

SFWR ENG 4E03

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Note: material covered in [Stats 3Y03 Summary](#) will not be covered in this summary. To find a unit CTRL-F “[<unit>]”, e.g. for Number of jobs in system, CTRL-F “[N]”

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Statistics

Poisson parameter $[\lambda]$: rate

Service rate $[\mu]$:

Continuous Random Variable (CRV):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Think chemistry, i.e. cancelling units

Probability Density Function (PDF) $[f(x)]$:

Cumulative Density Function (CDF) $[F(x)]$:

Second Moment $E[x^2]$

Variance

$$\text{var}(X) = E[X^2] - (E[X])^2$$

- Don't change probability, but square X for calculation only

$$\begin{aligned} \text{var}(x) &= E[(X - \mu)^2] \\ &= \sigma^2 \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ &= \int x^2 f(x) dx - \mu^2 \end{aligned}$$

The higher your variance, the worse your system will perform.

Exponential

- **Mean** $[E[X]]$: $1/\lambda$
 - a.k.a. Expected value
- **Variance**: $1/\lambda^2$

- **Probability Distribution Function (PDF)** $[P(X=x)]: \lambda e^{-\lambda x} / x!$
- **Cumulative Distribution Function (CDF)** $[f(x)]: \text{CDF} = \int \text{PDF}, \text{ i.e. } 1 - e^{-\lambda x}$
- Memoryless
- not always for time

Uniform

- **Variance:** $(b-a)^2/12$
- **Mean:** $(a+b)/2$
- **PDF:** $1 / (b-a), a \leq x \leq b$
- **CDF:** $x-a/b-a$
- **Uniform Distribution:** no memoryless property

Binomial

- **Mean** $[E[X]]: n \times \text{probability}$
- **Variance:** $n \times p \times (1 - p)$
- **Probability Distribution Function (PDF)** $[P(X)]: (n \text{ c } x)p^x(1-p)^{n-x}$
- **Cumulative Distribution Function (CDF)** $[f(x)]: \sum_{i=0}^{\lfloor x \rfloor} \binom{n}{i} p_i (1-p)^{n-i}$

Operations Analysis

Device $[i]$: units that are in terms of i are specific to an individual device or node within a system

Total devices $[k]$:

Service Time $[S]$: time per specific job

$$1/\mu$$

Visitation $[V]$: given or projected visits/jobs (closed system); cannot be calculated; basically a probability

$[E(V)]$: calculated visit/job ratio

$P(\text{visit}) \cdot \text{total visits in previous node}$

Demand $[D]$: total service time for all jobs

$$D_i = E[S_i] \cdot V_i$$

$$D = \sum_{i=0}^k D_i$$

Bottleneck $[D_{\max}]$: device with largest demand, utilization

Time in system $[T]$: time the job is in the system

$$E[T] = \frac{N}{X}$$

$$E[T] \geq \max(D, ND_{\max} - E[Z])$$

If $E[Z] = 0, T = R$

Response Time $[R]$: time the job is *being processed* in the system

If $E[Z] = 0, R = T$

M/M/1: $E[R] = 1/(\mu - \lambda)$

M/M/1/N: $E[R] = E[N]/\lambda$

M/M/C: $E[R] = E[R_Q] + E[S]$

Users [M]:

Optimal users [M*]:

$$M^* = \frac{D + E[Z]}{D_{\text{bottleneck}}}$$

Total Jobs [N]: $N=M$ in a closed system

- *Little's Law:* $E[N] = \lambda E[T], \lambda = X$
- $E[N] = \lambda E[R], \lambda = X$
- M/M/1:
 - $E[N] = \lambda/(\mu - \lambda) = \rho/(1 - \rho)$, if you have overall system λ
 - $E[N] = \sum_{i=0}^{\infty} i \pi_i \leftarrow$ probability \times #jobs, if your λ or μ is different for each state
- M/M/1/N: $E[N]$ is expected # jobs, N is max # jobs
$$E[N] = \sum_{i=0}^N i \pi_i = \pi_0 \frac{\lambda}{\mu} \left(\frac{N \left(\frac{\lambda}{\mu} \right)^{N-1} - (N+1) \left(\frac{\lambda}{\mu} \right)^N + 1}{1 - \left(\frac{\lambda}{\mu} \right)^2} \right)$$
- M/M/C: go through Little's law
 - $E[N] = E[N_Q] + \rho$
- M/M/ ∞ :
- Jackson Network: $E[N] = \sum E[N_i] = \sum \rho \lambda / (\mu_i - \rho \lambda) = \sum (\lambda_i / (\mu_i - \lambda_i))$

Think time [Z]: time it takes the user to put a request in and start, it's kinda like the frequency that users put in requests (seconds / request)

$$E[Z] = E[T] - E[R]$$

Throughput [X]: out-rate, max jobs / hour of full system

$$X \leq \min \left(\frac{1}{D_{\text{max}}}, \frac{N}{D + E[Z]} \right)$$

Note: $\frac{1}{D_{\text{max}}}$ and $\frac{N}{D + E[Z]}$ converge at their lowest point, so equate them

$$X_i = E[V_i] X$$

Utilization [ρ]: ratio that the time is busy

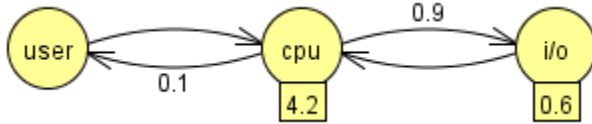
$$\rho_i = X_i E[S_i]$$

$$\rho_i = X D_i$$

$$\rho = \lambda / c_i \mu$$

Visitation Trick

If determining visitation at a node, establish a reference node from one of the incoming nodes, usually the user node, that has a returning percentage



$$V_{\text{user}} = 1 = 0.1 \cdot V_{\text{CPU}}$$

Summation Equations

Geometric Series: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, where $0 \leq r \leq 1$ (because otherwise it would be unstable)

$$\sum_{i=0}^{\infty} r^{i+1} = \sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$$

$$\sum_{i=1}^{\infty} r^i = \sum_{i=0}^{\infty} r^i - 1$$

Geometric Sequence: $S_n = \sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r}$

$$S_n = \sum_{i=1}^n r^i = \frac{r(1-r^n)}{1-r}$$

Removing the annoying factors:

$$\sum_{i=0}^{\infty} i(i+1)\rho^i$$

Take out a value so the integral takes out the i and i+1

$$= \rho \sum_{i=0}^{\infty} i(i+1)\rho^{i-1}$$

$$= \rho \frac{d\rho}{di} \left(\sum_{i=0}^{\infty} (i+1)\rho^i \right)$$

$$= \rho \frac{d\rho^2}{d^2i} \left(\sum_{i=0}^{\infty} \rho^{i+1} \right)$$

$$= \rho \frac{d\rho^2}{d^2i} \left(\sum_{i=1}^{\infty} \rho^i \right)$$

$$= \rho \frac{d\rho^2}{d^2i} \left(\sum_{i=0}^{\infty} \rho^i - \rho^0 \right)$$

$$= \rho \frac{d\rho^2}{d^2i} \left(\frac{1}{1-\rho} - 1 \right)$$

DTMC

Discrete Time Markov Chains (DTMC):

[n]: number of tasks in queue / system

Steady state: $n \rightarrow \infty$

For discrete: use the sum of the X 's, so $E[X] = \sum (P(X=i) \cdot X_i)$ and $E[X^2] = \sum (P(X=i) \cdot X_i^2)$

Balance Equations

$$\pi_n = \frac{\prod_{i=0}^n \lambda_i}{\prod_{i=0}^n \mu_i} \pi_0$$

^think of it like series / parallel, where you add multiple connections out in different directions (parallel) and multiply connections stacked onto each other (series)

OR $\text{jobs}_{\text{in}} = \text{jobs}_{\text{out}}$

Matrices

Rows: equations for nodes going out (add up to 1)

Columns: equations for nodes coming in

CTMC

Poisson Process

Counting Process: a way of determining the time between consecutive occurrences of an event

Poisson Process: a *counting process*, whose time between arrivals uses Exponential Distribution

- $\lambda_{\text{total}} = \sum \lambda_i$
 - you can also split up λ into multiple λ s
- Not only do you see each second as time independent, each stream of probabilities is independent
- $P(x; \lambda) = e^{-\lambda} \lambda^x / x!$
 - [x]: things will happen
 - [λ]: rate; $\lambda = \alpha t$
- [α]: expected number of events during unit interval
- [t]: time interval length
- $P_x(t) = \frac{e^{-\alpha t} \cdot (\alpha t)^x}{X!}$

Kendall notation

Job Processing time [μ]: rate of jobs leaving system (jobs/sec)

$\mu = 1 / \text{processing_time_per_job}$

M/M/1 Queue

[M]: time between arrivals is Markovian (Memoryless) $\sim \exp(\lambda)$

[M]: job processing times are Markovian (Memoryless) $\sim \exp(\mu)$

[1]: single server

$$(\sum p_{out}) \times \pi_i = \sum p_j \pi_j, j=0..n, j \neq i$$

π_0 : percent of time that the queue is empty

Attributes:

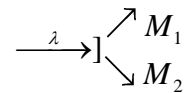
- FIFO
- Infinite buffer

Variations

- M/M/2 Queue: same, except 2 servers
- M/M/C Queue: C servers
- M/E_k/C: Erlang k, i.e. series of exponential
- H()/M/C: hyperexponential distribution
- PH/M/C: phase type, i.e. any combination of any number of exponentials with any rate
- M/G/C: Memoryless, general distribution of service time
- G/G/1: has not been solved yet
- M/M/1/1: 1 server, 1 job

[c]: number of servers

Think: one queue goes to multiple servers



Steady State

M/M/1

$$\pi_0 = 1 - \lambda/\mu$$

$$\pi_i = \rho^i (1 - \rho)$$

$$\pi_{n_1..n_k} = \prod_{i=1}^k \rho_i^{n_i} (1 - \rho_i)$$

M/M/1/N

When you can only have up to N jobs in system queue.

[λ']: rate jobs enter the system, until the queue is full

$$\lambda' = \lambda(1 - \pi_N)$$

$$\pi_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^N} = \frac{1}{1 + \sum_{i=1}^N \left(\frac{\lambda}{\mu}\right)^i}$$

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i$$

Waiting: jobs put into the queue

Blocked: jobs not allowed in the queue

M/M/C

Useful if multiple jobs are sharing the same queue

Does the μ you use for equations double in M/M/2? No, but you'll see jobs coming out of a system at a rate of $c \cdot \mu$.

$$\pi_0 = \left[1 + \sum_{i=1}^{c-1} \frac{1}{i!} \left(\frac{\lambda}{\mu} \right)^i + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left(\frac{1}{1-\rho} \right) \right]^{-1}$$
$$\pi_i = \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \pi_0, & n < c \\ \frac{1}{c! c^{n-c}} \left(\frac{\lambda}{\mu} \right)^n \pi_0, & n \geq c \end{cases}$$

M/M/ ∞

Same as M/M/C, except:

$$\pi_0 = e^{-\frac{\lambda}{\mu}}$$

and just find the unit

M/G/1

General Distribution of service time

Heaviside function: 1 if not zero

$[E[A]]$: arrivals

$E[A] = \rho$

Queuing

Blocking Probability $[P_Q]$: probability that a process will be blocked when entering the system and be placed in the queue

$$\text{Erlang-C Equation: } P_Q = \sum_{i=0}^{\infty} \pi_i = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left(\frac{1}{1-\rho} \right) \pi_0$$

Given λ and μ , what should c be so $P_Q < \rho$

Waiting time in queue $[R_Q]$: response time of queue

$$E[R_Q] = \frac{1}{\lambda} P_Q \left(\frac{\rho}{1-\rho} \right)$$

$$\text{M/M/1: } E[R_Q] = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$\text{M/M/C: } E[R_Q] = \left(\frac{(\lambda / \mu)^c \mu}{(c-1)!(c\mu - \lambda)^2} \right) \pi_0$$

$$\text{M/M}/\infty: E[R_Q] = 0$$

Number of jobs in queue [N_Q]:

M/M/1: $\rho^2/(1 - \rho)$

M/M/C: $E[N_Q] = \pi_0 \frac{\lambda \mu \rho^{c+1}}{(c-1)!(c\mu - \lambda)^2}$

M/M/∞: $E[N_Q] = 0$

You need to know what is in the progression of each step

e.g.

When you have varying

$$\pi_n = (n+1) \left(\frac{\lambda}{\mu} \right)^n \pi_0$$

$$1 = \sum_{i=0}^{\infty} (i+1) \rho^i \pi_0$$

$$= \pi_0 \sum_{i=1}^{\infty} (i+1) \rho^i$$

$$= \pi_0 \frac{d}{d\rho} \left(\sum_{i=0}^{\infty} \rho^{i+1} \right)$$

$$= \pi_0 \frac{d}{d\rho} \left(\sum_{i=1}^{\infty} \rho^i \right)$$

$$= \pi_0 \frac{d}{d\rho} \left(\sum_{i=0}^{\infty} \rho^{i+1} \right)$$

Square Root Staffing Rule

Given an M/M/c queue with arrival rate, λ , server speed, μ , and ρ is *large* (assume this means over 100, but we don't actually know what it means), α is a bound on P_Q , let c_α^* denote the least # of servers needed to ensure that $P_Q < \alpha$. Then

$c_\alpha^* \approx \rho + k\sqrt{\rho}$, where k is the solution to

$\frac{k\Phi(k)}{\phi(k)} = \frac{1-\alpha}{\alpha}$, where $\Phi(\cdot)$ is the CDF of the standard normal and $\phi(\cdot)$ is its pdf

[K]: minimum # servers to stay stable λ/μ or ρ

[k]: a constant...just assume 1 for now

Essentially, the perfect number of servers is $\rho + \sqrt{\rho}$

e.g.)

α	k	$\rho + k\sqrt{\rho}$
0.8	0.178	10, 018
0.5	0.506	10, 051
0.2	1.06	10, 106

0.1	1.42	10, 142
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[Q]: transition matrix

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

$$q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{P\{X_{t+\Delta t} = j \mid X_t = i\}}{\Delta t}$$

Replace $i \leftrightarrow j$ to get q_{ji} and q_{ji} .

Jackson Networks

Open Loop

$P(N_1 = n_1) =$

$$\pi_{\vec{n}} = \rho_1^{n_1} (1 - \rho_1) \rho_2^{n_2} (1 - \rho_2) \cdots \rho_k^{n_k} (1 - \rho_k)$$

$$\pi_{\vec{n}} = P(\text{state of system } \vec{n}) = \prod_{i=1}^k P(n \text{ jobs at node } i)$$

$$= \prod_{i=1}^k \rho_i^{n_i} (1 - \rho_i)$$

Poisson Arrivals See Time Averages property (PASTA): the probability of a state (i.e. π_i) as seen by an outside random observer is the same as the probability of the state seen by an arriving customer. It is the open loop counterpart to arrival theorem

$$\lambda_{\text{total}} = \sum \lambda_{in,i}$$

Traffic Equations

For each node, what is the number of jobs entering?

$$\lambda_x = R + \sum P_{i, \text{entering}} \cdot \lambda_{i, \text{entering}}$$

response rate + probability of each job entering

Closed Loop

Since your values will become linearly independent, you cannot simply use your regular traffic equations. You need to estimate a fake value for one of your λ 's and evaluate your probabilities using them.

Mean Value Analysis

(MVA): Finds $E[R]$ of each node of a **closed Jackson network**.

I think it is n^2 , whereas other methods are n^n

Visit Ratio [v]: based on a reference node, usually set $v_{\text{ref}} = c$

$$p_i = \frac{v_i}{\sum_{j=1}^k v_j}, \text{ e.g. } p_i = \frac{c}{c + 0.3c + 0.7c}$$

1.

2. Base case: $\lambda^{(1)} = 1/\mu$

3. For $k = 1, \dots, K$, compute: $E[R_i^{(M)}] = \frac{1}{\mu_i} + \frac{p_i \lambda^{(M-1)} E[R_i^{(M-1)}]}{\mu_i}$

4. Little's Law: $\lambda^{(M)} = \frac{M}{\sum_{i=1}^k p_i E[R_i^{(M)}]}$

5. Plug it in: $E[N_i] = \lambda_i E[R_i]$

- Performs better than balance equations or Jackson Network, but can't find steady state distribution or PDF
- Recursive algorithm
- Only finds $E[N]$, i.e. mean queue length

The higher your variance, the worse your system will perform.

Arrival Theorem: when a job arrives at a node within a closed Jackson network, there will be a number of jobs at the node, $M - 1$, where M is the expected number of jobs in the given node.

Pareto Power $[\alpha]$: $0 < \alpha < 2$

Pareto distribution: an exponential which doesn't start at 0 (a.k.a. **zipfian**)

$$\text{CDF: } F(x) = 1 - \left(\frac{K_{\min}}{x} \right)^\alpha$$

$$\text{PDF: } f(x) = \frac{\alpha K_{\min}^\alpha}{x^{\alpha+1}}, x > K_{\min}$$

$$\text{Var} = \begin{cases} \frac{K_{\min}^2 \alpha}{(\alpha-1)^2 (\alpha-2)}, & \alpha > 2 \\ \infty, & \alpha \leq 2 \end{cases}$$

Just think: 99% controls 50% and 1% controls the rest
integral of the density function between k and p come out to 1

Inspection Paradox:

General Distribution

Baskett, Chandy, Muntz and Palacios (BCMP) theorem: named after the authors of the paper

[First Come, First Serve](#)

First Come First Serve (FCFS):

[Last Come, First Serve](#)

Last Come First Serve (LCFS):

Processor Sharing

Processor Sharing (PS): fair, overhead, switching costs, thrashing

Failure

Failure / Hazard Rate [h]:

$$h(t) = \frac{f(t)}{1 - F(t)}$$

Increasing Failure Rate (IFR):

Decreasing Failure Rate (DFR):

Remaining Processing Time: