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On an Intuitionistic Modal Logic

Abstract. In this paper we consider an intuitionistic variant of the modal logic **S4** (which we call **IS4**). The novelty of this paper is that we place particular importance on the natural deduction formulation of **IS4** — our formulation has several important metatheoretic properties. In addition, we study models of **IS4** — not in the framework of Kripke semantics, but in the more general framework of category theory. This allows not only a more abstract definition of a whole *class* of models but also a means of modelling proofs as well as provability.

Keywords: intuitionistic logic, modal logic, proof theory, categorical models.

1. Introduction

Modal logics are traditionally extensions of *classical* logic with new operators, or modalities, whose operation is intensional. Modal logics are most commonly justified by the provision of an intuitive semantics based upon 'possible worlds', an idea originally due to Kripke. Kripke also provided a possible worlds semantics for intuitionistic logic, and so it is natural to consider intuitionistic logic extended with intensional modalities. Indeed much work already exists on this topic (a good account is given by Simpson [37]). In fact there is an almost bewildering number of intuitionistic modal logics — most of which consist of extensions of intuitionistic logic with some selection of modal rules and axioms. Much of this work is justified by reference to a Kripke model — the number of choices arises from the large number of combinations of conditions one can impose on the accessibility relations.

In this paper we take a different approach. Rather than use Kripke models, we are motivated by the much more general language of category theory. One reason is that, unlike the situation for Kripke semantics, we are interested in modelling not just provability but also the proofs themselves. This approach is often termed categorical proof theory, or simply categorical logic [31]. Category theory provides a language for describing abstractly what is required of a model or, more precisely, which structures are needed for an arbitrary category to model the logic. Checking that a candidate is a concrete model then simplifies to checking that it satisfies the abstract definition. Thus soundness, for example, need only be checked once and for

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all, for the abstract definition. All concrete models which satisfy the abstract definition are guaranteed to be sound. In this way categorical semantics provide a general and often simple formulation of what it is to be a model. This is of interest because it is often the case that more traditional models lack any generality or are quite complicated to describe (or both). In addition categorical semantics enable one to model some very powerful logics such as impredicative type theories and various higher order logics.

Another important aspect of our approach is our interest in proof theory. Most work on (intuitionistic) modal logics simply gives the logic using an axiomatic formulation, the primary interest being provability. In contrast we are interested in (primarily) natural deduction and sequent calculus formulations, and their metatheoretic properties. (Interestingly, Satre [35] gives natural deduction formulations of (classical) modal logics, but does not study their metatheoretic properties. This point will be addressed more fully in §4.)

This paper is not the place for a philosophical discussion of the significance, or importance, of intuitionistic modal logics in general. However it is worth highlighting a number of important applications of intuitionistic modal logics in computer science. Stirling [38] uses an intuitionistic modal logic to capture a notion of bisimilarity of divergent processes. Fairtlough and Mendler [15] use an intuitionistic modal logic [4] to reason about the behaviours of hardware circuits. Davies and Pfenning [13] have used part of the logic described in this paper to define a programming language with explicit binding time constructors.

In this paper we study an intuitionistic version of the modal logic **S4**. As we shall see, given our interests in proof theory and categorical models, there is quite a lot to say about this logic. Extensions of our approach to other intuitionistic modal logics remains future work. The paper is organised as follows. In §§ 2–3 we give axiomatic and sequent calculus formulations of **IS4**, respectively. In §4 we consider the definition of a natural deduction formulation due to Satre [35] and show that it is not closed under substitution. We also consider Prawitz's proposal [32], which we also reject. We give our formulation and demonstrate a number of important properties. In §4.1 we consider the question of proof normalisation and consider the question of the subformula property for our natural deduction formulation. We give details of commuting conversions which are sufficient to derive a subformula property. In §5 we sketch some details of an alternative natural deduction formulation which uses two assumption sets — one for modalised formulae, and one for non-modalised formulae. In §6 we define the λ^{S4}-calculus, which is given by the Curry-Howard correspondence from our natural deduction formulation. In §7 we give in detail our categorical analysis of both the necessity and possibility modalities. We give a sound definition of a categorical model for **IS4**. In §8 we give details of the serious proof and model theoretic problems of Prawitz's natural deduction formulation of the necessity modality. In §9 we give some details of an alternative possibility modality.

2. An axiomatic formulation of IS4

In this paper we shall only consider propositional **IS4**. Consequently formulae are given by the grammar

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A,$$

where p is taken from some countable set of propositional constants.

In this section we give an axiomatic, or Hilbert-style, formulation of **IS4**. As discussed in the introduction, this is the method favoured by most modal logicians. An axiomatic formulation uses a set of axioms and a handful of rules. To tie up with subsequent sections on sequent calculus and natural deduction formulations, we shall present deduction in an axiomatic formulation using the notation $\Gamma \vdash A$, where Γ is a set of formulae. The interpretation of $\Gamma \vdash A$ is that from the assumptions contained in Γ one can deduce A. (Clearly if Γ is empty, then A is a *theorem*.) We first fix our collection of axioms, which consist of those for intuitionistic logic (see, for example [12]) along with the following axioms for the modalities.

$$\begin{array}{l}
\Box(A \supset B) \supset (\Box A \supset \Box B) \\
\Box A \supset A \\
\Box A \supset \Box\Box A \\
\Box(A \supset \Diamond B) \supset (\Diamond A \supset \Diamond B) \\
A \supset \Diamond A
\end{array}$$

Two deduction rules are immediate, i.e.

$$\frac{}{\Gamma, A \vdash A}\text{Identity} \qquad \text{and} \qquad \frac{}{\Gamma \vdash A}\text{Axiom}$$

The Identity rule is clear: if we assume A, then we can deduce A. The Axiom rule is only permitted when A is (a substitution instance of) an axiom. We also have the familiar rule

$$\frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\text{Modus Ponens.}$$

The rule for introducing a necessity modality is as follows.

$$\frac{\vdash A}{\Gamma \vdash \Box A}\text{Box}$$

Axioms:

1	$A \supset (B \supset A)$
2	$(A \supset B \supset C) \supset ((A \supset B) \supset (A \supset C))$
3	$A \supset (B \supset A \wedge B)$
4	$A \wedge B \supset A$
5	$A \wedge B \supset B$
6	$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
7	$A \supset A \vee B$
8	$B \supset A \vee B$
9	$\perp \supset A$
10	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
11	$\Box A \supset A$
12	$\Box A \supset \Box\Box A$
13	$\Box(A \supset \Diamond B) \supset (\Diamond A \supset \Diamond B)$
14	$A \supset \Diamond A$

Rules:

$$\frac{}{\Gamma, A \vdash A}\ \text{Identity} \qquad \frac{}{\Gamma \vdash A}\ \text{Axiom}$$

$$\frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \text{Modus Ponens}$$

$$\frac{\vdash A}{\Gamma \vdash \Box A}\ \text{Box}$$

Figure 1. Axiomatic formulation of **IS4**.

Notice that we require A to be a theorem. The complete axiomatisation of **IS4** is given in Figure 1. It is relatively straightforward to demonstrate three important properties of our formulation.[1]

THEOREM 1. *1. If* $\Gamma \vdash B$ *then* $\Gamma, A \vdash B$.

2. If $\Gamma, A \vdash B$ *then* $\Gamma \vdash A \supset B$.

3. If $\Box\Gamma \vdash A$ *then* $\Box\Gamma \vdash \Box A$.

The second of these properties is often called the *deduction theorem.* Where it is not obvious by context, a deduction in the axiomatic formulation will be prefixed by an annotated turnstile, $\vdash_A$.

[1] These properties hold by a straightforward induction. To save space, in the rest of this paper, facts which can be proven by a simple induction will be stated without proof.

$$\frac{}{\Delta, A \vdash A}\,\text{Identity}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\,\text{Cut}$$

$$\frac{}{\Gamma, \perp \vdash A}\,\perp\mathcal{L}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\,\vee\mathcal{L} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee\mathcal{R} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee\mathcal{R}$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge\mathcal{L} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge\mathcal{L} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge\mathcal{R}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}\,\supset\mathcal{L} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset\mathcal{R}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\,\Box\mathcal{L} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A}\,\Box\mathcal{R}$$

$$\frac{\Box\Gamma, A \vdash \Diamond B}{\Delta, \Box\Gamma, \Diamond A \vdash \Diamond B}\,\Diamond\mathcal{L} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\,\Diamond\mathcal{R}$$

Figure 2. Sequent calculus formulation of **IS4**.

3. A sequent calculus formulation of IS4

Gentzen's sequent calculus is an important tool for formulating logics. However, as Bull and Segerberg point out [9, §2], it has never really flourished in modal logic (some important exceptions include [11] and [22]). Deductions in the sequent calculus are also written $\Gamma \vdash A$ where Γ is a set of formulae. There are rules for introducing each connective on the left and the right of the turnstile. The complete formulation is given in Figure 2.

An important property of this formulation is that we have an admissible 'weakening' rule (this explains the presence of the Δ in the Identity, $\Box\,\mathcal{R}$ and $\Diamond\,\mathcal{L}$ rules).

Proposition 1. *If* $\Gamma \vdash B$ *then* $\Gamma, A \vdash B$.

The sequent calculus formulation, where we use the symbol $\vdash_S$ to represent a sequent deduction, can be shown to be equivalent to the axiomatic presentation given in the previous section.

THEOREM 2. $\vdash_S \Gamma \vdash A$ *iff* $\vdash_A \Gamma \vdash A$.

An important property of sequent calculus formulations is the so-called cut-elimination theorem — a concept due to Gentzen [18]. Instances of the Cut rule are systematically replaced with instances on smaller proofs (the technical details are a little delicate; Gallier [17] gives a nice explanation). The two important new important cases for **IS4** are as follows.

$$\dfrac{\dfrac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\Box\mathcal{R} \qquad \dfrac{\Delta, A \vdash B}{\Delta, \Box A \vdash B}\,\Box\mathcal{L}}{\Box\Gamma, \Delta \vdash B}\,\text{Cut}$$

which is rewritten to

$$\dfrac{\Box\Gamma \vdash A \qquad \Delta, A \vdash B}{\Box\Gamma, \Delta \vdash B}\,\text{Cut}$$

and

$$\dfrac{\dfrac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\,\Diamond\mathcal{R} \qquad \dfrac{\Box\Delta, A \vdash \Diamond B}{\Box\Delta, \Diamond A \vdash \Diamond B}\,\Diamond\mathcal{L}}{\Gamma, \Box\Delta \vdash \Diamond B}\,\text{Cut}$$

which is rewritten to

$$\dfrac{\Gamma \vdash A \qquad \Box\Delta, A \vdash \Diamond B}{\Gamma, \Box\Delta \vdash \Diamond B}\,\text{Cut.}$$

It is possible to show that all occurrences of the Cut rule can be systematically removed from an **IS4** sequent calculus deduction. This is a version of Gentzen's so-called Hauptsatz.

THEOREM 3. *Given a derivation π of $\Gamma \vdash A$, a derivation π' of $\Gamma \vdash A$ can be found which contains no instances of the* Cut *rule.*

PROOF. By a routine adaptation of Gentzen's method [18]. ■

An important consequence of this theorem is the *subformula property*. A fact about our sequent calculus formulation is that every rule except the Cut rule has the property that the premises are made up of subformulae of the conclusion.

PROPOSITION 2. *In a cut-free proof of $\Gamma \vdash A$, all the formulae which occur within the proof are contained in the set of subformulae of Γ and A.*

A number of interesting facts can be derived from this property — these have been studied by Schwichtenberg [36]. As we shall see in the next section, this property is not so straightforward for a natural deduction formulation.

4. A natural deduction formulation of IS4

In a natural deduction system, originally due to Gentzen [18], but subsequently expounded by Prawitz [32], a deduction is a derivation of a proposition from a finite set of assumptions using some predefined set of inference rules. Assumptions are assumed to have been labelled by markers, which we will write as A^x. Assumptions of the same form with the same labels form an assumption *class*. Within a deduction, we may 'discharge' any number of assumption classes. The applications of these inference rules are annotated with the labels of those classes which they discharge.

The natural deduction formulation of intuitionistic logic is well-known and uncontroversial. The difficulties arise in dealing with the modalities. For example, take the paper of Satre [35]. There he defines the necessity rules of **S4** as follows [35, p468].

$$\begin{array}{c}\Gamma\\ \vdots\\ \dfrac{A}{\Box A}\,\Box\mathcal{I}\end{array} \qquad\text{and}\qquad \begin{array}{c}\vdots\\ \dfrac{\Box A}{A}\,\Box\mathcal{E}\end{array}$$

with the proviso that the introduction rule is only well-formed if the formulae contained in Γ are all of the form $\Box C$, for some formula C. Whilst adequate with respect to provability, this rule suffers from a serious defect: it is not closed under substitution of deductions. For example take the deduction

$$\begin{array}{c}\Box A_1 \cdots \Box A_k\\ \vdots\\ \dfrac{B}{\Box B}\,\Box\mathcal{I}.\end{array}$$

Consider substituting for the open assumption $\Box A_1$, the deduction

$$\dfrac{C \supset \Box A_1 \qquad C}{\Box A_1} \supset\mathcal{E}.$$

We then get the following deduction

$$\begin{array}{c}\dfrac{C \supset \Box A_1 \qquad C}{\Box A_1} \supset\mathcal{E} \qquad \cdots \Box A_k\\ \vdots\\ \dfrac{B}{\Box B}\,\Box\mathcal{I}\end{array}$$

which is clearly *not* a valid deduction as the assumptions are not all boxed.

This defect was anticipated by Prawitz [32, Chapter VI] (surprisingly, Prawitz was not cited by Satre [35]). He highlights the problem as a lack of closure under proof normalisation, but this problem actually arises because of the lack of closure under substitution. Prawitz provides a solution using the notion of "essentially modal formulae". What this amounts to is a relaxation of the restriction on the introduction rule that all undischarged formulae must be modal. It is replaced with the restriction that for each formula C in Γ, such that A depends on C, there is a modal formula $\Box B$ in the path from A to C. Diagrammatically this can be given as

$$\begin{array}{c} \begin{array}{ccc} \Delta_1 & & \Delta_k \\ \vdots & & \vdots \\ \Box B_1 & \cdots & \Box B_k \end{array} \\ \vdots \\ \dfrac{A}{\Box A}\ \Box\mathcal{I}. \end{array}$$

It is important to notice that there may be several modal formulae in any given path. One of the contributions of this paper is to show that this fact has serious proof and model theoretic consequences — these are detailed in §8.

Consequently, we shall use a different formulation which is both closed under substitution of deductions and has pleasant proof and model theoretic properties. Our introduction rule is as follows.

$$\dfrac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\ \ \ldots\ \ \begin{array}{c}\vdots\\ \Box A_k\end{array}\qquad \begin{array}{c}[\![\Box A_1^{x_1}\cdots\Box A_k^{x_k}]\!]\\ \vdots\\ B\end{array}}{\Box B}\ \Box\mathcal{I}_{x_1,\ldots,x_k}$$

Thus we insist not only that all the undischarged assumptions of the deduction of B are modal, but we then discharge them all (this is emphasised by the use of semantic braces $[\![\cdots]\!]$) and reintroduce them. (As mentioned earlier, we annotate the inference rules with the labels of the assumption classes which have been discharged.)

Remark 1. Curiously, our introduction rule actually appears in both [35] and [27, Page 242] (although we were unaware of this at the time). However neither author seems aware of its proof theoretic significance, or of the problems with Satre's formulation [35]. ∎

There are similar difficulties in formulating the elimination rule for the possibility modality. Our formulation is as follows.

$$\dfrac{\begin{array}{c}\vdots\\ \Diamond B\end{array}\quad \begin{array}{c}\vdots\\ \Box A_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ \Box A_k\end{array}\qquad \begin{array}{c}[\![\Box A_1^{x_1}\cdots\Box A_k^{x_k} B^{x_{k+1}}]\!]\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\ \Diamond\,\mathcal{E}_{x_1,\ldots,x_{k+1}}$$

The complete set of natural deduction rules for **IS4** is given in Figure 3. It is possible to present natural deduction rules in a 'sequent-style', where given a sequent $\Gamma \vdash A$, then Γ represents all the undischarged assumptions and A represents the conclusion of the deduction. This formulation should not be confused with the sequent calculus formulation, which differs by having rules which act on the left and right of the turnstile, rather than rules for the introduction and elimination of the connectives. The 'sequent-style' formulation of natural deduction for **IS4** is given in Figure 4.

PROPOSITION 3. *The following rules are admissible*

1. *If* $\Gamma \vdash B$ *then* $\Gamma, A \vdash B$.
2. *If* $\Gamma \vdash A$ *and* $\Gamma, A \vdash B$ *then* $\Gamma \vdash B$.

Our natural deduction formulation (where we use the prefix $\vdash_N$) is equivalent to the axiomatic formulation given in §2.

THEOREM 4. $\vdash_N \Gamma \vdash A$ *iff* $\vdash_A \Gamma \vdash A$.

4.1. Normalisation

With a natural deduction formulation we can produce so-called detours in a deduction, which arise where we introduce a logical connective only to eliminate it immediately afterwards. We can define a reduction relation, written $\rightsquigarrow_\beta$ and called β-reduction, by considering each case in turn. The treatment of the intuitionistic connectives is entirely standard and the reader is referred to other works e.g. [32]. There are two new cases for the modalities. The first is where $\Box\,\mathcal{I}$ is followed by $\Box\,\mathcal{E}$. The deduction

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Box A_1\end{array}\ \ldots\ \begin{array}{c}\mathcal{D}_k\\ \vdots\\ \Box A_k\end{array}\quad \begin{array}{c}[\![\Box A_1\ldots\Box A_k]\!]\\ \vdots\\ B\end{array}}{\Box B}\ \Box\,\mathcal{I}}{B}\ \Box\,\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{A}\;\bot\,\mathcal{E}$$

$$\dfrac{\begin{array}{c}[A^x]\\ \vdots\\ B\end{array}}{A \supset B}\;\supset\,\mathcal{I}_x \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \supset B\end{array}\quad\begin{array}{c}\vdots\\ A\end{array}}{B}\;\supset\,\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad\begin{array}{c}\vdots\\ B\end{array}}{A \wedge B}\;\wedge\,\mathcal{I} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{A}\;\wedge\,\mathcal{E} \quad \dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{B}\;\wedge\,\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B}\;\vee\,\mathcal{I} \quad \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B}\;\vee\,\mathcal{I} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array}\quad\begin{array}{c}[A^x]\\ \vdots\\ C\end{array}\quad\begin{array}{c}[B^y]\\ \vdots\\ C\end{array}}{C}\;\vee\,\mathcal{E}_{x,y}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\quad \dots \quad\begin{array}{c}\vdots\\ \Box A_k\end{array}\quad\begin{array}{c}\llbracket \Box A_1^{x_1} \cdots \Box A_k^{x_k} \rrbracket\\ \vdots\\ B\end{array}}{\Box B}\;\Box\,\mathcal{I}_{x_1,\dots,x_k} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \Box B\end{array}}{B}\;\Box\,\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\Diamond A}\;\Diamond\,\mathcal{I}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\quad \dots \quad\begin{array}{c}\vdots\\ \Box A_k\end{array}\quad\begin{array}{c}\vdots\\ \Diamond B\end{array}\quad\begin{array}{c}\llbracket \Box A_1^{x_1} \cdots \Box A_k^{x_k} B^{x_{k+1}} \rrbracket\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\;\Diamond\,\mathcal{E}_{x_1,\dots,x_{k+1}}$$

Figure 3. Natural deduction formulation of **IS4**.

$$\overline{\Gamma, A \vdash A}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \bot \mathcal{E}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset \mathcal{I} \qquad \frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B} \supset \mathcal{E}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge \mathcal{I} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge \mathcal{E} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge \mathcal{E}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee \mathcal{I} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee \mathcal{I}$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C} \vee \mathcal{E}$$

$$\frac{\Gamma \vdash \Box A_1 \ \cdots \ \Gamma \vdash \Box A_k \qquad \Box A_1, \ldots, \Box A_k \vdash B}{\Gamma \vdash \Box B} \Box \mathcal{I}$$

$$\frac{\Gamma \vdash \Box A}{\Gamma \vdash A} \Box \mathcal{E}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A} \Diamond \mathcal{I}$$

$$\frac{\Gamma \vdash \Box A_1 \ \cdots \ \Gamma \vdash \Box A_k \ \ \Gamma \vdash \Diamond B \qquad \Box A_1, \ldots, \Box A_k, B \vdash \Diamond C}{\Gamma \vdash \Diamond C} \Diamond \mathcal{E}$$

Figure 4. Natural deduction formulation of **IS4** in sequent-style.

is reduced to

$$\begin{array}{c} \mathcal{D}_1 \qquad \mathcal{D}_k \\ \vdots \qquad\; \vdots \\ [\![\Box A_1 \ \dots \ \Box A_k]\!] \\ \vdots \\ B. \end{array}$$

The second is where $\Diamond\,\mathcal{I}$ is followed by $\Diamond\,\mathcal{E}$. The deduction

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Box A_1\end{array} \quad \begin{array}{c}\mathcal{D}_k\\ \vdots\\ \Box A_k\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_{k+1}\\ \vdots\\ B\end{array}}{\Diamond B}\,\Diamond\,\mathcal{I} \qquad \begin{array}{c}[\![\Box A_1 \cdots \Box A_k B]\!]\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\,\Diamond\,\mathcal{E}$$

is reduced to

$$\begin{array}{c} \mathcal{D}_1 \qquad\quad \mathcal{D}_k \quad \mathcal{D}_{k+1} \\ \vdots \qquad\quad\;\; \vdots \qquad \vdots \\ [\![\Box A_1 \ \dots \ \Box A_k \quad B]\!] \\ \vdots \\ \Diamond C. \end{array}$$

A proof containing no instances of a β-reduction is said to be in *β-normal form.* Our formulation of **IS4** has the following property.

PROPOSITION 4. *If* $\Gamma \vdash A$ *in* **IS4** *then there is a natural deduction of* A *from* Γ *which is in* β-normal form.

An interesting question is whether there is a subformula property for a given natural deduction formulation. As discussed in the previous section, this is a simple fact for most sequent calculus formulations. A good discussion of this issue for natural deduction can be found in the book by Girard [21]. For intuitionistic logic, as explained by Girard, it is not sufficient that a deduction be in β-normal form for it to satisfy the subformula property. We are forced to introduce a new reduction relation, written $\leadsto_c$ (and called c-reduction), which Girard calls a *commuting conversion* (they are called *permutative reductions* by Prawitz [33]). These deal with what Girard calls the "bad" elimination rules ($\vee\,\mathcal{E}$ and $\perp\mathcal{E}$). A deduction of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\,\vee\,\mathcal{E} \qquad \vdots}{D}\, r\,\mathcal{E}$$

is reduced to

$$\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}\quad \vdots}{D}\,r\,\mathcal{E}\qquad \dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}\quad \vdots}{D}\,r\,\mathcal{E}}{D}\,\vee\mathcal{E}$$

and a deduction of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}\,\bot\,\mathcal{E}\qquad \vdots}{B}\,r\,\mathcal{E}$$

is reduced to

$$\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{B}\,\bot\,\mathcal{E}$$

where $r\,\mathcal{E}$ represents *any* elimination rule. A proof containing no instances of a c-reduction is said to be in c-normal form. A β-normal form which is also a c-normal form is said to be a (β, c)-normal form. For intuitionistic logic we can shown the following subformula property.

THEOREM 5. (Prawitz) *If the natural deduction derivation of $\Gamma \vdash A$ (in intuitionistic logic) is in (β, c)-normal form, then all formulae in the deduction are contained in the set of subformulae of Γ and A.*

As one may have expected, the form of the rules for the modalities means that we require further commuting conversions for a subformula property to hold. Consider the introduction rule for the necessity modality.

$$\dfrac{\begin{matrix}\vdots\\ \Box A_1\end{matrix}\quad \ldots\quad \begin{matrix}\vdots\\ \Box A_k\end{matrix}\qquad \begin{matrix}[\![\Box A_1^{x_1}\cdots\Box A_k^{x_k}]\!]\\ \vdots\\ B\end{matrix}}{\Box B}\,\Box\,\mathcal{I}$$

The problem is that there is no reason why the $\Box A_i$ should be subformula of $\Box B$ or an undischarged assumption. For example, the A_i could be the conclusion of a bad elimination rule. Consider the following deduction which is in (β, c)-normal form.

$$\dfrac{\begin{array}{c}\Box(C\supset A\supset B)\vee\bot \quad \Box C\\ \vdots\,\mathcal{D}_1\\ \Box(A\supset B)\end{array} \qquad \Box A \qquad \dfrac{\dfrac{[\Box(A\supset B)]}{A\supset B}\,\Box\,\mathcal{E} \quad \dfrac{[\Box A]}{A}\,\Box\,\mathcal{E}}{B}\supset\mathcal{E}}{\Box B}\,\Box\,\mathcal{I}$$

where $\mathcal{D}_1$ is the deduction

$$\dfrac{\Box(C\supset A\supset B)\vee\bot \qquad \dfrac{[\Box(C\supset A\supset B)]\Box C \qquad \dfrac{\dfrac{[\Box(C\supset A\supset B)]}{C\supset A\supset B}\quad\dfrac{[\Box C]}{C}}{A\supset B}\supset\mathcal{E}}{\Box(A\supset B)}\,\Box\,\mathcal{I} \qquad \dfrac{[\bot]}{\Box(A\supset B)}}{\Box(A\supset B)}\vee\mathcal{E}$$

Unfortunately the formula $\Box(A\supset B)$ is not a subformula of either the conclusion ($\Box B$) or of the undischarged assumptions ($\{\Box(C\supset A\supset B)\vee\bot, \Box C, \Box A\}$). Consequently we shall add the following commuting conversion.

$$\dfrac{\begin{array}{c}\vdots\\\Box A_1\end{array}\ \dots\ \dfrac{\vdots\quad \begin{array}{c}\vdots\\ \Box A_i\end{array}}{\Box A_i}\,b\,\mathcal{E} \qquad \begin{array}{c}\vdots\\\Box A_k\end{array}\qquad \begin{array}{c}[\![\Box A_1\cdots\Box A_k]\!]\\\vdots\\B\end{array}}{\Box B}\,\Box\,\mathcal{I}$$

which is reduced to

$$\dfrac{\vdots \qquad \dfrac{\begin{array}{c}\vdots\\\Box A_1\end{array}\ \dots\ \begin{array}{c}\vdots\\\Box A_i\end{array}\ \dots\ \begin{array}{c}\vdots\\\Box A_k\end{array}\qquad \begin{array}{c}[\![\Box A_1\cdots\Box A_k]\!]\\\vdots\\B\end{array}}{\Box B}\,\Box\,\mathcal{I}}{\Box B}\,b\,\mathcal{E}$$

where $b\,\mathcal{E}$ is any bad elimination rule ($\vee\,\mathcal{E}$ or $\bot\,\mathcal{E}$).

Unfortunately this is still not enough; an A_i could be the result of the $\Box\,\mathcal{I}$ rule. Consider the following deduction which is a (β, c)-normal form.

$$\dfrac{\begin{array}{c}\Box(C\supset A\supset B)\Box C\\ \vdots\,\mathcal{D}_2\\ \Box(A\supset B)\end{array} \qquad \Box A \qquad \dfrac{\dfrac{[\Box(A\supset B)]}{A\supset B}\,\Box\,\mathcal{E} \quad \dfrac{\Box A}{A}\,\Box\,\mathcal{E}}{B}}{\Box B}\,\Box\,\mathcal{I}$$

where $\mathcal{D}_2$ is the deduction

$$\frac{\Box(C\supset A\supset B)\Box C \qquad \dfrac{\dfrac{[\Box(C\supset A\supset B)]}{C\supset A\supset B}\,\Box\mathcal{E} \quad \dfrac{[\Box C]}{C}\,\Box\mathcal{E}}{A\supset B}\,\supset\mathcal{E}}{\Box(A\supset B)}\,\Box\mathcal{I}$$

Again $\Box(A\supset B)$ is neither a subformula of the conclusion ($\Box B$) nor of the undischarged assumptions ($\{\Box(C\supset A\supset B),\Box C,\Box A\}$). Consequently we add another commuting conversion. The deduction

$$\frac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\;\ldots\; \dfrac{\begin{array}{c}\vdots\\ \Box C_1\end{array}\;\ldots\;\begin{array}{c}\vdots\\ \Box C_j\end{array}\qquad \begin{array}{c}[\![\Box C_1\cdots\Box C_j]\!]\\ \vdots\\ A_i\end{array}}{\Box A_i}\,\Box\mathcal{I}\;\ldots\;\begin{array}{c}\vdots\\ \Box A_k\end{array}\qquad \begin{array}{c}[\![\Box A_1\cdots\Box A_k]\!]\\ \vdots\\ B\end{array}}{\Box B}\,\Box\mathcal{I}$$

is reduced to

$$\frac{\begin{array}{c}\vdots\\ \Box A_1\end{array}\;\ldots\;\begin{array}{c}\vdots\\ \Box C_1\end{array}\;\ldots\;\begin{array}{c}\vdots\\ \Box C_j\end{array}\;\ldots\;\begin{array}{c}\vdots\\ \Box A_k\end{array}\qquad \begin{array}{c}[\![\Box A_1\cdots]\!]\quad \dfrac{[\![\Box C_1\cdots\Box C_j]\!]\quad \begin{array}{c}[\![\Box C_1\cdots\Box C_j]\!]\\ \vdots\\ A_i\end{array}}{\Box A_i}\,\Box\mathcal{I}\quad [\![\cdots\Box A_k]\!]\\ \vdots\\ B\end{array}}{\Box B}\,\Box\mathcal{I}$$

Similar rules exist for the rules describing the possibility modality.

THEOREM 6. *If the natural deduction derivation of* $\Gamma\vdash A$ *(in* **IS4***) is in* (β,c)*-normal form, then all formulae in the deduction are contained in the set of subformulae of* Γ *and* A.

PROOF. By a relatively straightforward adaptation of Prawitz's proof for intuitionistic logic [33, §II.3]. ■

5. Multi-context formulation

It is possible to give alternative natural deduction formulations of **IS4**. In this section we shall sketch some details of one such alternative — some others are listed in §11. This alternative originates from work on intuitionistic linear logic [1].

Recall that the difficulties in formulating the introduction rule for the necessity modality was, in part, in ensuring that all the undischarged assumptions (the context) were modal. An alternative would be to split the context into two, where one part contains only modal formulae and the other the non-modal formulae. The condition then becomes a check that the non-modal context is empty. More precisely, judgements are of the form

$$\Gamma; \Delta \vdash A$$

where Γ is the modal context and Δ the non-modal context. To reiterate: the idea is that this judgement corresponds to the judgement

$$\Box\Gamma, \Delta \vdash A$$

in our formulation of the previous section. Thus the $\Box\,\mathcal{I}$ rule is given by

$$\frac{\Gamma; - \vdash A}{\Gamma; \Delta \vdash \Box A}\;\Box\,\mathcal{I}$$

To ensure closure under substitution, the elimination rule is slightly more complicated.

$$\frac{\Gamma; \Delta \vdash \Box A \qquad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta \vdash B}\;\Box\,\mathcal{E}$$

The complete formulation, which we call a multi-context formulation is given in Figure 5.

PROPOSITION 5. *The following rules are admissible.*

1. *If* $\Gamma; \Delta \vdash B$ *then* $\Gamma, A; \Delta \vdash B$.
2. *If* $\Gamma; \Delta \vdash B$ *then* $\Gamma; \Delta, A \vdash B$.
3. *If* $\Gamma; \Delta \vdash A$ *and* $\Gamma; \Delta, A \vdash B$ *then* $\Gamma; \Delta \vdash B$.
4. *If* $\Gamma; - \vdash A$ *and* $\Gamma, A; \Delta \vdash B$ *then* $\Gamma; \Delta \vdash B$.

THEOREM 7. *The following rules are derivable.*

1. *If* $\Gamma, A; \Delta \vdash B$ *then* $\Gamma; \Delta, \Box A \vdash B$.
2. *If* $\Gamma; \Delta, \Box A \vdash B$ *then* $\Gamma, A; \Delta \vdash B$.
3. *If* $\Gamma, A; \Delta \vdash B$ *then* $\Gamma; \Delta \vdash \Box A \supset B$.

$$\frac{}{\Gamma, A; \Delta \vdash A} \qquad \frac{}{\Gamma; \Delta, A \vdash A}$$

$$\frac{\Gamma; \Delta, A \vdash B}{\Gamma; \Delta \vdash A \supset B} \supset \mathcal{I} \qquad \frac{\Gamma; \Delta \vdash A \supset B \qquad \Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash B} \supset \mathcal{E}$$

$$\frac{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \wedge B} \wedge \mathcal{I} \quad \frac{\Gamma; \Delta \vdash A \wedge B}{\Gamma; \Delta \vdash A} \wedge \mathcal{E} \quad \frac{\Gamma; \Delta \vdash A \wedge B}{\Gamma; \Delta \vdash B} \wedge \mathcal{E}$$

$$\frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash A \vee B} \vee \mathcal{I} \quad \frac{\Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \vee B} \vee \mathcal{I}$$

$$\frac{\Gamma; \Delta \vdash A \vee B \qquad \Gamma; \Delta, A \vdash C \qquad \Gamma; \Delta, B \vdash C}{\Gamma; \Delta \vdash C} \vee \mathcal{E}$$

$$\frac{\Gamma; - \vdash A}{\Gamma; \Delta \vdash \Box A} \Box \mathcal{I}$$

$$\frac{\Gamma; \Delta \vdash \Box A \qquad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta \vdash B} \Box \mathcal{E}$$

$$\frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash \Diamond A} \Diamond \mathcal{I}$$

$$\frac{\Gamma; \Delta \vdash \Diamond A \qquad \Gamma; A \vdash \Diamond C}{\Gamma; \Delta \vdash \Diamond C} \Diamond \mathcal{E}$$

Figure 5. Multi-context formulation of **IS4**.

PROOF. 1.

$$\dfrac{\dfrac{}{\Gamma; \Box A, \Delta \vdash \Box A} \quad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta, \Box A \vdash B} \Box \mathcal{E}$$

2.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A; - \vdash A}}{\Gamma, A; \Delta \vdash \Box A} \Box \mathcal{I} \quad \Gamma; \Delta, \Box A \vdash B}{\Gamma, A; \Delta \vdash B}$$

The last rule used is a substitution, shown to be admissible in Proposition 5.
3.

$$\dfrac{\dfrac{\dfrac{}{\Gamma; \Box A, \Delta \vdash \Box A} \quad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta, \Box A \vdash B} \Box \mathcal{E}}{\Gamma; \Delta \vdash \Box A \supset B} \supset \mathcal{I}$$

■

6. Term assignment for IS4

The Curry-Howard correspondence [23] relates natural deduction formulations of logics with typed λ-calculi. It essentially annotates each stage of a deduction with a 'term', which is an encoding of the construction of the deduction so far. Consequently a logic can be viewed as a type system for a term assignment system. The correspondence also links proof normalisation to term reduction. We shall use this term assignment in the next section to devise a categorical model of **IS4**.

The Curry-Howard correspondence can be applied to the natural deduction formulation to obtain the term assignment system given in Figure 6. (In places we have used the shorthand $\vec{M}$ instead of the sequence $M_1, \ldots M_k$.) The resulting calculus we shall call the λ^{S4}-calculus.

The syntax has been chosen so that the non-modal fragment is similar to several functional programming languages, and that used by Gallier in his tutorial article [17]. An important property of our system is that substitution is well-defined in the following sense.

THEOREM 8. *If* $\Gamma \triangleright N\colon A$ *and* $\Gamma, x\colon A \triangleright M\colon B$ *then* $\Gamma \triangleright M[x := N]\colon B$.

The reduction rules discussed in §4.1 can be given at the level of terms. In Figure 7 we give the β-reduction rules denoted by $\leadsto_\beta$. It is also possible to give the reduction rules associated with the commuting conversions. We shall omit them for reasons of space.

$$\frac{}{\Gamma, x{:}\,A \rhd x{:}\,A}$$

$$\frac{\Gamma \rhd M{:}\,\bot}{\Gamma \rhd \nabla_A(M){:}\,A}\ \bot\mathcal{E}$$

$$\frac{\Gamma, x{:}\,A \rhd M{:}\,B}{\Gamma \rhd \lambda x{:}\,A.M{:}\,A \to B} \to \mathcal{I} \qquad \frac{\Gamma \rhd M{:}\,A \to B \qquad \Gamma \rhd N{:}\,A}{\Gamma \rhd MN{:}\,B} \to \mathcal{E}$$

$$\frac{\Gamma \rhd M{:}\,A \qquad \Gamma \rhd N{:}\,B}{\Gamma \rhd \langle M, N\rangle{:}\,A \times B} \times\mathcal{I} \quad \frac{\Gamma \rhd M{:}\,A \times B}{\Gamma \rhd \mathsf{fst}(M){:}\,A} \times\mathcal{E} \quad \frac{\Gamma \rhd M{:}\,A \times B}{\Gamma \rhd \mathsf{snd}(M){:}\,B} \times\mathcal{E}$$

$$\frac{\Gamma \rhd M{:}\,A}{\Gamma \rhd \mathsf{inl}(M){:}\,A + B} +\mathcal{I} \qquad \frac{\Gamma \rhd M{:}\,B}{\Gamma \rhd \mathsf{inr}(M){:}\,A + B} +\mathcal{I}$$

$$\frac{\Gamma \rhd M{:}\,A + B \qquad \Gamma, x{:}\,A \rhd N{:}\,C \qquad \Gamma, y{:}\,B \rhd P{:}\,C}{\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P{:}\,C} +\mathcal{E}$$

$$\frac{\Gamma \rhd M_1{:}\,\Box A_1 \ \cdots\ \Gamma \rhd M_k{:}\,\Box A_k \qquad x_1{:}\,\Box A_1, \ldots, x_k{:}\,\Box A_k \rhd N{:}\,B}{\Gamma \rhd \mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}{:}\,\Box B} \Box\mathcal{I}$$

$$\frac{\Gamma \rhd M{:}\,\Box A}{\Gamma \rhd \mathsf{unbox}(M){:}\,A} \Box\mathcal{E}$$

$$\frac{\Gamma \rhd M{:}\,A}{\Gamma \rhd \Diamond M{:}\,\Diamond A} \Diamond\mathcal{I}$$

$$\frac{\begin{array}{ll}\Gamma \rhd M_1{:}\,\Box A_1 \cdots \Gamma \rhd M_k{:}\,\Box A_k & \\ \Gamma \rhd N{:}\,\Diamond B & x_1{:}\,\Box A_1, \ldots, x_k{:}\,\Box A_k, y{:}\,B \rhd P{:}\,\Diamond C\end{array}}{\Gamma \rhd \mathsf{let}\ \Diamond y \Leftarrow N\ \mathsf{in}\ P\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}{:}\,\Diamond C} \Diamond\mathcal{E}$$

Figure 6. Term assignment for **IS4**

$$
\begin{array}{lll}
(\lambda x{:}\,A.M)N & \leadsto_\beta & M[x := N] \\[2ex]
\mathsf{fst}(\langle M, N\rangle) & \leadsto_\beta & M \\[2ex]
\mathsf{snd}(\langle M, N\rangle) & \leadsto_\beta & N \\[2ex]
\mathsf{case}\ \mathsf{inl}(M)\ \mathsf{of}\ \mathsf{inl}(x)\ \rightarrow\ N \parallel \mathsf{inr}(y)\ \rightarrow\ P & \leadsto_\beta & N[x := M] \\[2ex]
\mathsf{case}\ \mathsf{inr}(M)\ \mathsf{of}\ \mathsf{inl}(x)\ \rightarrow\ N \parallel \mathsf{inr}(y)\ \rightarrow\ P & \leadsto_\beta & P[y := M] \\[2ex]
\mathsf{unbox}(\mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}) & \leadsto_\beta & N[\vec{x} := \vec{M}] \\[2ex]
\mathsf{let}\ \Diamond y \Leftarrow \Diamond N\ \mathsf{in}\ P\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x} & \leadsto_\beta & P[\vec{x} := \vec{M}, y := N]
\end{array}
$$

Figure 7. β-reduction rules.

An important property of these reduction rules is given by the so-called subject reduction theorem.

THEOREM 9. *If* $\Gamma \rhd M{:}\,A$ *and* $M \leadsto_\beta N$ *then* $\Gamma \rhd N{:}\,A$.

(Of course, this theorem also extends to the commuting conversions.) It should be noted that the Curry-Howard correspondence can be applied to the multi-context formulation given in §5, to yield a very compact term calculus.

7. The categorical model

The fundamental idea of a categorical treatment of proof theory is that propositions should be interpreted as the objects of a category and proofs should be interpreted as morphisms. The proof rules correspond to natural transformations between appropriate hom-functors. The proof theory gives a number of reduction rules, which can be viewed as equalities between proofs. In particular these equalities should hold in the categorical model.

Other categorical studies have been carried out, notably by Flagg [16]; Meloni and Ghilardi [19] and Reyes and Zolfaghari [34]. However these have been mainly concerned with categorical *model* theory, rather than categorical *proof* theory. In particular, they all assume an isomorphism, $\Box\Box A \cong \Box A$, whereas we see no reason to assume such a strong condition. In this work we have morphisms in both directions, but we do *not* collapse the model so that they are isomorphic.

Let us fix some notation. We shall follow the convention common in computer science and write composition of morphisms in a left-to-right fashion, e.g. $f;g$, rather than the mathematical convention of right-to-left, e.g. $g \circ f$.

The interpretation of a proof is represented using semantic braces, $[\![-]\!]$, making the usual simplification of using the same letter to represent a proposition as its interpretation. Given a term $\Gamma \triangleright M: A$ where $M \rightsquigarrow_\beta N$, we shall write $\Gamma \triangleright M = N: A$.

DEFINITION 1. A category, $\mathbb{C}$, is said to be a categorical model of a logic/term calculus iff

1. For all proofs $\Gamma \triangleright M: A$ there is a morphism $[\![M]\!]: \Gamma \to A$ in $\mathbb{C}$; and
2. For all proof equalities $\Gamma \triangleright M = N: A$ it is the case that $[\![M]\!] =_{\mathbb{C}} [\![N]\!]$ (where $=_{\mathbb{C}}$ represents equality of morphisms in the category $\mathbb{C}$).

Given this definition we simply analyse the introduction and elimination rules for each connective. Both this and consideration of the reduction rules should suggest a particular categorical structure to model the connective. The case for intuitionistic logic is well known; the reader is referred to Lambek and Scott's book [24] for a good discussion. Essentially the categorical model of intuitionistic logic (with disjunction) is a cartesian closed category (CCC) with coproducts. Hence all we need do here is consider the two modalities, which we shall do in some detail. The less-categorically minded reader may wish simply to skip to Definition 5.

The introduction rule for the necessity modality is of the form

$$\frac{\Gamma \triangleright M_1: \Box A_1 \quad \cdots \quad \Gamma \triangleright M_k: \Box A_k \qquad x_1: \Box A_1, \ldots, x_k: \Box A_k \triangleright N: B}{\Gamma \triangleright \mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}: \Box B}\ \Box\mathcal{I}$$

To interpret this rule we need a natural transformation with components

$$\Phi_\Gamma: \mathbb{C}(\Gamma, \Box A_1) \times \cdots \times \mathbb{C}(\Gamma, \Box A_k) \times \mathbb{C}(\Box A_1 \times \cdots \times \Box A_k, B) \to \mathbb{C}(\Gamma, \Box B)$$

Given morphisms $e_i: \Gamma \to \Box A_i$, $c: \Gamma' \to \Gamma$ and $d: \Box A_1 \times \cdots \times \Box A_k \to B$, naturality gives the equation

$$c; \Phi_\Gamma(e_1, \ldots, e_k, d) \quad = \quad \Phi_{\Gamma'}((c; e_1), \ldots, (c; e_k), d).$$

In particular if we have morphisms $m_i: \Gamma \to \Box A_i$ then we take $c = \langle m_1, \ldots, m_k\rangle$, e_i to be the i-th product projection, written π_i, and d to be some morphism $p: \Box A_1 \times \cdots \times \Box A_k \to B$, then by naturality we have

$$\langle m_1, \ldots, m_k\rangle; \Phi_{\Box A_1, \ldots, \Box A_k}(\pi_1, \ldots, \pi_k, p) = \Phi_{\Box A_1, \ldots, \Box A_k}(m_1, \ldots, m_k, p).$$

Thus $\Phi(m_1,\ldots,m_k,p)$ can be expressed as the composition $\langle m_1,\ldots,m_k\rangle$; $\Psi(p)$, where Ψ is a transformation

$$\Psi\colon \mathbb{C}(\Box A_1 \times \cdots \times \Box A_k, B) \to \mathbb{C}(\Box A_1 \times \cdots \times \Box A_k, \Box B).$$

For the moment, the effect of this transformation will be written as $(-)^*$ and so we can make the preliminary definition

$$\begin{array}{l}[\![\Gamma \triangleright \mathsf{box}\, N\, \mathsf{with}\, M_1,\ldots,M_k\, \mathsf{for}\, x_1,\ldots,x_k\colon \Box B]\!] \stackrel{\mathrm{def}}{=} \\ \quad \langle([\![\Gamma \triangleright M_1\colon \Box A_1]\!]),\ldots,([\![\Gamma \triangleright M_k\colon \Box A_k]\!])\rangle; \\ \quad ([\![x_1\colon \Box A_1,\ldots,x_k\colon \Box A_k \triangleright N\colon B]\!])^*\end{array}$$

The elimination rule for the necessity modality is of the form

$$\frac{\Gamma \triangleright M\colon \Box A}{\Gamma \triangleright \mathsf{unbox}(M)\colon A}\ \Box\,\mathcal{E}$$

To interpret this rule we need a natural transformation with components

$$\Phi_\Gamma\colon \mathbb{C}(\Gamma, \Box A) \to \mathbb{C}(\Gamma, A).$$

It follows from the Yoneda Lemma [25, Page 61] that there is the bijection

$$[\mathbb{C}^{op}, \mathbf{Sets}](\mathbb{C}(\Gamma, \Box A), \mathbb{C}(\Gamma, A)) \cong \mathbb{C}(\Box A, A).$$

By constructing this isomorphism one can see that the components of Φ are induced by post-composition with a morphism $\varepsilon\colon \Box A \to A$. Thus we make the definition

$$[\![\Gamma \triangleright \mathsf{unbox}(M)\colon A]\!] \stackrel{\mathrm{def}}{=} [\![\Gamma \triangleright M\colon \Box A]\!]; \varepsilon.$$

From Figure 7 we have the term equality

$$\frac{\Gamma \triangleright M_1\colon \Box A_1 \quad \cdots \quad \Gamma \triangleright M_k\colon \Box A_k \qquad x_1\colon \Box A_1,\ldots,x_k\colon \Box A_k \triangleright N\colon B}{\Gamma \triangleright \mathsf{unbox}(\mathsf{box}\, N\, \mathsf{with}\, \vec{x}\, \mathsf{for}\, \vec{M}) = N[\vec{x} := \vec{M}]\colon B}$$

Taking morphisms $m_i\colon \Gamma \to \Box A_i$ and $p\colon \Box A_1 \times \cdots \times \Box A_k \to B$, say, this term equality amounts to the categorical equality

$$\langle m_1,\ldots,m_k\rangle; (p)^*; \varepsilon \quad = \quad \langle m_1,\ldots,m_k\rangle; p. \tag{1}$$

We can certainly define an operation

$$\begin{array}{l}\Box\colon \mathbb{C}(\Gamma, A) \to \mathbb{C}(\Box\Gamma, \Box A), \\ f \mapsto (\varepsilon; f)^*.\end{array}$$

We shall make the simplifying assumption that this operation is a *functor*. However, notice that if Γ is the object $A_1 \times \cdots \times A_k$, then $\Box\Gamma$ will be represented by $\Box(A_1 \times \cdots \times A_k)$, but clearly we mean $\Box A_1 \times \cdots \times \Box A_k$. Thus we shall make the further simplifying assumption that $\Box$ is a *symmetric monoidal functor*, $(\Box, \mathsf{m}_{A,B}, \mathsf{m}_1)$. This notion is originally due to Eilenberg and Kelly [14]. In essence this provides a natural transformation

$$\mathsf{m}_{A,B}\colon \Box A \times \Box B \to \Box(A \times B)$$

and morphism

$$\mathsf{m}_1\colon 1 \to \Box 1$$

which satisfy a number of conditions which are detailed below.

Definition 2. A *monoidal functor*, $(\Box, \mathsf{m}_{A,B}, \mathsf{m}_1)$, on a CCC $\mathbb{C}$ satisfies the four following equations.

1. $\mathsf{id}_{\Box A} \times \mathsf{m}_1; \mathsf{m}_{A,1}; \Box\pi_1 = \pi_1$
2. $\mathsf{m}_1 \times \mathsf{id}_{\Box A}; \mathsf{m}_{1,A}; \Box\pi_2 = \pi_2$
3. $\alpha; \mathsf{m}_{A,B} \times \mathsf{id}_{\Box C}; \mathsf{m}_{A\times B,C} = \mathsf{id}_{\Box A} \times \mathsf{m}_{B,C}; \mathsf{m}_{A,B\times C}; \Box(\alpha)$
4. $\gamma; \mathsf{m}_{B,A} = \mathsf{m}_{A,B}; \Box(\gamma)$

where

$$\begin{aligned} \alpha &\stackrel{\text{def}}{=} \langle\langle \pi_1, (\pi_2;\pi_1)\rangle, (\pi_2;\pi_2)\rangle\colon A \times (B \times C) \to (A \times B) \times C \\ \gamma &\stackrel{\text{def}}{=} \langle \pi_2, \pi_1\rangle\colon A \times B \to B \times A \end{aligned}$$

Equation 1 gives

$$(\varepsilon_A; f)^*; \varepsilon_B = \varepsilon_A; f$$

for any morphism $f\colon A \to B$; or, in other words the diagram

$$\begin{tikzcd} \Box A \arrow[r, "\Box f"] \arrow[d, "\varepsilon"'] & \Box B \arrow[d, "\varepsilon"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Given the assumption that $\Box$ is a symmetric monoidal functor, this diagram suggests that ε is a monoidal natural transformation.

We have that from the identity morphism $\mathsf{id}_{\Box A}\colon \Box A \to \Box A$, we can form the canonical morphism $\delta_A \stackrel{\mathrm{def}}{=} (\mathsf{id}_{\Box A})^*$. Equation 1 gives

$$\delta_A; \varepsilon_{\Box A} = \mathsf{id}_{\Box A}.$$

The categorically-minded reader will recognise this equation as one of the three for a *comonad*. We shall make the simplifying assumption that not only does $(\Box, \varepsilon, \delta)$ form a comonad but that δ is also a monoidal natural transformation. Hence the comonad is actually a *monoidal* comonad.

DEFINITION 3. 1. The triple $(\Box, \varepsilon, \delta)$ forms a *comonad* on a CCC $\mathbb{C}$, if $\Box$ is an endofunctor on $\mathbb{C}$, and $\varepsilon\colon \Box A \to A$ and $\delta\colon \Box A \to \Box\Box A$ are natural transformations which satisfy the following three equations.

(a) $\delta_A; \varepsilon_{\Box A} = \mathsf{id}_{\Box A}$

(b) $\delta_A; \Box(\varepsilon_A) = \mathsf{id}_{\Box A}$

(c) $\delta_A; \delta_{\Box A} = \delta_A; \Box(\delta_A)$

2. A comonad $(\Box, \varepsilon, \delta)$ is in addition a *monoidal comonad* if $(\Box, \mathsf{m}_{A,B}, \mathsf{m}_1)$ is a monoidal functor and the following four equations hold.

(a) $\mathsf{m}_{A,B}; \varepsilon_{A\times B} = \varepsilon_A \times \varepsilon_B$

(b) $\mathsf{m}_1; \varepsilon_1 = \mathsf{id}_1$

(c) $\mathsf{m}_{A,B}; \delta_{A\times B} = \delta_A \times \delta_B; \mathsf{m}_{\Box A,\Box B}; \Box(\mathsf{m}_{A,B})$

(d) $\mathsf{m}_1; \delta_1 = \mathsf{m}_1; \Box\mathsf{m}_1$

We can now finalise the interpretation of the introduction rule for the necessity modality.

$$\begin{aligned}[\![\Gamma \triangleright \mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}\colon \Box B]\!] \stackrel{\mathrm{def}}{=} \; & \langle [\![\Gamma \triangleright M_1\colon \Box A_1]\!], \ldots, [\![\Gamma \triangleright M_k\colon \Box A_k]\!]\rangle; \\ & \delta_{A_1} \times \cdots \times \delta_{A_k}; \mathsf{m}_{\Box A_1,\ldots,\Box A_k}; \\ & \Box[\![x_1\colon \Box A_1, \ldots, x_k\colon \Box A_k \triangleright N\colon B]\!]\end{aligned}$$

The introduction rule for the possibility modality is of the form

$$\frac{\Gamma \triangleright M\colon A}{\Gamma \triangleright \Diamond M\colon \Diamond A}\ \Diamond\mathcal{I}$$

To interpret this rule we need a natural transformation

$$\Phi\colon \mathbb{C}(-, A) \to \mathbb{C}(-, \Diamond A)$$

It follows from the Yoneda Lemma [25, Page 61] that there is the bijection

$$[\mathbb{C}^{op}, \mathbf{Sets}](\mathbb{C}(-, A), \mathbb{C}(-, \Diamond A)) \cong \mathbb{C}(A, \Diamond A).$$

By constructing this isomorphism one can see that the components of Φ are induced by post-composition with a morphism $\eta\colon A \to \Diamond A$. Thus we make the definition

$$[\![\Gamma \triangleright M\colon A]\!] \stackrel{\text{def}}{=} [\![\Gamma \triangleright M\colon \Box A]\!]; \eta.$$

The elimination rule for the possibility modality is of the form

$$\frac{\begin{array}{ll}\Gamma \triangleright M_1\colon \Diamond A_1 \cdots \Gamma \triangleright M_k\colon \Diamond A_k & \\ \Gamma \triangleright N\colon \Diamond B & x_1\colon \Diamond A_1, \ldots, x_k\colon \Diamond A_k, y\colon B \triangleright P\colon \Diamond C\end{array}}{\Gamma \triangleright \mathsf{let}\ \Diamond y \Leftarrow N\ \mathsf{in}\ P\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}\colon \Diamond C}\ \Diamond\mathcal{E}$$

To interpret this rule we need a natural transformation with components

$$\begin{array}{rl}\Phi_\Gamma\colon & \mathbb{C}(\Gamma, \Box A_1) \times \cdots \times \mathbb{C}(\Gamma, \Box A_k) \times \\ & \mathbb{C}(\Gamma, \Diamond B) \times \mathbb{C}(\Box A_1 \times \cdots \times \Box A_k \times B, \Diamond C) \to \mathbb{C}(\Gamma, \Diamond C)\end{array}$$

Given morphisms $e_i\colon \Gamma \to \Box A_i$, $c\colon \Gamma' \to \Gamma$, $d\colon \Gamma \to \Diamond B$, and $f\colon \Box A_1 \times \cdots \times \Box A_k \times B \to \Diamond C$, naturality gives the equation

$$c; \Phi_\Gamma(e_1, \ldots, e_k, d, f) \quad = \quad \Phi_{\Gamma'}((c; e_1), \ldots, (c; e_k), (c; d), f).$$

In particular if we have morphisms $m_i\colon \Gamma \to \Box A_i$ and $n\colon \Gamma \to \Diamond B$, then we take $c = \langle m_1, \ldots, m_k, n\rangle$, e_i to be the i-th product projection, written π_i, and f to be some morphism $p\colon \Box A_1 \times \cdots \times \Box A_k \times B \to \Diamond C$, then by naturality we have

$$\langle m_1, \ldots, m_k, n\rangle; \Phi(\pi_1, \ldots, \pi_k, \pi_{k+1}, p) = \Phi(m_1, \ldots, m_k, n, p).$$

Thus $\Phi(m_1, ..., m_k, n, p)$ can be expressed as the composition $\langle m_1, ..., m_k, n\rangle$; $\Psi(p)$, where Ψ is a transformation

$$\Psi\colon \mathbb{C}(\Box A_1 \times \cdots \times \Box A_k \times B, \Diamond C) \to \mathbb{C}(\Box A_1 \times \cdots \times \Box A_k \times \Diamond B, \Diamond C).$$

For the moment, the effect of this transformation will be written as $(-)^*$ and so we can make the preliminary definition

$$\begin{array}{l}[\![\Gamma \triangleright \mathsf{let}\ \Diamond y \Leftarrow N\ \mathsf{in}\ P\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}\colon \Diamond C]\!] \stackrel{\text{def}}{=} \\ \quad \langle([\![\Gamma \triangleright M_1\colon \Box A_1]\!]), \ldots, ([\![\Gamma \triangleright M_k\colon \Box A_k]\!])([\![\Gamma \triangleright N\colon \Diamond B]\!])\rangle; \\ \quad ([\![x_1\colon \Box A_1, \ldots, x_k\colon \Box A_k, y\colon B \triangleright N\colon \Diamond C]\!])^*\end{array}$$

From Figure 7 we have the term equality

$$\frac{\begin{array}{ll}\Gamma \triangleright M_1\colon \Box A_1 \cdots \Gamma \triangleright M_k\colon \Box A_k & \\ \Gamma \triangleright N\colon B & x_1\colon \Box A_1, \ldots, x_n\colon \Box A_k, y\colon B \triangleright P\colon \Diamond C\end{array}}{\Gamma \triangleright \mathsf{let}\ \Diamond y \Leftarrow \Diamond N\ \mathsf{in}\ P\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x} = P[\vec{x} := \vec{M}, y := N]\colon \Diamond C}$$

Taking, for example, morphisms $m_i\colon \Gamma \to \Box A_i$, $n\colon \Gamma \to B$, and $p\colon \Box A_1 \times \cdots \times \Box A_k \times B \to \Diamond C$, the term equality amounts to the categorical equality

$$\langle m_1, \ldots, m_k, (n;\eta)\rangle; (p)^* \quad = \quad \langle m_1, \ldots, m_k, n\rangle; p \qquad (2)$$

We can then define an operation

$$\begin{array}{l} \Diamond\colon \mathbb{C}(\Gamma, A) \to \mathbb{C}(\Diamond\Gamma, \Diamond A), \\ f \mapsto (f;\eta)^*. \end{array}$$

Again we shall make the simplifying assumption that this operation is a *functor*. Equation 2 gives

$$\eta; (f;\eta)^* \quad = \quad (f;\eta)$$

for any morphism $f\colon A \to B$, or, in other words, the diagram

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "f"'] & \Diamond A \arrow[d, "\Diamond f"] \\ B \arrow[r, "\eta"'] & \Diamond B \end{tikzcd}$$

commutes. Given our assumption that $\Diamond$ is a functor, this diagram suggests that η is a natural transformation. Given the identity morphism $\mathsf{id}_{\Diamond C}$, we can form the canonical morphism $\mu_C \stackrel{\text{def}}{=} (\mathsf{id}_{\Diamond C})^*$. Equation 2 gives

$$\eta;\mu \quad = \quad \mu.$$

Again the categorically-minded reader will recognise this equation as one of the three for a *monad*. Thus we shall expect that $(\Diamond, \eta, \mu)$ forms some sort of monad. A first guess might be that it should be the dual of the structure required for the necessity modality, i.e. a *comonoidal monad*. However this seems far too strong a requirement — for example, it requires a natural transformation $\Diamond(A+B) \to \Diamond A + \Diamond B$, which is not even a theorem of **IS4**!

Recalling the form of the $\Diamond\,\mathcal{E}$ rule, it seems clear that the meaning of the possibility modality is actually related to the necessity modality. Fortunately, category theory provides a means to describe such a situation.

Definition 4. 1. The triple $(\Diamond, \eta, \mu)$ forms a *monad* on a CCC $\mathbb{C}$ if $\Diamond$ is an endofunctor on $\mathbb{C}$, and $\eta\colon A \to \Diamond A$ and $\mu\colon \Diamond\Diamond A \to \Diamond A$ are natural transformations which satisfy the following three equations.

(a) $\eta_{\Diamond A}; \mu_A = \mathsf{id}_{\Diamond A}$

(b) $\Diamond(\eta_A); \mu_A = \mathsf{id}_{\Diamond A}$

(c) $\mu_{\Diamond A}; \mu_A = \Diamond(\mu_A); \mu_A$

2. Given a monoidal comonad $(\Box, \varepsilon, \delta, \mathsf{m}_1, \mathsf{m}_{A,B})$ and a monad $(\Diamond, \eta, \mu)$ on a CCC $\mathbb{C}$, we say that $\Diamond$ is a $\Box$-*strong monad*, if there exists a natural transformation

$$\mathsf{st}_{A,B}: \Box A \times \Diamond B \to \Diamond(\Box A \times B)$$

which satisfies the following four equations.

(a) $\varepsilon_1 \times \mathsf{id}_{\Diamond A}; \pi_2 = \mathsf{st}_{1,A}; \Diamond(\varepsilon_1 \times \mathsf{id}_A); \Diamond(\pi_2)$

(b) $\alpha; \mathsf{m}_{A,B} \times \mathsf{id}_{\Diamond C}; \mathsf{st}_{A\times B,C} = \mathsf{id}_{\Box A} \times \mathsf{st}_{B,C}; \mathsf{st}_{A,\Box B\times C}; \Diamond(\alpha); \Diamond(\mathsf{m}_{A,B} \times \mathsf{id}_C)$

(c) $\mathsf{id}_{\Box A} \times \eta_B; \mathsf{st}_{A,B} = \eta_{\Box A\times B}$

(d) $\mathsf{st}_{A,\Box B}; \Diamond(\mathsf{st}_{A,B}); \varepsilon = \mathsf{st}_{A,B}$

Clearly we can define n-ary versions of the strength. For example the 3-ary version

$$\mathsf{st}_{A,B,C}: \Box A \times \Box B \times \Diamond C \to \Diamond(\Box A \times \Box B \times C)$$

can be defined as

$$\mathsf{id}_{\Box A} \times \mathsf{st}_{B,C}; \mathsf{st}_{A,\Box B\times C}.$$

We can now fix the interpretation of the $\Diamond\,\mathcal{E}$ rule as follows.

$$\begin{aligned}
&[\![\Gamma \triangleright \mathsf{let}\ \Diamond y \Leftarrow N\ \mathsf{in}\ \vec{M}\ \mathsf{with}\ \vec{x}\ \mathsf{for} : \Diamond C]\!] \stackrel{\text{def}}{=} \\
&\quad \langle [\![\Gamma \triangleright M_1 : \Box A_1]\!], \ldots, [\![\Gamma \triangleright M_k : \Box A_k]\!], [\![\Gamma \triangleright N : \Diamond B]\!]\rangle; \\
&\quad \mathsf{st}_{A_1,\ldots,A_k,B}; \\
&\quad \Diamond([\![x_1 : \Box A_1, x_n : \Box A_k, y : B \triangleright P : \Diamond C]\!]); \mu_C
\end{aligned}$$

Thus our definition of a categorical model for **IS4** is as follows.

Definition 5. A categorical model for **IS4** consists of a cartesian closed category with coproducts, $\mathbb{C}$, together with a monoidal comonad $(\Box, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_1)$ and a $\Box$-strong monad $(\Diamond, \eta, \mu, \mathsf{st}_{A,B})$.

To reiterate, this notion of a categorical model satisfies the following properties.

Proposition 6. *Given a category, $\mathbb{C}$, which satisfies Definition 5.*

1. *For every deduction $\mathcal{D}$ of $\Gamma \vdash A$ in* **IS4** *there is a morphism $[\![\mathcal{D}]\!]: \Gamma \to A$ in $\mathbb{C}$.*
2. *If a deduction $\mathcal{D}$ β-reduces to $\mathcal{D}'$, then $[\![\mathcal{D}]\!] = [\![\mathcal{D}']\!]$.*
3. *If a deduction $\mathcal{D}$ c-reduces to $\mathcal{D}'$, then $[\![\mathcal{D}]\!] = [\![\mathcal{D}']\!]$.*

In other words, any concrete model which satisfies the conditions given in Definition 5, is a *sound* model of **IS4**.

8. Prawitz's formulation and the categorical model

Although Prawitz's formulation has the appearance of being equivalent to that presented in this paper, it has rather unfortunate proof and model theoretic consequences. Consider the following deduction in Prawitz's formulation.

$$\dfrac{\dfrac{\dfrac{\dfrac{(1)\ \Box\Box\Box A}{(2)\ \Box\Box A}\,\Box\mathcal{E}}{(3)\ \Box A}\,\Box\mathcal{E}}{A}\,\Box\mathcal{E}}{\Box A}\,\Box\mathcal{I}$$

As explained in §4, the application of the $\Box\mathcal{I}$ rule is valid as there is a modal formula in the path from the conclusion, A, to the undischarged assumption, $\Box\Box\Box A$. In fact there are three candidates for this modal formula, which are numbered in the deduction. In our formulation these alternatives represent distinct deductions. For example, alternatives (2) and (3) are given by the deductions

$$\dfrac{\dfrac{\Box\Box\Box A}{\Box\Box A}\,\Box\mathcal{E} \qquad \dfrac{\dfrac{[\![\Box\Box A]\!]}{\Box A}\,\Box\mathcal{E}}{A}\,\Box\mathcal{E}}{\Box A}\,\Box\mathcal{I}$$

and

$$\dfrac{\dfrac{\dfrac{\Box\Box\Box A}{\Box\Box A}\,\Box\mathcal{E}}{\Box A}\,\Box\mathcal{E} \qquad \dfrac{[\![\Box A]\!]}{A}\,\Box\mathcal{E}}{\Box A}\,\Box\mathcal{I}$$

It is important to note that both these deductions are in (β, c)-normal form, and so are *distinct* deductions. Prawitz's formulation essentially collapses these two deductions into one. In other words his formulation forces a seemingly unnecessary identification of deductions. Let us consider the consequences of this identification with respect to the categorical model. The two derivations above are modelled by the morphisms

$$\varepsilon_{\Box\Box A}; \delta_{\Box A}; \Box(\varepsilon_{\Box A}); \Box(\varepsilon_A)\colon \Box\Box\Box A \to \Box A$$

and

$$\varepsilon_{\Box\Box A};\varepsilon_{\Box A};\delta_A;\Box\varepsilon_A\colon\Box\Box\Box A\to\Box A$$

respectively. Insisting on these being equal amounts to the equality

$$\varepsilon_{\Box\Box A};\Box\varepsilon_A=\varepsilon_{\Box\Box A};\varepsilon_{\Box A}.$$

Precomposing this equality with the morphism $\delta_{\Box A}$ gives

$$\Box\varepsilon_A=\varepsilon_{\Box A}.$$

It is easy to see that this is sufficient to make the comonad *idempotent*, i.e. $\Box A\cong\Box\Box A$. In other words, all sound models of Prawitz's proof theory must have an idempotent comonad, which is a very strict requirement. It is worth reiterating that our natural deduction formulation does *not* impose this identification of proofs and consequently does not force an idempotency.

9. An alternative possibility modality

In §7 we saw that the possibility modality is modelled by a $\Box$-strong monad. A natural weakening of this condition is to only insist that it be a *strong* monad.

DEFINITION 6. Given a monad $(\Diamond,\eta,\mu)$ on a CCC $\mathbb{C}$, we say that it is a *strong monad* if there exists a natural transformation

$$\mathsf{s}_{A,B}\colon A\times\Diamond B\to\Diamond(A\times B)$$

which satisfies the following four equations.

1. $\mathsf{s}_{1,A};\Diamond(\pi_2)=\pi_2$
2. $\mathsf{id}_A\times\eta_B;\mathsf{s}_{A,B}=\eta_{A\times B}$
3. $\alpha;\mathsf{id}_A\times\mathsf{s}_{B,C};\mathsf{s}_{A,B\times C}=\mathsf{s}_{A\times B,C};\Diamond(\alpha)$
4. $\mathsf{s}_{A,\Diamond B};\Diamond(\mathsf{s});\mu_{A\times B}=\mathsf{id}_A\times\mu_B;\mathsf{s}_{A,B}$

This weaker possibility modality has the following natural deduction rules.

$$\frac{\Gamma\vdash A}{\Gamma\vdash\Diamond A}\,\Diamond\mathcal{I}\qquad\frac{\Gamma\vdash\Diamond A\qquad\Gamma,A\vdash\Diamond B}{\Gamma\vdash\Diamond B}\,\Diamond\mathcal{E}$$

It can also be axiomatised by the following two axioms.

$$A\supset\Diamond A$$
$$\Diamond A\supset((A\supset\Diamond B)\supset\Diamond B)$$

Somewhat surprisingly, this rather odd possibility modality has been discovered before [10, 4]. Indeed, in terms of computer science, this modality has many important applications [30, 15].

10. Additional properties

The computationally-inclined reader may have expected some additional reduction rules for our λ^{S4}-calculus. Two interesting additional reduction rules are as follows.

$$
\begin{array}{rl}
\mathsf{box}\, M \,\mathsf{with} & N_1, \dots, N_{i-1}, N_i, N_{i+1}, \dots, N_k \\
\mathsf{for} & x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k \\
\rightsquigarrow_\beta & \\
\mathsf{box}\, M \,\mathsf{with} & N_1, \dots, N_{i-1}, N_{i+1}, \dots, N_k \\
\mathsf{for} & x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k \\
& \qquad\qquad \text{where } x_i \notin \mathrm{FV}(M)
\end{array} \tag{G}
$$

$$
\begin{array}{l}
\begin{array}{rl}
\mathsf{box}\, M \,\mathsf{with} & N_1, \dots, N_{i-1}, P, P, N_{i+2}, \dots, N_k \\
\mathsf{for} & x_1, \dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots, x_k
\end{array} \\
\rightsquigarrow_\beta \\
\begin{array}{rl}
\mathsf{box}\, M[x_i, x_{i+1} := y] \,\mathsf{with} & N_1, \dots, N_{i-1}, P, N_{i+2}, \dots, N_k \\
\mathsf{for} & x_1, \dots, x_{i-1}, y, x_{i+2}, \dots, x_k
\end{array}
\end{array} \tag{C}
$$

The first of these rules, (G), is a sort of 'garbage collection' rule, the second, (C), a 'coalescing' of identical explicit substitutions.

These rules are also interesting from our categorical perspective in that we can characterise them quite succinctly.[2]

PROPOSITION 7. (Schalk) *A categorical model for* **IS4** *(as given in Definition 5), which in addition satisfies the property that the maps* $!_{\Box A}\colon \Box A \to 1$ *and* $\Delta_{\Box A}\colon \Box A \to \Box A \times \Box A$ *are (free) coalgebra morphisms, satisfies the two reduction rules* (G) *and* (C).

This extra condition on the terminal and diagonal morphisms amounts to requiring that the following two equations hold.

$$
\begin{array}{rcl}
!_{\Box A}; \mathsf{m}_1 & = & \delta_A; \Box(!_{\Box A}) \\
\delta_A; \Box(\Delta_{\Box A}) & = & \Delta_{\Box A}; \delta_A \times \delta_A; \mathsf{m}_{\Box A, \Box A}
\end{array}
$$

This richer notion of a categorical model of **IS4** actually amounts to a linear category (in the sense of Bierman [5]) where the underlying symmetric monoidal category is, in fact, a cartesian closed category. Further investigation of this categorical model remains future work.

[2] This fact was proved by A. Schalk (personal communication).

11. Conclusions

In this paper we have considered the propositional, intuitionistic modal logic **IS4**. We have given equivalent axiomatic, sequent calculus and natural deduction formulations, the corresponding term assignment system and the definition of a general categorical model.

As mentioned in the introduction we place particular importance on a natural deduction formulation and its metatheoretic properties. We have shown that the formulation proposed by Satre [35] is not closed under substitution, and the formulation proposed by Prawitz [32] introduces seemingly unnecessary identifications of proofs, which in the model forces an idempotency.

A number of authors have considered the question of providing natural deduction formulations of (intuitionistic) modal logics (some in response to our earlier work [6]). These include Benevides and Maibaum [2], Bull and Segerberg [9, pages 29–30], Davies and Pfenning [13], Martini and Masini [26], Mints [27, Pages 221–294] and Simpson [37]. However they all use extensions of one form or another to the nature of natural deduction (for example, by indexing formulae with possible worlds information). Again we reiterate the conceptual simplicity of our proposal — we use no new features of natural deduction.

The techniques used in this paper to formulate modalities, also apply to Girard's linear logic [20]. Conversely there are a number of papers concerning linear logic which could be usefully studied in the context of modal logics, e.g. [3, 29, 28]. We leave this to future work.

We also prefer the use of categorical models over traditional Kripke-style possible worlds models. Unlike other categorical work we have placed emphasis on modelling the proof theory not just provability. Our resulting model is considerably simpler than other proposals.

In the future we should like to consider other modal logics within our framework. It is clear that not all of the hundreds of modal logics will fit into our framework. However we do not view this as a weakness of our work. Rather we feel it is important to identify those modal logics which have simple, but interesting proof theories and mathematically appealing classes of (categorical) models.

Acknowledgements

This paper has suffered a little at the hands of time. The material (concerning just the necessity modality) was first presented at the Logic at Work conference in Amsterdam in December 1992 and appears in draft form in

the informal proceedings [6]. For the next three years it was to appear as a chapter of a collected works, before being dropped at the final minute by the editors. It was then simply issued as [8]. The details of the possibility modality were presented at the 1994 European meeting of the ASL — an abstract appears as [7]. We are grateful to Professor Mints for inspiring us to finally write up our material.

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