The Linear Advection Equation and The Fourier Transform

The Simplest Model For a Wave is:

$$\eta_t + c\eta_x = 0$$

$$\eta(x,0) = \eta_0(x)$$

To solve it, we introduce the coordinate transformation

$$\xi = x - ct\,, \quad \chi = x + ct$$

$$\Rightarrow \partial_t = -c\,\partial_\xi + c\,\partial_\chi$$

$$\partial_x = \partial_\xi + \partial_\chi$$

$$\Rightarrow \eta_t + c\eta_x = 0$$

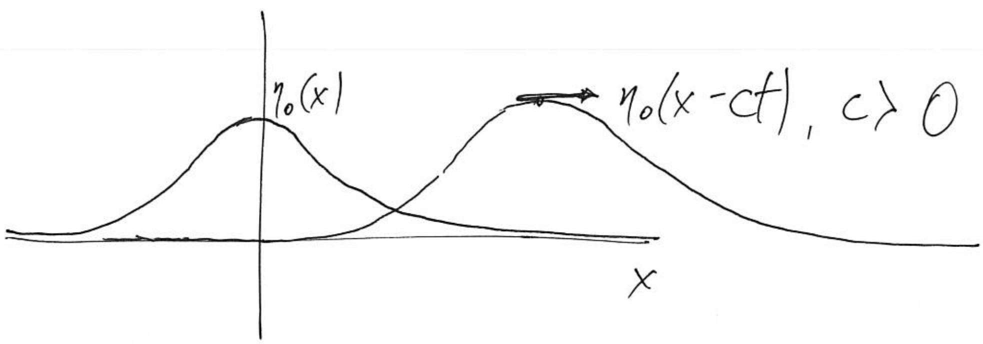
$$\Rightarrow -c\eta_\xi + c\eta_\chi + c(\eta_\xi + \eta_\chi) = 0$$

$$\Rightarrow 2c\,\eta_\chi = 0$$

$$\Rightarrow \eta = \eta(\xi) = \eta(x - ct).$$

If we match this at $t=0$ to $\eta_0(x) \longrightarrow$

$$\eta(x,t) = \eta_0(x-ct)$$



$\hookrightarrow$ Initial condition propagates without change of shape.

A slightly more complicated model is

$$\eta_{tt} = c^2 \eta_{xx}$$

$$\eta(x,0) = \eta_0(x)$$

$$\eta_t(x,0) = v_0(x)$$

Again, let $\xi = x-ct$; $\chi = x+ct$

$$\hookrightarrow \partial_t^2 = (-c\partial_\xi + c\partial_\chi)^2 = c^2\partial_\xi^2 - 2c^2\partial_{\xi\chi}^2 + c^2\partial_\chi^2$$

$$\partial_x^2 = (\partial_\xi + \partial_\chi)^2 = \partial_\xi^2 + 2\partial_{\xi\chi}^2 + \partial_\chi^2$$

↳ $4c^2 \partial_{\xi\chi}^2 \eta = 0$

↳ $\partial_\chi (\partial_\xi \eta) = 0$

↳ $\partial_\xi \eta = f(\xi)$

↳ $\eta = \int^\xi f(s)\,ds + g(\chi)$

$= \tilde{f}(\xi) + g(\chi)$

or: $\eta(x,t) = f(x-ct) + g(x+ct)$

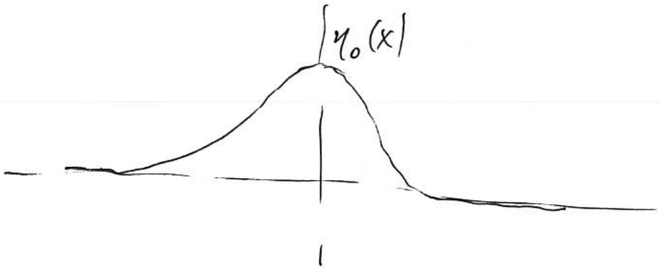
↳ $\eta_0 = f(x) + g(x)$

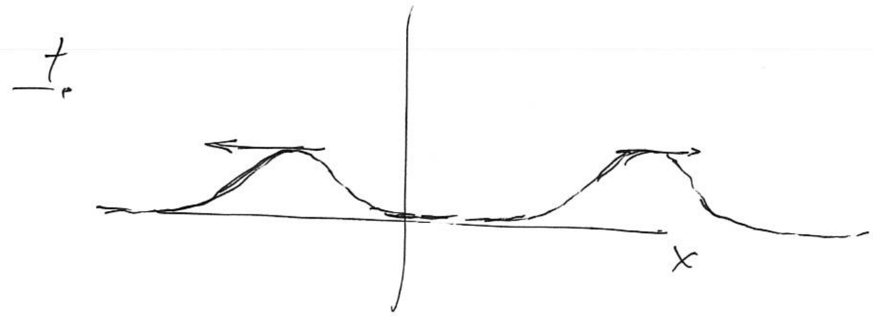
$v_0 = -cf'(x) + cg'(x)$

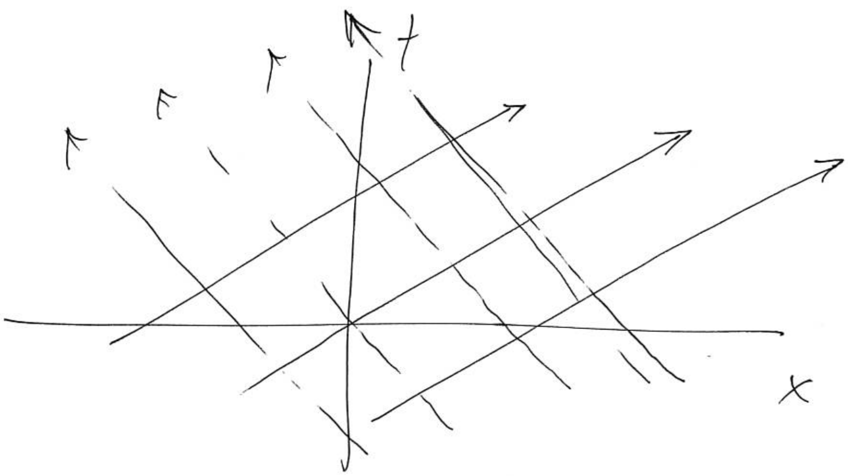
↳ $\eta_0' = f' + g'$ $\longrightarrow$ $c\eta_0' + v_0 = 2cg'$

$v_0 = -cf' + cg'$ $\qquad$ $c\eta_0' - v_0 = 2cf'$

$$\eta(x,t) = \frac{1}{2}\eta_0(x-ct) + \frac{1}{2}\eta_0(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,ds$$







so now information propagates in both directions, but again, functions essentially just get transported about.

Now let's see all of this from a very different viewpoint.

The Fourier Transform:

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x)\,dx$$

And its inverse

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \hat{f}(k)\,dk$$

The most important property of a Fourier-Transform is what it does to derivatives

$$(\partial_x f)^\wedge = \int_{\mathbb{R}} e^{-ikx} \partial_x f\,dx = \cancel{e^{-ikx} f \Big|_{-\infty}^{\infty}} \to 0 + ik \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

$$= ik\,\hat{f}(k)$$

i.e. $(\partial_x)^\wedge = ik \longrightarrow (\partial_x^n)^\wedge = (ik)^n$

or derivatives become polynomials upon transformation.

So, let's go back to

$$\eta_t + c\eta_x = 0$$
$$\eta(x,0) = \eta_0(x)$$

We transform $\eta_t + c\eta_x = 0$

$\hookrightarrow$ $\hat{\eta}_t + ikc\,\hat{\eta} = 0$

$\hookrightarrow$ $\hat{\eta}_t = -ikc\,\hat{\eta}$

$\hookrightarrow$ $\hat{\eta}(k,t) = \hat{\eta}(k,0)\,e^{-ikct}$

or $\hat{\eta}(k,t) = \hat{\eta}_0(k)\,e^{-ikct}$

$\hookrightarrow$ $$\eta(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}(k,t)\,e^{ikx}\,dk$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}_0(k)\,e^{ik(x-ct)}\,dk = \eta_0(x-ct).$$

So, same answer, but now we have a very different understanding of what is going on.