CMPSCI 240: Reasoning about Uncertainty

Lecture 4: Sequential experiments

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Outline

1. Recap
2. Sequential Experiments
3. Total Probability and Bayes Theorem

Conditional Probability Laws

- Conditional probability laws specify the conditional probability $P(A \mid B)$ of any event A given that we know with certainty that the true outcome is contained in event B.
- Conditional probability of event A given event B where $P(B) > 0$:

$$P(A \mid B) = P(A \cap B)/P(B)$$

  and so $P(A \cap B) = P(B)P(A|B)$.
- Basic Properties:
  1. $P(A|B) \geq 0$
  2. $P(\Omega|B) = 1$
  3. $P(A^C|B) = 1 - P(A|B)$
  4. If $A_1, ..., A_N$ are mutually disjoint events

$$P(A_1 \cup \ldots \cup A_N|B) = P(A_1|B) + \ldots + P(A_N|B)$$

Outline

Sequential Experiments

- A sequential experiment is an experiment that involves several different steps that happen in sequence.
- Each step is an event that narrows down the possible outcomes until the final step, which assigns a unique outcome.
- Probability laws for sequential experiments are often much more intuitive to specify using conditional probability.

Example: Radar Detection

- **Problem:** We're in charge of air traffic control. We know that if a plane is near the airport, the radar will correctly detect it with probability 0.99. If a plane is not present, the radar still generates a detection with probability 0.1. We know that an aircraft is near the airport with probability 0.05. What is the probability that no plane is present and the radar generates a detection?
- Steps in the sequential experiment:
  1. Is an aircraft present? Define event A="an aircraft is present"
  2. Does the radar generate a detection? Define event D="radar generates detection"
- Outcomes of the sequential experiment:
  1. $A \cap D$: Aircraft and detection
  2. $A \cap D^C$: Aircraft and no detection
  3. $A^C \cap D$: No aircraft and detection
  4. $A^C \cap D^C$: No aircraft and no detection

Example: Radar Detection

- **Problem:** If a plane is near the airport, the radar will correctly detect it with probability 0.99. If a plane is not present, the radar still generates a detection with probability 0.1. We know that an aircraft is near the airport with probability 0.05.

- **Question:** What probabilities can we deduce?
- **Answer:**
  1. $P(D|A) = 0.99$: "If a plane is near the airport, the radar will correctly detect it with probability 0.99"
  2. $P(D|A^C) = 0.1$: "If a plane is not present, the radar still generates a detection with probability 0.1"
  3. $P(A) = 0.05$: "An aircraft is near the airport with probability 0.05"

Example: Radar Detection

- **Question:** If $P(D|A) = 0.99$, $P(D|A^C) = 0.1$ and $P(A) = 0.05$, what is the probability that no plane is present and the radar generates a detection? In other words, what is $P(A^C \cap D)$?
- **Answer:**

$$\begin{aligned} P(A^C \cap D) &= P(D|A^C)P(A^C) = P(D|A^C)(1 - P(A)) \\ &= 0.1 \cdot 0.95 = 0.095 \end{aligned}$$

Example: Radar Detection

- **Question:** If $P(D|A) = 0.99$, $P(D|A^C) = 0.1$ and $P(A) = 0.05$, what is the probability that a plane is present and the radar does not generate a detection?

- **Answer:** The probability we want to compute is $P(A \cap D^C)$. Using the multiplication and complement rules, we find:

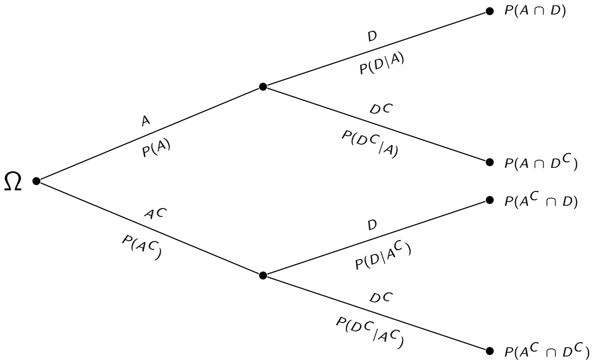
$$\begin{aligned} P(A \cap D^C) &= P(D^C|A)P(A) = (1 - P(D|A))P(A) \\ &= 0.01 \cdot 0.05 = 0.0005 \end{aligned}$$

Conditional Probability Trees for Sequential Experiments

- One convenient way to describe these types of sequential experiments is to use a *conditional probability tree*.
  1. The steps in the experiment correspond to edges in the tree.
  2. A path from the root to a leaf is an outcome.
  3. The probability of any outcome can be computed by multiplying the edge probabilities on the path from the outcome to the root.

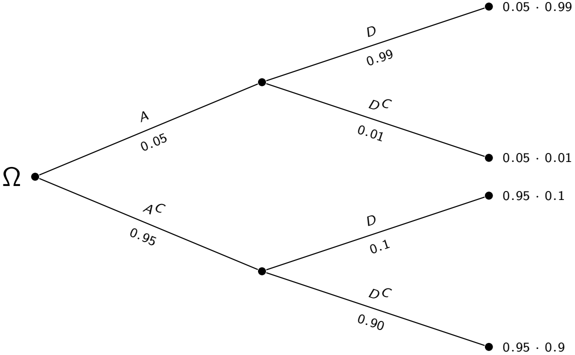
Example: Radar Detection

Given the conditional probabilities $P(D|A) = 0.99$, $P(D|A^C) = 0.1$ and $P(A) = 0.05$, we construct the corresponding tree as shown below.



Example: Radar Detection

We then substitute the numerical values derived from the probabilities $P(D|A) = 0.99$, $P(D|A^C) = 0.1$ and $P(A) = 0.05$ into the tree.



To get the probability of an outcome, just multiply the probabilities along the corresponding path.

Extending to more steps: N-Step Multiplication Rule

- Now suppose we have N events $A_1, ..., A_N$ and we're interested in the probability of the conjunction of these events

$$P(\cap_{n=1}^{N} A_n) = P(A_1 \cap \cdots \cap A_N) \ .$$

- The **N-step multiplication rule** states that:

$$\begin{aligned} P(\cap_{n=1}^{N} A_n) &= \prod_{n=1}^{N} P(A_n | \cap_{i=1}^{n-1} A_i) \\ &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_N|A_1 \cap \cdots \cap A_{N-1}) \end{aligned}$$

N-Step Multiplication Rule

We can prove the result using repeated application of the multiplication rule for conditional probabilities.

$$
\begin{aligned}
&P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\cdots P(A_N|A_1 \cap \cdots \cap A_{N-1}) \\
&= P(A_1) \cdot \frac{P(A_1 \cap A_2)}{P(A_1)} \cdot \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \cdots \frac{P(A_1 \cap A_2 \cap \ldots A_N)}{P(A_1 \cap A_2 \cap \ldots \cap A_{N-1})} \\
&= P(A_1 \cap A_2 \cap \ldots A_N)
\end{aligned}
$$

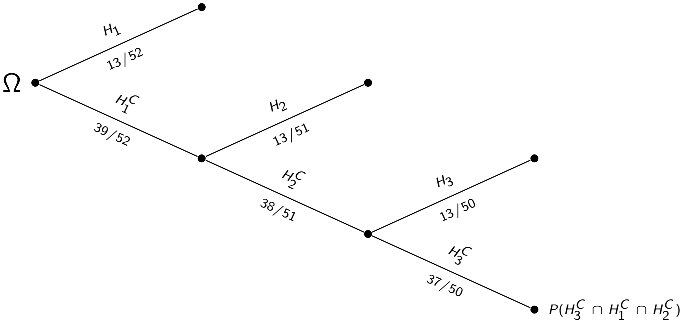
Example: Pick Three Cards

- Consider a card game where you draw three consecutive cards from a deck without replacement and win if none of the cards is a heart.
- **Question:** What is the probability that you win the game?
  1. Let H_1, H_2, H_3 be the events that the first, second, and third cards that you draw are hearts.
  2. The probability that we want to compute is thus $P(H_1^C \cap H_2^C \cap H_3^C)$.
  3. Using the multiplication rule, we can compute it as

$$P(H_1^C)P(H_2^C|H_1^C)P(H_3^C|H_1^C \cap H_2^C) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \approx 0.41$$

Example: Pick Three Cards

- Consider a simple card game where you win if you draw three consecutive cards from a deck without replacement and none of the cards is a heart. **Question:** What does the conditional probability tree look like?



Clicker Questions

Suppose we draw two cards at random without replacement from a standard deck of cards. Let H_1 and H_2 be the events that the first and second cards that you draw are hearts.

Q1: What is the probability $P(H_2^C | H_1)$?

A: 1/2 B: 38/52 C: 39/52 D: 38/51 E: 39/51

Answer: E

Q2: What is the probability $P(H_1 \cap H_2)$?

A: $\frac{13}{52} \times \frac{12}{52}$ B: $\frac{13}{52} \times \frac{13}{52}$ C: $\frac{13}{52} \times \frac{12}{51}$ D: $\frac{13}{52} \times \frac{13}{51}$

Answer: C

Outline

Total Probability and Bayes Theorem

- Total Probability If $A_1, \ldots, A_n$ partition Ω then for any event B:

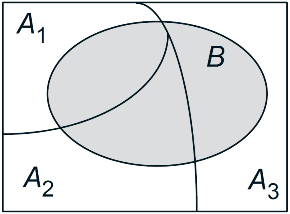
$$P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_n)P(A_n) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i)$$

- Bayes Theorem If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}$$

Proof of Total Probability Theorem

- Since $A_1, \ldots, A_n$ partition Ω then $(B \cap A_1), \ldots, (B \cap A_n)$ partition B.



- For any A_i, the multiplication rule gives us

$$P(B \cap A_i) = P(B|A_i)P(A_i) \ .$$

- Putting these results together we have the following proof:

$$\begin{aligned} P(B) &= P((B \cap A_1) \cup \cdots \cup (B \cap A_n)) \\ &= P(B \cap A_1) + \cdots + P(B \cap A_n) \\ &= P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n) \end{aligned}$$