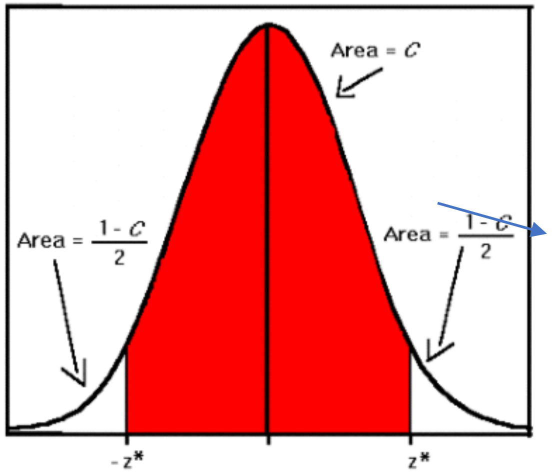
Confidence Interval

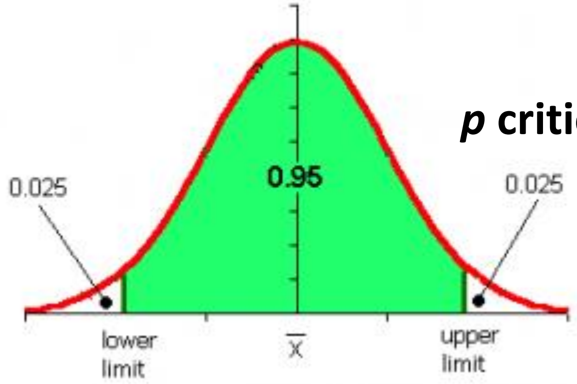
An example including **confidence interval , hypothesis , Z-score, T-score** and Distribution



C=0.95, **C is Confidence Level**

Critical Value, like z-score, t-score

p **critical value**



Get Z-score and corresponding p value: Z table

Confidence Intervals

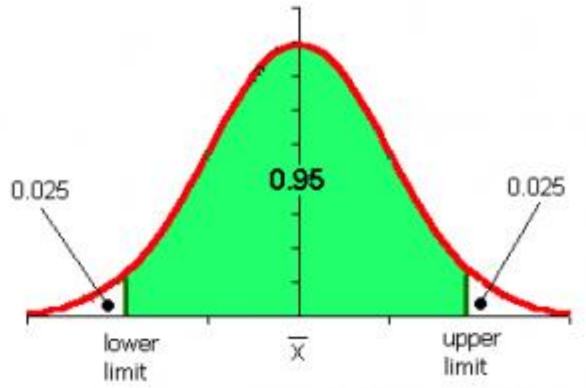
- A ***confidence interval*** gives an estimated range of values which is likely to include an unknown population parameter, the estimated range being calculated from a given set of sample data.

- The ***level C*** of a confidence interval gives the probability that the interval produced by the method employed includes the true value of the parameter .

- The higher the level C, the greater confidence that the true value of the population parameter will be included in the established interval.

The 95% Confidence Interval Explained

A 95% confidence interval gives you a very specific set of numbers for your confidence level. For example, let's suppose you were surveying a local school to see what the student's state test scores are. You set a 95% confidence level and find that the 95% confidence interval is (780,900). That means if you repeated this over and over, 95 percent of the time the scores would fall somewhere between 780 and 900.



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Assumptions behind our Confidence Intervals

- The sample was randomly selected (independence assumption).
- The sample size is large enough to insure that the sampling distribution of the sample means is normally distributed.
- There are no outliers (extreme high or low values).

Degrees of freedom (df)

- Degrees of freedom of an estimate is the number of independent pieces of information that went into calculating the estimate.

- The number of degrees of freedom is df = number of observations in sample – number of parameters estimated from the sample.

You could use 4 people, giving 3 degrees of freedom (4 – 1 = 3)

Degrees of Freedom: Two Samples

If you have two <u>samples</u> and want to find a <u>parameter</u>, like the <u>mean</u>, you have two “n” s to consider (sample 1 and sample 2). Degrees of freedom in that case is:

Degrees of Freedom (Two Samples): $(N_1 + N_2) - 2$.

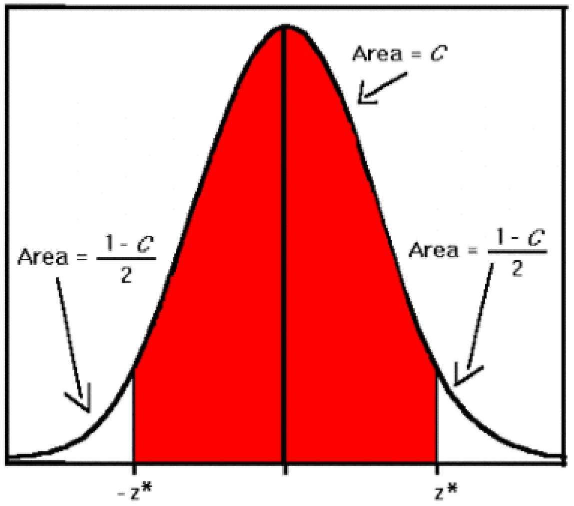
Example

Suppose a student measuring the boiling temperature of a certain liquid observes the readings (in degrees Celsius) 102.5, 101.7, 103.1, 100.9, 100.5, and 102.2 on 6 different samples of the liquid. He calculates the sample mean to be 101.82. If he knows that the standard deviation for this procedure is 1.2 degrees, what is the confidence interval for the population mean at a 95% confidence level?

In other words, the student wishes to estimate the true mean boiling temperature of the liquid using the results of his measurements. If the measurements follow a normal distribution, then the sample mean will have the distribution $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$. Since the sample size is 6, the standard deviation of the sample mean is equal to $\frac{\sigma}{\sqrt{n}}$= 1.2/sqrt(6) = 0.49.

Confidence Intervals for Unknown Mean and Known Standard Deviation (z-score)

For a population with unknown mean μ and known standard deviation σ , **a confidence interval for the population mean**, based on a simple random sample (SRS) of size *n*, is $\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$, where z^* is the upper (1-*C*)/2 critical value for the standard normal distribution.



z = (x – μ) / (σ / √n)

±z = (x – μ) / (σ / √n)

u = x ± z(σ / √n)

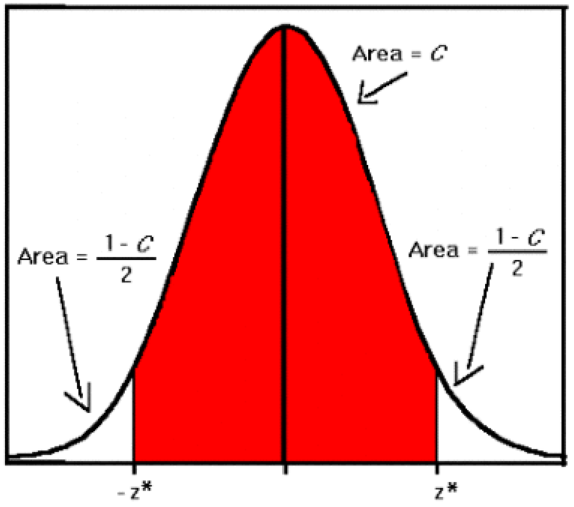
$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

$$z = \frac{X - \mu}{\sigma}$$

$$\text{sample mean} = N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

In the example above, the student calculated the sample mean of the boiling temperatures to be 101.82, with standard deviation 0.49. The critical z value for a 95% confidence interval is 1.96, where (1-0.95)/2 = 0.025. A 95% confidence interval for the unknown mean is ((101.82 - (1.96*0.49)), (101.82 + (1.96*0.49))) = (101.82 - 0.96, 101.82 + 0.96) = (100.86, 102.78).

95% confident intervals: (100.86, 102.78)



$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

As the level of confidence decreases, the size of the corresponding interval will decrease. Suppose the student was interested in a 90% confidence interval for the boiling temperature. In this case, $C = 0.90$, and $(1-C)/2 = 0.05$. The critical value z^* for this level is equal to 1.645, so the 90% confidence interval is ((101.82 - (1.645*0.49)), (101.82 + (1.645*0.49))) = (101.82 - 0.81, 101.82 + 0.81) = (101.01, 102.63)

An increase in sample size will decrease the length of the confidence interval without reducing the level of confidence. This is because the standard deviation decreases as *n* increases. The ***margin of error m*** of a confidence interval is defined to be the value added or subtracted from the sample mean which determines the length of the interval: $\boldsymbol{m = z^{*}\frac{\sigma}{\sqrt{n}}}$, therefore n is calculated in this way:

$$n = \left(\frac{z^{*}\sigma}{m}\right)^{2}$$

Suppose in the example above, the student wishes to have a margin of error equal to 0.5 with 95% confidence. Substituting the appropriate values into the expression for *m* and solving for *n* gives the calculation $n = (1.96*1.2/0.5)^2 = (2.35/0.5)^2 = 4.7^2 = 22.09$. To achieve a 95% confidence interval for the mean boiling point with total length less than 1 degree, the student will have to take 23 measurements.

Confidence Intervals for Unknown Mean and Unknown Standard Deviation (t-score)

In most practical research, the standard deviation for the population of interest is not known. In this case, the standard deviation σ is replaced by the estimated standard deviation *s*, also known as the ***standard error***. Since the standard error is an estimate for the true value of the standard deviation, the distribution of the sample mean $\bar{x}$ is no longer normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

Instead, the sample mean follows the ***t distribution*** with mean μ and standard deviation $\frac{s}{\sqrt{n}}$.

The *t* distribution is also described by its ***degrees of freedom***. **For a sample of size *n*, the *t* distribution will have *n-1* degrees of freedom.** The notation for a *t* distribution with *k* degrees of freedom is *t(k)*. As the sample size *n* increases, the *t* distribution becomes closer to the normal distribution, since the standard error approaches the true standard deviation $\sigma$for large *n*.

For a population with unknown mean μ and unknown standard deviation , a confidence interval for the population mean, based on a simple random sample (SRS) of size *n*, is $\bar{x} \pm t^* \frac{s}{\sqrt{n}}$, where *t is the upper (1-*C*)/2 critical value for the *t* distribution with *n-1* degrees of freedom, *t(n-1)*.**

The specific confidence interval formulas for the t-tests we have covered are as follows:

Same, only use different letters

$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$

$\bar{x}$ *is* M, $s_M = \frac{s}{\sqrt{n}}$

- For one-sample t-tests:

$\mathrm{LL} = \mathrm{M} - (\mathrm{t})(\mathrm{s_M})$ and $\mathrm{UL} = \mathrm{M} + (\mathrm{t})(\mathrm{s_M})$

- For two-sample t-tests, independent measures design:

$\mathrm{LL} = (\mathrm{M_1} - \mathrm{M_2}) - (\mathrm{t})(\mathrm{s_{M_1 - M_2}})$ and $\mathrm{UL} = (\mathrm{M_1} - \mathrm{M_2}) + (\mathrm{t})(\mathrm{s_{M_1 - M_2}})$

- For two-sample t-tests, repeated measures design:

$\mathrm{LL} = \mathrm{M_D} - (\mathrm{t})(\mathrm{s_{M_D}})$ and $\mathrm{UL} = \mathrm{M_D} + (\mathrm{t})(\mathrm{s_{M_D}})$

Find the critical value by the degree of freedom

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$$

Example

The dataset "Normal Body Temperature, Gender, and Heart Rate" contains 130 observations of body temperature, along with the gender of each individual and his or her heart rate.

t^* for 129 degrees of freedom. This value is approximately 1.962, the critical value for 100 degrees of freedom. The estimated standard deviation for the sample mean is 0.733/sqrt(130) = 0.064. A 95% confidence interval, then, is approximately ((98.249 - 1.962*0.064), (98.249 + 1.962*0.064)) = (98.249 - 0.126, 98.249+ 0.126) = (98.123, 98.375).

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Descriptive Statistics

Variable          N     Mean    Median  Tr Mean    StDev  SE Mean
TEMP            130   98.249    98.300   98.253    0.733    0.064

Variable        Min      Max        Q1       Q3
TEMP         96.300  100.800    97.800   98.700
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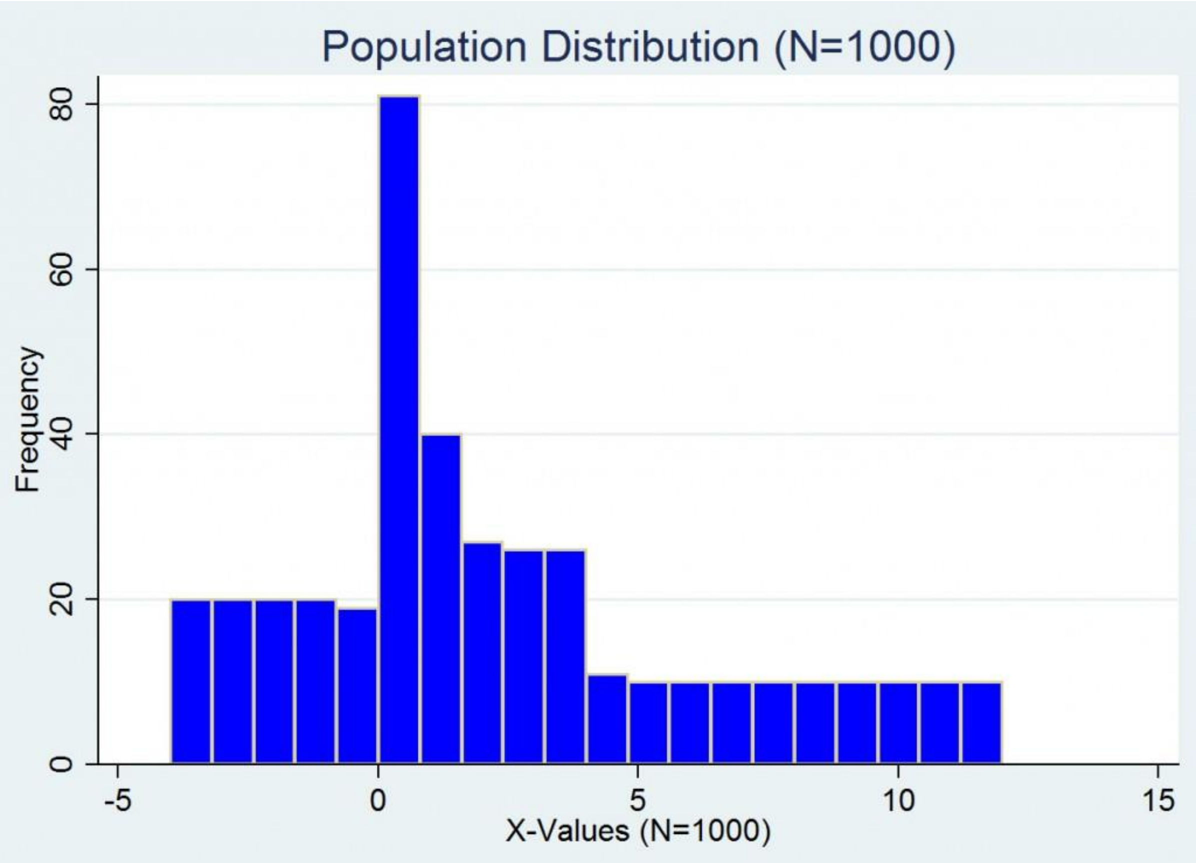
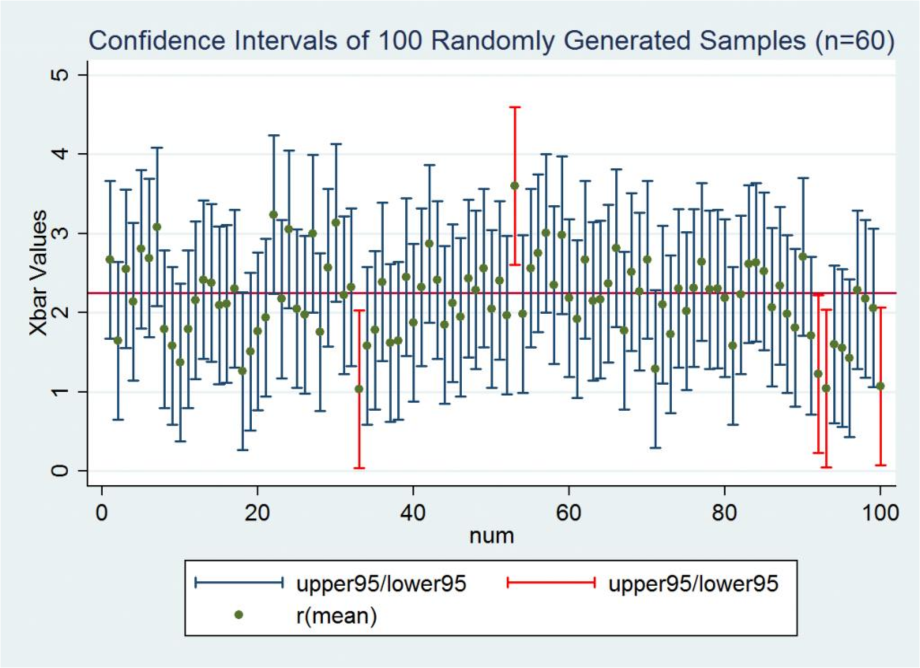
Interpretation through a Simulation



u=2.25

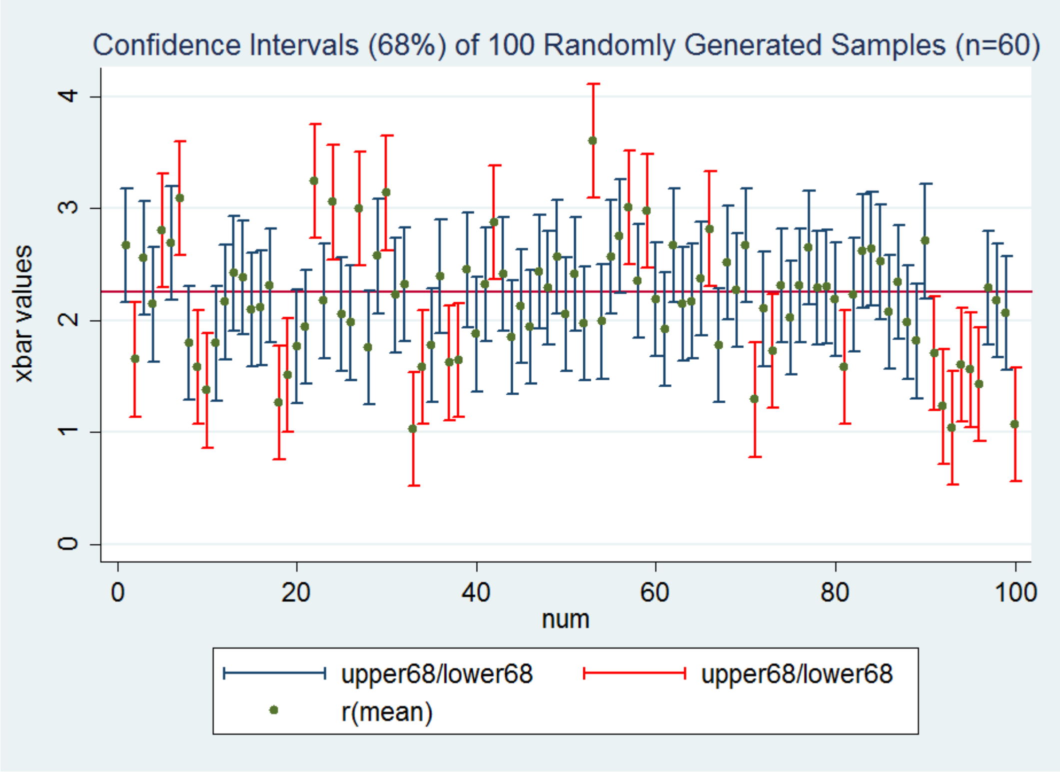
- Using the population distribution, we will now sample (size n = 60) 100 times from this distribution and calculate 100 distinct confidence intervals (95%)



- Calculate 68% confidence intervals:

As you can see the length of each interval (68%) has decreased in comparison to the 95% confidence intervals.

Because we have changed our multiplier (z) from 1.96 to 1.*



Example for
One Sample

Research Question. The average typing speed for the secretaries of a large company is 52 words per minute. A long time secretary has developed finger dexterity exercises which she reports have improved her speed dramatically. The owner of the company thus hires a researcher to test the effectiveness of the exercise program. After four weeks of training in the finger exercises, the typing speed of 17 secretaries is measured. The mean number of words per minute was M = 57 with SS = 3600.

a) What would the point estimate of μ be after using the finger exercise program?

b) Establish a 90% confidence interval for the mean and write a sentence of interpretation.

a) The point estimate of μ is $M = 57$.

b) $LL = M - (t)(s_M)$ and $UL = M + (t)(s_M)$

To use the formulas for establishing confidence intervals, we need values for M, t, and s_M. The value for $M = 57$. For a 90% confidence interval with $df = 16$, the value for $t = \pm 1.746$. We now have to calculate the estimated standard error (s_M).

$$s = \sqrt{\frac{SS}{n-1}} = \sqrt{\frac{3600}{17-1}} = 15$$

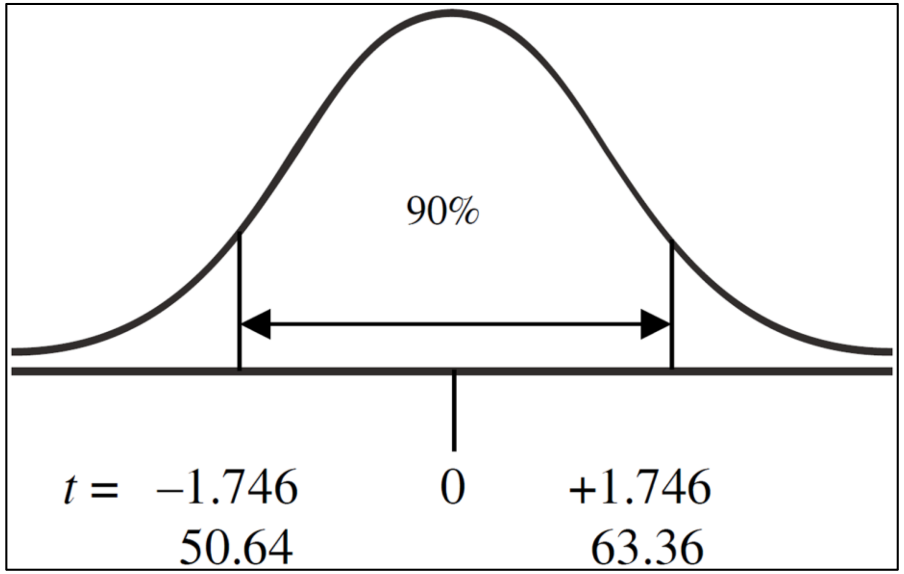
$$s_M = \frac{s}{\sqrt{n}} = \frac{15}{\sqrt{17}} = 3.64$$

$LL = 57 - (1.746)(3.64)$
$= 57 - 6.36$
$= 50.64$

$UL = 57 + (1.746)(3.64)$
$= 57 + 6.36$
$= 63.36$

Interpretation. We can be 90% confident that the population mean (after the finger dexterity program) would be between 50.64 and 63.36.

This is what the t-distribution looks like for this interval:



The t-values of ±1.746 form the boundaries of the middle 90% of the distribution when df = 16.

Example for
Two Samples, Independent Samples Design

Research Question. A high school counselor has developed a pamphlet of memory-improving techniques designed to help students in preparing for exams. One group of students is given the pamphlets and instructed to practice the techniques for three weeks, after which their memories are tested. The memories of another group of students, who did not receive the pamphlets, are also tested at this time.

Memory Pamphlet	No Pamphlet Group
$n_1 = 36$	$n_2 = 40$
$M_1 = 79$	$M_2 = 68$
$SS_1 = 478$	$SS_2 = 346$

a) Make a point estimate of how much improvement in memory results from following the techniques.

b) Establish a 90% confidence interval around the value for the difference between means and write a sentence of interpretation.

a) The point estimate for the population difference between means ($\mu_1 - \mu_2$) is: $M_1 - M_2 = 79 - 68 = 11$.

b) $LL = (M_1 - M_2) - (t)(s_{M_1-M_2})$ and $UL = (M_1 - M_2) + (t)(s_{M_1-M_2})$

$$s_{M_1-M_2} = \sqrt{\left(\frac{SS_1 + SS_2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$= \sqrt{\left(\frac{478 + 346}{36 + 40 - 2}\right)\left(\frac{1}{36} + \frac{1}{40}\right)}$$

$$= .75$$

$$\boxed{\begin{aligned} LL &= 11 - (1.671)(.75) \\ &= 9.75 \end{aligned}}$$

and

$$\boxed{\begin{aligned} UL &= 11 + (1.671)(.75) \\ &= 12.25 \end{aligned}}$$

Interpretation. We can be 90% confident that the difference between population means would be between 9.75 and 12.25.

Example for
Two Sample t Test, Repeated Measures Design

Research Question: A promising math tutoring program has been developed by a senior math major who also tutors students. His math professors are impressed and decide to test its effectiveness. N = 45 students who were struggling in math in the Spring semester were given a comprehensive math test at the end of the semester. Over the summer, they were tutored in the new program and then re-tested on an alternate form of the test. The mean for the sample of difference scores was $M_D = 12$ with SS = 1648.

a) Make a point estimate of μ_D.

b) Establish a 90% confidence interval around the difference value and write a sentence of interpretation.

a) The point estimate for μ_D is $M_D = 12$

b) $LL = M_D - (t)(s_{M_D})$ and $UL = M_D + (t)\left(s_{M_D}\right)$

But first, we need the standard deviation and estimated standard error of the mean difference.

$$s_D = \sqrt{\frac{SS}{n-1}} = \sqrt{\frac{1648}{45-1}} = 6.12$$

$$s_{M_D} = \frac{s_D}{\sqrt{n}} = \frac{6.12}{\sqrt{45}} = .91$$

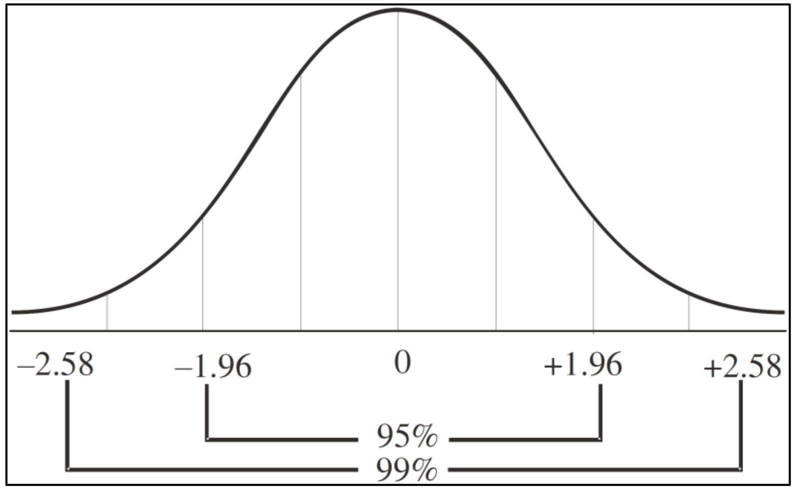
$$\begin{aligned} LL &= 12 - (1.684)(.91) \\ &= 12 - 1.53 \\ &= 10.47 \end{aligned}$$

$$\begin{aligned} UL &= 12 = (1.684).91) \\ &= 12 + 1.53 \\ &= 13.53 \end{aligned}$$

<u>Interpretation</u>. We can be 90% confident that the increase in scores will be between 10.47 and 13.53.

Degree of Confidence versus Degree of Specificity

- When we increase the level of confidence, we lose specificity. The range of values becomes broader and is thus more likely to include the true population parameters.
- Conversely, when we reduce the level of confidence, we gain specificity but lose confidence in our estimation.



The 95% confidence level establishes a narrower (and more specific) interval of values while the 99% level provides a wider (and less specific) interval of values.