6.4 Specialization to parabolic problems

When solving parabolic problems, we can use the methods described in the previous sections to solve the resulting discrete elliptic systems. However, we have some additional structures that we can exploit to improve the efficiency of the solvers.

- If a constant time step size is used, we can compute the factorization of the system matrix once and reuse it in each time step if a direct solver is used.
- If an iterative solver is used, we can use the solution of the previous time step as an initial guess for the current time step. This is particularly useful if the solution changes only slightly between two time steps. For higher-order methods, there have been efforts to design even better starting values for the iterative solvers.
- If an error estimate of the time integration method is available, we can adapt the tolerance of the iterative solvers accordingly. Typically, one would like to have an error of the linear system solvers that is smaller than the error of the time integration method — but not too small to avoid unnecessary computational effort. For example, a typical choice is to have an error of the linear system solvers that is one order of magnitude smaller than the error of the time discretization method.

Moreover, there are some specialized methods for parabolic problems, e.g., the *alternating direction implicit (ADI)* method. We describe the version introduced by

D. W. Peaceman and H. H. Rachford Jr. "The numerical solution of parabolic and elliptic differential equations." In: *Journal of the Society for industrial and Applied Mathematics* 3.1 (1955), pp. 28–41. DOI: 10.1137/0103003.

Consider a regular finite difference grid in 3D. Then, we have an ODE of the form

$$\partial_t \underline{u} = A_1 \underline{u} + A_2 \underline{u} + A_3 \underline{u},$$

where $A_i \approx \partial_{x_i}^2$. The ADI method splits the time integration into three steps:

$$\begin{aligned}
\frac{\underline{u}^{n+1,x} - \underline{u}^n}{\Delta t} &= A_1 \underline{u}^{n+1,x} + A_2 \underline{u}^n + A_3 \underline{u}^n, \\
\frac{\underline{u}^{n+1,y} - \underline{u}^{n+1,x}}{\Delta t} &= A_1 \underline{u}^{n+1,x} + A_2 \underline{u}^{n+1,y} + A_3 \underline{u}^n, \\
\frac{\underline{u}^{n+1} - \underline{u}^{n+1,y}}{\Delta t} &= A_1 \underline{u}^{n+1,x} + A_2 \underline{u}^{n+1,y} + A_3 \underline{u}^{n+1}.
\end{aligned}$$

Each step can be solved efficiently by tridiagonal solver. However, the method is restricted to regular grids and second-order finite differences in space to get the tridiagonal structure. Moreover, the method is also only second-order accurate in time.

There are also some explicit methods that have been designed specifically for parabolic problems, e.g., *Runge-Kutta Chebyshev (ROCK)* methods developed in

B. P. Sommeijer, L. F. Shampine, and J. G. Verwer. "RKC: An explicit solver for parabolic PDEs." In: *Journal of Computational and Applied Mathematics* 88.2 (1998), pp. 315–326. DOI: 10.1016/S0377-0427(97)00219-7

A. Abdulle and A. A. Medovikov. "Second order Chebyshev methods based on orthogonal polynomials." In: *Numerische Mathematik* 90 (2001), pp. 1–18. DOI: 10.1007/s002110100292

A. Abdulle. "Fourth order Chebyshev methods with recurrence relation." In: *SIAM Journal on Scientific Computing* 23.6 (2002), pp. 2041–2054. DOI: 10.1137/S1064827500379549