# Reinforcement Learning

## Lecture 3: Markov Decision Processes

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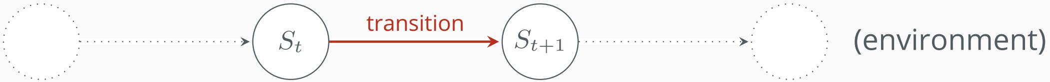
# Lecture Overview

Lecture covers Chapter 3 in Sutton & Barto [3] and uses David Silver's examples [2]

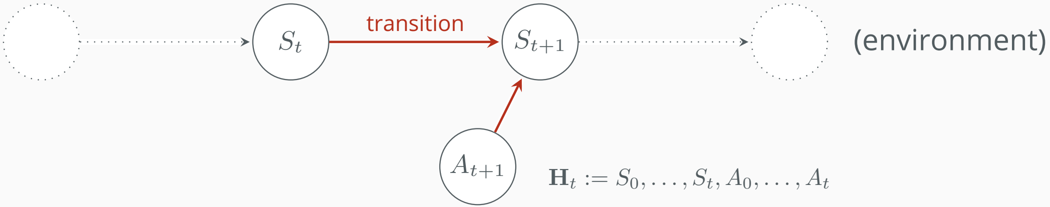
## Markov Process / Markov Chain




$$\mathbf{H}_t := S_0, \dots, S_t$$

Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t}) \doteq p(S_{t+1} \mid \underline{S_t})$

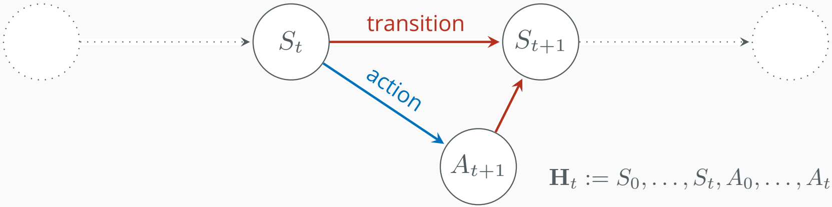
# Overview different Markov processes




$$\mathbf{H}_t := S_0, \dots, S_t, A_0, \dots, A_t$$

Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \mid \underline{S_t, A_{t+1}})$

# Overview different Markov processes



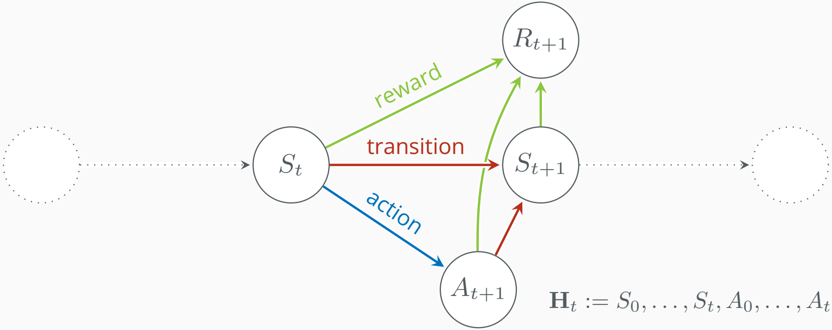

(environment)

$$\mathbf{H}_t := S_0, \ldots, S_t, A_0, \ldots, A_t$$

Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \mid \underline{S_t, A_{t+1}})$

Policy: $p(A_{t+1} \mid \underline{\mathbf{H}_t}) \doteq p(A_{t+1} \mid \underline{S_t})$

## Markov Decision Process (MDP) $\longleftarrow$ we are mainly working with this model




(environment)

$$\mathbf{H}_t := S_0, \dots, S_t, A_0, \dots, A_t, R_0, \dots, R_t$$

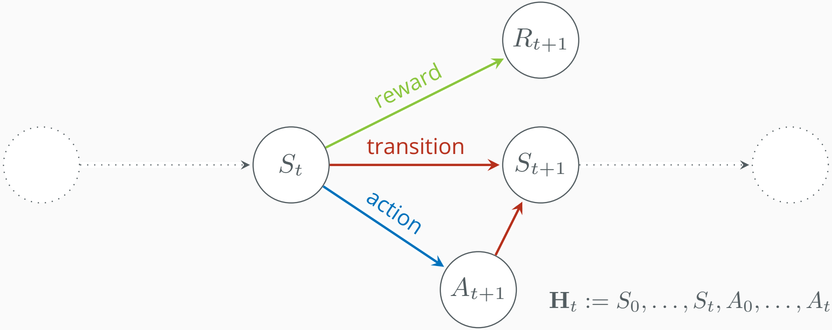
Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \mid \underline{S_t, A_{t+1}})$

Policy: $p(A_{t+1} \mid \underline{\mathbf{H}_t}) \doteq p(A_{t+1} \mid \underline{S_t})$

Reward Function: $p(R_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}, S_{t+1}}) \doteq p(R_{t+1} \mid \underline{S_t, A_{t+1}, S_{t+1}})$

# Overview different Markov processes

## Markov Decision Process (MDP) $\longleftarrow$ we are mainly working with this model




(environment)

$$\mathbf{H}_t := S_0, \ldots, S_t, A_0, \ldots, A_t, R_0, \ldots, R_t$$

Transition Function: $p(S_{t+1} \,|\, \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \,|\, \underline{S_t, A_{t+1}})$

Policy: $p(A_{t+1} \,|\, \underline{\mathbf{H}_t}) \doteq p(A_{t+1} \,|\, \underline{S_t})$

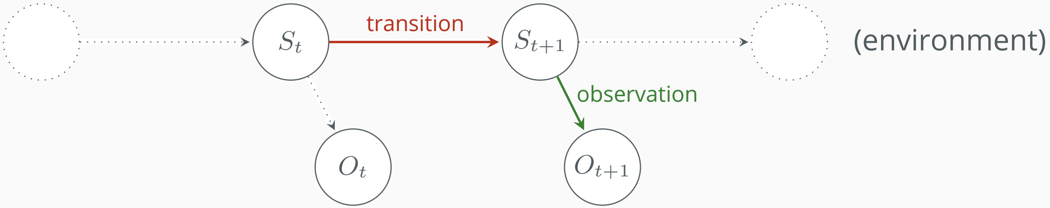
Reward Function: $p(R_{t+1} \,|\, \underline{\mathbf{H}_t, A_{t+1}, S_{t+1}}) \doteq p(R_{t+1} \,|\, \underline{S_t})$

# Overview different Markov processes

## Hidden Markov Model (HMM)

Transition Function: $p(S_{t+1} \,|\, \underline{\mathbf{H}_t}) \doteq p(S_{t+1} \,|\, \underline{S_t})$

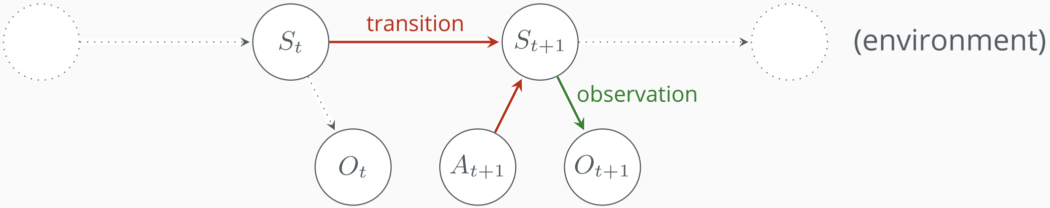
Observation Function: $p(O_{t+1} \,|\, \underline{\mathbf{H}_t}) \doteq p(O_{t+1} \,|\, \underline{S_{t+1}})$



## Partially Observable Markov Decision Process (POMDP)

Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \mid \underline{S_t, A_{t+1}})$

Observation Function: $p(O_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(O_{t+1} \mid \underline{S_{t+1}})$



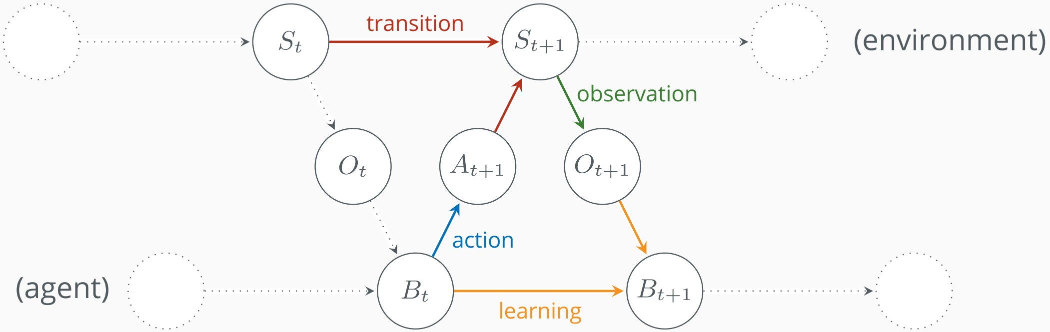
## POMDP / Markov Interaction Process

Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \mid \underline{S_t, A_{t+1}})$

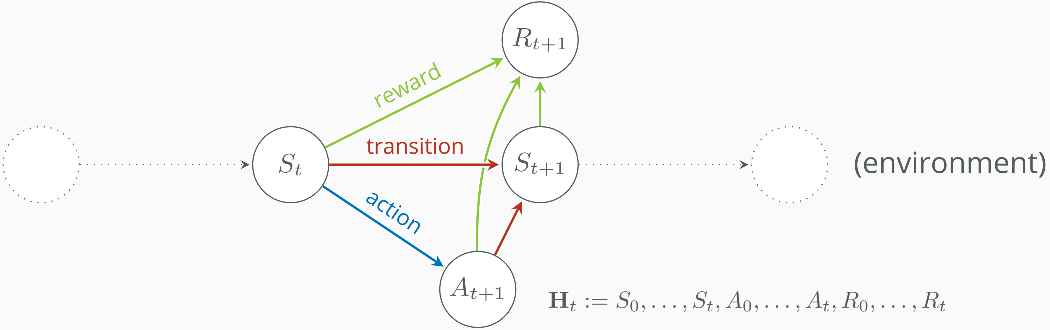
Observation Function: $p(O_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(O_{t+1} \mid \underline{S_{t+1}})$

Policy: $p(A_{t+1} \mid \underline{\mathbf{H}_t}) \doteq p(A_{t+1} \mid \underline{B_t})$

Belief Update: $p(B_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}, O_{t+1}}) \doteq p(A_{t+1} \mid \underline{B_t, O_{t+1}})$



## Markov Decision Process (MDP) $\longleftarrow$ we are mainly working with this model




$$\mathbf{H}_t := S_0, \ldots, S_t, A_0, \ldots, A_t, R_0, \ldots, R_t$$

Transition Function: $p(S_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}}) \doteq p(S_{t+1} \mid \underline{S_t, A_{t+1}})$

Policy: $p(A_{t+1} \mid \underline{\mathbf{H}_t}) \doteq p(A_{t+1} \mid \underline{S_t})$

Reward Function: $p(R_{t+1} \mid \underline{\mathbf{H}_t, A_{t+1}, S_{t+1}}) \doteq p(R_{t+1} \mid \underline{S_t, A_{t+1}, S_{t+1}})$

# Markov Chain markov property recap

With the **Markov property** , we can throw away the history and just use the agent's state:

**Definition:** Markov property

A state $S_t$ is **Markov** if and only if

$$P(S_{t+1} \mid S_t) = P(S_{t+1} \mid S_1, S_2, ..., S_t)$$

- For example, a chess board
  - We don't need to know how the game was played up to this point
- The state fully characterises the distribution over future events:

$$H_{1:t} \rightarrow S_t \rightarrow H_{t+1:\infty}$$

# Markov Chain state transition matrix

The probability of transitioning from state $s$ to $s'$ for a Markov state is:

$$\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s),$$

where the **state transition probability** for all states to all successor states can be expressed as a large matrix:

$$\mathcal{P} = \begin{array}{c} \overbrace{\hphantom{\mathcal{P}_{11} \quad \cdots \quad \mathcal{P}_{1n}}}^{\text{to}} \\ \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \end{array},$$

and each row sums to $1$.

Click to try a demo [1]

# Markov Chain definition

A **Markov chain** (also called Markov Process) is a set of states and a state-transition matrix

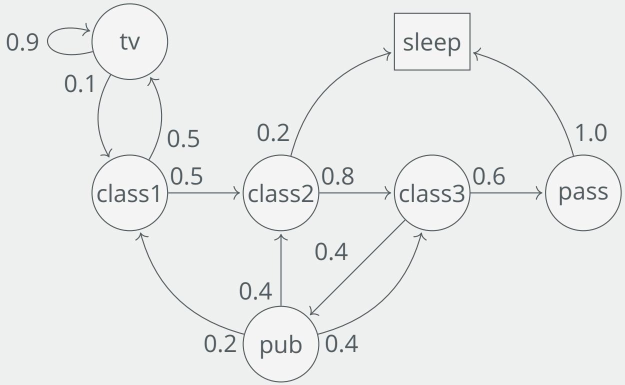
**Definition:** Markov chain

A **Markov chain** is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$

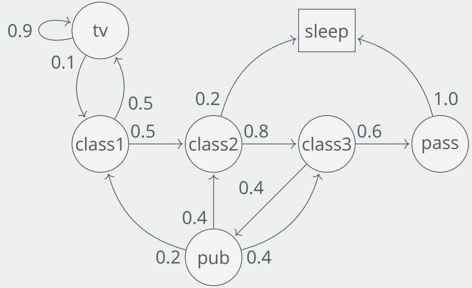
# Markov Chain example

**Example:** Markov Chain




Example from [2]

### **Example:** Markov Chain

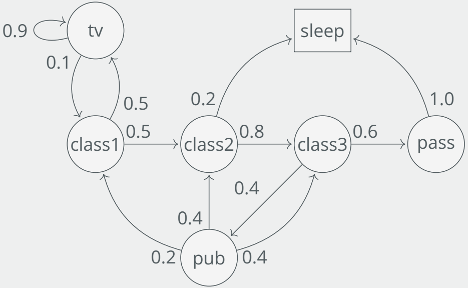



### State Transition Matrix

$$
\mathcal{P} = \begin{array}{c} \\ c1 \\ c2 \\ c3 \\ pass \\ pub \\ tv \\ sleep \end{array}
\begin{array}{c}
\begin{array}{ccccccc} c1 & c2 & c3 & pass & pub & tv & sleep \end{array} \\
\begin{bmatrix}
 & 0.5 & & & & 0.5 & \\
 & & 0.8 & & & & 0.2 \\
 & & & 0.6 & 0.4 & & \\
 & & & & & & 1.0 \\
0.2 & 0.4 & 0.4 & & & & \\
0.1 & & & & & 0.9 & \\
 & & & & & & 1.0
\end{bmatrix}
\end{array}
$$

Example from [2]

## Example: Markov Chain




## Episode

An episode is a varying-length sample of a Markov chain:

$$S_1, S_2, ..., S_T,$$

for example starting from $S_1 =$ class1:

| Episode samples |
|---|
| c1,c2,c3,pass,sleep |
| c1,tv,tv,tv,c1,c2,c3,pub,c2,sleep |

Example from [2]

# Markov Reward Process definition

A Markov **reward** process is a Markov Chain with a **reward** function

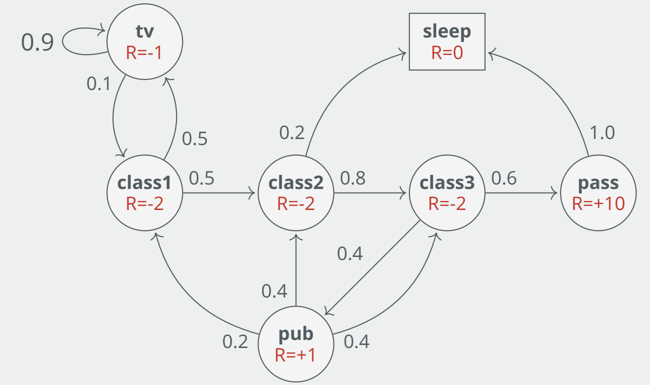
**Definition:** Markov reward process

A **Markov reward process** is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$
- $\mathcal{R}$ is a **reward** function where $\mathcal{R}_s = \mathbb{E}[R_{t+1} \mid S_t = s]$
- $\gamma$ is the **discount** rate $\gamma \in [0, 1]$

# Markov Reward Process example

**Example:** Markov Reward Process




Example from [2]

# Markov Reward Process the return

The **return** $G_t$, in the simplest case, is the total future reward:

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \ldots + R_T$$

In practice, we discount rewards into the future by the *discount rate* $\gamma \in [0, 1]$.

**Definition:** The return

The return $G_t$ is the discounted total future reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

## **Definition:** The state value function

The **state value function** $v(s)$ in an MRP is the long-term value of a state:

$$v(s) = \mathbb{E}[G_t \mid S_t = s],$$

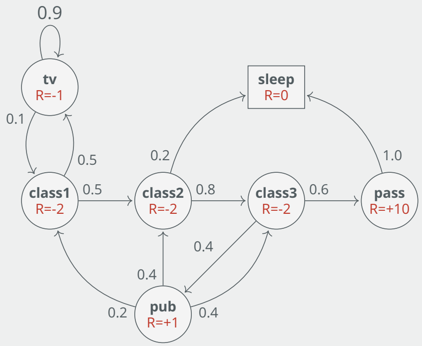
for example calculated by sampling episodes...

| Sample episodes |
|---|
| c1,c2,c3,pass,sleep |
| c1,tv,tv,tv,c1,c2,c3,pub,c2,sleep |
| c1,c2,sleep |
| ... |

## **Example:** Puppy

# Markov Reward Process state value function

## Example: MRP




## Example: The state value function

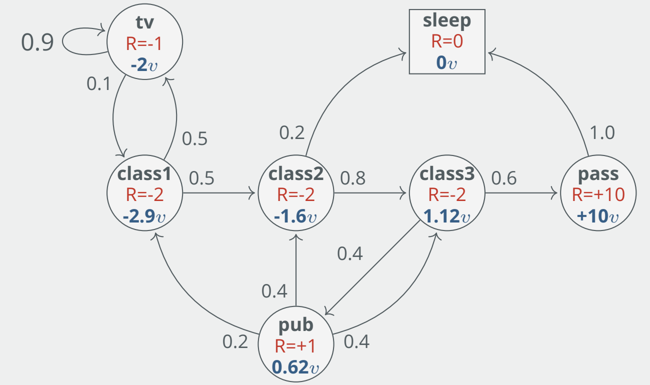
This is an example $v(s)$ with $s =$ 'class1' and $\gamma = \frac{1}{2}$:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots$$
$$= R_{t+1} + \frac{1}{2} R_{t+2} + \frac{1}{4} R_{t+3} + \ldots$$

| Episode samples | Value function |
|---|---|
| c1,c2,c3,pass,sleep | $v_1 = -2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 10 = -2.25$ |
| c1,tv,tv,c1,c2,c3,pub,c2,sleep | $v_1 = -2 - \frac{1}{2} \cdot 1 - \frac{1}{4} \cdot 1 + \frac{1}{8} \cdot \ldots = -3.125$ |
| c1,c2,sleep | $v_1 = -2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 0 + \frac{1}{8} \quad = -3$ |
| ... | $\ldots = -2.9$ |

# Markov Reward Process value function example

**Example:** Markov Reward Process for $\gamma = 0.5$




Example from [2]

# Markov Reward Process the Bellman equation

Through a series of identities, we can decompose the value function into the **immediate reward** $R_{t+1}$ and the discounted **value of the next state** $\gamma v(S_{t+1})$.

**Definition:** Bellman equation for MRP

The Bellman equation is:

$$\begin{aligned} v(s) &= \mathbb{E}[G_t \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \ldots \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \gamma^2 R_{t+4} + \ldots) \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s], \end{aligned}$$

which is equivalent to:

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

The Bellman equation can be expressed with matrices:

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix},$$

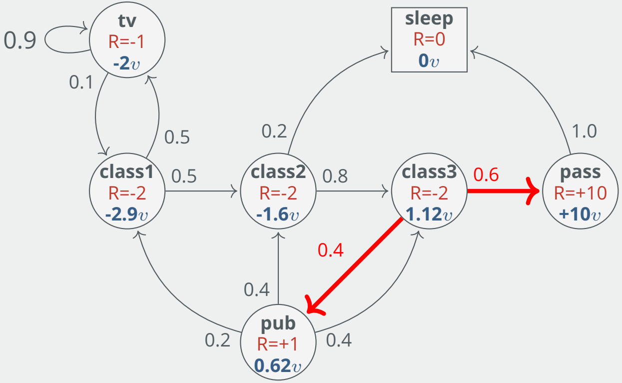
which is a linear equation that can be solved:

$$\begin{aligned} v &= \mathcal{R} + \gamma \mathcal{P} v \\ (\mathrm{I} - \gamma \mathcal{P}) v &= \mathcal{R} \\ v &= (\mathrm{I} - \gamma \mathcal{P})^{-1} \mathcal{R}, \end{aligned}$$

where $\mathrm{I}$ is the identity matrix. Unfortunately this matrix inversion is too slow, except for small MDPs, so we use iterative methods for larger MDP (MC evaluation and TD learning).

**Verification:** MRP for $\gamma = 0.5$

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s') = -2 + 0.5 * (0.6 * \mathbf{10} + 0.4 * \mathbf{0.62}) = \mathbf{1.12}$$



# Markov Decision Process definition

A Markov **decision** process adds 'actions' so the transition probability matrix now depends on which action the agent takes.

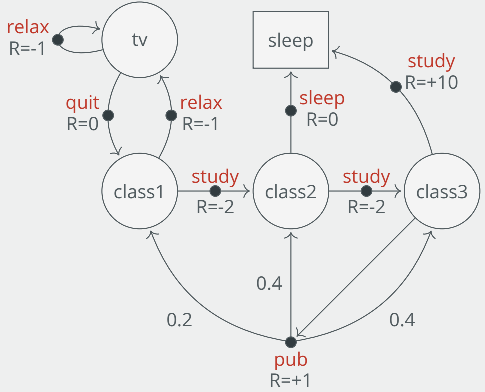
**Definition:** Markov decision process

A **Markov decision process** is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}^a_{ss'} = P(S_{t+1} = s' \mid S_t = s, A_t = a)$
- $\mathcal{R}$ is a **reward** function where $\mathcal{R}^a_s = \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a]$
- $\gamma$ is the **discount** rate $\gamma \in [0, 1]$

# Markov Decision Process example

**Example:** Markov Decision Process




Example from [2]

# Markov Decision Process policies

A policy is a distribution over actions which determines how an agent behaves in the environment.

- A lazy agent will sample relaxing actions more than frequently than studying
- A high-performing agent will study at all classes, then study more at home!

**Definition:** Policy

A policy $\pi$ is a distribution over actions given a state:

$$\pi(a|s) = P(A_t = a \mid S_t = s)$$

**Definition:** The state-value function

The **state-value function** $v_\pi(s)$ is the same, but its the return when following a given policy $\pi$:

$$v_\pi(s) = \mathbb{E}_\pi[G_t \mid S_t = s]$$

**Definition:** The action-value function

The **action-value function** is the long term-value of a state when choosing an action with policy $\pi$:

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$

**Example:** Arizona trail

# Markov Decision Process the Bellman equation

Similarly to MRPs, the state-value function can be decomposed into the immediate reward and the discounted value of the next state:

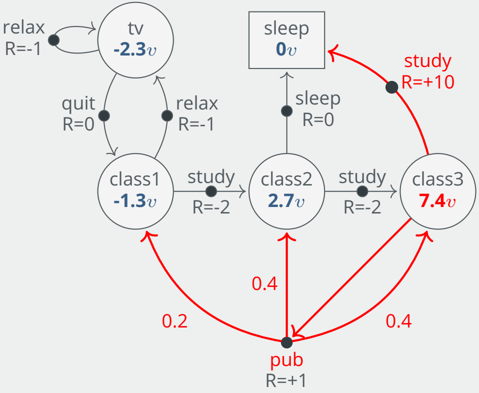
$$
\begin{aligned}
v_\pi(s) &= \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s, a),
\end{aligned}
$$

which is also the case for the action-value function, where:

$$
\begin{aligned}
q_\pi(s, a) &= \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a] \\
&= \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s').
\end{aligned}
$$

# Markov Decision Process example verifying the Bellman equation

## Verification: MDP with average policy




## Verification

Under the policy $\pi$ where we do everything {study,pub} with $50\%$ probability and $\gamma = 1$:

$$
\begin{aligned}
v_\pi(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a) \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) \left( \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right) \\
&= \frac{1}{2} * 10 \\
&+ \frac{1}{2}(1 + 0.2(-1.3v) + 0.4(2.7v) + 0.4(7.4v)) \\
&= 7.4v
\end{aligned}
$$

**Definition:** The optimal state-value function

The **optimal state-value function** $v_*(s)$ is the maximum value function over all policies:

$$v_*(s) = \max_{\pi} v_{\pi}(s)$$

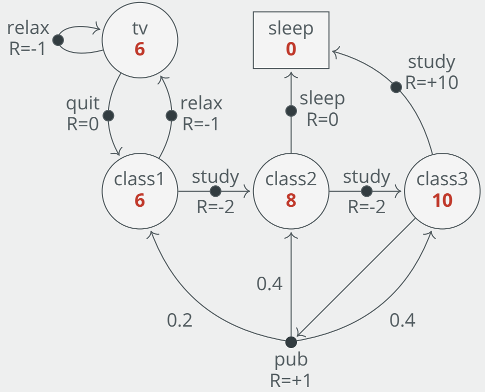
**Definition:** The optimal action-value function

The **optimal action-value function** is the maximum action value function over all policies:

$$q_*(s, a) = \max_{\pi} q_{\pi}(s, a)$$

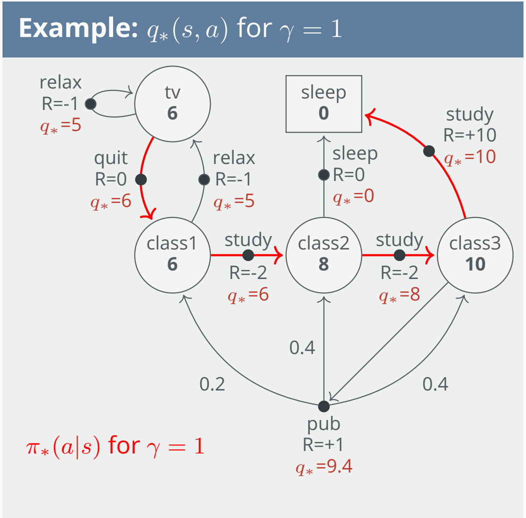
**Example:** Mo Farah

**Example:** $v_*(s)$ for $\gamma = 1$




Example from [2]

# Markov Decision Process optimal action-value and optimal policy




Example from [2]

# Markov Decision Process the Bellman optimality equations for $v*$ and $q_*$

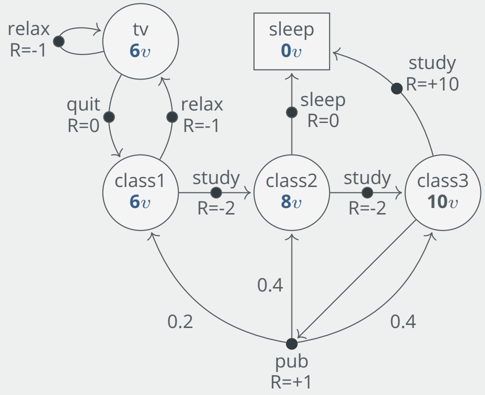
The optimal value functions are similarly recursively related by the Bellman optimality equations, where:

$$
\begin{aligned}
v_*(s) &= \max_{\pi} v_\pi(s) \\
&= \max_{a} q_*(s, a),
\end{aligned}
$$

and the optimal action-value function:

$$
\begin{aligned}
q_*(s, a) &= \max_{\pi} q_\pi(s, a) \\
&= \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s').
\end{aligned}
$$

## Verification: MDP with average policy




## Verification

The optimal state-value for class3 following $\gamma = 1$ requires $q_*$ for the pub action:

$$
\begin{aligned}
v_*(s) &= \max_a q_*(s,a) \\
&= \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s') \\
&= \max \Big\{ 10 + 1 * (0v), \\
&\quad (1 + 0.2(6v) + 0.4(8v) + 0.4(10v)) \Big\} \\
&= \max\{q_* = 10, q_* = 9.4\} \\
&= \mathbf{10}v
\end{aligned}
$$

# References I


[1] V. Powell.
**Markov chains: A visual explanation.**
Available online, 2020.

[2] D. Silver.
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https://www.davidsilver.uk/teaching/, 2015.

[3] R. S. Sutton and A. G. Barto.
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