

Nissim Francez

DOES THE IMPLICATION ELIMINATION RULE NEED A MINOR PREMISE?

Abstract. The paper introduces NJ^g , a variant of Gentzen's NJ natural deduction system, in which the implication elimination rule has no minor premise. The NJ^g -systems extends traditional ND-system with a new kind of action in derivations, assumption incorporation, a kind of dual to the assumption discharge action. As a result, the implication (I/E)-rules are invertible and, almost by definition, harmonious and stable, a major condition imposed by proof-theoretic semantics on ND-systems to qualify as meaning-conferring. There is also a proof-term assignment to NJ^g -derivations, materialising the Curry-Howard correspondence for this system.

Keywords: natural deduction; NJ ; proof-theoretic semantics; grounds for assertion; term assignment

1. Introduction

Ever since Gentzen's seminal work [8, 7] inventing natural deduction, NJ for intuitionistic logic and NK for classical logic, the traditional introduction and elimination rules (I/E -rules) for the (material) implication connective (in both those systems) are the following (displayed in Prawitz's style).

$$\frac{\begin{array}{c} [\varphi]_i \\ \vdots \\ \psi \end{array}}{\varphi \supset \psi} (\supset I^i) \qquad \frac{\varphi \supset \psi \quad \varphi}{\psi} (\supset E)$$

The premise of $(\supset I)$ is not a formula, but a sub-derivation, deriving the consequent ψ from a *discharged* assumption of the antecedent φ .

The square brackets indicate the discharge of the assumption, and the index i indicates which application of $(\supset I)$ discharges this instance of the assumption. The possibility of discharging assumptions, *possibly vacuously*, is a landmark of natural-deduction (ND) proof-systems.

The $(\supset E)$ -rule is the rule known from antiquity as *modus ponens*.

Gentzen used also an equivalent *logistic* style, using sequents of the form $\Gamma \rightarrow \varphi$, where the *context* Γ is a collection¹ of formulas, that makes explicit the (open) assumptions on which the conclusion φ depends. The *sequent-separator* ‘ \rightarrow ’ separates the context Γ , the antecedent of the sequent, from the succedent φ . Expressed in this style, those rules have the following form²

$$\frac{\Gamma, \varphi \rightarrow \psi}{\Gamma \rightarrow \varphi \supset \psi} (\supset I) \qquad \frac{\Gamma \rightarrow \varphi \supset \psi \quad \Gamma \rightarrow \varphi}{\Gamma \rightarrow \psi} (\supset E)$$

The notation $\vdash_{\mathcal{N}} \Gamma \rightarrow \varphi$ indicates the provability of the sequent $\Gamma \rightarrow \varphi$ in some ND-system \mathcal{N} (mostly NJ or its alternative NJ^g proposed below), expressing derivability of φ from Γ .

The switch to presenting ND in the logistic format, besides having a technical advantage (to manifest itself below), carries also philosophical significance. It stresses that ND-systems are concerned with *hypothetical reasoning*, deduction from open assumptions, in contrast to *categorical reasoning*, obtaining formal theses in a calculus. The latter is merely a special case of the former where the context Γ is empty and notationally dismissed. See [22] for an extensive discussion of this topic.

The question I would like to pose, and answer *negatively*, is the one in the title of the paper:

Q_{mp}: Does an implication elimination rule need a minor premise?

This negative answer is accompanied by an alternative ND-system³ NJ^g ,

¹ The exact nature of this collection is immaterial at this point.

² Strictly speaking, this notation is not transparent regarding vacuous discharge. One way out is to formulate another (I)-rule, namely

$$\frac{\Gamma \rightarrow \psi}{\Gamma \rightarrow \varphi \supset \psi} (\supset I)$$

or introduce a notation for *optional* occurrence of a formula in an antecedent of a sequent. To avoid cluttering the notation, I will leave vacuity to be, here and elsewhere, implicitly understood, similar to the Prawitz notation.

³ I will focus on intuitionistic logic, or even on minimal logic [15] (without the explosion rule for \perp), but the arguments apply to classical logic as well.

a *grounds-retrieving* variant of NJ , incorporating an (E) -rule for the material implication without a minor premise. There are two issues I want to point out:

- The system NJ^g arises from a novel view of the role of E -rules in ND-systems, switching from drawing *immediate conclusions* from a formula to *retrieving the grounds of assertion* of a formula.
- The “price” of the above switch is the inclusion of a restricted form of a (Cut) -rule, typical to sequent calculi, as a primitive, non-admissible ND-rule, the task of which is to restore transitivity of deduction. The latter gets lost by the above switch.

The paper is structured as follows. Section 2 provides an analysis of traditional $(\supset I/E)$ -rules, an analysis leading to a reanalysis of such rules in Section 3. The system NJ^g is introduced in Section 4. Section 5 extends NJ^g with λ^g proof-terms, a variant of the simply-typed λ -calculus. Negation (as a special case of implication) is considered in Section 6. Section 7 ends with conclusions.

2. Analysing the $(\supset I/E)$ -rules

Before answering Q_{mp} , let us consider the rationale and justification of the $(\supset I/E)$ -rules of NJ .

The idea behind the ND proof-systems was to mimic the reasoning of a mathematician when proving theorems; hence the name of those systems.

In case of implication, the $(\supset I)$ -rule reflects the proof procedure known as *conditional proof*. In order to prove that φ (with possibly additional auxiliary assumptions) implies ψ , add φ temporarily to the arsenal of already made assumptions and prove ψ from this extended arsenal. If succeeding, this constitutes a proof of the implication $\varphi \supset \psi$ not depending anymore on the temporary assumption φ .

Thus, $(\supset I)$ *records* the existence of a conditional proof of ψ from φ (and the other assumptions).

As for the $(\supset E)$ -rule, given the implication $\varphi \supset \psi$, understood as recording the existence of the corresponding conditional proof, the conclusion ψ follows as an *immediate conclusion* by applying this implicit existing proof to φ , where the latter has itself been established (as a minor premise) instead of assumed.

It is possible to summarise the deductive roles in traditional ND-systems (and in NJ in particular) as follows:

I -rules: Establish a formula dominated by the introduced connective as an *immediate conclusion* from other formulas or sub-derivations.

E -rules: Establish some formula as an *immediate conclusion* of a formula dominated by the eliminated connective.

Note the asymmetry in the above roles of I/E -rules when formulated in Prawitz's style: a premise can be either a formula or a sub-derivation, while a conclusion can only be a formula.

Consider the following typical example of a derivation using the $(\supset I/E)$ -rules, to be contrasted with another example presented later.

Example 2.1 (transitivity of \supset). The following derivation establishes

$$\vdash_{NJ} \varphi \supset \psi, \psi \supset \chi \rightarrow \varphi \supset \chi$$

$$\frac{\frac{[\varphi]_i \quad \varphi \supset \psi}{\psi} (\supset E) \quad \psi \supset \chi}{\frac{\chi}{\varphi \supset \chi} (\supset I^i)} (\supset E)$$

Note the *forward chaining* embodied in the above derivation.

The above mentioned view of the $(\supset I/E)$ -rules manifests itself via the proof-theoretic *justification* embodied in the *harmony* (and *stability*) [3], establishing a *balance* between the $(\supset I/E)$ -rules, shown by the following *reduction* and *expansion* [17].

$$\frac{\frac{[\varphi]_i}{\mathcal{D}_1} \quad \psi}{\varphi \supset \psi} (\supset I^i) \quad \frac{\mathcal{D}_2}{\varphi} (\supset E) \quad \rightsquigarrow_r \quad \frac{\mathcal{D}_2}{\varphi} (\supset E) \quad \frac{\varphi}{\psi} (\supset I^i)$$

(relying on closures of NJ -derivations under composition, see Proposition 2.1 below), and

$$\frac{\mathcal{D}}{\varphi \supset \psi} \rightsquigarrow_e \quad \frac{\frac{\mathcal{D}}{\varphi \supset \psi} \quad [\varphi]_i}{\psi} (\supset E) \quad \frac{\psi}{\varphi \supset \psi} (\supset I^i)$$

The reduction is a step leading to (either weak or strong) *normalisation* of derivations in intuitionistic logic [17]. It also is, by the Curry-Howard correspondence, the β step in normalisation in the simply-typed λ -calculus [10]. The expansion is the η -step in this calculus.

2.1. Transitivity of deduction

An important property of ND-systems is the transitivity of deduction, expressed by⁴ *closure under derivation composition*. This composition is obtained by “pasting” a derivation \mathcal{D}_2 with conclusion φ to a leaf in a derivation \mathcal{D}_1 of ψ , a leaf labelled with an assumption φ . This procedure does not necessarily produce a legal derivation for arbitrary ND-systems. For example, if derivations are required to be *normal*, the result of pasting two normal derivations need not be normal. For another example, where some modification needs to be done before pasting in order to get a legal derivation, in the context of *relevant logic*, see [5]. Thus, closure under derivation composition, expressing transitivity of deduction, has to be proved whenever relied upon.

PROPOSITION 2.1 (closure of *NJ* under derivation composition). *If*

$$\begin{array}{c} \Gamma_1, \varphi \\ \mathcal{D}_1 \\ \psi \end{array} \quad \text{and} \quad \begin{array}{c} \Gamma_2 \\ \mathcal{D}_2 \\ \varphi \end{array}$$

are *NJ*-derivations, so is

$$\begin{array}{c} \Gamma_2 \\ \mathcal{D}_2 \\ \Gamma, \varphi \\ \mathcal{D}_1 \\ \psi \end{array}$$

PROOF. See [16, p. 171] and the accompanying discussion. \square

When the Γ s are left implicit, as in Prawitz-notation, the result of the composition is depicted as (cf. the outcome of the reduction step in (4.2)):

$$\begin{array}{c} \mathcal{D}_2 \\ \varphi \\ \mathcal{D}_1 \\ \psi \end{array}$$

⁴ Sometimes this operation is also called *closure under substitution*, since in the λ -terms corresponding to the two argument derivations of this operation, it is a substitution of a term for a variable.

In the logistic style, such a composition of derivations can be depicted in the form

$$\frac{\Gamma_1, \varphi \rightarrow \psi \quad \Gamma_2 \rightarrow \varphi}{\Gamma_1, \Gamma_2 \rightarrow \psi} \quad (2.1)$$

which is the form of the (*Cut*)-rule expressing transitivity of derivability in sequent calculi. By the closure of *NJ*-derivations under composition (Proposition 2.1), this rule is *admissible* in *NJ*, not needed as a primitive rule. Relating to this depiction, the main reason stated in [16, p. 172] as to why this pasting procedure works is

In natural deduction in sequent calculus style [logistic style – N.F.], there are no principal formulas in the antecedent, and therefore the substitution formula in the right premise also appears in at least some premise of the rule concluding the right premise.

As will be realised in the next section, the important point is not the absence of any principal formulas in the antecedent, but the absence of *incorporated* assumptions, as defined below. The absence of the latter enforces the occurrence of the substituted formula in a premise, so that composition can be propagated upwards.

3. A reanalysis of the roles of *I/E*-rules

In the theory of meaning known as *Proof-Theoretic Semantics* (see [23] for a brief overview and [4] for a detailed exposition), the meaning of a logical connective is intimately connected with *grounds for assertion* of sentences dominated by that connective, instead of with the truth-conditions of such sentences. For the purpose of this paper, I will not define in detail grounds for assertion. I will only assume that those grounds are established by means of the premises of *I*-rules for the connective considered. See [6] or [4] for a specific definition of grounds for assertion; see also [18, 19] for such an approach, based on a different notion of grounds for assertion.

This leads to the following characterisation of the roles of *I/E*-rules in ND-systems (in contrast with the roles specified in Section 2).

***I*-rules:** Establish the grounds for assertion of formulas dominated by the introduced connective.

***E*-rules:** *Retrieve*⁵ the grounds for assertion of formulas dominated by the eliminated connective.

Thus, the role of an *E*-rule changes from its role in *NJ* of establishing *immediate* consequences of formulas to a new role, that of retrieving the grounds for introduction of those formulas.

Those two tasks need not coincide!

Note, in addition, that as shown in the following section, the symmetry between premises and conclusions is restored in the proposed system *NJ*⁹: *both* are sub-derivations. In a sense, the kind of (*E*)-rule proposed here could be considered a dual to the generalisation of ND-systems in [20]: the latter allows rules as premises, while I allow also rules as conclusions.

3.1. Assumption incorporation

In order to express the retrieved grounds for assertion, determined by I-rules, that can be in the general case of the form of a sub-derivation in addition to being formulas, I extend ND-systems with a notion of *assumption incorporation*, a kind of dual operation to assumption discharge. The discussion is in terms of logistic presentations of rules. Assumption discharge manifests itself in a rule by a certain *shrinking* of the context of the premise, to yield the context of the conclusion. The rule thus has the form (in case of a one-premise rule, say an *(*I)*-rule for a generic connective ‘*’).

$$\frac{\Gamma, \alpha \rightarrow \beta}{\Gamma \rightarrow \gamma} (*I)$$

Above, the assumption α is present in the context of the premise but not present in the context of the conclusion. This is the meaning of *discharging* α by that rule.

Consider next the following form⁶ of an *E*-rule for the generic ‘*’, to which I refer as the *(*E^g)*-rule. The superscript *g* alludes to the grounds-retrieving role of an elimination rule.

$$\frac{\Gamma \rightarrow \gamma}{\Gamma, \alpha \rightarrow \beta} (*E^g) \tag{3.1}$$

⁵ I share this view of *E*-rules with a similar view mentioned by Bruno Jacinto and Stephen Read [12].

⁶ This form is clearly inspired by rule-form in a sequent calculus, though it acts here the way ND-rules act.

In this form of a rule, the context of the premise *grows* instead of shrinking, to yield the context of the conclusion. The formula α is present in the context of the conclusion while not being present in the context of the premise. I refer to this phenomenon as the *incorporation* of the assumption α (into the context of the conclusion) by the rule $(*E^g)$.

The form of the rule in (3.1) turns out to be convenient for expressing grounds for assertion whenever those grounds constitute of the existence of a sub-derivation. The conclusion of $(*E^g)$ exactly expresses the existence of a sub-derivation of β from the incorporated assumption α and any lateral assumptions Γ .

A natural qualm has to be answered at this point: *is such a rule still an ND-rule?*

It does not seem to be directly related to any step in an informal proof of a mathematical theorem. Still, I want to claim that it is an ND-rule, provided α is *fully schematic*, featuring no specific logical constant!

A possible proof-theoretic-internal reason for doubting such a rule being “ND-kosher” is that it adds material to the antecedent of a sequent. However, by the above provision this addition is *schematic* only; no logical constant is displayed in the expanded context of the conclusion of such a rule. The *essential* characteristic of ND-rules, at least in their capacity of constituting meaning-conferring definitional tools, is that they only introduce and eliminate formulas featuring a constant in the succedent of a sequent, and assumption incorporation forms no exception to this essential characteristic.

The justification of classifying the proposed system as a natural-deduction system and not as a sequent-calculus can also be phrased in terms of Schroeder-Heister’s [21] terminology. Schroeder-Heister distinguishes between two ways of introducing an assumption into an antecedent of a sequent: *specific*, i.e., according to the assumptions meaning, and *non-specific*. The sequent-calculi left-rules all introduce assumption into an antecedent specifically. On the other hand, the incorporation operation introduces an assumption into an antecedent non-specifically, independently of the form (and, hence, of the meaning) of that assumption.

3.2. Invertibility

An advantage of the ground-retrieving $(*E^g)$ -rules is that the combination of the $(*I/E^g)$ -rules can be presented as one *invertible* rule, known

also as a *double-line* rule.

$$\frac{\Gamma \rightarrow \gamma}{\Gamma, \alpha \rightarrow \beta} (*I/E^g)$$

See [2] (and further references therein) for a similar presentation of the rules for implication in a multiple-conclusion sequent calculus. The rationale for invertibility there, though, is different than the one put forward here, a rationale based on the roles of *I/E*-rules with regards to grounds for assertion. In particular, no appeal is made there to assumption incorporation as an act in ND-derivations.

The big advantage of such invertible rules is that they have trivial reductions and expansions, being balanced by definition.

$$\frac{\frac{\Gamma, \alpha \rightarrow \beta}{\Gamma \rightarrow \gamma} (*I)}{\Gamma, \alpha \rightarrow \beta} (*E^g) \quad \rightsquigarrow_r \quad \Gamma, \alpha \rightarrow \beta$$

and

$$\Gamma \rightarrow \gamma \quad \rightsquigarrow_e \quad \frac{\frac{\Gamma \rightarrow \gamma}{\Gamma, \alpha \rightarrow \beta} (*E^g)}{\Gamma \rightarrow \gamma} (*I)$$

Thus, the proof-theoretic justification of double-line rules is obtained “for free”.

4. A one-premise ($\supset E^g$)-rule

In this section, I show how these ideas are used for obtaining a one-premise (E^g)-rule for implication.

4.1. Defining ($\supset E^g$)

A special case of the non-coincidence of the two roles of (*E*)-rules is that of the material implication. While the immediate consequence is indeed captured by Gentzen’s ($\supset E$)-rule (i.e., (*MP*)), the grounds for asserting the conditional *are not* captured by this rule. The ground for asserting the conditional is *the existence of a conditional proof* of the consequent from the assumed antecedent, the premise of ($\supset I$), and *not* the consequent of the conditional, ψ .

Based on the above considerations, I propose to replace Gentzen's $(\supset E)$ -rule (i.e., modus ponens) by the following (E) -rule $(\supset E^g)$ for the material implication. The presentation here is in the logistic style, as there is no convenient representation of the assumption incorporation involved in the rule (and in derivations based on it) in Prawitz' style.

$$\frac{\Gamma \rightarrow \varphi \supset \psi}{\Gamma, \varphi \rightarrow \psi} (\supset E^g)$$

As is clearly manifested by $(\supset E^g)$, its conclusion *retrieves* the ground for assertion of $\varphi \supset \psi$, embodied in the premise of $(\supset I)$, namely, the existence of a conditional proof of ψ from φ (and any lateral assumptions).

In accordance to the discussion in Section 3.2, we get an invertible, double-line rule for ' \supset ':

$$\frac{\Gamma, \varphi \rightarrow \psi}{\Gamma \rightarrow \varphi \supset \psi} (\supset I/E^g) \quad (4.1)$$

This double-line rule exhibits the symmetry lacking in *NJ*. *Both* a premise and a conclusion can be sub-derivations.

The relationship expressed in (4.1), in various analogical forms, is of course not new.

- It is expressed by the deduction theorem in Hilbert-like axiomatic systems.

$$\Gamma \vdash_H \varphi \supset \psi \text{ iff } \Gamma, \varphi \vdash_H \psi$$

While in *NJ* only one direction of this theorem is internalised as a rule, here *both* directions are.

- The view of an implication connective as an internalisation of the structural meta-linguistic derivability relation ' \vdash ' is discussed in [2] as well as in [24], where a *coherence* double-line rule called *confusion* (*C*)

$$\frac{A \vdash B}{\vdash A \Rightarrow B} (C)$$

is proposed as a means of such an internalisation.

- It is used by Avron [1] as a definition of a binary operator as an *internal implication* of a logic.

What *is* new here is the use of this rule as a natural-deduction *E*-rule (for the material implication) and the motivation of doing so.

The rules ($\supset I/E^g$) are trivially locally-sound and locally-complete, i.e., enjoying both reduction and expansion. The form of the reduction is

$$\frac{\frac{\Gamma, \varphi \rightarrow \psi}{\Gamma \rightarrow \varphi \supset \psi} (\supset I)}{\Gamma, \varphi \rightarrow \psi} (\supset E^g) \quad \rightsquigarrow_r \quad \Gamma, \varphi \rightarrow \psi \quad (4.2)$$

A natural question arising at this point is the following: what is the status of the occurrence of φ in the sequent resulting by the reduction? Clearly, this sequent looks like the premise of the ($\supset I$)-rule application in the source derivation. However, the latter may be an open assumption, not necessarily a result of an ($\supset E$)-rule application. In contrast, the maximal sequent eliminated *is* the result of an application of an ($\supset E$)-rule. Therefore, the occurrence of φ in its context is incorporated.

For a reason to become apparent below, the occurrence of φ in the sequent in the reduced derivation needs to be considered also as incorporated. This leads to the following definition.

DEFINITION 4.1 (incorporation). The occurrence of φ within a sequent $\Gamma, \varphi \rightarrow \psi$ (in a derivation) is *incorporated* iff one of the following two conditions holds:

1. φ entered the sequent as a result of an application of ($\supset E^g$).
2. The sequent is the outcome of a reduction as in (4.2).

Here is an example of a derivation using the proposed rule, having *only one* open assumption (where the importance of this fact will be revealed in the next section).

Example 4.1. $\vdash \varphi \supset (\psi \supset \chi) \rightarrow \psi \supset (\varphi \supset \chi)$

$$\frac{\frac{\frac{\frac{\varphi \supset (\psi \supset \chi) \rightarrow \varphi \supset (\psi \supset \chi)}{\varphi \supset (\psi \supset \chi), \varphi \rightarrow \psi \supset \chi} (\supset E^g)}{\varphi \supset (\psi \supset \chi), \psi, \varphi \rightarrow \chi} (\supset E^g)}{\varphi \supset (\psi \supset \chi), \psi \rightarrow \varphi \supset \chi} (\supset I)}{\varphi \supset (\psi \supset \chi) \rightarrow \psi \supset (\varphi \supset \chi)} (\supset I)$$

In this derivation, φ , and then ψ , are incorporated as assumption during successive eliminations of conditionals, and then discharged in the reverse ordering.

4.2. But what about transitivity of deduction?

Recall that for NJ , transitivity was obtained by pasting derivations, relying on closure under composition (proposition 2.1). Furthermore, recall the comment of [16] as to why pasting derivations is possible. Does the same remain true for the new rule? Unfortunately, not!

To see that, consider the following two derivations \mathcal{D}_1 and \mathcal{D}_2

$$\mathcal{D}_1 : \frac{\varphi \supset \psi \rightarrow \varphi \supset \psi}{\varphi, \varphi \supset \psi \rightarrow \psi} (\supset E^g) \quad \mathcal{D}_2 : \frac{\psi \supset \chi \rightarrow \psi \supset \chi}{\psi, \psi \supset \chi \rightarrow \chi} (\supset E^g)$$

Those two derivations ought to be composable, as they are instances of the premises of the rule in (2.1): they have ψ both as a (succedent of a) conclusion of \mathcal{D}_1 and an open assumption in the antecedent of the conclusion of \mathcal{D}_2 . Composing them according to (2.1) yields:

$$\frac{\frac{\varphi \supset \psi \rightarrow \varphi \supset \psi}{\varphi, \varphi \supset \psi \rightarrow \psi} (\supset E^g) \quad \frac{\psi \supset \chi \rightarrow \psi \supset \chi}{\psi, \psi \supset \chi \rightarrow \chi} (\supset E^g)}{\varphi, \varphi \supset \psi, \psi \supset \chi \rightarrow \chi} ()$$

However, attempting to propagate this “cut” upwards results in failure! The assumption ψ , *incorporated* in the conclusion of \mathcal{D}_2 , *does not occur* anymore as a premise of $(\supset E^g)$. This leads to the following conclusion.

PROPOSITION 4.1 (non-closure under composition). *The system with $(\supset E^g)$ is not closed under composition.*

In order to regain transitivity of deduction, we have to introduce the non-admissible (*Cut*)-rule as a *primitive* rule. Recall that the general form of such a rule is:

$$\frac{\Gamma, \varphi \rightarrow \psi \quad \Gamma \rightarrow \varphi}{\Gamma \rightarrow \psi} (Cut) \tag{4.3}$$

Refer to the occurrence of the cut-formula φ in the second premise as the *concluded occurrence* of φ and to the other occurrence the *assumed occurrence*. The *restricted* (*Cut*)-rule needed here, (*Cut_i*), adheres to the following restriction, the importance of which is the assurance of the *subformula property* (cf. Proposition 4.3):

(A): the assumed occurrence of the cut-formula is an incorporated assumption.

We now can see why the outcome of a reduction removing a maximal sequent needs to be considered incorporated. Consider the following

initial sequents:

$$\frac{}{\Gamma, \varphi \rightarrow \varphi} (Ax)$$

I/E -rules:

$$\frac{\Gamma, \varphi \rightarrow \psi}{\Gamma \rightarrow \varphi \supset \psi} (\supset I) \quad \frac{\Gamma \rightarrow \varphi \supset \psi}{\Gamma, \varphi \rightarrow \psi} (\supset E^g)$$

Cut -rule:

$$\frac{\Gamma_1, \varphi \rightarrow \psi \quad \Gamma_2 \rightarrow \varphi}{\Gamma_1, \Gamma_2 \rightarrow \psi} (Cut_i), \varphi \text{ incorporated in the first premise}$$

Figure 1. The system NJ^g

derivation.

$$\frac{\frac{\frac{\Gamma, \varphi \rightarrow \psi}{\Gamma \rightarrow \varphi \supset \psi} (\supset I)}{\Gamma, \varphi \rightarrow \psi} (\supset E^g) \quad \Gamma \rightarrow \varphi}{\Gamma \rightarrow \psi} (Cut_i)$$

By a reduction removing the maximal sequent $\Gamma \rightarrow \varphi \supset \psi$, the remainder is just the instance of Cut_i . For this instance to be a legal application of the rule, the assumed occurrence of φ in the first premise has to be an incorporated occurrence of φ . Otherwise, the reduction would result in an illegal derivation.

4.3. The system NJ^g

We now can summarise the above discussion by defining the natural-deduction proof-system NJ^g in Figure 1. An immediate conclusion from the structure of the above reduction is the following.

PROPOSITION 4.2 (strong normalizability). *NJ^g is strongly normalisable.*

PROOF. Immediate, since a reduction involves no substitution of derivations, every reduction reduces the number of maximal sequents. \square

PROPOSITION 4.3 (subformula property). *If $\vdash_{NJ^g} \Gamma \rightarrow \varphi$, then any formula χ occurring in an NJ^g -derivation of $\Gamma \rightarrow \psi$ is either a subformula of φ or of some $\psi \in \Gamma$.*

PROOF. Immediate by induction on the NJ^g -derivation, as each rule, including Cut_i , preserves this property. \square

4.4. More examples

Next, consider an example with two open assumptions, putting the primitive (Cut_i) -rule to work. For convenience, although (Cut_i) is formulated as a context-sharing rule, I apply it also with different contexts in the two premises, leaving out some implicit adjustment of the initial sequents. Recall that weakening is admissible in NJ and NJ^g . Contrast this example with Example 2.1.

Example 4.2 (transitivity of implication – again).

$$\begin{array}{c}
 \vdash_{NJ^g} \varphi \supset \psi, \psi \supset \chi \rightarrow \varphi \supset \chi \\
 \\
 \frac{\frac{\psi \supset \chi \rightarrow \psi \supset \chi}{\psi \supset \chi, \psi \rightarrow \chi} (\supset E^g) \quad \frac{\varphi \supset \psi \rightarrow \varphi \supset \psi}{\varphi \supset \psi, \varphi \rightarrow \psi} (\supset E^g)}{\frac{\varphi, \varphi \supset \psi, \psi \supset \chi \rightarrow \chi}{\varphi \supset \psi, \psi \supset \chi \rightarrow \varphi \supset \chi} (\supset I)} (Cut_i)
 \end{array} \tag{4.4}$$

This example exhibits the *backward chaining*, typical to NJ^g , in contrast to the forward chaining exhibited in NJ . Note that the application of (Cut_i) in the above example adheres to the restriction **(A)**, as the assumed occurrence of ψ is indeed an incorporated assumption.

Below is another example, of a formal proof in NJ^g (with no open assumptions).

Example 4.3 (an Hilbert axiom). Below is a derivation for

$$\begin{array}{c}
 \vdash_{NJ^g} \rightarrow ((\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))) \\
 \\
 \frac{\frac{(\varphi \supset (\psi \supset \chi)) \rightarrow (\varphi \supset (\psi \supset \chi))}{(\varphi \supset (\psi \supset \chi)), \varphi \rightarrow (\psi \supset \chi)} (\supset E^g)}{\frac{(\varphi \supset (\psi \supset \chi)), \varphi, \psi \rightarrow \chi}{\varphi, (\varphi \supset \psi), ((\varphi \supset (\psi \supset \chi)) \rightarrow \chi)} (\supset E^g)} (\supset I) \\
 \frac{\frac{\varphi, (\varphi \supset \psi), ((\varphi \supset (\psi \supset \chi)) \rightarrow \chi)}{(\varphi \supset \psi), ((\varphi \supset (\psi \supset \chi)) \rightarrow (\varphi \supset \chi))} (\supset I)}{\frac{(\varphi \supset (\psi \supset \chi)) \rightarrow ((\varphi \supset \psi) \supset (\varphi \supset \chi))}{((\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi)))} (\supset I)} (Cut_i)
 \end{array}$$

4.5. Deductive equivalence of NJ and NJ^g

A natural question to ask at this stage is, what is the relative deductive strength of NJ^g in comparison to NJ . In particular, what about modus-ponens? One would hate to lose such a fundamental rule.

The answer is given by the following propositions, establishing the deductive equivalence of the two system. The two systems share their initial sequents and their $(\supset I)$ -rules. Therefore, to show deductive equivalence, what is needed is:

- Show that the rules $(\supset E)$ and $(\supset E^g)$ are each derivable in the other system.
- Show that (Cut_i) is derivable in NJ .

Thus, while (MP) is no longer the “official” (E) -rule for ‘ \supset ’, it is a *derivable rule*, and the ability to infer the consequent of an implication when given its antecedent is not lost.

PROPOSITION 4.4 (derivability of (MP) in NJ^g). *The rule*

$$\frac{\Gamma \rightarrow \varphi \supset \psi \quad \Gamma \rightarrow \varphi}{\Gamma \rightarrow \psi} (MP)$$

is derivable in NJ^g .

PROOF. The derivation is:

$$\frac{\frac{\Gamma \rightarrow \varphi \supset \psi}{\Gamma, \varphi \rightarrow \psi} (\supset E^g) \quad \Gamma \rightarrow \varphi}{\Gamma \rightarrow \psi} (Cut_i)$$

Note the dependency of this derivation on (Cut_i) , observing restriction **(A)**. □

PROPOSITION 4.5 (derivability of $(\supset E^g)$ in NJ). *The rule*

$$\frac{\Gamma \rightarrow \varphi \supset \psi}{\Gamma, \varphi \rightarrow \psi} (\supset E^g)$$

is derivable in NJ .

PROOF. The derivation is

$$\frac{\frac{\Gamma \rightarrow \varphi \supset \psi}{\Gamma, \varphi \rightarrow \varphi \supset \psi} (W) \quad \frac{}{\Gamma, \varphi \rightarrow \varphi} (Ax)}{\Gamma, \varphi \rightarrow \psi} (MP)$$

□

PROPOSITION 4.6 (derivability of (Cut_i) in NJ). *The rule*

$$\frac{\Gamma_1, \varphi \rightarrow \psi \quad \Gamma_2 \rightarrow \varphi}{\Gamma_1, \Gamma_2 \rightarrow \psi} (Cut_i)$$

is derivable in NJ . Actually, the stronger *Cut* (without the restriction **(A)**) is derivable.

PROOF. The derivation is

$$\frac{\frac{\frac{\Gamma_1, \varphi \rightarrow \psi}{\Gamma_1 \rightarrow \varphi \supset \psi} (\supset I) \quad \frac{\Gamma_2 \rightarrow \varphi}{\Gamma_1, \Gamma_2 \rightarrow \varphi} (W)}{\Gamma_1, \Gamma_2 \rightarrow \varphi \supset \psi} (W) \quad \frac{\Gamma_2 \rightarrow \varphi}{\Gamma_1, \Gamma_2 \rightarrow \varphi} (W)}{\Gamma_1, \Gamma_2 \rightarrow \psi} (MP) \quad \square$$

Note the use of (W) (Weakening), admissible in NJ , in the above derivations.

5. Term assignment to NJ^g

5.1. Introduction

In this section, I propose a term-assignment system λ^g -terms for NJ^g . Traditionally, term-systems are essentially viewed in two ways, both incorporating the Curry-Howard correspondence:

proof-terms: Under this view, terms *encode derivations*, so that given a context assigning variables to the open assumptions and a term, possibly having free occurrences of the variables in the context, for the conclusion, it is possible to reconstruct a derivation of the conclusion from the assumptions (in the proof-system to which terms are assigned). The assignment of a term M to a formula φ under this view is denoted $\varphi : M$.

typing system: Under this view, the formulas are viewed as types, and the terms are inhabitants of the type. The judgement is usually expressed as $M : \varphi$, and understood as M having the type φ in the context assigning types (of assumptions) to the free variables of M .

I am interested here in the first view and therefore use the notation $\varphi : M$ for specifying term-assignments.

5.2. λ^g -terms

The λ^g -terms differ from the traditional λ -terms in the simply-typed λ -calculus [11] (the latter serving as proof-terms for NJ) in the employment of an additional term-constructor κ (corresponding to (Cut_i)) and an additional reduction rule. Application terms, while present, serve a role somewhat different than in the λ -calculus.

We are given a set of variables, V , ranged over by x, y etc., possibly subscripted. I use M, N as meta-variables over λ^g -terms.

DEFINITION 5.1 (λ^g -terms). The set T_λ^g of λ^g -terms is the smallest set satisfying the following clauses.

1. If $x \in V$ then $x \in T_\lambda^g$.
2. If $M \in T_\lambda^g$ and $x \in V$, then $\lambda^g x.M \in T_\lambda^g$.
3. If $M, N \in T_\lambda^g$, then $(MN) \in T_\lambda^g$.
4. If $M, N \in T_\lambda^g$ and $x \in V$, then $\kappa(M, x, N) \in T_\lambda^g$.

Free and bound occurrences of variables in λ^g -terms is like the usual definition in the λ -calculus, with the extra provision that free occurrences of x in M are bound by $\kappa(M, x, N)$.

DEFINITION 5.2 (term association, context). 1. A *term-association* is a pair $\varphi : M$, where $M \in T_\lambda^g$. The term M is the *subject* of the term-association.
 2. A *context* Γ is a finite (possibly empty) collection of term-associations $\varphi_1 : x_1, \dots, \varphi_n : x_n$, $n \geq 0$, with pairwise distinct x_i s.

DEFINITION 5.3 (term equalities). The following term-equation is added in T_λ^g to the usual equalities for λ -terms.

- $\kappa(M, x, N) = M[x := N]$, substituting N for all free occurrences of x in M .

5.3. Term-decorated NJ^g -rules

Below are the term-decorated rules of NJ^g . Here Γ is a context. An initial sequent is of the form

$$\begin{array}{c} \Gamma, \varphi : x \rightarrow \varphi : x \\[10pt] \frac{\Gamma, \varphi : x \rightarrow \psi : M}{\Gamma \rightarrow \varphi \supset \psi : \lambda^g x.M} (\supset I) \quad \frac{\Gamma \rightarrow \varphi \supset \psi : M}{\Gamma, \varphi : z \rightarrow \psi : (Mz)} (\supset E^g), \quad z \text{ fresh} \\[10pt] \frac{\Gamma_1, \varphi : x \rightarrow \psi : M \quad \Gamma_2 \rightarrow \varphi : N}{\Gamma_1, \Gamma_2 \rightarrow \psi : \kappa(M, x, N)} (Cut_i) \end{array}$$

Note how the decoration of the premises with proof-terms respects the restriction **(A)** on (Cut_i) . I use NJ^g ambiguously for both the original ND -system and its term-decorated version.

Below I repeat the previous example derivations, this time with the associated proof-terms.

Example 5.1.

$$\vdash_{NJ^g} \varphi \supset (\psi \supset \chi) : x \rightarrow \psi \supset (\varphi \supset \chi) : \lambda^g v. \lambda^g u. ((xu)v)$$

$$\begin{array}{c}
\frac{\varphi \supset (\psi \supset \chi) : x \rightarrow \varphi \supset (\psi \supset \chi) : x}{\varphi \supset (\psi \supset \chi) : x, \varphi : u \rightarrow \psi \supset \chi : (xu)} (\supset E^g) \\
\frac{\varphi \supset (\psi \supset \chi) : x, \varphi : u, \psi : v, \rightarrow \chi : ((xu)v)}{\varphi \supset (\psi \supset \chi) : x, \psi : v \rightarrow \varphi \supset \chi : \lambda^g u.((xu)v)} (\supset I) \\
\frac{\varphi \supset (\psi \supset \chi) : x \rightarrow \psi \supset (\varphi \supset \chi) : \lambda^g v. \lambda^g u.((xu)v)}{\varphi \supset (\psi \supset \chi) : x \rightarrow \psi \supset (\varphi \supset \chi) : \lambda^g v. \lambda^g u.((xu)v)} (\supset I)
\end{array}$$

Example 5.2 (transitivity of implication – with proof-terms).

$$\vdash_{NJ^g} \varphi \supset \psi : x, \psi \supset \chi : y \rightarrow \varphi \supset \chi : \lambda^g z. (y(xz))$$

Abbreviate $\varphi \supset \psi : x, \psi \supset \chi : y$ to Γ .

$$\begin{array}{c}
\frac{\Gamma \rightarrow \psi \supset \chi : y}{\Gamma, \psi : v \rightarrow \chi : (yv)} (\supset E^g) \quad \frac{\Gamma \rightarrow \varphi \supset \psi : x}{\Gamma, \varphi : z \rightarrow \psi : (xz)} (\supset E^g) \\
\frac{\Gamma, \varphi : u \rightarrow \chi : \kappa((yv), v, (xu))}{\Gamma, \varphi : u \rightarrow \chi : \kappa((yv), v, (xu))} (Cut_i) \\
\frac{\Gamma, \varphi : u \rightarrow \chi : \kappa((yv), v, (xu))}{\varphi \supset \psi : x, \psi \supset \chi : y \rightarrow \varphi \supset \chi : \lambda^g z. \kappa((yv), v, (xz))} (\supset I)
\end{array}$$

The required final term associated with the conclusion, $\lambda^g z. (y(xz))$, results after simplification of the κ -subterm according to the equality from Definition 5.3.

The following *subject construction* theorem⁷ expresses the matching between the structure of a λ^g -term M with a derivation of a conclusion ψ decorated with M from a context determined by the variables in M . In other words, M allows reconstructing the NJ^g -derivation from Γ . The theorem is an extension of the corresponding theorem for TA_λ , modifying the ‘ \supset ’-elimination case and adding a clause for κ -terms.

THEOREM 5.1 (subject construction). *Consider an NJ^g -derivation \mathcal{D} establishing $\vdash_{NJ^g} \Gamma \rightarrow \psi : M$ for some $M \in T_\lambda^g$. Then:*

1. *If M is $x \in V$, then for some Γ_1 , $\Gamma = \Gamma_1, \psi : x$ and \mathcal{D} is the initial sequent $\Gamma_1, \psi : x \rightarrow \psi : x$.*
2. *If for some $N \in T_\lambda^g$, $x \in V$ and Γ in which x is not free, it holds that M is $\lambda^g x. N$, then for some φ, χ it holds that $\psi = \varphi \supset \chi$ and the last rule applied in \mathcal{D} is*

$$\frac{\Gamma, \varphi : x \rightarrow \psi : N}{\Gamma \rightarrow \varphi \supset \psi : \lambda^g x. N} (\supset I)$$

⁷ I preserve the name of the theorem to keep the analogy with the simply-typed λ -calculus.

3. If for some $N \in T_\lambda^g$ and $x \in V$ M is (Nx) , then for some Γ', φ, χ it holds that $\Gamma = \Gamma', \varphi : x$ and the last rule applied in \mathcal{D} is

$$\frac{\Gamma' \rightarrow \varphi \supset \psi : N}{\Gamma', \varphi : x \rightarrow \psi : (Nx)} (\supset E^g)$$

4. If for some $P, Q \in T_\lambda^g$ and $x \in V$ it holds that M is $\kappa(P, x, Q)$, then for some $\Gamma_1, \Gamma_2, \varphi$ s.t. x is not free in $\Gamma_1 \Gamma_2$, it holds that $\Gamma = \Gamma_1, \Gamma_2$, and the last rule applied in \mathcal{D} is

$$\frac{\Gamma_2, \varphi : x \rightarrow \psi : P \quad \Gamma_1 \rightarrow \varphi : Q}{\Gamma_1, \Gamma_2 \rightarrow \psi : \kappa(P, x, Q)} (Cut_i)$$

The proof is routine and is left out. Note that in the absence of (MP) as an E -rule, the terms (MN) for N not a variable do not code derivations.

6. Adding negation

Recall the usual NJ -strategy of defining negation by

$$\neg \varphi := \varphi \supset \perp$$

This definition leads to the following $(\neg I/E)$ -rules.

$$\frac{[\varphi]_i \quad \vdots \quad \perp}{\varphi \supset \perp} (\neg I^i) \qquad \frac{\neg \varphi \quad \varphi}{\perp} (\neg E)$$

where $(\neg E)$ is an instance of (MP) .

By adopting this convention into NJ^g , we get the following double-line $(\neg I/E^g)$ -rule, again avoiding a minor premise for $(\neg E)$:

$$\frac{\Gamma, \varphi \rightarrow \perp}{\Gamma \rightarrow \neg \varphi} (\neg I/E^g)$$

where the $(\neg E^g)$ -rule retrieves the grounds for asserting $\neg \varphi$: the existence of a derivation of absurdity from φ , as expressed by the $(\neg I)$ -rule.

Clearly, the deductive equivalence of NJ^g and NJ is preserved by this inclusion of negation.

With the above definition of negation, The rule Modus-Tollens (MT) is derivable in NJ^g .

PROPOSITION 6.1 (derivability of (MT)). *The rule*

$$\frac{\Gamma \rightarrow \varphi \supset \psi \quad \Gamma \rightarrow \neg\psi}{\Gamma \rightarrow \neg\varphi} (MT)$$

is derivable in NJ^g .

PROOF. The derivation is an instance of (4.4) with χ being \perp . \square

7. Conclusions

The paper introduces NJ^g , a variant of Gentzen's NJ natural deduction system, in which the implication elimination rule has no minor premise. The NJ^g -systems extends traditional ND-system with a new kind of action in derivations, assumption incorporation, a kind of dual to the assumption discharge action. As a result, the implication (I/E) -rules are invertible and, almost by definition, harmonious and stable, a major condition imposed by PTS on ND-systems to qualify as meaning-conferring. There is also a proof-term assignment to NJ^g -derivations, materialising the Curry-Howard correspondence for this system.

There is a more general observation emerging from NJ^g about ND-systems in general. Traditionally, they are seen (according to Gentzen himself) as a wish to embody in formal system the activity of a mathematician informally proving theorems. However, since the emergence of PTS, according to the Dummett-Prawitz methodological views, ND-systems are now seen as a meaning-conferring, definitional tool. Their role as such need not coincide with their role as a formalisation of mathematical reasoning. For example, Gentzen's strict partition of ND-rules to I-rules and E-rules is not so essential for formalising mathematical proof, but is of utmost methodological importance for meaning-conferring. This can be seen by comparing Gentzen's ND-systems with those of J askowski [13, 14], where the latter are not so strictly divided into (I/E) -rules. For a detailed discussion of those differences (viewed only from the traditional view of formalising mathematics), see [9].

It is possible to extend the current work to the full intuitionistic propositional logic. However, the natural grounds-retrieving $(\vee E^g)$ -rule, assuming Gentzen's $(\vee I)$ -rules, would be

$$\frac{\Gamma \rightarrow \varphi \vee \psi}{\Gamma \rightarrow \varphi, \psi} (\vee E^g)$$

involving a transition to a logistic multiple-conclusions ND-system which I will not do here.

Acknowledgments. I thank Roy Dyckhoff and Shay Logan for useful feedback on earlier drafts of this paper. I thank two anonymous referees of this journal for some critical remarks that helped to improve the presentation.

References

- [1] Avron, A., “Simple consequence relations”, *Information and Computation* 92, 1 (1991): 105–139. DOI: [10.1016/0890-5401\(91\)90023-U](https://doi.org/10.1016/0890-5401(91)90023-U)
- [2] Došen, K., “Logical constants s punctuation marks”, *Notre Dame Journal of Formal Logic* 30, 3 (1989): 362–381. PDF: https://projecteuclid.org/download/pdf_1/euclid.ndjfl/1093635154
- [3] Dummett, M., *The Logical Basis of Metaphysics*, Harvard University Press, Cambridge, MA., 1993 (paperback); hard copy 1991.
- [4] Francez, N., *Proof-theoretic Semantics*, College Publications, London, 2015.
- [5] Francez, N., “Relevant harmony”, *Journal of Logic and Computation* 26, 1 (2016): 235–245. Special issue *Logic: Between Semantics and Proof Theory*, in honor of Arnon Avron’s 60th birthday. DOI: [10.1093/logcom/ext026](https://doi.org/10.1093/logcom/ext026)
- [6] Francez, N., “Views of proof-theoretic semantics: Reified proof-theoretic meanings”, *Journal of Computational Logic* 26, 2 (2016): 479–494. Special issue in honour of Roy Dyckhoff. DOI: [10.1093/logcom/exu035](https://doi.org/10.1093/logcom/exu035)
- [7] Gentzen, G., “The consistency of elementary number theory”, pages 493–565 in M. E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, 1935. English translation of the 1935 paper in *Mathematische Annalen* (in German).
- [8] Gentzen, G., “Investigations into logical deduction”, pages 68–131 in M. E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, 1935. English translation of the 1935 paper in German.
- [9] Hazen, A. P., and F. J. Pelletier, “Gentzen and Jaśkowski natural deduction: Fundamentally similar but importantly different”, *Studia Logica* 102, 6 (2014): 1–40. Special Issue: “Gentzen’s and Jaśkowski’s Heritage 80 Years of Natural Deduction and Sequent Calculi”, edited by A. Indrzejczak. DOI: [10.1007/s11225-014-9564-1](https://doi.org/10.1007/s11225-014-9564-1)

- [10] Hindley, J. R., *Basic Simple Type Theory*, Cambridge University Press, 1997. DOI: [10.1017/CB09780511608865](https://doi.org/10.1017/CB09780511608865)
- [11] Hitzmann, J., “Semantic partition and the ambiguity of sentences containing temporal adverbials”, *J. of Semantics*, in press, 1997.
- [12] Jacinto, B., and S. Read, “General-elimination stability”, *Studia Logica* 105, 2 (2017): 361–405. DOI: [10.1007/s11225-016-9692-x](https://doi.org/10.1007/s11225-016-9692-x)
- [13] Jaśkowski, S., “Teoria dedukcji oparta na regułach założeniowych” (Theory of deduction based on suppositional rules), page 36 in *Księga pamiątkowa pierwszego polskiego zjazdu matematycznego* (Proceedings of the First Polish Mathematical Congress), Polish Mathematical Society, Kraków, 1929.
- [14] Jaśkowski, S., “On the rules of suppositions in formal logic”, *Studia Logica*, 1 (1936): 5–32. Reprinted: pages 232–258 in S. McCall, *Polish Logic 1920–1939*, Oxford UP, 1967.
- [15] Johansson, I., “Der minimalkalkül, ein reduzierter intuitionistischer formalismus”, *Compositio Mathematica* 4 (1936): 119–136.
- [16] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge University Press, Cambridge, UK, 2001. DOI: [10.1017/CB09780511527340](https://doi.org/10.1017/CB09780511527340)
- [17] Prawitz, D., *Natural Deduction: A Proof-Theoretical Study*, Almqvist and Wicksell, Stockholm, 1965. Soft cover edition by Dover, 2006.
- [18] Prawitz, D., “Meaning approached via proofs”, *Synthese* 148, 3 (2006): 507–524. DOI: [10.1007/s11229-004-6295-2](https://doi.org/10.1007/s11229-004-6295-2)
- [19] Prawitz, D., “Explaining deductive inference”, pages 233–248 in H. Wansing (ed.), *Dag Prawitz on proofs and meaning*, Springer, 2014. Volume 7 of the series “Outstanding Contributions to Logic”. DOI: [10.1007/978-3-319-11041-7_3](https://doi.org/10.1007/978-3-319-11041-7_3)
- [20] Schroeder-Heister, P., “A natural extension of natural deduction”, *Journal of Symbolic Logic* 49, 4 (1984): 1284–1300. DOI: [10.2307/2274279](https://doi.org/10.2307/2274279)
- [21] Schroeder-Heister, P., “On the notion of *assumption* in logical systems”, in R. Bluhm and C. Nimtz (eds.), *Philosophy-Science-Scientific Philosophy*, Mentis, Paderborn, 2004. Selected papers of the 5th int. congress of the society for Analytic Philosophy, Bielfield, September 2003.
- [22] Schroeder-Heister, P., “The categorical and the hypothetical: A critique of some fundamental assumptions of standard semantics”, *Synthese* 187, 3 (2012): 925–942. DOI: [10.1007/s11229-011-9910-z](https://doi.org/10.1007/s11229-011-9910-z)
- [23] Schroeder-Heister, P., “Proof-theoretic semantics”, in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Spring 2013 edition, 2013. <https://plato.stanford.edu/archives/win2016/entries/proof-theoretic-semantics/>

- [24] Scott, D., “Rules and derived rules”, pages 101–118 in S. Stenlund (ed.), *Logical Theory and Semantic Analysis*, Reidl, Dordrecht, 1974. DOI: [10.1007/978-94-010-2191-3_13](https://doi.org/10.1007/978-94-010-2191-3_13)

NISSIM FRANCEZ
Computer Science Department
the Technion-IIT
Haifa, Israel
`francez@cs.technion.ac.il`