Rough volatility of Bitcoin

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Abstract

Recent studies have found that the log-volatility of asset returns exhibit roughness. This study investigates roughness or the anti-persistence of Bitcoin volatility. Using the multifractal detrended fluctuation analysis, we obtain the generalized Hurst exponent of the log-volatility increments and find that the generalized Hurst exponent is less than $1/2$, which indicates log-volatility increments that are rough. Furthermore, we find that the generalized Hurst exponent is not constant. This observation indicates that the log-volatility has multifractal property. Using shuffled time series of the log-volatility increments, we infer that the source of multifractality partly comes from the distributional property.

Keywords: Rough volatility, Bitcoin, Hurst exponent, Multifractality
JEL classification: G10, G14

1. Introduction

Studies have intensively examined the statistical properties of asset prices and confirmed the existence of universal properties across various asset returns. These properties are now classified as "stylized facts," which include (i)fat-tailed distributions, (ii)volatility clustering (iii) slow decay of autocorrelation in absolute returns, and so on, see for example, Cont (2001). The stylized fact (iii) also characterizes volatility to be long memory, and more generally, the power transformed absolute returns $|r_t|^d$ have high autocorrelation for long lags(Taylor, 1986). The power d, which gives the highest autocorrelation, is dependent on assets, and the autocorrelation is highest for stocks when d is around 1(Ding et al., 1993). For other assets, see, Granger and Ding (1995); Ding and Granger (1996); Dacorogna et al. (2001); Takaishi and Adachi (2018).

It is important to model volatility with these properties to estimate or forecast an accurate volatility value, such as option pricing, and risk management of assets. The most successful volatility models might be the autoregressive conditional heteroscedasticity(ARCH)(Engle, 1982) and generalized ARCH (GARCH)(Bollerslev, 1986) models, which are often used in empirical analysis.

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However, they fail to capture the property of long memory in volatility. To incorporate long memory in volatility, studies have proposed several models, such as long memory stochastic volatility(Breidt et al., 1998; Harvey, 2007), fractionally integrated ARCH(Baillie et al., 1996), and fractional stochastic volatility (FSV)Comte and Renault (1998). The FSV model uses fractional Brownian motion with the Hurst parameter greater than $1/2$, which ensures long memory.

Recently, Gatheral et al. (2018) analyzed log-volatility using the realized volatility(RV) as a proxy of true volatility and claimed that the time series of the log-volatility increments for stock and bond prices show rough behavior, that is, the Hurst exponent is smaller than $1/2$. They also claim that the time series shows monofractal behavior. From these empirical observations and the requirement of a small Hurst exponent for at-the-money skewFukasawa (2011), they considered the log-volatility model by a fractional Brownian motion with $H < 1/2$, which is a variant of the FSV model and referred to as rough fractional stochastic volatility(RFSV) models. Further empirical studies confirmed the roughness of the log-volatility for thousands of stocks(Bennedsen et al., 2016) and implied volatility(Livieri et al., 2018).

This study aims to provide further evidence of roughness of log-volatility in Bitcoin. Many studies have investigated the statistical properties of Bitcoin, showing that stylized facts are also present in Bitcoin returns, see for example, Bariviera et al. (2017); Chu et al. (2015); Takaishi (2018). In this study, we use the multifractal detrended fluctuation analysis (MF-DFA) (Kantelhardt et al., 2002) to calculate the generalized Hurst exponent $h(q)$ of the log-volatility increments. In the MF-DFA, $h(q)$ is obtained from the exponent of qth order fluctuation function. Gatheral et al. (2018) calculate $h(q)$ from the qth order structure function(SF) in a range of $q = (0, 3]$ and find that $h(q)$ is constant, indicating that the time series is monofractal. Here, we calculate $h(q)$ in a wide range of $q = [-25, 25]$ using the MF-DFA and investigate whether $h(q)$ is independent of q. In fact, we find evidence that $h(q)$ varies with q, which shows the multifractal nature of the log-volatility increments.

Section 2 in this letter describes the data and methodology, while Section 3 presents the empirical results and Section 4 concludes.

2. Data and Methodology

In this study, we use Bitcoin Tick data (in dollars) traded on COINBASE from January 28, 2015 to January 6, 2019 and downloaded from Bitcoincharts[1]. These data are used to construct the RV(Andersen and Bollerslev, 1998; Barndorff-Nielsen and Shephard, 2001; McAleer and Medeiros, 2008) and we use the RV as a proxy of volatility. Let $p_{t_n}; t_n = n\delta t; n = 0, 1, .., M$ be the nth Bitcoin prices with sampling period δt on day t, where $M = 1440\text{min}/\delta t\text{min}$. We define the return r_{t,t_j} by the logarithmic price difference, namely,

$$r_{t,t_j} = \log p_{t,t_j} - \log p_{t,t_{j-1}}. \tag{1}$$

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[1]http://api.bitcoincharts.com/v1/csv/

The daily RV on day t with sampling period δt is given by

$$RV_t^{\delta t} = \sum_{j=1}^{M} r_{t,t_j}^2. \tag{2}$$

In an ideal situation, in the limit of $\delta t \to 0$, the RV is expected to converge to the integrated volatility

$$IV_t = \int_t^{t+1440min} \sigma^2(\mu)d\mu, \tag{3}$$

where $\sigma(\mu)$ is the spot volatility. Usually, the ideal situation is violated by the market microstructure noise (MMS), which has many sources, such as the discreteness of the price, the bid-ask bounce, and properties of the trading mechanism. The existence of MMS biases the RV, and especially, the bias strongly dominates at high frequency, which can be visualized by the volatility signature plot(Andersen et al., 2000). Here, we use a moderate 5-min sampling frequency to avoid strong bias at high-frequency by maintaining reasonable accuracy(Bandi and Russell, 2006; Liu et al., 2015).

To obtain a more accurate RV, we could introduce a modification factor, such as the Hansen-Lunde (HL) factor(Hansen and Lunde, 2005). The HL factor is a multiplicative factor that corrects the RV, so that the average of RV matches the daily return variance. Since the multiplicative factor does not change the Hurst exponent of the log volatility increments here, we use unmodified RV in our analysis.

To estimate the generalized Hurst exponent, we use the MF-DFA, which may be applied to non-stationary time series (Kantelhardt et al., 2002). The MF-DFA has become a popular method to study the multifractal properties of various time series, and studies on Bitcoin have already applied this method, for example, Takaishi (2018); El Alaoui et al. (2018). Let us consider the time series $x_i : i = 1, 2, ...N$. The MF–DFA consists of the following steps.

(i) Determine the profile $Y(i)$,

$$Y(i) = \sum_{j=1}^{i} (x_j - \langle x \rangle), \tag{4}$$

where $\langle x \rangle$ stands for the average of x_i.

(ii) Divide the profile $Y(i)$ into N_s non-overlapping segments of equal length s, where $N_s \equiv int(N/s)$. Since the length of the time series is not always a multiple of s, a short time period at the end of the profile may remain. To utilize this part, we repeat the same procedure starting from the end of the profile. Therefore, in total, we obtain $2N_s$ segments.

(iii) Calculate the variance

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} (Y[(\nu-1)s+i] - P_\nu(i))^2, \tag{5}$$

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for each segment $\nu, \nu = 1, ..., N_s$ and

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} (Y[N - (\nu - N_s)s + i] - P_\nu(i))^2, \qquad (6)$$

for each segment $\nu, \nu = N_s + 1, ..., 2N_s$. Here, $P_\nu(i)$ is the fitting polynomial to remove the local trend in segment ν; we use a cubic order polynomial.

(iv) Average over all segments and obtain the qth order fluctuation function

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} (F^2(\nu, s))^{q/2} \right\}^{1/q}. \qquad (7)$$

(v) Determine the scaling behavior of the fluctuation function. If the time series r_{t_i} are long-range power law correlated, $F_q(s)$ is expected to be the following functional form for large s.

$$F_q(s) \sim s^{h(q)}. \qquad (8)$$

The scaling exponent $h(q)$ is called the generalized Hurst exponent. $h(2)$ corresponds to the usual Hurst exponent.

We also determine $h(q)$ by the SF method used in Gatheral et al. (2018). The SF or moments $m(q, \Delta)$ of the log-volatility increments is defined as

$$m(q, \Delta) = \frac{1}{N}' \sum_{k=1}^{N'} |log(\sigma_{k\Delta}) - log(\sigma_{(k-1)\Delta})|^q, \qquad (9)$$

and we expect the following relationship,

$$m(q, \Delta) \sim c_q \Delta^{\zeta(q)}, \qquad (10)$$

where the exponent $\zeta(q)$ is assumed as $\zeta(q) = qh(q)$. $h(q)$ can be obtained through $\zeta(q)$.

3. Empirical Results

In Figure 1(left), we show the time series of the RV calculated at $\delta t = 5$min. Using the RV, the log-volatility on day t is defined by $log(\sigma_t) \equiv log(RV_t^{1/2})$. Then the log-volatility increments LV_t^Δ with Δ separation is defined by $LV_t^\Delta = log(\sigma_{t\Delta}) - log(\sigma_{(t-1)\Delta})$. Figure 1(right) shows the time series of LV_t^Δ at $\Delta = 1$ day.

In the MF-DFA, $h(q)$ is obtained as the exponent of the fluctuation function $F_q(S)$. Figure 2 shows $F_q(S)$ as a function of s in the log-log plots. We obtain $h(q)$ by fitting $F_q(S)$ to a linear function in $s = [80, 280]$. Figure 3 shows the results of $h(q)$ and clearly, $h(q)$ is smaller than $1/2$, which indicates that the time series is anti-persistent, that is, rough. The Hurst exponent $h(2)$ is estimated to be 0.144. This value is similar to those obtained for other assets(Gatheral
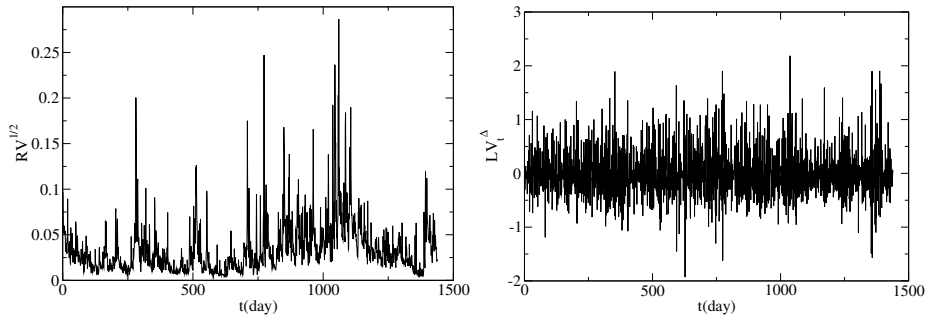
Figure 1: (left): Time series of the RV at $\delta = 5$min. (right): Time series of the log-volatility increments LV_t^{Δ} at $\Delta = 1$ day.
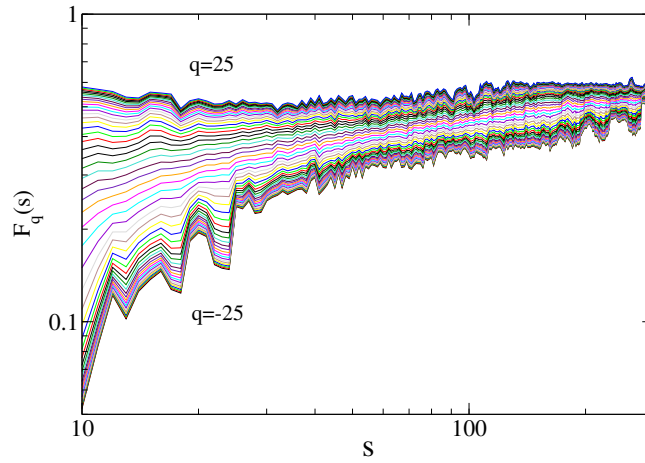

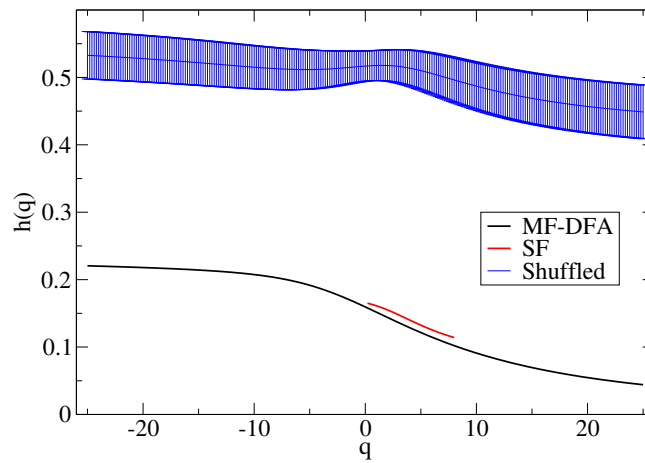Figure 2: Fluctuation function $F_q(s)$.

Figure 3: Generalized Hurst exponent $h(q)$ for the log-volatility increments of Bitcoin.

Table 1: Results of $h(q)$ from the MF-DFA and the SF method. The results indicated by "Shuffled" show $h(q)$ from 20 shuffled time series.

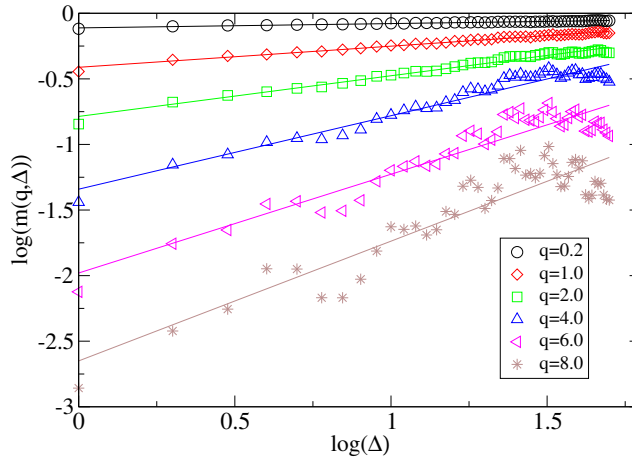| Bitcoin	h(-1)	h(0.2)	h(1)	h(1.6)	h(2)	h(3)	h(4)
MF-DFA	0.167	0.158	0.152	0.147	0.144	0.136	0.129
SF	—	0.165	0.161	0.159	0.155	0.148	0.140
Shuffled	0.515(21)	0.517(20)	0.517(20)	0.517(20)	0.516(21)	0.515(21)	0.512(22)
SPX	h(-1)	h(0.2)	h(1)	h(1.6)	h(2)	h(3)	h(4)
MF-DFA	0.142	0.139	0.136	0.134	0.133	0.129	0.125
SF	—	0.143	0.141	0.139	0.137	0.133	0.129
Shuffled	0.486(18)	0.485(19)	0.485(19)	0.484 (19)	0.484(19)	0.483(19)	0.482(19)

Figure 4: The structure function $m(q, \Delta)$, as a function of Δ. For better visibility, we have shifted some results vertically.

et al., 2018; Bennedsen et al., 2016; Livieri et al., 2018). Table 1 lists several selected values of $h(q)$.

In addition to roughness, we recognize that $h(q)$ varies as a function of q , which is evidence of multifractality in the time series of the log-volatility increments. We also calculate $h(q)$ using the SF method and Figure 4 displays $m(q, \Delta)$. We restrict the parameter q in a range of $q = (0, 8]$, since the SF becomes extremely noisy for $q > 8$ within the current statistics. We obtain $h(q)$ by making a linear fit in the range of $\Delta = [1, 40]$. The results of $h(q)$ are plotted in Figure 3 together with those from the MF-DFA and we find that the results are consistent with those from the MF-DFA.

The sources of multifractality are examined in Kantelhardt et al. (2002), who claim that two sources contribute to the appearance of multifractality: (i)temporal correlations and (ii)broad distributions. The distributions of the log-volatility increments are found to be close to Gaussian(Gatheral et al., 2018; Livieri et al., 2018). However, Bennedsen et al. (2016) claim that non-Gaussian behavior of log-volatility is observed for a significant number of stocks. Within limited statistics, it is difficult to confirm Gaussian for our data set. Figure

6(left) depicts the distribution of log-volatility increments of Bitcoin. Although it seems close to Gaussian, the kurtosis is calculated to be 4.4, which is greater than 3, the value of Gaussian. Thus, it is possible that the distribution of log-volatility increments is slightly leptokurtic.
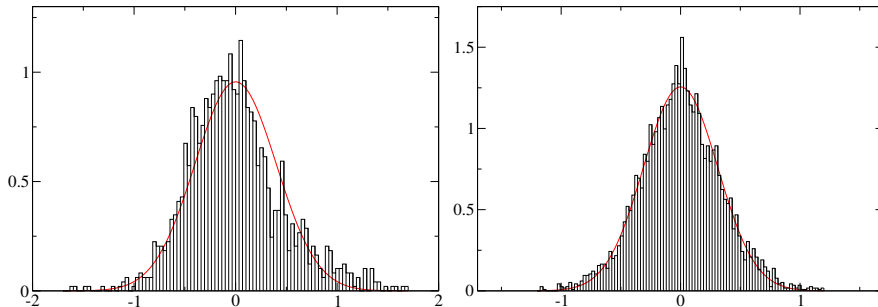

Figure 5: (left): Distribution of the log-volatility increments for Bitcoin. (right): Distribution of the log-volatility increments for SPX. The solid lines show a fit to Gaussian function. The kurtosis is calculated to be 4.4(3.9) for Bitcoin(SPX)

To investigate the origins of roughness and multifractality, we calculate $h(q)$ for the shuffled time series of the log-volatility increments. The shuffling process can kill any temporal correlations; therefore, if both roughness and multifractality originate from temporal correlations, we could expect that roughness and multifractality disappear for the shuffled time series. In Figure 7, we show $h(q)$ from 20 shuffled time series of the log volatility increments, and find that $h(q)$ comes close to 0.5. Since roughness disappears in the shuffling process, it turns out that the temporal correlations contribute to roughness. On the other hand, it seems that $h(q)$ of the shuffled time series still varies slightly with q, which means that the shuffled time series have weak multifractality. To quantify the degree or strength of multifractality, we measure $\Delta(h) = h(q_{min}) - h(q_{max})$ used in Zunino et al. (2008). We also use the singularity spectrum $f(\alpha)$(Kantelhardt et al., 2002) defined by

$$\alpha = h(q) + qh'(q), \tag{11}$$

$$f(\alpha) = q[\alpha - h(q)] + 1. \tag{12}$$

The range of variability of α in $f(\alpha)$, that is, $\Delta\alpha = max(\alpha) - min(\alpha)$, also offers a degree of multifractality. Figure 8 shows $f(\alpha)$ as a function of α, and Table 2 lists the results of Δh and $\Delta\alpha$. Although both Δh and $\Delta\alpha$ decrease for the shuffled time series, they still remain finite. Thus, we conclude that the multifractality of log-volatility increments originates partly from the distributional property of log-volatility increments, and this observation supports the leptokurtic distribution for log-volatility increments.

Previously, the monofractal behavior of the log-volatility has been observed in a narrow range of q, that is, $q = (0, 3]$. It might be difficult to identify the variability of $h(q)$ in such a narrow range. We perform this same analysis for the Standard & Poor's 500 Index (SPX) volatility and try to obtain $h(q)$ in a wide

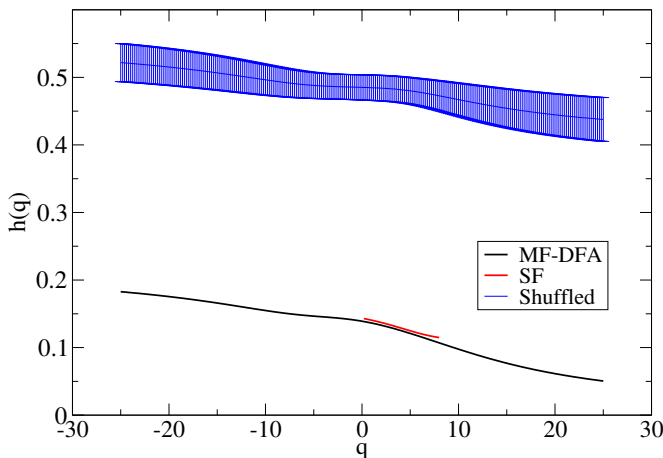|	Δh	Δh(Shuffled)	$\Delta \alpha$	$\Delta \alpha$(Shuffled)
Bitcoin	0.232	0.099	0.232	0.155
SPX	0.132	0.084	0.209	0.142

Table 2: Results of Δh and $\Delta \alpha$.

Figure 6: Generalized Hurst exponent $h(q)$ of the log-volatility increments for SPX.

range of q. The 5min RV data of SPX from January 3, 2000 to February 27, 2019 are downloaded from the Oxford-Man Institute of Quantitative Finance Realized Library[2]. Figure 5 (right), 6, and 8 display the distribution of the log-volatility increments, $h(q)$ and $f(\alpha)$, respectively. Typically, the results are very similar to those of Bitcoin, and importantly, we recognize the variability of $h(q)$, that is, multifractality for SPX.

4. Conclusions

We investigate the generalized Hurst exponent $h(q)$ of the log-volatility increments for Bitcoin using the MF-DFA. We find that $h(q)$ is less than $1/2$, which is consistent with the previous results empirically observed for other assets. Furthermore, we also find that $h(q)$ varies with q, which indicates the existence of multifractality in the time series of the log-volatility increments. Using a shuffled time series, we confirm that while roughness is related to temporal correlations, multifractality originates, in part, from the distributional properties of log-volatility increments. From a rough volatility perspective, Neuman et al. (2018) consider a fractional Brownian motion when the Hurst exponent goes to zero and show that it converges to a Gaussian random distribution close to a log-correlated Gaussian field related to some multifractal processes(Mandelbrot
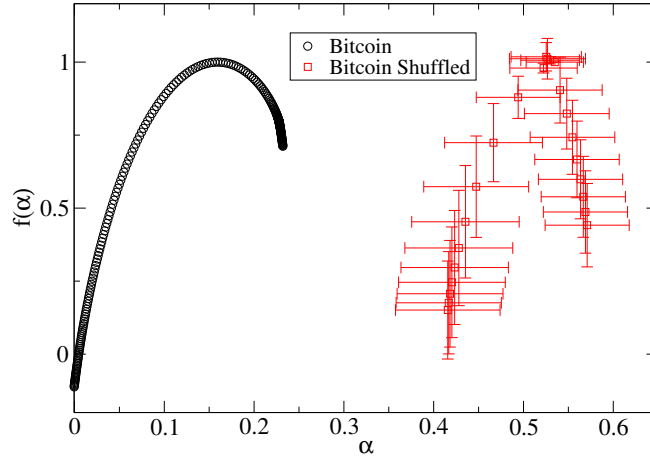
[2]http://realized.oxford-man.ox.ac.uk/data/download

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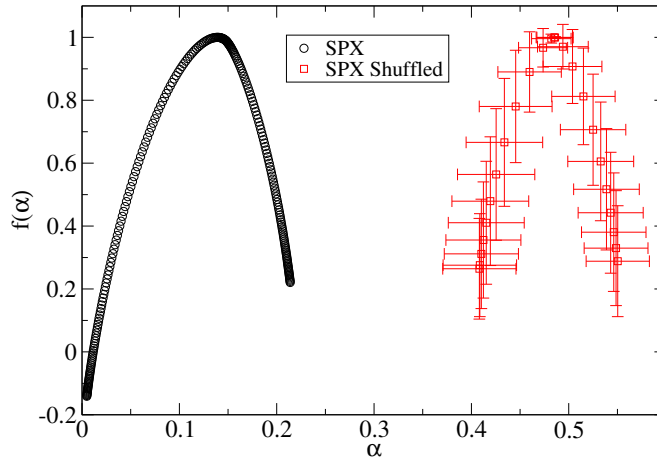Figure 7: The singularity spectrum $f(\alpha)$ for Bitcoin.

Figure 8: The singularity spectrum $f(\alpha)$ for SPX.

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et al., 1997). Our finding of the existence of multifractality in log-volatility increments supports a more serious consideration of such a volatility model, including multifractality.

Acknowledgment

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