The Bernoulli Equation in Action: (1)

Again, if $\vec{\omega} = 0 \rightarrow \vec{u} = \nabla\phi$, and $\rho = \rho_0$

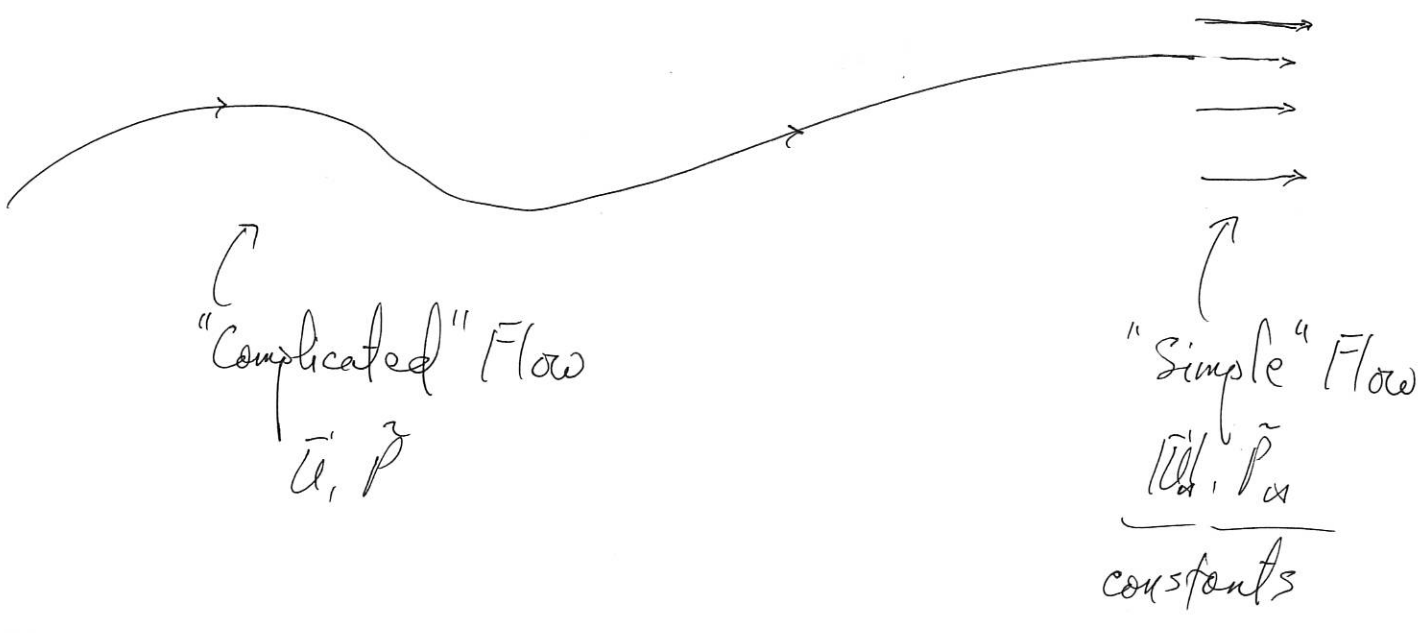
then $\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{\rho_0}p + gz = 0$

If the flow is steady, we have

$$\frac{1}{2}|\nabla\phi|^2 + \frac{1}{\rho_0}p + gz = 0$$

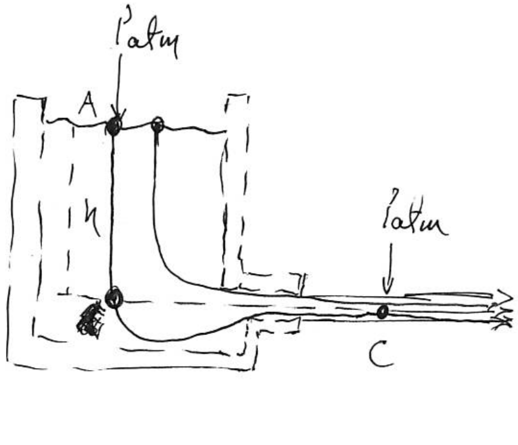
or: $\frac{1}{2}|\vec{u}|^2 + \frac{1}{\rho_0}\tilde{p}(x,y,z) = 0$, $\tilde{p} = \frac{1}{\rho_0}p + gz$

So along a streamline / path-line



$$\Rightarrow \frac{1}{2}|\vec{u}|^2 + \frac{1}{\rho_0}\tilde{P}(x,y,z) = \frac{1}{2}|\vec{u}_\infty|^2 + \frac{1}{\rho_0}\tilde{P}_\infty$$

Orifice in a tank:



? What is the speed u of the jet coming from the tank?

at A on the free surface: $\vec{u}_\infty = 0$, $\tilde{P} = \frac{1}{\rho_0}\bar{P}_{atm} + gh \equiv \tilde{P}_\infty$

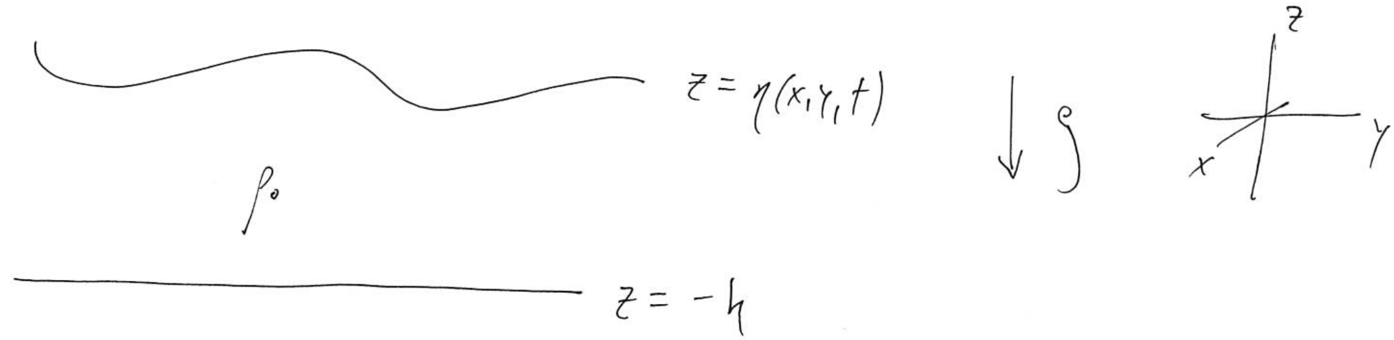
at C in the jet: $\tilde{P} = \frac{1}{\rho_0}\bar{P}_{atm}$, $\vec{u} = (u,0,0)$

$$\Rightarrow \frac{1}{\rho_0}\bar{P}_{atm} + gh = \frac{1}{\rho_0}\bar{P}_{atm} + \frac{1}{2}u^2$$

$$\Rightarrow u = \sqrt{2gh}$$

Again, not perfect, but now, you have an estimate for a speed that would be tricky to measure otherwise.

Speaking of free surfaces:



So we have an undulatory surface $z=\eta(x,y,t)$. How do we describe its motion?

Inviscid, Incompressible, Irrotational: $\vec{u}=\nabla\phi$

$$\Delta\phi = 0\,; \qquad -h < z < \eta(x,y,t)$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0\,; \qquad z=\eta(x,y,t)$$

If we do not allow flow through $z=-h$

$$\vec{u}=\nabla\phi \longrightarrow \phi_z\big|_{z=-h} = 0$$

But what about η_t?

Here, we make a very Lagrangian move:

The surface moves with the flow, or the surface is a fluid path line

↳ $z(t) = \eta(x(t), y(t), t)$

↳ $\dfrac{dz}{dt} = \eta_x \dfrac{dx}{dt} + \eta_y \dfrac{dy}{dt} + \eta_t$

↳ $\phi_z = \eta_x \phi_x + \eta_y \phi_y + \eta_t \quad$ at $z = \eta$

Thus, the full system of equations we must solve is:

$$\Delta \phi = 0 \qquad -h < z < \eta(x, y, t)$$

$$\phi_z = 0 \qquad z = -h$$

$$\eta_t = \phi_z - \eta_x \phi_x - \eta_y \phi_y \qquad z = \eta(x, y, t)$$

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + g\eta = 0 \qquad z = \eta(x, y, t)$$

Boundary condition is part of equation!

So, we will get back to that beast in time. Now, let's focus on

Potential Flow: $\bar{u} = \nabla \phi$, $\Delta \phi = 0$

We go to the plane $\phi = \phi(x,y)$, $\bar{u} = (u,v)$

Note: from $\nabla \cdot \bar{u} = 0$, we can also introduce a streamfunction

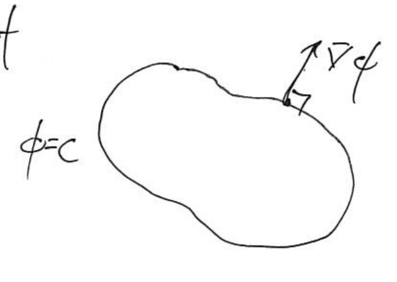
i.e. $u = \psi_y$; $v = -\psi_x$

So in potential flow:

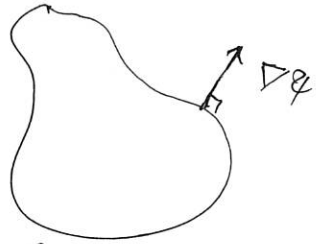
$$u = \phi_x \; ; \; v = \phi_y$$

$$\hookrightarrow \begin{cases} \phi_x = \psi_y \\ \phi_y = -\psi_x \end{cases} \longrightarrow \text{Cauchy-Riemann Equations or CR Equations}$$

$\hookrightarrow$ So if for $\phi(x,y) = c$, $\nabla\phi$ is normal to the level set



Likewise $\psi(x,y) = \tilde{c}$ $\rightarrow$



$\psi = \tilde{c}$

↳ using the CR Equations: $\nabla\phi \cdot \nabla\psi = \phi_x \psi_x + \phi_y \psi_y = \psi_y \psi_x - \psi_x \psi_y = 0$

So $\psi = \tilde{c}$ is orthogonal to $\phi = c$ $\rightarrow$ if $\vec{u} = \nabla\phi \Rightarrow \psi = \tilde{c}$ traces out stream/path lines of fluid.

Example:

$$\phi(x,y) = \frac{1}{4\pi} \ln(x^2+y^2) = \frac{1}{4\pi} \ln r^2 = \frac{1}{2\pi} \ln r$$

↳ $\Delta\phi = \frac{1}{r}\partial_r(r\partial_r\phi) + \frac{1}{r^2}\partial_\theta^2 \phi = 0.$

CR Equations:

$$\phi_x = \psi_y \quad \longrightarrow \quad \cos(\theta)\phi_r - \frac{\sin(\theta)}{r}\phi_\theta = \sin(\theta)\psi_r + \frac{\cos(\theta)}{r}\psi_\theta$$

$$\phi_y = -\psi_x \qquad \sin(\theta)\phi_r + \frac{\cos(\theta)}{r}\phi_\theta = -\cos(\theta)\psi_r + \frac{\sin(\theta)}{r}\psi_\theta$$

↳ $\phi_r = \frac{1}{r}\psi_\theta \,;\; -\psi_r = \frac{1}{r}\phi_\theta$

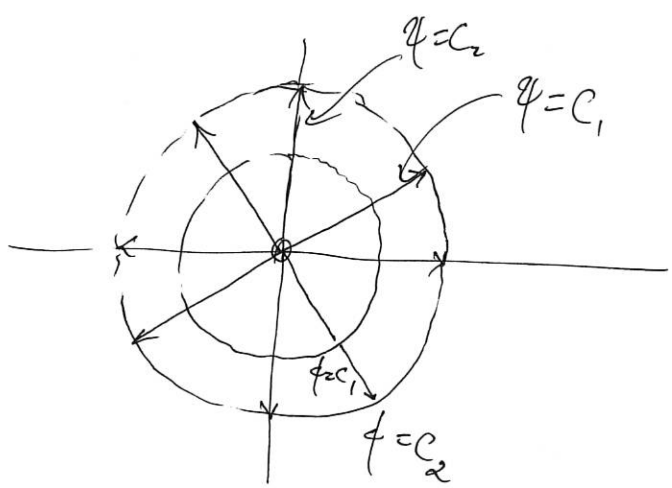
↳ $\phi_\theta = 0 \rightarrow \psi_r = 0$

$\phi_r = \frac{1}{2\pi r} \rightarrow \psi_\theta = \frac{1}{2\pi} \rightarrow \psi = \frac{\theta}{2\pi}$

$\phi(x,y) = c \rightarrow r = e^{2\pi c}$

$\psi(x,y) = \tilde{c} \rightarrow \theta = 2\pi\tilde{c}$

↳



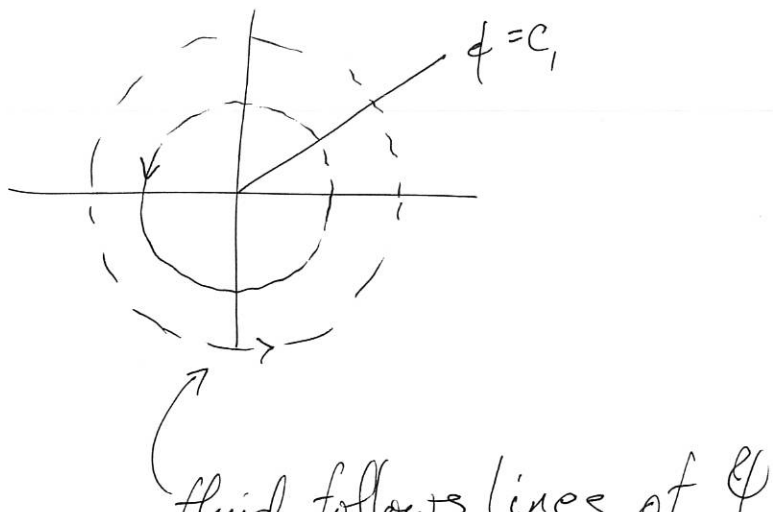
i.e. we have a source/sink since fluid follows lines of ψ

And flip side:

$$\phi = \frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) = \frac{\Gamma}{2\pi}\theta$$

So $\phi_r = 0 \rightarrow \psi_\theta = 0$

$$\phi_\theta = \frac{\Gamma}{2\pi} \rightarrow \psi_r = -\frac{\Gamma}{2\pi r} \longrightarrow \psi = -\frac{\Gamma}{2\pi} \ln r$$



So, what all of this tells us is that everytime I choose an analytic function

$$f(z) = \underbrace{\phi(x,y) + i\psi(x,y)}$$

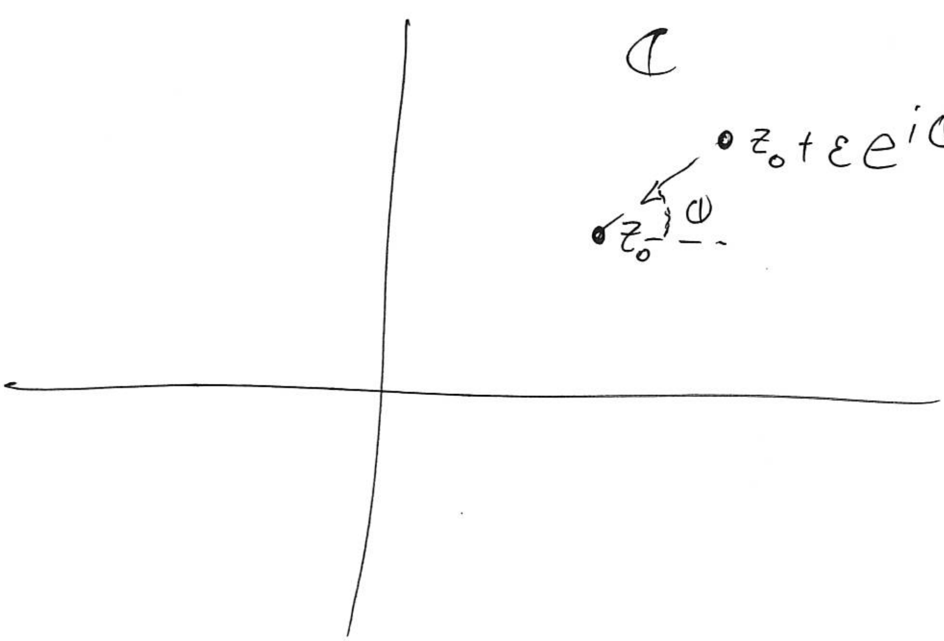
Seperate into real and imaginary parts

↳ $\vec{u} = \nabla \phi$,

Fluid follows $\psi = \tilde{C}$.

Note: $\vec{u} = \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} \overset{C.R.}{=} \begin{pmatrix} \phi_x \\ -\psi_x \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

So by analytic, we mean $\frac{df}{dz}$ exists around some point $z_0 \in \mathbb{C}$



→ $\frac{df}{dz}$ is a directional derivative i.e. it cannot depend on what path we take to get to z_0.

So:

$$\left.\frac{df}{dz}\right|_{z=z_0} = \lim_{\varepsilon\to 0} \frac{f(z_0+\varepsilon e^{i\theta}) - f(z_0)}{\varepsilon e^{i\theta}}$$

$\theta = 0$: $$\left.\frac{df}{dz}\right|_{z=z_0} = \lim_{\varepsilon\to 0}\left\{\left(\frac{\phi(x_0+\varepsilon, y_0)-\phi(x_0,y_0)}{\varepsilon}\right) + i\left(\frac{\psi(x_0+\varepsilon, y_0)-\psi(x_0,y_0)}{\varepsilon}\right)\right\}$$

$$= \phi_x + i\psi_x$$

$$= u - iv$$

$\hookrightarrow$ $$\frac{df}{dz} = u - iv$$

So, we can, if we have our potential ϕ and stream function ψ, by putting them together in $f(z) = \phi + i\psi$ get the velocity by differentiating f and taking real and imaginary parts i.e.

$$u = \mathrm{Re}\left\{\frac{df}{dz}\right\}; \quad v = -\mathrm{Im}\left\{\frac{df}{dz}\right\}.$$

Note:

If $\theta = \pi/2 \to \left.\frac{df}{dz}\right|_{z=z_0} = \lim_{\varepsilon \to 0} \frac{1}{i\varepsilon}\Big[\big(\phi(x_0, y_0+\varepsilon) - \phi(x_0,y_0)\big) + i\big(\psi(x_0,y_0+\varepsilon) - \psi(x_0,y_0)\big)\Big]$

$$= -i\phi_y + \psi_y$$

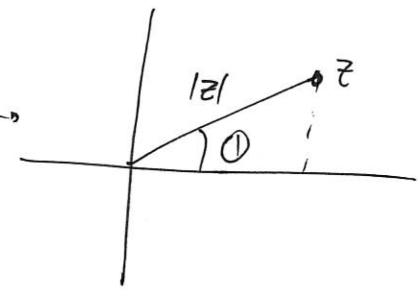
$\Rightarrow \quad \phi_x + i\psi_x = -i\phi_y + \psi_y$

$\Rightarrow$ CR Equat: $\begin{cases} \phi_x = \psi_y \\ \phi_y = -\psi_x \end{cases}$

So, now we just start choosing $f(z)$ so that it is differentiable, and this guarantees we will find a potential flow.

Example:

$f(z) = z^n$. Let $z = |z| e^{i\theta}$ →



$= |z| (\cos(\theta) + i \sin(\theta))$

↳ $f(z) = |z|^n e^{in\theta}$

$= |z|^n (\cos(n\theta) + i \sin(n\theta))$

Setting $r = |z|$

↳ $\phi(r, \theta) = r^n \cos(n\theta)$

$\psi(r, \theta) = r^n \sin(n\theta)$

note: if you want, you can set

$r = (x^2 + y^2)^{1/2}$; $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

We also readily find

$$f'(z) = n z^{n-1} = n\left(r^{n-1}\cos((n-1)\theta) + i r^{n-1}\sin((n-1)\theta)\right)$$

$\rightarrow$ $u = n r^{n-1}\cos((n-1)\theta)$

$v = -n r^{n-1}\sin((n-1)\theta)$

As for flow lines: $\psi(r,\theta) = \tilde{C}$

$\rightarrow$ $r^n \sin(n\theta) = \tilde{C}$

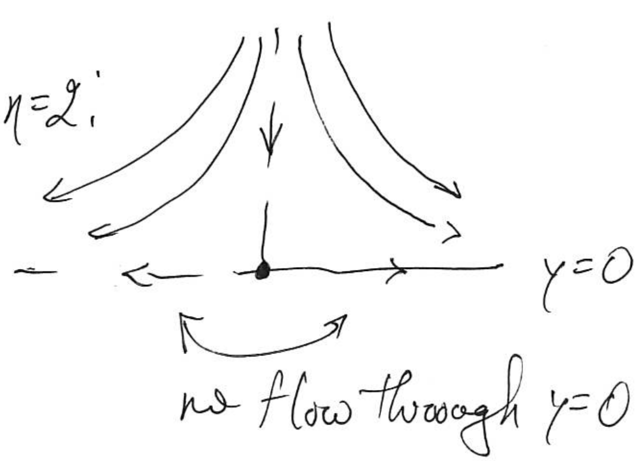
n even: $n = 2k$

$\theta = 0$: $u = n r^{n-1}$; $v = 0$

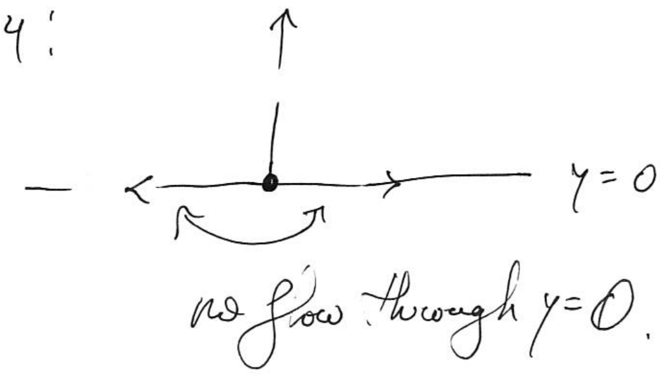
$\theta = \pi$: $u = -n r^{n-1}$; $v = 0$

$\theta = \frac{\pi}{2}$: $u = 0$; $v = -n r^{n-1}(-1)^{k-1} = n r^{n-1}(-1)^k$

$\rightarrow$ $n=2$:

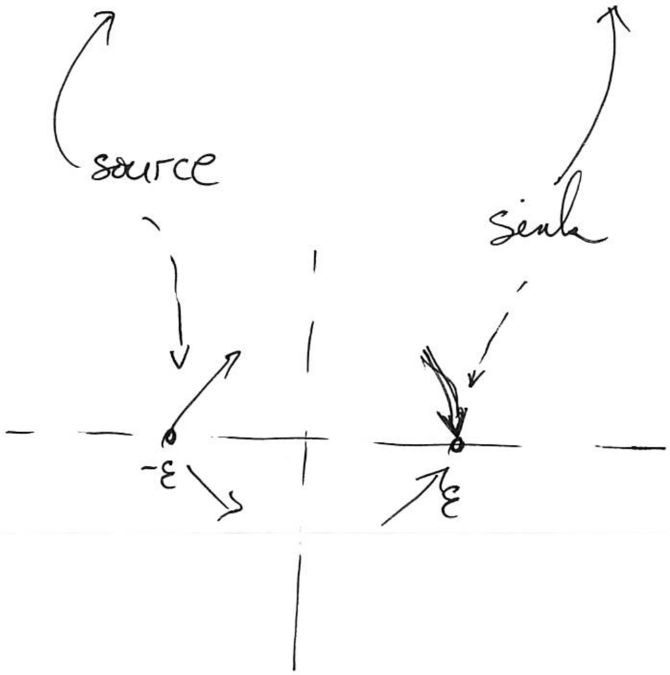


$n=4$:



Mix and Match: The Doublet

$$f(z) = \frac{m}{2\pi}\ln(z+\varepsilon) - \frac{m}{2\pi}\ln(z-\varepsilon)$$



Let: $\frac{m\varepsilon}{\pi} \to \mu$ as $\varepsilon \to 0^+$ (i.e. let strength m increase accordingly)

$$\hookrightarrow f(z) = \frac{m}{2\pi}\ln\left(\frac{z+\varepsilon}{z-\varepsilon}\right) = \frac{m}{2\pi}\ln\left(\left(1+\frac{\varepsilon}{z}\right)\left(1+\frac{\varepsilon}{z}+O(\varepsilon^2)\right)\right)$$

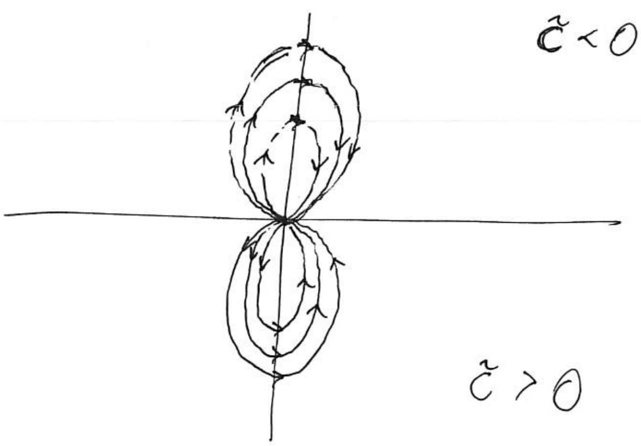
$$= \frac{m}{2\pi}\ln\left(1+\frac{2\varepsilon}{z}+O(\varepsilon^2)\right)$$

$$= \frac{m}{2\pi}\left(\frac{2\varepsilon}{z}+O(\varepsilon^2)\right) \overset{\varepsilon\to 0}{=} \frac{\mu}{z} = \frac{\mu}{r}e^{-i\phi}$$

So, then we see that

$$\phi = \frac{\mu \cos(\theta)}{r} \quad ; \quad \psi = -\frac{\mu \sin(\theta)}{r}$$

$\hookrightarrow$ $\psi = \tilde{C}$ $\rightarrow$ $-\frac{\mu \sin(\theta)}{r} = \tilde{C}$ or $r = -\frac{\mu \sin(\theta)}{\tilde{C}}$



So we get a source and sink to collapse onto one another.

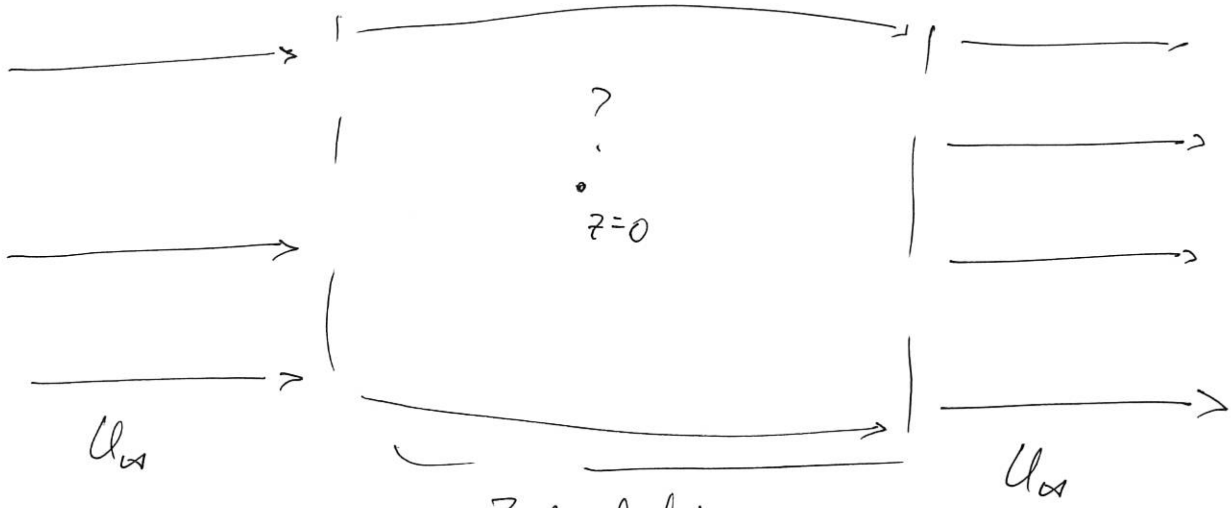
Flow Past a Cylinder without Circulation:

$$f(z) = U_\infty z + \frac{\mu}{z}$$

$\hookrightarrow$ as $|z| \to \infty$, $f(z) \sim U_\infty z$ $\rightarrow$ $(u - iv) = f'(z) \sim U_\infty$ $\rightarrow$

$u = U_\infty$, $v = 0$

So we know now:



But what happens in the middle?

$$f(z) = U_\infty r e^{i\theta} + \frac{\mu}{r} e^{-i\theta}$$

$$= \left(U_\infty r + \frac{\mu}{r}\right)\cos(\theta) + \left(U_\infty r - \frac{\mu}{r}\right)\sin(\theta)$$

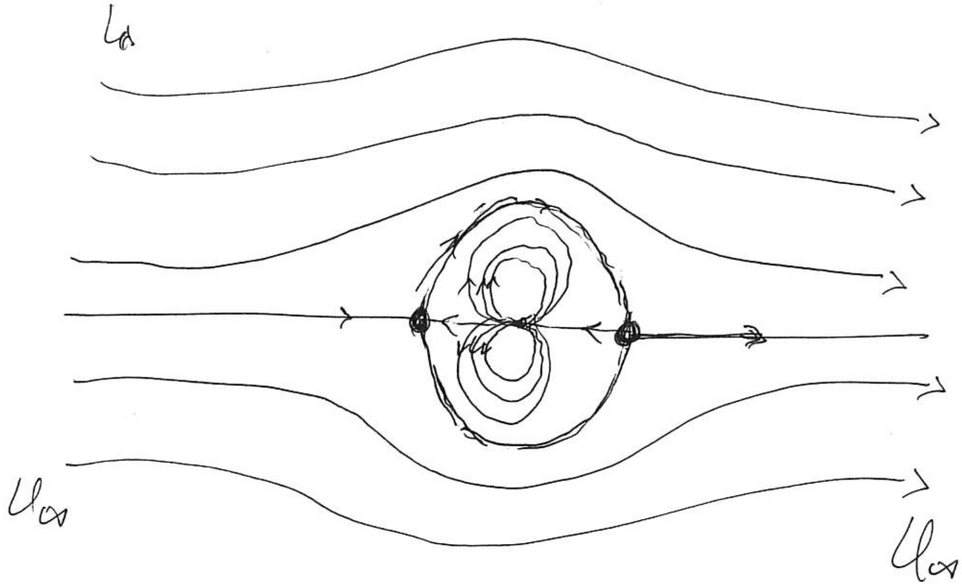
$\hookrightarrow$ $\phi(r,\theta) = \left(U_\infty r + \frac{\mu}{r}\right)\cos(\theta)$

$\psi(r,\theta) = \left(U_\infty r - \frac{\mu}{r}\right)\sin(\theta)$

So, when we look at $\psi(r,\theta) = \tilde{c}$, if we set $\tilde{c} = 0 \rightarrow$

$\theta = 0, \pi,$ and so forth.

So if we note the doublet dominates around $z=0$



From Bernoulli Equation, if we set the pressure at "∞" to be $\bar{p}_\infty$

↳

$$\frac{\rho_0}{2}|f'(z)|^2 + \bar{p} = \frac{\rho_0}{2}U_\infty^2 + \bar{p}_\infty$$

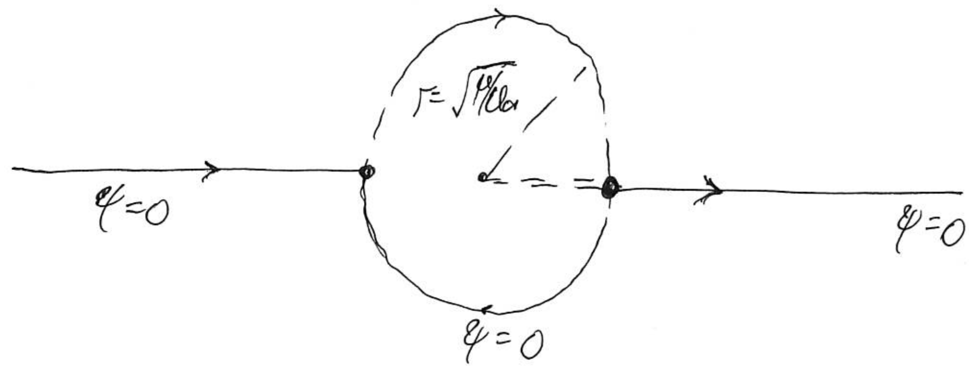
↳ $$|f'(z)|^2 = \left(U_\infty - \frac{\mu}{r^2}\cos(2\theta)\right)^2 + \frac{\mu^2}{r^4}\sin^2(2\theta)$$

$$= U_\infty^2 - \frac{2\mu U_\infty}{r^2}\cos(2\theta) + \frac{\mu^2}{r^4}$$

↳ $$\bar{p} = \bar{p}_\infty + \frac{\rho_0 \mu U_\infty}{r^2}\cos(2\theta) - \frac{\rho_0\mu^2}{2r^4}$$

But we also end up with

$$U_\infty r - \frac{\mu}{r} = 0 \quad \text{or} \quad r = \sqrt{\frac{\mu}{U_\infty}}$$

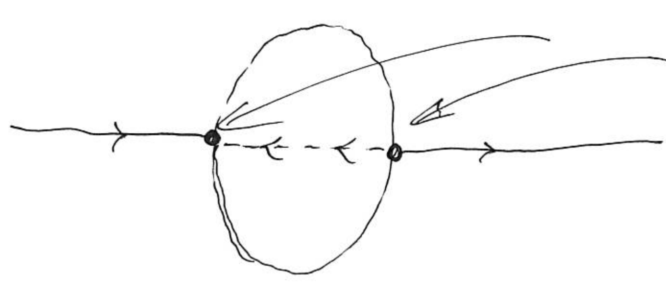


Note: $(u - iv) = f'(z) = U_\infty - \frac{\mu}{z^2} = U_\infty - \frac{\mu}{r^2} e^{-2i\theta}$

$$\left\{ \begin{aligned} u &= U_\infty - \frac{\mu}{r^2} \cos(2\theta) \\ v &= \frac{\mu}{r^2} \sin(2\theta) \end{aligned} \right.$$

So note, for $\theta = 0, \pi$, $v = 0$

but $u = U_\infty - \frac{\mu}{r^2} \rightarrow$ at $r = \sqrt{\frac{\mu}{U_\infty}} \rightarrow u = 0$



} Fluid velocity is exactly zero

Called stagnation pts.

We can then find the pressure on the cylinder to be

$$\bar{p}\Big|_{r=\sqrt{\mu/U_\infty}} = \bar{p}_\infty + \frac{1}{2}\rho_0 U_\infty^2 \cos^2(\theta)$$

↳ Note, symmetry of pressure distribution ⇒ no drag

↳ Known as D'Alembert's Paradox

↳ Need viscosity to resolve this.