Module 2

Probability Distributions

Spring 2026

Learning goals

- ❑ Review key probability distributions, activities focus on Gaussian PDFs in multiple dimensions.
  - ❑ Discrete probability distributions such as Bernoulli or Binomial, or continuous PDFs such as Student-t, are discussed in chapter 2. They will not be used in the current set of activities, but are relevant to future work.
- ❑ Reflect on why the Gaussian Distribution is so widely used to model the distribution of continuous variables. We will introduce PDF representing correlated variables

Why probability distributions matter

- ❑Measurements are noisy
- ❑Models are uncertain
- ❑Predictions require uncertainty

Probability as uncertainty, not randomness

- Describes incomplete information
- Applies to:
  - experiments
  - Parameters
  - predictions

- Probability is bookkeeping for uncertainty!

Discrete Probability

Example: coin toss

$$p(x = \text{heads}|\mu) = \mu$$
$$p(x = \text{tails}|\mu) = 1 - \mu$$

Described by Bernoulli distribution

$$\text{Bern}(x|\mu) = \mu^x (1 - \mu)^{1-x}$$

$$\sum_i p(x_i) = 1$$

Binomial Distribution

- How many times would we expect m heads in N coin flips?

$$\mathrm{B}in(m|N,\mu) = \binom{N}{m} \mu^{m} (1-\mu)^{N-m}$$

$$\binom{N}{m} = \frac{N!}{(N-m)!m!}$$

-> number of ways of choosing m objects out of N identical objects

From probabilities to densities

- What if parameters are continuous? e.g. voltage, position, etc
- Probability of *exact* value is *zero*
- We must define a probability *density* instead

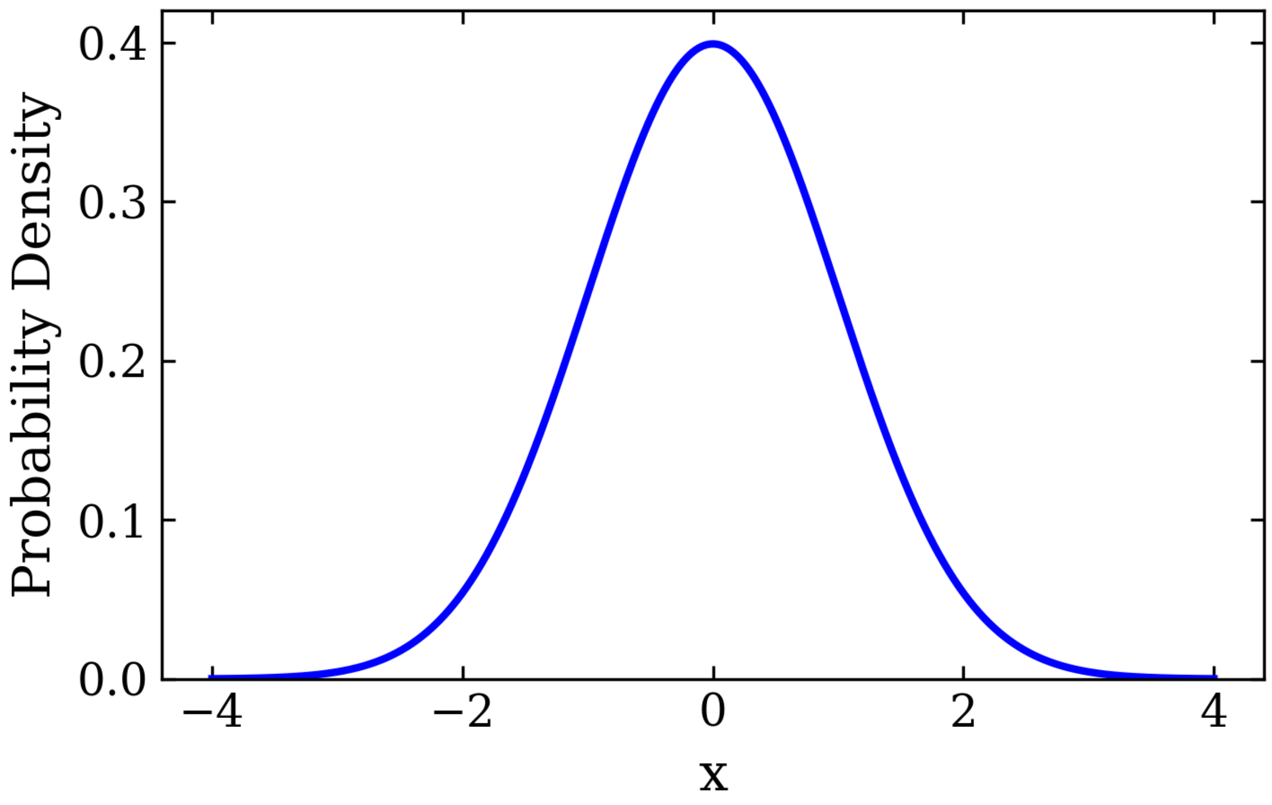
- Think about probability not at fixed point, but over an interval

Probability density function (PDF)

Probability density function

$$p(x) \geq 0$$

$$\int_{-\infty}^{\infty} p(x) dx = 1$$



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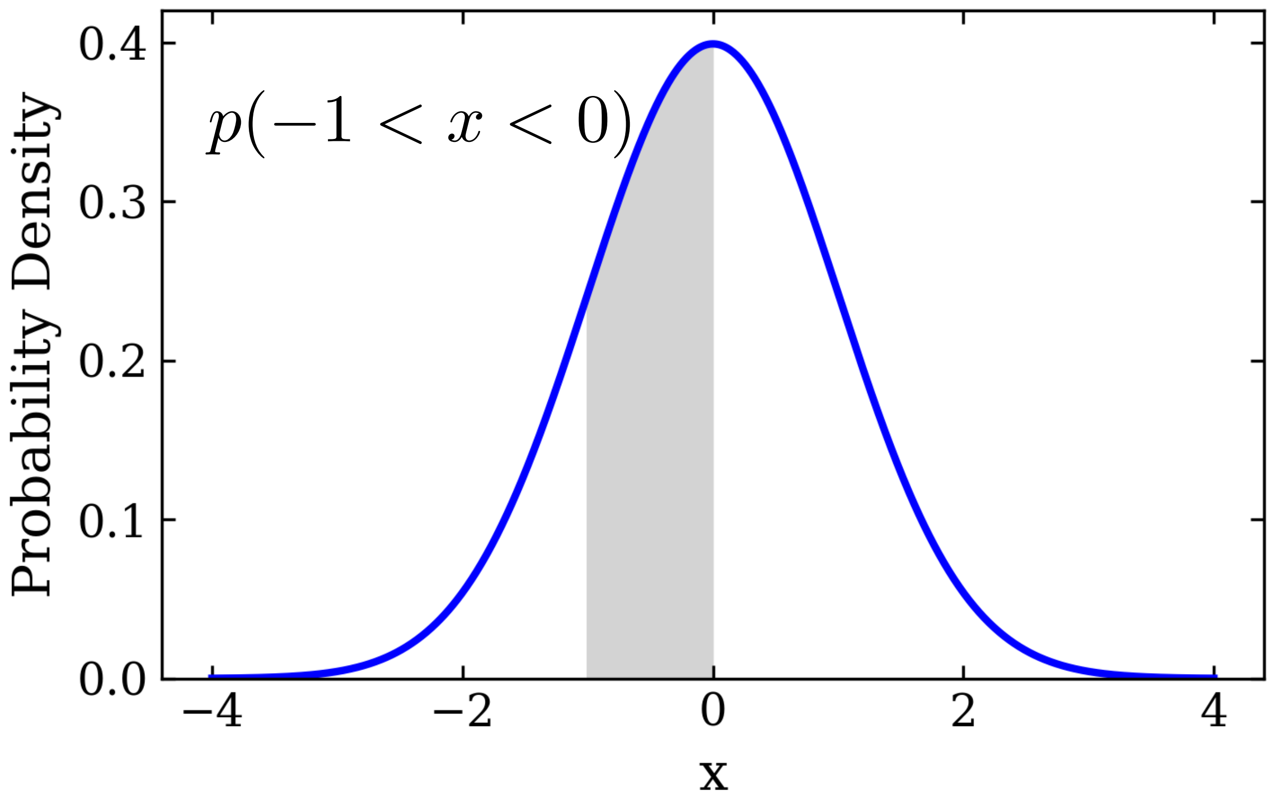
$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Area = probability

Hight = density (NOT probability)



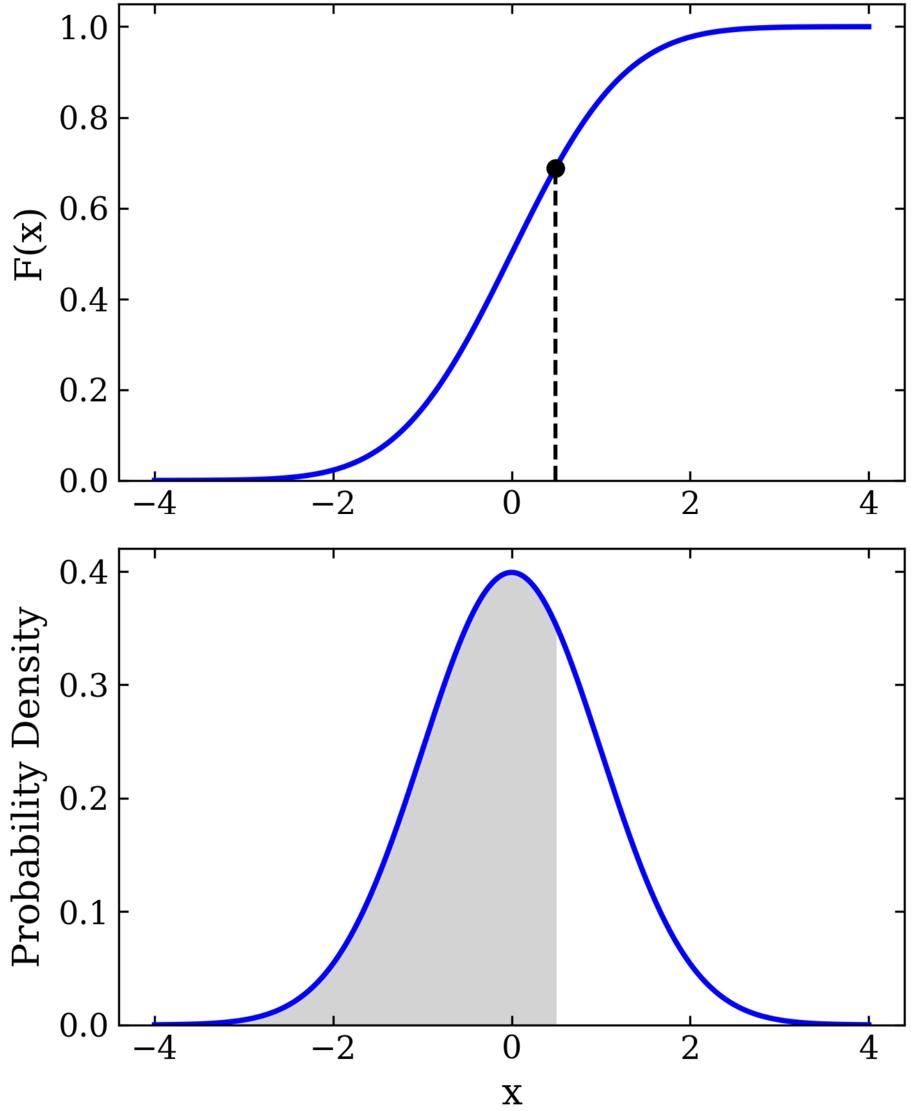
Cumulative distribution function (CDF)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(x')dx'$$

- ❑ Monotonic function describing total of probabilities

- ❑ Useful for calculating probability within an interval, hypothesis testing, etc

$$P(a < X \leq b) = F(b) - F(a)$$



Expectation values

- <u>Expectation value</u>: weighted average of a function $f(x)$ under probability distribution $p(x)$

Discrete: $$\mathbb{E}[f] = \sum_x p(x) f(x)$$

Continuous: $$\mathbb{E}[f] = \int p(x) f(x) \mathrm{d}x$$

Variance

- Variance: measure of how much variability there is in $f(x)$ around its mean value $\mathbb{E}[f]$

$$\mathrm{var}[f] = \mathbb{E}[(f(x) - \mathbb{E}[f(x)])^2]$$

$$\mathrm{var}[f] = \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$

Covariance

- Covariance quantifies the extent in which the variance of multiple variables is related

$$\mathrm{cov}[X_i, X_j] = \mathbb{E}[\{X_i - \mathbb{E}[X_i]\}\{X_j - \mathbb{E}[X_j]\}]$$

- Variance can be thought of as a specific case of the covariance matrix

$$\mathrm{cov}[\mathbf{X}] = \begin{bmatrix} \mathrm{var}(X_i) & \mathrm{cov}(X_i, X_j) \\ \mathrm{cov}(X_j, X_i) & \mathrm{var}(X_j) \end{bmatrix}$$

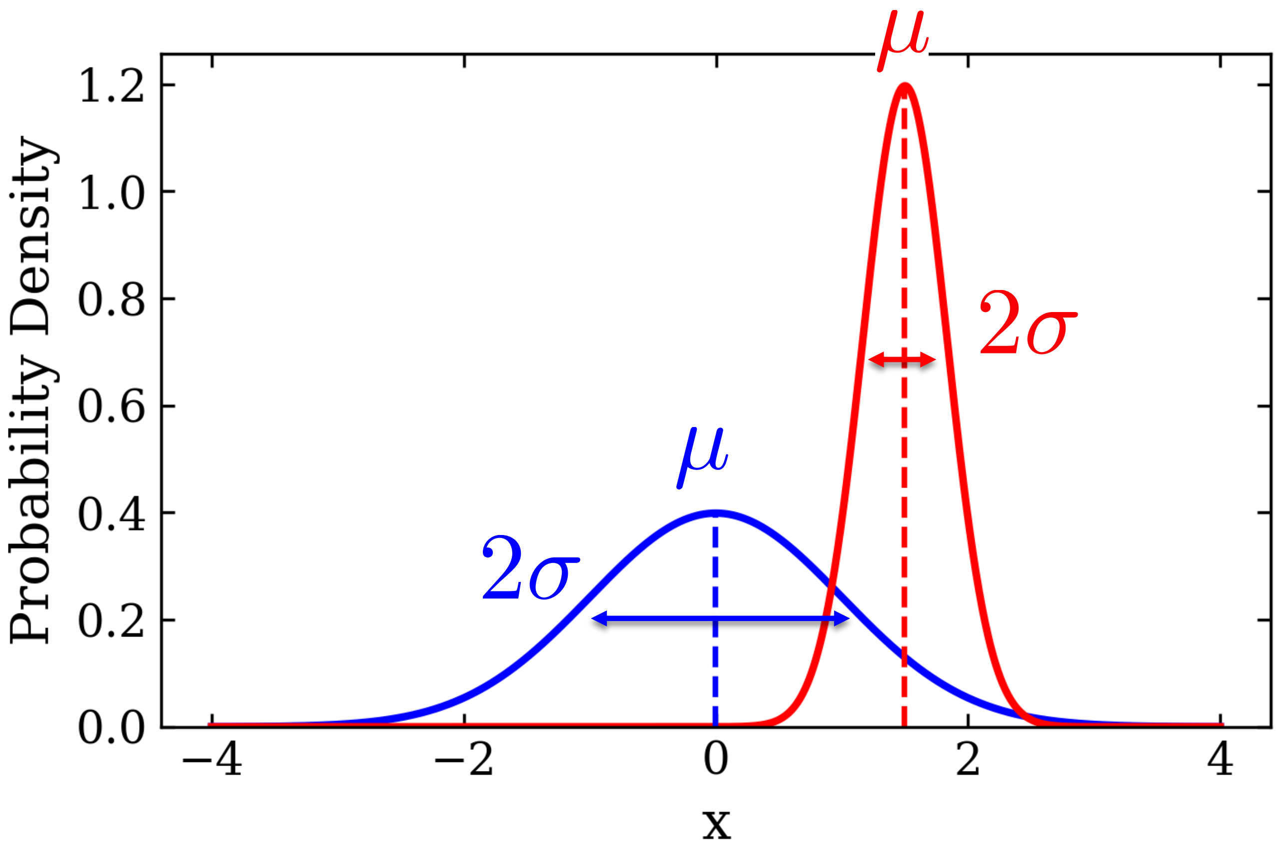
Normal (Gaussian) Distribution

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- ❑ Parameterized by μ and σ^2
- ❑ μ: center of distribution
- ❑ σ: width of distribution

$$\mu = \mathbb{E}[x]$$

$$\sigma^2 = \mathrm{var}[x]$$



Why do we care about Gaussians?

- ❑ Gaussian distributions occur everywhere in physics
  - Thermal noise
  - Electronics noise
  - Measurement errors
  - etc…
- ❑ But why?

Central Limit Theorem

- **The scaled sum of a sequence of i.i.d. random variables with finite mean and variance converges in distribution to the normal distribution.**

Let $x_1, \ldots, x_N$ be independent random variables with finite mean μ and variance σ^2. Then the distribution of the sample mean:

$$\bar{x} = \frac{1}{N}\sum_i x_i$$

approaches a Gaussian distribution as $N \to \infty$, <u>regardless of the original distribution</u>.

Central Limit Theorem

Assume you have N independent measurements $x_1, \ldots, x_N$ drawn from a distribution with

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Define the sample mean: $$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Calculate the variance of the mean

$$\mathrm{Var}(\bar{x}) = \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) = \frac{1}{N^2}\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right)$$

Central Limit Theorem

$$\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right) = \sum_{i=1}^{N} \mathrm{Var}(x_i) + 2\sum_{i<j} \mathrm{Cov}(x_i, x_j)$$

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If the x_i are independent, $\mathrm{Cov}\left(x_i, x_j\right) = 0$ $for\ i \neq j$. Then:

$$\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right) = \sum_{i=1}^{N} \sigma^2 = N\sigma^2 \longrightarrow \mathrm{Var}(\bar{x}) = \frac{1}{N^2}(N\sigma^2) = \frac{\sigma^2}{N}$$

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Averaging N independent noisy measurements cancels fluctuations, so the noise amplitude shrinks like $\sqrt{N}$

$$\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}}$$

Central Limit Theorem

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However, most *measurements* are

Central Limit Theorem

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For a given sample of noise V_{noise} in an electrical system, you are really measuring the *sum* of many physical processes

$$V_{\mathrm{noise}} = \delta V_1 + \delta V_2 + \delta V_3 + \cdots$$

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Gaussian random variable

- thermal motion of many electrons
- microscopic scattering
- phonons
- shot noise contributions
- …. Not necessarily gaussian

You will see this in the coding activity

Lecture 2

What about sums of Gaussians?

What if we want to add multiple Gaussian variables?

E.g. $Z = X + Y$

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$$\boxed{\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2}$$

Random Event Processes

Not all randomness comes from noise

- Many measurements involve discrete random events

Examples:

- radioactive decays
- photon detections
- dark counts in detectors
- cosmic ray hits

We want a statistical model for:

- when events occur
- how many events occur

Constant Rate Assumption

- Let us assume events occur independently
- And assume constant average rate λ

  λ = average events per unit time
- In a small time interval dt: $P(\text{event in } dt) = \lambda\, dt$

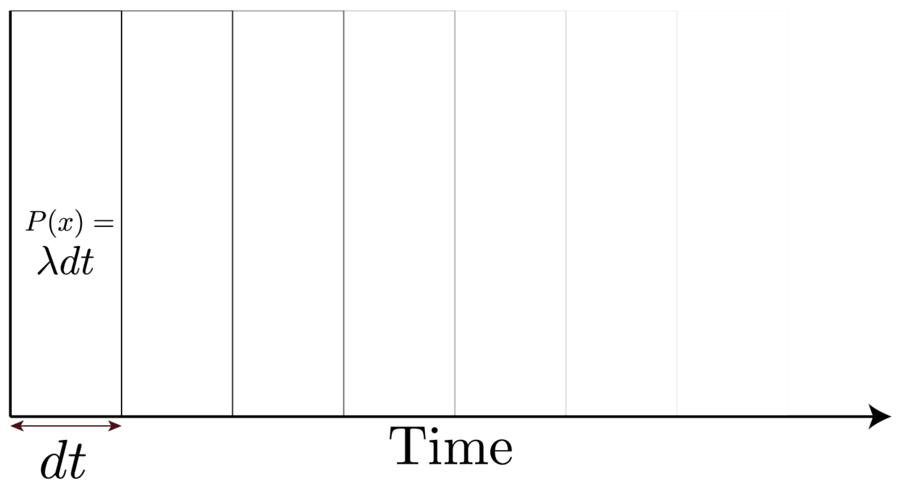
- This is the defining assumption of a ***Poisson process***

  → No memory, no aging, no buildup

We will motivate this as follows:

Discrete Time Approximation

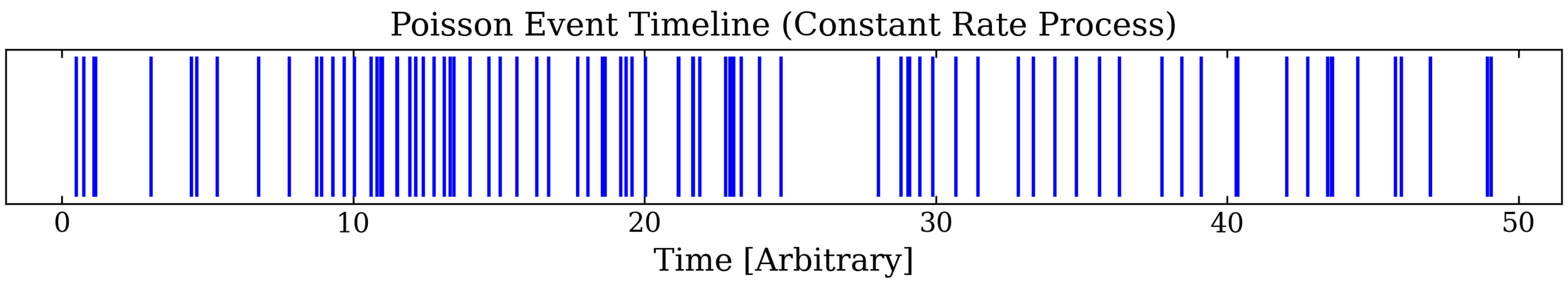
- Let us divide time into small bins dt
- In each bin event occurs with probability λdt



- Each bin is a Bernoulli trial

Simulated Event Timeline

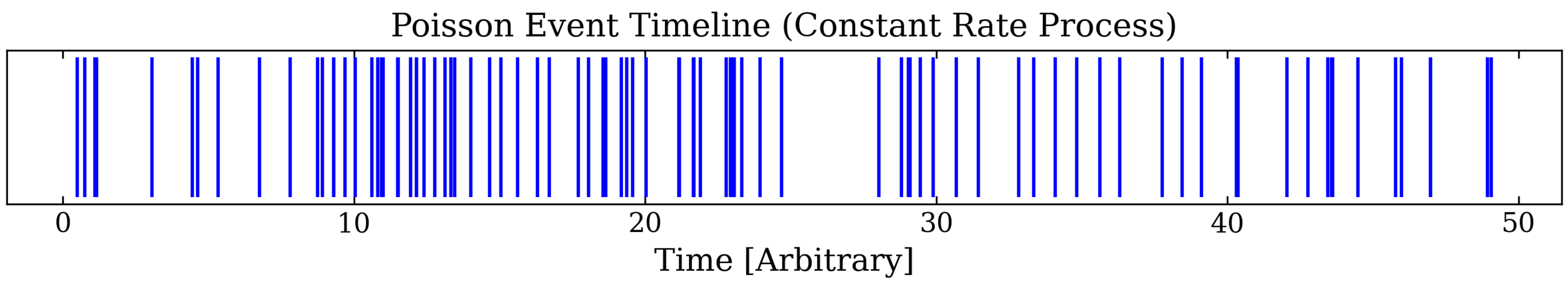
- For every bin dt we sample u from a random uniform distribution (0,1)
- If $u > \lambda\, dt$ then we say an ‘event’ occurred → repeat for all dt



Each line = detected event

- Arrival times are irregular and unpredictable -> average density is constant

Counting Events in Fixed Windows



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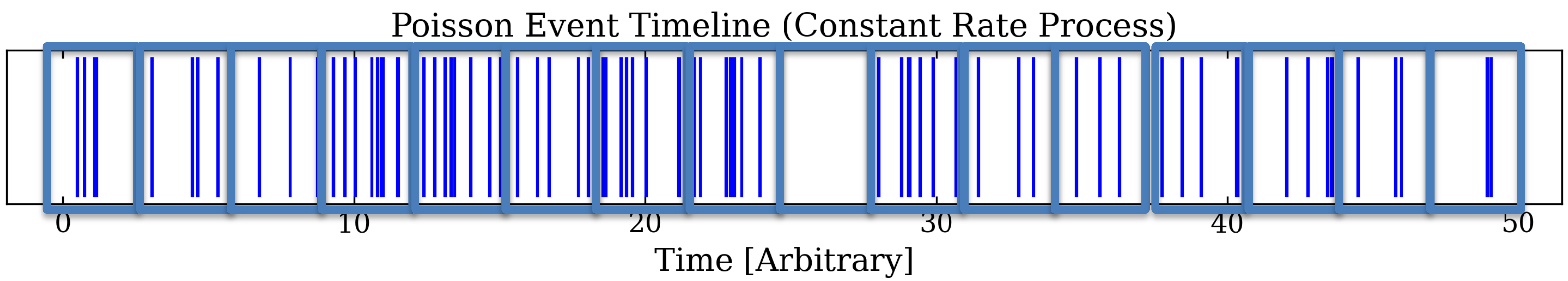

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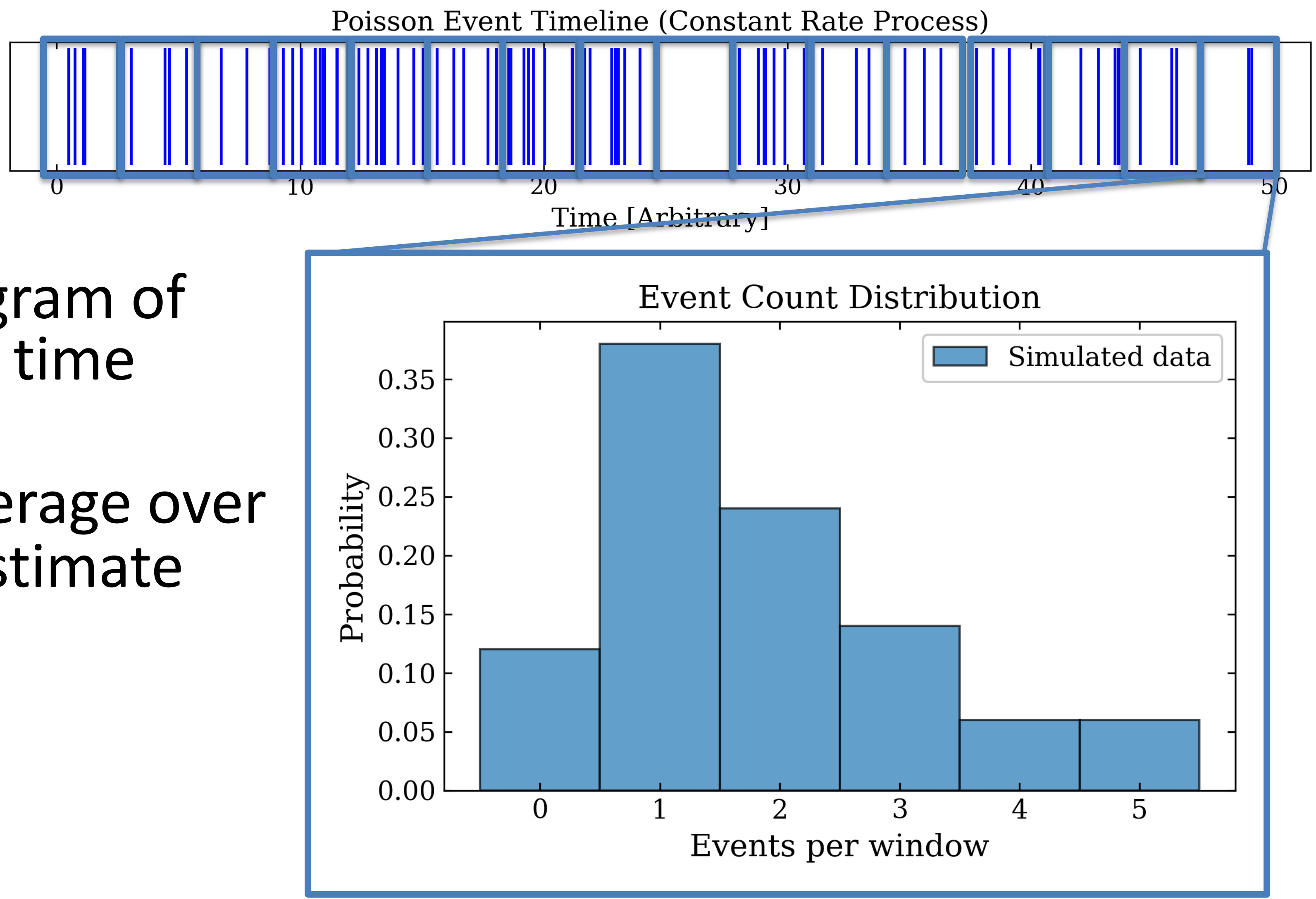

- Then build histogram of event counts per time window
- -> repeat and average over all windows to estimate probability

What describes this process?

- Recall, over short interval Δt:
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- Probability of one event in a bin is now:

$$p(1) = \lambda \Delta t = \frac{\lambda T}{N}$$

Binomial Selection

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$\rightarrow$ use the binomial distribution! $P_N(k) = \binom{N}{k} p^k (1-p)^{N-k}$

Substitute in $p = \frac{\lambda T}{N}$

$$P_N(k) = \binom{N}{k} \left(\frac{\lambda T}{N}\right)^k \left(1 - \frac{\lambda T}{N}\right)^{N-k}$$

Large N limit

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- Binomial coefficient → $$\binom{N}{k} \approx \frac{N^k}{k!}$$

- Power term → $$\left(\frac{\lambda T}{N}\right)^k = \frac{(\lambda T)^k}{N^k}$$

- Exponential term → $$\lim_{N\to\infty}\left(1 - \frac{\lambda T}{N}\right)^N = e^{-\lambda T}$$

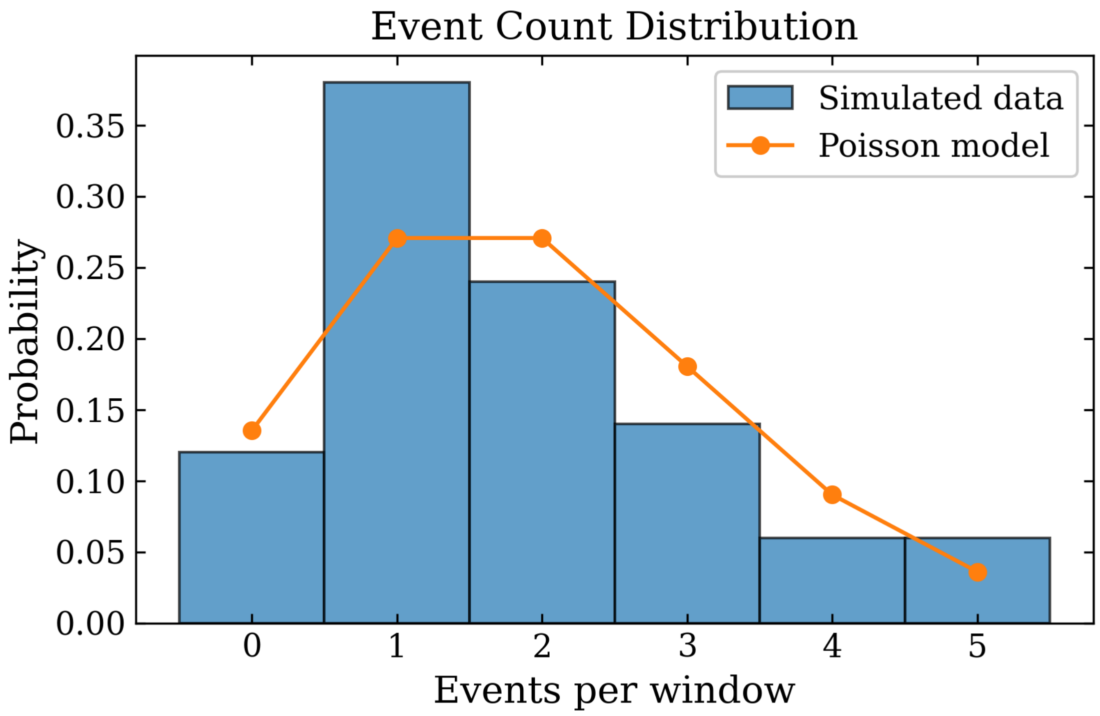
Large N limit

Let $\mu = \lambda T$

Arrive at the Poisson distribution:

$$P(k \mid \mu) = \frac{\mu^k}{k!} e^{-\mu}$$

$$\mathbb{E}(k) = \mathrm{var}(k) = \mu$$



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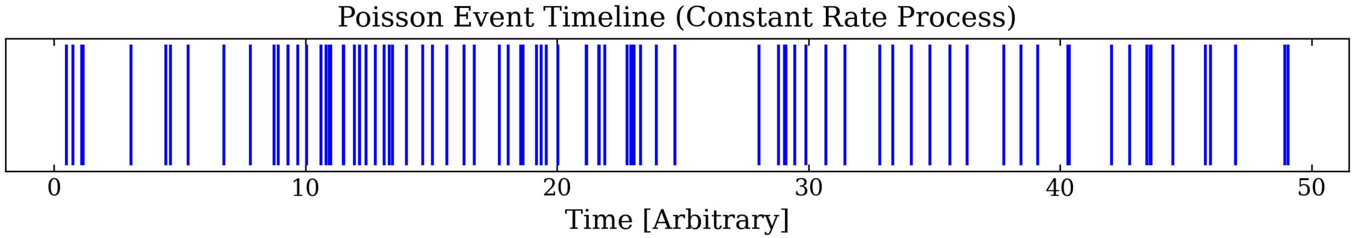
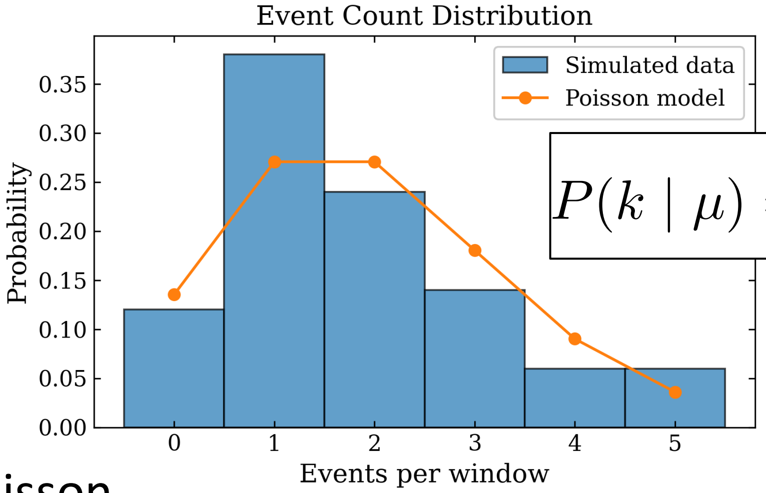
- $F(t) = p(t_1 \leq t) = 1 - p(t_1 > t) = 1 - e^{-\lambda t}$
- PDF is thus:

$$p(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t}$$

Poisson and Exponential Distribution





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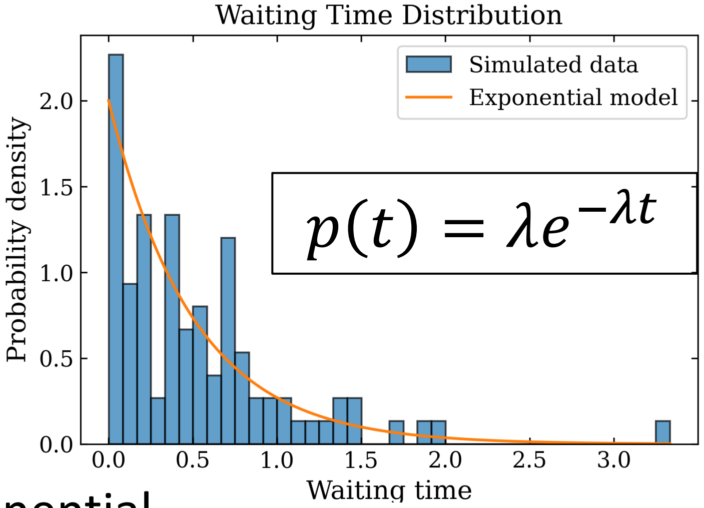


$$p(t) = \lambda e^{-\lambda t}$$

Poisson

- Discrete distribution
- "**how many** events in fixed time"

$$\mathbb{E}(k) = \mathrm{var}(k) = \mu$$

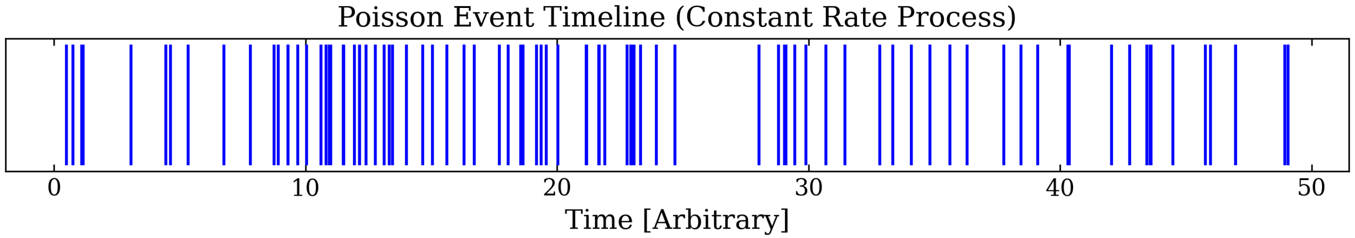
Exponential

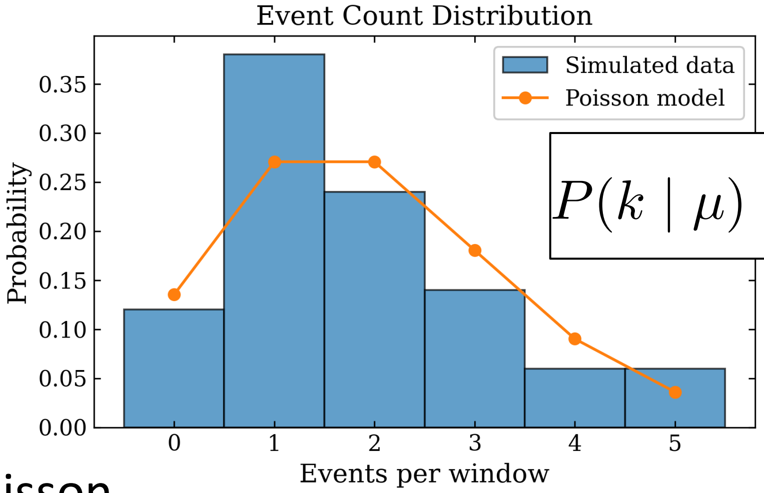
- Continuous distribution
- **"when** the next event happens"

$$\mathbb{E}(t) = \frac{1}{\lambda}$$

$$\mathrm{var}(t) = \frac{1}{\lambda^2}$$

Poisson and Exponential Distribution





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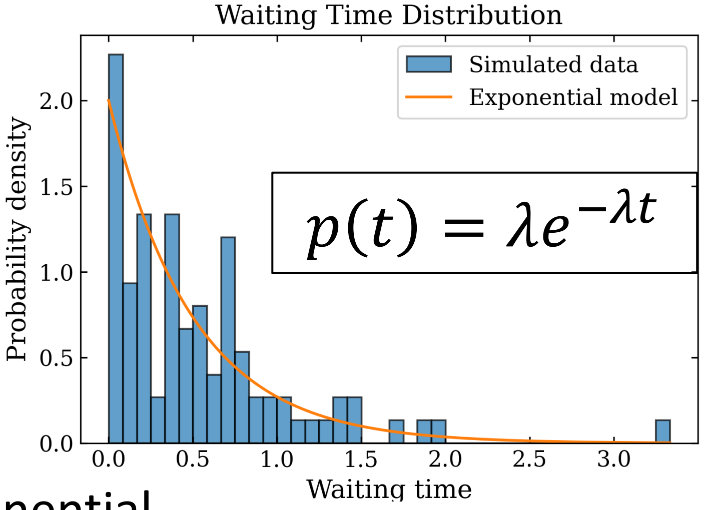

$$p(t) = \lambda e^{-\lambda t}$$

Poisson

- Discrete distribution
- "**how many** events in fixed time"

$$\boxed{\mathbb{E}(k) = \mathrm{var}(k) = \mu}$$

Big deal!

Exponential

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- "**when** the next event happens"

$$\mathbb{E}(t) = \frac{1}{\lambda}$$

$$\mathrm{var}(t) = \frac{1}{\lambda^2}$$

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- $\mu = \lambda T \gg 1$ means:
  - High event rate
  - Long observation period
- →accumulating many statistically independent event opportunities
- Total observed events is a *sum*

$$\boxed{\mathrm{Poisson}(\mu)_{\mu\to\infty} \to \mathcal{N}(\mu, \mu)}$$

Large μ limit

$$\boxed{\text{Poisson}(\mu)_{\mu \to \cancel{\infty}} \to \mathcal{N}(\mu, \mu)}$$

> 10

Now we get the best of both words

- Can take advantage of techniques for gaussian estimation
- **Mean and variance however still describe underlying Poisson physics!!**