IDEAS AND RESULTS IN PROOF THEORY

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In this lecture I shall give an exposition of certain themes in proof theory. I do not intend to give a general survey of proof theory but shall concentrate on the following topics:

(i) the ideas behind what may be called Gentzen-type analysis in proof theory, where in particular, I want to draw attention to the fact that they constitute the embryo to a general proof theory;

(ii) extensions of the results obtained by Gentzen to more powerful theories;

(iii) the connection between proofs and the terms used in functional interpretations of intuitionistic logic, in particular, the connection between Gentzen-type analysis and the Gödel-type analysis that originated with Gödel's so-called Dialectica interpretation.

In an appendix, I develop a notion of validity of derivations, which may be contemplated as a possible explication of Gentzen's ideas about an operational interpretation of the logical constants.

Proofs of the results are usually left out but in the appendix mentioned, it is shown how this notion of validity may be used as a convenient tool to establish the main result about strong normalization in first order logic. In a second appendix, it is shown how to extend this notion and this result to second order logic.

To start with, I shall make some general comments about proof theory.

To simplify the reading, I list the content in more detail:

I want to mention that many of the view points of this paper evolved during several conversations that I have had with Georg Kreisel and Per Martin-Löf to both of whom I am much indebted. Georg Kreisel also read the draft of the paper and made valuable suggestions.

I. General and reductive proof theory

1. As the name suggests, proof theory studies proofs. In other words, it studies not only the theorems of a theory, *what* we know in the theory, but also *how* we know the theorems.

Such studies may differ considerably with respect to their aims, however. Accordingly, I shall distinguish between *general* and *reductive proof theory*. This is a distinction which (although not called by these names) is in effect made in some of Georg Kreisel's writings.

In *general proof theory*, we are interested in the very notion of proof and its properties and impose no restriction on the methods that may be used in the study of this notion.

In *reductive proof theory*, we are interested in analysing mathematical theories to attain a reduction of them. The study of the proofs of a theory is the tool used to obtain this end, and hence, this study has to use more elementary principles than those occurring in the theory in question.

2. The subject matter of *general proof theory* is thus proofs considered as a process by which we get to know the theorems of a theory or the validity of an argument, and this process is studied here in its own right – the study is not only a tool for another analysis. Obvious topics in general proof theory are:

2.1. The basic question of defining the notion of proof, including the question of the distinction between different kinds of proofs such as constructive proofs and classical proofs.

2.2. Investigation of the structure of (different kinds of) proofs, including e.g. questions concerning the existence of certain normal forms.

2.3. The representation of proofs by formal derivations. In the same way as one asks when two formulas define the same set or two sentences express the same proposition, one asks when two derivations represent the same proof; in other words, one asks for identity criteria for proofs or for a "synonymity" (or equivalence) relation between derivations.

2.4. Applications of insights about the structure of proofs to other logical questions that are not formulated in terms of the notion of proof.

3. *Reductive proof theory* must of course be based upon some conception of an order between different principles with respect to how elementary they are. Typically, one singles out one kind of principles like the finitistic ones or the constructive ones and then tries to interprete a theory which at least on the surface contains also other principles into a theory that contains only the principles in question. An interpretation of this kind may also be obtained by other methods than the proof theoretical ones, i.e. without essentially using the notion of proof, but proof theoretical interpretations often differ radically from interpretations usually considered in logic in some respects (besides the special epistemological status of the theory in which the interpretation is made).

3.1. Firstly, it may not be possible to interprete every sentence in the interpreted theory directly. The sentences in the interpreted theory are thus divided into two groups: the *real sentences* which are directly given a meaning and the *ideal sentences* which lack such a direct meaning. The problem is then to show that the theory as a whole can be understood as a theory about the real sentences in which the ideal sentences only serve (as so-called ideal elements) to facilitate operations within the theory or to round off the theory. To this end, one has to show that every derivable real sentence – i.e. also when the proof proceeds over the meaningless ideal sentences – is true according to the meaning given to the real sentences. A typical proof theoretical demonstration of this kind is obtained when it is shown how to transform a derivation of a real sentence that proceeds over ideal sentences to a derivation containing only real sentences. In reductive proof theory, it is required that this demonstration is given by using only the principles occurring in the theory in which the interpretation is made.

3.2. Secondly, the interpretation itself may depend not only on the form of the sentences but also on their derivations. For instance, there may be a mapping f from formulas in a theory $\mathbf{T}_1$ to formulas in a theory $\mathbf{T}_2$, a mapping g from derivations in $\mathbf{T}_1$ into terms in $\mathbf{T}_2$, and an operation ϕ such that when π is a derivation in $\mathbf{T}_1$ of a sentence A, then $\phi(f(A),g(\pi))$ is a sentence holding in $\mathbf{T}_2$.

4.1. As an early example (perhaps the first one) of an investigation in general proof theory, we may take *Frege's* formal definition of derivability. His definition may be understood as an extensional characterization of logical proofs (i.e. a characterization with respect to the set of theorems) within certain languages: his notion of derivability is such that to every (intuitive) proof of a proposition within a certain domain, there exists a derivation of the corresponding formula; but the characterization is only extensional since the formal derivation may use quite different methods of proof and have a structure different from the intuitive proof.

4.2. The founder of reductive proof theory is of course *Hilbert*. In his various proof theoretical publications, Hilbert considered two main problems. The first one is to give an analysis of mathematical theories which at least on the surface does not seem to go very far, namely, to demonstrate their formal consistency with the use of finitary means. The second one, which is philosophically more mature, is to reduce classical mathematics to finitary mathematics using the idea of a division of mathematical sentences into real and ideal sentences, the real sentences here being the ones that can be given a finitary meaning directly. The two problems are not always distinguished although Hilbert was often careful enough to do so. The second problem is clearly the most interesting one, but under some rather general conditions, the two problems happen to be equivalent: the truth of the derivable real sentences implies the consistency of the theory provided every sentence is derivable from a contradiction (and some real sentence is not true); and conversely, the consistency implies the truth of the derivable real sentences provided each true real sentence is derivable (and the negation of a false real sentence is a true real sentence).

4.3. Later examples of works in reductive proof theory are *Kreisel*'s no counter example interpretation and *Gödel*'s interpretations in terms of computable functionals (the so-called Dialectica interpretation). These two interpretations, which give a constructive meaning to *every* derivable sentence in first order arithmetic, show the second characteristic feature of a proof theoretical interpretation described above where the interpretation depends on the derivations.

4.4. *Gentzen*'s works to which I shall now turn are to a great extent part of both general and reductive proof theory.

II. Gentzen's analysis of first order proofs

A most basic question in general proof theory, which is also of fundamental importance for reductive proof theory, is of coure the question how the notion of logical proof within first order languages is to be analysed. The work by Gentzen (1935) may be viewed as an answer to this question. The answer was given in two steps: In a first analysis, Gentzen showed how the notion could be defined in terms of certain formal systems constructed by him. Then, by a deeper analysis of the structure of these proofs, he showed that they could be written in a very special form, which altogether gave a very satisfactory understanding of these proofs.

The first step was carried out for the so-called systems of natural deduction, while the second step was carried out for the so-called calculi of sequents. There are certain advantages (which will soon become apparent) in carrying out both steps for the systems of natural deduction and I shall briefly do this here.

II.1. Gentzen's systems of natural deduction

1.1. *Main idea*

Gentzen's systems of natural deduction arise from a particular analysis of deductive inferences by which the deductive role of the different logical constants are separated. The inferences are broken down into atomic steps in such a way that each step involves only one logical constant. The steps are of two kinds, and for each logical constant there are inferences of both kinds: steps that allow the *introduction* of the logical constant (i.e., the conclusion of the inference has the constant as outermost symbol) and steps that allow the *elimination* of the logical constant (i.e., the premiss or one of the premisses of the inference has the constant as outermost symbol).

The proofs start from *assumptions*, which at certain steps in the proof may be discharged or *closed*; typically, an assumption A is closed at an introduction of an implication $A \supset B$.

1.2. *Inference rules*

It is suitable to represent the proofs as *derivations* written in tree form. The top formulas of the tree are then the assumptions, and the other formulas of the tree are to follow from the one(s) immediately above by one of the *inference rules* that formalizes the atomic inferences mentioned above. A formula A in the tree is said to *depend* on the assumptions standing above A that have not been closed by some inference preceeding A. The *open assumptions* of a derivation are the assumptions on which the last formula depends.

I state the inference rules in the form of schemata in the usual way. A formula written within square brackets above a premiss is to indicate that assumptions of this form occurring above the premiss are discharged at this inference. An inference rule is labelled with the logical constant that it deals with followed by "I" when it is an introduction and "E" when it is an elimination.

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \qquad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$

⊃I) $\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

$$\forall\text{I})\quad \frac{A(a)}{\forall x A(x)} \qquad\qquad \forall\text{E})\quad \frac{\forall x A(x)}{A(t)}$$

$$\exists\text{I})\quad \frac{A(t)}{\exists x A(x)} \qquad\qquad \exists\text{E})\quad \frac{\exists x A(x) \quad \begin{matrix}[A(a)]\\ B\end{matrix}}{B}$$

1.2.1. *Restrictions*. Obvious conventions about substitution are to be understood. The rules $\forall$I and $\exists$E are with the following further restrictions concerning the parameter a, called the *proper parameter* (or Eigenparameter) of the inference: In $\forall$I, a is not to accur in the assumtions that $A(a)$ depends on; in $\exists$E, a is not to accur in B nor in the assumptions that the premiss B depends on except those of the form $A(a)$ (closed by the inference).

1.2.2. *Negation*. We assume that the first order languages contain a constant Λ for absurdity (or falsehood) and that $\sim A$ is understood as shorthand for $A \supset \Lambda$. The obvious introduction and elimination rules for negation

$$\sim\text{I})\quad \frac{\begin{matrix}[A]\\ \Lambda\end{matrix}}{\sim A} \qquad\qquad \sim\text{E})\quad \frac{A \quad \sim A}{\Lambda}$$

are then special cases of $\supset$I and $\supset$E, respectively.

1.2.3. *Major and minor premisses*. In an inference by an application of an E-rule, the premiss in which the constant in question is exhibited (in the figures above) is called the *major premiss* of the inference and the other premiss(es) if any, we call the *minor premiss(es)*.

1.2.4. *Convention about proper parameters*. To simplify certain formal details in the sequel, it shall tacitly be assumed that a parameter in a derivation is the proper parameter of at most one inference, that the proper parameter of an $\forall$I-inference occurs only above the conclusion of the inference, and that the proper parameter of an $\exists$E-inference occurs only above the minor premiss of the inference.

1.3. *The systems* M, I, C

1.3.1. *Minimal logic*. The rules given above determine the system of natural deduction for (first order) *minimal logic,* abbreviated **M**.

1.3.2. *Intuitionistic logic*. By adding the rule Λ_I (intuitionistic absurdity rule)

$$\frac{\Lambda}{A}$$

where A is to be atomic and different from Λ, we get the system of natural deduction for *(first order) intuitionistic logic* (**I**).

1.3.3. *Classical logic*. The system of nautral deduction for (first order) *classical logic* (**C**) is obtained by

(i) considering languages without the constant v and $\exists$ and leaving out the rules for these constants (they become derived rules when v and $\exists$ are defined in the usual way) and

(ii) adding to the rules of **M** the rule Λ_C (classical absurdity rule)

$$\frac{\begin{matrix}[\sim A]\\ \Lambda\end{matrix}}{A}$$

where A is atomic and different from Λ.

1.4. *Derivations and derivability*

We say that Π is a *derivation* in **M**(**I** or **C**) of A from a set of formulas Γ when Π is a tree formed according to the above explanations using the rules of **M**(**I** or **C**) with an end formula A depending on formulas all belonging to Γ. If there is such a derivation, we say that A is *derivable* in the system in question from Γ; in short: $\Gamma \vdash A$. When A is derivable from the empty set of assumptions, we may say simply that A is *derivable*.

1.5. *Extensions by the additions of atomic systems*

It is also of interest to consider extensions of the three systems defined above by adding further rules for atomic formulas. There may also be reason to specify the languages of the systems exactly. By an *atomic system*, I shall understand a system determined by a set of *descriptive constants*, (i.e. individual, operational, and predicate constants) and a set of *inference rules* for atomic sentences with these constants (i.e. both the premisses and the conclusion are to be atomic formulas of this kind). A rule may lack premisses and is then called an *axiom*. Let **S** be an (atomic) system of this kind. By a *formula over* **S**, we shall understand a formula whose descriptive constants are those of **S**. By **M(S)**, we shall understand the system of natural deduction whose language is the first order language determined by the descriptive constants of **S** and whose rules are the rules of **S** and **M**. We define the systems **I**(**S**) and **C**(**S**) similarly. Note that **I**(**S**) is the same as **M**($\mathbf{S}^+$) where $\mathbf{S}^+$ is obtained from **S** by adding the rule Λ_{I}.

In many contexts, it is not essential how the rules of a system **S** are specified and we make no restriction of that kind. Of special interest, however, are the *Post systems* where the inference rules are determined as the instances of a finite number of schemata of the form

$$\frac{A_1 \quad A_2 \quad \dots \quad A_n}{B}$$

where A_1, A_2, A_n, and B are atomic formulas. One may also require that B contains no parameter that does not occur in some A_i.

1.6. *Remark*

A trivial reformulation of the systems described above is obtained if we make explicit the assumptions that a formula in a derivation depends on by writing these assumptions in a sequence followed by an arrow in front of the formula. The tree of formulas is then replaced by a tree of so-called *sequents* of the form $\Gamma \to A$ where Γ is a sequence of formulas. If the inference rules are now formulated with these sequents as the basic objects, they get a somewhat different look. For instance, $\supset$I and $\supset$E now become

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B} \qquad\qquad \frac{\Gamma \to A \quad \Delta \to A \supset B}{\Gamma, \Delta \to B}$$

and the derivations now start not from assumptions but from axioms of the form $A \to A$. Clearly, there is only an inessential notational difference between the first formulation and this reformulation; and the new formulation is not to be confused (as is sometimes done) with the calculus of sequents, which differs more essentially by having, instead of elimination rules, rules for operating on the formulas to the left of the arrow (cf. for instance, the new formulation of $\supset$E with the rule for introducing $\supset$ in the antecedent in the calculus of sequents). (For a remark concerning the relationship between natural deduction and the calculus of sequents, see sec. 5.1.)

II.2. The significance of Gentzen's systems

The most noteworthy properties of Gentzen's systems of natural deduction seems to be (2.1) the analysis of deductive inferences into atomic steps, by which the deductive role of the different logical constants is separated and (2.2) the discovery that these atomic steps are of two kinds, viz. introductions and eliminations, standing in a certain symmetrical relation to each other.

2.1. This analysis may be understood as an attempt to characterize the notion of proof, not only provability as first done by Frege (sec. I.4.1), in the sense that it is an attempt (2.1.1) to isolate the essential deductive operations and (2.1.2) to break them down as far as possible.

2.1.1. The deductive operations that are isolated are to begin with constructive (or intuitionistic) ones. There is a close correspondence between the constructive meaning of a logical constant and its introduction rule. For instance, an implication $A \supset B$ is constructively understood as the assertion of the existence of a construction of B from A, and in accordance with this meaning $\supset$I allows the inference of $A \supset B$ given a proof of B from A. Of course, a proof of B from A is not the same as a construction of B from A; it is rather a special kind of such a construction (cf. sections 2.2.2 and IV.1). There is thus not a complete agreement but a close correspondence between the constructive meaning of the constants and the introduction rules; sometimes the correspondence is very close: some of Heyting's explanation, see e.g. Heyting (1956) 97–99 and 102, may almost be taken as a reading of Gentzen's introduction rules.

This correspondence between the introduction rules and the constructive interpretation of the logical constants is a strong indication that the essential *constructive* deductive operations have been isolated. By a (first order) *positive proof*, I shall in this paper understand a (first order) proof that uses intuitionistically valid deductive operations but no operation assuming any special properties of Λ; an *intuitionistic proof* is then simply a proof using intuitionistically valid deductive operations. The claim is thus that **M** constitutes an analysis of first order positive proofs and **I**, an analysis of first order intuitionistic proofs.

The *classical* deductive operations are then analysed as consisting of the constructive ones plus a principle of indirect proof for atomic sentences, which may be understood as stating as a special assumption that the atomic sentences are decidable[1]. One may doubt that this is the proper way of analysing classical

[1] In the formulation by Gentzen (1935), one adds instead of the rule Λ_C the axiom of the excluded middle to get classical logic from intuitionistic logic. Our choice of Λ_C is motivated by the subsequent development.

inferences, and it is true that the rules of the classical calculus of sequents or some variants of it (like the one by Schütte (1951) or the one by Tait (1968)) are closer to the classical meaning of the logical constants. But this possibility of analysing classical inferences as a special case of constructive ones (applicable to decidable sentences) provides a way of constructively understanding classical reasoning (which is the essential fact behind Kolmogoroff's (1925) and Gödel's (1932) interpretation of classical logic in intuitionistic logic; see also Prawitz and Malmnäs (1968)) and also explains the success in carrying over to classical logic the deeper results concerning the structure of proofs, which at first sight are evident only for constructive proofs (cf. sec. 2.2.3).

The claim that the essential deductive operations have been isolated is not to be understood as a claim that these operations mirror all informal deductive practices, which would be an unreasonable demand in view of the fact that informal practices may sometimes contain logically insignificant irregularities. What is claimed is that the essential logical content of intuitive logical operations that can be formulated in the languages considered can be understood as composed of the atomic inferences isolated by Gentzen. It is in this sense that we may understand the terminology *natural* deduction.

Nevertheless, Gentzen's systems are also natural in the more superficial sense of corresponding rather well to informal practices; in other words, the structure of informal proofs are often preserved rather well when formalized within the systems of natural deduction. This may seem surprising in view of what was said about the correspondence between the inference rules and the *constructive* meaning of the logical constants. But it is a fact that actual reasoning often makes use of a mixture of constructive and non-constructive principles with predominance of constructive principles, the non-constructive ones typically amounting to an occasional use of the principle of indirect proof. For instance, it is a fact that an implication $A \supset B$ most often is proved by deducing B from A, a procedure that is not singled out by the classical truth functional meaning of $\supset$ as the most natural one.

2.1.2. It seems fair to say that a proof built up from Gentzen's atomic inference is *completely analysed* in the sense that one can hardly imagine the possibility of breaking down his atomic inferences into some simpler inferences.

The separation of the deductive role of the different logical constants was partly achieved already by Hilbert in some of his axiomatic formulations of sentential logic. Gentzen is able to complete this separation by separating also the role of implication from that of the other constants.

We may note in passing that the derivations in Gentzen's systems are completely analysed also in the less important sense that for each formula in the derivation, it is uniquelly determined from what premisses and by which inference rule it is inferred; these are properties that Gentzen's systems share also with certain other logical calculi.

2.1.3. From what has been said above, it should be clear that Gentzen's systems of natural deduction are not arbitrary formalizations of first order logic but constitutes a significant analysis of the proofs in this logic[1].

The situation may be compared to the attempts to characterize the notion of computation where e.g. the formalism of μ-recursive functions or even the general recursive functions may be regarded as an extensional characterization of this notion while Turing's analysis is such that one may reasonably assert the thesis that every computation when sufficiently analysed can be broken down in the operations described by Turing.

2.2. What makes Gentzen's systems especially interesting is the discovery of a certain symmetry between the atomic inferences, which may be indicated by saying that the corresponding introductions and eliminations are *inverses* of each other. The sense in which an elimination, say, is the inverse of the corresponding introduction is roughly this: the conclusion obtained by an elimination does not state anything more than what must have already been obtained if the major premiss of the elimination was inferred by an introduction. For instance, if the premiss of an &E was inferred by introduction, then the conclusion of the &E must already occur as one of the premisses of this introduction. Similarly, if the major premiss $A \supset B$ of an $\supset$E was inferred by an introduction, then a proof of the conclusion B of the $\supset$E is obtained from the proof of the major premiss of the $\supset$E by simply replacing its assumption A by the proof of the minor premiss.

2.2.1. In other words, a proof of the conclusion of an elimination is already "contained" in the proofs of the premisses when the major premiss is inferred

[1] It seems fair to say that no other system is more convincing in this respect. The axiomatic systems introduced by Frege and continued by Russell, Hilbert, and others clearly do not usually have this aim at all. There exists other systems that have been called systems of natural deduction, often proposed for didactical pruposes. To the extent that they are not to be considered as mere notational variants of Gentzen's systems, it must be said that they do not even approximately match Gentzen's analysis. (Their alleged pedagogical merits also seem doubtful, see Prawitz (1965) 103–105 and Prawitz (1967).) Gentzen's calculi of sequents may be considered as (and were historically) derived from his systems of natural deduction.

by introduction. We shall refer to this by saying that the pairs of corresponding introductions and eliminations satisfy the *inversion principle*. We shall see very soon how to make this principle more precise, but let us first consider another aspect of the principle.

2.2.2. The inversion principle seems to allow a *reinterpretation* of the logical symbols. With Gentzen, we may say that the introductions represent, as it were, the "definitions" of the logical constants. An introduction rule states a sufficient condition for introducing a formula with this constant as outermost symbol (which, as we saw, was in very close agreement with the constructive meaning of the constant but did not express this meaning completely), and this condition may now be taken as the "meaning" of the logical constant. For instance, $A \supset B$ is now to mean that there is a deduction of B from A. The eliminations, on the other hand, are "justified" by this very meaning given to the constants by the introductions. As guaranteed by the inversion principle, the conclusion of an elimination only states what must hold in view of the meaning of the major premiss of the elimination. The examples with &E and $\supset$E given above already illustrates this. As a further illustration, consider the inference

$$\frac{\sim\forall x A(x)}{\exists x \sim A(x)}$$

This inference is clearly not justified by the meaning given to the constants by the introductions: The sufficient condition for introducing $\sim\forall xA$ i.e. a proof of Λ from the assumption $\forall xA$, in no way guarantees that $\exists x \sim A$ holds (in the sense of the introductions rules) since such a proof may not at all contain a proof of $\sim A(t)$ for some term t.

These ideas of Gentzen's are of course quite vague since it is not meant that the introductions are literally to be understood as "definitions". In appendix A, I shall consider one possible way of making the ideas precise.

2.2.3. Here, I shall consider a more direct way of making the inversion principle precise. Since it says that nothing new is obtained by an elimination immediately following an introduction (of the major premiss of the elimination), it suggests that such sequences of inferences can be dispensed with. From this observation, it is possible to obtain quite simply Gentzen's results about the structure of proofs, the second step in his analysis. Indeed, the whole idea of this analysis is contained in the observations made above. Note, however, that only the rules of minimal logic are governed by the inversion principle; both the rule Λ_I and the rule Λ_C clearly fall outside the pattern of introductions and eliminations. Note in particular that the principle of indirect proof when not restricted to atomic formulas constitute quite a new principle for inferring compound formulas (which is not at all justified in the terminology of sec. 2.2.2 by the meaning given to the constants by the introduction rules). Our restriction that the conclusions of applications of the rules Λ_I and Λ_C are to be atomic (1.3.1 and 1.3.2) are motivated by these considerations. It is by this restriction that these extra rules do not disturb the pattern of introductions and eliminations or the development in the next section (cf. what was said in sec. 2.1.1 about the value of reducing classical logic in this way).

II.3. Normal derivations and the reducibility relation

3.1. *Definition of normal derivation*

3.1.1. *Maximum formulas.* A formula occurrence in a derivation that stands at the same time as the conclusion of an introduction and as the major premiss of an elimination is called a *maximum formula.* As is seen by inspecting the inference rules, such a formula is of greater complexity than the surrounding formulas; it constitutes a local maximum (in its path as defined in sec. 3.2.1). The inversion principle implies that a maximum formula is an unnecessary detour in a derivation which can be removed.

3.1.2. *Maximum segments.* Note: at applications of the rules for vE and $\exists$E, formulas of the same form occur immediately below each other. A sequence of formula occurrences $A_1, A_2, ..., A_n$ in a derivation is said to be a *maximum segment* in the derivation when they are of the same form, A_{i+1} stands immediately below A_i, A_1 is the conclusion of an introduction, and A_n is the major premiss of an elimination (A_i, for $i < n$, is then a minor premiss of an vE or $\exists$E). Also maximum segments constitutes detours that can be removed. When $\vee$ and $\exists$ are not present (as in our formulation of classical logic) no such segments can appear.

3.1.3. *Normal derivations.* A derivations in any of the systems considered above is now defined as *normal* or said to be in *normal form* when it contains no maximum formula and no maximum segment.

3.1.4. *Fully normal derivations.* Also a normal derivation may contain certain redundancies that one wants to remove. An application of vE is said to be *redundant* if the two minor premisses do not stand below assumptions that are closed by the inference. Similarly, an application of ∃E is said to be *redundant* if no assumption is closed by the inference. An application of $\wedge_C$ is said to be *redundant* if there is an assumption $\sim A$ closed by the inference that is major premiss of an elimination and all of the assumptions on which the minor premiss A depends are assumptions on which also the conclusion A of the application of $\wedge_C$ depends. Redundant applications of ∨E, ∃E, and $\wedge_C$ constitute unnecessary complications and are easily removed (see sec. 3.4). A derivation is said to be *fully normal* or to be in *full normal form* when it is normal and contains no such rendundant application of vE, ∃E, or $\wedge_C$.

3.2. *The form of normal derivations*

A normal derivation has quite a perspicuous form: it contains two parts, one *analytical part* in which the assumptions are broken down in their components by use of the elimination rules, and one *synthetical part* in which the final components obtained in the analytical part are put together by use of the introduction rules. Between the analytical and the synthetical part there is a *minimum part* in which operations on atomic formulas may occur. To state this form of the normal derivations in a more precise way, we introduce a certain terminology.

3.2.1. *Branches and paths*. A *branch* in a derivation is a sequence $A_1, A_2, \ldots, A_n$ of formula occurrences in the derivation where A_1 is an assumption not discharged by ∨E or ∃E, A_{i+1} is the formula occurrence immediately below A_i, and A_n is either the end formula of the derivation or the minor premiss of an ⊃E but no A_i with $i < n$ is such a minor premiss. A *path* is like a branch except that the formula occurrence immediately succeeding the major premiss of an ∨E or ∃E is not the formula occurrence immediately below but one of the assumptions discharged by the inference in question.

3.2.2. For derivations not containing any ∨E or ∃E, we can now state the form of normal derivations in the following

Theorem. *A branch in a normal derivation in* **M(S)**, **I(S)**, or **C(S)** *not containing* ∨E *or* ∃E *can be divided into three unique parts:*
(i) one *analytical part,* $A_1, A_2, \ldots, A_{m-1}$, in which each formula is the major premiss of an elimination and contains the immediately succeeding formula as a subformula;
(ii) one *minimum part,* $A_m, A_{m+1}, \ldots, A_{m+k}$, in which each formula except the last one is premiss of $\wedge_I$, $\wedge_C$, or a rule of **S**;

(iii) one *synthetical part,* $A_{m+k}, A_{m+k+2}, \ldots, A_n$, in which each formula is the conclusion of an introduction and is a subformula of the immediately preceding one.

Both the analytical and the synthetical part may be empty ($m = 1$ or $m + k = n$, respectively). In the case of the system **M**, the minimum part contains exactly one formula A_m, called the *minimum formula* of the branch. The minimum formula is the conclusion of an elimination, the premiss of an introduction, and a subformula of both A_1 and A_n, and is the only formula of the branch with all these properties. In the case of the other systems, the minimum part may contain more than one formula, which then all have to be atomic. When all the minima formulas in a fully normal derivation are atomic, I shall say that the derivation is in *expanded normal form.*

3.2.3. For derivations containing $\vee$E or $\exists$E, we consider paths instead of branches. The paths in such a derivation are divided into *segments* consisting of all the consecutive formula occurrences of the same form. The statement of the form of normal derivations in sec. 3.2.2 then extends literally to the case when $\vee$E and $\exists$E are present if we replace "branch" by "path" and "formula" by "segment" and agree to the following conventions: a *segment* is said to be the *conclusion* (*(major) premiss*) of the inference of which the first (last) occurrence in the segment is the conclusion ((major) premiss); if the first occurrence in the segment is an assumption closed by an inference $\vee$E or $\exists$E, the segment is said to be the conclusion of this inference; one segment is said to be a *subsegment* of another segment, if the formula occurring in the first segment is a subformula of the formula occurring in the second segment. In order that every formula in a derivation is to belong to one path, we have to require that no $\vee$E or $\exists$E is redundant and therefore, we consider fully normal derivations.

3.2.4. From this detailed description of the normal derivations, we easily obtain some more specific corollaries:

3.2.4.1. **Corollary.** *The last inference in a fully normal derivation in* **M**, **I**, *or* **C** *without open assumptions is an introduction.*

3.2.4.2. **Corollary.** *A fully normal derivation in* **M(S)**, **I(S)**, *or* **C(S)** *of an atomic formula from a set of atomic formulas is a derivation in* **S**.

3.2.4.3. *Subformula principle for* **M** *and* **I**: Each formula in a fully normal derivation in **M** or **I** of A from Γ is a subformula of A or of some formula of Γ.

3.2.4.4. *Subformula principle for* **C**: Each formula in a normal derivation in **C** of A from Γ either is a subformula of A or of some formula of Γ or is an assumption $\sim B$ closed by Λ_C (in which case B is a subformula of the kind mentioned or is an occurrence of Λ immediately below such an assumption).

3.2.4.5. The subformula principle also holds when the logical systems are extended by the addition of the rules of a system **S** for atomic formulas but then with an exception for the atomic formulas. The principle may furthermore be strengthened by saying somewhat more about how the different formula occurrences are related to A or Γ (see Prawitz (1965) 43–44 and 53–54).

3.3. *Reductions, immediate simplifications and expansions*

3.3.1. We shall now show how to remove maximal formulas from a derivation. This means simply that we shall make the inversion principle explicit for the different cases that can arise. The derivation obtained from a derivation Π by removing (in the way already exemplified in sec. 2.2) a maximum formula whose outermost symbol is α we shall call an *α-reduction* of Π. Below we state the five reductions corresponding to the five possible forms of a maximum formula. These five reductions are said to be *proper reductions*. To be able to remove also maximal segments, we add two *permutative reductions*, viz. $\vee$E- and $\exists$E-*reductions*, which decrease the length of maximal segments.

To state the reductions in a convenient way, I shall use Π (with or without indices) to range over derivations and Σ to range over finite sequences of derivations. Furthermore, I shall use a *concatenation operation* $(\Sigma/\Gamma/\Pi)$ where Γ is a set of top formulas in Π to denote the result obtained from Π by writing the deductions in Σ above each of top formulas in Γ (i.e., in such a way that the top formulas in question come immediately below the end formulas of the derivations in Σ in the given order). When Γ is a set of top formulas of the form A, I often denote the set by $[A]$ and write $(\Sigma/[A]/\Pi)$ in the more graphic notation

$$\begin{array}{c} \Sigma \\ \hline [A] \\ \Pi \end{array}$$

When this notation is used, it is always tacitly assumed that the proper parameters in Π do not occur in the formulas in Σ.

In the cases below, the right derivation is a reduction (of the kind in question) of the derivation to the left.

3.3.1.1. *&-reductions*

$$\dfrac{\dfrac{\dfrac{\Sigma_1}{A_1}\quad\dfrac{\Sigma_2}{A_2}}{A_1 \,\&\, A_2}}{A_i} \qquad\qquad \dfrac{\Sigma_i}{A_i} \qquad\qquad i = 1 \text{ or } 2\,.$$

3.3.1.2. v-*reductions*

$$\dfrac{\dfrac{\dfrac{\Sigma}{A_i}}{A_1 \vee A_2}\quad \begin{matrix}[A_1]\\ \dfrac{\Sigma_1}{B}\end{matrix}\quad \begin{matrix}[A_2]\\ \dfrac{\Sigma_2}{B}\end{matrix}}{B} \qquad\qquad \begin{matrix}\dfrac{\Sigma}{[A_i]}\\ \dfrac{\Sigma_i}{B}\end{matrix}$$

Here $[A_i]$ denotes the set of assumptions in Σ_i that are closed by the $\vee$E in question (i=1 or 2).

3.3.1.3. $\supset$-*reductions*

$$\dfrac{\dfrac{\Sigma_1}{A_1}\quad \dfrac{\begin{matrix}[A_1]\\ \dfrac{\Sigma_2}{A_2}\end{matrix}}{A_1 \supset A_2}}{A_2} \qquad\qquad \begin{matrix}\dfrac{\Sigma_1}{[A_1]}\\ \dfrac{\Sigma_2}{A_2}\end{matrix}$$

Here $[A]$ denotes the set of assumptions in Σ_2 that are closed by the $\supset$I.

3.3.1.4. $\forall$-*reductions*

$$\dfrac{\dfrac{\dfrac{\Sigma(a)}{A(a)}}{\forall x A(x)}}{A(t)} \qquad\qquad \dfrac{\Sigma(t)}{A(t)}$$

We write $\Sigma(a)$ to indicate that the formulas in the part of the derivation above $A(a)$ may contain the parameter a. The deduction $\Sigma(t)$ is to be obtained from $\Sigma(a)$ by replacing every occurrence of a by the term t. Note that the restriction on $\forall$I and the tacit assumption about the proper parameters (sec. 1.2.4) together guarantee that the right derivation is correct.

3.3.1.5. $\exists$-*reductions*

$$\dfrac{\dfrac{\Sigma_1}{A(t)}}{\exists x A(x)} \quad \dfrac{\dfrac{[A(a)]}{\Sigma_2(a)}}{B} \atop \overline{\qquad\quad B \qquad\quad} \qquad\qquad \dfrac{\dfrac{\Sigma_1}{[A(t)]}}{\dfrac{\Sigma_2(t)}{B}}$$

The remark made above in connection with the $\forall$-reduction applies also here mutatis mutandis. $[A(a)]$ denotes the set of assumptions in $\Sigma_2(a)$ which are closed by the $\exists$E.

3.3.1.6. $\vee$E-*reductions*

$$\dfrac{\dfrac{\dfrac{\Sigma_1}{A \vee B} \quad \dfrac{\Sigma_2}{C} \quad \dfrac{\Sigma_3}{C}}{C} \quad \Sigma_4}{D} \qquad\qquad \dfrac{\dfrac{\Sigma_1}{A \vee B} \quad \dfrac{\dfrac{\Sigma_2}{C} \quad \Sigma_4}{D} \quad \dfrac{\dfrac{\Sigma_3}{C} \quad \Sigma_4}{D}}{D}$$

The lowest occurrence of C in the left deduction is to be the last occurrence in a maximum segment. Σ_4 is thus a sequence of deductions of the minor premisses of the (elimination) inference of which the C mentioned is major premiss; hence, Σ_4 may be empty (viz., in the case of &E and $\forall$E) and (to be quite correct) may have to be written to the right of C (viz., in the case of $\supset$E).

3.3.1.7. $\exists$E-*reductions*

$$\dfrac{\dfrac{\dfrac{\Sigma_1}{\exists x A x} \quad \dfrac{\Sigma_2}{C}}{C} \quad \Sigma_4}{D} \qquad\qquad \dfrac{\dfrac{\Sigma_1}{\exists x A x} \quad \dfrac{\dfrac{\Sigma_2}{C} \quad \Sigma_4}{D}}{D}$$

The remark above in connection with $\exists$E-reductions applies also here.

3.3.1.8. *Remark*: It has been remarked by Martin-Löf that it is only necessary to require in the $\vee$E- and $\exists$E-reductions that the lowest occurrence of C is the major premiss of an elimination. A reduction of this kind can then always be carried out and we can sharpen the requirements as to the normal form accordingly.

3.3.2. *Immediate simplifications.* To bring a derivation to full normal form, we also have to get rid of redundant applications of $\vee$E, $\exists$E, and Λ_C. I shall say that Π_2 is an *immediate simplification* of Π_1 if Π_2 can be obtained from Π_1 by replacing a subtree Π in one of the following three ways:

3.3.2.1. Π has the form shown to the left below where no assumption in Σ_i ($i = 1$ or 2) is closed by the $\vee$E in question and Π is replaced by the derivation shown to the left below

$$\frac{\dfrac{\Sigma}{A_1 \vee A_2} \quad \dfrac{\Sigma_1}{B} \quad \dfrac{\Sigma_2}{B}}{B} \qquad\qquad \frac{\Sigma_i}{B}$$

3.3.2.2. Π has the form shown to left below where no assumption in Σ' is closed by the $\exists$E in question and Π is replaced by the derivation shown to the right below

$$\frac{\dfrac{\Sigma}{\exists x A(x)} \quad \dfrac{\Sigma'}{B}}{B} \qquad\qquad \frac{\Sigma'}{B}$$

3.3.2.3. Π has the form shown to the left below where no assumption in Σ is closed in Σ' and Π is replaced by the derivation shown to the right below.

$$\cfrac{\cfrac{\cfrac{\cfrac{\dfrac{\Sigma}{A} \quad \sim A}{[\Lambda]}}{\Sigma'}}{\Lambda}}{A} \qquad\qquad \frac{\Sigma}{A}$$

3.3.3. *Immediate expansions.* There is an obvious way of bringing a derivation in full normal form to expanded normal form (i.e. so that all the minima formulas are atomic, see 3.2.2). I shall say that Π_2 is an *immediate* expansion of Π_1 if Π_2 is obtained from Π_1 by applying one of these obvious operations. For instance, if Π_1 contains a minimum formula A & B and the derivation to the left below is the subtree of Π_1 that ends with this formula, we may replace this subtree by the derivation shown to the right below

$$\frac{\Sigma}{A \,\&\, B} \qquad\qquad \frac{\dfrac{\dfrac{\Sigma}{A \,\&\, B}}{A} \quad \dfrac{\dfrac{\Sigma}{A \,\&\, B}}{B}}{A \,\&\, B}$$

3.4. *Reducibility, equivalence, and convertibility*

3.4.1. *Immediate reducibility.* A derivation Π is said to *reduce immediately* to Π' if Π' is obtained from Π by replacing a subtree of Π by a reduction of it.

3.4.2. *Reducibility.* A derivation Π is said to *reduce* to Π' if there is a sequence $\Pi_1, \Pi_2, ..., \Pi_n$, $n \geqslant 1$, where $\Pi_1 = \Pi$, Π_i reduces immediately to Π_{i+1}, for each $i < n$, and $\Pi_n = \Pi'$.

A sequence $\Pi_1, \Pi_2, ...$ where for each i, Π_i reduces immediately to Π_{i+1} and the last derivation of the sequence if any is normal is said to be a *reduction sequence starting from* Π_1.

An *irreducible* derivation is one to which there is an immediate reduction, in other words, a derivation that reduces only to itself. A derivation is clearly irreducible if and only if it is in normal form.

3.4.3. *Equivalence.* Equivalence between derivations is defined in the same way as reducibility but allowing also Π_{i+1} to reduce to Π_i. This relation is clearly reflexive, transitive, and symmetric.

3.4.4. *Convertibility.* A derivation Π_1 is said to *convert* to Π_2 if there is a derivation Π' such that Π_1 reduces to Π' and Π_2 is obtained from Π' by successive immediate simplifications. When we want to consider derivations in expanded normal form, we redefine this notion by requiring instead that Π_2 is obtained from Π' by successive immediate simplifications or expansions.

3.5. *Results and conjecture*

We shall distinguish two kinds of theorems, normal form theorems and normalization theorems[1], of which the last kind is a strengthening of the first kind.

3.5.1. **Normal form theorem.** *If A is derivable in* **M(S)**, **I(S)**, *or* **C(S)** *from* Γ, *then there is a (fully, expanded) normal derivation in the system in question of A from* Γ.

[1] This terminology was suggested by Kreisel (where I used the less felicitous terminology weak and strong normal form theorem).

Thus, the normal form theorem asserts that to every derivation there exists one in normal form (of the same formula from the same assumptions) but does not say anything about how the normal derivation is related to the given one. There are also proofs of the normal form theorem, (viz. well-known semantical completeness proofs) which does not give any information of this kind either. Such information is however supplied by the following theorem.

3.5.2. **Normalization theorem.** *Every derivation in* **M(S)**, **I(S)**, *or* **C(S)** *reduces (converts) to (full, expanded) normal form.*

The theorem can be proved by the following observation: When Π is immediately reduced to Π' by a proper reduction, there is at least one maximum formula that disappears although other ones may arise, and when one of the two permutative reductions are used, the length of at least one maximum segment is decreased. Although it is true that new maximal formulas and maximal segments may arise by a reduction, it can be seen that the reductions always can be choosen so that the formulas occurring as new maximal formulas or occurring in new maximal segments are of less degree than the removed one. Thus, we can use an induction over these measures.

The normalization theorem may be strengthened further as follows.

3.5.3. **Strong normalization theorem.** *Every derivation* Π *in* **M(S)**, **I(S)**, *or* **C(S)** *reduces to a unique normal derivation* Π' *and every reduction sequence starting from* Π *terminates (in* Π'*).*

3.5.4. **Corollary about $\vee$ and $\exists$.** *$A \vee B$ is derivable in* **M(S)** *or* **I(S)** *only if either A or B is derivable in* **M(S)** *or* **I(S)**, *respectively.* *$\exists x A(x)$ is derivable in* **M(S)** *or* **I(S)** *only if for some term t, $A(t)$ is derivable in* **M(S)** *or* **I(S)**, *respectively.*

The corollary follows immediately from the normal form theorem together with corollary 3.2.4.1.

3.5.5. **Corollary about atomic formulas.** *If an atomic formula A is derivable in* **M(S)**, **I(S)** *or* **C(S)** *from a set of atomic formulas* Γ, *then A is derivable from* Γ *already in* **S**.

The corollary is an immediate consequence of the normal form theorem and corollary 3.2.4.2.

3.5.6. *Identity between proofs*. Each derivation clearly represents a proof (we have also argued for the converse of this, see sections 2.1 and 4.1.1). But when do two derivations represent the same proof? A first possible answer to this question is the following

Conjecture. *Two derivations represent the same proof if and only if they are equivalent.*

That two equivalent derivations represent the same proof seems to be a reasonable thesis. It seems evident from our discussion above of the inversion principle that a proper reduction does not effect the identity of the proof represented. There may be some doubts concerning the permutative ∨E- and ∃E-reductions in this connection, and there may be reasons to consider a more direct way of removing maximal segments than the one choosen above. With this reservation, one half of the conjecture seems unproblematic.

It is more difficult to find facts that would support the other half of the conjecture; the possibility of finding adequacy conditions for an identity criteria such as the one above is recently discussed by Kreisel (1970) in this volume.

It should be noted that the strong normalization theorem gives a certain coherence to the conjecture. It implies that two derivations are equivalent only if the normal derivations to which they reduce are identical, and hence, that two different normal derivations are never equivalent. As an example of two derivations in **M** of the same formula that are not equivalent, we may thus consider the following two derivations:

$$\cfrac{\cfrac{\cfrac{A}{A \supset A}}{B \supset (A \supset A)}}{(A \supset A) \supset (B \supset (A \supset A))} \qquad \cfrac{\cfrac{A \supset A}{B \supset (A \supset A)}}{(A \supset A) \supset (B \supset (A \supset A))}$$

The proofs that they represent are clearly based on different ideas and are hence different, which is thus in agreement with the conjecture.

Nevertheless, the conjecture as stated above is clearly in need of certain refinements. Firstly, derivations that only differ with respect to proper parameters should obviously be counted as equivalent. Secondly, one may ask whether not also the expansion operations preserve the identity of the proofs represented. It seems unlikely that any interesting property of proofs is sensitive to differences created by an expansion.

II.4. A summary of the analysis

To connect the analysis presented above with the introductory remarks about general and reductive proof theory, we may try to summarize what seem to be the most significant aspects of the analysis.

4.1. The examples of topics in general proof theory given in sec. I.2.1.-I.2.4 are all exemplified here.

4.1.1. We have argued at length (sec. 2.1) for the claim that Gentzen's systems of natural deduction constitutes a *characterization* (cf. I.2.1) of (different kinds of) first order proofs. We may summarize this claim in the thesis:

Every first order positive, intuitionistic, or classical proof can be represented in **M**, **I**, *or* **C**, *respectively.*

4.1.2. The second step in Gentzen's analysis (presented in sec. 3) clearly constitutes a further significant analysis of the *structure* (cf. I.2.2) of first order proofs inasmuch as the normal form (sec. 3.1) has quite distinctive features (sec. 3.2). With Gentzen, we may say that the proof represented by a normal derivation makes no detours ("*es macht keine Umwege*"); or, having formulated the normal form for natural deductions, we may say somewhat more pregnantly: the proof is *direct* in the sense that it proceeds from the assumptions to the conclusions by first only using the meaning of the assumptions by breaking them down in their components (the analytical part), and then only veryfying the meaning of the conclusions by building them up from there components (the synthetical part)[1].

4.1.2.1. If we accept the thesis in 4.1.1 and also accept as a thesis what was called the unproblematic part of the conjecture about the identity of proofs, viz., that two equivalent derivations represent the same proof, we may give the normalization theorem the following formulation:

Every first order proof can be written in normal form, i.e. can be represented by a normal derivation.

4.1.2.3. If we accept Gentzen's ideas about a reinterpretation of the logical constants in which the elimination rules are justified by the meaning given to

[1] As "conclusions", we must here count also the hypothesis of a premiss, and as "premisses" also the hypothesis of a conclusion; a more precise statement can be obtained by the use of the notion of positive and negative subformula (cf. Prawitz (1965) 43).

the logical constants by the introduction rules (sec. 2.2.2), we may note another aspect of the normalization (or normal form) theorem; lacking a more precise formulation of Gentzen's ideas, we have to remain somewhat vague (but cf. appendix A):

A derivation proceeding via introductions followed by eliminations (i.e., with a structure opposite that of a normal deduction) may leave some doubts about whether the conclusion obtained is really valid (in the sense here discussed), i.e., one may question whether the condition for introducing the conclusion by introduction is satisfied. The inversion principle states that each *particular* elimination following an introduction is justified since by a reduction, the conclusion can also be obtained directly without *this* detour; but of course, new maximum formulas may arise by this reduction. The normalization theorem strengthens this by showing that *all* maximum formulas can be removed from a derivation and thus *justifies the logical system as a whole.*

4.1.2.3.1. The corollary 3.5.4 about disjunction and existential quantification may be seen as just an expression of this, showing that a derivation of $A \vee B$ is justified since it can be transformed either to a derivation of A or to a derivation of B as required by the meaning of $\vee$ and that a derivation of $\exists x A(x)$ is justified since it can be transformed to a derivation of $A(t)$ for some term t as required by the meaning of $\exists$.

4.1.2.3.2. Similarly, the corollary 3.5.5 can also be seen in terms of this justification, showing as it does that the logical rules are not creative: they do not allow derivations of atomic formulas that cannot also be obtained directly without the use of the logical rules.

4.1.3. The equivalence relation obtained in the analysis (sec. 3.4) seems to allow also an approach to the problem of when two derivations represents the same proof (cf. I.2.3). As we saw (sec. 3.5.6), certain considerations strengthened by the strong normalization theorem makes it reasonable to propose a *conjecture stating an identity criterion for proofs.*

4.1.4. Not surprisingly, the analysis allows also *applications* to problems in other areas, i.e., problems not formulated in terms of proofs (cf. I.2.4). As an illustration, we have quoted the corollary about disjunction and existential quantification (cf. also 4.1.2.3.1). Many other applications, e.g. the interpolation theorem to mention one, could have been quoted.

4.2. For reductive proof theory, the analysis allows a certain reduction of the logical part of first order systems **M(S)**, **I(S)**, and **C(S)**. Suppose that we accept the atomic sentences in these systems as real sentences (cf. sec. I2.1) with the meaning given to them by **S** and thus consider an atomic sentence as true if and only if it is derivable in **S**. Corollary 3.5.5 (obtained just in the way suggested in sec. I.2.1 by transforming derivations that proceed over ideal sentences to derivations that use only real sentences) then asserts:

every derivable real sentence is true;

in other words, the systems in question are conservative extensions of the system **S** (cf. also 4.1.2.3.2). To be a complete reduction, one also has to show that the proof of this corollary uses something essentially less than the whole system of first order logic.

II.5. Historical remarks

5.1. Gentzen presented his analysis of first order proofs in 1935. The systems of natural deduction were stated at this time, but as already remarked above, the analysis of the structure of proofs (here presented in sec. 3) was carried out for his calculi of sequents. It is clear from his writings, however, that he discovered the analysis by reflecting upon his systems of natural deduction. Although the inversion principle was not stated explicitly by Gentzen, its idea was clearly recognized as the basic idea underlying his results for the calculus of sequents (see especially Gentzen (1938) 26–27). Here we have used this idea to obtain the results directly in a very natural way. According to what he says, Gentzen preferred to formulate and prove his results for the calculus of sequents because the axiom of the excluded middle (which he used instead of Λ_C) presented special problems (Gentzen (1935) 177). As we have presented it here, the classical case presents no special difficulties but comes out rather as a special case of minimal logic; however, this is at the cost of taking $\vee$ and $\exists$ in classical logic as defined symbols, and it is true that some complications in the proof arise when one wants to have them as primitives.

In the way Gentzen presented his analysis, it was technically summed up in his so-called Hauptsatz, which states that all applications of the so-called cut rule can be eliminated from the proofs. This Hauptsatz and our normal form theorem are equivalent in the sense that one can be obtained from the other by a suitable translation between the two kinds of systems (as described in some detail in Prawitz (1965) 90–93, where also some other comments about the relations between the systems are made).

There are certain advantages in carrying out the analysis for natural deduction besides the fact that the development (including the proofs of the theorems) flows very natural from the underlying idea. The main advantages is that the significance of the analysis, as I have tried to describe it above, seems to become more visible[1]. Furthermore it has recently been possible to extend the analysis of first order proofs to the proofs of more comprehensive systems when they are formulated as systems of natural deduction (as will be described in III), while an analogous analysis with a calculus of sequents formulation does not suggest itself as easily. Finally, the connection between this Gentzen-type analysis and functional interpretations such as Gödel's Dialectica interpretation becomes very obvious when the former is formulated for natural deduction (as will be seen in IV).

5.2. The approach used above was introduced in Prawitz (1965) to which I may refer for further details. A somewhat weaker version of the normal form theorem was independently obtained by Raggio (1965).

The extension of the logical systems with the rules of an atomic system **S** is a minor present addition, which gives a convenient way to state certain points systematically; it covers also such an extension of pure predicate logic as is obtained by the addition of rules for identity.

The theorems in Prawitz (1965) are stated as normal form theorems but their proofs give also the corresponding normalization theorems; however, this viewpoint is not sufficiently stressed. The strong normalization theorems are a later addition. Theorems of a similar kind were originally obtained for terms such as those used in connection with the functional interpretation of intuitionistic logic. The uniqueness property corresponds to the so-called Church-Rosser property for reductions in the λ-calculus. For results for terms to the effect that all reduction sequences terminates, see e.g. Sanchis (1967) and Howard (1970).

The conjecture about identity between proofs (3.5.6) is due to Martin-Löf and is also influenced by similar ideas in connection with terms (see Tait (1967)).

[1] It is an historical fact that Gentzen's result when technically summed up in the Hauptsatz has given rise to many misunderstandings. Since the cut rule may be looked upon as a generalization of modus ponens or as stating a transitive law of implication, there has been a not to infrequent misconception that the significance of the result was that modus ponens or the transitive law could be eliminated from proofs. As Kreisel remarked in his lecture at this symposium, one has then to explain what is dubious about these old, respectable principles. Clearly, this belief is not only superficial but quite mistaken: modus ponens is present also in normal derivations in the systems of natural deduction (and it is a triviality that the principle of transitivity holds in these systems although it does not occur as a primitive rule); it is really also present in cut-free derivations in the calculus of sequents in the form of introduction of $\supset$ in the antecedent.

III. Extensions to other systems

III.1. First order Peano arithmetic

1.1. Definition of the system **P**

Let $\mathbf{S^A}$ be the Post system (sec. I.1.5) determined by

(i) the descriptive constants 0, $'$, =, Q, and R where 0 is an individual constant (denoting zero), $'$ is an 1-place operational constant (denoting the successor function), = is a 2-place predicate constant (denoting identity), and Q and R are 3-place predicate constants to express addition and multiplication, and

(ii) the usual rules for identity, rules corresponding to Peano's third and fourth axioms, rules corresponding to the recursive definitions of addition and multiplication, and rules expressing the functional property of Q and R; e.g., the rule corresponding to Peano's fourth axiom is

$$\frac{a' = b'}{a = b}\;;$$

note that all the axioms of first order arithmetic except the induction axioms can be written as rules of a Post system.

Let $\mathbf{P_M}$, $\mathbf{P_I}$, and $\mathbf{P_C}$ be the system obtained from $\mathbf{M(S^A)}$, $\mathbf{I(S^A)}$, and $\mathbf{C(S^A)}$, respectively, by the addition of the *rule of induction*:

$$\frac{A(0) \quad \begin{matrix}[A(a)] \\ A(a')\end{matrix}}{A(t)}$$

whose applications are with the restriction that a, the proper parameter of the application, is not to occur in other assumptions that $A(a')$ depends on except those of the form $A(a)$, which are closed by the application of the rule.

I shall use **P** ambiguously for $\mathbf{P_M}$, $\mathbf{P_I}$, or $\mathbf{P_C}$. Note that if Λ is defined as $0 = 1$, then $\mathbf{P_M}$ and $\mathbf{P_I}$ have the same set of theorems and the theorems of $\mathbf{P_C}$ coincide with the theorems of $\mathbf{P_M}$ that do not contain $\vee$ or $\exists$.

The system **P** thus constitutes a formalization of first order Peano arithmetic. I shall not discuss to what extent proofs of first order arithmetic can be represented in such a system.

In the next sections, I shall instead describe how some of the results about the structure of first order proofs can be extended to **P**.

1.2. *The status of the rule of induction*

If one is to extend the analysis of the structure of first order proofs to the proofs represented by derivations in the system **P**, one has to account for how the rule of induction fits into the pattern of introductions and eliminations. Since the validity of the rule depends on the understanding that the individual domain consists of the natural numbers, it would be natural to reformulate the rule as an introduction rule for $\forall$:

$$\frac{A(0) \quad \begin{array}{c}[A(a)]\\ A(a')\end{array}}{\forall x A(x)}$$

with the same restrictions as before. (The introduction rule for $\forall$ in first order logic can be derived from this rule.)

One may then hope to obtain a normal form theorem where normal derivations are defined literally as in II.3.1. Indeed, when an application of the introduction rule for $\forall$ formulated above is followed by an $\forall$E where the term t is a numeral representing the number n, i.e. in the situation shown to the left below, we have a reduction as shown to the right below

$$\frac{\dfrac{\dfrac{\Sigma_1}{A(0)} \quad \begin{array}{c}[A(a)]\\ \dfrac{\Sigma_2(a)}{A(a')}\end{array}}{\forall x A(x)}}{A(t)} \qquad \qquad \begin{array}{c} \dfrac{\Sigma_1}{[A(0)]} \\ \dfrac{\Sigma_2(0)}{[A(0')]} \\ \dfrac{\Sigma_2(0')}{[A(0'')]} \\ \vdots \\ A(t) \end{array} \qquad n \text{ times}$$

If we call this an $\forall$-reduction, we may define the notion of reducibility exactly as in II.3.4. However, when t contains a parameter, a maximum formula of the kind shown to the left above cannot be eliminated in this way, and in fact, it can be shown that there is no normal form theorem of this kind for **P**; see Kreisel (1965) 163. (This confirms the well-known situation where one feels that one has to prove a stronger theorem than the theorem one is interested in to be able to carry out the induction step.)

1.3. *Gentzen's result*

Gentzen's (1938) result for first order Peano arithmetic can be described in our present terminology as a proof by finitistic means enlarged by induction up to ϵ_0 of the theorem: Every proof of an atomic sentence in **P** reduces to normal form. Since a normal derivation of an atomic sentence in **P** is a derivation already in $\mathbf{S}^{\mathbf{A}}$, this entails a result of the kind sought for in reductive proof theory (provided the induction up to ϵ_0 is accepted as sufficiently elementary), and as we see, it is a direct extension of the result described in II.3.5.5 or II.4.2.

Gentzen's own formulation is in terms of a calculus of sequents.

An earlier result by Gentzen (1936) is in terms of a natural deduction system for arithmetic but is of a different character.

Formally, the present formulation for natural deduction is a considerable simplification. It also suggests a stronger result that will be described below.

1.4. *A weak normal form for derivations in* **P**

Although all maximal formulas that arise by applications of the induction rule cannot be eliminated, one may consider the derivations obtained by carrying out the reduction described above in 1.2 and the reductions defined in II.3.3 as far as possible. These derivations will be said to be in normal form but it should be remembered that this normal form is considerably weaker than the one suggested above in section 1.2.

Since all maximal formulas cannot be avoided, it is better to keep the rule of induction in its first formulation in section 1.1. We shall thus regard this rule as falling outside the pattern of introductions and eliminations, and in a normal derivation, applications of this rule is restricted to diminish its disturbance of this pattern.

More precisely, we define a derivation in **P** as *normal* when it contains no maximal formulas and no maximal segments and the term t in an application of the induction rule is not 0 or of the form u' (it follows that t is then a parameter). A derivation in **P** is said to be in *full normal form* when it (i) is in normal form, (ii) satisfies the conditions in section II.3.1.4, and (iii) contains only parameters that are either proper parameters or have occurrences in open assumptions or the end-formula.

To the reductions described in II.3.3, we now add induction-reductions, which are of two forms. In both cases, the right derivation below is said to be an (*induction-*) *reduction* of the derivation to the left.

$$\dfrac{\dfrac{\Sigma_1}{A(0)} \quad \dfrac{\Sigma_2}{A(a')}}{A(0)} \qquad\qquad \dfrac{\Sigma_1}{A(0)}$$

$$\dfrac{\dfrac{\Sigma_1}{A(0)} \quad \dfrac{\begin{array}{c}[A(a)]\\ \Sigma_2(a)\end{array}}{A(a')}}{A(t')} \qquad\qquad \dfrac{\begin{array}{c}\dfrac{\Sigma_1}{A(0)} \quad \dfrac{\begin{array}{c}[A(a)]\\ \Sigma_2(a)\end{array}}{A(a')}\end{array}}{\begin{array}{c}[A(t)]\\ \dfrac{\Sigma_2(t)}{A(t')}\end{array}}$$

$[A(a)]$ is the set of assumptions closed by the application of the induction rule. Repeated applications of these induction-reductions cover the reduction described above in section 1.2.

Reducibility and *equivalence* between derivations in **P** is now defined as in II.3.4 (including the induction-reductions among the reductions). Note that a derivation is in normal form if and only if it is irreducible. A derivation Π_1 in **P** is said to *convert* to Π_2 if there is a derivation Π' obtained from Π_1 by substituting 0 for all parameters in Π_1 that do not satisfy condition (iii) in the definition of full normal form above and Π' converts to Π_2 in the sense defined in section II.3.4.4.

Many of the results in section II.3 can now be extended to **P**. In particular, we can prove the (strong) normalization theorem as formulated in II.3.5.3.

In a normal derivation, we have thus normalized all applications of the logical rules and reduced the applications of the induction rule as far as possible. In this respect, we have strengthened Gentzen's result as formulated in 1.3, which was concerned only with derivations of atomic sentences.

It should be noted that the theorem about the form of normal derivations in section II.3.2 does not extend to **P** (unless the paths are defined so that they end at premisses of applications of the induction rule and may start at the conclusions of such applications). Also the subformula principle fails in **P**. However, the corollary II.3.2.4.1 about the last inference in a derivation in full normal form extends to **P** (replacing **S** by $\mathbf{S}^{\mathbf{A}}$). Hence, besides the corollary II.3.5.5 about atomic formulas (which is the result by Gentzen already noted above), the corollary II.3.5.4 about disjunction and existential quantification extends to **P**. In addition, we note the following *corollary*: If $\forall x \exists y A(x,y)$ is derivable in $\mathbf{P_I}$, then there is a mechanically computable function f such that for each natural number n, $A(n^*,f(n)^*)$ is derivable in $\mathbf{P_I}$ where n^* and $f(n)^*$ are the numerals that denote n and $f(n)$. The proof is immediate from the results above: We can first mechanically convert the derivation of

$\forall x \exists y A(x,y)$ to full normal form. By a corollary referred to above, its last inference is then $\forall$I whose premiss is of the form $\exists y \forall(a,y)$. Omitting the last formula of the derivation and substituting n^* for a, we have a derivation of $\exists y A(n^*,y)$, which again may be converted to full normal form. By the same corollary, the last inference of the derivation is now $\exists$I, whose premiss is of the form $A(n^*,m^*)$. The value m of f for the argument n can thus be read off from this derivation.

The normalization theorem for **P**, which was suggested by Per Martin-Löf, is proved by Jervell (1970) in this volume; or rather, a certain weaker variant of the normalization theorem as formulated above is proved by Jervell. The notion of validity described in appendix A seems to provide the most convenient way of proving the (strong) normalization theorem. For applications in reductive proof theory, one would have to use more elementary methods, however.

The corollaries mentioned above have also been obtained by Scarpellini (1969) using somewhat similar methods. They were obtained earlier by other methods, e.g. by Kleene's notion of realizability.

III.2. First order arithmetic with infinite induction

Schütte (1951b) seems to have been the first one to realize that by replacing the rule of induction by an infinite rule for the introduction of $\forall$, so-called infinite induction, one obtains a formulation of arithmetic that allows a normal form for derivations without the weakness noted above. I shall briefly describe how this may be done in the present framework. To keep the description short, I consider only languages without $\exists$.

$\mathbf{P}^{\mathrm{i}}$ is to be a system that is like **P** except for the following two differences:
(i) the individual parameters are left out, and hence, the numerals are the only individual terms;
(ii) the $\forall$I-rule of II.1.2 and the rule of induction are replaced by an infinite $\forall$I-rule (or a rule of infinite induction) indicated by the following schema

$$\frac{A(0) \quad A(1) \quad A(2) \quad \ldots}{\forall x A(x)}$$

If $\Pi_1, \Pi_2, \Pi_3, \ldots$ or more compactly, $\{\Pi_i\}_i$ is an infinite sequence of derivations such that for each n, Π_n is a derivation of $A(n^*)$, then the new $\forall$I-rule allows us to combine these trees by writing $\forall x A(x)$ immediately below all the end-formulas of the derivations in $\Pi_1, \Pi_2, \Pi_3, \ldots$ to get the derivation

$$\frac{\Pi_1 \quad \Pi_2 \quad \Pi_3 \quad \dots}{\forall x A(x)} \qquad \text{or} \qquad \frac{\{\Pi_i\}_i}{\forall x A(x)}$$

where $\forall x A(x)$ depends on the union of all the sets Γ_i of open assumptions in Π_i. The derivation in $\mathbf{P}^{\mathbf{i}}$ may thus consist of infinite trees but in minimal and intuitionistic $\mathbf{P}^{\mathbf{i}}$ and for applications in constructive proof theory, the new $\forall$I-rule is restricted by the requirement that the sequence $\{\Pi_i\}_i$ is to be constructively given.

It is easily verified that all the sentences derivable in **P** are also derivable in $\mathbf{P}^{\mathbf{i}}$.

Note that the inversion principle holds also for the new pair of $\forall$I and $\forall$E. We may thus formulate a new *$\forall$-reduction* for $\mathbf{P}^{\mathbf{i}}$: The derivation to the right below is said to be an ($\forall$-) reduction of the derivation to the left

$$\cfrac{\cfrac{\cfrac{\Sigma_1}{A(0)} \quad \cfrac{\Sigma_2}{A(1)} \quad \cfrac{\Sigma_3}{A(2)} \quad \dots}{\forall x A(x)}}{A(n)} \qquad\qquad \cfrac{\Sigma_n}{A(n)}$$

If $\exists$ is also present, we only have to replace the finite $\exists$E by an obvious infinite one and modify the $\exists$-reductions accordingly.

With this change in the definition of $\forall$- (and $\exists$-) reductions, the whole of section II.3 can be carried over to $\mathbf{P}^{\mathbf{i}}$. In particular, we note that in contrast to **P**, the normal derivations have the form described in II.3.2 and satisfy the subformula principle.

Infinite systems of this kind are clearly of significance for reductive proof theory. In contrast to the case of **P**, there is no effort in verifying that the proof of the normalization theorem requires only induction up to ϵ_0 in addition to ordinary finitary arguments.

It is less clear whether these systems are of any great interest for general proof theory. Kreisel (1967) and (1968) argues that infinite thoughts seem to be much better represented by infinite objects than by the words we use to communicate them. On the other hand, it must be remembered that an infinite derivation of the kind described above is in certain respects an incomplete representation of an argument. In order to be conclusive, each application of $\forall$I in such a derivation should be supplemented by an argument showing that for each n, Π_n is a derivation of $A(n^*)$. It is by leaving out this supplementary argument in the representations of the proofs that the derivations get such a simple structure.

III.3. Infinite sentential logic

Tait (1968) showed that Schütte's results for first order arithmetic with infinite induction as well as the results for some other systems could be considered as special cases of results for a system for infinite sentential logic. His results were formulated for a calculus of sequents (or rather for a variant of such a calculus where Gentzen's sequents are simplified so that they contain only a succedent). A natural deduction formulation was given by Martin-Löf (1969), who also (in contrast to Schütte and Tait) considers intuitionistic (or minimal) infinite sentential logic.

The formulas of Martin-Löf's infinite sentential logic are built up from sentential parameters with the help of implications and one operation that allows us to form infinite conjunctions. We may also add an operation allowing the formation of infinite disjunctions. The *formulas* can then be defined as follows:

(1) A sentential parameters is a formula.

(2) If A and B are formulas, then so is $(A \supset B)$.

(3) If $\{A_i\}_i$ is an infinite sequence of formulas, then $\underset{i}{\&} A_i$ and $\underset{i}{\vee} A_i$ are formulas.

In reductive proof theory and in intuitionistic or minimal infinite sentential logic, clause (3) is restricted by requiring that $\{A_i\}_i$ is a constructive sequence.

Inferences with these infinite formulas may, as in first order logic, be analysed in introductions and eliminations for the three operations that build up the formulas. As in first order arithmetic, the inferences may contain infinitely many premisses and the derivations may thus consist of infinite trees. The rules for $\supset$I and $\supset$E are exactly as in first order logic. The other four rules are indicated by the following figures:

$\underset{i}{\&}$I) $\dfrac{\{A_i\}_i}{\underset{i}{\&} A_i}$ $\qquad$ $\underset{i}{\&}$E) $\dfrac{\underset{i}{\&} A_i}{A_j}$ for any j

$\underset{i}{\vee}$I) $\dfrac{A_j}{\underset{i}{\vee} A_i}$ for any j $\qquad$ $\underset{i}{\vee}$E) $\dfrac{\underset{i}{\vee} A_i \quad \left\{\begin{matrix}[A_i] \\ B\end{matrix}\right\}_i}{B}$

The &I-rule allows the inference of the infinite conjunction $\underset{i}{\&} A_i$ given a sequence $\{\Pi_i\}_i$ of derivations such for each i, Π_i is a derivation of A_i (cf. infinite induction in section 2).

The $\underset{i}{\&}$E-rule allows the inference of any A_j from $\underset{i}{\&} A_i$.

The $\bigvee_i$I-rule allows the inference of $\bigvee_i A_i$ from any A_j.

The $\bigvee_i$E-rule allows assumptions to be closed as follows: given a derivation of $\bigvee_i A_i$ depending on (the formulas of) Γ and a sequence $\{\Pi_i\}_i$ of derivations such that for each i, Π_i is a derivation of B depending on Δ_i, we may infer B that now depends on the union of Γ and the union of all sets $\Delta_i - \{A_i\}$.

As before, we restrict in certain contexts the $\&_i$I- and $\bigvee_i$E-rules by requiring that the sequences of derivations in question are constructive.

The natural deduction system for *minimal infinite sentential logic,* $\mathbf{M^i}$, is determined by these six inference rules. To get corresponding intuitionistic and classical system, $\mathbf{I^i}$ and $\mathbf{C^i}$, we may add the $\Lambda_\mathbf{I}$- and $\Lambda_\mathbf{C}$-rules, respectively. However, if Λ is defined as the conjunction of all sentential parameters, the $\Lambda_\mathbf{I}$-rule holds as a derived rule in $\mathbf{M^i}$.

It is easily seen that also the new pairs of introductions and eliminations satisfy the inversion principle and accordingly we can define $\&_i$- and $\bigvee_i$-reductions as shown below where the right derivation is a reduction of the kind in question of the left derivation:

$\&_i$*-reductions*

$$\dfrac{\dfrac{\left\{\dfrac{\Sigma_i}{A_i}\right\}_i}{\underset{i}{\&} A_i}}{A_j} \qquad\qquad \dfrac{\Sigma_j}{A_j}$$

$\bigvee_i$*-reductions*

$$\dfrac{\dfrac{\dfrac{\Sigma}{A_j}}{\underset{i}{\bigvee} A_i} \quad \left\{\begin{matrix}[A_i]\\ \Sigma_i \\ \hline B\end{matrix}\right\}_i}{B} \qquad\qquad \dfrac{\begin{matrix}\dfrac{\Sigma}{[A_j]}\\ \Sigma_j\end{matrix}}{B}$$

The results of section II.3 can now be carried over to $\mathbf{M^i}$, $\mathbf{I^i}$, and $\mathbf{C^i}$.

III.4. Theories of (iterated) inductive definitions

If the rules of a Post system **S** are understood as completely determining the predicates (as the sets of terms for which the predicates can be proved to hold), i.e. if the rules of **S** are understood as inductive definitions of the predicates in question, then there are certain valid inferences that are not allowed in any of the systems **M(S)**, **I(S)**, and **C(S)**. Let us e.g. make the assumption that a certain formula $A(a)$ satisfies as a value of a one-place predicate constant P all the rules of **S** in which P occurs; i.e., we assume that if we replace every occurrence of the form Pt in these rules by $A(t)$ (possibly replacing also other predicate symbols by other formulas throughout), then for each rule, we can infer the formula obtained from the conclusion (by this replacement) from the formulas obtained from the premisses (by this replacement). For instance, if we have a rule

$$\frac{Pa}{Pa+2}$$

then the assumption is that we can infer $A(a+2)$ from $A(a)$. Under this assumption, we can infer $A(t)$ from Pt.

We can formulate an inference rule for inferences of this kind. There is one major premiss Pt and the minor premisses express the fact that $A(a)$ satisfies as a value of P all the rules in which P occurs (and if other predicate constants are replaced by formulas, the minor premisses are also to express the corresponding thing for these formulas and predicates). This rule may naturally be thought of as an *elimination rule for the predicate constant P* while the rules in **S** in which P occurs in the conclusion may be thought of as *introduction rules for P.*

It can be seen that the inversion principle holds for these new pairs of introductions and eliminations and we can thus add additional reductions accordingly.

The system that arise from **M(S)** by the additions of these elimination rules may be called the *first order minimal theory of inductive definitions based on* **S**, abbreviated $\mathbf{M(S)}^{\mathrm{ind}}$. As indicated above, Gentzen's analysis of first order proofs can be extended to these theories of inductive definitions. This extension is due to Martin-Löf (1970), who in particular proves the normalization theorem for the systems $\mathbf{M(S)}^{\mathrm{ind}}$. By starting from **I(S)** or **C(S)**, we obtain corresponding intuitionistic or classical system, respectively[1].

Let $\mathbf{S}^{\mathbf{N}}$ be the Post system that is like $\mathbf{S}^{\mathbf{A}}$ (sec. 1.1) except that it contains an additional one-place predicate constant **N** (for the property of being a natural number) and rules corresponding to Peano's first and second axioms, i.e. the rule consisting of just the conclusion $N0$ (or the axiom $N0$) and the rule

[1] It should be noted however that the subformula principle fails for all these systems.

$$\frac{Na}{Na'}$$

The elimination rule for N thus allows inferences of the form

$$\frac{Nt \quad A(0) \quad \begin{matrix}[A(a)] \\ A(a')\end{matrix}}{A(t)}$$

which is just another formulation of mathematical induction. (Note that induction inferences are here eliminations because they depend on the meaning of N, while in **P** they may be thought of as introductions depending on the intended restriction of the individual domain (cf. sec. 1.2).) $\mathbf{M}(\mathbf{S}^{\mathbf{P}})^{\mathrm{ind}}$ thus constitutes an alternative formulation of $\mathbf{P}_{\mathbf{M}}$ and similarly for the intuitionistic and classical variants.

Martin-Löf further shows how to extend this analysis to theories of iterated inductive definitions by starting from certain generalized Post systems. Rather strong mathematical theories, viz. subsystems of second order arithmetic, can be formulated in this way.

These results, which are presented in Martin-Löf's paper in this volume, are of considerable interest for general proof theory.

III.5. Second order logic

5.1. *The second order systems*

Introduction and elimination inferences can also be formulated for second order quantification as was done in Prawitz (1965). It is sometimes convenient to extend the formalism by the addition of a logical constant λ, used in second order (or abstraction) terms

$$\{\lambda x_1 x_2 ... x_n A(x_1, x_2, ..., x_n)\}$$

to denote the relation that holds between the $x_1, x_2, ..., x_n$ such that $A(x_1, x_2, ..., x_n)$. The following rules (where X is an n-ary predicate variable and P is an n-ary predicate parameter, $n = 0, 1, 2, ...$) can then be stated

$$\forall_2\text{I)} \quad \frac{A(P)}{\forall X A(X)} \qquad\qquad \forall_2\text{E)} \quad \frac{\forall X A(X)}{A(T)}$$

$\exists_2$I) $\dfrac{A(T)}{\exists XA(X)}$ $\exists_2$E) $\dfrac{\exists XA(X) \quad \begin{matrix}[A(P)] \\ B\end{matrix}}{B}$

λI) $$\frac{A(t_1,t_2,\dots,t_n)}{\{\lambda x_1x_2\dots x_nA(x_1,x_2,\dots,x_n)\}\,t_1t_2\dots t_n}$$

λE) $$\frac{\{\lambda x_1x_2\dots x_nA(x_1,x_2,\dots,x_n)\}\,t_1t_2\dots t_n}{A(t_1,t_2,\dots,t_n)}$$

In an inference by $\forall_2$I, P, the proper parameter of the inference, is not to occur in any assumption on which $A(P)$ depends. In an inference by $\forall_2$E or $\exists_2$I, T is an n-ary second order term, i.e. either an n-ary predicate parameter or constant or an abstraction $\{\lambda x_1x_2\dots x_nB(x_1,x_2,\dots,x_n)\}$ where $B(x_1,x_2,\dots,x_n)$ is a formula. In an inference by $\exists_2$E, P, which is called the proper parameter of the inference, is not to occur in B nor in any assumption that the premiss B depends on except those of the form $A(P)$ that are closed by the inference.

The system of natural deduction for second order minimal, intuitionistic, and classical logic, denoted $\mathbf{M}^2$, $\mathbf{I}^2$, and $\mathbf{C}^2$, arise from **M**, **I**, and **C**, respectively, by the addition of these rules. However, conjunction, disjunction and existential quantification turn out to be definable by implication and universal quantification also in minimal and intuitionistic second order logic why these constants and their rules can be left out. Absurdity is definable as $\forall XX$ (where X is a 0-place predicate variable), which makes the Λ_I-rule derivable in $\mathbf{M}^2$; hence there is no interesting distinction between intuitionistic and minimal second order logic. $\mathbf{M}^2$ is also referred to as (the natural deduction system for) the (intuitionistic) *theory of species.*

5.2. *Mathematical analysis*

Let $\mathbf{S^P}$ be the Post system determined by the descriptive constants 0, ′, and =, the ordinary rules for =, and the rules corresponding to Peano's third and fourth axioms, i.e. $\mathbf{S^P}$ is like $\mathbf{S^A}$ in section 1.1 except for not containing the addition and multiplication predicates and their rules. $\mathbf{C}^2(\mathbf{S^P})$, i.e. the system whose language is the second order language determined by the descriptive constants of $\mathbf{S^P}$ and whose rules are the rules of $\mathbf{S^P}$ and $\mathbf{C}^2$, is then a formulation of classical second order Peano arithmetic, also written $\mathbf{P}^2_\mathbf{C}$. The system $\mathbf{M}^2(\mathbf{S^P})$ or $\mathbf{P}^2_\mathbf{M}$ is the intuitionistic analogue to that theory, i.e. Peano arithmetic built upon the theory of species. As in the first order case, I shall use $\mathbf{P}^2$ ambiguously for $\mathbf{P}^2_\mathbf{M}$ and $\mathbf{P}^2_\mathbf{C}$. (Note however that $\mathbf{P}^2_\mathbf{C}$ is not a subtheory of $\mathbf{P}^2_\mathbf{M}$.)

In $\mathbf{P}^2$, we may in a well-known manner define the notion of natural number and then prove all the Peano axioms, including the induction axiom. Recursively definable functions such as additions and multiplication are explicitly definable in $\mathbf{P}^2$. Furthermore, real numbers are definable in $\mathbf{P}^2$ as certain sets (or relations) of natural numbers after which the existence of a least upper bound to each bounded set of real numbers is provable in $\mathbf{P}^2$ as a theorem schemata. In this way, theorems in traditional mathematical analysis can be proved in $\mathbf{P}^2$.

Thus, unlike the first order case where additional considerations are necessary to extend results for the logical systems to the arithmetical ones, proof theoretical results for second order logic (extended with atomic systems) carry over directly to second order arithmetic and mathematical analysis. This seems first to have been realized by Takeuti (1954) (although in a much more complicated way by what he calls a theory of restrictions). He formulated a calculus of sequents for second (and higher) order logic and conjectured that Gentzen's Hauptsatz could be extended to that system.

5.3. *Normal form and normalization in second order logic*

We note that also the new introductions and eliminations satisfy the inversion principle, which allows us to formulate the following additional reductions below, where the derivation to the right is a reduction of the kind in question of the derivation to the left:

$\forall_2$-reduction

$$\dfrac{\dfrac{\dfrac{\Sigma(P)}{A(P)}}{\forall X A(X)}}{A(T)} \qquad\qquad \dfrac{\Sigma(T)}{A(T)}$$

$\exists_2$-reduction

$$\dfrac{\dfrac{\dfrac{\Sigma_1}{A(T)}}{\exists X A(X)} \quad \dfrac{\begin{matrix}[A(P)]\\ \Sigma'(P)\end{matrix}}{B}}{B} \qquad\qquad \dfrac{\dfrac{\dfrac{\Sigma_1}{[A(T)]}}{\Sigma'(T)}}{B}$$

λ-reduction

$$\dfrac{\dfrac{\dfrac{\Sigma}{A(t_1,t_2,\ldots,t_n)}}{\{\lambda x_1 x_2 \ldots x_n A(x_1 x_2,\ldots,x_n)\}\; t_1 t_2 \ldots t_n}}{A(t_1,t_2,\ldots,t_n)} \qquad\qquad \dfrac{\Sigma}{A(t_1,t_2,\ldots,t_n)}$$

We may thus define the notions of normal derivations, reducibility etc. in the same way as in first order (section II.3). Note also that a second order normal derivation has the same form as a first order normal derivation, i.e. the theorem in II.3.2 holds also here, and the corollaries (in II.3.4) about normal derivations extend thus directly to second order. The subformula principle, however, is of no interest in second order logic since if the subformula notion is defined for second order formulas in analogy with the way it is defined for first order formulas, every formula is a subformula of $\forall X A(X)$ and $\exists X A(X)$. But e.g. the corollary II.3.2.4.2 about atomic formulas is just the fact noted by Takeuti (but formulated by him in a different terminology) as already remarked in the preceding section and means that a sufficiently elementary proof of the normal form theorem for $\mathbf{M}^2$ or $\mathbf{C}^2$ contains a reductive analysis (and thus a consistency proof) of $\mathbf{P}^2$.

However, although the inversion principle and the notions in section II.3 extends to second order logic, the proof of the normalization theorem as outlined in II.3.5.2 does not. The impredicativity of second order logic clearly turns up in $\forall_2$- and $\exists_2$-reductions: a maximum formula $\forall X A(X)$, say, in a derivation Π_1 may be replaced by a new maximum formula $A(T)$ in Π_2 when Π_1 reduces immediately to Π_2 by one of these reductions, and $A(T)$ may be considerably more complex than $\forall X A(X)$; in particular, T may contain $\forall X A(X)$ as a part, in which case a series of immediate reductions may again present us with a maximum formula $\forall X A(X)$. Hence, no measure of the complexity of the maximum formulas (considered isolated) can show that all of them will disappear after a finite number of immediate reductions.

Takeuti's conjecture has, however, been proved by three different semantical methods, viz. (i) by Tait (1966), (ii) by Prawitz (1967a), and (iii) by Takahashi (1967) and Prawitz (1968); the method (iii) was developed for classical higher order logic (i.e. simple type theory). All these proofs established a certain semantical completeness of a cut-free calculus of sequents for classical second order logic from which the normal form theorem for $\mathbf{C}^2$ follows as a corollary.

A similar result was proved for a calculus of sequents of intuitionistic second order logic by Prawitz (1968a) (and was recently extended to higher order intuitionistic logic by Takahashi (1970)) from which the normal form theorem for $\mathbf{M}^2$ (or $\mathbf{I}^2$) follows.

Since all these proofs use methods that are in no way more elementary than those formalized in $\mathbf{P}^2$ (for a discussion of the three methods, see Prawitz (1970)), they are of no immediate interest for reductive proof theory (except that one may be more optimistic about finding a constructive proof of the normal form theorem once an abstract proof has been given). Takeuti (1967) was able to prove the Hauptsatz for a subtheory of classical second order logic, which corresponds roughly to restricting $\forall_2$I and $\exists_2$E to terms T that contain essentially only one second order quantifier, by using an induction over certain recursive ordinals. However, the interest of this result for reductive proof theory is limited by the fact that the arguments needed to establish the well-ordering of the ordering in question is not very elementary.

For general proof theory, these results leaves a great deal to be desired. In particular, they establish only a normal form theorem and not a normalization theorem, i.e. they leave open the question whether there is a reduction sequence for every derivation that terminates; indeed, these semantical proofs give no information about how a normal derivation is obtained from a given derivation. Takeuti's proof for the subtheory of second order logic mentioned above gives some information of this kind. But although he shows how the cuts can be successively removed, the removal of the cuts involves an essential rebuilding of the derivations. Therefore, also his proof leaves open the question whether the derivations in his subsystem reduce to normal derivations.

Added after the Symposium in Oslo: Ideas presented by Girard (1970), for a system of terms intended for an interpretation of $\mathbf{P}^2$ (cf. chapter IV) have changed this situation. It has been shown independently by Girard (1970), Martin-Löf (1970a), and myself (appendix B) that these ideas can be carried over to derivations which yields the normalization theorem for $\mathbf{M}^2$. By some additions, one can also prove the strong normalization theorem and extend the result to $\mathbf{C}^2$ (see appendix B). This extends thus the whole analysis of section III.3 to second order logic. The result yields also other applications (see appendix B.3). For applications in reductive proof, it should be noted, however, that these new proofs use essentially the principles formalized in $\mathbf{P}^2$, and there is thus still no reduction of e.g. the impredicative character of second order logic.

IV. Functional interpretations

IV.1. An interpretation of positive and intuitionistic logic

1.1. *Introductory considerations*

The operational interpretation of the logical constants suggested by Gentzen (section II.2.2.2) is, as remarked above (II.2.1.1), partly narrower than the usual constructive interpretation. For instance, constructively, $A \supset B$ is interpreted as asserting the existence of a construction by which any construction of A can be transformed to a construction of B, which is weaker than asserting the existence of a proof of B from A; such a proof gives a particular kind of a uniform construction transforming constructions of A to constructions of B, which is not required by the usual constructive interpretation.

A systematic account of the constructive interpretation of the logical constants, along the lines of the explanations by Heyting (1956) e.g., can be given by defining inductively when something is a construction of a sentence. As a starting point for such a definition, we may consider the clauses:

(i) k is a construction of $A \supset B$ if and only if k is a constructive function such that for each construction k' of A, $k(k')$ (i.e. the value of k for the argument k') is a construction of B;

(ii) k is a construction of $\forall x A(x)$ if and only if k is a constructive function such that for each term t, $k(t)$ is a construction of $A(t)$.

A definition of this kind has of course to be based or relativized to something that determines the constructions of atomic formulas. In accordance with constructive intentions, I shall assume that the constructions of atomic formulas are recursively enumerable, and the notion of a construction can then be relativized conveniently to Post systems (II.1.5).

Below, I shall tacitly assume that the Post systems contain at least one individual constant.

I shall thus speak of a construction k of a sentence A relative or *over* a Post system **S**. When A is atomic such a construction k will simply be a derivation of A in **S**. In accordance to clause (i) when relativized to **S**, a construction k of $A_1 \supset A_2$ over **S** where A_1 and A_2 are atomic will be a constructive (or with Church's thesis: recursive) function that transforms every derivation of A_1 in **S** to a derivation of A_2 in **S**. However, a consequence of such a definition would be that if A_1 is not constructible over **S** (i.e. not derivable in **S**), $A_1 \supset A_2$ is automatically constructible over **S** since any constructive function would vacously satisfy the condition in clause (i). In particular, provided Λ is not constructible over **S** and $\sim A$ is a shorthand for $A \supset \Lambda$ as usual, it follows that there is no system **S** over which $\sim\sim A \supset A$ is not constructible (hence, classically, $\sim\sim A \supset A$ is constructible in every **S**), which is clearly contrary to the constructive interpretation of implication and negation.

The notion of implication described by (i) is thus quite weak. Note also that an implication in the sense of (i) relativized to Post systems may be constructible over a system **S** but not over an *extension* **S**$'$ of **S** obtained by adding some new inference rules to **S** (e.g., if $A \supset B$ was constructible over **S** just because A was not constructible over **S** but A is constructible over **S**$'$). A stronger notion of implication is obtained by requiring that the construction k transforms not only constructions of A over **S** to constructions of B over **S** but also all constructions of A over extensions $\mathbf{S}'$ of $\mathbf{S}$ to constructions of B over $\mathbf{S}'$. This stronger requirement is particularly appropriate if the Post systems are thought of as approximations of our knowledge about the atomic sentences and not as a complete description of it (as in the theory of inductive definitions in III.4).

I shall adopt these ideas in the definitions below. As a further explanation of these ideas, note that constructions of iterated implications require constructive functions of higher type. For instance, when A and B are atomic, a construction of $(A \supset B) \supset C$ has to be a constructive functional defined for certain constructive functions. We have thus to consider a hierarchy of constructive objects.

1.2. *Definitions*

1.2.1. *Finite types.* The finite types are defined by the following induction:

(i) 0 is a type and i is a type.

(ii) If τ and σ are types and $\sigma \neq i$, then (τ,σ) is a type.

1.2.2. *Constructive objects of finite types.* To simplify the situation, I shall assume that all the symbols of Post systems are drawn from a common stock of symbols so that the totality of *names*, i.e. terms without parameters, in Post systems and the totallity of derivations in Post systems are limited. We can then define the constructive objects of finite types by the following induction:

(i) The derivations in Post systems are the constructive objects of type 0.

(ii) The names in Post systems are the constructive objects of type i.

(iii) The constructive functions from objects of type τ to objects of type σ are the constructive objects of type (τ,σ).

General constructive (or computable) functions of finite types seem first to have been considered by Gödel (1958).

1.2.3. *Types of formulas.* For simplicity, we restrict ourselves to the fragments of first order languages containing only the logical constants $\wedge$, $\supset$, and $\forall$. The type of a formula in these languages are defined by the following induction:

(i) Atomic formulas have the type 0.

(ii) If $A(a)$ is of type τ, then $\forall x A(x)$ is of type (i,τ).

(iii) If A is of type τ and B is of type σ, then $A \supset B$ is of type (τ,σ).

1.2.4. *Constructions of sentences.* Still restricting ourselves to the $\wedge$-, $\supset$-, $\forall$-fragment of first order languages, we define the notion of a construction of a sentence over a Post system **S** by the induction:

(i) k is a construction of an atomic sentence A over **S** if and only if k is a derivation of A in **S**.

(ii) k is a construction of a sentence $A \supset B$ over **S** if and only if k is a constructive object of the type of $A \supset B$ and for each extension **S**$'$ of **S** and for each construction k' of A over **S**$'$, $k(k')$ is a construction of B over **S**$'$.

(iii) k is a construction of a sentence $\forall x A(x)$ over **S** if and only if k is a constructive object of the type of $\forall x A(x)$ and for each name t in **S**, $k(t)$ is a construction of $A(t)$ over **S**.

1.2.5. *Construction of formulas from formulas.* Let $a_1, a_2, ..., a_m$ be the individual parameters that occur in the formulas $A_1, A_2, ..., A_n$, and A (ordered e.g. after their first occurrence). k is said to be a construction of A from A_1, $A_2, ..., A_n$ over **S** if k is an n+m-ary constructive function with the following property: if **S**$'$ is an extension of **S**, if $t_1, t_2, ..., t_m$ are names in **S**, and if $k_1, k_2, ..., k_n$ are constructions of $A_1^*, A_2^*, ..., A_r^*$ over **S**$'$ respectively where A_i^* is the result of replacing every a_j in A_i by t_j, then $k(t_1,t_2,...,t_m,k_1,k_2,...,k_n)$ is a construction of A^* over **S**$'$ where A^* is again the result of replacing every a_j in A by t_j.

1.2.6. *Truth and validity.* We may define a sentence A as *positively true in* a Post system **S** when there is a construction of A over **S** and define a formula A as a *positive consequence* of formulas $A_1, A_2, ..., A_n$ in **S** when there is a construction of A from $A_1, A_2, ..., A_n$ over **S**. A may be defined as *positively valid* when it is positively true in every Post system **S** over which it is a formula. If in all the definitions above we consider only consistent Post systems, i.e. Post systems in which $\wedge$ is not derivable, then we may speak about *intuitionistic constructions, truth*, etc.

1.3. *Remarks*

1.3.1. The definitions above were suggested by Prawitz (1968b) (concerning their relation to Kleene's notion of realizability see this paper) where also the other logical constants were considered, and it was shown that A is positively or intuitionistically valid if A is derivable in **M** or **I**, respectively. It was also seen that the axioms of first order arithmetic are positively and intuitionistically true in the Post system $\mathbf{S}^{\mathbf{A}}$ (or, to be quite correct, the Post system that is like $\mathbf{S}^{\mathbf{A}}$ except for containing operational constants instead of predicates for addition and multiplication).

Although classically valid formulas such as $\sim\sim A \supset A$ are not in general positively or intuitionistically valid, it should be noted that minimal and intuitionistic logics are not complete with respect to this interpretation.

1.3.2. We may note that there is a simpler interpretation of intuitionistic (or minimal) logic where we define "$A \supset B$ is true" as simply meaning "if A is true, then B is true", understanding the phrase "if..., then..." intuitionistically. The interpretation in 1.2 above has a richer structure than this simple interpretation but it is nevertheless not reductive: we do not avoid the use of implication and universal quantification in clauses 1.2.4(ii) and 1.2.4(iii). An interpretation with a still richer structure which is reductive is obtained if we demand that a construction of $A \supset B$ is to consist not only of a function k as described above but also of a proof demonstrating that k has the properties required in clause 1.2.4(ii). Such an interpretation was first suggested by Kreisel (1960) and has later been developed by Goodman (1968). This interpretation is superior to the one described here also in other essential respects. The main purpose of the present chapter is to establish a connection between derivations and terms in certain λ-calculi and the present section is only to serve as an introduction to that.

IV.2. Terms that define constructions

The functions needed as constructions of derivable formulas are of a very simple kind and can be defined in a certain λ-calculus, which I shall call **K** and which is described below. I shall also consider certain extensions of this calculus.

2.1. *The calculus* K

2.1.1. *Types and symbols in* **K**. The *types* are the same as those defined in 1.2.1.

The *symbols* of **K** are *variables*, viz. for each type τ there are denumerably many variables of type τ, denoted $\alpha^\tau, \beta^\tau, \ldots$, the *abstraction operator* λ, and parenthesis.

2.1.2. *Terms in* **K**. The terms in **K** are defined by the following induction:

(i) Variables of type τ are terms of type τ.

(ii) If Φ is a term of type τ and α is a variable of type σ, then $\lambda\alpha\Phi$ is a term of type (σ,τ).

(iii) If Φ is a term of type (τ,σ) and Ψ is a term of type τ, then $\Phi(\Psi)$ is a term of type σ.

λ binds the variable in question in the usual way.

2.1.3. *Interpretation of the terms in* **K**. A closed term of type τ can then be interpreted as a constructive object of type τ in the following way. By a *valuation*, we understand a function V from the variables in **K** such that $V(\alpha^\tau)$ is a constructive object of type τ. We define W_V, viz. the *value* of the terms in **K** with respect to a valuation V, by induction over the length of the terms:

2.1.3.1. $W_V(\alpha) = V(\alpha)$.

2.1.3.2. $W_V(\lambda\alpha^\tau\Phi)$ is the function f such that f is defined for all constructive objects of type τ and when k is such an object, $f(k) = W_{V'}(\Phi)$ where V' is the valuation that is like V except for assigning k to α.

2.1.3.3. $W_V(\Phi(\Psi))$ is the value of the function $W_V(\Phi)$ for the argument $W_V(\Psi)$.

The value of a closed term with respect to V is clearly independent of V and is a constructive object of the same type as the term. A closed term is said to *denote* this value.

An open term Φ with the free variables $\alpha_1^{\tau_1}, \alpha_2^{\tau_2}, \ldots, \alpha_n^{\tau_n}$ is said to *denote* the n-ary function f such that $f(k_1,k_2,\ldots,k_n)$ is defined when each k_j is an object of type τ_j and in this case, $f(k_1,k_2,\ldots,k_n) = W_{V'}(\Phi)$ where V' is like V except for assigning k_j to $\alpha_j^{\tau_j}$ (j=1,2,...,n).

2.1.4. *Result*. Let A be a sentence without operational constants that is derivable in **M** (or **I**). Then, there is a closed term Φ in **K** that denotes an (intuitionistic) construction of A over every Post system over which A is a formula.

2.1.5. *A mapping of derivations on terms.* The result stated above is in effect proved in Prawitz (1968b) by defining a function F from derivations and showing that if Π is a derivation of A from $A_1, A_2, ..., A_n$ then $F(\Pi)$ is a construction of A from $A_1, A_2, ..., A_n$.

2.2. *The calculi* **K(S)**

The results stated above can be extended to systems **M(S)** by extending the calculus **K**. Given a Post system **S**, we define an extension **K(S)** of **K** as follows.

The calculus **K(S)** contains above the symbols of **K** the individual and the operational constants in **S** as symbols. In addition, for each rule in the system **S**, there is a *rule constant* r in **K(S)**.

The terms in **K(S)** are formed as in **K** but with the following additional clauses: The individual constants in **S** are terms in **K(S)** of type i. If $t_1, t_2, ..., t_n$ are terms in **K(S)** of type i and f is an n-ary operational constant in **S**, then $f(t_1,t_2,...,t_n)$ is a term in **K(S)** of type i. If r is the rule constant corresponding to a rule in **S** with n parameters and m premisses and $t_1, t_2, ..., t_n$ are terms in **K(S)** of type i and $\Phi_1, \Phi_2, ..., \Phi_m$ are terms in **K(S)** of type 0, then $r(t_1,t_2,...,t_n,\Phi_1,\Phi_2,...,\Phi_m)$ is a term in **K(S)** of type 0.

The definition of the value (with respect to a valuation) of a term is extended so that the individual constants in **S** have themselves as values and the value of a term $f(t_1,t_2,...,t_n)$ (of type i) is the term obtained by writing f followed by the value of the terms $t_1, t_2, ..., t_n$. If r is the rule constant corresponding to the rule defined by the schema

$$\frac{A_1 \quad A_2 \quad \cdots \quad A_m}{A}$$

then the value of $r(t_1,t_2,...,t_n,\Phi_1,\Phi_2,...,\Phi_m)$ is the tree

$$\frac{\Pi_1 \quad \Pi_2 \quad \cdots \quad \Pi_m}{A^*}$$

where Π_j is the value of Φ_j (j=1,2,...,m) and A^* is the result of simultaneously replacing the jth parameter in A (the parameters being ordered after their first occurrences) by the value of t_j (j=1,2,...,n).

The mapping F in 2.1.5 above can then be extended so that for each derivation Π in $\mathbf{M(S)}$, $F(\Pi)$ is a construction with respect to $\mathbf{S}$ of the end-formula of Π from the open premisses in Π.

2.3. *The calculus* $\mathbf{T}$

The calculus $\mathbf{T}$ is an extension of the calculus $\mathbf{K(S^A)}$ (where $\mathbf{S^A}$ is the Post system described in III.1), which for each type $\tau \neq i$ contains an additional constant I^τ such that if Φ is a term of type τ in $\mathbf{T}$, if Ψ is a term of type $(i,(\tau,\tau))$ in $\mathbf{T}$, and if t is a term of type i in $\mathbf{T}$, then $I^\tau(\Phi,\Psi,t)$ is a term of type τ in $\mathbf{T}$.

In connection with $\mathbf{T}$, we may modify the definition of constructive objects so that the objects of type i are just the numerals and the objects of type 0 are just the derivations in $\mathbf{S^A}$ without parameters.

The value $W_V(\Phi)$ of a term Φ in $\mathbf{T}$ with respect to a valuation V that assigns constructive objects of this modified kind to the variables, is then defined as above with the additional clause:

If $W_V(t) = 0$, then $W_V(I^\tau(\Phi,\Psi,t)) = W_V(\Phi)$; if $W_V(t)$ is a numeral of the form u', then $W_V(I^\tau(\Phi,\Psi,t))$ = the value of the function $W_V(\Psi)(u)$ for the argument $W_V(I^\tau(\Phi,\Psi,u))$.

The mapping F in 2.1.5 can then be extended so that for each derivation Π in $\mathbf{P}$, $F(\Pi)$ is a construction with respect to $\mathbf{S^A}$ of the end-formula of Π from the open assumptions in Π.

2.4. *Reductions and normal terms in* $\mathbf{K}$, $\mathbf{K(S)}$, *and* $\mathbf{T}$

The calculus $\mathbf{K}$ differs from Church's λ-calculus only by having a certain type-structure. As in Church's λ-calculus, we have also here obvious reductions and a normal form. We shall say for terms in $\mathbf{K}$ or $\mathbf{K(S)}$ that $\Phi(\Psi)$ is a (λ-) *reduction* of $\lambda\alpha\Phi(\alpha)(\Psi)$, that Φ *reduces immediately* to Ψ if Ψ is obtained from Φ by replacing one of its subterms by its reduction, that Φ *reduces* to Ψ if Ψ is obtained from Φ by a series of immediate reductions, and that Φ is in *normal form* when it contains no reducible subterm, i.e. no subterm of the form $\lambda\alpha\Phi(\alpha)(\Psi)$.

We make the same definitions for $\mathbf{T}$ except that we add two more reductions: Φ is said to be a reduction of $I^\tau(\Phi,\Psi,0)$ and $\Psi(t)(I^\tau(\Phi,\Psi,t))$ is a reduction of $I^\tau(\Phi,\Psi,t')$.

2.5. *A connection between derivations and terms*

There is an obvious similarity between the definitions for derivations in sections II.3 and III.1.4 and the definitions for terms in $\mathbf{K}$ and $\mathbf{T}$ in section 2.4 above. More precisely, it can be seen that the mapping F mentioned in 2.1.5 is an homomorphism (onto its range) with respect to immediate reducibility, i.e. if Π_1 reduces immediately to Π_2, then $F(\Pi_1)$ reduces immediately to $F(\Pi_2)$, furthermore, if $F(\Pi_1)$ reduces immediately to Φ, then there is a Π_2 such that $F(\Pi_2) = \Phi$ and Π_1 reduces immediately to Π_2. From results about normalization for terms in **K**, **K(S)** or **T**, we may thus infer corresponding results for derivations in **M**, **M(S)** and **P**.

Conversely, one can easily define an homomorphism with respect to immediate reducibility from terms in **K** onto a subset of derivations in **M**, and hence from the results about derivations in II.3, we can infer corresponding results about terms in **K**. It does not seem to be known at present whether there is such an homomorphism from terms in **T** onto a subset of derivations in **P**.

Curry and Feys (1958) seem first to have noted the similarity between derivations in propositional logic and terms in certain λ-calculi. Howard (1969) extended their observations to predicate logic and Martin-Löf (1969) made a further extension to infinite sentential logic (see III.3). All these observations are made for λ-calculi with infinitely many ground types (one for each atomic formula in the logical system in question). It is then possible to establish an isomorphism (instead of the homomorphism) between terms and derivations.

IV.3. **Gödel's functional interpretation**

3.1. *The general form of Gödel's interpretation*

The interpretation of intuitionistic logic by Gödel (1958) (referred to in I.4.3) is an example of a reductive proof theoretical analysis of the kind described in section I.3.2. The general form of Gödel's result may be described as follows: With every formula $A(\bar{a})$ in **P** where $\bar{a}$ is a list of the parameters in the formula, there is associated a formula A of the form $\exists\bar{y}\forall\bar{z}B(\bar{y},\bar{z},\bar{a})$ where $\exists\bar{y}$ and $\forall\bar{z}$ are abbreviations of sequences of $\exists$- and $\forall$-quantifiers, respectively, and $\bar{y}$ and $\bar{z}$ are abbreviations of the variables quantified by these quantifiers, and $B(\bar{y},\bar{z},\bar{a})$ is built up by sentential connectives from equations between terms which may contain in addition to ordinary arithmetical operators variables for constructive functions of finite types (cf. sec. 1.1.2). It is then shown that if $A(\bar{a})$ is provable in $\mathbf{P}_\mathbf{I}$, then there is a sequence of functions abbreviated $\bar{F}$ which is definable in a calculus **T** and for which $\bar{B}(\bar{F}(\bar{a}),\bar{b},\bar{a})$ is provable in **T**. The calculus **T** is here like the one defined in section 2.3 except that there is no type i and no rule constants, i.e. the language are the same with these differences and the axioms are obvious quantifier free axioms which are in harmony with the interpretation stated in section 2. The functions $\bar{F}$ are obtainable uniformly from the proof of $A(\bar{a})$ in $\mathbf{P}_\mathbf{I}$. (Thus, the functions

f and g in section I.3.2 may be written $f(A(\bar{a})) = B(\bar{c},\bar{b},\bar{a})$ and $g(\Pi) = \bar{F}$. The operation Φ in I.3.2 is then simply the operation of substituting the second argument for certain variables in the first argument.

3.2. *Tait's and Howard's analysis of* **T**

Since the proof of Gödel's result is strictly finitary, it constitutes a reduction of **P** provided the theory **T** is admitted as a more elementary theory than **P**. However, Tait (1965) and (1967) has suggested that also the theory **T** should be analysed. In particular, Tait (1967) shows that every term in **T** is what he calls convertible (see appendix A.5) and reduces to the normal form defined above in 2.4, which among other things establishes the consistency of **T**.

Howard (1968) shows that the proof of this result can be given by the strictly finitary means of Skolem arithmetic extended with induction up to ϵ_0. One obtains thus the same epistemological reduction as Gentzen (sec. III. 1.3).

But as noted above in 2.4 and 2.5, not only are the same principles needed to carry out Gentzen's analysis of **P** and this analysis of Gödel's **T**, the result is essentially of the same structure.

Appendix A. Validity of derivations

A.1. Validity based on the introduction rules

1.1. *Introductory discussion*

Gentzen's idea that the introduction rules may be looked upon as a kind of definitions that give the meaning of the logical constants in terms of which the elimination rules are justified (section II.2.2.2) is very suggestive, it seems to me. Of course, as already remarked, the introduction rules are not literal definitions and the question thus arises how to make Gentzen's idea precise.

Since the premiss(es) of a first order introduction rule is (are) subformula(s) of the conclusion, a natural idea would be to try to use the condition for inferring a formula A by introduction as a clause in a definition by induction of a notion "A is valid". One may then be able to verffy that the elimination rules preserve this validity.

The basic idea in this definition of validity would thus be that A is valid if A can be built up or constructed by introductions, in other words, if there is a *construction* of A by the use of introductions; note, however, that the constructions considered here are quite different from the ones in section IV.1. Instead of defining the validity of A directly, we may thus define inductively what constitutes a construction of A.

Such an inductive definition would have to be based on some given constructions of the atomic formulas (since the conclusions of Gentzen's I-rules are all compound formulas). These constructions can be given by an atomic system **S**(sec. II.1.5) and as in IV.1, we may thus relativize the notion of a construction to such systems.

Inductive clauses defining constructions of conjunctions, disjunctions, universal quantifications, and existential quantifications relative an atomic system **S** do not offer any difficulties. Constructions of formulas that contain only &, v, $\forall$, and $\exists$ as logical constants will thus consist simply of derivations of these formulas using only the introduction rules for these constants and the rules of **S**.

But implication constitutes a problem since the condition for inferring $A \supset B$ by introduction is a derivation of B from A and this derivation may have to use not only successive introductions constructing B but also eliminations operating on A, which are the very rules that we want to justify on the basis of the meaning given to the constants by the introductions; it would of course be circular to define a construction of $A \supset B$ as simply a derivation of B from A.

The brief discussion by Gentzen (1934) does not give any hint as to how implication is to be handled. A natural idea, which seems to be consonant with the general idea that we are trying to explicate, is to understand by a construction of $A \supset B$ a derivation of B from A *that togehter with a construction of A yields a construction of B.*

To make this idea precise, we have to specify what "yield" is to mean here. Remembering the conjecture about identity between proofs (sec. I.3.5.6), it seems natural to require that the derivation together with a construction of A *reduces* to a construction of B and to take this as the meaning of "yield".

However, one modification of this idea seems desirable. The condition on a construction of $A \supset B$ as formulated above would be vacuously satisfied if there is no construction of A relative to the system **S** in question. The conditions is thus quite weak and can be strengthened as in IV.1 by requiring that the condition above is satisfied also relative to every extension of the system in question.

I shall summarize these ideas in a definition below. I shall then say that a derivation of A that constitutes a construction in the above sense is *valid*; however, in view of the conjecture about identity between proofs, a derivation that reduces to such a construction shall also be counted as valid. Rather than defining the notion of construction discussed above, I shall define this notion of validity inductively. In this definition of validity of derivations, we shall thus require in all the clauses (not only the clause for implication) that the derivation reduces to a derivation of a particular kind.

1.2. *Definitions*

1.2.1. *Validity of closed derivations*. Let Π be a *closed* derivation in a system **M(S)** of a sentence A, i.e. Π has no open assumptions and no parameters that are not proper. We assume that **S** contains at least one individual constant. We define: Π is *valid in* **S** if and only if

1.2.1.1. A is atomic and Π reduces to a derivation in **S**; or

1.2.1.2. A is of the form $A_1 \& A_2$ and Π reduces to a derivation of the form

$$\frac{\Pi_1 \quad \Pi_2}{A_1 \& A_2}$$

and for $i = 1$ and 2, Π_i is a derivation of A_i and is valid in **S**; or

1.2.1.3. A is of the form $A_1 \vee A_2$ and Π reduces to a derivation of the form

$$\frac{\Pi'}{A_1 \vee A_2}$$

where Π′ is a derivation of A_1 or A_2 and is valid in **S**; or

1.2.1.4. A is of the form $A_1 \supset A_2$ and Π reduces to a derivation of the form

$$\frac{\Pi_2}{A_1 \supset A_2}$$

such that for each extension **S′** of **S** and for each closed derivation

$$\frac{\Sigma_1}{A_1}$$

in **M(S′)** that is valid in **S′**, it holds that

$$\begin{array}{c}\underline{\Sigma_1}\\ [A_1]\\ \Pi_2\end{array}$$

is a derivation of A_2 and is valid in **S′** ($[A_1]$ is here the set of open assumptions in Π_2 of the form A_1); or

1.2.1.5. A is of the form $\forall x B(x)$ and Π reduces to a derivation of the form

$$\frac{\Pi'(a)}{\forall x B(x)}$$

whose last inference is an $\forall$I with a as proper parameter and for each name t (i.e. individual term without parameters), $\Pi'(t)$ obtained from $\Pi(a)$ by substituting t for a is (a derivation of $B(t)$) valid in $\mathbf{S}$, or

1.2.1.6. A is of the form $\exists x B(x)$ and Π reduces to a derivation of the form

$$\frac{\Sigma'}{\exists x B(x)}$$

where Σ' is a derivation of $B(t)$ for some name t in $\mathbf{S}$ and is valid in $\mathbf{S}$.

1.2.2. *Validity of open derivations.* A derivation Π in $\mathbf{M}(\mathbf{S})$ (with open assumptions or not proper assumptions) is valid in $\mathbf{S}$ if and only if for each result Π' of substituting names in $\mathbf{S}$ for not proper parameters in Π it holds: if $\mathbf{S}'$ is an extension of $\mathbf{S}$ and Π^* is the result of replacing every assumption A in Π' by a closed derivation of A in $\mathbf{M}(\mathbf{S}')$ that is valid in $\mathbf{S}'$, then Π^* is valid in $\mathbf{S}$.

1.3. *Discussion*

1.3.1. Does the notion of validity defined above capture Gentzen's idea about an operational interpretation of the logical constants (as discussed in section II.2.2.2)? Clearly, it follows immediately from the definition that a derivation whose last inference is an introduction is valid (in $\mathbf{S}$) if the derivation(s) of the premiss(es) is (are) valid (in $\mathbf{S}$). Furthermore, it can be seen by just the kind of argument that Gentzen had in mind that eliminations preserve validity. More precisely, we may first note as a lemma (proved by induction over the degree of the end formula) that if Π_1 reduces to Π_2 and Π_2 is valid, then Π_1 is valid. Assume now that Π is a derivation whose last inference is an elimination and that the derivation(s) of the premiss(es) of this elimination is (are) valid. From the definition of validity, it follows that the derivation of the major premiss reduces to a derivation whose last inference is an introduction. Hence, the derivation Π as a whole reduces to a derivation Π' with the same elimination as last inference and the major premiss of this elimination is now a maximum formula. It can then be seen that the reduction of Π' is valid, which is just a precise version of Gentzen's informal argument. From the lemma it now follows that Π is valid.

It seems thus fair to say that the notion of validity defined above constitutes one possible explication of Gentzen's idea. But of course, this does not exclude the possibility that there exist other and more interesting ways of developing this idea.

1.3.2. Although the notion of validity defined above may explicate Gentzen's idea about an operational interpretation of the logical constants, one may ask whether such an interpretation is at all reasonable. As remarked in section IV.1.1, the operational interpretation of implication is much stronger than the usual constructive interpretation. And the same holds for universal quantification. It must be admitted that such a strong meaning of $\supset$ and $\forall$ is seldom used. The interest of this operational interpretation is rather that the rules of minimal (or intuitionistic, see 1.3.3 below) logic are sound also for this very strong interpretation. In this context, we may say that a formula A is valid in a system $\mathbf{S}$ if there is a derivation of A that is valid in $\mathbf{S}$ and that A is logically valid if it is valid in every $\mathbf{S}$ over which it is a formula. The proof outlined in 1.3.1 above then shows that every derivable formula in $\mathbf{M}$ is logically valid in this strong sense. However, this notion of validity of a formula in a system does not have natural mathematical applications. For instance, the induction axioms are not generally valid in the system $\mathbf{S}^{\mathbf{A}}$ (defined in III.1.1): Although a derivation of $A(0)$ valid in $\mathbf{S}^{\mathbf{A}}$ and a derivation of $\forall x(A(x) \supset A(x'))$ valid in $\mathbf{S}^{\mathbf{A}}$ guarantee together the existence of a derivation of $A(t)$ valid in $\mathbf{S}^{\mathbf{A}}$ for every numeral t, there may be no uniformly valid derivation of this kind, i.e. no derivation of $A(a)$ valid in $\mathbf{S}^{\mathbf{A}}$, as required if $\forall x A(x)$ is to be valid in $\mathbf{S}^{\mathbf{A}}$.

In section 2 below, I shall consider a variant of the validity notion, which in contrast to the one above will allow an interpretation of the axioms of first order arithmetic.

1.3.3. The validity notion is defined above only for derivations in systems $\mathbf{M}(\mathbf{S})$. But intuitionistic logic is of course also covered since a system $\mathbf{I}(\mathbf{S})$ is identical to the system $\mathbf{M}(\mathbf{S}^+)$ where $\mathbf{S}^+$ is the extension of $\mathbf{S}$ obtained by adding the rule Λ_{I}. However, if we add derived rules to $\mathbf{M}(\mathbf{S})$ involving compound formulas and want to extend the validity notion to such a system, we have also to extend the notion of reducibility. Such an extension seems always possible also when the added rule only preserves derivability (from null assumptions). For instance, the rule Λ_{C} (principle of indirect proof for atomic formulas) is such a rule and by adding the simplification defined in II.3.3.2.3 as a new reduction, we can extend the notion to systems $\mathbf{C}(\mathbf{S})$. But note that the notion cannot be extended in this fashion to systems containing the rule mentioned in II.2.2.2. It would be desirable to define a more general notion of validity which is applicable without changes to extensions of **M(S)** by the addition of derived rules and by which the invalidity of a rule such as the one mentioned in II.2.2.2 could be demonstrated.

1.3.4. Not surprisingly, the above notion of validity is strongly connected with the notion of normalizability. Although the validity of every derivation was easily shown in 1.3.1 above, it also follows directly from the normalization theorem (by an induction over the degree of the end formulas). But certain aspects of the normalization theorem, in particular the one discussed in II.4.1.2.2, are better expressed by saying that each derivation is valid. Note also that the corollaries about atomic formulas, disjunctions and existential quantifications also follow immediately from the validity of derivations (cf. II.4.1.2.2.1 and II.4.1.2.2.2).

Conversely, if we strengthen the notion of validity by requiring not only that Π reduces to a derivation of a certain kind but that every sequence of immediate reductions if continued far enough reduces Π to a derivation of this kind, we get a notion that may be called strong validity and that provides a convenient tool for proving the strong normalization theorem. In section 3 below, I shall modify this notion slightly to make it even more eonvenient for this purpose.

A.2. Validity based on the elimination rules

Since the introductions and eliminations are inverses of each other, Gentzen's idea to justify the eliminations by the meaning given to the constants by the introductions may be reversed. Instead of interpreting the constants in the way above as asserting the existence of certain construction that build up formulas with these constants, we may interprete them as stating the performability of certain operations. A derivation will then be valid when it can be used to obtain certain valid derivations of the subformulas.

Hence, while the clause 1.2.1 would be left unchanged, a derivation Π of $A_1 \,\&\, A_2$ would now be defined as valid in **S** if

$$\frac{\Pi}{A_i}$$

is valid in **S** for $i = 1$ and 2.

A derivation Π of $A_1 \supset A_2$ would be defined as valid in **S** if for each extension **S**$'$ of **S** and for each derivation Π' of A_1 that is valid in **S**$'$,

$$\frac{\Pi' \quad \Pi}{A_2}$$

is valid in $\mathbf{S}'$.

Similarly, a derivation Π of $\forall x A(x)$ would be defined as valid when for each name t in $\mathbf{S}$

$$\frac{\Pi}{A(t)}$$

is valid in $\mathbf{S}$.

One may say that while a valid derivation in the sense of the preceding section 1 guarantees the existence of a certain construction of the end formula, a valid derivation in the sense of this section constitutes a rule for inferring certain formulas.

The validity of a derivation whose last inference is an elimination is now immediate from the definition. But to show that validity is preserved by the introductions, we have now essentially to show that a construction of the kind considered in section 1 also constitutes a rule for obtaining valid derivations (in the sense of this section) of other formulas. This is again shown by essential use of the reductions, and the basic idea is thus the same in the two cases.

We may note however that the universal quantifier is now not interpreted as narrowly as in section 1 since we do not require a uniformly valid derivation of $A(a)$ to have a valid derivation of $\forall x A(x)$. In particular, we note that every derivation in the system **P** for first order arithmetic is valid in the sense of this section.

However, disjunctions and existential quantifications seem impossible to handle in this way since the induction over the complexity of the end formulas used in the definition of validity breakes down in these cases.

A.3. Validity used in proofs of normalizability

3.1. *Introductory remarks*

The clauses 1.2.2 – 1.2.6 in the definition 1.2 of validity can be changed without affecting the meaning of the notion by separating two cases, viz. (i) the case when the last inference of Π is an introduction and (ii) the case when the last inference of Π is an elimination. In case (i) we may require in the different cases not that Π reduces to a derivation of a certain kind but that Π is already of this kind. In case (ii), we may simply require that Π reduces to a valid derivation.

When the validity notion is used as a tool in proving normalizability, it is suitable to make this separation and to define the notion directly for all derivations, i.e. not only for the closed ones. Furthermore, when case (ii) applies it is suitable to define also the normal derivations as valid. This agrees with the definition 1.2 when the end formula of the derivation is atomic but is a deviation when the end formula is not atomic. It is then not necessary to relativize the validity notion to atomic systems. Note also that with this modification, there will always exist a normal derivation of each formula and hence we do not get the problem noted in connection with implication in section 1.1 that led us to consider extensions of atomic systems.

To prove the strong normalization theorem, we strengthen the notion of validity to strong validity by requiring when case (ii) applies that *every* derivation to which the given derivation immediately reduces is valid.

3.2. *Definitions*

3.2.1. *Strong validity*. Let Π be a derivation of A in any system **M(S)**, **I(S)**, or **C(S)**. We define: A is strongly valid if and only if one of the following cases applies:

3.2.1.1. The last inference of Π is an &I, $\vee$I, or $\exists$I and the derivation(s) of the premiss(es) of the introduction is (are) strongly valid;

3.2.1.2. The last inference of Π is an $\supset$I, in which case Π is of the form

$$\frac{\begin{matrix}[A_1]\\ \Pi_2\end{matrix}}{A_1 \supset A_2}$$

with $[A_1]$ as the set of assumptions closed by the $\supset$I, and for each strongly valid derivation

$$\frac{\Sigma_1}{A_1}$$

it holds that

$$\frac{\Sigma_1}{\begin{matrix}[A_1]\\ \Pi_2\end{matrix}}$$

is strongly valid;

3.2.1.3. The last inference of Π is $\forall$I with proper parameter a and for each individual term t, it holds that the result of substituting t for a in the derivation of the premiss of the $\forall$I is strongly valid;

3.2.1.4. The last inference of Π is not an introduction and the following conditions hold:

(i) Π is normal or each derivation Π' to which Π immediately reduces is strongly valid;

(ii) if Π is of the form

$$\dfrac{\dfrac{\Sigma}{B_1 \vee B_2} \quad \dfrac{\begin{matrix}[B_1]\\ \Sigma_1\end{matrix}}{A} \quad \dfrac{\begin{matrix}[B_2]\\ \Sigma_2\end{matrix}}{A}}{A}$$

where $[B_i]$ is the set of assumptions in Σ_i closed by this vE (i=1 and 2), then the derivation of the minor premisses of the vE are strongly valid and for each derivation

$$\dfrac{\Sigma_i'}{B_i}$$

which is a part of a derivation Π' to which the derivation of the major premiss $B_1 \vee B_2$ reduces and where B_i is a formula immediately above either the end formula of Π' or an end segment of Π', it holds that

$$\dfrac{\dfrac{\Sigma_i'}{[B_i]}\\ \Sigma_i}{A}$$

is strongly valid;

(iii) if Π is of the form

$$\dfrac{\dfrac{\Sigma_1}{\exists x B(x)} \quad \dfrac{\begin{matrix}[B(a)]\\ \Sigma(a)\end{matrix}}{A}}{A}$$

where a is the proper parameter of this $\exists$E and $[B(a)]$ is the set of assumptions closed by the $\exists$E, then the derivation of the minor premiss of the $\exists$E is strongly valid and for each derivation

$$\frac{\Sigma'}{B(t)}$$

which is a part of a derivation Π' to which the derivation of the major premiss $\exists x B(x)$ reduces and where $B(t)$ is a formula immediately above either the end formula of Π' or an end segment of Π', it holds that

$$\frac{\dfrac{\Sigma'}{[B(t)]}}{\dfrac{\Sigma(t)}{A}}$$

is strongly valid.

The definition is to be understood as a generalized inductive definition, i.e., it proceeds by induction over the complexity of the end formula of the derivation and for each fixed complexity, a derivation is valid if this follows by a finite number of applications of the clauses 3.2.1.1 – 3.2.1.4.

3.2.2. *Strong validity under substitution.* We define: A derivation Π is strongly valid under substitution if and only if for each substitution of terms for not proper parameters in Π and for each way of replacing open assumptions in the derivation after this substitution by strongly valid derivations of the assumptions (not necessarily replacing each open assumption) it holds that the resulting derivation is strongly valid.

3.3. *Results*

We can now state the two main results:

3.3.1. **Theorem.** *Every derivation in* **M(S)**, **I(S)** *or* **C(S)** *is strongly valid under substitution.*

3.3.2. **Theorem.** *Every reduction sequence starting from a strongly valid derivation terminates (in a normal derivation).*

From these two theorems follow the strong normalization theorem in II.3.5.3 if we also verify the more trivial property (similar to the so-called

Church-Rosser property in the λ-calculus) that two reduction sequences that start from the same derivation and terminate always terminate in the same derivation.

3.4. *Proofs*

We first state two lemmata concerning reducibility, which follow immediately from the definitions.

3.4.1. **Lemma.** *If the last inference of* Π *is an introduction and is thus of the form*

$$\frac{\Pi'}{A} \qquad \textit{or} \qquad \frac{\Pi' \quad \Pi''}{A}$$

and $\Pi_1, \Pi_2, \ldots$ *is a reduction sequence starting from* Π, *then each* Π_i *ends with the same introduction and is thus of the form*

$$\frac{\Pi_i'}{A} \qquad \textit{or} \qquad \frac{\Pi_i' \quad \Pi_i''}{A}$$

respectively, and $\Pi_1', \Pi_2', \ldots$ *and* $\Pi_1'', \Pi_2'', \ldots$ *with omissions for possible repetitions are also reduction sequences.*

3.4.2. **Lemma.** *If* $\Pi_1(a)$ *reduces to* $\Pi_2(a)$, *a is not a proper parameter in* $\Pi_1(a)$, *and* $[A(a)]$ *is a set of open assumptions in* $\Pi_1(a)$, *then every derivation*

$$\begin{array}{c} \Sigma \\ \hline [A(t)] \\ \Pi_1(t) \end{array} \qquad \textit{reduces to} \qquad \begin{array}{c} \Sigma \\ \hline [A(t)] \\ \Pi_2(t) \end{array}$$

By the use of these two lemmata and an induction over the definition of strong validity, we prove

3.4.3. **Lemma.** *If* Π_1 *reduces to* Π_2 *and* Π_1 *is strongly valid, then so is* Π_2.

3.4.4. *Proof of theorem* 3.3.2. We can now prove theorem 3.3.2 by an induction over the definition of strong validity. When the last inference of the derivation is not an introduction, the assertion follows immediately from the induction hypothesis. When the last inference of the derivation is an introduction, it can be seen that the derivation(s) of the premiss(es) of this introduction is (are) also strongly valid and that the induction hypothesis can be applied to them. The assertion then follows from lemma 3.4.1.

3.4.5. *Proof of theorem* 3.3.1. We prove theorem 3.3.1 by an induction over the length of the derivation. The base of the induction is trivial. Also the induction step is immediate in the case when the last inference of the derivation is an introduction: the induction assumption implies immediately the defining clause of strong validity in question. When the last inference is not an introduction, we make use of the following

Lemma. *A derivation* Π *whose last inference is not an introduction is strongly valid when the following conditions are satisfied:*

(i) *every reduction sequence starting from a derivation of a premisss of the last inference of* Π *terminates;*

(ii) *if the last inference of* Π *is* &E, $\supset$E, *or* $\forall$E, *then the derivation(s) of the premiss(es) of this inference is (are) strongly valid;*

(iii) *if the last inference is* $\vee$E *or* $\exists$E, *then condition* (ii) *or* (iii), *respectively, in clause* 3.2.1.4 *in the definition of strong validity is satisfied.*

We shall see that this lemma implies that a derivation Π of the form

$$\frac{\Pi_1 \quad \Pi_2 \quad \dots \quad \Pi_n}{A}$$

whose last inference is not an introduction is strongly valid under substitution if the derivations Π_1, Π_2, ..., and Π_n are strongly valid under substitution. We have thus to show that each result Π^* of substituting individual terms for individual parameters and derivations for assumptions in Π as described in the definition of strong validity under substitution (3.2.2) is strongly valid. Π^* has the form

$$\frac{\Pi_1^* \quad \Pi_2^* \quad \dots \quad \Pi_n^*}{A^*}$$

where Π_i^* is the result of carrying out the substitution in Π_i (i=1,2,...,n), and hence, each Π_i^* is strongly valid. Condition (ii) of the lemma is thus satisfied by Π^*. In view of theorem 3.2.2, condition (i) is also satisfied. Too see that also condition (iii) is satisfied, we use the fact the derivation(s) of the minor premiss(es) of the last inference of Π was (were) strongly valid under substitution and apply lemma 3.4.2. We can thus apply the lemma to Π^* and conclude that Π^* is strongly valid. It remains only to prove the lemma.

Proof of the lemma. To prove the lemma, we shall use an induction over the length of reduction trees of derivations. The reduction tree of a derivation Π is the tree whose threads are the reduction sequences starting from Π (we may also represent them as a graph but that is immaterial here). The reduction tree of an n-tuple of derivations is the tree consisting of n roots, the ith root being the end node of the reduction tree of the ith derivation.

We shall assign an *induction value* to each derivation Π of the kind described in the lemma. It will consist of a triple (α,β,γ) where α is the length of the reduction tree of the derivation of the major premiss of the last inference of Π and β is the length of the derivation of this premiss if there is such a premiss, otherwise $\alpha = \beta = 0$, and γ is the length of the reduction tree of the n-tuple of derivation(s) of the premiss(es) of the last inference of Π (taken in order from the left to the right). Because of clause (i) in the lemma, α and γ are finite. The induction values are ordered lexicographically, and we shall prove the lemma assuming that it has been proved for all derivations with lower induction value.

To prove that Π of the kind described in the lemma is strongly valid, we only need to verify clause (i) in the clause 3.2.1.4 of the definition of strong validity since the other clauses are satisfied according to condition (iii) of the lemma. If Π is normal there is thus nothing to prove. If Π is not normal, we have to show that each Π' to which Π immediately reduces is strongly valid. We consider three cases.

Case (a). Π' is obtained from Π by replacing a proper subtree of Π by its reduction. If Π is

$$\frac{\Pi_1 \quad \Pi_2 \quad \dots \quad \Pi_n}{A}$$

then Π' is in this case of the form

$$\frac{\Pi_1' \quad \Pi_2' \quad \dots \quad \Pi_n'}{A}$$

where for one i $(\leqslant n)$, Π_i reduces immediately to Π_i' and for other j $(i \neq j \leqslant n)$, $\Pi_j' = \Pi_j$. Hence, the reduction value (α',β',γ') of Π' is lower than the induction value (α,β,γ) of Π since either $\alpha' < \alpha$ or $\alpha' = \alpha$, $\beta' = \beta$ but $\gamma' < \gamma$. Π' obviously satisfies condition (i) of the lemma. Conditions (ii) and (iii) are satisfied because of lemma 3.4.3; in the case of condition (iii), we have also to apply lemma 3.4.2 and the fact that reducibility is transitive. By the induction hypothesis, it thus follows that Π' is strongly valid.

Case (b). Π' is a proper reduction of Π. Then the major premiss B of the last inference of Π is the conclusion of an introduction. When B is a conjunction, implication, or universal quantification, we can then (because of clause (ii) in the lemma) apply the definition of strong validity for derivations whose last inference is an introduction and conclude that Π' is strongly valid. When B is a disjunction or existential quantification, the strong validity of Π' follows from clause (iii) of the lemma (and the reflexivity of reducibility).

Case (c). Π' is a commutative reduction of Π. We consider the case of $\vee$E-reductions, the case of $\exists$E-reductions being similar. Π and Π' are then of the form shown to the left and right below, respectively, wheren $[B_i]$ is the set of assumptions in Σ_i closed by the $\vee$E:

$$
\dfrac{\dfrac{\dfrac{\Sigma}{B_1 \vee B_2} \quad \dfrac{\begin{matrix}[B_1]\\ \Sigma_1\end{matrix}}{C} \quad \dfrac{\begin{matrix}[B_2]\\ \Sigma_2\end{matrix}}{C}}{C} \qquad \Sigma_3}{A}
\qquad\qquad
\dfrac{\dfrac{\Sigma}{B_1 \vee B_2} \quad \dfrac{\dfrac{\begin{matrix}[B_1]\\ \Sigma_1\end{matrix}}{C} \quad \Sigma_3}{A} \quad \dfrac{\dfrac{\begin{matrix}[B_2]\\ \Sigma_2\end{matrix}}{C} \quad \Sigma_3}{A}}{A}
$$

The induction value $(\alpha', \beta', \gamma')$ of Π' is less than the induction value (α,β,γ) of Π since $\alpha' \leqslant \alpha$ and $\beta' < \beta$. To see that the lemma can be applied to Π' it is now sufficient to verify that condition (iii) is satisfied by Π': this together with theorem 3.3.2 implies that condition (i) is satisfied for the derivation of the minor premisses of the last inference of Π'; for the major premiss this is obvious in view of assumptions about Π.

We have thus to show that

$$
\dfrac{\dfrac{\Sigma_i}{C} \quad \Sigma_3}{A} \quad \text{and} \quad \dfrac{\dfrac{\begin{matrix}\dfrac{\Sigma_i'}{[B_i]}\\ \Sigma_i\end{matrix}}{C} \quad \Sigma_3}{A} \tag{1}
$$

are strongly valid, the latter under the assumption that

$$
\dfrac{\Sigma_i'}{B_i}.
$$

is a part of a derivation to which

$$\frac{\Sigma}{B_1 \vee B_2}$$

reduces and B_i is the formula immediately above either the end formula or an end segment of this derivation.

To show this, we shall first apply the induction hypothesis to these derivations in (1). The induction value $(\alpha_1, \beta_1, \gamma_1)$ of any of these derivations in (1) is lower than the induction value (α, β, γ) of Π because if the derivation is of the kind shown to the left in (1) then $\alpha_1 \leqslant \alpha$ and $\beta_1 < \beta$ and in the other case $\alpha_1 < \alpha$. Furthermore, these derivations obviously satisfy condition (i) of the lemma since Π satisfies this condition. If C is a conjunction, implication, or universal quantification, then because of the assumptions about Π, viz. condition (ii) of the lemma, the derivation of the major premiss C of the last inference of Π is strongly valid, and hence, by clause (ii) in 3.2.1.4 in the definition of strong validity, the derivation of the major premiss C of a derivation in (1) is strongly valid. In other words, the derivations in (1) satisfy also condition (ii) of the lemma. It remains to see that they also satisfy condition (iii) of the lemma. This condition concerns Σ_3 and from the fact that this condition is satisfied by Π, one can infer that it is also satisfied by the derivations in (1).

We can thus apply the lemma to the derivations in (1) to conclude that they are strongly valid and then apply the lemma to Π' to conclude that also Π' is strongly valid.

We have thus shown that Π is strongly valid and have hence proved the lemma.

3.5. *Historical remark*

Although we have developed the validity notion here as an explication of Gentzen's idea about an operational interpretation of the logical constants and have shown how this notion slightly modified can be used in proofs of normalization theorems, a notion of this kind, called convertibility, was originally developed by Tait (1967) as a tool in establishing normalizations for terms in the calculus **T**, among other things (cf. IV.2.2).

The notion was carried over to derivations by Martin-Löf (1970) and was used by him in establishing a normalization theorem for the theory of iterated inductive definitions (cf. III.4). His notion and the use he makes of it differ from ours in this section 3 in the following two respects: firstly, Marin-Löf does not consider permutative reductions and shows thus that maximal formulas but not maximal segments can be removed from a derivation; secondly, he considers a particular reduction sequence and shows that it terminates where we show that all reduction sequences terminate. Certain complications that arise in the proof of theorem 3.3.1 (particularly in the lemma) as compared to Martin-Löf's proof are due to the first difference; the second difference does not essentially complicate the proof.

Appendix B. Proof of the strong normalization theorem in 2nd order logic

B.1. Intuitive idea

Attempts to extend the ideas of Appendix A to second order logic immediately encounters the following difficulty: the induction of the definition of validity in A1 or A2 or of strong validity in A3 breaks down at formulas $\forall XA(X)$ and $\exists XA(X)$ since $A(T)$ may be a more complex formula than $\forall XA(X)$ and $\exists XA(X)$. In other words, if we put down clauses defining the validity of derivations ending with $\forall XA(X)$ in analogy with the clauses for $\forall xA(x)$ and ask whether a derivation with $\forall XA(X)$ is valid, then it is true that in a finite number of steps, we can resolve this question into questions concerning the validity of derivations ending with formulas of the form $Tt_1t_2...t_n$; but when T is an abstraction term $\{\lambda x_1x_2...x_nB(x_1,x_2,...,x_n)\}$ we have thus to ask questions concerning derivations ending with $B(t_1,t_2,...,t_n)$ and B may contain the formula $\forall XA(X)$ (cf. III.5.3).

Girard's idea is now, when carried over to the present framework, that we assign an arbitrary meaning to the assertion that a derivation ending with $Tt_1t_2...t_n$ is valid: it is to be valid if and only if it belongs to an arbitrarily choosen set N. The formula $Tt_1t_2...t_n$ may thus be considered as a formula with lowest complexity since we shall never have to break down this formula to answer questions about validity.

In other words, we can define a notion of validity of derivations *relative* an assignment $\mathcal{N}$ to second order terms T of sets $\mathcal{N}(T)$ determining the meaning of the notion for derivations ending with $Tt_1t_2...t_n$.

A derivation

$$\dfrac{\dfrac{\Sigma(P)}{A(P)}}{\forall XA(X)}$$

is then defined as valid when for each term T and for (almost) each such assignment $\mathcal{N}$ to T,

$$\frac{\Sigma(T)}{A(T)}$$

is valid relative $\mathcal{N}$. In order to ensure that validity relative such assignments $\mathcal{N}$ implies normalizability, we will have to put certain requirements on $\mathcal{N}$ and therefore we consider only "almost" all assignments; more precisely, we consider all assignments that assign *regular sets* as defined below.

The definition of validity will thus contain a quantification over all regular sets. We shall prove that the set of valid derivations is such a regular set. In other words, when we define the notion of validity for the case when the derivation ends with a formula $\forall XA(X)$, there will be a quantification over sets among which is the very notion that we are defining. We have thus here a splendid example of an impredicativity. And this impredicativity will be used in the proof that each derivation is valid, where we shall consider in particular the assignment that assigns the set of valid derivations to a second order term T.

B.2. **Formal development**

We shall develop the idea described above in more detail in the same general way as in Appendix A.3. For the sake of shortness, we leave out existential quantification of second order variables; the reader should have no difficulty in extending the treatment of first order existential quantification to second order in the same general way as we shall here extend the treatment of first order universal quantification to second order.

2.1. *Definitions*

2.1.1. *Regular sets.* We define: A set N of second order derivations is regular if and only if the following two conditions are satisfied:

2.1.1.1. if $\Pi \in N$, then every reduction sequence starting with Π terminates;

2.1.1.2. if $\Pi \in N$ and Π reduces to Π', then $\Pi' \in N$.

2.1.2. *Strong validity relative assignments* $\mathcal{N}$. Let Π be a derivation of A in $\mathbf{M(S)}$ and let $\mathcal{N}$ be an assignment of regular sets to occurrences of second order terms in A; different occurrences of the same term may be assigned different sets and all occurrences need not have an assignment[1]. We shall define

[1] For footnote, see next page.

the notion: Π is strongly valid relative $\mathcal{N}$. The definition is by a main induction on the "degree of A relative $\mathcal{N}$" by which we understand the measure we get if the degree of A is calculated in the usual way except that occurrence of $Tt_1t_2...t_n$ where T is in the domain of $\mathcal{N}$ is given the same measure as the atomic formulas. For each such complexity, we define the notion by inductive clauses in exactly the same way as in A.3.2.1 except that we replace "strong validity" by "strong validity relative $\mathcal{N}$" (with the understanding in clauses A.3.2.1.1 – 3 that $\mathcal{N}$ is to assign to an occurrence of a second order term in the immediate subformulas of A in question the same value (if any) that $\mathcal{N}$ assigns to the corresponding occurrence in A) and add the following clauses:

2.1.2.1. the last inference of Π is λI, A is of the form $Tt_1t_2...t_n$ where this occurrence of T is in the domain of $\mathcal{N}$ and $\Pi \in \mathcal{N}(T)$;

2.1.2.2. the last inference of Π is λI, A is of the form $Tt_1t_2...t_n$ where T is not in the domain of $\mathcal{N}$, and the derivation of the premiss of the last inference of Π is strongly valid relative $\mathcal{N}$;

2.1.2.3. the last inference of Π is $\forall_2$I in which case Π is of the form

$$\cfrac{\cfrac{\Sigma(P)}{A(P)}}{\forall XA(X)}$$

where P is an n-ary predicate parameter, and it holds for each n-ary second order term T and for each regular set N that

$$\frac{\Sigma(T)}{A(T)}$$

[1] Since we must allow assignments of different sets to different occurrences of the same term, one may prefer to describe the relativization of validity of a derivation Π to an assignment $\mathcal{N}$ as a relativization to a triple $(A,\mathcal{T},\mathcal{N}')$ instead. $\mathcal{T}$ is here an assignment of predicate terms to predicate parameters, $A^{\mathcal{T}}$, i.e. the result of simultaneously substituting these terms for these parameters in A, is the end formula of Π, and $\mathcal{N}'$ is an assignment of regular sets to the same predicate parameters to which $\mathcal{T}$ assigns terms. For instance, if $\mathcal{N}$ is an assignment that assigns N_1 to certain occurrences of T and N_2 to other occurrences of T, then A is formed from the end formula of Π by replacing the first occurrences of T by a predicate parameter P and the other occurrences by a different parameter Q. $\mathcal{T}$ then assigns T to both P and Q but $\mathcal{N}'$ assigns N_1 to P and N_2 to Q.

is strongly valid relative $\mathcal{N}+\binom{T}{N}$, by which is meant the assignment that assigns N to the occurrences of T in $A(T)$ that is substituted for P in $A(P)$ and assigns to other occurrences of second order terms in $A(T)$ the same value (if any) that $\mathcal{N}$ assigns to the corresponding occurrence in $\forall XA(X)$.

2.1.3. *Strong validity under substitution.* We define for derivations in any system $\mathbf{M}_2(\mathbf{S})$: Π is strongly valid under substitution if and only if for each substitution of individual terms for individual variables in Π, for each substitution of n-ary (n=0,1,...) second order terms for n-ary second order parameters in Π, and for each assignment $\mathcal{N}$ of regular sets to occurrences of these substituted terms in the open premisses and end formula of the derivation after the substitution, where the same set is assigned to occurrences that replace the same parameters, it holds: if Π^* is obtained from Π by carrying out these substitutions and then simultaneously replacing open assumptions by derivations in $\mathbf{M}_2(\mathbf{S})$ of the same assumptions that are strongly valid relative $\mathcal{N}$, then Π^* is strongly valid relative $\mathcal{N}$.

2.2. *Results*

The results in A.3.3 can now be extended to second order:

2.2.1. **Theorem.** *Each derivation in* $\mathbf{M}_2(\mathbf{S})$ *is strongly valid under substitution.*

2.2.2. **Theorem.** *If* Π *is a strongly valid derivation in* $\mathbf{M}_2(\mathbf{S})$ *relative as assignment* $\mathcal{N}$ *of regular sets, then each reduction sequence from* Π *terminates.*

The strong normalization theorem now follows for $\mathbf{M}_2(\mathbf{S})$ in the same way as for $\mathbf{M}(\mathbf{S})$.

2.3. *Proofs*

The lemmata A.3.4.1 – A.3.4.3 hold also for second order logic; in lemma 3.4.3, we replace "strong validity" by "strong validity relative $\mathcal{N}$". When proving this last lemma by induction over the definition of strong validity relative $\mathcal{N}$, we note that condition 2.1.1.2 in the definition of regular sets is needed.

We prove theorem 2.2.2 by induction over the definition of strong validity relative $\mathcal{N}$ in the same way as we proved theorem A.3.3.2 in A.3.4.4 except that in the new case corresponding to clause 2.1.2.1, we have to use condition 2.1.1.1 in the definition of regular set, and in the new case corresponding to clause 2.1.2.3, we note that there exists regular sets (e.g. the empty set) and hence that strong validity relative $\mathcal{N}$ also of derivations whose last inference is $\forall_2$I implies strong validity of the derivation of the premiss of the inference relative $\mathcal{N} + \binom{P}{N}$ for some N.

The proof of theorem 2.2.2 has the same structure as the proof of theorem A.3.3.1 in A.3.4.5. The base and the induction step in the case when the last inference is an introduction are again immediate. The lemma in the proof is stated in the same way except that we add λE and $\forall_2$E to condition (ii). There is essentially only one new case that arises, viz. in case (b) in the proof of the lemma, we have to consider the case of $\forall_2$-reductions. In this case, we know that the derivation of the premiss of the last inference of Π, which is of the form

$$\cfrac{\cfrac{\cfrac{\Sigma(P)}{C(P)}}{\forall X C(X)}}{C(T)}$$

is strongly valid relative $\mathcal{N}$ and hence by definition that

$$\frac{\Sigma(T)}{C(T)} \; ,$$

which is just the reduction Π' of Π, is strongly valid relative $\mathcal{N} + \binom{T}{N}$ for any regular N. Since the set of valid derivations relative $\mathcal{N}$ is regular according to theorem 2.2.2 and the analogue to lemma A.3.4.3, we may set N equal to this set. We then apply the following lemma and conclude that Π' is strongly valid also relative $\mathcal{N}$.

Lemma. *Let N be the set of strongly valid derivations relative $\mathcal{N}$. Then,* Π *is strongly valid relative $\mathcal{N}$ if and only if* Π *is strongly valid relative $\mathcal{N} + \binom{T}{N}$, where $\mathcal{N} + \binom{T}{N}$ is an extension of $\mathcal{N}$ that assigns N to certain occurrences of T in the end formula of* Π.

The lemma is proved by a trivial induction over the definition of strong validity relative $\mathcal{N}$.

B.3. Extensions to other systems

Unlike the situation in first order logic, the results for minimal (or intuitionistic) logic in section 2 do not immediately extend to classical logic. The reason for this is that by an $\forall_2$-reduction or an $\exists_2$-reduction, an application of the Λ_C-rule may not any longer satisfy the restriction that we have imposed on this rule, viz. that the conclusion is atomic (I.1.3.3). However, by a simple, separate argument one can show that the derivation can be transformed so that this restriction becomes satisfied (cf. Prawitz (1965), 40–41, 70–71). Alternatively, we can add these transformations as new reductions and then proceed as in section 2. The results thus hold also for classical second order logic.

If we stay in minimal (or intuitionistic) second order logic, we can apply the results of section 2 to show that a rule corresponding to an axiom of choice holds as a rule that preserves derivability (from null assumptions). The rule is

$$\frac{\forall x \exists X A(x,X)}{\exists Y \forall x A(x, \lambda x_1 x_2 \ldots x_n Y x x_1 x_2 \ldots x_n)}$$

where X is an n-ary and Y is an $(n+1)$-ary predicate variable. To see that this rule preserves derivability, assume that the premiss is derivable. From the analogue of lemma II.3.5.4 for second order logic, we can conclude that $A(a,T(a))$ is derivable for some $T(a)$ where $T(a)$ is an n-ary second order term which may contain the parameter a. By use of the λ-rules, we may infer the formula

$$A(a, \lambda x_1 x_2 \ldots x_n \{\lambda x x_1 x_2 \ldots x_n T(x) x_1 x_2 \ldots x_n\} a x_1 x_2 \ldots x_n)$$

from which the conclusion follows by an $\forall$I and $\exists$I.

In agreement with the remark made in A.1.3.3, it is possible to add this rule, which is to be counted as an elimination rule, to the system $\mathbf{M}^2$, define a new reduction, and extend the results of section 2 to this system. The reduction in question is shown below where the second derivation is a reduction of the first derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Sigma}{A(a,T(a))}}{\exists X A(a,X)}}{\forall x \exists X A(x,X)}}{\exists Y \forall x A(x, \lambda x_1 x_2 \ldots x_n Y x x_1 x_2 \ldots x_n)}$$

$$\cfrac{\cfrac{\cfrac{\Sigma}{A(a, \lambda x_1 x_2 \ldots x_n \{\lambda x x_1 x_2 \ldots x_n T(x) x_1 x_2 \ldots x_n\} a x_1 x_2 \ldots x_n)}}{\forall x A(x, \lambda_1 x_2 \ldots x_n \{\lambda x x_1 x_2 \ldots x_n T(x) x_1 x_2 \ldots x_n\} x x_1 x_2 \ldots x_n)}}{\exists Y \forall x A(x, \lambda x_1 x_2 \ldots x_n Y x x_1 x_2 \ldots x_n)}$$

where X is an n-ary and Y an $(n+1)$-ary predicate variable and $T(a)$ is an n-ary second order term that may contain the parameter a. In order that the second figure is to be a correct derivation, the abstraction terms should be understood as abbreviations on the meta-level, i.e. $A(\{\lambda x_1 x_2 ... x_n B(x_1,x_2,...,x_n)\} t_1 t_2 ... t_n)$ is to be understood as denoting $A(B'(t_1,t_2,...,t_n))$ where $B'(t_1,t_2,...,t_n)$ is obtained from $B(t_1,t_2,...,t_n)$ by renaming bound variables if necessary to avoid conflicts (cf. version 1 of the second order systems in Prawitz (1965)).

The proofs in section 2 may be extended without difficulties to this new system with this new reduction. The result is of interest since it shows that in an intuitionistic framework, the axiom of choice does not destroy definability, i.e. the property that $A(T)$ is derivable for some term T if $\exists X A(X)$ is derivable.

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