CMPSCI 240: Reasoning about Uncertainty

Lecture 13: Coupon Collecting and Correlation and Causation

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Outline

Expectation and Variance Review

- The expected value $E[X]$ of a random variable X is a probability-weighted average of the possible values of X:

$$E[X] = \sum_k k\, P(X = k)$$

- If X is a random variable and $f : \mathbb{R} \to \mathbb{R}$ then $Y = f(X)$ is also a random variable with expectation

$$E(Y) = \sum_k f(k) P(X = k)$$

- The variance is quantifies how close to $\mu = E[X]$ we expect X to be:

$$\mathrm{var}(X) = \sum_k (k - \mu)^2 P(X = k) = E[X^2] - \mu^2.$$

and the standard deviation of X is $\sigma_X = \sqrt{\mathrm{var}(X)}$

Multiple Random Variables

- Given two random variables, X and Y mapping from Ω to $\mathbb{R}$, we can define events of the form

$$\{X = i, Y = j\} = \{X{=}i\} \cap \{Y{=}j\} = \{o \in \Omega \mid X(o){=}i \text{ and } Y(o){=}j\}$$

- The probabilities of these events give the joint PMF of X and Y:

$$P(X = i, Y = j) = P(\{X = i, Y = j\})$$

- Given the joint PMF, we can compute the marginal probabilities:

$$P(X = i) = \sum_j P(X = i, Y = j)$$

$$P(Y = j) = \sum_i P(X = i, Y = j)$$

Functions of Multiple Random Variables

- Given random variables $X_1, X_2, \ldots, X_N$ and $f : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \to \mathbb{R}$,

$$Z = f(X_1, X_2, \ldots, X_N)$$

is a new random variable with expectation

$$E(Z) = \sum_{a_1, a_2, \ldots, a_N} f(a_1, a_2, \ldots, a_N) P(X_1 = a_1, X_2 = a_2, \ldots, X_N = a_N)$$

- Linearity of Expectation: If $Z = \sum_{i=1}^{N} c_i X_i$,

$$E(Z) = E(\sum_{i=1}^{N} c_i X_i) = \sum_{i=1}^{N} c_i E(X_i)$$

- Linearity of Variance: If $Z = \sum_{i=1}^{N} c_i X_i$,

$$var(Z) = var(\sum_{i=1}^{N} c_i X_i) = \sum_{i=1}^{N} c_i^2 var(X_i)$$

if $X_1, \ldots, X_N$ are pairwise independent, i.e., for all i, j, a, b

$$P(X_i = a, X_j = b) = P(X_i = a) P(X_j = b) \; .$$

Outline

Independence

- Two discrete random variables X and Y are independent if and only if $P(X = a, Y = b) = P(X = a)P(Y = b)$ for all a and b.
- When two random variables are not independent, it's natural to want to measure how dependent they are.

Quantifying Dependence: Covariance

- The **covariance** between X and Y is one measure of dependence that quantifies the degree to which there is a **linear relationship** between X and Y.

$$\begin{aligned} cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

- The covariance of X and Y is positive if when X is large, Y is also large. It's negative if when X is large, Y is small.
- If X and Y are independent then $cov(X, Y) = 0$ but $cov(X, Y) = 0$ does not necessarily imply that X and Y are independent.
- We can write $var(X + Y) = var(X) + var(Y) + 2cov(X, Y)$.

Example

P(X,Y)		
X\Y	$Y = 0$	$Y = 1$
$X = 0$	0.4	0.1
$X = 1$	0.2	0.3

- $P(X = 0) = 0.5, P(X = 1) = 0.5$ and so $E[X] = 0.5$
- $P(Y = 0) = 0.6, P(Y = 1) = 0.4$ and so $E[Y] = 0.4$
- $E[XY]$ can be computed as follows

$$\begin{aligned} E[XY] &= 0 \times 0 \times P(X = 0, Y = 0) + 0 \times 1 \times P(X = 0, Y = 1) + \\ &\quad 1 \times 0 \times P(X = 1, Y = 0) + 1 \times 1 \times P(X = 1, Y = 1) \\ &= 0.3 \end{aligned}$$

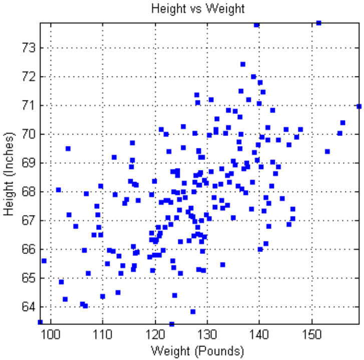
- $cov(X, Y) = E(XY) - E(X)E(Y) = 0.3 - 0.5 \times 0.4 = 0.1$

Quantifying Dependence: Correlation

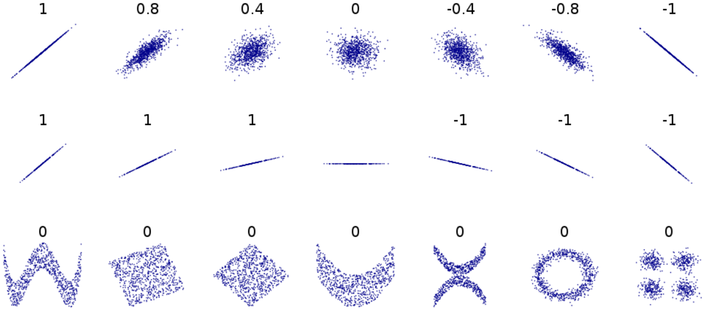
- The maximum magnitude of the covariance depends on the variance of X and the variance of Y.
- The **correlation** between X and Y is closely related to the covariance, but is normalized to the range $[-1, 1]$. 1 indicates maximum positive covariance and -1 indicates maximum negative covariance:

$$\rho(X, Y) = corr(X, Y) = \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}}$$

Visualizing Correlations: Height vs Weight ($\rho = 0.56$)



Visualizing Correlations: Linear vs Non-Linear



Causation

- **Question:** When two random variables are correlated does this mean one random variable causes the other?
- **Example:** There are more fireman at the scene of larger fires? Do fireman cause an increase in the size of a fire.
- **Example:** More people drown on days where a lot of ice cream is sold. Does ice cream cause drowning?
- **Example:** In the height/weight example, height and weight were positively correlated. Does increasing your weight make you taller?
- **Example:** When you see a wind turbine turning it is usually windy. Do wind turbines create wind?

Causation

Given two correlated random variables X and Y:

- X might cause Y (i.e., causation)
- Y might cause X (i.e., reverse causation)
- A third random variable Z might cause X and Y (i.e., common cause)
- A combination of all of these (e.g., self-reinforcement)
- The correlation might be spurious due to small sample size

Outline

Coupon Collecting/Shuffle Mode

- You have n songs on your phone.
- In shuffle mode, the player picks songs uniformly at random.
- Let T be the total number of songs played until every song is played.
- T could be infinite or as small as n.
- For this section, recall that if X is a geometric random variable with parameter p then $P(X = k) = (1 - p)^{k-1}p$ and has expectation $1/p$.

What's the probability that $T = n$?

- What's the probability that $T = n$?
- Number of possible sequences of n songs: n^n
- Number of possible sequences of n songs including every song: $n!$
- Therefore, probability is:

$$\frac{n!}{n^n} = \frac{n}{n} \times \frac{n-1}{n} \times \ldots \times \frac{1}{n} \leq 2^{-n/2}$$

Expected Value of T

- To analyze $E[T]$ we define $C_1, C_2, \ldots, C_n$ where

  $C_i =$ songs played after $(i-1)$-th new song until i-th new song is played

  and note that $T = \sum_{i=1}^{n} C_i$
- By linearity of expectation:

$$E[T] = \sum_{i=1}^{n} E[C_i]$$

- C_i is a geometric random variable with

$$P(C_i = j) = p_i(1-p_i)^{j-1} \quad \text{for } j = 1, 2, \ldots$$

  where $p_i = \frac{n-i+1}{n}$
- $E[C_i] = \frac{1}{p_i} = \frac{n}{n-i+1}$
- So

$$E[T] = \frac{n}{n} + \frac{n}{n-1} + \ldots + \frac{n}{1} = nH_n \approx n \ln n$$

Outline

Secrets of the Chebyshev Bound

- Chebyshev Bound:

$$P(X \leq E(X)-c)+P(X \geq E(X)+c) = P(|X-E(X)| \geq c) \leq Var(X)/c^2$$

- The bound is useful when we are trying to bound the probability that X is much smaller or larger than it's expectation.
- However, it also implies bounds on just one tail.
- For example, if $E(X) = 10$ and $var(X) = 2$ then

$$P(X \geq 15) = P(X \geq E(X) + 5) \leq P(|X - E(X)| \geq 5) \leq 2/25$$

Poisson Expectation

For a Poisson random variable, $P(X = k) = \frac{e^{-\lambda}}{k!}\lambda^k$. Hence,

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}}{k!}\lambda^k \\
&= \lambda \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda}}{k!}\lambda^{k-1} \\
&= \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda}}{(k-1)!}\lambda^{k-1} \\
&= \lambda(P(X=0) + P(X=1) + P(X=2) + \dots) \\
&= \lambda
\end{aligned}
$$

The last line follows because the events $\{X = 0\}, \{X = 1\}, \{X = 2\}, \dots$ partition the sample space and hence the probabilities sum up to 1.

Geometric Expectation

For a Geometric random variable, $P(X = k) = (1 - p)^{k-1}p$. You'll prove in the homework that:

- $E[X] = P(X \geq 1) + P(X \geq 2) + P(X \geq 3) \ldots$
- $P(X \geq k) = (1 - p)^{k-1}$

Using these,

$$\begin{aligned} E[X] &= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) \ldots \\ &= 1 + (1 - p) + (1 - p)^2 + \ldots \\ &= 1/p \end{aligned}$$

Alternative Expression for Expectation

- If $Y = f(X)$, we can write $E[Y] = \sum_k f(k)P(X = k)$.
- Use the fact that $P(Y = r) = \sum_{k:f(k)=r} P(X = k)$ and then,

$$
\begin{aligned}
E[Y] &= \sum_r rP(Y = r) \\
&= \sum_r r \sum_{k:f(k)=r} P(X = k) \\
&= \sum_r \sum_{k:f(k)=r} rP(X = k) \\
&= \sum_r \sum_{k:f(k)=r} f(k)P(X = k) \\
&= \sum_k f(k)P(X = k)
\end{aligned}
$$

Secrets of Pairwise Independence

- Suppose we have some bernoulli random variables $X_1, X_2, \ldots, X_n$ where for all $i < j$ the joint probabilities are given in the following table:

$X_i \backslash X_j$	0	1
0	0.25	0.25
1	0.25	0.25

- Are the variables pairwise independent? I.e., for all $i < j$ and $a, b \in \{0, 1\}$, $P(X_i = a, X_j = b) = P(X_i = a)P(X_j = b)$. Yes.
- Are they also three-wise independent? I.e., for all $i < j < k$ and $a, b, c \in \{0, 1\}$

$$P(X_i = a, X_j = b, X_k = c) = P(X_i = a)P(X_j = b)P(X_k = c)$$

- Not necessarily, e.g., let X_1 and X_2 be the result of tossing two independent coins and $X_3 = X_1 + X_2 \pmod 2$.