So now ignoring non-linearity means setting $\varepsilon = 0$. We have $\gamma = H/L$ alone.

↳ $$\partial_x^2 \phi + \partial_y^2 \phi + \frac{1}{\gamma^2}\partial_z^2 \phi = 0, \qquad -1 < z < 0$$

$$\phi_z = 0, \qquad z = -1$$

$$\eta_t = \frac{1}{\gamma^2}\phi_z, \qquad z = 0$$

$$\phi_t + \eta = \tilde{\sigma}\Delta\eta, \qquad z = 0$$

↳ $$\hat{\phi}(\vec{k}, z, t) = \alpha(\vec{k})\cosh(\gamma|\vec{k}|(z+1))e^{\pm i\omega(\vec{k})t}$$

$$\hat{\eta}(\vec{k}, t) = \beta(\vec{k})e^{\pm i\omega(\vec{k})t}$$

$$\omega^2(\vec{k}) = |\vec{k}|(1+\tilde{\sigma}|\vec{k}|^2)\frac{\tanh(\gamma|\vec{k}|)}{\gamma}$$

~~[illegible]~~ Case I: $\gamma|\vec{k}| \ll 1$ or $H|\vec{k}| \ll L$

↳ $$\omega^2(\vec{k}) = |\vec{k}|^2 + O(|\vec{k}|^4)$$

Leading order $\eta_{tt} \simeq \eta_{xx}$

↳ $$c_p = \omega(\vec{k})/|\vec{k}| \simeq 1 + O(|\vec{k}|^3)$$

(Dispersion becomes a relatively weaker effect.)

Okay, anyway, we get $c_s = 1$ to leading order.

What does that mean in units?

$$\tilde{x} = \frac{x}{L} \; ; \quad \tau = \frac{\sqrt{gH}}{H} t \; ; \quad \tilde{t} = \delta\tau = \frac{\sqrt{gH}}{L} t$$

So $$\frac{x}{t} = \frac{L\tilde{x}}{L\tilde{t}} \sqrt{gH} = \sqrt{gH}\; \tilde{x}/\tilde{t}$$

So $c_s = 1 \Rightarrow c_s = \sqrt{gH}$ in units.

Thus for a tsunami, the leading wave speed is $\sqrt{gH}$

So $g \sim 9.8\ m/s^2 \sim 10\ m/s^2$

$H \sim 5\ km \sim 5 \times 10^3\ m$

So $c_s \simeq \sqrt{gH} \simeq (\sqrt{5} \times 10^2)\ m/s \simeq 223\ m/s \simeq 502\ mph$

Case II: $\gamma|\vec{k}| \to \infty$ or $\gamma|\vec{k}| \gg 1$ or $H|\vec{k}| \gg L$

↳ $\omega^2(\vec{k}) \simeq \frac{1}{\gamma}|\vec{k}|(1+\tilde{\sigma}|\vec{k}|^2)$

↳ $c_p \simeq$ ... mmm... not so clear. I mean that γ is awkward.

And what if we took $H \to -\infty$? Then what?

↳ $\tilde{z} = z/L$ since we have no H anymore $\to \gamma = 1$ so that we have

$$\Delta\phi = 0 \qquad -\infty < z < \varepsilon\eta$$

$$\phi_z \to 0 \qquad z \to -\infty$$

$$\eta_t = -\varepsilon(\eta_x\phi_x + \eta_y\phi_y) + \phi_z, \qquad z = \varepsilon\eta$$

$$\phi_t + \frac{\varepsilon}{2}\left\{|\nabla_{x,y}\phi|^2 + \phi_z^2\right\} + \eta = \tilde{\sigma}\nabla\cdot\left(\frac{\nabla\eta}{(1+\varepsilon^2\|\nabla\eta\|^2)^{1/2}}\right), \qquad z = \varepsilon\eta$$

Setting $\varepsilon = 0$ and taking FT in xy gives

$$-|\vec{k}|^2\hat{\phi} + \partial_z^2\hat{\phi} = 0, \qquad -\infty < z < 0$$

$$\hat{\phi}_z \to 0 \quad \text{as } z \to -\infty$$

↳ $\hat{\phi}(k,z,t) = \tilde{\beta}(k,t)\, e^{+|\vec{k}|z}$

$\hat{\eta}_t = \hat{\phi}_z = +|\vec{k}|\, \tilde{\beta}(k,t)$ since $z=0$

$\hat{\phi}_t + \hat{\eta} = -\tilde{\sigma}|\vec{k}|^2 \hat{\eta}$

↳ $\tilde{\beta}_{tt} = -|\vec{k}|(1+\tilde{\sigma}|\vec{k}|^2)\tilde{\beta}$

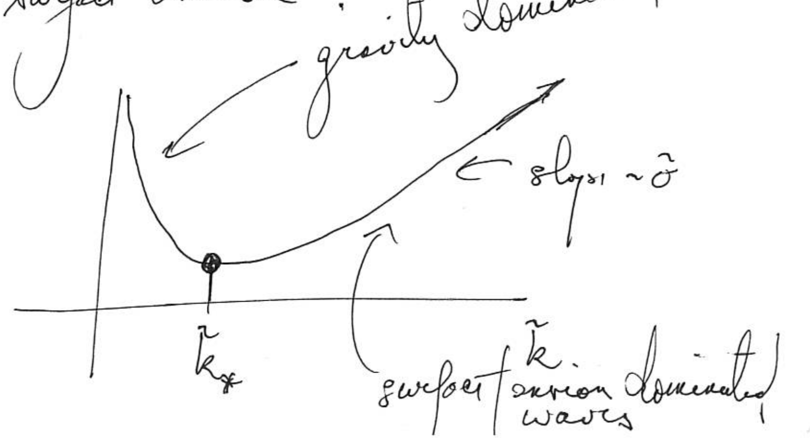
↳ $\omega^2(\vec{k}) = |\vec{k}|(1+\tilde{\sigma}|\vec{k}|^2)$

↳ we see in deep water, dispersion is to be expected to leading order

$$\vec{c}_p = \frac{\omega(\vec{k})}{|\vec{k}|}\left(\frac{\vec{k}}{|\vec{k}|}\right) = \pm\left(\frac{1+\tilde{\sigma}|\vec{k}|^2}{|\vec{k}|}\right)^{1/2}\hat{k}$$

↳ we also now see the impact of surface tension :

$\varphi(\tilde{k}) = \frac{1}{\tilde{k}} + \tilde{\sigma}\tilde{k}$ —



So, returning to the case in which we pay attention to depth

$$\hat{\phi}(\vec{k}, z, t) = \alpha(\vec{k}) \cosh(\gamma|\vec{k}|(z+1))\, e^{\pm i\omega(\vec{k})t}$$

$$\hat{\eta}(\vec{k}, z, t) = \beta(\vec{k})\, e^{\pm i\omega(\vec{k})t}$$

From: $\hat{\eta}_t = \frac{1}{\gamma^2} \hat{\phi}_z$ at $z=0$

$\hookrightarrow$ $\beta(\vec{k})(\pm i\omega) = \frac{\alpha|\vec{k}|}{\gamma} \sinh(\gamma|\vec{k}|)$

or: $\alpha(\vec{k}) = \frac{\pm i\omega\gamma}{|\vec{k}|\sinh(\gamma|\vec{k}|)} \beta(\vec{k})$

So if, using

$$\eta(x,y,t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \beta(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} \pm \omega(\vec{k})t)}\, dk_x\, dk_y$$

which we should really write as

$$\eta = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left\{ \beta_1(\vec{k})\, e^{i\omega(\vec{k})t} + \beta_2(\vec{k})\, e^{-i\omega(\vec{k})t} \right\} e^{i\vec{k}\cdot\vec{x}}\, dk_x\, dk_y$$

then if we enforce that η is real i.e. $\eta = \eta^*$

$\hookrightarrow$ $$\eta^* = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left\{ \beta_1^*(\vec{k}) e^{-i\omega(\vec{k})t} + \beta_2^*(\vec{k}) e^{i\omega(\vec{k})t} \right\} e^{-i\vec{k}\cdot\vec{x}} dk_x dk_y$$

let $k_{(x)} \to -k_{(x)}$; $k_{(y)} \to -k_{(y)}$

$\hookrightarrow$ $$\eta^* = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left\{ \beta_1^*(-\vec{k}) e^{-i\omega(\vec{k})t} + \beta_2^*(-\vec{k}) e^{i\omega(\vec{k})t} \right\} e^{i\vec{k}\cdot\vec{x}} dk_{(x)} dk_{(y)}$$

$\hookrightarrow$ $\eta = \eta^* \implies \beta_1(\vec{k}) = \beta_2^*(-\vec{k})$; $\beta_2(\vec{k}) = \beta_1^*(-\vec{k})$

So if we look at $\eta(x,t)$ mode by mode:

$$\eta = \left(\tilde{\beta}_1(\vec{k}) e^{i\omega(\vec{k})t} + \tilde{\beta}_1^*(-\vec{k}) e^{-i\omega(\vec{k})t}\right) e^{i\vec{k}\cdot\vec{x}}$$

$$+ \left(\tilde{\beta}_1(-\vec{k}) e^{i\omega(\vec{k})t} + \tilde{\beta}_1^*(\vec{k}) e^{-i\omega(\vec{k})t}\right) e^{-i\vec{k}\cdot\vec{x}} \;;\; \tilde{\beta}_i = \beta_i/(2\pi)^2$$

$$\hookrightarrow \eta = \tilde{\beta}_1(\vec{k})e^{i(\vec{k}\cdot\vec{x}+\omega(\vec{k})t)} + \tilde{\beta}_1^*(\vec{k})e^{-i(\vec{k}\cdot\vec{x}+\omega(\vec{k})t)}$$

$$+ \tilde{\beta}_1^*(-\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)} + \tilde{\beta}_1(-\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)}$$

$$= 2|\tilde{\beta}_1(\vec{k})|\cos(\vec{k}\cdot\vec{x}+\omega(\vec{k})t + \varphi(\vec{k}))$$

$$+ 2|\tilde{\beta}_1(-\vec{k})|\cos(\vec{k}\cdot\vec{x}-\omega(\vec{k})t - \varphi(-\vec{k}))$$

where we let: $\begin{cases} \tilde{\beta}_1(\vec{k}) = |\tilde{\beta}_1(\vec{k})| e^{i\varphi(\vec{k})} \\ \tilde{\beta}_1(-\vec{k}) = |\tilde{\beta}_1(-\vec{k})| e^{i\varphi(-\vec{k})} \end{cases}$

~~Apply ... we have~~

$$\cancel{\phi(x,y,z,t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left(\frac{i\omega\eta}{|\vec{k}|\sinh(\eta|\vec{k}|)}\beta_1(\vec{k})e^{i\omega(\vec{k})t} - \frac{i\omega\eta}{|\vec{k}|\sinh(\eta|\vec{k}|)}\beta_1^*(-\vec{k})e^{-i\omega(\vec{k})t}\right)e^{i\vec{k}\cdot\vec{x}}\,dk_{(x)}\,dk_{(y)}}$$

$$\cancel{\text{let } \tilde{\beta}_1 = \sinh(\eta|\vec{k}|)\beta_1, \quad i = e^{i\pi/2}}$$

$$\cancel{\hookrightarrow \phi = \frac{\omega\eta}{|\vec{k}|}|^2}$$

As for ϕ_1 we have

$$\phi(x,y,z,t) = \frac{i\omega\gamma}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\cosh(\gamma|\vec{k}'|(z+1))}{|\vec{k}'|\sinh(\gamma|\vec{k}'|)}\left(\beta_1(\vec{k})e^{i\omega(\vec{k}')t} - \beta_1^*(-\vec{k})e^{-i\omega(\vec{k}')t}\right)e^{i\vec{k}'\cdot\vec{x}'}dk_{(x)}dk_{(y)}$$

$\Rightarrow$

$$\phi = \frac{i\omega\gamma}{|\vec{k}'|}\frac{\cosh(\gamma|\vec{k}'|(z+1))}{\sinh(\gamma|\vec{k}'|)}\left\{\tilde{\beta}_1(\vec{k})e^{i(\vec{k}\cdot\vec{x}'+\omega(\vec{k}')t)} - \tilde{\beta}_1^*(+\vec{k})e^{-i(\vec{k}\cdot\vec{x}'+\omega(\vec{k}')t)}\right.$$

$$\left. - \left(\tilde{\beta}_1^*(-\vec{k})e^{i(\vec{k}'\cdot\vec{x}'-\omega(\vec{k}')t)} - \tilde{\beta}(-\vec{k})e^{-i(\vec{k}'\cdot\vec{x}'-\omega(\vec{k}')t)}\right)\right\}$$

$$= -\frac{2\omega\gamma}{|\vec{k}'|}\frac{\cosh(\gamma|\vec{k}|(z+1))}{\sinh(\gamma|\vec{k}|)}\left\{|\tilde{\beta}|\sin(\vec{k}\cdot\vec{x}'+\omega(\vec{k})t+\varphi(\vec{k}))\right.$$

$$\left. - |\tilde{\beta}_1(-\vec{k})|\sin(\vec{k}'\cdot\vec{x}'-\omega(\vec{k}')t-\varphi(-\vec{k}))\right\}$$

So, we can now focus on a direction and simplify things to:

$$\eta = \tilde{a}\cos(\vec{k}\cdot\vec{x}+\omega(\vec{k})t+\varphi)$$

$$\phi = -\frac{\tilde{a}\omega\gamma}{|\vec{k}|}\frac{\cosh(\gamma|\vec{k}|(z+1))}{\sinh(\gamma|\vec{k}|)}\sin(\vec{k}\cdot\vec{x}+\omega(\vec{k})t+\varphi)$$

But more interesting is the following: in η and ϕ, we see that we have in $\bar{x}'$ and t periodic behavior. To wit, if I define the phase

$$\Theta(\bar{x}',t) = \bar{k}' \cdot \bar{x}' + \omega(\bar{k}')t$$

$\Rightarrow \cos(\Theta(\bar{x}',t) + 2n\pi) = \cos(\Theta(\bar{x},t))$ and so forth.

Define the average $\langle\ \rangle$ so that

$$\langle f \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\Theta)\,d\Theta$$

Let $T = \frac{\rho_0}{2}\int_{-H}^{\eta} |\bar{u}|^2 dz \overset{z=H\tilde{z}}{=} \frac{\rho_0 g H^2}{2}\varepsilon^2 \int_{-1}^{\varepsilon\eta} \left(|\bar{\nabla}_{x,y}\phi|^2 + \frac{1}{\alpha^2}\phi_z^2 \right) dz$

So that this gives us the total kinetic energy under a particular point of the surface.

What is even more interesting though is

$$\overline{T} = \langle T \rangle \cong \frac{\varepsilon^2 \rho_0 g H^2}{2} \int_{-1}^{0} \left\langle |\nabla_{x,y}\phi|^2 + \frac{1}{\eta^2}\phi_z^2 \right\rangle dz$$

So:

$$\left(\phi_x, \phi_y, \frac{1}{\eta}\phi_z\right) = -\frac{\tilde{a}\omega\eta}{|\vec{k}|} \frac{\cosh(\eta|\vec{k}|(z+1))}{\sinh(\eta|\vec{k}|)} \left(k_x \cos(\theta), k_y \cos(\theta), |\vec{k}| \sin(\theta)\right)$$

$\hookrightarrow$

$$|\nabla_{x,y}\phi|^2 + \frac{1}{\eta^2}\phi_z^2 = \frac{\tilde{a}^2\omega^2\eta^2}{\cancel{|\vec{k}|^2}} \frac{\cosh^2(\eta|\vec{k}|(z+1))}{\sinh^2(\eta|\vec{k}|)} \cancel{|\vec{k}|^2}$$

$\hookrightarrow$

$$\overline{T} = \frac{\varepsilon^2 \rho_0 g H^2 \tilde{a}^2 \omega^2 \eta^2}{\sinh^2(\eta|\vec{k}|)} \int_{-1}^{0} \cosh^2(\eta|\vec{k}|(z+1))\, dz$$

From here, we can essentially derive everything up to at least leading order for isolated waves. For example, we can now find

$$\vec{u} = \nabla \phi$$

note: $\tilde{x} = x/L$; $\tilde{y} = y/L$; $\tilde{z} = z/H$; $\phi = \varepsilon L \sqrt{gH}\, \tilde{\phi}$; $\tau = \frac{\sqrt{gH}}{L} t$

$\hookrightarrow$ $$\vec{u} = \varepsilon \sqrt{gH} \left(\phi_x, \phi_y, \tfrac{1}{\delta} \phi_z \right) \qquad -1 < z < 0$$

$\hookrightarrow$ Momentum / Bernoulli Equation:

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + g z = -\frac{1}{\rho_0} P \qquad -1 < z < 0$$

$\hookrightarrow$ $$\varepsilon g H \phi_t + \frac{\varepsilon^2 g H}{2} \left(|\nabla_{x,y} \phi|^2 + \frac{1}{\delta^2} \phi_z^2 \right) + g H z = -\frac{1}{\rho_0} \rho_0 g H \tilde{P}$$

$\hookrightarrow$ $$-\tilde{P} = z + \varepsilon \phi_t + \frac{\varepsilon^2}{2} \left(|\nabla_{x,y} \phi|^2 + \frac{1}{\delta^2} \phi_z^2 \right)$$

$$\simeq z + \varepsilon \phi_t + \mathcal{O}(\varepsilon^2)$$