SFWR ENG 4E03

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Note: material covered in Stats 3Y03 Summary will not be covered in this summary. To find a unit CTRL-F "[<unit>]", e.g. for Number of jobs in system, CTRL-F "[N]"

Contents

Statistics

Poisson parameter [λ]: rate
Service rate [μ]:
Continuous Random Variable (CRV):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Think chemistry, i.e. cancelling units

Variance

$$\operatorname{var}(X) = E\left[X^2\right] - \left(E[X]\right)^2$$

- Don't change probability, but square X for calculation only

$$\begin{aligned}\operatorname{var}(x) &= E\left[(X-\mu)^2\right] \\ &= \sigma^2 \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,\mathrm{d}x \\ &= \int x^2 f(x)\,\mathrm{d}x - \mu^2\end{aligned}$$

The higher your variance, the worse your system will perform.

Exponential

- **Mean** [E[X]]: 1/λ
  - a.k.a. Expected value
- **Variance**: $1/\lambda^2$
- **Probability Distribution Function (PDF)** [P(X=x)]: $\lambda e^{-\lambda x}/x!$
- **Cumulative Distribution Function (CDF)** [f(x)]: CDF = ∫PDF, i.e. $1 - e^{-\lambda x}$
- Memoryless
- not always for time

Uniform

- **Variance**: $(b-a)^2/12$
- **Mean**: (a+b)/2
- **PDF**: 1 / (b–a) , a ≤ x ≤ b
- **CDF**: x–a/b–a
- **Uniform Distribution**: no memoryless property

Binomial

- **Mean** [E[X]]: n × probability
- **Variance**: n × p × (1 – p)
- **Probability Distribution Function (PDF)** [P(X)]: $(n \text{ c } x)p^x(1-p)^{n-x}$
- **Cumulative Distribution Function (CDF)** [f(x)]: $\sum_{i=0}^{\lfloor x \rfloor} \binom{n}{i} p_i (1-p)^{n-i}$

Operations Analysis

Device [i]: units that are in terms of *i* are specific to an individual device or node within a system
Total devices [k]:

Service Time [S]: time per specific job
1/μ

Visitation [V]: given or projected visits/jobs (closed system); cannot be calculated; basically a probability
[E(V)]: calculated visit/job ratio
P(visit)·total visits in previous node

Demand [D]: total service time for all jobs

$$D_i = E[S_i] \cdot V_i$$

$$D = \sum_{i=0}^{k} D_i$$

Bottleneck [D_{max}]: device with largest demand, utilization

Time in system [T]: time the job is in the system

$$E[T] = \frac{N}{X}$$

$$E[T] \geq \max(D, ND_{\max} - E[Z])$$

If E[Z] = 0, T = R

Response Time [R]: time the job is *being processed* in the system
If E[Z] = 0, R = T

M/M/1: E[R] = 1/(μ–λ)
M/M/1/N: E[R] = E[N]/λ'
M/M/C: E[R] = E[R_Q] + E[S]

Users [M]:
Optimal users [M*]:

$$M^* = \frac{D + E[Z]}{D_{\text{bottleneck}}}$$

Total Jobs [N]: N=M in a closed system

- *Little's Law*: $E[N] = \lambda E[T], \lambda = X$
- $E[N] = \lambda E[R], \lambda = X$
- M/M/1:
    - E[N] = λ/(μ–λ) = ρ/(1–ρ), if you have overall system λ
    - E[N] = $\sum_{i=0}^{\infty} i\pi_i$ ← probability × #jobs, if your λ or μ is different for each state
- M/M/1/N: E[N] is expected # jobs, N is max # jobs

$$E[N] = \sum_{i=0}^{N} i\pi_i = \pi_0 \frac{\lambda}{\mu}\left(\frac{N\left(\frac{\lambda}{\mu}\right)^{N-1} - (N+1)\left(\frac{\lambda}{\mu}\right)^{N} + 1}{1 - \left(\frac{\lambda}{\mu}\right)^{2}}\right)$$

- M/M/C: go through Little's law
    - E[N] = $E[N_Q]$ + ρ
- M/M/∞:
- Jackson Network: E[N] = Σ$E[N_i]$ = ΣPλ/(μ_i–Pλ) = Σ(λ_i/(μ_i–λ_i))

Think time [Z]: time it takes the user to put a request in and start, it's kinda like the frequency that users put in requests (seconds / request)

$$E[Z] = E[T] - E[R]$$

Throughput [X]: out-rate, max jobs / hour of full system

$$X \le \min\left(\frac{1}{D_{\max}}, \frac{N}{D + E[Z]}\right)$$

Note: $\frac{1}{D_{\max}}$ and $\frac{N}{D + E[Z]}$ converge at their lowest point, so equate them

$$X_i = E[V_i]X$$

Utilization [ρ]: ratio that the time is busy

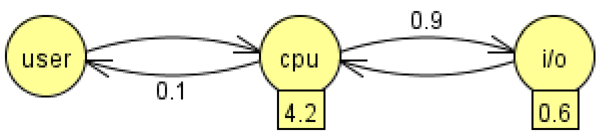
$$\rho_i = X_i E[S_i]$$

$$\rho_i = XD_i$$

ρ = λ/c_iμ

Visitation Trick

If determining visitation at a node, establish a reference node from one of the incoming nodes, usually the user node, that has a returning percentage



$V_{user} = 1 = 0.1 \cdot V_{CPU}$

Summation Equations

Geometric Series: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, where 0 ≤ r ≤ 1 (because otherwise it would be unstable)

$$\sum_{i=0}^{\infty} r^{i+1} = \sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$$

$$\sum_{i=1}^{\infty} r^i = \sum_{i=0}^{\infty} r^i - 1$$

Geometric Sequence: $S_n = \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

$$S_n = \sum_{i=1}^{n} r^i = \frac{r\left(1-r^n\right)}{1-r}$$

Removing the annoying factors:

$$\sum_{i=0}^{\infty} i\left(i+1\right)\rho^i$$

Take out a value so the integral takes out the i and i+1

$$= \rho \sum_{i=0}^{\infty} i\left(i+1\right)\rho^{i-1}$$

$$= \rho \frac{\mathrm{d}\rho}{\mathrm{d}i}\left(\sum_{i=0}^{\infty}\left(i+1\right)\rho^i\right)$$

$$= \rho \frac{\mathrm{d}\rho^2}{\mathrm{d}^2 i}\left(\sum_{i=0}^{\infty}\rho^{i+1}\right)$$

$$= \rho \frac{\mathrm{d}\rho^2}{\mathrm{d}^2 i}\left(\sum_{i=1}^{\infty}\rho^{i}\right)$$

$$= \rho \frac{\mathrm{d}\rho^2}{\mathrm{d}^2 i}\left(\sum_{i=0}^{\infty}\rho^{i} - \rho^0\right)$$

$$= \rho \frac{\mathrm{d}\rho^2}{\mathrm{d}^2 i}\left(\frac{1}{1-\rho} - 1\right)$$

DTMC

Discrete Time Markov Chains (DTMC):

[n]: number of tasks in queue / system

Steady state: n->∞
For discrete: use the sum of the X's, so $E[X] = \Sigma(P(X=i)\cdot X_i)$ and $E[X^2] = \Sigma(P(X=i)\cdot X_i^2)$

Balance Equations

$$\pi_n = \frac{\prod_{i=0}^{n} \lambda_i}{\prod_{i=0}^{n} \mu_i} \pi_0$$

^think of it like series / parallel, where you add multiple connections out in different directions (parallel) and multiply connections stacked onto each other (series)

OR $jobs_{in} = jobs_{out}$

Matrices

Rows: equations for nodes going out (add up to 1)
Columns: equations for nodes coming in

CTMC

Poisson Process

Counting Process: a way of determining the time between consecutive occurrences of an event
Poisson Process: a *counting process*, whose time between arrivals uses Exponential Distribution

- $\lambda_{total} = \Sigma\lambda_i$
  - you can also split up λ into multiple λs
- Not only do you see each second as time independent, each stream of probabilities is independent
- $P(x;\lambda) = e^{-\lambda}\lambda^x/x!$
  - [x]: things will happen
  - [λ]: rate; $\lambda = \alpha t$
- [α]: expected number of events during unit interval
- [t]: time interval length
- $P_X(t) = \frac{e^{-\alpha t} \cdot (\alpha t)^X}{X!}$

Kendall notation

Job Processing time [μ]: rate of jobs leaving system (jobs/sec)
μ = 1/ processing_time_per_job

M/M/1 Queue
[M]: time between arrivals is Markovian (Memoryless) ~ exp(λ)
[M]: job processing times are Markovian (Memoryless) ~ exp(μ)
[1]: single server

$(\Sigma p_{out})\times\pi_i = \Sigma p_j\pi_j$, j=0..n, j≠i

π_0: percent of time that the queue is empty

Attributes:

- FIFO
- Infinite buffer

Variations

- M/M/2 Queue: same, except 2 servers
- M/M/C Queue: *C* servers
- M/E_k/C: Erlang k, i.e. series of exponential
- H()/M/C: hyperexpontial distribution
- PH/M/C: phase type, i.e. any combination of any number of exponentials with any rate
- M/G/C: Memoryless, general distribution of service time
- G/G/1: has not been solved yet
- M/M/1/1: 1 server, 1 job

[c]: number of servers

Think: one queue goes to multiple servers

$$\xrightarrow{\lambda}] \begin{matrix} \nearrow M_1 \\ \searrow M_2 \end{matrix}$$

Steady State

M/M/1

$\pi_0 = 1 - \lambda/\mu$

$\pi_i = \rho^i(1 - \rho)$

$$\pi_{n_1..n_k} = \prod_{i=1}^{k} \rho_i^{n_i}\left(1-\rho_i\right)$$

M/M/1/N

When you can only have up to N jobs in system queue.

[λ']: rate jobs enter the system, until the queue is full

$\lambda' = \lambda(1 - \pi_N)$

$$\pi_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^N} = \frac{1}{1+\sum_{i=1}^{N}\left(\frac{\lambda}{\mu}\right)^i}$$

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i$$

Waiting: jobs put into the queue

Blocked: jobs not allowed in the queue

M/M/C

Useful if multiple jobs are sharing the same queue

Does the μ you use for equations double in M/M/2? No, but you'll see jobs coming out of a system at a rate of c·μ.

$$\pi_0 = \left[1 + \sum_{i=1}^{c-1} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left(\frac{1}{1-\rho}\right)\right]^{-1}$$

$$\pi_i = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0, & n < c \\ \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n \pi_0, & n \geq c \end{cases}$$

M/M/∞

Same as M/M/C, except:

$$\pi_0 = e^{-\frac{\lambda}{\mu}}$$

and just find the unit

M/G/1

General Distribution of service time

Queuing

Blocking Probability [P_Q]: probability that a process will be blocked when entering the system and be placed in the queue

Erlang-C Equation: $P_Q = \sum_{i=0}^{\infty} \pi_i = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left(\frac{1}{1-\rho}\right)\pi_0$

Given λ and μ, what should c be so $P_Q < \rho$

Waiting time in queue [R_Q]: response time of queue

$$E\left[R_Q\right] = \frac{1}{\lambda} P_Q \left(\frac{\rho}{1-\rho}\right)$$

M/M/1: $E\left[R_Q\right] = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$

M/M/C: $E\left[R_Q\right] = \left(\frac{(\lambda / \mu)^c \mu}{(c-1)!(c\mu - \lambda)^2}\right)\pi_0$

M/M/∞: $E[R_Q] = 0$

Number of jobs in queue [N_Q]:

M/M/1: $\rho^2/(1-\rho)$

M/M/C: $E\left[N_Q\right] = \pi_0 \frac{\lambda\mu\rho^{c+1}}{(c-1)!(c\mu - \lambda)^2}$

M/M/∞: $E[N_Q] = 0$

You need to know what is in the progression of each step *e.g.*

When you have varying

$$\pi_n = (n+1)\left(\frac{\lambda}{\mu}\right)^n \pi_0$$

$$1 = \sum_{i=0}^{\infty}(i+1)\rho^i \pi_0$$

$$= \pi_0 \sum_{i=1}^{\infty}(i+1)\rho^i$$

$$= \pi_0 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\sum_{i=0}^{\infty}\rho^{i+1}\right)$$

$$= \pi_0 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\sum_{i=1}^{\infty}\rho^{i}\right)$$

$$= \pi_0 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\sum_{i=0}^{\infty}\rho^{i+1}\right)$$

Square Root Staffing Rule

Given an M/M/c queue with arrival rate, λ, server speed, μ, and ρ is *large* (assume this means over 100, but we don't actually know what it means), α is a bound on P_Q, let c_α^* denote the least # of servers needed to ensure that $P_Q < \alpha$. Then

$c_\alpha^* \approx \rho + k\sqrt{\rho}$, where k = is the solution to

$\frac{k\Phi(k)}{\phi(k)} = \frac{1-\alpha}{\alpha}$, where Φ(·) is the CDF of the standard normal and ϕ() is its pdf

[K]: minimum # servers to stay stable λ/μ or ρ
[k]: a constant…just assume 1 for now

Essentially, the perfect number of servers is ρ + √ρ

e.g.)

α	k	ρ + k√ρ
0.8	0.178	10, 018
0.5	0.506	10, 051
0.2	1.06	10, 106
0.1	1.42	10, 142

[Q]: transition matrix

$$q_{ii} = -\sum_{j=i} q_{ij}$$

$$q_{ij} = \lim_{\Delta t \to 0} \frac{P\{X_{t+\Delta t} = j \mid X_t = i\}}{\Delta t}$$

Replace i <--> j to get q_{ij} and q_{ji}.

Jackson Networks

Open Loop

$P(N_1 = n_1) =$

$$\pi_{\vec{n}} = P_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\cdots\rho_k^{n_k}(1-\rho_k)$$

$\pi_{n\sim}$ = P(state of system n~) $= \prod_{i=1}^{k}$ P(n jobs at node i)

$$= \prod_{i=1}^{k} \rho_i^{n_i}(1-\rho_i)$$

Poisson Arrivals See Time Averages property (PASTA): the probability of a state (i.e. π_i) as seen by an outside random observer is the same as the probability of the state seen by an arriving customer. It is the open loop counterpart to arrival theorem

$\lambda_{total} = \Sigma\lambda_{in,i}$

Traffic Equations

For each node, what is the number of jobs entering?

$\lambda_x = R + \Sigma P_{i,entering} \cdot \lambda_{i,entering}$

response rate + probability of each job entering

Closed Loop

Since your values will become linearly independent, you cannot simply use your regular traffic equations. You need to estimate a fake value for one of your λ's and evaluate your probabilities using them.

Mean Value Analysis

Finds E[R] of each node of a **closed Jackson network**.

1. Base case: $L_k(0) = 0$
2. For k = 1, ..., K, compute $E[R_Q] = \dfrac{L_k(m-1)+1}{\mu_k}$
3. Little's Law: $\lambda_m = \dfrac{m}{\sum_{k=1}^{K} W_k(m)v_k}$
4. Plug it in: $L_k(m) = v_k\lambda_m W_k(m)$

$$\lambda_{m-1} = \frac{M-1}{\sum_{i=1}^{k} p_i E\left[R_i^{(M-1)}\right]}$$

$$E\left[R_i^{(M)}\right] = \frac{1}{\mu_i} + \frac{p_i \lambda^{(M-1)} E\left[R_i^{(M-1)}\right]}{\mu_i}$$

- Performs better than balance equations or Jackson Network, but can't find steady state distribution or PDF
- Recursive algorithm
- Only finds E[N], i.e. mean queue length

The higher your variance, the worse your system will perform.

Arrival Theorem: when a job arrives at a node within a closed Jackson network, there will be a number of jobs at the node, M – 1, where M is the expected number of jobs in the given node.

Pareto distribution: an exponential which doesn't start at 0 (a.k.a. **zipfian**)

Just think: 99% controls 50% and 1% controls the rest

Inspection Paradox:

General Distribution

Baskett, Chandy, Muntz and Palacios (BCMP) theorem: named after the authors of the paper

First Come, First Serve

First Come First Serve (FCFS):

Last Come, First Serve

Last Come First Serve (LCFS):

Processor Sharing

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