Quiz 2 on October 12, 2021

You may write on both sides of the paper.

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An image can be *blurred* by doing a convolution between a kernel and that image. Mathematically, the convolution of f and g is defined as $(f*g)(x)=\int_{-\infty}^{\infty} f(y)g(x-y)dy$.

Let's consider f as a one-dimensional image and use $g(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$ as the response function to apply a Gaussian blur. Generally speaking, fine features of a length scale smaller than σ will be suppressed and effectively removed by convolving the image f with the Gaussian function g. This can be seen by convolving $f(x)=e^{ikx}$ and comparing it with $(f*g)(x)$ to see how it is suppressed. (a) Derive the explicit expression for $(f*g)(x)$ and find the dependence of its amplitude on k. (b) Draw a sketch of this dependence on k.

Hint: To answer this question, you need the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}, \qquad \int_{-\infty}^{\infty} e^{-a(x+b)^2}dx=\sqrt{\frac{\pi}{a}}.$$

Solution:

$$(f*g)(x)=\int_{-\infty}^{\infty} f(y)g(x-y)dy$$

$$=\int_{-\infty}^{\infty} e^{iky}\cdot\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-y)^2}{2\sigma^2}}dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-x^2}{2\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(y^2-2xy-2\sigma^2 iky)}dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-x^2}{2\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(y^2-2y(x+\sigma^2 ik)+(x+\sigma^2 ik)^2-(x+\sigma^2 ik)^2)}dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-x^2}{2\sigma^2}}\int_{-\infty}^{\infty} e^{\frac{-(y-(x+\sigma^2 ik))^2}{2\sigma^2}}\cdot e^{\frac{(x+\sigma^2 ik)^2}{2\sigma^2}}dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{1}{2\sigma^2}(-x^2+x^2+2x\sigma^2 ik-\sigma^4k^2)}\cdot\int_{-\infty}^{\infty} e^{-\frac{(y-(x+\sigma^2 ik))^2}{2\sigma^2}}dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{ikx-\frac{1}{2}\sigma^2k^2}\cdot\sqrt{2\pi\sigma^2}$$

$$=e^{ikx-\frac{1}{2}\sigma^2k^2}$$

$$h(k)=|(f*g)(x)|=\left|e^{ikx-\frac{1}{2}\sigma^2k^2}\right|=e^{-\frac{\sigma^2k^2}{2}}$$