

SFWR ENG 4E03

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Note: material covered in [Stats 3Y03 Summary](#) will not be covered in this summary

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Statistics

Poisson parameter $[\lambda]$: rate

Service rate $[\mu]$:

Continuous Random Variable (CRV):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Think chemistry, i.e. cancelling units

Variance

$$\text{var}(X) = E[X^2] - (E[X])^2$$

- Don't change probability, but square X for calculation only

$$\begin{aligned}\text{var}(x) &= E[(X - \mu)^2] \\ &= \sigma^2 \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ &= \int x^2 f(x) dx - \mu^2\end{aligned}$$

Exponential

- **Mean** $[E[X]]$: $1/\lambda$
 - a.k.a. Expected value
- **Variance**: $1/\lambda^2$
- **Probability Distribution Function (PDF)** $[P(X=x)]$: $\lambda e^{-\lambda x}/x!$
- **Cumulative Distribution Function (CDF)** $[f(x)]$: $\text{CDF} = \int \text{PDF}$, i.e. $1 - e^{-\lambda x}$
- Memoryless
- not always for time

Uniform

- **Variance**: $(b-a)^2/12$
- **Mean**: $(a+b)/2$
- **PDF**: $1/(b-a)$, $a \leq x \leq b$
- **CDF**: 1
- **Uniform Distribution**: no memoryless property

Binomial

- **Mean** $[E[X]]$: $n \times \text{probability}$
- **Variance**: $n \times p \times (1-p)$
- **Probability Distribution Function (PDF)** $[P(X)]$: $(n \text{ c } x)p^x(1-p)^{n-x}$
- **Cumulative Distribution Function (CDF)** $[f(x)]$: $\sum_{i=0}^{\lfloor x \rfloor} \binom{n}{i} p_i (1-p)^{n-i}$

Operations Analysis

Device $[i]$: units that are in terms of i are specific to an individual device or node within a system

Total devices $[k]$:

Service Time $[S]$: time per specific job

$1/\mu$

Visitation $[V]$: given or projected visits/jobs (closed system); cannot be calculated; basically a probability

$[E(V)]$: calculated visit/job ratio

$P(\text{visit}) \cdot \text{total visits in previous node}$

Demand [D]: total service time for all jobs

$$D_i = E[S_i] \cdot V_i$$

$$D = \sum_{i=0}^k D_i$$

Bottleneck [D_{\max}]: device with largest demand

Time in system [T]: time the job is in the system

$$E[T] = \frac{N}{X}$$

$$E[T] \geq \max(D, ND_{\max} - E[Z])$$

If $E[Z] = 0$, $T = R$

Response Time [R]: time the job is *being processed* in the system

If $E[Z] = 0$, $R = T$

$$E[R] = E[R_Q] + E[S]$$

Users [M]:

Optimal users [M^*]:

$$M^* = \frac{D + E[Z]}{D_{\text{bottleneck}}}$$

Total Jobs [N]: $N=M$ in a closed system

- Little's Law: $E[N] = \lambda E[T]$, $\lambda = X$
- $E[N] = \lambda E[R]$, $\lambda = X$
- Steady state probability
 - M/M/1:
 - $E[N] = \lambda / (\mu - \lambda) = \rho / (1 - \rho)$, if you have overall system λ
 - $= \sum_{i=0}^{\infty} i \pi_i \leftarrow \text{probability} \times \# \text{jobs}$, if your λ or μ is different for each state
 - M/M/C: $E[N] = \Sigma E[N_i] = \Sigma P \lambda / (\mu_i - P \lambda) = \Sigma (\lambda_i / (\mu_i - \lambda_i))$

Think time [Z]: time it takes the user to put a request in and start, it's kinda like the frequency that users put in requests (seconds / request)

$$E[Z] = E[T] - E[R]$$

Throughput [X]: out-rate, max jobs / hour of full system

$$X \leq \min\left(\frac{1}{D_{\max}}, \frac{N}{D + E[Z]}\right)$$

Note: $\frac{1}{D_{\max}}$ and $\frac{N}{D + E[Z]}$ converge at their lowest point, so equate them

$$X_i = E[V_i]X$$

Utilization $[\rho]$: ratio that the time is busy

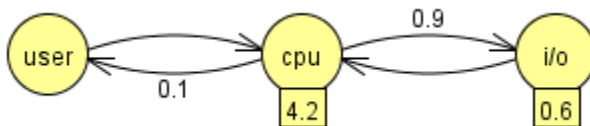
$$\rho_i = X_i E[S_i]$$

$$\rho_i = X D_i$$

$$\rho = \lambda / c_i \mu$$

Visitation Trick

If determining visitation at a node, establish a reference node from one of the incoming nodes, usually the user node, that has a returning percentage



$$V_{\text{user}} = 1 = 0.1 \cdot V_{\text{CPU}}$$

Summation Equations

Geometric Series: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, where $0 \leq r \leq 1$ (because otherwise it would be unstable)

$$\sum_{i=0}^{\infty} r^{i+1} = \sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$$

$$\sum_{i=1}^{\infty} r^i = \sum_{i=0}^{\infty} r^i - 1$$

Geometric Sequence: $S_n = \sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r}$

$$S_n = \sum_{i=1}^n r^i = \frac{r(1-r^n)}{1-r}$$

Removing the annoying factors:

$$\sum_{i=0}^{\infty} i(i+1)\rho^i$$

Take out a value so the integral takes out the i and $i+1$

$$\begin{aligned}
&= \rho \sum_{i=0}^{\infty} i(i+1) \rho^{i-1} \\
&= \rho \frac{d\rho}{di} \left(\sum_{i=0}^{\infty} (i+1) \rho^i \right) \\
&= \rho \frac{d\rho^2}{d^2i} \left(\sum_{i=0}^{\infty} \rho^{i+1} \right) \\
&= \rho \frac{d\rho^2}{d^2i} \left(\sum_{i=1}^{\infty} \rho^i \right) \\
&= \rho \frac{d\rho^2}{d^2i} \left(\sum_{i=0}^{\infty} \rho^i - \rho^0 \right) \\
&= \rho \frac{d\rho^2}{d^2i} \left(\frac{1}{1-\rho} - 1 \right)
\end{aligned}$$

DTMC

Discrete Time Markov Chains (DTMC):

[n]: number of tasks in queue / system

$$\pi_n = \frac{\prod_{i=0}^n \lambda_i}{\prod_{i=0}^n \mu_i} \pi_0$$

^think of it like series / parallel, where you add multiple connections out in different directions (parallel) and multiply connections stacked onto each other (series)

OR $\text{jobs}_{\text{in}} = \text{jobs}_{\text{out}}$

Steady state: $n \rightarrow \infty$

For discrete: use the sum of the X's, so $E[X] = \sum (P(X=i) \cdot X_i)$ and $E[X^2] = \sum (P(X=i) \cdot X_i^2)$

Matrices

Rows: equations for nodes going out (add up to 1)

Columns: equations for nodes coming in

CTMC

Poisson Process

Counting Process: a way of determining the time between consecutive occurrences of an event

Poisson Process: a *counting process*, whose time between arrivals uses Exponential Distribution

- $\lambda_{\text{total}} = \sum \lambda_i$
 - you can also split up λ into multiple λ s

- Not only do you see each second as time independent, each stream of probabilities is independent
- $P(x;\lambda) = e^{-\lambda}\lambda^x/x!$
 - $[x]$: things will happen
 - $[\lambda]$: rate; $\lambda = \alpha t$
- $[\alpha]$: expected number of events during unit interval
- $[t]$: time interval length
- $$P_x(t) = \frac{e^{-\alpha t} \cdot (\alpha t)^x}{X!}$$

Kendall notation

Job Processing time $[\mu]$: rate of jobs leaving system (jobs/sec)

$\mu = 1/\text{processing_time_per_job}$

M/M/1 Queue

$[M]$: time between arrivals is Markovian (Memoryless) $\sim \exp(\lambda)$

$[M]$: job processing times are Markovian (Memoryless) $\sim \exp(\mu)$

$[1]$: single server

$$(\sum p_{out}) \times \pi_i = \sum p_j \pi_j, j=0..n, j \neq i$$

π_0 : percent of time that the queue is empty

Attributes:

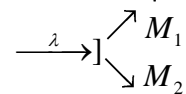
- FIFO
- Infinite buffer

Variations

- M/M/2 Queue: same, except 2 servers
- M/M/C Queue: C servers
- M/E_k/C: Erlang k, i.e. series of exponential
- H()/M/C: hyperexponential distribution
- PH/M/C: phase type, i.e. any combination of any number of exponentials with any rate
- M/G/C: General distribution
- G/G/1: has not been solved yet
- M/M/1/1: 1 server, 1 job

$[c]$: number of servers

Think: one queue goes to multiple servers



$$\pi_n = (n+1) \left(\frac{\lambda}{\mu} \right)^n \pi_0$$

$$\begin{aligned}
1 &= \sum_{i=0}^{\infty} (i+1) \rho^i \pi_0 \\
&= \pi_0 \sum_{i=1}^{\infty} (i+1) \rho^i \\
&= \pi_0 \frac{d}{d\rho} \left(\sum_{i=0}^{\infty} \rho^{i+1} \right) \\
&= \pi_0 \frac{d}{d\rho} \left(\sum_{i=1}^{\infty} \rho^i \right) \\
&= \pi_0 \frac{d}{d\rho} \left(\sum_{i=0}^{\infty} \rho^{i+1} \right)
\end{aligned}$$

M/M/C Provisioning

Useful if multiple jobs are sharing the same queue

Blocking Probability $[P_Q]$: probability that a process will be blocked when entering the system and be placed in the queue

Erlang-C Equation:
$$P_Q = \sum_{i=0}^{\infty} \pi_i = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left(\frac{1}{1-\rho} \right) \pi_0$$

Given λ and μ , what should c be so $P_Q < p$

Waiting time in queue $[R_Q]$: response time of queue

$$E[R_Q] = \frac{1}{\lambda} P_Q \left(\frac{\rho}{1-\rho} \right)$$

$$\text{M/M/1: } E[R_Q] = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$\text{M/M/C: } E[R_Q] = \left(\frac{(\lambda / \mu)^c \mu}{(c-1)!(c\mu - \lambda)^2} \right) \pi_0$$

$[Q]$: transition matrix

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

$$q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{P\{X_{t+\Delta t} = j \mid X_t = i\} - \delta_{ij}}{\Delta t}$$

Replace $i \leftrightarrow j$ to get q_{ji} and q_{ji} .

Traffic Equations

For each node, what is the number of jobs entering?

$$\lambda_x = R + \sum P_{i, \text{entering}} \cdot \lambda_{i, \text{entering}}$$

response rate + probability of each job entering

Jackson Networks

hi

Questions

- Assignment 5, Q2 states in ready queue??
- Assignment 6, Q3, M/M/C