Connor Johnson

Assignment 5

1. $\Pr(Y=1|X=x) = x$ $\qquad f_1(x) = \begin{cases} 6x(1-x) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ $\qquad f(x,y) = g(y|x)f_1(x)$

$\Pr(Y=0|X=x) = 1-x$

$$g(y|x) = \begin{cases} x & y=1 \\ 1-x & y=0 \\ 0 & \text{otherwise} \end{cases} \qquad f(x,y) = \begin{cases} 6x^2(1-x) & 0<x<1,\ y=1 \\ 6x(1-x)^2 & 0<x<1,\ y=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr(Y=y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 6x^2(1-x)\,dx$$

$$= \frac{6x^3}{3} - \frac{6x^4}{4}\Big|_0^1 = \frac{6}{3} - \frac{6}{4} = \boxed{\frac{1}{2}}$$

$$g_2(x|y=1) = \frac{f(x,y=1)}{\Pr(Y=1)} = \frac{6x^2(1-x)}{1/2} = \boxed{12x^2(1-x)}$$

2. a) $2\int_0^{1/2}\int_x^1 x+y\,dy\,dx = 2\int_0^{1/2} xy + \frac{y^2}{2}\Big|_x^1 = 2\int_0^{1/2}\left(x+\frac{1}{2}\right) - \frac{3}{2}x^2\,dx$

$$= 2\cdot \frac{-x^3}{2} + \frac{x^2}{2} + \frac{x}{2}\Big|_0^{1/2} = \frac{-1}{16} + \frac{1}{8} + \frac{1}{4} = \frac{5}{16} * 2 = \boxed{\frac{5}{8}}$$

b) $f_1(x) = \int_y f(x,y)\,dy = 2\int_x^1 x+y\,dy = xy + \frac{y^2}{2}\Big|_x^1 = -3x^2+2x+1$

$$\boxed{f_1(x) = -3x^2+2x+1}$$

c) $g(y|x) = \dfrac{2(x+y)}{-3x^2+2x+1}$

3. a) $g(y|x) = \begin{cases} \frac{1}{x} & 0<y<x \\ 0 & \text{otherwise} \end{cases}$

$g(y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} g(y_i|x)$ since y is iid

$= \frac{1}{x^n}$ $\quad g(y_1, y_2, \ldots, y_n) = \begin{cases} \frac{1}{x^n} & \text{for } 0<y_i<x \\ 0 & \text{otherwise} \end{cases}$

b) $h(y) = \int_{y}^{\infty} f(x) g(y|x)\, dx$

$h(y_1, y_2, \ldots, y_n) = \int_{\max(y_i)}^{\infty} \frac{1}{n!} x^n e^{-x} \frac{1}{x^n}\, dx = \frac{1}{n!}\left[-e^{-x}\right]_{\max(y_i)}^{\infty} = \frac{1}{n!} e^{-\max(y_i)}$