So, in order to understand dispersion, we need two basic tools:

The Riemann - Lebesgue Lemma:

Let $I(t) = \int_a^b f(k)\, e^{itk}\, dk$

If $f(k)$ is continuous on $[a,b]$ i.e. $f \in C([a,b])$

↳ $\lim\limits_{t \to \infty} I(t) = 0$.

Further, if f is piecewise differentiable, then

$$I(t) \sim 1/t \quad \text{for } t \to \infty.$$

Pf) If we assume for convenience that $f \in C'([a,b])$,

then from integration by parts we have:

$$\int_a^b f(k)\, e^{itk}\, dk = \frac{1}{it} f(k)\, e^{itk} \Big|_a^b - \frac{1}{it} \int_a^b f'(k)\, e^{itk}\, dk$$

Noting that

$$\left| \int_a^b f'(k) e^{itk} \, dk \right| \leq \int_a^b |f'(k)| \, dk < \infty \quad \left(\text{if } f \in C^1([a,b]) \right)$$

more or less gives us the result. $\blacksquare$

Likewise, we have the Stationary phase Method:

$$I(t) = \int_a^b f(k) e^{it\omega(k)} \, dk$$

If for $k \in [a,b]$, $\omega(k)$ does not have a critical point

↳ by Inverse Function Theorem, we have

$$\tilde{k} = \omega(k)$$

↳ $$I(t) = \int_{\omega(a)}^{\omega(b)} f(\omega^{-1}(\tilde{k})) \frac{dk}{d\tilde{k}} e^{it\tilde{k}} \, d\tilde{k}$$

Thus, using Riemann-Lebesgue, we anticipate

$$I(t) \sim 1/t \quad \text{as } t \to \infty.$$

But, let $k_* \in (a,b)$; $\omega'(k_*) = 0$

$$\Rightarrow I(t) = \int_{k_*-\varepsilon}^{k_*+\varepsilon} f(k)\, e^{it\omega(k)}\,dk + \underbrace{\int_a^{k_*-\varepsilon} + \int_{k_*+\varepsilon}^{b}}_{\sim 1/t \text{ by R.L.}}$$

$$\int_{k_*-\varepsilon}^{k_*+\varepsilon} f(k)\, e^{it\omega(k)}\,dk \doteq e^{it\omega(k_*)} \int_{-\varepsilon}^{\varepsilon} f(k_*+k)\, e^{it\omega''(k_*)k^2/2}\,dk$$

let $\tilde{k} = \left(\dfrac{t\,|\omega''(k_*)|}{2}\right)^{1/2} k$

$$\Rightarrow \int_{k_*-\varepsilon}^{k_*+\varepsilon} f(k)\, e^{it\omega(k)}\,dk \doteq \left(\frac{2}{t\,|\omega''(k_*)|}\right)^{1/2} e^{it\omega(k_*)} \int_{-\varepsilon\left(\frac{t|\omega''|}{2}\right)^{1/2}}^{\varepsilon\left(\frac{t|\omega''|}{2}\right)^{1/2}} f\left(k_* + \left(\frac{2}{t|\omega''|}\right)^{1/2} \tilde{k}\right) e^{i\,\mathrm{sgn}(k_*)\tilde{k}^2}\,d\tilde{k}$$

So we see, as $t \to \infty$

$$I(t) = \left(\frac{2}{t|\omega''(k_*)|}\right)^{1/2} e^{it\omega(k_*)} f(k_*) \int_{-\infty}^{\infty} e^{i\,\mathrm{sgn}(k_0)k^2}\,dk + O\left(\frac{1}{t}\right)$$

$$= \left(\frac{2}{t|\omega''(k_*)|}\right)^{1/2} e^{it\omega(k_*)} f(k_*) \sqrt{\pi}\, e^{i\pi\,\mathrm{sgn}(k_*)/4}$$

$$+ O\left(\frac{1}{t}\right)$$

So, who cares?

Linear Schrödinger Equation:

$$i\eta_t = \eta_{xx}$$

$$\eta(x,0) = \eta_0(x)$$

Okay, so we use the FT

los $i\hat{\eta}_t = (ik)^2\hat{\eta} = -k^2\hat{\eta}$

Thus:

$$\hat{\eta}(k,t) = \hat{\eta}_0(k) e^{ik^2 t}$$

↳ $$\eta(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\eta}_0(k) e^{ikx + ik^2 t} \, dk$$

So, if we try to write $e^{i(kx + k^2 t)} = e^{ik(x - ct)}$

↳ $c = -k$ ... So that's not a constant.

Moreover, if we look at our solution as

$$\eta(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\eta}_0(k) e^{i\left(k\frac{x}{t} + k^2\right)t} \, dk$$

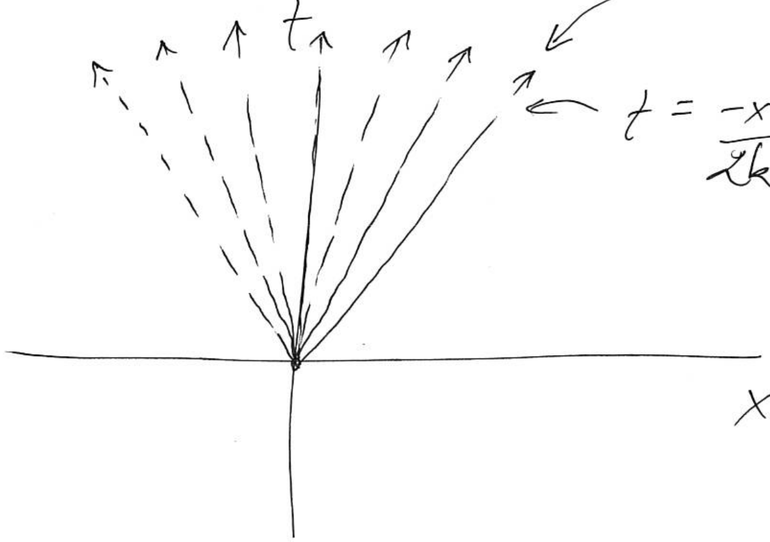
↳ By stationary phase, we see we have a critical k value at:

$$\partial_k \left(k \frac{x}{t} + k^2\right) = 0 \longrightarrow \frac{x}{t} + 2k = 0 \text{ or } k_* = -\frac{x}{2t}.$$

What does this mean?

Along the fixed line $k_* = \frac{-x}{2t}$ or $t = \frac{-x}{2k^*}$ or $x = -2k_* t$

↳ $\eta(x,t) \simeq \left(\frac{2}{2t}\right)^{1/2} \frac{1}{2\pi} \hat{\eta}_0\left(\frac{-x}{2t}\right) e^{-ix^2/4t^2 + i\pi/4} + O\left(\frac{1}{t}\right)$



$t = \frac{-x}{2k^*}$ or $x_* = -2k_* t$

Thus, we get motion along any direction, and we have two different speeds in the system:

Phase Speed: $c_p = -k$ → speed of an individual wave $e^{i(kx+k^2t)}$

Group Speed: $c_g = -2k$ → speed of a group of waves around $k_* = -x/2t$.

↳ System is Dispersive if $c_p \neq c_g$

So, we rewrite $\eta(x,t)$ so that

$$\eta(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\eta}_0(k) e^{i(kx - \omega(k)t)} dk$$

So that $\omega(k) = -k^2$

$\hookrightarrow$ $c_p = \omega(k)/k$

$$c_g = \frac{d\omega}{dk}$$

We call $\omega(k)$ the Dispersion Relation

Linear KdV

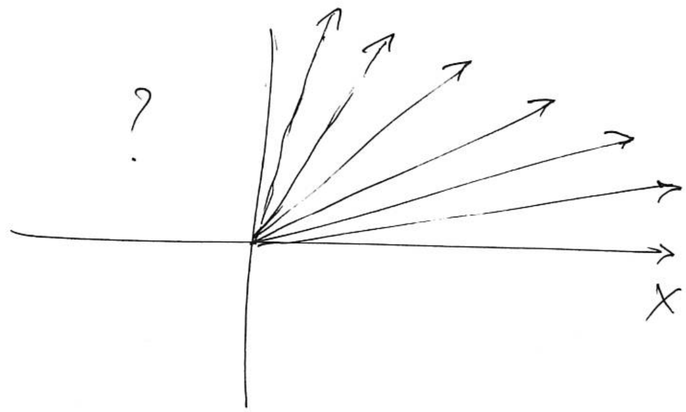
$$u_t = u_{xxx}$$
$$u(x,0) = u_0(x)$$

FT $\hookrightarrow$ $\hat{u}_t = (ik)^3 \hat{u} = -ik^3 \hat{u} \rightarrow \hat{u}(k,t) = \hat{u}_0(k) e^{-ik^3 t}$

$\hookrightarrow$ $u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx - k^3 t)} dk \rightarrow \omega(k) = k^3$.

↳ $c_p = k^2$, $c_g = 3k^2$

↳ We follow along lines $\frac{x}{t} = 3k^2$

↳ 

↳ $k_* = \pm \left(\frac{x}{3t}\right)^{1/2}$; $\omega''(k_*) = 6k_*$

↳ $u(x,t) = \ldots$ your turn during lecture.