Quiz 1 on September 21, 2021

You may write on both sides of the paper.

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1. Consider a one-dimensional random walk starting from the origin $x=0$. The step size is δ, i.e. $x(t+1)-x(t)=\pm\delta$ where $t=0,1,2,3,\cdots$ measures the number of steps completed. Find the probability density function (PDF) of x at $t=N$ when N steps have been completed. Here we assume that the step size δ is very small and the number of steps N is very large, hence the distribution of x can be treated as a continuous one.

Solution: Let n = number of forward steps, $N-n$ = number of backward steps.

After N steps, net moves $= n\delta - (N-n)\delta = (2n-N)\delta$.

Denote $m=(2n-N)\delta$

$$\Rightarrow \quad n=\frac{N+\frac{m}{\delta}}{2} \quad \text{and} \quad N-n=\frac{N-\frac{m}{\delta}}{2}$$

$$P(n) = C^N_n\left(\frac{1}{2}\right)^N = \frac{N!}{n!(N-n)!}\left(\frac{1}{2}\right)^N$$

As N is very large,

$$\ln\frac{N!}{n!(N-n)!} \approx N\ln N - N - n\ln n + n - (N-n)\ln(N-n) + (N-n)$$

$$= N\ln N - n\ln n - (N-n)\ln(N-n)$$

$$= N\ln N - \left(\frac{N+\frac{m}{\delta}}{2}\right)\ln\left(\frac{N+\frac{m}{\delta}}{2}\right) - \left(\frac{N-\frac{m}{\delta}}{2}\right)\ln\left(\frac{N-\frac{m}{\delta}}{2}\right)$$

$$= g(m)$$

$$g'(m) = -\frac{1}{2\delta}\ln\left(\frac{N+\frac{m}{\delta}}{2}\right) - \left(\frac{N+\frac{m}{\delta}}{2}\right)\left(\frac{1}{\frac{N+\frac{m}{\delta}}{2}}\right)\left(\frac{1}{2\delta}\right) + \frac{1}{2\delta}\ln\left(\frac{N-\frac{m}{\delta}}{2}\right) - \left(\frac{N-\frac{m}{\delta}}{2}\right)\left(\frac{1}{\frac{N-\frac{m}{\delta}}{2}}\right)\left(-\frac{1}{2\delta}\right)$$

$$= -\frac{1}{2\delta}\ln\left(\frac{N+\frac{m}{\delta}}{2}\right) + \frac{1}{2\delta}\ln\left(\frac{N-\frac{m}{\delta}}{2}\right)$$

$$= \frac{1}{2\delta}\ln\left(\frac{N-\frac{m}{\delta}}{N+\frac{m}{\delta}}\right) \quad \Rightarrow \quad g'(0) = 0$$

$$g''(m) = \frac{1}{2\delta}\left(\left(\frac{1}{N-\frac{m}{\delta}}\right)\left(-\frac{1}{\delta}\right) - \left(\frac{1}{N+\frac{m}{\delta}}\right)\left(\frac{1}{\delta}\right)\right) = -\frac{1}{2\delta}\left(\frac{1}{N\delta-m} + \frac{1}{N\delta+m}\right)$$

$$\Rightarrow \quad g''(0) = -\frac{1}{2\delta}\left(\frac{1}{N\delta}+\frac{1}{N\delta}\right) = -\frac{1}{N\delta^2} < 0$$

$\Rightarrow$ $g(m)$ has a maximum at $m=0$.

Applying Taylor Expansion to $g(m)$ at $m=0$, we have:

$$g(m) \approx g(0) + g'(0)m + \frac{1}{2}g''(0)m^2 = N\ln N - \frac{N}{2}\ln\frac{N}{2} - \frac{N}{2}\ln\frac{N}{2} - \frac{m^2}{2N\delta^2}$$

$$= N\ln 2 - \frac{m^2}{2N\delta^2}$$

$$\Rightarrow P(m) = e^{g(m)} \left(\frac{1}{2}\right)^N \approx e^{N\ln 2 - \frac{m^2}{2N\delta^2}} \left(\frac{1}{2}\right)^N = e^{-\frac{m^2}{2N\delta^2}}$$

$\therefore$ The PDF of X at $t=N$ is given by:

$$\frac{1}{\sqrt{2\pi N\delta^2}} e^{-\frac{m^2}{2N\delta^2}}$$

2. Consider the stochastic equation of motion $\frac{d}{dt}x(t)=\xi(t)$, in which ξ is a white noise satisfying the autocorrelation $\langle \xi(t_2)\xi(t_1)\rangle = 2D\delta(t_2-t_1)$. (a) Find the mean square displacement $\langle [x(t)]^2 \rangle$ as a function of time t with the initial condition $x(0)=0$. (b) Find the probability density function of x at time t based on $\langle [x(t)]^2 \rangle$ found in (a).

Solution:

$$\frac{d}{dt}x(t) = \xi(t)$$

$$x(t) - x(0) = \int_0^t \xi(t)\,dt$$

$$\langle [x(t)]^2 \rangle = \langle \int_0^t d\tau_1 \int_0^t d\tau_2\, \xi(\tau_1)\xi(\tau_2) \rangle$$

$$= \int_0^t d\tau_1 \int_0^t d\tau_2\, \langle \xi(\tau_1)\xi(\tau_2) \rangle$$

$$= \int_0^t d\tau_1 \int_0^t d\tau_2\, 2D\delta(\tau_2 - \tau_1)$$

$$= \int_0^t d\tau_1\, 2D$$

$$= 2Dt$$

$$\langle x(t) \rangle = \int_0^t \langle \xi(t) \rangle\, dt = 0$$

$\Rightarrow$ the p.d.f. of x at time t is given by $\frac{1}{\sqrt{4\pi Dt}}\, e^{-\frac{x^2}{4Dt}}$