Given $X{\sim}N(0,1)$, $Y{\sim}N(1,2)$ and Z= $X+Y$,

we have $f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $f_Y(y)=\frac{1}{\sqrt{4\pi}}e^{-\frac{(y-1)^2}{4}}$.

To show $Z{\sim}N(1,3)$, we need to show $f_Z(z)=\frac{1}{\sqrt{6\pi}}e^{-\frac{(z-1)^2}{6}}$.

$$F_Z(z)=P(Z\le z)=P(X+Y\le z)=\int_{-\infty}^{\infty}P(X+Y\le z|X=x)f_X(x)dx$$
$$=\int_{-\infty}^{\infty}P(Y\le z-x)f_X(x)dx=\int_{-\infty}^{\infty}\int_{-\infty}^{z-x}f_Y(y)\,dyf_X(x)dx$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{z-x}f_X(x)f_Y(y)\,dydx$$

$$f_Z(z)=\frac{dF_Z(z)}{dz}=\frac{d}{dz}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{z-x}f_X(x)f_Y(y)\,dydx\right)=\frac{d}{dz}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{z-x}f_Y(y)\,dyf_X(x)dx\right)$$
$$=\int_{-\infty}^{\infty}\frac{d}{dz}(F_Y(z-x))f_X(x)dx=\int_{-\infty}^{\infty}f_Y(z-x)f_X(x)dx$$

$$\int_{-\infty}^{\infty}f_Y(z-x)f_X(x)dx=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi}}e^{-\frac{(z-x-1)^2}{4}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}e^{-\frac{(z-x-1)^2+2x^2}{4}}dx$$
$$=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}e^{-\frac{(z-1)^2-2x(z-1)+3x^2}{4}}dx$$
$$=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2-\frac{2x(z-1)}{3}+\frac{(z-1)^2}{9}-\frac{(z-1)^2}{9}+\frac{(z-1)^2}{3}}{\frac{4}{3}}}dx$$
$$=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{\infty}e^{-\frac{\left[x-\frac{z-1}{3}\right]^2+\frac{2(z-1)^2}{9}}{\frac{4}{3}}}dx=\frac{1}{\sqrt{6\pi}}e^{\frac{(z-1)^2}{6}}\frac{1}{\sqrt{2\pi\left(\frac{2}{3}\right)}}\int_{-\infty}^{\infty}e^{-\frac{\left[x-\frac{z-1}{3}\right]^2}{2\left(\frac{2}{3}\right)}}dx$$
$$=\frac{1}{\sqrt{6\pi}}e^{\frac{(z-1)^2}{6}}$$

as $\frac{1}{\sqrt{2\pi(\frac{2}{3})}}e^{-\frac{[x-\frac{z-1}{3}]^2}{2(\frac{2}{3})}}$ is the pdf of $N(\frac{z-1}{3},\frac{2}{3})$.

We therefore have $Z{\sim}N(1,3)$.