A Stylized History of Quantitative Finance

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Outline

The evolution of a quantitative approach to finance has proceeded through many small but significant steps and occasional large epiphanies.

This talk outlines how, over the past 70 years, financial models have quantified the notion of derivatives, diffusion, risk, volatility, the riskless rate, diversification, hedging, replication, and the principle of no riskless arbitrage.

Modern Finance in One Sentence

Feynman summarizes physics in one sentence:

Everything is made out of atoms.

IF

you can hedge away all correlated risk

AND

you can then diversify over all uncorrelated risk

THEN

you should expect to earn only the riskless rate

This sentence leads to CAPM, APT, Black-Scholes, …

Derivatives as a Method of Understanding

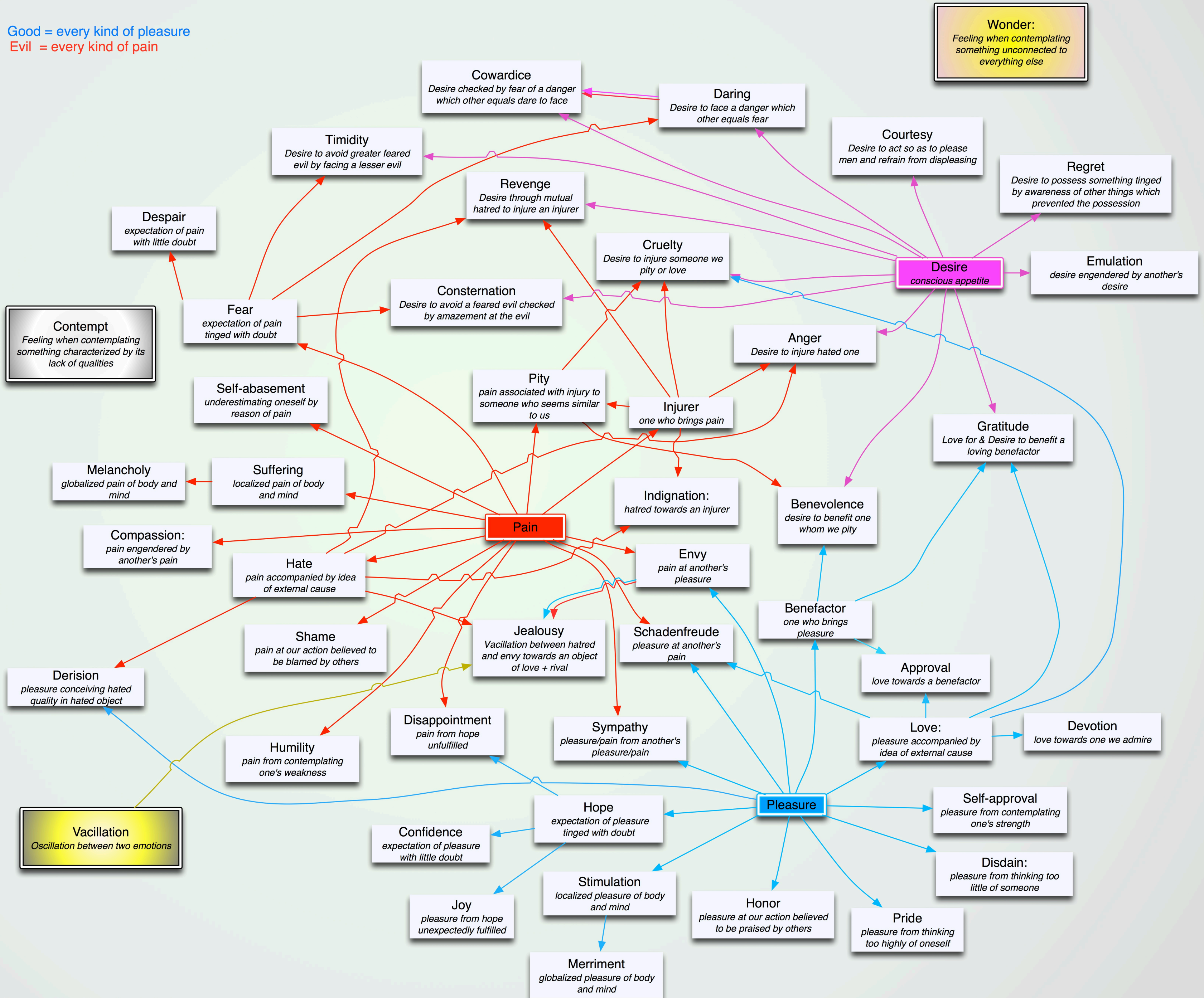
- Euclid axiomatized geometry, starting with point, lines, planes and then proving theorems.
- Spinoza tried to axiomatize human emotions by starting with the primitive visceral affects:

  Desire, Pleasure, Pain.

- Of Human Bondage
- *Good* = everything that brings pleasure.
- *Evil* = everything that brings pain.
- *Love* = Pleasure associated with an external object. (Equity)
- *Hate* = Pain associated with an external object.
- *Envy* = Pain at another's Pleasure. (Equity and Debt)
- Schadenfreude
- Cruelty: Desire to inflict Pain on a someone you Love. (Equity, Debt, Credit)
- Three more primitives: Vacillation, Wonder, Contempt.

Pleasure Pain Desire: A Map of the Emotions*

Emanuel Derman



*according to *Ethics*, Benedict de Spinoza

Derivatives *sans* Diffusion

- Spinoza believed that human behavior follows laws, that nothing is random.
- But his scheme and definitions are static: there is almost no possibility of motion, except for

  Vacillation: the cyclic alternation between two different passions.

- *Jealousy is the oscillation between Hatred and Envy in relation to a Love object and a rival*

  Hatred is Pain associated with an external Person

  Envy is Pain at another's Pleasure.

- Vacillation involves **volatility** — the more rapidly and intensely one Vacillates, the greater the Jealousy.
- Spinoza has no Anxiety in his system. Various opinions:
  - Anxiety is a vacillation between Hope and Fear.
  - Anxiety is not a Passion.
  - There was no Anxiety in the 17th Century.

Diffusion *sans* Derivatives

- 1831: Thomas Graham
- "…gases… when brought into contact, do not arrange themselves according to their density, … but they spontaneously diffuse, mutually and equally, through each other, and so remain in the intimate state of mixture for any length of time."
- 1858: James Clerk Maxwell

  the first microscopic theory of atoms moving in gases

- 1872: Ludwig Boltzmann
- Boltzmann equation: kinetic theory — derives the properties of matter from the properties of atoms and their distributions
- Early 20th Century: Albert Einstein, Marian Smoluchowski and Jean-Baptiste Perrin

  Confirm atomic theory of matter.

- Physicists understood diffusion but deal only with underliers/atoms, but not functions of underliers. The exception: Bachelier in 1900 analyzed the behavior of options that are dependent on stocks that diffuse according to arithmetic Brownian motion. (Ahead of his time, rediscovered in the 1960s)

Defining Risk as Volatility

- Investors were traditionally interested only in how much return they might earn.

- But return is uncertain.

- 1952: Harry Markowitz emphasized the statistics of returns, and emphasized looking at portfolios rather than individual stocks.

- Look at the relation between risk σ and return μ.

  Risk = the standard deviation of returns. *Volatility* σ

  Suggests finding the portfolio with the most return for a given risk.

The Question: What is the Relation Between Risk σ and Return μ ?

- Given that returns on stocks are uncertain, what is the appropriate relation between the risk we expose ourselves to and the return we expect?
- The key question of finance:
  What μ *should* one expect to earn, on average, for taking on a particular future risk σ?

- To answer this, note that there is one security whose returns has no uncertainty at all:
  the riskless bond, whose return is **guaranteed** to be r (a T Bill, say)

- This serves as a touchstone for measuring all other returns.
- We can denote every stock by the doublet (μ,σ) denoting its expected return and its volatility.
- The riskless bond is $(r,0)$.

A Strategy for Answering the Question: Replicate a Riskless Bond

- 1958: Modigliani and Miller introduced **replication as a strategy for valuation**:
  - To value a security (μ,σ), reduce its risk to zero by combining it with other securities into a portfolio P **that has zero risk**.
  - Then P has the risk of a riskless bond.
  - By the *Law of One Price*, P must be guaranteed to earn the riskless rate *r*.
  - Imposing this on P leads to relation between μ and σ for the stock.
- To do this one must know how to reduce risk.

How Can One Reduce Risk?

- Dilution:
  Combine a security with a riskless bond

- Diversification:
  Combine a security with many other uncorrelated securities

- Hedging:
  Combine a security with a correlated security

- Apply this in three successively more realistic toy worlds.

Simple World 1
A few uncorrelated stocks and a riskless bond: All stocks have same Sharpe Ratio

- Dilution: Combine weight *w* of a risky stock *S* (μ,σ) with a weight (1 – *w*) of a riskless bond *B* (*r*,0) to create a new security with lower risk & return

$$[w\mu + (1-w)r,\ w\sigma] = [r + w(\mu - r),\ w\sigma]$$

- Law of One Price:
  All uncorrelated stocks with risk wσ earn excess return w(μ – r)

- One parameter fixes everything

- Same Sharpe ratio for all stocks!

$$\frac{\mu - r}{\sigma} = \lambda$$

- More risk, more return

Less Simple World 2:

Many Uncorrelated Stocks: Diversify!

The Sharpe Ratio is zero

Every stock is expected to earn the riskless rate r

- Suppose there are countless uncorrelated stocks (μ_i, σ_i)
- Put them all in a portfolio with weight: $P = \sum w_i S_i$
- Then the portfolio risk σ diversifies to zero.

$$\mu - r = \lambda\sigma = 0$$

- Thus the portfolio is riskless: $\mu = r$
- But the portfolio return is the sum of individual returns:

$$\mu = \sum w_i \mu_i = \sum w_i (r + \lambda\sigma_i) = r + \lambda \sum \sigma_i \quad \text{therefore} \quad \lambda = 0$$

- Thus every stock is expected to earn the riskless rate!

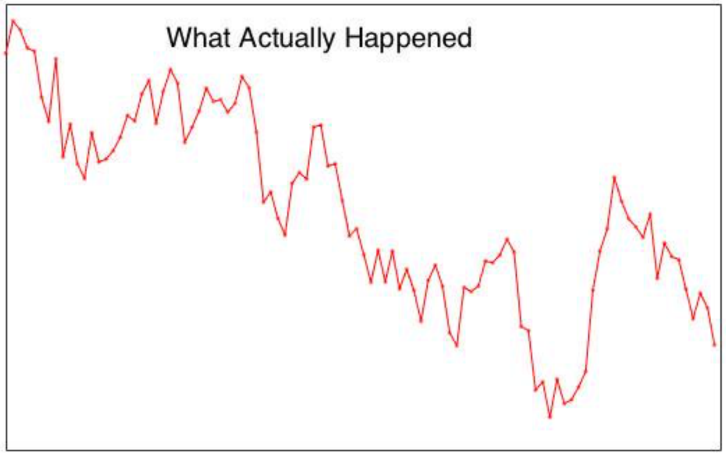
More Realistic World 3: CAPM

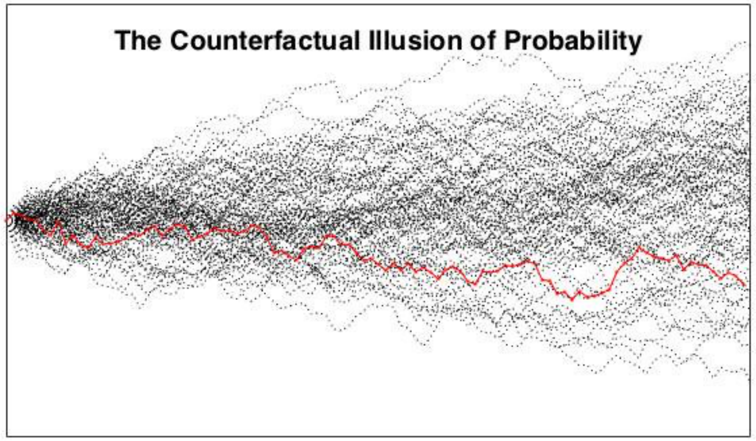
All Stocks are Correlated with the Market: Hedge the Market Risk, then Diversify! Every market-neutral stock must earn the riskless rate.

- Suppose there are countless stocks S_i correlated with the market M
- Then the market-neutral stock $S_i^M = S_i - \beta_i\left(\left|\frac{S_i}{M}\right|\right)M$ is uncorrelated with M
- After diversification each market-neutral stock S_i^M arns riskless rate.
- Which means $(\mu_i - r) = \beta_i(\mu_M - r)$
- CAPM just says that if you hedge every stock with the market, and then diversify over all remaining risk, you should earn only the riskless rate.

Why is CAPM Bad?

- Because Risk is Not Really the Standard Deviation of Returns.
- Because the market M and the stock S are not really stably correlated.
- Markets are not exactly like flipping coins. There isn't a well-defined *a priori* probability of a market crash. Probability is a bit of an illusion





1960s: Early Options Models: Diffusion and Volatility but No Replication

- Samuelson, Sprenkel, Ayres, Boness … value call options actuarially, as the expected discounted payoff of the option under a lognormal distribution with an unknown future growth rate and a known volatility.

- But at what rate does the stock grow?

- And what discount rate to use?

1973: Putting everything together
Black-Scholes-Merton
Diffusion+Volatility+Hedging+Replication

- Create a portfolio long the call, short Δ shares of stock: $P = C - \Delta\, S$
- Calculate change in value $dP = dC - \Delta\, dS$
- Use diffusion for the move in the underlying stock price dS
- Use stochastic calculus to find the move in the derivative $dC(S)$
- Choose Δ to eliminate stock risk in $dP = dC - \Delta\, dS$
- Require that hedged portfolio, which is riskless, earns the known riskless rate r:

$$dC - \Delta\, dS = r(C - \Delta\, dS)$$

- Then we get the same formula for the price of the call C as the actuarial one, but where all growth and discount rates are replaced by the riskless rate r.

Black-Scholes-Merton

- You can replicate/hedge an option with stock
- Option C and stock S must have same Sharpe ratio

$$\frac{\mu_S - r}{\sigma_S} = \frac{\mu_C - r}{\sigma_C}$$

- Ito's Lemma applied to a call *C* leads to the Black-Scholes pde:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

- A unified treatment of BSM and CAPM from one principle

Why is BSM better than CAPM?

- Because you more realistically can hedge an option with a stock, because the correlation is really close to 1.
- So even if you don't believe the risk is the standard deviation of returns, the two securities really are connected, unlike the statistical connection between two different stocks.
- The Caveat: We have assumed that volatility is unchanging! If volatility is random, then the derivative is not really a derivative except at expiration

1970s: Using the BS Equation

- Now, to value an option, instead of forecasting the return of the stock, traders must forecast the volatility of the stock.

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

- Black and Scholes set about using the equation by using historical volatilities to estimate future volatilities. But who knows what future volatility will be?

1976: Calibration
The Invention of Implied Volatility

- Latane and Rendelman suggested fitting option market prices to the Black-Scholes formula and extracting the implied future volatility of the stock that fits the option market price. They then suggest calculating hedge ratios from the model using the implied volatility.
- Implied volatility is a parameter, not a statistic.
- But implied volatilities are unstable.
  This process must be repeated as the implied volatility keeps changing, so there is *something not quite right*.
- Implied volatilities tell you that, given an option price, if you believe the model, this is what the future must be like. But the future doesn't turn out that way.
- Nevertheless, from now on everyone calibrates models.
- Most people don't even realize it was an invention.

Because the Model Doesn't Work, Trading Volatility is Now a Possibility

- In fact:
  - Volatility is stochastic.
  - So you can't **really** replicate an option
  - Instead, you can speculate on the **volatility parameter**, using options.
- If you take the model seriously, hedged options now become a way of trading the volatility parameter rather than speculating on the stock price.
- Volatility as an asset class.

Social Science vs. Physical Science

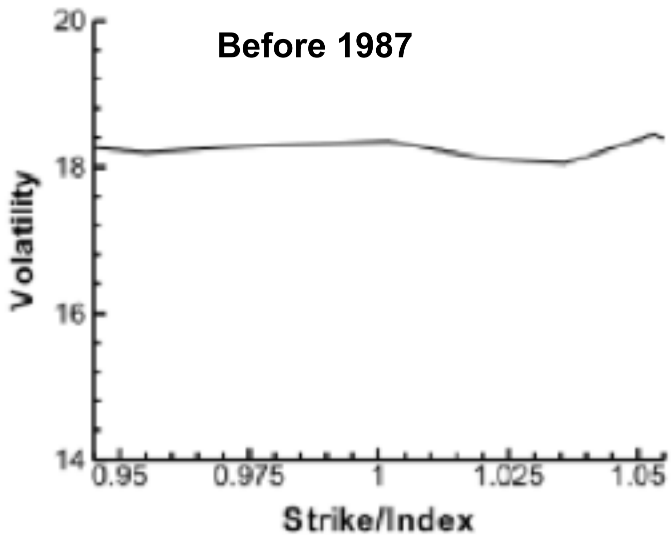
- Black-Scholes is a social science model.
- It makes an analogy between a social/mental phenomenon and a physical one: it compares stock price changes to the Brownian motion of randomly diffusing particles.
- The analogy driving the model is not absolutely true or even accurate, but the framework and concepts it provides have reified the notion of risk as volatility.

- In physics you start from the present and predict what will happen in the future. You're always moving from present to future.
- In finance, you calibrate the future to known present prices. Then you use that future to hedge, and to predict other present prices.
- You move from the present to the future to the present.

1977: Yield Curve Modeling: Parameters

- Modeling the yield curve — an extension of Black-Scholes to bonds rather than stocks, to rates rather than securities. Initiated by Vašíček.
- Focus on evolution of parameters rather than securities.
- The difficulty is avoiding future arbitrages in the model when there is more than one security, as with the yield curve.
- You can value options on two stocks independently, but you cannot value options on two different-maturity bonds independently:
    - There are no-arbitrage constraints on bond prices — no negative forward rates.
- And so on … to other extensions of the hedging paradigm for 45 years …

1987: The Smile



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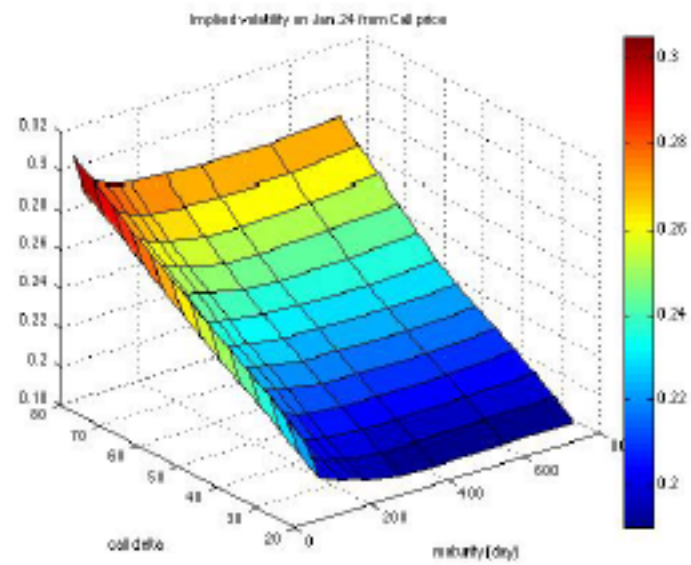
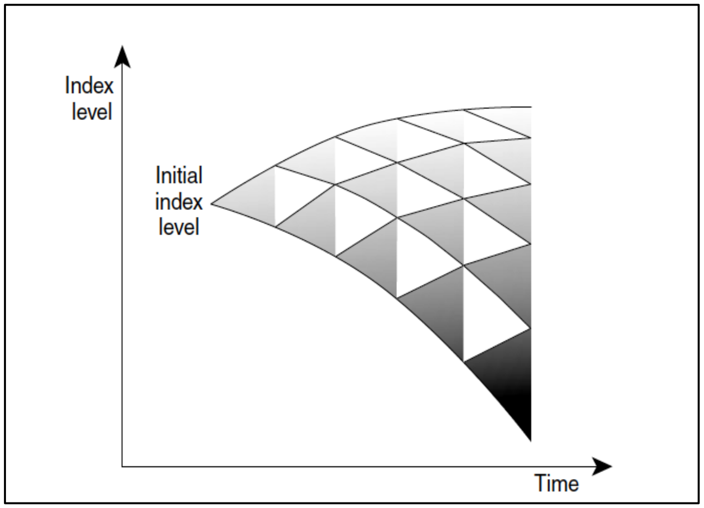
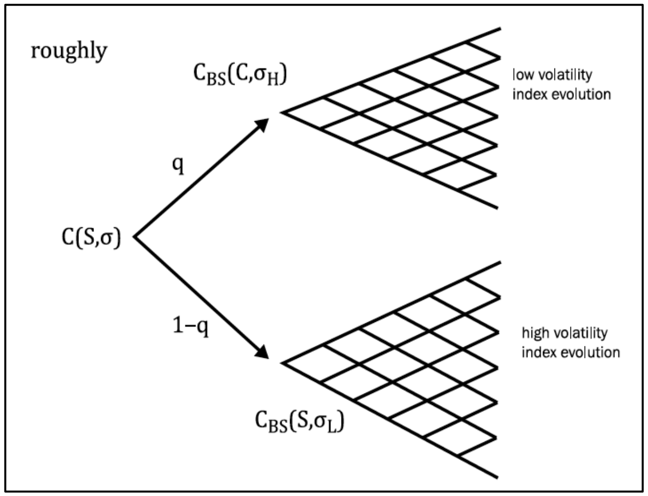


- When you fit the BS model to different option strikes, each one implies a different future volatility for the underlying stock.
- *Now something is really wrong* — the BS model cannot accommodate different volatilities for the same stock.
- Nevertheless, people keep using the model inconsistently to estimate hedge ratios as they calibrate the model to a particular option price.

1994 - present: The Smile





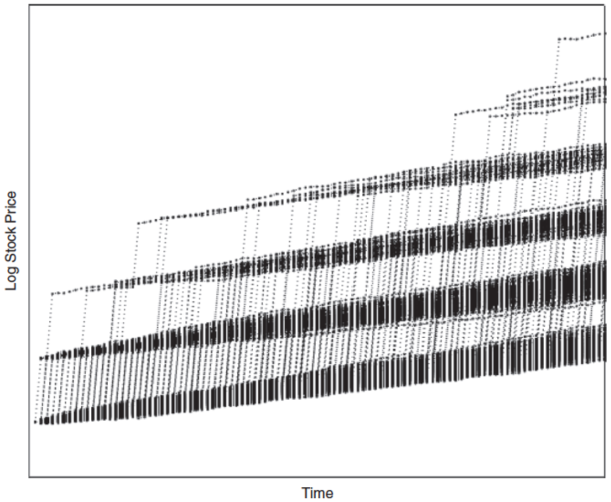


FIGURE 24.3 A Monte Carlo Simulation of the Log Stock Prices in the Jump-Diffusion Model

- BS Extensions: local volatility, stochastic volatility, jumps plus diffusion …
- More complexity without accurate knowledge of the parameters.
- Use calibration-to-market-prices to imply the parameters — e.g. the volatility of volatility in a calibrated stochastic volatility model.
- But markets change and these implied parameters are themselves unstable and random, and there are now more of them. So, for example, volatility of volatility is stochastic.
- So now, *using a "better" model* you have a market for trading volatility of volatility

Where We Are
1 …

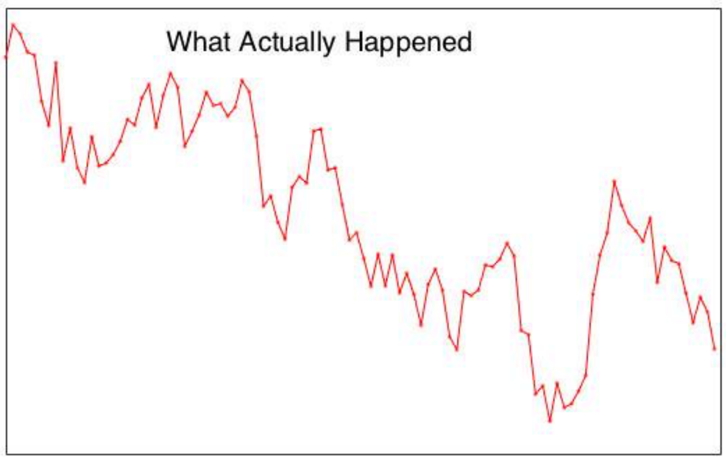
- The BS Model is genuinely useful yet always inadequate. Derivatives are not truly derivative, except at expiration.
- The analogy with diffusion is inaccurate, but the framework has reified the notion of risk as volatility.
- This reification has dominated the modeling of financial markets and provided a foundation for extensions of the Black-Scholes model.
- Each new model is inadequate too. It introduces new parameters, which, as the model is embraced and yet fails, become new parameters the market can speculate and trade on.
- When a widely embraced reification is proved to be inaccurate by the behavior of the market, systemic market collapses can occur.
- It takes all of these securities —

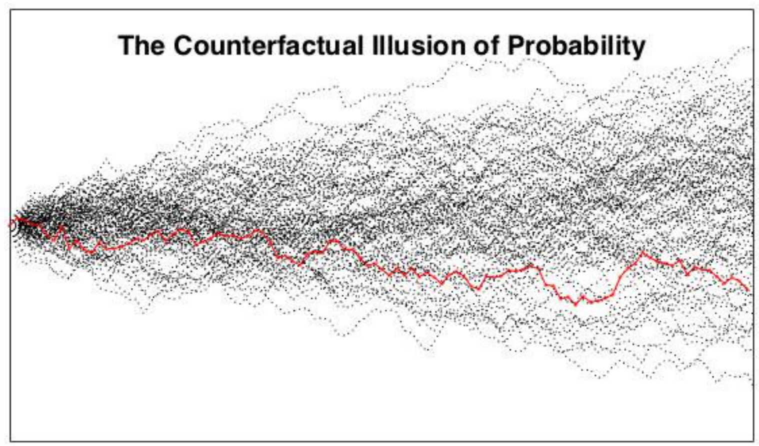
  stocks, options, options on options, volatility, volatility of volatility …

  to define the independent possibilities of the market.

… Where We Are
2

- The probabilistic approach to distributions is a fallacy. The “probabilities” only come into existence after an event, which is offering a price. (Ayache, *The Medium of Contingency*)
- Probabilistic models … are only internal episodes that we require in order locally and always imperfectly to hedge something. We have to keep in mind that …(these)… are useful only insofar as they will be recalibrated.





Post-modern Finance: The Dangerous Reification of Parameters

- One used to eat food, now one eats nutrients, and thinks of food items as baskets of nutrients.
- One used to buy stocks and portfolios of stocks. Now one thinks of stocks as combinations of factors: momentum, volatility, popularity …
- One used to trade securities; now one trades parameters.
  - Rates instead of bond prices.
  - Volatility instead of options prices.
  - Credit, VIX, momentum, smart beta, low vol, popularity …
- Models turn prices into parameters.
- The danger is that this reification makes it easy for unskilled crowds to trade subtle things that previously took skill.
  - CDS make it easy to trade credit
  - VIX derivatives made it easy to trade vol

The Dominance of Axiomatization and Formalism in Academic Finance …

- The fundamental theorem of finance:

Theorem 3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be countably generated and* $X = L^p(\Omega, \mathcal{F}, \mathbb{P})$ *endowed with the norm topology* τ, *if* $1 \leq p < \infty$, *or the Mackey topology induced by* $L^1(\Omega, \mathcal{F}, \mathbb{P})$, *if* $p = \infty$.

Let $S = (S_t)_{0 \leq t \leq T}$ *be a stochastic process taking values in* X. *Define* $M_0 \subseteq X$ *to consist of the simple stochastic integrals* $\sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}})$ *as in (2).*

Then the "no free lunch" condition (3) is satisfied if and only if there is a probability measure Q *with* $\frac{dQ}{d\mathbb{P}} \in L^q(\Omega, \mathcal{F}, \mathbb{P})$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *such that* $(S_t)_{0 \leq t \leq T}$ *is a* Q*-martingale.*

- Finance vs Physics: Efficacy inversely proportional to formality.
- There are no reliable theorems in finance; it's not math, it's the world.

A Plea for Less *Faux* Formalism

- Gian Carla Rota on Mark Kaç:

  “Throughout his life he remained skeptical of abstraction, of techniques, of axiomatics. Instead he inspired the first generation of scientists who learned to think probabilistically. He warned them that axioms will change with the whims of time, but an application is forever.”

- Paul Dirac: “I am not interested in proofs, I am interested only in what nature does.”
- Barenblatt in a book on “Scaling”:

  Of special importance is the following fact: the construction of models, like any genuine art, cannot be taught by reading books and/or journal articles (I assume that there could be exceptions, but they are not known to me). The reason is that in articles and especially in books the ‘scaffolding’ is removed, and the presentation of results is shown not in the way that they were actually obtained but in a different, perhaps more elegant way. Therefore it is very difficult, if not impossible, to understand the real ‘strings’ of the work: how the author really came to certain results and how to learn to obtain results on your own.