Evolution of Narrow-Band Spectra in Deep-Water Constant Vorticity Flows

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Introduction

To describe the evolution of a nearly linear wave-train in a constant-vorticity-shear current running over infinitely deep water, we use the form of the nonlinear Schrödinger equation derived in [1], though see also [2], given by

$$i\partial_\tau \eta_1 + \alpha_d \partial_\xi^2 \eta_1 + \alpha_{nl} \left|\eta_1\right|^2 \eta_1 = 0,$$

where $\tau = \epsilon^2 t$, $\xi = \epsilon(x + c_g t)$, and

$$c_g = \frac{1 + 3\tilde{\sigma}k_0^2}{2s\Omega - \omega}, \; s = \text{sgn}(k_0),$$

$$\begin{aligned}
\alpha_d(k_0, \omega) =& \frac{(c_g^2 - 3|k_0|\tilde{\sigma})}{2\Omega - s\omega}, \\
\alpha_{nl}(k_0, \omega) =& \frac{k_0 \left(sk_0^3 \left(8 + \tilde{\sigma}k_0^2 + 2(\tilde{\sigma}k_0^2)^2\right) + \omega\alpha_v\right)}{(2s\Omega - \omega)\left(1 + c_g\omega\right)\left(4\Omega^2 - s(2k_0(1 + 4\tilde{\sigma}k_0^2) + 2\omega\Omega)\right)},
\end{aligned}$$

where Ω denotes the dispersion relationship and ω is the non-dimensionalized magnitude of the vorticity of the flow, and $\tilde{\sigma}$ is the surface tension. In a now seminal paper, [3] studies families of profiles with relatively narrow banded spectra around k_0 and modulational instability (MI) is either manifested or supressed. The results in [3] were confirmed numerically in [4, 5] by examing the mean properties associated with ensembles of initial conditions. This was done by starting from the $2L$ periodic-initial condition

$$\eta_1(\xi, 0) = \epsilon_{rms}\sqrt{\frac{\delta\tilde{k}}{\sigma\sqrt{\pi}}} \sum_{k=-K+1}^{K} e^{-\tilde{k}^2/2\sigma^2} e^{i\theta_k} e^{i\tilde{k}\xi}, \; \tilde{k} = \frac{\pi k}{L}$$

where $\delta\tilde{k} = \pi/L$ and the phases θ_k are randomly chosen uniformly between 0 and 2π. We can then readily show that, for $L \gg 1$,

$$\epsilon_{rms} \approx \overline{\left|\eta_1\right|^2}^{1/2},$$

where $\overline{()}$ denotes the ensemble average. In our coordinates then, the work in [3] and later confirmed in [4] shows that MI is suppressed for spectral widths σ such that

$$\sigma \geq \epsilon_{rms}\sqrt{\frac{2\alpha_{nl}}{\alpha_d}}$$

However, all of these results are for the case of zero vorticity and surface tension.

Thus, in this paper we establish the validity of the above bound on the spectral width for the appearance or suppression of MI. We likewise examine the impact that vorticity plays in controlling the effective width for the onset of MI. Moving beyond just examining the properties of the NLS equation with vorticity, we look at the statistical properties of solutions to a higher-order model, the vor-Dysthe equation derived in [1].

References

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