Coordinate Transformations, Jacobians, and Eulerian / Lagrangian points of view

The Gradient of a function f:

$$\nabla f = (f_x, f_y, f_z)$$

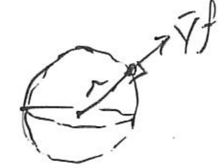
- Direction of Steepest Ascent

$f(x,y,z) = C_1 \rightarrow$

$f(x,y,z) = C_2 > C_1 \rightarrow$

- ∇f is normal to $f(x,y,z) = C$

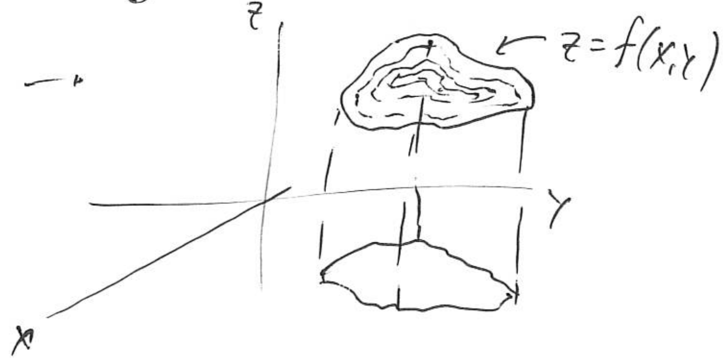
e.g. $f(x,y,z) = x^2+y^2+z^2 \rightarrow f(x,y,z) = r^2$



$\rightarrow \nabla f = 2\langle x,y,z \rangle$

So in $\int_S \vec{F} \cdot \hat{\eta}\, d\sigma$, if we can think of S over some region of space as say a graph where:

$z = f(x,y) \to$



↳ $\hat{\eta} = (-\nabla_{x,y} f, 1) / (1 + |\nabla_{x,y} f|^2)^{1/2}$

↳ note, used $z - f(x,y) = 0$

$\nabla(z - f(x,y)) = (-\nabla_{x,y} f, 1)$

note: $|\nabla_{x,y} f| = (f_x^2 + f_y^2)^{1/2}$

Keep in mind, could also have

$x = g(y,z) \to \hat{\eta} = (1, -\nabla_{y,z} g) / (1 + |\nabla_{y,z} g|^2)^{1/2}$

$|\nabla_{y,z} g|^2 = (g_y^2 + g_z^2)^{1/2}$

Okay, that is all well and good, but what if we want to work in another coordinate system?

Polar Coordinates / Cylindrical Coordinates

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = G(r,\theta,z) = \begin{pmatrix} g_1(r,\theta) \\ g_2(r,\theta) \\ z \end{pmatrix}$

or : $G : \mathbb{R}^3 \to \mathbb{R}^3$

Anyway: what is ∇ in these coordinates, and why do we care?

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}$$

So: $x^2 + y^2 = r^2 \longrightarrow \begin{cases} x = r r_x \to r_x = \cos(\theta) \\ y = r r_y \to r_y = \sin(\theta) \end{cases}$

Likewise from $x = r\cos(\theta)$

$$\Rightarrow 1 = r_x \cos(\theta) - r\theta_x \sin(\theta)$$

$$\Rightarrow 1 = \cos^2(\theta) - r\theta_x \sin(\theta)$$

$$\Rightarrow \theta_x = -\frac{1}{r}\sin(\theta)$$

$$y = r\sin(\theta)$$

$$\Rightarrow 1 = r_y \sin(\theta) + r\theta_y \cos(\theta)$$

$$\Rightarrow 1 = \sin^2(\theta) + r\theta_y \cos(\theta)$$

$$\Rightarrow \theta_y = \frac{1}{r}\cos(\theta)$$

$$\Rightarrow \partial_x = \cos(\theta)\partial_r - \frac{\sin(\theta)}{r}\partial_\theta$$

$$\partial_y = \sin(\theta)\partial_r + \frac{\cos(\theta)}{r}\partial_\theta$$

$$\Rightarrow \nabla_{x,y} = \partial_x \hat{i} + \partial_y \hat{j}$$

$$\hookrightarrow \nabla_{x,y} = \left(\cos(\theta)\,\partial_r - \frac{\sin(\theta)}{r}\partial_\theta\right)\hat{i} + \left(\sin(\theta)\,\partial_r + \frac{\cos(\theta)}{r}\partial_\theta\right)\hat{j}$$

$$= \left(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}\right)\partial_r + \frac{1}{r}\left(-\sin(\theta)\hat{i} + \cos(\theta)\hat{j}\right)\partial_\theta$$

$$= \hat{e}_r\,\partial_r + \frac{1}{r}\hat{e}_\theta\,\partial_\theta$$

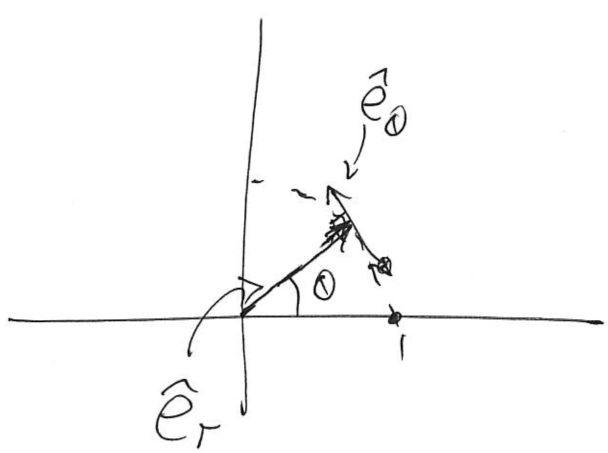
where $\hat{e}_r = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}$

$\hat{e}_\theta = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j}$

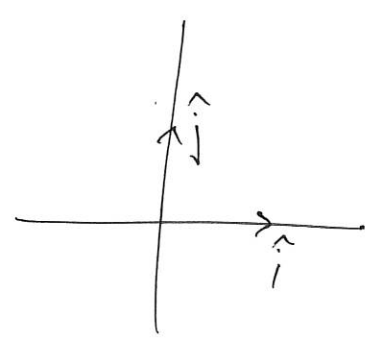
We see: $|\hat{e}_r| = |\hat{e}_\theta| = 1$

$\hat{e}_r \cdot \hat{e}_\theta = -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) = 0$

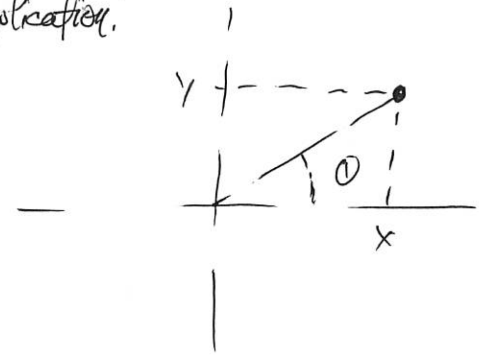
$\hookrightarrow \hat{e}_r \perp \hat{e}_\theta$



vs.



Application:



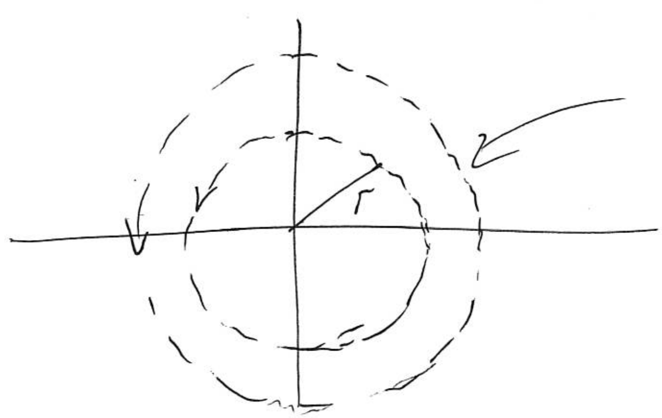
$$\longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \phi(x,y)$$

So as we will show, some fluid flows are well described by the velocity field

$$\vec{u} = \nabla_{x,y}\phi = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

But, if we compute the gradient in polar coordinates, we get

$$\vec{u} = \hat{e}_r \partial_r(\theta) + \frac{1}{r}\hat{e}_\theta \partial_\theta(\theta) = \frac{1}{r}\hat{e}_\theta$$



$|\vec{u}| = \frac{1}{r}$ so decreases as r increases

Lagrangian vs. Eulerian Coordinates

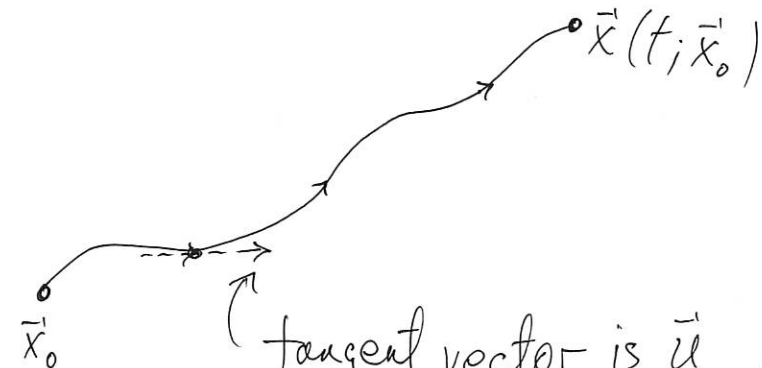
The Lagrangian Point of View:

I give you a velocity field: $\vec{u}(\vec{x},t)$

You solve: $\dfrac{d\vec{x}}{dt} = \vec{u}(\vec{x},t)$; $\vec{x}(0) = \vec{x}_0$

↳ This makes a path $\vec{x}(t;\vec{x}_0)$ where

$$\vec{x}(0;\vec{x}_0) = \vec{x}_0$$



tangent vector is $\vec{u}$ or

$$\frac{d}{dt}\vec{x}(t;\vec{x}_0) = \vec{u}(\vec{x}(t;\vec{x}_0),t)$$

While in ODE's we call $\vec{x}_0$ an initial condition, in fluids we call it a Lagrangian marker, or just marker.

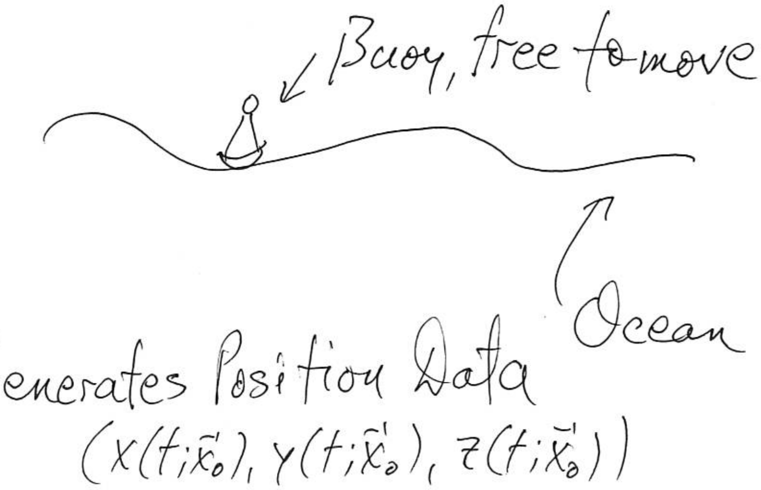
In the Lagrangian point-of-view, or coordinates, we follow the lines traced out by the fluid.

Said another way, if $\vec{x}_0 \in \mathbb{R}_3$ then Lagrangian coordinates are given by the map

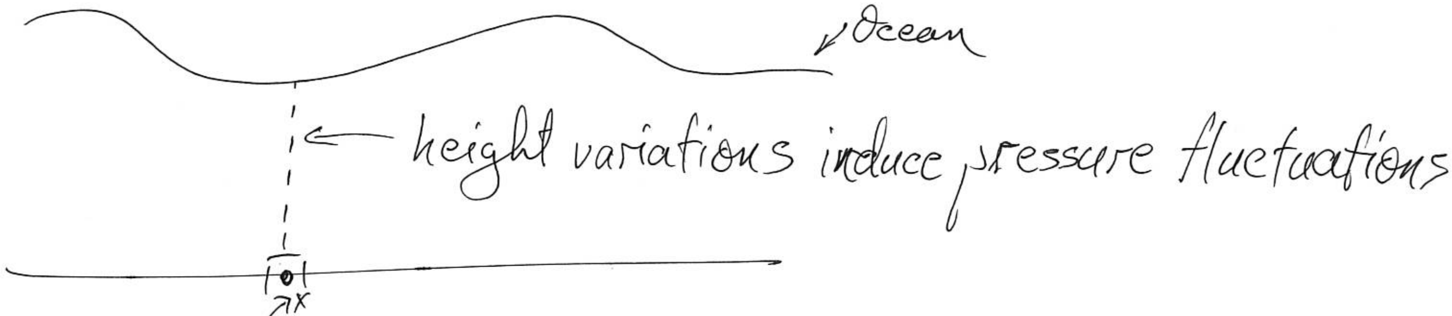
$$\varphi_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

where $\varphi_t(\vec{x}_0) = \vec{x}(t;\vec{x}_0)$

Lagrangian Data:



In the Eularian point of view, we fix our position in space, say $\bar{x}$, and watch the fluid go by.



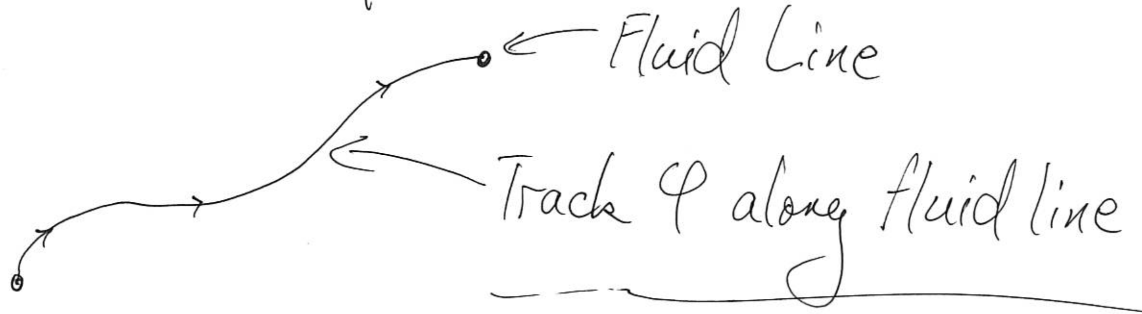
Pressure Plate, fixed in space at $\underline{x}$

$\hookrightarrow P(x,t) \equiv$ pressure record in time at fixed position $\underline{\underline{x}}$

So say I give you Lagrangian Data of the form

$$\varphi(t) = \varphi(x(t), y(t), z(t), t)$$

Say φ is temperature, salinity, wave height, whatever



How do I transform Lagrangian to Eularian data?

(4)

Through: $\frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t) = (u, v, \omega)$

$= (u_1, u_2, u_3)$

$\hookrightarrow \frac{d\varphi}{dt} = \frac{d}{dt}\, \varphi(x(t), y(t), z(t), t)$

$= \varphi_x \dot{x} + \varphi_y \dot{y} + \varphi_z \dot{z} + \varphi_t$

$= u\varphi_x + v\varphi_y + \omega\varphi_z + \varphi_t$

$= \vec{u} \cdot \nabla\varphi + \varphi_t$

$$\hookrightarrow \frac{d\varphi}{dt} = \underbrace{\vec{u} \cdot \nabla\varphi + \varphi_t}_{}$$

!!!

Eularian derivative at fixedpoints.

Lagrangian derivative along path

We "upgrade" $d\varphi/dt$ to $D\varphi/Dt$ and call it "the material derivative" or

$$\boxed{\frac{D}{Dt} = \underbrace{\vec{u}\cdot\nabla + \partial_t}}$$

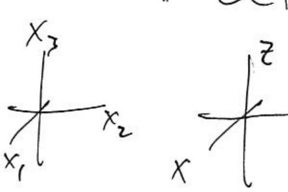
Lagrangian Path

Eularian fixed point

Deformations in a fluid: I give you $\vec{u}(\vec{x},t)$ everywhere.

Note: let $\vec{u}(\vec{x},t) = \left(u_1(x_1,x_2,x_3,t), u_2(x_1,x_2,x_3,t), u_3(x_1,x_2,x_3,t)\right)$

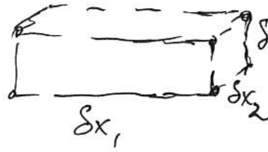
↳ Let's us work w/ terms like



$\partial_{x_j} u_i$ and so we have for example

$$\nabla\cdot\vec{u} = \sum_{i=1}^{3} \partial_{x_i} u_i, \quad \text{not } \nabla\cdot\vec{u} = u_x + v_y + w_z$$

You have $\vec{u}$:



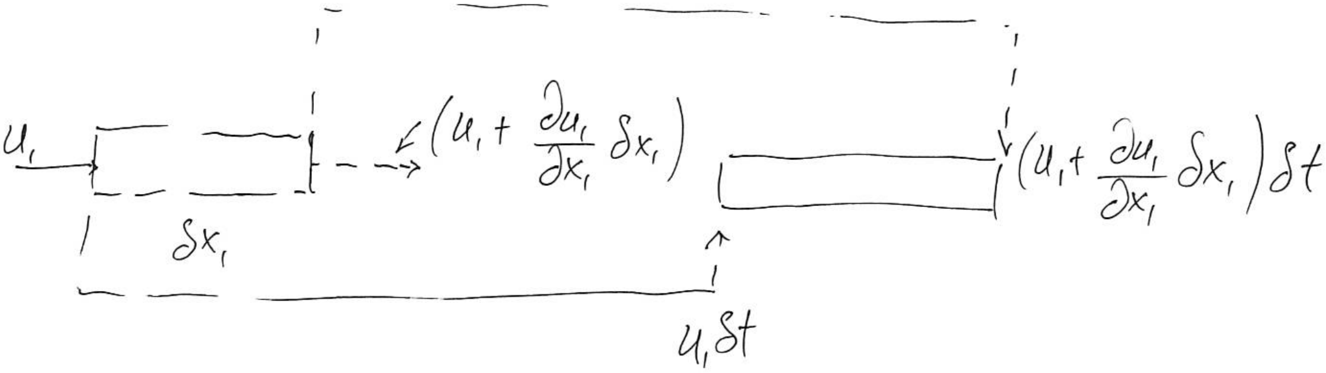
$\xrightarrow{t+\delta t}$ What does the cuboid look like?

Linear Strain Rate:

How to compute: $\dfrac{1}{\delta x_1}\underline{\dfrac{D}{Dt}(\delta x_1)}$?

time rate of change of fluid parcel along x_1/x direction following the fluid

per length along x_1/x direction



Thus :

$$\frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) \simeq \frac{1}{\delta x_1}\frac{1}{\delta t}\left(\left(u_1 + \frac{\partial u_1}{\partial x_1}\delta x_1\right)\delta t - u_1 \delta t\right)$$

$$\simeq \frac{\partial u_1}{\partial x_1} \quad \text{or} \quad \partial_{x_1} u_1$$

So for a volume: $\delta V = \delta x_1 \delta x_2 \delta x_3$

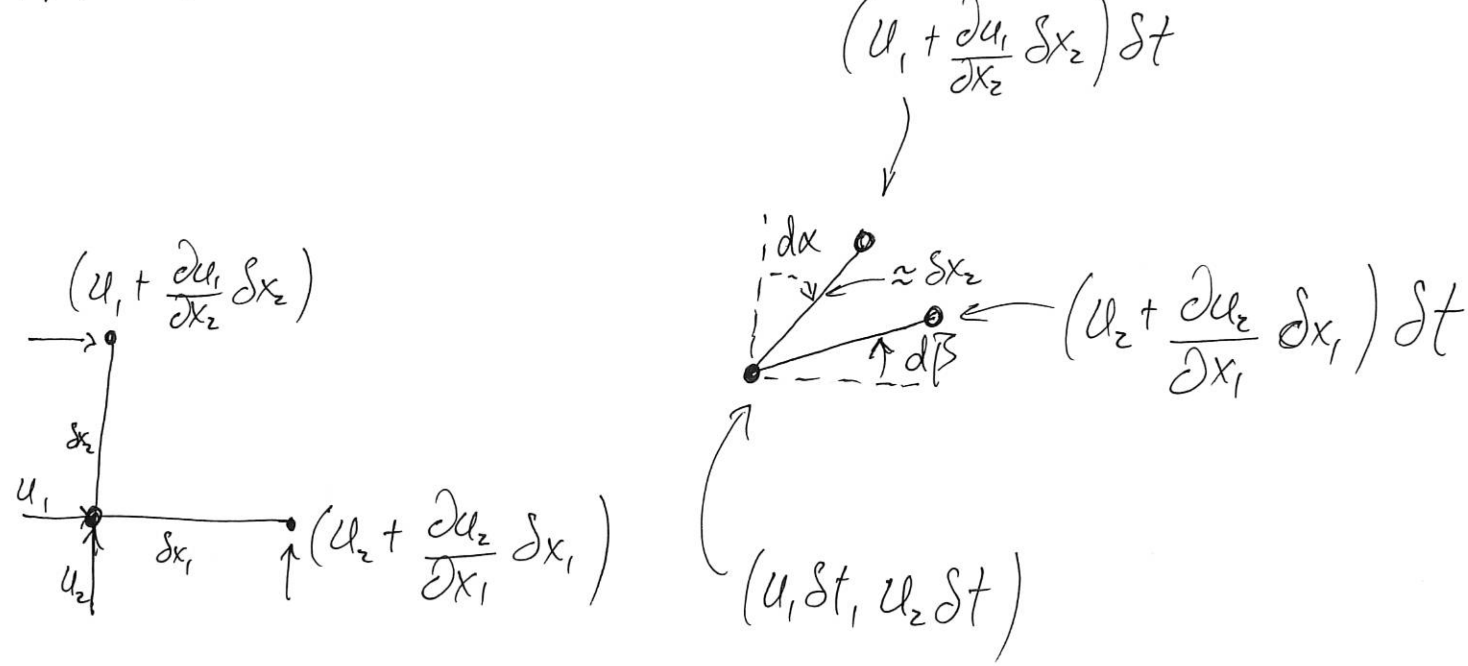
$$\Rightarrow \frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x_1 \delta x_2 \delta x_3}\left\{\delta x_2 \delta x_3 \frac{D}{Dt}(\delta x_1) + \delta x_1 \delta x_3 \frac{D}{Dt}(\delta x_2) + \delta x_1 \delta x_2 \frac{D}{Dt}(\delta x_3)\right\}$$

$$= \frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) + \frac{1}{\delta x_2}\frac{D}{Dt}(\delta x_2) + \frac{1}{\delta x_3}\frac{D}{Dt}(\delta x_3)$$

$$\simeq \sum_{j=1}^{3} \partial_{x_j} u_j = \nabla \cdot \bar{u}$$

i.e. Divergence measures how we contract/grow along each axis.

But this is not the only way to deform a shape in a fluid.



So if we treat $d\alpha \ll 1 \rightarrow \sin(d\alpha) \simeq d\alpha$

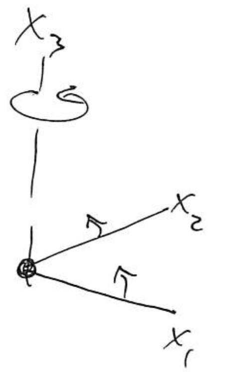
$$\hookrightarrow d\alpha \simeq \left(\left(u_1 + \frac{\partial u_1}{\partial x_2}\delta x_2\right)\delta t - u_1 \delta t\right) \Big/ \delta x_2$$

$$\simeq \frac{\partial u_1}{\partial x_2}\delta t \quad \text{or} \quad \frac{d\alpha}{dt} = \frac{\partial u_1}{\partial x_2}$$

$$\hookrightarrow \frac{d\beta}{dt} = \frac{\partial u_2}{\partial x_1} \longrightarrow \frac{d}{dt}(\alpha + \beta) = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.$$

We can just as well measure the rate of rotation:

$$-\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

↳ 

From this, we define the vorticity $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$

↳ $\omega_3 = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$

↳ $\omega_2 = \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1}$ or $\vec{\omega} = \nabla \times \vec{u}$

$\omega_1 = \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3}$