# CMPSCI 240: Reasoning about Uncertainty

## Lecture 6: Counting

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# Outline


1 Discrete Probability

2 Counting

3 Permutations

4 k-Permutations

5 Examples

6 Combinations

# Discrete Probability Laws

- If $\Omega$ is finite and all outcomes are equally likely, then $P(A) = \frac{|A|}{|\Omega|}$.
- Sometimes it's challenging to compute $|A|$ and $|\Omega|$ and they are too large work out by hand...

# Shortcuts for Counting

- *Permutations:* There are $n! = n \times (n-1) \times \ldots \times 2 \times 1$ ways to permute $n$ objects. E.g., permutations of $\{a, b, c\}$ are

$$abc, acb, bac, bca, cab, cba$$

- *k-Permutations:* There are $n \times (n-1) \times \ldots \times (n-k+1)$ ways to choose the first $k$ elements of a permutation of $n$ objects. E.g., 2-permutations of $\{a, b, c, d\}$ are

$$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc$$

- *Combinations:* There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to choose a subset of size $k$ from a set of $n$ objects. E.g., the subsets of $\{a, b, c, d\}$ of size 2 are

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

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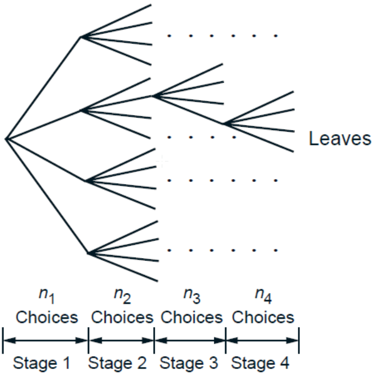
# The Counting Principle

- Consider a sequential process with $s$ stages. At each stage $i$, there are $n_i$ possible results. How many outcomes does the process have?




- This gives us $n_1 \times n_2 \times \cdots \times n_s = \prod_{i=1}^{s} n_i$ possible outcomes.

# Example: Phone Numbers

- **Question:** How many different 7 digit phone numbers are there?
- **Answer:** This is an $s = 7$ stage experiment with $n_i = 10$ possible events per stage. This gives $10^7$ possible phone numbers.
- **Question:** If your new cell number is randomly assigned, what's the probability that the last two digits are your birthday?
- **Answer:** If the last two digits are your birthday (e.g. 05, 31, etc...) then there is only one choice for these two digits and $10^5$ choices for the remaining 5 digits. The probability is thus $10^5/10^7$ or 1 in 100.

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# Counting Permutations

- Let $S$ be a set of $r$ objects. Consider an $r$-stage experiment: at each stage we choose one object without replacement.
- This process produces an ordering or "permutation" of the $r$ objects. For example, if $r = 3$ and $S = \{a, b, c\}$, one ordering is $bac$.
- This is an $r$ stage process. We have $n_1 = r$, $n_2 = r - 1$,..., $n_r = 1$. By the counting principle, the number of permutations is

$$r(r-1)(r-2)\cdots 1 = r!\ .$$

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# Counting $k$-Permutations

- Let $S$ be a set of $r$ objects. Consider an $k$-stage experiment: at each stage we choose one object without replacement.
- This process produces an ordering of the $k$ objects. For example, if $r = 3, k = 2$, and $S = \{a, b, c\}$, one ordering is $ba$ and another is $ab$. These are also called $k$-permutations.
- This is a $k$-stage process where $n_1 = r$, $n_2 = r - 1$,..., $n_k = r - k + 1$. By the counting principle, the number of permutations is

$$r(r-1)(r-2)\cdots(r-k+1) = r!/(r-k)!\ .$$

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# Example: Sports Rankings

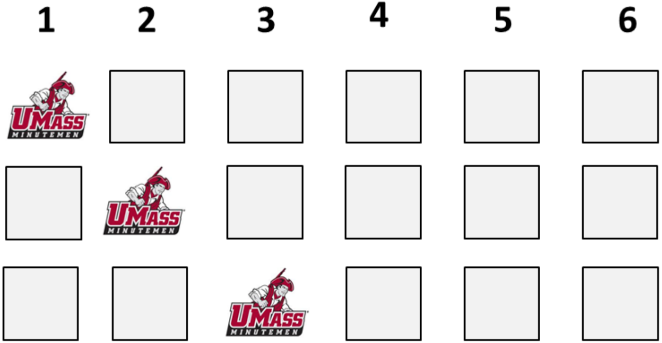
- **Question:** Suppose your sports team is in a league with 5 other teams. At the end of the season all teams are ranked. How many different ranking are there?

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|

- **Answer:** There are $r = 6$ teams. A ranking is just an ordering or permutation of the teams, so the number of rankings is $6! = 720$.

# Example: Sports Rankings

- **Question:** Suppose all rankings are equally likely. What's the probability that your team finishes in the top 3?

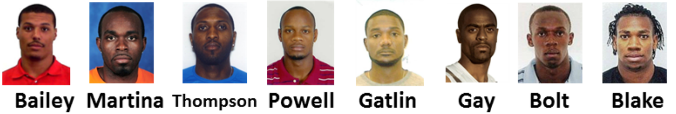


# Example: Sports Rankings

- **Answer:** If your team is 1st, there are 5! possible orderings for the other teams. If your team is 2nd, there are 5! possible orderings for the other teams. If your team is 3rd, there are 5! possible orderings for the other teams.
- There are thus $3 \cdot 5!$ orderings where your team is in the top 3 and the probability is $3 \cdot 5!/6! = 3/6 = 1/2$.

# Example: Olympic Medalists

- **Question:** There are 8 runners in the mens 100 meter final. How many different medal orderings are there?





- **Answer:** This is a $k = 3$ stage process with $r = 8$ objects. Since the ordering matters, this is a $k$-permutation problem. The answer is thus $8!/(8-3)! = 336$.

# Example: Olympic Medalists

- **Question:** If all medal orderings are equally likely, what's the probability that Bolt gets a medal?
- **Answer:** If Bolt takes the first spot, there are $7!/(7-2)!$ ways the other 7 runners could be assigned to the remaining spots. This is also true if Bolts takes the second or third spots. So, final answer is:

$$\frac{3 \cdot 7!/5!}{8!/5!} = 3/8$$

# Poker Hands

- Suppose we pick five cards from a deck of cards. What's the probability of getting a full house, i.e., three cards of the same rank and two cards of another rank.
- 13 ways to pick which rank gets picked three times.
- 12 remaining ways to pick which rank gets picked two times.
- $\binom{4}{3} = 4$ ways to pick three cards of the chosen rank.
- $\binom{4}{2} = 6$ ways to pick two cards of the chosen rank.
- In total:

$$13 \times 12 \times 4 \times 6 = 3744$$

- So probability of getting a full house is

$$\frac{3744}{\binom{52}{5}} = 0.00144057623\ldots$$

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# Counting Combinations

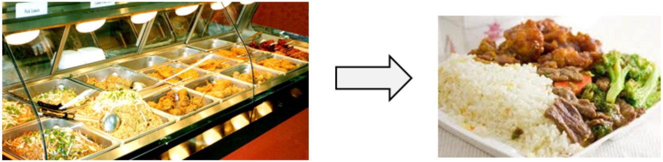
- Let $S$ be a set of $n$ objects. How many subsets of size $k$ are there?
- The number of $k$-permutations is $n!/(n-k)!$ but this over counts the number of subsets, e.g., $ab$ and $ba$ are different 2-permutations of $\{a, b, c\}$, but the same subset $\{a, b\}$.
- Every subset of $k$ objects corresponds to $k!$ different $k$-permutations, so the number of "$k$-combinations" is

$$\frac{n!/(n-k)!}{k!} = \frac{n!}{(n-k)!k!}$$

  which is denoted $\binom{n}{k}$, pronounced "$n$ choose $k$".

# Example: Flaming Wok

- **Question:** The Flaming Wok restaurant sells a 3-item lunch combo. You can choose from 10 different items. How many different lunch combos are there?



- **Answer:** Since your lunch is the same regardless of the order the items are put on the plate, this is a combination problem. The number of lunch combos is thus $\binom{10}{3}$ or 120.

# Example: Flaming Wok

- **Question:** Suppose you ask for a random combo and there's one item you don't like. What's the combo you order has items you like?
- **Answer:** The probability that you will like a random combo is the number of combos you like, which is $\binom{9}{3} = 84$, divided by the number of combos, which is $\binom{10}{3}$. This gives $84/120 = 0.7$.