CMPSCI 240: Reasoning about Uncertainty

Lecture 5: Total Probability and Independence

Andrew McGregor

University of Massachusetts

Last Compiled: February 8, 2017

Outline

Total Probability and Bayes Theorem

- Total Probability If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_n)P(A_n) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i)$$

- Bayes Theorem If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}$$

Example: Taking the bus

Every morning you leave your house and take the first bus that goes to the university. There's a 25% chance that the first bus that comes will be a red bus and a 75% chance it will be a blue. If you take the red bus, you get to class late 20% of the time. If you take the blue bus, you get to class late 55% of the time. What's the probability that you get to class late?

- **Question:** What events are specified in the problem?
- **Answer:**

  B_{red}= "red bus is first", B_{blue}= "blue bus is first", L= "get to class late"

- **Question:** What probabilities are specified in the problem?
- **Answer:**

  $P(B_{red}) = 0.25\,,\ \ P(B_{blue}) = 0.75\,,\ \ P(L|B_{red}) = 0.2\,,\ \ P(L|B_{blue}) = 0.55\,.$

- Need to compute $P(L)$: Since B_{blue} and B_{red} partition Ω:

$$P(L) = P(L|B_{blue})P(B_{blue}) + P(L|B_{red})P(B_{red}) = 0.4625$$

Example: Taking the bus 2

As before,

$$P(B_{red}) = 0.25\ ,\quad P(B_{blue}) = 0.75$$

$$P(L|B_{red}) = 0.2\ ,\quad P(L|B_{blue}) = 0.55\ .$$

Suppose the lecturer observes that you are late. What's the probability you caught the blue bus?

- Need to compute $P(B_{blue}|L)$:

$$P(B_{blue}|L) = \frac{P(B_{blue} \cap L)}{P(L)} = \frac{P(L|B_{blue})P(B_{blue})}{P(L)} = 0.891891891\ldots$$

Example: Testing Stressed Students

Suppose that 1/5 of students are stressed when doing online quizzes. A faculty member at UMass develops a systems for recognizing stressed students during these quizzes. The test can correctly identify positive cases 5/6 of the time and correctly identify negative cases 3/4 of the time. What's the probability that a student is recognized as stressed?

- **Events:** $S =$"Stressed" and $T =$"Test positive".
- **Relationships:** S and S^C partition Ω.
- **Probabilities:** $P(S) = 1/5$, $P(T|S) = 5/6$, $P(T^C|S^C) = 3/4$.
- **Question:** What is $P(T)$?
- **Answer:**

$$\begin{aligned} P(T) &= P(T|S)P(S) + P(T|S^C)P(S^C) \\ &= 5/6 \cdot 1/5 + 1/4 \cdot 4/5 = 11/30 \ . \end{aligned}$$

Example: Testing Stressed Students 2

As before, $P(S) = 1/5$, $P(T|S) = 5/6$, $P(T^C|S^C) = 3/4$ and we've deduced that $P(T) = 11/30$. What's the probability that a student is stressed given that the test is positive?

- **Question:** What is $P(S|T)$?
- **Answer:**

$$P(S|T) \quad = \quad \frac{P(S \cap T)}{P(T)} = \frac{1/6}{11/30} = \frac{5}{11} \; .$$

Outline

Example: Flipping Two Coins

- Consider flipping a fair coin twice in a row.
- If we know the coin is fair, does knowing the result of the first flip give us any information about the result of the second flip?
- What's the probability the coin comes up heads on the second flip?
- What's the probability the coin comes up heads on the second flip given that it came up heads on the first flip?

Probabilistic Independence

- Intuitively, when knowing that one event occurred doesn't change the probability that another event occurred or will occur, we say that the two events are *probabilistically independent*.
- We say that two events A and B are independent

$$P(A \cap B) = P(A)P(B) \ .$$

and this implies that $P(A|B) = P(A)$ and $P(B|A) = P(B)$ assuming $0 < P(A) < 1$ and $0 < P(B) < 1$.

Rolling Two Dice

- **Question:** Suppose you roll two fair four sided dice. Is the event $A =$ "first roll is 3" independent of the event $B =$ "second roll is 4"?
- **Answer 1:** Intuitively, like the coin flip, the two rolls have nothing to do with each other so the events A and B should be independent.
- **Answer 2:** Formally, $P(A \cap B) = 1/16$ since there are 16 possible outcomes and the event $A \cap B$ refers to exactly one of them. $P(A) = 1/4$ since there's a $1/4$ chance that the first roll is a 3. Similarly, $P(B) = 1/4$. Thus,

$$P(A)P(B) = (1/4)(1/4) = 1/16$$

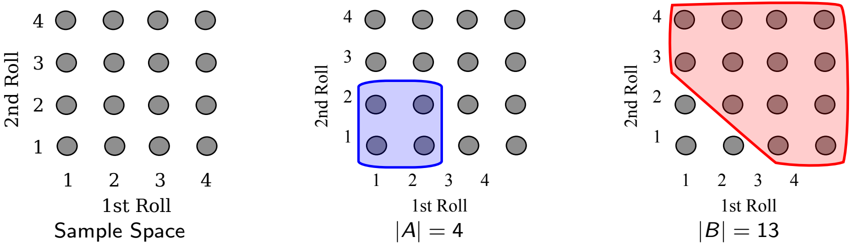
so the events are independent.

Rolling Two Dice

- **Question:** Suppose you roll two fair four sided dice. Are the events $A =$“maximum is less than 3” and $B =$“sum is greater than 3” independent?
- **Answer 1:** Intuitively, the answer is no. If the maximum was low it would appear that this should reduce the probability of the sum being greater than 3.

Rolling Two Dice

- **Question:** Suppose you roll two fair four sided dice. Are the events $A =$"maximum is less than 3" and $B=$"sum is greater than 3" independent?



- **Answer 2:** Formally,

$$P(A \cap B) = 1/16\ ,\ \ P(A) = 1/4 \ \text{ and } \ P(B) = 13/16\ .$$

Since $1/16 \neq 1/4 \cdot 13/16$, the events are not independent.

An Event and Its Complement

- **Question:** Are A and A^c independent if $0 < P(A) < 1$?
- **Answer 1:** Intuitively, no. If you know A happens, then you know A^C does not happen.
- **Answer 2:** Formally, $P(A \cap A^C) = P(\emptyset) = 0$. If $0 < P(A) < 1$, then

$$P(A)P(A^C) \neq 0 \ .$$

Independence of Three Events

- Three events A, B, and C are independent if and only if:

$$P(A \cap B) = P(A)\, P(B)$$
$$P(A \cap C) = P(A)\, P(C)$$
$$P(B \cap C) = P(B)\, P(C)$$
$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

- Note that pairwise independence does not imply independence.
- Suppose we have a finite collection of events $\mathcal{A} = \{A_1, ..., A_N\}$. The events in $\mathcal{A}$ are said to be independent if and only if for any subset $\mathcal{B} \subseteq \mathcal{A}$ containing two or more events we have:

$$P(\cap_{B \in \mathcal{B}} B) = \prod_{B \in \mathcal{B}} P(B)$$

Conditional Independence

- A and B are conditionally independent given C if and only if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

- If $P(B \mid C) > 0$ this is equivalent to $P(A \mid B \cap C) = P(A \mid C)$
- If $P(A \mid C) > 0$ this is equivalent to $P(B \mid A \cap C) = P(B \mid C)$

Conditional Independence Example

- I have one fair coin and one biased coin that lands heads with probability $2/3$.
- I pick a coin with equal probability: let F be the event it's the fair coin and let F^c be the event it's the biased coin.
- I toss the chosen coin twice: let A be the event the first toss is heads and let B be the event the second toss is heads.
- Are A and B independent? No.

$$P(A) = P(A|F)P(F) + P(A|F^c)P(F^c) = \frac{1}{2}\times\frac{1}{2} + \frac{2}{3}\times\frac{1}{2} = \frac{7}{12} = P(B)$$

$$P(A \cap B) = P(A \cap B|F)P(F) + P(A \cap B|F^c)P(F^c) = \frac{1}{4}\times\frac{1}{2} + \frac{4}{9}\times\frac{1}{2} = \frac{25}{72}$$

- Are A and B independent conditioned on F? Yes.

$$P(A|F) = P(B|F) = 1/2 \quad \text{and} \quad P(A \cap B|F) = 1/4 \ .$$