PRECISE ASYMPTOTICS: ROBUST STOCHASTIC VOLATILITY MODELS

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ABSTRACT. We present a new methodology to analyze large classes of (classical and rough) stochastic volatility models, with special regard to short-time and small noise formulae for option prices. Our main tool is the theory of regularity structures, which we use in the form of [Bayer et al; A regularity structure for rough volatility, 2017]. In essence, we implement a Laplace method on the space of models (in the sense of Hairer), which generalizes classical works of Azencott and Ben Arous on path space and then Aida, Inahama–Kawabi on rough path space. When applied to rough volatility models, e.g. in the setting of [Forde-Zhang, Asymptotics for rough stochastic volatility models, 2017], one obtains precise asymptotic for European options which refine known large deviation asymptotics.

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1. Introduction

We revisit the large deviation theory of option pricing under stochastic volatility. More specifically, we are interested in the short dated regime where resulting asymptotic expansions are widely used. In classical (Markovian) stochastic volatility models, such expansions are typically derived from existing heat-kernel expansions, and this approach was followed by many authors, see e.g. [35, 47, 43, 34, 16, 17, 4] and the monographs [37, 32] with many references. Viscosity solutions to non-linear partial differential equations provide an alternative route to some of these results [11].

A main feature of this work is to provide a sufficiently general setup to treat the novel class of rough volatility models, term coined in the seminal work [31]: volatility follows an anomalous diffusion with (negative) long range correlations described by a Hurst parameter H < 1/2 (with sample paths rougher than those of Brownian motion). The statistical evidence in [31] was subsequently explained from a market microstructure model [18]. Evaluation of Wiener functionals ("pricing") under rough volatility goes back to [2], followed by [29, 7, 30, 22, 19] and now many others. It was recently seen [8] that Hairer's theory of regularity structures [36], a major extension of Lyons' rough path theory [46], provides a robust formulation of rough volatility models. This was used to derive large deviation estimates, extending the results of [22].

The contribution of this paper is a general methodology, applicable to large classes of (classical and rough) stochastic volatility models in a unified way, to go beyond large deviations and compute precise asymptotics. In essence, we achieve this by a carefully designed Laplace method on the space of models (in the sense of [36]), which generalized the notion of rough path space, itself a generalization of classical pathspace. Let us also emphasize that we deal with "call price" Wiener functionals of the form " $E(f(X_1^{\varepsilon}))$ ", with $f(x) \sim (e^x - e^k)^+$ which is not at all of the form " $E(\exp F(X^{\varepsilon})/\varepsilon^2)$ " discussed in Ben Arous [3] (see also [5, 12]), later revisited by the Japanese stochastic / rough analysis community [1, 39, 40, 38]. Perhaps closest in spirit, in a classical diffusion setting, Azencott [6] studies " $P(X_1^{\varepsilon} > k)$ " using, among others, anticipating stochastic calculus (something we shall elegantly bypass with pathwise methods). As is typical for the Laplace method, a rigorous treatment of the remainder term requires great care. In this regard, Azencott [6] notes "Il serait très intéressant d'avoir une justification systématique générale de la validité des développements formels de ce type et nous avons (à moyen terme!) une vue assez optimiste sur l'existence d'un formalisme indolore et garanti mathématiquement." The here proposed use of regularity structures (or in the H=1/2 diffusion case: rough paths) provides exactly this type of formalism.

Our results cover general stochastic volatility models as studied in [11, 16, 17]. For the sake of clarity, we leave the full statement to the main text (Theorem 6.1) and here give a loose formulation:

Theorem 1.1. Consider a, possibly rough with $H \in (0,1/2]$, stochastic volatility model (SVM), with arbitrary number of factors, which can be robustified in the sense of rough paths or regularity structures. Consider European call prices c = c(t,k) with (out-of-money) log-strikes $k_{\varepsilon} = x\varepsilon^{1-2H} > 0$. Under a non-degeneracy assumption for the most-likely path, there exists a rate function $\Lambda = \Lambda(x)$, regular near x, and a function $A = A(x) \sim 1$ as $x \downarrow 0$, such that, with $\sigma_x^2 = 2\Lambda(x)/\Lambda'(x)^2$, we have small noise asymptotics of the form

(1.1)
$$c(\varepsilon^2, k_{\varepsilon}) \sim \exp\left(-\frac{\Lambda(x)}{\varepsilon^{4H}}\right) \varepsilon^{1+4H} \frac{A(x)}{(\Lambda'(x))^2 \sigma_x \sqrt{2\pi}} \quad as \ \varepsilon \downarrow 0.$$

Let us point out that short-time asymptotics are obtained from small-noise via the substitution $\varepsilon^2 = t$. In case H = 1/2, Theorem 1.1 deal with classical SVMs and is in agreement with Kusuoka–Osajima [43, 47]. In this case, the energy function $\Lambda(x)$ has a geometric interpretation (shortest square-distance to some arrival manifold determined by log-strike k = x). For strictly positive spot-volatility and sufficiently small x, the non-degeneracy condition is satisfied and the expansion (1.1) is valid. In the geometric (H = 1/2) setting the non-degeneracy can be formulated in terms of focal points [16, 17]. This also provides a computational framework (Hamiltonian differential equations) to construct examples where such and related expansions break down for critical OTM level $x^* > 0$, see e.g. [10]. (In particular, one cannot hope for validity of such an expansion to hold true for all x without the said non-degeneracy assumption).

The "robustness" assumption (in the sense of rough paths or regularity structures) essentially requires some smoothness of the coefficients in the model, which seems to rule out (classical and rough) Heston-type model because of square-root coefficients. While such degeneracies are not at all the focus of this work, we point out that that our expansion is determined by a neighbourhood of the most-likely path, in uniform (and even stronger) metrics. Hence, any "initial" localization to a uniform neighbourhood of the most-likely path, obtained by pathwise large deviations (such as [48] in the Heston case), essentially allows to ignore the square-root issues and to apply our theorem, consistent with known Heston results [21, Thm 3.1].

The main interest of our theorem, of course, lies in the rough regime $H \in (0,1/2)$, where in particular it refines RoughVol large deviations studied by Forde-Zhang [22]. Granted some minimal moments assumptions, it applies to rough volatility models as discussed in [7, 8] and notably the rough Bergomi model (with log-normal fractional volatility and negative correlation). The previous remark on Heston applies mutatis mutandis to the rough Heston model [20]. As a sanity check, let us also point out that Black-Scholes corresponds to H = 1/2, $\Lambda(x) = x^2/(2\sigma^2)$ and $\sigma_x \equiv \sigma$. In case of zero rates, the log-price has mean $\mu = -\frac{1}{2}\sigma^2$ so that Theorem 6.1 (with computable $A(x) = e^{-x/2}$) gives the correct Black-Scholes asymptotics,

$$c(\varepsilon^2, x) \sim \exp\left(-\frac{x^2}{2\varepsilon^2\sigma^2}\right) \varepsilon^3 \frac{\sigma^3 e^x}{x^2\sqrt{2\pi}} e^{-x/2}$$
 as $\varepsilon \downarrow 0$.

There is little hope, of course, to obtain explicit formulae for $\Lambda = \Lambda(x)$ and A = A(x) in the case of generic (rough or classical) stochastic volatility models. That said, once a model has been specified, and Theorem 1.1 specifies the general form of the asymptotic expansion, one can try to expand Λ and A around x = 0. It is clear that such an expansion only depends on the coefficients of the specified model near the startpoint (i.e. spot and spot-vol). This has been done in the classical Heston case in [21] and for smooth SVMs in [47]. For rough volatility models of Forde-Zhang type, an expansion of $\Lambda = \Lambda(x)$ to third order was given in [9]; this work also dealt with "moderate option" pricing [28] under rough volatility, which is refined and generalized in the present

work. Implications for implied volatility asymptotics were then seen to complement those given in [2, 29, 30]. Further explicit computations in the rough volatility case (starting with refined expansions of Λ , A) are possible (but lengthy) and left to [27], together with consequences for implied volatility and numerical evidence.

2. Notation

Wiener space $C([0,1],\mathbb{R}^m)$ supports m-dimensional standard Brownian motion $W=W(t,\omega)$ and contains the Cameron-Martin space $H \equiv H^1 \subset C([0,1],\mathbb{R}^m)$ of square-integrable paths. It is tacitly assumed that all Brownian and Cameron-Martin paths start at the origin. When m=2, write $W = (W, \overline{W})$ and $h = (h, \overline{h})$ accordingly; given a correlation parameter $\rho \in [1, 1]$, and $\rho^2 + \overline{\rho}^2 = 1$, a scalar Brownian motion (ρ -correlated with W) is given by $\widetilde{W} = \rho W + \overline{\rho} \overline{W}$. Similarly, write $hat{h} = \rho h + \overline{\rho} h$. By Hurst parameter we mean a scalar $H \in (0, 1/2]$. Convolution with the singular kernel $K^H(s,t) = \sqrt{2H}|t-s|^{H-1/2}\mathbf{1}_{0 < s < t}$ is known to β -regularize, with $\beta = 1/2 + H$. (When H = 1/2, this is precisely indefinite integration.) Applied to scalar white noise \dot{W} , a.s. in the negative Hölder space $C^{-1/2}$, we get a (Riemann-Liouville or Volterra) fractional Brownian motion $\widehat{W} = K^H * \dot{W} \in C^{H^-}$ a.s. We use the same notation and write $\widehat{h} = K^H * \dot{h} \in H^{\beta}$, where $\dot{h} \in L^2 \equiv H^0$ and H^{β} denotes a fractional Sobolev space. Throughout, X = (X,Y) is a n-dimensional stochastic process, with scalar first component $X = X(t, \omega)$, given in terms of an m-dimensional Brownian motion W, and possibly the Volterrification of some components (such as \widehat{W}). We shall assume a robust form, which allows to write X as continuous function of $\mathbf{W} = \mathbf{W}(\omega)$, which is a suitable enhancement of W to a random element in rough path space \mathcal{C}^{α} , $\alpha \in (1/3, 1/2)$ or more generally, a random element in the space of models $\mathcal{M} = \mathcal{M}^{\kappa}$, defined on a regularity structure whose precise form depends on the dynamics at hand. (The case of rough volatility is reviewed in detail in the Appendix.)

3. Preliminaries on Black-Scholes asymptotics

The Black-Scholes log-price at time $t = \varepsilon^2$ is given by

$$X_1^{\varepsilon} = \varepsilon^2 \mu + \varepsilon \sigma B_1 \sim N\left(t\mu, \sigma^2 t\right)$$
.

Consider log-strike k > 0 and set

$$Z_1^{\varepsilon} = \varepsilon^2 \mu + \varepsilon \sigma (B_1 + k/\varepsilon \sigma) = k + \varepsilon \sigma B_1 + \varepsilon^2 \mu.$$

Then the call price is given by

$$E\left[\left(e^{X_1^{\varepsilon}} - e^k\right)^+\right] = E\left[\left(e^{Z_1^{\varepsilon}} - e^k\right)^+ \exp\left(-\frac{kB_1}{\varepsilon\sigma} - \frac{k^2}{2\varepsilon^2\sigma^2}\right)\right]$$

$$= e^{-\frac{k^2}{2\varepsilon^2\sigma^2}} e^k E\left[e^{-\frac{kB_1}{\varepsilon\sigma}} \left(e^{\varepsilon\sigma B_1 + \varepsilon^2\mu} - 1\right)^+\right]$$

$$\sim e^{-\frac{k^2}{2\varepsilon^2\sigma^2}} e^k E\left[e^{-\frac{kB_1}{\varepsilon\sigma}} \left(\varepsilon\sigma B_1 + \varepsilon^2\mu\right)^+\right]$$

$$= e^{-\frac{k^2}{2\varepsilon^2\sigma^2}} \varepsilon\sigma e^k E\left[e^{-\frac{kB_1}{\varepsilon\sigma}} \left(B_1 + \varepsilon\mu/\sigma\right)^+\right].$$

The justification of \sim comes from a Laplace type argument: the asymptotic behaviour is determined on the event $\{|\varepsilon B_1| < \delta\}$, for any fixed $\delta > 0$, which in turn allows to replace $e^{\varepsilon B_1} - 1$ by εB_1 .

(Details are left to the reader.) The next lemma, applied with $\tilde{\varepsilon} = \varepsilon \sigma/k$ (so that $\varepsilon \mu/\sigma = \tilde{\varepsilon} k \mu/\sigma^2$) then gives

(3.1)
$$E\left[\left(e^{X_1^{\varepsilon}} - e^k\right)^+\right] \sim e^{-\frac{k^2}{2\varepsilon^2\sigma^2}} \varepsilon^3 \frac{\sigma^3 e^k}{k^2 \sqrt{2\pi}} e^{k\mu/\sigma^2}.$$

Lemma 3.1. As $\varepsilon \to 0$,

$$E\left[e^{-\frac{B_1}{\varepsilon}}\left(B_1+\varepsilon\alpha\right)^+\right]\sim \frac{\varepsilon^2}{\sqrt{2\pi}}e^{\alpha}$$
.

One can prove Lemma 3.1 with an elementary Laplace type argument, noting that the relevant contribution comes from $0 < B_1 + \varepsilon \alpha < \delta$, any $\delta > 0$. (This also explains why changing $B_1 + \varepsilon \alpha$ by, say, $B_1 + \varepsilon \alpha + \varepsilon B_1 + \varepsilon B_1^2$, will not change the asymptotics.) That said, Lemma 3.1 is also an immediate consequence of the following (non-asymptotic) estimate, which offers some flexibility in the sequel.

Lemma 3.2. Let $\alpha \in \mathbb{R}$, $\gamma \in [0,1)$, ε strictly positive, and $N \sim N(0,1)$. Then for some C > 0, it holds that

$$\min \left[(1 - \gamma)e^{\frac{\alpha}{1 - \gamma}} + 2\gamma, (1 + \gamma)e^{\frac{\alpha}{1 + \gamma}} \right] - \varepsilon^{2} \left(C(1 + \alpha^{2}) \max\{e^{\frac{\alpha}{1 - \gamma}}, e^{\frac{\alpha}{1 + \gamma}}\} + 6\gamma \right)$$

$$(3.2) \qquad \leq \sqrt{2\pi}\varepsilon^{-2}\mathbb{E} \left[\exp\left(-\varepsilon^{-1}N\right) \left(N + \gamma|N| + \varepsilon\alpha\right)^{+} \right]$$

$$\leq \max \left[(1 - \gamma)e^{\frac{\alpha}{1 - \gamma}} + 2\gamma, (1 + \gamma)e^{\frac{\alpha}{1 + \gamma}} \right].$$

Proof. The middle expression (3.2) equals

$$\varepsilon^{-2} \int_{-\infty}^{+\infty} e^{-y^2/2} e^{-\frac{y}{\varepsilon}} \left(y + \gamma |y| + \varepsilon \alpha \right)^+ dy = \int_{-\infty}^{+\infty} e^{-v^2 \varepsilon^2/2} e^{-v} \left(v + \gamma |v| + \alpha \right)^+ dv.$$

The elementary $1 - y^2/2 \le \exp(-y^2/2) \le 1$ then leads to the stated bounds. Indeed,

$$(3.2) \le \int_{-\infty}^{+\infty} e^{-v} (v + \gamma |v| + \alpha)^{+} dv$$

which is computable: when $\alpha < 0$, the right-hand side equals

$$(1+\gamma) \int_{-\frac{\alpha}{1+\alpha}}^{\infty} \left(v + \frac{\alpha}{1+\gamma} \right) e^{-v} dv = (1+\gamma) e^{\frac{\alpha}{1+\gamma}}$$

whereas when $\alpha \geq 0$ we have

$$\int_{-\frac{\alpha}{1-\gamma}}^{0} e^{-v} \left(v(1-\gamma) + \alpha \right) dv + \int_{0}^{\infty} e^{-v} \left(v(1+\gamma) + \alpha \right) dv$$
$$= (1-\gamma)e^{\frac{\alpha}{1-\gamma}} + \int_{0}^{\infty} e^{-v} 2\gamma v dv$$
$$= (1-\gamma)e^{\frac{\alpha}{1-\gamma}} + 2\gamma.$$

To obtain the lower bound, use $e^{-y^2/2} \ge 1 - y^2/2$ and split the integral to obtain

$$(3.2) \ge \int_{-\infty}^{+\infty} e^{-v} (v + |\gamma|v + \alpha)^{+} dv - \frac{\varepsilon^{2}}{2} \int_{-\infty}^{\infty} e^{-v} v^{2} (v + |\gamma|v + \alpha)^{+} dv.$$

The first integral is computed as before. For the second one, we again distinguish according to the sign of α . When $\alpha < 0$ we obtain

$$e^{\frac{\alpha}{1+\gamma}} \int_0^\infty e^{-u} \left(u - \frac{\alpha}{1+\gamma} \right)^2 u du \le C e^{\frac{\alpha}{1+\gamma}} (1+\alpha^2)$$

whereas in the case $\alpha \geq 0$ we have

$$\begin{split} &\int_{-\frac{\alpha}{1-\gamma}}^{0}e^{-v}v^{2}\left(v(1-\gamma)+\alpha\right)dv+\int_{0}^{\infty}e^{-v}v^{2}\left(v(1+\gamma)+\alpha\right)dv\\ =&e^{\frac{\alpha}{1-\gamma}}(1-\gamma)\int_{0}^{\infty}e^{-u}\left(u-\frac{\alpha}{1+\gamma}\right)^{2}udu+\int_{0}^{\infty}e^{-v}2\gamma v^{3}dv\\ \leq&Ce^{\frac{\alpha}{1-\gamma}}+12\gamma. \end{split}$$

4. Unified large, moderate and rough deviations

We now put forward our basic large deviation assumption. The object of interest is a scalar process (X^{ε}) , with the interpretation of a log-price process run at speed ε^2 . The reader can have in mind an Itô diffusion: classical StochVol models assume that (X^{ε}) is one component of a higher-dimensional Markov diffusion; RoughVol models have additional components driven (or given) by fractional Brownian motion. We further note that our setup includes pricing in the moderate deviation regime.

4.1. Basic large deviation assumption (A1).

(A1a) The family $\{\overline{X}_1^{\varepsilon}: 0 < \varepsilon \leq 1\}$, given as unit time marginal of a rescaled process X^{ε} ,

$$\overline{X}^{\varepsilon} := \frac{\overline{\varepsilon}}{\varepsilon} X^{\varepsilon}$$

satisfies a LDP with rate function (a.k.a. energy) Λ and speed $\bar{\varepsilon}^2 \geq \varepsilon^2$.

(A1b) The (in general only lower semicontinuous) rate function Λ is continuous.¹

Note that a large deviation principle for (X^{ε}) may or may not hold. Remark that condition (A1b) is known in all the StochVol / RoughVol setting interesting to us (cf. references below).

Example 4.1 (Black-Scholes / Schilder LDP). Let $X_1^{\varepsilon} = \varepsilon \sigma B_1 + \varepsilon^2 \mu$ (with $\mu = -\sigma^2/2$ in absence of rates) and a LDP holds with

$$\overline{\varepsilon} = \varepsilon, \quad \Lambda(x) = \frac{x^2}{2\sigma^2}.$$

Example 4.2 (Classical StochVol, Freidlin-Wentzell LDP). Here Λ is in general non-explicit, but has an interpretation in terms of geodesic distance from arrival log-spot/spot-vol $(0, y_0)$ to the arrival manifold (x, \cdot) . In a locally elliptic setting, the rate function is viscosity solution to eikonal equation hence continuous, see e.g. [11, Thm 2.3]. It is shown in [16] that Λ is smooth "away from focality points" which is always the case for x close to zero.

Example 4.3 (RoughVol, Forde-Zhang LDP [22]). Let $H \in (0, 1/2]$. With

$$X_1^{\varepsilon} = \int_0^1 \sigma\left(\widehat{\varepsilon}\widehat{W}\right) \varepsilon d\left(\overline{\rho}\overline{W} + \rho W\right) - \frac{1}{2}\varepsilon^2 \int_0^1 \sigma^2\left(\widehat{\varepsilon}\widehat{W}\right) dt ,$$

¹In the sequel, we will formulate conditions that guarantee that Λ is even smooth.

where $\widehat{W} = K^H * \dot{W}$, $\widehat{\varepsilon} := \varepsilon^{2H}$, the LDP assumption holds for

$$\overline{X}_{1}^{\varepsilon} = \int_{0}^{1} \sigma\left(\widehat{\varepsilon}\widehat{W}\right) \widehat{\varepsilon} d\left(\overline{\rho}\overline{W} + \rho W\right) - \frac{1}{2}\varepsilon\widehat{\varepsilon} \int \sigma^{2}\left(\widehat{\varepsilon}\widehat{W}\right) dt ,$$

i.e. with $\overline{\varepsilon} = \widehat{\varepsilon} = \varepsilon^{2H}$, and with continuous [22, Cor. 4.6.] rate function

(4.1)
$$\Lambda(x) = \mathcal{J}(x) := \inf\{\frac{1}{2} \|h, \overline{h}\|_{H^1}^2 : \int_0^1 \sigma(\widehat{h}) d(\overline{\rho} \overline{h} + \rho h) = x\} \equiv \frac{1}{2} \|h^x, \overline{h}^x\|_{H^1}^2.$$

We note that [22] assumes a (linear) growth condition on $\sigma(.)$; it was later seen that the growth condition of $\sigma(.)$ can be removed so that the exponential form of the volatility function ("rough Bergomi" [7]) is covered. In fact, this follows immediately from assumption (A3a-c) below, which hold in the RoughVol setting, due to [8]. But see also [41, 33] for related results.

Example 4.4 (Black-Scholes MDP). Take $\beta \in [0, 1/2)$ and $\overline{\varepsilon} = \varepsilon^{1-2\beta}$. Then the LDP assumption holds, with speed $\overline{\varepsilon}^2$ and rate function $\Lambda(x) = \frac{x^2}{2\sigma^2}$, for

$$\overline{X}_1^{\varepsilon} = \overline{\varepsilon}\sigma B_1 + \overline{\varepsilon}\varepsilon\mu = \frac{\overline{\varepsilon}}{\varepsilon}(\varepsilon\sigma B_1 + \varepsilon^2\mu) \ .$$

Example 4.5 (Classical StochVol, moderate deviations regime [28]). With $\beta \in (0, 1/2)$ and $\overline{\varepsilon} = \varepsilon^{1-2\beta}$ as above, one finds that $\overline{X}_1^{\varepsilon} = \frac{\overline{\varepsilon}}{\varepsilon} X_1^{\varepsilon}$ satisfies the same MDP as Brownian motion, with Black-Scholes paramter σ replaced by spot-volatility σ_0 . For $\beta = 0$ we are in the realm of Freidlin-Wentzell LDP, $\beta = 1/2$ is the central limit scaling.

Example 4.6 (RoughVol, moderate deviations regime [9]). Consider the log-price X^{ε} under RoughVol, as introduced in Example 4.3. Let $0 < \beta < H \le 1/2$ and set $\overline{\varepsilon} = \varepsilon^{2H-2\beta}$. Then the LDP assumption holds, with speed $\overline{\varepsilon}^2$ and rate function $\Lambda(x) = \frac{x^2}{2\sigma_0^2}$, for

$$\overline{X}_1^{\varepsilon} = \frac{\overline{\varepsilon}}{\varepsilon} X_1^{\varepsilon} .$$

For $\beta = 0$, we are back in the large deviation setting, $\beta = H$ is the central limit scaling.

4.2. Moment assumption (A2) and large deviation option pricing. Fix $x \ge 0$ and consider (possibly rescaled) log-strike

$$0 \le k_{\varepsilon} = (\varepsilon/\overline{\varepsilon}) x \le x$$
.

Note: $k_{\varepsilon} \equiv x$ in classical Stoch Vol large deviations, $k_{\varepsilon} \to 0$ in rough and moderate cases. Note

$$(4.2) P[\overline{X}_1^{\varepsilon} > x] = P[X_1^{\varepsilon} > k_{\varepsilon}] = \exp\left(-\frac{\Lambda(x) + o(1)}{\overline{\varepsilon}^2}\right) ,$$

which has an interpretation in terms of out-of-the money digital option prices. We are interested in call-prices of the form

$$E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+] = E[(\exp(\overline{X}_1^{\varepsilon} \varepsilon / \overline{\varepsilon}) - \exp(x \varepsilon / \overline{\varepsilon}))^+].$$

Since a LDP for random variables (X^{ε}) says nothing about their integrability, we need a (very mild) moment assumption.

(A2) There exists p > 1 such that $\limsup_{\varepsilon \to 0} E[(\exp(X_1^{\varepsilon})^p] =: m_p < \infty$ Here is a typical way to check (A2). The proof is left to the reader.

Lemma 4.7. Assume $e^{-rt} \exp(X_t) \equiv S_t$ is martingale, with $X_t \equiv X_1^{\varepsilon}$ and $\varepsilon^2 \equiv t$. Assume there exists p > 1 and t > 0 such that $E[S_t^p] < \infty$. Then (A2) holds.

Remark 4.8. Condition (A2) has been carefully crafted not to rule out models with log-normal volatility. Specifically, it is satisfied by the "log-normal" SABR model with $\beta=1$ and with correlation parameter $\rho<0$, cf. [49, 42, 45].

Proposition 4.9. Assume (A1-A2). Fix x > 0 and set $k_{\varepsilon} = (\varepsilon/\overline{\varepsilon}) x$. Then

$$E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+] = \exp\left(-\frac{\Lambda(x) + o(1)}{\overline{\varepsilon}^2}\right).$$

Proof. We prove (and later: need) only the upper bound. Fix y > x and consider $\varepsilon \in (0,1]$ so that $\overline{\varepsilon}/\varepsilon = \nu_{\varepsilon} \ge 1$ and then

$$E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+] = E[(\exp(\overline{X}_1^{\varepsilon}/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^+ 1_{\{\overline{X}_1^{\varepsilon} \in (x,y]\}}]$$

$$+ E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+ 1_{\{\overline{X}_1^{\varepsilon} > y\}}]$$

$$\leq (e^{y/\nu_{\varepsilon}} - e^{x/\nu_{\varepsilon}})^+ P[\overline{X}_1^{\varepsilon} > x] + E[(\exp(X_1^{\varepsilon})^p]^{1/p} P(\overline{X}_1^{\varepsilon} > y)^{1/q}$$

$$\leq (e^y - e^x)^+ P[\overline{X}_1^{\varepsilon} > x] + E[(\exp(X_1^{\varepsilon})^p]^{1/p} P(\overline{X}_1^{\varepsilon} > y)^{1/q}$$

where we have taken Hölder's inequality for the second term, thanks to (A2), with Hölder conjugate $q = p' < \infty$, and, uniformly over small ε , with $E[(\exp(X_{\varepsilon}^{\varepsilon})^p)^{1/p} < \infty$. Use (A1) to obtain

$$\limsup_{\varepsilon \to 0} \overline{\varepsilon}^2 \log \left(E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+] \right) \le \max \left(-\Lambda(x), -\frac{\Lambda(y)}{q} \right)$$

and we conclude by letting $y \to +\infty$. (Thanks to goodness of the rate function, one cannot reach infinity at finite cost so that $\Lambda(y) \to \infty$ as $y \to \infty$.)

When applied to classic StochVol option pricing in the short-time limit, $t = \varepsilon^2$, Proposition 4.9 is a rigorous formulation of what is often loosely written as

$$E[(S_t - K)^+] \approx e^{-\Lambda(k)/t}$$

where $\Lambda(k)$ is the rate function, $k = \log(K/S_0)$. Similarly, under RoughVol, with energy function Λ , this relation becomes

$$E[(S_t - K_t)^+] \approx e^{-\Lambda(x)/t^{2H}}$$

where $K_t = S_0 \exp(xt^{1/2-H})$. In the corresponding moderate regime one has

$$E[(S_t - L_t)^+] \approx \exp\left(-\frac{x^2}{2\sigma_0^2 t^{2H - 2\beta}}\right),$$

with $L_t = S_0 \exp(xt^{1/2-(H-\beta)})$ where $0 < \beta < H < 1/2$. (The "moderate" approximation formula under classical StochVol is exactly of the same form, with H = 1/2.)

The remainder of this paper is devoted to replace \approx by a honest asymptotic equivalence, as seen in the Black-Scholes example: for fixed x > 0, as $t \downarrow 0$,

$$E[(e^{\sigma B_t + \mu t} - e^x)^+] \sim e^{-x^2/(2\sigma^2 t)} t^{3/2} \frac{\sigma^3 e^x e^{x\mu/\sigma^2}}{x^2 \sqrt{2\pi}}$$
.

As we shall see, our methods apply in great generality to obtain such "precise" large deviation for StochVol and RoughVol. A detour is necessary in the moderate regime where the presence of another scale ε^{β} rules out the stochastic Taylor expansions in $\overline{\varepsilon} = \varepsilon^{2H}$. Nonetheless, we have precise enough control, that the moderate expansion is obtained, in essence, from uniformity of our precise large deviation estimates.

5. Exact call price formula

Under assumptions (A1), (A2) Proposition 4.9 tells us that

$$c(\varepsilon^2, k_{\varepsilon}) := E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+] \approx \exp\left(-\frac{\Lambda(x)}{\overline{\varepsilon}^2}\right).$$

We are interested in refined asymptotics. To this end, we have to be more explicit about the construction of the X^{ε} . The basic setup, is an m-dimensional Wiener space $C([0,1],R^m)$, equipped with the Wiener measure, as common underlying probability space for the processes (X^{ε}) , constructed, for instance, as strong solution to stochastic Itô or Volterra differential equations. In all these examples, (X^{ε}) has somewhat obvious meaning when the white noise \dot{W} is replaced (or perturbed) by some Cameron–Martin element $\dot{h} \in L^2([0,1],R^m)$ – which in turn underlies large deviation (and precise) asymptotics. If one wants to be less specific about the origin of the Wiener functionals under consideration, exhibiting the right abstract condition is no easy matter and leads to various notions of "regular" Wiener functionals [44] (applied to option pricing in [43, 47]). Our abstract assumptions, cf. (A3) and (A4) are different: in a sense we avoid working with a regular Wiener functionals by imposing stability w.r.t. a suitable enhancement of the noise which restores analytical control.

5.1. **The case of SDEs and (classical) StochVol.** Follow the setting of [5, 6, 3]. Consider an *n*-dimensional diffusion given in Itô SDE form, in the small noise regime

$$dX^{\varepsilon} = b(\varepsilon, X^{\varepsilon})dt + \sigma(X^{\varepsilon})\varepsilon dW,$$

with fixed initial data, driven by an m-dimensional Brownian motion W. In a (classical) StochVol setting, $X^{\varepsilon} = (X^{\varepsilon}, Y^{\varepsilon}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ where X^{ε} has the interpretation of price (or log-price) process, together with (n-1) volatility factors. Fix $\varepsilon > 0$. Under standard assumptions on b, σ there exists a measurable map ("Itô-map") ϕ^{ε} on Wiener space,

$$X^{\varepsilon}(\omega) =: \phi^{\varepsilon}(\varepsilon \mathbf{W}(\omega))$$
.

Provided that $b(\varepsilon,\cdot)$ converges suitably to $b(0,\cdot)$ as $\varepsilon \downarrow 0$, the family (X^{ε}) is exponentially equivalent to $\phi^0(\varepsilon W(\omega))$; a small noise LDP in this setting is provided by Freidlin-Wentzell theory, with (good) rate function given by formal application of the contraction principle and Schilder's LDP for Brownian motion. Stochastic Taylor expansions of the form

(5.1)
$$\phi_1^{\varepsilon}(\varepsilon W(\omega) + h) = \phi_1^0(h) + \varepsilon g_1(\omega) + \varepsilon^2 g_2(\omega) + \dots + \varepsilon^{n-1} g_{n-1}(\omega) + r_n^{\varepsilon}(\omega)$$

are easily obtained by a formal computation (and can subsequently be justified, including remainder estimates). In Lyons' rough path theory one enhances noise W to a (random) rough path of the form $\mathbf{W} = (W, \int W \otimes dW) \in \mathcal{C}^{\alpha}, \alpha \in (1/3, 1/2)$, with Itô integration, such that

$$\phi^{\varepsilon}(W(\omega)) = \Phi^{\varepsilon}(\mathbf{W}(\omega))$$

where Φ^{ε} is the Itô-Lyons map, known to be locally Lipschitz continuous with respect to suitable rough path metrics. Scaling εW lifts to rough path dilation $\delta_{\varepsilon} \mathbf{W} := (\varepsilon W, \varepsilon^2 \int W \otimes dW)$ so that $\phi^{\varepsilon}(\varepsilon W(\omega)) = \Phi^{\varepsilon}(\delta_{\varepsilon} \mathbf{W}(\omega))$. (A LDP for $\delta_{\varepsilon} \mathbf{W}$ in rough path topology then provides an easy proof of Freidlin-Wentzell large deviations for SDEs, see e.g. [24, Ch.9].) Last but not least, Cameron-Martin perturbation $W \to W + h$ lifts to rough path translation, formally given by, $T_h \mathbf{W} = (W + h, \int (W + h) \otimes d(W + h))$, so that, for fixed $h \in H$ and any n = 1, 2, ...,

$$\phi_1^{\varepsilon}(\varepsilon W(\omega) + h) = \Phi_1^{\varepsilon}(T_h(\delta_{\varepsilon} \mathbf{W})) = \Phi_1^0(h) + \varepsilon G_1(\mathbf{W}) + \dots + \varepsilon^{n-1} G_{n-1}(\mathbf{W}) + R_n^{\varepsilon}(\mathbf{W}).$$

We insist that the second equality is a purely deterministic "rough Taylor expansion", which becomes random only after inserting the random rough path $\mathbf{W} = \mathbf{W}(\omega)$, constructed from Brownian motion. The point of this construction (developed in [1, 39, 40, 38]) is the robustification of all terms in (5.1), i.e.

$$g_1(\omega) = G_1(\mathbf{W}(\omega)), \ g_2(\omega) = G_2(\mathbf{W}(\omega)), \dots, r_n^{\varepsilon}(\omega) = R_n^{\varepsilon}(\mathbf{W}(\omega)).$$

with precise (deterministic) rough path estimates, notably (cf. [39, Thm. 5.1])

$$(5.2) |R_n^{\varepsilon}(\mathbf{W})| \lesssim \varepsilon^n (1 + |||\mathbf{W}|||^n) ,$$

valid on bounded sets of $\delta_{\varepsilon} \mathbf{W}$, uniformly over $\varepsilon \in (0,1]$. As will be seen, expansion to order 2 is sufficient for our purposes.

We need to understand the behaviour of G_1, G_2 under scaling and translation. The situation is somewhat simplified by assuming that the drift vanishes to first order, as is typical in the setting of short-time asymptotics.

Assume $\partial_{\varepsilon}b(0,\cdot)=0$. Then $\partial_{\varepsilon}\Phi(0,\cdot)\equiv\partial_{\varepsilon}|_{\varepsilon=0}\Phi^{\varepsilon}(\cdot)=0$ and, for all $\varepsilon\in(0,1]$, $h\in H$, and $\mathbf{W}\in\mathcal{C}^{\alpha}$,

$$\Phi_1^0(T_h(\delta_{\varepsilon}\mathbf{W})) = \Phi_1^0(h) + \varepsilon G_1^0(\mathbf{W}) + \varepsilon^2 G_2^0(\mathbf{W}) + R_3^{0,\varepsilon}(\mathbf{W})$$

with continuous linear $G_1^0 := G_1$ and continuous quadratic

$$G_2^0 := G_2 - \frac{1}{2} \partial_{\varepsilon\varepsilon}^2 \Phi(0, \mathbf{h})$$

and remainder estimate for $R_3^{0,\varepsilon}(\mathbf{W})$ exactly as in (5.2). Here we call a map $G: \mathcal{C}^{\alpha} \mapsto C[0,1]$ continuous linear if there exists a continuous linear map $G_1: C^{\alpha} \to C[0,1]$ so that $G(\mathbf{W}) = G_1(\mathbf{W})$. (As a trivial consquence, $\sup\{G(\mathbf{W}): |||\mathbf{W}||| \leq 1\} < \infty$ and $G(T_k\delta_{\varepsilon}\mathbf{W}) = G(\mathbf{k}) + \varepsilon G(\mathbf{W})$.) Similarly, call a continuous map $G: \mathbf{W} \mapsto C[0,1]$ continuous quadratic if $\sup\{G(\mathbf{W}): |||\mathbf{W}||| \leq 1\} < \infty$ and

$$G(T_{\mathbf{k}}\delta_{\varepsilon}\mathbf{W}) = G(\mathbf{k}) + \varepsilon G(\mathbf{W}, \mathbf{k}) + \varepsilon^{2}G(\mathbf{W})$$

where $G(\mathbf{W}, \mathbf{k}) = G_2(\mathbf{W}, \mathbf{k})$, for a continuous bilinear map $G_2 : \mathbf{H} \times C^{\alpha} \to C[0, 1]$.

Example 5.1 (Black-Scholes). Let $X_1^{\varepsilon} = \phi_1(\varepsilon, \varepsilon B)$ with $\phi_1(\varepsilon, B) = \varepsilon^2 \mu + \sigma B_1$. (There is no need for a rough path lift here.) After a Girsanov shift $\varepsilon B \to \varepsilon B + h$, have

$$Z_1^{\varepsilon} := \phi_1(\varepsilon, \varepsilon B + h) = \sigma h_1 + \varepsilon \sigma B_1 + \varepsilon^2 \mu$$

which allows to read off g_1, g_2 (and hence G_1, G_2 , in obvious "pathwise" robust form) and zero remainders. Note $\phi_1(\varepsilon, h) = \sigma h_1 + \varepsilon^2 \mu$, so that $\frac{1}{2} \partial_{\varepsilon\varepsilon}^2 \phi_1(0, h) = \mu$. (There is no need here to distinguish between ϕ and Φ). In particular, $G_2^0 \equiv 0$.

5.2. The case of rough volatility. Following the notation of Example 4.3, we consider the small noise setting for rough volatility. Let $\sigma(.)$ be a scalar function. There is interest in avoiding too restrictive growth conditions on $\sigma(.)$ such as to include exponential functions (cf. RoughBergomi [7]), whereas one can safely assume $\sigma(.)$ to be smooth. Recall $\hat{\varepsilon} = \varepsilon^{2H}$, $H \in (0, 1/2]$. Recall that

$$W = (W, \overline{W})$$

consists of two independent (standard) Brownian motions. These are used to construct

$$\widetilde{W} = \overline{\rho} \overline{W} + \rho W \quad \text{and} \quad \widehat{W} = K^H * \dot{W} \ ,$$

so that \widetilde{W} is again a standard Brownian motion (ρ -correlated with W) whereas \widehat{W} is a fractional Brownian motion, only dependent on W. Note that $\widetilde{W} = \overline{W}$ in the uncorrelated case; note also

 $\widehat{W} = W$ in the case of H = 1/2, which falls under the classical StochVol setting. We will use identical notation when dealing with Cameron–Martin paths $h = (h, \overline{h})$, so that $\widetilde{h} = \overline{\rho}\overline{h} + \rho h$, $\widehat{h} = K^H * h$. Under rough volatility, the (rescaled) log-price process has the form

$$\overline{X}_{1}^{\varepsilon} = \int_{0}^{1} \sigma\left(\widehat{\varepsilon}\widehat{W}\right) \widehat{\varepsilon}d\widetilde{W} - \frac{1}{2}\varepsilon\widehat{\varepsilon} \int_{0}^{1} \sigma^{2}\left(\widehat{\varepsilon}\widehat{W}\right)dt .$$

A stochastic Taylor expansion, after a Girsanov shift $\widehat{\varepsilon}(W, \overline{W}) \mapsto \widehat{\varepsilon}(W + h, \overline{W} + \overline{h})$, gives,

$$\overline{Z}_{1}^{\varepsilon,h} = \int_{0}^{1} \sigma\left(\widehat{\varepsilon}\widehat{W} + \widehat{h}\right) d\left[\widehat{\varepsilon}\widetilde{W} + \widetilde{h}\right] - \frac{\varepsilon\widehat{\varepsilon}}{2} \int_{0}^{1} \sigma^{2}\left(\widehat{\varepsilon}\widehat{W} + \widehat{h}\right) dt$$

$$\equiv g_{0} + \widehat{\varepsilon}g_{1}(\omega) + \widehat{\varepsilon}^{2}g_{2}(\omega) + r_{3}(\omega) .$$
(5.3)

We can read off g_0 and $g_1(\omega)$ as zero and first order terms (in $\hat{\varepsilon}$) of the expansion

$$\int_{0}^{1} \sigma\left(\widehat{\varepsilon}\widehat{W} + \widehat{h}\right) d\left[\widehat{\varepsilon}\widetilde{W} + \widetilde{h}\right] = \int_{0}^{1} \sigma\left(\widehat{h}\right) d\widetilde{h} + \widehat{\varepsilon}\left(\int_{0}^{1} \sigma\left(\widehat{h}\right) d\widetilde{W} + \int_{0}^{1} \sigma'\left(\widehat{h}\right) \widehat{W} d\widetilde{h}\right) \\
+\widehat{\varepsilon}^{2}\left(\int_{0}^{1} \sigma'\left(\widehat{h}\right) \widehat{W} d\widetilde{W} + \frac{1}{2} \int_{0}^{1} \sigma''\left(\widehat{h}\right) \widehat{W}^{2} d\widetilde{h}\right) + \dots$$
(5.4)

but the precise form of $g_2(\omega)$ - and thus the remainder $r_3(\omega)$ implicitly defined in (5.3) - requires the following distinction:

Case of H = 1/2. In this case $\hat{\varepsilon} = \varepsilon, \hat{h} = h$ and so

$$g_2(\omega) = \int_0^1 \sigma'(h)\widehat{W}d\widehat{W} + \frac{1}{2} \int_0^1 \sigma''(h)\widehat{W}^2 d\widehat{h} - \frac{1}{2} \int_0^1 \sigma^2(h)dt$$

Remark that in this case the remainder $r_3(\omega)$ is the sum of the implicit (dots) remainder in (5.4) and ε^2 times the difference

$$\frac{1}{2} \int_0^1 \sigma^2 (\varepsilon W + h) dt - \frac{1}{2} \int_0^1 \sigma^2(h) dt = O(\varepsilon \|W\|_{\infty; [0,1]}) ,$$

valid on $\{\omega : \varepsilon ||W(\omega)||_{\infty;[0,1]} < \delta\}$, for any finite δ , which shows that this term contributes $C\varepsilon^3 ||W||_{\infty;[0,1]}$ to the remainder, where C can be taken as C^1 -norm of σ^2 , restricted to a δ -fattening of the set $\{h(t) : 0 \le t \le 1\}$.

Case of H < 1/2. In this case the second order (in $\hat{\varepsilon}$) term is given by

$$g_2(\omega) = \int \sigma'(\widehat{h}) \widehat{W} d\widetilde{W} + \frac{1}{2} \int_0^1 \sigma''(\widehat{h}) \widehat{W}^2 d\widetilde{h}$$

whereas the remainder $r_3(\omega)$ is the sum of the implicit (dots) remainder in (5.4) and

$$-\frac{\varepsilon \widehat{\varepsilon}}{2} \int_{0}^{1} \sigma^{2} \left(\widehat{\varepsilon} \widehat{W} + \widehat{h}\right) dt = O(\varepsilon \widehat{\varepsilon}) = o(\widehat{\varepsilon}^{2}) ,$$

valid on $\{\omega : \varepsilon ||W(\omega)||_{\infty;[0,1]} < \delta\}$, as before.

Robust form. The noise $W = W(\omega)$ can be lifted to a (random) model

$$\mathbf{W} = \left(\mathbf{W}, \int \widehat{W} dW, \int \widehat{W}^2 dW, \dots \right) \in \mathcal{M}$$

and the (log)price process under rough volatility can be written as its continuous image $\Phi(\mathbf{W})$, see [8]. (All this is reviewed in Appendix A.1.2 for the reader's convenience.) The Cameron-Martin

space acts naturally by translation. For any $h \in H$, have homeomorphism $T_h : \mathcal{M} \to \mathcal{M}$, with inverse T_{-h} , which "lifts" the meaning of $W \mapsto W + h$, with an estimate (Lemma A.2), of the form

$$|||T_{h}\mathbf{W}|| \lesssim |||\mathbf{W}||| + ||h||_{H}.$$

This translation is moreover consistent with Cameron–Martin shift of Wiener paths in the sense that

$$\forall h \in H : \mathbf{W}(\omega + h) = T_h \mathbf{W}(\omega) \text{ a.s. }.$$

One further defines dilation of models, similar to rough paths, by

$$\delta_{\varepsilon} \mathbf{W} = \left(\varepsilon \mathbf{W}, \varepsilon^2 \int \widehat{W} dW, \varepsilon^3 \int \widehat{W}^2 dW, \dots \right) \in \mathcal{M} .$$

We then have

$$\overline{Z}^{\varepsilon,h}(\omega) = \Phi^{\varepsilon}(T_h \delta_{\widehat{\varepsilon}} \mathbf{W}(\omega)) ;$$

where we introduced, for an arbitrary $\mathbf{M} \in \mathcal{M}$,

$$\Phi^0(\mathbf{M}) := \int \sigma\left(\widehat{M}\right) d\mathbf{M} \ ,$$

and then, for every $\varepsilon \geq 0$,

$$\Phi^{\varepsilon}(\mathbf{M}) \equiv \Phi(\widehat{\varepsilon}, \mathbf{M}) := \Phi^{0}(\mathbf{M}) - \frac{\varepsilon \, \widehat{\varepsilon}}{2} \int \sigma^{2} \left(\widehat{M}\right) dt \ .$$

Observe that $\partial_{\hat{\epsilon}}\Phi(0,\cdot)\equiv 0$. We see in Theorem A.6 that

$$\Phi^0_1(T_{\mathbf{h}}\delta_{\widehat{\varepsilon}}\mathbf{W}(\omega)) = \Phi^0_1(\mathbf{h}) + \widehat{\varepsilon}G^0_1(\mathbf{W}(\omega)) + \widehat{\varepsilon}^2G^0_2(\mathbf{W}(\omega)) + R^0_3(\mathbf{W}(\omega))$$

where G_1^0, G_2^0 are continuous linear and quadratic, respectively, in the obvious adaption of these notions to the present setting of regularity structures and

$$|R_3^0(\mathbf{W})| \lesssim \widehat{\varepsilon}^3 |||\mathbf{W}|||^3$$

on bounded sets of $\widehat{\varepsilon}||\mathbf{W}||$. Mind that the corresponding expansion

$$\Phi_1^{\varepsilon}(T_{\mathbf{h}}\delta_{\varepsilon}\mathbf{W}(\omega)) = \Phi_1^{0}(\mathbf{h}) + \widehat{\varepsilon}G_1(\mathbf{W}(\omega)) + \widehat{\varepsilon}^2G_2(\mathbf{W}(\omega)) + R_3(\mathbf{W}(\omega))$$

may be different, with

$$G_1^0 = G_1$$
 and $G_2^0 = G_2 - \frac{1}{2} \partial_{\widehat{\epsilon}\widehat{\epsilon}}^2 \Phi(0, \mathbf{h})$

and

$$|R_3(\mathbf{W})| \lesssim o(\widehat{\varepsilon}^2) + \widehat{\varepsilon}^3 |||\mathbf{W}|||^3$$
.

(When H<1/2 we have $G_2^0=G_2$ and the $o(\widehat{\varepsilon}^2)$ -term can be written more quantitatively as $\widehat{\varepsilon}^2\times O(\widehat{\varepsilon}^{1-2H})$. In case H=1/2, have $G_2^0\neq G_2$, in turn the $o(\widehat{\varepsilon}^2)$ -term is $O(\widehat{\varepsilon}^3)$.)

A LDP for rescaled enhanced noise (in the space of models) is also given in [8] and thus induces Forde-Zhang type larges deviations for $\Phi^{\varepsilon}(\delta_{\widehat{\varepsilon}}\mathbf{W})$ with speed $\widehat{\varepsilon}^2$ s, without any growth assumptions on the volatility function $\sigma(.)$. The energy function Λ is smooth near zero [9]; moreover $\Lambda(x) = \frac{1}{2} \|\mathbf{h}^x\|_{\mathbf{H}}^2$ in terms of the unique non-degenerate minimizing control path $\mathbf{h}^x = (h^x, \overline{h}^x)$.

- 5.3. Assumptions (A3-5): robust model specification and control theory. Following the two running examples above, we now present our general conditions. Recall that \mathcal{M} is the space of models, in the case of rough volatility (in the setting of Section 5.2), the construction is detailed in the appendix, but our setup is much more general (and for instance, also, covers generic diffusion StochVol).
- (A3a) There exists a continuous map $\Phi: (\varepsilon, \mathbf{M}) \mapsto \Phi^{\varepsilon}(\mathbf{M})$, from $[0,1] \times \mathcal{M} \to C[0,1]$, such that we have the robust representation

$$\overline{X}^{\varepsilon}(\omega) = \Phi^{\varepsilon}(\delta_{\overline{\varepsilon}}\mathbf{W}(\omega)) \text{ a.s.}$$

where $\delta_{\overline{\varepsilon}}\mathbf{W}$ is the $\overline{\varepsilon}$ -dilation of the random model $\mathbf{W} = \mathbf{W}(\omega)$ which enhances (multidimensional) Brownian motion $W = W(\omega)$, and on which H acts by translation T_h , $h \in H$. We further assume that \mathbf{W} is the (possibly renormalized) limit of canonically lifted Wong-Zakai approximations.²

(A3b) A Schilder-type LDP holds in model topology for $\delta_{\overline{\varepsilon}}\mathbf{W}$ with (good) rate function given by $\mathbf{h} \mapsto \frac{1}{2} \|\mathbf{h}\|_{\mathbf{H}}^2$ when $\mathbf{h} = (\mathbf{h}, ...)$ is the canonical lift of $\mathbf{h} \in \mathbf{H}$, and ∞ else.

Note that condition (A3), by application of the contraction principle, implies a LDP for $(\overline{X}_1^{\varepsilon})$, and thus condition (A1a), with good rate function

(5.5)
$$\Lambda(x) = \inf_{h \in H} \{ \frac{1}{2} \|h\|_{H}^{2} : \Phi_{1}(h) = x \} \equiv \inf_{h \in \mathcal{K}^{x}} \frac{1}{2} \|h\|_{H}^{2} ,$$

where $\mathcal{K}^x \subset H$ denotes the space of x-admissible controls, i.e. elements $h \in H : \Phi(h) = x$ where we abuse notation, in terms of the canonical lift **h** of h, by writing

$$\Phi(\mathbf{h}) = \Phi^0(\mathbf{h}).$$

We also assume a robust "stochastic" Taylor-like expansion, formulated deterministically in terms of models $\mathbf{M} = (M, ...) \in \mathcal{M}$ with $M \in C^{\alpha}$ for suitable α ; cf. Appendix. Care is necessary because of the distinct $\varepsilon, \overline{\varepsilon}$ scaling and

$$\Phi^{\varepsilon}(\delta_{\overline{\varepsilon}}\mathbf{M})) \neq \Phi^{0}(\delta_{\overline{\varepsilon}}\mathbf{M})$$
.

Let B a Banach space (typically C[0,1] or \mathbb{R}). We call a map $G: \mathcal{M} \mapsto B$ continuous linear if there exists a continuous linear map $G_1: C^{\alpha} \to B$ so that $G(\mathbf{M}) = G_1(M)$. As a (trivial) consequence, $\sup\{G(\mathbf{M}): |||\mathbf{M}||| \leq 1\} < \infty$ and, for all $\varepsilon \geq 0$ and $k \in H$,

$$G(T_{\mathbf{k}}\delta_{\varepsilon}\mathbf{M}) = G(\mathbf{k}) + \varepsilon G(\mathbf{M})$$
.

Similarly, call a continuous map $G: \mathbf{M} \mapsto B$ continuous quadratic if $\sup\{G(\mathbf{M}): |||\mathbf{M}||| \le 1\} < \infty$ and

$$G(T_{\mathbf{k}}\delta_{\varepsilon}\mathbf{M}) = G(\mathbf{k}) + \varepsilon G(\mathbf{M}, \mathbf{k}) + \varepsilon^{2}G(\mathbf{M})$$

for a continuous bilinear map $G_2: C^{\alpha} \times \mathbb{H} \to B$ so that $G_2(M, \mathbf{k}) =: G(\mathbf{M}, \mathbf{k})$.

(A4a) For every $h \in H$ there exists $(G_1^{0,h}, G_2^{0,h}, R_3^{0,h,\varepsilon})$ such that for every $\varepsilon \geq 0$ and model $\mathbf{M} = (M, ...) \in \mathcal{M}$ we have

$$\Phi_1^0(T_{\rm h}\delta_{\overline{\varepsilon}}\mathbf{M}) = G_0 + \overline{\varepsilon}G_1^{0,{\rm h}}(\mathbf{M}) + \overline{\varepsilon}^2G_2^{0,{\rm h}}(\mathbf{M}) + R_3^{0,{\rm h},\varepsilon}(\mathbf{M}) \ .$$

²In the (Itô) rough path case, this says precisely that $\mathbf{W} = \lim_{\eta \to 0} T_{-I/2} \mathbf{W}^{\eta}$, where T is a higher-order translation operator on rough path space that subtracts $1/2 \times$ identity matrix from the second order noise. This is a simple instance of the renormalization for the Itô rough volatility model [8], and also higher-order translations for branched rough path [14].

with $G_0^{\rm h}=\Phi_1^0({\rm h})$, continuous linear $G_1^{0,{\rm h}}:{\bf M}\to\mathbb{R}$, continuous quadratic $G_1^{0,{\rm h}}:{\bf M}\to\mathbb{R}$ and order three remainder $R_3^{0,\varepsilon}$, meaning the validity of: whenever $\overline{\varepsilon}||{\bf W}|||\leq \delta$, then

$$|R_3^{0,\varepsilon}(\mathbf{W})| \lesssim o(\overline{\varepsilon}^2) + \overline{\varepsilon}^3 |||\mathbf{W}|||^3$$
;

all of which is assumed to hold uniformly over bounded h. We also assume that $(h, \mathbf{M}) \mapsto G_i^{0,h}(\mathbf{M})$ is continuous for i = 1, 2.

(A4b) Similarly, assume existence of $(G_1^{\rm h},G_2^{\rm h},R_3^{\rm h,\varepsilon})$ such that

$$\Phi_1^{\varepsilon}(T_{\rm h}\delta_{\overline{\varepsilon}}\mathbf{M}) = G_0^{\rm h} + \overline{\varepsilon}G_1^{\rm h}(\mathbf{M}) + \overline{\varepsilon}^2G_2^{\rm h}(\mathbf{M}) + R_3^{{\rm h},\varepsilon}(\mathbf{M}) \ .$$

with same remainder estimate and

$$G_1(\mathbf{M}) \equiv G_1^0(\mathbf{M}) \text{ and } G_2(\mathbf{M}) \equiv G_2^0(\mathbf{M}) + K_2.$$

Remark 5.2. View (A4b) as consequence of C^{2+} -regularity of $\Phi_1^{\varepsilon}(\mathbf{M}) = \Phi_1(\overline{\varepsilon}, \mathbf{M})$ in $\overline{\varepsilon}$, with

$$\partial_{\overline{\varepsilon}}\Phi_1(0,\cdot) \equiv 0$$
 and $\frac{1}{2}\partial_{\overline{\varepsilon}\overline{\varepsilon}}^2\Phi_1(0,h) =: K_2$.

Note that Assumption (A4a) is easily seen to imply C^2 (Fréchet) differentiability of Φ on H, so that it makes sense to consider $D\Phi$ and $D^2\Phi$ at h^x .

The next set of conditions is of control theoretic nature, with x > 0 fixed, and only concern $\Phi = \Phi^0$ as map from H to C[0,1]. Recall that $\mathcal{K}^x \subset H$ denotes the space of elements $h \in H : \Phi_1(h) = x$.

(A5a) There exists a unique minimizer $h^x \in \mathcal{K}^x$, with

$$\Lambda(x) = \frac{1}{2} \|\mathbf{h}^x\|_{\mathbf{H}}^2.$$

(A5b) $\Phi_1: H \to \mathbb{R}$ has, at h^x , a surjective differential

$$D\Phi_1(\mathbf{h}^x) \in L(\mathbf{H} \to \mathbb{R})$$
.

(A5c) The minimizer h^x is non-degenerate, namely

$$q_x D_h^2 \Phi_1(h^x) < \mathrm{Id}$$

(in form sense), where $q_x \in \mathbb{R}$ is such that $h^x = D\Phi_1(h^x)^*q^x$.

The existence of the Lagrange multiplier q^x is discussed in Lemma B.1. Note that assumption (A5) is classical in the SDE context, cf. for instance [6, 43, 16]. We discuss further Assumption (A5c) in Appendix B.2, in particular it can be seen as strict positivity of the Hessian of $I(h) := \frac{1}{2} \|h\|^2$ when restricted to \mathcal{K}^x . Also note that in fact assumptions (A5a)-(A5c) imply that Λ is C^1 at x and then one simply has $q_x = \Lambda'(x)$, cf. Lemma B.5.

We now assume (A4) and (A5). Fix $h = h^x$ as supplied by (A5) and the corresponding rough Taylor expansion $(x, G_1^x, G_2^x, R_3^{x,\varepsilon})$ supplied by Assumption (A4b), using that $\Phi(h^x) = x$. Let $\mathbf{W} = \mathbf{W}(\omega)$ be the Itô lift of Brownian motion. We define the (probabilistic) objects

$$(5.6) g_1^x(\omega) := G_1^{\mathbf{h}^x}(\mathbf{W}(\omega)), \quad g_2^x(\omega) := G_2^{\mathbf{h}^x}(\mathbf{W}(\omega)), \quad r_3^{x,\varepsilon}(\omega) := R_3^{\mathbf{h}^x,\varepsilon}(\mathbf{W}(\omega)).$$

Then the stochastic Taylor expansion for the h^x Girsanov shift of $\overline{X}^{\varepsilon}$ is given by

$$\overline{Z}^{\varepsilon,x}(\omega) = x + \overline{\varepsilon}g_1^x(\omega) + \overline{\varepsilon}^2g_2^x(\omega) + r_3^{x,\varepsilon}(\omega) .$$

We saw in Section 5.1 and 5.2 that condition (A4a-b) holds in the classical StochVol (SDE) case, as well as the RoughVol case.

5.4. The call price formula. We can now state a key formula which generalizes the RoughVol call price formula considered in [9] and which is at the heart of our analysis. It applies both for classical StochVol situations (with H=1/2) where $\Lambda=\mathcal{I}$ has the geometric interpretation of shortest square-distance to some arrival manifold determined by the strike, and to RoughVol with the Forde-Zhang rate function $\Lambda=\mathcal{J}$ as given in (4.1).

Theorem 5.3. Assume (A1-A5) for fixed x > 0. Let $k_{\varepsilon} = x\varepsilon/\overline{\varepsilon}$. Let g_1^x, g_2^x and $r_3^{\varepsilon,x}$ be the stochastic Taylor coefficients / remainder defined in (5.6). Then

$$c(\varepsilon^2, k_{\varepsilon}) := \exp\left(-\frac{\Lambda(x)}{\overline{\varepsilon}^2}\right) e^{k_{\varepsilon}} J(\varepsilon, x)$$

with

$$(5.7) J(\varepsilon, x) = E\left[\exp\left(-\frac{\Lambda'(x)g_1^x}{\overline{\varepsilon}}\right)\left(\exp\left(\varepsilon g_1^x + \overline{\varepsilon}\varepsilon g_2^x + (\varepsilon/\overline{\varepsilon})r_3^{\varepsilon, x}\right) - 1\right)^+\right]$$

Moreover, writing σ_x^2 for the variance of g_1^x one has the differential relation

(5.8)
$$\sigma_x^2 = \frac{2\Lambda(x)}{\Lambda'(x)^2} .$$

Remark 5.4. Assumption (A2) is not really needed here (one just needs exponential integrability of $\overline{X}^{\varepsilon}$ to have finite call prices), and (A5b) is only needed for (5.8) (to ensure that Λ is C^1 at x).

Proof. By the very definition of call price function c and $X_t \sim X_1^{\varepsilon}$ with $t = \varepsilon^2$,

$$c(t,k) = E[(\exp(X_1^{\varepsilon}) - \exp(k)^+].$$

Since $X^{\varepsilon} := \overline{X}^{\varepsilon} \varepsilon / \overline{\varepsilon}$, consider $k = k_{\varepsilon}$ and thus, with $1/\nu_{\varepsilon} = \varepsilon / \overline{\varepsilon}$,

$$c(\varepsilon^2, k_{\varepsilon}) = E[(\exp(\overline{X}_1^{\varepsilon}/\nu_{\varepsilon}) - \exp(k_{\varepsilon}))^+].$$

Our assumptions imply that there is a LDP for $\overline{X}_1^{\varepsilon}(\omega) = \phi_1^{\varepsilon}(\overline{\varepsilon}\omega)$ such that

$$-\varepsilon^2 \log P[\overline{X}_1^\varepsilon > x] = \tfrac{1}{2} \|\mathbf{h}^\mathbf{x}\|_{\mathbf{H}^1}^2 = \Lambda(\mathbf{x})$$

for some unique minimizer of the control problem $\phi_1^0(h) = x$. By Assumption (A4) we have the stochastic Taylor expansion of the form

$$\overline{Z}_1^{\varepsilon}(\omega) = \phi_1^{\varepsilon}(\overline{\varepsilon}\omega + \mathbf{h}^x) = x + \overline{\varepsilon}g_1^x + \overline{\varepsilon}^2g_2^x + r_3^{x,\varepsilon}$$

with the same notations as in (5.6). Apply Girsanov's theorem, $\overline{\varepsilon}W \to \overline{\varepsilon}W + h^x = \overline{\varepsilon}(W + h^x/\overline{\varepsilon})$, and obtain

$$c(\varepsilon^2, k_{\varepsilon}) = E\left[e^{\frac{1}{\overline{\varepsilon}} \int_0^1 \dot{\mathbf{h}}^x d\mathbf{W} - \frac{1}{2\overline{\varepsilon}^2} \|\mathbf{h}^x\|_{H^1}^2 \left(\exp(\overline{Z}_1^{\varepsilon}/\nu_{\varepsilon}) - \exp(k_{\varepsilon})\right)^+\right].$$

We then use first order optimality of h^x to see that (see Appendix B.1)

(5.9)
$$\int_0^1 \dot{\mathbf{h}}^x d\mathbf{W} = \Lambda'(x) g_1^x(\omega)$$

and this establishes the call price formula. At last, using Itô isometry we can write, from (5.9),

$$||h^x||_{H^1}^2 = ||\dot{h}^x||_{L^2}^2 = (\Lambda'(x))^2 Var(g_1^x)$$

and conclude with $||h^x||_{H^1}^2 = 2\Lambda(x)$.

6. Precise asymptotic pricing

We now come to the main result of this paper.

6.1. Precise large deviations.

Theorem 6.1. Assume (A1-A5) for fixed x > 0. Let $k_{\varepsilon} = x\varepsilon/\overline{\varepsilon}$. Then there exists a function $A = A(x) \sim 1$ as $x \downarrow 0$, such that, with $\sigma_x^2 = 2\Lambda(x)/\Lambda'(x)^2$ as earlier,

(6.1)
$$c(\varepsilon^2, k_{\varepsilon}) \sim \exp\left(-\frac{\Lambda(x)}{\overline{\varepsilon}^2}\right) \varepsilon \overline{\varepsilon}^2 \frac{A(x)}{(\Lambda'(x))^2 \sigma_x \sqrt{2\pi}} \quad as \ \varepsilon \downarrow 0.$$

We here derive the (correct) formula with a *formal* computation that ignores the remainder term. The real proof, found in Section 8, relies in particular on the "robustification", cf. Assumption (A4), to handle the remainder. This robustification holds in great generality, including the RoughVol situation.

Proof. In view of the exact call price formula in Theorem 5.3, it suffices to analyse

$$(6.2) J(\varepsilon,x) = E\left[\exp\left(-\frac{\Lambda'(x)g_1^x}{\overline{\varepsilon}}\right)\left(\exp\left(\varepsilon g_1^x + \overline{\varepsilon}\varepsilon\,g_2^x + (\varepsilon/\overline{\varepsilon})\,r_3^{\varepsilon,x}\right) - 1\right)^+\right].$$

We ignore the remainder. With high probability, $(\varepsilon g_1^x + \overline{\varepsilon} \varepsilon g_2^x)$ is small when $\varepsilon, \overline{\varepsilon} \to 0$, hence we expect

$$E\left[\exp\left(-\frac{\Lambda'(x)g_1^x}{\overline{\varepsilon}}\right)\left(\exp(\varepsilon g_1^x + \overline{\varepsilon}\varepsilon g_2^x) - 1\right)^+\right] \sim \varepsilon E\left[\exp\left(-\frac{\Lambda'(x)g_1^x}{\overline{\varepsilon}}\right)\left(g_1^x + \overline{\varepsilon}g_2^x\right)^+\right] \ .$$

With x fixed, write $g_i \equiv g_i^x$. Recall (first order optimality)

$$\Lambda'(x) g_1(\omega) = \int_0^1 \dot{\mathbf{h}}^x d\mathbf{W}$$

and we can then decompose $g_2 \in C_0 \oplus C_2^3$,

$$g_2 = \Delta_2 + g_1 \Delta_1 + g_1^2 \Delta_0$$

where $\Delta = (\Delta_0, \Delta_1, \Delta_2)$ is independent from g_1 , and where $\Delta_0 \in \mathcal{C}_0$, $\Delta_1 \in \mathcal{C}_1$ and $\Delta_2 \in \mathcal{C}_0 \oplus \mathcal{C}_2$. This leaves us with the computation of

$$E\left[\exp\left(-\frac{\Lambda'(x)g_1}{\overline{\varepsilon}}\right)\left(g_1+\overline{\varepsilon}(\Delta_2+g_1\Delta_1+g_1^2\Delta_0)\right)^+\right]=E\left[E(....|\Delta)\right].$$

The inner (conditional) expectation is a simple Gaussian integral for which a finite-dimensional Laplace analysis (cf. Lemma 3.1) gives

$$E(....|\Delta) \sim \frac{\overline{\varepsilon}^2}{(\Lambda'(x))^2 \sigma_x \sqrt{2\pi}} \exp\left(\Lambda'(x)\Delta_2\right) \quad \text{as } \overline{\varepsilon} \to 0.$$

The asymptotic behaviour of the full expectation is then indeed obtained, as one would hope, by averaging over $\Delta_2 = \Delta_2^x(\omega)$, so that

$$E\left[\exp\left(-\frac{\Lambda'(x)g_1}{\overline{\varepsilon}}\right)\left(g_1+\overline{\varepsilon}(\Delta_2+g_1\Delta_1+g_1^2\Delta_0)\right)^+\right]\sim \frac{\overline{\varepsilon}^2}{(\Lambda'(x))^2\sigma_x\sqrt{2\pi}}E\left[\exp\left(\Lambda'(x)\Delta_2^x\right)\right].$$

 $^{{}^{3}}C_{i}$ denotes the *i*-th homogeneous Gaussian chaos

Clearly, such a formula requires $\exp(\Lambda'(x)\Delta_2^x) \in L^1(P)$; in fact, the proof requires $L^{1+}(P)$ and we see in Section 8.2 that this is precisely the case because h^x is a non-degenerate minimizer, cf. assumption (A5). Taking into account the factor $e^{k_{\varepsilon}} = e^{x\varepsilon/\overline{\varepsilon}}$ from Theorem 5.3, see that

(6.3)
$$A(x) = \begin{cases} E\left[\exp(\Lambda'(x)\Delta_2^x)\right], & \text{if } H < 1/2\\ e^x E\left[\exp(\Lambda'(x)\Delta_2^x)\right], & \text{if } H = 1/2 \end{cases}$$

Remark 6.2. Sanity check: Black-Scholes H=1/2, with $\sigma^x\equiv\sigma$, $\Lambda(x)=x^2/(2\sigma^2)$. We then have

$$c(\varepsilon^2, x) \sim \exp\left(-\frac{x^2}{2\varepsilon^2\sigma^2}\right) \varepsilon^3 \frac{\sigma^3}{x^2\sqrt{2\pi}} A(x)$$

which matches precisely the previously derived Black-Scholes expansion (3.1), with $A(x) = e^{x(1+\mu/\sigma^2)}$ as predicated by (6.3) with $\Delta_2 = \mu$. Remark that in the Black-Scholes case, assumptions (A1-A5) are indeed satisfied for any x > 0.

6.2. Precise moderate deviations.

Theorem 6.3. Assume that Assumptions (A1)-(A5) hold for x=0 with $h^0=0$. Let $k_{\varepsilon}=x_{\varepsilon}\varepsilon/\overline{\varepsilon}$ with $x_{\varepsilon}\to 0$, $x_{\varepsilon}/\overline{\varepsilon}\to \infty$. Then

(6.4)
$$c(k_{\varepsilon}, \varepsilon^{2}) \sim_{\varepsilon \to 0} \exp\left(-\frac{\Lambda(x_{\varepsilon})}{\overline{\varepsilon}^{2}}\right) \varepsilon \overline{\varepsilon}^{2} \frac{\sigma_{0}^{3}}{x_{\varepsilon}^{2} \sqrt{2\pi}}.$$

Remark 6.4. Formally, this follows from our precise large deviations (6.1) by replacing x by x_{ε} , and using $x_{\varepsilon} \to 0$. The rigorous proof follows along similar lines as the proof of Theorem 6.1 and is postponed to Section 8.

Remark 6.5. Let $\overline{\varepsilon} = \varepsilon^{2H}$ and consider $\beta \in (2H/3, H)$. Then $x_{\varepsilon} = x\varepsilon^{2\beta}$ falls in the regime of the above theorem and moreover,

$$\Lambda(x_{\varepsilon})/\varepsilon^{4H} \sim \frac{x^2}{2\sigma_0^2 \varepsilon^{4H-4\beta}}$$
 as $\varepsilon \to 0$

and in fact the expansion in (6.4) is nothing else than the Black-Scholes expansion run in the moderate scale, with speed function $\varepsilon^{4H-4\beta}$, instead of the large deviation speed ε^2 . More generally, one can obtain for arbitrary $\beta \in (0, H)$ the expansion

$$\Lambda(x_{\varepsilon})/\varepsilon^{4H} = \sum_{k=2}^{M} \frac{\Lambda^{(k)}(0)}{k!} \frac{x^k}{\varepsilon^{4H-2k\beta}} + o(1)$$
 as $\varepsilon \to 0$

where M is such that $(M+1)\beta > 2H$, if we know that Λ is C^M at 0. (Note that under assumption (A5), C^M regularity of Λ at x simply requires C^{M+1} regularity of Φ on H, cf Lemma B.5.)

7. The case of RoughVol

Recall the RoughVol model as introduced in Example 4.3

$$X_1^{\varepsilon} = \int_0^1 \sigma\left(\widehat{\varepsilon}\widehat{W}\right) \varepsilon d\left(\overline{\rho}\overline{W} + \rho W\right) - \frac{1}{2}\varepsilon^2 \int_0^1 \sigma^2\left(\widehat{\varepsilon}\widehat{W}\right) dt ,$$

with smooth volatility function $\sigma(.)$, and Forde-Zhang energy function \mathcal{J} as given in (4.1). Write $\sigma_0 = \sigma(0)$ for spot-vol and also set $\sigma'_0 = \sigma'(0)$.

Our main result, applied to this model, yields the following result.

Corollary 7.1 (RoughVol). Let $H \in (0, 1/2]$ and $k_{\varepsilon} = x \varepsilon^{1-2H} > 0$. Assume that assumption (A2) is satisfied. Then, for x small enough, $\mathcal{J} = \mathcal{J}(x)$ is continuously differentiable and

(7.1)
$$c(\varepsilon^2, k_{\varepsilon}) \sim \exp\left(-\frac{\mathcal{J}(x)}{\varepsilon^{4H}}\right) \varepsilon^{1+4H} \frac{A(x)}{(\mathcal{J}'(x))^2 \sigma_x \sqrt{2\pi}} \quad as \ \varepsilon \downarrow 0,$$

for some function A(x) with $A(x) \to 1$ as $x \to 0$

Remark 7.2. It is known that Assumption (A2) holds when σ has linear growth, cf. [22]. In the case H=1/2, (A2) holds under much weaker assumptions, e.g. for σ of exponential growth and correlation $\rho < 0$ [49, 42, 45]. We expect similar results to hold in the rough regime but they are not known at the moment.

Proof. It suffices to check that all the assumptions of the theorem are satisfied for x small enough. As mentioned in Section 4, Assumption (A1) follows from [22, 8]. Assumptions (A3) and (A4) (i.e. the regularity structure framework) essentially follow from the analysis in [8], details are given in Appendix A.

Finally we discuss Assumption (A5). Note that if these assumptions hold at $x \ge 0$ they actually hold in a neighborhood of x (using the fact that Φ_1 has continuous second derivatives). In particular it is enough to check that they hold at x = 0. (A5a) is obvious with $h^0 = 0$. By direct computation one has

$$D\Phi_1(0)[k] = \sigma_0 \widetilde{k}_1$$

so that $\sigma_0 \neq 0$ implies that (A5b) holds. Finally, since $q^0 = 0$, (A5c) is obvious since it reduces to the trivial inequality 0 < Id.

8. Proof of main result

We complete the proof of Theorem 6.1.

8.1. Localization of J. We first introduce a localized version of J as given in (6.2). That is, we set

(8.1)
$$J_{\delta}(\varepsilon, x) = E_{\delta} \left[\exp \left(-\frac{\Lambda'(x)g^{x}}{\overline{\varepsilon}} \right) \left(e^{(\overline{\varepsilon}/\nu_{\varepsilon})g_{1}^{x} + (\overline{\varepsilon}^{2}/\nu_{\varepsilon})\mu + (\overline{\varepsilon}^{2}/\nu_{\varepsilon})R^{\varepsilon, x}} - 1 \right)^{+} \right].$$

where the expectation is with respect to the sub-probability

$$P_{\delta}(A) = P(A \cap \{\overline{\varepsilon}||\mathbf{W}|| < \delta\})$$
.

Proposition 8.1. Fix $\delta > 0$. Then there exists $c = c_{x,\delta} > 0$ such that

$$|J_{\delta}(\varepsilon, x) - J(\varepsilon, x)| = O(\exp(-c/\overline{\varepsilon}^2)).$$

Hence any "algebraic expansion" of J (i.e. in powers of $\overline{\varepsilon}$) is unaffected by switching to J_{δ} .

Proof. We revert to the respective expression of J and J_{δ} before the Girsanov shift. To this end, introduce

$$B := \{ \mathbf{M} \in \mathcal{M} : |||T_{-\mathbf{h}^x}\mathbf{M}||| \ge \delta \} ,$$

noting that B^c is a neighbourhood (in model topology) of the canonical lift of h^x . This allows to write, using (6.2), (8.1) and the Girsanov transform in Theorem 5.3, we have

(8.2)
$$J(\varepsilon, x) - J_{\delta}(\varepsilon, x) = \exp\left(\frac{\Lambda(x)}{\overline{\varepsilon}^2}\right) e^{-k_{\varepsilon}} E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+ 1_B] > 0$$

where h^x is (by assumption) the *unique* minimizer. We only need to upper bound this expression, since it is always positive. Let us write

$$E[(\exp(X_1^{\varepsilon}) - \exp(k_{\varepsilon}))^+ 1_B] = E[(\exp(\overline{X}_1^{\varepsilon}/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^+ 1_B]$$

We localize the call payoff $\psi(z) := (e^z - 1)^+$ to an ATM neighbourhood. For fixed b > x we split the expectation over the two sets $\{\overline{X}_1^{\varepsilon} \geq b\}$ and $\{\overline{X}_1^{\varepsilon} < b\}$. We have

$$\begin{split} &E[(\exp(\overline{X}_{1}^{\varepsilon}/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^{+}1_{B}] \\ &\leq E(\exp(\overline{X}_{1}^{\varepsilon}/\nu_{\varepsilon}) - \exp(b/\nu_{\varepsilon}))^{+} + E[(\exp(b/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^{+}1_{\overline{X}_{1}^{\varepsilon} \geq b}] \\ &+ E[(\exp(\overline{X}_{1}^{\varepsilon}/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^{+}1_{\overline{X}_{1}^{\varepsilon} < b}1_{B}] \end{split}$$

Because of the call price (upper) large deviation estimate in Proposition 4.9

(8.3)
$$E[(\exp(\overline{X}_1^{\varepsilon}/\nu_{\varepsilon}) - \exp(b/\nu_{\varepsilon}))^+] \le \exp(-(\Lambda(b) + o(1))/\overline{\varepsilon}^2)$$

and since b > x it is clear that $\Lambda(b) > \Lambda(x)$. The same is true for

(8.4)
$$E[(\exp(b/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^{+} 1_{\overline{X}_{1}^{\varepsilon} > b}] \leq \frac{(\exp(b) - \exp(x))^{+}}{\nu_{\varepsilon}} P(\overline{X}_{1}^{\varepsilon} \geq b) .$$

It remains to deal with the localized term, using $\nu_{\varepsilon} \geq 1$ for $\varepsilon \leq 1$,

$$(8.5) E[(\exp(\overline{X}_1^{\varepsilon}/\nu_{\varepsilon}) - \exp(x/\nu_{\varepsilon}))^{+} 1_{\overline{X}_{1}^{\varepsilon} < b} 1_{B}] \le e^{b} P[\{\overline{X}_{1}^{\varepsilon} \in [x,b)\} \cap B] .$$

An upper bound on this is given by e^b times

$$P[\{\overline{X}_1^{\varepsilon} \geq x\} \cap B] = P[\Phi_1^{\varepsilon}(\overline{\varepsilon}\mathbf{W}) \geq x; |||T_{-\mathbf{h}^x}\delta_{\overline{\varepsilon}}\mathbf{W}||| \geq \delta]$$

Introduce the set

$$A^{x,\delta} = \{ \mathbf{M} : \Phi_1^0(\mathbf{M}) \ge x; |||T_{-\mathbf{h}^x}\mathbf{M}||| \ge \delta \}$$

By assumption (A3) we find that

$$P[\Phi_1^{\varepsilon}(\bar{\varepsilon}\mathbf{W}) \geq x; |||T_{-\mathbf{h}^x}\delta_{\overline{\varepsilon}}\mathbf{W}||| \geq \delta] \leq e^{-(\|\overline{\mathbf{h}}_{x,\delta}\|_{H^1}^2 + o(1))/(2\bar{\varepsilon}^2)} \ .$$

where (as before, **h** denotes the canonical lift of $h \in H$ to a model)

$$\mathbf{h}^{x,\delta} \in \mathop{\mathrm{argmin}} \inf_{\mathbf{h} \in \mathcal{H}} \{ \tfrac{1}{2} \| \mathbf{h} \|_{H^1}^2 : \mathbf{h} \in A_{\delta,x} \}$$

(The infimum over a closed set of a good rate function is attained, although there may be many minimizers.) Since $h^{x,\delta}$ is clearly x-admissible, we have $\|h^{x,\delta}\|_{H} \ge \|h^{x}\|_{H}$. But this inequality must be strict, for otherwise the assumed uniqueness of the minimizer implies

$$\mathbf{h}^{x,\delta} = \mathbf{h}^x$$

which is not possible since $h^{x,\delta} \in A^{x,\delta} \subset B$, i.e. outside a neighbourhood of h^x . Set $2\eta := \|h^{x,\delta}\|_{\mathcal{H}}^2 - \|h^x\|_{\mathcal{H}}^2$. The proof is then finished by setting $c = \min\{(\Lambda(b) - \Lambda(x))/2, \eta/2\} > 0$.

8.2. **Local analysis.** With $h^x \in H$ fixed, assumption (A4) provides us with $(G_1^x, G_2^x, R_3^{x,\varepsilon})$ so that, restricted to H,

$$G_1^x = D\Phi_1(\mathbf{h}^x), \ \ G_2^x = D^2\Phi_1(\mathbf{h}^x)$$
 (as quadratic form) .

On the other hand, with K_2 as in Assumption (A4b),

$$g_1^x(\omega) = G_1^x(\mathbf{W}(\omega)), \ g_2^x(\omega) = G_2^x(\mathbf{W}(\omega)) = G_2^{0,x}(\mathbf{W}(\omega)) + K_2$$

Clearly g_1^x is a zero mean Gaussian, $N(0, \sigma_x^2)$, say. We then proceed as in [6] and introduce the zero mean Gaussian process $V = V^x$

$$V_t(\omega) := W_t(\omega) - g_1^x(\omega)v_t$$

where v is chosen so that V is independent from g_1^x . We now describe such a v explicitely, which also requires identifying g_1^x as a certain Wiener integral.

Lemma 8.2. (i) Identifying $D\Phi_1(h^x)$ with an element of H, one has the equality of random variables

(8.6)
$$g_1^x(\omega) = \langle D\Phi_1(\mathbf{h}^x), \mathbf{W} \rangle$$

where the r.h.s. is a Wiener integral.

(ii) Let

(8.7)
$$\mathbf{v} = \frac{D\Phi_1(\mathbf{h}^x)}{\|D\Phi_1(\mathbf{h}^x)\|^2}, \quad \mathbf{V}(\omega) = \mathbf{W}(\omega) - \mathbf{v}g_1^x(\omega)$$

Then V is independent from g_1^x .

Proof. (i) By definition, for all k in H one has $\langle D\Phi(\mathbf{h}^x), \mathbf{k} \rangle_{\mathbf{H}} = G_1^x(\mathbf{k})$. Hence we have by assumption (A3) and (A4)

$$G_1^x(\mathbf{W}) = \lim_{\eta \to 0} G_1^x(\mathfrak{R}^{\eta} \mathbf{W}^{\eta})$$

$$= \lim_{\eta \to 0} G_1^x(W^{\eta})$$

$$= \lim_{\eta \to 0} \langle D\Phi_1(\mathbf{h}^x), \mathbf{W}^{\eta} \rangle$$

$$= \langle D\Phi_1(\mathbf{h}^x), \mathbf{W} \rangle.$$

(ii) It suffices to show that for all $\mathbf{k} \in \mathbf{H}$, $E\left[\langle \mathbf{V}, \mathbf{k} \rangle g_1^x\right] = 0$, which is an easy consequence of Itô isometry and the definition of \mathbf{v} .

Note that V is not adapted to the filtration generated by W so that a random model which lifts V cannot be constructed by elementary Itô-integration. Instead, we give a pathwise construction, by applying

$$\mathbf{V} := T_{-G_1^x(\mathbf{W})_{\mathbf{V}}} \mathbf{W}$$
,

with $\mathbf{W} = \mathbf{W}(\omega)$. (The resulting random $\mathbf{V}(\omega)$ is then a "Gaussian model", cf. [36].)

Lemma 8.3. (i) The Gaussian model $\mathbf{V} = \mathbf{V}(\omega)$ is measurably determined from V and hence independent from $g_1^x(\omega)$.

(ii) There exists a $\Delta^x = (\Delta_0^x, \Delta_1^x, \Delta_2^x)$ P-independent from $g_1^x(\omega)$, such that (omit superscript x)

$$g_2 = \Delta_2 + g_1 \Delta_1 + g_1^2 \Delta_0$$

(iii) The rescaled family $\delta_{\varepsilon} \mathbf{V}$ satisfies a LDP in model topology, with good rate function given by $\mathbf{v} \mapsto \frac{1}{2} \|\mathbf{h}\|_{H}^{2}$ whenever \mathbf{v} is the canonical lift of $\mathbf{h} \in \mathbf{H}_{0}$, and $+\infty$ else.

(iv)
$$\exp(|||\mathbf{V}|||^2) \in L^{0+}$$
.

Proof. (i) Let W^{η} , V^{η} denote convolution with a mollifier function, with rescale parameter $\eta > 0$. Of course, $V^{\eta} \to V$ uniformly with uniform Hölder bounds. Using the same notation for mollification of W and v, have

$$V^{\eta} = W^{\eta} - g_1^x(\omega) v^{\eta} .$$

Call \mathbf{W}^{η} the canonical model lift of \mathbf{W}^{η} . By assumption (A3) there exists a renormalized approximation $\widehat{\mathbf{W}}^{\eta} = \mathfrak{R}^{\eta} \mathbf{W}^{\eta}$ which converges as $\eta \to 0$, in probability and model topology, to the Itô-model \mathbf{W} . Since translation commutes with renormalisation

$$\mathfrak{R}^{\eta}T_{-G_{1}^{x}(\mathbf{W})\mathbf{v}^{\eta}}\mathbf{W}^{\eta} = T_{-G_{1}^{x}(\mathbf{W})\mathbf{v}^{\eta}}\widehat{\mathbf{W}}^{\eta} \to T_{-G_{1}^{x}(\mathbf{W})}\mathbf{W} = \mathbf{V} \text{ as } \eta \to 0$$
.

It now suffices to note that $T_{-G_1^x(\mathbf{W})\mathbf{v}^{\eta}}(\mathbf{W}^{\eta})$ is precisely the canonical model lift of, and hence measurably determined by, $\mathbf{V}^{\eta} = \mathbf{W}^{\eta} - g_1^x(\omega)\mathbf{v}^{\eta} = \mathbf{W}^{\eta} - G_1^x(\mathbf{W})\mathbf{v}^{\eta}$.

(ii) Straightforward from the definition $g_2^x(\omega) = G_2(\mathbf{W}(\omega)) = G_2^0(\mathbf{W}(\omega)) + K_2$. By assumption (A4), the map G_2^0 is continuous quadratic; applied with $\mathbf{W} = T_{g_1(\omega)\mathbf{v}}\mathbf{V}$ this gives the claimed decomposition of g_2 , with

$$\Delta_2(\omega) = G_2(\mathbf{V}(\omega)) + K_2$$
, $\Delta_1(\omega) = G_2(\mathbf{V}(\omega), \mathbf{v})$ and $\Delta_0 = G_2(\mathbf{v})$.

(iii) Recall that $G_1(\mathbf{W}) = \langle D\Phi_1(\mathbf{h}^x), W \rangle$, so that for h in H, by the definition (8.7) of v,

$$h - G_1(\mathbf{h})v = h - \frac{\langle D\Phi_1(\mathbf{h}^x), \mathbf{h} \rangle}{\|D\Phi_1(\mathbf{h}^x)\|^2} h^x = P_{\mathbf{H}_0} h.$$

Recalling the LDP satisfied by \mathbf{W} , by the contraction principle this implies that $\delta_{\varepsilon}\mathbf{V} = T_{-G_1^x(\delta_{\varepsilon}\mathbf{W})v}\mathbf{W}$ satisfies a LDP with rate function given by

$$I(\Pi) = \inf\{\frac{1}{2}||h||^2, h \in \mathcal{H}_0, \Pi = \mathbf{h}\}.$$

(iv) **W** is a Gaussian model and we have a Fernique estimate for its homogenous norm $|||\mathbf{W}|||$, cf. Appendix A.3. Since $|||\mathbf{V}||| \lesssim |||\mathbf{W}||| + ||\mathbf{v}^x|||g_1||$ where g_1 is a Gaussian, the claim follows. \square

We finally show that the non-degeneracy assumption (A5c) is actually equivalent to an exponential integrability property for the Wiener functional Δ_2^x (defined in Lemma 8.3 (ii) above)

Proposition 8.4. Let $A^x = D^2\Phi_1(h^x)$, then

$$\Lambda'(x)A^x < \text{Id (as form on H_0)} \Leftrightarrow \exp(\Lambda'(x)\Delta_2) \in L^{1+}$$

Proof. Note that $\exp(\Lambda'(x)\Delta_2) \in L^{1+}$ if and only if

(*): there exists $C > \Lambda'(x)$ such that, for all r large enough $P(\Delta_2^x \ge r) \le \exp(-Cr)$.

Recall that

$$\Delta_2 = G_2(\mathbf{V}) - K_2,$$

where by assumption G_2 is quadratic, so that $\varepsilon^2 G_2(\mathbf{V}) = G_2(\delta_{\varepsilon} \mathbf{V})$ where G_2 is a continuous function. By Lemma 8.3 (iii), we know that $\delta_{\varepsilon} \mathbf{V}$ satisfies a LDP, so that by the contraction principle one has

$$P\left(\Delta_2 \ge \varepsilon^{-2}\right) \sim P\left(\varepsilon^2 G_2(\mathbf{V}) \ge 1\right) = \exp\left(-\frac{C^* + o(1)}{\varepsilon^2}\right)$$

where

$$C^* = \inf_{\mathbf{h} \in \mathcal{H}'} \left\{ \frac{1}{2} \left\| \mathbf{h} \right\|^2 : \frac{1}{2} \partial_{\varepsilon}^2 |_{\varepsilon = 0} \Phi \left(\varepsilon \mathbf{h} + \mathbf{h}^x \right) \ge 1 \right\} = \frac{1}{2} \| \mathbf{h}^* \|^2$$

(the existence of $h^* \in H_0 \setminus \{0\}$ comes from a compactness argument). Hence (*) above is reduced $(r = 1/\varepsilon^2....)$ to the question $C^* > \Lambda'(x)$.

Note also

$$\frac{1}{2}A^{x}\left(h^{*},h^{*}\right)=\partial_{\varepsilon}^{2}|_{\varepsilon=0}\Phi_{1}\left(\varepsilon h^{*}+h^{x}\right)\geq1$$

and in fact equals one. But then

$$C^* - \Lambda'(x) = \frac{1}{2} \|\mathbf{h}^*\|^2 - \frac{1}{2} \Lambda'(x) \mathbf{A}^x (\mathbf{h}^*, \mathbf{h}^*)$$

so that the non-degeneracy condition clearly implies that $C^* > \Lambda'(x)$. On the other hand, if the non-degeneracy condition fails there exists h with $1 = \frac{1}{2} A^x(h,h) \ge \frac{1}{2\Lambda'(x)} \|h\|^2$ which implies that $C \le \Lambda'(x)$.

Proposition 8.5. Fix x > 0. Then one has as $\varepsilon \to 0$

$$J(\varepsilon, x) \sim \varepsilon \overline{\varepsilon}^2 \frac{1}{\sqrt{2\pi}} E\left[\exp\left(\Lambda'(x) \Delta_2^x\right)\right].$$

Proof. In this proof we denote by C positive constants, whose value may change from line to line. By Proposition 8.1 we can work with the sub-probability $P_{\delta} = P(...; \overline{\varepsilon}||\mathbf{W}|| < \delta)$, i.e. on the part of the probability space where the model remainder estimates are available. We want to estimate

$$J_{\delta}(\varepsilon, x) = E_{\delta} \left[\exp \left(-\frac{\Lambda'(x)g_1^x}{\overline{\varepsilon}} \right) \left(e^{\varepsilon g_1^x + \varepsilon \overline{\varepsilon} g_2^x + (\varepsilon/\overline{\varepsilon})r_3^{\varepsilon, x}} - 1 \right)^+ \right]$$

Recall $\varepsilon = \overline{\varepsilon}/\nu_{\varepsilon}$. Recall also $r_3^{\varepsilon,x}(\omega) = R_3^{\varepsilon,x}(\mathbf{W}(\omega))$. For this "robustified" remainder Assumption (5b) applies so that $|R_3^{x,\varepsilon}(\mathbf{W})| \lesssim o(\overline{\varepsilon}^2) + \overline{\varepsilon}^3 |||\mathbf{W}|||^3$ whenever $\overline{\varepsilon}|||\mathbf{W}||| \leq \delta$. Thus, for $\overline{\varepsilon}$ small enough (depending on δ), and for a suitable positive constant C,

$$|R_3^{x,\varepsilon}(\mathbf{W})| \le C\delta\overline{\varepsilon}^2 + \overline{\varepsilon}^3|||\mathbf{W}|||^3 \le \delta\overline{\varepsilon}^2(C + |||\mathbf{W}|||^2)$$
.

Since we also have $\overline{\varepsilon}g_1(\omega) = \overline{\varepsilon}G_1(\mathbf{W}(\omega)) = O(\delta)$ and $\overline{\varepsilon}^2g_2(\omega) = \overline{\varepsilon}^2G_2(\mathbf{W}(\omega)) = O(\delta^2)$ when working with P_{δ} , we see that for some constant C > 0 one has

$$J_{\delta}(\varepsilon, x) \in \varepsilon(1 \pm C\delta) E_{\delta} \left[\exp \left(-\frac{\Lambda'(x)g_1^x}{\overline{\varepsilon}} \right) (g_1^x + \overline{\varepsilon}[g_2^x \pm \delta(C + |||\mathbf{W}|||^2)])^+ \right].$$

Recall by Lemma 8.3 one has

$$g_2 = \Delta_2 + g_1 \Delta_1 + g_1^2 \Delta_0$$

where the Δ_i are independent from g_1 , and we let

$$\widetilde{\Delta}_0 := \Delta_0 + C\delta \|\mathbf{v}\|_{\mathbf{H}}^2,$$

$$\widetilde{\Delta}_2^{\pm} := \Delta_2 \pm \delta(C + |||\mathbf{V}|||),$$

where $\widetilde{\Delta}_2^{\pm}$ is also *P*-independent from g_1 . Note also that

$$|\overline{\varepsilon}\Delta_1| = |G_2(\delta_{\overline{\varepsilon}}\mathbf{V}, \mathbf{v}) \le C\overline{\varepsilon}||\mathbf{V}|| \le C\delta$$

when $\overline{\varepsilon}||\mathbf{W}|| \leq \delta$. Thus, the asymptotic behaviour of $J_{\delta}(\varepsilon, x)$ is sandwiched by $\varepsilon(1 \pm C\delta)$ times

$$\dots \in E_{\delta} \left[\exp \left(-\frac{\Lambda'(x)g_1}{\overline{\varepsilon}} \right) \left(g_1 + \overline{\varepsilon} (\widetilde{\Delta}_2^{\pm} + g_1 \Delta_1 \pm g_1^2 \widetilde{\Delta}_0^{\pm})^+ \right] \right]$$

$$\in E_{\delta} \left[\exp \left(-\frac{\Lambda'(x)g_1}{\overline{\varepsilon}} \right) \left(g_1 + \overline{\varepsilon} \widetilde{\Delta}_2^{\pm} \pm C(1 + \widetilde{\Delta}_0) \delta |g_1| \right)^+ \right]$$

We now prove the upper bound for the asymptotics. Clearly,

$$E_{\delta}\left[\exp\left(-\frac{\Lambda'(x)g_1}{\overline{\varepsilon}}\right)\left(g_1+\overline{\varepsilon}\widetilde{\Delta}_2^+\pm C(1+\widetilde{\Delta}_0)\delta|g_1|\right)^+\right]\leq E\left[\cdots\right]$$

where \cdots means the same argument. Set

$$\gamma_{\delta} := C(1 + \widetilde{\Delta}_0)\delta$$

and assume that δ is small enough that $\gamma_{\delta} < 1$. By Lemma 3.2, we have that

$$E\big[\cdots|\Delta_2,\mathbf{V}\big] \leq \frac{\overline{\varepsilon}^2}{\sqrt{2\pi}}\sigma_x \max\left[(1-\gamma_\delta)e^{\frac{\Lambda'(x)\left(\Delta_2+\delta\left(C+|||\mathbf{V}|||^2\right)\right)}{1-\gamma_\delta}} + 2\gamma_\delta, (1+\gamma_\delta)e^{\frac{\Lambda'(x)\left(\Delta_2+\delta\left(C+|||\mathbf{V}|||^2\right)\right)}{1+\gamma_\delta}} \right].$$

By Proposition 8.4 and Assumption (A5c), $\exp(\Lambda'(x)\Delta_2) \in L^{1+}$ and by Lemma 8.3 (iv) $\exp(|||\mathbf{V}|||^2) \in L^{0+}$, so that by letting successively $\bar{\varepsilon}$ and δ go to 0 we obtain that

$$\limsup_{\varepsilon \to 0} \overline{\varepsilon}^{-2} E \left[\, \cdots \, \right] \leq \frac{1}{\sqrt{2\pi}} E \left[\exp \left(\Lambda' \left(x \right) \Delta_2 \right) \right].$$

The lower bound is proved in the same way using the lower bound in Lemma 3.2.

8.3. **Proof of Theorem 6.3.** The proof is similar to that of Theorem 6.1 (but simpler since we only need to expand to first order), and we keep the same notations. By Lemma B.5, for x small enough the minimizer h^x is unique and is C^2 as a function of x, in particular Λ is C^1 at 0^+ . Note also that equation (8.7) and the fact that $g_1^x = \langle D\Phi(h^x), W \rangle$ imply that

$$\sigma_x \|\mathbf{v}^x\| = 1.$$

The proof proceeds as in the large deviation case, and after the same Girsanov transform, we are left with

$$E\left[\exp\left(-\frac{\Lambda'(x_{\varepsilon})\sigma_{x_{\varepsilon}}N_{1}}{\overline{\varepsilon}}\right)\left(e^{\varepsilon\sigma_{x_{\varepsilon}}N_{1}+\varepsilon\overline{\varepsilon}R_{2}^{x}}-1\right)^{+}\right]$$

where $N_1 \sim \mathcal{N}(0,1)$. Note that on $\|\varepsilon \mathbf{W}\| \leq 1$ we have (uniformly in x near 0)

(8.8)
$$R_2^x \le C(1 + \|\mathbf{W}\|^2) \le C(1 + \|\mathbf{V}^x\|^2 + \sigma_x |N_1| \|\mathbf{v}^x\|)$$

and in addition by the LDP for W the term

$$E\left[\dots 1_{\{\|\widehat{\varepsilon}\mathbf{W}\|\geq 1\}}\right] \leq \exp\left(-\frac{c}{\widehat{\varepsilon}^2}\right)$$

is negligible compared to what we want. We are therefore left with

$$E\left[\exp\left(-\frac{\Lambda'(x_{\varepsilon})\sigma_{x}N_{1}}{\widehat{\varepsilon}}\right)\left(\exp\left(\varepsilon\sigma_{x}N_{1}\pm\varepsilon\widehat{\varepsilon}C\left(1+\|\mathbf{V}^{x}\|^{2}+\sigma_{x}|N_{1}|\|\mathbf{v}^{x}\|\right)\right)-1\right)^{+}\right]$$

which by using Lemma 3.2 as in the LDP case is equivalent to

$$\varepsilon \frac{\widehat{\varepsilon}^{2}}{\sqrt{2\pi}\Lambda'(x_{\varepsilon})} E\left[\exp\left(\pm C\Lambda'(x_{\varepsilon})\left(1+\|\mathbf{V}^{x}\|^{2}\right)\right)\right].$$

Finally we use that

$$\|\mathbf{V}^x\| \lesssim \|\mathbf{W}\| + \|\mathbf{v}^x\||g_1^x| \leq \|\mathbf{W}\| + |N_1|$$

has Gaussian tails, uniformly in x, so that since $\Lambda'(0^+) = 0$ we obtain

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\pm C\Lambda'(x_{\varepsilon})\left(1 + \|\mathbf{V}^x\|^2\right)\right)\right] = 1,$$

which concludes the proof.

APPENDIX A. ELEMENTS OF REGULARITY STRUCTURES

A.1. The rough vol model. We review the essentials of [8].

A.1.1. Basic pricing setup. Recall $\widehat{W}_t = \int_0^t K^H(s,t) dW_s$, with $K^H(s,t) = \sqrt{2H}|t-s|^{H-1/2}$, $H \in (0,1/2]$. Given a (sufficiently) smooth scalar function f, we are interested in robust integration of $\int f(\widehat{W}) dW$. The Hölder exponent of \widehat{W} is $H - \kappa$, any $\kappa > 0$. Let M be the smallest integer such that (M+1)H - 1/2 > 0 and then pick κ small enough such that

(A.1)
$$(M+1)(H-\kappa) - 1/2 - \kappa > 0.$$

(In case H=1/2, have M=1 and then $1/2-\kappa\in(1/3,1/2)$ which is the rough path case.) The model space is defined as

(A.2)
$$\mathcal{T} = \left\langle \left\{ \Xi, \Xi \mathcal{I}(\Xi), \dots, \Xi \mathcal{I}(\Xi)^M, \mathbf{1}, \mathcal{I}(\Xi), \dots, \mathcal{I}(\Xi)^M \right\} \right\rangle,$$

where $\langle \ldots \rangle$ denotes the vector space generated by the (purely abstract) symbols in $\{\ldots\} =: S$. The symbol $\mathcal{I}(\ldots)$ represents "integration against the kernel K^H ", so that $\mathcal{I}(\Xi)$ represents fractional Brownian motion \widehat{W} . Symbols like $\Xi \mathcal{I}(\Xi)^m = \Xi \cdot \mathcal{I}(\Xi) \cdot \ldots \cdot \mathcal{I}(\Xi)$ or $\mathcal{I}(\Xi)^m = \mathcal{I}(\Xi) \cdot \ldots \cdot \mathcal{I}(\Xi)$ should be read as products between the objects above. Every symbol $\tau \in S$ has a homogeneity, defined as

$$|\Xi \mathcal{I}(\Xi)^m| = -1/2 - \kappa + m(H - \kappa), \ m \ge 0$$
$$|\mathcal{I}(\Xi)^m| = m(H - \kappa), \ m > 0$$
$$|\mathbf{1}| = 0.$$

Introduce the set of homogeneities $A := \{ |\tau| | \tau \in S \}$, with minimum $|\Xi| = -1/2 - \kappa$. Note that the homogeneities are multiplicative in the sense that, $|\tau \cdot \tau'| = |\tau| + |\tau'|$ for $\tau, \tau' \in S$.

At last, we have the *structure group* G, an (abstract) group of linear operators on the model space \mathcal{T} which should satisfy $\Gamma \tau - \tau \in \bigoplus_{\tau' \in S: |\tau'| < |\tau|} \mathbb{R}\tau'$ and $\Gamma \mathbf{1} = \mathbf{1}$ for $\tau \in S$ and $\Gamma \in G$. We will choose $G = \{\Gamma_h \mid h \in (\mathbb{R}, +)\}$ given by

$$\Gamma_h \mathbf{1} = \mathbf{1}, \ \Gamma_h \Xi = \Xi, \ \Gamma_h \mathcal{I}(\Xi) = \mathcal{I}(\Xi) + h \mathbf{1}.$$

and $\Gamma_h(\tau' \cdot \tau) = \Gamma_h \tau' \cdot \Gamma_h \tau$ for τ' , $\tau \in S$ for which $\tau \cdot \tau' \in S$ is defined.

The Itô model (Π, Γ) . To give a meaning to the product terms $\Xi \mathcal{I}(\Xi)^k$ we follow the idea from rough paths and define "iterated integrals" for $s, t \in \mathbb{R}, s \leq t$ as

(A.3)
$$\mathbb{W}^m(s,t) = \int_s^t (\widehat{W}(r) - \widehat{W}(s))^m dW(r) .$$

We are now in the position to define a model (Π, Γ) that gives a rigorous meaning to the interpretation we gave above for $\Xi, \mathcal{I}(\Xi), \Xi \mathcal{I}(\Xi), \ldots$ Recall that in the theory of regularity structures a model is a collection of linear maps $\Pi_s : \mathcal{T} \to C_c^1(\mathbb{R})'$, $\Gamma_{st} \in G$ such that

$$\Pi_t = \Pi_s \Gamma_{st},$$

$$(A.5) |\Pi_s \tau(\varphi_s^{\lambda})| \lesssim \lambda^{|\tau|}.$$

(A.6)
$$\Gamma_{st}\tau = \tau + \sum_{\tau' \in S: |\tau'| < \tau} c_{\tau'}(s,t)\tau', |c_{\tau'}(s,t)| \lesssim |s-t|^{|\tau|-|\tau'|}$$

where the bounds hold uniformly for $\tau \in S$, any s,t in a compact set and for $\varphi_s^{\lambda} := \lambda^{-1} \varphi(\lambda^{-1}(\cdot - s))$ with $\lambda \in (0,1]$ and $\varphi \in C^1$ with compact support in the ball B(0,1). We define the following "Itô" model $(\Pi, \Gamma) = (\Pi^{\text{Itô}}, \Gamma^{\text{Itô}})$.

$$\begin{split} &\Pi_s \mathbf{1} = 1 & \Gamma_{ts} \mathbf{1} = \mathbf{1} \\ &\Pi_s \Xi = \dot{W} & \Gamma_{ts} \Xi = \Xi \\ &\Pi_s \mathcal{I}(\Xi)^m = \left(\widehat{W}(\cdot) - \widehat{W}(s)\right)^m & \Gamma_{ts} \mathcal{I}(\Xi) = \mathcal{I}(\Xi) + (\widehat{W}(t) - \widehat{W}(s)) \mathbf{1} \\ &\Pi_s \Xi \mathcal{I}(\Xi)^m = \left\{t \mapsto \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{W}^m(s,t)\right\} & \Gamma_{ts} \tau \tau' = \Gamma_{ts} \tau \cdot \Gamma_{ts} \tau' \,, \quad \text{for } \tau, \tau' \in S \text{ with } \tau \tau' \in S \end{split}$$

We extend both maps from S to \mathcal{T} by imposing linearity.

Lemma A.1. [8] The pair $(\Pi^{It\hat{o}}, \Gamma^{It\hat{o}})$ defines (a.s.) a model on (\mathcal{T}, A) .

A.1.2. Full regularity structure for rough volatility. Rough volatility is specified in terms of two independent Brownians (W, \overline{W}) . Writing $\overline{\Xi}$ for the abstract symbol that corresponds to (the Schwartz derivative) of \overline{W} this leads to

(A.7)
$$\overline{\mathcal{T}} = \mathcal{T} + \left\langle \left\{ \overline{\Xi}, \overline{\Xi} \mathcal{I}(\Xi), \dots, \overline{\Xi} \mathcal{I}(\Xi)^M \right\} \right\rangle.$$

Again we fix $|\Xi| = -1/2 - \kappa$ and the homogeneity of the other symbols are defined multiplicatively as before. We extend the Itô model (Π, Γ) to this regularity structure by defining

$$\Pi_s \overline{\Xi} \mathcal{I}(\Xi)^m = \left\{ t \mapsto \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_s^t \left(\widehat{W}(u) - \widehat{W}(s) \right)^m d\overline{W}(u) \right) \right\}$$

(the above integral being in Itô sense), and ⁴

$$\Gamma_{ts}\left(\overline{\Xi}\mathcal{I}(\Xi)^m\right) = \overline{\Xi}\Gamma_{ts}\left(\mathcal{I}(\Xi)^m\right).$$

Hairer's reconstruction theorem ([36], [24, Sec. 13]) then yields a robust view on Itô-integration and, in particular, identifies objects such as $\int \sigma(\widehat{W}) d\widetilde{W}$ and asset price give as stochastic exponential of $\int \sigma(\widehat{W}) d\widetilde{W}$, with $\widetilde{W} = \rho W + \overline{\rho} \overline{W}$ as continuous function Φ of the Itô model.

⁴Upon setting $\Gamma_{ts}(\overline{\Xi}) = \overline{\Xi}$, the given relation is precisely implied by multiplicativity of Γ .

A.1.3. Rough path like formalism. We note that the Itô model $(\Pi^{\text{Itô}}, \Gamma^{\text{Itô}})$ constructed above is in one-one corresponds with objects of the form

$$\mathbf{W}(\omega) = \left(W, \overline{W}, \int \widehat{W} dW, \int \widehat{W} d\overline{W}, \int \widehat{W}^2 dW,, \int \widehat{W}^M d\overline{W}\right) ,$$

with model norm of $(\Pi^{\text{It\^{o}}}, \Gamma^{\text{It\^{o}}})$ comparable with

$$(\mathrm{A.8}) \qquad \qquad \|\mathbf{W}\| := \|W\|_{1/2-\kappa} + \|\overline{W}\|_{1/2-\kappa} + \ldots + \left\| \int \widehat{W}^M d\overline{W} \right\|_{M(H-\kappa)+1/2-\kappa} \,,$$

generalizing the α -Hölder norm (with $\alpha = 1/2 - \kappa$) of the Brownian noise W = (W, \overline{W}) . Here, for instance, and working on [0, T], the final summand is spelled out as

$$\sup_{0 < s < t < T} \frac{\left| \int_s^t \widehat{W}_{s,r}^M d\overline{W}_r \right|}{|t - s|^{M(H - \kappa) + 1/2 - \kappa}} \ .$$

The model metric is given by $\|\mathbf{W}; \mathbf{V}\| := \|\mathbf{W} - \mathbf{V}\|$. The reconstruction theorem [36, 8] shows that Itô integration (and much more ...) is locally Lipschitz in this metric. More precisely, for sufficiently smooth f,

$$\left(\int f(\widehat{W})d\mathbf{W}\right)(\omega) = \int f(\widehat{W}(\omega))d\mathbf{W}(\omega)$$
 a.s.

where the left-hand side is a classical Itô integral, and the right-hand side defined (thanks to reconstruction) in a robust pathwise fashion, as locally Lipschitz function of $\mathbf{W} = \mathbf{W}(\omega)$.

A.2. **Model translation.** We need a generalization of the translation map known from rough paths (e.g. [24], see also [15] for the case of a genuine model in a singular SPDE situation) to the rough volatility regularity structure [8].

We have to be careful in "lifting" the meaning of "W + h" to a rough path or model. Given a two-dimensional Brownian motion $W(\omega) = (W, \overline{W})$, with $\widehat{W} = K^H * W$, first recall the associated (random) model of the form (cf. [8] for details)

$$\mathbf{W}(\omega) = \left(W, \overline{W}, \widehat{W}, \int \widehat{W} dW, \int \widehat{W} d\overline{W}, \int \widehat{W}^2 dW, \dots \right) .$$

One naturally defines, with $h = (h, \overline{h}),$

$$T_{\rm h}(\mathbf{W}) = \left(W + h, \overline{W} + \overline{h}, \widehat{W} + \widehat{h}, \int (\widehat{W} + \widehat{h}) d(W + \overline{h}), \ldots\right).$$

We warn the reader that the "translate" $T_{-h}(\mathbf{W})$ is different from

$$\mathbf{W} - \mathbf{h} = (W - h, \overline{W} - \overline{h}, \widehat{W} - \widehat{h}, \int \widehat{W} dW - \int \widehat{h} d\overline{h}, ...),$$

which one uses to define the model distance $|||\mathbf{W}; \mathbf{h}||| := |||\mathbf{W} - \mathbf{h}|||$ where \mathbf{h} is the canonical model associated to $\mathbf{h} \in \mathbf{H}$.

The following lemma can be seen as variation of rough path results found in [24, Sec 11.1].

Lemma A.2. (i) For fixed $h = (h, \overline{h}) \in H$ and $(deterministic^5)$ model $\mathbf{W} = (W, \overline{W}, \int \widehat{W} dW, ...)$ with homogeneities $\{1/2 - \kappa, 1/2 - \kappa, H - \kappa, (H + 1/2) - 2\kappa, ...\}$, the translated model $T_h(\mathbf{W})$ is

⁵Consider here $\int \widehat{W} dW$ as an analytic object, supplied a priori, and not as stochastic integral. In a rough path setting like [24], one would denote this by \mathbb{W} .

of the same type and T_h is a continuous map (with respect to model distance), with (continuous) inverse given by T_{-h} .

(ii) For all such h and W,

(A.9)
$$|||T_h \mathbf{W}|| \lesssim |||\mathbf{W}|| + ||h||_{\mathbf{H}}$$

with a multiplicative constant that only depends on H and κ).

(iii) For fixed $h \in H$ and all W,

$$||T_{-h}(\mathbf{W})|| \lesssim \delta \quad iff \quad ||\mathbf{W}; \mathbf{h}|| \lesssim \delta$$

where the multiplicative constant also depends on h.

Proof. (i) Recall $h \in H^1 \subset C^{1/2}$, so that W - h, $\overline{W} - \overline{h}$ are again $(1/2 - \kappa)$ -Hölder. More interestingly,

$$\widehat{W} - \widehat{h} = \{t \mapsto \widehat{W}_t - \int_0^t K^H(t, s) \dot{h}_s ds\}$$

is again $C^{H-\kappa}$ using that K^H convolution not only maps $C^{-1/2-\kappa} \to C^{H-\kappa}$ (which explains $\widehat{W} \in C^{H-\kappa}$), but also $L^2 = W^{0,2} \to W^{1/2+H,2} \subset C^H$, so that $\widehat{h} \in C^H$.

We now move to higher level objects, such as $\int \widehat{W}^k dW$, with translate given by

$$\int (\widehat{W} + \widehat{h})^k d(W + h) = \sum_{i=0}^k \binom{k}{j} \int \widehat{W}^j \widehat{h}^{k-j} dW + \int \widehat{W}^j \widehat{h}^{k-j} dh.$$

We need to check a certain Hölder type regularity on the arising integrals. Let us explain for example how to prove

$$\int_{0}^{t} \widehat{h}_{s,r}^{k} \widehat{W}_{s,r}^{l} dW_{r} \lesssim |t-s|^{(k+l)H+1/2-(k+l+1)\kappa}$$

for positive integers k and l and small fixed κ .

We first prove that

(A.10)
$$\left\| \int_{s}^{\cdot} \widehat{W}_{s,r}^{l} dW_{r} \right\|_{p-var,[s,t]} \lesssim |t-s|^{lH+\frac{1}{2}-(l+1)\kappa}$$

for $p \ge (1/2 - \kappa)^{-1}$.

Indeed, for a given partition $\{t_i\}$ of [s, t], one has

$$\begin{split} \left| \int_{t_{i}}^{t_{i+1}} \widehat{W}_{s,r}^{l} dW_{r} \right|^{p} \lesssim \left| \int_{t_{i}}^{t_{i+1}} \widehat{W}_{t_{i},r}^{l} dW_{r} \right|^{p} + \left| \widehat{W}_{s,r}^{l} - \widehat{W}_{t_{i},r}^{l} \right|^{p} \left| \int_{t_{i}}^{t_{i+1}} dW_{r} \right|^{p} \\ \lesssim \left| t_{i+1} - t_{i} \right|^{p(lH + \frac{1}{2} - (l+1)\kappa)} + \left| t - s \right|^{p(lH - l\kappa)} \left| t_{i+1} - t_{i} \right|^{p(\frac{1}{2} - \kappa)} \end{split}$$

and summing over i we obtain (A.10). We then use the fact that, by the Besov variation embedding of [26], there exists q with q + p > 1 such that

$$\|\hat{h}\|_{q-\text{var};[s,t]} \lesssim |t-s|^{H-\kappa} \|h\|_{H^1([0,T])}$$

so that by Young's inequality it holds that

$$\left| \int_{s}^{t} \widehat{h}_{s,r}^{k} \widehat{W}_{s,r}^{l} dW_{r} \right| \lesssim \|\widehat{h}_{s,\cdot}^{k}\|_{q-\operatorname{var};[s,t]} \left\| \int_{s}^{\cdot} \widehat{W}_{s,r}^{l} dW_{r} \right\|_{p-\operatorname{var},[s,t]} \lesssim |t-s|^{(k+l)H+\frac{1}{2}-(k+1)\kappa} .$$

Continuity on model distance of the translation operator is an easy consequence of the above analysis.

(ii) is straightforward by keeping track of which powers of ||h|| and $|||\mathbf{W}|||$ appear in the computation, for instance in the case studied above one has

$$\left(\frac{\left|\int_{s}^{t} \widehat{h}_{s,r}^{k} \widehat{W}_{s,r}^{l} dW_{r}\right|}{|t-s|^{(k+l)H+\frac{1}{2}-(k+1)\kappa}}\right)^{\frac{1}{k+l+1}} \lesssim \|\mathbf{h}\|_{\mathbf{H}}^{\frac{k}{k+l+1}} |||\mathbf{W}|||^{\frac{l}{k+l+1}} \lesssim \|\mathbf{h}\|_{\mathbf{H}} + |||\mathbf{W}|||.$$

(iii) This follows from straightforward computations using Young's inequality as in the proof of Lemma A.2. For instance, considering the first nonlinear term, one has

$$\int (\widehat{W} - \widehat{h}) d(W - h) - \left(\int \widehat{W} dW - \int \widehat{h} dh \right) = 2 \int \widehat{h} dh - \int \widehat{h} dW - \int \widehat{W} dh$$
$$\lesssim \|h - W\|_{C^{1/2 - \kappa}} + \|\widehat{h} - \widehat{W}\|_{C^{H - \kappa}}$$

by Young's inequality and the variation properties of h, \hat{h} .

Lemma A.3 (Robust Cameron–Martin shift). In the context of the random model $\mathbf{W}(\omega)$ above, built on two-dimensional Brownian motion: for all $h \in H$, with probability one, $\mathbf{W}(\omega + h) = T_h \mathbf{W}(\omega)$.

Proof. This is similar to Theorem [24, Thm 11.5] in the rough path setting or [15] in the setting of the gPAM model. \Box

A.3. Homogenous model norms and Fernique estimates. As above, we regard models as objects of the form

$$\mathbf{W}(\omega) = \left(W, \overline{W}, \widehat{W}, \int \widehat{W} dW, \int \widehat{W} d\overline{W}, \int \widehat{W}^2 dW, \dots \right) .$$

We cannot expect that $\|\mathbf{W}(\omega)\|$, with model norm as introduced in (A.8), has Gaussian concentration. Hence introduce the *homogenous model norm*

$$|||\mathbf{W}||| := \|W\|_{1/2-\kappa} + \|\overline{W}\|_{1/2-\kappa} + \ldots + \|\int \widehat{W}^2 dW\|_{2H+1/2-3\kappa}^{1/3} + \ldots$$

for which one can show Gaussian concentration (a.k.a. Fernique estimates).

Theorem A.4. We have $\exp(|||\mathbf{W}|||^2) \in L^{0+}(P)$.

Proof. This follows from the generalized Fernique theorem [25], noting that T_h provides a robust Cameron–Martin shift with (deterministic) estimate $|||T_h \mathbf{W}|| \leq |||\mathbf{W}|| + ||h||_{\mathbf{H}}$.

A.4. Renormalized Wong-Zakai type result. Let W^{η} be the mollification of W obtained by convolutions with a mollifier function at scale $\eta > 0$ and similar for \overline{W} . There is a canonical lift to a model \mathbf{W}^{η} , which does *not* converge when H < 1/2 (cf. [8]) and converges to the Stratonovich Brownian rough path when H = 1/2. Increments $\mathbf{W}^{\eta}_{s,t}$ take values in certain (truncated) Hopf algebra, which - as vector space - we can identify with $\mathbb{R}^{M(H)}$.

Theorem A.5. There exists a family of linear maps on $M^{\eta}: \mathbb{R}^{M(H)} \to \mathbb{R}^{M(H)}$ so that $\widehat{\mathbf{W}}$, with pointwise definition

$$\widehat{\mathbf{W}}_{s,t}^{\eta} := M^{\eta} \mathbf{W}_{s,t}^{\eta}$$

converges (in probability and model topogy) to the Itô model W. Moreover the action of M^{η} commutes with the translation operator from Section A.2.

Proof. This is a (non-quantitative) formulation of [8, Thm 3.14]. The commutation relation is easy to check by hand and fully consistent with [14], which identifies (in a general branched rough path context) renormalization with higher order translation, with resulting *abelian* renormalization group.

A.5. "Stochastic" Taylor remainder estimates via model norms. We need a variation of the rough path results [1, 39, 40, 38] in the setting of regularity structures for rough volatility, as recalled in Appendix A . Recall from [8] that there is a well-defined dilation δ_{ε} acting on the relevant models. Formally, it is obtained by replacing reach occurance of $W, \overline{W}, \widehat{W}$ with ε times that quantity. As a consequence, dilation works well with homogenous model norms,

$$|||\delta_{\varepsilon}\mathbf{W}||| = \varepsilon |||\mathbf{W}|||$$
.

The following theorem is purely deterministic.

Theorem A.6 (Stochastic Taylor-like expansion). Let f be a (sufficiently) smooth function. Fix $h \in H^1$ and $\varepsilon > 0$. If \mathbf{W} is a model (as in the previous section), then so is $T_h(\delta_{\varepsilon}\mathbf{W})$. The pathwise "rough/model" integral

$$\Psi(\varepsilon) := \int_0^1 f\left(\varepsilon \widehat{W}_t + \mathbf{h}_t\right) d(T_{\mathbf{h}}(\delta_{\varepsilon} \mathbf{W}))_t$$

is well-defined, continuously differentiable in ε and we have the remainder estimate

$$|\Psi(\varepsilon) - \Psi(0) - \varepsilon \Psi'(0) - (1/2)\varepsilon^2 \Psi''(0)| = O(\varepsilon^3 |||\mathbf{W}|||^3),$$

valid on bounded sets of $\varepsilon |||\mathbf{W}|||$.

Proof. (Sketch) Differentiating formally with respect to ε yields

$$\Psi'(\varepsilon) = \int_0^1 f(\varepsilon \widehat{W}_t + \widehat{h}_t) d\mathbf{W}_t + f'(\varepsilon \widehat{W}_t + \widehat{h}_t) \widehat{W}_t d(T_h \delta_\varepsilon \mathbf{W}_t)$$

and

$$\Psi''(\varepsilon) = \int_0^1 2f'(\varepsilon \widehat{W}_t + \widehat{h}_t) \widehat{W}_t d\mathbf{W}_t + f''(\varepsilon \widehat{W}_t + \widehat{h}_t) \widehat{W}_t^2 d(T_h \delta_\varepsilon \mathbf{W}_t)$$

and similarly for Ψ''' which we do not spell out. All integrals here are defined by Hairer's reconstruction, i.e. as limit of suitable Riemann-sum approximations involving the "elementary" objects in the model e.g. $\int \widehat{W}^k dW$; one also needs to use the regularity of h as in the proof of Lemma A.2. One then just needs to check that $\Psi'''(\varepsilon) = O(|||\mathbf{W}|||^3)$, uniformly over $\varepsilon \in [0, 1]$, and here one uses precisely the assumption that $\varepsilon|||\mathbf{W}|||$ remains bounded.

Remark A.7. Theorem A.6 is really a special case of forthcoming work with Y. Boutaib [13] where we discuss a general form of this result, applicable to the reconstruction of modelled distributions and thus wide classes of singular stochastic partial differential equations.

APPENDIX B. LOCAL ANALYSIS AROUND THE MINIMIZER

Recall that we are interested in $\Phi : (\varepsilon, \mathbf{M}) \mapsto \Phi^{\varepsilon}(\mathbf{M})$, from $(0, 1] \times \mathcal{M} \to C[0, 1]$. Here we restrict to $\varepsilon = 0$ and $\mathbf{H} \subset \mathbf{M}$, where elements $\mathbf{h} \in \mathbf{H}$ are identified with their canonical lift $\mathbf{h} \in \mathbf{M}$. Hence, by abuse of notation, we regard

$$\Phi(h) := \Phi^0(h)$$

as map $\Phi: H \to C[0,1]$. Then $\mathcal{K}^x \subset H$ is the space of *x-admissible controls*, i.e. elements $h \in H^1: \Phi_1(h) = x$. Whenever \mathcal{K}^x is non-empty the energy

$$\Lambda(x) = \inf_{h \in H} \left\{ \frac{1}{2} \int_0^1 |\dot{h}|^2 dt : \Phi_1(h) = x \right\} = \inf_{h \in \mathcal{K}^{\times}} \frac{1}{2} ||h||_H^2$$

is finite.

B.1. First order optimality. In this section we make the standing assumption that $h \mapsto \Phi_1(h)$ is (Fréchet) C^1 , and consider a minimizing x-admissible control h^x such that

$$D\Phi_1(\mathbf{h}^x) \in L(\mathbf{H} \to \mathbb{R})$$

is surjective. This entails (cf. [12, p.25]) that \mathcal{K}^x to be a Hilbert manifold near h^x with tangent space

(B.1)
$$\mathfrak{Ker}D\Phi_{1}\left(\mathbf{h}^{x}\right)=T_{\mathbf{h}^{x}}\mathcal{K}^{x}=\left\{\mathbf{h}\in\mathbf{H}:\left\langle D\Phi_{1}\left(\mathbf{h}^{x}\right),\mathbf{h}\right\rangle =0\right\} =:\mathbf{H}_{0}\ .$$

Lemma B.1 (First order optimality, Lagrange multiplier). For each such optimal control h^x there exists a unique $q^x = q(h^x) \in \mathbb{R}$ (think: tangent space at $x \in \mathbb{R}$) such that

$$\mathbf{h}^x = D\Phi_1(\mathbf{h}^x)^* q^x$$

where we recall that $D\Phi_1(h^x): H \to \mathbb{R}$ so that its adjoint maps $\mathbb{R} \to H$ where we identify \mathbb{R}^*, H^* with \mathbb{R}, H respectively.

Proof. The map $D\Phi_1(h^x s)^* : \mathbb{R} \to H_0^{\perp}$ is one-one. On the other hand, because h^x is a minimizer, the differential of I at h^x must be zero on H_0 , i.e.

$$\langle h^x, k \rangle = 0$$
 for all $k \in H_0$

so that $h^x \in H_0^{\perp}$. We conclude that there exists a (unique) value $q^x \in \mathbb{R}$ s.t. $D\Phi_1(h^x)^* q^x = h$. \square

Whenever the energy Λ is C^1 near x, we can see that $q^x = \Lambda'(x)$.

Lemma B.2 (First order optimality and energy). Assume Λ is C^1 near x.

(i) any optimal control $h^x \in \mathcal{K}^x$ is a critical point of

$$h \mapsto -\Lambda \left(\Phi_1^h\right) + \frac{1}{2} \left\|h\right\|_H^2$$

so that for all $h \in H$,

(B.2)
$$\langle \mathbf{h}^x, \mathbf{h} \rangle = \Lambda'(x) \langle D\Phi_1(\mathbf{h}^x), \mathbf{h} \rangle .$$

(ii) with
$$g_1(\omega) = G_1(\mathbf{W}) = \partial_{\varepsilon}|_{\varepsilon=0} \Phi_1(\mathbf{h}^x + \varepsilon \mathbf{W})$$
, we have

(B.3)
$$\int_0^1 \dot{\mathbf{h}}^x d\mathbf{W} = \Lambda'(x) g_1.$$

Proof. Write $\Phi^{h} \equiv \Phi(h)$. For fixed $h \in H$, define

$$u(t) := -\Lambda \left(\Phi_1^{h^x + th}\right) + \frac{1}{2} \|h^x + th\|_{\mathcal{H}}^2 \ge 0$$

with equality at t=0 (since $x=\Phi_1^{\mathbf{h}^x}$ and $\Lambda\left(x\right)=\frac{1}{2}\left\|\mathbf{h}^x\right\|_{\mathbf{H}}^2$) and non-negativity for all t because $\mathbf{h}^x+t\mathbf{h}$ is an admissible control for reaching $\widetilde{x}=\Phi_1^{\mathbf{h}^x+t\mathbf{h}}$ (so that $\Lambda\left(\widetilde{x}\right)=\inf\left\{\ldots\right\}\leq\frac{1}{2}\left\|\mathbf{h}^x+t\mathbf{h}\right\|_{\mathbf{H}}^2$.) By assumption, Λ is C^1 in a neighbourhood of x and since Φ is also (Fréchet) C^1 it follows that

u = u(t) is C^1 near t = 0. Since u(0) = 0 and $u \ge 0$ we must have $\dot{u}(0) = 0$. In other words, h^x is a critical point for

$$H\ni h\mapsto -\Lambda\left(\Phi_{1}^{h}\right)+\frac{1}{2}\left\Vert h\right\Vert _{H}^{2}.$$

The functional derivative of this map at h^x must hence be zero. In particular, for all $h \in H$,

$$0 \equiv -\Lambda' \left(\Phi_1^{h^x} \right) \langle D\Phi_1 \left(\mathbf{h}^x \right), \mathbf{h} \rangle + \langle \mathbf{h}^x, \mathbf{h} \rangle$$
$$= -\Lambda' \left(x \right) \langle D\Phi_1 \left(\mathbf{h}^x \right), \mathbf{h} \rangle + \langle \mathbf{h}^x, \mathbf{h} \rangle.$$

Point (ii) is then a consequence of (i) and Lemma 8.2.

Remark B.3 (Tangent space of admissible controls at h^x). Combination of (B.1) with (B.2) shows that

$$T_{h^x} \mathcal{K}^x =: \{h^x\}^{\perp} =: H_0$$

B.2. Non-degeneracy. To this, recall that h^x was (by assumption) an energy minimizer which led to the first order optimality condition (B.2). As in calculus, being a minimizer tells us something about the sign of the second derivative. Namely, if we let

$$I: \mathbf{h} \in \mathbf{H}_0 \mapsto \frac{1}{2} \|\mathbf{h}\|_{\mathbf{H}}^2$$

and recalling that \mathcal{K}^x admits a (Hilbert) manifold structure one can define the second derivative $I''(\mathbf{h}^x)$ (as a quadratic form on H'). More precisely: let $\mathbf{k} \in \mathbf{H}'$, then there exists a family κ^{ε} such that $\kappa^0 = \mathbf{h}^x$ and $\mathbf{k} = \frac{d\kappa^{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}$, and we let

$$I''(\mathbf{h}^x)[\mathbf{k},\mathbf{k}] := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\|\kappa^\varepsilon\|_{\mathbf{H}}^2 - \|\mathbf{h}^x\|_{\mathbf{H}}^2 \right).$$

Note that by the second order optimality condition $I''(h^x) \geq 0$.

We then let $A = A^x$ be the bilinear operator on H given by

$$\mathbf{A}[\mathbf{k}, \mathbf{l}] = D_{\mathbf{h}}^{2} \Phi_{1}(\mathbf{h}^{x}) \left[P_{\mathbf{h}^{x}}^{\perp} \mathbf{k}, P_{\mathbf{h}^{x}}^{\perp} \mathbf{l} \right],$$

where $P_{\mathbf{h}^x}^{\perp}$ is the projection on $\mathbf{H}_0 = \{\mathbf{h}^x\}^{\perp}$. In other words, like in differential geometry, the Hessian $\mathbf{A} = \mathbf{A}^x$ is a quadratic form on the tangent-space $T_{\mathbf{h}^x}\mathcal{K}^x = \mathbf{H}_0$.

We then have a more explicit description for $I''(h^x)$ in terms of A.

Lemma B.4. Let $k \in H_0 = T_{h^x} \mathcal{K}^x$, then letting $\Lambda(h) = \frac{1}{2} \|h\|_{\mathcal{K}^x}^2$, one has

$$I''(\mathbf{h}^x)[k,k] = ||k||^2 - qA[k,k]$$

where q is given by Lemma B.1 (if Λ is C^1 at x then $q = \Lambda'(x)$).

Proof. Fix $k \in H'$ and $\kappa^{\varepsilon} \in \mathcal{K}^{x}$ with

$$\kappa^{\varepsilon} = h^{x} + \varepsilon \mathbf{k} + r^{\varepsilon}$$

with $||r^{\varepsilon}|| = o(\varepsilon)$. Then the constraint $\Phi(k^{\varepsilon}) = x$ implies by Taylor expansion at order 2 that

$$D\Phi(\mathbf{h}^x)[r^{\varepsilon}] + \varepsilon^2 \frac{1}{2} D^2 \Phi_1(\mathbf{h}^x)[\mathbf{k}, \mathbf{k}] = o(\varepsilon^2)$$

and by Lemma B.1

$$\varepsilon^{-2} \langle \mathbf{h}^x, r^{\varepsilon} \rangle = -\frac{q}{2} D^2 \Phi_1(\mathbf{h}^x)[\mathbf{k}, \mathbf{k}] + o(1).$$

Hence we have that

$$I''(\mathbf{h}^x)[k,k] = \lim_{\varepsilon \to 0} \varepsilon^{-2} \left(\|\varepsilon \mathbf{k} + r^\varepsilon\|^2 + 2 \langle \mathbf{h}^x, \varepsilon \mathbf{k} + r^\varepsilon \rangle \right) = \|k\|^2 - qD^2 \Phi_1(\mathbf{h}^x)[k,k].$$

Recall that in this paper we further make a non-degeneracy assumption on $I''(h^x)$ as part of Assumption (A5).

Under this assumption, one actually has that the minimizer h (and therefore Λ) is C^1 in a neighborhood of x:

Lemma B.5. Assume that h^x is the unique minimizer in \mathcal{K}^x , that Φ_1 is C^{k+1} -Fréchet differentiable in a neighborhood of h^x , and that h^x is nondegenerate in the sense that $I''(h^x) > 0$. Then there exists a neighborhood V of x and a C^k map $\hat{h}: V \to H$ such that for all $y \in V$

$$\hat{h}(y) = h^y = \arg\min\{\|h\|^2, \Phi_1(h) = y\}.$$

In particular, Λ is C^k on V.

Proof. This is an application of the inverse function theorem, cf. [43, Proposition 5.1]. \Box

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