

# Vortex Patches under Cnoidal Waves

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## Introduction

### Model

Throughout, we are attempting to describe the simultaneous evolution of a free surface  $y = \eta(x, t) + H$ , and a compactly supported patch of vorticity  $\omega(x, y, t)$  underneath the free surface. We suppose along the curve  $z = 0$  that we have a solid boundary so that the normal velocity is identically zero. In an inviscid, incompressible fluid, we can represent the fluid velocity  $\mathbf{u}(x, y, t)$  generated by a vortex patch characterized by vorticity profile  $\omega(\mathbf{x}, t)$  over the compact domain  $\Omega(t)$  via the integral equation

$$\mathbf{u}(\mathbf{x}, t) = \int_{\Omega(t)} \mathbf{K}(\mathbf{x} - \tilde{\mathbf{x}}) \omega(\tilde{\mathbf{x}}, t) d\tilde{\mathbf{x}} + \nabla \tilde{\phi}, \quad \Delta \tilde{\phi} = 0.$$

where  $\omega$  is the vorticity, and  $\mathbf{K}$  is the standard Biot-Savart law kernel. The harmonic function  $\tilde{\phi}$  is used to address boundary conditions as explained in [1]. An attractive means for discretizing this equation as summarized in [2] is to approximate the vorticity  $\omega$  by a collection of  $N$  point-vortices at positions  $\mathbf{x}_l(t)$  via the expansion

$$\omega(\tilde{\mathbf{x}}, t) = \sum_{j=1}^N \frac{\Gamma_j}{\delta^2} \chi\left(\frac{\tilde{\mathbf{x}} - \mathbf{x}_l(t)}{\delta}\right), \quad \mathbf{x}_l(t) = (x_l(t), y_l(t)), \quad (1)$$

where  $\chi$  is some appropriately chosen mollifier, see [3], and  $\Gamma_j$  is the circulation associated with the point vortex at  $\mathbf{x}_l(t)$ . Thus, we can reduce the problem of tracking the evolution of the vortex patch to describing the motion of the point vortices via the system of ODE's

$$\frac{d\mathbf{x}_j}{dt} = \sum_{l \neq j}^N \Gamma_l \mathbf{K}_\delta(\mathbf{x}_j - \mathbf{x}_l) + \nabla \tilde{\phi}(\mathbf{x}_j, t), \quad \mathbf{K}_\delta(\mathbf{x}) = \frac{1}{\delta^2} \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \tilde{\mathbf{x}}) \chi\left(\frac{\tilde{\mathbf{x}}}{\delta}\right) d\tilde{\mathbf{x}}.$$

Choosing, as in [3], the mollifier  $\chi$  to be the fourth-order kernel

$$\chi(r) = 2e^{-r^2} - \frac{1}{2}e^{-r^2/2},$$

introducing periodic boundary conditions in the lateral direction and a solid boundary along the curve  $z = 0$  then modifies the above dynamical system to be

$$i \frac{dz_j^*}{dt} = \frac{1}{2\pi} \left( \sum_{l \neq j}^N \Gamma_l \sum_{m=-\infty}^{\infty} \frac{\tilde{\chi}(z_j - z_l - 2Lm; \delta)}{z_j - z_l - 2Lm} - \sum_{l=1}^N \Gamma_l \sum_{m=-\infty}^{\infty} \frac{\tilde{\chi}(z_j - z_l^* - 2Lm; \delta)}{z_j - z_l^* - 2Lm} \right) + \partial_y \tilde{\phi} + i \partial_x \tilde{\phi},$$

where  $z_j = x_j + iy_j$ , the period in  $x$  is given by  $2L$ , and

$$\tilde{\chi}(r; \delta) = \left(1 - e^{-r^2/2\delta^2}\right) \left(1 + 2e^{-r^2/2\delta^2}\right).$$

As can be seen, the presence of the mollifier prevents from the closed form evaluation of the sums in  $m$ , thereby potentially adding significant overhead in numerical computations, even if fast Fourier transforms are used to evaluate the sums. We note however that

$$\tilde{\chi}(r; \delta) = 1 + \bar{\chi}(r), \quad \bar{\chi}(r) = \left(1 - 2e^{-r^2/2\delta^2}\right) e^{-r^2/2\delta^2}$$

which tacitly explains the role of mollification, which is to remove singularities in the determination of particular velocities when  $|z_j - z_l| \lesssim \delta$ . Thus, when we know that  $|z_j - z_l| > \delta$ , we take  $\tilde{\chi}(r; \delta) \sim 1$  so that

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{\tilde{\chi}(z_j - z_l - 2Lm; \delta)}{z_j - z_l - 2Lm} \approx \frac{1}{4L} \cot\left(\frac{\pi}{2L}(z_j - z_l)\right),$$

where the sum is taken in the principal value sense. In the case that  $|z_j - z_l| \lesssim \delta$ , we use instead

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{\tilde{\chi}(z_j - z_l - 2Lm; \delta)}{z_j - z_l - 2Lm} \approx \frac{1}{4L} \cot\left(\frac{\pi}{2L}(z_j - z_l)\right) + \frac{1}{2\pi} \frac{\bar{\chi}(z_j - z_l; \delta)}{z_j - z_l}.$$

The error incurred in these approximations is only exponentially small. We evaluate the corresponding sums over the image points  $z_j - z_l^*$  so as to keep the zero flow through  $z = 0$  condition strictly enforced. Our use of a Fast-Multipole Method for the evaluation of the velocities  $\dot{z}_j$  in effect determines all points either far or close to  $z_j$ , and thus the approximation above is a very natural and easy one to use in our numerical scheme.

Following the arguments in [4], and again emphasizing the compact support of the vorticity  $\omega(x, y, t)$ , we then have at the free surface the coupled nonlinear system

$$\eta_t = -\partial_x \eta \partial_x \tilde{\phi} + \partial_z \tilde{\phi} + P_v,$$

and

$$\tilde{\phi}_t + \frac{1}{2} \left| \nabla \tilde{\phi} \right|^2 + \text{Im} \{Q_v\} \partial_x \tilde{\phi} + \text{Re} \{Q_v\} \partial_z \tilde{\phi} + g\eta = E_v - \frac{1}{2} |Q_v|^2 + \frac{\sigma}{\rho_0} \partial_x \left( \frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right)$$

where we have defined

$$c(\eta, z_j) = \cot\left(\frac{\pi}{2L}(\eta + H - z_j)\right),$$

so that

$$P_v = \text{Re} \{Q_v\} - \text{Im} \{Q_v\} \partial_x \eta,$$

$$Q_v = \frac{1}{4L} \sum_{j=1}^N \Gamma_j (c(\eta, z_j) - c(\eta, z_j^*)),$$

and

$$E_v = \frac{1}{4L} \sum_{j=1}^N \Gamma_j (\dot{x}_j \text{Im} \{c(\eta, z_j) - c(\eta, z_j^*)\} + \dot{z}_j \text{Re} \{c(\eta, z_j) + c(\eta, z_j^*)\})$$

Note, we have ignored the mollification given the separation between the surface and the point vortices used to approximate the vortex patch.

Defining  $q = \tilde{\phi}|_{z=\eta+H}$ , standard arguments [5, 4] allow for the derivation of series representations to the Dirichlet-to-Nuemann operator  $G(\eta)$  so that

$$\eta_t = G(\eta)q + P_v,$$

and

$$\begin{aligned} q_t + \frac{1}{2} (\partial_x q)^2 + g\eta - E_v + \frac{1}{2} |Q_v|^2 - \frac{\sigma}{\rho_0} \partial_x \left( \frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right) = \\ - \frac{1}{1 + (\partial_x \eta)^2} \left( \left( P_v + \text{Re} \{Q_v\} - \frac{1}{2} (Gq + \partial_x \eta \partial_x q) \right) (Gq + \partial_x \eta \partial_x q) + \text{Im} \{Q_v\} (\partial_x q - \partial_x \eta Gq) \right) \end{aligned}$$

Thus, the surface boundary conditions can be recast entirely in terms of surface variables alone. This then leaves the problem of evaluating the derivatives of  $\tilde{\phi}$  at the vortex positions thereby allowing us to computing the speeds of the point vortices and closing the system of equations in terms of  $\eta$ ,  $q$ , and  $z_j$ . To do this, we repeat the arguments in [4], where it was shown that

$$\partial_y \tilde{\phi} + i \partial_x \tilde{\phi} \Big|_{z_j} = -\frac{1}{4L} \int_{-L}^L ((c(\eta, z_j) - c^*(\eta, z_j^*)) \partial_x q - i(c(\eta, z_j) + c^*(\eta, z_j^*)) G(\eta) q) dx$$

## Results

## Conclusion

## Appendix

## References

- [1] P.G. Saffman. *Vortex Dynamics*. Cambridge University Press, Cambridge, 1992.
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- [3] J.T. Beale and A. Majda. High order accurate vortex methods with explicit velocity kernels. *J. Comp. Phys.*, 58:188–208, 1985.
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- [5] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108:73–83, 1993.