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1. a) minimum MSE when predicted value of Y is $E(Y)$

$$E(Y|X=\tfrac{1}{2}) = E\left(\frac{f(x,y)}{f_X(x)}\right)$$

$$f_X(x) = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$$

$$= E\left(\frac{x+y}{x+\frac{1}{2}}\right)$$

$$= E\left(y + \tfrac{1}{2}\right)$$

$$\boxed{\hat{y} = \frac{7}{12}}$$

$$= \int_0^1 y\left(y+\tfrac{1}{2}\right)dy$$

$$= \frac{y^3}{3} + \frac{y^2}{4}\Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

b) MSE is $Var(Y|X)$

$$Var(Y|X) = E(Y^2|X) - \underbrace{E(Y|X)}_{7/12}{}^2$$

$$E(Y^2|X_{x=\frac{1}{2}}) = \int_0^1 y^2\left(y+\tfrac{1}{2}\right)dy = \frac{y^4}{4} + \frac{y^3}{6}\Big|_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$Var(Y|X=\tfrac{1}{2}) = \frac{5}{12} - \left(\frac{7}{12}\right)^2$$

$$= \frac{60}{144} - \frac{49}{144} = \frac{11}{144}$$

$$\cancel{Var(Y)} = \boxed{Var(Y|X=\tfrac{1}{2}) = \frac{11}{144}}$$

2. X = num failed components

$$P(X \geq 1) = 1-(.8)^{10} = .893$$

$$P(X \geq 2) = 1-(.8^{10} + 10(.2)(.8)^9) = .625$$

$$P(X \geq 2 \mid X \geq 1) = \frac{P(X \geq 2 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{.625}{.893} \approx \boxed{.700}$$

3. X = num diseased children

$$P(X \geq 1) = 1-(1-p)^n \qquad f(x) = p$$

$$P(X=x \mid X \geq 1) = \frac{f(X)}{1-(1-p)^n}$$

$$E(X=x \mid X \geq 1) = \sum_{x \geq 1}^{n} \frac{x f(x)}{1-(1-p)^n}$$

$$= \frac{1}{1-(1-p)^n} \sum_{x=1}^{n} x f(x) \qquad \sum_{x=1}^{n} x f(x) = E(X) = np$$

$$\boxed{E(X=x \mid X \geq 1) = \frac{np}{1-(1-p)^n}}$$

4. $$P_r(X_1) = {}_{n_1}C_x\, p^x (1-p)^{n_1-x}$$

$$P_r(X_2) = {}_{n_2}C_x\, p^x (1-p)^{n_2-x}$$

$$P_r(X_1+X_2=k) = {}_{n_1+n_2}C_k\, p^k (1-p)^{n_1+n_2-k}$$

$$\cancel{P_r(X_1=x)=}$$

$$P_r(X_2 = k-x) = {}_{n_2}C_{k-x}\, p^{k-x} (1-p)^{n_2-(k-x)}$$

$$P_r(X_1 \mid X_1+X_2=k) = \frac{{}_{n_1}C_x \cancel{p^x (1-p)^{n_1-x}}\ {}_{n_2}C_{k-x} \cancel{p^{k-x} (1-p)^{n_2-(k-x)}}}{{}_{n_1+n_2}C_k \cancel{p^k (1-p)^{n_1+n_2-k}}}$$

$$= \frac{{}_{n_1}C_x\, {}_{n_2}C_{k-x}}{{}_{n_1+n_2}C_k} = \frac{\binom{n_1}{x}\binom{n_2}{k-x}}{\binom{n_1+n_2}{k}}$$

$$\boxed{P_r(X_1 \mid X_1+X_2=k) = \text{hypergeom}(x; n_1+n_2, k, n_1)}$$

$\square$

5. $P(X_1 = x) = \dfrac{\lambda_1^x e^{-\lambda_1}}{x!}$

$P(X_2 = x) = \dfrac{\lambda_2^x e^{-\lambda_2}}{x!}$

$P(X_1 + X_2 = k) = \dfrac{(\lambda_1+\lambda_2)^k e^{-(\lambda_1+\lambda_2)}}{k!}$

$P(X_2 = k-x) = \dfrac{\lambda_2^{k-x} e^{-\lambda_2}}{(k-x)!}$

$$P(X_1 = x \mid X_1 + X_2 = k) = \frac{P(X_1 = x \text{ and } X_2 = \cancel{x}\ k-x)}{P(X_1+X_2=k)}$$

$$= \frac{\dfrac{\lambda_1^x e^{-\lambda_1}}{x!} \cdot \dfrac{\lambda_2^{k-x} e^{-\lambda_2}}{(k-x!)}}{\dfrac{(\lambda_1+\lambda_2)^k e^{-\lambda_1-\lambda_2}}{k!}}$$

$$= \frac{k!}{x!(k-x!)} \cdot \frac{\lambda_1^x \lambda_2^{k-x}}{(\lambda_1+\lambda_2)^k}$$

$$P(X_1 = x \mid X_1 + X_2 = k) = \binom{k}{x} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k-x}$$

$$\boxed{P(X_1 = x \mid X_1 + X_2 = k) = \text{binomial}\left(x;\ k,\ \frac{\lambda_1}{\lambda_1+\lambda_2}\right)}$$

6. X = number of missing passengers

$$P(X \geq 2) = 1 - \left(200(.01)^1(.99)^{199} + (.99)^{200}\right) = 1 - (.270666 + .1339797) = \cancel{\boxed{.4046}}$$

$$= \boxed{.5954}$$