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AN INTUITIONISTIC COMPLETENESS THEOREM FOR INTUITIONISTIC PREDICATE LOGIC[1]

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Introduction. The problem of treating the semantics of intuitionistic logic within the framework of intuitionistic mathematics was first attacked by E. W. Beth [1]. However, the completeness theorem he thought to have obtained, was not true, as was shown in detail in a report by V. H. Dyson and G. Kreisel [2]. Some vague remarks of Beth's, for instance in his book, ***The foundations of mathematics***, show that he sustained the hope of restoring his proof. But arguments by K. Gödel and G. Kreisel gave people the feeling that an intuitionistic completeness theorem would be impossible [3]. (A (strong) completeness theorem would imply

$$\neg\neg \bigvee_x A(x) \rightarrow \bigvee_x A(x)$$

for any primitive recursive predicate A of natural numbers, and one has no reason to believe this for the usual intuitionistic interpretation.)

Nevertheless, the following contains a correct intuitionistic completeness theorem for intuitionistic predicate logic. So the old arguments by Gödel and Kreisel should not work for the proposed semantical construction of intuitionistic logic. They do not, indeed. The reason is, loosely speaking, that negation is treated positively.

Although Beth's semantical construction for intuitionistic logic was not satisfying from an intuitionistic point of view, it proved to be useful for the development of classical semantics for intuitionistic logic. A related and essentially equivalent classical semantics for intuitionistic logic was found by S. Kripke [4].

In this paper we start from the classical definition of a Kripke-model and modify this slightly. For a classical mathematician, the change is not interesting: to every model of the new kind one easily constructs a model of the old kind, in which the same formulas are valid, and vice versa. But in intuitionistic mathematics we can prove the existence of a universal modified Kripke-model, i.e. a model such that every sentence, true in the model, is derivable in intuitionistic predicate calculus.

Of course the question arises whether we can claim to have found a reasonable semantics of intuitionistic logic; i.e. if the construction gives us a clearer concept "intuitionistically true sentence". I would not say so. Rather, the construction looks like a technical device, which should work for a big class of formal systems. For it asks us to see some complicated object built up from the formal system itself as the intended meaning of the formal system.

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For some of the technical details I am indebted to P. G. Aczel's paper [5].

The paper is organized as follows: In §1 we define the modified Kripke-models. In §2 we introduce the concept of a semiregular set of sentences of IPC. In §3 we consider a set of semiregular sets of sentences of IPC and use this to construct a universal modified Kripke-model. In §4 we give the proofs of two lemmas which were assumed in §3. §5 contains some concluding remarks.

§1. Modified Kripke-models.

1.1. Formulas and sentences of intuitionistic predicate calculus (IPC) are built up in the usual way from the following material:

A countable set of individual constants X.

For any natural number n, countably many n-ary predicate letters $P_1^n, P_2^n, \cdots$.

A 0-ary predicate letter $\perp$.

Connectives and quantifiers $\wedge, \vee, \rightarrow, \bigwedge, \bigvee$.

Countably many individual variables $x_0, x_1, x_2, \cdots$.

The set of formulas is called Form, the set of sentences is called Sent. For every formula $\mathfrak{A}$ we use $\neg\mathfrak{A}$ as an abbreviation for $\mathfrak{A} \rightarrow \perp$.

It does not matter which axiomatization we use for IPC. For $\Gamma \subseteq$ Sent and $\mathfrak{A} \in$ Sent, we say $\Gamma \vdash \mathfrak{A}$ for: there exists a derivation of $\mathfrak{A}$ from Γ.

1.2. DEFINITION. A modified Kripke-model for IPC is a triple $\mathfrak{M} = \langle \Omega_{\mathfrak{M}}, \leq_{\mathfrak{M}}, M \rangle$ where:

(i) Ω is a set.

(ii) $\leq_{\mathfrak{M}}$ is a partial ordering of $\Omega_{\mathfrak{M}}$.

(iii) For every $\alpha \in \Omega_{\mathfrak{M}}$ we have $M(\alpha)$ is a pair $\langle X_\alpha, V_\alpha \rangle$ such that:

(a) $X_\alpha \subseteq X$.

(b) V_α is a function which assigns to every predicate letter of IPC except $\perp$ a relation on X_α of the appropriate type.

(c) V_α assigns to $\perp$ a subset $V_\alpha(\perp)$ of $\{\perp\}$.

(d) If $\alpha, \beta \in \Omega_{\mathfrak{M}}$ and $\alpha \leq_{\mathfrak{M}} \beta$, then $X_\alpha \subseteq X_\beta$, $V_\alpha(\perp) \subseteq V_\beta(\perp)$ and, for any predicate letter P, $V_\alpha(P) \subseteq V_\beta(P)$.

1.21. REMARK. The definition is rather general. We do not require $\Omega_{\mathfrak{M}}$ to be a countable set as is usual in intuitionistic considerations on Kripke-models. If we do so, and add some more restrictions, we could form a spread of modified Kripke-models. Perhaps that is more or less the idea Beth had in mind. However, for the universal Kripke-model we will construct, it is essential that $\Omega_{\mathfrak{M}}$ is continuum-like.

1.22. REMARK. An (unmodified) Kripke-model is a model in the sense of Definition 1.2, fulfilling the condition $\bigwedge_{\alpha \in \Omega_{\mathfrak{M}}}[V_\alpha(\perp) = \varnothing]$. One can think of points $\alpha \in \Omega_{\mathfrak{M}}$ for which $V_\alpha(\perp) \neq \varnothing$ as sick points, which should be cut away. However, when considering the definition of validity for implicational formulae (cf. 1.3) and doing so as an intuitionist, one sees that the sick points influence the scene at the sound points too. $V_\alpha(\perp)$ need not be a decidable subset of $\{\perp\}$; in fact, for the universal model we are going to construct, it is important that we have no effective method for deciding which points are sound and which are sick.

1.23. REMARK. Every element of X_α is an individual constant in the language of IPC. So elements of X_α are identified with their name in the language of IPC.

1.3. DEFINITION. Let $\mathfrak{M} = \langle \Omega_{\mathfrak{M}}, \leq_{\mathfrak{M}}, M \rangle$ be a modified Kripke-model. We define $\mathfrak{M} \Vdash_\alpha \mathfrak{A}$ for any $\alpha \in \Omega_{\mathfrak{M}}$ and $\mathfrak{A} \in \mathrm{Sent}$. First suppose $\mathfrak{A}$ is atomic.

$$\begin{array}{lll} \mathfrak{M} \Vdash_\alpha \perp & \text{iff} & \perp \in V_\alpha(\perp), \\ \mathfrak{M} \Vdash_\alpha P(a_1, \cdots, a_n) & \text{iff} & \langle a_1, \cdots, a_n \rangle \in V_\alpha(P) \text{ or } \mathfrak{M} \Vdash_\alpha \perp. \end{array}$$

Now suppose $\mathfrak{A}$ is nonatomic. There are several cases.

$$\begin{array}{lll} \mathfrak{M} \Vdash_\alpha \mathfrak{A}_1 \wedge \mathfrak{A}_2 & \text{iff} & \mathfrak{M} \Vdash_\alpha \mathfrak{A}_1 \text{ and } \mathfrak{M} \Vdash_\alpha \mathfrak{A}_2. \\ \mathfrak{M} \Vdash_\alpha \mathfrak{A}_1 \vee \mathfrak{A}_2 & \text{iff} & \mathfrak{M} \Vdash_\alpha \mathfrak{A}_1 \text{ or } \mathfrak{M} \Vdash_\alpha \mathfrak{A}_2. \\ \mathfrak{M} \Vdash_\alpha \mathfrak{A}_1 \to \mathfrak{A}_2 & \text{iff} & \bigwedge_{\beta \geq_{\mathfrak{M}} \alpha} [\text{if } \mathfrak{M} \Vdash_\beta \mathfrak{A}_1, \text{ then } \mathfrak{M} \Vdash_\beta \mathfrak{A}_2]. \\ \mathfrak{M} \Vdash_\alpha \bigvee_x \mathfrak{A}_1 & \text{iff} & \bigvee_{a \in X_\alpha} [\mathfrak{M} \Vdash_\alpha S^x_a \mathfrak{A}_1]. \\ \mathfrak{M} \Vdash_\alpha \bigwedge_x \mathfrak{A}_1 & \text{iff} & \bigwedge_{\beta \geq_{\mathfrak{M}} \alpha} [\text{for all } a \in X_\beta\colon \mathfrak{M} \Vdash_\beta S^x_a \mathfrak{A}_1]. \end{array}$$

($S^x_a \mathfrak{A}_1$ is the formula resulting from $\mathfrak{A}_1$ when every free occurrence of x is replaced by an occurrence of a.) Finally, we define, for any $\mathfrak{A} \in \mathrm{Sent}$, $\mathfrak{M} \Vdash \mathfrak{A} =_D \bigwedge_{\alpha \in \Omega_{\mathfrak{M}}} [\mathfrak{M} \Vdash_\alpha \mathfrak{A}]$.

1.4. THEOREM. *Let* $\mathfrak{M} = \langle \Omega_{\mathfrak{M}}, \leq_{\mathfrak{M}}, M \rangle$ *be a modified Kripke-model. Let* $\Gamma \subseteq \mathrm{Sent}$ *and* $\mathfrak{A} \in \mathrm{Sent}$. *If* $\Gamma \vdash \mathfrak{A}$ *and* $\bigwedge_{\mathfrak{B} \in \Gamma}[\mathfrak{M} \Vdash \mathfrak{B}]$, *then* $\mathfrak{M} \Vdash \mathfrak{A}$.

The proof of the theorem is easy, and left to the reader.

§2. Semiregular sets of sentences.

2.1. DEFINITION. Let $\mathfrak{A} \in \mathrm{Sent}$. Then $\mathrm{IC}(\mathfrak{A}) =_D$ the set of individual constants occurring in $\mathfrak{A}$.

Let $\Gamma \subseteq \mathrm{Sent}$. Then $\mathrm{IC}(\Gamma) =_D \bigcup_{\mathfrak{A} \in \Gamma} \mathrm{IC}(\mathfrak{A})$.

2.2. DEFINITION. Let $\Gamma \subseteq \mathrm{Sent}$. Γ is called *semiregular* iff the following holds:

(i) For any $\mathfrak{A} \in \mathrm{Sent}$ such that $\mathrm{IC}(\mathfrak{A}) \subseteq (\Gamma)$, if $\Gamma \vdash \mathfrak{A}$, then $\mathfrak{A} \in \Gamma$.

(ii) For any $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathrm{Sent}$, if $\mathfrak{A}_1 \vee \mathfrak{A}_2 \in \Gamma$, then $\mathfrak{A}_1 \in \Gamma$ or $\mathfrak{A}_2 \in \Gamma$.

(iii) For any variable x and any $\mathfrak{A}(x) \in \mathrm{Form}$ in which x is the only free variable, if $\bigvee_x \mathfrak{A}(x) \in \Gamma$, then there is $a \in \mathrm{IC}(\Gamma)$, such that $S^x_a \mathfrak{A} \in \Gamma$.

Thus a semiregular set of sentences is a disjunctive and existential theory. It is not necessarily consistent.

§3. A spread whose elements generate semiregular sets of sentences.

3.1. $\mathrm{Nat} =_D$ the set of natural numbers.

$\mathrm{Seq} =_D$ the set of finite sequences of natural numbers.

For $a, b \in \mathrm{Seq}$ we define: $b \subseteq a =_D a$ is an initial segment of b. We wish to define a spread $\sigma = \langle \Sigma, \Gamma \rangle$. The spread law Σ is a function from Seq to $\{0, 1\}$. When $a \in \mathrm{Seq}$ and $\Sigma(a) = 0$ we say that a is admitted by Σ. The complementary law Γ is a function from $\{a \in \mathrm{Seq} \mid \Sigma(a) = 0\}$ to the set of finite subsets of Sent. It fulfills the condition:

$$\bigwedge_{a, b \in \mathrm{Seq}} [\Sigma(a) = 0 \wedge \Sigma(b) = 0 \wedge b \subseteq a \to \Gamma(a) \subseteq \Gamma(b)].$$

Let α be an element of the universal spread (that is, α is an arbitrary infinite sequence of natural numbers). If $n \in \mathrm{Nat}$, then $\bar{\alpha}n$ is the finite sequence $\langle \alpha(1), \cdots, \alpha(n-1) \rangle$. If $\bigwedge_{n \in \mathrm{Nat}}[\Sigma(\bar{\alpha}n) = 0]$, we write $\alpha \in \Sigma$ and we define $\Gamma_\alpha =_D \bigcup_{n \in \mathrm{Nat}} \Gamma(\bar{\alpha}n)$.

Our definition will be such that:

(i) $\bigwedge_\alpha [\alpha \in \Sigma \to \Gamma_\alpha$ is a semiregular set of sentences].

(ii) If Γ is a semiregular set of sentences and $\bigwedge_{\mathfrak{A} \in \mathrm{Sent}} \bigvee_{\alpha \in \sigma_{01}} [\mathfrak{A} \in \Gamma$ if and only if

$\bigvee_n[\alpha(n) = 1]]$ then there exists $\alpha \in \Sigma$ such that $\Gamma = \Gamma_\alpha$. The condition on Γ is fulfilled if for any $\mathfrak{A} \in$ Sent the statement: $\mathfrak{A} \in \Gamma$ does not depend on a free choice parameter. For in that case we can apply the Brouwer-Kripke-principle. (As in [6], σ_{01} is the binary fan.)

Thus $\langle \Sigma, \Gamma \rangle$ is the set of all "reasonable" semiregular subsets of Sent.

3.2. How do we make sure Γ_α is a theory? First we make a list of all derivations. Then, suppose, for some natural number n and $\mathfrak{A} \in$ Sent, $\Gamma(\bar{\alpha}n) \vdash \mathfrak{A}$ (and $\mathrm{IC}(\mathfrak{A}) \subseteq \mathrm{IC}(\Gamma(\bar{\alpha}n))$). We look for the first derivation of $\mathfrak{A}$ from $\Gamma(\bar{\alpha}n)$ in our list. We calculate the number of its place in the list of derivations, say m. We then take $\mathfrak{A} \in \Gamma(\bar{\alpha}(n + 3m))$.

We want to keep track carefully of $\mathrm{IC}(\Gamma_\alpha)$. We introduce a function D from $\{a \in \mathrm{Seq} \mid \Sigma(a) = 0\}$ to the set of finite subsets of X. If $\alpha \in \Sigma$ we define $D_\alpha =_D \bigcup_{n \in \mathrm{Nat}} D(\bar{\alpha}n)$ and D will be defined such that $D_\alpha = \mathrm{IC}(\Gamma_\alpha)$. Σ, Γ and D will be defined simultaneously by recursion to the length of the finite sequences for which they are defined.

3.31. *Preliminaries for the construction of* Σ, Γ *and* D. $X = X_0 \cup X_1 \cup X_2 \cup \cdots$. $X_n =_D \{a_{n1}, a_{n2}, a_{n3}, \cdots\}$. We split the set of individual constants X into a denumerable sequence of denumerable pairwise disjoint sets. Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be an enumeration of Sent. Let $d_1, d_2, \cdots$ be an enumeration of the set of derivations. Let $\langle\langle\ \rangle\rangle$ be a bijective mapping from $\mathrm{Nat}^2 \times \{0, 1, 2\}$ to Nat such that, for all $m, n \in \mathrm{Nat}$,

$$\langle\langle n, m, 0\rangle\rangle + 2 = \langle\langle n, m, 1\rangle\rangle + 1 = \langle\langle n, m, 2\rangle\rangle.$$

3.32. DEFINITION OF Σ, Γ AND D. We first define Σ, Γ and D for the empty sequence $\langle\ \rangle$:

$$\Sigma(\langle\ \rangle) = 0, \quad \Gamma(\langle\ \rangle) = \varnothing, \quad D(\langle\ \rangle) = X_0.$$

Now suppose, for some $\langle a_1, \cdots, a_p\rangle \in \mathrm{Seq}$, $\Sigma(\langle a_1, \cdots, a_p\rangle) = 0$ and $\Gamma(\langle a_1, \cdots, a_p\rangle)$ and $D(\langle a_1, \cdots, a_p\rangle)$ have been defined. We call $a =_D \langle a_1, \cdots, a_p\rangle$ and $a * n =_D \langle a_1, \ldots, a_p\rangle$ for any $n \in \mathrm{Nat}$ and distinguish three cases:

Case 1. $p + 1 = \langle\langle n, m, 0\rangle\rangle$ for some (uniquely determined) $m, n \in \mathrm{Nat}$. Then $\Sigma(a * 0) = 0$ provided: $\neg \bigvee_{i \le p}$ [d_i is derivation of $\mathfrak{A}_n$ from $\Gamma(a)$] or $\mathrm{IC}(\mathfrak{A}_n) \nsubseteq D(a)$; $\Sigma(a * 1) = 0$ provided $\mathrm{IC}(\mathfrak{A}_n) \subseteq D(a)$; $\Sigma(a * n) = 1$ if $n > 1$ or if $n \le 1$ and the above mentioned conditions are not satisfied. Further $\Gamma(a * 0) = \Gamma(a)$, $\Gamma(a * 1) = \Gamma(a) \cup \{\mathfrak{A}_n\}$, $D(a * 1) = D(a * 0) = D(a)$. So at this step we consider the formula $\mathfrak{A}_n$ for the mth time. We are obliged to leave it out, if it is not about the constants we are interested in. We are obliged to take it in, if it is about the constants we are interested in and is, by one of the first p derivations, a consequence of the formulas we already have. We can do as we like, if it is about the constants we are interested in, and is not, by one of the first p derivations, a consequence of the formulas we already have.

Case 2. $p + 1 = \langle\langle n, m, 1\rangle\rangle$ for some (uniquely determined) $m, n \in \mathrm{Nat}$. Then $\Sigma(a * 0) = 1$; $\Sigma(a * q) = 0$ if $q > 0$. Further $\Gamma(a * q) = \Gamma(a)$ and $D(a * q) = D(a) \cup \{a_{p1}, \cdots, a_{pq}\}$. So at this step we extend the set of individual constants by at least one element.

Case 3. $p + 1 = \langle\langle n, m, 2\rangle\rangle$ for some (uniquely determined) $m, n \in \mathrm{Nat}$. At this step we look again at the formula we (possibly) took in two steps ago. If it is a disjunction or an existential formula, we have to take some measures.

Subcase 3.1. $\mathfrak{A}_n$ is not a disjunction or an existential formula, or $a_{p-1} \neq 1$ (so we did not take it in). Then $\Sigma(a * 0) = 0, \Sigma(a * q) = 1$ if $q > 1$. Further $\Gamma(a * 0) = \Gamma(a)$, $D(a * 0) = D(a)$. So we simply leave things as they are.

Subcase 3.2. $\mathfrak{A}_n$ is a disjunction and $a_{p-1} = 1$. Then $\Sigma(a * 1) = 0$ and $\Sigma(a * 2) = 0$; $\Sigma(a * q) = 1$ if $q \neq 1, 2$. Further, let us write $\mathfrak{A}_n = (\mathfrak{A}_n)_1 \vee (\mathfrak{A}_n)_2$, $\Gamma(a * 1) =_D \Gamma(a) \cup \{(\mathfrak{A}_n)_1\}$, $\Gamma(a * 2) =_D \Gamma(a) \cup \{(\mathfrak{A}_n)_2\}$, $D(a * 1) =_D D(a * 2) =_D D(a)$.

Subcase 3.3. $\mathfrak{A}_n$ is an existential formula and $a_{p-1} = 1$. Then $\Sigma(a * 0) = 1$ and $\Sigma(a * q) = 0$ if $q > 1$. Further, let $t_1, t_2, t_3, \cdots$ be a fixed enumeration of $D(a)$. Let us write $\mathfrak{A}_n = \bigvee_x (\mathfrak{A}_n)_1(x)$, $\Gamma(a * q) =_D \Gamma(a) \cup \{S^x_{t_q}(\mathfrak{A}_n)_1(x)\}$ for $q > 1$. This concludes the definition of Σ, Γ and D. It is easy to verify $D_\alpha = \mathrm{IC}(\Gamma_\alpha)$. (If $c \in D_\alpha$, then the formula $P^1_1(c) \to P^1_1(c)$ will belong to Γ_α.)

3.4. *The universal modified Kripke-model.*

3.41. We define a modified Kripke-model $\mathfrak{U} = \langle \Omega_{\mathfrak{U}}, \leq_{\mathfrak{U}}, M \rangle$ in the following way: $\Omega_{\mathfrak{U}} =_D \{\alpha \mid \alpha \in \Sigma\}$ where Σ is the spread defined in the previous paragraph.

The partial ordering $\leq_{\mathfrak{U}}$ on $\Omega_{\mathfrak{U}}$ is defined by $\alpha \leq_{\mathfrak{U}} \beta \equiv_D \Gamma_\alpha \subseteq \Gamma_\beta$.

For $\alpha \in \Sigma$ we define $M_\alpha =_D \langle X_\alpha, V_\alpha \rangle$ where $X_\alpha =_D \mathrm{IC}(\Gamma_\alpha) = D_\alpha$, $V_\alpha(P) =_D \{\langle a_1, \cdots, a_n \rangle \mid \langle a_1, \cdots, a_n \rangle \in X^n_\alpha$ and $P(a_1, \cdots, a_n) \in \Gamma_\alpha\}$ for any n-ary predicate symbol P, $V_\alpha(\perp) =_D \{\perp\} \cap \Gamma_\alpha$.

We end this section by stating the two lemmas required to give our completeness theorem. The proofs can be found in §4.

3.42. LEMMA. *Suppose* $\mathfrak{A} \in$ Sent *and* $\alpha \in \Sigma$. *Then* $\mathfrak{U} \models_\alpha \mathfrak{A}$ *if and only if* $\mathfrak{A} \in \Gamma_\alpha$.

The proof of this lemma is by induction on the length of $\mathfrak{A}$. Most cases are straightforward, except when $\mathfrak{A}$ is an implication or a universal formula (cf. 4.1 and 4.2).

3.43. LEMMA. *Suppose* $\mathfrak{A} \in$ Sent *and* $\bigwedge_{\alpha \in \Sigma}[\mathfrak{A} \in \Gamma_\alpha]$. *Then* $\vdash \mathfrak{A}$.

For the proof of this lemma see 4.3.

The completeness theorem is an easy consequence of Lemmas 3.42 and 3.43. If $\mathfrak{A} \in$ Sent and $\mathfrak{U} \models \mathfrak{A}$, then $\vdash \mathfrak{A}$.

§4. Some important lemmas. We now prove two properties of the spread $\langle \Sigma, \Gamma \rangle$ defined in 3.3.

4.1. LEMMA. *Let* $\alpha \in \Sigma$ *and* $\mathfrak{A}$, $\mathfrak{B} \in$ Sent. *Then* $\mathfrak{A} \to \mathfrak{B} \in \Gamma_\alpha$ *if and only if* $\bigwedge_{\beta \in \Sigma}$ [*if* $\Gamma_\beta \supseteq \Gamma_\alpha$ *and* $\mathfrak{A} \in \Gamma_\beta$, *then* $\mathfrak{B} \in \Gamma_\beta$].

PROOF. From left to right: Immediate, every Γ_β being a theory. From right to left: Let $\mathfrak{A}$ be $\mathfrak{A}_n$ (the nth formula in the enumeration of Sent, introduced in 3.31), and let $\mathfrak{B}$ be $\mathfrak{A}_j$. We will define a subfan $\Sigma^{n,j}_\alpha$ of Σ with the following properties:

(i) For any $\beta \in \Sigma^{n,j}_\alpha$, $\mathfrak{A}_n \in \Gamma_\beta$ and $\Gamma_\alpha \subseteq \Gamma_\beta$.

(ii) For any $b \in$ Seq and $\mathfrak{p} \in$ Sent, if $\bigwedge_{q \in \mathrm{Nat}}$ [if $\Sigma^{n,j}_\alpha (b * q) = 0$, then $\mathfrak{p} \in \Gamma(b * q)$] then $\Gamma(b) \vdash \mathfrak{p}$ or $\mathfrak{p} = \mathfrak{A}_n$ or $\mathfrak{p} \in \Gamma_\alpha$.

(iii) For any $\beta \in \Sigma^{n,j}_\alpha$ and any natural number r, if, according to the definition of Σ and Γ, $\Gamma(\bar\beta(r + 1)) \backslash \Gamma(\bar\beta r)$ contains an instantiation $S^x_a\mathfrak{p}$ of an existential sentence $\bigvee_x \mathfrak{p} \in \Gamma(\bar\beta_r)$, then:

if $\bigvee_x \mathfrak{p} \in \Gamma(\bar\alpha r)$, the constant a fulfills the condition $S^x_a \mathfrak{p} \in \Gamma_\alpha$;

if $\bigvee_x \mathfrak{p} \notin \Gamma(\bar\alpha r)$, the constant a fulfills the condition $a \notin D_\alpha = \mathrm{IC}(\Gamma_\alpha)$, $a \notin \mathrm{IC}(\mathfrak{B})$, $a \notin \mathrm{IC}(\mathfrak{A}_n)$, $a \notin \mathrm{IC}(\Gamma(\bar\beta r))$.

The definition of $\Sigma^{n,j}_\alpha$ will be postponed until 4.11. From the property (i) of $\Sigma_\alpha^{n,j}$ and the assumption of the theorem we have $\bigwedge_{\beta\in\Sigma_\alpha^{n,j}}[\mathfrak{B}\in\Gamma(\beta)]$. Thus $\bigwedge_{\beta\in\Sigma_\alpha^{n,j}}\bigvee_{m\in\mathrm{Nat}}[\mathfrak{B}\in\Gamma(\bar\beta m)]$. But $\bigwedge_{\beta\in\Sigma_\alpha^{n,j}}\bigwedge_{m\in\mathrm{Nat}}[\Gamma(\bar\beta m)\subseteq\Gamma(\bar\beta(m+1))]$. So, using the fan theorem, we conclude

$$\bigvee_{m\in\mathrm{Nat}}\ \bigwedge_{\beta\in\Sigma_\alpha^{n,j}}[\mathfrak{B}\in\Gamma(\bar\beta m)].$$

So we calculate a number m_0 such that $\bigwedge_{\beta\in\Sigma_\alpha^{n,j}}[\mathfrak{B}\in\Gamma(\bar\beta m_0)]$. Now, we prove that for any $b\in\mathrm{Seq}$: If $\Sigma_\alpha^n(b)=0$ and $l(b)\leq m_0$, then $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. (If $b\in\mathrm{Seq}$, then $l(b)=_D$ the length of b.) The proof is by induction on $m_0-l(b)$.

Basic step. $l(b)=m_0$. Then $\mathfrak{B}\in\Gamma_{(b)}$; so $\Gamma(b)\vdash\mathfrak{B}$; so $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Induction step. $l(b)=m<m_0$. Hypothesis: $\bigwedge_{b\in\mathrm{Seq}}$ [if $\Sigma_\alpha^{n,j}(b)=0$ and $l(b)\geq m+1$, then $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$].

We distinguish three cases:

Case 1. For some (uniquely determined) $m_1, m_2{:}m+1=\langle\langle m_1, m_2 0\rangle\rangle$.

Subcase 1.1. $\Sigma_\alpha^{n,j}(b*0)=0$. Now $\Gamma(b*0)=\Gamma(b)$. From the hypothesis we know $\Gamma(b*0),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$; so $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Subcase 1.2. $\Sigma_\alpha^{n,j}(b*0)=1$. Now it is not allowed to leave out the formula $\mathfrak{A}_{m_1}$. We use the property (ii) of $\Sigma_\alpha^{n,j}$. There are three possibilities:

$$\Gamma(b)\vdash\mathfrak{A}_{m_1}\quad\text{or}\quad\mathfrak{A}_{m_1}=\mathfrak{A}_n\quad\text{or}\quad\mathfrak{A}_{m_1}\in\Gamma_\alpha.$$

Further $\Gamma(b*1)=\Gamma(b)\cup\{\mathfrak{A}_{m_1}\}$ and (hypothesis) $\Gamma(b*1),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. In all cases we easily conclude $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Case 2. For some (uniquely determined) $m_1, m_2, m+1=\langle\langle m_1, m_2, 1\rangle\rangle$. For some $p\in\mathrm{Nat}$, $\Sigma_\alpha^{n,j}(b*p)=0$. Let us choose such a number and call it p_0. Then $\Gamma(b*p_0)=\Gamma(b)$ and (hypothesis) $\Gamma(b*p_0),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. So $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Case 3. For some (uniquely determined) $m_1, m_2, m+1=\langle\langle m_1, m_2, 2\rangle\rangle$.

Subcase 3.1. $\Sigma_\alpha^{n,j}(b*0)=0$. Then $\Gamma(b)=\Gamma(b*0)$ and (hypothesis) $\Gamma(b*p)$, $\Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. We immediately have $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Subcase 3.2. $\Sigma_\alpha^{n,j}(b*0)=1$ and $\mathfrak{A}_{m_1}=\bigvee_x\mathfrak{p}$ for some $\mathfrak{p}\in\mathrm{Form}$. For some p, $\Sigma_\alpha^{n,j}(b*0)=0$. Let us choose such a p and call it p_0. $\Gamma(b*p_0)=\Gamma(b)\cup\{S_{t_{p_0}}^x\mathfrak{p}\}$. According to the hypothesis, $\Gamma(b*p_0),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. Now we use the property (iii) of $\Sigma_\alpha^{n,j}$. There are two possibilities.

First possibility. $\bigvee_x\mathfrak{p}\in\Gamma(\bar\alpha n)$. We then know $S_{t_{p_0}}^x\mathfrak{p}\in\Gamma_\alpha$. So $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Second possibility. $\bigvee_x\mathfrak{p}\notin\Gamma(\bar\alpha n)$. We then know t_{p_0} does not occur in Γ_α, $\mathfrak{A}_n$ or $\mathfrak{B}$. So again $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

Subcase 3.3. $\Sigma_\alpha^{n,j}(b*0)=1$ and $\mathfrak{A}_{m_1}=\mathfrak{p}\vee\mathfrak{q}$ for some $\mathfrak{p},\mathfrak{q}\in\mathrm{Form}$. First suppose $\Sigma_\alpha^{n,j}(b*1)=1$. Then $\bigwedge_{q\in\mathrm{Nat}}[\Sigma_\alpha^{n,j}(b*q)=0\to\mathfrak{q}\in\Gamma(b*q)]$. (For $b*2$ now is the only extension of b, admitted by $\Sigma_\alpha^{n,j}$.) So, using property (ii) of $\Sigma_\alpha^{n,j}$, we conclude $\Gamma(b)\vdash\mathfrak{q}$. Now $\Gamma(b*2)=\Gamma(b)\cup\{\mathfrak{q}\}$ and (hypothesis) $\Gamma(b*2),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. So we have $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. The case $\Sigma_\alpha^{n,j}(b*2)=1$ is treated similarly.

Now suppose $\Sigma_\alpha^{n,j}(b*1)=0$ and $\Sigma_\alpha^{n,j}(b*2)=0$. Then $\Gamma(b*1),\ \Gamma_\alpha,\mathfrak{A}_n\vdash\mathfrak{B}$ and $\Gamma(b*2),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. So $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n,\ \mathfrak{p}\vdash\mathfrak{B}$ and $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n,\ \mathfrak{q}\vdash\mathfrak{B}$. So $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n$, $\mathfrak{p}\vee\mathfrak{q}\vdash\mathfrak{B}$. But $\mathfrak{p}\vee\mathfrak{q}=\mathfrak{A}_{m_1}\in\Gamma(b)$. So $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$.

This concludes the proof of the statement: If $\Sigma_\alpha^{n,j}(b)=0$ and $l(b)\leq m_0$ then $\Gamma(b),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. By consequence, $\Gamma(\langle\ \rangle),\ \Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. So $\Gamma_\alpha,\ \mathfrak{A}_n\vdash\mathfrak{B}$. So $\Gamma_\alpha\vdash\mathfrak{A}_n\to\mathfrak{B}$.

If we take the definition of $\Sigma_\alpha^{n,j}$ for granted, this concludes the proof of Lemma 4.1.

4.11. DEFINITION OF $\Sigma_\alpha^{n,j}$. Let $\alpha \in \Sigma$ and $n, j \in \mathrm{Nat}$. We define the subfan $\Sigma_\alpha^{n,j}$ by $\beta \in \Sigma_\alpha^{n,j}$ if and only if $\beta(\langle\langle n, 0, 0\rangle\rangle) = 1$ (this guarantees $\mathfrak{A}_n \in \Gamma_\beta$) *and*

$$\bigwedge_{p,q\in\mathrm{Nat}} [\beta(\langle\langle p, q, 0\rangle\rangle) \geq \alpha(\langle\langle p, q, 0\rangle\rangle)]$$

(this guarantees $\Gamma_\alpha \subseteq \Gamma_\beta$) *and* $\bigwedge_{p,q\in\mathrm{Nat}}[\beta(\langle\langle p, q, 1\rangle\rangle) = \alpha(\langle\langle p, q, 1\rangle\rangle) + s_0(\bar{\beta}(\langle\langle p, q, 1\rangle\rangle))]$ (this guarantees $D_\alpha \subseteq D_\beta$ and D_β is sufficiently bigger than D_α) *and* $\bigwedge_{p,q\in\mathrm{Nat}}$ [if $\beta(\langle\langle p, q, 2\rangle\rangle) \neq 0$, then $\beta(\langle\langle p, q, 2\rangle\rangle) = \alpha(\langle\langle p, q, 2\rangle\rangle) + r_0(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle))$ or $\beta(\langle\langle p, q, 2\rangle\rangle) = \alpha(\langle\langle p, q, 2\rangle\rangle) + r_1(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle))$] (this guarantees that good instantiations of disjunctions and existential formulas are chosen).

The function s_0 is defined in such a way that: At least one element of $D(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle))$ does not occur in Γ_α, $\mathfrak{A}_n$, $\mathfrak{B} = \mathfrak{A}_j$ or $D(\bar{\beta}(\langle\langle p, q, 1\rangle\rangle))$. (As only finitely many constants occur in $\mathfrak{A}_n$ and $\mathfrak{A}_j$, s_0, from a given moment on, can be taken constantly equal to 1.) The functions r_0 and r_1 are defined in such a way that: If $\alpha(\langle\langle p, q, 2\rangle\rangle) \neq 0$ then $r_0(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle)) = r_1(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle)) = 0$. If $\alpha(\langle\langle p, q, 2\rangle\rangle) = 0$ and $\mathfrak{A}_p$ is a disjunction, then $r_0(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle)) = 1$ and $r_1(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle)) = 2$. If $\alpha(\langle\langle p, q, 2\rangle\rangle) = 0$ and $\mathfrak{A}_p$ is an existential formula, then t_{r_0} and t_{r_1} are constants which do not occur in Γ_α, $\mathfrak{A}_n$, $\mathfrak{B}$ or $D(\bar{\beta}(\langle\langle p, q, 1\rangle\rangle))$ (t_{r_0} is the r_0th element of $D(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle))$ in a fixed enumeration of the set).

It is a matter of patience to verify that the above guidelines give a subfan $\Sigma_\alpha^{n,j}$ fulfilling the conditions (i), (ii) and (iii) of 4.1.

4.2. LEMMA. *Let* $\alpha \in \Sigma$ *and* $\bigwedge_x \mathfrak{A}(x) \in \mathrm{Sent}$. *Then* $\bigwedge_x \mathfrak{A}(x) \in \Gamma_\alpha$ *if and only if* $\bigwedge_{\beta\in\Sigma} \bigwedge_{a\in X}$ [*if* $\Gamma_\alpha \subseteq \Gamma_\beta$ *and* $a \in D_\beta = \mathrm{IC}(\Gamma_\beta)$, *then* $S_a^x\mathfrak{A} \in \Gamma_\beta$].

PROOF. From left to right: Immediate, every Γ_β being a theory. From right to left: Choose $a \in X$ such that $a \notin D_\alpha = \mathrm{IC}(\Gamma_\alpha)$. Construct a subfan Σ_α^a of Σ, like the subfan we constructed in the proof of Lemma 4.1, but replace the first condition on the subfan, mentioned in that proof, by

$$\bigwedge_{\beta\in\Sigma_\alpha^a} [\Gamma_\alpha \subseteq \Gamma_\beta \text{ and } a \in D_\beta = \mathrm{IC}(\Gamma_\beta)].$$

We then know $\bigwedge_{\beta\in\Sigma_\alpha^a}[S_a^x\mathfrak{A} \in \Gamma_\beta]$. Reasoning along the same lines as in the proof of 4.1 we conclude $\Gamma_\alpha \vdash S_a^x\mathfrak{A}$. So (since a does not occur in Γ_α) $\Gamma_\alpha \vdash \bigwedge_x \mathfrak{A}(x)$.

4.3. *Conclusion.* The proof of Lemma 3.42 now is a bookkeeping affair. We leave it to the reader.

As for Lemma 3.43, we prove it as follows: Suppose $\mathfrak{A} \in \mathrm{Sent}$ and $\bigwedge_{\alpha\in\Sigma}[\mathfrak{A} \in \Gamma_\alpha]$. Determine $\beta \in \Sigma$ such that $\bigwedge_{p\in\mathrm{Nat}} \bigwedge_{q\in\mathrm{Nat}}[D(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle))$ contains a constant which does not occur in $\mathfrak{A}$ or in $D(\bar{\beta}(\langle\langle p, q, 1\rangle\rangle))]$. Determine a sequence $(n(p, q))_{p,q\in\mathrm{Nat}}$ of natural numbers such that for any $p, q \in \mathrm{Nat}$, the constant $t_{n(p,q)}$ (in the fixed enumeration of $D(\bar{\beta}(\langle\langle p, q, 2\rangle\rangle)))$ does not occur in $\mathfrak{A}$ or in $D(\bar{\beta}(\langle\langle p, q, 1\rangle\rangle))$. Now consider $\{\alpha \mid \alpha \in \Sigma \mid D_\alpha = D_\beta \wedge \bigwedge_{p\in\mathrm{Nat}} \bigwedge_{q\in\mathrm{Nat}}[\alpha(\langle\langle p, q, 2\rangle\rangle) = 0 \vee \alpha(\langle\langle p, q, 2\rangle\rangle) = n(p, q) \vee \alpha(\langle\langle p, q, 2\rangle\rangle) = 1 \vee \alpha(\langle\langle p, q, 2\rangle\rangle) = 2]\}$. This is a subfan of Σ, call it $\Sigma^{\mathfrak{A}}$. Remembering that $\bigwedge_{\alpha\in\Sigma^{\mathfrak{A}}}[\mathfrak{A} \in \Gamma_\alpha]$ we apply an argument similar to the one used in 4.1 to conclude $\vdash\mathfrak{A}$.

We can state the completeness theorem somewhat more generally than we did at the end of 3.43.

4.4. THEOREM. *Suppose* $\Gamma \subseteq \mathrm{Sent}$ *and* $\bigvee_{\alpha\in\Sigma}[\mathrm{IC}(\Gamma) \subseteq D_\alpha]$ *and* $\bigwedge_{\mathfrak{B}\in\mathrm{Sent}} \bigvee_{\beta\in\sigma_{01}}$ [$\mathfrak{B} \in \Gamma$ *if and only if* $\bigvee_{n\in\mathrm{Nat}}[\beta(n) = 1]$]. *Then, for any sentence* $\mathfrak{A}$: *If* $\bigwedge_{\alpha\in\Sigma}$ [*if* $\bigwedge_{\mathfrak{B}\in\Gamma}[\models_\alpha \mathfrak{B}]$, *then* $\models_\alpha \mathfrak{A}$], *then* $\Gamma \vdash \mathfrak{A}$.

(The condition on Γ means that the statement $\mathfrak{B} \in \Gamma$ does not depend on free choice parameters for any $\mathfrak{B} \in \text{Sent}$.)

PROOF. Determine $\alpha \in \Sigma$ such that $\text{IC}(\mathfrak{A}) \subseteq D_\alpha$ and such that $\bigwedge_{p \in \text{Nat}} \bigwedge_{q \in \text{Nat}} [D(\bar{\alpha}(\langle\langle p, q, 2\rangle\rangle))$ contains a constant which does not occur in $\mathfrak{A}$ or in $D(\bar{\alpha}(\langle\langle p, q, 1\rangle\rangle))]$. Determine a sequence $(n(p, q))_{p,q \in \text{Nat}}$ of natural numbers such that, for any p, $q \in \text{Nat}$, the constant $t_{n(p,q)}$ (in the fixed enumeration of $D(\bar{\alpha}(\langle\langle p, q, 2\rangle\rangle))$ does not occur in $\mathfrak{A}$ or in $D(\bar{\alpha}(\langle\langle p, q, 1\rangle\rangle))$. Let $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ be the enumeration of Sent, introduced in 3.31. Determine sequences $\beta_1, \beta_2, \cdots$ from σ_{01} such that $\bigwedge_{m \in \text{Nat}} [\mathfrak{A}_m \in \Gamma$ if and only if $\bigvee_{n \in \text{Nat}}[\beta_m(n) = 1]]$. Now consider

$$\{\gamma \mid \gamma \in \Sigma \mid D_\gamma = D_\alpha \wedge \bigwedge_{p,q \in \text{Nat}}[(\beta_p(q) = 1 \to \gamma(\langle\langle p, q, 0\rangle\rangle) = 1) \\ \wedge\ \gamma(\langle\langle p, q, 2\rangle\rangle) \leq \max(2, n(p, q))]\}.$$

This is a subfan of Σ, call it Σ^Γ. We know $\bigwedge_{\gamma \in \Sigma^\Gamma}[\Gamma \subseteq \Gamma_\gamma]$, so $\bigwedge_{\gamma \in \Sigma^\Gamma}[\models_\gamma \mathfrak{A}]$. Reasoning as in 4.1 we find $\Gamma \vdash \mathfrak{A}$.

§5. Concluding remark. In what way did we modify the usual Kripke-models? A modified Kripke-model can be thought of as a Kripke-model in the usual sense, extended by some strange "exploding" points (points at which $\perp$ is valid). The usual condition for validity of a negative sentence $\neg\mathfrak{A}$ is replaced by a stronger one: Formerly one said: the formula $\neg\mathfrak{A}$ is valid in α, if $\mathfrak{A}$ is not valid at any (nonexploding) successor of α. Now we say: the formula $\neg\mathfrak{A}$ is valid in α, if all successors of α, at which $\mathfrak{A}$ is valid, explode. Thinking classically, one can cut away from a given modified Kripke-model the exploding points. It does not change the situation at the nonexploding points.

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