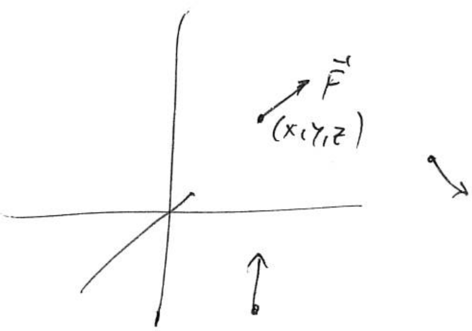
Flux, Divergence, and Gauss' THM: ①

Vector Fields: $\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$
$= M\hat{i} + N\hat{j} + P\hat{k}$



and so forth.

Flux: How much of $\vec{F}$ "moves" across a surface?

} at each point on S, we have a local unit normal $\hat{n}$ and some amount of surface area $d\sigma$

Flux: $\vec{F}\cdot\hat{\eta}\, d\sigma$ →

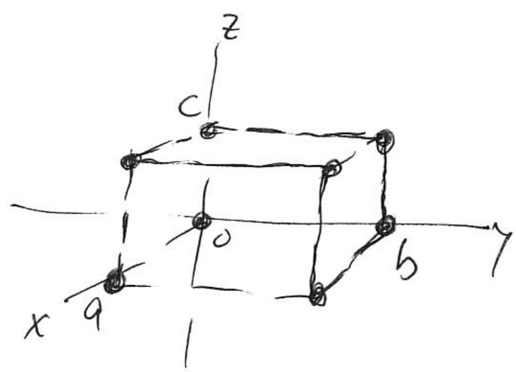
$\vec{F}\cdot\hat{\eta}$ i.e. projection of $\vec{F}$ along unit normal

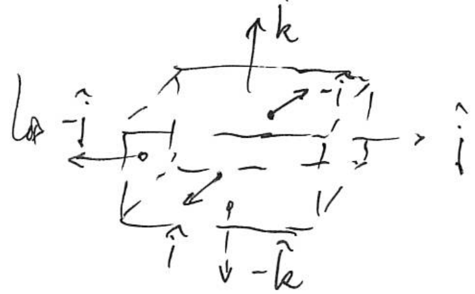
where $\vec{F}\cdot\hat{\eta} = |\vec{F}|\cos(\theta)$

($|\hat{\eta}| = 1$ by definition)

Total / Global Flux: $\displaystyle\int_S \vec{F}\cdot\hat{\eta}\, d\sigma$

Flux Across a ~~cube~~: Rhomboid



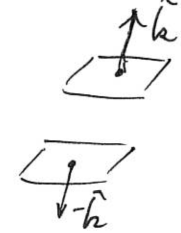


, $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

$$\hookrightarrow \iint_S \vec{F}\cdot\hat{\eta}\, d\sigma = \int_0^c\int_0^b \{ M(a,y,z) - M(0,y,z) \}\, dy\, dz$$

$$+ \int_0^a\int_0^c \{ N(x,b,z) - N(x,0,z) \}\, dz\, dx$$

$$+ \int_0^a\int_0^b \{ P(x,y,c) - P(x,y,0) \}\, dy\, dx$$



Fundamental THM of Calculus:

$$M(a,y,z) - M(0,y,z) = \int_0^a \frac{\partial M}{\partial x}\, dx$$

$$N(x,b,z) - N(x,0,z) = \int_0^b \frac{\partial N}{\partial y}\, dy$$

$$P(x,y,c) - P(x,y,0) = \int_0^c \frac{\partial P}{\partial z}\, dz$$

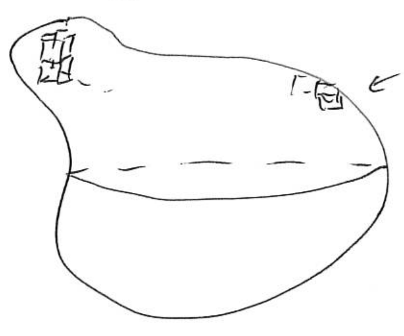
$$\Rightarrow \int_S \vec{F}\cdot\hat{\eta}\,d\sigma = \int_0^a\int_0^b\int_0^c \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dz\,dy\,dx$$

$$= \int_0^a\int_0^b\int_0^c \nabla\cdot\vec{F}\,dz\,dy\,dx$$

where we have defined the "divergence" of $\vec{F}$ to be

$$\nabla\cdot\vec{F} = M_x + N_y + P_z$$

So, if we imagine any "reasonable" surface as a bunch of "little" boxes:



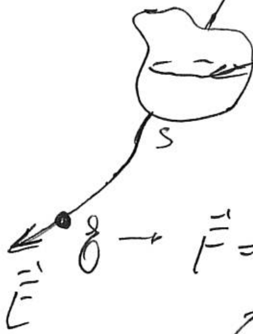
← fill out with boxes of decreasing size.

$\Rightarrow$ Gauss' / Divergence THM: $\displaystyle\int_S \vec{F}\cdot\hat{\eta}\,d\sigma = \int_V \nabla\cdot\vec{F}\,dV$

outward facing unit normal

volume contained in S.

Electrostatics:



$\vec E$ ← Vector Field: Electric Flux or Field.

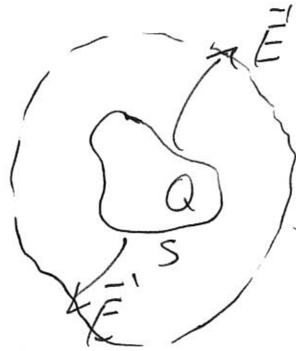
Charge $Q = \int_V \rho(x,y,z)\,dV$

Charge Density

$q \to \vec F = q\vec E$

i.e. force on test charge is found by multiplying with $\vec E$.

↳ Basic Law:



No matter where I measure the field flux $\int_{S'} \vec E \cdot \hat n \, d\sigma$,

so long as it contains all of Q, I must have the same net flux.

Thus: for any $S' \supseteq S$

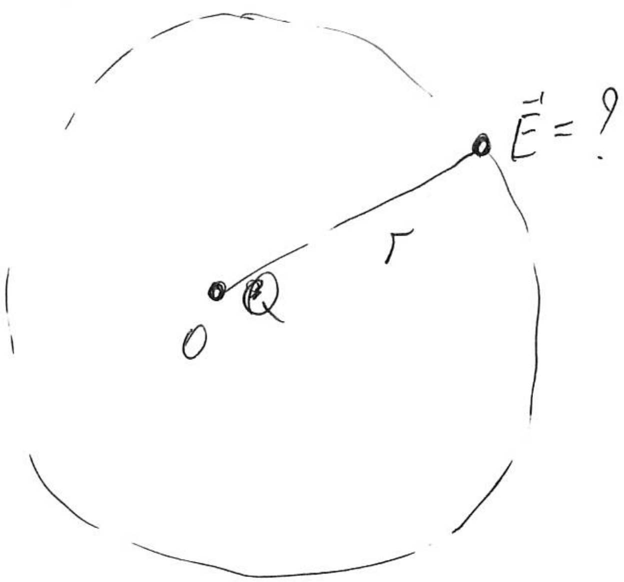
$$\int_{S'} \vec E \cdot \hat n \, d\sigma = cQ$$

c is physical constant.

$$\Rightarrow \int_S \vec{E}\cdot\hat{\eta}\, d\sigma = cQ = c\int_V \rho\, dV$$

$$\Rightarrow \int_V \nabla\cdot\vec{E}\, dV = c\int_V \rho\, dV$$

$\Rightarrow$ Maxwell's 1st law: $\underline{\nabla\cdot\vec{E} = c\rho}$



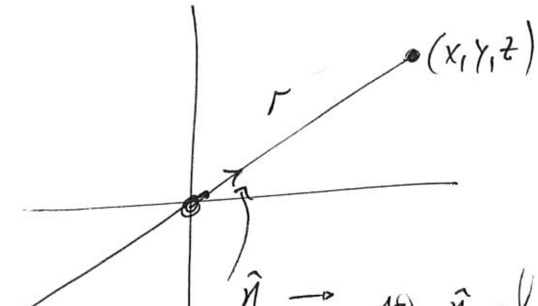
$$\Rightarrow cQ = \int_S \vec{E}\cdot\hat{\eta}\, d\sigma$$

For a sphere of radius r

$\Rightarrow x^2 + y^2 + z^2 = r^2$

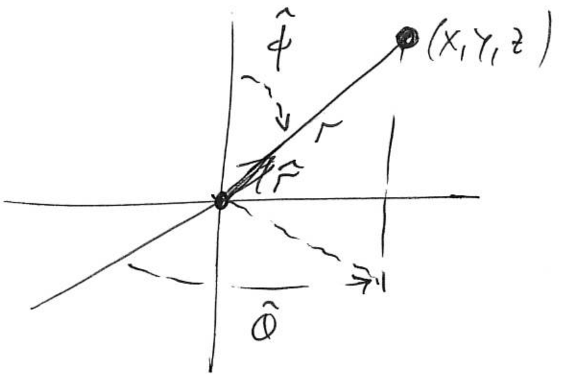
$\Rightarrow \hat{\eta} = (x, y, z)/r$

from $\vec{\eta} = \nabla(x^2 + y^2 + z^2 - r^2) = 2(x, y, z)$

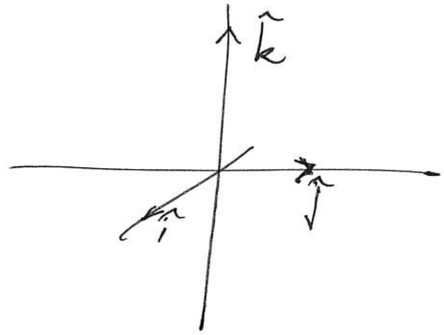


$\hat{n} \to$ so $\hat{n}$ always points parallel to the radius of the sphere.

$\Rightarrow \hat{n} = \hat{r}$



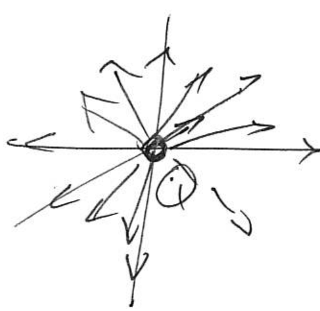
or



By symmetry we argue $\vec{E} = E\hat{r}$, and E is constant along a spherical surface.

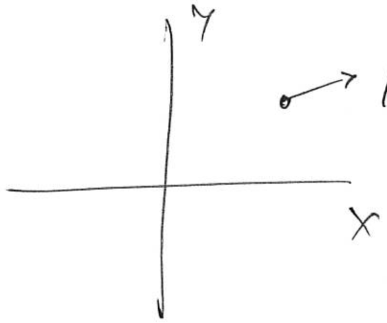
$$\Rightarrow cQ = E\int_S d\sigma = E(4\pi r^2) \to E = \frac{cQ}{4\pi r^2}$$

or $\vec{E} = \dfrac{cQ}{4\pi r^2}\hat{r}$



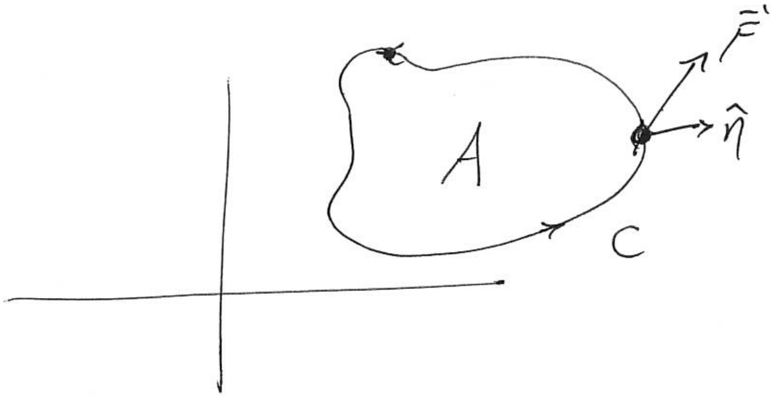
and so forth.

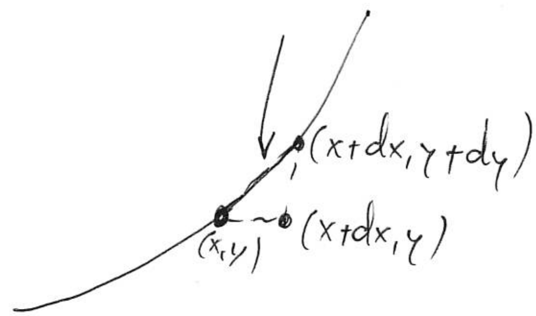
Green's THM: Gauss' THM in the plane



$\vec{F} = M\hat{i} + N\hat{j}$

$$ds = (dx^2 + dy^2)^{1/2}$$





arc-length

→ Flux across C: $\displaystyle\oint_C \vec{F}\cdot\hat{n}\, ds = \int_A \nabla\cdot\vec{F}\, dA$

→ we talk about parametrizations of curves, which is how we compute line integrals.

→ $$ds = \frac{ds}{dt}\,dt = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} dt$$

Goofy Example:

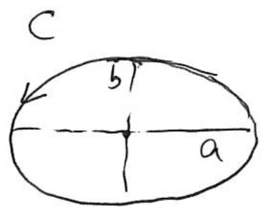
$\leftarrow s = r\theta \rightarrow ds = r\,d\theta$

i.e. "t" $= \theta$

↳ $\oint_C ds = \int_0^{2\pi r} ds = 2\pi r$

$\int_0^{2\pi} \frac{ds}{d\theta}\, d\theta = r \int_0^{2\pi} d\theta = 2\pi r$

Elliptic Integrals:



$$L = \int_C ds = \int_0^{2\pi} \frac{ds}{d\theta}\, d\theta = \int_0^{2\pi} \left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \right)^{1/2} d\theta \quad \left(t = \theta \text{ here} \right)$$

let $x = a\cos(\theta)$, $y = b\sin(\theta)$

↳ $L = \int_0^{2\pi} \left(a^2 \sin^2(\theta) + b^2 \cos^2(\theta) \right)^{1/2} d\theta$