CMPSCI 240: Reasoning about Uncertainty

Lecture 2: Sets and Events

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Outline

1 Recap

2 Experiments and Events

3 Probabilistic Models

4 Probability Rules

Recap

Given two sets A and B then:

- The intersection of the sets, denoted $A \cap B$, consists of all elements that are in both A and B.
- The union of the sets, denoted $A \cup B$, consists of all elements that are in at least one of A or B.

Given a universal set Ω and a set $A \subseteq \Omega$ then:

- The complement of A, denoted A^C, consists of all elements in Ω that aren't in A.

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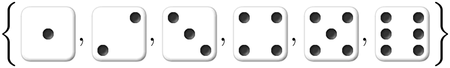
Experiments and Sample Spaces

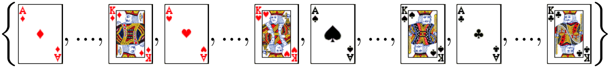
- In probability theory, an **experiment** is a process that results in exactly one of several possible **outcomes**.
- The set of possible outcomes of a probabilistic experiment is called the **sample space** Ω of the experiment.

Experiments and Sample Spaces

- Rolling a single die.

$\Omega =$ 

- Drawing a card from a deck.

$\Omega =$ 

- Flipping a single coin.

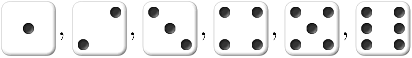
$\Omega =$ 

- Flipping a single coin twice.

$\Omega =$ 

Events

- An **event** is a subset of the sample space Ω.
- Consider rolling a die and define the sample space to be:

$\Omega = \{$  $\}$

- Consider the event "the dice comes up even." What subset of elements from Ω does this correspond to?

$\{x | x \in \Omega \text{ and } x \text{ is even}\} = \{$  $\}$

Atomic Events

- An **atomic event** is a subset of the sample space Ω that only contains a single outcome.

- Consider rolling a die and define the sample space to be:

$$\Omega = \{ ⚀, ⚁, ⚂, ⚃, ⚄, ⚅ \}$$

- $\{ ⚀ \}$ **is an atomic event**.

- $\{ ⚀, ⚁ \}$ is an event, but **not an atomic event** since it contains **two** outcomes.

Constructing Arbitrary Events From Atomic Events

- Events are sets so any event can be expressed in terms of a union of atomic events

- Let $A =$ "x is an odd number" $= \{$⚀, ⚂, ⚄$\}$. Then

$$A = \{\text{⚀}\} \cup \{\text{⚂}\} \cup \{\text{⚄}\}$$

- Let $B =$ "x is less than 3" $= \{$⚀, ⚁$\}$. Then

$$B = \{\text{⚀}\} \cup \{\text{⚁}\}$$

Constructing Complex Events From Simple Events

- We can construct complex events from unions or intersection of simpler ones. Consider the case of a single dice roll:

- Let $C =$ "x is an odd number" and "x is less than 3".
- Then $C = A \cap B = \{$  $\}$

- Let $D =$ "x is an odd number" or "x is less than 3".
- Then $D = A \cup B = \{$  $\}$

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1. Recap
2. Experiments and Events
3. Probabilistic Models
4. Probability Rules

Probability Laws and Probability Axioms

- A probability law is a function $P(A)$ that assigns a number between 0 and 1 to events $A \subseteq \Omega$. It encodes our assumptions about the likelihood of an event: if A is likely $P(A)$ will be near 1 and if A unlikely $P(A)$ will be near 0.
- To be a mathematically valid probability law, the function $P(\cdot)$ must satisfy the three **axioms of probability theory**:
  1. Nonnegativity: $P(A) \geq 0$ for every $A \subseteq \Omega$
  2. Normalization: $P(\Omega) = 1$
  3. Additivity: $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint

Probabilistic Models

- An abstract probabilistic model consists of two basic elements:
  1. A sample space Ω defining the outcomes of interest and
  2. A valid probability law defining the probability $P(A)$ of each event of of interest $A \subseteq \Omega$.
- A probability model is considered a “model” because it is intended to capture the most relevant features of a potentially much more complex real-world process.

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Useful Probability Rules

- Probability of the Empty Event:

$$P(\emptyset) = 0$$

- Probability of Event Complement:

$$P(A^c) = 1 - P(A)$$

- Probability of Unions: If $A_1, \ldots, A_N$ are mutually disjoint,

$$P(A_1 \cup \ldots \cup A_N) = P(A_1) + P(A_2) + \ldots + P(A_N) \ .$$

Probability of Event Complement

Question: What can we say about the probability $P(A^C)$ of the complement A^C of event A?

1. By definition $A \cup A^C = \Omega$ and $A \cap A^C = \emptyset$.
2. We also know that $P(\Omega) = 1$ by the normalization axiom.
3. Using these results we have:

$$\begin{aligned} P(A \cup A^c) &= P(\Omega) = 1 && \text{... Normalization} \\ P(A \cup A^c) &= P(A) + P(A^c) && \text{... Additivity} \\ P(A) + P(A^c) &= P(\Omega) = 1 \\ P(A^c) &= 1 - P(A) \end{aligned}$$

Probability of the Empty Event

Question: What's the probability of the empty event $\emptyset$?

1. Since Ω and $\emptyset$ are disjoint, the additivity axiom implies

$$P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset)$$

2. Next note that $\Omega \cup \emptyset = \Omega$ and so

$$P(\Omega \cup \emptyset) = P(\Omega)$$

3. Therefore

$$P(\Omega) + P(\emptyset) = P(\Omega)$$

   and hence $P(\emptyset) = 0$.

Probability of Finite Unions

Question: Suppose we have a finite collection of mutually disjoint events $A_1, \dots, A_N$. What can we say about the probability of the union of these events $P(A_1 \cup \dots \cup A_N)$?

$$
\begin{aligned}
P(A_1 \cup \dots \cup A_N) &= P(A_1 \cup (A_2 \cup \dots \cup A_N)) \; \dots \text{ Associativity} \\
&= P(A_1) + P(A_2 \cup \dots \cup A_N) \; \dots \text{ Additivity} \\
&= P(A_1) + P(A_2 \cup (A_3 \cup \dots \cup A_N)) \; \dots \text{ Associativity} \\
&= P(A_1) + P(A_2) + P(A_3 \cup \dots \cup A_N) \; \dots \text{ Additivity} \\
&\vdots \\
&= P(A_1) + \dots + P(A_N)
\end{aligned}
$$

Describing Discrete Probability Laws

- Consider an experiment where we ask people whether they like strawberries, kiwis, or green apples the most. The sample space is $\Omega = \{\text{🍓}, \text{🥝}, \text{🍏}\}$. The probability law needs to specify:

$$P(\{\text{🍓}, \text{🥝}, \text{🍏}\}) = ?$$

$$P(\{\text{🍓}, \text{🥝}\}) = ? \ , \ P(\{\text{🥝}, \text{🍏}\}) = ? \ , \ P(\{\text{🍓}, \text{🍏}\}) = ?$$

$$P(\{\text{🍓}\}) = ? \ , \ P(\{\text{🥝}\}) = ? \ , \ P(\{\text{🍏}\}) = ? \ , \ P(\{\}) = ?$$

- Do we need to specify all 8 probabilities? No! If suffices to specify the probability for the atomic events since the union rule allows to determine the probability of the other events.
- E.g., $P(\{\text{🍓}\}) = 0.75$, $P(\{\text{🥝}\}) = 0.2$ and $P(\{\text{🍏}\}) = 0.05$ implies

$$P(\{\text{🍓}, \text{🥝}\}) = P(\{\text{🍓}\}) + P(\{\text{🥝}\}) = 0.75 + 0.2 = 0.95 \ .$$

- Probabilities of the atomic events should sum to 1 so that $P(\Omega) = 1$.

Clicker Questions

Q1: If $P(A) = 0.2$, $P(B) = 0.4$ and A and B are disjoint events. What is $P(A \cup B)$?

A: 0 B: 0.2 C: 0.4 D: 0.6 E: 0.8

Answer: D

Q2: If $P(A) = 0.2$, $P(B) = 0.4$ and $A \subseteq B$. What is $P(A \cup B)$?

A: 0.2 B: 0.4 C: 0.6 D: 0.8 E: 2

Answer: B

Q3: If $P(A) = 0.2$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$. What is $P(A \cup B)$?

A: 0.1 B: 0.2 C: 0.5 D: 0.6 E: 0.7

Answer: C

Bonus Slide

The last quiz question can be solved using the “inclusion-exclusion rule”:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \ .$$

Note that if A and B are disjoint events, then $P(A \cap B) = 0$ and this simplifies to the additivity rule. To prove the formula, let

$$C_1 = A \cap B^c \qquad C_2 = A \cap B \qquad C_3 = A^c \cap B$$

be a partition of $A \cup B$. Then note that

$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(C_1) + P(C_2) + P(C_2) + P(C_3) - P(C_2) \\ &= P(C_1) + P(C_2) + P(C_3) \\ &= P(A \cup B) \end{aligned}$$