HW - D.
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Problem 2

Six-sided die. X is the number of times 6 occurs over n throws of the die.

Single die.

$$Y: \begin{cases} P(Y=1) = 1/6 \\ P(Y=0) = 5/6 \end{cases}$$

$$E(X) = n\,E(Y) = n\,\tfrac{1}{6}$$

$$VAR(X) = n\,VAR(Y).$$

$$VAR(Y) = p(1-p)^2 + (1-p)(0-p)^2$$
$$p(1-p)^2 + p^2(1-p)$$
$$p(1-p)(1-p+p)$$
$$p(1-p)$$

(i.e. the variance of a binomial variable)

$$VAR(X) = np(1-p)$$

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Problem 2 (continued)

Markov Inequality: where $a = n/4$

$$p(X \geq a) \leq \frac{E[X]}{a} = \frac{n/6}{n/4} = \boxed{\frac{2}{3}}$$

Chebyshev Inequality:

$$P(|X-\mu| \geq a) \leq \frac{var(X)}{a^2} = \frac{np(1-p)}{(n/4)^2} = \frac{16\,p(1-p)}{n}$$

$$= \frac{16(.17)(.83)}{n} = \boxed{\frac{2.22}{n}}$$

Chernoff Bounds -

$$p(X \geq a) \leq e^{-ta} M(T)$$

$$M = E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p_i = (\text{as above}) = \frac{2}{3}$$

Problem 3

What is expected number of times "proof" appears in 1 000 000 letters randomly chosen from $\{a, .., z\}$

Typing proof is $= p(\text{"p"}) \cdot p(\text{"r"}) \cdot p(\text{"o"}) \dots p(\text{"f"})$

$$= \frac{1}{26} \cdot \frac{1}{26} \cdots \frac{1}{26}$$

$$= \left(\frac{1}{26}\right)^5$$

$$X_n = \begin{cases} 1 & \text{if the character is the start of the sequence} \\ 0 & \text{otherwise.} \end{cases}$$

$$E(X) = \sum_{n=1}^{1000\,000-4} \left(\frac{1}{26}\right)^5 \cdot 1$$

$$= (1^M - 4)\left(\frac{1}{26}\right)^5$$

$$\boxed{= .084}$$

if we had 100 monkeys we would see 8.4 occurrences

Problem 4

Single DIE

$$E[X_1] = \sum_{1}^{6} \frac{1}{6} i = \frac{1}{6} \frac{6(6+1)}{2} = \frac{7}{2} = \boxed{3.5}$$

by $\sum_{1}^{n} x = \frac{n(n+1)}{2}$

$$E[X_n^2] = \sum_{1}^{6} \frac{1}{6} i^2 = \frac{1}{6} \frac{(2 \cdot 6+1)6(6+1)}{6}$$

$$= \frac{1}{6}(13)(7) = \frac{91}{6} = \boxed{15.2}$$

by $\sum_{1}^{N} x^2 = \frac{(2n+1)n(n+1)}{6}$

$$VAR[X_1] = E[X_1^2] - E[X_1]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{182}{12} - \frac{147}{12} = \boxed{\frac{35}{12}}$$

100 Rolls

$$Y = X_1 + X_2 + \dots X_{100}$$

$$E[Y] = 100\, E[X_1] = 100 \; 3.5 = 350$$ by Linearity

$$VAR[Y] = 100\, VAR[X_1] = 100 \frac{35}{12} = \frac{3500}{12} = \boxed{292}$$

Applying Chebyshev

$$P(|X - \mu| \geq k) \leq \frac{VAR(X)}{k^2}$$

where $k = 50$

$$P(|X - 350| \geq 50) \leq \frac{292}{50^2} = .12 = \boxed{12\%}$$