CMPSCI 240: Reasoning about Uncertainty

Lecture 3: Probability Rules and Uniform Probability

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Last Compiled: January 31, 2017

Outline

Experiments, Sample Spaces, Events

- Experiment: a process that results in exactly one of several possible outcomes, e.g., rolling a dice
- Sample space: the set of all possible outcomes of an experiment, e.g., $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Event: a subset of Ω, e.g., $A =$ "odd number" $= \{1, 3, 5\}$
- Atomic event: event consisting of a single outcome, e.g., $\{1\}$
- Probability law: A function $P(\cdot)$ that maps event to a number between 0 and 1 that satisfies the probability axioms:
  1. Nonnegativity: $P(A) \geq 0$ for every $A \subseteq \Omega$
  2. Normalization: $P(\Omega) = 1$
  3. Additivity: $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint

Useful Probability Rules

- Probability of the Empty Event:

$$P(\emptyset) = 0$$

- Probability of Event Complement:

$$P(A^c) = 1 - P(A)$$

- Probability of Unions: If $A_1, \ldots, A_N$ are mutually disjoint,

$$P(A_1 \cup \ldots \cup A_N) = P(A_1) + P(A_2) + \ldots + P(A_N) \ .$$

- Inclusion-Exclusion: For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \ .$$

Describing Discrete Probability Laws

- Consider an experiment where we ask people whether they like strawberries, kiwis, or green apples the most. The sample space is $\Omega = \{\text{🍓}, \text{🥝}, \text{🍏}\}$. The probability law needs to specify:

$$P(\{\text{🍓}, \text{🥝}, \text{🍏}\}) = ?$$

$$P(\{\text{🍓}, \text{🥝}\}) = ?\ ,\ P(\{\text{🥝}, \text{🍏}\}) = ?\ ,\ P(\{\text{🍓}, \text{🍏}\}) = ?$$

$$P(\{\text{🍓}\}) = ?\ ,\ P(\{\text{🥝}\}) = ?\ ,\ P(\{\text{🍏}\}) = ?\ ,\ P(\{\}) = ?$$

- Do we need to specify all 8 probabilities? No! If suffices to specify the probability for the atomic events since the union rule allows to determine the probability of the other events.
- E.g., $P(\{\text{🍓}\}) = 0.75$, $P(\{\text{🥝}\}) = 0.2$ and $P(\{\text{🍏}\}) = 0.05$ implies

$$P(\{\text{🍓}, \text{🥝}\}) = P(\{\text{🍓}\}) + P(\{\text{🥝}\}) = 0.75 + 0.2 = 0.95\ .$$

- Probabilities of the atomic events should sum to 1 so that $P(\Omega) = 1$.

Outline

Discrete Uniform Probability Law

- Discrete uniform probability law: If Ω is finite and all outcomes are equally likely, then $P(A) = \frac{|A|}{|\Omega|}$. **Why?**
- If each outcome o_i is equally likely and there are k of them, then each atomic event $\{o_i\}$ must have probability $P(\{o_i\}) = 1/k$ since

$$1 = P(\Omega) = P(\{o_1\} \cup \{o_2\} \cup \ldots \cup \{o_k\}) = \sum_{i=1}^{k} P(\{o_i\})$$

- If there are r outcomes in an event A are $\{a_1, ..., a_r\}$, then

$$P(A) = \sum_{i=1}^{r} P(\{a_i\}) = \sum_{i=1}^{r} \frac{1}{k} = \frac{r}{k}$$

Example: Flipping a Coin

- Suppose we consider flipping a coin. We don't know anything about the coin so we assume that the two possible outcomes

{ ,  }

  are both equally likely.
- **Question:** What is the probability law implied by these assumptions?
- **Answer:** Since there are two possible outcomes, $|\Omega| = 2$. The probability of each outcome is thus: $1/|\Omega| = 1/2$.

Example: Rolling a Die

- Suppose we have a six sided die. We assume the die is fair so that all six outcomes

  {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}

  are equally likely.
- **Question:** What is the probability that the roll is even?
- **Answer:** The event "the roll is even" corresponds to

  $A =$ {⚁, ⚃, ⚅} .

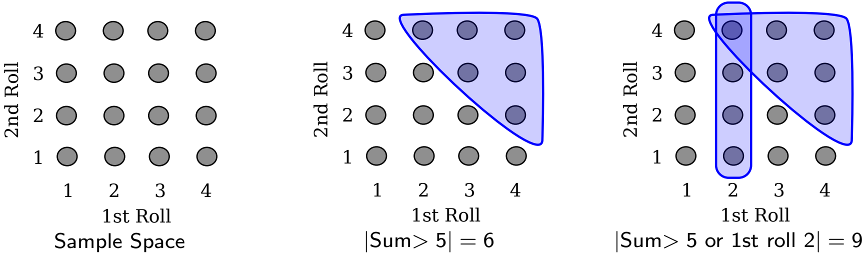
  Since $|A| = 3$, the probability is: $P(A) = 3/6 = 1/2$.

Example: Rolling Four-sided Dice Twice

- Suppose we roll a four-sided dice twice. We assume the die is fair so that all outcomes

$$\{(1,1),(1,2),...,(4,4)\}$$

  are equally likely.
- **Question:** What is the probability that the sum of the rolls is greater than 5 or first roll is 2?
- **Answer:** The probability is: 9/16.



Clicker Questions

Suppose we roll a four-sided dice twice. We assume the die is fair so that all outcomes $\{(1,1),(1,2),\ldots,(4,4)\}$ are equally likely.

Q1: What is the probability both dice are equal?

A: $4/16$ B: $1/16$ C: $8/16$ D: $7/16$ E: $2/16$

Answer: A

Q2: What is the probability the second roll is strictly larger than the first?

A: $4/16$ B: $6/16$ C: $7/16$ D: $2/16$ E: $10/16$

Answer: B

Q3: What is the probability that the product of the rolls is odd?

A: $4/16$ B: $6/16$ C: $7/16$ D: $2/16$ E: $10/16$

Answer: A

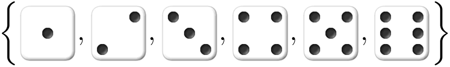
Outline

Conditional Probability Laws

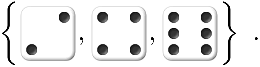
- Conditional probability provides a way to reason about the probabilities of different outcomes of an experiment if we have partial information about the outcome that occurred.

Example: Rolling a Die

- Suppose we have a six sided die. We assume the die is fair so the following outcomes are equally likely



- **Question:** What is the probability that we rolled a six if we know the roll was even?
- **Answer:** If we know the roll is an even number, then we know it must be in the set



Since any of these three outcomes is equally likely to occur, the probability that the roll is six if we know the roll is even must be $1/3$.

Conditional Probability Laws

- Conditional probability law: specifies the conditional probability $P(A \mid B)$ of any event A given that we know with certainty that the true outcome is contained in event B.

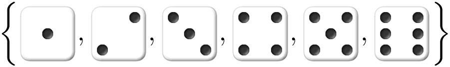
- Conditional probability of event A given event B:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ assuming } P(B) > 0$$

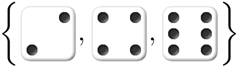
- If all outcomes are equally likely, $P(A \mid B) = |A \cap B| \,/\, |B|$

Example: Rolling a Die

- Suppose we have a six sided die. We assume the die is fair so the following six outcomes are equally likely



- **Question:** What is the probability that we rolled a six if we know the roll was even?
- **Answer:** Define the events

$B =$  and $A =$  .

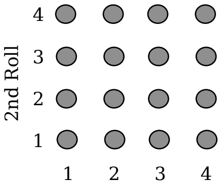
According to the formula for the uniform case, the conditional probability $P(A|B) = |A \cap B|/|B| = 1/3$.

Example: Rolling Two Four-sided Dice

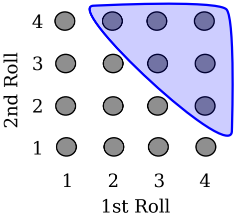
- Suppose we have two four-sided dice that we roll together. We assume the die are fair so the following 16 outcomes are equally likely

$$\{(1,1),(1,2),...,(4,4)\}$$

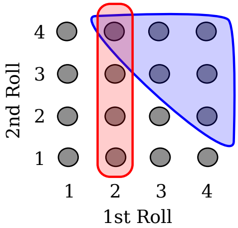
- **Question:** What is the probability that the first roll was 2 if the sum of the rolls is greater than 5?
- **Answer:** The probability is: $1/6$.



Sample Space



$|B| = |\text{Sum} > 5| = 6$



$|A \cap B| = |\text{Sum} > 5 \text{ \& 1st roll } 2| = 1$

Example: Preferences for Fruit

- Suppose $P(\text{🍓}) = 0.75$, $P(\text{🥝}) = 0.2$ and $P(\text{🍏}) = 0.05$.
- **Question:** What is the probability that someone prefers green apples if we know they prefer green fruit?
- **Answer:** Let $B = \{\text{🥝}, \text{🍏}\}$ and $A = \{\text{🍏}\}$

$$
\begin{aligned}
P(A|B) = \frac{P(A \cap B)}{P(B)} &= \frac{P(\text{🍏})}{P(\{\text{🥝}, \text{🍏}\})} \\
&= \frac{P(\text{🍏})}{P(\text{🥝}) + P(\text{🍏})} \\
&= \frac{0.05}{0.2 + 0.05} = 0.2
\end{aligned}
$$

Conditional Probability Laws and Probability Axioms

- If $P(\cdot)$ satisfies the probability axioms and $B \subseteq \Omega$ satisfies $P(B) > 0$, then the conditional probability $P(\cdot|B)$ also satisfies the axioms.
- Nonnegativity: $P(A|B) \geq 0$ for all $A \subseteq \Omega$
  1. $P(A|B) = P(A \cap B)/P(B)$ by the definition of conditional probability
  2. $P(B) > 0$ by assumption and $P(A \cap B) \geq 0$
  3. Therefore, $P(A|B) = P(A \cap B)/P(B) \geq 0$
- Normalization: $P(\Omega|B) = 1$
  1. $P(\Omega|B) = P(\Omega \cap B)/P(B)$ by the definition of conditional probability
  2. $\Omega \cap B = B$ since by definition of B, $B \subseteq \Omega$
  3. Therefore, $P(\Omega|B) = P(\Omega \cap B)/P(B) = P(B)/P(B) = 1$
- Additivity: ...

Conditional Probability Laws and Probability Axioms

- **Additivity:** $P(A_1 \cup A_2 | B) = P(A_1 | B) + P(A_2 | B)$ if A_1, A_2 disjoint
  1. By the definition of conditional probability
  $$P(A_1 \cup A_2 | B) = P((A_1 \cup A_2) \cap B)/P(B)$$
  2. By distributivity of intersection we have
  $$(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$$
  3. Since A_1 and A_2 are disjoint, $(A_1 \cap B)$ and $(A_2 \cap B)$ are disjoint and by additivity we have
  $$P((A_1 \cap B) \cup (A_2 \cap B)) = P(A_1 \cap B) + P(A_2 \cap B)$$
  4. Using this in the conditional probability yields
  $$\begin{aligned} P(A_1 \cup A_2 | B) &= (P(A_1 \cap B) + P(A_2 \cap B))/P(B) \\ &= P(A_1 \cap B)/P(B) + P(A_2 \cap B)/P(B) \\ &= P(A_1 | B) + P(A_2 | B) \end{aligned}$$

Rules with Conditional Probability

- Since conditional probability laws satisfy the probability axioms, results derived from axioms still apply after conditioning:
  1. $P(A^C|B) = 1 - P(A|B)$
  2. If $A_1, ..., A_N$ are mutually disjoint events

$$P(A_1 \cup \ldots \cup A_N|B) = P(A_1|B) + \ldots + P(A_N|B)$$

- **Multiplication Rule:** If $P(A) > 0$ then

$$P(A \cap B) = P(A)P(B|A)$$

- We say two events A and B are **independent** iff

$$P(A \cap B) = P(A)P(B) \; .$$

This is equivalent to $P(B|A) = P(B)$ assuming $P(A) > 0$.