MSDM5004 Spring 2021
Homework 3 (Part I)
Due Apr. 11

1. Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

is second order both in time and space.

2. Write a code using MATLAB (or some other software) to evaluate the integral using composite trapezoidal rule

$$\int_0^1 e^{-x^2} dx.$$

Please evaluate the integral using different numbers of subintervals to show the convergence of the numerical result.

3. Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = u_0(x) & 0 \le x \le 1 \end{cases}$$

where

$$u_0(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} < x \le 1. \end{cases}$$

(1) The analytical solution is given by

$$u(x,t) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 t} \sin m\pi x, \quad \text{where} \quad a_m = 2\int_0^1 u_0(x) \sin m\pi x dx.$$

Compute the coefficient a_m.

(2) Write a code using MATLAB (or some other software) to obtain numerical solution using the explicit scheme. Use $J = 20$, $\Delta x = 0.05$, and (i) $\Delta t = 0.0012$ (ii) $\Delta t = 0.0013$. Plot the numerical solution and the analytical solution at $t = 0, \Delta t, 25\Delta t, 50\Delta t$ (To plot the analytical solution, you can stop the summation at a large number N, when you cannot see difference in the solution curve if N is increased).

4. Write a code to solve the PDE in problem 3 using the Crank-Nicolson method.

Use $J = 20$, $\Delta x = 0.05$, and (i) $\Delta t = 0.0012$ (ii) $\Delta t = 0.0013$, (iii) $\Delta t = 0.012$. Plot the solutions at $t = 0, \Delta t, 25\Delta t, 50\Delta t$.