

Robust Hedging of Volatility Derivatives

Columbia Financial Engineering Seminar

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Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Realized variance

For a price process S (e.g. a stock, stock index, currency) where

$$dS_t = \sigma_t S_t dW_t,$$

a contract that pays the *realized variance*

$$\int_0^T \sigma_t^2 dt$$

can be created/synthesized/replicated/hedged by combining a static position in T -expiry calls and puts, with dynamic trading in S .

Essentially no assumptions on dynamics of *instantaneous volatility* σ .

See Dupire (92), Neuberger (94), Carr-Madan (98), Derman et al (99).

Realized volatility

But how does one create a contract that pays, say,

$$\text{realized volatility} := \sqrt{\text{realized variance}}$$

Previous research has done this by specifying the dynamics of σ .

We do not specify a process for σ . In this sense, replication is *robust*.

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Static hedging of path-independent payoffs

For general f , for any expansion point κ , any S ,

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + \int_{\kappa}^{\infty} f''(K)(S - K)^+ dK + \int_0^{\kappa} f''(K)(K - S)^+ dK$$

In terms of the bond price B_0 , and call and put prices $C_0(K)$ and $P_0(K)$, therefore, a claim on any “European-style” payoff $f(S_T)$ has time-0 value

$$f(\kappa)B_0 + f'(\kappa)[C_0(\kappa) - P_0(\kappa)] + \int_{\kappa}^{\infty} f''(K)C_0(K)dK + \int_0^{\kappa} f''(K)P_0(K)dK.$$

No restrictions on the underlying price process.

See Breeden-Litzenberger (1978), Carr-Madan (1998).

Example: $\log(S_T/S_0)$

Suppose the payoff to be replicated is $X_T = \log(S_T/S_0)$.

Then expand $f(S) := \log(S/S_0)$ about S_0 .

The value of a claim on X_T is therefore

$$-\int_0^{S_0} \frac{1}{K^2} P_0(K) dK - \int_{S_0}^{\infty} \frac{1}{K^2} C_0(K) dK.$$

So hold $-(1/K^2)dK$ units of each out-of-the-money option.

Useful for synthesizing a variance swap.

Example: $(S_T/S_0)^p$

Again $X_T := \log(S_T/S_0)$.

For $p \in \mathbb{R}$, the power payoff

$$e^{pX_T} = (S_T/S_0)^p$$

decomposes, as above, into calls and puts on S_T .

For $p = a + bi \in \mathbb{C}$, the power “payoff”

$$e^{pX_T} = e^a [\cos(bX_T) + i \sin(bX_T)]$$

has Re part $e^a \cos(b \log(S_T/S_0))$ and Im part $e^a \sin(b \log(S_T/S_0))$,
each of which decomposes again into payoffs of calls and puts on S_T .

Useful for synthesizing general volatility derivatives.

Example of a variance swap

Bank of America Securities LLC

(For Discussion Only)

Indicative Terms

October 8, 1999

S&P 500 Index Realized Variance Swap

Equity Payer: Bank of America, N.A. ("BoA")
Equity Receiver: Merrill Lynch International
Trade Date: October 8, 1999
Maturity Date: May 7, 2003
Underlying Index: The Standard & Poor's 500 Composite Stock Price Index
Equity Calculations:

- (a) "Initial Price" means 0.305
 (b) "Final Price" means the actual realized index Variance defined in accordance with the following formula and definition:

$$\sqrt{\frac{\sum_{i=1}^n \left[\ln \left(\frac{P_{i+1}}{P_i} \right) \right]^2}{n-2}} \times \sqrt{52}$$

- (c) "Natural Logarithm" means for any Daily Quotient, as determined by the Calculation Agent, the exponential number which equates 2.718281828 to such Daily Quotient;
 (d) "n" means the total number of Valuation Dates;
 (e) "P_i" means the closing level of the index on the ith valuation date (i.e.: P₁ is the closing level of the index on October 6, 1999, P₂ is the closing level of the index on the first Wednesday that is an Exchange Business Day following the Trade Date and P_n is the closing level of the index on the Final Valuation Date.
 (f) "Valuation Dates" means, commencing on October 6, 1999, and each Wednesday thereafter up to and including the Final Valuation Date and if any such date is not an Exchange Business Day, the next following day that is an Exchange Business Day, subject to the Market Disruption Events as set forth in the 1996 ISDA Equity Derivatives Definitions.
 (g) $\sum_{i=1}^n$ means the summation from i=1 to i=n.

Notional: 111,230,666

Equity Payment: Notional * [Final Price² - Initial Price²]
 If the Equity Payment is a positive value, then the Equity Payer pays the Equity Receiver this value.
 If the Equity Payment is a negative value, then the Equity Receiver pays the Equity Payer the absolute value of this number.

Credit Terms: n/a

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NOT FIDC INSURED MAY LOSE VALUE

NO BANK GUARANTEE

Introduction

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Robust hedging

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Robustness to correlation

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Discrete approaches

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Assumptions

Assume zero rates for simplicity.

Assume a positive price process S_t for $t \in [0, T]$, where

$$dS_t = \sigma_t S_t dW_t$$

Here W is Brownian motion under a risk-neutral probability measure.

In particular, S is continuous, but σ need not be.

Let $X_t := \log(S_t/S_0)$. We want to create payoff

$$\langle X \rangle_T \equiv \int_0^T \sigma_t^2 dt$$

Realized variance and log payoffs

By Itô's rule,

$$dX_t = \frac{1}{S_t} dS_t + \frac{1}{2} \left(\frac{-1}{S_t^2} \right) \sigma_t^2 S_t^2 dt.$$

Rewrite as

$$X_T = \int_0^T \frac{1}{S_t} dS_t - \frac{1}{2} \int_0^T \sigma_t^2 dt.$$

Rearranging,

$$\langle X \rangle_T = -2X_T + \int_0^T \frac{2}{S_t} dS_t.$$

This is the sum of a **European-style payoff** and the **gains from a dynamic trading strategy**.

Replicating a variance swap

Create the payoff $-2X_T = -2 \log(S_T/S_0)$ with a static options position with initial value

$$\int_0^{S_0} \frac{2}{K^2} P_0(K) dK + \int_{S_0}^{\infty} \frac{2}{K^2} C_0(K) dK,$$

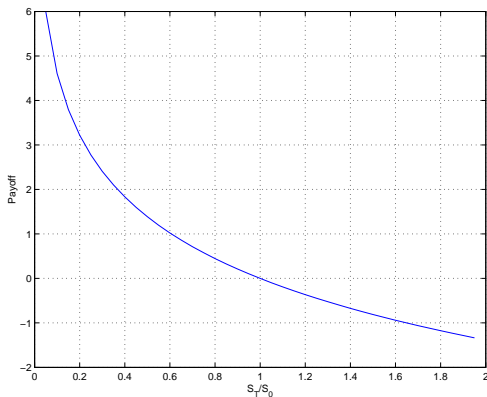
Create the payoff $\int_0^T (2/S_t) dS_t$ by dynamically holding

$2/S_t$ shares

at each time t .

European hedge of a variance swap

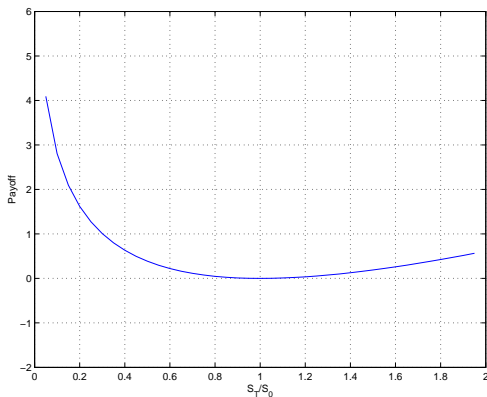
$$-2 \log(S_T/S_0)$$



European hedge of a variance swap

Including also the share position:

$$-2 \log(S_T/S_0) + 2S_T - 2S_0$$



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Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

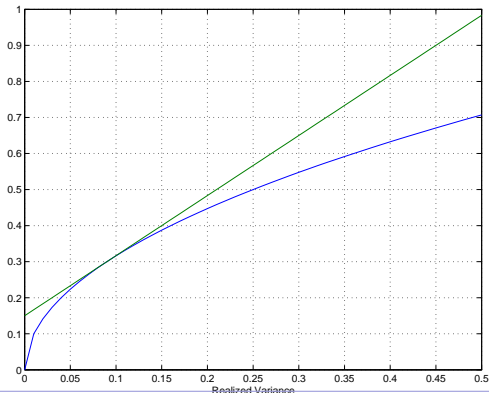
Distributional inference

Without a model, you can bound from above

By Jensen, $\mathbb{E}\sqrt{\langle X \rangle_T} \leq \sqrt{\mathbb{E}\langle X \rangle_T}$.

To enforce this bound, superreplicate the $\sqrt{\langle X \rangle_T}$ payoff using:

$\sqrt{\mathbb{E}\langle X \rangle_T}$ in cash, plus $1/(2\sqrt{\mathbb{E}\langle X \rangle_T})$ var swaps with fixed leg $\mathbb{E}\langle X \rangle_T$.



But can you price and hedge?

All of the following rely on a specification of the dynamics of σ .

Brockhaus-Long (1999)	Brenner-Ou-Zhang (2001)
Detemple-Osakwe (2000)	Howison-Rafailidis-Rasmussen (2002)
Heston-Nandi (2000)	Javaheri-Wilmott-Haug (2002)
Matytsin (2000)	Windcliff-Forsyth-Vetzal (2003)

The majority are one-dimensional diffusion models, in which one can replicate via dynamic trading in the underlying and one option.

Difficulties with stochastic volatility diffusion models

Simple SV models may be misspecified

- ▶ Trouble simultaneously fitting long- and short-dated option prices.
- ▶ Out-of-sample pricing errors.
- ▶ Hedges fail if volatility jumps.

Even if correctly specified, the model depends on parameters which are not directly observable.

Conventional wisdom: vol swaps are model-dependent

A managing director at a leading Wall Street dealer (Nov 2002):

“[T]here is no replicating portfolio for a volatility swap and the magnitude of the convexity adjustment is **highly model-dependent**.”

“As a consequence, market makers’ prices for volatility swaps are both wide (in terms of bid-offer) and widely dispersed.”

“[P]rice takers such as hedge funds may occasionally have the luxury of being able to cross the bid-offer – that is, buy on one dealer’s offer and sell on the other dealer’s bid.”

An equity derivatives quant at another leading dealer (Aug 2003):

“The pricing of vol swaps is **very model-sensitive**”

The need for robust hedging

RISK (August 2003):

“While variance swaps - where the underlying is volatility squared - can be perfectly replicated under classical derivatives pricing theory, this has not generally been thought to be possible with volatility swaps. So while a few equity derivatives desks are comfortable with taking on the risk associated with dealing volatility swaps, many are not.”

Introduction

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Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Robust hedging by dynamic trading in options

- ▶ Assume frictionless trading in S and in T -expiry Europeans on S .
- ▶ Continuously trade stock and options to replicate volatility derivatives.

In reality, such strategies may be expensive at the individual contract level, but more practical at the aggregate portfolio level.

- ▶ In particular, let us replicate the payoff $e^{\lambda \langle X \rangle_T}$, for $\lambda \in \mathbb{C}$.
Combine such exponentials to produce general functions of $\langle X \rangle_T$.
- ▶ Work under probability measure wrt which asset prices are martingales. So $\text{price}_t = \mathbb{E}_t(\text{payoff})$.

Independence and conditioning

Let $dS_t = \sigma_t S_t dW_t$, with σ and W independent. (We'll relax this.)

Conditional on \mathcal{F}_T^σ , the W is still a Brownian motion. So conditionally

$$X_T = \int_0^T \sigma_t dW_t - \frac{1}{2} \langle X \rangle_T \sim \text{Normal} \left(-\frac{1}{2} \langle X \rangle_T, \langle X \rangle_T \right).$$

For each $p \in \mathbb{C}$, therefore,

$$\begin{aligned} \mathbb{E} e^{pX_T} &= \mathbb{E} \left[\mathbb{E} (e^{pX_T} | \mathcal{F}_T^\sigma) \right] \\ &= \mathbb{E} \left[e^{\mathbb{E}(pX_T | \mathcal{F}_T^\sigma) + \text{Var}(pX_T | \mathcal{F}_T^\sigma)/2} \right] \\ &= \mathbb{E} \left[e^{(p^2/2 - p/2) \langle X \rangle_T} \right] = \mathbb{E} e^{\lambda \langle X \rangle_T}, \end{aligned}$$

where $\lambda = p^2/2 - p/2$. Equivalently, $p = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$.

Sanity check

We have related the distributions of X and $\langle X \rangle$ via

$$\mathbb{E}e^{(p^2/2-p/2)\langle X \rangle_T} = \mathbb{E}e^{pX_T}.$$

Consistency check:

Differentiate wrt p , and then evaluate at $p = 0$. Obtain

$$\mathbb{E}\langle X \rangle_T = -2\mathbb{E}X_T,$$

the familiar result.

European and variance claims

The same argument holds at time t instead of time 0, so

$$\mathbb{E}_t e^{p(X_T - X_t)} = \mathbb{E}_t e^{\lambda(\langle X \rangle_T - \langle X \rangle_t)}.$$

Then the exponential variance claim and the European power claim

$$L_t := \mathbb{E}_t e^{\lambda \langle X \rangle_T}$$

$$P_t := \mathbb{E}_t e^{pX_T}$$

are related by

$$L_t = N_t P_t,$$

where

$$N_t := e^{\lambda \langle X \rangle_t - pX_t}.$$

Dynamics of the exponential variance claim

We have $L_t = N_t P_t = e^{\lambda \langle X \rangle_t - p X_t} P_t$, so

$$\begin{aligned} dL_t &= N_t dP_t + P_t dN_t + dP_t dN_t \\ &= N_t dP_t + P_t N_t \left(-p \frac{dS_t}{S_t} \right) \end{aligned}$$

where all drift terms vanish because L is a martingale. Hence

$$L_T = P_0 + \int_0^T N_t dP_t - \int_0^T \frac{p P_t N_t}{S_t} dS_t.$$

So for $p, \lambda \in \mathbb{R}$, replicate $L_T = e^{\lambda \langle X \rangle_T}$ via the self-financing strategy

N_t claims on e^{pX_T}

$-p P_t N_t / S_t$ shares

$p P_t N_t$ bonds

The complex case

For complex λ and p , and complex $\alpha = \alpha(\lambda)$,

$$\begin{aligned}\operatorname{Re}(\alpha L_T) = & \operatorname{Re}(\alpha P_0) + \int_0^T \operatorname{Re}(\alpha N_t) d\operatorname{Re}(P_t) - \int_0^T \operatorname{Im}(\alpha N_t) d\operatorname{Im}(P_t) \\ & - \int_0^T \frac{\operatorname{Re}(p\alpha P_t N_t)}{S_t} dS_t\end{aligned}$$

so replicate $\operatorname{Re}(\alpha L_T)$ by trading cosine and sine claims:

$\operatorname{Re}(\alpha N_t)$ claims on $\operatorname{Re}(e^{pX_T})$

$-\operatorname{Im}(\alpha N_t)$ claims on $\operatorname{Im}(e^{pX_T})$

$-\operatorname{Re}(p\alpha P_t N_t)/S_t$ shares

$\operatorname{Re}(p\alpha P_t N_t)$ bonds

The share holdings delta-hedge the options

Note that each power claim (with payoff e^{pX_T}) has price

$$P_t = e^{pX_t} \mathbb{E}_t e^{\lambda(\langle X \rangle_T - \langle X \rangle_t)}$$

So if $\sigma_t = \sigma(V_t)$ where V_t is a multidimensional Markov process, then we have, for some function f ,

$$P_t = (S_t/S_0)^p f(V_t, t) =: F(S_t, V_t, t).$$

Therefore

$$\Delta_t := \frac{\partial F}{\partial S}(S_t, V_t, t) = pP_t/S_t.$$

Hence the $-pP_t N_t/S_t$ shares *delta-neutralize* the N_t power claims.

General payoffs via Laplace inversion

Think of the $\exp(\lambda \langle X \rangle_T)$ payoffs as basis functions.

A general function $q(\langle X \rangle_T)$ can be expressed in terms of this basis, with coefficients given by the Laplace transform H :

$$q(\langle X \rangle_T) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(\lambda) e^{\lambda \langle X \rangle_T} d\lambda$$

for appropriately chosen a . So in principle,

$$\mathbb{E} q(\langle X \rangle_T) = \mathbb{E} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(\lambda) e^{p \pm (\lambda) X_T} d\lambda.$$

(But mind integrability issues.)

Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Example: Volatility swap

For $v \geq 0$, a contour shift of the Laplace inversion formula gives

$$\sqrt{v} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sv}}{s^{3/2}} ds.$$

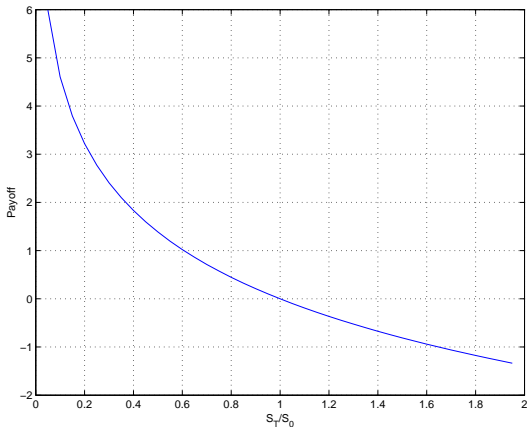
Therefore a claim on

$$\sqrt{\langle X \rangle_T} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-s\langle X \rangle_T}}{s^{3/2}} ds$$

has, according to our replication strategy, the same value as a claim on

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - \operatorname{Re}[e^{(1/2 - \sqrt{1/4 - 2s})X_T}]}{s^{3/2}} ds.$$

Recall: hedge a variance swap with contract on $-2\log$



But what does the hedge of a volatility swap look like?

Introduction

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Robust hedging

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Robustness to correlation

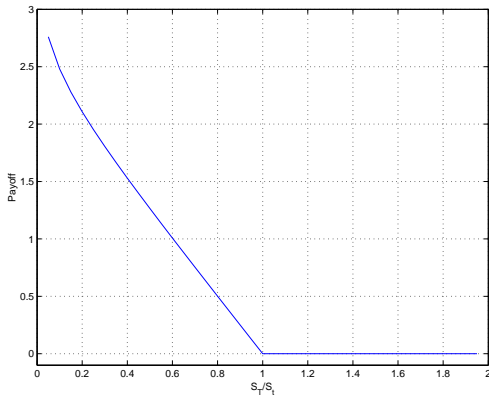
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Discrete approaches

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Hedge a volatility swap ...

by numerically evaluating the messy integral to get the payoff ...



... of a plain vanilla put !?

The payoff profile of the vol swap hedge

No, not exactly a put payoff, but remarkably close.

- ▶ Payoff is zero for $S \geq S_0$.
- ▶ The jump in the first derivative at $S = S_0$ is $\sqrt{2\pi}/S_0$.
- ▶ Payoff is not linear for $S < S_0$, despite appearances. Near $S = S_0$, payoff is approximately $-\sqrt{2\pi}[(S/S_0 - 1) + \frac{1}{48}(S/S_0 - 1)^3]$.

A good approximation: the hedge is $= \sqrt{2\pi}/S_0$ ATM puts.

(Friz-Gatheral 04: formula in terms of a Bessel function)

Surprising ... but also reasonable ...

So you want to be a financial engineer

No calculators allowed. Your interviewer says to you:

Spot is 100. No dividends. What's the Black-Scholes price of a 1-year at-the-money vanilla option with 20% volatility?

So you want to be a financial engineer

No calculators allowed. Your interviewer says to you:

Spot is 100. No dividends. What's the Black-Scholes price of a 1-year at-the-money vanilla option with 20% volatility?

Black-Scholes: If $dS_t = \sigma S_t dW_t$

then a claim on $(S_T - K)^+$ must have price $C_0 = C^{BS}(S_0, \sigma)$ where

$$C^{BS}(S, \sigma) = S\Phi(d_1) - K\Phi(d_2).$$

and

$$d_{1,2} := d_{+,-} := \frac{\log(S/K)}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$$

ATM option prices are almost linear in Black vol...

Brenner-Subrahmanyam (88): ATM, so

$$d_{1,2} = 0 \pm \frac{\sigma\sqrt{T}}{2}.$$

For small x ,

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x + 0 + O(x^3).$$

Black-Scholes:

$$S_0\Phi(d_1) - K\Phi(d_2) = S_0(\Phi(\sigma\sqrt{T}/2) - \Phi(-\sigma\sqrt{T}/2)) \approx \frac{1}{\sqrt{2\pi}}S_0\sigma\sqrt{T}.$$

The linearity constant is $S_0\sqrt{T}/\sqrt{2\pi} \approx 0.4 \times S_0\sqrt{T}$.

(Your answer: 8 dollars)

...hence vol swap value \approx constant \times ATM option value...

Let C be call price, so $C_T = (S_T - K)^+$. Let $\bar{\sigma} := \sqrt{\frac{1}{T} \int_0^T \sigma_t^2 dt}$.

Condition and use independence (Hull-White 87):

$$C_0 = \mathbb{E}C_T = \mathbb{E}\left[\mathbb{E}(C_T|\mathcal{F}_T^\sigma)\right] = \mathbb{E}\left[C^{BS}(\bar{\sigma})\right].$$

Having aced the interview, you know that for K at the money,

$$C_0 \approx \mathbb{E}\left[\frac{1}{\sqrt{2\pi}}S_0\bar{\sigma}\sqrt{T}\right] = \frac{S_0}{\sqrt{2\pi}}\mathbb{E}\left[\bar{\sigma}\sqrt{T}\right].$$

So it's reasonable that $\sqrt{2\pi}/S_0$ vanilla options ATM have about the same value as a newly issued volatility contract.

...and vol swap value \approx ATM implied vol

Another approximation (Feinstein 89): Let I be ATM implied vol.

Then

$$C^{BS}(I) = C_0 = \mathbb{E}C^{BS}(\bar{\sigma}) \approx C^{BS}(\mathbb{E}\bar{\sigma}),$$

ignoring second-order effects. So, under independence,

$$I \approx \mathbb{E}\bar{\sigma}$$

More specifically, the vol swap value has upper and lower bounds

$$I \leq \mathbb{E}\bar{\sigma} \leq \sqrt{\mathbb{E}\bar{\sigma}^2}$$

and we can find a second-order approximation

$$\mathbb{E}\bar{\sigma} \approx \left(1 + \frac{(\mathbb{E}\bar{\sigma}^2 - I^2)T}{2I^2T + 8}\right)I.$$

Limitations of back-of-the-envelope calculations

These back-of-the-envelope approaches are useful but they

- ▶ do not give exact answers
- ▶ do not establish the hedging strategy
- ▶ do not apply at times after inception
- ▶ do not allow scenario analysis for risk management
- ▶ do not apply to general volatility derivatives
- ▶ do not suggest what to do under correlation

Our theory does.

So you want to be a financial philosopher

Let me conclude this section with some idle speculation.

Why were puts and calls destined to be the canonical option payoffs?

- ▶ Because of their simplicity?
- ▶ Because their second strike derivative is a Dirac delta function?
- ▶ Because they capture the notions of downside and upside risk?
- ▶ *Because their payoff profile nearly matches the robust hedge of realized volatility?*

Introduction

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Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Nonzero correlation

Assume that

$$dS_t = \sigma_t S_t \sqrt{1 - \rho^2} dW_{1t} + \sigma_t S_t \rho dW_{2t}$$

$$d\sigma_t = \alpha(\sigma_t)dt + \beta(\sigma_t)dW_{2t},$$

where W_1 and W_2 are independent BMs. So

$$dS_t = \sigma_t S_t dW_{3t}$$

where σ and $W_3 := \sqrt{1 - \rho^2}W_1 + \rho W_2$ have correlation ρ .

We've written σ as a 1-factor diffusion, but this is not essential.

Conditioning

Then

$$\begin{aligned}dX_t &= -\frac{1}{2}\sigma_t^2 dt + \sigma_t\sqrt{1-\rho^2}dW_{1t} + \sigma_t\rho dW_{2t} \\&= -\frac{1-\rho^2}{2}\sigma_t^2 dt + \sigma_t\sqrt{1-\rho^2}dW_{1t} + \frac{\rho^2}{2}\sigma_t^2 + \sigma_t\rho dW_{2t}\end{aligned}$$

So conditional on $\mathcal{F}_T^{W_2}$,

$$X_T \sim \text{Normal}\left(X_0 + \log M_T(\rho) - \frac{1-\rho^2}{2}\bar{\sigma}^2, (1-\rho^2)\bar{\sigma}^2\right)$$

where

$$M_T(\rho) := \exp\left(-\frac{\rho^2}{2}\int_0^T \sigma_t^2 dt + \rho\int_0^T \sigma_t W_{2t}\right)$$

The mixing formula

Romano-Touzi (1996), Willard (1996):

Price of option paying $G_T = G(S_T)$ is

$$\begin{aligned} G_0 &= \mathbb{E}G_T = \mathbb{E}(\mathbb{E}(G_T|\mathcal{F}_T^{W_2})) \\ &= \mathbb{E}G^{BS}(S_0M_T(\rho), \bar{\sigma}\sqrt{1-\rho^2}) \end{aligned}$$

Option price is expectation of the Black-Scholes formula for that option, evaluated at a randomized stock price and random volatility.

Taylor expansion

Expand about $\rho = 0$

$$\begin{aligned} G_0 &= \mathbb{E} G^{BS}(S_0 M_T(\rho), \bar{\sigma} \sqrt{1 - \rho^2}) \\ &\approx \mathbb{E} G^{BS}(S_0, \bar{\sigma}) + \rho S_0 \mathbb{E} \left[\frac{\partial G^{BS}}{\partial S}(S_0, \bar{\sigma}) \int_0^T \sigma_t dW_{2t} \right] + O(\rho^2) \end{aligned}$$

Neutralize the **first-order effect of correlation**:

Suppose the payoff function G has zero B-S delta, for *all* σ .

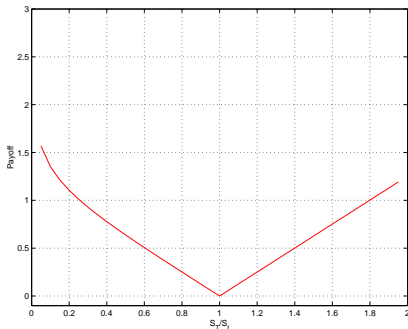
Then **the ρ term** vanishes and

$$G_0 \approx \mathbb{E} G^{BS}(S_0, \bar{\sigma})$$

We say that the European claim on G is [first-order] *ρ -neutral*.

ρ -Neutralization: merely adding forwards does *not* suffice

Put-like vol swap hedge + Forwards = **Straddle-like vol swap hedge**.



But, under any dynamics, this has the same value, and hence the same ρ -sensitivity, as the original hedge.

Instead we want hedges that are B-S delta-neutral for *all* B-S σ .

ρ -Neutralization: basis functions

Consider $G_T := u_+ e^{p_+ X_T} + u_- e^{p_- X_T}$, where $p_{\pm} := \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$.

If $\rho = 0$ and $u_+ + u_- = 1$, this hedges the $e^{\lambda \langle X \rangle_T}$ contract.

For robustness to $\rho \neq 0$, choose $u_+ p_+ + u_- p_- = 0$. This makes the B-S delta vanish because

$$G_T = u_+ (S_t Z / S_0)^{p_+} + u_- (S_t Z / S_0)^{p_-}$$

where the Z distribution does not depend on S_t , so B-S delta is

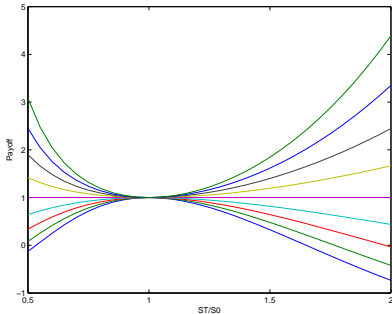
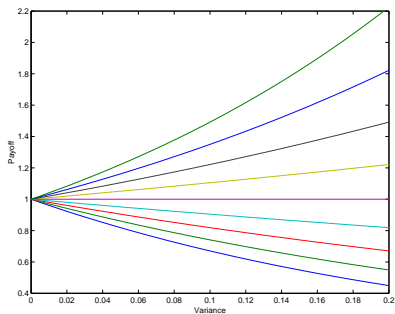
$$u_+ \frac{p_+}{S_t} \mathbb{E}_t (S_t Z / S_0)^{p_+} + u_- \frac{p_-}{S_t} \mathbb{E}_t (S_t Z / S_0)^{p_-} = \frac{u_+ p_+ + u_- p_-}{S_t} \mathbb{E}_t e^{p_+ X_T}$$

Combining the two conditions on u_{\pm} , we have

$$u_{\pm} = \frac{1}{2} \mp \frac{1}{2\sqrt{1+8\lambda}}.$$

Basis functions and their ρ -neutral hedges

Basis functions $\exp(\lambda \langle X \rangle_T)$ and their European hedges, $-4 \leq \lambda \leq 4$



ρ -Neutralization: general functions

Consider general $q(\langle X \rangle_T)$, not necessarily a basis function.

We can aggregate ρ -neutral basis function hedges to create a ρ -neutral hedge of q .

In principle,

$$\begin{aligned}\mathbb{E}q(\langle X \rangle_T) &= \mathbb{E} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(\lambda) e^{\lambda \langle X \rangle_T} d\lambda \\ &= \mathbb{E} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(\lambda) [u_+ e^{p_+ X_T} + u_- e^{p_- X_T}] d\lambda,\end{aligned}$$

where u_{\pm} and p_{\pm} are the functions of λ specified previously.

(But mind integrability issues.)

Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

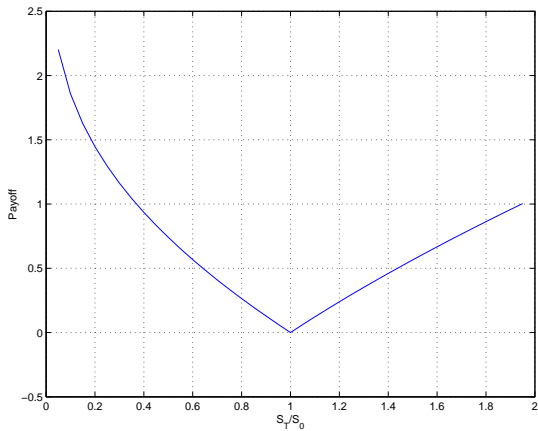
Volatility swaps

Discrete approaches

Hedging

Distributional inference

First-order ρ -neutral vol swap hedge



Introduction

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Robust hedging

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Robustness to correlation

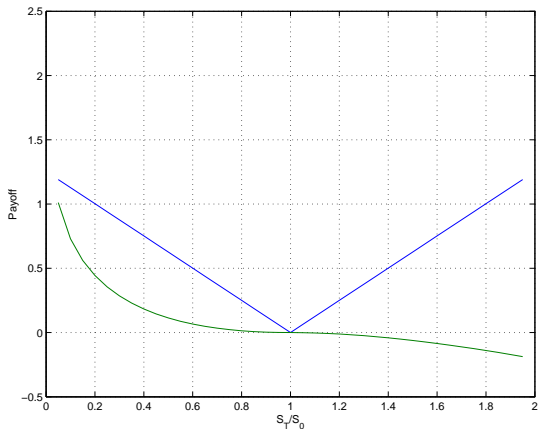
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Discrete approaches

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First-order ρ -neutral vol swap hedge

equals $\sqrt{\pi/2}/S_0$ ATM straddles, plus a correction



Closed-form formula for the hedge

Let I_n be the modified Bessel function of order n . Take $S_0 = 1$.

We can show that the correlation-robust hedge consists of:

$$\sqrt{\pi/2} \quad \text{ATM straddles}$$

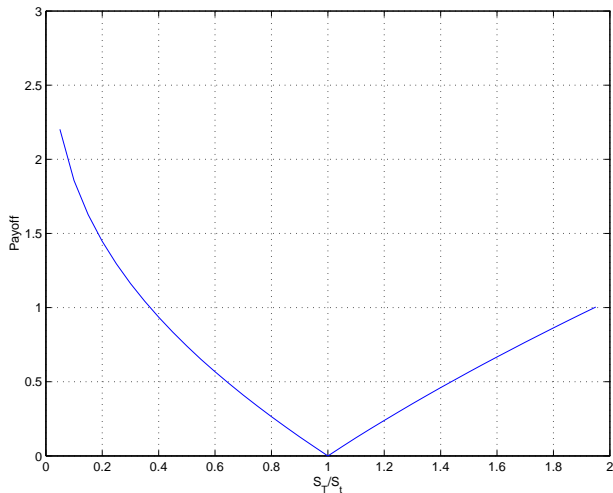
$$\sqrt{\frac{\pi}{8}} e^{-k/2} (I_1(k/2) - I_0(k/2)) dk \quad \text{calls at log-strikes } k > 0$$

$$\sqrt{\frac{\pi}{8}} e^{-k/2} (I_0(k/2) - I_1(k/2)) dk \quad \text{puts at log-strikes } k < 0$$

This is the *initial* hedge – for a newly-issued vol swap.

As variance accumulates during the life of the vol swap, the hedge evolves.

Evolution of the hedge: $\langle X \rangle_t = 0.0$



Introduction

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Robust hedging

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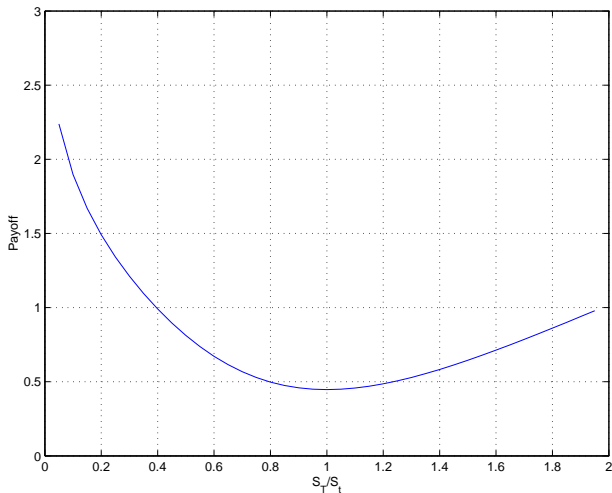
Robustness to correlation

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Discrete approaches

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Evolution of the hedge: $\langle X \rangle_t = 0.2$



Introduction

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Robust hedging

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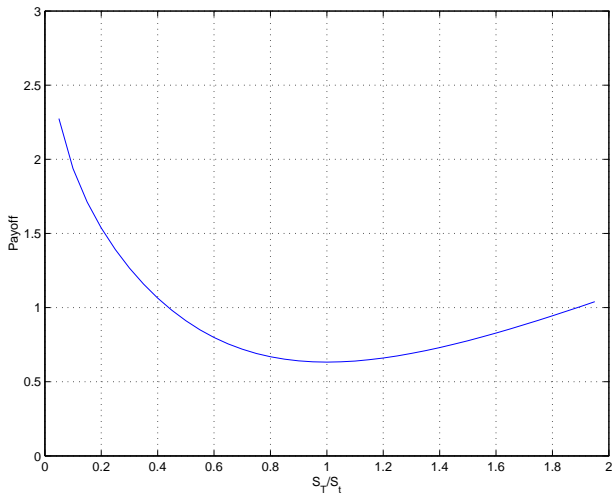
Robustness to correlation

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Discrete approaches

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Evolution of the hedge: $\langle X \rangle_t = 0.4$



Introduction

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Robust hedging

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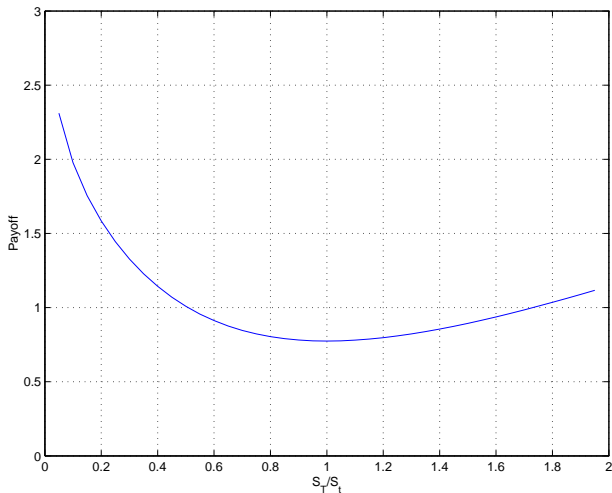
Robustness to correlation

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Discrete approaches

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Evolution of the hedge: $\langle X \rangle_t = 0.6$



Introduction

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Robust hedging

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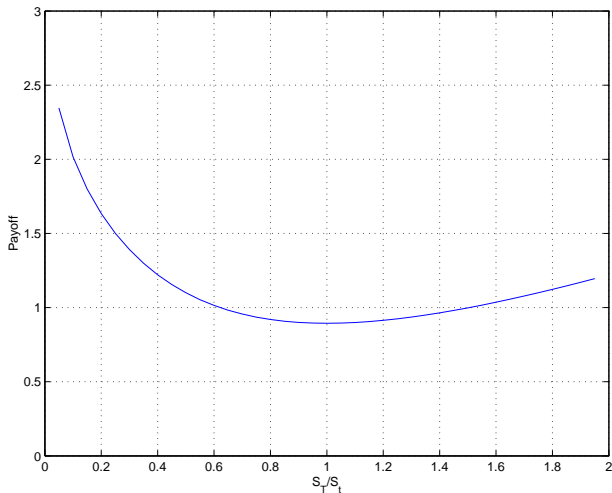
Robustness to correlation

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Discrete approaches

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Evolution of the hedge: $\langle X \rangle_t = 0.8$



Introduction

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Robust hedging

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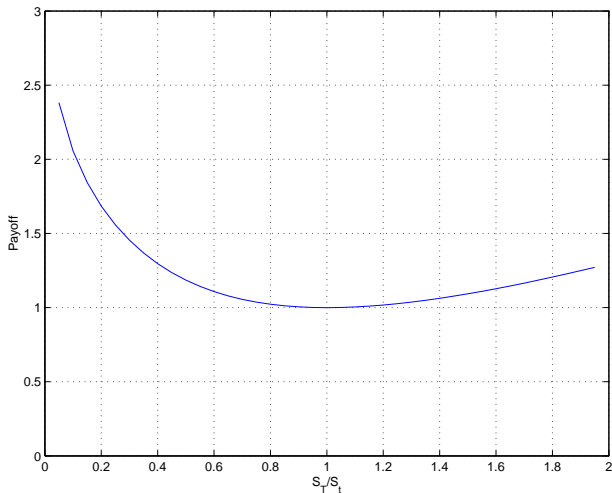
Robustness to correlation

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Discrete approaches

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Evolution of the hedge: $\langle X \rangle_t = 1.0$



Introduction

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Robust hedging

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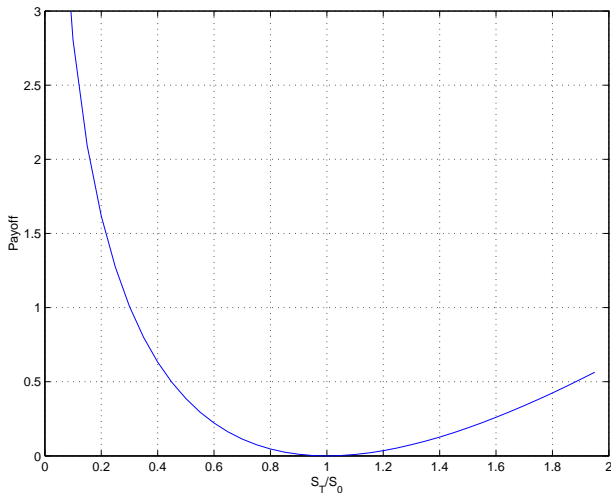
Robustness to correlation

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Discrete approaches

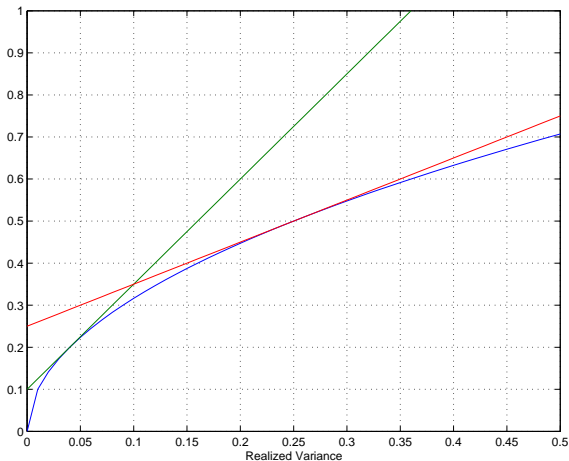
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... which resembles ... the hedged $-2\log$ payoff



Sanity check

As we roll rightward, **sqrt** loses concavity, becoming more **linear**.



Back of the envelope, again

Suppose $\langle X \rangle_t = 1$ is large compared to $R_T := \langle X \rangle_T - \langle X \rangle_t$

Volatility contract pays

$$\sqrt{\langle X \rangle_T} = \sqrt{1 + R_T} \approx 1 + \frac{1}{2}R_T,$$

but from variance swap theory, we know

$$\mathbb{E}_t R_T = \mathbb{E}_t \left[-2 \log(S_T/S_t) + 2(S_T - S_t) \right].$$

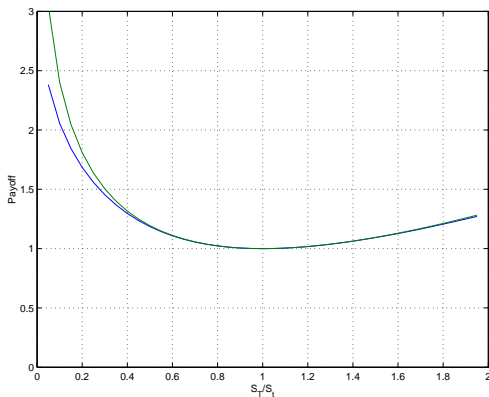
So perhaps at time t , the replicating time- T payoff resembles

$$1 - \log(S_T/S_t) + (S_T - S_t)$$

for S_T near S_t .

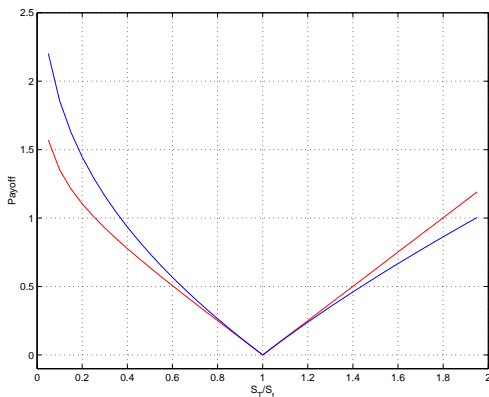
Payoff comparison

At $\langle X \rangle_t = 1.0$, compare the **true hedge** and the **log approximation**.

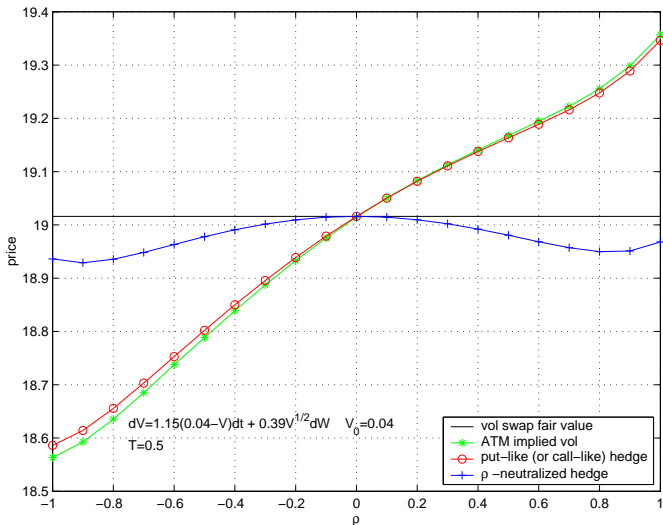


How accurately do these hedges price a vol swap?

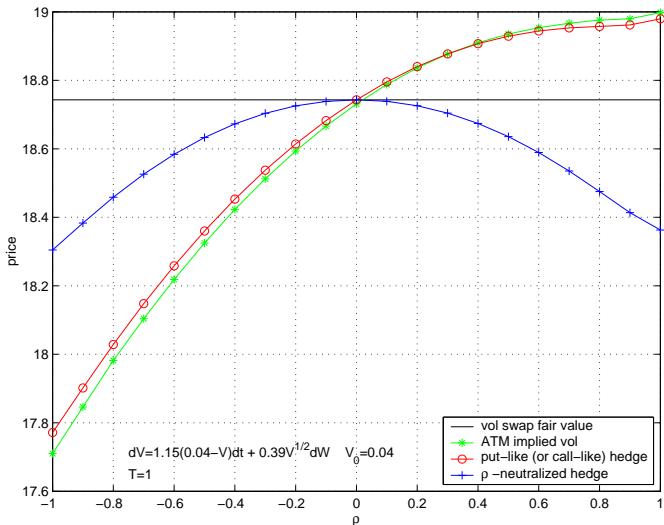
Let's go back to time 0 and test how well the **simple robust hedge** and the ρ -neutral robust hedge price a vol swap.



Heston dynamics and $T = 0.5$



Heston dynamics and $T = 1.0$



Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Discrete setting

For some payoffs, analytic methods are unsuccessful.

So do a twofold discretization:

- ▶ Let $\{v_1, \dots, v_J\}$ approximate the $\langle X \rangle_T$ distribution's support.
- ▶ Use only N “basis” functions $\exp(\lambda_n \langle X \rangle_T)$ for $n = 1, \dots, N$.

We have in mind $J > N$.

Two applications:

- ▶ Hedge a variance payoff $q(\langle X \rangle_T)$.
- ▶ Infer distribution of $\langle X \rangle_T$.

Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Matrix notation

We want

$$q(v_j) \approx \sum_{n=1}^N h_n e^{\lambda_n v_j}, \quad j = 1, \dots, J.$$

In matrix form,

$$\begin{pmatrix} e^{\lambda_1 v_1} & \dots & e^{\lambda_N v_1} \\ \vdots & & \vdots \\ e^{\lambda_1 v_J} & \dots & e^{\lambda_N v_J} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} \approx \begin{pmatrix} q(v_1) \\ \vdots \\ q(v_J) \end{pmatrix},$$

or

$$\mathbf{A}\mathbf{h} \approx \mathbf{q}.$$

Solve for \mathbf{h} . Then valuation of variance contract is $\mathbf{h} \cdot \mathbf{g}$ where \mathbf{g} has components $g_n := \mathbb{E}(u_n^+ e^{p_n^+ X_T} + u_n^- e^{p_n^- X_T})$ for $n = 1, \dots, N$.

Choosing the coefficients

Choose \mathbf{h} such that $\|\mathbf{A}\mathbf{h} - \mathbf{q}\|$ is small.

But also make $\|\mathbf{h}\|$ small. Why:

- ▶ Tame the behavior of $\tilde{q}(v)$ at $v \notin \{v_1, \dots, v_J\}$.
- ▶ Suppose

$$g_n = \mathbb{E}e^{\lambda_n \langle X \rangle_T} + \varepsilon_n \quad n = 1, \dots, N.$$

Under the independence assumption, $\varepsilon_n = 0$. If independence does not hold, our valuation of $\tilde{q}(\langle X \rangle_T)$ has error $\mathbf{h} \cdot \boldsymbol{\varepsilon}$.

Tikhonov regularization: choose \mathbf{h} to minimize

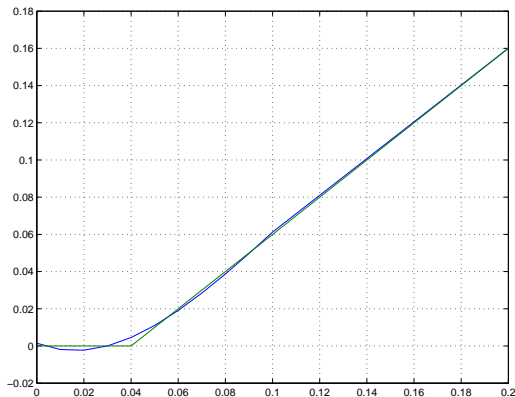
$$\|\mathbf{A}\mathbf{h} - \mathbf{q}\|^2 + \alpha \|\mathbf{h}\|^2,$$

where $\alpha > 0$ is a regularization parameter.

Example: Variance call

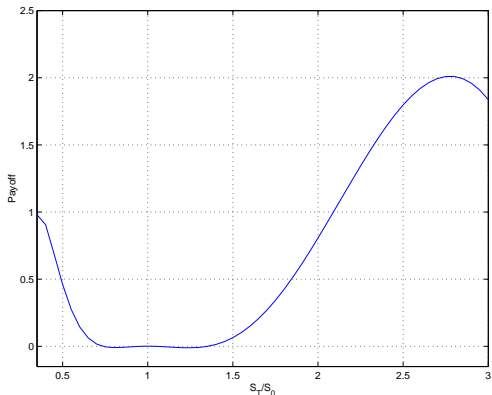
Example: $q(v) := (v - 0.04)^+$ gives a call on variance struck at 0.04.

Let's approximate q using 5 basis functions: $\lambda = 0, -1, -2, -3, -4$.



Variance call: European hedge of approximation

$$\sum_{n=1}^5 h_n (u_n^+ e^{p_n^+ X_T} + u_n^- e^{p_n^- X_T})$$



Introduction

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Robust hedging

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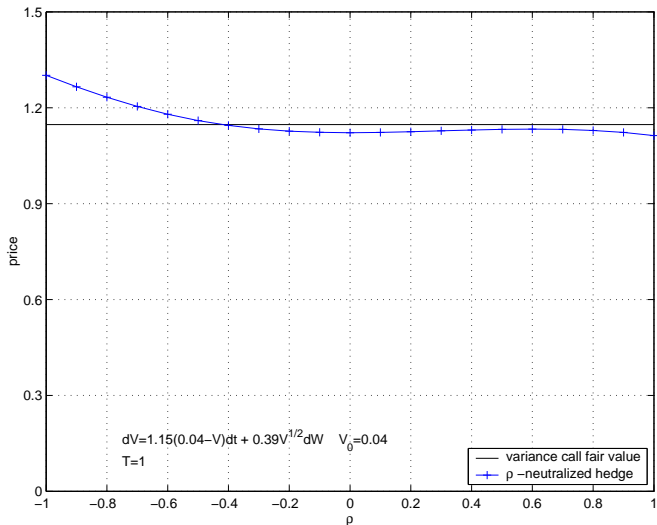
Robustness to correlation

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Discrete approaches

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Variance call: $T = 1.0$ and Heston dynamics



Introduction

Europeans and variance swaps are robustly hedgeable.

Are general functions of variance robustly hedgeable?

Robust hedging

General variance payoffs

Volatility swaps

Robustness to correlation

General variance payoffs

Volatility swaps

Discrete approaches

Hedging

Distributional inference

Use finitely many Europeans to infer the $\langle X \rangle_T$ distribution

Friz-Gatheral (04): Discretize H-W to infer the $\langle X \rangle_T$ distribution.

Observe expiry- T call prices $C(K_1), \dots, C(K_N)$. Under *independence*,

$$\mathbb{E}C^{BS}(K_n, \langle X \rangle_T) = C(K_n).$$

Find probabilities $p_j := \mathbb{P}(\langle X \rangle_T = v_j)$ for $j = 1, \dots, J$ by solving

$$\begin{pmatrix} C^{BS}(K_1, v_1) & \cdots & C^{BS}(K_1, v_J) \\ \vdots & & \vdots \\ C^{BS}(K_N, v_1) & \cdots & C^{BS}(K_N, v_J) \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_J \end{pmatrix} = \begin{pmatrix} C(K_1) \\ \vdots \\ C(K_N) \end{pmatrix}.$$

Distribution of $\langle X \rangle_T$ then yields prices for general contracts on $\langle X \rangle_T$.

Check accuracy

Friz-Gatheral (04):

Generate European call prices, under Heston dynamics, at log-strikes

0, 0.14, 0.28, 0.42, 0.56.

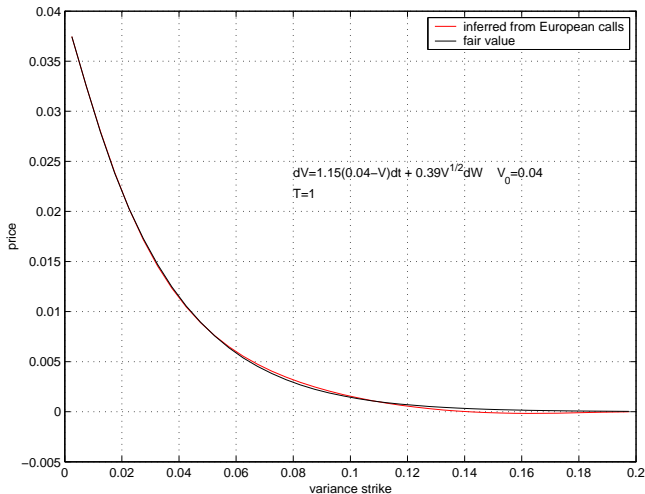
Infer $\langle X \rangle_T$ distribution.

Price variance calls at all desired variance strikes.

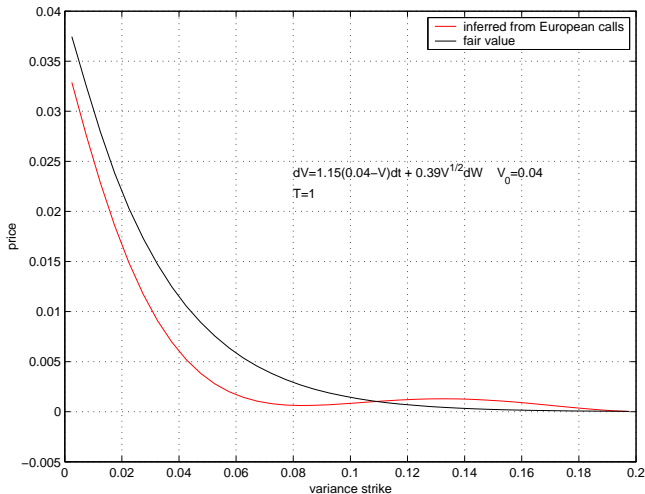
Compare to true variance call prices at all desired variance strikes.

F-G do for $\rho = 0$. Not recommended for ρ nonzero!

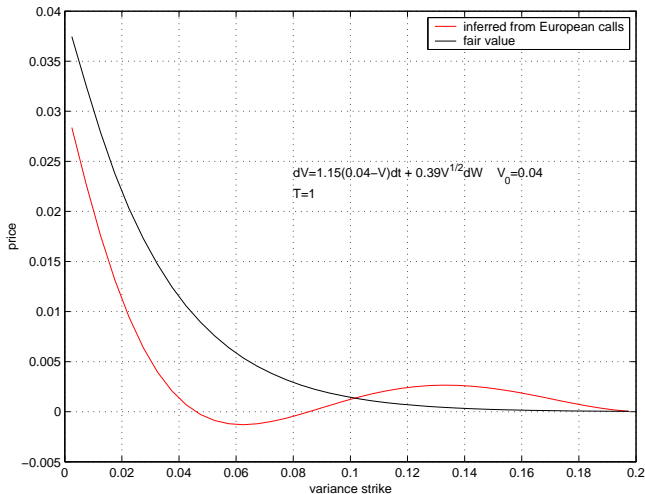
Heston dynamics, $\rho = 0$



Heston dynamics, $\rho = -0.2$



Heston dynamics, $\rho = -0.4$



Solution: use ρ -neutral Europeans

Problem: under nonzero correlation, $\mathbb{E}C^{BS}(K_n, \langle X \rangle_T) \neq C(K_n)$.

Solution: instead of European calls, we *use ρ -neutral* Europeans.

For example, let $G(\lambda)$ be the value of our ρ -neutral European hedge of $\exp(\lambda \langle X \rangle_T)$. Observe $G(\lambda_1), \dots, G(\lambda_N)$. Then solve

$$\begin{pmatrix} G^{BS}(\lambda_1, v_1) & \cdots & G^{BS}(\lambda_1, v_J) \\ \vdots & & \vdots \\ G^{BS}(\lambda_N, v_1) & \cdots & G^{BS}(\lambda_N, v_J) \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_J \end{pmatrix} \approx \begin{pmatrix} G(\lambda_1) \\ \vdots \\ G(\lambda_N) \end{pmatrix}.$$

which we can write as

$$\mathbf{A}^\top \mathbf{p} \approx \mathbf{g}$$

where \mathbf{A} is the matrix defined earlier.

Inferring the probabilities

Solve underdetermined system by imposing regularity on solution.

Tikhonov regularization: choose unique solution

$$\mathbf{p}_\alpha := \arg \min_{\mathbf{p}} (\|\mathbf{A}^\top \mathbf{p} - \mathbf{g}\|_2^2 + \alpha \|\mathbf{p}\|_2^2),$$

where α is a carefully chosen regularization parameter.

(Why not just choose $\alpha \rightarrow 0$? The matrix \mathbf{A} is ill-conditioned: small perturbations in \mathbf{g} can cause large errors in \mathbf{p}_0 .)

More generally, the penalty term can be replaced with $\alpha D(\mathbf{p}, \mathbf{p}^*)$, where D is some measure of deviation between \mathbf{p} and some prior \mathbf{p}^* .

Check accuracy

We generate prices of ρ -neutral T -expiry Europeans, under Heston dynamics, for λ values

$$0, -1, -2, -3, -4.$$

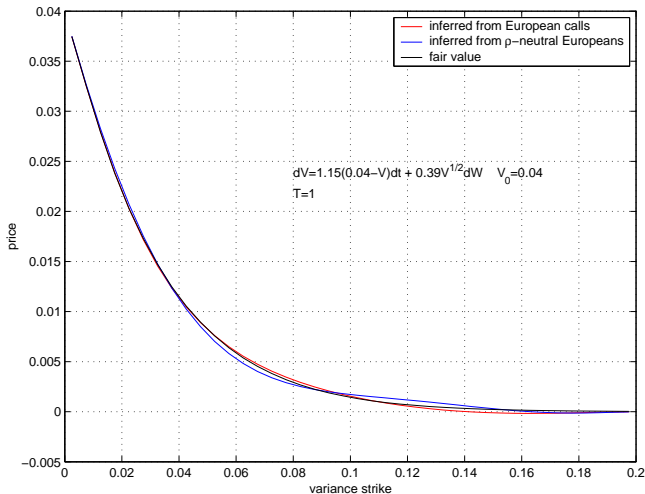
Infer $\langle X \rangle_T$ distribution, using Tikhonov-style regularization.

Price variance calls at all desired variance strikes.

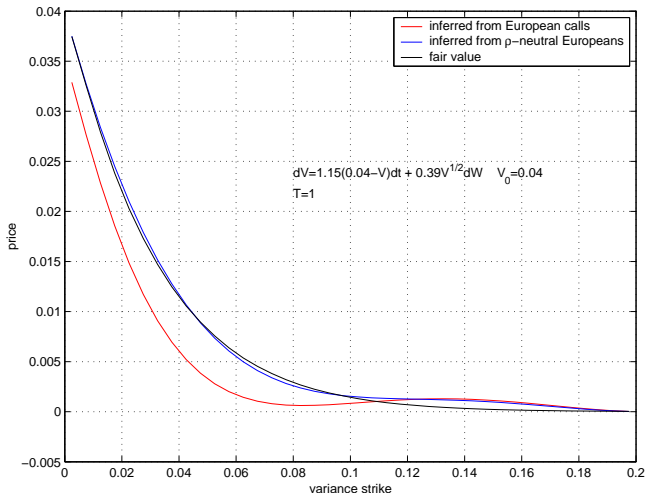
Compare to true variance call prices at all desired variance strikes.

Conclusion: we have gained robustness to correlation.

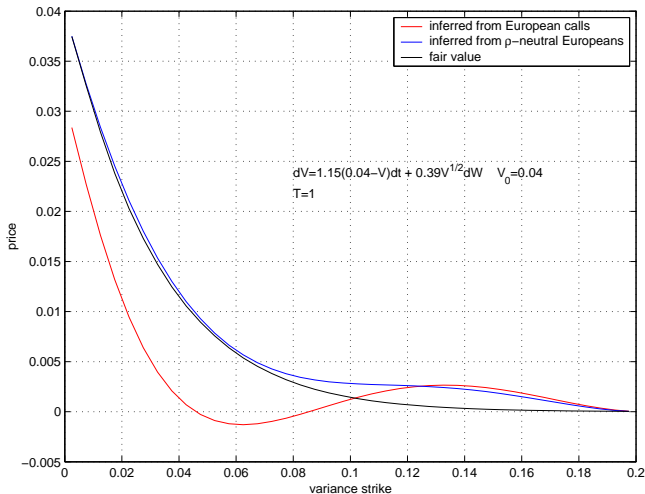
Heston dynamics, $\rho = 0$



Heston dynamics, $\rho = -0.2$



Heston dynamics, $\rho = -0.4$



Consistency of discrete hedge and distribution analyses

In the discrete setting with λ taking N values and $\langle X \rangle_T$ taking J values, we've seen two ways to price a variance payoff \mathbf{q} .

- ▶ Replicate \mathbf{q} using a mixture of exponential variance claims, with coefficient vector \mathbf{h} .

Value the replicating portfolio as the sum of its parts = $\mathbf{h} \cdot \mathbf{g}$.

- ▶ Find vector of probabilities \mathbf{p} .

Value the payoff by taking expectation of payoff = $\mathbf{q} \cdot \mathbf{p}$.

If we want to solve for a hedge \mathbf{h} *and* the distribution \mathbf{p} , can we ensure consistency of the associated valuations?

Consistency of discrete hedge and distribution analyses

Regularized solution of $\mathbf{A}\mathbf{h} \approx \mathbf{q}$ gives coeffs (wrt exponential basis)

$$\mathbf{h}_\alpha = \mathbf{R}_{\mathbf{A},\alpha}(\mathbf{q}),$$

using a regularization scheme $\mathbf{R}_{\mathbf{A}}$ (such as Tikhonov or pseudoinverse) with a parameter α .

- ▶ Example: Tikhonov is $\mathbf{R}_{\mathbf{A},\alpha}(\mathbf{g}) := (\mathbf{A}^\top \mathbf{A} + \alpha \mathbf{I})^{-1} \mathbf{A}^\top \mathbf{g}$.
- ▶ Example: Pseudoinverse is the $\alpha \rightarrow 0$ limit of Tikhonov.

Regularized solution of $\mathbf{A}^\top \mathbf{p} \approx \mathbf{g}$ gives estimated probabilities

$$\mathbf{p}_{\alpha'} = \mathbf{R}_{\mathbf{A}^\top, \alpha'}(\mathbf{g})$$

using a regularization scheme $\mathbf{R}_{\mathbf{A}^\top}$ with a parameter α' .

Consistency of discrete hedge and distribution analyses

If both schemes are linear, and $\mathbf{R}_{\mathbf{A}^\top, \alpha'} = \mathbf{R}_{\mathbf{A}, \alpha}^\top$, then hedge has value

$$\mathbf{h}_\alpha \cdot \mathbf{g} = (\mathbf{q}^\top \mathbf{R}_{\mathbf{A}, \alpha}^\top) \mathbf{g} = \mathbf{q}^\top (\mathbf{R}_{\mathbf{A}^\top, \alpha'} \mathbf{g}) = \mathbf{q} \cdot \mathbf{p}_{\alpha'}$$

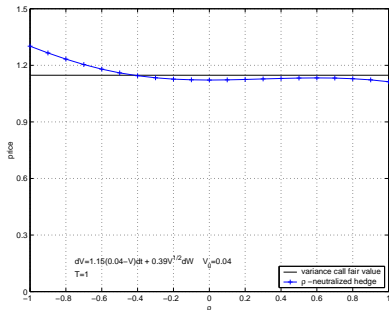
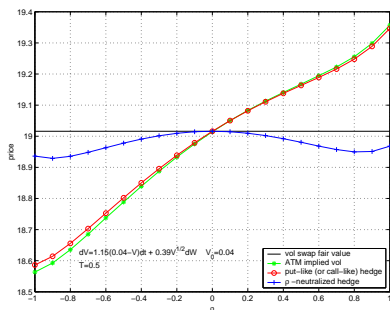
which is consistent with the inferred distribution.

Key condition: The scheme we use to infer the distribution is the *adjoint* of the scheme we use to construct the hedge.

In the case of Tikhonov regularization, this condition says to choose the *same regularization parameter* α for both distributional inference and hedge construction.

Conclusion

Without specifying a stochastic process for volatility, we have synthesized claims paying functions of realized variance.



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