MA3237 : Assignment #3

Due on Wednsday, 4 Apr 2014

Chen Yu
A0077976E
Engineering Science Programme

Homework Part I

(1)

Case 1:

$$\hat{V}_{B1} = \frac{\#\text{hits in B}}{\#\text{hits in A}} V_A$$

Let $\hat{\theta}_1 = \frac{\#\text{hits in B}}{\#\text{hits in A}}$, which follows a binomial distribution with $N = n$, $p = \frac{V_B}{V_A}$
Then

$$\sigma_1 = \frac{\theta_1(1-\theta_1)}{n} V_A^2 = \frac{V_A V_B (V_A - V_B)}{n}$$

Case 2:

$$\hat{V}_{B2} = \frac{\#\text{hits in BD}}{\#\text{hits in AD}} (V_A - V_D) + V_{BD}$$

Let $\hat{\theta}_2 = \frac{\#\text{hits in BD}}{\#\text{hits in AD}}$, which follows a binomial distribution with $N = n$, $p = \frac{V_{BD}}{V_{AD}}$
Then

$$\begin{aligned}
\sigma_2 &= \frac{\theta_2(1-\theta_2)}{n}(V_A - V_D)^2 \\
&= \frac{(V_A - V_D) V_{B\setminus D}(V_A - V_{B\setminus D} - V_D)}{n} \\
&= \frac{(V_A - V_D)(V_B - V_{B\cap D})(V_A - V_{D\setminus B} - V_B)}{n}
\end{aligned}$$

Since

$$\begin{aligned}
V_A &> V_A - V_D \\
V_B &> V_B - V_{B_{B\cap D}} \\
V_A - V_B &> V_A - V_B - V_{D\setminus B}
\end{aligned}$$

we get $\sigma_1 > \sigma_2$

(2)

a) In this method, the estimator follows a binomial distribution with $n = N$ and $p = \frac{I}{1} = I$
Hence the variance is

$$\frac{1}{N} I(1 - I) = \frac{1}{N}(I - I^2)$$

b) The variance of this Monte Carlo Integration is

$$\begin{aligned}\mathrm{Var}[\frac{1}{N}\sum_{i=1}^{n} f(U_i)] &= \frac{1}{N^2} N \mathrm{Var}[f(U_i)] \\ &= \frac{1}{N}(\mathrm{E}[f(U_i)^2] - I^2)\end{aligned}$$

Since

$$\forall x \in [0, 1], f(x) \in (0, 1)$$

we have

$$f(x)^2 < f(x)$$

, thus

$$\mathrm{E}[f(U_i)^2] < \mathrm{E}[f(U_i)]$$

Therefore the method in b) has a smaller variance.

(3)

$$
\begin{aligned}
(n-1)S_n^2 &= \sum_{i=1}^{n} (X_i - \bar{X})^2 \\
&= \sum_{i=1}^{n} (X_i^2 - 2\bar{X}X_i + \bar{x}^2) \\
&= \sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{x}^2 \\
&= \sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{x}^2 \\
&= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[S_n^2] &= \frac{1}{n-1}\mathrm{E}[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2] \\
&= \frac{1}{n-1}(n\mathrm{E}[X_i^2] - n\mathrm{E}[\bar{X}^2]) \\
&= \frac{n}{n-1}(\mathrm{E}[X_i^2] - \mathrm{E}[\bar{X}^2])
\end{aligned}
$$

$$
\mathrm{E}[X_i^2] = (\mathrm{E}[X_i])^2 + \mathrm{Var}[X_i]
$$

$$
\begin{aligned}
\mathrm{E}[\bar{X}^2] &= (\mathrm{E}[\bar{X}])^2 + \mathrm{Var}[\bar{X}] \\
&= (\mathrm{E}[\frac{1}{n}\sum_{i=1}^{n} X_i])^2 + \mathrm{Var}[\frac{1}{n}\sum_{i=1}^{n} X_i] \\
&= (\mathrm{E}[X_i])^2 + \frac{1}{n}\mathrm{Var}[X_i] && \text{(i.i.d)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{E}[S_n^2] &= \frac{n}{n-1}((\mathrm{E}[X_i])^2 + \mathrm{Var}[X_i] - (\mathrm{E}[X_i])^2 - \frac{1}{n}\mathrm{Var}[X_i]) \\
&= \mathrm{Var}[X_i]
\end{aligned}
$$

Homework Part II(1)

Let $P_{x,y}$ be the transition probablity function. The transition probablity function generated by the new Metropolis algorithm is

$$Q_{x,y} = \min\{1, \frac{i}{P_{y,x} f(y)} P_{x,y} f(x)\} P_{x,y}$$

$$\begin{aligned} f(x) Q_{x,y} &= f(x) \min\{1, \frac{i}{P_{y,x} f(y)} P_{x,y} f(x)\} P_{x,y} \\ &= \min\{f(x) P_{y,x}, f(y) P_{y,x}\} \end{aligned}$$

$$\begin{aligned} f(y) Q_{y,x} &= f(y) \min\{1, \frac{i}{P_{x,y} f(x)} P_{y,x} f(y)\} P_{y,x} \\ &= \min\{f(x) P_{y,x}, f(y) P_{y,x}\} \end{aligned}$$

$$\implies f(x) Q_{x,y} = f(y) Q_{y,x}$$

, thus satisfying the detailed balance, which ensures that the Metropolis-Hasting chain generated has invariant distribution.

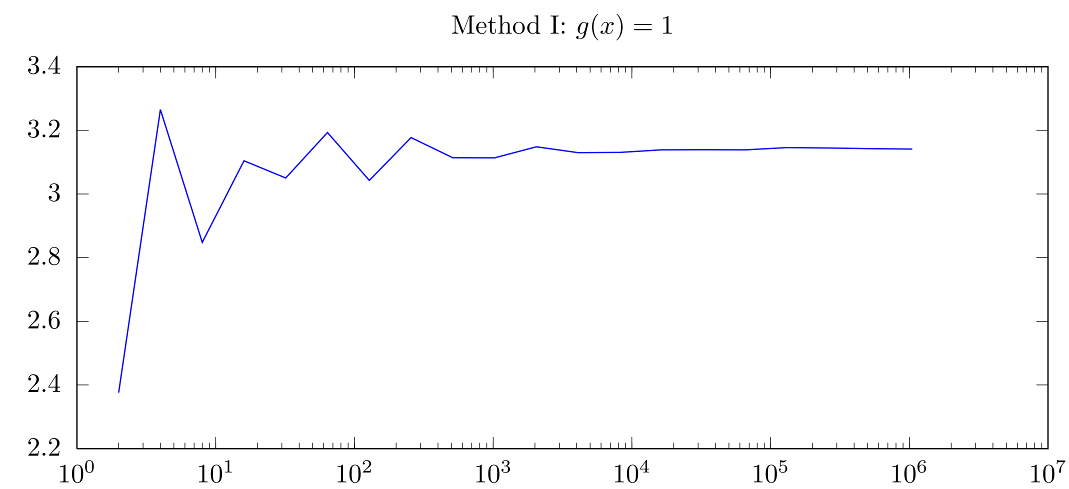
Computer Project Part I

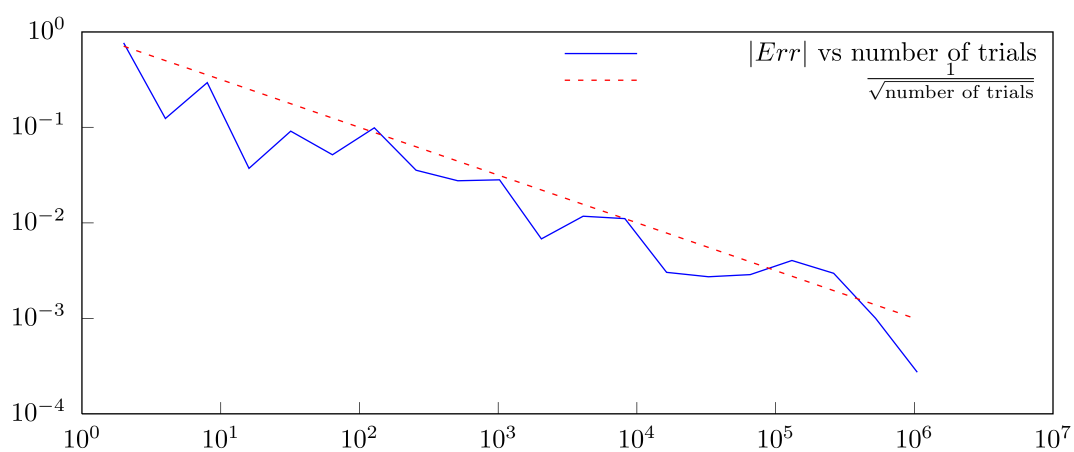
Code

```
clear;
close all;
graphics_toolkit("gnuplot");

function I = repI(i)
     I = [];
     for j = 1:i
               I = [I,'I'];
     end
end

nTrials=2.^(1:20); m=length(nTrials);

vEst = cell(3,1);
for i = 1:3
     vEst{i} = zeros(1,m);
end

rand('state',0);
for i=1:m
     n=nTrials(i);
     x = rand(1,n);
     vEst{1}(i) = sum(4*sqrt(1-x.^2))/n;
     x = -sqrt(4 - 3*rand(1,n)) + 2;
     vEst{2}(i) = sum(12*sqrt(1-x.^2)./(4-2*x))/n;
     x = -sqrt(1 - rand(1,n)) + 1;
     vEst{3}(i) = sum(4*sqrt(1-x.^2)./(2-2*x))/n;
end

err = cell(3,1);
for i = 1:3
     err{i} = abs(vEst{i} - pi);
end

equation = cell(3,1);
equation{1} = '$g(x) = 1$';
equation{2} = '$g(x) = \\frac{4-2x}{3}$';
equation{3} = '$g(x) = 2-2x$';

for i = 1:3
     the_plot = figure(i);
     subplot(2,1,1);
     semilogx(nTrials,vEst{i}, 'LineWidth', 2);
     title(['Method ',repI(i),': ', equation{i}]);

     subplot(2,1,2);
     hold on;
     loglog(nTrials, err{i}, 'LineWidth', 2);
     loglog(nTrials, 1./sqrt(nTrials),'r:','LineWidth', 2);
     legend('$\abs{Err}$ vs number of trials',
```

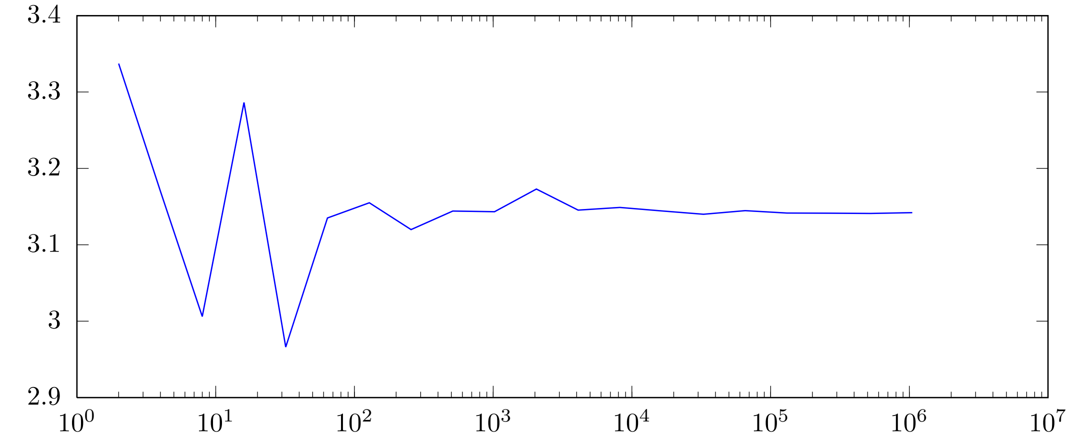
```
            '$\frac{1}{\sqrt{\text{number of trials}}}$');
    legend boxoff;

    print(the_plot,['Method_',repI(i),'.tex'],'-S470,420','-dtex')
end

err_plot = figure(4);
err_diff = cell2mat(err).*sqrt(nTrials);
boxplot(err_diff', 1, '+', 0);
title(['$\abs{Err} \times \sqrt{\text{number of trials}}$',
      ' for different methods'])
set(gca (), 'ytick', [1 2 3], 'yticklabel', equation)
print(err_plot,'comparison','-S470,150','-dtex')
```

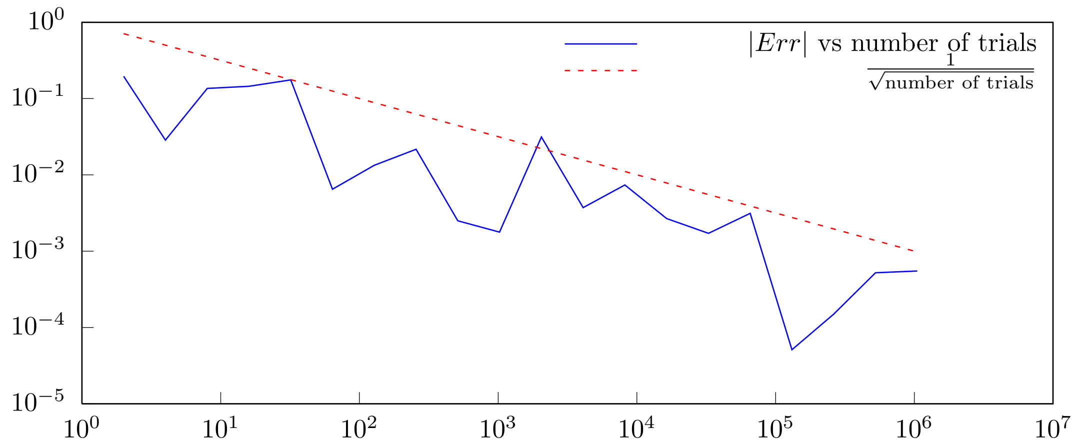
Figures

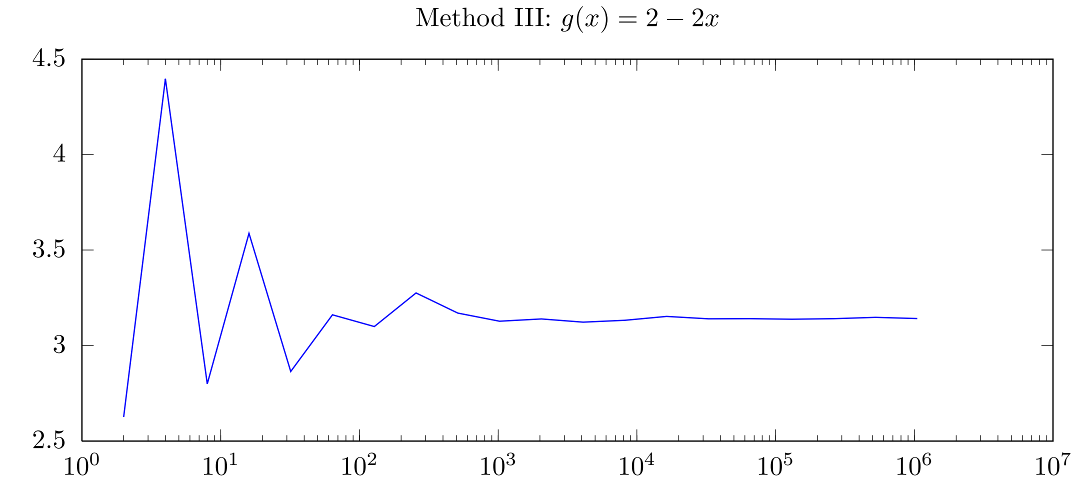


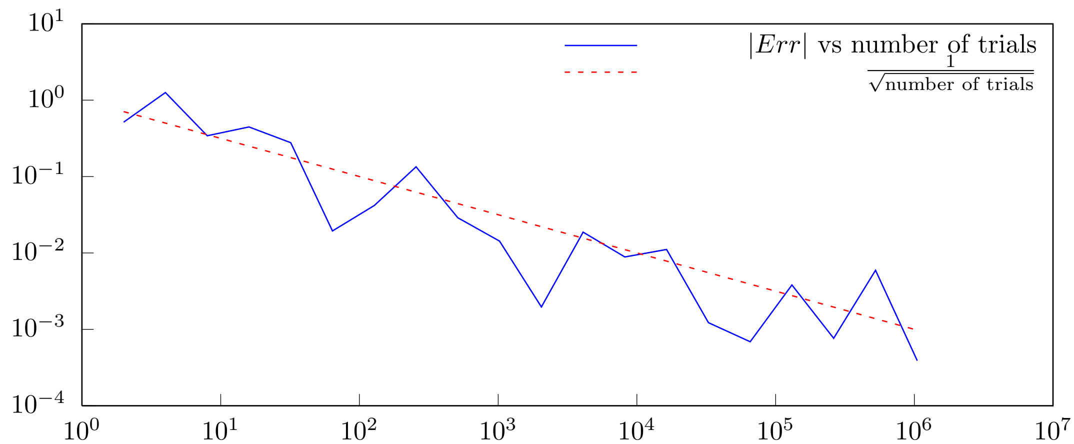


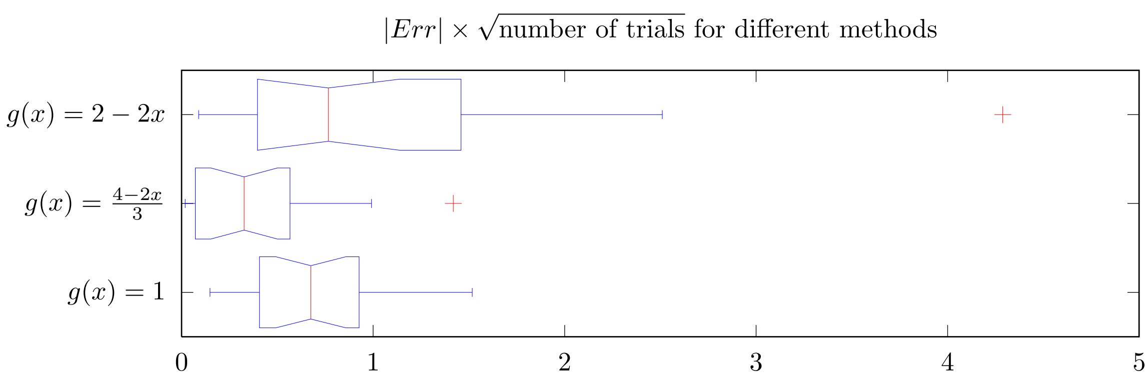
Method II: $g(x) = \frac{4-2x}{3}$






Method III: $g(x) = 2 - 2x$
4.5
4
3.5
3
2.5
10^0 10^1 10^2 10^3 10^4 10^5 10^6 10^7


$|Err|$ vs number of trials
$\frac{1}{\sqrt{\text{number of trials}}}$
10^1 10^0 10^{-1} 10^{-2} 10^{-3} 10^{-4}
10^0 10^1 10^2 10^3 10^4 10^5 10^6 10^7


$|Err| \times \sqrt{\text{number of trials}}$ for different methods
$g(x) = 2 - 2x$
$g(x) = \frac{4-2x}{3}$
$g(x) = 1$
0 1 2 3 4 5

Computer Project Part II

Code

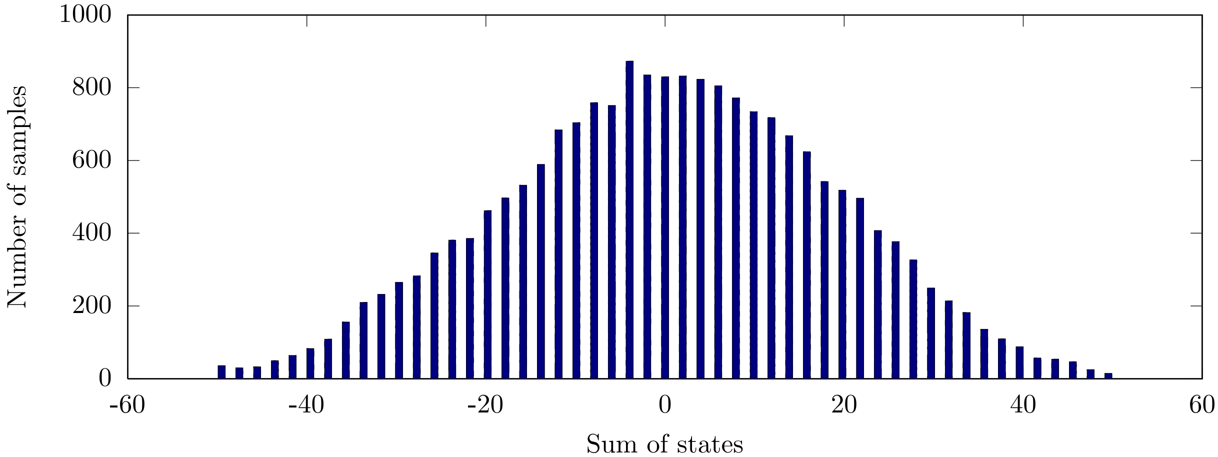
```
clear; close all;
graphics_toolkit("gnuplot");

N_bins = 101;

function M = Metro_Ising(mu, N_steps, sample_rate, N)
     x = ones(N, 1);
     M = zeros(0, N_steps/sample_rate);
     for i = 1:N_steps
          j = ceil(N*rand);
          if j == 1
               h = exp(-2*mu*x(1)*(x(2)));
          elseif j == N
               h = exp(-2*mu*x(N)*(x(N-1)));
          else
               h = exp(-2*mu*x(j)*(x(j-1)+x(j+1)));
          end
          U = rand;
          if U <= h
               x(j) = -x(j);
          end
          if mod(i, 50) == 0
               M(i/50) = sum(x);
          end
     end
end

N_steps = 1e6;
sample_rate = 50;
N = 50;
%mesh = -N:floor(2*N/(N_bins - 1)):N;
%[n, h] = hist(M, mesh);
the_plot = figure();
M_h = Metro_Ising(1, N_steps, sample_rate, N);
M_l = Metro_Ising(2, N_steps, sample_rate, N);

h_plot = subplot(2,1,1);
hist(M_h, 101);
title(['Histogram of sum of states of high temperature ($\mu = 1$)',
       '1D Ising model with $20,000$ from $1,000,000$ states']);
xlabel('Sum of states');
ylabel('Number of samples');

l_plot = subplot(2,1,2);
hist(M_l, 101);
title(['Histogram of sum of states of low temperature ($\mu = 2$)',
       '1D Ising model with $20,000$ from $1,000,000$ states']);
xlabel('Sum of states');
ylabel('Number of samples');

```

Computer Project Part II [Code] continued on next page. . .

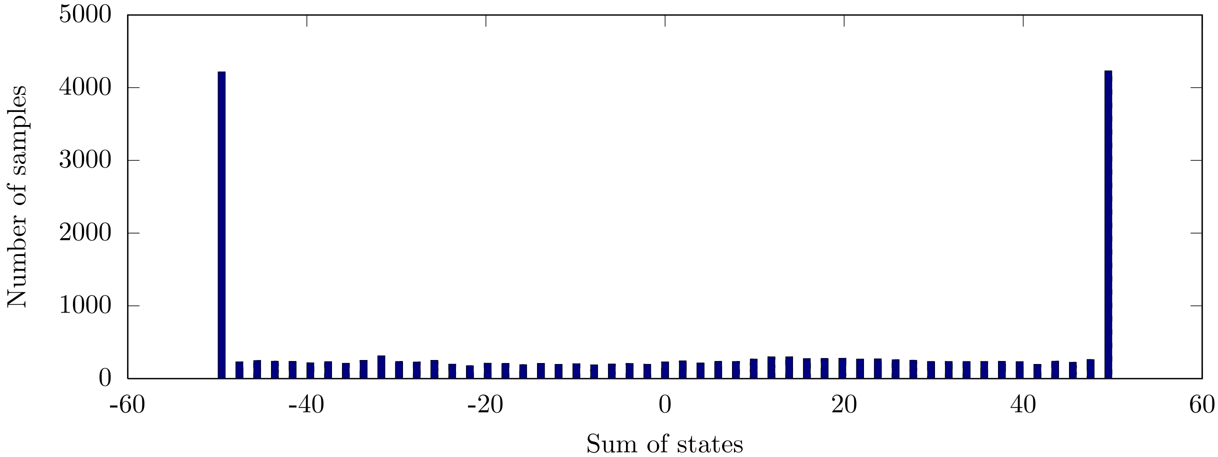
```
print(the_plot, ['MetropolisIsing','.tex'],'-S520,400','-dtex')
```

Figures

Histogram of sum of states of high temperature ($\mu = 1$) 1D Ising model with $20,000$ from $1,000,000$ states



Histogram of sum of states of low temperature ($\mu = 2$) 1D Ising model with $20,000$ from $1,000,000$ states



Homework Part II(2)

f is the distribution of all microstates in the state space of 2^N possible micro states of $(x_1, x_2, ..., x_N)$, where $x_i = -1$ or 1.
$P_{x,y} = \frac{1}{N}$ if $x_i = y_i$ for all but one component; which results in a proposed state y where only the spin of one particle is reversed.