Project C-B Project

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Claim (A version of Scheffé's theorem).

Let $(X, \mathcal{M}, \mu)$ be a measure space and (f_n) a sequence of (a) integrable functions (b) uniformly bounded from below (i.e. there is some M such that $M < f_n$ for all n). Suppose that (c) $f_n \to f$ a.e. and (d) $\int f_n \, d\mu \to \int f \, d\mu$ where (e) f is integrable. Then $f_n \to f$ in L^1_μ, i.e.

$$\int |f_n - f| \, d\mu \to 0.$$

Possible counterexample?

Consider $f = 0$ and $f_n = \mathbb{1}_{(-n,-n-1)} - \mathbb{1}_{(n,n+1)}$ on $\mathbb{R}$ with Lebesgue measure.

Then:

- $f_n \to f$ a.e., since $f_n(x) = 0$ for all $n > |x|$, (c)
- $\int f_n \, dx = 1 - 1 = 0 = \int f \, dx$ for all n, (a, d, e)
- $f_n \geq -1$ for all n. (b)

However, $|f_n - f| = \mathbb{1}_{(-n,-n-1)} + \mathbb{1}_{(n,n+1)}$, so that

$$\int |f_n - f| \, dx = 2$$

for all n.

Remedy? I think that adding the assumption $\int |f_n| \, d\mu \to \int |f| \, d\mu$ would be enough, and I was wondering what you think about that assumption. Is it too much?