac{dV}{dx}$

Thus we look for something a bit more general:

eg:

$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

is invariant under $t \to -t$; $y \to -y$, we call the system <u>reversible</u>.

Note: $t \to -t$

$$\Rightarrow \quad -\dot{x} = f(x,y)$$

$$-\dot{y} = g(x,y)$$

$$y \to -y$$

$$\Rightarrow \quad \dot{x} = -f(x,-y)$$

$$\dot{y} = g(x,-y)$$

So, for this to work, we need

$$-f(x,-y) = f(x,y) \qquad \text{i.e. odd in } y \text{ for } f$$

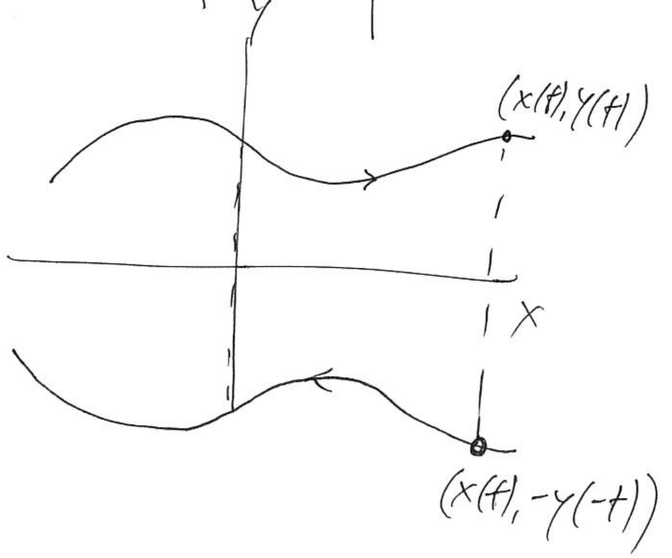
$$g(x,-y) = g(x,y) \qquad \text{even in } y \text{ for } g$$

$$\hookrightarrow \; f(x,y) = \alpha_1 x^{m_1} y + \alpha_2 x^{m_2} y^3 + \cdots$$

$$g(x,y) = \beta_0 + \beta_1 x^{n_1} y^2 + \beta_2 x^{n_2} y^4 + \cdots$$

So very general requirement compared to $\ddot{x} = -\frac{1}{m}\frac{dV}{dx}$.

So, invariant with respect to $t \to -t$, $y \to -y$ means



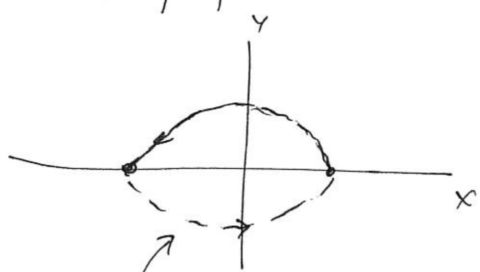
i.e. if we have the curve above, we must have the curve below and vice versa.

THM: Suppose the origin $\vec{x}_* = 0$ is a linear center for the system

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

and this system is reversible. Then for a sufficiently small neighborhood around the origin, all orbits are closed.

Pf | So the graphical idea is that if we "swirl" at all, then eventually our curve must touch the x-axis twice.



To make this a bit more formal, we would write

$$\dot{x} = -\omega y + \alpha x^m y^{2k+1} + \cdots = -\omega y \left(1 - \frac{\alpha}{\omega} x^m y^{2k} + \cdots\right)$$
$$\dot{y} = \omega x + \beta x^n y^{2j} + \cdots = \omega x \left(1 + \frac{\beta}{\omega} x^{n-1} y^{2j} + \cdots\right)$$

So leading order is $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \to \lambda^2 = -\omega^2$ ✓.

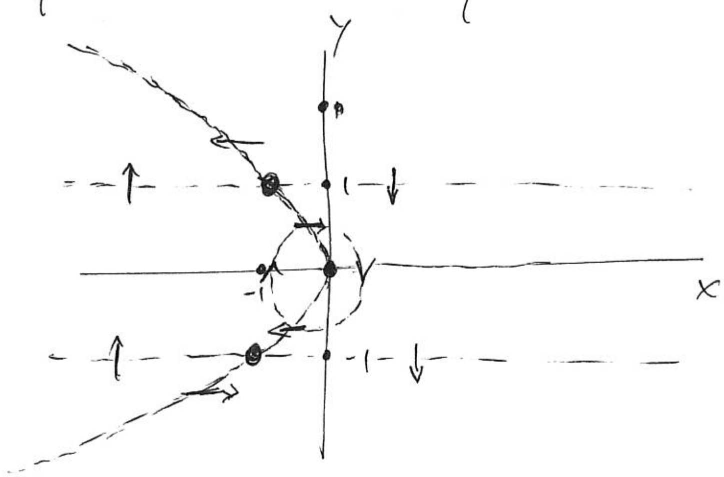
Look at:

$$\dot{x} = y - y^3$$

$$\dot{y} = -x - y^2$$

↳ Clearly has linear center at origin → clearly is non-linear center

$$\left.\begin{aligned}\dot{x} = 0 &\to y(1-y^2) = 0 \\ \dot{y} = 0 &\to x = -y^2\end{aligned}\right\}$$ Plot nullclines



↳ Fixed points at $(0,0)$, $(-1,\pm 1)$

↳ $J = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix} \to J\big|_{(-1,\pm 1)} = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix}$

↳ $\Delta = -2 < 0$ → both are saddles.

Given:

$$\dot{x} = y$$
$$\dot{y} = x - x^2$$

Let's show we have a homoclinic orbit in the half plane $x \geq 0$;

So, clearly $(0,0)$ is a fixed point and

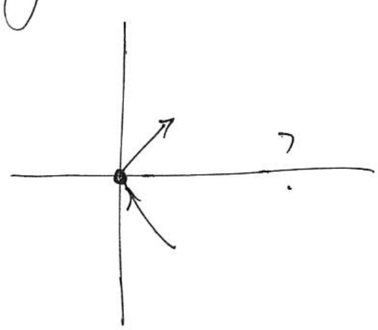
$$J = \begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix} \rightarrow J\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so $\lambda^2 = 1 \rightarrow \lambda = \pm 1$

$\lambda = 1 \rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1 \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So just from this we get:



we see the other fixed point is at $(1,0) \to J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to$ so that's not a lot of help ...

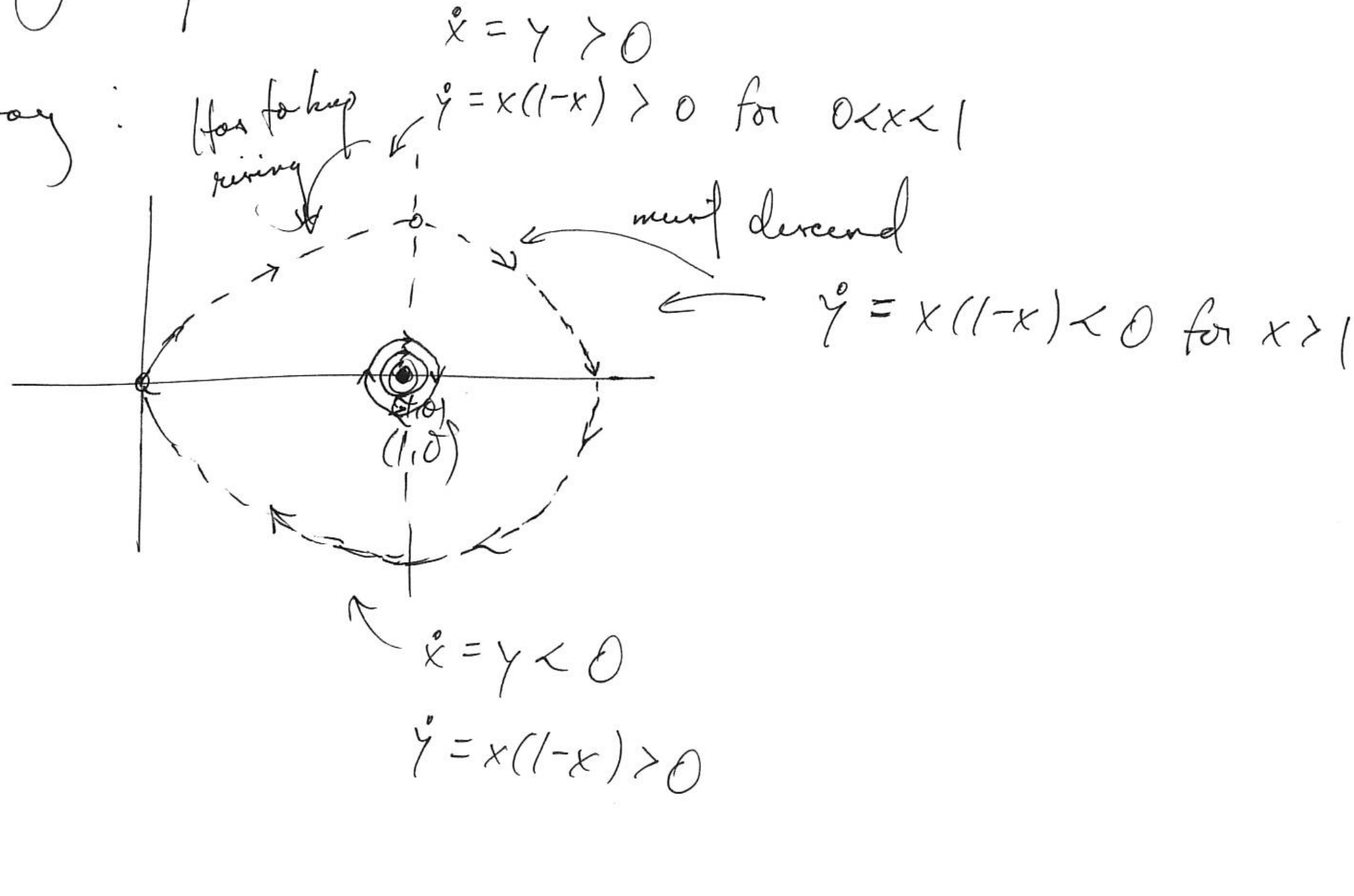
well actually let's do this ...

let $\tilde{x} = x - 1 \to \dot{\tilde{x}} = \dot{x} = y$

$$\dot{y} = \tilde{x} + 1 - (\tilde{x}+1)^2 = (\tilde{x}+1)(1-(\tilde{x}+1))$$
$$= -\tilde{x}(\tilde{x}+1)$$

So this system is also reversible $\to$ we have a non linear center at the other fixed point.

So anyway :



And sometimes we just don't have any structure at all...

Lotka - Voltera Models

$x(t)$ - Rabbits

$y(t)$ - Sheep

$\hookrightarrow \quad \dot{x} = r_r x(N_r - x) - c_r xy$

$\dot{y} = r_s y(N_s - y) - c_s xy$

So we see this is two separate logistic equations with coupling provided via the competition terms $c_r xy$ and $c_s xy$

So, we make some assumptions

$r_r >> r_s \longrightarrow$ Rabbits out produce sheep

$N_r >> N_s \longrightarrow$ There can be many more rabbits than sheep

$c_r >> c_s \longrightarrow$ Competition is far more difficult on rabbits than sheep.

So we can take a state of non-dimensionalization

$$\tau = t/t_s \;;\quad \tilde{x} = x/N_r \;;\quad \tilde{y} = y/N_s$$

so $\dot{x} = r_r \cancel{t_s} N_r\, x(1-x) - c_r N_s t_s\, xy$

$$\dot{y} = r_s t_s N_s\, y(1-y) - c_s N_r t_s\, xy$$

If we let $t_s = \dfrac{1}{r_r N_r}$

so $\dot{x} = x(1-x) - \dfrac{c_r N_s}{r_r N_r}\, xy$

$$\dot{y} = \frac{r_s N_s}{r_r N_r}\, y(1-y) - \frac{c_s}{r_r}\, xy$$

Set: $\dfrac{r_s N_s}{r_r N_r} = \varepsilon \ll 1$

so $\dot{x} = x(1-x) - \varepsilon \dfrac{c_r}{r_s}\, xy$

$$\dot{y} = \varepsilon y(1-y) - \frac{c_s}{r_r}\, xy$$

Let $\tilde{C}_r = C_r/\Gamma_s$; $\tilde{C}_s = C_s/\Gamma_r$

Thus $$\dot{x} = x(1-x) - \varepsilon\tilde{C}_r xy$$

$$\dot{y} = \varepsilon y(1-x) - \tilde{C}_s xy$$

Lastly, if we want competition to more adversely affect the rabbits —

we need $\varepsilon\tilde{C}_r \gg \tilde{C}_s$.