CMPSCI 240: Reasoning about Uncertainty

Lecture 9: Expectation, Variance, and Functions of Random Variables

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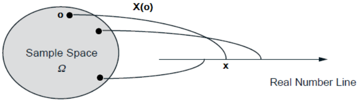
Last Compiled: March 9, 2017

Outline

Random Variables



- Formally, a random variable X is a mapping from Ω to $\mathbb{R}$.
- Given a random variable $X : \Omega \rightarrow \mathbb{R}$, we can construct a different event for each value of $k \in \mathbb{R}$:

$$\{X = k\} = \{o | o \in \Omega \text{ and } X(o) = k\}$$

- The probability of these events defines the *probability mass function*:

$$P(X = k) = \sum_{o \in \Omega \text{ such that } X(o)=k} P(o)$$

Discrete Random Variables

- **Uniform:** For $k = a, \ldots, b$:

$$P(X = k) = \frac{1}{b - a + 1}$$

- **Bernoulli:** For $k = 0$ or 1:

$$P(X = k) = \begin{cases} 1 - p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}$$

- **Binomial:** For $k = 0, \ldots, N$

$$P(X = k) = \binom{N}{k} p^k (1 - p)^{N-k}$$

- **Geometric:** For $k = 1, 2, 3, \ldots$, $P(X = k) = (1 - p)^{k-1} \cdot p$
- **Poisson:** $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$

Example

- Let X be the number of times you toss a dice until you see a six. Then X is

  A) Binomial B) Geometric C) Uniform D) Poisson E) Bernoulli

- The answer is Geometric.
- Suppose you toss a dice ten times and let X be the number of times you saw a six. Then X is

  A) Binomial B) Geometric C) Uniform D) Poisson E) Bernoulli

- The answer is Binomial.

Outline

Expected Value

- The expected value $E[X]$ is a probability-weighted average of the possible values of X:

$$E[X] = \sum_{k \in \mathbb{R}} k\, P(X = k)$$

- $E[X]$ is also called the expectation or mean of X.
- For example, if X maps to $\{2, 3, 5\}$ and

$$P(X = 2) = 1/2 \quad , \quad P(X = 3) = 1/4 \quad , \quad P(X = 5) = 1/4$$

then

$$E[X] = 2 \times 1/2 + 3 \times 1/4 + 5 \times 1/4 = 3$$

Expected Value: Question

- The expected value $E[X]$ is a probability-weighted average of the possible values of X:

$$E[X] = \sum_{k \in \mathbb{R}} k\, P(X = k)$$

- If X maps to $\{1, 2, 6\}$ and

$$P(X = 1) = 1/3 \quad , \quad P(X = 2) = 1/2 \quad , \quad P(X = 6) = 1/6$$

then is the expectation:

$$A)\ 2 \qquad B)\ 2.33\ldots \qquad C)\ 3 \qquad D)\ 3.5 \qquad E)\ 3.66\ldots$$

- Answer is $E[X] = 1 \times 1/3 + 2 \times 1/2 + 6 \times 1/6 = 2.33\ldots$

Expectations of Standard Random Variables

- **Uniform on $\{a, a+1, \ldots, b\}$:** $E[X] = \frac{a+b}{2}$
- **Bernoulli:** $E[X] = (1-p) \cdot 0 + p \cdot 1 = p$
- **Binomial:** $E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = np$
- **Geometric:** $E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} p = \frac{1}{p}$
- **Poisson:** $E[X] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}}{k!} \lambda^k = \lambda$

Uniform Expectation

$$
\begin{aligned}
E[X] &= \sum_{k=a}^{b} kP(X = k) \\
&= \sum_{k=a}^{b} k \times \frac{1}{b-a+1} \\
&= \frac{1}{b-a+1} \sum_{k=a}^{b} k \\
&= \frac{1}{b-a+1} \frac{(a+b)(b-a+1)}{2} \\
&= \frac{a+b}{2}
\end{aligned}
$$

Outline

Functions of Random Variables

- If X is a random variables and $f : \mathbb{R} \to \mathbb{R}$ then
$$Y = f(X)$$
is also a random variable with PMF:
$$P(Y = k) = P(f(X) = k) = \sum_{o \in \Omega \text{ with } f(X(o))=k} P(o)$$

- Expectation of Y:
$$E(Y) = \sum_k kP(Y = k) = \sum_r f(r)P(X = r)$$

- For example, if X maps to $\{2, 3, 5\}$ where
$$P(X = 2) = 1/2 \quad , \quad P(X = 3) = 1/4 \quad , \quad P(X = 5) = 1/4$$
and $Y = X^2$ then
$$E[Y] = 2^2 \times 1/2 + 3^2 \times 1/4 + 5^2 \times 1/4 = 10.5$$

Variance

- Variance measures how far we expect a random variable to be from its average:

$$\text{var}(X) = E[(X - E[X])^2] = \sum_k (k - E[X])^2 \cdot P(X = k)$$

- An equivalent definition is

$$\text{var}(X) = E[X^2] - E[X]^2$$

- The **standard deviation** is the positive square root of the variance:

$$\text{std}(X) = \sqrt{\text{var}(X)}\ .$$