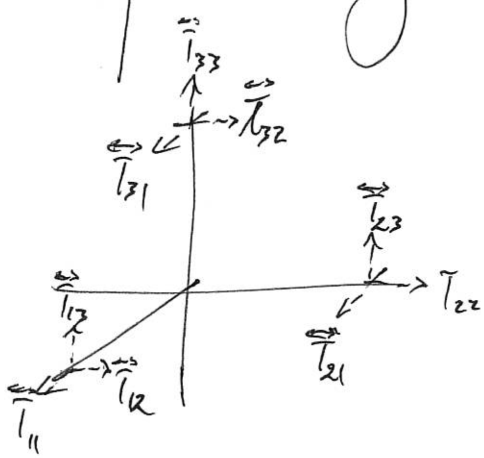
The Strain-Rate Tensor $\overleftrightarrow{T}$

$$\overleftrightarrow{T}_{ij} = \frac{\partial u_i}{\partial x_j}, \quad \begin{matrix} i = 1,2,3 \\ j = 1,2,3 \end{matrix}$$

↳ 9 components in general

↳



Allows us to understand different ways that a velocity field $\vec{u}$ deforms a region of fluid.

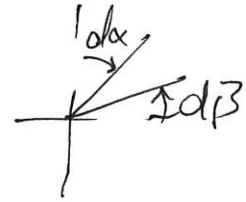
So: $\overleftrightarrow{T}_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial u_i}{\partial x_j}$

$$= \frac{1}{2}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial u_j}{\partial x_i} + \frac{1}{2}\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$$

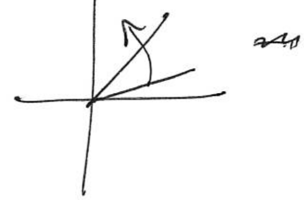
$$= \overleftrightarrow{e}_{ij} + \frac{1}{2}\overleftrightarrow{r}_{ij}$$

Where:

$$\overleftrightarrow{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right)$$



$$\overleftrightarrow{r}_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \frac{d\beta}{dt} - \frac{d\alpha}{dt}$$



Again, using the vorticity $\vec{\omega} = \nabla \times \vec{u} = \left(\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1\right)$

↳ $$\overleftrightarrow{r} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \rightarrow \overleftrightarrow{r}^T = -\overleftrightarrow{r}$$

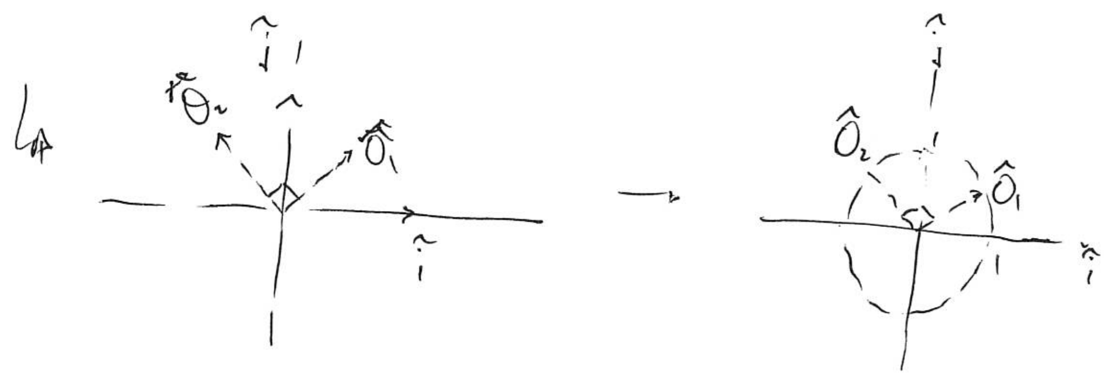
and $\overleftrightarrow{e}_{ij} = \overleftrightarrow{e}_{ji}$ or $\left(\overleftrightarrow{e}\right)^T = \overleftrightarrow{e}$

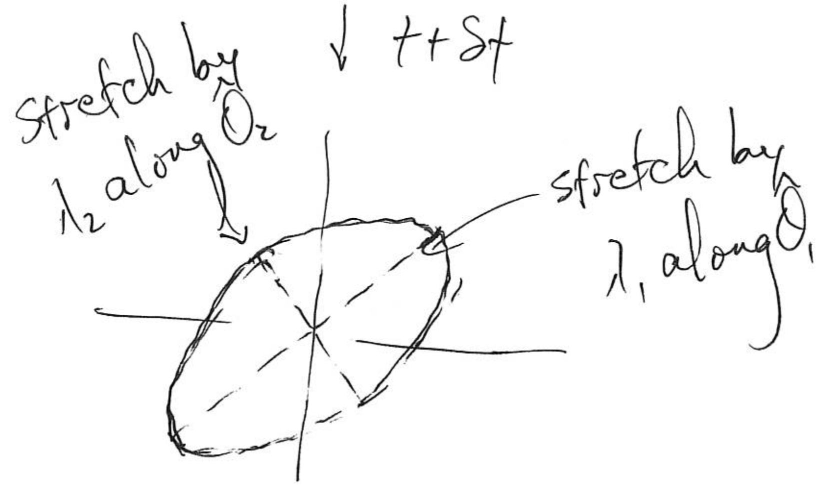
↳ $\overleftrightarrow{T} = \overleftrightarrow{e} + \frac{1}{2}\overleftrightarrow{r}$ ← anti-symmetric, note $\overleftrightarrow{r}$ is basically $\vec{\omega}$, angular velocity is $\frac{1}{2}\vec{\omega}$ → thus, $\frac{1}{2}\overleftrightarrow{r}$

symmetric

Since $(\overleftrightarrow{e})^T = \overleftrightarrow{e} \rightarrow \overleftrightarrow{e} = O \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} O^T$

where $O = \left(\hat{O}_1 \middle| \hat{O}_2 \middle| \hat{O}_3 \right)$, $OO^T = \underline{I} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $\lambda_j \in \mathbb{R}$



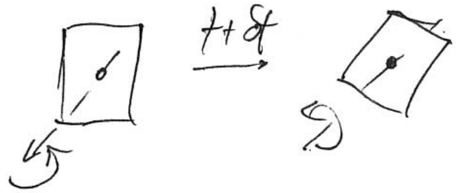


$\lambda_j > 0 \rightarrow$ elongate along $\hat{O}_j$

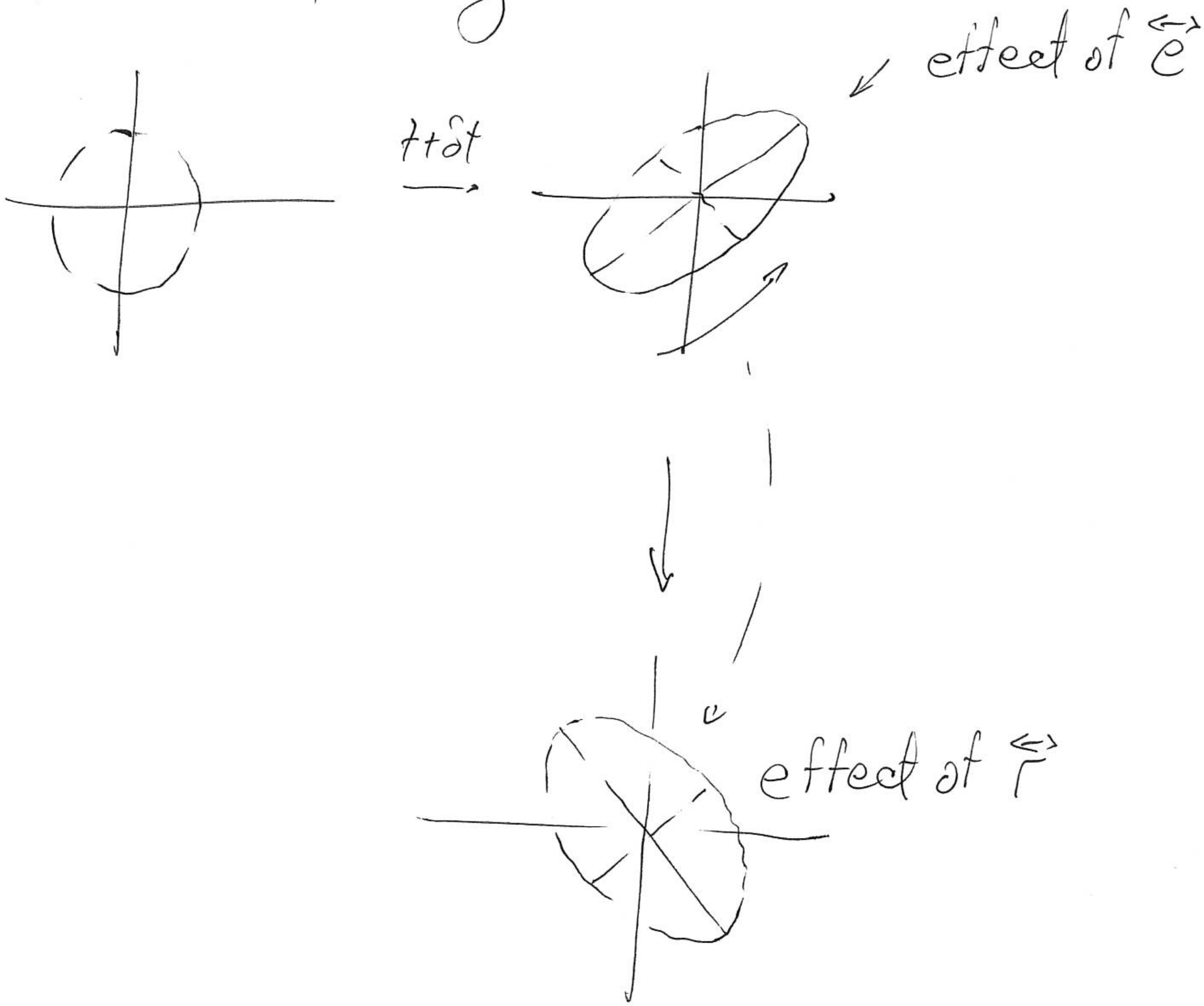
$\lambda_j < 0 \rightarrow$ compress along $\hat{O}_j$

$\lambda_j = 0 \rightarrow$ neutral along $\hat{O}_j$

In contrast $\overleftrightarrow{r}$ describes the tendency for a shape to turn:

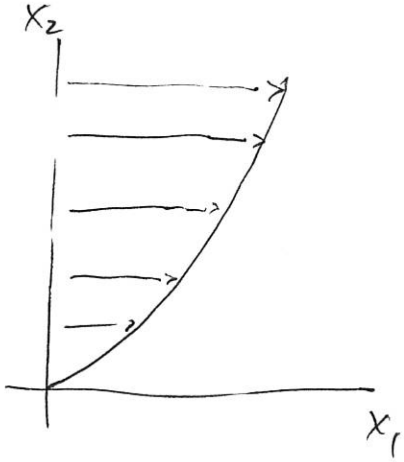


So from above, we might have



Parallel Shear Flows:

$$\vec{u} = (u_1(x_2), 0, 0) \quad \left(\text{or } (u(y), 0, 0)\right)$$



$$\to \overleftrightarrow{T} = \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ u_1(x_2) & 0 & 0 \end{vmatrix} = -u_1'(x_2)\,\hat{k}$$

$$\hookrightarrow \overleftrightarrow{r} = \begin{pmatrix} 0 & \partial u_1/\partial x_2 & 0 \\ -\partial u_1/\partial x_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hookrightarrow \overleftrightarrow{T} = \overleftrightarrow{e} + \frac{1}{2}\overleftrightarrow{r} \to \overleftrightarrow{e} = \begin{pmatrix} 0 & \frac{1}{2}\partial u_1/\partial x_2 & 0 \\ \frac{1}{2}\partial u_1/\partial x_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↳ $\gamma \equiv \partial u_1 / \partial x_2$

↳ $$\overleftrightarrow{T} = \frac{1}{2}\begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↳ ignore x_3 since clearly nothing happens

↳ $$\begin{pmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{pmatrix} \rightarrow \lambda^2 - \frac{\gamma^2}{4} \rightarrow \lambda = \pm \frac{|\gamma|}{2}$$

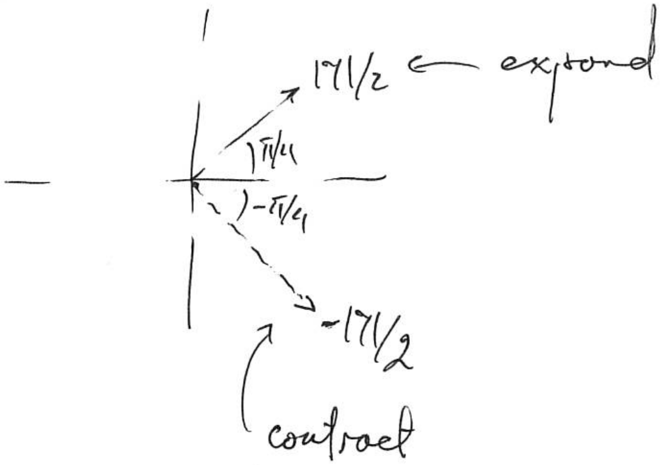
$$\rightarrow \left(\begin{array}{cc|c} \mp|\gamma|/2 & \gamma/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

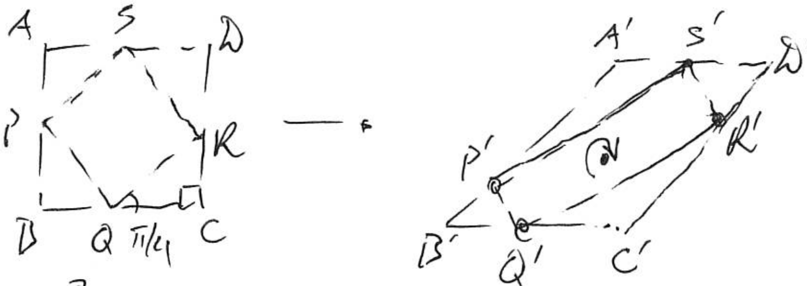
$$\rightarrow \mp \frac{|\gamma|}{2} x_1 + \frac{\gamma}{2} x_2 = 0$$

$$\rightarrow x_2 = \pm \operatorname{sgn}(\gamma) x_1$$

take $\gamma > 0$ for sake of argument

↳ $$\overleftrightarrow{e} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |\gamma|/2 & & \\ & -|\gamma|/2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}^T$$

↳ 

↳ 

Angular Velocity $= \frac{1}{2}\omega = -\frac{\gamma}{2}$

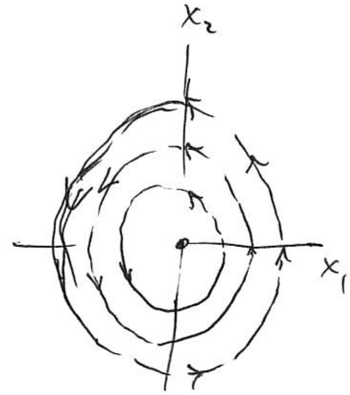
Shear in ABCD

Normal Stresses in PQRS

note $\overline{BC}$ doesn't turn, but $\overline{BA}$ does

Solid Body Rotation:

in polar coordinates:

$\vec{u} = \omega_0 r \hat{e}_\theta \rightarrow$ 

↳ $\vec{\omega} = \omega_3 \hat{k} = (\partial_{x_1} u_2 - \partial_{x_2} u_1)\hat{k}$

$x_1 = r\cos(\theta)$, $\quad x_2 = r\sin(\theta)$

↳ $\partial_{x_1} = \cos(\theta)\,\partial_r - \dfrac{\sin(\theta)}{r}\,\partial_\theta$

$\partial_{x_2} = \sin(\theta)\,\partial_r + \dfrac{\cos(\theta)}{r}\,\partial_\theta$

$$\vec{u} = u_1\,\hat{i} + u_2\,\hat{j} = u_1\left(\cos(\theta)\hat{e}_r - \sin(\theta)\hat{e}_\theta\right) + u_2\left(\sin(\theta)\hat{e}_r + \cos(\theta)\hat{e}_\theta\right)$$

$$= \left(u_1\cos(\theta) + u_2\sin(\theta)\right)\hat{e}_r + \left(-u_1\sin(\theta) + u_2\cos(\theta)\right)\hat{e}_\theta$$

$$= u_r\,\hat{e}_r + u_\theta\,\hat{e}_\theta$$

↳ $\partial_{x_1}u_2 - \partial_{x_2}u_1 = \partial_r u_\theta - \dfrac{\sin(\theta)}{r}\partial_\theta u_2 - \dfrac{\cos(\theta)}{r}\partial_\theta u_1$, $\quad \sin(\theta)\partial_\theta u_2 = \partial_\theta(\sin(\theta)u_2) - \cos(\theta)u_2$

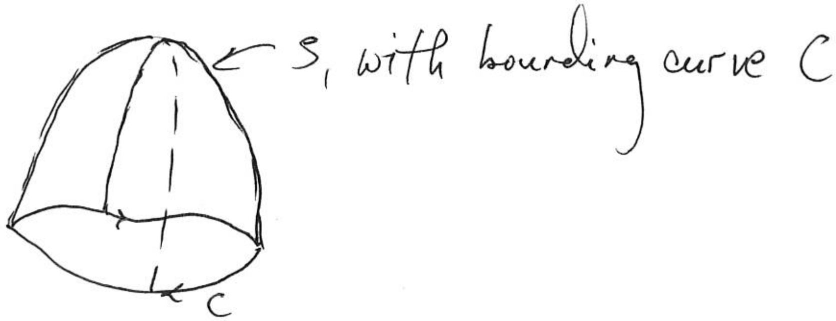
$$= \partial_r u_\theta - \frac{1}{r}\partial_\theta u_r + \frac{1}{r}u_\theta$$

$$= \frac{1}{r}\partial_r(r u_\theta) - \frac{1}{r}\partial_\theta u_r$$

↳ $\omega_3 = 2\omega_0$

Your last bit of Vector Calc:

Stoke's THM:



$$\hookrightarrow \oint_C \vec{F}\cdot\hat{t}\,ds = \oint_C \vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot\hat{\eta}\,d\sigma$$

Note: S can be any surface connected to C

So if C is in a plane, say x_1, x_2

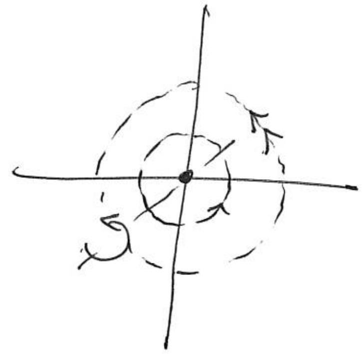
$$\hookrightarrow \oint_C \vec{F}\cdot\hat{t}\,ds = \oint_C \vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot\hat{k}\,dA$$

So for solid body rotation:

$$u_r = 0$$

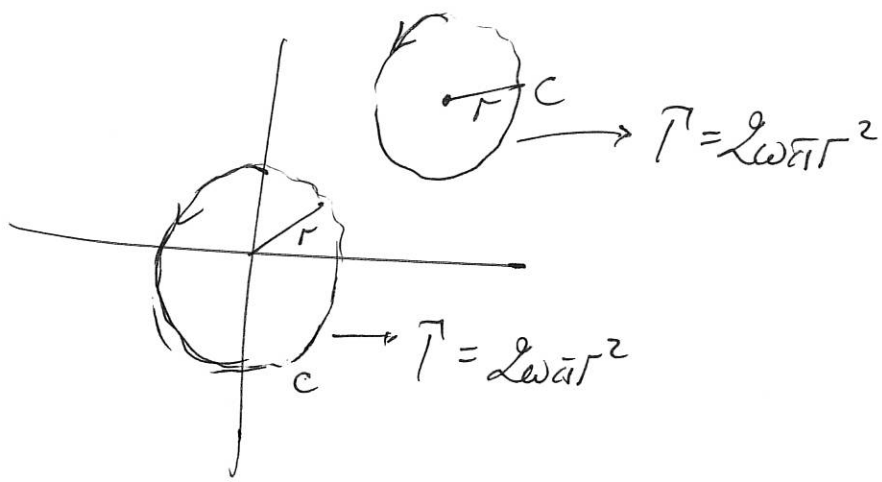
$$u_\theta = \omega r$$

$\hookrightarrow \vec{\omega} = \nabla \times \vec{u} = 2\omega \hat{k}$,



$$\hookrightarrow \Gamma = \oint_C \vec{u} \cdot d\vec{s} = 2\omega \iint dA = 2\omega A$$

$\hookrightarrow$



i.e. no matter where I place the circle,

$$\Gamma = 2\pi\omega r^2 = \oint_C \vec{u} \cdot d\vec{s} \rightarrow \vec{u} \text{ is always turning!}$$

In contrast, we have the irrotational point vortex: (13)

$$u_r = 0$$

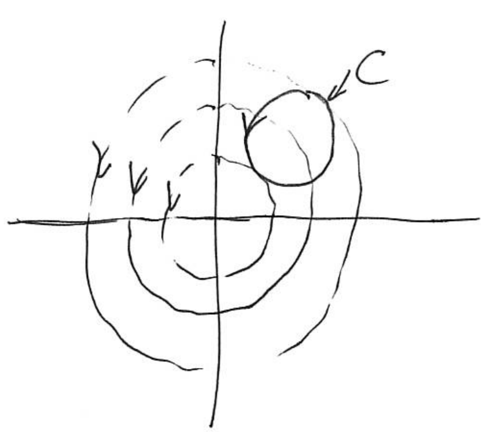
$$u_\theta = \frac{\alpha}{r}$$

So, ill defined @ $r=0$, but otherwise

$$\vec{\omega} = \left(\frac{1}{r}\partial_r(r u_\theta) - \frac{1}{r}\partial_\theta u_r\right)\hat{k}$$

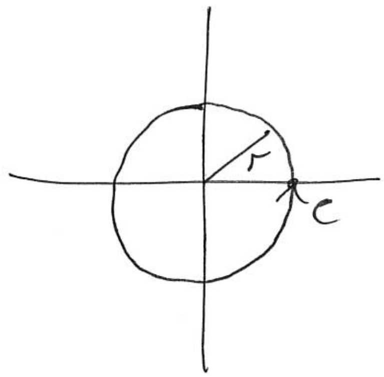
$$= 0\hat{k}\,!$$

↳ So if circle C does not enclose the origin



↳ $\Gamma = \oint_C \vec{u}\cdot d\vec{s} = \iint 0\, dA = 0$

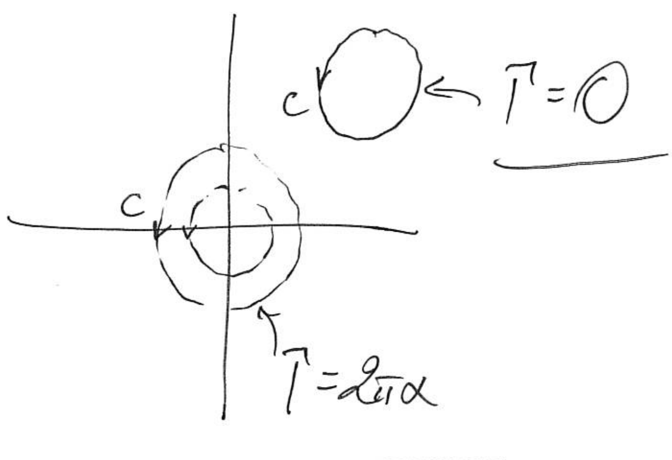
But, what if C is centered at the origin



$\hookrightarrow$ $$\Gamma = \oint_C \vec{u} \cdot d\vec{s} = \int_0^{2\pi} \left(\frac{\alpha}{r} \hat{e}_\theta \right) \cdot \left(r \, d\theta \, \hat{e}_\theta \right) = \alpha \int_0^{2\pi} d\theta = 2\pi\alpha$$

So now $\Gamma = 2\pi\alpha$, which is independent of r!

$\hookrightarrow$

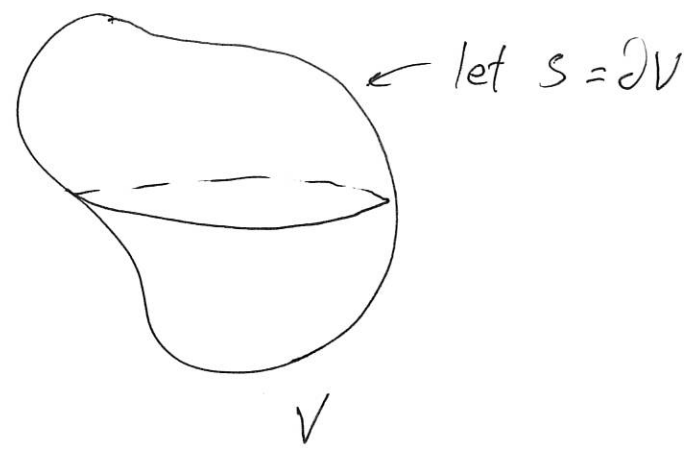


$\hookrightarrow$ we say $\vec{\omega} = \alpha \delta(\vec{x})$

Dirac Delta Function

Conservation of Mass and Incompressibility:

In an Eularian sense, fix a volume within a fluid:



If $\rho(\vec{x},t) = \rho(x_1,x_2,x_3,t)$ is fluid density $\rightarrow$

$$M(t) = \int_V \rho(\vec{x},t)\, d^3\vec{x} \quad \text{is total mass within } V$$

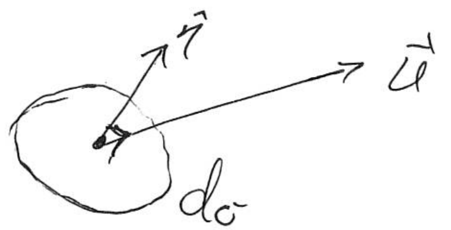
$\vec{x}$ is Eularian here, and thus independent of time.

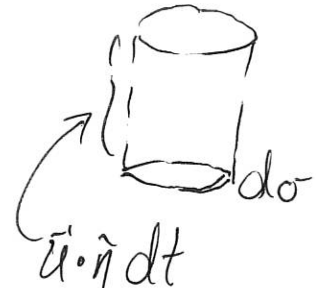
$$\Rightarrow \frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t}\, d^3\vec{x}$$

Now we assume:

Within V, mass is neither created nor destroyed.

↳ $\dfrac{dM}{dt}$ = mass which crosses ∂V

↳ 

↳ So we find a "cylinder": 

↳ $dM = -\rho\,\vec{u}\cdot\hat{n}\,dt\,d\sigma$

↳ $\dfrac{dM}{dt} = -\rho\,\vec{u}\cdot\hat{n}\,d\sigma \rightarrow \dfrac{dM}{dt} = -\iint_S \rho\,\vec{u}\cdot\hat{n}\,d\sigma$

Thus:

$$\int_V \frac{\partial \rho}{\partial t}\, d^3\vec{x} = -\iint_S \rho\vec{u}\cdot\hat{n}\, d\sigma$$

$$= -\int_V \nabla\cdot(\rho\vec{u})\, d^3\vec{x}$$

$\Rightarrow$ $$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) \right\} d^3\vec{x} = 0$$

$\Rightarrow$ V was arbitrary, so

$$\boxed{\rho_t + \nabla\cdot(\rho\vec{u}) = 0}$$

Rembering $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$

$\Rightarrow$ $\rho_t + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$

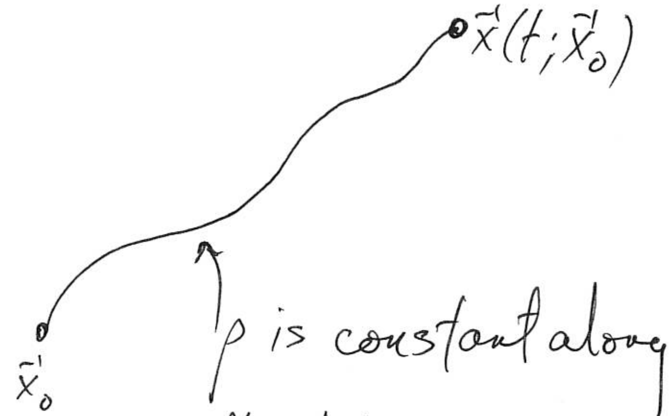
And thus:

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{u}$$

Most common assumption in fluids:

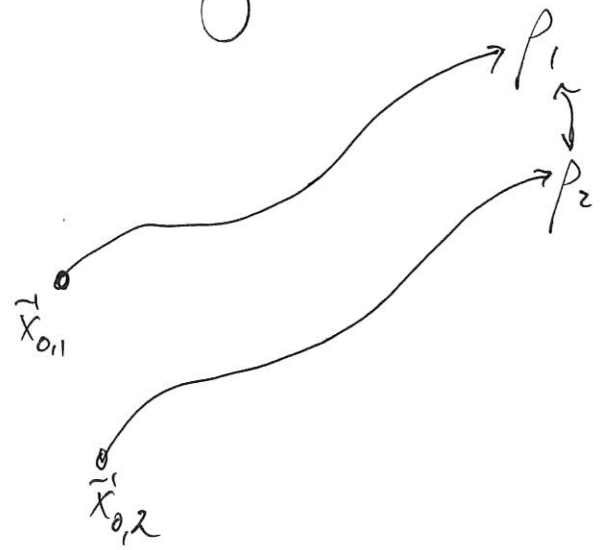
$$\nabla\cdot\vec{u} = 0$$

$$\hookrightarrow \frac{D\rho}{Dt} = 0 \rightarrow$$



ρ is constant along fluid lines

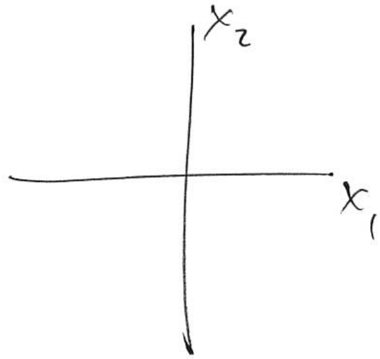
though



density need not be constant

Streamfunctions:

if we restrict to the plane



$\hookrightarrow$ $\nabla \cdot \vec{u} = 0 \rightarrow \partial_{x_1} u_1 + \partial_{x_2} u_2 = 0$

Introduce streamfunction $\psi(x_1, x_2, t)$ such that

$$u_1 = \partial_{x_2} \psi, \quad u_2 = -\partial_{x_1} \psi$$

$$\hookrightarrow \partial_{x_1} u_1 + \partial_{x_2} u_2 = \partial^2_{x_1 x_2} \psi - \partial^2_{x_2 x_1} \psi = 0.$$

Okay, so? $\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1$

$$\hookrightarrow \omega = \partial_{x_1}(-\partial_{x_1} \psi) - \partial_{x_2}(\partial_{x_2} \psi)$$

$$\hookrightarrow \Delta \psi = -\omega, \quad \Delta = \partial^2_{x_1} + \partial^2_{x_2}.$$