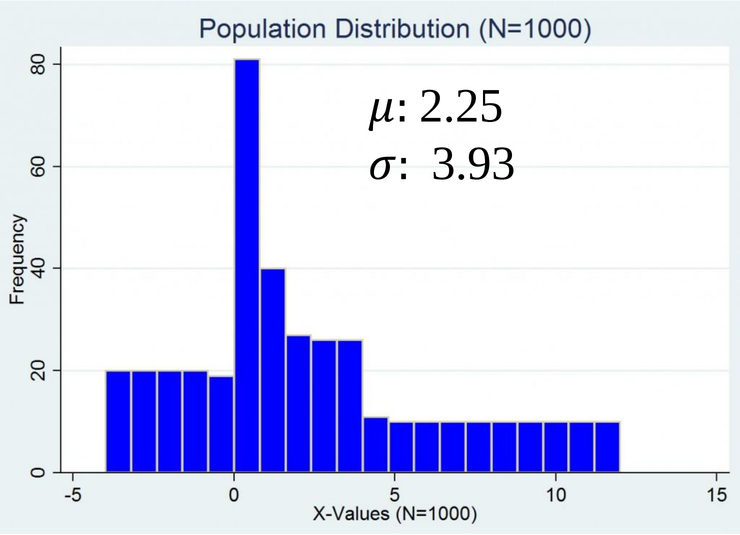
The Central Limit Theorem, z-Score and t-Score

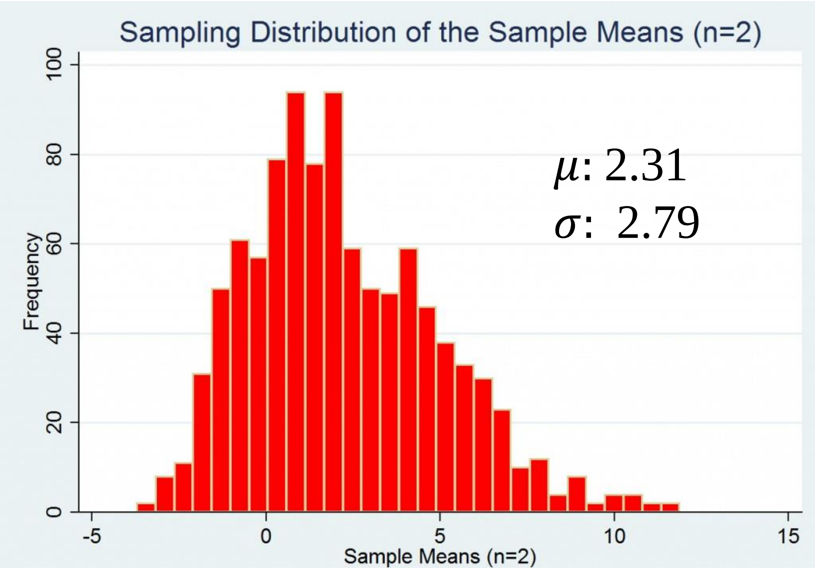
Central Limit Theorem

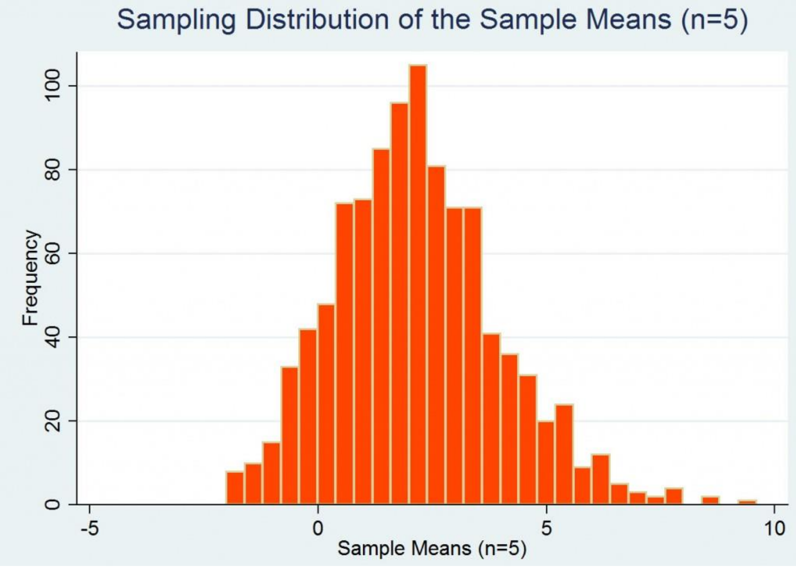
- What happens when the sample comes from a population that is not normally distributed? This is where the Central Limit Theorem comes in.
- The Central Limit Theorem applies to a sample mean from any distribution. As long as the sample size is large, the distribution of the sample means will follow an approximate Normal distribution.

The Central Limit Theorem!

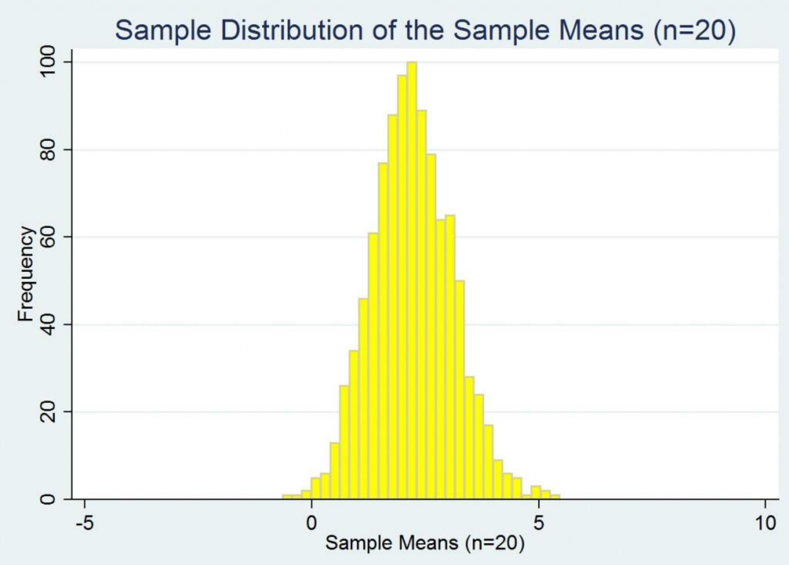
- The essence of the Central Limit Theorem: As the sample size increases, the sampling distribution of the sample mean (x bar) concentrates more and more around μ (the population mean).
- The shape of the distribution also gets closer and closer to the normal distribution as sample size n increases.

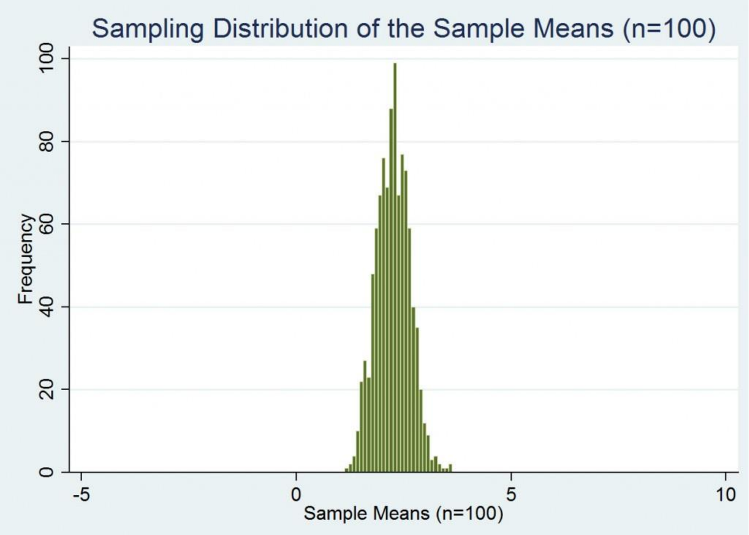






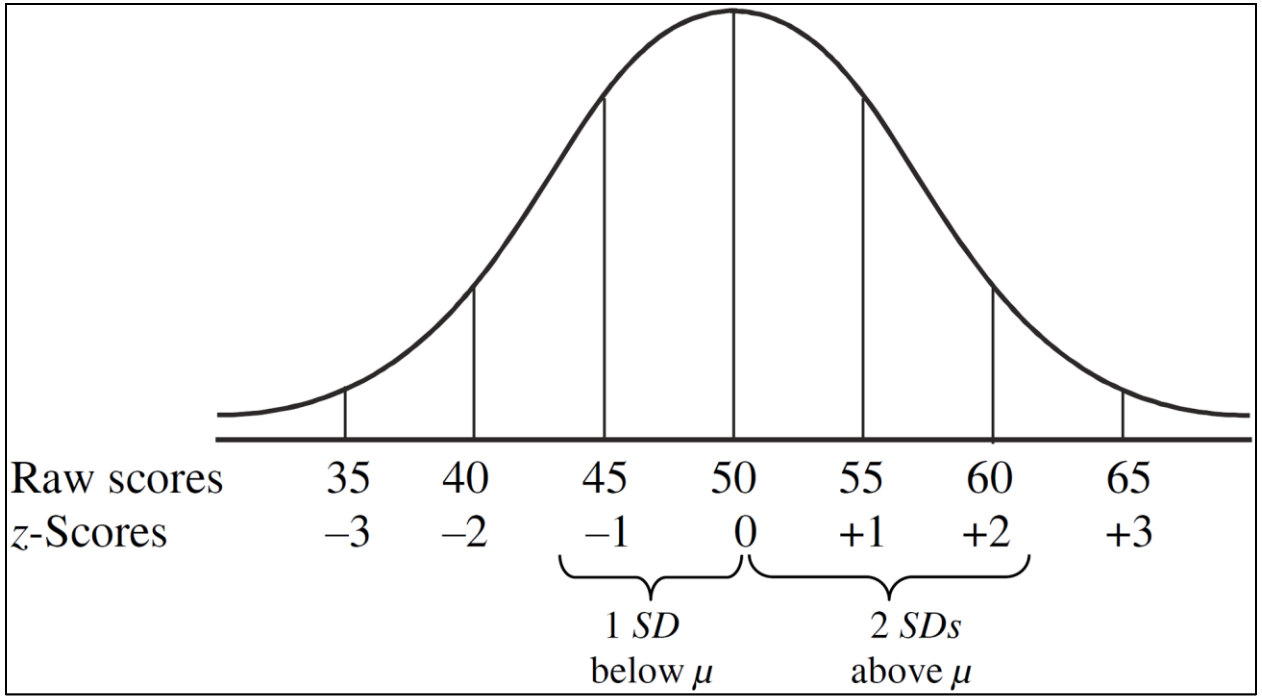
As you can see, as sample size increases, the distribution gets increasingly narrow and increasingly approaches a normal distribution. This is the essence of the **Central Limit Theorem**.





Example is from Elizabeth A. Albright, PhD, A Croaker from the Neuse River at Minesott Beach, NC.

Relationship between raw scores and z-scores



A **z-score distribution** has a mean of zero and a standard deviation of one.

a standard normal distribution

Converting Raw Scores to Z-Scores

- Raw scores need comparative information to be meaningful.
  - They are best interpreted in light of the mean and standard deviation.
  - *z*-scores help us determine the relative standing of a raw score in comparison to the rest of the group.

<u>*z*-Score Formula</u>

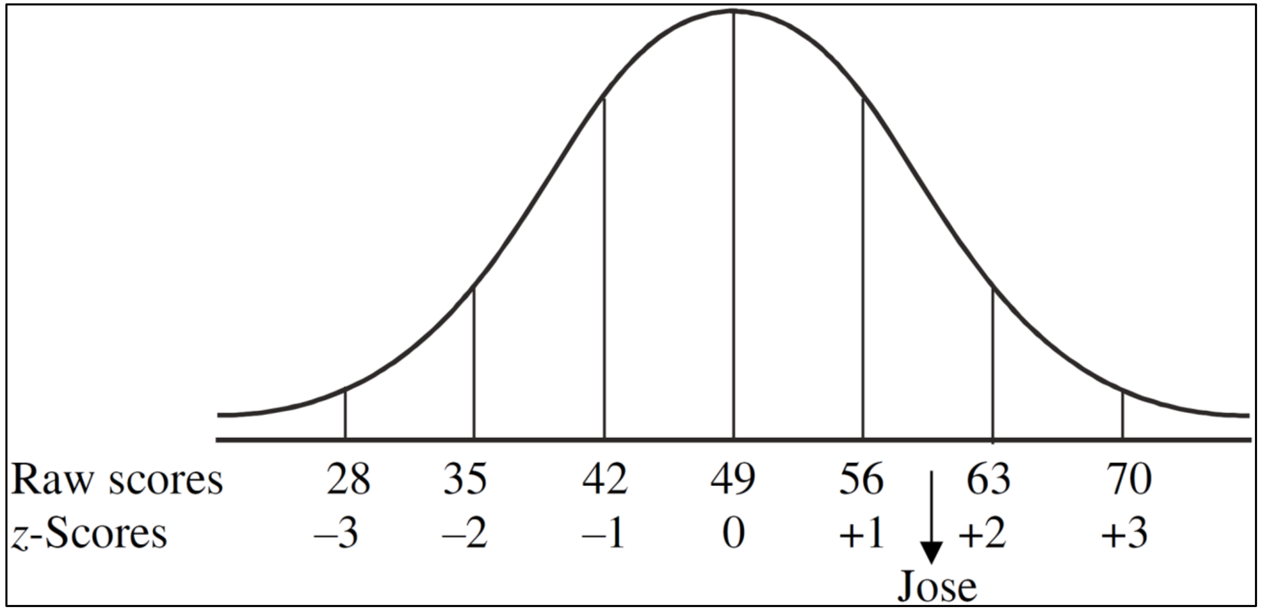
$$z = \frac{X - \mu}{\sigma}$$

This formula tells us to find out how much a particular raw score deviates from the mean in standard deviation units.

For Example,

Jose scored 60 on his history test. The class mean was 49 and the standard deviation was 7. What is Jose’s equivalent z-score?

$$z = \frac{60 - 49}{7} = +1.57$$



Using z-scores to compare scores from different distributions.

Example,

Jason wants two master's degrees, one in psychology and one in business administration. He, thus, had to take two departmental entrance exams. His score for the psychology exam was 109. The mean for all psychology applicants this year was 93 with a standard deviation of 12. Jason's score in business administration was 56. The mean for this group was 52 with a standard deviation of 6. On which test was Jason's performance superior?

Psychology

$$z = \frac{109 - 93}{12} = +1.33$$

Business Admin.

$$z = \frac{56 - 52}{6} = +.67$$

We can also determine raw scores from z-scores, using the formula below:

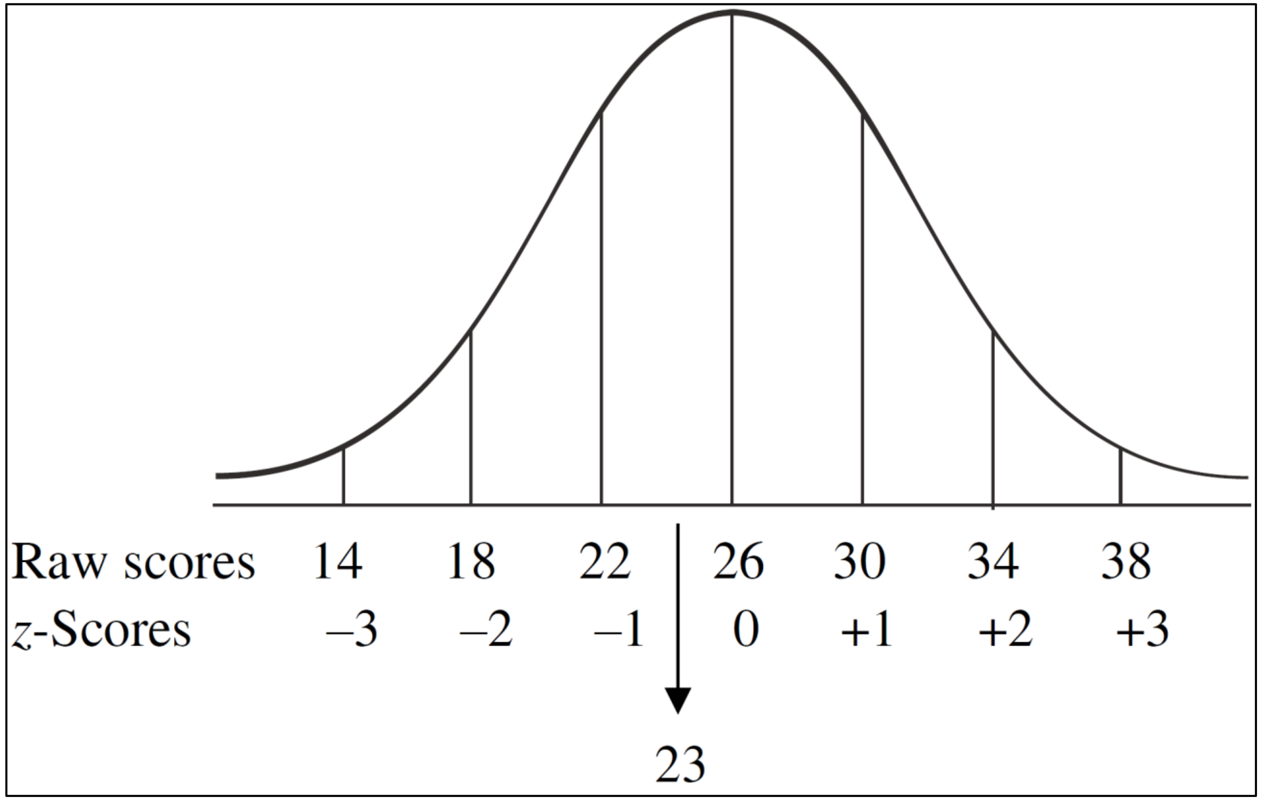
$$X = \mu + z(\sigma)$$

For Example,

Given: $z = -.75$
$\sigma = 4$
$\mu = 26$

What is the equivalent raw score?

$$\boxed{\begin{aligned} X &= \mu + z(\sigma) \\ &= 26 + (-.75)(4) \\ &= 23 \end{aligned}}$$



A z-score of −.75 translates into a raw score of 23 for this distribution.

Standard scores

- are scores expressed relative to a specified mean and standard deviation.
- *z*-scores specify a mean of 0 and a standard deviation of 1.
- But, z-scores can be awkward to work with because:
  - half of the scores will be negative.
  - and most will require decimals.

Other Standard Scores

- *z*-scores can be transformed into other types of standard scores.
- Doing so does not alter shape of the distribution or relative position of the scores.

z-score	$\mu = 0$	$\sigma = 1$
Wechsler IQ	$\mu = 100$	$\sigma = 15$
SAT subscales	$\mu = 500$	$\sigma = 100$
T-score	$\mu = 50$	$\sigma = 10$

Transformed Standard Scores (TSS)

Z-scores can be transformed into any standard scores in which a new mean and standard deviation are specified.

$$TSS = \text{specified } \mu + z(\text{specified } \sigma)$$

Hint: Raw scores will first have to be converted to z-scores before using this formula.

For Example,

<u>Wechsler IQ (μ = 100, σ = 15)</u>

Suppose a class of students taking a course in psychological measurement took the Wechsler IQ test. The class mean was 102 and the standard deviation was 12. Conrad's raw score was 98. What is his equivalent standardized Wechsler IQ score?

Raw scores first have to be converted to *z*-scores

$$z = \frac{98 - 102}{12} = -.33$$

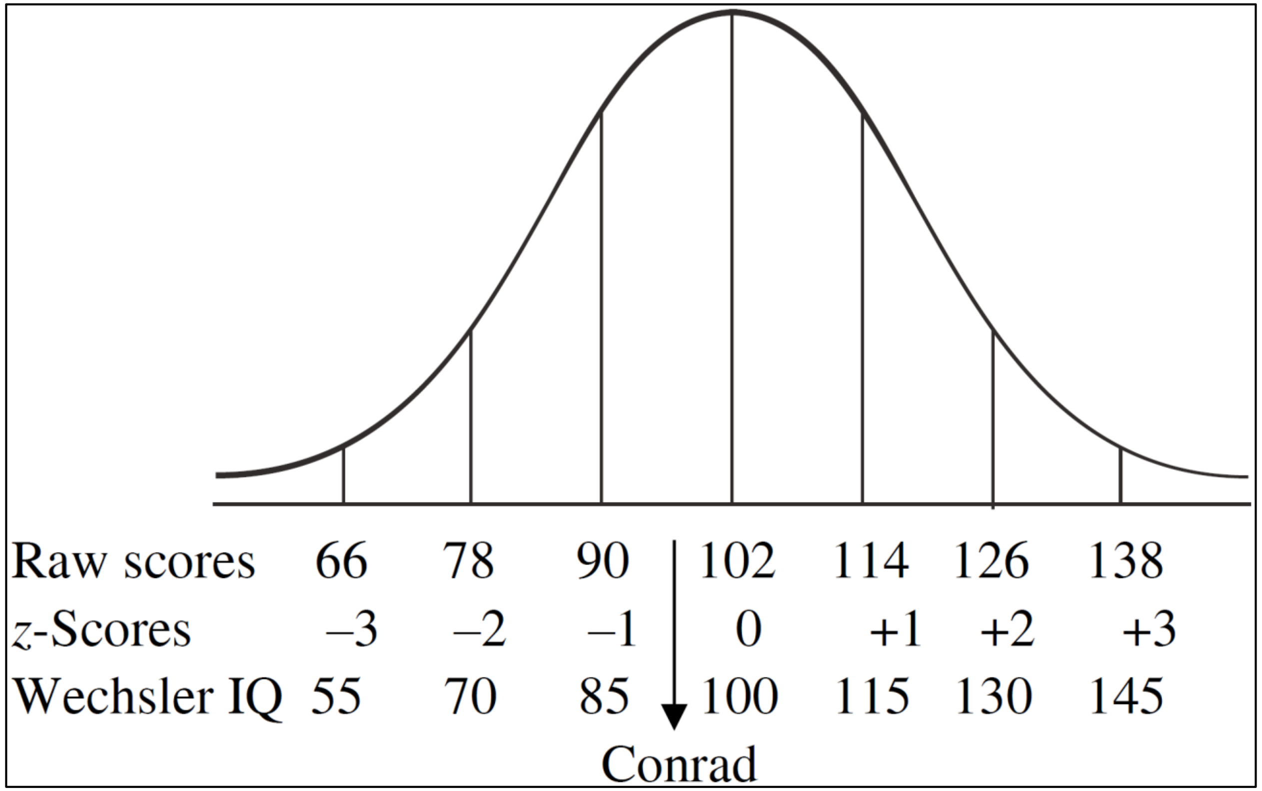
specified μ

specified σ

Wechsler IQ = 100 + (−.33)(15)

= 95.05

Conrad's position in the distribution has not changed.



SAT (μ = 500, σ = 100)

For a given distribution of scores with a mean of 148 and a standard deviation of 18, Najma scored 166. What is her equivalent standardized SAT score?

$$z = \frac{166 - 148}{18} = +1.00$$

SAT = 500 + 1.00(100)
= 600

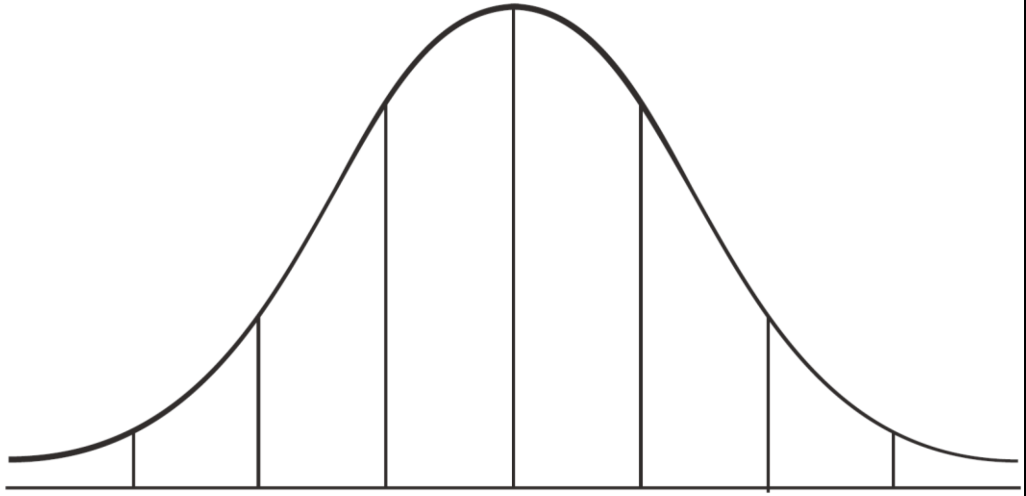
T-Scores (μ = 50, σ = 10)

For a given distribution of scores with a mean of 60 and a standard deviation of 5, Marie scored 78. What is her equivalent T-score?

$$z = \frac{78 - 60}{5} = +3.60$$

T = 50 + 3.60(10)
= 86

Relationship between the transformed scores



z-Scores	–3	–2	–1	0	+1	+2	+3
Wechsler IQ	55	70	85	100	115	130	145
SAT	200	300	400	500	600	700	800
T-scores	20	30	40	50	60	70	80

Z-Scores

Standard deviation

- Average amount that an entire distribution of scores deviates from the mean.

Z-scores

- How far a particular raw score deviates from the mean in standard deviation units.
- Positive z-scores reflect deviations above the mean.
- Negative z-scores reflect deviations below the mean.

T-score

Student's t distribution (t distribution)

- Similar to the normal distribution
- Normal distribution: more scores in the center
- t distribution: more in the tails.
- t distribution is *leptokurtic*.
- t distribution approaches the normal distribution as the degrees of freedom increase.

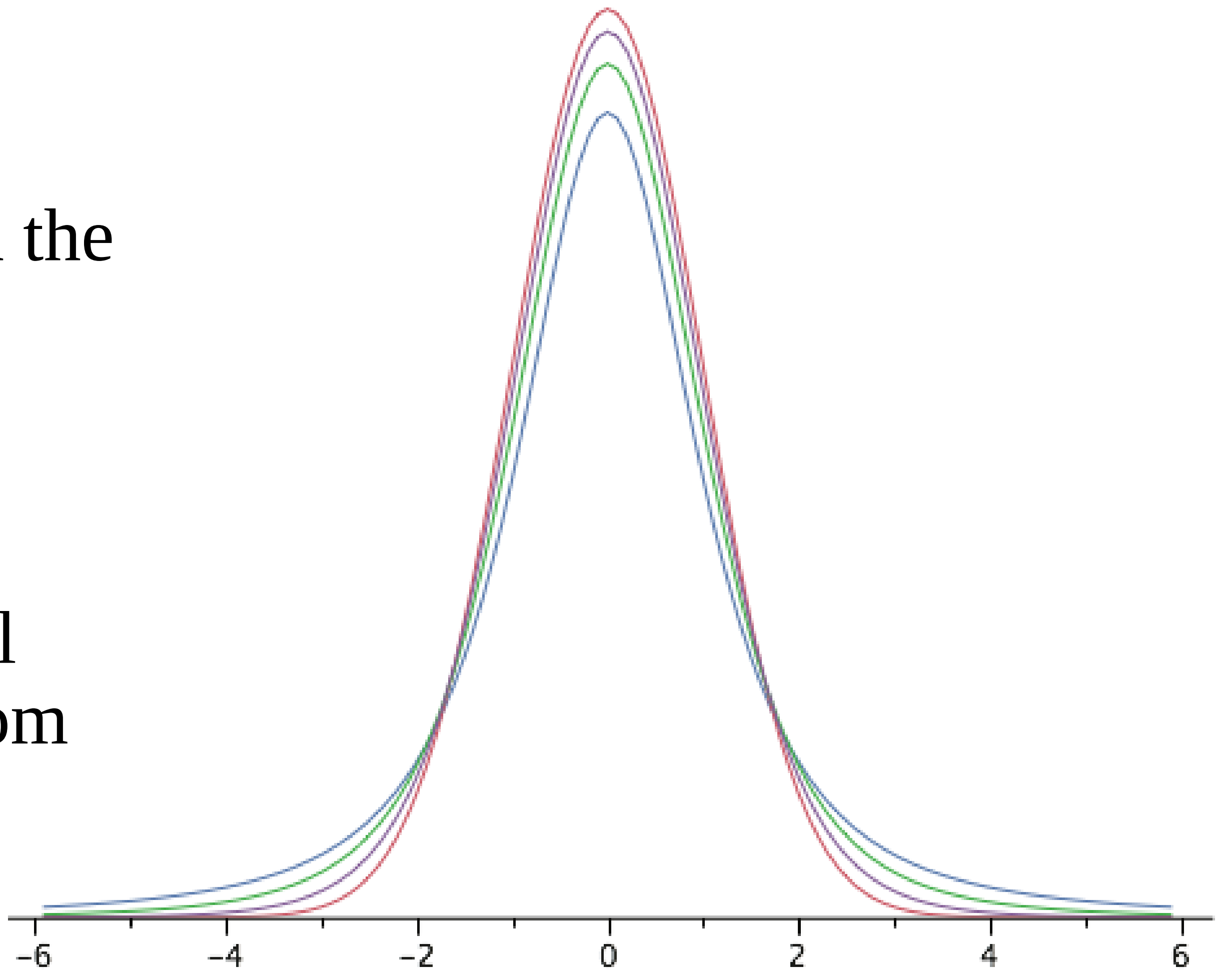


Figure 1. A comparison of t distributions with 2, 4, and 10 df and the standard normal distribution. Th distribution with the highest peak is the 2 df distribution, the next highest is 4 df, the highest after tha is 10 df, and the lowest is the standard normal distribution.

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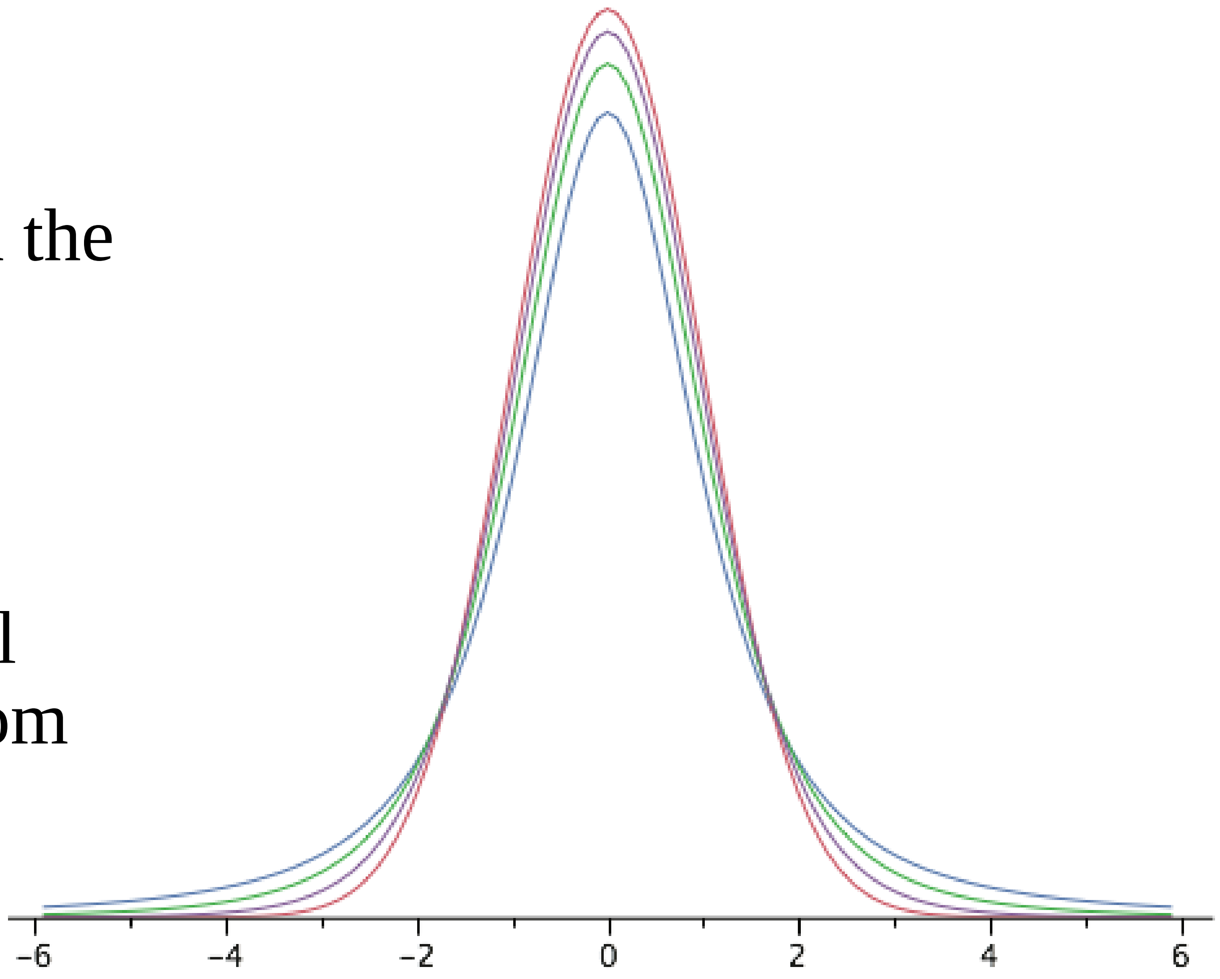

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z-Score Formula

$$z = \frac{X - \mu}{\sigma}$$

The t-Statistic

- We previously, used the z-statistic in testing hypotheses about μ.
- However, the z-test requires knowing σ.
- if we don't know σ, we can estimate the population standard deviation using n–1.

$$s = \sqrt{\frac{SS}{n-1}}$$

This is the formula for the t-statistic:

$$t_{obt} = \frac{M - \mu}{s_M}$$

instead of using $\sigma_M = \frac{\sigma}{\sqrt{n}}$

we use $s_M = \frac{s}{\sqrt{n}}$

$$t_{obt} = \frac{M - \mu}{s_M}$$

Plug the information into the formula and solve:

M = <u>sample mean</u> = 280

μ_0 = <u>population mean</u> = 300

s = sample standard deviation = 50

n = <u>sample size</u> = 15

t = (280 – 300)/ (50/√15) = -20 / 12.909945 = -1.549.

The t-Distribution

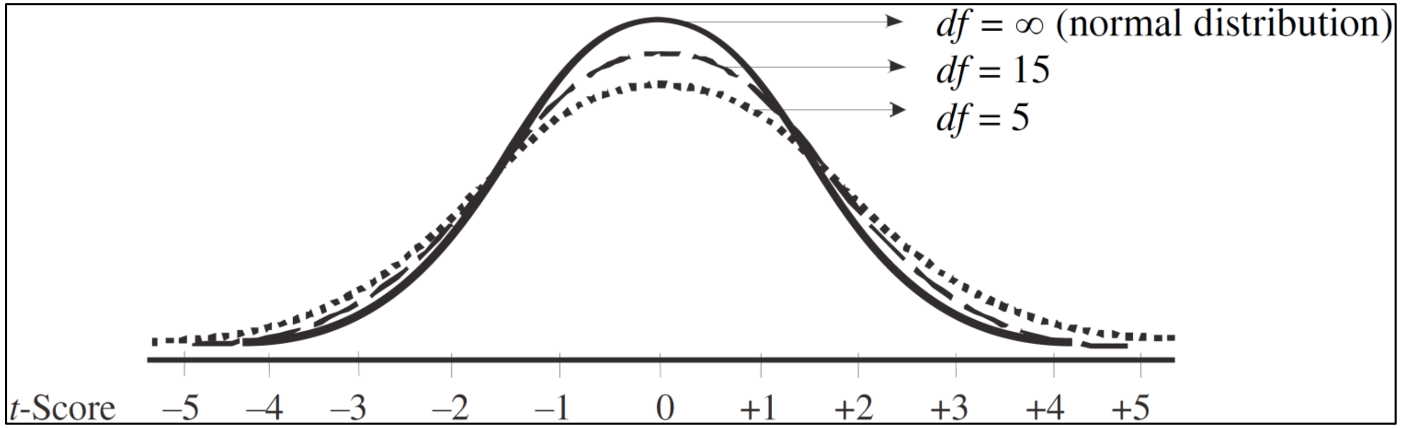
- The **t-distribution** is theoretical, symmetrical and bell-shaped.
- However, with small samples the curve is shaped differently.
- Is actually a family of curves, one for each sample size.
- The particular t-distribution that we use will be based on the degree of freedom associated with the sample.

When using the t-distribution for a single sample, df will also be n-1.

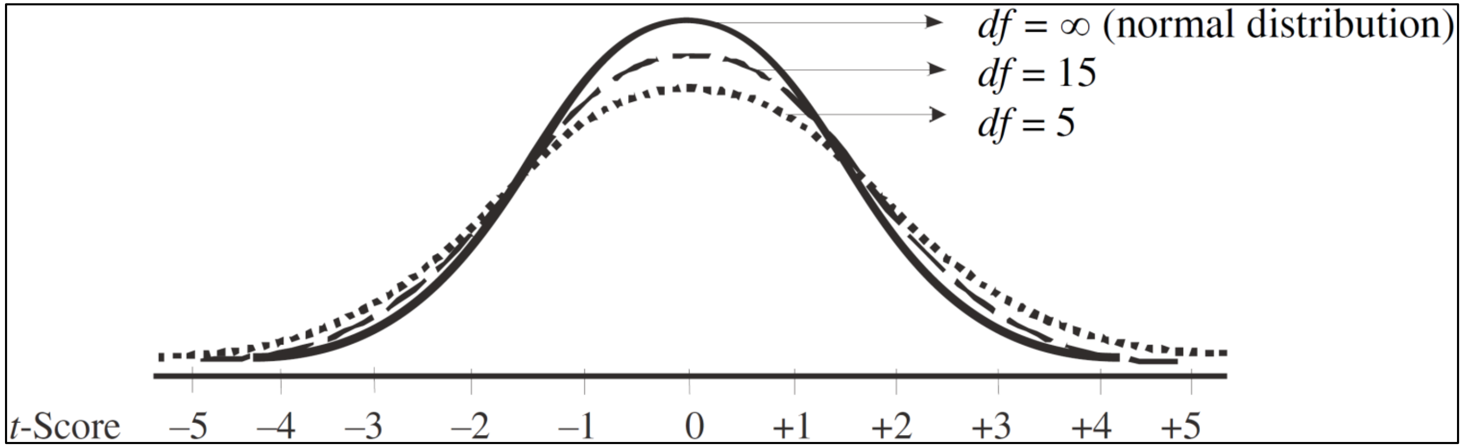
For example, if your sample size is 16, df will be 15.

Shape of the t-distribution

Illustration for 3 different df → 5, 15, and infinity



- The curve is flatter with smaller df and the t-values extend further into the tails.
- As df increase, the t-distribution looks more and more like the normal distribution.
- When df are infinite, there is no difference in shape between the t-curve and the normal distribution curve.



Since smaller samples result in a distribution that is flatter and more spread out, larger t-values will be required to reject H_0 (i.e., we have to go further into the tails to reach significance).

The t-Distribution Table

Level of significance for one-tailed test						
	.10	.05	.025	.01	.005	.0005
Level of significance for two-tailed test						
df	.20	.10	.05	.02	.01	.001
1	3.708	6.314	12.706	31.821	63.657	636.619
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460

- If the df for your particular research problem are not shown, use the critical values associated with the *next lowest* df.
- For a t test, it is not necessary to divide the proportion associated with the alpha level in half for two-tailed tests. The t-table already accommodates for this.

Degrees of freedom (df)

- Degrees of freedom of an estimate is **the number of independent pieces of information that went into calculating the estimate**.

The number of degrees of freedom (df) = number of observations in sample – number of parameters estimated from the sample.

- N: the number of items in the sample and only one parameter estimated from the samples

  Df=N-1

- For example, you were finding the mean weight loss for a low-carb diet. You could use 4 people, giving 3 degrees of freedom (4 – 1 = 3), or you could use one hundred people with df = 99.

Degrees of Freedom: Two Samples

If you have two samples and want to find a parameter, like the mean, you have two “n” s to consider (sample 1 and sample 2). Degrees of freedom in that case is:

Degrees of Freedom (Two Samples): $(N_1 + N_2) - 2$.

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The number of degrees of freedom (df) = number of observations in sample – number of parameters estimated from the sample.

- Since the t distribution is leptokurtic, the percentage of the distribution within 1.96 standard deviations of the mean is less than the 95% for the normal distribution.
- Table 1 shows the number of standard deviations from the mean required to contain 95% and 99% of the area of the t distribution for various degrees of freedom. These are the values of t that you use in a confidence interval. The corresponding values for the normal distribution are 1.96 and 2.58 respectively.