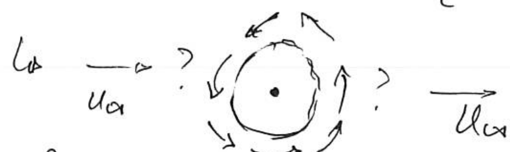
Flow Past a Cylinder w/ Circulation:

$$f(z) = U_\infty z + \frac{\mu}{z} + \frac{i\Gamma}{2\pi} \ln\left(\frac{z}{a}\right); \quad a = \sqrt{\frac{\mu}{U_\infty}}, \text{ let } \Gamma > 0$$

for sake of argument.



↳ $$f = U_\infty r e^{i\theta} + \frac{\mu}{r} e^{-i\theta} + \frac{i\Gamma}{2\pi}\left\{ \ln\left(\frac{r}{a}\right) + i\theta \right\}$$

↳ $$\phi = U_\infty r \cos(\theta) + \frac{\mu}{r}\cos(\theta) - \frac{\Gamma\theta}{2\pi}$$

$$\psi = U_\infty r \sin(\theta) - \frac{\mu}{r}\sin(\theta) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)$$

↳ So we see, by design, that $r = \sqrt{\frac{\mu}{U_\infty}}$ is still a streamline corresponding to $\psi = 0$

Note though that $\psi\big|_{\theta=0,\pi} = \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)$

So we no longer have a horizontal line as a streamline.

So how to understand this flow?

↳ $f'(z) = U_\infty - \frac{\mu}{z^2} + \frac{i\Gamma}{2\pi z}$

$= U_\infty - \frac{\mu}{r^2} e^{-2i\theta} + \frac{i\Gamma}{2\pi r} e^{-i\theta}$

$= U_\infty - \frac{\mu}{r^2}\cos(2\theta) + \frac{\Gamma}{2\pi r}\sin(\theta)$

$+ i\left(\frac{\mu}{r^2}\sin(2\theta) + \frac{\Gamma}{2\pi r}\cos(\theta) \right)$

↳ $u = U_\infty - \frac{\mu}{r^2}\cos(2\theta) + \frac{\Gamma}{2\pi r}\sin(\theta)$

$v = -\left(\frac{\mu}{r^2}\sin(2\theta) + \frac{\Gamma}{2\pi r}\cos(\theta) \right)$

↳ $\dot{x} = u(x,y)$ $\longrightarrow$ So a stagnation pt. i.e. $u = v = 0$

$\dot{y} = v(x,y)$

↳ is a fixed point of the flow.

↳ Can study this like any other dynamical system.

$v = 0$: $\frac{\cos(\theta)}{r}\left\{ \frac{2\mu\sin(\theta)}{r} + \frac{\Gamma}{2\pi} \right\} = 0$

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$\to \theta = \pi/2,\ 3\pi/2$

$\Gamma = -\frac{4\pi\mu\sin(\theta)}{r}$

If $\theta = \pi/2,\ 3\pi/2 \to u = U_\infty + \frac{\mu}{r^2} \pm \frac{\Gamma}{2\pi r} = 0$, "+" $= \frac{\pi}{2}$; "−" $= \frac{3\pi}{2}$

$\to r^2 + a^2 \pm \tilde{\Gamma} r = 0, \quad \tilde{\Gamma} = \Gamma/2\pi U_\infty$

$a^2 = \mu/U_\infty$

$\to r = \frac{1}{2}\left\{ \mp\tilde{\Gamma} \pm \left(\tilde{\Gamma}^2 - 4a^2\right)^{1/2} \right\}$

So, if $|\tilde{\Gamma}| > 2a \longrightarrow$ at $r_* = \frac{1}{2}\left(\tilde{\Gamma} + \left(\tilde{\Gamma}^2 - 4a^2\right)^{1/2} \right), \theta = \frac{3\pi}{2}$

we have a stagnation point, and we can readily show $r_* > a$.

Keeping in mind that for polar coordinates we have

$r_x = \cos(\theta)\ ;\ r_y = \sin(\theta) \xrightarrow{@\theta = \frac{3\pi}{2}} r_x = 0;\ r_y = -1$

$\theta_x = -\frac{\sin(\theta)}{r};\ \theta_y = \frac{\cos(\theta)}{r} \longrightarrow \theta_x = \frac{1}{r};\ \theta_y = 0$

$$\to J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Bigg|_{\left(r=r_*,\ \theta = \frac{3\pi}{2}\right)} = \begin{pmatrix} \cancelto{0}{\frac{2\mu\sin(2\theta)}{r^3}} + \cancelto{0}{\frac{\Gamma}{2\pi r^2}\cos(\theta)} & \frac{-2\mu}{r^3} - \frac{\Gamma}{2\pi r^2} \\ +\frac{2\mu}{r^3} - \frac{\Gamma}{2\pi r^2} & 0 \end{pmatrix}$$

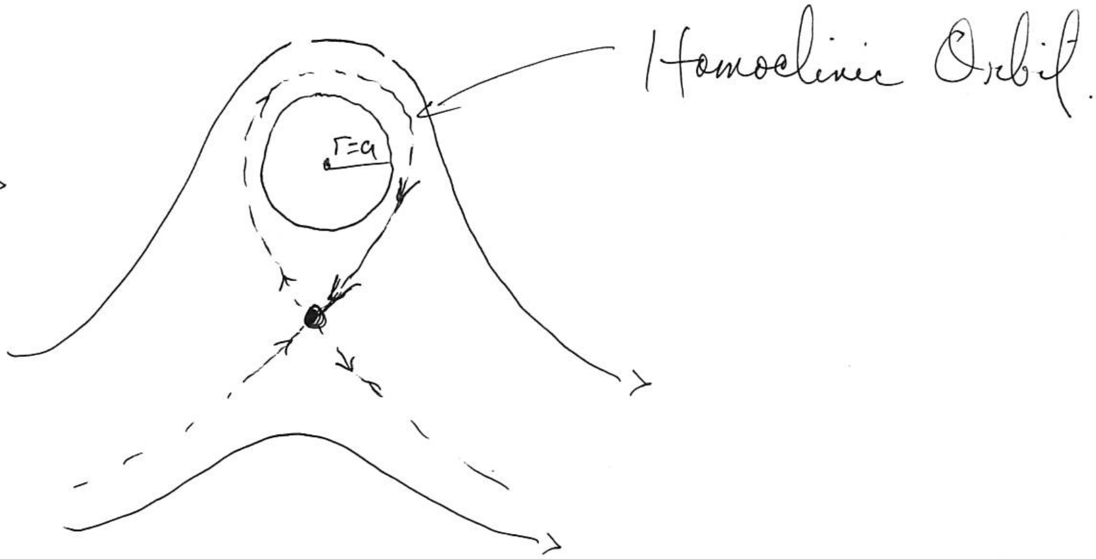
↳

$$\lambda^2 + \left(\frac{2\mu}{r_*^3} + \frac{\Gamma}{2\pi r_*^2}\right)\left(\frac{2\mu}{r_*^3} - \frac{\Gamma}{2\pi r_*^2}\right) = 0$$

↳ $$\lambda^2 = \frac{1}{r_*^4}\left\{\left(\frac{\Gamma}{2\pi}\right)^2 - \frac{4\mu^2}{r_*^2}\right\}$$

if $\dfrac{\Gamma}{2\pi} > \dfrac{2\mu}{r_*}$ $\longrightarrow$ stagnation point is a saddle.

↳



How to see that closed path connecting the stagnation point to itself:

$$\psi\Big|_{\left(r=r_*,\ \theta=\frac{3\pi}{2}\right)} = -U_\infty r_* + \frac{\mu}{r_*} + \frac{\Gamma}{2\pi}\ln\left(\frac{r_*}{a}\right) = \tilde{C}.$$

i.e. the stagnation point sets the value of the level set.

So, if we change our perspective, we remember that in Polar coordinates the C-R equations are: (5)

$$\phi_r = \frac{1}{r}\psi_\theta \; ; \; \frac{1}{r}\phi_\theta = -\psi_r$$

So, if $u = \phi_x = -\psi_y$; $v = \phi_y = \psi_x$

↳ $\vec{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta$

↳ $u_r = \phi_r$, $u_\theta = \frac{1}{r}\phi_\theta = -\psi_r$

↳ $$u_\theta = -\left(U_\infty \sin(\theta) + \frac{\mu}{r^2}\sin(\theta) + \frac{\Gamma}{2\pi a}\right)$$

$$u_r = U_\infty \cos(\theta) - \frac{\mu}{r^2}\cos(\theta) = \left(U_\infty - \frac{\mu}{r^2}\right)\cos(\theta)$$

So at $r = a = \sqrt{\frac{\mu}{U_\infty}}$, $u_r = 0$ as we would expect since $r = \sqrt{\frac{\mu}{U_\infty}}$ is a stream line.

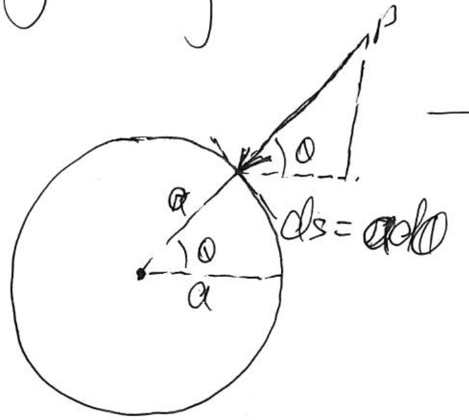
So: $$u_\theta\Big|_{r=\sqrt{\frac{\mu}{U_\infty}}} = -\left(\frac{\Gamma}{2\pi a} + 2U_\infty \sin(\theta)\right)$$

~~From~~ B.

From Bernoulli's Eqn: along $r = a = \sqrt{\frac{\mu}{U_\infty}}$

$$\bar{P}\Big|_{r=a} = \bar{P}_\infty + \frac{\rho_0}{2}\left(U_\infty^2 - U_\theta^2\right)$$

So, as we go along the circle:



→ The pressure $\bar{P}$ by assumption always acts normal to a surface in a fluid.

↳ Vertical Force on Cylinder: $-\bar{P}\Big|_{r=a}\sin(\theta)\,a\,d\theta$

Horizontal Force on Cylinder: $-\bar{P}\Big|_{r=a}\cos(\theta)\,a\,d\theta$

↳ Lift: $L = -a\int_0^{2\pi}\bar{P}\Big|_{r=a}\sin(\theta)\,d\theta$

Drag: $D = -a\int_0^{2\pi}\bar{P}\Big|_{r=a}\cos(\theta)\,d\theta$

So we note that

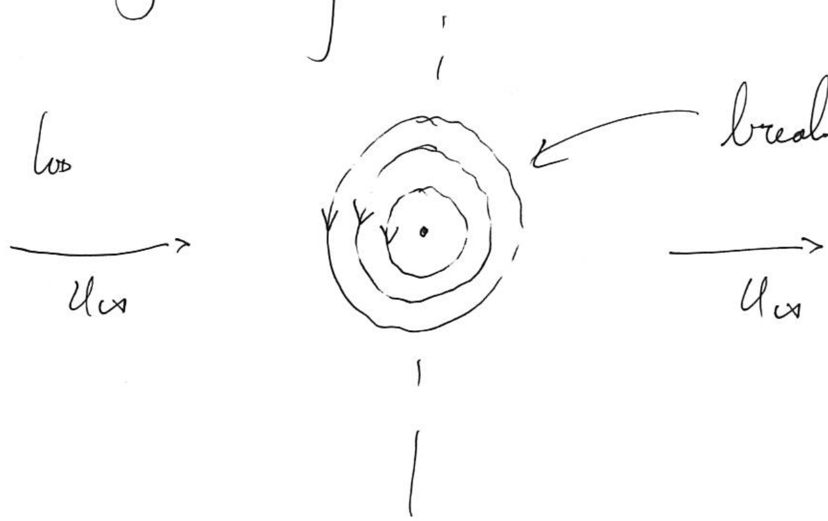
$$U_\theta^2 = \frac{\Gamma^2}{4\pi^2 a^2} + \frac{2\Gamma U_\infty}{\pi a}\sin(\theta) + 2U_\infty^2\left(1-\cos(2\theta)\right) \; ; \quad \sin^2(\theta) = \frac{1}{2}\left(1-\cos(2\theta)\right)$$

So

$$L = + \frac{\rho_0 \Gamma U_\infty}{\pi}\int_0^{2\pi} \sin^2(\theta)\, d\theta = \rho_0 \Gamma U_\infty$$

As for drag; we readily see that is still $\underline{\underline{D = 0}}$.

So we see: by including circulation Γ



breaks symmetry along y axis.

$\uparrow L = \rho_0 U_\infty \Gamma \leftarrow$ circulation generates lift!