SuperCDMS Cross Channel Correlated Noise Analysis

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Abstract

The goal of this analysis is to separate the uncorrelated noise from the cross channel correlated noise. The uncorrelated noise can then be used to better estimate the true energy resolution of the detector.

1 Method

Let us assume that the measured signal on any channel can be written as a standard linear model,

$$y_i(t) = \mathrm{A}_i n^c(t) + n_i^{\mathrm{r}}(t), \tag{1}$$

Where $y(t)$ is the measured current as a function of time, $n^{\mathrm{r}}(t)$ is the random intrinsic noise as a function of time, $n_c(t)$ is the correlated noise, A is the weighting coefficient, and the i subscript is the channel index.
Let us work in frequency space by taking the Fourier transform of Eq. 1.

$$Y_{Ti}(\omega) = \mathrm{A}_i n_T^c(\omega) + n_{Ti}^{\mathrm{r}}(\omega). \tag{2}$$

Let us assume that A is complex, and calculate the cross correlation,

$$S_{ij}(\omega) = \lim_{T\to\infty} \frac{2}{T} \left\langle Y_{Ti}(\omega) Y_{Tj}^*(\omega) \right\rangle. \tag{3}$$

$$S_{ij}(\omega) = \lim_{T\to\infty} \frac{2}{T} \left\langle \left[\mathrm{A}_i n_T^c(\omega) + n_{Ti}^{\mathrm{r}}(\omega)\right] \left[\mathrm{A}_j n_T^c(\omega) + n_{Tj}^{\mathrm{r}}(\omega)\right]^* \right\rangle, \tag{4}$$

$$S_{ij}(\omega) = \lim_{T\to\infty} \frac{2}{T} \left\langle \mathrm{A}_i \mathrm{A}_j^* |n_T^c(\omega)|^2 + n_{Ti}^{\mathrm{r}}(\omega) \left(n_{Tj}^{\mathrm{r}}(\omega)\right)^* + n_{Ti}^{\mathrm{r}} \left(\mathrm{A}_j n_T^c(\omega)\right)^* + \mathrm{A}_i n_T^c(\omega) \left(n_{Tj}^{\mathrm{r}}(\omega)\right)^* \right\rangle, \tag{5}$$

$$\begin{aligned} S_{ij}(\omega) = \lim_{T\to\infty} \frac{2}{T} \Big[&\left\langle \mathrm{A}_i \mathrm{A}_j^* |n_T^c(\omega)|^2 \right\rangle + \left\langle n_{Ti}^{\mathrm{r}}(\omega) \left(n_{Tj}^{\mathrm{r}}(\omega)\right)^* \right\rangle \\ &+ \left\langle n_{Ti}^{\mathrm{r}} \left(\mathrm{A}_j n_T^c(\omega)\right)^* \right\rangle + \left\langle \mathrm{A}_i n_T^c(\omega) \left(n_{Tj}^{\mathrm{r}}(\omega)\right)^* \right\rangle \Big]. \end{aligned} \tag{6}$$

By definition there is no correlation between the cross channel correlated noise and the intrinsic noise on each channel, thus we claim that

$$\left\langle n_T^c(\omega) \left(n_{Tj}^{\mathrm{r}}(\omega)\right)^* \right\rangle = \left\langle n_{Ti}^{\mathrm{int}} \left(n_T^c(\omega)\right)^* \right\rangle = 0. \tag{7}$$

Also, the covariance between the the intrinsic noise for any two channels should be

$$\lim_{T\to\infty} \frac{1}{T} \left\langle n_{Ti}^{\mathrm{int}}(\omega) \left(n_{Tj}^{\mathrm{int}}(\omega)\right)^* \right\rangle = \left(\sigma_i^{\mathrm{int}}\right)^2 \delta_{ij}, \tag{8}$$

Where σ_i^2 is the variance on the i^{th} channel. Similarly, by definition,

$$\lim_{T\to\infty} \frac{1}{T} \left\langle |n_T^c(\omega)|^2 \right\rangle = \sigma_c^2. \tag{9}$$

Using these definitions the cross correlation becomes,

$$S_{ij}(\omega) = 2\mathrm{A}_i\mathrm{A}_j^*\sigma_c^2 + 2\left(\sigma_i^{\mathrm{int}}\right)^2\delta_{ij}. \tag{10}$$

Since we are letting A be complex, we write $\mathrm{A} = \alpha + i\beta$.

$$S_{ij}(\omega) = 2(\alpha_i + i\beta_i)(\alpha_j - i\beta_j)\sigma_c^2 + 2\left(\sigma_i^{\mathrm{int}}\right)^2\delta_{ij}, \tag{11}$$

$$S_{ij}(\omega) = 2(\alpha_i\alpha_j + \beta_i\beta_j + i(\beta_i\alpha_j - \alpha_i\beta_j))\sigma_c^2 + 2\left(\sigma_i^{\mathrm{int}}\right)^2\delta_{ij}. \tag{12}$$

We can look at the Real and Imaginary parts of the cross correlation,

$$\boxed{\begin{array}{l} \mathrm{Re}\left(S_{ij}(\omega)\right) = 2(\alpha_i\alpha_j + \beta_i\beta_j)\sigma_c^2 + 2\left(\sigma_i^{\mathrm{int}}\right)^2\delta_{ij} \\ \mathrm{Im}\left(S_{ij}(\omega)\right) = 2(\beta_i\alpha_j - \alpha_i\beta_j)\sigma_c^2 \end{array}}\,. \tag{13}$$

For a detector with n channels, we will have n^2 different equations. We have 3 parameters to fit per channel, plus one additional parameter for σ_c^2, so $3n+1$ DOF. However, since we can only measure the relative phase of the real and imaginary parts of A, we fix the phase of one of the terms by setting one of the imaginary components to zero. This reduces the number of DOF to 3n. Therefor we must have at least 3 channels on a device to use this analysis.
Without loss of generality, we continue this analysis with $n = 4$
Since we have 4 channels, this means 16 equations and 12 unknowns.
We can construct a matrix of S_{ij}'s,

$$\bar{\mathbf{S}} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{S}_{13} & \mathbf{S}_{14} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{S}_{23} & \mathbf{S}_{24} \\ \mathbf{S}_{31} & \mathbf{S}_{32} & \mathbf{S}_{33} & \mathbf{S}_{34} \\ \mathbf{S}_{41} & \mathbf{S}_{42} & \mathbf{S}_{43} & \mathbf{S}_{44} \end{pmatrix} \tag{14}$$

Since the elements of this matrix are symmetric under exchange of index, we can define 4^2 unique equations by making the lower triangle real and the upper triangle imaginary (and discarding the duplicate data).

$$\bar{\mathbf{S}} = \begin{pmatrix} \mathbf{S}_{11} & \mathrm{Im}\,\mathbf{S}_{12} & \mathrm{Im}\,\mathbf{S}_{13} & \mathrm{Im}\,\mathbf{S}_{14} \\ \mathrm{Re}\,\mathbf{S}_{21} & \mathbf{S}_{22} & \mathrm{Im}\,\mathbf{S}_{23} & \mathrm{Im}\,\mathbf{S}_{24} \\ \mathrm{Re}\,\mathbf{S}_{31} & \mathrm{Re}\,\mathbf{S}_{32} & \mathbf{S}_{33} & \mathrm{Im}\,\mathbf{S}_{34} \\ \mathrm{Re}\,\mathbf{S}_{41} & \mathrm{Re}\,\mathbf{S}_{42} & \mathrm{Re}\,\mathbf{S}_{43} & \mathbf{S}_{44} \end{pmatrix}, \tag{15}$$

where it is understood that the diagonal components will inherently be real.
We can estimate the cross spectral density using the `scipy.signal.csd()` function, which estimates the CSD using the Welch method.
We have N measurements of for each value, so each member of $\bar{\mathbf{S}}$ is a vector of length N. We estimate the CSD of each N elements, calculate the mean and the variance of the CSD's.
We then construct a χ^2 statistic of the form,

$$\chi^2 = [\mathbf{S}_{ij} - \langle\mathbf{S}_{ij}\rangle]^\top \mathbf{C}_{ijk\ell}^{-1}[\mathbf{S}_{k\ell} - \langle\mathbf{S}_{k\ell}\rangle], \tag{16}$$

If we assume there is no autocorrelation within a single channel as a function of frequency, this equation simplifies and we can write the covariance matrix for the $\bar{\mathbf{S}}$ matrix as

$$\mathbf{C}^{-1} = \frac{1}{\sigma_i^2\sigma_j^2}\mathbf{I}. \tag{17}$$

where the $\sigma_{i/j}^2$ are estimated by the variance of the CSD's for the N traces.
Since we assert there is no frequency dependence, we can construct a system of coupled equations at each frequency bin and minimize the χ^2 at each frequency using the `scipy.optimize.least_squares()` function, which is a non-linear least squares solver using the Levenberg-Marquardt method.