Q1.1 solution

$$f(\mathrm{x}) = \frac{e}{2}e^x + \frac{2^{-x}}{4} + \cos(x+1) - 3 = \frac{e^{x+1}}{2} + 2^{-(x+2)} + \cos(x+1) - 3$$

$$f'(\mathrm{x}) = \frac{e^{x+1}}{2} - \ln 2 \cdot 2^{-(x+2)} - \sin(x+1)$$

To solve $f(\mathrm{x}) = 0$ by Newton's method, we have:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_0 = 0.5$$

1st iteration:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{\frac{e^{1.5}}{2} + 2^{-2.5} + \cos(1.5) - 3}{\frac{e^{1.5}}{2} - \ln 2 \cdot 2^{-2.5} - \sin(1.5)} = 0.956$$

2nd iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.956 - \frac{\frac{e^{1.956}}{2} + 2^{-2.956} + \cos(1.956) - 3}{\frac{e^{1.956}}{2} - \ln 2 \cdot 2^{-2.956} - \sin(1.956)} = 0.841$$

3rd iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.841 - \frac{\frac{e^{1.841}}{2} + 2^{-2.841} + \cos(1.841) - 3}{\frac{e^{1.841}}{2} - \ln 2 \cdot 2^{-2.841} - \sin(1.841)} = 0.829$$

4th iteration:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.829 - \frac{\frac{e^{1.829}}{2} + 2^{-2.829} + \cos(1.829) - 3}{\frac{e^{1.829}}{2} - \ln 2 \cdot 2^{-2.829} - \sin(1.829)} = 0.829$$

After 4 iterations, we obtain $x = 0.829$

Q1.2 Matlab code (Newton's method):

```
syms x
f = exp(x+1)/2+2^(-x)/4+cos(x+1)-3
f1 = diff(f)
n=4;
xn=zeros(n+1,1)
xn(1)=0.5
for i=2:n+1
   xn(i)=xn(i-1)-vpa(subs(f,x,xn(i-1)))/vpa(subs(f1,x,xn(i-1)))
end;
```

Q1.2 Matlab code (Secant method):

```
syms x
f = exp(x+1)/2+2^(-x)/4+cos(x+1)-3
n=4;
xn=zeros(n+2,1)
xn(1)=0.5
xn(2)=1
for i=3:n+2
   xn(i)=xn(i-1)-vpa(subs(f,x,xn(i-1)))*(xn(i-1)-xn(i-2))/(vpa(subs(f,x,xn(i-1)))-vpa(subs(f,x,xn(i-2))))
end;
```

Q2 solution

Define

$$F\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}f_1(x_1,x_2)\\f_2(x_1,x_2)\end{pmatrix},$$

where

$$f_1(x_1,x_2) = 4x_1 + 6x_1^2 + 4x_1^3 - 2x_2 - 2$$
$$f_2(x_1,x_2) = -2x_1 + 2x_2 + 2$$

The Jacobian matrix $J(x)$ for this system is

$$J\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}4 + 12x_1 + 12x_1^2 & -2\\ -2 & 2\end{pmatrix}$$

To solve the system by Newton's method, we have:

$$x^{(k)} = x^{(k-1)} - J\left(x^{(k-1)}\right)^{-1} F\left(x^{(k-1)}\right)$$

1st Iteration:

$$x^{(0)} = \begin{pmatrix}0.5\\-0.4\end{pmatrix}$$

$$F\left(x^{(0)}\right) = \begin{pmatrix}4(0.5) + 6(0.5)^2 + 4(0.5)^3 - 2(-0.4) - 2\\ -2(0.5) + 2(-0.4) + 2\end{pmatrix} = \begin{pmatrix}2.8\\2\end{pmatrix}$$

$$J\left(x^{(0)}\right) = \begin{pmatrix}11 & -2\\ -2 & 2\end{pmatrix}$$

$$J\left(x^{(0)}\right)^{-1} = \frac{1}{(11)(2) - (-2)(-2)}\begin{pmatrix}2 & 2\\ 2 & 11\end{pmatrix} = \begin{pmatrix}0.111 & 0.111\\ 0.111 & 0.611\end{pmatrix}$$

$$x^{(1)} = x^{(0)} - J\left(x^{(0)}\right)^{-1} F\left(x^{(0)}\right) = \begin{pmatrix}0.5\\-0.4\end{pmatrix} - \begin{pmatrix}0.111 & 0.111\\ 0.111 & 0.611\end{pmatrix}\begin{pmatrix}2.8\\2\end{pmatrix} = \begin{pmatrix}0.167\\-0.833\end{pmatrix}$$

2nd Iteration:

$$F\left(x^{(1)}\right) = \begin{pmatrix}4(0.167) + 6(0.167)^2 + 4(0.167)^3 - 2(-0.833) - 2\\ -2(0.167) + 2(-0.833) + 2\end{pmatrix} = \begin{pmatrix}0.519\\0\end{pmatrix}$$

$$J\left(x^{(1)}\right) = \begin{pmatrix}3 & -2\\ -2 & 2\end{pmatrix}$$

$$J\left(x^{(1)}\right)^{-1} = \frac{1}{(3)(2) - (-2)(-2)}\begin{pmatrix}2 & 2\\ 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1.5\end{pmatrix}$$

$$x^{(2)} = x^{(1)} - J\left(x^{(1)}\right)^{-1} F\left(x^{(1)}\right) = \begin{pmatrix}0.167\\-0.833\end{pmatrix} - \begin{pmatrix}1 & 1\\ 1 & 1.5\end{pmatrix}\begin{pmatrix}0.167\\-0.833\end{pmatrix} = \begin{pmatrix}-0.352\\-1.352\end{pmatrix}$$

After 2 iterations, we have $x = \begin{pmatrix}-0.352\\-1.352\end{pmatrix}$

Q3 Matlab code (Newton's method):

```
syms x1

syms x2

f1 = 1+x1^2/4-x2^2+exp(x1/2)*cos(x2);

f2 = x1*x2+exp(x1/2)*sin(x2);

F = [f1;f2];

J = [diff(f1,x1),diff(f1,x2);diff(f2,x1),diff(f2,x2)];

n=5;

xn(1,1:2)=[-2 4]

for i=1:n

   Jxninv = inv(vpa(subs(J,[x1,x2],xn(i,1:2))));

   Fxn = vpa(subs(F,[x1,x2],xn(i,1:2)));

   xn(i+1,1:2)=xn(i,1:2)'-Jxninv*Fxn

end;
```

Q4.1 Solution

x	-2	0	1
$f(x)$	1	2	0

The required Lagrange interpolating polynomial is:

$P(x) = L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2)$

where

$$L_0 = \frac{(x-0)(x-1)}{(-2-0)(-2-1)} = \frac{1}{6}x(x-1) = \frac{1}{6}(x^2 - x)$$

$$L_1 = \frac{(x+2)(x-1)}{(0+2)(0-1)} = \frac{-1}{2}(x+2)(x-1) = \frac{-1}{2}(x^2 + x - 2)$$

$$L_2 = \frac{(x+2)(x-0)}{(1+2)(1-0)} = \frac{1}{3}x(x+2) = \frac{1}{3}(x^2 + 2x)$$

$$P(x) = \frac{1}{6}(x^2 - x)(1) - \frac{1}{2}(x^2 + x - 2)(2) + \frac{1}{3}(x^2 + 2x)(0) = \frac{-5}{6}x^2 - \frac{7}{6}x + 2$$

Q4.2 Solution

$$f(-1) \approx P(-1) = \frac{-5}{6}(-1)^2 - \frac{7}{6}(-1) + 2 = \frac{7}{3} = 2.333$$

Q5 Solution

i	0	1	2	3	4	5
x_i	1.0	1.1	1.3	1.5	1.9	2.1
y_i	1.77	1.89	2.14	2.38	2.87	3.11

Let $P(x) = mx + c$ be the required least square polynomial of degree 1

The problem could be represented as:

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1.1 & 1 \\ 1.3 & 1 \\ 1.5 & 1 \\ 1.9 & 1 \\ 2.1 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1.77 \\ 1.89 \\ 2.14 \\ 2.38 \\ 2.87 \\ 3.11 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} m \\ c \end{pmatrix}$

Let $f(m,c) = \|\mathbf{Ax}-\mathbf{b}\|^2 = \sum_{i=0}^{5}(mx_i + c - y_i)^2$ be the sum of squared error of the system

As a minimizer, we have:

$$\begin{cases} \dfrac{\partial f}{\partial m} = \displaystyle\sum_{i=0}^{5} 2x_i(mx_i + c - y_i) = 0 \\ \dfrac{\partial f}{\partial c} = \displaystyle\sum_{i=0}^{5} 2(mx_i + c - y_i) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 14.17m + 8.9c = 22.185 \\ 8.9m + 6c = 14.16 \end{cases}$$

$$\Rightarrow \begin{cases} m = 1.22 \\ c = 0.551 \end{cases}$$

The required polynomial is $P(x) = 1.22x + 0.551$

The error $E = \sum_{i=0}^{5}|1.22x_i + 0.551 - y_i| = 11.171$