Appendix

Shear Profiles over Infinitely Deep Fluids

As noted in the text, the assumption that the fluid velocity is to leading order given by

$$\mathbf{u} \approx \omega z \hat{\mathbf{i}}$$

is clearly unrealistic insofar as it leads to currents of infinite speed as one descends through the fluid. A more realistic, though also more complicated, ansatz is to suppose that to leading order we have that $\mathbf{u} \approx u(z)\hat{\mathbf{i}}$ where

$$u(z) = \begin{cases} \omega z, & -h_0 \leq z < 0 \\ \frac{-\omega h_0}{h_1 - h_0}(z + h_1), & -h_1 \leq z < -h_0 \\ 0, & z < -h_1 \end{cases}$$

where $h_1 > h_0 \gg 1$, so that we are looking at a deep, continuous shear profile which is zero at or near the surface $z = \epsilon\eta(x,t)$ and at the depth $z = -h_1$, after which the fluid is to leading order quiescent. Note, we can also see this profile as satisfying to leading order two 'no-slip' conditions, one near the free surface at $z = 0$ and one near $z = -h_1$.

While a full, nonlinear description of the above shear profile would require two more freely evolving interfaces, and thus is beyond the scope of this paper, we can readily find the dispersion relationship affiliated with this profile. This then allows us to provide some analytic argument for why we study an otherwise unphysical velocity profile in the main body of the text. Likewise, information from the dispersion relationship provides us with a better understanding of how depth varying shear profiles induce both surface and internal waves.

Following relatively classical approaches, we introduce three fluid velocities

$$\mathbf{u}_1 = \omega z \hat{\mathbf{i}} + \epsilon \nabla \phi_1, \ -h_0 + \epsilon\eta_2 < z < \epsilon\eta_1$$

$$\mathbf{u}_2 = \frac{-\omega h_0}{h_1 - h_0}(z + h_1)\hat{\mathbf{i}} + \epsilon \nabla \phi_2, \ -h_1 + \epsilon\eta_3 < z < -h_0 + \epsilon\eta_2$$

$$\mathbf{u}_3 = \epsilon \nabla \phi_3, \ z < -h_1 + \epsilon\eta_3$$

so that after linearizing around the small disturbances, we have the following system of evolution equations describing the behavior of small disturbances

$$\partial_t \phi_1 + g\eta_1 + \omega \partial_t \partial_x^{-1} \eta_1 = 0, \ z = 0$$

$$\partial_t \eta_1 = \partial_z \phi_1, \ z = 0,$$

$$\partial_t \phi_2 - \frac{\omega h_1}{h_1 - h_0} \partial_t \partial_x^{-1} \eta_2 = \partial_t \phi_1, \ z = -h_0$$

$$\partial_t \eta_2 = \partial_z \phi_2, \; z = -h_0,$$
$$\partial_t \eta_2 = \partial_z \phi_1, \; z = -h_0,$$

and

$$\partial_t \phi_3 = \partial_t \phi_2 - \frac{\omega h_0}{h_1 - h_0} \partial_t \partial_x^{-1} \eta_3, \;\; z = -h_1$$

$$\partial_t \eta_3 = \partial_z \phi_3, \; z = -h_1,$$
$$\partial_t \eta_3 = \partial_z \phi_2, \; z = -h_1.$$

Taking each function ϕ_j to be harmonic and letting $\partial_z \phi_3 \to 0$ as $z \to -\infty$ gives us the following solutions for each of the scalar velocity potentials

$$\phi_1(x, z, t) = (\alpha_{11} \cosh(kz) + \alpha_{12} \sinh(kz)) \, e^{i\theta(x,t)} + \mathrm{cc}$$

$$\phi_2(x, z, t) = (\alpha_{21} \cosh(k(z + h_0)) + \alpha_{22} \sinh(k(z + h_0))) \, e^{i\theta(x,t)} + \mathrm{cc}$$

and

$$\phi_3(x, z, t) = \alpha_3 e^{|k|(z+h_1)} e^{i\theta(x,t)} + \mathrm{cc}$$

where $\theta(x, t) = kx + \Omega(k, \omega)t$, and 'cc' denotes the complex conjugate.

This then gives us the linear system of equations

$$\begin{aligned}
\Omega \alpha_{11} &= \left(\frac{gk}{\Omega} + \omega \right) \alpha_{12} \\
\Omega \left(\alpha_{11} \cosh(kh_0) - \alpha_{12} \sinh(kh_0) \right) &= \Omega \alpha_{21} + \frac{\omega h_1}{\delta h} \alpha_{22} \\
\Omega \left(\alpha_{21} \cosh(k\delta h) - \alpha_{22} \sinh(k\delta h) \right) &= \left(\Omega - \frac{\omega h_0 s}{\delta h} \right) \alpha_3 \\
\alpha_{22} &= -\alpha_{11} \sinh(kh_0) + \alpha_{12} \cosh(kh_0) \\
\alpha_3 &= s_k \left(-\alpha_{21} \sinh(k\delta h) + \alpha_{22} \cosh(k\delta h) \right)
\end{aligned}$$

where $\delta h = h_1 - h_0$ and $s = \mathrm{sgn}(k)$. From this we derive the dispersion relationship

$$\frac{gk}{\Omega} + \omega - \Omega \tanh(kh_0) + \left(\frac{1}{\Omega} \left(\frac{gk}{\Omega} + \omega \right) \tanh(kh_0) - 1 \right) \left(\frac{\omega h_1}{\delta h} + \Omega \Omega_1(k\delta h) \right) = 0$$

where

$$\Omega_1(k\delta h) = \frac{s \left(\Omega - \frac{\omega h_0}{\delta h} s \right) + \Omega \tanh(k\delta h)}{\Omega + s \left(\Omega - \frac{\omega h_0}{\delta h} s \right) \tanh(k\delta h)}.$$

While in general we would have to find the roots of a fifth-order polynomial to determine the values of Ω, we see for $h_1 \gg h_0$ that $\Omega_1 \sim s$, and thus we get that the dispersion relationship simplifies to

$$\frac{gk}{\Omega} + \omega - \Omega \tanh(kh_0) + \left(\frac{1}{\Omega} \left(\frac{gk}{\Omega} + \omega \right) \tanh(kh_0) - 1 \right) (\tilde{\omega} + s\Omega) \sim 0,$$

where

$$\tilde{\omega} = \frac{\omega h_1}{\delta h}.$$

Since $\tilde{\omega}$ only approaches ω at an algebraic rate, it seems appropriate to keep it included in the analysis. Letting

$$|\tanh(kh_0)| = 1 - \tilde{\epsilon},$$

and noting that $\tilde{\epsilon}$ vanishes to zero exponentially fast as we increase h_0, we see the reduced dispersion relationship factors into the form

$$(2\Omega + s\tilde{\omega})\left(g\,|k| + s\omega\Omega - \Omega^2\right) + \tilde{\epsilon}\left(\Omega^3 - (s\tilde{\omega} + \Omega)(g|k| + \omega s\Omega)\right) \sim 0.$$

Thus we have a completely regular perturbation problem for the roots, which we readily see are given by

$$\Omega \sim \frac{1}{2}\left(-s\omega \pm \sqrt{\omega^2 + 4g|k|}\right), \quad -\frac{s\tilde{\omega}}{2}.$$

Thus of the three roots we find, two give the dispersion relationship we find in the body of the text using $\mathbf{u} \sim \omega z\hat{\mathbf{i}}$. Looking at the affiliated disturbances of the relevant free surface and internal wave, we find that

$$\eta_1 \sim \alpha_{12}\frac{ik}{\Omega}e^{i\theta} + \mathrm{cc}, \; \eta_2 \sim \alpha_{12}\frac{ik}{\Omega^3}\cosh(kh_0)\left(\Omega^2 - (1-\tilde{\epsilon})\,(g|k| + s\omega\Omega)\right)e^{i\theta} + \mathrm{cc}.$$

Thus if we choose Ω so that $\Omega^2 - (g|k| + s\omega\Omega) = 0$, the magnitude of the internal wave essentially vanishes, thereby localizing dynamics along the free surface near $z = 0$. Therefore, while not necessarily physically justifiable throughout the bulk of the fluid, our simplified shear profile assumption produces results which are asymptotically consistent with a more sophisticated treatment of the shear profile.

We note though that the more physically realistic shear profile shows that there is a choice of Ω which corresponds to a non-trivial internal mode near $z = -h_0$. We also note that this choice of Ω makes many of the terms in our modulation theory singular, thus showing that this case would require a markedly different treatment. While interesting, this issue is beyond the scope of the current paper and will be addressed in future research.

Higher-Order-Dispersive Corrections to the VDE

To better understand why we must include the higher-order corrections to the dispersion used in the VDE, we examine how such terms are found from first principles. To leading order from the kinetic equation over an infinitely deep fluid we have that

$$\eta_t = -\mathcal{H}Q,$$

where $q(x,t) = \phi(x, \eta(x,t), t)$, $Q = q_x$, and $\mathcal{H}$ is the Hilbert transform. Likewise, from the Bernoulli equation we have to leading order that

$$Q_t + \omega\eta_t + \eta_x = 0,$$

so that by combining the two expressions we have the wave equation in η alone given by

$$\eta_{tt} - \omega\mathcal{H}\eta_t - \mathcal{H}\eta_x = 0. \tag{1}$$

Using Fourier transforms, we can readily write a solution to the affiliated initial-value problem in the form

$$\eta(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}_0(k) e^{ikx + i\Omega_+(k)t} dk + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}_0(k) e^{ikx + i\Omega_-(k)t} dk$$

where

$$\Omega_\pm(k) = \frac{1}{2}\left(\omega s \pm \sqrt{\omega^2 + 4|k|}\right).$$

If we assume, as per usual when deriving NLS-type models, that

$$\hat{\eta}_0(k) = \frac{1}{\epsilon}\hat{A}\left(\frac{k-k_0}{\epsilon}\right),$$

representing the assumption that the initial conditions are narrowly banded around a carrier wave number k_0, then we see that, taking only the plus branch of $\Omega_\pm$, that

$$\begin{aligned}\eta(x,t) =& \frac{e^{i\theta(x,t,k_0)}}{2\pi}\int_{\mathbb{R}} \hat{A}(\tilde{k}) e^{i\tilde{k}\xi} e^{i\tau\tilde{\Omega}(k_0,\tilde{k})} d\tilde{k} \\ \sim& \frac{e^{i\theta(x,t,k_0)}}{2\pi}\int_{\mathbb{R}} \hat{A}(\tilde{k}) e^{i\tilde{k}\xi} e^{i\tau\left(\frac{\Omega''(k_0)}{2}\tilde{k}^2 + \frac{\epsilon}{6}\Omega'''(k_0)\tilde{k}^3 + \frac{\epsilon^2}{24}\Omega''''(k_0)\tilde{k}^4\right)} d\tilde{k},\end{aligned}$$

where

$$\theta(x,t,k_0) = k_0 x + \Omega(k_0)t,\ c_g(k_0) = \Omega'(k_0),\ \xi = \epsilon(x + c_g t),\ \tau = \epsilon^2 t,$$

and where

$$\tilde{\Omega}(k_0,\tilde{k}) = \frac{1}{\epsilon^2}\left(\Omega(k_0 + \epsilon\tilde{k}) - \Omega(k_0) - \Omega'(k_0)\epsilon\tilde{k}\right)$$

and where we have expanded the dispersion relationship up to $\mathcal{O}(\epsilon^3)$ as is done in the main body of the text. This expansion corresponds to the affiliated linear-evolution equation for the slowly-evolving envelope $A(\xi,\tau)$

$$i\partial_\tau A = \left(\frac{\Omega''(k_0)}{2}\partial_\xi^2 - \frac{i\epsilon}{6}\Omega'''(k_0)\partial_\xi^3 - \frac{\epsilon^2}{24}\Omega''''(k_0)\partial_\xi^4\right)A.$$

As to the issue of why to include the ϵ^2 term, we note that the affiliated group velocity of the slowly-evolving envelope is given by

$$\begin{aligned}\frac{d}{d\tilde{k}}\tilde{\Omega} =& \frac{1}{\epsilon}\left(\Omega'(k_0+\epsilon\tilde{k}) - \Omega'(k_0)\right) \\ \sim& \Omega''(k_0)\tilde{k}\left(1 - \frac{3\epsilon c_g^2}{s}\tilde{k} + 10\epsilon^2 c_g^4 \tilde{k}^2\right).\end{aligned}$$

We see that the full dispersion relationship has exactly one root at $\tilde{k} = 0$, so that this root is the only point of stationary phase in the corresponding integral representation of the solution $\eta(x,t)$. However, using our Taylor series expansion, were we to ignore the ϵ^2 term, we would introduce a new stationary-phase point at $\tilde{k}_*$ where

$$\tilde{k}_* = \frac{s}{3\epsilon c_g^2}.$$

The presence of this stationary-phase point allows for the accumulation of energy at high-frequencies, which without a corresponding band-limiting requirement introduces non-physical effects into the model. By including the ϵ^2 term though, we remove this spurious high-frequency stationary-phase point, thereby avoiding the inclusion of an otherwise unnecessary bandwidth condition on the slowly-evolving envelope.