Conservation of Momentum or $F = ma$ in a fluid

Momentum: $\vec{p} = m\vec{u} \rightarrow \rho(\vec{x},t)\,\vec{u}(\vec{x},t)\,d\vec{x}$

So, from physics we have:

$$\frac{D\vec{p}}{Dt} = \vec{F} \longleftarrow \text{forces on a fluid element}$$

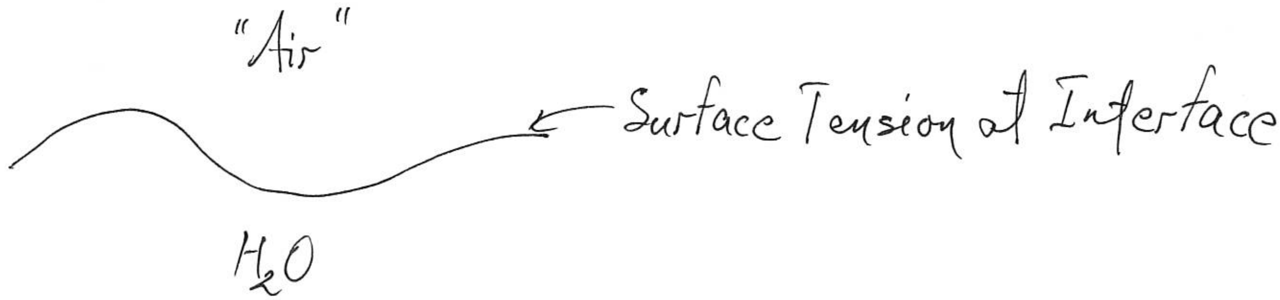
time rate of change of momentum

So what are the forces on a fluid?

a) External forces i.e. gravity: $-g\hat{k}$ $\downarrow$

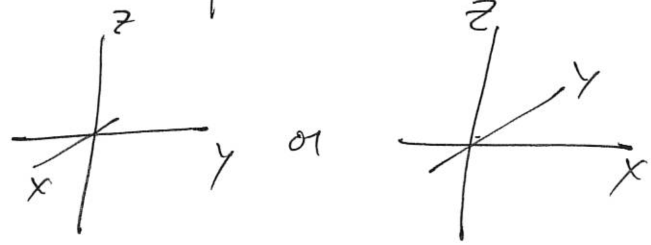
$\hookrightarrow -g\hat{k} = -\nabla V, \; V = gz$

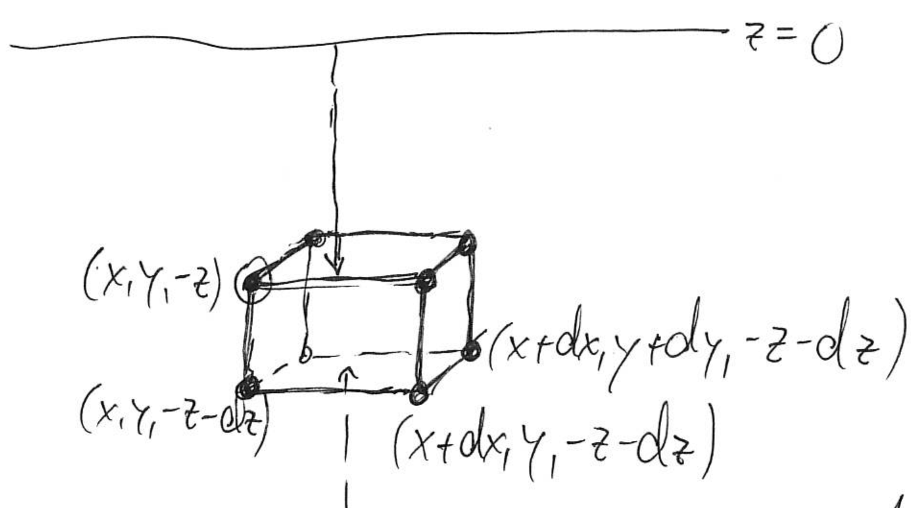
b) Line forces like surface tension:



c) Internal Forces: And it gets complicated

Pressure: Suppose a fluid is perfectly at rest on the planet Earth.



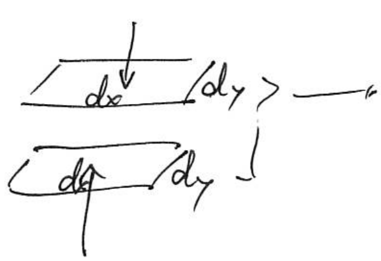


Assume constant density

Gravitational Force: $-\rho g \, dx \, dy \, dz$

To balance this, we introduce "Pressure" $P(x,y,z)$

So that:

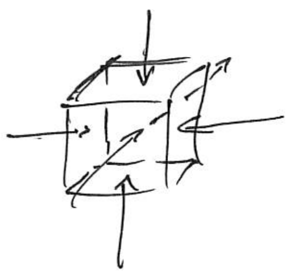


$$\left(P(x,y,-z-dz) - P(x,y,-z)\right)dx\,dy - \rho g \, dx\,dy\,dz = 0$$

$\Rightarrow -\frac{\partial P}{\partial z} = \rho g$, $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0$ in this case

$\Rightarrow P(z) = +P_{atm} - \rho g z$

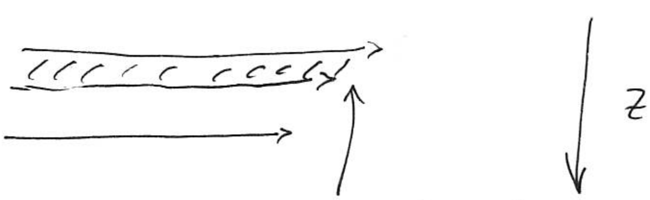
In general, if fluid is in motion $P(x,y,z,t)$ depends on all variables, but always is normal to fluid elements.



or

could expand out in principle... though usually removed i.e. no "negative pressure".

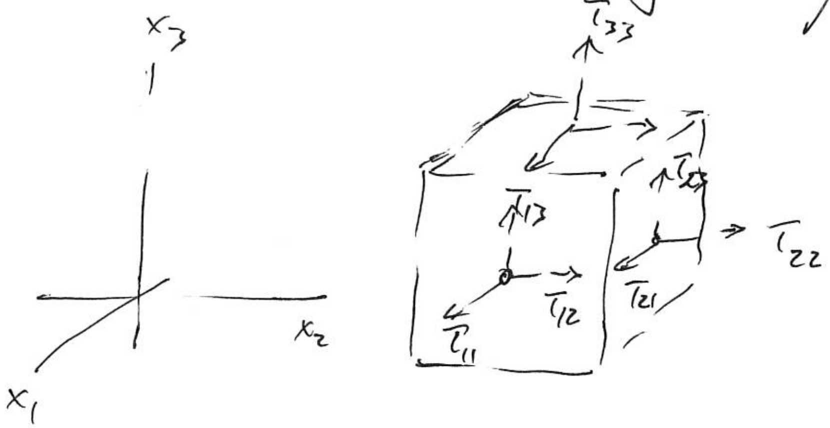
But, what about friction?

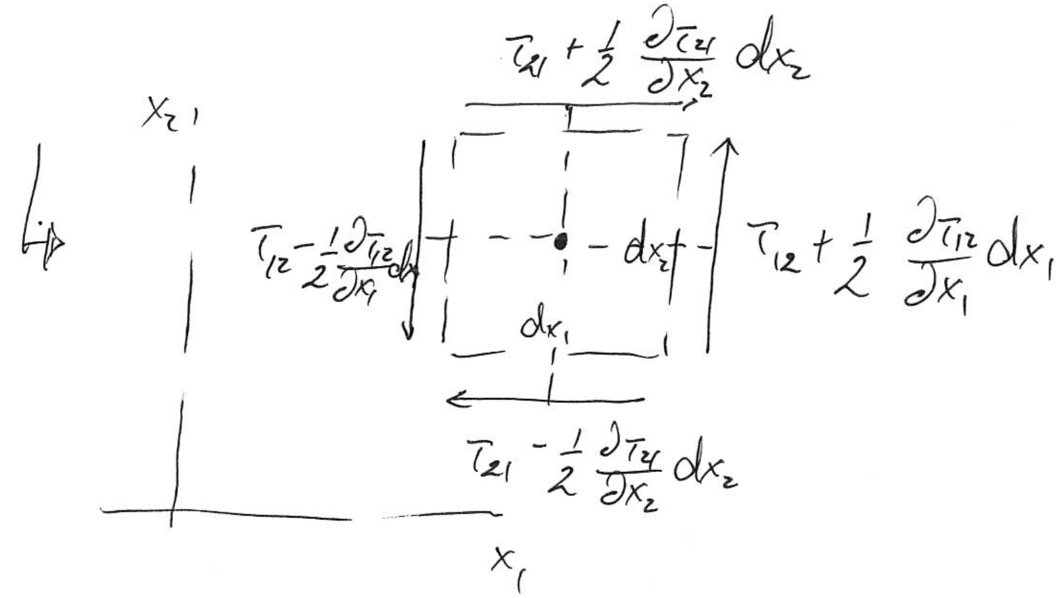


This fluid region feels regions above and below i.e. there is friction.

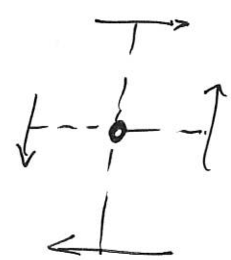
So we now introduce an all purpose:

Stress Tensor: $\tau_{ij} \sim \frac{F}{A}$





Torque: $\vec{T} = \vec{r} \times \vec{F}$ 

$\Rightarrow$ 

Net Torque:

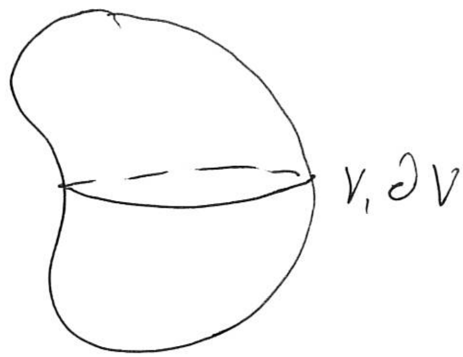
$\hookrightarrow$ $T = (\tau_{21} - \tau_{12})\, dx_1\, dx_2$

if $T_{21} \neq T_{12}$ $\rightarrow$ we get non-trivial ~~accel~~ acceleration for arbitrarily small regions $\rightarrow$ small regions of fluid can spin arbitrarily fast! Not Physical

$\hookrightarrow$ So $T_{21} = T_{12}$!

$\hookrightarrow$ $T_{ij} = T_{ji}$

So, fix volume V:



$\vec{p} = (p_1, p_2, p_3) \equiv$ net momentum in V

$\hookrightarrow$ $p_j = \int_V \rho u_j \, d^3\vec{x} \rightarrow \frac{dp_j}{dt} = \int_V \partial_t(\rho u_j) \, d^3\vec{x}$

~~XXX~~ $\frac{dp_j}{dt}$ = momentum flux along j^{th} direction + net forces on surface along j^{th} direction

So for momentum flux:

$$-\int_S \rho u_j \vec{u}\cdot\hat{\eta}\, d\sigma = -\int_V \nabla\cdot(\rho u_j \vec{u})\, dV$$

Net forces on surface along j^{th} direction:

Let: $\vec{\tau}_j = (\tau_{j1}, \tau_{j2}, \tau_{j3})$ all forces/area along j due to fluid

$F_{e,j} = -\rho\, \partial_{x_j} \tilde{V}\, d^3\vec{x}$ external "potential" forces

$$\text{Net Force} = \int_S \vec{\tau}_j\cdot\hat{\eta}\, d\sigma - \int_V \rho\, \partial_{x_j} \tilde{V}\, d^3\vec{x}$$

$$\int_S \vec{\tau}_j \cdot \hat{n}\, d\sigma = \int_V \nabla \cdot \vec{\tau}_j \, d^3\vec{x}$$

$$\to \partial_t(\rho u_j) = -\nabla \cdot (\rho u_j \vec{u}) + \nabla \cdot \vec{\tau}_j - \rho \partial_{x_j} \tilde{V}$$

$$\partial_t(\rho u_j) + \nabla \cdot (\rho u_j \vec{u})$$

$$= u_j \underbrace{\cancel{(\partial_t \rho + \nabla \cdot (\rho \vec{u}))}}_{\text{C.O.M.}}^{0} + \rho(\partial_t u_j + \vec{u} \cdot \nabla u_j) = \rho \frac{D u_j}{D t}$$

$$\to \rho \frac{D u_j}{D t} = -\rho \partial_{x_j} \tilde{V} + \nabla \cdot \vec{\tau}_j$$

$$\to \underline{\underline{m \cdot a = F}} \quad \text{i.e. Newton's } 2^{nd} \text{ Law}$$

At this point, we write

$$T_{ij} = -\bar{p}\,\delta_{ij} + \sigma_{ij}$$

$\hookrightarrow$ $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

pressure

i.e. isotropic stress

σ_{ij} represents the impact of viscosity

$\hookrightarrow$ Oh God the HORROR!

The One thing we will say about this is:

If fluid is incompressible: $\underline{\nabla \cdot \bar{u} = 0}$

$\hookrightarrow$ $\rho \dfrac{D\bar{u}}{Dt} = -\rho \nabla \tilde{V} - \nabla \bar{p} + \sigma \Delta \bar{u}$ $\longleftarrow$ "Kinematic Viscosity" <u>Constant !!!</u>

This Equation:

$$\nabla \cdot \bar{u} = 0$$

$$\rho \frac{D\bar{u}}{Dt} = -\rho \nabla \tilde{V} - \nabla \bar{p} + \sigma \Delta \bar{u}$$

Navier-Stokes Equation

Note: Pressure $\bar{p}$ is not thermodynamic as in

$\bar{p} V = nRT$ for an ideal gas

Instead it kind of gets "found" via $\bar{u}$ and $\tilde{V}$

↳ $\nabla \cdot \left(\rho \frac{D\bar{u}}{Dt} + \rho \nabla \tilde{V} \right) = -\Delta \bar{p} \qquad \left(\nabla \cdot \sigma \Delta \bar{u} = \sigma \Delta (\nabla \cdot \bar{u}) = 0 \right)$

But this starts a long, very difficult story...

An Example:

So from our conservation of momentum derivation worked

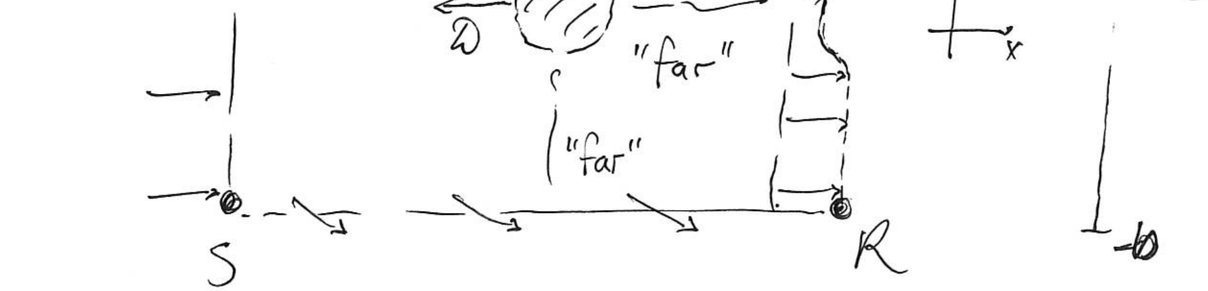
for $p_i(t) = \int_V \rho u_i \, d^3\vec{x}$

$$\to \frac{dp_i}{dt} = \underbrace{\int_V \rho\, \partial_{x_i} V \, d^3\vec{x} + \int_{\partial V} \vec{\tau}_i \cdot \hat{n} \, d\sigma}_{\text{Forces}} - \underbrace{\int_{\partial V} \rho u_i \vec{u} \cdot \hat{n} \, d\sigma}_{\text{Flux of momentum across } \partial V}$$

$$\to \frac{dp_i}{dt} = F_i - \dot{p}_i^{out}$$

momentum loss/gain along boundary ∂V

total forces on V



So, how can we find the drag force ~~along~~ on the cylinder? 2

Along PS: Steady horizontal speed U_∞

Fixed pressure $\bar{P}_\infty$

Along QR: Measured shear profile $u(y)$

↳ Note, assume far from cylinder and flow is "steady" in time

Along PQ/SR: "Mostly" horizontal velocity U_∞

Assume ρ is constant.

1st/Mass Balance:

Mass in: $2bU_\infty \rho dt$ or $\dot{m}_{in} = 2bU_\infty \rho$

Mass out: $\dot{m}_{PQ} + \dot{m}_{SR} + \rho \int_{-b}^{b} u(y)\,dy$

$$\hookrightarrow \quad 2bU_\infty \rho = \dot{m}_{PQ} + \dot{m}_{SR} + \rho \int_{-b}^{b} u(y)\,dy$$

2^{nd}/Momentum Balance:

Momentum In: $\dot{p}^{(in)} = 2b\rho U_\infty^2$

Momentum Out: $\dot{p}^{(out)} = \dot{p}_{PQ} + \dot{p}_{SR} + \rho \int_{-b}^{b} u^2(y)\,dy$

Note: If along PQ, SR we assume mostly horizontal velocity U_∞

$$\hookrightarrow \quad \dot{p}_{PQ} + \dot{p}_{SR} = U_\infty(\dot{m}_{PQ} + \dot{m}_{SR}) = U_\infty\left(2bU_\infty\rho - \rho \int_{-b}^{b} u(y)\,dy\right)$$

So, if flow is steady $\rightarrow \frac{dP}{dt} = 0$

And only force is $\mathcal{D}$:

$$\mathcal{D} = \dot{P}^{(in)} - \dot{P}^{(out)}$$

$$= 2b\rho U_\infty^2 - U_\infty\left(2b\rho U_\infty - \rho\int_{-b}^{b} u(y)\,dy\right) - \rho\int_{-b}^{b} u^2(y)\,dy$$

$$\mathcal{D} = \rho\int_{-b}^{b} u(y)\left(U_\infty - u(y)\right)dy$$

Compute

Known

Measure

Water on Earth: Let's ignore viscosity

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{u} = 0$$

$$\rho\frac{D\vec{u}}{Dt} = -\rho g\hat{k} - \nabla p$$

Most Common Assumption:

$$\rho = \text{constant}$$

↳ $\nabla \cdot \vec{u} = 0$

$$\frac{D\vec{u}}{Dt} = -g\hat{k} - \frac{1}{\rho}\nabla p$$

Though we note, the Boussinesq Approximation is also popular:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{D\vec{u}}{Dt} = -\frac{(\rho_0 + \delta\rho(x,y,z))}{\rho_0} g\hat{k} - \frac{1}{\rho_0}\nabla p$$

Assume density is constant <u>except</u> on gravity term

↳ Allows us to study buoyancy effects.

Bernoulli's Equation:

Basic Identity:

$$\frac{D\vec{u}}{Dt} = \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} = \partial_t \vec{u} + \frac{1}{2}\nabla|\vec{u}|^2 + \vec{\omega}\times\vec{u}$$

$\hookrightarrow$

$$\partial_t \vec{u} + \frac{1}{2}\nabla|\vec{u}|^2 + \vec{\omega}\times\vec{u} = -g\hat{k} - \frac{1}{\rho}\nabla\bar{P}$$

So: $+g\hat{k} = \nabla(gz)$

And ~~sel~~ suppose: $\rho = \rho(\bar{P})$ i.e. density only depends on pressure.

$\hookrightarrow$ $\bar{P} = \rho RT$ or $\rho = \bar{P}/RT$; mmm... T scares us

Adiabatic Gas Dynamics: $\underline{\bar{p} = c\rho^{\gamma}}$ ← but not if we never exchange heat.

or

~~_~~ Constant ρ