CMPSCI 240: Reasoning about Uncertainty

Lecture 7: More Advanced Counting

Andrew McGregor

University of Massachusetts

Last Compiled: February 21, 2017

Outline

Shortcuts for Counting

- *Permutations:* There are $n! = n \times (n-1) \times \ldots \times 2 \times 1$ ways to permute n objects. E.g., permutations of $\{a, b, c\}$ are

$$abc, acb, bac, bca, cab, cba$$

- *k-Permutations:* There are $n \times (n-1) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$ ways to choose the first k elements of a permutation of n objects. E.g., 2-permutations of $\{a, b, c, d\}$ are

$$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc$$

- *Combinations:* There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to choose a subset of size k from a set of n objects. E.g., the subsets of $\{a, b, c, d\}$ of size 2 are

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

Counting Combinations

- Let S be a set of n objects. How many subsets of size k are there?
- Every subset corresponds to $k!$ different k-permutations, so the number of "k-combinations" is

$$\frac{\text{the number of } k\text{-permutations}}{k!} = \frac{n!/(n-k)!}{k!} = \frac{n!}{(n-k)!k!}$$

which is denoted $\binom{n}{k}$, pronounced "n choose k".

Counting Sequences

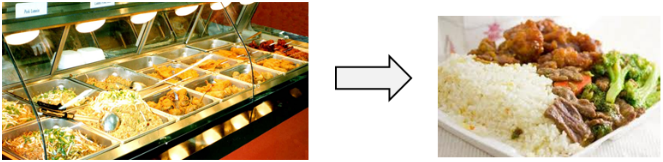
Example: Let $S = \{a, b, c, d\}$ be a set of size $n = 4$ and consider subsets of size $k = 2$

Subsets of size 2	2-permutations
{a,b}	ab, ba
{a,c}	ac, ca
{a,d}	ad, da
{b,c}	bc, cb
{b,d}	bd, db
{c,d}	cd, dc

We know there are $n!/(n-k)! = 12$ different 2-permutations and these can be arranged into groups of size $k! = 2$ such that each 2-permutation in the same group corresponds to the same subset of size 2. Hence, the total number of groups is $n!/(n-k)! \times \frac{1}{k!} = \binom{4}{2} = 6$.

Example: Flaming Wok

- **Question:** The Flaming Wok restaurant sells a 3-item lunch combo. You can choose from 10 different items. How many different lunch combos are there?



- **Answer:** Since your lunch is the same regardless of the order the items are put on the plate, this is a combination problem. The number of lunch combos is thus $\binom{10}{3}$ or 120.

Example: Flaming Wok

- **Question:** Suppose you ask for a random combo and there's one item you don't like. What's the combo you order has items you like?
- **Answer:** The probability that you will like a random combo is the number of combos you like, which is $\binom{9}{3} = 84$, divided by the number of combos, which is $\binom{10}{3}$. This gives $84/120 = 0.7$.

Outline

Counting Partitions

- Consider an experiment where we divide r objects into l groups with sizes $k_1, k_2, ..., k_l$ such that $r = \sum_{i=1}^{l} k_i$. The order of items within each group doesn't matter.
- A combination divides items into one group of k and one group of $r - k$ and is thus a 2-partition.
- How many partitions are there?
- There are $\binom{r}{k_1}$ ways to choose the objects for the first group. This leaves $r - k_1$ objects. There are $\binom{r-k_1}{k_2}$ ways to choose objects for the second group. There are $\binom{r-k_1-k_2-\ldots-k_{l-1}}{k_l}$ ways to choose the objects for the last group.

Counting Partitions

- Using the counting principle, the number of partitions is thus:

$$\binom{r}{k_1} \cdot \binom{r-k_1}{k_2} \cdots \binom{r-k_1-k_2-\ldots-k_{l-1}}{k_l}$$

$$= \frac{r!}{k_1!(r-k_1)!} \cdot \frac{(r-k_1)!}{k_2!(r-k_1-k_2)!} \cdots \frac{(r-k_1-k_2-\ldots-k_{l-1})!}{k_l!(r-k_1-k_2-\ldots-k_l)!}$$

- Canceling terms yields the final result:

$$\frac{r!}{k_1!\cdots k_l!}$$

Example: Discussion Groups

- **Question:** How many ways are there to split a discussion section of 12 students into 3 groups of 4 students each?
- **Answer:** This is a partition problem with 3 partitions of 4 objects each and 12 objects total. Using the partition counting formula, the answer is:

$$\frac{12!}{4! \cdot 4! \cdot 4!} = \frac{12!}{(4!)^3} = 34,650$$

Outline

Summary of Counting Problems

Structure	Description	Order Matters	Formula
Permutation	Number of ways to order n objects	Yes	$n!$
k-Permutation	Number of ways to form a sequence of size k using k different objects from a set of n objects	Yes	$\frac{n!}{(n-k)!}$
Combination	Number of ways to form a set of size k using k different objects from a set of n objects	No	$\frac{n!}{k!(n-k)!}$
Partition	Number of ways to partition n objects into l groups of size $k_1, \ldots, k_l$	No	$\frac{n!}{k_1!\cdots k_l!}$

Example: Grade Assignments

- **Question:** Suppose a professor decides at the beginning of the semester that in a class of 10 students, 3 A's, 4 B's, 2 C's and one C- will be given. How many different ways can the students be assigned grades at the end of the semester?
- **Answer:** This is a partition problem. There are 10 objects and 4 groups. The group sizes are 3,4,2,1. The answer is thus:

$$\frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!}$$

Example: Top of the class

- **Question:** A computer science program is considering offering three senior year scholarships to their top three incoming seniors worth $\$10,000$, $\$5,000$ and $\$2,000$. If there are 100 incoming seniors, how many ways are there for the scholarships to be awarded?
- **Answer:** This is a k-permutation problem. There are 100 students and 3 distinct scholarships. The number of assignments of students to scholarships is thus:

$$\frac{100!}{(100-3)!} = 100 \cdot 99 \cdot 98$$

Example: Binary Strings

- **Question:** How many length five binary strings are there with exactly two ones? E.g., 00011, 00101, 00110
- **Answer:** This is a combination problem is disguise! Consider numbering the positions of a binary string $\{1, 2, 3, 4, 5\}$. If you pick a subset of these of size two and set the corresponding bits to one and the other bits to zero, you get a binary string with exactly two ones. Then the number of strings is

$$\binom{5}{2} = \frac{5!}{3! \times 2!} = 10$$

Example: Overbooked

- **Question:** Suppose a class with 50 students is scheduled in a room with only 40 seats. How many ways are there for 40 of the 50 students to get a seat.
- **Answer:** This is a combination problem. We only care if a student gets a seat, not which seat they get. The answer is thus:

$$\binom{50}{40} = \frac{50!}{40!10!}$$

Outline

PIN Codes 1

- Suppose you bank insists you use a four digit PIN code. How many possible codes are there:

$$A : 10!/4! \qquad B : 4^{10} \qquad C : \binom{10}{4} \qquad D : 10^4 \qquad E : \binom{4}{10}$$

- Correct answer is D: Consider a 4 stage counting process with 10 choices at each stage.

PIN Codes 2

- Suppose you bank insists you use a four digit PIN code where every number is odd. How many possible codes are there:

$$A : 1000 \qquad B : 5^{10} \qquad C : 5^4 \qquad D : \binom{5}{4} \qquad E : 10!/5!$$

- Correct answer is C: Consider a 4 stage counting process with 5 choices at each stage.

PIN Codes 3

- Suppose you bank insists you use a four digit PIN code where every number is different from the one before. How many possible codes are there:

  $A : 10 \times 9^3 \qquad B : 10^9 \qquad C : 9^4 \qquad D : \binom{10}{4} \times 9 \times 9 \times 9 \qquad E : 10!/9!$

- Correct answer is A: Consider a 4 stage counting process with 10 choices at the first stage and 9 choices are each subsequent stage.

PIN Codes 4

- Suppose you bank insists you use a four digit PIN code where all the numbers are distinct. How many possible codes are there:

  $A : 10 \times 9^3 \qquad B : 9^4 \qquad C : \binom{10}{4} \qquad D : 1 \qquad E : 10 \times 9 \times 8 \times 7$

- Correct answer is E: This is the number of 4-permutations of $\{0, 1, \ldots, 9\}$.

PIN Codes 5

- Suppose you bank insists you use a four digit PIN code that are palindromes, i.e., are the same read forward and backward, e.g., 1221. How many possible codes are there:

$$A : 45 \qquad B : 89 \qquad C : 90 \qquad D : 100 \qquad E : 120$$

- Correct answer is D: Consider the 2-stage counting process with 10 choices at both stages that specifies the first two values.

PIN Codes 6

- Suppose you bank insists you use a four digit PIN code where every digit is strictly bigger than the previous digit. How many possible codes are there:

  $A : \binom{10}{4}$ $\quad B : 10 \times 9 \times 8 \times 7$ $\quad C : 100$ $\quad D : 10 \times 9^3$ $\quad E : 10!/6!$

- Correct answer is A: Any set of four numbers could appear in your PIN and once you've chosen your set of four numbers, only one ordering of these numbers is valid.

Outline

Example: Independent Coin Flips

- Suppose we have a biased coin that lands heads with probability p and tails with probability $(1-p)$.
- In the next section of the talk, we'll show that if we toss the coin n times then the probability that it lands heads k times is

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Outline

Example: Coins

- The commonly used coins in the USA are $1, 5, 10$, and 25 cents.
- In a multiset, elements can be repeated, e.g., $\{1, 1, 5, 10\}$.
- How many multisets of 7 coins are there? E.g., $\{1, 1, 1, 1, 1, 1, 1\}$ or $\{1, 1, 1, 5, 10, 25\}$ ...
- General Problem: In how many ways may we choose k elements out of a set of size n if we're allowed to repeat elements?
- Answer: $\binom{n-1+k}{n-1}$.

Bijective Functions

Definition

A function f defined on domain A with range B maps each $a \in A$ to exactly one element $f(a) \in B$.

Definition

A function $f : A \to B$ is bijective if for every $b \in B$ there exists a unique a such that $f(a) = b$.

Lemma

If there exists a bijection between two sets A and B then $|A| = |B|$.

One way to show f is a bijection is to find an inverse function g such that for any $b \in B$,

$$g(b) = a \Leftrightarrow f(a) = b$$

Back to the the harder counting problem...

- Let S be the set of size k subsets of $\{a_1, \ldots, a_n\}$ where elements are chosen with repetition.
- Define a function $f : S \to T$ where T is the set of binary strings of length $n - 1 + k$ with $n - 1$ ones. For a set $\mathbf{s} \in S$

$$f(\mathbf{s}) = \underbrace{0\ldots0}_{\ell_1}1\underbrace{0\ldots0}_{\ell_2}1\ldots1\underbrace{0\ldots0}_{\ell_n}$$

  where ℓ_i is the number of copies of a_i in $\mathbf{s} \in S$.
- f is a bijection because there exists an inverse g defined by

$$g(\mathbf{t}) = \bigcup_i \{m_i \text{ copies of } a_i\}$$

  where m_i is the number of 0's between $(i-1)$th and ith 1 in $\mathbf{t} \in T$.
- Since f is a bijection we know $|S| = |T| = \binom{n+k-1}{n-1}$

Example

Suppose $S = \{a, b, c\}$ and we want to choose 4 elements and we're allowed to pick elements multiple times.

Sequence $f(\mathbf{s})$	Multiset of four items $\mathbf{s}$
000011	{a,a,a,a}
000101	{a,a,a,b}
000110	{a,a,a,c}
001001	{a,a,b,b}
001010	{a,a,b,c}
001100	{a,a,c,c}
010001	{a,b,b,b}
010010	{a,b,b,c}
010100	{a,b,c,c}
011000	{a,c,c,c}
100001	{b,b,b,b}
100010	{b,b,b,c}
100100	{b,b,c,c}
101000	{b,c,c,c}
110000	{c,c,c,c}

Example: Coins

- The commonly used coins in the USA are $1, 5, 10$, and 25 cents.
- How many multisets of 7 coins are there?

$$\binom{n-1+k}{n-1} = \binom{4-1+7}{4-1} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

where k is the number of coins being chosen and n is the number of different types.

PIN Codes 7

- Suppose you bank insists you use a four digit PIN code where every digit is at least as large as the previous digit. How many possible codes are there:

$A : \binom{10}{9}$ $B : 10\times9\times8\times7$ $C : \binom{10}{4}$ $D : \binom{13}{9}$ $E : \binom{13}{10}$

- Correct answer is D: Any set multiset of four numbers could appear in your PIN and once you've chosen these, only one ordering of these numbers is valid. There are $\binom{10-1+4}{10-1}$ such multisets.