BEYOND ENDOSCOPY FOR THE RELATIVE TRACE FORMULA II: GLOBAL THEORY

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ABSTRACT. For the group $G = \operatorname{PGL}_2$ we perform a comparison between two relative trace formulas: on one hand, the relative trace formula of Jacquet for the quotient $T \setminus G/T$, where T is a non-trivial torus, and on the other the Kuznetsov trace formula (involving Whittaker periods), applied to non-standard test functions. This gives a new proof of the celebrated result of Waldspurger on toric periods, and suggests a new way of comparing trace formulas, with some analogies to Langlands' "Beyond Endoscopy" program.

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1. Introduction.

1.1. The result of Waldspurger. The celebrated result of Waldspurger [Wal85], relating periods of cusp forms on GL_2 over a nonsplit torus (against a character of the torus, but here we will restrict ourselves to the trivial character) with the central special value of the corresponding quadratic base

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change L-function, was reproven by Jacquet [Jac86] using the relative trace formula. Both of the proofs, however, rely on coincidences that are particular to this case, and do not generalize to most higher-rank cases of the Gross-Prasad conjectures [II10] and their generalizations [GGP12, SV]. In the case of Waldspurger, the coincidence is the appearance of this period in the setting of the theta correspondence; in the case of Jacquet, it is the appearance of the same L-value in periods over a split torus, previously studied by Hecke. None of these coincidences exist in arbitrary higher rank.

The purpose of the present article is to provide yet another proof of the result of Waldspurger, developing a new method, also based on the relative trace formula, which might admit generalization. Although one would need many more examples of this method in order to talk of any serious evidence of possible generalization, some of its features are pleasantly aligned with the formulation of the general conjectures of [SV]. The present paper is based on the local results proven in [Sak13a], and as was the case there, I feel free to use the work of Jacquet (and hence the aforementioned coincidences) in order to shorten proofs of a few local statements that could also be proven "by hand". However, the global argument is completely independent.

The method relies on a "non-standard" comparison of relative trace formulas equipped with "non-standard" test functions and, hence, has similarities to the "Beyond Endoscopy" project of Langlands [Lan04]. Among others, our methods give an independent, trace formula-theoretic proof of the meromorphic continuation of quadratic base change L-functions. It should be kept in mind, though, that our comparisons, spectrally, correspond to some version of relative functoriality [SV] for the identity map of dual groups, as opposed to the much more ambitious goal of Langlands involving arbitrary maps of L-groups.

1.2. The relative trace formula and its conjectural spectrum. The relative trace formula (RTF) of Jacquet should be seen as a potential generalization of the invariant trace formula of Arthur and Selberg, as well as of the twisted trace formula. In the most general setting, one starts with two (homogeneous, quasi-affine) spherical varieties X_1, X_2 for a given (connected) reductive group G over a global field k, and constructs a distribution on the adelic points of $X_1 \times X_2$, invariant under the diagonal action of $G(\mathbb{A}_k)$, which is naively (i.e., ignoring analytic issues) defined as

$$\Phi_1 \otimes \Phi_2 \mapsto \int_{[G]} \Sigma \Phi_1(g) \cdot \Sigma \Phi_2(g) dg,$$

where Σ denotes the morphism

$$S(X_i(\mathbb{A}_k)) \ni \Phi \mapsto \sum_{\gamma \in X_i(k)} \Phi(\gamma g) \in C^{\infty}([G]),$$

¹In fact, this is not quite the most general setting, since we can also introduce line bundles defined by complex adele class characters of unipotent groups – as in the Kuznetsov trace formula used here.

[G] denotes the automorphic quotient $G(k)\backslash G(\mathbb{A}_k)$, and S denotes the space of Schwartz functions (we will call them "standard test functions").

The presentation here is, actually, oversimplifying: instead of considering a $G^{\text{diag}}(\mathbb{A}_k)$ -invariant distribution on the adelic points of $X_1 \times X_2$, one should talk about distributions on the adelic points of the quotient stack $(X_1 \times X_2)/G^{\text{diag}}$. This is necessary, even in the simple cases that we are considering here, in order to include "pure inner forms" of the spaces under consideration into the picture and get a complete comparison between relative trace formulas. The appropriate notions for harmonic analysis on stacks were developed in [Sak16]; however in this paper we will only use the notion of stacks symbolically, and explicitly define the spaces of test functions that we need, without making use of that theory.

In the special case $X_1 = X_2 = a$ group H, under the action of $G = H \times H$ by left and right multiplication, the relative trace formula specializes to the Arthur-Selberg trace formula, while if we twist the action of $\{1\} \times H \subset G$ on the second copy by an automorphism of H we get the twisted trace formula. Notice the stack-theoretic isomorphism of $(H \times H)/G$ (diagonal action of G) with the quotient of H by itself via conjugation.

Let us concentrate on the case $X_1 = X_2 =: X$. The relative trace formula admits a geometric and a spectral expansion. The conjectures proposed in [SV] imply that the *most tempered* part of the spectral expansion is supported on the set of automorphic representations with Arthur parameter (assuming the existence of the hypothetical Langlands group \mathcal{L}_k) of the form

$$\mathcal{L}_k \times \mathrm{SL}_2 \xrightarrow{\varphi \times \mathrm{Id}} {}^L G_X \times \mathrm{SL}_2 \to {}^L G,$$

where LG_X is the "L-group of X". (This was defined in [SV], based on the work of Gaitsgory and Nadler [GN10], only when the group is split; the general case of L-groups for spherical varieties has not been developed yet, although there are many examples where the answer is clear.) The map ${}^LG_X \times \operatorname{SL}_2 \to {}^LG$ is a canonical one (up to conjugacy by the canonical Cartan subgroup of LG). In particular, when X,Y are spherical varieties for groups G,H, respectively, and $r:{}^LG_X \to {}^LH_Y$ is an L-homomorphism between their dual groups, by an extension of Langlands' "Beyond Endoscopy" philosophy this should induce comparisons between their (stable) relative trace formulas. The problem is highly non-trivial already when the dual groups are isomorphic, which is the case at hand here.

The spectral side involves periods of automorphic forms over stabilizers of points on X, and the values of periods are expected in a wide variety of cases to be related to special values of L-functions. More precisely, according to the conjectures of [SV, Chapter 17], under some assumptions on X the contribution of such a parameter φ to the spectral expansion will involve a certain quotient $L_X(\varphi)$ of special values of automorphic L-functions associated to the L-group LG_X , whose Euler factor at an unramified place v is related to the Plancherel measure for $L^2(X_v)$ (normalized in some canonical

way that will not be explained here). In the group case one has $L_X = 1$, which is why one does not see L-functions on the spectral side. Comparisons between relative trace formulas give rise to relations between periods and the associated "relative characters", in exactly the same way that character relations arise in the endoscopic comparison of trace formulas.

1.3. The limitations of standard comparisons. The geometric side of the relative trace formula is, roughly (i.e., ignoring "pure inner forms") a sum of orbital integrals over G(k)-orbits on $(X_1 \times X_2)(k)$; these orbits correspond, at least generically and in the stable case, to k-points on the "base" $\mathcal{B} = (X_1 \times X_2) /\!\!/ G := \operatorname{spec} k[X_1 \times X_2]^G$. The latter is very often an affine space, so summation over G(k)-orbits becomes some kind of "Poisson sum".

Experience suggests that when the dual groups and the related L-values for two relative trace formulas match, then usually there is a natural matching of the geometric sides: a map between stable, closed rational orbits, and an identification of $G(\mathbb{A}_k)$ -coinvariants of the spaces of standard test functions on $(X_1 \times X_2)(\mathbb{A}_k)$ with the corresponding space of the second RTF which preserves (up to scalar "transfer factors") the corresponding orbital integrals. "Standard test functions", by definition, are generated by functions of the form $\prod_v \Phi_v$, where Φ_v is the characteristic function of $X_i(\mathfrak{o}_v)$ for almost every place v (where \mathfrak{o}_v denotes the integers of k_v), and an arbitrary Schwartz function at remaining places.

While there are many known cases of such comparisons, it is clear from the multitude of different L-functions attached to spherical varieties (s. the table at the end of [Sak13b]) that one cannot hope that every RTF will have one or more "matching" ones, in the above sense. For example, no relative trace formula comparison has been proposed for attacking the Gross-Prasad conjecture for the space $X = SO_n^{\text{diag}} \setminus (SO_n \times SO_{n+1})$. On the other hand, it is clear that the relative trace formula provides the correct setting for understanding the – still far from being understood – fine points behind the general period conjecture of [SV], such as how many elements inside of an L-packet contribute to the spectral expansion. For example, in the case of Gross-Prasad or Whittaker periods there is, locally, only one distinguished representation inside of each (Vogan) L-packet, which corresponds to the fact that the corresponding relative trace formulas are stable (i.e., there is, at least generically, no distinction between orbits and stable orbits). The appearance of pure inner forms in the conjectures can also be understood in terms of the relative trace formula, and more precisely in terms of the quotient stack $(X \times X)/G$, cf. [SV, §16.5], [Sak16].

 $^{^2}$ We are really referring to stable or "quasi-stable" trace formulas here, e.g., in the case of the Arthur-Selberg trace formula the individual summands of the invariant trace formula which are matched with stable trace formulas of endoscopic groups; these summands can be considered as "quasi-stable" trace formulas with their own L-group, namely the corresponding endoscopic L-group.

This poses the dilemma: How can the relative trace formula be on one hand fundamental for the correct statement of the conjectures but on the other hand insufficient for their proof? My hope is that the answer will be given through a "non-standard" comparison of relative trace formulas, in the setting where dual groups match but the associated L-values are unequal. In order to set up an equality between two such trace formulas one has to replace the characteristic function of $X_i(\mathfrak{o}_v)$ by a non-standard unramified function which (as in Langlands' "Beyond Endoscopy" program) will force a correction by suitable L-values on the spectral side. Again as in the "Beyond Endoscopy" program, one should not expect an orbit-by-orbit comparison of the geometric sides in this case. Instead, the transfer factors will be integral operators between the two spaces of orbits; let us call them "transfer operators". The biggest conceptual difficulty here, as I see it, is to show that these operators are automorphic, i.e., that they preserve "Poisson sums".

This is not the first time that such a "non-standard" comparison has been performed. Rudnick's thesis [Rud90], predating the "Beyond Endoscopy" program by more than a decade, can be seen as such a comparison between the Petersson–Kuznetsov formula and the Selberg trace formula for GL₂ (restricted to holomorphic cusp forms). The success and similarity of these different cases provides grounds for optimism.

1.4. A non-standard comparison. In this article I employ the ideas above in the simplest possible case, namely the comparison of the Kuznetsov trace formula (i.e., the one associated to the Whittaker period) with the relative trace formula of Jacquet [Jac86] for the case $X = T \setminus G$. In both cases, G is the group PGL₂, and T denotes a non-trivial torus in it, split or nonsplit. In the split case the corresponding relative trace formula has never appeared in print to the best of my knowledge, and is quite interesting analytically.

The local comparisons of trace formulas needed here were performed in the article [Sak13a], including a local matching theorem [Sak13a, Theorem 5.1] and a fundamental lemma [Sak13a, Theorem 5.4]. Here I show that they induce a global equality of relative trace formulas, and perform the spectral analysis. The end result is a new proof (see Theorem 7.4.1) of the celebrated result of Waldspurger on toric periods with trivial character on the torus:

1.4.1. **Theorem** ([Wal85]). Let $\pi \hookrightarrow L^2([G'])$ be a cuspidal automorphic representation of G', an inner form of G, and write it as a restricted tensor product $\pi = \otimes'_v \pi_v$ of unitary representations of $G'(k_v)$. Let T be a nonsplit torus in G', splitting over a quadratic extension with associated idele class character η . We endow the groups with the Tamagawa measures, and we endow π with the norm induced from $L^2([G'])$. Then, for $\phi = \otimes_v \phi_v \in \pi$,

$$\left| \int_{[T]} \phi(t) dt \right|^2 = \frac{1}{2} \prod_{v}' \int_{T(k_v)} \langle \pi_v(t) \varphi_v, \varphi_v \rangle dt, \tag{1.1}$$

where the Euler product, outside of a finite set of places, should be understood as a partial L-value; the corresponding Euler factors are

$$\frac{\zeta_v(2)L(\pi_v, \frac{1}{2})L(\pi_v \otimes \eta_v, \frac{1}{2})}{L(\eta_v, 1)^2 L(\pi_v, \text{Ad}, 1)}.$$
(1.2)

The determination, in terms of epsilon factors, of whether π_v is G_v -distinguished, which is a result of Tunnell and Saito, can also be obtained from this relative trace formula-based approach, see Theorem 7.5.2, although we need to rely on the proof of Jacquet, combined with Jacquet–Langlands [JL70], in order to verify that the root numbers are the correct ones. The proof includes an independent proof of the meromorphic continuation of the base change L-function $L(\pi, \frac{1}{2} + s)L(\pi \otimes \eta, \frac{1}{2} + s)$, see Corollary 5.3.4, in the spirit of "Beyond Endoscopy".

The case of a split torus amounts to a classical result of Hecke, and will also be included in the results of this paper; in that case, the "Tamagawa measure" on [T] (induced, formally, from a k-rational volume form) has to be multiplied by a factor of $\zeta_v(1)$ at almost every place, and, correspondingly, for those places the factor $L(\eta_v, 1)^2 = \zeta_v(1)^2$ will disappear from the denominator of (1.2), ensuring that the result is formally the same. Notice that the factor $\frac{1}{2}$ appears because we normalize the inner product on π according to the inner product on $L^2([PGL_2])$; if, instead, we were using the norm of the space $L^2(GL_2(k)\backslash GL_2(\mathbb{A}_k)^1)$, where $GL_2(\mathbb{A}_k)^1$ denotes the elements whose determinant has adelic absolute value 1, then this factor would not appear.

The input for proving this formula is the analogous formula for Whittaker periods, cf. [LM15, Theorem 4.1] or [SV, Theorem 18.3.1], where the local L-factors $L(\pi_v, \frac{1}{2})L(\pi_v \otimes \eta_v, \frac{1}{2})$ from the numerator of (1.2) are missing. Whittaker periods appear on the spectral side of the Kuznetsov trace formula, and the missing L-factors will be "added" to the Kuznetsov formula. Classically, this corresponds to a series of Kuznetsov formulas according to the Dirichlet series of this product of L-functions, as, for example, in the thesis of Rudnick [Rud90]. Adelically, it corresponds to using a space of "non-standard" Whittaker functions, strictly larger than the usual Schwartz space, that was explained in [Sak13a, §4.5]. At almost every place, this space contains a distinguished unramified "basic vector", corresponding to generating function of the unramified Euler factor of the above L-function.

To go from the formula for Whittaker periods to the formula for torus periods, we need to compare two "trace formulas": the relative trace formula for $T\backslash G/T$, and the Kuznetsov trace formula for G, corresponding to the two-sided quotient of G by a unipotent subgroup N equipped with a non-trivial character ψ . Symbolically, we write $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ for the space of orbital integrals of test functions for $T\backslash G/T$ (thinking of $T\backslash G/T$ as an algebraic stack \mathcal{Z}), and similarly $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$ for the space of orbital integrals of (our non-standard) test functions for the Kuznetsov formula (with the symbol \mathcal{W} coming from "Whittaker"). The invariant-theoretic quotients $T\backslash G/T$ and

 $N\backslash G /\!\!/ N$ are both affine lines, to be denoted by \mathcal{B} (for "base"). Choosing appropriate coordinates, and a trivialization of the line bundle corresponding to the character ψ in the case of the Kuznetsov formula, we can think of the spaces $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ and $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$ of orbital integrals as functions on the adelic points of a Zariski dense open subset of "regular" points on the affine line. (This subset is different for \mathcal{Z} and \mathcal{W} .)

The non-standard matching theorem and a fundamental lemma between those spaces of test functions were proved in [Sak13a], using a certain "transfer operator", that is, an explicit linear isomorphism

$$|\bullet|\mathcal{G}:\mathcal{S}(\mathcal{Z}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{W}(\mathbb{A}_k))$$
 (1.3)

obtained from local isomorphisms under which (at non-Archimedean places) the elements corresponding to orbital integrals of the "basic vectors" correspond, as well as the orbital integrals of their convolutions by the same element of the unramified Hecke algebra of PGL_2 .

In this article, I compare the global "trace formulas". I will consider the relative trace formula for the quotient $\mathcal{Z} = T \backslash G/T$ as a functional on the space $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$, denoted by RTF, and similarly the Kuznetsov formula as a functional on $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$, denoted by KTF. I take the geometric sides of the trace formulas as the definitions of these functionals; these are, essentially, the Poisson sums of the above functions of orbital integrals over the "base" \mathcal{B} :

$$RTF(f) \text{ or } KTF(f) := \sum_{\xi \in \mathcal{B}(k) = k} f(\xi). \tag{1.4}$$

The above expression has to be taken with a grain of salt, as the evaluations of f at singular points of the base have to be interpreted appropriately. The precise definitions are given in (2.5), (2.6).

As mentioned previously, now that we have a "transfer operator" \mathcal{G} instead of scalar transfer factors, the comparison of trace formulas cannot simply be obtained by the triviality of some scalars when evaluated on rational elements. Rather, to show that the transfer operator \mathcal{G} preserves the sums (1.4) amounts to a Poisson summation formula. What makes the argument work in this case is that the transfer operator is explicitly described as a consecutive application of Fourier transforms and birational maps on the base, thus "in principle" allowing for an application of the (usual) Poisson summation formula for Fourier transform. This is the first "miracle" on which the global method is based, and it eventually leads to a proof of equality between the two trace formulas. The word "eventually" conceals a lot of analytic deviousness: since elements of $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k)), \mathcal{S}(\mathcal{W}(\mathbb{A}_k))$ are not defined as functions but in a meager set of the adeles (namely, the adelic points of a Zariski open subset), the Poisson summation formula for Fourier transform cannot be applied directly. I give an overview of the technicalities involved in the next subsection.

Relative trace formulas are averages over automorphic representations, and it is quite straightforward in the case of standard comparisons to isolate those (if one has the fundamental lemma for Hecke algebras) in order to obtain a representation-by-representation (or at least packet-by-packet) comparison. However, isolating the representations is much deeper in the case of our non-standard comparison, as the Kuznetsov formula for nonstandard test functions is not given by a convergent sum, but by analytic continuation. This is to be expected, of course, since the L-value that we inserted is not given by a convergent Euler product. Separating representations amounts to showing that this analytically continued formula is still a measure on the space of Satake parameters, which could be established by appealing to analytic estimates for the L-function $L(\pi, \frac{1}{2} + s)L(\pi \otimes \eta, \frac{1}{2} + s)$. However, such an approach would beat the purpose, in view of possible applications of this method in higher rank. Indeed, one would like to extract properties of the pertinent L-functions from the relative trace formula, and not vice versa. Thus, I follow a different approach that is made possible by appealing to a second "miracle"; the existence of another explicit transform (also satisfying some form of Poisson summation) between certain spaces of orbital integrals; this transform is essentially a reflection of the functional equation of this L-function at the level of orbital integrals.

While this article introduces several analytic methods which might work quite generally, the most important question that needs to be resolved, in my opinion, is of algebraic nature: What makes the aforementioned two miracles possible, that is: why are the comparison of trace formulas and the functional equation represented by Fourier transforms and birational maps at the level of orbital integrals? Can these miracles be generalized to higher rank? Clearly, we will not know the answer to these questions before more examples of non-standard RTF comparisons are examined.

1.5. Poisson summation formula. Making the principle "Fourier transforms and birational maps (i.e., the constituents of the transfer operator \mathcal{G}) preserve Poisson summation" into a proof, namely, proving the formula

$$RTF(f) = KTF(| \bullet | \mathcal{G}f), \text{ for all } f \in \mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$$
 (1.5)

involves a good deal of adelic analysis, which occupies the first part of this article. The problem is that the elements of the spaces $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ and $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$ of orbital integrals are only defined on a Zariski dense open subset \mathcal{B}^{reg} of the base \mathcal{B} (in our case, the affine line). (The Zariski open subset \mathcal{B}^{reg} is different for each of the two RTFs that we are considering.) Locally, the subset $\mathcal{B}^{\text{reg}}(k_v)$ of $\mathcal{B}(k_v)$ is dense; globally, its adelic points, however, are of measure zero in the adelic points of \mathcal{B} .

A first approach to the Poisson summation formula would be to replace the local factors outside of a finite set of places S by standard Schwartz functions on the affine space $\mathcal{B}(k_n)$; this will make them honest functions on a subset of $\mathcal{B}(\mathbb{A}_k)$ of full measure, and one could hope to take a limit with S.

However, this still does not work, because it leads to logarithmically divergent terms. The solution lies in "deforming" the spaces $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$, $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$ of orbital integrals. In our case, the key fact is that the germs of orbital integrals (locally) at the "irregular" points of $\mathcal{B}(k_v)$ have singularities which behave like generalized eigenfunctions for the multiplicative group of a local parameter. For instance, they may be of the form

$$C_1(\xi)\log|\xi|_v + C_2(\xi),$$

where C_1, C_2 are smooth functions, and ξ is a coordinate for $\mathcal{B}(k_v)$; this is a generalized eigenfunction for the multiplicative group k_v^{\times} , of degree two. We may continuously deform its eigencharacter so that our function obtains the form $D_1(t) + D_2(t)|\xi|_v^t$, where $t \in \mathbb{C}$ is a parameter; this eliminates some logarithmic divergence and opens the way for an application of the Poisson summation formula (when $\Re(t) \gg 0$). (The precise forms of germs and their deformations are described in §4.1.)

To demonstrate this argument, and establish several useful facts that we need, we first prove a Poisson summation formula for a "baby case" in sections 2 and 3, where the relative trace formulas get replaced by the quotient of a quadratic extension by the group of its elements of norm one. This is a good "infinitesimal" model for our theory, and proving some theorems in this setting makes it easier to understand the argument, and saves us from a lot of heavy notation. However, for the comparison of relative trace formulas that we are interested in, things are more complicated because the L-values inserted in the Kuznetsov trace formula are not represented by convergent Euler products. Analytically, this is reflected by the fact that the orbital integrals $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$ of our non-standard test functions for the Kuznetsov formula are not of sufficiently fast decay at infinity, thus the sum (1.4) does not converge. Therefore, in section 4 we vary the L-values that we insert in the Kuznetsov formula with a second parameter s, constructing a family of spaces of non-standard orbital integrals $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$, which for $s \in \mathbb{C}$ is tailored to produce the L-value $L(\pi_v, \frac{1}{2} + s)L(\pi_v \otimes \eta_v, \frac{1}{2} + s)$ (at least at unramified places). This is to be compared with a similar deformed space $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$ for the torus RTF, which is not interpreted as a space of orbital integrals, but just as a space of functions on $\mathcal{B}^{\text{reg}}(\mathbb{A}_k)$, specializing to our space of orbital integrals when s=0. For $\Re(s)\gg 0$, the technique described above proves a Poisson summation formula:

$$RTF(f) = KTF(| \bullet |^{s+1}\mathcal{G}f), \tag{1.6}$$

s. Theorem 4.7.1.

On the side of $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$, now, our test functions are of rapid decay, and the Poisson sum (1.4) converges for arbitrary s. This proves the analytic continuation of the Kuznetsov formula for non-standard test functions given by an arbitrary value of s. The next goal is to isolate the contributions of

individual L-packets to (1.6). The notion of (global) L-packets is used in analogy to the local L-packets of Vogan [Vog93], and is due to the fact that the "stack" $\mathcal{Z} = T \setminus G/T$ includes contributions from inner forms of G. Conveniently, global packets in our case are determined by strong multiplicity one: automorphic representations for PGL_2 and its inner forms belong to the same packet if and only if they are locally equivalent almost everywhere.

1.6. Spectral decomposition. The fact that the Kuznetsov trace formula for s=0 is only described as the analytic continuation of some expression which converges for large values of $\Re(s)$ makes the spectral analysis much more complicated than in usual trace formula comparisons. However, it is important to stress that going beyond the domain of convergence does not rely on hard analytic number theory: for the sum over automorphic representations that constitutes the Kuznetsov formula, it is a direct outcome of the above argument, where the Kuznetsov sum is equated to the analogous Poisson sum for the deformation $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$ of the space of orbital integrals of the torus trace formula.

Nonetheless, this just proves the analytic continuation of a weighted average of L-functions over all automorphic representations, and is not enough to separate the equality of trace formulas representation-by-representation. The problem here is that while in usual RTFs the spectral expansion (or the interesting part of it) is absolutely convergent and hence, under the action of the Hecke algebra on test functions, a measure on the set of Satake parameters, here it is not a priori so. To exhibit it as a measure on the set of Satake parameters (s. the theorems stated in §5.3), and to obtain period relations for each individual packet out of the equality (1.6) between the two trace formulas, we use what was before called the "second miracle", which is a reflection of the functional equation of $L(\pi, \frac{1}{2} + s)L(\pi \otimes \eta, \frac{1}{2} + s)$ at the level of orbital integrals. This is an explicit linear isomorphism

$$\mathcal{T}: \mathcal{S}(\mathcal{W}^{-s}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k)) \tag{1.7}$$

which again satisfies the "fundamental lemma for all elements of the unramified Hecke algebra" and preserves Poisson sums, s. Theorem 5.5.1. An application of the Phragmén-Lindelöf principle now allows us to bound the Kuznetsov formula as a functional on the Hecke algebra inside of the critical strip for the L-function, and to isolate terms with different Satake parameters in the equality (1.6). This spectral analysis is performed in sections 5 and 6.

1.7. The result of Waldspurger. Finally, in section 7 I use this packet-by-packet comparison that is obtained from the previous section to deduce the result of Hecke and Waldspurger, Theorem 1.4.1, on toric periods with trivial character.

The starting point is the packet-by-packet identity obtained from (1.6), which has the form

$$\mathcal{J}_{\varphi}(f) = \mathcal{I}_{\varphi}(|\bullet|\mathcal{G}f).$$

Here φ denotes a collection of Hecke eigenvalues outside of a finite set of places, corresponding to a generic automorphic representation. The distributions \mathcal{J}_{φ} and \mathcal{I}_{φ} are "relative characters" (also called spherical characters or Bessel distributions) obtained from the period functionals

$$F \mapsto \int_{[T]} F(t)dt$$

and

$$F \mapsto \int_{[N]} F(n)\psi^{-1}(n)dn$$

on automorphic forms. The operator $|\bullet|\mathcal{G}$ is the transfer operator, as before. On the Kuznetsov side, one has a well-known Euler factorization into local functionals, that I alluded to in §1.4, which has the form

$$\mathcal{I}_{arphi} = rac{1}{2} \prod_{v}' I_{arphi_v},$$

where the Euler product is not literally convergent (which is why it is denoted by \prod'), but can be interpreted using partial L-functions.

This gives an Euler factorization of the relative character \mathcal{J}_{φ} for the torus period, but it is quite indirect: the local factors are described as pull-backs, via the transfer operators, of the local functionals I_{φ_v} on the Kuznetsov side. To obtain Waldspurger's result, one needs to describe them intrinsically as T_v -biinvariant distributions on G_v (or an inner form). This is a usual problem with the relative trace formula: what is the transfer of relative characters? Obtaining the answer is often quite involved (see, for example, [Zha14]).

It is very encouraging that this method seems tailored to give the correct local factors without any complicated arguments. Namely, both I_{φ_v} and the desired factors J_{φ_v} can be characterized in terms of the Plancherel formula for the pertinent homogeneous spaces, and their matching via the transfer operators is an immediate consequence of the fact that transfer operators preserve L^2 -inner products! Thus, the present method is directly fitted to the framework of the general period conjecture [SV, Conjecture 17.4.1].

1.8. Relation to other methods. The two "miracles" that make the method of this paper possible, both of local nature, are reflections at the level of orbital integrals of methods that have been used before to prove the same final result. I explained this briefly in [Sak13a, section 5], and explain it again in the proof of the "second miracle" in §6.5; this fact is used to avoid local calculations – calculations, to be sure, that can in a straightforward albeit tedious way be performed directly. In a nutshell, orbital integrals for the relative trace formula for $T \setminus G/T$ are equal to orbital integrals for the relative trace formula $A \setminus G/(A, \eta)$ by the work of Jacquet [Jac86], where A is a split torus and η is the quadratic character associated to the splitting field of T, and the "unfolding" method of Hecke provides a passage from $C_c^{\infty}(A \setminus G)$

(or $C_c^{\infty}(A\backslash G, \eta)$) to Whittaker functions (non-compactly supported). This passage descends, roughly, to our "transfer operator" $|\bullet|\mathcal{G}$ at the level of orbital integrals.

Because of this fact, I do not know if this method can be generalized to higher rank – where the methods of Jacquet and Hecke certainly do not generalize. I certainly hope so, and if the adequacy of the relative trace formula for expressing the fine details of distinction, or the direct relevance of this method to the general Plancherel-theoretic setting are any indication, one has reasons to be optimistic. After the first version of this paper was written and submitted, I noticed that this method can be extended to prove the full result of Waldspurger, with a character on the torus, while Jacquet's comparison of [Jac86] cannot. (Jacquet eventually used an entirely different comparison in [Jac87] to address the general case.) In any case, one needs to examine many more examples of non-standard comparisons, which is something I plan to do in the near future.

In any case, it is important to remark that we make absolutely no use of the methods of Jacquet and Hecke in global arguments. The equality between the two relative trace formulas is obtained by completely independent means, namely the Poisson summation formula that we described above. Given that the local calculations can also be done "by hand", our method is completely self-contained. We also don't make use of any hard facts about L-functions, except for the meromorphic continuation and polynomial growth in bounded vertical strips of partial abelian L-functions.

1.9. Relation to "Beyond Endoscopy". The "Beyond Endoscopy" project of R. Langlands [Lan04] is a very ambitious project aiming at proving functoriality to its full extent. The vision, very simplistically, is to compare, for any embedding ${}^LG_1 \to {}^LG_2$ of L-groups, the stable trace formula for G_1 with that part of the stable trace formula of G_2 which corresponds, spectrally, to the expected lift of representations. To isolate the latter, one uses non-standard test functions³ in the trace formula of G_2 to introduce suitable L-functions on the spectral side; and one hopes to be able to calculate residues that will "capture" functorial lifts.

I repeat that the above is a very simplistic presentation of the proposed project. However, even the smallest steps give rise to tremendous difficulties. In particular, a lot of effort has been focused on obtaining analytic continuation of the expressions obtained when introducing L-functions. Some of the papers doing this for various L-functions include [Ven04, Her11, Whi14]; in particular, [Whi14] treats the same L-functions that we treat in this paper, showing analytic continuation in a strip beyond the domain of convergence

³In classical language, these non-standard test functions correspond to "series of trace formulas"; for example, some version of the Kuznetsov trace formula can isolate the n-th Fourier coefficient of automorphic forms on GL_2 , and one takes a weighted sum over n corresponding to the Dirichlet series expressing the desired L-function in terms of Fourier coefficients.

of the Euler product. In applications to functoriality, one usually faces the problem of isolating individual representations in a trace formula comparison, which has not been successfully resolved: a common recourse is estimates for L-functions obtained by other methods [Ven04], something which clearly should be avoided for completion of the project. An exception is [Her12], where the analytic continuation of the standard L-function for GL_2 is obtained by an analog of the functional equation at the level of orbital integrals; this is similar to the method we use here.

Comparing the present paper to the above methods, several similarities and differences should be observed:

- I also use non-standard test functions to introduce *L*-functions into the trace formulas.
- However, I obtain the full analytic continuation of these L-functions without any hard analysis; rather, the method is more conceptual, and relies on being able to compare the Kuznetsov formula with non-standard test functions depending on a parameter s (and convergent for $\Re(s) \gg 0$) with a deformation of another relative trace formula with standard test functions, which therefore has meromorphic continuation for every s.
- Here, as in Beyond Endoscopy, the comparisons are not via scalar transfer factors, but by a series of Fourier transforms and other operations, i.e., by *Poisson summation*. Poisson summation has been used in [FLN10, Alt15] to isolate the contribution of the trivial representation, and in various other references to obtain estimates for the "series of trace formulas" that allow continuation beyond the domain of convergence. However, to the best of my knowledge, a full comparison of trace formulas using Poisson summation formulas has not appeared before, with the notable exception of Rudnick's thesis [Rud90].
- Despite these similarities in method, it should be emphasized that the type of "relative functoriality" that I prove here is closer in spirit to endoscopy rather than "Beyond Endoscopy". Indeed, if one admits the point of view that I alluded to above, namely: each invariant trace formula is a sum of its "quasi-stable" parts, and each has its own L-group, then endoscopy is a matter of comparing quasi-stable trace formulas with the same L-group. Here, too, I am comparing the relative trace formula for $T\backslash G/T$ with the Kuznetsov trace formula for $G(=PGL_2)$; both are stable, and their L-group is SL_2 . In contrast, the goal of Langlands is to "extract" from a given stable trace formula the contribution of a smaller L-group. While this is far from my scope, the non-standard comparisons that I introduce here, and hope to study in the future, may give some indication of more conceptual ways to proceed with the desired comparisons of "Beyond Endoscopy".

- 1.10. Acknowledgements. I would like to thank Akshay Venkatesh for pointing my attention to his thesis [Ven04] as a possible source of ideas for attacking the period conjectures of [SV]. I would also like to thank Joseph Bernstein, who taught me the correct way to think about several aspects of the relative trace formula. Finally, I am very grateful to the anonymous referee for a careful reading and numerous small corrections and suggestions. This work was supported by NSF grants DMS-1101471 and DMS-1502270.
- 1.11. **Notation.** Some of the notation is local, redefined in every section; for example, X and Y are reserved for varieties which change throughout the text. Here we give a summary of the symbols that are used globally; more notation, used in the second part of the paper, is introduced and summarized in §5.1.
 - k is a global field with ring of adeles \mathbb{A}_k , E a quadratic etale extension, hence either a quadratic field extension of k or the ring $k \oplus k$. The ring of integers of k at a non-Archimedean place v will be denoted by \mathfrak{o}_v , its residue field degree by q_v and a uniformizing element by ϖ_v . We will denote $E_v = E \otimes_k k_v$, and at non-Archimedean places \mathfrak{o}_{E_v} will denote the ring of integers of E_v . The quadratic idele class character associated to E is denoted by η . If it is clear from the context that ξ is an element of k_v^{\times} , We feel free to write $\eta(\xi)$ for the evaluation of η via the embedding $k_v^{\times} \hookrightarrow \mathbb{A}_k^{\times}$; but sometimes, for emphasis, we write η_v instead. The same comment holds for absolute values, as well as zeta- and L-functions: we write $L(\eta_v, s)$ or $L_v(\eta_v,s)$ when we want to emphasize that we are referring to the local factors, etc. The usual, unnormalized, absolute values which satisfy the product formula are being used on the completions k_n ; thus, for non-Archimedean places the absolute value is q_v raised to the opposite of the valuation, while for complex places the absolute value is the square of the usual one. For a variety X over k, we denote $X_v := X(k_v)$.
 - We fix throughout a complex character ψ of \mathbb{A}_k/k and a factorization $\psi = \prod_v \psi_v$, such that outside of a finite set of places the conductor of ψ_v is the ring of integers of k_v .
 - We fix the standard Tamagawa measure dx on \mathbb{A}_k , together with the factorization: $dx = \prod_v dx_v$ into self-dual measures with respect to the characters ψ_v . For non-Archimedean places unramified over \mathbb{Q}_p or $\mathbb{F}_p(t)$, when the conductor of ψ_v is \mathfrak{o}_v this measure is such that $dx_v(\mathfrak{o}_v) = 1$.
 - Fourier transform on k_v is defined as:

$$\mathcal{F}(\Phi)(x) = \hat{\Phi}(x) := \int_{k_v} \Phi(y) \psi_v^{-1}(xy) dy, \tag{1.8}$$

where dy and ψ_v are the aforementioned measure and character.

• The "transfer operator" 4 \mathcal{G} is defined between certain spaces of densely defined functions on k_v , considered as tempered distributions (sometimes by analytic continuation). It is given by:

$$\mathcal{G} = \mathcal{F} \circ \iota \circ \mathcal{F}$$
.

where \mathcal{F} is the Fourier transform (1.8), and ι is the transformation:

$$\iota f(x) = \frac{\eta_v(x)}{|x|_v} f\left(\frac{1}{x}\right).$$

- The action of the multiplicative group k_v^{\times} on functions on k_v is normalized in (3.2) in order to be unitary on L^2 ; this normalization makes Fourier transform on k_v anti-equivariant with respect to the action of k_v^{\times} .
- A finite set S of places of k will always, implicitly, include all Archimedean places, all places which are ramified over \mathbb{Q}_p or $\mathbb{F}_p((t))$, and all finite places where ψ_v does not have the ring of integers as its conductor, together with any other places specified in the text. Statements about "almost all" places will, implicitly, exclude such a set S. The S-integers of k will be denoted by k_S .
- For a finite set S of places, expressions of the form $\zeta^S(s)$, $L^S(\eta, s)$, $L^S(\pi, s)$ will denote the partial L-functions indicated (i.e., the L-functions with the Euler factors at places of S omitted). We will use $(\zeta^S(s))^*$, $(L^S(\eta, s))^*$, etc. to denote the leading term in the Laurent expansion of this partial L-function at s.
- For p-adic groups, the usual notion of "smooth" vectors and representations typically gives rise to LF vector spaces, i.e., strict inductive limits of Fréchet spaces. To achieve uniformity with the Archimedean case, I described in [Sak13a, Appendix A] a notion of "almost smooth" vectors which gives rise to Fréchet space representations. For uniformity of presentation, we work with such Fréchet spaces both in the Archimedean and non-Archimedean cases, calling these vectors (by abuse of language) "smooth"; however, the reader can ignore this and focus on smooth vectors in the traditional sense, replacing the Fréchet spaces that we consider with the corresponding LF spaces of their smooth vectors.
- S generally denotes spaces of test functions or their orbital integrals. Unless otherwise stated, for a smooth variety Y we denote by $S(Y_v)$ the space of Schwartz functions on $Y_v = Y(k_v)$, namely the space of rapidly decaying, smooth functions on Y_v . Here, again, at non-Archimedean places one can consider the usual LF space of locally constant, compactly supported functions, or the Fréchet space of "almost smooth", rapidly decaying functions, defined in [Sak13a,

⁴This is the transfer operator for the "baby case". For the comparison between the two relative trace formulas, the transfer operator is $| \bullet | \mathcal{G}$, as in (1.3).

Appendix A]. The reader can choose to consider either of the two, but for uniformity of language we will be talking about Fréchet spaces.

The spaces denoted by S are always sections of "Schwartz cosheaves" in the language of [Sak13a, Appendix B]. Thus, for example, $S(k_v^{\times})$ denotes smooth functions that vanish faster than positive and negative powers of x both close to 0 and close to ∞ .

- The notion of "stalk" for Schwartz cosheaves over a closed (semial-gebraic) subset $Z \subset Y_v$ is again that of [Sak13a, Appendix B]: by definition, the stalk is the quotient of sections over $Y_v \setminus Z$. For example, two Schwartz functions have the same germ over Z if they differ by a Schwartz function on the complement of Z.
- For certain families of Fréchet spaces parametrized by a complex parameter we introduce in Appendix A notions of "polynomial families of seminorms" and of "sections of polynomial growth/rapid decay". These notions always refer to their behavior as the parameter varies in bounded vertical strips.
- The action of an element h in the Hecke algebra of smooth, compactly supported measures on a p-adic or real group G on a vector v in a smooth representation is denoted by $h \star v$; we denote by h^{\vee} the linear dual of h, i.e., $h^{\vee}(g) = h(g^{-1})$. (No topology on the full Hecke algebra will be used in this paper, and we will only need locally constant, compactly supported measures at non-Archimedean places, but again one can instead consider the Fréchet Hecke algebra of "almost smooth" Schwartz measures, both at Archimedean and non-Archimedean places.)
- G denotes the group PGL_2 over k. We let $[G] = G(k) \setminus G(\mathbb{A}_k)$ (and similarly for other groups), and $[G]_{\emptyset} = [G]_{\emptyset} = A(k)N(\mathbb{A}_k) \setminus G(\mathbb{A}_k)$, where B = AN is a Borel subgroup of G with a Levi decomposition, with N the unipotent radical of B.
- The (smooth) principal series representation of G_v unitarily induced from a character χ_v of a Borel subgroup $B(k_v)$ is denoted by $I(\chi_v)$; that is, $I(\chi_v)$ is the space of smooth function on G_v satisfying: $f(bg) = \chi_v \delta^{\frac{1}{2}}(b) f(g)$ for every $b \in B_v$ and $G \in G_v$, where δ denotes the modular character of the Borel (i.e., the quotient of right by left Haar measure).
- T denotes a k-torus in G, associated to the quadratic extension E that we fixed before; that is, T is split if $E = k \oplus k$, and it splits over E, if E is a field. It can be identified with the quotient $\mathrm{Res}_{E/k}\mathbb{G}_m/\mathbb{G}_m$, or with the group of norm one elements of $\mathrm{Res}_{E/k}\mathbb{G}_m$. For an isomorphism class β of T-torsors (over k or a completion k_v), we let T^{β} , G^{β} denote the isomorphism classes of the groups $\mathrm{Aut}_T(R^{\beta})$ and $\mathrm{Aut}_G(G \times^T R^{\beta})$, where R^{β} is a representative of β ; here and throughout, \times^T denotes the quotient of the product

of the two varieties by the (free) diagonal action of T. Of course, T^{β} is isomorphic to T, while G^{β} is an inner form of G, and such a realization (fixing R^{β}) gives rise to an embedding: $T^{\beta} \hookrightarrow G^{\beta}$.

- \mathcal{Z} is used as a symbol for the quotient stack $T \setminus G/T$. The stack-theoretic point of view is not necessary for reading this paper; what matters is the space $\mathcal{S}(\mathcal{Z}_v)$ of orbital integrals for the $T \setminus G/T$ -relative trace formula at a place v of k. This space was introduced in [Sak13a], s. references in §2.1. I note that it encodes orbital integrals not only for a single pair (G,T), but of a class of such pairs parametrized by the first Galois cohomology group of T (i.e., by the set of isomorphism classes of T-torsors).
- Similarly, W is a symbol for the stack $N \setminus G/N$, but equipped with a line bundle that is determined by the non-degenerate character ψ of $N(\mathbb{A}_k) \simeq \mathbb{A}_k$, where N is identified with \mathbb{G}_a over k. Again, the stack-theoretic point of view is not necessary, and instead what matters is a space $\mathcal{S}(W_v)$ of orbital integrals (s. again 2.1). These are orbital integrals of a space of non-standard Whittaker functions tailored to produce the L-value $L(\pi_v, \frac{1}{2})L(\pi_v \otimes \eta_v, \frac{1}{2})$. This space will also be denoted by $\mathcal{S}(W_v^0)$, where $\mathcal{S}(W_v^s)$ is, more generally, a space of orbital integrals of Whittaker functions tailored to produce the L-value $L(\pi_v, \frac{1}{2} + s)L(\pi_v \otimes \eta_v, \frac{1}{2} + s)$.
- the L-value $L(\pi_v, \frac{1}{2} + s)L(\pi_v \otimes \eta_v, \frac{1}{2} + s)$.

 Similarly, \mathcal{X} is a symbol for the stack $\operatorname{Res}_{E/k}\mathbb{G}_a/T$, where T, E are as above, with T now identified with the group of elements of E^{\times} of norm one. This is the "baby case" an infinitesimal version of $T \setminus G/T$ discussed in sections 2 and 3, and again it is not the stack-theoretic point of view that matters, but the associated space $\mathcal{S}(\mathcal{X}_v)$ of orbital integrals.
- \mathcal{B} denotes one-dimensional affine space, the "base" of our quotient stacks $\mathcal{X}, \mathcal{Z}, \mathcal{W}$. In each of these cases, \mathcal{B} is identified with the associated invariant theoretic quotient, except that in the Kuznetsov case (\mathcal{W}) we also invert the variable. (So, the invariant-theoretic quotient spec $k[N\backslash G]^N$ is isomorphic to the affine line, but identified with $\mathbb{G}_m \cup \{\infty\}$, with $\mathbb{G}_m \subset \mathcal{B}$, while in the other cases $\mathcal{B} \simeq \operatorname{spec} k[T\backslash G]^T \simeq \operatorname{spec} k[\operatorname{Res}_{E/k}\mathbb{G}_a]^T$ through specific isomorphisms that we fix.)

In each of these cases, there is an open dense subset \mathcal{B}^{reg} of \mathcal{B} (different in each case), which is identified with the "regular" set of the corresponding quotient. We will be using this notation when it is clear which quotient space we are referring to, and the notation $\mathcal{B}^{\text{reg}}_{\mathcal{F}}$, $\mathcal{B}^{\text{reg}}_{\mathcal{F}}$, etc. when we want to indicate the quotient space.

 $\mathcal{B}_{\mathcal{Z}}^{\text{reg}}$, $\mathcal{B}_{\mathcal{W}}^{\text{reg}}$, etc. when we want to indicate the quotient space. The Schwartz spaces of orbital integrals $\mathcal{S}(\mathcal{X}_v)$, $\mathcal{S}(\mathcal{Z}_v)$, $\mathcal{S}(\mathcal{W}_v)$ are identified with functions on the pertinent open subset $\mathcal{B}_v^{\text{reg}} = \mathcal{B}^{\text{reg}}(k_v)$ of regular points on the base.

Part 1. Poisson summation

2. Generalities and the baby case.

2.1. Global Schwartz spaces. If v is any place of k, I introduced in [Sak13a] certain local "Schwartz spaces" of measures and functions on a dense open subset \mathcal{B}^{reg} (depending on the case considered) of $\mathcal{B}(k_v) = k_v$, denoted (here with the appropriate subscript v) by

$$\mathcal{M}(\mathcal{X}_v), \ \mathcal{M}(\mathcal{Z}_v), \ \mathcal{M}(\mathcal{W}_v^s),$$

for the measures, resp.

$$\mathcal{S}(\mathcal{X}_v), \ \mathcal{S}(\mathcal{Z}_v), \ \mathcal{S}(\mathcal{W}_v^s)$$

for the functions. The reader may restrict their attention to the quotient \mathcal{X} and then read the treatment of the "baby case" in this and the next section, before returning here for the general definitions.

All of these spaces are obtained as push-forwards of measures, resp. regular orbital integrals, for certain quotients: the first is for the quotient (symbolically written \mathcal{X}) of a two-dimensional quadratic space by $T = \mathrm{SO}_2$ (or, equivalently, a two-dimensional etale algebra over our base field, divided by the kernel of the norm map) [Sak13a, §2.4, 2.5, 2.9], the second (denoted \mathcal{Z}) for the quotient $(T \setminus G \times T \setminus G)/G = T \setminus G/T$, where $G = \mathrm{PGL}_2$ [Sak13a, §3.5], and the third for the quotient associated to the Kuznetsov trace formula for PGL_2 , but equipped with non-standard test functions depending on a parameter s, as explained in [Sak13a, §4.3, 4.5, 4.6, 6.1]. When s = 0, we will also be using the notation $\mathcal{S}(\mathcal{W}_v)$, without the s-exponent. All of these spaces are Fréchet and, more precisely, sections of "Schwartz cosheaves" over $\mathbb{P}^1(k_v)$, in the language of [Sak13a, Appendix B].

Let us for now use the symbol \mathcal{Y} to stand for either of \mathcal{X}, \mathcal{Z} or \mathcal{W}^s . The spaces of measures are more canonical (since they are obtained as pushforwards of measures "upstairs", although there are some choices involved in the case of \mathcal{W}^s in order to trivialize the bundles associated to a character of a unipotent subgroup), but in any case we have defined, by choosing suitable measures on $\mathcal{B}(k_v)$, linear isomorphisms

$$\mathcal{M}(\mathcal{Y}_v) \xrightarrow{\sim} \mathcal{S}(\mathcal{Y}_v),$$
 (2.1)

These isomorphisms are not important for the present article, since we will not be working with measures.

Now assume that the above quotients are defined over a global field k. That is, there is a quadratic etale algebra E/k and, letting T = the kernel of the norm map from E to k, we have $\mathcal{X} = \operatorname{Res}_{E/k}\mathbb{G}_a/T$, and $\mathcal{Z} = T \setminus \operatorname{PGL}_2/T$, where T has been embedded in PGL_2 . For the Kuznetsov quotient \mathcal{W}^s , we need to identify a unipotent subgroup N of PGL_2 with \mathbb{G}_a (over k), and then use the additive adele class character ψ and its factorization that we fixed in the notation section 1.11.

We will now define global Schwartz spaces as restricted tensor products of these local Schwartz spaces. For this, we will need to choose a "basic vector" at almost every place. It would seem more natural at first, for $\mathcal{Y} = \mathcal{X}$ or \mathcal{Z} , to choose the characteristic measure of the integral points of the space "upstairs", and take its image in $\mathcal{S}(\mathcal{Y}_v)$ to be the basic vector. For example, for $\mathcal{Y} = \mathcal{X} = \mathrm{Res}_{E/k}\mathbb{G}_a$, this vector at a non-Archimedean place v would be the image in $\mathcal{M}(\mathcal{X}_v)$ of the probability Haar measure on the ring of integers \mathfrak{o}_{E_v} . Let us temporarily call this the "characteristic measure of $\mathcal{Y}(\mathfrak{o}_v)$ ". (The case $\mathcal{Y} = \mathcal{W}^s$ should be treated separately.)

One can see that the restricted tensor product $\otimes'_v \mathcal{M}(\mathcal{Y}_v)$ with respect to this "basic vector" is a vector space of well-defined measures on $\mathcal{B}(\mathbb{A}_k)$. However, these push-forwards are not absolutely continuous with respect to Haar measure on $\mathcal{B}(\mathbb{A}_k) = \mathbb{A}_k$, and hence do not naturally give rise to functions on that space. Therefore, this basic vector is not well-suited for evaluation at rational points.

Instead, we will be working with the spaces $\mathcal{S}(\mathcal{Y}_v)$ of Schwartz functions and define, in all three cases, a "global Schwartz space of functions"

$$S(\mathcal{Y}(\mathbb{A}_k)) := \widehat{\otimes}_v' S(\mathcal{Y}_v), \tag{2.2}$$

defined with respect to some "basic vectors/functions" $f_{\mathcal{X}_v}^0$, $f_{\mathcal{Z}_v}^0$, $f_{\mathcal{W}_v^s}^0$.

For $\mathcal{Y} = \mathcal{X}$ or \mathcal{Z} , the basic vector is *not* the function corresponding under (2.1) to the characteristic measure of $\mathcal{Y}(\mathfrak{o}_v)$; instead it will be taken with respect to the multiple thereof, characterized by the the property:

$$f_{\mathcal{Y}_v}^0|_{\mathcal{B}^{\mathrm{reg}}(\mathfrak{o}_v)} \equiv 1.$$
 (2.3)

This way, elements of the restricted tensor product can be considered as functions on $\mathcal{B}^{\text{reg}}(\mathbb{A}_k)$. Explicitly, the basic vector for $\mathcal{Y}_v = \mathcal{X}_v$ and \mathcal{Z}_v is obtained from the $T(k_v)$ -orbital integrals of the characteristic function of \mathfrak{o}_{E_v} , resp. $T \setminus \text{PGL}_2(\mathfrak{o}_v)$, with the measure on $T(\mathfrak{o}_v)$ normalized to be equal to 1. In the case of \mathcal{Z}_v , if we think of it as the quotient of $(T \setminus G)^2$ by the diagonal action of G, then the basic vector $f_{\mathcal{Z}_v}^0$ is obtained from the $G(k_v)$ -orbital integrals of the characteristic function of $(T \setminus G)^2(\mathfrak{o}_v)$ with the measure on $G(\mathfrak{o}_v)$ equal to 1.

I remind that coordinates have been chosen in [Sak13a] so that the regular sets are $\mathcal{B}_{\mathcal{X}}^{\text{reg}} = \mathcal{B} \setminus \{0\}$ and $\mathcal{B}_{\mathcal{Z}}^{\text{reg}} = \mathcal{B} \setminus \{0, -1\}$. Notice that $\mathcal{B}^{\text{reg}}(\mathbb{A}_k)$ is a set of additive measure zero in $\mathcal{B}(\mathbb{A}_k)$, and that elements of these restricted tensor products do not make sense as functions on $\mathcal{B}(\mathbb{A}_k)$.

For $\mathcal{S}(\mathcal{W}_v^0)$ the basic vector $f_{\mathcal{W}_v^s}^0$ will be a multiple of what was called "basic vector" and denoted by f_v^s in [Sak13a, §6.2, (6.3)]. I postpone the precise definition of this scalar (which does not depend on s), and point the reader to §4.5. The value of $f_{\mathcal{W}_v^s}^0$ on $\mathcal{B}_{\mathcal{W}}^{\text{reg}}(\mathfrak{o}_v) = \mathfrak{o}_v^{\times}$ is that of the function $f_{4,v}^{0,0}$ in table (4.15). The Euler product of these regular values does not make sense unless $\Re(s) \gg 0$, which is enough for us because it is only for such values of s that the Kuznetsov trace formula will be expressible as a

convergent sum. For s=0, the basic vector $f_{\mathcal{W}_v^s}^0$ is the image of the basic vector $f_{\mathcal{Z}_v}^0$ under the transfer operator

$$|\bullet|\mathcal{G}:\mathcal{S}(\mathcal{Z}_v) \xrightarrow{\sim} \mathcal{S}(\mathcal{W}_v^s)$$

that we will recall later.

Notice that for a finite number of factors, there is a canonical notion of completed tensor product, since the spaces are nuclear. The infinite, restricted tensor product over all places is therefore an LF topological vector space.

2.2. Relative trace formulas and their comparison. The relative trace formula for each of the cases $\mathcal{Y} = \mathcal{X}, \mathcal{Z}, \mathcal{W}^s$ will be a functional

$$\mathcal{S}(\mathcal{Y}(\mathbb{A}_k)) \to \mathbb{C}$$

defined, roughly, as "summation over k-points of the base \mathcal{B} ". We are not ready to give a self-contained definition, but we will define it with references to the following sections.

In the baby case $\mathcal{Y} = \mathcal{X}$, where $\mathcal{B}_{\mathcal{X}}^{\text{reg}} = \mathcal{B} \setminus \{0\}$, the functional is defined as

$$f \mapsto \sum_{\xi \neq 0} f(\xi) + \tilde{O}_0(f), \tag{2.4}$$

where \tilde{O}_0 denotes an "irregular orbital integral" to be defined in §2.4.

In the torus case, $\mathcal{Y} = \mathcal{Z}$, where $\mathcal{B}_{\mathcal{Z}}^{\text{reg}} = \mathcal{B} \setminus \{0, -1\}$, the functional will be denoted by RTF and is defined as

$$RTF(f) = \sum_{\xi \neq 0, -1} f(\xi) + \tilde{O}_0(f) + \tilde{O}_{-1}(f).$$
 (2.5)

Because the local behavior of orbital integrals around $\xi = 0, -1$ is the same as in the baby case around zero (for $\xi = -1$, just change the variable ξ to $\xi + 1$), the definition of \tilde{O}_0 and \tilde{O}_{-1} is completely analogous to that of \tilde{O}_0 in the baby case and will not be given separately.

Finally, in the Kuznetsov case $\mathcal{Y} = \mathcal{W}^s$ the functional will be denoted by KTF. The definition here cannot be given by an explicit sum except when $\Re(s) \gg 0$. For other values of s, it will be obtained by analytic continuation in §4.7. For $\Re(s) \gg 0$ we define:

$$KTF(f) = \sum_{\xi \in k^{\times}} f(\xi) + \tilde{O}_0(f) + \tilde{O}_{\infty}(f).$$
 (2.6)

The irregular orbital integral at 0 is defined in the same way as for the previous cases, and the one at ∞ is defined in §4.3.

The goal of the first part of the paper, except for proving the analytic continuation of KTF to s=0, is to show that our "transfer operators" preserve these functionals. More precisely, in the baby case we have the transfer operator (which will be recalled in §2.4):

$$\mathcal{G}: \mathcal{S}(\mathcal{X}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{X}(\mathbb{A}_k)).$$

Between the Schwartz spaces of \mathcal{Z} and \mathcal{W} , we have according to [Sak13a] the "transfer operator":

$$| \bullet | \cdot \mathcal{G} : \mathcal{S}(\mathcal{Z}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{W}(\mathbb{A}_k)).$$

In both cases, our main goal is to show that these transfer operators preserve the corresponding relative trace formulas; this amounts to a "Poisson summation formula" for the transfer operators.

2.3. **Preliminaries on tori.** We recall a few facts about Galois cohomology: By Kottwitz's interpretation of Tate-Nakayama duality [Kot84], for every torus T over k and any place v of k we have isomorphisms:

$$H^{1}(k_{v},T) = \pi_{0} \left(\check{T}^{\Gamma_{v}}\right)^{*}, \tag{2.7}$$

where \check{T} denotes the connected dual torus of T, Γ_v is the decomposition group at v of the absolute Galois group Γ of k, π_0 denotes the connected components and star denotes the Pontryagin dual. Moreover, the restriction maps

$$\pi_0 \left(\check{T}^{\Gamma_v}\right)^* \to \left(\check{T}^{\Gamma}\right)^*$$

give rise to an exact sequence

$$1 \to \ker^{1}(k,T) \to H^{1}(k,T) \to \prod_{v}' H^{1}(k_{v},T) \to \left(\check{T}^{\Gamma}\right)^{*}, \qquad (2.8)$$

where \prod' indicates that almost all factors should be equal to 1 and $\ker^1(k,T)$ is defined by this sequence. (In our case, both split and nonsplit, $\ker^1(k,T) = 1$ and the last map is surjective, but with a view towards possible generalizations we won't use that.)

Now we introduce a notion of "average volume" for $T(k_v)$ or [T]. Most of it was introduced in $[Sak13a, \S 2.5]$ already, but it was not called so.

If F is a local field and T is a torus over F, we have a canonical "absolute value" map

$$T(F) \xrightarrow{\mathrm{abs}} V_T := \mathrm{Hom}(\mathcal{X}^*(T)_F, \mathbb{R}_+^{\times})$$

given by the pairing: $(t,\chi) \mapsto |\chi(t)|$ $(t \in T(F), \chi \in \mathcal{X}^*(T)_F)$, where $\mathcal{X}^*(T)_F$ denotes the group of F-rational characters of T. This character group being free of some rank r, if we choose generators we have an identification of V_T with $(\mathbb{R}_+^\times)^r$, so we can endow it with the Haar measure corresponding to the standard Haar measure $d^\times x = \frac{dx}{x}$ on \mathbb{R}_+^\times ; this measure on V_T does not depend on the choice of generators.

Now, given a Haar measure dt on T we define:

$$\operatorname{AvgVol}(T(F)) = \lim_{N \to \infty} \frac{dt(\operatorname{abs}^{-1}\left(\left[\frac{1}{N}, N\right]^r\right))}{d^{\times}x\left(\left[\frac{1}{N}, N\right]^r\right)},$$

where we have used the identification of V_T with $(\mathbb{R}_+^{\times})^r$. (This is the same number as what was denoted by $\text{Vol}(T(F)_0)$ in [Sak13a, §2.5].)

Globally, now, if T is defined over a global field k, we similarly have a canonical map

$$T(\mathbb{A}_k) \to [T] \xrightarrow{\text{abs}} V_T := \text{Hom}(\mathcal{X}^*(T)_k, \mathbb{R}_+^{\times}),$$

where as usual: $[T] = T(\mathbb{A}_k)/T(k)$. Thus, given a Haar measure dt on [T], by choosing a similar isomorphism of the right hand side with $(\mathbb{R}_+^{\times})^r$, we define:

$$\operatorname{AvgVol}([T]) = \lim_{N \to \infty} \frac{dt(\operatorname{abs}^{-1}\left(\left[\frac{1}{N}, N\right]^r\right))}{d^{\times}x\left(\left[\frac{1}{N}, N\right]^r\right)}.$$

Of course, the "average volume" is just the volume if T is anisotropic (both locally and globally).

- 2.4. Global irregular distributions, global transfer operators, and statement of Poisson summation in the baby case. Let us now focus our attention to the "baby case" of [Sak13a, section 2], namely E/k a quadratic etale extension (with associated idele class character η) and $\mathcal{X} := X/T$, where $X = \operatorname{Res}_{E/k}\mathbb{G}_a$ and $T = \ker N_k^E$ (kernel of the norm map from E^{\times} to k^{\times}). In this case, the Schwartz space $\mathcal{S}(\mathcal{X}_v)$ constists of sections of the Schwartz cosheaf over $\mathcal{B}(k_v) = k_v$ consisting of functions on $\mathcal{B}^{\text{reg}}(k_v) = k_v^{\times}$ with the following description:
 - over k_v^{\times} , the cosheaf coincides with the usual cosheaf of Schwartz functions;
 - in a neighborhood of zero, the functions are of the form

$$f(\xi) = \begin{cases} -C_1(\xi) \log |\xi|_v + C_2(\xi), & \text{if } \eta_v = 1\\ C_1(\xi) + C_2(\xi)\eta_v(\xi), & \text{if } \eta_v \neq 1 \end{cases}$$

where the C_i 's are smooth functions.

The map

$$\mathcal{S}(k_v) \oplus \mathcal{S}(k_v) \to \mathcal{S}(\mathcal{X}_v)$$

given by

$$(C_1, C_2) \mapsto$$
 the function f as above

identifies $\mathcal{S}(\mathcal{X}_v)$ with a quotient of the Fréchet space $\mathcal{S}(k_v) \oplus \mathcal{S}(k_v)$. The induced quotient topology coincides with the topology obtained by thinking of $\mathcal{S}(\mathcal{X}_v)$ as a coinvariant space.

Recall that the "transfer operator" \mathcal{G} is an automorphism of the local Schwartz space $\mathcal{S}(\mathcal{X}_v)$; it is defined as the composition

$$\mathcal{G} = \mathcal{F} \circ \iota \circ \mathcal{F},\tag{2.9}$$

where \mathcal{F} is usual Fourier transform with respect to a fixed character as described in the notation section of the introduction and ι is the operator

$$\iota(f)(\bullet) = \frac{\eta_v(\bullet)}{|\bullet|} f\left(\frac{1}{\bullet}\right).$$

Moreover, by [Sak13a] Propositions 2.10 and 2.16, it relates the orbital integrals of a function "upstairs" with those of its Fourier transform. (Here

"upstairs" does not refer only to E_v but also to a second copy of it in the nonsplit case – we will recall how to lift elements of $\mathcal{S}(\mathcal{X}_v)$ later in §2.5.) In particular, since the characteristic function of \mathfrak{o}_{E_v} is stable under Fourier transform (at almost every non-Archimedean place v), the transform \mathcal{G} preserves the "basic vectors", and hence gives rise to an automorphism of the global Schwartz space:

$$\mathcal{G}: \mathcal{S}(\mathcal{X}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{X}(\mathbb{A}_k)).$$
 (2.10)

In [Sak13a] Propositions 2.5 and 2.14 I defined local "irregular distributions" \tilde{O}_{0_v} , \tilde{O}_{u_v} (when E_v is split) and $\tilde{O}_{0_v,+}$, $\tilde{O}_{0_v,-}$ (when E_v is inert) on $\mathcal{S}(\mathcal{X}_v)$; we have added here an index v to clarify that we are talking about $0 \in \mathcal{B}_v$. We recall the definitions:

In the split case,

$$f(\xi) = -C_1(\xi) \cdot \log |\xi|_v + C_2(\xi), \tag{2.11}$$

we set $\tilde{O}_{0_v}(f) = C_1(0)$ and $\tilde{O}_{u_v}(f) = C_2(0)$.

In the nonsplit case,

$$f(\xi) = C_1(\xi) + C_2(\xi)\eta_v(\xi), \tag{2.12}$$

we set $\tilde{O}_{0_v,+}(f) = C_1(0)$ and $\tilde{O}_{0_v,-}(f) = C_2(0)$.

All the distributions with index 0_v can be unified by the notation $\tilde{O}_{0_v,\kappa}$, where $\kappa \in H^1(k_v,T)^*$ (and * denotes the Pontryagin dual); in the split case this cohomology group is trivial, and in the nonsplit case we let the index "—" correspond to the nontrivial element of $H^1(k_v,T)^*$. Moreover, for every v we have a natural map

$$\check{T}^{\Gamma} \to \pi_0 \left(\check{T}^{\Gamma_v} \right) = H^1(k_v, T)^*,$$

so for every $\kappa \in \check{T}^{\Gamma}$ and every v we define $\tilde{O}_{0_v,\kappa}$ via the image of this map. In the *nonsplit case* we define globally, for $\kappa \in \check{T}^{\Gamma}$:

$$\tilde{O}_{0,\kappa} = \prod_{v}' \tilde{O}_{0_{v},\kappa}. \tag{2.13}$$

The Euler product on the right hand side is not absolutely convergent (which is why we denote it by \prod'), and outside of a finite number of places S it should be interpreted as a special value of a Dirichlet L-function. More precisely, recall that the basic vector is that multiple of the orbital integrals of $1_{\mathfrak{o}_{E_v}}$ under the norm map $E_v \to k_v = \mathcal{B}(k_v)$ which satisfies the normalization condition (2.3). It is easy to compute it:

2.4.1. **Lemma.** The basic vector $f_{\mathcal{X}_v}^0 \in \mathcal{S}(\mathcal{X}_v)$ (defined for almost every finite place v, with residual degree q_v) is supported on the set of $\xi \in \mathcal{B}_{\mathcal{X}}^{reg}(k_v) = k_v^{\times}$ with $|\xi|_v \leq 1$, and its value on those points is equal to

$$\begin{cases} 1 - \log_{q_v} |\xi|_v & \text{if } E_v/k_v \text{ is split,} \\ \frac{1 + \eta_v(\xi)}{2} & \text{if } E_v/k_v \text{ is non-split.} \end{cases}$$

Therefore, in the split case we have:

$$\tilde{O}_{0_v}(f_{\mathcal{X}_v}^0) = (\log q_v)^{-1}, \ \tilde{O}_{u_v}(f_{\mathcal{X}_v}^0) = 1,$$

while in the nonsplit case:

$$\tilde{O}_{0_v,+}(f^0_{\mathcal{X}_v}) = \frac{1}{2} = \tilde{O}_{0_v,-}(f^0_{\mathcal{X}_v}).$$

Notice that in each case the values of $\tilde{O}_{0_v,\kappa}(f_{\mathcal{X}_v}^0)$ are the leading coefficients of the local L-function $L_v^S(\eta_v,s)$ at s=0, and that the corresponding global L-function is, in the nonsplit case, regular at s=0. We now interpret the partial Euler product, in the nonsplit case:

$$\prod_{v \notin S}' \tilde{O}_{0_v,\kappa}(f_{\mathcal{X}_v}^0)$$

as the leading term of the Taylor series of the partial L-function:

$$L^S(\eta, t) \tag{2.14}$$

at t=0. It is clear that the full "Euler product" $\prod_{v}' \tilde{O}_{0_{v},\kappa}$, defined as the product of the above with the factors corresponding to the remaining places, is independent of the set S chosen to define it.

Finally, we are ready to define the contribution of 0 to the global Poisson sum in the nonsplit case. The notation \tilde{O}_0 that follows should not be confused with the local notation of [Sak13a], which we designate here by the additional index v:

Definition. In the nonsplit case we define the distribution \hat{O}_0 on the global Schwartz space $\mathcal{S}(\mathcal{X}(\mathbb{A}_k))$ as

$$\tilde{O}_0 := \sum_{\kappa \in \check{T}^{\Gamma}} \tilde{O}_{0,\kappa}. \tag{2.15}$$

The sum on the right consists, of course, of only two terms.

For the split case, we notice that at almost every place we have $O_{0_v}(f_{\mathcal{X}_v}^0) = (\log q_v)^{-1}$ = the leading term of the local Dedekind zeta function $\zeta_v(t)$ at t = 0, while $\tilde{O}_{u_v}(f_{\mathcal{X}_v}^0) = 1$ = twice the constant coefficient in the Laurent expansion of the local zeta function at s = 0. We will define a global irregular distribution $f \mapsto \tilde{O}_0(f)$ which, formally, is the limit with $t \to 0$ of a sum of Euler products with opposite residues, the first of which is outside of a finite number of places equal to the partial Dedekind zeta function $\zeta^S(t)$ and the second of which is equal to $\zeta^S(-t)$.

More precisely, take S a sufficiently large set of places, and let a_{-1}^S, a_0^S be the leading, resp. the second, coefficient in the Laurent expansion of the partial zeta function $\zeta^S(t)$ around t=0; notice that the order of zero of this partial zeta function is |S|-1. Then we have the following:

Definition. In the split case we define the distribution \tilde{O}_0 as

$$\tilde{O}_0(\prod_v f_v) = 2a_0^S \prod_{v \in S} \tilde{O}_{0_v}(f_v) + a_{-1}^S \prod_{v \in S} \tilde{O}_{0_v}(f_v) \cdot \left(\sum_{v \in S} \frac{\tilde{O}_{u_v}(f_v)}{\tilde{O}_{0_v}(f_v)} \right). \quad (2.16)$$

We are now ready to state the Poisson summation formula in the baby case:

2.4.2. **Theorem.** For $f \in \mathcal{S}(\mathcal{X}(\mathbb{A}_k))$, in either the split or the nonsplit case, we have

$$\sum_{\xi \neq 0} f(\xi) + \tilde{O}_0(f) = \sum_{\xi \neq 0} \mathcal{G}f(\xi) + \tilde{O}_0(\mathcal{G}f). \tag{2.17}$$

Remark. Although the Poisson summation formula for $X = \operatorname{Res}_{E/k} \mathbb{G}_a$ can be stated, equivalently, at the finite level (that is, for functions on $\prod_{v \in S} E_v$, for some finite set S of places), the Poisson summation formula of Theorem 2.4.2 for $\mathcal{X} = X/T$ is genuinely adelic, in the following sense: The sums over $\xi \in \mathcal{B}^{\operatorname{reg}}(k)$ can be restricted to $\xi \in \mathcal{B}^{\operatorname{reg}}(k) \cap \mathcal{B}(k_S)$, for a sufficiently large set of places S, since the support of the basic function is $\mathcal{B}(\mathfrak{o}_v)$; however, for a point ξ which is not in $\mathcal{B}^{\operatorname{reg}}(k_S)$ the functions appearing in the sum depend on the coordinates of ξ outside of S, since the basic function is not identically equal to 1 on $\mathcal{B}(\mathfrak{o}_v)$. If f is compactly supported then we can enlarge S so that all $\xi \in \mathcal{B}^{\operatorname{reg}}(k)$ in its support belong to $\mathcal{B}^{\operatorname{reg}}(k_S)$; however, this is not possible simultaneously for both sides of the formula. This comment is valid for all Poisson summation formulas that we will encounter in this paper; in fact, for one of them (the Kuznetsov trace formula with non-standard test functions) we will not even be able to restrict to $\mathcal{B}^{\operatorname{reg}}(k) \cap \mathcal{B}(k_S)$.

2.5. Indirect proof of Poisson summation in the nonsplit case. We will first deduce the Poisson summation of Theorem 2.4.2 from the Poisson summation formula for Fourier transform on the adeles of E. In the next section, we will prove it directly using the explicit expression (2.9) for \mathcal{G} , building the tools that we will need for the comparison of RTFs in later sections.

First of all, recall from [Sak13a, §2.10] that, locally, elements of $\mathcal{S}(\mathcal{X}_v)$ are obtained as orbital integrals of elements in $\bigoplus_{\alpha} \mathcal{S}(X^{\alpha})$; here $X = \operatorname{Res}_{E/k} \mathbb{G}_a$; α ranges over isomorphism classes of T-torsors over k_v (parametrized by $H^1(k_v,T)$), and for such a torsor R^{α} we set $X^{\alpha} = X \times^T R^{\alpha}$. Since $\operatorname{Aut}_T(R^{\alpha}) \simeq T$, the space X^{α} carries a T-action; it is non-canonically isomorphic to X. Given an element $f \in \mathcal{S}(\mathcal{X}_v)$ we will call "lift" of f an element $\Phi \in \bigoplus_{\alpha} \mathcal{S}(X_v^{\alpha})$, together with a Haar measure dt_v on T_v such that the orbital integrals of Φ give f, see [Sak13a, 2.10].

The sums over ξ in (2.17) are, in fact, over $\xi \in k_S \setminus \{0\}$ for some finite number of places S, since f will be equal to the basic vector outside of S, which is supported only on the image of $X(\mathfrak{o}_v)$. Thus, the sums will be only over those ξ whose preimage under $X^{\text{reg}} \to \mathcal{B}^{\text{reg}}$ is a T-torsor (over k) that is trivial outside of S. Since there is only a finite number of such torsors,

let us fix, for now, such an isomorphism class of T-torsors over k, denoted by β , and let us assume that

$$f = \prod_{v \notin S} f_{\mathcal{X}_v}^0 \cdot \prod_{v \in S} f_v$$

(since by continuity it is enough to prove the Poisson summation for a dense subspace), where f_v , for all v, f_v is in the image of $\mathcal{S}(X_v^{\beta})$. Fix lifts (Φ_v, dt_v) so that $\prod_v dt_v$ is a factorization of the Tamagawa measure on $T(\mathbb{A}_k)$, and Φ_v is the characteristic function of \mathfrak{o}_{E_v} for $v \notin S$. (Hence, $dt_v(T(o_v)) = 1$ for $v \notin S$.)

In [Sak13a, (2.25)], I defined an extension of Fourier transform to the space X^{β} ; I claim that $\Phi := \prod_{v \in S} \Phi_v$ satisfies the usual Poisson summation formula with respect to this Fourier transform, that is:

$$\sum_{\xi \in X^{\beta}(k_S)} \Phi(a\xi) = \frac{1}{|a|} \sum_{\xi \in X^{\beta}(k_S)} \hat{\Phi}\left(\frac{\xi}{a}\right). \tag{2.18}$$

Indeed, we can fix, over k_S , an isomorphism $\iota: X \to X^{\beta}$ mapping 1 to some element $e \in X^{\beta}(k_S)$, and let $\Phi_v^1 = \iota^* \Phi_v$, a function on E_v . Recall the formula [Sak13a, (2.27)]:

$$\iota^*\widehat{\Phi_v}(y) = |a|_v\widehat{\Phi_v^1}(ay),$$

where $a = N_k^E(e) \in k_S$. Taking into account that $\prod_{v \in S} |a|_v = 1$, and that ι induces a bijection between $X(k_S)$ and $X^{\beta}(k_S)$, the claim follows from Poisson summation on \mathbb{A}_E .

Now denote by $0_{\beta} \in X^{\beta}(k)$ the "zero" point in X^{β} . For an arbitrary $f = \otimes_v f_v$ (not necessarily supported on the X^{β} corresponding to a single torsor), with a lift $\Phi \in \otimes'_v \oplus_{\alpha \in H^1(k_v,T)} \mathcal{S}(X^{\alpha}_v)$ (with respect to the Tamagawa measure on $T(\mathbb{A}_k)$), integrating (2.18) over [T] we get:

$$\sum_{\xi \neq 0} f(\xi) + \text{Vol}([T]) \sum_{\beta \in H^1(k,T)} \Phi(0_{\beta}) = \sum_{\xi \neq 0} \mathcal{G}f(\xi) + \text{Vol}([T]) \sum_{\beta \in H^1(k,T)} \hat{\Phi}(0_{\beta}).$$
(2.19)

By the short exact sequence (2.8) and abelian Fourier analysis we have

$$\operatorname{Vol}([T]) \sum_{\beta \in H^1(k,T)} \Phi(0_{\beta}) =$$

$$= \frac{\left|\ker^{1}(k,T)\right|}{\left|\check{T}^{\Gamma}\right|} \operatorname{Vol}([T]) \sum_{\kappa \in \check{T}^{\Gamma}} \prod_{v} \sum_{\alpha_{v} \in H^{1}(k_{v},T)} \left\langle \alpha_{v}, \kappa \right\rangle \Phi_{v}(0_{\alpha_{v}}).$$

⁵We will consistently be using β for torsors defined over k and β_v for their localizations, while the symbols α , α_v will be reserved for torsors defined locally, not necessarily as restrictions of global torsors.

In the notation that we introduced previously, we have, by definition,

$$\tilde{O}_{0_v,\kappa}(f_v) = \frac{\operatorname{AvgVol}(T_v)}{|H^1(k_v,T)|} \sum_{\alpha_v \in H^1(k_v,T)} \langle \alpha_v, \kappa \rangle \, \Phi_v(0_{\alpha_v}),$$

and therefore there remains to show:

$$\frac{\left|\ker^{1}(k,T)\right|}{\left|\check{T}^{\Gamma}\right|}\operatorname{Vol}([T])\prod_{v}^{\prime}\frac{\left|H^{1}(k_{v},T)\right|}{\operatorname{AvgVol}(T_{v})}=1,\tag{2.20}$$

where the Euler product should be interpreted in an analogous way as in the definition of the global distribution \tilde{O}_0 , and is therefore denoted by \prod' . More precisely, if we fix a factorization of the Tamagawa measure on $T(\mathbb{A}_k)$ as $\prod_v dt_v$ with $dt_v(T(\mathfrak{o}_v)) = c_v$ for $v \notin S$ so that the Euler product $\prod_{v \notin S} c_v$ is convergent (for example, $c_v = 1$ as before), then for $v \notin S$ we have

$$\frac{|H^1(k_v, T)|}{\operatorname{AvgVol}(T_v)} = \frac{1}{c_v \cdot (L_v(\eta_v, 0))^*},$$

where, I remind, the exponent * denotes the leading term in the Laurent expansion. Thus, to interpret the formal Euler product \prod' , we set

$$\prod_{v}' \frac{|H^{1}(k_{v},T)|}{\operatorname{AvgVol}(T_{v})} = \frac{1}{(L^{S}(\eta,0))^{*} \prod_{v \notin S} c_{v}} \cdot \prod_{v \in S} \frac{|H^{1}(k_{v},T)|}{\operatorname{AvgVol}(T_{v})}.$$

We could deduce (2.20) directly from known formulas about Tamagawa measures of tori, i.e., essentially the Dirichlet class number formula, but since the formula does not really depend on choices of measures and explicit calculations, let us instead sketch its reduction to the "basics" (more precisely, to Tate's thesis) using only the minimum of necessary arguments:

For any tori T_i over k with a short exact sequence

$$1 \rightarrow T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow 1$$
,

and any compatible choice of Haar measures on their adelic points (in the obvious sense, i.e., integration over T_2 should be equal to integration over T_1 followed by integration over T_3), it is known that the quantities

$$\mu(T_i) := \frac{\left|\ker^1(k, T_i)\right|}{\left|\check{T}_i^{\Gamma}\right|} \operatorname{AvgVol}([T_i])$$

satisfy

$$\mu(T_2) = \mu(T_1)\mu(T_3);$$

see [Ono63, Theorem 4.4.1].

Similarly, for a similar sequence over a completion k_v we have a long exact sequence

$$1 \to T_1(k_v) \to T_2(k_v) \to T_3(k_v) \to H^1(k_v, T_1) \xrightarrow{\iota} H^1(k_v, T_2) \to \dots,$$

from which it follows that

$$\frac{\operatorname{AvgVol}(T_1(k_v))}{|\ker(\iota)|} = \frac{\operatorname{AvgVol}(T_2(k_v))}{\operatorname{AvgVol}(T_3(k_v))}.$$
 (2.21)

Notice that $|\ker(\iota)| = (k_v^{\times} : N_{k_v}^{E_v} E_v^{\times}).$ Applying these considerations to our case:

$$1 \to T \to \operatorname{Res}_{E/k} \mathbb{G}_m \to \mathbb{G}_m \to 1, \tag{2.22}$$

the statement (2.20) reduces to the statement

$$\frac{\operatorname{AvgVol}([\operatorname{Res}_{E/k}\mathbb{G}_m])}{\operatorname{AvgVol}([\mathbb{G}_m])} = \frac{(\zeta_E^S(0))^*}{(\zeta_k^S(0))^*} \prod_{v \in S} \frac{\operatorname{AvgVol}(E_v^{\times})}{\operatorname{AvgVol}(k_v^{\times})}.$$
 (2.23)

The statement is now a corollary of Tate's thesis, more precisely of the fact that the residue at s = 0 of a zeta integral

$$\int_{\mathbb{A}_k^{\times}} \Phi(x) |x|^s d^{\times} x$$

is equal to $\Phi(0)$ AvgVol($[\mathbb{G}_m]$) (and similarly for Tate integrals for E).

Remark. Tate's thesis shows that

$$\operatorname{AvgVol}[\mathbb{G}_m] = \operatorname{"Res"} \prod_{v} \operatorname{AvgVol}(k_v^{\times}), \tag{2.24}$$

where the expression on the right should not be taken literally but interpreted, again, as an expression of the form: $(\zeta_k^S(0))^*$ times a convergent product. Indeed, the residue of the zeta integral of a Schwartz function Φ at s=0 is on one hand equal to $\text{AvgVol}[\mathbb{G}_m]\Phi(0)$, and on the other hand the "Euler product" of the right hand side times $\Phi(0)$.

Further observing that, for a self-dual measure dx_v with respect to an additive character ψ_v^{-1} used to define Fourier transform, using the corresponding multiplicative measure

$$d^{\times} x_v = |x_v|^{-1} dx_v (2.25)$$

we have

$$\Phi_v(0)$$
AvgVol $(k_v^{\times}) = \operatorname{Res}_{s=0} \int_{k_v^{\times}} \Phi_v(x) |x|^s d^{\times} x =$

$$= \mathrm{Res}_{s=0} \gamma_v(1, 1-s, \psi_v^{-1}) \int_{k_v^\times} \hat{\Phi}_v(x) |x|^{1-s} d^\times x = \Phi(0) \mathrm{Res}_{s=0} \gamma_v(1, 1-s, \psi_v^{-1}),$$

where $\gamma_v(1, s, \psi_v^{-1})$ denotes the gamma factor for the trivial multiplicative character $\chi = 1$, we get that for such a measure and character we have

$$AvgVol(k_v^{\times}) = Res_{s=0}\gamma_v(1, 1 - s, \psi_v^{-1}) = Res_{s=0}\gamma_v(1, s, \psi_v)^{-1}.$$
 (2.26)

Globally, the triviality of gamma factors shows that $\text{AvgVol}[\mathbb{G}_m] = 1$ under the usual normalization for Tamagawa measure – i.e.,, multiplication by convergence factors corresponding to $\zeta_{\mathbb{Q}_p}(1)$ at the finite places over \mathbb{Q}_p . This, of course, is just a reformulation of the usual derivation of the class number formula from Tate's thesis.

Combining (2.26) (applied to both k_v^{\times} and E_v^{\times}) with (2.21), we get

$$\gamma_v^*(\eta_v, 0, \psi_v) \frac{\text{AvgVol}(T(k_v))}{(k_v^{\times} : N_{k_v}^{E_v} E_v^{\times})} = 1.$$
 (2.27)

Here, again, * denotes the dominant term in the Laurent expansion when 0 is replaced by s. This formula is identical to (2.26) when E_v/k_v is split, and when E_v is a quadratic field, the factor $\gamma_v^*(\eta_v, 0, \psi_v) = \gamma_v(\eta_v, 0, \psi_v)$ arises as the quotient

$$\frac{\operatorname{Res}_{s=0}\gamma_{k_v}(1,s,\psi_v)^{-1}}{\operatorname{Res}_{s=0}\gamma_{E_v}(1,s,\psi_v\circ\operatorname{tr})^{-1}}$$

(where we now indicate with an index the algebra to which each gamma factor is attached). Here the measures on k_v and E_v have been taken to be self-dual with respect to the characters ψ_v and $\psi_v \circ \text{tr}$, respectively, the measures on k_v^{\times} and E_v^{\times} are given by (2.25), and the measure on $T(k_v)$ is determined by compatibility with respect to the sequence (2.22).

Global triviality of gamma factors, applied to (2.27), shows that for Tamagawa measures

$$\prod_{v}' \frac{\operatorname{AvgVol}(T_v)}{|H^1(k_v, T)|} = 1. \tag{2.28}$$

2.6. Indirect proof of Poisson summation in the split case. Now we consider the case $E = k \oplus k$. Again we let $f = \prod_v f_v$ and fix preimages $\Phi_v \in \mathcal{S}(X_v)$ of f_v (with compatible measures on the torus), almost always equal to the characteristic function of \mathfrak{o}_v^2 . The left and right sides of (2.17) are continuous functionals on the LF-space $\mathcal{S}(\mathcal{X}(\mathbb{A}_k))$, and the transfer operator \mathcal{G} is continuous on it; therefore, we might prove equality just for a dense subspace. We will assume, therefore, that for all v we have: $\Phi_v(x,y) = \Phi_{1,v}(x)\Phi_{2,v}(y)$ for some Schwartz functions $\Phi_{1,v}, \Phi_{2,v}$ in one variable (equal to the characteristic function of \mathfrak{o}_v almost everywhere).

The function $\Phi := \prod_v \Phi_v$ satisfies the usual Poisson summation formula with respect to Fourier transform, that is:

$$\sum_{\xi \in X(k)} \Phi(a\xi) = \frac{1}{|a|} \sum_{\xi \in X(k)} \hat{\Phi}\left(\frac{\xi}{a}\right), \tag{2.29}$$

where $|a| = |a_1 \cdot a_2|$ for $a = (a_1, a_2)$. Integrating over $[T] = \mathbb{A}_k^{\times}/k^{\times}$ we get

$$\sum_{\xi \in k^{\times}} f(\xi) - \sum_{\xi \in k^{\times}} \mathcal{G}f(\xi) = \int_{\mathbb{A}_{k}^{\times}/k^{\times}} \left(\sum_{\gamma \in (k^{\times})^{2}} \Phi(a\gamma) - \sum_{\gamma \in (k^{\times})^{2}} \hat{\Phi}\left(\frac{\gamma}{a}\right) \right) =$$

$$= \int_{\mathbb{A}_{k}^{\times}/k^{\times}} \left(-\sum_{\gamma_{1} \in k^{\times}} \Phi_{1}(a\gamma_{1}) \Phi_{2}(0) - \sum_{\gamma_{2} \in k} \Phi_{1}(0) \Phi_{2}(a^{-1}\gamma_{2}) + \right. \\ \left. + \sum_{\gamma_{1} \in k} \hat{\Phi}_{1}(a^{-1}\gamma_{1}) \hat{\Phi}_{2}(0) + \sum_{\gamma_{2} \in k^{\times}} \hat{\Phi}_{1}(0) \hat{\Phi}_{2}(a\gamma_{2}) \right) da. \tag{2.30}$$

Notice that the integration over [T] giving rise to the left hand side is absolutely convergent, and hence so is the right.

By one-dimensional Poisson summation, the right hand side can be written

$$\int_{\mathbb{A}_k^{\times}/k^{\times}} \left(-\sum_{\gamma_1 \in k^{\times}} \Phi_1(a\gamma_1) \Phi_2(0) - \sum_{\gamma_2 \in k} |a| \Phi_1(0) \hat{\Phi}_2(a\gamma_2) + \cdots \right)$$

$$+ \sum_{\gamma_1 \in k} |a| \Phi_1(a\gamma_1) \hat{\Phi}_2(0) + \sum_{\gamma_2 \in k^{\times}} \hat{\Phi}_1(0) \hat{\Phi}_2(a\gamma_2) da.$$

Notice that the term corresponding to $\gamma_2 = 0$ in the second sum cancels the term with $\gamma_1 = 0$ in the third. By interpreting the remaining integrals as Tate integrals, we get

$$\lim_{s \to 0} \left(-\zeta(\Phi_1, s) \Phi_2(0) - \Phi_1(0) \zeta(\hat{\Phi}_2, s+1) + \hat{\Phi}_2(0) \zeta(\Phi_1, s+1) + \hat{\Phi}_1(0) \zeta(\hat{\Phi}_2, s) \right) = 0$$

$$= \lim_{s \to 0} \left(-\zeta(\Phi|_{x}, s) - \zeta(\Phi|_{y}, -s) + \zeta(\hat{\Phi}|_{x}, -s) + \zeta(\hat{\Phi}|_{y}, s) \right), \tag{2.31}$$

where $\Phi|_x$ and $\Phi|_y$ denote, respectively, the restrictions to y=0 and x=0, considered as functions of the variable x, resp. y. The last step is by the functional equation of Tate integrals.

Then we claim:

2.6.1. **Lemma.** Let L denote the functional on $S(X(\mathbb{A}_k))$:

$$L(\Phi) = \lim_{t \to 0} \left(\zeta(\Phi|_x, t) + \zeta(\Phi|_y, -t) \right),$$

then

$$L(\Phi) = \tilde{O}_0(f), \tag{2.32}$$

the "irregular" distribution of (2.16).

Proof. One just needs to check the definitions. Fixing a sufficiently large finite set of places S, we have

$$\zeta(\Phi|_x, t) = \frac{a_{-1}}{t} + a_0 + \text{higher order terms},$$

with

$$a_{-1} = a_{-1}^S \cdot \prod_{v \in S} \tilde{O}_{0_v}(f_v)$$

and

$$a_0 = \prod_{v \in S} \tilde{O}_{0_v}(f_v) \left(a_0^S + a_{-1}^S \cdot \sum_{v \in S} \frac{a_{0,v}}{\tilde{O}_{0_v}(f_v)} \right),$$

where $a_{0,v}$ = the constant term in the Laurent expansion of the local zeta integral $\zeta(\Phi|_x,t)$ at t=0. Similarly for $\zeta(\Phi|_y,t)$, with $a_{0,v}$ replaced by the constant term $b_{0,v}$ of the Laurent expansion of the local zeta integral $\zeta(\Phi|_y,t)$ at t=0. From this it follows that

$$\lim_{t \to 0} \left(\zeta(\Phi|_x, t) + \zeta(\Phi|_y, -t) \right) = \prod_{v \in S} \tilde{O}_{0_v}(f_v) \left(2a_0^S + a_{-1}^S \cdot \sum_{v \in S} \frac{a_{0,v} + b_{0,v}}{\tilde{O}_{0_v}(f_v)} \right),$$

and keeping in mind that $a_{0,v} + b_{0,v} = \tilde{O}_{u_v}(f_v)$ by [Sak13a, 2.10], this is equal to $\tilde{O}_0(f)$.

Hence, (2.31) can be written as

$$-\tilde{O}_0(f) + \tilde{O}_0(\mathcal{G}f),$$

and this implies the Poisson summation formula (2.17).

3. Direct proof in the baby case

3.1. **Motivation.** The proof of the Poisson summation formula for $\mathcal{X} = \operatorname{Res}_{E/k}\mathbb{G}_a/T$ which was presented in the previous section is unsatisfactory for two reasons: first, it is not a direct proof on the Schwartz space $\mathcal{S}(\mathcal{X}(\mathbb{A}_k))$, but it uses properties of the space "upstairs" $\mathcal{S}(\mathbb{A}_E)$. For the more complicated Poisson summation formulas that one will encounter, here and elsewhere, using properties of the space upstairs is precisely what one would like to avoid – in fact, we would like a direct proof at the level of the base \mathcal{B} , in order to deduce properties of the space upstairs. The fact that our transfer operator \mathcal{G} is given in terms of Fourier transforms and birational maps suggests that such a direct proof should be possible, using at some point the classical Poisson summation formula for Fourier transform.

Hence, in the present section we will discuss a direct proof of the Poisson summation formula that was proven in the previous section, using only the given spaces of orbital integrals and not the fact that they arise as coinvariants of the usual Schwartz space on a vector space.

The Poisson summation formula will be proven by a method of analytic continuation as the given "Schwartz spaces" vary according to a complex parameter t. Instead of directly focusing on the specific case of interest, we present an axiomatic approach to these Schwartz spaces in order to single out the properties that we are using in the proof. With some modifications, this approach will allow us to prove the Poisson summation formula for the comparison of relative trace formulas in the next section.

3.2. Local Schwartz spaces varying with a parameter. Throughout our discussion there will be a complex parameter t. We will say "for large t" (denoted: $\Re(t) \gg 0$) for statements that hold on a half-plane of the form $\Re(t) \geq \sigma$. For the discussion of non-Archimedean places with residual degree q_v , the parameter t is considered to be varying in $\mathbb{C}/\frac{2\pi i}{\log q_v}\mathbb{Z}$.

We will give axioms for four different "Schwartz spaces" $\mathcal{S}_1^t, \mathcal{S}_2^t, \mathcal{S}_3^t, \mathcal{S}_4^t$, depending on the parameter t, and will define certain integral transforms between them, again depending on t. In fact, for the baby case the first and fourth spaces, as well as the second and third spaces, will be identical, but in order to get used to the scheme, let us keep them in mind as different spaces. These spaces will be restricted tensor products, over all places, of local Fréchet spaces varying "analytically" as described in Appendix A. We will start by describing axioms for their local factors, and later (in §3.3) we will add some axioms on their basic vectors.

The (local) Schwartz spaces $\mathcal{S}_{1,v}^t$ and $\mathcal{S}_{4,v}^t$ are sections of a Schwartz cosheaf over $\mathcal{B}(k_v) = k_v$, which away from $0 \in \mathcal{B}$ coincides with the cosheaf of usual Schwartz functions. (In particular, we have rapid decay at ∞ .) We now describe the behavior close to $\xi = 0 \in \mathcal{B}(k_v)$.

As with the "model \mathcal{A}^{t} " of Appendix A, we will define the stalks of $\mathcal{S}_{1,v}^{t}$ and $\mathcal{S}_{4,v}^{t}$ over 0 in such a way that the *fibers* are annihilated by the operator

$$(\operatorname{Id} - \eta_v(a)|a|_v^{-t - \frac{1}{2}} a \cdot) (\operatorname{Id} - |a|_v^{-\frac{1}{2}} a \cdot),$$
 (3.1)

for every $a \in k_v^{\times}$, where $a \cdot$ we denote the normalized action of k_v^{\times} on functions on \mathcal{B}

$$(a \cdot f)(x) := |a|_v^{\frac{1}{2}} f(ax). \tag{3.2}$$

The annihilator of the fibers does not, of course, provide a complete description in the Archimedean case. The precise definition, for generic t, is that the elements of $\mathcal{S}_{1,v}^t$ and $\mathcal{S}_{4,v}^t$ are, in a neighborhood of zero, smooth functions on k_v^{\times} which have the form

$$C_1(\xi) + C_2(\xi)\eta_v(\xi)|\xi|_v^t,$$

where C_1 and C_2 extend to smooth functions in a neighborhood of zero, except:

• when t=0 and $\eta_v=1$, in which case the functions have the form

$$C_1(\xi) + C_2(\xi) \log |\xi|_v,$$

i.e., they specialize to elements of $\mathcal{S}(\mathcal{X}_v)$;

• when $t \in 2\mathbb{Z}$, $\eta_v = 1$ and $k_v = \mathbb{R}$, or $t \in (2\mathbb{Z} + 1)$, $\eta_v \neq 1$ and $k_v = \mathbb{R}$, or $t \in \mathbb{Z}$ and $k_v = \mathbb{C}$ in which case the functions have the form

$$\begin{cases} C_1(\xi) + C_2(\xi)\eta_v(\xi)|\xi|_v^t \log |\xi|_v, & \text{when } t \ge 0; \\ C_1(\xi)\eta_v(\xi)|\xi|_v^t + C_2(\xi) \log |\xi|_v, & \text{when } t < 0. \end{cases}$$

The (local) Schwartz spaces $\mathcal{S}_{2,v}^t$ and $\mathcal{S}_{3,v}^t$ consist of sections of the cosheaf over $\mathbb{P}^1(k_v)$ of functions $\mathcal{B} = k_v$ which away from ∞ coincides with the

cosheaf of Schwartz functions, and in a neighborhood of infinity the functions have the form

$$C(\frac{1}{\xi}) \cdot \eta_v(\xi) |\xi|_v^{-t-1},$$

for some smooth function C.

Clearly, these spaces are stable under the involution

$$\iota_t: f \mapsto \frac{\eta_v(\bullet)}{|\bullet|^{t+1}} f\left(\frac{1}{\bullet}\right). \tag{3.3}$$

Appendix A includes a long discussion of these spaces, including their topology, their analytic structure as t varies, and a notion of "polynomial families of seminorms" (as t varies in bounded vertical strips). We summarize the results that we need, noting first that there is some exceptional behavior at certain values of t (A.7), which will not be included in the result that follows. These exceptions have to do with poles of local gamma factors, where Fourier transforms of characters on k_v^{\times} are not characters on k_v^{\times} .

3.2.1. **Proposition.** For t different than the values of (A.7), Fourier transform \mathcal{F} induces an isomorphism between the Fréchet spaces $\mathcal{S}_{1,v}^t$ and $\mathcal{S}_{2,v}^t$ (or $\mathcal{S}_{3,v}^t$ and $\mathcal{S}_{4,v}^t$).

The composition

$$\mathcal{G}_t: \mathcal{F} \circ \iota_t \circ \mathcal{F}: \mathcal{S}_1^t \to \mathcal{S}_4^t \tag{3.4}$$

makes sense, by analytic continuation, for every t.

Both \mathcal{F} and \mathcal{G}_t are bounded by polynomial families of seminorms on the corresponding spaces, as t varies, and preserve analytic sections.

For t outside of the values of (A.4), (A.5), analytic sections of $S_{i,t}^t$ (where i=1 or 4) are of the form: $\xi \mapsto C_1^t(\xi) + C_2^t(\xi) \eta_v(\xi) |\xi|_v^t$ where $t \mapsto C_1^t$, $t \mapsto C_2^t$ are strongly meromorphic sections into the Fréchet space $S(k_v)$ of Schwartz functions on k_v . Such a section extends to t=0 iff C_1^t and C_2^t have simple poles with opposite residues at t=0, with the residue an element of $S(k_v^\times)$ when $\eta_v \neq 1$, and in that case C_1^t , C_2^t can be chosen to be holomorphic at t=0.

Recall that a strongly meromorphic section into a Fréchet space is one which, in a neighborhood of any point t_0 , becomes (weakly=strongly) holomorphic after multiplication by a power of $(t - t_0)$.

For the last statement of the proposition, notice that given an element of $S_{i,t}^t$, i=1,4, the pair (C_1^t,C_2^t) is only defined up to an element of $S(k_v^\times)$, embedded as $f\mapsto (f,f\eta_v|\bullet|_v^{-t})$. One can formulate similar statements about when a meromorphic section extends to a holomorphic one at the other exceptional values (A.4), (A.5) which involve logarithms. I leave this description to the reader, as it will not be needed.

We can also relate the asymptotic constants of elements of these Schwartz spaces and their Fourier transforms:

3.2.2. **Lemma.** Let $t \notin \mathbb{Z}$. If $f_1 \in \mathcal{S}_{1,v}^t$ is equal to

$$C_1(\xi) + C_2(\xi)\eta_v(\xi)|\xi|_v^t$$

in a neighborhood of $\xi = 0$, with C_1 and C_2 smooth functions, and $\mathcal{F} f_1 \in \mathcal{S}^t_{2,v}$ is of the form

$$D(\frac{1}{\xi}) \cdot \eta_v(\xi) |\xi|_v^{-t-1}$$

in a neighborhood of infinity, then

$$D(0) = \gamma(\eta_v, -t, \psi_v) \cdot C_2(0) \tag{3.5}$$

(local abelian gamma-factors).

Similarly, if $f_2 \in \mathcal{S}_{2,v}^t$ and $\mathcal{F} \circ \iota_t(f_2) \in \mathcal{S}_{4,v}^t$ is of the form $E_1(\xi) + E_2(\xi)\eta_v(\xi)|\xi|_v^t$ in a neighborhood of 0 then

$$f_2(0) = \gamma(\eta_v, -t, \psi_v^{-1}) \cdot E_2(0). \tag{3.6}$$

Proof. Recall that for (almost) every character χ of k_v^{\times} , considered as a tempered distribution on k_v by meromorphic continuation according to Tate's thesis, we have a relation

$$\widehat{\chi(\bullet)} = \gamma(\chi^{-1}, 0, \psi) \cdot | \bullet | \cdot \chi^{-1}(\bullet). \tag{3.7}$$

(We omit the index v for this proof.) Indeed, this is just a reformulation of the functional equation for zeta integrals; in what follows, we denote the obvious bilinear (not hermitian) pairing by angular brackets, and use the exponent ψ when Fourier transform is taken with respect to the character ψ , instead of ψ^{-1} which is our standard convention,

$$\begin{split} \langle \phi, \widehat{\chi} \rangle &= \left\langle \widehat{\hat{\phi}}^{\psi}, \widehat{\chi} \right\rangle = \left\langle \widehat{\phi}, \chi \right\rangle = Z(\widehat{\phi}, \chi, 1) = \\ &= \gamma(\chi^{-1}, 0, \psi) Z(\phi, \chi^{-1}, 0) = \gamma(\chi^{-1}, 0, \psi) \left\langle \phi, \chi^{-1}(\bullet) \cdot | \bullet |^{-1} \right\rangle. \end{split}$$

To prove the desired relations between the asymptotic coefficients, it suffices to relate them for one element in the Schwartz space for which they are non-zero. We obtain such an element by multiplying $\chi(\xi) = \eta(\xi)|\xi|^t$ by the characteristic function of a neighborhood of the identity, thus smoothening its Fourier transform – but leaving it invariant in a neighborhood of infinity. The claim now follows.

- 3.3. Basic vectors and global Schwartz spaces. We now assume that our local spaces $\mathcal{S}_{i,v}^t$ are endowed, for almost every (non-Archimedean) v, with analytic sections of "basic vectors" $f_{i,v}^{t,0}$, $t \in \mathbb{C}/\frac{2\pi i}{\log q_v}\mathbb{Z}$, which satisfy the following axioms:
 - (1) The value of $f_{i,v}^{t,0}$ on $\mathcal{B}^{\text{reg}}(\mathfrak{o}_v) = \mathfrak{o}_v^{\times}$ is a constant $c_{i,v}^t$ such that the partial Euler product: $\prod_{v \notin S} c_{i,v}^t$ converges for t large, locally uniformly in $\Re(t)$, and for i = 1, 4 admits analytic continuation to all values of t;

(2)
$$\mathcal{F}\left(f_{1,v}^{t,0}\right) = f_{2,v}^{t,0}, \ \iota_t\left(f_{2,v}^{t,0}\right) = f_{3,v}^{t,0} \text{ and } \mathcal{F}\left(f_{3,v}^{t,0}\right) = f_{4,v}^{t,0}.$$

(3) For i = 1, 4 and every t, the basic functions $f_{i,v}^{t,0}$ are supported on $\mathcal{B}(\mathfrak{o}_v) \cap \mathcal{B}^{\text{reg}}(k_v)$. In a neighborhood of $\xi = 0$, for $t \neq 0$, they are equal to

$$f_{i,v}^{t,0}(\xi) = L_v(\eta_v, t) + L_v(\eta_v, -t)\eta(\xi)|\xi|^t,$$

where $L_v(\eta_v, t)$ is the local Dirichlet L-function; notice that this extends analytically to t = 0 (cf. Proposition 3.2.1). Finally, there is a constant $r_i^t \geq 0$, independent of v and uniformly bounded in bounded vertical strips, such that the function

$$|\xi|^{r_i^t} \frac{|f_{i,v}^{t,0}(\xi)|}{|f_{i,v}^{t,0}(\mathfrak{o}_v^{\times})|}$$

is ≤ 1 .

(4) For i = 2, 3 and for $\Re(t) \gg 0$ there is a constant $r_i \geq 0$, independent of v or t, such that the function

$$|\xi|^{r_i} \frac{|f_{i,v}^{t,0}(\xi)|}{|f_{i,v}^{t,0}(\mathfrak{o}_v^{\times})|}$$

• ≤ 1 for $|\xi| \leq 1$; • $\leq |\xi|_v^{-M}$ for $|\xi|_v > 1$, where M is a prescribed large integer (depending on the global field k, s. the proof of Proposition

In our application, we will actually have $f_{1,v}^{t,0}=f_{4,v}^{t,0}$ and $f_{2,v}^{t,0}=f_{3,v}^{t,0}$, but this will not be the case for the relative trace formula and therefore we consider them as different vectors, to fix ideas.

The relations between the basic vectors and the asymptotic behavior of $f_{1,v}^{t,0}$ and $f_{4,v}^{t,0}$ around $\xi=0$ also determine the asymptotic behavior of $f_{2,v}^{t,0}$ and $f_{3,v}^{t,0}$ around $\xi=0$ and $\xi=\infty$ by Lemma 3.2.2. The following table summarizes the regular values and asymptotic behavior of basic vectors:

i	$f_{i,v}^{t,0}$ around $\xi = 0$	$f_{i,v}^{t,0}$ around $\xi = \infty$	$f_{i,v}^{t,0}(\mathfrak{o}_v^{ imes})$
1 or 4	$L_v(\eta_v, t) + L_v(\eta_v, -t)\eta_v(\xi) \xi _v^t$	0	$c_{i,v}^t$
2 or 3	$L_v(\eta_v, t+1)$	$L_v(\eta_v, t+1) \cdot \eta_v(\xi) \xi _v^{-t-1}$	$c_{i,v}^t$
			(3.8)

The axioms allow us to make sense of the restricted, completed tensor products of local Schwartz spaces with respect to the basic vectors as functions on:

$$\mathcal{B}^{\mathrm{reg}}(\mathbb{A}_k) = \mathbb{A}_k^{\times},$$

for every t when i = 1 or 4, and for large t when i = 2 or 3. We will denote these global Schwartz spaces by \mathcal{S}_i^t , i.e., dropping the index v from the local notation. The parameter t now varies in \mathbb{C} , in the number field case, and in $\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z}$, in the function field case (with base field \mathbb{F}_q). Moreover,

the axioms allow us to interpret Fourier transforms and the operators ι_t as isomorphisms between the global spaces:

$$\begin{split} \mathcal{F} : \mathcal{S}_1^t & \xrightarrow{\sim} & \mathcal{S}_2^t \\ \iota_t : \mathcal{S}_2^t & \xrightarrow{\sim} & \mathcal{S}_3^t \\ \mathcal{F} : \mathcal{S}_3^t & \xrightarrow{\sim} & \mathcal{S}_4^t \\ \mathcal{G}_t : \mathcal{S}_1^t & \xrightarrow{\sim} & \mathcal{S}_4^t. \end{split}$$

Recall that Fourier transform makes sense when t (resp. -t) does not belong to the values (A.7), while \mathcal{G}_t makes sense for every t.

3.4. Irregular distributions. We define the functional O_0 on the global Schwartz spaces \mathcal{S}_1^t and \mathcal{S}_4^t which, formally, for $t \notin \mathbb{Z}$ assigns to an element $f = \bigotimes_v f_v$ with asymptotics: $f_v(\xi) = C_{1,v}^t(\xi) + C_{2,v}^t(\xi) \eta_v(\xi) |\xi|^t$ the value

$$C_1^t + C_2^t = \prod_v C_{1,v}^t(0) + \prod_v C_{2,v}^t(0).$$

The rigorous definition is as follows:

$$\tilde{O}_0(f) := L^S(\eta, t) \prod_{v \in S} C_{1,v}^t(0) + L^S(\eta, -t) \prod_{v \in S} C_{2,v}^t(0), \tag{3.9}$$

where S is large enough so that outside of S we have $f_v = f_{i,v}^0$. It extends continuously to all elements of S_i^t .

We similarly define functionals \tilde{O}_0 and \tilde{O}_{∞} on \mathcal{S}_2^t and \mathcal{S}_3^t when t is large. Since there is no term of the form $\eta_v(\xi)|\xi|_v^t$ in a neighborhood of zero here, for an element $f = \bigotimes_v f_v \in \mathcal{S}_2^t$ or \mathcal{S}_3^t , we have

$$\tilde{O}_0(f) := L^S(\eta, t+1) \prod_{v \in S} f_v(0), \tag{3.10}$$

while if $f_v(\xi) = D_v^t(\frac{1}{\xi})\eta_v(\xi)|\xi|_v^{-t-1}$ for ξ in a neighborhood of ∞ , we have

$$\tilde{O}_{\infty}(f) = L^{S}(\eta, t+1) \prod_{v \in S} D_{v}^{t}(0).$$
 (3.11)

The factor $L^S(\eta, t+1)$ has to do with the asymptotic behavior of the basic function, cf. Table (3.8).

We will now verify that for an analytic section of S_1^t or S_4^t the functional \tilde{O}_0 extends at t=0 to the "irregular orbital integral" \tilde{O}_0 defined in §2.4.

3.4.1. **Proposition.** For i = 1 or 4, and an analytic section $t \mapsto f^t \in \mathcal{S}_i^t$, the function $t \mapsto \tilde{O}_0(f^t)$ extends holomorphically to all t, and its value is bounded by polynomial seminorms on any bounded vertical strip; in particular, its value at some t_0 depends only on f^{t_0} and not on the section. At t = 0 it coincides with the functional denoted by $\tilde{O}_0(f^0)$ in §2.4.

Proof. We start by proving the assertions for t=0. The issue is, of course, that as $t\to 0$ some of the local factors may blow up, according to Proposition 3.2.1.

In the non-split case, the product $L^S(\eta,t)\prod_{v\in S}C^t_{1,v}(0)$ is holomorphic at t=0; indeed, this is the case for the full Dirichlet L-function $L(\eta,t)$, and the Euler factors of our product have at most the order of pole of the local L-factors, as follows from Proposition 3.2.1. Hence, the value of this expression at t=0 is equal to the product of the leading coefficients of its factors, which is precisely equal to

$$L^{S}(\eta,0)^{*} \cdot \prod_{v \in S} \tilde{O}_{0_{v},+}(f_{v}^{0}),$$

where $L^S(\eta,0)^*$ denotes the leading term of $L^S(\eta,t)$ at t=0. Similarly for the other term of (3.9); the sum of the two terms coincides with the definition of \tilde{O}_0 in (2.15).

Regarding the bound by polynomial seminorms: If the order of zero of $L^S(\eta,t)$ at t=0 is r, then both the functions $L^S(\eta,-t)t^{-r}$ and $t^r\prod_{v\in S}C^t_{2,v}(0)$ are holomorphic in a vertical strip around zero; the first is of polynomial growth by standard properties of abelian L-functions, and the second is bounded by polynomial seminorms on $\otimes_{v\in S}\mathcal{S}^t_{i,v}$ by the definition of those in Appendix A: namely, if we take Fourier transforms of the local factors f^t_v , then by Lemma 3.2.2 those will be of the form $D^t_v(\frac{1}{\xi})\eta_v(\xi)|\xi|^{-t-1}_v$ in a neighborhood of infinity, with $D^t_v(0) = \gamma(\eta, -t, \psi) \cdot C^t_{2,v}(0)$. Recall that the factor $D^t_v(0)$ is by definition bounded by polynomial seminorms, hence so is the product $t^r\prod_{v\in S}C^t_{1,v}(f^t_v)$ is bounded by polynomial seminorms, we recall, again from the appendix, that multiplication by $\eta_v(\bullet)|\bullet|^{-t}_v$ defines an isomorphism between the spaces $\mathcal{S}^t_{i,v}$ and $\mathcal{S}^{-t}_{i,v}$ which preserves the structures of polynomial seminorms, so this reduces the problem to the previous case.

In the split case, $\zeta^S(t)$ has a zero of order |S|-1 at t=0, while the factors $C_{1,v}^t(0), \prod_{v \in S} C_{2,v}^t(0)$ each have a simple pole (at most) with opposite residues. Thus, the residue of

$$c_1(t) := \zeta^S(t) \prod_{v \in S} C_{1,v}^t(0)$$

is opposite to the residue of

$$c_2(t) := \zeta^S(-t) \prod_{v \in S} C_{2,v}^t(0),$$

and the sum of the two terms is regular at t=0. The proof of boundedness by polynomial seminorms is similar to the nonsplit case and is left to the reader. We now verify that the extension of the functional to t=0 coincides with that of §2.4.

In what follows, we set $A_v(t) = tC_{1,v}^t(0)$ and $B_v(t) = tC_{2,v}^t$; then $A_v(0) = -B_v(0) = \tilde{O}_0(f_v^0)$, and $A_v'(0) + B_v'(0) = \tilde{O}_u(f_v^0)$. We denote by $\zeta^S(0)^*$ the leading term of $\zeta^S(t)$ at t = 0, and we write -0 instead of 0 to signify that

we are replacing t by -t.

$$\lim_{t \to 0} (c_1(t) + c_2(t)) = \lim_{t \to 0} \left[(tc_1(t)) \cdot \frac{1}{t} \left(1 + \frac{c_1(t)}{c_2(t)} \right) \right] =$$

$$= \zeta^S(0)^* \prod_{v \in S} \tilde{O}_0(f_v^0) \cdot \frac{d}{dt} \left. \frac{c_2(t)}{c_1(t)} \right|_{t=0} =$$

$$= \zeta^S(0)^* \prod_{v \in S} \tilde{O}_0(f_v^0) \cdot \frac{c_2(0)}{c_1(0)} \cdot \left(\partial \log \frac{\zeta^S(-0)}{\zeta^S(0)} + \sum_{v \in S} \partial \log \frac{c_{2,v}(0)}{c_{1,v}(0)} \right) =$$

$$= \zeta^S(0)^* \prod_{v \in S} \tilde{O}_0(f_v^0) \cdot \left(\partial \log \frac{\zeta^S(0)}{\zeta^S(-0)} + \sum_{v \in S} \left(\frac{A_v'(0)}{A_v(0)} - \frac{B_v'(0)}{B_v(0)} \right) \right) =$$

$$= \zeta^S(0)^* \prod_{v \in S} \tilde{O}_0(f_v^0) \cdot \left(\partial \log \frac{\zeta^S(0)}{\zeta^S(-0)} + \sum_{v \in S} \frac{\tilde{O}_u(f_v^0)}{\tilde{O}_0(f_v^0)} \right).$$

This is precisely the term $\tilde{O}_0(f^0)$ of (2.16), which completes the proofs for t=0.

I leave the proof for other integer values of t to the reader. I remark that, for example, when $k_v = \mathbb{R}$ and $\eta_v = 1$ the limit of $C_{2,v}^t(0)$ as t approaches a positive even integer may be infinite, but this coincides with a trivial zero of the partial L-function $L^S(\eta, -t)$ representing the formal product: $\prod_{v \notin S} C_{2,v}^t(0)$.

3.5. The Poisson sum. We define the following functionals on the global Schwartz spaces S_i^t , all denoted by PS for "Poisson sum". When i = 1 or 4:

$$PS_i: \mathcal{S}_i^t \ni f \mapsto \tilde{O}_0(f) + \sum_{\xi \in k^{\times}} f(\xi), \tag{3.12}$$

When i = 2 or 3 and t is large:

$$PS_i: \mathcal{S}_i^t \ni f \mapsto \tilde{O}_0(f) + \tilde{O}_{\infty}(f) + \sum_{\xi \in k^{\times}} f(\xi). \tag{3.13}$$

The following is immediate (assuming convergence, which will be proved right afterwards):

3.5.1. **Lemma.** Consider the map $\iota_t : \mathcal{S}_2^t \xrightarrow{\sim} \mathcal{S}_3^t$ defined in (3.3). It preserves Poisson sums, i.e., the pull-back of PS_3 via this map is the functional PS_2 .

Now we discuss convergence:

3.5.2. **Proposition.** The functional $S_i^t \ni f^t \mapsto \sum_{\xi \in k^\times} f^t(\xi)$ converges absolutely for every t when i = 1, 4, and for $\Re(t) \gg 0$ when i = 2, 3. For such values of t, the sum is bounded in vertical strips by polynomial seminorms on the spaces S_i^t ; in particular, for an analytic section $t \mapsto f^t \in S_i^t$ the value of the functional is analytic in t.

Moreover, if we replace the basic functions by 1_{o_v} (the characteristic functions of the integers) outside of a finite set S of places, the assertion remains true for $\Re(t) \gg 0$, and on any vertical strip there is a bound by polynomial seminorms which is uniform in S.

Proof. Let $f = \prod_{v \notin T} f_v^{t,0} \cdot f_T \in \mathcal{S}_i^t$, where T is a finite set of places and $f_T \in \widehat{\otimes}_{v \in T} \mathcal{S}_{i,v}^t$.

In cases i = 1 or 4, by the axioms of §3.3, the basic function $f_{i,v}^{t,0}$ is supported on the integers of k_v and on every vertical strip we have a bound

$$\left| \prod_{v \notin T} f_{i,v}^{t,0}(\xi_T) \right| \le C(t) \cdot |\xi_T|^{-r_i},$$

where $r_i \geq 0$ is a constant and C(t) is of polynomial growth in vertical strips. Notice that for $\Re(t) \gg 0$, where the Euler product of the regular values is convergent, such an estimate holds, uniformly in S, if we replace $f_{i,v}^{t,0}$ by $1_{\mathfrak{o}_v}$ for $v \notin S$.

On the other hand, f_T is of rapid decay, i.e., $f_T(\xi_T)$ vanishes faster than any power of $|\xi_T|$, and from this it easily follows that the sum over k^{\times} is absolutely convergent.

For S_2^t and S_3^t I refer the reader to the more general Proposition 4.2.1. \square

Combining this with Proposition 3.4.1 we get:

- 3.5.3. Corollary. For an analytic section $t \mapsto f^t \in \mathcal{S}_i^t$, the number $PS_i(f^t)$ varies analytically for $\Re(t) \gg 0$ when i = 2, 3, and for all t when i = 1, 4.
- 3.6. **Poisson summation formula.** We are ready to prove the main result of this section:
- 3.6.1. **Proposition.** For $\Re(t) \gg 0$, f_1 an element of the global Schwartz space S_1^t and f_3 an element of S_3^t , we have:

$$PS_1(f_1) = PS_2(\mathcal{F}f_1),$$
 (3.14)
 $PS_3(f_3) = PS_4(\mathcal{F}f_3).$

Both equations amount to the same, of course.

This immediately implies:

3.6.2. Corollary. For every t and $f \in \mathcal{S}_1^t$:

$$PS_1(f) = PS_4(\mathcal{G}_t f). \tag{3.15}$$

Proof. Given $f \in \mathcal{S}_1^{t_0}$ (for a fixed t_0), it can be realized as is the specialization of an analytic section $f^t \in \mathcal{S}_1^t$, as t varies in the parameter space. Since $PS_i(f^t)$ is analytic in t (Lemma 3.5.3), it suffices to prove it for large t; but then it follows from the above proposition and the fact that Poisson sums are preserved under ι_t (Lemma 3.5.1).

Proof of Proposition 3.6.1. We can approximate both sides of (3.14) by expressions which depend on a finite set of places T, in the limit as T tends to include all places.

Indeed, we notice:

3.6.3. Lemma. If $f = \bigotimes_{i=1}^{t} f_{i} \in \mathcal{S}_{i}^{t}$, $\Re(t) \gg 0$, then

$$\lim_{T} \sum_{\xi \in k} \prod_{v \notin T} 1_{\mathfrak{o}_{v}}(\xi) \prod_{v \in T} f_{v}(\xi) = \sum_{\xi \in k} f(\xi). \tag{3.16}$$

Notice that the sums on both sides include $\xi = 0$; since we have only explained how to think of elements of \mathcal{S}_i^t as functions on \mathbb{A}_k^{\times} , this needs some explanation. Recall from Table (3.8) that the asymptotic behavior of $f_{i,v}^{t,0}$ around $\xi = 0$ is of the form

$$f_{i,v}^{t,0}(\xi) = L_v(\eta_v, t) + L_v(\eta_v, -t)\eta_v(\xi)|\xi|_v^t$$

when i = 1 or 4, and of the form

$$f_{i,v}^{t,0}(\xi) = L_v(\eta_v, 1+t)$$

when i=2 or 3. Moreover, for $\Re(t)>0$ the elements of $\mathcal{S}^t_{i,v}$ extend continuously to $\xi=0$. Therefore it is natural, for large t, to extend the evaluation of elements of \mathcal{S}^t_i to $\xi=0$ by taking the Euler product of their local extensions to $\xi=0$.

To prove the lemma, notice that for any given ξ , we clearly have

$$\lim_{T} \prod_{v \notin T} 1_{\mathfrak{o}_{v}}(\xi) \prod_{v \in T} f_{v}(\xi) = f(\xi).$$

By Proposition 3.5.2 we may interchange the sum over ξ and the limit over T, and this proves the lemma.

Now for every given T, the function $\prod_{v \notin T} 1_{\mathfrak{o}_v} \prod_{v \in T} f_v$ satisfies conditions for the usual Poisson summation formula: it is continuous, decays faster than $|\xi|^{-1-\delta}$ at infinity, and its Fourier transform $\prod_{v \notin T} 1_{\mathfrak{o}_v} \prod_{v \in T} \mathcal{F}(f_v)$ also has the same properties (since $\mathcal{F}f_v$ belongs to $\mathcal{S}_{2,v}^t$ if f_v belongs to $\mathcal{S}_{1,v}^t$ and vice versa). Hence we have (say, for $f \in \mathcal{S}_1^t$)

$$\sum_{\xi \in k} \prod_{v \notin T} \mathbf{1}_{\mathfrak{o}_v}(\xi) \prod_{v \in T} f_v(\xi) = \sum_{\xi \in k} \prod_{v \notin T} \mathbf{1}_{\mathfrak{o}_v}(\xi) \prod_{v \in T} \mathcal{F} f_v(\xi).$$

Taking the limit with T we get

$$\sum_{\xi \in k} f(\xi) = \sum_{\xi \in k} \mathcal{F}f(\xi). \tag{3.17}$$

Finally, if we add to the above the relations (3.5) and (3.6) and take into account the fact that the global gamma factors $\gamma(\eta, -t, \psi)$ and $\gamma(\eta, -t, \psi)$ are equal to 1, we get the desired result.

3.7. **Application: the baby case.** Finally, we describe a deformation $S_1^t = S_4^t$ of the global Schwartz space $S(\mathcal{X}(\mathbb{A}_k))$ of the baby case, and verify that the spaces S_i^t $(i = 1 \dots 4$, where the spaces $S_2^t = S_3^t$ are obtained by Fourier transform from S_1^t) satisfy the postulated axioms.

First of all, it is clear from the definitions that the local Schwartz spaces $S_{1,v}^t$, $S_{4,v}^t$ specialize to $S(\mathcal{X}_v)$ when t=0. We now endow them with the following basic function, which, as we will see, on one hand coincides with the basic function of $S(\mathcal{X}_v)$ when t=0, and on the other satisfies the axioms of §3.3:

$$f_{1,v}^{t,0}(\xi) := \begin{cases} \frac{1 - \eta(\varpi_v) q_v^{-t} \cdot \eta_v(\xi) |\xi|_v^t}{1 - \eta_v(\varpi_v) q_v^{-t}}, & \text{when } t \neq 0 \text{ or } \eta_v \neq 1; \\ 1 - \log_{q_v} |\xi|_v, & \text{when } t = 0 \text{ and } \eta_v = 1. \end{cases}$$
(3.18)

This function can be obtained by suitably normalized orbital integrals corresponding to $(k_v^{\times}, \eta_v(\bullet)|\bullet|_v^t)$ -coinvariants of the characteristic function of $1_{\mathfrak{o}^2}$; however, this is not important for us here. What is important is the following:

3.7.1. **Lemma.** Let $X = \operatorname{Res}_{E/k} \mathbb{G}_a$ as in the previous section, and consider the action of T = U(1) on X; thus $\mathcal{X} = X/T$. Then for t = 0 and for a suitable Haar measure on $T(k_v)$, the above basic function is equal to image (i.e., the orbital integrals) in $\mathcal{S}(\mathcal{X}(k_v))$ of the characteristic function of $X(\mathfrak{o}_v)$.

The proof is an easy calculation and will be omitted. This lemma shows that for t=0 we get, indeed, the global Schwartz space $\mathcal{S}(\mathcal{X}(\mathbb{A}_k))$ of the "baby case".

A calculation as in the proof of Lemma 3.2.2 shows that for large t and $|\xi| \leq 1$

$$\mathcal{F}(f_{1,v}^{t,0})(\xi) = \frac{1}{1 - \eta(\varpi_v)q_v^{-t}} \left(1 - \eta(\varpi_v)q_v^{-t} \frac{1 - q_v^{-1}}{1 - \eta(\varpi_v)q_v^{-t-1}} \right) = L(\eta_v, t+1), \tag{3.19}$$

while for large t and $|\xi| > 1$

$$\mathcal{F}(f_{1,v}^{t,0})(\xi) = \frac{-\eta(\varpi_v)q_v^{-t}}{1 - \eta(\varpi_v)q_v^{-t}} \cdot \frac{1 - \eta(\varpi_v)q_v^t}{1 - \eta(\varpi_v)q_v^{-t-1}} \cdot \eta(\xi)|\xi|^{-t-1} = L(\eta_v, t+1)\eta(\xi)|\xi|^{-t-1}.$$
(3.20)

In particular, the function $f_{2,v}^{t,0} = \mathcal{F}(f_{1,v}^{t,0})$ is equal to $L_v(\eta_v, t+1)$ on \mathfrak{o}_v^{\star} , and is preserved by the transformation ι_t . Hence $f_{3,v}^{t,0} = f_{2,v}^{t,0}$ and $f_{4,v}^{t,0} = f_{1,v}^{t,0}$. It is immediate to verify that these functions satisfy the axioms of §3.3, hence Corollary 3.6.2 holds for $\mathcal{S}_1^t = \mathcal{S}_4^t = \mathcal{S}(\mathcal{X}(\mathbb{A}_k))$. This gives a direct proof of Theorem 2.4.2.

4. Poisson summation for the relative trace formula

We now return to our main problem, namely proving a Poisson summation formula between the torus relative trace formula (2.5) (viewed as a functional

on the global Schwartz space $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$, defined in §2.1) and the Kuznetsov formula with non-standard sections (2.6) (viewed as a functional on the global Schwartz space $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$). As explained in the introduction, this will not be possible "on the nose", because the sum (2.6) does not converge at the desired point of evaluation s=0; therefore, we will also need to deform the space $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ with a parameter s, and prove an identity for $\Re(s) \gg 0$.

Thus, here we will have two complex parameters, s and t. The parameter t will, as in the previous chapter, help us deform an exponent of the asymptotics of Schwartz functions by "larger" exponents so that our functions vanish faster and are suitable for Poisson summation. The parameter s will be more than just a technical tool, and it parametrizes the space $S(W^s(\mathbb{A}_k))$ of non-standard test functions for the Kuznetsov trace formula in such a way that (when t = 0) this space corresponds to the L-function

$$L(\pi, \frac{1}{2} + s)L(\pi \otimes \eta, \frac{1}{2} + s),$$

as we saw in $\S 2.1$.

The main result of this section is Theorem 4.7.1. It is a Poisson summation formula for large values of the parameter s. Its Corollary 4.7.3 shows that the functional KTF of (2.6) can be analytically continued to arbitrary values of s; however, we will have no explicit expression for this functional at s=0, and we will continue to work in a domain of convergence of the Euler product for the above L-function ($\Re(s) \gg 0$) when we perform the spectral analysis in the next sections.

As in the baby case, we proceed axiomatically by defining various Schwartz cosheaves on the projective line, before applying the theory to the relative trace formulas. To avoid heavy notation, throughout our discussion in most of this section, the parameter s will be fixed and will usually not appear explicitly in the notation. Moreover, "the parameters" refers to the parameters $s, t, 2s \pm t$, and "large values of the parameters" means large values of their real parts.

4.1. Axioms for the local Schwartz spaces. As in the baby case, we introduce four local Schwartz spaces $\mathcal{S}_{i,v}^t$ $(i=1,\ldots,4)$ (the parameter s will be implicit). For this section we denote by \mathcal{B}^{reg} the complement of $\{-1,0,\infty\}$ in one-dimensional projective space. (The point -1 will not literally be irregular in all cases, but to condense and unify notation we consider it as such.) The cosheaves $\mathcal{S}_{i,v}^t$, restricted to $\mathcal{B}^{\text{reg}}(k_v)$, all coincide with the cosheaf of Schwartz functions. The definition of these four spaces, for generic values of the parameters, is completed by the following table, which describes the asymptotic behavior of their elements close to the "singular points". Here, the C_i 's and D_i 's are smooth functions.

i	around $\xi = 0$	around $\xi = -1$
1 or 4	$C_1(\xi) + C_2(\xi)\eta(\xi) \xi ^t$	$C_3(\xi) + C_4(\xi)\eta(\xi+1) \xi+1 ^{2s}$
2 or 3	$D_1(\xi) + D_2(\xi) \xi ^{-t+2s}\psi\left(\frac{1}{\xi}\right)$	$D_3(\xi)$
(Continued) i	around $\xi = \infty$	
1 or 4	$C_5(\xi^{-1}) \xi ^{-s+\frac{t}{2}-1}\cdot\mathcal{K}(\xi)$	
2 or 3	$ D_4(\xi^{-1})\eta(\xi) \xi ^{-t-1} + D_5(\xi^{-1})\psi(\xi)\eta(\xi) \xi ^{-2s-1} $	

(4.1)

Here and later, \mathcal{K} (for "Kloosterman"), at non-Archimedean places, denotes the function which is supported on $|\xi| > 1$ and equal to

$$\int_{|x|^2 = |\xi|} \psi(x - \xi x^{-1}) dx \tag{4.2}$$

there. For the Archimedean case, the analogous "Kloosterman germ" at infinity is the germ of functions as in [Sak13a, (4.27)]. This definition is actually correct for the space $\mathcal{S}_{4,v}^t$; for the space $\mathcal{S}_{1,v}^t$ we need to replace ξ by $-\xi$; however, since in our application to the relative trace formula these Kloosterman germs will not appear for i=1 (i.e., the germ of the function C_5 at zero will be zero), we do not introduce new notation for this minor modification.

In the limit when the exponents become integers (notably, when s = t = 0 and $\eta_v = 1$), we may also have logarithmic terms; the limiting behavior in those cases has been described in Appendix A, and is completely analogous to the baby case. The Fréchet structure on these spaces is also described in the appendix.

Again, the spaces $S_{2,v}^t$ and $S_{3,v}^t$ are mapped isomorphically onto each other by the operator ι_t of (3.3):

$$\iota_t(f) = \frac{\eta(\bullet)}{|\bullet|^{t+1}} f\left(\frac{1}{\bullet}\right).$$

In analogy to Proposition 3.2.1 and Lemma 3.2.2, we have:

4.1.1. **Proposition.** For t and 2s outside the exceptional values (A.7), Fourier transform carries $S_{1,v}^t$ isomorphically to $S_{2,v}^t$, and $S_{3,v}^t$ to $S_{4,v}^t$. If $f \in S_{1,v}^t$ has asymptotics denoted by C_i as in table (4.1), the corresponding asymptotic coefficients for its Fourier transform are:

$$D_4(0) = \gamma(\eta_v, -t, \psi_v) C_2(0),$$

$$D_2(0) = C_5(0),$$

$$D_5(0) = \gamma(\eta_v, -2s, \psi_v) C_4(0)$$
(4.3)

Similarly, if $f \in \mathcal{S}_{3,v}^t$ has asymptotics denoted as in table (4.1), then its asymptotic coefficients D_i are related to the asymptotic coefficients of its

Fourier transform as follows:

$$D_4(0) = \gamma(\eta_v, -t, \psi_v^{-1})C_2(0),$$

$$D_2(0) = C_5(0),$$

$$D_5(0) = \gamma(\eta_v, -2s, \psi_v^{-1})C_4(0)$$
(4.4)

The transformation \mathcal{G}_t of (3.4) carries $\mathcal{S}_{1,v}^t$ isomorphically to $\mathcal{S}_{4,v}^t$, even for values of t and 2s as in (A.7).

Both \mathcal{F} and \mathcal{G}_t are bounded by polynomial families of seminorms on the corresponding spaces, as t varies, and preserve analytic sections.

In our application to the relative trace formula, unlike in the baby case, the spaces $S_{1,v}^t$ and $S_{4,v}^t$ will not have the same basic functions; actually, we will restrict to the subspace of $S_{1,v}^t$ where the germ of C_5 is zero (i.e., elements of $S_{1,v}^t$ will standard Schwartz functions away from 0, -1), and accordingly for $S_{4,v}^t$ we will have $C_4 = 0$ (sections of $S_{4,v}^t$ will be smooth at -1). This is very important in order to be able to continue to s = 0 (continuation will be a result of the Poisson sum for $S_{1,v}^t$ being the sum of a rapidly decaying function), but it is not important in proving the Poisson summation formula for large values of the parameters, and therefore we use this more general approach which applies to S_1^t and S_4^t simultaneously.

Finally, we note that the Schwartz space $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ for the Kuznetsov trace formula with non-standard test functions depending on a parameter s (defined in [Sak13a], section 6) will belong to the space

$$|\bullet|^{s+1}\cdot\mathcal{S}_4^0.$$

4.2. Global Schwartz spaces and convergence of the Poisson sum. As before, we define global Schwartz spaces S_i^t as restricted tensor products with respect to basic vectors $f_{i,v}^{t,0}$ (again, the parameter s is implicit in the notation). These basic vectors satisfy the compatibility relations:

$$f_{2,v}^{t,0} = \mathcal{F}(f_{1,v}^{t,0}),$$

$$f_{3,v}^{t,0} = \iota_t f_{2,v}^{t,0}$$

$$f_{4,v}^{t,0} = \mathcal{F}(f_{3,v}^{t,0}).$$

$$(4.5)$$

Besides, they are required to satisfy, for large values of the parameters, the following axioms:

- (1) Basic functions are constant on $\mathcal{B}^{\text{reg}}(\mathfrak{o}_v)$ and their values there (their "regular values") are such that the corresponding Euler products converge, locally uniformly in the real parts of the parameters.
- (2) The constant terms of the asymptotics of basic functions (i.e., $C_1(0)$, $C_3(0)$, $D_1(0)$, in the notation of Table 4.1) give convergent Euler products, locally uniformly in the real parts of the parameters. For the rest of the asymptotic constants (the rest of the numbers $C_i(0), D_i(0)$) we only need to assume that one representative in each

implicit chain of identities given by (4.3), (4.4) and (4.5) has a convergent Euler product, locally uniformly in the real parts of the parameters. For example, the constants $C_5(0)$ of $f_{1,v}^{t,0}$, $D_2(0)$ of $f_{2,v}^{t,0}$, $D_5(0)$ of $f_{3,v}^{t,0}$ and $C_4(0)$ of $f_{4,v}^{t,0}$ are all determined by each other via these formulas, and it is enough to have a convergent Euler product for one of them.

Remark. In the baby case, §3.3, we explicitly gave the values of those asymptotic constants in our axioms for the basic functions. The reader can consult table 4.15 to see the values that will be used for the relative trace formula.

(3) There is a constant $r_i \geq 0$ (possibly different for each of the spaces), independent of the place, with the property that the function:

$$|1 + \xi|_v^{r_i} \cdot |\xi|^{r_i} \cdot \frac{|f_{i,v}^{t,0}(\xi)|}{|f_{i,v}^{t,0}(\mathcal{B}^{\text{reg}}(\mathfrak{o}_v))|}$$
 is

- ≤ 1 for $|\xi| \leq 1$;
- $\leq |\xi|^{-M}$ for $|\xi| > 1$, where M is a prescribed large positive number (depending on the global field, s. the proof of Proposition 4.2.1).

I repeat that these axioms hold for large values of the parameters, and they will be enough to prove a Poisson summation formula for such values, Proposition 4.4.2. We will then specialize to the specific basic functions showing up in our relative trace formulas, which will have additional properties allowing analytic continuation to other values of the parameters (§4.7). (In the baby case, §3.3, we added these additional properties to the axioms, but that would be too cumbersome to do in this more complicated case.)

Given the above axioms, we have the following generalization of Proposition 3.5.2:

4.2.1. **Proposition.** For large values of the parameters and f in one of the global Schwartz spaces S_i^t , the functional

$$f \mapsto \sum_{\xi \in \mathcal{B}^{\text{reg}}(k)} f(\xi)$$

converges absolutely, and can be bounded by polynomial seminorms on the space S_i^t in any bounded vertical strip. Moreover, if we replace the basic functions by 1_{o_v} outside of a finite set S of places, the assertion remains true, and the polynomial bound can be taken to be independent of the set S.

Proof. I prove it for number fields, and leave the case of function fields to the reader. Up to a convergent Euler product, the sum is the same if we replace f_v by

$$|1 + \xi|_v^{r_i} \cdot |\xi|_v^{r_i} \frac{f_v(\xi)}{f_v(\mathcal{B}^{\text{reg}}(\mathfrak{o}_v))}$$

(where in a finite set of places we interpret the denominator as 1), therefore we may assume that the basic functions themselves are ≤ 1 for $|\xi|_v \leq 1$, $\leq |\xi|_v^{-M}$ for $|\xi|_v > 1$, and their regular value is 1.

We define a "height function" on every k_v by: $r_v(\xi) = \max(1, |\xi|_v)$. Notice that for $\xi \in k^{\times}$ we have $r_v(\xi) = 1$ for almost every place, therefore $r(\xi) = \prod_v r_v(\xi)$ makes sense. If $k = \mathbb{Q}$ then, for (m, n) = 1 we have

$$r\left(\frac{m}{n}\right) = \max(1, \left|\frac{m}{n}\right|_{\mathbb{R}}) \cdot |n|_{\mathbb{R}} = \max(|m|_{\mathbb{R}}, |n|_{\mathbb{R}}).$$

In general, we claim:

4.2.2. Lemma. There is a positive number N such that

$$\#\{\xi \in k | r(\xi) < T\} < T^N \text{ for all } T \gg 0.$$
 (4.6)

Indeed, choose a basis $(v_i)_{i=1}^n$ of k over $\mathbb Q$ and define modified height functions:

$$r'_v(x = \sum a_i v_i) = \prod_{i=1}^n r_v(a_i),$$

where the height functions on the right hand side are those of \mathbb{Q}_v . At almost every place k_v/\mathbb{Q}_v is unramified, and the elements v_i are integral and a basis for the unramified residue field extension; for those places: $|x|_v = \max_i |a_i|_{k_v} = \max_i |a_i|_{\mathbb{Q}_v}^n$ and therefore $r_v'(x) \leq r_v(x) \leq r_v'(x)^n$. A similar relation holds for the remaining finite set of places, but with certain positive constants: $m_{1,v}r_v'(x) \leq r_v(x) \leq m_{2,v}r_v'(x)^n$, and therefore also globally on k-points:

$$m_1 r'(\xi) \le r(\xi) \le m_2 r'(\xi)^n, \ \xi \in k.$$

This reduces the lemma to the case of $k = \mathbb{Q}$, where it is obvious.

We continue with the proof of the proposition. By our axioms for the basic functions, we have the following estimate on f:

$$|f(\xi)| \le C \cdot r(\xi)^{-M}$$

for $\xi \in \mathcal{B}^{\text{reg}}(k)$ and some constant C bounded by polynomial seminorms. (We assume that the real values of the parameters are large enough, so that the bound by a multiple of $|\xi|_v^{-M}$ when $|\xi|_v > 1$ holds at every place, not just for the basic functions.)

Hence

$$\sum_{\xi} |f(\xi)| \ll \sum_{T \in \mathbb{N}} T^N \cdot T^{-M},$$

which converges absolutely for large M, and the implicit constant is bounded by polynomial seminorms.

Notice that if we replace f_v by $f_v(\mathcal{B}^{reg}(\mathfrak{o}_v)) \cdot 1_{\mathfrak{o}_v}$ outside of a finite number of places S, the same estimates are true, uniformly in S. Finally, by the (locally uniform in the real part of the parameters) absolute convergence of the partial Euler product $\prod_{v \notin S} f_v(\mathcal{B}^{reg}(\mathfrak{o}_v))$, we deduce that the same is true if we replace f_v by $1_{\mathfrak{o}_v}$.

4.3. Irregular distributions. In complete analogy with the previous section, we define functionals \tilde{O}_0 , \tilde{O}_{-1} and \tilde{O}_{∞} on the global Schwartz spaces \mathcal{S}_i^t which formally, in the notation of Table 4.1 but adding an index v, are (for generic values of the parameters):

For i = 1 or 4:

$$\tilde{O}_{0}(f) = \prod_{v}' C_{1,v}(0) + \prod_{v}' C_{2,v}(0),$$

$$\tilde{O}_{-1}(f) = \prod_{v}' C_{3,v}(0) + \prod_{v}' C_{4,v}(0),$$

$$\tilde{O}_{\infty}(f) = \prod_{v}' C_{5,v}(0).$$

For i = 2 or 3:

$$\tilde{O}_0(f) = \prod_{v}' D_{1,v}(0) + \prod_{v}' D_{2,v}(0),$$

$$\tilde{O}_{-1}(f) = \prod_{v}' D_{3,v}(0),$$

$$\tilde{O}_{\infty}(f) = \prod_{v}' D_{4,v}(0) + \prod_{v}' D_{5,v}(0).$$

The rigorous definition is by using partial Euler products as in (3.9), which is possible by our axioms for the basic functions. These functionals extend analytically to all values of the parameters, as in §3.4.

4.4. Poisson summation for large values of the parameters. The functionals PS_i will be defined on the global Schwartz spaces S_i^t in a completely analogous way to the baby case, namely:

$$PS_i(f) = \tilde{O}_0(f) + \tilde{O}_{-1}(f) + \tilde{O}_{\infty}(f) + \sum_{\xi \in k \setminus \{-1,0\}} f(\xi).$$
 (4.7)

They are well-defined when the parameters are large. Again, the following is clear:

4.4.1. **Lemma.** The transformation $\iota_t: \mathcal{S}_2^t \xrightarrow{\sim} \mathcal{S}_3^t$ preserves Poisson sums.

We are now ready to prove the Poisson summation formula for large parameters:

4.4.2. **Proposition.** For large values of the parameters, f_1 an element of the global Schwartz space S_1^t and f_3 an element of S_3^t we have:

$$PS_1(f_1) = PS_2(\mathcal{F}(f_1));$$
 (4.8)

$$PS_3(f_3) = PS_4(\mathcal{F}(f_3)).$$
 (4.9)

Of course, both formulas amount to the same.

Proof. We essentially go over the same steps as in the baby case, Proposition 3.6.1. First, the analog of Lemma 3.6.3 holds: for large values of the parameters and $f = \bigotimes_v f_v \in \mathcal{S}_i^t$ we have:

$$\lim_{T} \sum_{\xi \in k} \prod_{v \notin T} 1_{\mathfrak{o}_{v}} \prod_{v \in T} f_{v}(\xi) = \sum_{\xi \in k} f(\xi), \tag{4.10}$$

where T is a finite set of places and the limit is taken as T includes every place. Again, we take summation over all points, not just regular ones, and here is where we use the fact that the asymptotic constant terms at irregular points give convergent Euler products. The argument is the same, using this time Proposition 4.2.1.

The rest of the proof is also the same: we apply the usual Poisson summation formula to the sum: $\sum_{\xi \in k} \prod_{v \notin T} 1_{\mathfrak{o}_v} \prod_{v \in T} f_v(\xi)$, taking the limit over T, and finally add the contributions of the rest of the terms to the Poisson sum, which match because of Proposition 4.1.1 and the global triviality of gamma factors.

Combining Proposition 4.4.2 and Lemma 4.4.1 we get:

4.4.3. Corollary. For large values of the parameters and $f \in \mathcal{S}_1^t$ we have:

$$PS_1(f) = PS_4(\mathcal{G}f).$$

4.5. The spaces of the relative trace formula. In [Sak13a], section 6, I defined a local "Schwartz space" of functions (here with an index v) $S(W_v^s)$, corresponding to the orbital integrals for a space of non-standard test functions for the Kuznetsov trace formula, depending on a parameter s. We also endowed this space (at almost every place) with a basic vector (which we will modify here by a volume factor, see below).

The following completely characterizes the space $\mathcal{S}(\mathcal{W}_{v}^{s})$:

4.5.1. **Lemma.** Multiplication by $|\bullet|^{-s-1}$ is an isomorphism:

$$S(\mathcal{W}_v^s) \xrightarrow{\sim} (S_{4,v}^0)',$$
 (4.11)

where $(S_{4,v}^0)'$ denotes the closed subspace of $S_{4,v}^0$ consisting of elements for which, in the notation of Table 4.1, the stalk of C_4 at -1 is zero (i.e., those elements of $S_{4,v}^0$ which are smooth at $\xi = -1$).

Moreover, the transform \mathcal{G} is an isomorphism:

$$(\mathcal{S}_{1,v}^0)' \xrightarrow{\sim} (\mathcal{S}_{4,v}^0)', \tag{4.12}$$

where $(S_{1,v}^0)'$ denotes the closed subspace of $S_{1,v}^0$ consisting of elements for which, in the notation of Table 4.1, the stalk of C_5 at 0 is zero (i.e., those elements of $S_{1,v}^0$ which are of rapid decay at infinity).

The first statement is a straightforward generalization of the description of $\mathcal{S}(\mathcal{W}_v^0)$ in [Sak13a], section 4, and the second statement is easy; we omit the proofs. For later use, we denote the space $(\mathcal{S}_{1,v}^0)'$ by $\mathcal{S}(\mathcal{Z}_v^s)$; at s=0 it specializes to the space $\mathcal{S}(\mathcal{Z}_v)$ of orbital integrals for the torus RTF, as proven in the matching theorem [Sak13a, Theorem 5.1].

To apply the Poisson summation formula that we proved to this space, we need to define the degeneration of its basic function with a parameter t and verify that it, and the corresponding basic functions for the other spaces $\mathcal{S}_{i,v}^t$, satisfy the axioms of §4.2. In this paper we will use the basic function for $\mathcal{S}(\mathcal{W}_v^s)$ in [Sak13a], Lemma 6.4, but **divided by a volume term** (for convenience, since such a volume term does not affect the Poisson summation formula); this modification, as we will see, will be compatible with our requirement for $f_{\mathcal{Z}_v}^0$ to have regular value 1. Here is that function, translated to the space $\mathcal{S}_{4,v}^0$ (i.e., multiplied by $|\bullet|_v^{-s-1}$):

$$f_{4,v}^{0,0}(\xi) := L(\eta_v, 2s+1) \left((I - q_v^{-2s-1} \varpi_v^2 \cdot) f(\xi) + |\xi|_v^{-s-1} 1_{|\xi|_v = q_v^2} + |\xi|_v^{-s-1} \mathcal{K}(\xi) \right), \tag{4.13}$$

where

- $\mathcal{K}(\xi)$ was defined in (4.2);
- f is the basic function of the "baby case" Schwartz space (supported on $|\xi|_v \leq 1$):

$$f(\xi) = \begin{cases} 1 - \log_{q_v} |\xi|_v, & |\xi|_v \le 1, \text{ in the split case,} \\ \frac{1 + \eta_v(\xi)}{2}, & |\xi|_v \le 1, \text{ in the non-split case;} \end{cases}$$

• the action of ϖ_v^2 is normalized as in (3.2).

Now we modify this to define the basic function $f_{4,v}^{t,0}$ as follows:

• we replace the function f by the basic function f^t of the baby case corresponding to arbitrary t, which for $t \neq 0$ is equal to:

$$(L(\eta_v, t) + L(\eta_v, -t)\eta_v(\xi)|\xi|_v^t) \cdot 1_{\mathfrak{o}_v};$$

• we replace the factor $|\xi|_v^{-s-1}$ by $|\xi|_v^{-s+\frac{t}{2}-1}$.

For the calculations that follow, we split $f_{4,v}^{t,0}$ into a sum $F_1 + F_2$ (we hide t and v from this notation, as we do throughout for the parameter s), where the summands are:

$$F_{1}(\xi) = L(\eta_{v}, 2s+1)(I - q_{v}^{-2s-1}\varpi_{v}^{2}\cdot)f_{t}(\xi)$$

$$F_{2}(\xi) = L(\eta_{v}, 2s+1)\left(q_{v}^{-2s+t-2}1_{|\xi|_{v}=q_{v}^{2}} + |\xi|_{v}^{-s+\frac{t}{2}-1}\mathcal{K}(\xi)\right). (4.14)$$

Remark. It can easily be computed that the function F_1 is equal, for small $|\xi|_v$, to

$$\left(\frac{L(\eta_v, 2s+1)}{\zeta_v(2s+2)}L(\eta_v, t) + \frac{L(\eta_v, 2s+1)}{\zeta_v(2s+2t+2)}L(\eta_v, -t)\eta_v(\xi)|\xi|_v^t\right).$$

- 4.6. Verification of the axioms. Here we verify that the basic function $f_{4,v}^{t,0}$ defined in the previous subsection gives rise to functions $f_{i,v}^{t,0}$ (i = 1...4) which satisfy the axioms of §4.2. For this subsection we drop the index v and set: $\epsilon = \eta_v(\varpi_v)$. (Recall that we are defining the basic functions outside of a finite set of places which includes the places of ramification of η ; therefore, $\eta_v(\varpi_v)$ makes sense.)
- 4.6.1. **Lemma.** The basic function $f_{4,v}^{t,0}$ defined in the previous subsection, and the basic functions $f_{i,v}^{t,0}$, i = 1, 2, 3, obtained from $f_{4,v}^{t,0}$ via (4.5), satisfy the axioms of §4.2.

Proof. The summand F_1 of (4.14) has inverse Fourier transform

$$L(\eta, 2s+1) \cdot (I-q^{-2s-1}\varpi^{-2})\mathcal{F}^{-1}f.$$

In particular, from (3.20) we deduce that its value on \mathfrak{o}^{\times} is equal to

$$\frac{L(\eta, 2s+1)}{\zeta(2s+2t+2)}L(\eta, t+1).$$

Notice that $\iota_t(a \cdot f) = \eta(a)|a|^{-t}a^{-1} \cdot (\iota_t f)$. Therefore

$$\iota_t \mathcal{F}^{-1}(F_1) = L(\eta, 2s+1) \cdot (I - q^{-2s-2t-1} \varpi^2 \cdot) \iota_t \mathcal{F}^{-1} f.$$

In particular, from (3.19) we deduce that its value on \mathfrak{o}^{\times} is equal to

$$\frac{L(\eta, 2s+1)}{\zeta(2s+2t+2)}L(\eta, t+1).$$

Finally, by applying inverse Fourier transform once more we get

$$\mathcal{G}_t^{-1}(F_1) = L(\eta, 2s+1) \cdot (I - q^{-2s-2t-1}\varpi^{-2})\mathcal{G}_t^{-1}f,$$

which by (3.18) takes on \mathfrak{o}^{\times} the value

$$L(\eta, 2s+1)$$
.

It can be seen that the inverse Fourier transform of $|x|^{-\frac{\alpha}{2}-1} \cdot \mathcal{K}(x) + q^{-\alpha-2}1_{|x|=q}(x)$ is

$$q^{-\alpha}1_{|x| \le q^{-2}} + |x|^{\alpha}\psi(\frac{1}{x})1_{|x| < 1}.$$

Hence,

$$\mathcal{F}^{-1}F_2(x) = L(\eta, 2s+1)q^{-2s+t}1_{|x| \le q^{-2}} + L(\eta, 2s+1)|x|^{2s-t}\psi\left(\frac{1}{x}\right)1_{|x| < 1}.$$

Its value on \mathfrak{o}^{\times} is equal to 0.

Now.

$$\iota_t \mathcal{F}^{-1} F_2(x) = L(\eta, 2s+1) q^{-2s+t} \eta(x) |x|^{-t-1} 1_{|x| \ge q^2} + L(\eta, 2s+1) \eta(x) |x|^{-2s-1} \psi(x) 1_{|x| > 1},$$
 with value on \mathfrak{o}^{\times} equal to 0.

Finally, it can be seen that $\eta(x)|x|^{-\delta-1}1_{|x|>1}$ is the Fourier transform of

$$\left(\frac{L(\eta,-\delta)}{L(\eta,\delta+1)}\eta(x)|x|^{\delta}-\frac{L(\eta,-\delta)}{\zeta(1)}\right)1_{\mathfrak{o}}(x).$$

Notice that $\eta(x)|x|^{-t-1}1_{|x|\geq q^2}=\epsilon q^{-t-\frac{1}{2}}\varpi\cdot \left(\eta(x)|x|^{-t-1}1_{|x|>1}(x)\right)$. Therefore,

$$\begin{split} \mathcal{G}_t^{-1} F_2(x) &= L(\eta, 2s+1) q^{-2s} \epsilon \left(\frac{L(\eta, -t)}{L(\eta, t+1)} \epsilon q^t \eta(x) |x|^t - \frac{L(\eta, -t)}{\zeta(1)} \right) \mathbf{1}_{|x| < 1}(x) + \\ &+ L(\eta, 2s+1) \left(\frac{L(\eta, -2s)}{L(\eta, 2s+1)} \eta(x+1) |x+1|^{2s} - \frac{L(\eta, -2s)}{\zeta(1)} \right) \mathbf{1}_{\mathfrak{o}}(x). \end{split}$$

Its value at integral points of $\mathbb{A}^1 \setminus \{-1,0\}$ is:

$$-\epsilon q^{-2s-1}L(\eta, 2s+1).$$

Based on these calculations, the axioms for the basic functions $f_{i,v}^{t,0}$ obtained from $f_{4,v}^{t,0}$ via (4.5) are easily checked. We present their regular values and asymptotic behavior in a table, which is the analog of (3.8), using the notation of Table (4.1). By "reg.val" we mean the value $f_{i,v}^{t,0}(\mathcal{B}^{\text{reg}}(\mathfrak{o}_v))$.

	ct.0	ct.0 1 c 4
i	$f_{i,v}^{t,0}$ around $\xi = 0$	$f_{i,v}^{t,0}$ around $\xi = -1$
4	$C_1(0) = \frac{L(\eta, 2s+1)L(\eta, t)}{\zeta(2s+2)}$	$C_3(0) = \text{Reg. val.}$
	$C_2(0) = \frac{L(\eta, 2s+1)L(\eta, -t)}{\zeta(2s+2t+2)}$	$C_4 = 0$
3	$D_1(0) = L(\eta, 2s+1)L(\eta, t+1)$	$D_3(0) = \text{Reg. val.}$
	$\cdot (1 - q^{-2s} + q^{-2s+t} - \epsilon q^{-2s-1})$	
	$D_2(0) = L(\eta, 2s+1)$	
2	$D_1(0) = \frac{L(\eta, 2s+1)L(\eta, t+1)}{\zeta(2s+2t+2)}$	$D_3(0) = \text{Reg. val.}$
	$D_2 = 0$	
1	$C_1(0) = L(\eta, 2s+1)L(\eta, t).$	$C_3(0) = L(\eta, 2s)$
	$(1 - q^{-2s - 2t} + q^{-2s - t} - \epsilon q^{-2s - 1})$	
	$C_2(0) = L(\eta, 2s+1)L(\eta, -t).$	$C_4(0) = L(\eta, -2s)$
	$\cdot (1 - q^{-2s} + q^{-2s+t} - \epsilon q^{-2s-1})$	
(Continued) i	$f_{i,v}^{t,0}$ around $\xi = \infty$	$f_{i,v}^{t,0}(\mathcal{B}^{\mathrm{reg}}(\mathfrak{o}_v))$
4	$C_5(0) = L(\eta, 2s+1)$	$L(\eta, 2s+1)$.
		$\left[\cdot [1 - q^{-2s-2}(1 + \epsilon q^{-t} + q^{-2t})] \right]$
3	$D_4(0) = \frac{L(\eta, 2s+1)L(\eta, t+1)}{\zeta(2s+2t+2)}$	$\frac{L(\eta,2s+1)L(\eta,t+1)}{\zeta(2s+2t+2)}$
	$D_5 = 0$	$\zeta(2s+2t+2)$
2	$D_4(0) = L(\eta, 2s+1)L(\eta, t+1)$	$\frac{L(\eta,2s+1)L(\eta,t+1)}{\zeta(2s+2t+2)}$
	$\cdot (1 - q^{-2s} + q^{-2s+t} - \epsilon q^{-2s-1})$	$\zeta(2s+2t+2)$
	$D_5(0) = L(\eta, 2s+1)$	
1	$C_5 = 0$	1
		(4.15)

I remark the following:

(1) The constant terms $(C_1(0), C_3(0), D_1(0))$, as well as the regular values, give convergent Euler products when $\Re(s), \Re(t), \Re(s \pm t) \gg 0$

- (i.e., "for large values of the parameters"). They admit meromorphic continuation to t=0, as long as the rest of the parameters remain large.
- (2) For the rest of the (non-zero) terms, the non-convergent factors of the Euler products can always be interpreted as an L-function. They, too, admit meromorphic continuation to t=0 when the rest of the parameters remain large.
- (3) In the limit when $t \to 0$, $f_{1,v}^{t,0}$ converges to the following function supported on \mathfrak{o}_v :

$ \xi _v < 1$	$ \xi + 1 _v < 1$	$ \xi _v = \xi + 1 _v = 1$
$\begin{cases} \frac{1+\eta_v(\xi)}{2}, & \text{if } \eta_v \neq \\ 1-\log_q \xi _v, & \text{if } \eta_v = \end{cases}$	$ \left L(\eta_v, 2s) + L(\eta_v, -2s) \cdot \eta_v(\xi + 1) \xi + 1 _v^{2s} \right $	1
	(2	4.16)

4.7. Continuation to t=0 and s=0. We remain in the setting of the relative trace formula, i.e., the basic functions discussed in the previous subsection. From the calculations of §4.6 it is easy to see that the basic functions $f_{1,v}^{t,0}$ and $f_{4,v}^{t,0}$ satisfy the assumptions of §4.2 even for t around zero, as long as $s\gg 0$. Therefore, Proposition 4.2.1 continues to hold for the spaces $\mathcal{S}_1^t, \mathcal{S}_4^t$, i.e., summation over the regular points of $\mathcal{B}(k)$ converges absolutely.

Hence, by analytic continuation we get that Corollary 4.4.3 continues to hold for t=0 and large values of s; we state it only for the subspace of \mathcal{S}_4^0 which corresponds to the space with parameter s: $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ of the Kuznetsov trace formula. We recall once again that $|\bullet|^{-s-1}\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ can be identified with a subspace $(\mathcal{S}_4^0)'$ of \mathcal{S}_4^0 , cf. Lemma 4.5.1, and the preimage of this in \mathcal{S}_1^0 under \mathcal{G} is denoted by $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$. For later use, we introduce the following notation for their basic functions:

$$\begin{split} f_{\mathcal{Z}_v^s}^0 &= f_{1,v}^{t,0} \text{ (with implicit parameter } s), \\ f_{\mathcal{W}_v^s}^0 &= |\bullet|^{s+1} f_{4,v}^{t,0} \text{ (with implicit parameter } s). \end{split}$$

By the Matching Theorem 5.1 and the Fundamental Lemma 5.4 of [Sak13a], the space $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$ specializes at s=0 to the Schwartz space $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ of the relative trace formula for $T\backslash G/T$. (I remark here that the basic functions here differ by the basic functions of [Sak13a] by a scalar so that the regular value of $f_{\mathcal{Z}_v}^0$ is one.) Thus, the space $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$ is a degeneration of a space of orbital integrals, which carries no representation-theoretic information and no Hecke action. It will only be used to analytically continue

⁶The space can be actually obtained from the split-torus relative trace formula of Jacquet, by twisting torus periods by a continuous family of characters; thus, it can be endowed with a Hecke action, but of course our purpose is to ignore this action, and any relevance to the split-torus RTF, in hopes that the method can be generalized. For local purposes, in order to shorten calculations, we feel free to use this fact (such as in the proof of Proposition 6.5.2).

the Kuznetsov trace formula with parameter s to s=0; thus, a problem of analytic continuation which has a flavor of Langlands' "Beyond Endoscopy" is being treated by algebraic means, by transforming the non-compactly supported orbital integrals of the Kuznetsov trace formula to the essentially compactly supported ones of the spaces $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$. But, first, let us formulate the Poisson summation formula for large values of s.

We set, for large values of s,

$$RTF(f) := PS_1(f) \text{ for } f \in \mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k)), \tag{4.17}$$

$$KTF(f) := PS_4(|\bullet|^{-s-1}f) \text{ for } f \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k)). \tag{4.18}$$

These are the same definitions for the torus relative trace formula and the Kuznetsov trace formula as given in (2.5), (2.6), except that the definition for RTF is now being applied to deformations of the space $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$. The analytic continuation of Corollary 4.4.3 to t=0, applied to those spaces, reads:

4.7.1. **Theorem.** For $\Re(s) \gg 0$ and $f \in \mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$ we have:

$$RTF(f) = KTF(| \bullet |^{s+1} \mathcal{G}f).$$

Explicitly, this means

$$\tilde{O}_0(f) + \tilde{O}_{-1}(f) + \sum_{\xi \in k \setminus \{-1,0\}} f(\xi) = \tilde{O}_0(\mathcal{G}f) + \tilde{O}_{\infty}(\mathcal{G}f) + \sum_{\xi \in k \setminus \{0\}} \mathcal{G}f(\xi). \tag{4.19}$$

Now suppose that f varies in an analytic section of $s \mapsto f_s \in \mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$, The basic functions $f_{\mathcal{Z}^s_v}^0$ satisfy the assumptions of §4.2 for all s and their stalks at ∞ are trivial, i.e., they are of rapid decay in a neighborhood of infinity. Thus, Proposition 4.2.1 continues to hold for elements of the global Schwartz space $\mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$:

4.7.2. **Proposition.** For $f_s \in \mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$, any value of s, the sum

$$\sum_{\xi \in k \setminus \{-1,0\}} f_s(\xi)$$

converges absolutely and is bounded by polynomial seminorms on $\hat{\otimes}_{v \in T} \mathcal{S}(\mathcal{Z}_v^s)$ in vertical strips.

4.7.3. Corollary. For $s \mapsto F_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ an analytic section, KTF (F_s) can be analytically continued to all $s \in \mathbb{C}$ and is bounded by polynomial seminorms in vertical strips. In particular, if F_s is of polynomial growth or rapid decay in a vertical strip, then so is KTF (F_s) , and the value of KTF (F_s) at any specific s depends only on F_s and not on the section.

Proof. The section $f_s = \mathcal{G}^{-1}\left(|\bullet|^{-s-1}F_s\right) \in \mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k))$ is also analytic by Proposition 4.1.1, and bounded by polynomial seminorms on $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$. Applying to it the series that appears on the left hand side of (4.19) we get an analytic function of s, bounded by polynomial seminorms of the original section $s \mapsto F_s$.

We now study the other two terms on the left hand side of (4.19), $\tilde{O}_0(F_s)$ and $\tilde{O}_{-1}(F_s)$. The reader should keep in mind the asymptotic behavior of basic functions described in (4.16), as well as Lemma A.3.2 and Proposition 3.4.1 which describe the form that analytic sections and irregular orbital integrals have when $s \to 0$; as remarked, similar descriptions hold for other "bad" values of s, i.e., when s is among the values of s.

The term $O_0(f_s)$ is clearly defined for all values of s, since the behavior of basic functions in the neighborhood of zero is identical to that in the baby case; the corresponding irregular orbital integrals for s = 0 were defined in (2.15), (2.16).

The behavior in a neighborhood of $\xi = -1$ is analogous to the behavior of orbital integrals in the baby case, §3.3, with the parameter t replaced by 2s. Thus, again $\tilde{O}_{-1}(f_s)$ is defined for every s. Therefore, the expression $PS_4(F_s) = PS_1(f_s)$ can be analytically continued to every s.

By the polynomial growth of partial abelian L-functions on vertical strips, $\tilde{O}_0(f_s)$ and $\tilde{O}_{-1}(f_s)$ are bounded by polynomial seminorms of f_s , and hence by polynomial seminorms of F_s .

Notice that the continuation is not at all obvious from the definition of KTF, as even the individual terms in the sum do not admit analytic continuation. As we shall see, this corollary is equivalent to the analytic continuation of the weighted sum of L-functions $L(\pi, \frac{1}{2} + s)L(\pi \otimes \eta, \frac{1}{2} + s)$ which appear on the spectral side of the Kuznetsov trace formula with nonstandard sections.

Part 2. Spectral analysis.

5. Main theorems of spectral decomposition

In this section we use several results that will be proven in following sections in order to deduce the main conclusions of spectral analysis:

- (1) For the Kuznetsov trace formula with parameter s, the contribution of each cuspidal automorphic representation admits analytic continuation to all $s \in \mathbb{C}$.
- (2) For matching functions, the contributions of each cuspidal automorphic representation to the relative trace formula for the torus and the Kuznetsov trace formula (with s=0) coincide.
- 5.1. **Notation.** For a finite set S of places (including the Archimedean ones, as per our standard assumptions from $\S 1.11$), denote by $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))_S$, $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))_S$ etc. the subspaces of the global Schwartz spaces that we have seen thus far, consisting of vectors of the form:

$$f^{0,S}\otimes f_S$$
,

where $f^{0,S}$ denotes the tensor product over $v \notin S$ of the corresponding basic vectors and f_S belongs to the (completed) tensor product of the local spaces over $v \in S$. For our introductory discussion, let us denote one of these

spaces by S_S , and denote by RTF the corresponding version of the relative trace formula (which for the spaces $S(W^s(\mathbb{A}_k))$ was denoted by KTF). By abuse of notation, we may consider f_S as an element of S_S , by tensoring it with $f^{0,S}$.

As explained in [Sak13a, 5.4], if v is a non-Archimedean place, f_v^0 denotes the basic vector of $\mathcal{S}(\mathcal{Z}_v)$ or $\mathcal{S}(\mathcal{W}_v^s)$ and h belongs to the unramified Hecke algebra $\mathcal{H}(G(k_v), G(\mathfrak{o}_v))$ of $G(\mathfrak{o}_v)$ -biinvariant, compactly supported measures on $G(k_v)$, then it makes sense to write

$$h \star f_v^0$$

for the orbital integrals of the convolution by h of the corresponding basic function "upstairs"; equivalently, h is considered as an element of component of the Bernstein center where unramified representations live, and the Bernstein center acts naturally on $\mathcal{S}(\mathcal{Z}_v)$ and $\mathcal{S}(\mathcal{W}_v^s)$.

Thus, the unramified Hecke algebra outside of S

$$\mathcal{H}^S := \otimes'_{v \notin S} \mathcal{H}(G(k_v), G(\mathfrak{o}_v)) \tag{5.1}$$

acts by mapping S_S into the corresponding space $S(\mathcal{Z}(\mathbb{A}_k))$ or $S(\mathcal{W}^s(\mathbb{A}_k))$; hence, we get a functional

$$RTF^{S}: \mathcal{H}^{S} \otimes \mathcal{S}_{S} \ni h \otimes f_{S} \mapsto RTF(h \star f^{0,S} \otimes f_{S}) \in \mathbb{C}, \tag{5.2}$$

which in the case of the Kuznetsov formula we will, correspondingly, denote by $\mathrm{KTF}^S.$

By the Satake isomorphism, we have

$$\mathcal{H}(G(k_v), G(\mathfrak{o}_v)) = \mathbb{C}[\check{G} /\!\!/ \check{G}],$$

where $/\!\!/$ denotes the invariant-theoretic quotient, i.e., $\mathbb{C}[\check{G} /\!\!/\check{G}] = \mathbb{C}[\check{G}]^{\check{G}}$, and \check{G} acts here by conjugaction. Hence:

$$\mathcal{H}^{S} = \mathbb{C}\left[\prod_{v \notin S} (\check{G} /\!\!/ \check{G})\right], \tag{5.3}$$

where by definition regular functions on an infinite product are the restricted tensor product of regular functions on the factors with respect to the constant function 1. This isomorphism will be denoted: $h \mapsto \hat{h}$.

Notice that every automorphic representation which is unramified outside of S determines a unique point on $\prod_{v \notin S} (\check{G} /\!\!/ \check{G})$ and, by strong multiplicity one, is determined by it (so we will identify π as a point on this product space). We let

$$U^S \subset \prod_{v \notin S} (\check{G} /\!\!/ \check{G})$$

denote the subset corresponding to unitary unramified representations; it is a compact subset with respect to the product Hausdorff topology. We will be writing $\mathbb{C}[U^S]$ for the restrictions of polynomial functions to U^S – of course, U^S is Zariski dense, so $\mathbb{C}[U^S] = \mathbb{C}\left[\prod_{v \notin S}(\check{G} /\!\!/ \check{G})\right]$.

Our goal is to express the functionals RTF^S, KTF^S defined by (5.2) as $(S_S)^*$ -valued measures on U^S . (In particular, for any fixed f_S this would make them scalar-valued measures on U^S , allowing us to extend them from the Hecke algebra – polynomials on U^S – to continuous functions on U^S , and thus to separate points.) This is not quite possible in the split case, but it is possible up to a derivative of a delta distribution. There is no clear a priori reason why "unitary" should play a role here; however, polynomials satisfy the conditions of the Stone-Weierstrass theorem when restricted to the unitary spectrum – and this is what allows us to decompose a comparison of trace formulas spectrally. We include a proof of this application of the Stone-Weierstrass theorem for the sake of completeness:

5.1.1. **Lemma.** $\mathbb{C}[U^S]$ is dense on the space $C(U^S)$ of continuous functions on U^S .

Proof. To apply Stone-Weierstrass, we need to show that as functions on U^S the polynomials are closed under complex conjugation. For every Hecke element h we let $h^*(g) = \overline{h(g^{-1})}$, and then for any unramified unitary representation π (considered as a point on U^S), if $v \in \pi$ is a unitary unramified vector we have:

$$\widehat{h^*}(\pi) = \langle \pi(h^*)v, v \rangle = \langle v, \pi(h)v \rangle = \left\langle v, \hat{h}(\pi)v \right\rangle = \overline{\hat{h}(\pi)}.$$

We will denote by \mathbb{C}_1 , $\mathbb{C}_\eta \in U^S$ the points corresponding to the characters $1, \eta$. For any unramified character χ of the Borel subgroup $B(\mathbb{A}^S_k)$, we will denote by χ the point on $\prod_{v \notin S} (\check{G} /\!\!/ \check{G})$ corresponding to the unramified principal series representation unitarily induced from χ . In particular, $\delta^{\pm \frac{1}{2}}$ and \mathbb{C}_1 denote the same point on U^S (where δ denotes the modular character of the Borel). If $\pi \in \prod_{v \notin S} (\check{G} /\!\!/ \check{G})$ is a point corresponding to a character χ of the Borel, by an evaluation (of polynomials) of order d at π we will mean any linear combination of the value and the first d-1 derivatives of the polynomials along the complex one-parameter family $s \mapsto \chi \delta^s$; notice that the choice of χ vs. a Weyl group-conjugate character $^w\chi$ does not make a difference for this notion. When the point is mentioned many times, the analogous multiplicity is implied, e.g., an "evaluation at \mathbb{C}_1 , \mathbb{C}_η ", when $\eta = 1$, is an evaluation at \mathbb{C}_1 of order 2. Notice that an evaluation of order d is also an evaluation of any larger order, in this language.

We will denote by \hat{G}^{aut} the set of automorphic representations of $G(\mathbb{A}_k)$ which appear in the Plancherel formula for $L^2([G])$, that is: the points corresponding to cuspidal representations, unitary idele class characters of the Borel, and residual representations (in this case, idele class characters of G). We consider \hat{G}^{aut} as a subset of U^S – to be precise, it is a subset of $\lim_{\to} U^S$ as S becomes larger, but we will freely talk about it as a subset of U^S , meaning its intersection with U^S . We denote by $\hat{G}^{\text{aut}}_{\text{Ram}}$ the subset of those which are not residual discrete series (i.e., characters, in the case of

 PGL_2). These are the representations of "Ramanujan type", but of course we do not use their expected temperedness at any point. For PGL_2 they coincide with the generic elements of \hat{G}^{aut} , but since we will also use these sets for inner forms, we prefer the name "Ramanujan" to "generic".

Hence we have:

$$\hat{G}^{\mathrm{aut}} \supset \hat{G}^{\mathrm{aut}}_{\mathrm{Ram}} = \hat{G}^{\mathrm{aut}}_{\mathrm{cusp}} \sqcup \hat{G}^{\mathrm{aut}}_{\mathrm{Eis}},$$

where $\hat{G}_{\text{cusp}}^{\text{aut}}$ denotes cuspidal representations and $\hat{G}_{\text{Eis}}^{\text{aut}}$ denotes principal series unitarily induced from unitary idele class characters of the Borel subgroup. These sets are clearly measurable with respect to the standard Borel structure on U^S (notice that we are not using the Fell topology anywhere, but the topology induced from the space $\prod_{v \notin S} (\check{G} /\!\!/ \check{G})$), since $\hat{G}_{\text{cusp}}^{\text{aut}}$ consists of a countable number of points and $\hat{G}_{\text{Eis}}^{\text{aut}}$ is a countable union of "lines". Any reference to "measures" on U^S or $\hat{G}_{\text{Ram}}^{\text{aut}}$ will imply regular Borel measures.

Here is a delicate point involving inner forms: we will also need to consider the sets $\widehat{G}^{\alpha}^{\text{aut}}$, for inner forms of G corresponding to nontrivial torsors of the torus T (when it is not split). Of course, as a subset of U^S it belongs to \widehat{G}^{aut} by the global Jacquet–Langlands correspondence. We will therefore never use $\widehat{G}^{\alpha}^{\text{aut}}$ explicitly – only \widehat{G}^{aut} will appear. However, as much as it simplifies notation, one should not assume that the Jacquet–Langlands correspondence is being used: the comparison of trace formulas *shows*, a posteriori, that those elements of $\widehat{G}^{\alpha}^{\text{aut}}$ which have nonzero contribution to the torus RTF correspond, as points of U^S , to points of \widehat{G}^{aut} . We point the reader to the proof of Proposition 7.1.2, which clarifies the issue.

5.2. Lifts and relative characters. Up to now, except for the baby case, we have avoided talking about the spaces "upstairs", which give rise to our Schwartz spaces via orbital integrals. We now recall from [Sak13a, section 3] that any element $f \in \mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ of the form $f = \otimes_v f_v$ is obtained by the $G(\mathbb{A}_k)$ -orbital integrals (diagonal action) of an element $\Phi = \otimes \Phi_v$ of the space

$$\bigotimes' \bigoplus_{\alpha} \mathcal{S}(Y_v^{\alpha} \times Y_v^{\alpha}), \tag{5.4}$$

where, for each place v, the sum over α runs over isomorphism classes of torsors R^{α} of the torus T over k_v , and Y^{α} denotes the "pure inner form" $Y^{\alpha} \simeq T^{\alpha} \backslash G^{\alpha}$ of $Y = T \backslash G$, where $Y^{\alpha} = Y \times^{T} R^{\alpha}$, $T^{\alpha} = \operatorname{Aut}_{T} R^{\alpha} \simeq T$ and $G^{\alpha} = \operatorname{Aut}_{G}(G \times^{T} R^{\alpha})$. The restricted tensor product is taken with respect to the characteristic functions of the \mathfrak{o}_{v} -points of $(Y_{v} \times Y_{v})(\mathfrak{o}_{v})$ (at non-Archimedean places).

An element Φ of (5.4) whose orbital integrals give f as a function on $\mathcal{B}_{\mathcal{Z}}^{\text{reg}}(\mathbb{A}_k)$ will be called a *lift* of f. (Such an element is non-unique.) In order for this notion to be meaningful, we need to fix Haar measures on the groups G_v^{α} , and we start by fixing any choice of measures on G_v which

factorize the Tamagawa measure on $G(\mathbb{A}_k)$. Since inner twists preserve rational volume forms, this factorization also determines Haar measures on the inner forms G_v^{α} , in such a way that if R^{α} is a globally defined torsor, the product volume on $G^{\alpha}(\mathbb{A}_k)$ is also the Tamagawa measure. We can also talk about local lifts of $f_v \in \mathcal{S}(\mathcal{Z}_v)$, but then we will implicitly mean an element of $\bigoplus_{\alpha} \mathcal{S}(Y_v^{\alpha} \times Y_v^{\alpha})$ together with a choice of Haar measure on G_v (and the induced Haar measures on its inner forms).

Similarly, an element $f_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ can be obtained by the orbital integrals of an element $\Phi_s \in \mathcal{S}^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ in the notation of [Sak13a, §6.1]. Here $X \simeq N \backslash PGL_2$, \mathcal{L}_{ψ} denotes the complex line bundle whose sections are functions on $PGL_2(\mathbb{A}_k)$ such that $f(ng) = \psi(n)f(g)$, and \mathcal{L}_{ψ}^{-1} denotes the inverse line bundle (described similarly using the character ψ^{-1}). The exponent s denotes a certain non-standard space of test functions defined in [Sak13a]. These spaces also have the natural structure of a Fréchet bundle over C, as their local components are Schwartz cosheaves which are isomorphic away from "infinity", and they are also isomorphic in a neighborhood of infinity where they are, up to smooth functions, equal to elements of generalized principal series varying analytically with s; we leave the details to the reader. The notion of polynomial families of seminorms (in bounded vertical strips) is defined in a completely analogous way as for $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ (cf. Appendix A), and one can easily see that polynomial seminorms on $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ are the $G(\mathbb{A}_k)$ -invariant polynomial seminorms on $S^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1}).$

The generalization of the notion of a character from the adjoint quotient of a group H/(H-conj) to a quotient $X_1 \times X_2/G$ such as in the relative trace formula can be found in the literature under the names relative character, spherical character or Bessel distribution. We will use the former term. Unlike the case of the group, there is no canonical normalization of relative characters in general, and they depend on the functionals chosen to define them.

Definition. A relative character on $\mathcal{S}(\mathcal{Z}_v)$ is a functional which (for any choice of Haar measure on G_v) factors through a sequence of morphisms:

$$\bigoplus_{\alpha} \mathcal{S}(Y_v^{\alpha} \times Y_v^{\alpha}) \to \bigoplus_{\alpha} \pi_{\alpha} \otimes \tilde{\pi}_{\alpha} \to \mathbb{C},$$

where, for each α , π_{α} is an irreducible admissible representation of G^{α} and the last arrow is the canonical pairing.

An automorphic relative character on $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ is a functional which factors through a sequence of morphisms:

$$\bigotimes_{v}' \bigoplus_{\alpha} \mathcal{S}(Y_{v}^{\alpha} \times Y_{v}^{\alpha}) \to \bigoplus_{\beta} \pi_{\beta} \otimes \tilde{\pi}_{\beta} \to \mathbb{C},$$

where β runs over isomorphism classes⁷ of k-rational torsors of T, for each β , π_{β} is an irreducible automorphic representation of G^{β} , and it is understood that the projection to the β -summand factors through $\bigotimes_{v}' \mathcal{S}(Y_{v}^{\beta} \times Y_{v}^{\beta})$ and is equivariant with respect to $G^{\beta}(\mathbb{A}_{k})$.

The same definitions (without the need for torsors) apply to the spaces $\mathcal{S}(\mathcal{W}_v^s)$, $\mathcal{S}(\mathcal{W}(\mathbb{A}_k))$.

Now we come to the definition of global periods and the corresponding relative characters. We fix a point $\varphi \in \hat{G}^{\mathrm{aut}}_{\mathrm{disc}}$. Let β be a k-rational T-torsor, and let V_{φ}^{β} be the φ -isotypic subspace of $L^2([G^{\beta}])$. At this point we do not need to use the Jacquet–Langlands correspondence or strong multiplicity one for inner forms; however, if we do not use strong multiplicity one, we should define "corresponds to φ " as the limit, over finite sets S of places, of the subspaces of $L^2([G^{\beta}])$ on which the unramified Hecke algebra outside of S acts as φ .

Then, with the exception of the case T =split and φ =the trivial representation, the *period integral* over [T] is defined, possibly after regularization (cf. Lemma 6.3.5):

$$V_{\varphi}^{\beta} \ni \phi \mapsto \int_{[T^{\beta}]}^{*} \phi(t)dt. \tag{5.5}$$

By Frobenius reciprocity this gives a map:

$$V_{\alpha}^{\beta} \to C^{\infty}(Y^{\beta}(\mathbb{A}_k))$$

(notice, in this case, that $Y^{\beta}(\mathbb{A}_k) = T^{\beta}(\mathbb{A}_k) \setminus G^{\beta}(\mathbb{A}_k)$), and conjugate-dually:

$$S(Y^{\beta}(\mathbb{A}_k)) \to V_{\varphi}^{\beta}.$$
 (5.6)

We make a remark on measures here: We use measures defined by global volume forms throughout, including to define a pairing between $C^{\infty}(Y^{\beta}(\mathbb{A}_{k}))$ and $S(Y^{\beta}(\mathbb{A}_{k}))$, but the map (5.6) depends only on the measure on $G^{\beta}(\mathbb{A}_{k})$, not on the measures on $T^{\beta}(\mathbb{A}_{k})$ and $Y^{\beta}(\mathbb{A}_{k})$ (as long as they are chosen compatibly); thus, it is well-defined even in the case when T is a split torus, in which case global volume forms do not give well-defined measures on [T] and $Y(\mathbb{A}_{k})$.

Similar definitions hold when we replace the period over T by the period over N against an idele class character ψ or ψ^{-1} , and $\mathcal{S}(Y(\mathbb{A}_k))$ by $\mathcal{S}^s(\overline{X}(\mathbb{A}_k), \mathcal{L}_{\psi})$, in the above notation, provided that the integrals make sense (which they do, as we will see, in Proposition 6.4.1, when $\Re(s) \gg 0$).

Finally, for φ in the continuous spectrum, the same definitions hold (with β necessarily the trivial torsor), except that V_{φ} is not a subspace of $L^2([G])$. However, any choice of "continuous" Plancherel measure $d\varphi$ will endow the spaces V_{φ} with a unitary structure $\langle \, , \, \rangle_{\varphi}$. Instead of fixing such a Plancherel measure, we will use the canonical product $\langle \, , \, \rangle_{\varphi} \, d\varphi$.

⁷As before, we use β for globally defined torsors and β_v for their localizations, while the symbols α , α_v are reserved for torsors defined locally.

Definition. For $\varphi \in \hat{G}^{\mathrm{aut}}_{\mathrm{disc}}$ (except for the trivial representation when Tis split) the period relative character is the relative character obtained by combining (5.6) with its dual, for all classes β of T-torsors over k:

$$\mathcal{J}_{\varphi}: \bigotimes_{v}' \bigoplus_{\alpha} \mathcal{S}(Y_{v}^{\alpha} \times Y_{v}^{\alpha}) \to \bigoplus_{\beta} V_{\varphi}^{\beta} \otimes \overline{V_{\varphi}^{\beta}} \to \mathbb{C}, \tag{5.7}$$

where, by definition, the projection to the β -summand factors through $\bigotimes_{v}' \mathcal{S}(Y_v^{\beta} \times Y_v^{\beta}).$

We similarly define (without the use of nontrivial torsors) the period relative character on $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ (it makes sense, as we will see, when $\Re(s) \gg 0$ and $\varphi \in \hat{G}_{Ram}^{aut}$), and denote it by \mathcal{I}_{φ} .

For the continuous spectrum we similarly define measures $\mathcal{J}_{\varphi}d\varphi$, $\mathcal{I}_{\varphi}d\varphi$ valued in the space of functionals on $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$, resp. $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$.

We will see an alternate, more direct definition of these relative characters in §5.5.

5.3. Spectral decomposition: results.

- 5.3.1. **Theorem.** Consider the case $S_S = S(\mathcal{Z}(\mathbb{A}_k))_S$. Fix an element $f \in$ \mathcal{S}_S . Then the functional RTF^S, (5.2), on \mathcal{H}^S is the sum of the following two summands:
 - (1) a finite complex measure ν_f on \widehat{G}_{Ram}^{aut} ;
 - (2) an evaluation of \hat{h} at $\mathbb{C}_1, \mathbb{C}_\eta$ and (in the split case) \mathbb{C}_1 .

The measure ν_f is equal to the period relative character $\mathcal{J}_{\varphi}(f)d\varphi$, and its norm (in the Banach space of finite measures on \widehat{G}_{Ram}^{aut}) is bounded by seminorms on S_S .

- 5.3.2. **Theorem.** Consider the case $S_S = S(W^s(\mathbb{A}_k))_S$, where $\Re s \gg 0$. Fix an element $f \in \mathcal{S}_S$. Then the functional KTF^S, s. (5.2), on \mathcal{H}^S is the sum of the following two summands:

 - (1) a finite complex measure μ_f on Ĝ^{aut}_{Ram};
 (2) an evaluation of ĥ at δ^{1/2+s} and η · δ^{1/2+s}.

The measure μ_f is equal to the period relative character $\mathcal{I}_{\varphi}(f)d\varphi$, and its norm is bounded by polynomial seminorms on S_S . In particular, for an analytic section of rapid decay in vertical strips $s \mapsto f_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))_S$, the corresponding measures μ_{f_s} are also of analytic of rapid decay (in the Banach space of finite measures on $\widehat{G}_{\mathrm{Ram}}^{\mathrm{aut}}$).

Analytic continuation. We will then use a second "miracle", which corresponds to the reflection of the functional equation of the L-function $L(\pi, \frac{1}{2} +$ $s)L(\pi \otimes \eta, \frac{1}{2} + s)$ at the level of orbital integrals, to prove:

5.3.3. **Theorem.** Consider the case $S_S = S(W^s(\mathbb{A}_k))_S$, with arbitrary s. The expression of KTF^S as a sum of a measure μ_f on \widehat{G}_{Ram}^{aut} bounded by polynomial seminorms and an evaluation, as in Theorem 5.3.2, holds whenever $\Re(s) \neq \pm \frac{1}{2}$, with the possible modification that the evaluation is at the set of four points: $\delta^{\frac{1}{2}\pm s}$ and $\eta \cdot \delta^{\frac{1}{2}\pm s}$.

Moreover, for an analytic $s \mapsto f_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))_S$ the measure μ_{f_s} is well-defined as a measure on $\widehat{G}^{\mathrm{aut}}_{\mathrm{Ram}} \setminus \{\delta^{\frac{1}{2} \pm s}, \eta \cdot \delta^{\frac{1}{2} \pm s}\}$ (possibly infinite if $\Re(s) = \pm \frac{1}{2}$), and its restriction to any closed subset not including $\delta^{\frac{1}{2} \pm s}, \eta \cdot \delta^{\frac{1}{2} \pm s}$ is finite and varies analytically in s.

The reason that points with $\Re(s) = \pm \frac{1}{2}$ are excluded is that the evaluations and the continuous spectrum of \widehat{G}_{Ram}^{aut} are not disjoint in this case, and their contribution cannot be separated. It is easy to see (cf. the argument in the proof of Theorem 5.4.1) that the measure μ_f is uniquely defined.

In particular, for cuspidal representations which are always disjoint from the points $\delta^{\frac{1}{2}\pm s}$ and $\eta \cdot \delta^{\frac{1}{2}\pm s}$, we get the following:

5.3.4. Corollary. For every holomorphic section $s \mapsto f_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))_S$ and every $\varphi \in \hat{G}^{\mathrm{aut}}_{\mathrm{cusp}}$ (the cuspidal automorphic spectrum) the function $s \mapsto \mathcal{I}_{\varphi}(f_s)$ (defined, originally, for $\Re(s) \gg 0$) extends to a holomorphic function in the domain of f_s .

This implies, in particular, the meromorphic continuation of the partial L-function $L^S(\pi, \frac{1}{2} + s)L^S(\pi \otimes \eta, \frac{1}{2} + s)$, which is a factor of $\mathcal{I}_{\varphi}(f_s)$ (see (7.13)).

- 5.4. **Comparison.** The above theorems allow us to spectrally decompose the comparison between the two relative trace formulas:
- 5.4.1. **Theorem.** Fix an element $f \in \mathcal{S}(\mathcal{Z}(\mathbb{A}_k))_S$ and let $f' = | \bullet | \cdot \mathcal{G}f \in \mathcal{S}(\mathcal{W}^0(\mathbb{A}_k))_S$. If ν_f , $\mu_{f'}$ are the measures on $\hat{G}^{\mathrm{aut}}_{\mathrm{Ram}}$ obtained by Theorems 5.3.1 and 5.3.3, then $\nu_f = \mu_{f'}$.

Proof. By Theorem 5.3.1, the functional $h \mapsto \operatorname{RTF}(h \star f)$ is the sum of the integral against ν_f and an evaluation at \mathbb{C}_1 , \mathbb{C}_η , \mathbb{C}_1 , while the functional $h \mapsto \operatorname{KTF}(h \star f')$ is the sum of the integral against $\mu_{f'}$ and an evaluation at \mathbb{C}_1 , \mathbb{C}_1 , \mathbb{C}_η and \mathbb{C}_η . (Of course, a posteriori it will turn out that some repetitions are superfluous.)

Essentially by definition (see Corollary 4.7.3), the two functionals above coincide. In particular, for every $h_1 \in \mathcal{H}^S$ such that \hat{h}_1 vanishes at \mathbb{C}_1 , \mathbb{C}_1 , \mathbb{C}_η and \mathbb{C}_η (with the implied multiplicity) we have an equality of functionals on \mathcal{H}^S :

$$h \mapsto \text{RTF}(h \star h_1 \star f) = \text{KTF}(h \star h_1 \star f').$$

Both functionals are represented by measures:

$$RTF(h \star h_1 \star f) = \int \hat{h} \cdot \hat{h}_1 \nu_f,$$

$$KTF(h \star h_1 \star f') = \int \hat{h} \cdot \hat{h}_1 \mu_{f'}.$$

By the Stone-Weierstrass theorem (see Lemma 5.1.1), the functions of the form $\hat{h}, h \in \mathcal{H}^S$, are dense in the space of continuous functions on U^S . Therefore, the two measures $\hat{h}_1\nu_f$ and $\hat{h}_1\mu_{f'}$ coincide. Since h_1 was arbitrary, with only requirement its vanishing (with multiplicity) at the points $\mathbb{C}_1, \mathbb{C}_\eta$ which don't belong to $\hat{G}_{\text{Ram}}^{\text{aut}}$, it follows that $\nu_f = \mu_{f'}$.

5.5. Outline of the proofs. We will give the proofs of Theorems 5.3.1, 5.3.2, 5.3.3 using results that will be proven in the next section.

We start with Theorem 5.3.1:

Outline of the proof of Theorem 5.3.1. Let $f \in \mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$ and lift it to an element $\Phi \in \bigotimes_v' \bigoplus_{\alpha} \mathcal{S}(Y_v^{\alpha} \times Y_v^{\alpha})$, as in §5.2. Let $\Sigma \Phi$ denote the corresponding automorphic function on $\sqcup_{\beta} [G^{\beta}]^2$ (here β runs over torsors defined over k) obtained by summation over all k-points and Frobenius reciprocity:

$$\Sigma\Phi(g_1, g_2) = \sum_{(\gamma_1, \gamma_2) \in (Y^{\beta} \times Y^{\beta})(k)} \Phi(\gamma_1 g_1, \gamma_2 g_2), \text{ when } g_1, g_2 \in G^{\beta}(\mathbb{A}_k).$$

We will see (Proposition 6.3.1 and Corollary 6.3.2) that in the nonsplit case $\Sigma\Phi$ is of rapid decay; more precisely, the map Σ is a continuous map from $\mathcal{S}((Y^{\beta} \times Y^{\beta})(\mathbb{A}_{k}))$ to $\mathcal{S}([G^{\beta}]^{2})$. In the split case, it is asymptotically B-finite in both variables with simple exponents $\delta^{\frac{1}{2}}$; more precisely, the map Σ is a continuous map from $\mathcal{S}((Y \times Y)(\mathbb{A}_{k}))$ to a Fréchet space $\mathcal{S}^{+}_{[\delta^{\frac{1}{2}}]}([G]) \hat{\otimes} \mathcal{S}^{+}_{[\delta^{\frac{1}{2}}]}([G])$ of asymptotically B-finite functions on $[G]^{2}$ with the given exponent (in both variables). This notion will be defined in detail in §6.2; it means that at the cusp the function is equal to a rapidly decaying function plus an element of a principal series with the given exponent.

In any case we will see (Proposition 6.3.3) that the relative trace formula has the following expression:

$$RTF(f) = \langle \Sigma \Phi \rangle^*$$
,

where $\langle \Sigma \Phi \rangle^*$ is the sum over all β of the integral over the diagonal copy of $[G^{\beta}]$, suitably regularized in the split case.

In the nonsplit case, this immediately implies the stated spectral decomposition of the theorem, by the Plancherel formula for $L^2([G^{\beta}])$: the function $\Sigma\Phi$ defines a finite signed measure ν_{Φ} on $\hat{G}^{\rm aut}$ such that

$$\langle \Sigma \Phi \rangle^* = \langle \Sigma \Phi \rangle = \int_{\hat{G}^{\text{aut}}} \nu_{\Phi},$$

and, more generally,

$$\langle h \star \Sigma \Phi \rangle^* = \int_{\hat{G}^{\text{aut}}} \hat{h}(\pi) \nu_{\Phi}(\pi)$$
 (5.8)

for all $h \in \mathcal{H}^S$, acting on the first variable. Apart from the evaluations at \mathbb{C}_1 and \mathbb{C}_η , other characters of some of the inner forms $[G^\beta]$ do not contribute because they are nontrivial on $T(\mathbb{A}_k)$ – and, clearly, Φ belongs to a representation induced from the trivial character of $T(\mathbb{A}_k)$. Therefore,

the integral (5.8) can be split into the sum of evaluations at \mathbb{C}_1 , \mathbb{C}_η and a measure on \hat{G}_{Ram}^{aut} . (I remark again that, if one does not want to use the Jacquet–Langlands correspondence, one should describe it as a measure on the union of $\widehat{G}_{Ram}^{\beta}$, considered as subsets of U^S ; after the comparison of Theorem 5.4.1, this turns out to be a subset of $\hat{G}_{Ram}^{\alpha ut}$.) Moreover, the total mass of ν_{Φ} is bounded by seminorms on $\oplus_{\beta} \mathcal{S}([G^{\beta}]^2)$, hence by seminorms on $\oplus_{\beta} \mathcal{S}((Y^{\beta} \times Y^{\beta})(\mathbb{A}_k))$. It is easy to see from the definitions that the measure ν_{Φ} coincides with the measure $\mathcal{J}_{\varphi}(f)d\varphi$ defined in §5.2.

In the split case, one needs to extend the Plancherel formula to asymptotically B-finite functions with exponents $\delta^{\frac{1}{2}}$. We will do this in Proposition 6.2.6, and will describe the topology on the space of such functions in §6.2. The rest of the steps are the same, and again the only character contributing will be the trivial one, as others are nontrivial on $T(\mathbb{A}_k)$. This completes the proof of Theorem 5.3.1.

We now outline the proof of Theorem 5.3.2:

Outline of the proof of Theorem 5.3.2. Lift an element (or a section of polynomial growth on vertical strips) $f_s \in \mathcal{S}(W^s(\mathbb{A}_k))$ to an element (resp. a section of polynomial growth) $\Phi_s \in \mathcal{S}^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ as in §5.2.

We will introduce algebraic height functions r and R on the adelic points of X, resp. on [G] in §6.1. We will see (Lemma 6.4.2) that for every integer N and any $\Re(s)$ large enough, the elements of $\mathcal{S}^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ belong to the Banach space $C(X \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})_{-N}$ of continuous sections Φ which satisfy

$$\sup_{(x_1, x_2) \in X \times X(\mathbb{A}_k)} |\Phi(x_1, x_2)| r(x_1)^N r(x_2)^N < \infty.$$

(In fact, recall that in the second variable the elements of $\mathcal{S}^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ are of rapid decay.) More precisely, the map

$$S^{s}(\overline{X} \times X(\mathbb{A}_{k}), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1}) \to C(X \times X(\mathbb{A}_{k}), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})_{-N}$$
 (5.9)

is bounded by polynomial seminorms on the former.

As before, we define the map $\Phi \mapsto \Sigma \Phi$ as:

$$\Sigma\Phi(g_1, g_2) = \sum_{(\gamma_1, \gamma_2) \in (X \times X)(k)} \Phi(\gamma_1 g_1, \gamma_2 g_2),$$

whenever it converges. We define the space $C([G]^2)_{-N}$ in a completely analogous way, using the height function R on [G]. Then (Proposition 6.4.1) there is a positive constant c such that for large N the map: $\Phi \mapsto \Sigma \Phi$ represents a continuous morphism:

$$C(X \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})_{-N} \to C([G]^2)_{-cN}.$$

We can now write the Kuznetsov trace formula KTF (f_s) , for $s \gg 0$, as the sum of two terms, the term $\tilde{O}_0(f_s)$ of (4.7) and the rest, which we will

denote by $\text{KTF}_{\emptyset}(f_s)$ (this is the classical Kuznetsov trace formula, without the contribution of "infinity"). As in the torus case, we will see (Proposition 6.4.3) that

$$KTF_{\emptyset}(f_s) = \langle \Sigma \Phi_s \rangle ,$$

where the angular brackets again denote the integral over the diagonal copy of [G].

The Plancherel formula for $L^2([G])$ now gives rise, as in the proof of Theorem 5.3.1, to a spectral decomposition of the functional $h \mapsto \operatorname{KTF}(h \star f_s)$. This spectral decomposition only includes generic representations, therefore the corresponding measure μ_{f_s} of Theorem 5.3.2 is concentrated on $\hat{G}^{\operatorname{aut}}_{\operatorname{Ram}}$. The measure μ_{f_s} is bounded by the L^2 -norm of $\Sigma\Phi$, in particular by its $C([G]^2)_{-cN}$ -norm; more precisely, since the measure is invariant under the diagonal $G(\mathbb{A}_k)$ -action on this space, it is bounded by the norm on the $G(\mathbb{A}_k)^{\operatorname{diag}}$ -coinvariants of $C([G]^2)_{-cN}$ (the largerst quotient on which the diagonal $G(\mathbb{A}_k)$ -action is trivial).

The composition of the inclusion (5.9) with Σ gives rise to a map of coinvariants:

$$S(\mathcal{W}^s(\mathbb{A}_k)) \to \left(C([G]^2)_{-cN}\right)_{G(\mathbb{A}_k)},$$

which is bounded by polynomial seminorms on the former, since those are the $G(\mathbb{A}_k)$ -invariant polynomial seminorms on $\mathcal{S}^s(\overline{X}\times X(\mathbb{A}_k),\mathcal{L}_\psi\boxtimes\mathcal{L}_\psi^{-1})$, and the map (5.9) is bounded by polynomial seminorms. Hence, the resulting map:

$$S(W^s(\mathbb{A}_k)) \ni f_s \mapsto \mu_{f_s} \in \mathcal{M}(\hat{G}_{Ram}^{aut})$$
 (5.10)

(finite measures on $\hat{G}^{\rm aut}_{\rm Ram}\subset U^S)$ is bounded by polynomial seminorms on family on spaces on the left.

Finally, consider the functional:

$$\mathcal{H}^S \ni h \mapsto \tilde{O}_0(h \star f_s).$$

We will prove in Lemma 6.4.4 that it is an evaluation at $\delta^{\frac{1}{2}+2}$, $\eta \cdot \delta^{\frac{1}{2}+2}$. This completes the proof of Theorem 5.3.2.

We have also established several useful facts towards the proof of Theorem 5.3.3, to which we come now. We first need to discuss the aforementioned "miracle", which is a reflection of the functional equation of L-functions at the level of orbital integrals:

5.5.1. **Theorem.** There is an isomorphism of Fréchet spaces

$$\mathcal{T}: \mathcal{S}(\mathcal{W}^{-s}(k_v)) \xrightarrow{\sim} \mathcal{S}(\mathcal{W}^s(k_v))$$

with the following properties:

• Polynomial families of seminorms on the right are bounded by polynomial families on the left, and vice versa.

• It preserves basic vectors (i.e., $\mathcal{T}f_{v,-s}^0 = f_{v,s}^0$, whenever they are defined) and is equivariant with respect to the action of the spherical Hecke algebra on those (i.e., $\mathcal{T}(h \star f_{v,-s}^0) = h \star f_{v,s}^0$ for all $h \in \mathcal{H}(G(k_v), G(\mathfrak{o}_v))$).

In particular, \mathcal{T} defines an isomorphism of global Schwartz spaces

$$\mathcal{S}(\mathcal{W}^{-s}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k)).$$

• It preserves the functional KTF.

This will be proven in $\S6.5$.

Now let us see how to deduce Theorem 5.3.3 from this – it will be by application of the Phragmén-Lindelöf theorem.

Proof of Theorem 5.3.3. For a bounded vertical strip $V \subset \mathbb{C}$, let us denote by ∂V^- its left boundary, and by ∂V^+ its right one. Let V be a sufficiently wide vertical strip which is symmetric around s = 0. Let $V \ni s \mapsto f_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))_S$ be an analytic section of rapid decay on V, and lift it to a section $s \mapsto \Phi_s$ of $\mathcal{S}^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi^{-1})$. Given $f \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))_S$ for a specific value of s (as in the Theorem), we can make sure that $f_s = f$ for that value of s.

For every $h \in \mathcal{H}^S$, the function $s \mapsto \text{KTF}(h \star f_s)$ is holomorphic of finite order on V (Corollary 4.7.3).

Now fix a holomorphic section $s \mapsto h_s \in \mathcal{H}^S$ such that:

- the evaluations of \hat{h}_s at the points $\delta^{\frac{1}{2}\pm s}$ and $\eta \delta^{\frac{1}{2}\pm s}$ (to the corresponding order, if these points coincide) vanish;
- $\|\hat{h}_s\|_{L^{\infty}(U^S)}$ is bounded on V.

By "holomorphic section" we mean that, as measures on $G(\mathbb{A}_k^S)$, they are supported on the same finite set of double $\prod_{v\notin S} G(\mathfrak{o}_v)$ -orbits, where they vary analytically. This is equivalent to saying that \hat{h}_s is holomorphic into the space of polynomials on U^S of degree bounded by a fixed number. It is clear that such sections exist; let $f'_s = h_s \star f_s$.

For any fixed $s \in \partial V^+$, consider the functional $h \mapsto \mathrm{KTF}(h \star f_s')$ on \mathcal{H}^S . By Theorem 5.3.2 it is represented by a finite measure $\mu_{f_s'}$ on $\hat{G}_{\mathrm{Ram}}^{\mathrm{aut}}$. Moreover, this measure is of rapid decay on ∂V^+ since f_s is and \hat{h}_s is supbounded. In particular, for every fixed h the holomorphic function $\mathrm{KTF}(h \star f_s')$ is of rapid decay on ∂V^+ .

Similarly, given $s \in \partial V^-$, the functional $h \mapsto \text{KTF}(h \star f'_s)$ is, by Theorem 5.5.1, equal to the functional $h \mapsto \text{KTF}(h \star \mathcal{T}f'_s)$; recall that $\mathcal{T}f'_s \in \mathcal{S}(\mathcal{W}^{-s}(\mathbb{A}_k))$ is of rapid decay on V, since \mathcal{T} preserves this property. Therefore, by the same argument, when $s \in \partial V^-$ it is represented by a finite measure $\hat{h}_s \cdot \mu_{f'_s}$ on $\hat{G}^{\text{aut}}_{\text{Ram}}$, and for every fixed h the holomorphic function $\text{KTF}(h \star f'_s)$ is of rapid decay on ∂V^- .

Therefore, by the Phragmén-Lindelöf principle, we have

$$\left| \operatorname{KTF}(h \star f_s') \right| \leq \sup_{t \in \partial V^+ \cup \partial V^-} \left\| \hat{h} \cdot \mu_{f_t'} \right\| \leq$$

$$\leq \|\hat{h}\|_{L^{\infty}(U^S)} \cdot \sup_{t \in \partial V^+ \cup \partial V^-} \left\| \mu_{f'_t} \right\|$$

for all $s \in V$. This shows that for every s the functionals:

$$\mathcal{H}^S \ni h \mapsto \mathrm{KTF}(h \star f_s')$$

can be continuously extended, by the Stone-Weierstrass theorem (Lemma 5.1.1), to the space of continuous functions on U^S , and hence are represented by a family $\mu_{f'_s}$ of measures which is weak-star holomorphic, i.e., for every $F \in C(U^S)$ the function $s \mapsto \int F \mu_{f'_s}$ is analytic.

It follows that this family is *strongly* analytic into the Banach space of finite measures on U^S . Since the standard references on vector-valued holomorphic functions usually mention weak (not weak-star) holomorphy as the assumption for this conclusion, we revisit the steps of the proof of strong holomorphy to verify that they apply here; for simplicity, let us denote $v_s = \mu_{f_s}$, and the integral above by $v_s(F)$.

The basic property is strong continuity of the section $s \mapsto v_s$. This follows by observing that $\frac{1}{t}(v_{s+t}(F) - v_s(F))$ can be written as a Cauchy integral and bounded, for all 0 < |t| < r by the maximum of $r^{-1}|v_z(F)|$ on the circle of radius 2r around s, where r is a small positive number. The uniform boundedness principle then implies that the collection of vectors

$$\frac{1}{t}(v_{s+t} - v_s), \ 0 < |t| < r,$$

is strongly bounded, and hence v is continuous at s. Then the vector-valued Cauchy integral

$$\frac{1}{2\pi i} \int (z-s)^{-1} f(z) dz$$

(along a small circle around s) makes sense and represents $v_s(F)$ for every F, therefore is equal to v_s , and v is holomorphic.

By strong analyticity, the mass of every measurable subset of U^S varies analytically in s, in particular the mass of $\mu_{f'_s}$ is concentrated on $\hat{G}^{\rm aut}_{\rm Ram}$, for every s.

Since \hat{h}_s was arbitrary with only requirement its vanishing at the points $\delta^{\frac{1}{2}\pm s}$ and $\eta\delta^{\frac{1}{2}\pm s}$, and given that, when $\Re s \neq \pm \frac{1}{2}$, those points do not meet $\hat{G}^{\mathrm{aut}}_{\mathrm{Ram}}$, it follows that for $\Re s \neq \pm \frac{1}{2}$ the measure $\mu_{f_s} = \hat{h}_s^{-1}\mu_{f_s'}$ is also well-defined (and, clearly, independent of the choice of h_s). In fact, we can think of μ_{f_s} as a (not necessarily finite) measure on $\hat{G}^{\mathrm{aut}}_{\mathrm{Ram}} \setminus \{\delta^{\frac{1}{2}\pm s}, \eta\delta^{\frac{1}{2}\pm s}\}$ for every s, and its restriction to a subset which doesn't contain the points $\delta^{\frac{1}{2}\pm s}$ and $\eta\delta^{\frac{1}{2}\pm s}$ in its closure is (finite and) locally analytic in s (and finite).

Now consider the functional $h \mapsto \text{KTF}(h \star f_s) - \int_{\hat{G}_{\text{Ram}}^{\text{aut}}} \hat{h} \mu_{f_s}$ on \mathcal{H}^S , when $\Re s \neq \pm \frac{1}{2}$; it is necessarily an evaluation at the points $\delta^{\frac{1}{2} \pm s}$ and $\eta \delta^{\frac{1}{2} \pm s}$. Indeed, if \hat{h} vanishes at those points (to the corresponding order, if some of those points coincide), then $h = h_s$ for some h_s as above, and the functional is zero on h.

This completes the proof of Theorem 5.3.3.

6. Completion of proofs

In this section I prove all the auxiliary results used in the previous section.

6.1. **Height functions.** For any reductive group G over a local field F, if we fix a faithful algebraic representation: $G \to \operatorname{GL}_N$, we get a natural algebraic height function $\| \bullet \|$ on G(F) by pulling back the maximum of the operator norms of g and g^{-1} with respect to the norm $(r_1, \ldots, r_N) \mapsto \max_i |r_i|$ on F^n . If G is defined over a global field k, we take this representation to be defined over k, and then we can define the height function $\| \bullet \|$ on $G(\mathbb{A}_k)$ (and, by restriction, on G(k)) as the product over all places of the local height functions; this product is finite for any element.

These height functions on the group are a special case of the following (if we replace X by G):

Let X be a homogeneous, strongly⁸ quasi-affine G-variety over a global field k. Choose finite sets $\{f_i\}_i$ of generators of k[X] and $\{h_j\}_j$ of generators for the ideal of $\bar{X}^{\mathrm{aff}} \setminus X$ and set, at every place v and for $x \in X(k_v)$:

$$r_v(x) = \max \{1, |f_i(x)|_v, (\max\{|h_j(x)|_v\})^{-1}\},\$$

and globally for $x \in X(\mathbb{A}_k)$:

$$r(x) = \prod_{v} r_v(x)$$

(almost all factors are equal to one).

We call r an "algebraic height function", or simply a height function, on X. The following are true:

6.1.1. **Lemma.** (1) Any two height functions r_1, r_2 defined as above are polynomially equivalent in the following sense: there are positive constants $c_1, c_2, m_1, m_2 > 0$ such that:

$$c_1 r_1(x)^{m_1} \le r_2(x) \le c_2 r_1(x)^{m_2} \text{ for all } x \in X(\mathbb{A}_k).$$

(2) There is a positive number m and a constant c such that for every $x \in X(\mathbb{A}_k)$ and $g \in G(\mathbb{A}_k)$ we have:

$$c^{-1}||g||^{-m} \le \frac{r(x \cdot g)}{r(x)} \le c||g||^m.$$
(6.1)

⁸Strongly quasi-affine means that the canonical morphism: $X \to \bar{X}^{\text{aff}} := \operatorname{spec} k[X]$ is an open immersion.

(3) There is a positive number M and a constant c such that:

$$\#\{\xi \in X(k)|r(\xi) < T\} \le cT^M, \text{ for all } T \ge 1.$$
 (6.2)

Proof. For the first statement (it is of course enough to prove one inequality), let us first consider the affine case. If f_i and F_i are generators, then $F_i = \sum_{\alpha} c_i^{\alpha} f_{\alpha}$, where α is a multiindex denoting a product, and if d is the highest degree of the multiindices appearing we get: $|F_j| \ll \max\{|f_i|, |f_i|^d\} \ll \max\{1, |f_i|^d\}$, with the implicit constant being equal to 1 at almost all places.

For the quasiaffine case, if we denote by h_j and H_j the generators of the ideal, then we similarly have: $\max |H_i| \leq c \cdot \max\{|h_i|, |h_i|^d\}$ for some positive integral power d and some constant $c \geq 1$ which is equal to 1 at almost every place, hence $\max\{|h_i|\}^{-1} \leq c \cdot \max\{\max\{|H_i|\}^{-1}, \max\{|H_i|\}^{-\frac{1}{d}}\} \leq c \cdot \max\{1, \max\{|H_i|\}^{-1}\}$. This proves the first claim.

For the second, it is again enough to prove one side of the equality. The fact that $|f_i(x \cdot g)|_v \ll ||g||_v^m r_v(x)$, where f_i is as in the definition of r(x), follows from the corresponding statement for a representation of G: one can embed X into the space of a G-representation and take the f_i 's to be coordinate functions. The power M is uniform in v, and the implied constant can be taken to be 1 at almost every place. To prove a similar bound for $(\max\{|h_j(x\cdot g)|_v\})^{-1}$, we may without loss of generality assume that the h_j 's span a G-stable vector subspace W of k[X], and that $\|\bullet\|$ was defined using the representation of G on W, endowed with this basis (we will assume that it is faithful, because if it isn't the operator norm it defines is bounded by the operator norm of a faithful one). Then

$$\max\{|h_j(x \cdot gg^{-1})|_v\} \le ||g||_v \cdot \max\{|h_j(xg)|_v\},\$$

hence indeed $\max\{|h_j(xg)|_v\}^{-1} \le ||g||_v r(x)$.

The third follows from the analogous statement for affine space, by embedding X again into the space of a representation of G.

Finally, recall the notation: $[G] = G(k) \backslash G(\mathbb{A}_k)$, $[G]_{\emptyset} = A(k) N(\mathbb{A}_k) \backslash G(\mathbb{A}_k)$. We also define natural height functions on [G] as follows: having fixed $\| \bullet \|$ on $G(\mathbb{A}_k)$, we set:

$$R([g]) = \inf_{\gamma \in G(k)} \|\gamma g\|.$$

This is polynomially equivalent (in the same sense as in Lemma 6.1.1) to the usual height function on a Siegel domain \mathcal{S} (s. below):

$$\mathscr{S} \ni nak \mapsto ||a||.$$

Similarly on $[G]_{\emptyset} = A(k)N(\mathbb{A}_k)\backslash G(\mathbb{A}_k)$ we denote:

$$R([g]) = \inf_{\gamma \in A(k)N(\mathbb{A}_k)} \|\gamma g\|.$$

6.2. Asymptotically B-finite automorphic functions and regularized inner product.

6.2.1. Asymptotically B-finite automorphic functions. Recall that the constant term of a function ϕ on [G] is the function

$$\phi_N(g) := \int_{[N]} \phi(ng) dn \tag{6.3}$$

on $[G]_{\emptyset}$.

Recall that a Siegel domain is a closed subset $\mathscr{S} = N_0 A_{t_0} K$ of $G(\mathbb{A}_k)$, where B = NA is a Borel subgroup, $N_0 \subset N(k_\infty)$ is compact, A_{t_0} is a certain closed subset of $A(k_\infty)$ and $K \subset G(\mathbb{A}_k)$ is compact; the data is chosen so that the map: $\mathscr{S} \to [G]$ is surjective and proper, and the map $\mathscr{S} \to [G]_{\emptyset}$ is proper onto a neighborhood of the cusp. (To fix a geometric picture, recall that $N \setminus \mathrm{SL}_2 \simeq \mathbb{A}^2 \setminus \{0\}$, and that the cusp in $[G]_{\emptyset}$ is the image of $0 \in \mathbb{A}^2$ under the map: $N \setminus \mathrm{SL}_2(\mathbb{A}_k) \to [G]_{\emptyset}$.)

The preimage in \mathscr{S} of any neighborhood of the cusp (in [G] or $[G]_{\emptyset}$) will be called a "Siegel neighborhood" (of the cusp); the reader should mark this distinction, as a Siegel domain is supposed to surject to [G], while a Siegel neighborhood is not. For a sufficiently small Siegel neighborhood \mathscr{S}' , we have the property:

$$\gamma \in G(k), g_1, g_2 \in \mathcal{S}', \gamma g_1 = g_2 \Rightarrow \gamma \in B(k). \tag{6.4}$$

Pullbacks of functions (on [G] or $[G]_{\emptyset}$) to $\mathscr S$ will be denoted by the same letter, without any notation for the pullback.

Following standard language, we will call Schwartz functions on [G] or $[G]_{\emptyset}$ "rapidly decaying functions" (denoted $\mathcal{S}([G])$, resp. $\mathcal{S}([G]_{\emptyset})$). The notion of "Schwartz space $\mathcal{S}(\mathscr{S})$ of a Siegel neighborhood \mathscr{S} " also makes sense, by considering the Siegel neighborhood as a closed semialgebraic subset of $G(\mathbb{A}_k)$ and defining $\mathcal{S}(\mathscr{S})$ as the stalk of $\mathcal{S}(G(\mathbb{A}_k))$ over this subset, in the language of [Sak13a, Appendix B] (i.e. as a quotient, by restriction of functions).

We will for brevity say "automorphic function" for a smooth automorphic function of uniformly moderate growth. That is, a function ϕ on either [G] or $[G]_{\emptyset}$ will be called an automorphic function if it is locally constant outside of a finite set T of places, and lives, for some r, in the space V_r^{∞} of smooth vectors under the action of $G(k_T)$ of the Banach space V_r of functions on [G] defined by the norm: $\sup_{g \in [G]} |\phi(g)| R(g)^r$, where R(g) denotes an algebraic height function on either [G] or $[G]_{\emptyset}$ – s. Appendix B for a discussion of smooth vectors and representations. Thus, if the function is locally constant at all non-Archimedean places, this is the usual notion of a function of "uniformly moderate growth together with its derivatives" – the reader can restrict their attention to this case, and we will not be explaining easy extensions of well-known theorems to the "almost smooth" case. The most important of those theorems is the approximation of an automorphic function by its constant term:

6.2.2. **Theorem** ([MW95, Corollaries I.2.8, I.2.11]). In a Siegel neighborhood \mathscr{S} , and for every $\phi \in V_r^{\infty}$ (where V_r is as above) the difference $\phi - \phi_N$ is

a rapidly decaying function on \mathscr{S} , i.e., an element of $\mathscr{S}(\mathscr{S})$, which can be bounded in terms of seminorms of V_r^{∞} . In other words, the map: $V_r^{\infty} \ni \phi \mapsto \phi - \phi_N \in \mathscr{S}(\mathscr{S})$ is continuous.

We consider the normalized left action of [A] on functions on $[G]_{\emptyset}$, i.e., the one that preserves inner products:

$$a \cdot \phi(g) = \delta^{-\frac{1}{2}}(a)\phi(ag). \tag{6.5}$$

We say that an automorphic function ϕ on $[G]_{\emptyset}$ is B-finite if it is a sum of generalized eigenfunctions under the [A]-action. The set of exponents of ϕ is the multiset of its [A]-characters (under the above normalized action). For an idele class character χ of $B(\mathbb{A}_k)$, we let $\Re(\chi) = \Re(s)$, where $s \in \mathbb{C}$ is such that $\chi \delta^{-s}$ is unitary.

We say that it an automorphic function ϕ on $[G]_{\emptyset}$ is asymptotically B-finite if in a neighborhood of the cusp it coincides, up to a rapidly decaying function, with a B-finite function. The exponents of ϕ are the exponents of the latter.

We say that an automorphic function ϕ on [G] is asymptotically B-finite if on a Siegel neighborhood it coincides, up to a rapidly decaying function, with a B-finite function on $[G]_{\emptyset}$; its exponents are the exponents of the latter. For a multiset M of exponents, we will be denoting by $\mathcal{S}_{M}^{+}([G])$ the corresponding space of asymptotically B-finite functions; approximation by the constant term (Theorem 6.2.2) implies that $\mathcal{S}_{M}^{+}([G])$ is $G(\mathbb{A}_{k})$ -invariant.

If we fix the level, i.e., if we fix a compact open subgroup J of the finite adeles of G (or, more generally, if we fix it outside of a finite number Sof places including the Archimedean ones, so that J is a subgroup of the adeles of G outside of S), the space $\mathcal{S}_{M}^{+}([G])^{J}$ has a natural Fréchet space structure and, more precisely, these spaces form a Fréchet bundle over the complex manifold of ordered sets $(\chi_i)_i$ of exponents (possibly with multiplicities). For simplicity, we only describe this structure along one-dimensional families of the form $(\chi_1, \ldots, \chi_r) \cdot \delta^s$, where s varies in \mathbb{C} , and we implicitly fix the level throughout. For a given ordered r-tuple (χ_1, \ldots, χ_r) , denote by $[\chi_i]_{i=1}^r$ (or, for simplicity, by $[\chi_i]$) the corresponding unordered multiset, and let $I([\chi_i]_{i=1}^r)$ denote the generalized principal series⁹ with the given multiset of exponents. The association $(\chi_1, \dots, \chi_r) \mapsto I([\chi_i]_{i=1}^r)$ naturally forms a Fréchet bundle over the space of exponents, which can be trivialized by restricting to an orbit of a good maximal compact subgroup K of $G(\mathbb{A}_k)$. There is a natural map: $\mathcal{S}^+_{[\chi_i]}([G]) \ni \phi \mapsto \phi_{[\chi_i]} \in I([\chi_i])$ obtained by extracting from the constant term the (uniquely defined) element of $I([\chi_i])$ to which it is asymptotically equal in a neighborhood of the cusp, and this

⁹The notion of "generalized" principal series here is defined with respect to the [A]-action, not the $A(\mathbb{A}_k)$ -action. That is, its elements are smooth functions on $[G]_{\emptyset}$, and generalized [A]-eigenfunctions with a given multiset of exponents. If the multiset contains a single element χ , of course, this is nothing but the principal series representation $I(\chi)$.

gives an embedding

$$S_{[\chi_i]}^+([G]) \to S(\mathscr{S}) \oplus I([\chi_i]),$$
 (6.6)

where \mathscr{S} is a Siegel domain and the map to the first summand is $\phi \mapsto \phi - \phi_{[\chi_i]}$. We postulate that this embedding is closed, thus defining the aforementioned "natural" Fréchet space structure on $\mathcal{S}^+_{[\chi_i]}([G])$.

The image of this map is the subspace of elements (ϕ, f) such that $\phi + f$ descends to a function on [G], and if we fix an element $f_{\delta} \in I(\delta)$ which is positive as a function, then the map: $(\phi, f) \mapsto (\phi \cdot f_{\delta}^s, f \cdot f_{\delta}^s)$ identifies those spaces along the family, i.e., we do indeed have a Fréchet bundle.

6.2.3. Regularized integral. We will define a regularized integral

$$\int_{[G]}^* \phi(g) dg$$

for asymptotically B-finite functions on [G] whose exponents do not include $\delta^{\frac{1}{2}}$. It will, in fact, be a meromorphic family of $G(\mathbb{A}_k)$ -invariant functionals on the Fréchet bundle consisting of the spaces $\mathcal{S}^+_{[\chi_i\delta^s]}([G])$ as s varies. For $\Re(s) \ll 0$ it is simply the integral of ϕ over [G].

6.2.4. Lemma. For every holomorphic section:

$$s \mapsto \phi_s \in \mathcal{S}^+_{[\chi_i \delta^s]}([G])$$

the integral

$$\int_{[G]} \phi_s(g) dg,$$

which converges for $\Re(s) \ll 0$, admits meromorphic continuation, with poles of order at most equal to the multiplicity of $\delta^{\frac{1}{2}}$ among the exponents $\chi_i \delta^s$, and is bounded by polynomial seminorms on the spaces $\mathcal{S}^+_{[\chi_i \delta^s]}$.

This meromorphically continued integral will be denoted by $\int_{[G]}^* \phi_s(g) dg$. The notion of polynomial seminorms (always: in bounded vertical strips) makes sense since the underlying Fréchet spaces have been identified as above for all values of s.

Proof. It suffices to choose a Siegel neighborhood \mathscr{S}' and show the same statement for the integral restricted to the image of \mathscr{S}' in [G] (let us denote it by $[\mathscr{S}']$). Then we may replace ϕ_s by the eigen-part $\phi_{s,[\chi_i\delta^s]}$ of its constant term (notation as around (6.6)) on \mathscr{S}' , because their difference is of rapid decay, locally uniformly in $\Re(s)$ (Theorem 6.2.2), and bounded by "constant" seminorms on $\mathcal{S}^+_{[\chi_i\delta^s]}$ (by definition of the Fréchet bundle structure). By (6.4), we may think of $[\mathscr{S}']$ as a neighborhood of the cusp in $B(k)\backslash G(\mathbb{A}_k)$, and the integral of $\phi_{s,[\chi_i\delta^s]}$ there clearly has the properties of the lemma.

We use this integral now to define a regularized inner product of two asymptotically B-finite functions on [G]:

$$\langle \phi_1, \phi_2 \rangle_{[G]}^* := \int_{[G]}^* \phi_1(g)\phi_2(g)dg.$$
 (6.7)

It makes sense as long as no pair of exponents (χ_1, χ_2) of ϕ_1 and ϕ_2 , respectively, satisfies: $\chi_1 \cdot \chi_2 = 1$, and it is $G(\mathbb{A}_k)$ -invariant. Notice that our pairings $\langle \ \rangle$ will all be bilinear, instead of hermitian; in particular, for a unitary representation π we will denote by $\langle \ , \ \rangle_{\pi}$ the bilinear pairing between π and $\bar{\pi}$.

6.2.5. Plancherel decomposition of the regularized inner product. We will now obtain a spectral decomposition of this regularized inner product, similar to the Plancherel decomposition for the inner product of two L^2 -functions. For simplicity, we restrict ourselves from now on to the case of interest, i.e., elements of the space $\mathcal{S}_{\frac{1}{2}}^+([G])$, (those which only have a simple exponent equal to $\delta^{\frac{1}{2}}$) – in particular, the exponent of the product of two such functions is $\delta^{\frac{3}{2}}$, so their regularized inner product is defined.

To formulate the "Plancherel decomposition", recall first the actual Plancherel decomposition for the inner product of, say, two rapidly decaying functions on [G]:

$$\langle \phi_1, \phi_2 \rangle_{([G])} = \int_{\hat{G}_{Ram}} \left\langle \hat{\phi}_1(\pi), \hat{\phi}_2(\bar{\pi}) \right\rangle_{\pi} d\pi + \sum_{\chi \in \widehat{G}^{ab}} \hat{\phi}_1(\chi) \hat{\phi}_2(\chi^{-1}). \quad (6.8)$$

Here the notation is as follows: First of all, for an automorphic character χ of G, we set:

$$\hat{\phi}(\chi) = (\text{Vol}[G])^{-\frac{1}{2}} \int_{[G]} \phi(g) \chi^{-1}(g) dg.$$

Recall that elements of $\hat{G}^{\rm aut}_{\rm Ram}$ are by definition unitary representations. Fixing a Plancherel measure $d\pi$ on this set (of course, it is natural to choose it to be counting measure on the cuspidal spectrum, but this will not make a difference for us here) gives rise to morphisms: $\phi \mapsto \hat{\phi}(\pi) \in \pi$, unique up to scalars of norm one, which make the above formula hold if we set $\hat{\phi}(\bar{\pi}) = \hat{\phi}(\pi)$ (considered as an element of $\bar{\pi}$). These morphisms can be explicated in terms of inner products with cusp forms or unitary Eisenstein series: for $v \in \pi$ we have

$$\langle \phi(\pi), \overline{v} \rangle_{\pi} = \int_{[G]} \phi(g) \overline{\ell(v)(g)} dg,$$

where $\ell: \pi \to C^{\infty}([G])$ is some embedding depending on the choice of Plancherel measure. The automorphic forms $\ell(v)$ are either cusp forms (and hence of rapid decay) or unitary Eisenstein series (and hence with normalized exponents of absolute value 1).

This leads to the following observation: Replacing inner products by the regularized inner products that we defined before, the maps $\phi \mapsto \hat{\phi}(\pi) \in \pi \in \hat{G}^{\mathrm{aut}}_{\mathrm{Ram}}$ and $\phi \mapsto \hat{\phi}(\chi)$ make sense for $\phi \in \mathcal{S}^+_{\frac{1}{\delta^2}}([G])$, except when $\chi = 1$. This is easily seen by checking the exponents of the inner products.

We are now ready to extend the Plancherel decomposition to $\mathcal{S}_{\frac{1}{\delta^{\frac{1}{2}}}}^+([G])$:

6.2.6. **Proposition.** Let $\phi_1, \phi_2 \in \mathcal{S}^+_{\frac{1}{\delta^2}}([G])$, unramified outside a finite set of places S. Then the integral:

$$\int_{\hat{G}_{\text{Ram}}} \left\langle \hat{\phi}_1(\pi), \hat{\phi}_2(\bar{\pi}) \right\rangle_{\pi} d\pi \tag{6.9}$$

is absolutely convergent, and the difference:

$$\langle h \star \phi_1, \phi_2 \rangle_{[G]}^* - \int_{\hat{G}_{Ram}^{aut}} \left\langle \hat{h}(\pi) \cdot \hat{\phi}_1(\pi), \hat{\phi}_2(\bar{\pi}) \right\rangle_{\pi} d\pi$$

$$- \sum_{\chi \in \widehat{G}^{ab}} \hat{h}(\chi) \hat{\phi}_1(\chi) \hat{\phi}_2(\chi^{-1}), \tag{6.10}$$

as a functional of $h \in \mathcal{H}^S$, is an evaluation of order 3 (s. §5.1) at \mathbb{C}_1 .

In the above expression the representations and characters which are ramified outside of S (for which the Satake transform $\hat{h}(\pi)$, $\hat{h}(\chi)$ does not make sense) can, of course, be ignored since their contribution is zero.

Proof. Let us denote by L the difference (6.10), considered as a functional of $h \in \mathcal{H}^S$.

By continuity of both sides, it is enough to prove the statement for ϕ_1 in the dense subspace of functions which can be represented as $h_1 \star \phi_1$, where ϕ_1 is asymptotically *B*-finite with *double* exponent $\delta^{\frac{1}{2}}$, and $\hat{h}_1(\mathbb{C}_1) = 0$. By invariance:

$$\langle \phi_1, \phi_2 \rangle^* = \langle \phi_1', h_1^{\vee} \star \phi_2 \rangle^*.$$

So, it is enough to show that the proposition for ϕ_1 replaced by ϕ_1' and ϕ_2 replaced by $\phi_2' := h_1^{\vee} \star \phi_2$, which is of rapic decay (i.e., $\phi_2' \in \mathcal{S}([G])$). The definitions still make sense, and obviously

$$\left\langle \hat{h}(\pi) \cdot \hat{\phi}'_1(\pi), \hat{\phi}'_2(\bar{\pi}) \right\rangle_{\pi} = \left\langle \hat{h}(\pi) \cdot \hat{\phi}_1(\pi), \hat{\phi}_2(\bar{\pi}) \right\rangle_{\pi}$$

again by invariance (and similarly for the Grössencharacters).

We claim that for $\phi \in \mathcal{S}^+_{[\delta^{\frac{1}{2}}, \delta^{\frac{1}{2}}]}([G])$ and $h \in \mathcal{H}^S$ such that the evaluations

of order 3 of \hat{h} at \mathbb{C}_1 vanish, the function $h \star \phi$ belongs to $\mathcal{S}([G])$ and has integral zero over [G]. Indeed, the order of the exponent $\delta^{\frac{1}{2}}$ is equal to the maximal order at r = 0, minus one, of the pole of an integral:

$$\int_{[G]}^* \phi(g) \mathcal{E}(f_r)(g) dg,$$

where $r \mapsto f_r \in I(\delta^{r-\frac{1}{2}})$ is an analytic section (which can be assumed unramified outside of S) and \mathcal{E} denotes its Eisenstein series. In particular, for $\phi \in \mathcal{S}^+_{[\delta^{\frac{1}{2}}, \delta^{\frac{1}{2}}]}([G])$ this order is at most 3, and therefore if \hat{h} vanishes to order 2 at \mathbb{C}_1 then

$$\int_{[G]}^{*} h \star \phi(g) \mathcal{E}(f_r)(g) dg = \int_{[G]}^{*} \phi(g) \mathcal{E}(h^{\vee} \star f_r)(g) dg =$$
$$= \hat{h}(\delta^{\frac{1}{2}-r}) \int_{[G]}^{*} \phi(g) \mathcal{E}(f_r)(g) dg$$

has at most a simple pole.

The contribution of the trivial representation to $h \star \phi$ is the residue of such an integral when f_r is chosen so that $\mathcal{E}(f_r)$ has constant residue 1, therefore if, in addition, \hat{h} vanishes of order 3 then this contribution is zero. This proves the claim.

This being so, we can apply the usual Plancherel decomposition (6.8) to the inner product $\langle h \star \phi_1', \phi_2' \rangle$, when ϕ_1', ϕ_2' are as above and \hat{h} vanishes to triple order at \mathbb{C}_1 , and we see that the integral in the definition of L(h) is in this case absolutely convergent, and equal to zero. Such an h can be chosen so that $\hat{h}(\pi)$ is bounded below on \hat{G}_{Ram}^{aut} , and therefore (6.9) is absolutely convergent. Thus, L(h) is well-defined for every h, and its vanishing on those h that vanish of order 3 at \mathbb{C}_1 shows that it is an evaluation of order 3 at \mathbb{C}_1 .

6.3. The torus RTF. For this subsection, for a T-torsor α over k, let $Y^{\alpha} = T^{\alpha} \backslash G^{\alpha}$ and $\Sigma : \mathcal{S}(Y^{\alpha}(\mathbb{A}_{k})) \to C^{\infty}([G^{\alpha}])$ be as before:

$$\Sigma\Phi(g) = \sum_{\gamma \in Y^{\alpha}(k)} \Phi(\gamma g)$$

(or the analogous functional in two variables, i.e., from $S(Y^{\alpha} \times Y^{\alpha}(\mathbb{A}_k))$ to $C^{\infty}([G^{\alpha}] \times [G^{\alpha}])$). The sum converges locally uniformly in the variable g, thus indeed it represents a continuous morphism into $C^{\infty}([G])$. By Lemma B.2.2, it actually maps continuously into a subspace V_r^{∞} of smooth vectors of moderate growth – notation as in Theorem 6.2.2.

In the split case, we fix a Borel subgroup of $G^{\alpha} = G$ containing T, B = TN, and let \mathring{Y} denote the open B-orbit on $Y = Y^{\alpha}$; in the nonsplit case, we let $\mathring{Y} = Y^{\alpha}$.

6.3.1. **Proposition.** In the split case, the constant term of $\Sigma\Phi$ is given by:

$$(\Sigma\Phi)_{N}(g) = \left(\int_{N(\mathbb{A}_{k})} \Phi(T \cdot ug) du + \int_{N(\mathbb{A}_{k})} \Phi(Twug) du\right) + \sum_{\gamma \in (\mathring{Y}/\!\!/ N)(k)} \int_{N(\mathbb{A}_{k})} \Phi(\gamma ug) du.$$
(6.11)

The term in brackets will be denoted by $(\Sigma\Phi)_{N,\delta}$; it belongs to $I_B^G(\delta^{\frac{1}{2}})$. The second term will be denoted by $(\Sigma\Phi)_{N,\text{rest}}$, and it is rapidly decaying in a neighborhood of the cusp.

In the nonsplit case, the constant term of $\Sigma\Phi$ is equal to $(\Sigma\Phi)_{N,\text{rest}}$, and is rapidly decaying in a neighborhood of the cusp.

Proof. The computation of the constant term in the above form, and the fact that $(\Sigma\Phi)_{N,\delta} \in I_R^G(\delta^{\frac{1}{2}})$, are immediate.

The fact that $(\Sigma\Phi)_{N,\text{rest}}$ is of rapid decay for $\delta(b)$ large follows from the fact that eventually, for $\delta(b)$ large, the points of $(\mathring{Y} /\!\!/ N)(k)$ avoid any compact subset of $(Y /\!\!/ N)(\mathbb{A}_k)$.

By Theorem 6.2.2, this implies:

6.3.2. Corollary. The map Σ represents a continuous morphism

$$\Sigma: \mathcal{S}(Y^{\alpha}(\mathbb{A}_k)) \to \mathcal{S}([G^{\alpha}])$$

when T is non-split, and a continuous morphism

$$\Sigma: \mathcal{S}(Y(\mathbb{A}_k)) \to \mathcal{S}^+_{\frac{1}{2}}([G])$$

when T is split.

Proof. When α is a non-trivial torsor, then $[G^{\alpha}]$ is compact and the statement is equivalent to the continuity of Σ as a map into $C^{\infty}([G^{\alpha}])$. We now assume that α is trivial, so $G^{\alpha} = G$.

Recall that by (6.6), the topology on $\mathcal{S}^+_{[\chi_i]}([G])$ is determined by the asymptotics map $\phi \mapsto \phi_{[\chi_i]}$ to the generalized principal series $I([\chi_i])$, and by the difference $\phi - \phi_{[\chi_i]}$, which lives in the Schwartz space $\mathcal{S}(\mathscr{S})$ of a Siegel domain.

In the case at hand, for $\phi = \Sigma \Phi$, the multiset of exponents $[\chi_i]$ consists of $\delta^{\frac{1}{2}}$ or is empty, and the difference of $\phi_{[\chi_i]}$ from the constant term ϕ_N is equal to the last term $(\Sigma \Phi)_{N,\text{rest}}$ of (6.11), which when pulled back to a Siegel domain clearly represents a continuous morphism

$$S(Y(\mathbb{A}_k)) \to S(\mathscr{S}).$$

By Theorem 6.2.2 and continuity of the morphism $\mathcal{S}(Y(\mathbb{A}_k)) \to V_r^{\infty}$ (Lemma B.2.2), this implies that the map $\Phi \mapsto (\phi - \phi_{[\chi_i]}) \in \mathcal{S}(\mathscr{S})$ is continuous.

For the split case, the map $S(Y(\mathbb{A}_k)) \to I(\delta^{\frac{1}{2}})$ is given by the term $(\Sigma\Phi)_{N,\delta}$ of (6.11), which again is continuous.

Now we will explicate the functional RTF using the regularized integral and inner product that we defined; recall that the functional RTF is defined in (2.5). Let $f \in \mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$, and choose a lift $\Phi \in \oplus'_v \oplus_\alpha \mathcal{S}(Y^\alpha(k_v) \times Y^\alpha(k_v))$ according to §5.2. According to Proposition 6.3.1, the function $\Sigma\Phi$ on $\sqcup_{\beta}[G^{\beta}] \times [G^{\beta}]$ (β running over global T-torsors) is of rapid decay in the nonsplit case, and asymptotically B-finite with simple exponent $\delta^{\frac{1}{2}}$ in both

variables in the split case. Thus, the regularized integral $\langle \Sigma \Phi \rangle^*$ over the diagonal of $\sqcup_{\beta} [G^{\beta}]$ makes sense, and we are now ready to express the RTF in terms of that.

6.3.3. **Proposition.** In the above setting:

$$RTF(f) = \langle \Sigma \Phi \rangle^*$$
.

To begin the proof of the proposition, let $\mathcal{S}(\mathcal{Z}_v)_{\alpha}$ denote the coinvariants of $\mathcal{S}(Y^{\alpha}(k_v) \times Y^{\alpha}(k_v))$ with respect to the diagonal $G^{\alpha}(k_v)$ -action.

6.3.4. **Lemma.** If $f \in \otimes'_v \mathcal{S}(\mathcal{Z}_v)_{\alpha_v}$, and the collection $(\alpha_v)_v$ does not correspond to a T-torsor over k, then RTF(f) = 0.

Proof. This is clear for the regular orbital integrals, i.e., the terms of (2.5) with $\xi \neq 0, -1$, each of which is an evaluation on $\otimes'_v \mathcal{S}(\mathcal{Z}_v)_{\alpha_v}$ for precisely one T-torsor α over k.

The proof of Theorem 2.4.2 also shows that $O_0(f)$ (and $O_{-1}(f)$) are zero unless the support of a preimage of f meets $Y^{\alpha}(\mathbb{A}_k) \times Y^{\alpha}(\mathbb{A}_k)$ for some globally defined torsor α .

Given the previous lemma, it is enough for the proof of Proposition 6.3.3, by continuity, to restrict to the subspace spanned by functions of the form $\Phi(x,y) = \Phi_1(x)\Phi_2(y)$, where $\Phi_2 = h \star \delta_{y_0}$, where δ_{y_0} denotes a delta function at a k-point on Y^{α} (for some α) (the notion of "delta function" depends on a choice of measure), and $h \in C_c^{\infty}(G(\mathbb{A}_k))dg$. In particular, by invariance we can denote:

$$RTF(\Phi) = RTF(\Phi_1, h \star \delta_{y_0}) = RTF(h^{\vee} \star \Phi_1, \delta_{y_0}),$$

and vice versa every expression of the form RTF(Φ_1', δ_{y_0}) can be interpreted as RTF(Φ_1, Φ_2) for some Φ_1, Φ_2 as before, since by the Theorem of Dixmier and Malliavin $\Phi_1' = h \star \Phi_1$ for some h, Φ_1 .

6.3.5. **Lemma.** Let $y_0 \in Y^{\alpha}(k)$ (for some α) with stabilizer T, choose a Haar measure on $T(\mathbb{A}_k)$, hence on $Y^{\alpha}(\mathbb{A}_k)$, by our standard choice of Tamagawa measure on $G(\mathbb{A}_k)$, thus defining a "delta function" δ_{y_0} . Let $\Phi \in \mathcal{S}(Y^{\alpha}(\mathbb{A}_k))$. In the split case, $RTF(\Phi, \delta_{y_0})$ is equal to the analytic continuation to t = 0 of

$$\int_{[T]} \Sigma \Phi(a) \cdot \min(|a|, |a|^{-1})^t da, \tag{6.12}$$

where $\Sigma \Phi(g) = \sum_{\gamma \in Y^{\alpha}(k)} \Phi(\gamma g)$ as before, and |a| denotes the adelic absolute value after some identification of T with \mathbb{G}_m over k.

In the nonsplit case, it is equal to the integral of $\Sigma\Phi$ over [T].

We will denote this regularized integral by $\int_{[T]}^*$. (Similarly we define $\int_{T(\mathbb{A}_k)}^*$.)

¹⁰At non-Archimedean places this is obvious when one is working with locally constant functions, but for "almost smooth" functions it requires a proof, which we omit.

Proof. Notice that the invariant-theoretic quotient $Y^{\alpha} /\!\!/ T = \mathcal{B}$, under which the preimage of any $\xi \in \mathcal{B}$ is a T-stable subscheme of Y^{α} .

If $\Phi \equiv 0$ on the preimage of the irregular points $\xi = 0, -1$, then

$$\int_{[T]} \sum_{\gamma \in Y^{\alpha}(k)} \Phi(\gamma a) da = \sum_{\xi \in Y^{\alpha}(k)/T(k)} \int_{T_{\tilde{\xi}}(k) \setminus T(\mathbb{A}_k)} \Phi(\tilde{\xi}a) da,$$

where $\tilde{\xi}$ denotes a preimage of ξ on $Y^{\alpha}(k)$, and the summands corresponding to $\xi = 0, -1$ vanish, while for the rest we have $T_{\tilde{\xi}} = 1$, so we obtain

$$\sum_{\xi \neq 0, -1} f(\xi),$$

and a posteriori the original integral was convergent.

Let us now study the terms $O_0(f)$ and $O_{-1}(f)$ in the general case.

There is a T-stable neighborhood of the preimage of 0 which is T-equivariantly isomorphic to \mathbb{G}_a^2 , under an action of T as the special orthogonal group of a quadratic form. In that case, the proof of Theorem 2.4.2 shows that in the nonsplit case we have

$$\tilde{O}_0(f) = \operatorname{Vol}([T])\Phi(y_0)$$

(really, a sum of such evaluations over all α 's, but under our assumption on the support of Φ only the given α contributes).

In the split case, Lemma 2.6.1 expresses \tilde{O}_0 in terms of zeta integrals for the action of $T = \mathbb{G}_m$ on \mathbb{G}_a^2 , and we leave it to the reader to check that the expression in terms of zeta integrals coincides with the limit (6.12) used to define $\int_{-\infty}^{\infty} dt$. The analysis for \tilde{O}_{-1} is completely analogous.

Thus, the expression
$$(2.5)$$
 is in every case equal to (6.12) .

The proof of Proposition 6.3.3 will be complete if we relate the regularized integral over [T] to the regularized inner products defined previously:

6.3.6. **Lemma.** Let Ψ be any asymptotically B-finite automorphic function, and assume that it does not have the exponent $\delta^{\frac{1}{2}}$. Let $\Phi = h \star \delta_{y_0} \in \mathcal{S}(Y(\mathbb{A}_k))$. Then:

$$\langle \Psi, \Sigma \Phi \rangle^* = \int_{[T]}^* h^{\vee} \star \Psi(a) da.$$
 (6.13)

Proof. Since this is clear, and there is no need for regularization, when T is nonsplit, we restrict to the split case and fix an identification: $T \simeq \mathbb{G}_m$, so that we have an adelic absolute value $|\bullet|$ on $T(\mathbb{A}_k)$.

Let us fix a smooth partition of unity $1 = f_1 + f_2$ on [T] so that $f_1(a) = 0$ for |a| small, and $f_2(a) = 0$ for |a| large. For $t \in \mathbb{C}$, let $f_1^t(a) = |a|^{-t} f_1(a)$ and $f_2^t(a) = |a|^t f_2(a)$. Consider them as generalized functions on $T(k) \backslash G(\mathbb{A}_k)$, by the measures we fixed to define δ_{u_0} . We have

$$\int_{[T]}^* h^{\vee} \star \Psi(a) da = \text{ the analytic continuation to } t = 0 \text{ of }$$

$$\int_{[T]} h^{\vee} \star \Psi(a) f_1^t(a) da + \int_{[T]} h^{\vee} \star \Psi(a) f_2^t(a) da,$$

so it suffices to show that the analytic continuation to t=0 of $\int_{[T]} h^{\vee} \star \Psi(a) f_i^t(a) da$ is equal to

$$\langle \Psi, \Sigma(h \star f_i) \rangle^*$$
,

where by abuse of notation, although $h \star f_i$ is not an element of $\mathcal{S}(Y(\mathbb{A}_k))$, we write $\Sigma(h \star f_i)(g) = \sum_{\gamma \in T(k) \backslash G(k)} h \star f_i(\gamma g)$.

Exactly as in Proposition 6.2.2, $\Sigma(h\star f_i^t)$ is *B*-finite with exponent δ^{1-t} ; more precisely, the functions $\Sigma(h\star f_i^t)$ form an analytic section of the Fréchet bundle with fibers $\mathcal{S}_{\delta^{1-t}}^+$. Therefore, by definition,

 $\langle \Psi, \Sigma(h \star f_i) \rangle^* = \text{ the analytic continuation to } t = 0 \text{ of } \langle \Psi, \Sigma(h \star f_i^t) \rangle$, and for t large:

$$\left\langle \Psi, \Sigma(h \star f_i^t) \right\rangle = \int_{T(k) \backslash G(\mathbb{A}_k)} \Psi(g) h \star f_i^t(g) dg = \int_{[T]} h^{\vee} \star \Psi(a) f_i^t(a) da.$$

6.4. The Kuznetsov trace formula. For this subsection we let X be as in [Sak13a, §4] (hence, isomorphic to $N \setminus PGL_2$ – we fix such an isomorphism over k), ¹¹ and \mathcal{L}_{ψ} the complex line bundle whose sections (locally at a place v) are functions on $PGL_2(k_v)$ satisfying $f(ng) = \psi(n)f(g)$ (in particular, the line bundle is trivialized over the point corresponding to $1 \in G$). Again, we define the morphism "summation over k-points" from continuous sections of \mathcal{L}_{ψ} over $N \setminus G(\mathbb{A}_k)$ to functions on [G], whenever it converges, as:

$$\Sigma\Phi(\bullet) := \sum_{\gamma \in N(k) \backslash G(k)} \Phi(\gamma g); \tag{6.14}$$

by the same symbol we will also denote the corresponding map from sections of $\mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1}$ on two copies of X to functions on $[G] \times [G]$.

Choose an algebraic height function r on X, as described in §6.1. For example, one may identify the affine closure of X with spec $k[x^2, y^2, xy]$ (the quotient of affine 2-space by the action of $\{\pm 1\}$), and define height functions on $X(k_v)$ by:

$$r_v((x,y) \mod \{\pm 1\}) = \max\{m_v, m_v^{-1}\}, \text{ where } m_v = \max\{|x^2|_v, |y^2|_v\}.$$

For every positive number N we consider the Banach space of continuous sections Φ of \mathcal{L}_{ψ} on X satisfying

$$\sup_{\xi \in X(\mathbb{A}_k)} |\Phi(\xi)| r(\xi)^N < \infty. \tag{6.15}$$

¹¹In [Sak13a] we took some care to not choose an isomorphism $X \simeq N \backslash G$; here we consider such an isomorphism as given globally over k. We also identify N with \mathbb{G}_a over k, so that the chosen adele class character ψ gives rise to a character, denoted by the same letter, on [N].

(Recall that the absolute value of \mathcal{L}_{ψ} is the trivial line bundle.) We will denote this space by $C_{-N}(X(\mathbb{A}_k), \mathcal{L}_{\psi})$.

Notice that for smooth sections of (uniformly) moderate growth, the estimate (6.15) is automatic, for every N, in a neighborhood of the cusp, i.e., in a neighborhood of the image of "zero" under the isomorphism $N \setminus SL_2 = \mathbb{A}^2 \setminus \{0\}$. Hence, for those, the growth condition is restrictive only in a neighborhood of "infinity".

Similarly, we fix a height function R(g) on [G], and let $C_{-N}([G])$ denote the Banach space of continuous functions ϕ on [G] which satisfy

$$\sup_{x \in X(\mathbb{A}_k)} |\phi(g)| R(g)^N < \infty.$$

6.4.1. **Proposition.** There is a positive constant c such that for large N the map: $\Phi \mapsto \Sigma \Phi$ represents a continuous morphism:

$$C_{-N}(X(\mathbb{A}_k), \mathcal{L}_{\psi}) \to C_{-cN}([G]).$$
 (6.16)

Proof. For any compact neighborhood U of the identity in $G(\mathbb{A}_k)$ there are positive constants c_1, c_2 such that $c_1r(x) \leq r(xg) \leq c_2r(x)$ for all $g \in U$ and $x \in X(\mathbb{A}_k)$ (Lemma 6.1.1). This shows that all our estimates below are locally uniform on [G], but we will not comment further on that.

Using constants as in Lemma 6.1.1, we have

$$\sum_{\xi \in X(k)} |\Phi(\xi)| \ll \sum_{\xi \in X(k)} |r(\xi)|^{-N} \ll \int_0^\infty T^{-N} T^M dT < \infty,$$

for large N, so convergence of the sum representing $\Sigma\Phi(g)$ (by the above, locally uniform in g) is not an issue.

To prove the asymptotic properties of $\Sigma\Phi$ we need an estimate in the opposite direction than that of Lemma 6.1.1: we need to show that when $g \in [G]$ becomes "large", $r(x \cdot g)$ also becomes "large". We may fix a Siegel domain $\mathscr{S} \subset G(k_{\infty})$ for [G], g = nak, and assume that R(gak) = ||a|| for large ||a||, where ||a|| represents some algebraic height function on the torus A. For simplicity, let us assume that $k = \mathbb{Q}$ (the general case is similar, with different constants). For simplicity of notation, also, we will present the case of $N \setminus \mathrm{SL}_2 \simeq \mathbb{A}^2 \setminus \{0\}$ instead of that of $N \setminus \mathrm{PGL}_2$, denoting it again with X.

We may choose coordinates (x,y) on \mathbb{A}^2 so that the torus $\mathbb{G}_m \simeq A$ acts as: $(x,y) \cdot a = (ax,a^{-1}y)$. We define the height function for $N \setminus \mathrm{SL}_2$ as $r_v(x,y) = \max(m_v,m_v^{-1})$, where $m_v = \max\{|x|_v,|y|_v\}$. We also define a height function for \mathbb{A}^2 as $r_v'(x,y) = m_v$. Our goal is to estimate

$$\sum_{\xi \in \mathbb{A}^2 \setminus \{0\}(k)} r(\xi \cdot a)^{-N}. \tag{6.17}$$

for $a \in A(\mathbb{R})$ in a Siegel neighborhood (see §6.2.1). We may choose an isomorphism: $A(\mathbb{R}) \simeq \mathbb{R}^{\times}$ so that points in the Siegel neighborhood have

 $|a| \ge 1$. We split the sum in those ξ with x = 0 and the rest, and since $r'(\xi) \le r(\xi)$ we have that (6.17) is

$$\leq \sum_{x \in k^{\times}, y \in k} r'((x, y) \cdot a)^{-N} + \sum_{y \in k^{\times}} r((0, y) \cdot a)^{-N}.$$

For $x \neq 0$ we have

$$\frac{r'_{\infty}(ax, a^{-1}y)}{r'_{\infty}(x, y)} \ge \frac{|ax|_{\infty}}{\max\{|x|_{\infty}, |y|_{\infty}\}} \cdot \frac{\max\{|x|_{\infty}, |y|_{\infty}\}}{\max\{1, |x|_{\infty}, |y|_{\infty}\}} \ge$$

$$\ge \begin{cases} |a||x|_{\infty}, & \text{if } |x|_{\infty}, |y|_{\infty} \le 1, \\ |a|, & \text{if } |x|_{\infty} \ge 1, |y|_{\infty}, \\ |a|\frac{|x|_{\infty}^{2}}{|xy|_{\infty}} & \text{if } |y|_{\infty} \ge |x|_{\infty}, 1. \end{cases}$$

Using the product formula to express $|x|_{\infty} = |x|_f^{-1} \ge r_f'(\xi)^{-1} = r_f'(\xi \cdot a)^{-1}$ (the index f denoting the product over all finite places, and we have used the fact that $a \in G(k_{\infty})$), in the first case we get

$$\frac{r'_{\infty}(ax, a^{-1}y)}{r'_{\infty}(x, y)} \ge \frac{|a|}{r_f(\xi \cdot a)} \Rightarrow r'(\xi \cdot a)^2 \ge |a|r'(\xi).$$

We have used here that $r_f'(ax, a^{-1}y) = r_f'(x, y)$ and $r_\infty'(ax, a^{-1}y) \ge 1$. In the second case we get, by multiplying with $r_f'(x, y)$,

$$r'(\xi \cdot a) \ge |a|r'(\xi),$$

and in the third, using the additional fact that $|xy|_{\infty} \leq 2r'_{\infty}(\xi \cdot a)^2$, we get

$$\frac{r'_{\infty}(ax, a^{-1}y)}{r'_{\infty}(x, y)} \ge \frac{|a|}{2r'_{f}(\xi \cdot a)^{2} \cdot r'_{\infty}(\xi \cdot a)^{2}} \Rightarrow r'_{f}(\xi \cdot a)^{4} \ge \frac{1}{2}|a|r'(\xi).$$

Since $r'(\xi \cdot a) \ge 1$ we have in all cases

$$r'(\xi \cdot a)^4 \ge \frac{1}{2}|a|r'(\xi).$$
 (6.18)

Hence, using constants as in (6.2):

$$\sum_{\xi \in X(k), x \neq 0} r'(\xi \cdot a)^{-N} \ll |a|_{\infty}^{-\frac{N}{4}} \sum_{\xi \in X(k), x \neq 0} r'(\xi)^{-\frac{N}{4}} \ll$$

$$\ll |a|_{\infty}^{-\frac{N}{4}} \cdot \int_{0}^{\infty} T^{-N} T^{M} dT \ll |a|_{\infty}^{-\frac{N}{4}}$$

for N large enough, with the implicit constant depending on N. When x is zero, we similarly have

$$\frac{r_{\infty}(ax, a^{-1}y)}{r_{\infty}(x, y)} \ge \frac{|ay^{-1}|_{\infty}}{\max\{|y|_{\infty}, |y|_{\infty}^{-1}\}}$$

which is equal to |a| when $|y|_{\infty} \leq 1$. When $|y|_{\infty} > 1$ it is

$$|a||y|_{\infty}^{-2} = |a||y|_f^2 \ge |a|r_f(\xi)^{-2} = |a|r_f(\xi \cdot a)^{-2}$$

hence, as before, we have in both cases:

$$r(\xi \cdot a)^3 \ge |a| r(\xi),$$

and we can estimate using (6.2):

$$\sum_{y \in k} r((0, y) \cdot a)^{-N} \ll |a|^{-\frac{N}{3}}$$

when N is large.

This gives the following estimate for (6.17):

$$\sum_{\xi \in \mathbb{A}^2 \setminus \{0\}(k)} r(\xi \cdot a)^{-N} \ll |a|^{-\frac{N}{4}},$$

which completes the proof.

Now we look at the spaces $\mathcal{S}^s(\overline{X}\times X(\mathbb{A}_k), \mathcal{L}_{\psi}\boxtimes \mathcal{L}_{\psi}^{-1})$ of non-standard Whittaker functions, whose orbital integrals give rise to the space $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$.

6.4.2. **Lemma.** For any given N > 0 there is $\sigma_0 \in \mathbb{R}$ such that for $\Re(s) \geq \sigma_0$ the space $\mathcal{S}^s(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ belongs to $C_{-N}(X \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$, and the embedding is bounded by polynomial seminorms on the former.

Proof. This is immediate from the definition of these spaces in [Sak13a, $\S 6.1$].

In particular, for $\Re(s)$ large, the integral $\langle \Sigma \Phi_s \rangle$ over the diagonal converges and can be decomposed spectrally by the Plancherel formula for $L^2([G])$. Now recall that the Kuznetsov trace formula was defined in (2.6); we claim:

6.4.3. **Proposition.** In the above setting, and for $\Re(s) \gg 0$:

$$KTF(f_s) = \langle \Sigma \Phi_s \rangle + \tilde{O}_0(f_s).$$

Proof. It is clear that:

$$\begin{split} \langle \Sigma \Phi_s \rangle &= \sum_{\xi \in (X(k) \times X(k))/G(k)} \int_{G_{\xi}(k) \backslash G(\mathbb{A}_k)} \Phi_s(\xi g) dg = \\ &= \sum_{\xi \in X \times X/G(k)} \int_{G_{\xi}(\mathbb{A}_k) \backslash G(\mathbb{A}_k)} \Phi_s(\xi g) dg \end{split}$$

(using the fact that the Tamagawa volume of [N] is 1)

$$= \sum_{\xi \in k^{\times}} f_s(\xi) + \int_{N \setminus G(\mathbb{A}_k)} \Phi_s(g, g) dg,$$

so it is enough to identify this last integral with the "irregular orbital integral" $\tilde{O}_{\infty}(f_s)$.

Suppose that $\Phi_s = \prod_v \Phi_{v,s}$ (hence $f_s = \prod_v f_{v,s}$), where $\Phi_{v,s}$ is a section of $S^s(\overline{X} \times X(k_v), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$. Notice that for every place $v, \int_{N \setminus G(k_v)} \Phi_{v,s}(g,g)$ is the "inner product" denoted by $\langle f_{v,s} \rangle$ in [Sak13a, §4.9]. (The fact that

 $s \neq 0$ plays no role in its definition.) By [Sak13a, Proposition 4.8], and the definition of \tilde{O}_{∞} in §4.3, this is equal to $\tilde{O}_{\infty}(f_{v,s})$.

The functional \tilde{O}_0 was defined as the analytic continuation to t=0 of the functional (3.9), and an explicit expression of this appears in (2.15), (2.16). The last piece that we need for the proof of Theorem 5.3.3 is:

6.4.4. **Lemma.** The functional $h \mapsto \tilde{O}_0(h \star f_s)$ is an evaluation at $\delta^{\frac{1}{2}+s}$, $\eta \cdot \delta^{\frac{1}{2}+s}$ (s. §5.1).

Proof. For every place v, by definition of the space $S^s(\overline{X}(k_v))$ (s. [Sak13a, §4.5 and 6.1]), there is a natural quotient map obtained by the asymptotics of sections

$$S^s(\overline{X} \times X(k_v), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1}) \twoheadrightarrow I([\delta^{\frac{1}{2}+s}, \eta \delta^{\frac{1}{2}+s}]) \otimes S(X(k_v), \mathcal{L}_{\psi}^{-1}),$$

where $I([\delta^{\frac{1}{2}+s}, \eta \delta^{\frac{1}{2}+s}])$ is the generalized principal series with the given multiset of exponents (as in §6.2, but here locally at v). Let us denote:

$$V_v := I([\delta^{\frac{1}{2}+s}, \eta \delta^{\frac{1}{2}+s}]) \otimes \mathcal{S}(X(k_v), \mathcal{L}_{\psi}^{-1}).$$

Correspondingly, there is quotient of coinvariants under the diagonal $G(k_v)$ -action:

$$\mathcal{S}(\mathcal{W}_v^s) \twoheadrightarrow (V_v)_G$$

and $(V_v)_G$ is two-dimensional, whose dual is spanned by the local distributions appearing in the definition of \tilde{O}_0 of (2.15) and (2.16). (In the nonsplit case, those are denoted by $\tilde{O}_{0_v,\pm}$, and in the split case they are denoted by \tilde{O}_{0_v} and \tilde{O}_{u_v} .)

In the nonsplit case the restricted tensor product $\bigotimes_{v}' V_v$ has a canonical quotient to:

$$\left(I(\delta^{\frac{1}{2}+s}) \oplus I(\eta\delta^{\frac{1}{2}+s})\right) \otimes \mathcal{S}(X(\mathbb{A}_k), \mathcal{L}_{\psi}^{-1}),$$

and it is clear from the definition of \tilde{O}_0 in (2.15) that \tilde{O}_0 factors through this quotient; in particular, the functional $h \mapsto \tilde{O}_0(h \star f_s)$ is an evaluation at is an evaluation at $\delta^{\frac{1}{2}+s}$, $\eta \cdot \delta^{\frac{1}{2}+s}$.

In the split case the argument is more complicated. From the two summands of (2.16), it is clear that the first one:

$$2a_0^{S_0} \prod_{v \in S_0} \tilde{O}_{0_v},$$

(where we use S_0 to denote a set of places strictly larger than S) when applied to $h \star f$, is an evaluation at $\delta^{\frac{1}{2}+s}$.

The problem lies in analyzing the second term:

$$a_{-1}^{S_0} \prod_{v \in S_0} \tilde{O}_{0_v} \cdot \left(\sum_{v \in S_0} \frac{\tilde{O}_{u_v}}{\tilde{O}_{0,v}} \right).$$

and showing that, when applied to $h \star f$, it is a linear combination of an evaluation of \hat{h} at $\delta^{\frac{1}{2}+s}$ and its derivative at t=0 along the one-parameter family: $t\mapsto \delta^{\frac{1}{2}+s+t}$.

For this it suffices to show the following:

6.4.5. **Lemma.** For every $v \in S_0 \setminus S$ the functional:

$$h \mapsto \tilde{O}_{u_v}(h \star f_v^0)$$

is equal to an evaluation of \hat{h} at $\delta^{\frac{1}{2}+s}$ minus:

$$\tilde{O}_{0_v}(f_v^0) \cdot \frac{d}{ds} \hat{h}\left(\delta^{\frac{1}{2}+s}\right).$$

For this we recall several facts about orbital integrals for elements of $S^s(\overline{X} \times X(k_v), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$. First of all, at non-Archimedean places the stalk of $S(\mathcal{W}^s)$ at zero is canonically isomorphic to a stalk of the (N_v, ψ_v^{-1}) -coinvariants of the generalized principal series $I([\delta^{\frac{1}{2}+s}, \delta^{\frac{1}{2}+s}])$. Indeed, the theory of asymptotics [SV, §5] provides a canonical, G-equivariant map $e^*_{\emptyset}: C^{\infty}(N\backslash G, \mathcal{L}_{\psi}) \to C^{\infty}(N\backslash G)$ which is characterized by the property that the restriction of any function to a Cartan subgroup A normalizing N is equal to the restriction of its image under e^*_{\emptyset} when restricted to a subset of the form $\delta(a) \ll 1$. This map allows us to canonically identify the stalk of the cosheaf $S^s(\overline{X}, \mathcal{L}_{\psi})$ "at infinity" with the corresponding stalk of $I([\delta^{\frac{1}{2}+s}, \delta^{\frac{1}{2}+s}])$, in a G-equivariant way, and hence also their coinvariants. Let us denote by $V_{v,0}$ the corresponding stalk of $I([\delta^{\frac{1}{2}+s}, \delta^{\frac{1}{2}+s}])_{(N,\psi^{-1})}$: it is the stalk at the point corresponding to "infinity" on X; so we have a canonical isomorphism

$$\mathcal{S}(\mathcal{W}_v^s)_0 \simeq V_{v,0}.$$

Secondly, we notice that the normalized k_v^{\times} -action on this stalk $\mathcal{S}(\mathcal{W}_v^s)_0$, tensored by the inverse absolute value, i.e., the action

$$a * f(\xi) := |a|^{-\frac{1}{2}} f(a\xi),$$

corresponds under the above isomorphism to the normalized k_v^{\times} -action on $I([\delta^{\frac{1}{2}+s}, \delta^{\frac{1}{2}+s}])$, where k_v^{\times} is identified with A_v via the character δ and the normalized action is

$$a \cdot \Phi(g) := \delta(a)^{-\frac{1}{2}} \Phi(ag).$$

Thirdly, the endomorphism ring of $C^{\infty}(N\backslash G)^{K_v}$ (where $K_v = G(\mathfrak{o}_v)$) generated by the unramified Hecke algebra of G_v is a subring of the endomorphisms generated by the unramified Hecke algebra of A_v ; the normalization of the action of A_v is compatible with the Satake isomorphism, i.e., $h \in \mathcal{H}(G_v, K_v)$ induces the same endomorphism as the image of \hat{h} under the maps: $\mathbb{C}[\check{A}]^W \hookrightarrow \mathbb{C}[\check{A}] \xrightarrow{\sim} \mathbb{C}[A_v/A(\mathfrak{o}_v)] \xrightarrow{\sim} \mathcal{H}(A_v, A(\mathfrak{o}_v))$.

Finally, we recall the way that $\tilde{O}_{0_v}(h \star f_v^0)$ and $\tilde{O}_{u_v}(h \star f_v^0)$ determine the germ of $h \star f_v^0$ in the stalk $\mathcal{S}(\mathcal{W}_v^s)_0$: the asymptotic behavior of $h \star f_v^0$ close

to $\xi = 0$ is of the form

$$|\xi|^{s+1} \left(-\tilde{O}_{0_v}(h \star f_v^0) \log |\xi| + \tilde{O}_{u_v}(h \star f_v^0) \right)$$

(cf. (2.11), but here with the extra factor $|\xi|^{s+1}$).

Given all that, it suffices to show that, under the *-action of $\mathcal{H}(A_v, A(\mathfrak{o}_V)) \simeq \mathcal{H}(k_v^{\times}, \mathfrak{o}_v^{\times})$ on the stalk at $\xi = 0$ of functions of the form

$$F(\xi) = |\xi|^{s+1} \left(-C_1 \log |\xi| + C_2 \right),$$

the functional " C_2 " has the property

$$C_2(h*F) = A\hat{h}(\delta^{\frac{1}{2}+s}) + C_1(F)\frac{d}{ds}\hat{h}\left(t \mapsto \delta^{\frac{1}{2}+s+t}\right).$$

It is easy to see that under the above normalized action of A_v :

$$a * F(\xi) = |\xi|^{s+1} (-C_1|a|^{\frac{1}{2}+s} \log |a|) + \text{ a term proportional to } |a|^{\frac{1}{2}+s}.$$

If h = the characteristic measure of $aA(\mathfrak{o}_v)$ then under the isomorphism $\mathcal{H}(A_v, A(\mathfrak{o}_v)) \simeq \mathbb{C}[t, t^{-1}]$ (chosen so that the "point" δ^s on spec $\mathbb{C}[t, t^{-1}]$ corresponds to the evaluation $t \mapsto |\varpi|^s$) we have: $\hat{h}(t) = t^{\text{val}(a)}$.

Therefore, we translate: $-C_1|a|^{\frac{1}{2}+s}\log|a| = -C_1\frac{d}{ds}\hat{h}\left(\delta^{\frac{1}{2}+s}\right)$, and our claim is proven.

- 6.5. Functional equation. We now come to what was called the "second miracle", namely Theorem 5.5.1. To prove it, recall that $\mathcal{S}(\mathcal{Z}_v^s)$ denotes, by definition, the image of $\mathcal{S}(\mathcal{W}^s(k_v))$ under $f \mapsto \mathcal{G}^{-1}\left(|\bullet|^{-s-1}f\right)$. We claim:
- 6.5.1. **Proposition.** Multiplication by $\eta(\xi)\eta(\xi+1)|\xi+1|^{2s}$ induces an isomorphism:

$$\mathcal{R}: \mathcal{S}(\mathcal{Z}_v^{-s}) \xrightarrow{\sim} \mathcal{S}(\mathcal{Z}_v^s)$$

which maps basic functions to each other. Hence, it also induces an isomorphism of global Schwartz spaces:

$$\mathcal{S}(\mathcal{Z}^{-s}(\mathbb{A}_k)) \xrightarrow{\sim} \mathcal{S}(\mathcal{Z}^s(\mathbb{A}_k)).$$

Proof. The isomorphism follows immediately from the asymptotic behavior of elements in the space $\mathcal{S}_{1,v}^0$ as described by Table (4.1); notice that both $\mathcal{S}(\mathcal{Z}_v^{-s})$ and $\mathcal{S}(\mathcal{Z}_v^{-s})$ correspond to the subspaces with $C_5 = 0$, in the notation of that table.

The fact that basic functions are carried over to each other is clear from (4.16).

Now let \mathcal{T} denote the following isomorphism:

$$\mathcal{T}: \mathcal{S}(\mathcal{W}_v^{-s}) \to \mathcal{S}(\mathcal{Z}_v^{-s}) \to \mathcal{S}(\mathcal{Z}_v^s) \to \mathcal{S}(\mathcal{W}_v^s),$$

where the middle arrow is that of the previous proposition, and the others are the isomorphisms we have been using throughout, the first given by $f \mapsto \mathcal{G}^{-1}(|\bullet|^{s-1}f)$ and the second given by $f \mapsto |\bullet|^{s+1}\mathcal{G}(f)$. To finish the proof of Theorem 5.5.1, we need to show that \mathcal{T} is compatible with

the action of the spherical Hecke algebra on the basic vectors, and that it preserves the property of sections to be analytic of rapid decay in vertical strips. The latter is clear, since this is a property of the transform $|\bullet|^{s+1}\mathcal{G}$ and its inverse, and is clearly true for the transform of Proposition 6.5.1.

6.5.2. **Proposition.** At almost every place v, and for every $h \in \mathcal{H}(G_v, K_v)$, the isomorphism \mathcal{T} takes $h \star f_{\mathcal{W}_v}^0$ to $h \star f_{\mathcal{W}_v}^0$.

Proof. We will leave the details of the proof to the reader; it can be done by (very unpleasant) explicit calculations, or by "cheating" as in the proof of the fundamental lemma [Sak13a, Theorem 5.4], by using the following fact: the space $\mathcal{S}(\mathcal{Z}_v^s)$ can be identified with the space obtained by orbital integrals for the quotient $(\mathbb{G}_m, |\bullet|_v^s) \setminus G/(\mathbb{G}_m, \eta_v(\bullet)|\bullet|_v^s)$, where \mathbb{G}_m denotes the torus of diagonal elements in PGL₂. Thus, the space *does* have an action of the Hecke algebra (although strictly speaking we would like to avoid using that, and indeed we avoid it in global considerations).

With respect to this action, the above transforms: $\mathcal{S}(\mathcal{W}_v^{\pm s}) \leftrightarrow \mathcal{S}(\mathcal{Z}_v^{\pm s})$, given by \mathcal{G} and a suitable power of the absolute value, are equivariant when restricted to images of unramified vectors; this is proven exactly as in the proof of the Fundamental Lemma, [Sak13a, Theorem 5.4], with the intervention of an intermediate "space" \mathcal{W}_1^s .

Finally, we have G_v -equivariant maps:

$$\mathcal{S}\left((\mathbb{G}_m, |\bullet|_v^{-s})\backslash \operatorname{PGL}_2\right) \ni f \mapsto f(w\bullet) \in \mathcal{S}\left((\mathbb{G}_m, |\bullet|_v^{s})\backslash \operatorname{PGL}_2\right)$$

and:

$$\mathcal{S}\left(\left(\mathbb{G}_{m}, \eta_{v}(\bullet)\right) \bullet |_{v}^{-s}\right) \backslash \operatorname{PGL}_{2}\right) \ni f \mapsto f(w\bullet) \in \mathcal{S}\left(\left(\mathbb{G}_{m}, \eta_{v}(\bullet)\right) \bullet |_{v}^{s}\right) \backslash \operatorname{PGL}_{2}\right).$$

It can easily be checked that at the level of orbital integrals, i.e., G_v coinvariants, these descend to the transform \mathcal{R} of Proposition 6.5.1.

Even if we want to forget about the Hecke action on $\mathcal{S}(\mathcal{Z}_v^s)$ (more precisely: on its "unramified" vectors) when $s \neq 0$, for s = 0 we clearly have a Hecke action that comes from its original definition in terms of coinvariants of $\mathcal{S}(T_v \setminus G_v \times T_v \setminus G_v)$ (and suitable inner forms, which however do not play a role when discussing unramified vectors). A corollary of the above and the Fundamental Lemma of [Sak13a] (or a direct corollary of the last proof) is:

6.5.3. Corollary. The transform \mathcal{R} of Proposition 6.5.1, when s = 0, satisfies:

$$\mathcal{R}\left(h \star f_{\mathcal{Z}_v}^0\right) = h \star f_{\mathcal{Z}_v}^0$$

for every $h \in \mathcal{H}(G_v, K_v)$.

7. The formula of Waldspurger

7.1. **Local periods.** We define local relative characters, i.e., the local analogs of the period relative characters defined in §5.2.

If α denotes an isomorphism class of T-torsors over k_v , G^{α} is the corresponding inner form of G, and π_v^{α} is a tempered irreducible representation of G^{α} , then the integral

$$\tilde{v} \otimes v \mapsto \int_{T_v^{\alpha}} \langle \tilde{\pi}_v^{\alpha}(h) \tilde{v}, v \rangle_{\tilde{\pi}_v^{\alpha}} dh$$
 (7.1)

converges and represents a T_v^{α} -biinvariant functional on $\tilde{\pi}_v^{\alpha} \otimes \pi_v^{\alpha}$ and hence, by Frobenius reciprocity, a morphism:

$$\tilde{\pi}_v^{\alpha} \otimes \pi_v^{\alpha} \to C^{\infty}(T_v^{\alpha} \backslash G_v^{\alpha} \times T_v^{\alpha} \backslash G_v^{\alpha}).$$

(Of course, $T^{\alpha} \simeq T$, but we include the exponent in analogy with the notation for G.)

Dualizing, with respect to a fixed invariant measure on $(T^{\alpha}\backslash G^{\alpha})(k_v)$, and composing with the pairing of duality, we get a local relative character

$$J_{\pi_v^{\alpha}}: \mathcal{S}(T_v^{\alpha} \backslash G_v^{\alpha} \times T_v^{\alpha} \backslash G_v^{\alpha}) \to \pi_v^{\alpha} \hat{\otimes} \tilde{\pi}_v^{\alpha} \to \mathbb{C}. \tag{7.2}$$

As shown in [SV, §6], these relative characters play a role in the Plancherel formula. To express it in a way suitable for our application, fix k-rational invariant volume forms on G and $T \setminus G$; since inner twists preserve rationality of invariant volume forms, this fixes volume forms, on all inner twists G^{α} , $T^{\alpha} \setminus G^{\alpha}$ over k_v , and hence invariant measures on the k_v -points.

Recall the "inner product" functional on $\mathcal{S}(\mathcal{Z}_v)$, defined in [Sak13a, §3.6]: If $(\Phi^{\alpha})_{\alpha}$ is a lift of $f \in \mathcal{S}(\mathcal{Z}_v)$ as in §5.2, then the inner product is essentially the integral over the diagonal:

$$\langle f \rangle := (k_v^{\times} : N_{k_v}^{E_v} E_v^{\times})^{-1} \cdot \sum_{\alpha} (-1)^{\alpha} \operatorname{AvgVol}(T(k_v)) \int_{T^{\alpha} \backslash G^{\alpha}(k_v)} \Phi^{\alpha}(x, x) dx,$$

$$(7.3)$$

where the "average volume" is the one defined in §2.3.

By the Plancherel formula, this admits a decomposition into relative characters:

$$\langle f \rangle = \frac{\operatorname{AvgVol}(T(k_v))}{(k_v^{\times} : N_{k_v}^{E_v} E_v^{\times})} \cdot \sum_{\alpha} (-1)^{\alpha} \int_{\widehat{G}_v^{\alpha}} \int_{\operatorname{temp}} J_{\pi_v^{\alpha}}(f) d\pi_v^{\alpha}.$$
 (7.4)

Then, it was shown in [SV, Theorem 6.2.1] that the measure $d\pi_v^{\alpha}$ can be taken to be the Plancherel measure for G_v^{α} corresponding to the chosen Haar measure, and then the relative characters $J_{\pi_v^{\alpha}}$ appearing in (7.4) are the same as the ones for (7.2), provided that the measure on T_v^{α} used to define them is compatible with the choices of measures on G_v^{α} , $T_v^{\alpha} \setminus G_v^{\alpha}$.

Remark. One easily checks that the product $\operatorname{AvgVol}(T(k_v))J_{\pi_v^{\alpha}}d\pi_v^{\alpha}$, as a measure on \widehat{G}^{α} valued in the dual of $\mathcal{S}(\mathcal{Z}_v)$, does not depend on choices of measures (as long as, of course, the measures on $G_v^{\alpha}, T_v^{\alpha}$ and $T_v^{\alpha} \setminus G_v^{\alpha}$ are chosen compatibly).

For the split form $G^{\alpha} = G$, and for $\tilde{\pi}_v \otimes \pi_v$ varying in the family of tempered principal series representations $I(\chi_v^{-1}) \otimes I(\chi_v)$, as χ_v varies in the characters of the Borel subgroup, the integral (7.1) is meromorphic in

 χ_v , and hence the relative characters $J_{\chi_v} := J_{\pi_v}$ extend by meromorphicity to a dense open set of χ_v 's. In fact, it is easy to see that the integral (7.1) continues to converge for $|\chi_v| = \delta^{\sigma}$ with $|\sigma| < \frac{1}{2}$, a neighborhood of the set of unitary characters of the Borel which after induction contains all generic unitary representations, therefore J_{π_v} is defined for all generic unitary representations of G_v .

We can repeat the above for tempered representations in the Whittaker case, and we will denote the corresponding local functionals by I_{π_v} ; the only difference is that the analog of (7.1):

$$\tilde{v} \otimes v \mapsto \int_{N_v} \langle \tilde{\pi}_v(n)\tilde{v}, v \rangle \, \psi_v(n) dn$$
 (7.5)

is not convergent and needs to be regularized as in [SV, §6]. (Of course, in this case we have no torsors or inner forms of G showing up.) A priori, these functionals are defined on the coinvariants of standard test functions $S(\overline{X} \times X(k_v), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$, however it is easy to see, using Lemma 6.4.2, that for $\Re(s) \gg 0$ the elements of $S^s(\overline{X} \times X(k_v), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ belong to the Harish-Chandra Schwartz space [Ber88], and hence the functionals I_{π_v} extend continuously to this space, i.e., to $S(\mathcal{W}_v^s)$.

Again, the relative characters I_{π_v} play a role in the Plancherel decomposition of the "inner product" functional on $\mathcal{S}(\mathcal{W}_v)$, defined in [Sak13a, §4.9]:

$$\langle f \rangle := \int_{N \setminus G(k_v)} \Phi(x, x) dx = \int_{\widehat{G_v}^{\text{temp}}} I_{\pi_v}(f) d\pi_v,$$
 (7.6)

where Φ is a lift of f to the space $\mathcal{S}^0(\overline{X} \times X(\mathbb{A}_k), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1})$ of non-standard Whittaker functions, and measures are chosen in the same way as in the torus case.

Moreover, if $\pi_v = I(\chi_v)$ then the functionals I_{π_v} extend to a meromorphic family I_{χ_v} . To see that these functionals make sense for π_v generic, unitary, but not necessarily tempered, it is well-known that the integral

$$[\tilde{W}, W] := \int \tilde{W} \begin{pmatrix} a \\ 1 \end{pmatrix} W \begin{pmatrix} a \\ 1 \end{pmatrix} da, \tag{7.7}$$

where W belongs to the Whittaker model of π_v and \tilde{W} belongs to the Whittaker model of $\tilde{\pi}_v$, is convergent for π_v unitary (generic) and represents a nonzero invariant pairing. It is known (cf. the proof of [SV, Theorem 18.3.1]) that if, for a tempered representation, the regularized (7.5), combined by Frobenius reciprocity, takes $\tilde{v} \otimes v$ to the Whittaker function $\tilde{W} \otimes W$, then $\langle \tilde{v}, v \rangle = [\tilde{W}, W]$. By meromorphicity, this continues to be true in the domain of convergence of (7.7), and therefore (7.5) remains regular (and nonzero) for all unitary generic representations.

The following is shown in [SV, Theorem 6.4.1]:

7.1.1. **Proposition.** If $\pi_v \in \hat{G}_v^{\text{temp}}$ then $I_{\pi_v} \neq 0$ if and only if π_v is generic; if $\pi_v^{\alpha} \in \widehat{G}_v^{\alpha}$ then $J_{\pi_v^{\alpha}} \neq 0$ if and only if $\text{Hom}_{T_v^{\alpha}}(\pi_v^{\alpha}, \mathbb{C}) \neq 0$.

We now have the matching of orbital integrals:

$$|\bullet|\mathcal{G}:\mathcal{S}(\mathcal{Z}_v)\xrightarrow{\sim}\mathcal{S}(\mathcal{W}_v^0)$$

and a basic question concerns the push-forward, resp. pull-back, of the relative characters $J_{\pi_v^{\alpha}}$, resp. I_{π_v} .

7.1.2. **Proposition.** For every $\pi_v \in \hat{G}_v^{\text{temp}}$ there is a unique inner form G^{α} corresponding to a T-torsor α over k_v , and a Jacquet–Langlands lift π_v^{α} of π_v to G^{α} , such that the pull-back $(|\bullet|\mathcal{G})^* I_{\pi_v}$ to $\mathcal{S}(\mathcal{Z}_v)$ is a nonzero multiple of a relative character attached to π_v^{α} .

This proposition will be proven in $\S7.3$, in order not to interrupt the local discussion. (The proof uses a global argument.)

The definition of Jacquet–Langlands lift that is used here is: those elements π_v^{α} of \widehat{G}_v^{α} for which there exist:

- an automorphic representation $\pi \simeq \otimes'_w \pi_w$ of G with $\pi_w \simeq \pi_v$ at the place w = v;
- and an automorphic representation $\pi^{\beta} \simeq \otimes'_w \pi^{\beta}_w$ of G^{β} , where β is a k-rational torsor of T with $\beta_v = \alpha$, $\pi^{\beta}_v \simeq \pi^{\alpha}_v$ and $\pi^{\beta}_w \simeq \pi_w$ for almost all places w (where G^{β_w} is split).

We will use the fact that there is at most one Jacquet–Langlands lift π_v^{α} of π_v for any inner form α , and that it is tempered if π_v is. We will also use strong multiplicity one for inner forms of G. It is plausible that some of these facts can be obtained independently (for T-distinguished representations) from our methods by refining the arguments.

Moreover, for the theorem that follows we will need to use equality of formal degrees for Jacquet–Langlands lifts: Recall that a Haar measure on G_v induces a Haar measure on any inner form G_v^{α} (since inner twists preserve rationality of volume forms), and it is such Haar measures that we fix on all relevant inner forms. Then [AP05, Theorem 7.2] (see also [HII08, §3.1]) states that for any discrete series π_v of G and a Jacquet–Langlands lift π_v^{α} of it, the corresponding Plancherel measures satisfy

$$d\pi_v(\pi_v) = d\pi_v^{\alpha}(\pi_v^{\alpha}). \tag{7.8}$$

7.1.3. **Theorem.** For $\pi_v \in \widehat{G_v}^{\text{temp}}$ we have:

$$(|\bullet|\mathcal{G})^* I_{\pi_v} = (-1)^{\alpha} \gamma_v^* (\eta_v, 0, \psi_v) \cdot \frac{\operatorname{AvgVol}(T(k_v))}{(k_v^{\times} : N_{k_v}^{E_v} E_v^{\times})} J_{\pi_v^{\alpha}}, \tag{7.9}$$

where π_v^{α} is the Jacquet-Langlands lift of Proposition 7.1.2. More generally, the same holds for every generic unitary representation (with $\pi_v^{\alpha} = \pi_v$ if it is not tempered), with the regularized relative characters I_{π_v} , J_{π_v} defined above.

Remark. The factor $\gamma_v^*(\eta_v, 0, \psi_v) \cdot \frac{\operatorname{AvgVol}(T(k_v))}{(k_v^{\times}:N_{k_v}^{E_v}E_v^{\times})}$ is trivial if the volumes are chosen appropriately, see (2.27).

It is important to notice that our non-standard matching *directly* implies this local result; the key point is that the transfer operator essentially preserves inner products, [Sak13a, (5.2)]:

$$\langle | \bullet | \mathcal{G}f \rangle = \gamma^*(\eta_v, 0, \psi_v) \langle f \rangle.$$
 (7.10)

This is a pleasant suggestion that such non-standard comparisons may be naturally suited for the proof of the global period conjecture [SV, Conjecture 17.4.1].

Proof. Combining (7.10) with the Plancherel formula (7.6) for $L^2(N_v \backslash G_v, \psi_v)$ we have, for $f \in \mathcal{S}(\mathcal{Z}_v)$,

$$\gamma^*(\eta_v, 0, \psi_v) \langle f \rangle = \langle | \bullet | \mathcal{G} f \rangle =$$

$$= \int_{\widehat{G}_v^{\text{temp}}} I_{\pi_v}(| \bullet | \mathcal{G} f) d\pi_v$$

$$= \int_{\widehat{G}_v^{\text{temp}}} J'_{\pi_v^{\alpha}}(f) d\pi_v,$$

where π_v^{α} is as in Proposition 7.1.2 and $J'_{\pi_v^{\alpha}}$ is the relative character asserted in that proposition. By uniqueness of the Plancherel formula (7.4), and equality of formal degrees for the Jacquet–Langlands correspondence (7.8) we get that for almost every π_v the following are true:

(1)
$$J'_{\pi_v^{\alpha}} = \gamma^*(\eta_v, 0, \psi_v) \cdot (-1)^{\alpha} \frac{\operatorname{AvgVol}(T(k_v))}{(k_v^{\times} : N_{k_v}^{E_v} E_v^{\times})} J_{\pi_v^{\alpha}},$$

and.

(2)
$$J_{\pi_v^{\beta}} = 0$$
 for $\beta \neq \alpha$. (Notice: β, α depend on π_v .)

The equality (7.9) actually extends to the whole continuous tempered spectrum, since both sides are continuous in the parameter, and more generally it extends to an equality of holomorphic functions in the parameter χ_v when the representation π_v is of the form $I(\chi_v)$. In particular, it extends to generic, nontempered unitary representations (which are isomorphic to some $I(\chi_v)$ with non-unitary χ_v).

7.2. Unramified calculation. We now compute the values of J_{π_v}, I_{π_v} on the basic vectors $f_{Z_v}^0$, resp. $f_{W_s}^0$.

We recall from (4.13) and [Sak13a, §6.4] that the basic vector $f_{\mathcal{W}_{v}^{s}}^{0}$ is equal to $\operatorname{Vol}(N \backslash G(\mathfrak{o}_{v}))^{-1}$ times the image of

$$(H_s \star 1_{x_0 K}) \otimes 1_{y_0 K} \in \mathcal{S}^s(\overline{X} \times X(k_v), \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi}^{-1}),$$

 $X = N \backslash G$, where H_s is series of elements in the unramified Hecke algebra with Satake transform

$$\widehat{H}_s(\pi_v) = L(\pi_v, \frac{1}{2} + s)L(\pi_v \otimes \eta_v, \frac{1}{2} + s),$$

and $1_{x_0K}, 1_{y_0K}$ are the sections of \mathcal{L}_{ψ} , resp. \mathcal{L}_{ψ}^{-1} (trivialized as functions on G_v which vary by a character of N_v) which are equal to 1 on $G(\mathfrak{o}_v)$ and equal to 0 off $N_vG(\mathfrak{o}_v)$.

Using the measures obtained by k-rational volume forms, at almost every place we have $\operatorname{Vol}(N\backslash G(\mathfrak{o}_v))^{-1} = \operatorname{Vol}(G(\mathfrak{o}_v))^{-1} = \zeta_v(2)$. For the function $1_{x_0K}\otimes 1_{y_0K}$ and an irreducible unramified representation π_v one easily checks, at almost every place:

$$I_{\pi_v}(1_{x_0K} \otimes 1_{y_0K}) = \frac{1}{\zeta_v(2)L(\pi_v, \text{Ad}, 1)}.$$
 (7.11)

Remark. For a functional $\ell \in \operatorname{Hom}_{N_v}(\pi_v, \mathbb{C}_{\psi})$ corresponding to I_{π_v} (i.e., I_{π_v} is obtained from ℓ by Frobenius reciprocity and dualizing, both for π_v and for $\overline{\pi_v}$) this calculation corresponds to the following; here ϕ_v^0 is an unramified vector of norm one in π_v :

$$|\ell(\phi_v^0)|^2 = \frac{\zeta_v(2)}{L(\pi_v, \text{Ad}, 1)}.$$
 (7.12)

Indeed, comparing with (7.11) one needs to divide by the square of the volume of $N \setminus G(\mathfrak{o}_v)$, which enters in the dualization.

Combining all the above, for our basic function we will have:

$$I_{\pi_v}(f_{\mathcal{W}_v^s}^0) = \frac{L(\pi_v, \frac{1}{2} + s)L(\pi_v \otimes \eta_v, \frac{1}{2} + s)}{L(\pi_v, \operatorname{Ad}, 1)}.$$
 (7.13)

We now come to the calculation of $J_{\pi_v}(f_{\mathcal{Z}_v}^0)$. Using Theorem 7.1.3 and the remark following it, it is of course equal to $I_{\pi_v}(f_{\mathcal{W}_v^0}^0)$ (at almost every place). Let us also see this directly, and discuss what this means for local T_v -invariant functionals:

By definition, $f_{\mathcal{Z}_v}^0$ has value 1 on $\mathcal{B}^{\text{reg}}(\mathfrak{o}_v)$, therefore it is equal to $\text{Vol}(N \setminus G(\mathfrak{o}_v))^{-1} = \text{Vol}(G(\mathfrak{o}_v))^{-1}$ times what was denoted by $f_{\mathcal{Z}}^0$ in [Sak13a, §6.4]. In other words, it is $\text{Vol}(G(\mathfrak{o}_v))^{-1}$ times the image of the characteristic function of $(T \setminus G \times T \setminus G)(\mathfrak{o}_v)$. (The reader can check from Table (4.15) that the regular value of $f_{\mathcal{Z}_v}^0$ is 1, while the regular value of the image of this characteristic function would, of course, be equal to $\text{Vol}(G(\mathfrak{o}_v))$.)

Based on this, the formula

$$J_{\pi_v}(f_{\mathcal{Z}_v}^0) = \frac{L(\pi_v, \frac{1}{2})L(\pi_v \otimes \eta_v, \frac{1}{2})}{L(\pi_v, \operatorname{Ad}, 1)}$$
(7.14)

can be inferred from [Wal85, §II], and also from the calculation of [II10, §5] or the unramified Plancherel formula of [Sak13b, Theorem 9.0.1]).

Remark. For a functional $\ell \in \operatorname{Hom}_{T_v}(\pi_v, \mathbb{C})$ corresponding to J_{π_v} (using local "Tamagawa measures" obtained from residually nonvanishing volume forms) this calculation corresponds to the following; here ϕ_v^0 is an unramified

vector of norm one in π_v :

$$|\ell(\phi_v^0)|^2 = \frac{\zeta_v(2)L(\pi_v, \frac{1}{2})L(\pi_v \otimes \eta_v, \frac{1}{2})}{L(\eta_v, 1)^2 L(\pi_v, \operatorname{Ad}, 1)}.$$
(7.15)

Indeed, comparing with the value of J_{π_v} on the characteristic function of $(T \setminus G \times T \setminus G)(\mathfrak{o}_v)$, one needs to divide by the square of the volume of $T \setminus G(\mathfrak{o}_v)$, which is equal to $\frac{L(\eta_v, 1)}{\zeta_v(2)}$.

7.3. Whittaker periods. In §5.2 we defined, for every $\varphi \in \widehat{G}_{Ram}^{aut}$ and $s \in \mathbb{C}$ with $\Re(s) \gg 0$, the *period relative character*

$$\mathcal{I}_{\varphi}: \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k)) \to \mathbb{C}$$

or, more precisely, the corresponding functional-valued measure: $\mathcal{I}_{\varphi}d\varphi$, which is absolutely continuous with respect to Plancherel measure on $\hat{G}_{\mathrm{Ram}}^{\mathrm{aut}}$.

We now fix a Plancherel measure on \hat{G}_{Ram}^{aut} , in order to have a well-defined \mathcal{I}_{φ} (for almost all φ); we will take it to be equal to counting measure on the discrete spectrum. It is a corollary of Theorem 5.3.3 that for an analytic section $s \mapsto f_s \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$, the function $s \mapsto \mathcal{I}_{\varphi}(f_s)$ is meromorphic in s and analytic at s = 0. The Euler factorization of this period is known (see, for instance, [LM15, Theorem 4.1] or [SV, Theorem 18.3.1]):

7.3.1. **Theorem.** For $\Re(s) \gg 0$ and $f_s = \prod_v f_{s,v} \in \mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ we have an Euler factorization:

$$\mathcal{I}_{\varphi}(f_s) = \frac{1}{2} \prod_{v}' I_{\varphi_v}(f_{s,v}) \tag{7.16}$$

when $\varphi \in \hat{G}_{\text{cusp}}^{\text{aut}}$. For the continuous spectrum, $\mathcal{I}_{\varphi}(f_s)$ is proportional to $\frac{1}{2} \left(\prod_{s}^{n} I_{\varphi_{n}}(f_s) \right)^{*}$ (to be explained).

Notice that the appearance of the factor $\frac{1}{2}$ is due to the fact that we define the inner product in $L^2([G])$ by integrating against the Tamagawa measure on $[G] = [\operatorname{PGL}_2]$. If, instead, we consider the cuspidal representation as one of GL_2 with trivial central character, and define the inner product by integrating against the Tamagawa measure on $\operatorname{GL}_2(k) \setminus \operatorname{GL}_2(\mathbb{A}_k)^1$, where $\operatorname{GL}_2(\mathbb{A}_k)^1$ is the set of elements with $|\det(g)| = 1$, then this factor would not appear.

The symbol \prod_{v}' means that the product should be interpreted in terms of partial L-functions. Namely, we saw in (7.13) that for v outside a large enough finite set S of places we have:

$$I_{\varphi_v}(f_{s,v}) = \frac{L(\pi_v, \frac{1}{2} + s)L(\pi_v \otimes \eta_v, \frac{1}{2} + s)}{L(\pi_v, \operatorname{Ad}, 1)},$$

and in the cuspidal case we interpret:

$$\prod_{v}' I_{\varphi_{v}}(f_{s,v}) := \frac{L^{S}(\pi, \frac{1}{2} + s)L^{S}(\pi \otimes \eta_{v}, \frac{1}{2} + s)}{L^{S}(\pi, \operatorname{Ad}, 1)} \prod_{v \in S} I_{\varphi_{v}}(f_{s,v}).$$
(7.17)

For the continuous spectrum, $L^S(\pi, \mathrm{Ad}, t)$ has a simple pole at t = 1, and we let $(\prod_v' I_{\varphi_v}(f_s))^*$ be the dominant term in the corresponding Taylor series.

The statement in the continuous case is actually a formal corollary of multiplicity one for Whittaker models. For a more precise and canonical relation between I_{φ} and the "Euler product" (7.16) in the continuous case, see [SV, §18.1].

By meromorphic continuation, this formula continues to hold for every s where \mathcal{I}_{φ} admits analytic continuation, in particular at s=0.

On the other hand, by Theorem 5.4.1 we have, for those φ and $f \in \mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$:

$$\mathcal{J}_{\omega}(f) = \mathcal{I}_{\omega}(|\bullet|\mathcal{G}f). \tag{7.18}$$

We will now prove Proposition 7.1.2. We will use the previous theorem, but strictly speaking it is not necessary, as multiplicity one for Whittaker models implies that the restriction of \mathcal{I}_{φ} to any place is a multiple of I_{π_v} (where π_v is the local component of the automorphic representation attached to φ), which is all that we use.

Proof of Proposition 7.1.2. We first claim the following: Suppose that α denotes the class of a T_v -torsor and π_v^{α} a Jacquet–Langlands lift of the given tempered representation π_v to G_v^{α} which is T_v^{α} -distinguished (equivalently, by Proposition 7.1.1, such that $J_{\pi_v^{\alpha}} \neq 0$). Then, we claim that $(|\bullet|\mathcal{G})^* I_{\pi_v}$ has to be a multiple of $J_{\pi_v^{\alpha}}$.

To prove this claim we may, without loss of generality, assume that T is globally nonsplit; indeed, T_v can be realized as the local factor of a globally nonsplit torus. Then, for a dense set of such π_v^{α} 's (i.e., for every such discrete series, and a dense set of unitary principal series), the representation π_v^{α} can be globalized to some non-residual automorphic representation π^{β} (corresponding to $\varphi \in \hat{G}_{Ram}^{aut}$) where the T-period is nonzero, in particular with $\mathcal{J}_{\varphi} \neq 0$. For π_v^{α} discrete this follows, for instance, from [SV, Theorem 16.3.2], combined with the fact that the local discrete spectrum is automorphically isolated. For π_v^{α} in the continuous spectrum, one can easily construct an Eisenstein series whose T-period integral does not vanish identically with the continuous parameter of the Eisenstein series. It is enough to prove that $(|\bullet|\mathcal{G})^* I_{\pi_v}$ is a multiple of $J_{\pi_v^{\alpha}}$ for this dense set of π_v^{α} 's; by continuity, it will follow for all. For such a π_v^{α} , fix a global parameter $\varphi \in \hat{G}_{Ram}^{aut}$ obtained from this construction.

We may then choose $f^v \in \bigotimes'_{w \notin v} \mathcal{S}(\mathcal{Z}_w)$ such that, viewed as a functional on the variable $f_v \in \mathcal{S}(\mathcal{Z}_v)$ with $f = f^v \otimes f_v$, the expression (7.18) is nonzero, and hence a nonzero multiple of $(| \bullet | \mathcal{G})^* I_{\pi_v}$. On the other hand, the left hand side of (7.18), considered as a functional on $\mathcal{S}(\mathcal{Z}_v)$ by fixing f^v , is by construction a linear combination of relative characters corresponding to Jacquet–Langlands lifts of π_v to groups of the form $G_v^{\beta'}$, where β' is a non-trivial T-torsor over k.

If T_v is split or when π_v^{α} is in the continuous spectrum, the only Jacquet–Langlands lift of π_v is $\pi_v = \pi_v^{\alpha}$ itself, of course, so the claim is proven in that case. In the nonsplit case and for π_v^{α} discrete, to show that there is a unique π_v^{α} contributing to the pullback, it suffices to choose the function f^v carefully: more precisely, since any two distinct inner forms over k differ at at least two places, and vice versa we have the Hasse principle: the local forms (outside of v!) determine the global one, we can find a test function $f^v \in \bigotimes_{w \notin v}' \mathcal{S}(\mathcal{Z}_w)$ so that the only global form $G^{\beta'}$ which contributes to the period relative character $\mathcal{J}_{\varphi}(f^v \otimes f_v)$ is the form G^{β} of the above construction; and hence (by strong multiplicity one for G^{β}) the only automorphic representation which contributes is the Jacquet–Langlands lift π^{β} with $\pi_v^{\beta} = \pi_v^{\alpha}$. Hence, the functional

$$f_v \mapsto \mathcal{J}_{\varphi}(f^v \otimes f_v)$$

on $\mathcal{S}(\mathcal{Z}_v)$ has to be a relative character for π_v^{α} .

This completes the proof of the initial claim. We have now proven Proposition 7.1.2 for those π_v which admit a T_v -distinguished Jacquet-Langlands lift π_v^{α} , for some α (let us call them "special" for the purpose of this proof). The proof of the proposition will be complete if we show that Plancherel-almost every π_v is special. Employing (7.10) and the Plancherel formula for the spaces $L^2(T_v^{\alpha} \setminus G_v^{\alpha})$, we get a decomposition of the inner product on $L^2(N_v \setminus G_v, \psi_v)$ as follows:

$$\langle | \bullet | \mathcal{G}f \rangle = \sum_{\alpha} \int_{\widehat{G_v^{\alpha}}^{\operatorname{temp}}} J_{\pi_v^{\alpha}}'(f) d_{\pi_v^{\alpha}}.$$

The right-hand side, on the other hand, can by what we just proved be expressed as an integral of multiples of the I_{π_v} 's over all "special" π_v 's. Uniqueness of the Plancherel formula now implies that Plancherel-almost all π_v are special.

7.4. **Toric periods.** This is the theorem of Hecke and Waldspurger [Wal85] for PGL₂:

7.4.1. **Theorem.** For $\varphi \in \hat{G}_{cusp}^{aut}$ we have, as functionals on $\mathcal{S}(\mathcal{Z}(\mathbb{A}_k))$:

$$\mathcal{J}_{\varphi} = \frac{1}{2} \prod_{v}^{\prime} J_{\varphi_{v}}. \tag{7.19}$$

The product is again interpreted as in (7.17). The analytic continuation of the implicit partial L-function to s=0 is guaranteed by the analytic continuation of \mathcal{I}_{φ} as a functional on $\mathcal{S}(\mathcal{W}^s(\mathbb{A}_k))$ (Corollary 5.3.4).

Remark. By dualizing, this says that, for a cuspidal automorphic representation π of G^{α} (for some global T-torsor α) and a vector $\phi = \otimes_v \phi_v \in \pi \simeq \otimes'_v \pi_v$:

$$\left| \int_{[T]} \phi(t)dt \right|^2 = \frac{1}{2} \prod_{v}' \int_{T(k_v)} \left\langle \pi_v(t)\phi_v, \overline{\phi_v} \right\rangle dt, \tag{7.20}$$

where the Euler product should be interpreted using the partial L-values corresponding to (7.15). Here the measures used are Tamagawa measures defined via global volume forms; notice that in the split case the factor $L^S(\eta,1)^2 = \zeta^S(1)^2$ which will be obtained from the denominator of (7.15) is undefined, and so is the Tamagawa measure on $[T]^2$; multiplying both sides locally by convergence factors $\zeta_v(1)^2$ we get a meaningful expression. Notice that \langle , \rangle keeps denoting a bilinear form here, and we have used the unitary structure on π to identify $\tilde{\pi} = \bar{\pi}$.

Proof. As was mentioned after Theorem 7.1.3, for suitable choices of measures (which can be taken to factorize global Tamagawa measures) the statement of the theorem reads:

$$(|\bullet|\mathcal{G})^* I_{\pi_v} = (-1)^{\alpha_v} J_{\pi_v^{\alpha}}.$$

We recall that the collection of torsors α_v is the one afforded by Proposition 7.1.2.

By (7.18) the statement of the theorem is true when $L^S(\pi, \frac{1}{2})L^S(\pi \otimes \eta, \frac{1}{2}) = 0$, in which case both sides are zero. Therefore, assume that φ is such that $L^S(\pi, \frac{1}{2})L^S(\pi \otimes \eta, \frac{1}{2}) \neq 0$, equivalently: \mathcal{I}_{φ} applied to the space $\mathcal{S}(\mathcal{W}^0(\mathbb{A}_k))$ is nonzero.

We know from Lemma 6.3.4 that \mathcal{J}_{φ} is supported on "global torsors", and from Proposition 7.1.2 that each local factor J_{φ_v} is supported on a unique local torsor. It follows that for every such φ there is a unique global torsor α such that J_{φ_v} is supported on α_v for all v.¹²

Thus, the collection $(\alpha_v)_v$ corresponds to a global torsor, and the product of factors $(-1)^{\alpha_v}$ is trivial (the torsor is nontrivial at an even number of places).

By dualizing and using (7.1), we get the statement of Theorem 1.4.1. Notice that the unramified factors for these local periods were given in (7.15).

7.5. Determination of the distinguished representation. Finally, we return to the local setting to discuss one of the most mysterious issues of this field, the relation between ϵ -factors and distinguished representations inside of a local L-packet. We would like, for any $\pi_v \in \widehat{G}_v^{\text{disc}}$, to describe the local torsor α_v of Proposition 7.1.2 for which the corresponding Jacquet–Langlands lift ϖ_v^{α} is distinguished by the torus T_v (equivalently: $J_{\pi_v} \neq 0$). (We only focus on discrete series, because for the continuous spectrum the answer is obviously the trivial torsor.)

The answer is well-known from the work of Tunnell [Tun83], Saito [Sai93] (s. also Prasad [Pra07]) in terms of ϵ -factors. The question is whether this can also be seen from trace formula-theoretic considerations. This is indeed the case, although we do not have a completely independent way of verifying that the ϵ -factor that we will define is the correct one – this will follow

¹²This also follows by characterizing the local torsor of Proposition 7.1.2 in terms of epsilon factors, Theorem 7.5.2.

from the work of Jacquet [Jac86] in combination with Jacquet–Langlands [JL70]. Thus, this subsection is more a comment on these papers than an independent treatment of the question.

Consider the automorphism of $\mathcal{S}(\mathcal{Z}(k_v))$ afforded by Proposition 6.5.1 (with s=0):

$$f \mapsto \mathcal{R}f(\xi) := \eta_v(\xi)\eta_v(\xi+1)f(\xi).$$

Of course, if $\eta_v = 1$ then this is the identity, but if $\eta_v \neq 1$ then we have two T-torsors over k_v , which give rise to the inner forms G, G^{α} . Recall that we have canonical maps

$$T(k_v)\backslash G(k_v)/T(k_v) \to \mathcal{B}(k_v),$$
 (7.21)

$$T^{\alpha}(k_v)\backslash G^{\alpha}(k_v)/T^{\alpha}(k_v) \to \mathcal{B}(k_v),$$
 (7.22)

induced from the isomorphism of stacks $T\backslash G/T\simeq T^{\alpha}\backslash G^{\alpha}/T^{\alpha}$ [Sak13a, (3.4)]. The following is easy to compute:

7.5.1. **Lemma.** A point $\xi \in \mathcal{B}^{reg}(k_v) = k_v \setminus \{-1, 0\}$ belongs to the image of (7.21), resp. (7.22), if and only if $\eta_v(\xi)\eta_v(\xi+1) = 1$, resp. $\eta_v(\xi)\eta_v(\xi+1) = -1$.

Now we can prove the result of Tunnell and Saito, based on the aforementioned works of Jacquet and Langlands:

7.5.2. **Theorem.** Let π_v be an irreducible, generic, unitary representation of G_v , and let $\epsilon_T(\pi_v, \frac{1}{2}) = \epsilon(\pi_v, \frac{1}{2}, \psi_v) \epsilon(\pi_v \otimes \eta_v, \frac{1}{2}, \psi_v)$ be the central value of its ϵ -factor.

Then $\epsilon_T(\pi_v, \frac{1}{2})$ is independent of ψ_v , and the local torsor α of Proposition 7.1.2 is trivial if $\epsilon_T(\pi_v, \frac{1}{2}) = 1$, and nontrivial if $\epsilon_T(\pi_v, \frac{1}{2}) = -1$. In other words (see Proposition 7.1.1), π_v is T_v -distinguished iff $\epsilon_T(\pi_v, \frac{1}{2}) = 1$, and its Jacquet-Langlands lift π_v^{α} to the nontrivial inner twist is T_v -distinguished iff $\epsilon_T(\pi_v, \frac{1}{2}) = -1$.

Remark. The result of Tunnell and Saito is more general; for example, it extends to all generic irreducible representations, and nontrivial characters of T_v .

Proof. The following is an obvious reinterpretation of the local functional equation of the special case of [JL70, Theorem 2.18] with $s = \frac{1}{2}$ (in the coordinates of [JL70]):

7.5.3. **Lemma.** Let A be the split torus of diagonal elements in PGL₂, and let $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Consider the automorphism R of order 2 of $C^{\infty}(A_v \backslash G_v)$ given by $R\phi(g) = \phi(wg)$. Let ϕ be in the image of $\pi_v \hookrightarrow C^{\infty}(A_v \backslash G_v)$. Then:

$$R\phi(1) = \epsilon(\pi_v, \frac{1}{2}, \psi_v)\phi(1).$$

Similarly if we replace π_v by $\pi_v \otimes \eta_v$.

Indeed, a priori this is true with $\epsilon(\pi_v, \frac{1}{2}, \psi_v)$ replaced by $\frac{L(\tilde{\pi}_v, \frac{1}{2})}{L(\pi_v, \frac{1}{2})} \epsilon(\pi_v, \frac{1}{2}, \psi_v)$, but since representations of PGL₂ are self-dual, the quotient of local L-functions is equal to 1.

This shows, in particular, that $\epsilon(\pi_v, \frac{1}{2}, \psi_v) = \pm 1$ and is independent of ψ_v , and we will henceforth denote it by $\epsilon(\pi_v, \frac{1}{2})$ (and similarly for $\pi_v \otimes \eta_v$). By dualizing, if we consider a relative character:

$$J'_{\pi_v}: C_c^{\infty}(A_v \backslash G_v) \otimes C_c^{\infty}(A_v \backslash G_v, \eta_v) \to \pi_v \otimes \tilde{\pi}_v \to \mathbb{C}$$

then the automorphism of order 2 taking each element $\phi_1 \otimes \phi_2$ to $\phi_1(w \bullet) \otimes \phi_2(w \bullet)$ acts on J'_{π_v} by the scalar $\epsilon_T(\pi_v) = \epsilon(\pi_v, \frac{1}{2})\epsilon(\pi_v \otimes \eta_v, \frac{1}{2})$.

By [Jac86], this can be applied to the automorphism \mathcal{R} of $\mathcal{S}(\mathcal{Z}_v)$ (cf. the proof of Proposition 6.5.2), and the relative character $J_{\pi_v^{\alpha}}$:

7.5.4. Corollary. The automorphism \mathcal{R} of $\mathcal{S}(\mathcal{Z}_v)$ acts on $J_{\pi_v^{\alpha}}$ by the scalar $\epsilon_T(\pi_v, \frac{1}{2})$.

On the other hand, the automorphism is described explicitly in Proposition 6.5.1. By Lemma 7.5.1 we deduce that the support of the distribution $J_{\pi_v^{\alpha}}$ is the image of k_v -points corresponding to the torsor described in the statement of the theorem.

APPENDIX A. FAMILIES OF LOCALLY MULTIPLICATIVE FUNCTIONS

A.1. **General formalism.** In this appendix we will discuss the analytic structure on certain families $t \mapsto \mathcal{A}^t$ of Fréchet spaces of functions on a local field F, including notions of analytic sections and of polynomial seminorms and polynomial growth or rapid decay on (bounded) vertical strips. As in the rest of the paper, we fix a non-trivial complex additive character of F and a self-dual Haar measure, following the conventions for Fourier transform that were explained in §1.11.

We will define analytic sections by embedding our spaces into $Fr\acute{e}chet$ bundles over the parameter space of t. The parameter space of t is $\mathbb{C}/2\pi i\mathbb{Z}\log q$ in the non-Archimedean case, where q is the residual degree of F. For Archimedean places (where most of the following work is focused), we set $\log q := \infty$, and hence the parameter space is \mathbb{C} .

We recall that (the total space of) a Fréchet bundle over a complex manifold M is a complex Fréchet manifold N, together with a holomorphic map $\pi: N \to M$, with the fibers having the structure of a vector space and an open covering of M by neighborhoods U such that the restriction of N over U is (biholomorphically and linearly) isomorphic to a direct product $U \times A$, where A is a Fréchet space. In particular, for a Fréchet bundle it makes sense to talk about analytic sections $M \to N$ or (weakly) analytic families of functionals (i.e., sections $L: M \to N^*$ which are weakly analytic: under an isomorphism with $U \times A$, locally, $\langle L(m), a \rangle$ is analytic on $m \in U$ for every $a \in A$). Thus, once we embed the spaces \mathcal{A}^t into the fibers N_t of a Fréchet bundle over $\mathbb{C}/\frac{2\pi i}{\log a}\mathbb{Z}$, by an analytic section $t \mapsto f_t \in \mathcal{A}^t$ we will mean a

section which is analytic as a section into N, and by an analytic family of functionals $t \mapsto L_t \in (\mathcal{A}^t)^*$ we will mean one such that $\langle f_t, L_t \rangle$ is analytic for every analytic section $t \mapsto f_t$.

The embedding into a Fréchet bundle will depend on some choices, but it is easy to check that the notions that we define do not. Namely, our spaces will be global sections of Fréchet cosheaves over $\mathbb{P}^1(F)$, and we take a cover of $\mathbb{P}^1(F)$ by small open (semialgebraic) subsets U_i , such that the restrictions $\mathcal{A}^t(U_i)$ of our Schwartz cosheaves are of the form that we will describe below, and have the structure of Fréchet bundles. Then we choose a partition of unity by Schwartz functions subordinate to the U_i 's, and multiplying each element of \mathcal{A}^t by those we get an injective map:

$$A^t \to \bigoplus_i A^t(U_i),$$
 (A.1)

which splits the obvious (extension) map in the other direction. This gives us the notion of analytic sections that we need. We will also endow the spaces $\mathcal{A}^t(U_i)$ with notions of "polynomial seminorms" and sections of "polynomial growth" or "rapid decay"; then the same notions carry over to sections of $t \mapsto \mathcal{A}^t$ by the above embedding. (The "polynomial" and "rapid" notions refer, implicitly, to behavior on bounded vertical strips; they will not be polynomial or of rapid decay on the whole complex plane. Moreover, in the non-Archimedean case, "bounded vertical strips" are compact since $t \in \mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z}$, so there will be no notion of rapid decay, and every continuous section is automatically of polynomial growth; this comment will be implicit, and will not be repeated, every time there is a discussion of those notions.)

Now we come to describing, axiomatically, the cosheaves \mathcal{A}^t over open sets U_i . By choosing the U_i 's small enough, the cosheaves will coincide with the cosheaves of Schwartz functions away from a point of U_i , which up to an automorphism of \mathbb{P}^1 we can identify with 0. In a neighborhood U of zero, we consider two possibilities for the behavior of these functions:

- (1) either $\mathcal{A}^t(U) = |\bullet|^t \mathcal{S}(U) \cdot K$ for all t, where K is a fixed function not depending on t; in this case, we also assume that K is such that the kernel of the map: $f \mapsto f \cdot K$ is closed in $\mathcal{S}(U)$;
- (2) or they are defined in such a way that their fibers over 0 are annihilated by the operator:

$$(\mathrm{Id} - \eta(a)|a|^{-t - \frac{1}{2}} a \cdot)(\mathrm{Id} - |a|^{-\frac{1}{2}} a \cdot),$$
 (A.2)

for every $a \in F^{\times}$, where a we denote the normalized action of F^{\times} on functions on \mathcal{B} :

$$(a \cdot f)(x) := |a|^{\frac{1}{2}} f(ax).$$
 (A.3)

More precisely, in this second case our functions will be of the form:

$$C_1(\xi) + C_2(\xi)\eta(\xi)|\xi|^t$$

(with η the quadratic character associated to a quadratic etale algebra E over F) in a neighborhood of $\xi = 0$, where C_1 and C_2 extend to smooth functions in a neighborhood of zero, except:

• when

$$t = 0 \text{ and } \eta = 1, \tag{A.4}$$

in which case the functions have the form:

$$C_1(\xi) + C_2(\xi) \log |\xi|;$$

• when

$$t \in 2\mathbb{Z}, \eta = 1 \text{ and } F = \mathbb{R},$$
 or $t \in (2\mathbb{Z} + 1), \eta \neq 1$ and $F = \mathbb{R},$ or $t \in \mathbb{Z}$ and $F = \mathbb{C},$

in which case the functions have the form:

$$\begin{cases} C_1(\xi) + C_2(\xi)\eta(\xi)|\xi|^t \log |\xi|, & \text{when } t \ge 0; \\ C_1(\xi)\eta(\xi)|\xi|^t + C_2(\xi) \log |\xi|, & \text{when } t < 0. \end{cases}$$

A.2. Fréchet bundle structure and polynomial seminorms in the first case. In the first case, by multiplying with $|\bullet|^{-t}$ we identify all spaces $\mathcal{A}^t(U)$ with the Fréchet space $|\bullet|^{13} \mathcal{S}(U)/\{f: fK=0\}$. A polynomial family of seminorms will be a family $t \mapsto \rho_t$ of seminorms on $\mathcal{A}^t(U)$ which on bounded vertical strips is bounded by seminorms of the form

$$|P(t)| \cdot \rho$$
,

where ρ is a fixed seminorm of this Fréchet space and P is a polynomial in t. A section $t \mapsto f_t \in \mathcal{A}^t(U)$ will be said to be of polynomial growth, resp. rapid decay on a bounded vertical strip $V := \{t | \sigma_1 \leq \Re(t) \leq \sigma_2\}$ if it is so with respect to polynomial seminorms, i.e., $\rho_t(f_t)$ should be bounded by a polynomial, resp. absolutely bounded on V for every polynomial family of seminorms ρ_t .

A.3. Fréchet bundle structure and polynomial seminorms in the second case. In the second case, for generic t, the space $\mathcal{A}^t(U)$ also has a natural Fréchet space structure, by identifying it as the quotient

$$(\mathcal{S}(U) \oplus \mathcal{S}(U))/\mathcal{S}(U),$$
 (A.6)

where S(U) is embedded by: $f \mapsto (f, -\eta(\bullet)|\bullet|^t f)$. The Fréchet structure in the case $t = 0, \eta = 1$ can again be described as the quotient of $S(U) \oplus S(U)$, as discussed in §2.4, and similarly for Archimedean cases when $t \in \mathbb{Z}$.

However, to describe these spaces as a Fréchet bundle as t varies, we will rework the definition of the Fréchet topology for every t in terms of Fourier transforms. In fact, we may embed $\mathcal{A}^t(U)$ in the larger space of functions

¹³Recall that in the non-Archimedean space we work throughout with "almost smooth" functions, cf. [Sak13a, Appendix A], for the sake of uniformity; with usual smooth functions we get LF-spaces instead.

on F^{\times} which coincide with Schwartz functions away from a compact neighborhood of zero; we call this space the "model" for \mathcal{A}^t . (Thus, strictly speaking, we compose the embedding (A.1) with the embeddings into these "model" spaces, and we endow those with the desired structures.) With the exception of the values of t described below, we can consider elements of the model \mathcal{A}^t as tempered distributions on F (recall that we have fixed a Haar measure on F); for example, the distribution $(C_1(\xi) + C_2(\xi)\eta(\xi)|\xi|^t)d\xi$ is a well-defined measure for $\Re(t) \gg 0$, and has meromorphic continuation to all but countably many values of t (finitely many in the p-adic case). Thus, we can apply Fourier transform to them. The exceptions are when t+1 is a pole of the local zeta function, i.e.:

$$t = -1 \text{ and } \eta = 1;$$

$$t \in -2\mathbb{N} - 1, \eta = 1, F = \mathbb{R};$$

$$t \in -2\mathbb{N} - 2, \eta \neq 1, F = \mathbb{R};$$

$$t \in -\mathbb{N} - 1, F = \mathbb{C}.$$
(A.7)

A.3.1. **Lemma.** Let t be outside the values of (A.7). Fourier transform defines a topological isomorphism between the model \mathcal{A}^t and the Fréchet space \mathcal{B}^t of those smooth functions on F which in a neighborhood of infinity are equal to $|\xi|^{-t-1}\eta(\xi)h\left(\frac{1}{\xi}\right)$, for some $h \in \mathcal{S}(F)$ (with the obvious topology, which can be inferred from the above discussion). Moreover, Fourier transform descends to a topological isomorphism between $\mathcal{A}^t/\mathcal{S}(F)$ and the stalk of \mathcal{B}^t at ∞ (with the obvious topology, given by the derivatives of h at 0).

We will prove this lemma in a moment. Notice that multiplication by a smooth, nonvanishing function which is equal to 1 in a neighborhood of 0 and equal to $|\bullet|^t$ in a neighborhood of ∞ identifies the spaces \mathcal{B}^t as Fréchet spaces, so indeed they form a Fréchet bundle. This allows us to identify the association $t \mapsto \mathcal{A}^t$ with the association $t \mapsto \mathcal{B}^t$ over values of t different from (A.7), which has an obvious, natural structure of a Fréchet bundle. We pull back this Fréchet bundle structure to the spaces \mathcal{A}^t , for example: analytic sections into \mathcal{A}^t (when t is not contained in (A.7)) are those sections whose Fourier transforms are analytic into \mathcal{B}^t .

One can show that the operation "multiplication by $\eta(\bullet)|\bullet|^{-t}$ ", which takes \mathcal{A}^t to \mathcal{A}^{-t} , preserves this structure of a Fréchet bundle for values of t outside of (A.7); this *allows* us to extend the Fréchet bundle structure to arbitrary t. More precisely, this result is an easy corollary of the following direct characterization of analytic sections in terms of \mathcal{A}^t :

A.3.2. **Lemma.** For t not among the values (A.4), (A.5), the analytic sections into \mathcal{A}^t can be described as those sections of the form $t \mapsto C_{1,t}(\xi) + C_{2,t}(\xi)\eta(\xi)|\xi|^t$, where $C_{i,t}$ are analytic maps into $\mathcal{S}(F)$ (of course, not uniquely defined in terms of the section).

If $\eta = 1$, such a section, defined in a punctured neighborhood of t = 0, extends to an analytic section at t = 0 if and only if $C_{1,t}$ and $C_{2,t}$ have at most simple poles with opposite residues at t = 0.

There is a similar description for the values of t as in (A.5), but it is left to the reader.

Proof of Lemma A.3.1. This is a generalization of [Sak13a, Corollary 2.11], and the proof is similar, so we only emphasize the necessary additions.

If $\Phi \in \mathcal{S}(F^2)$ then consider the following function in one variable, representing the orbital integrals of Φ with respect to the hyperbolic action of F^{\times} against a character:

$$f_t(\xi) = \int_{F^{\times}} \Phi(\xi a, a^{-1}) \eta(a) |a|^{-t} d^{\times} a.$$
 (A.8)

One shows (as in [Sak13a, Proposition 2.5]) that $f_t \in \mathcal{A}^t$, and this allows one to identify the (model) space \mathcal{A}^t with the Fréchet space of twisted coinvariants (with respect to the character $\eta(\bullet)|\bullet|^t$) of $\mathcal{S}(F^2)$. This applies to all $t \in \mathbb{C}/2\pi i\mathbb{Z} \log q$.

Usual Fourier transform on $S(F^2)$ is an automorphism which is antiequivariant with respect to the hyperbolic F^{\times} -action, and if f'_t denotes the corresponding function for the Fourier transform of Φ , then the map $f_t \mapsto f'_{-t}$ is a (well-defined) topological isomorphism between \mathcal{A}^t and \mathcal{A}^{-t} . Moreover, as in [Sak13a, Proposition 2.10], it can be explicitly described (when t does not belong to (A.7)) by the operator

$$|\bullet|^{-t}\mathcal{G}_t := |\bullet|^{-t}\mathcal{F} \circ \iota_t \circ \mathcal{F}, \tag{A.9}$$

where \mathcal{F} is usual Fourier transform in one variable and:

$$\iota_t: f \mapsto \frac{\eta(\bullet)}{|\bullet|^{t+1}} f\left(\frac{1}{\bullet}\right).$$
(A.10)

The operator (A.10) is an automorphism of the space of continuous functions h on F such that $\lim_{\xi \to \infty} h(\xi) \eta(\xi) |\xi|^{t+1}$ exists, and clearly the Fourier transforms of functions of the form f_t as above are smooth, contain the space S(F), and as in [Sak13a, Lemma 2.9] belong to this space of continuous functions. The fact that the operator (A.9) defines an isomorphism between \mathcal{A}^t and \mathcal{A}^{-t} now identifies the Fourier transform of \mathcal{A}^t as $S(F) + \iota_t S(F) =$ the space \mathcal{B}^t .

To show that Fourier transform is continuous with respect to the stated topologies, we first notice that this is obvious on the subspace $\mathcal{S}(F) \subset \mathcal{A}^t$. Since $\mathcal{A}^t = \mathcal{S}(F) + \mathcal{G}_t \mathcal{S}(F)$ (indeed: $\mathcal{A}^t = |\bullet|^t \mathcal{A}^{-t}$), this implies continuity.

Proof of Lemma A.3.2. Recall that the notion of "analytic section" is defined using the obvious Fréchet bundle structure on \mathcal{B}^t , when t is not among the exceptions (A.7). Let $\Phi \in \mathcal{S}(F^2)$. Its image in \mathcal{A}^t via t-twisted orbital

integrals (A.8), composed with Fourier transform, gives an element $F_t \in \mathcal{B}^t$. Following the proof of [Sak13a, Proposition 2.10], it is easy to relate the two:

$$F_t(\xi) = \int_{F^{\times}} \hat{\Phi}^1(\xi a, a) \eta(a) |a|^{t+1} d^{\times} a,$$
 (A.11)

where $\hat{\Phi}^1$ denotes the Fourier transform of Φ in the first variable, and the integral should be understood as a zeta integral (in particular, it makes sense when t is not in (A.7)).

The following diagram summarizes the relations between the various spaces and transforms:

$$\begin{array}{c|c}
\mathcal{S}(F)^2 & \xrightarrow{\mathcal{F}} & \mathcal{S}(F^2) \\
(A.8) \downarrow & & \downarrow \\
\mathcal{A}^t & \xrightarrow{\mathcal{F}} & \mathcal{B}^t & \xrightarrow{\iota_t} & \mathcal{B}^t & \xrightarrow{|\bullet|^{-t}\mathcal{F}} & \mathcal{A}^{-t}
\end{array} \tag{A.12}$$

The relation (A.11) shows that any analytic section into \mathcal{B}^t lifts to an analytic section $t \mapsto \hat{\Phi}_t^1 \in \mathcal{S}(F^2 \setminus \{0\})$ and hence, via inverse Fourier transform the first variable, to an analytic section into $t \mapsto \Phi_t \in \mathcal{S}(F^2)$. Vice versa, let such an analytic section into $\mathcal{S}(F^2)$ be given, so $t \mapsto \hat{\Phi}_t^1 \in \mathcal{S}(F^2)$ is also analytic. Taking into account that the zeta integrals (for $t \notin (A.7)$) form an analytic family of distributions on $\mathcal{S}(F)$, we conclude that the image of $t \mapsto \hat{\Phi}_t^1$ in \mathcal{B}^t via (A.11) is an analytic section.

Now it remains to show that analytic sections into $S(F^2)$ descend to precisely those sections into A^t as in the statement of the lemma. Let V^t denote the stalk of A^t at $\xi = 0$; for t outside the values of (A.4), (A.5), the germ of an element of the form $C_{1,t}(\xi) + C_{2,t}(\xi)\eta(\xi)|\xi|^t$ is determined by all the derivatives at zero of the smooth functions $C_{1,t}$ and $C_{2,t}$.

One can easily relate these derivatives to zeta integrals of derivatives of the original function Φ_t , restricted to the two axes. For example, if $F = \mathbb{R}$, from (A.8) we easily deduce that, for t away from integral real values,

$$C_{1,t}^{(n)}(0) = \int_{\mathbb{R}^{\times}} \frac{\partial}{\partial x} \Phi(0, a^{-1}) \eta(a) a^n |a|^{-t} d^{\times} a,$$

and similar expressions hold for the derivatives of $C_{2,t}$ by replacing the y-axis by the x-axis. (This Tate integral converges for $\Re(t) \gg 0$, and should be interpreted in terms of its meromorphic continuation otherwise.) From this we can deduce that, for an analytic section $t \mapsto \Phi_t \in \mathcal{S}(F^2)$, its image in V^t has the stated form (i.e., any given derivative of $C_{1,t}$ and $C_{2,t}$ is analytic away from (A.4), (A.5), and at t = 0 they have at most simple poles with opposite residues).

Now we claim that any pair: $t \mapsto (C_{1,t}, C_{2,t})$ of the stated form, or any section into V^t of the stated form, lifts to an analytic section of $\mathcal{S}(F^2)$, more

precisely, that there are lifts:

section of the stated form into $V^t \rightsquigarrow \text{pair } (C_{1,t}, C_{2,t})$ of the stated form $\rightsquigarrow \text{analytic section into } \mathcal{S}(F^2)$. (A.13)

Let h_t denote a section of the stated form, either into pairs $(C_{1,t}, C_{2,t})$, or into V^t .

Vice versa, given an analytic section $t \mapsto \Phi_t \in \mathcal{S}(F^2)$, we have already explained that its image in V^t is of the stated form and admits a section of the stated form into pairs $(C_{1,t}, C_{2,t})$, which reduces the problem to the case when the image of Φ_t in V^t is identically zero, i.e., its image $f_t \in \mathcal{A}^t$ lies in the subspace $\mathcal{S}(F^{\times})$. We need to prove that $t \mapsto f^t$ is analytic into $\mathcal{S}(F^{\times})$ or, what is equivalent, into $\mathcal{S}(F)$. But we have already seen that its image $\mathcal{F}(f_t) \in \mathcal{B}^t$ is analytic. Since $f_t \in \mathcal{S}(F^{\times})$, $\mathcal{F}(f_t)$ will lie in $\mathcal{S}(F) \subset \mathcal{B}^t$, so the inverse Fourier transform of that is also analytic into $\mathcal{S}(F)$. This concludes the proof of the lemma.

Although Fourier transform is not defined at the values of t of (A.7), it is easy to see that the transform (A.9), or equivalently \mathcal{G}_t , is defined and preserves analytic sections:

A.3.3. **Lemma.** The transform (A.9) defines an automorphism of the Fréchet bundle A^t .

Proof. Since the inverse of Fourier transform on F^2 is Fourier transform (with the inverse character), it follows that the inverse of the operator (A.9): $A^t \to A^{-t}$ is the operator $|\bullet|^t \mathcal{G}'_{-t}$, where \mathcal{G}'_{-t} is defined as \mathcal{G}_{-t} , but with inverse Fourier transform. Thus, for t as in (A.7), one can define \mathcal{G}_t as: $|\bullet|^t \mathcal{G}'^{-1}_{-t}| \bullet|^{-t}$.

Now we come to the notion of polynomial seminorms and polynomial sections, resp. sections of rapid decay on vertical strips. We have already seen what a polynomial family of seminorms on S(F) is (it is included in the first case): it is bounded on bounded vertical strips by finite sums of the form $|P(t)|\rho$, where ρ is a fixed seminorm on S(F). Now fix a Schwartz partition of unity on F as $u_1 + u_2$, with $u_1 \in S(F)$ and $u_2 \in S(\mathbb{P}^1(F) \setminus \{0\})$, and use it to define a splitting of the map:

$$S(F) \oplus S(F) \ni (F_1, F_2) \mapsto F_1(\bullet) + F_2(\frac{1}{\bullet}) |\bullet|^{-t-1} \in \mathcal{B}^t. \tag{A.14}$$

We define a polynomial family of seminorms on \mathcal{B}^t to be a family of seminorms which, when pulled back to $\mathcal{S}(F) \oplus \mathcal{S}(F)$ by (A.14) are bounded by polynomial seminorms on bounded vertical strips. We say that a section $t \mapsto F_t \in \mathcal{B}^t$ is of polynomial growth, resp. rapid decay, if this is the case for any polynomial family of seminorms applied to it (always, implicitly, on bounded vertical strips). We can use inverse Fourier transform to translate these notions to \mathcal{A}^t (away from the points (A.7)) and then we have:

A.3.4. Lemma. The sections of polynomial growth, resp. rapid decay, of \mathcal{A}^t are those of the form $t \mapsto C_{1,t}(\bullet) + C_{2,t}(\bullet)\eta(\bullet)|\bullet|^t$, where $t \mapsto C_{1,t}, C_{2,t}$ are sections of the same type into $\mathcal{S}(F)$.

Proof. As for Lemma A.3.2 we will argue by lifting to $\mathcal{S}(F^2)$. We may ignore the values of t in a neighborhood of bounded width of the real line. Our goal is to show that sections of polynomial growth (resp. rapid decay, but we will avoid repeating this in the rest of the argument) into \mathcal{B}^t are precisely the images, under (A.11), of sections of polynomial growth into $\mathcal{S}(F^2)$, a notion which is stable under (partial) Fourier transform; and, that the image in \mathcal{A}^t of sections of polynomial growth into $\mathcal{S}(F^2)$ under (A.8) consists precisely of the sections described in the statement of the lemma. For what follows, we apply the operation (A.11) to a function denoted by Φ , instead of $\hat{\Phi}^1$.

Clearly, a section $t \mapsto F_t \in \mathcal{B}^t$ of polynomial growth can be lifted via (A.11) to a section of polynomial growth into $\mathcal{S}(F^2)$ (in fact, the lift can be taken to be in $\mathcal{S}(F^2 \setminus \{(0,0)\})$, where F^{\times} acts freely).

Similarly, a section $t \mapsto C_{1,t}(\bullet) + C_{2,t}(\bullet)\eta(\bullet)|\bullet|^t$ of polynomial growth, as in the statement of the lemma, can be lifted via (A.8) to a section Φ of polynomial growth into $\mathcal{S}(F^2)$ (in fact, into $\mathcal{S}(F^2 \setminus \{(0,0)\})$) or even $\mathcal{S}(F \times F^{\times})$, where F^{\times} acts freely).

There remains to show that sections of polynomial growth: $t \mapsto \Phi_t \in \mathcal{S}(F^2)$ descend to sections of polynomial growth, in the above sense, into \mathcal{A}^t and \mathcal{B}^t .

This is clearly true if Φ_t lies in $\mathcal{S}(F^2 \setminus \{(0,0)\})$, where F^{\times} acts freely. This will turn out to be enough, since for notions of growth on bounded vertical strips we can ignore some values of t on the real line. The conceptual reason is that away from integral real values of t, there are no non-trivial $(F^{\times}, \eta(\bullet)|\bullet|^t)$ -equivariant distributions on the stalk at zero (through either of the two F^{\times} actions appearing in (A.8), (A.11)).

To turn this into a rigorous argument, denote by $*^t$ the action of F^{\times} on $\mathcal{S}(F^2)$ twisted by the appropriate character depending on the case under consideration, so that (A.8), resp. (A.11), is zero on elements of the form $\Phi - a *^t \Phi$ (where $a \in F^{\times}$). The stalk W of $\mathcal{S}(F^2)$ at (0,0) is stable under the F^{\times} -action, and it is easy to see that if we fix $a \in F^{\times}$ with $|a| \neq 1$, for t away from integral real values, the map

$$(I - a *^t) : W \to W$$

is bijective. (Recall that an element $\bar{\Phi}$ of W is determined by the values of all derivatives of a representative Φ at the origin.) Moreover, the inverse of this map preserves sections of moderate growth: if $t \mapsto \bar{\Phi}_t$ is a section of polynomial growth into W, then the section $t \mapsto \bar{\Phi}'_t \in W$ such that $\bar{\Phi}_t = \bar{\Phi}'_t - a *^t \bar{\Phi}'_t$ is also of polynomial growth.

Given a section Φ_t of polynomial growth into $\mathcal{S}(F^2)$, with image $\bar{\Phi}_t$ in W, we let $\bar{\Phi}'_t$ be as described and lift it to a section of polynomial growth

 $t \mapsto \Phi'_t \in \mathcal{S}(F^2)$. Then the section

$$t \mapsto \Phi_t - \Phi'_t + a *^t \Phi'_t \in \mathcal{S}(F^2 \setminus \{0\})$$

is of polynomial growth and has the same image as Φ_t under (A.8), resp. (A.11). This proves the claim.

Finally, we notice the following obvious fact:

A.3.5. **Lemma.** For any given $f_{t_0} \in \mathcal{A}^{t_0}$ one can find an analytic section $t \mapsto f^t \in \mathcal{A}^t$, of rapid decay in vertical strips, defined for all $t \in \mathbb{C}/2\pi i\mathbb{Z} \log q$, whose specialization to t_0 is the given element.

A.4. The global case. In the global case, we will give ourselves spaces \mathcal{A}^t which are restricted tensor products, with respect to a specified vector $f_{t,v}^0$ defined at almost every place, of local spaces \mathcal{A}_v^t as above. We say "tensor products" in the completed sense here (for each finite set of places), i.e.:

$$\mathcal{A}^t = \lim_{\to} \hat{\otimes}_{v \in S} \mathcal{A}_v^t,$$

the limit taken as the finite set S tends to include all places, and the identification for $S_1 \subset S_2$ being by multiplication with the basic vector. With the individual tensor products being Fréchet spaces, the limit becomes and LF-space.

We will make the following assumption:

The section
$$t \mapsto f_{t,v}^0$$
 is analytic as t varies in $\mathbb{C}/\frac{2\pi i}{\log q_v}\mathbb{Z}$. (A.15)

An analytic section $t \mapsto f_t \in \mathcal{A}^t$, $t \in \mathbb{C}$, will then be a section which is of the form

$$f_t = \bigotimes_{v \notin S} f_{t,v}^0 \otimes f_{S,t},$$

with $t \mapsto f_{S,t}$ an analytic section into $\hat{\otimes}_{v \in S} \mathcal{A}_v^t$. By taking tensor products for the embeddings (A.1), the notion of "analytic section" makes sense for the (completed) tensor product, and so do the notions of polynomial seminorms and sections of polynomial growth/rapid decay on vertical strips. These notions do not depend on the choice of S used to represent f_t , because of axiom (A.15).

APPENDIX B. F-REPRESENTATIONS

B.1. **Definitions.** Let G denote here the points of a reductive group over an Archimedean field, and fix a faithful algebraic representation: $G \to \operatorname{GL}_N$ to get a natural algebraic height function $\| \bullet \|$ on G by pulling back the maximum of the operator norms of g and g^{-1} with respect to the norm $(r_1, \ldots, r_N) \mapsto \max_i |r_i|$.

We recall the notion of F- and SF-representations of G, in the language of [BK14]. The same definitions can be given for p-adic groups, if one uses "almost smooth" vectors as we have done throughout this paper (calling them just "smooth"). The results that we quote from [BK14] extend to the

p-adic case, however we will omit this discussion, hoping to include it in future work.

An F-representation of G is a continuous representation on a Fréchet space such that there is a sequence of seminorms defining the topology with the property that the action of any $g \in G$ is bounded with respect to each one of them. In particular, Banach representations are F-representations, and the general F-representation is an inverse limit of Banach representations. We recall that an F-representation is equivalent to a continuous representation of moderate growth of G on a Fréchet space, that is, a representation with the property:

(*) For every (continuous) seminorm p there exists a seminorm q and a positive number N such that $p(gv) \leq ||g||^N q(v)$ for every $g \in G$ and $v \in V$.

The topological convolution algebra $\mathcal{R}(G)$ of rapidly decaying L^1 -measures on G (i.e., measures μ such that $||g||^N \mu$ is an L^1 -measure for every N) acts continuously on any F-representation.

When V is a continuous representation of G on a Fréchet space, we endow the subspace V^{∞} of smooth vectors with the topology induced by all the seminorms of V applied to all derivatives of a vector. We say that V is smooth if $V \simeq V^{\infty}$ as topological vector spaces. If V is an F-representation, then V^{∞} is a smooth F-representation, or SF-representation in the language of [BK14]. Notice that when p is a seminorm, V is a smooth F-representation and $v \in V$ then there is an N such that $\sup_{g \in G} \frac{p(gDv)}{\|g\|^N} < 0$ for every $D \in \mathfrak{U}(\mathfrak{g})$ (in fact, N can be chosen independently of v), which is why these vectors are sometimes said to be of "uniformly moderate growth", but we will just be saying "moderate growth". (On the other hand, notice that if N works for the seminorm p, then one needs a larger N, in general, for the seminorm $v \mapsto p(Dv)$, because of a factor accounting for the adjoint action of G on D.)

Any F-representation is a continuous $\mathcal{S}(G)$ -module (we identify Schwartz functions with Schwartz measures here by choosing a Haar measure, in order not to introduce extra notation). Moreover, a smooth F-representation V is a non-degenerate $\mathcal{S}(G)$ -module, i.e., $\mathcal{S}(G)V = V$. In fact, it is already non-degenerate under the action of compactly supported elements of $\mathcal{S}(G)$; this is a consequence of the theorem of Dixmier and Malliavin.

Something stronger is true: the category of smooth F-representations is equivalent to the category of non-degenerate continuous $\mathcal{S}(G)$ -module [BK14, Proposition 2.20]. As the proof of Proposition 2.20 shows, non-degeneracy can be understood in a stronger sense: the topology on V is the quotient topology for the action map:

$$S(G) \hat{\otimes} V \to V,$$
 (B.1)

where $\hat{\otimes}$ denotes the completed tensor product (projective, say, but since $\mathcal{S}(G)$ is nuclear it coincides with the injective one).

B.2. Morphisms. Let V be a continuous representation on a Fréchet space and $N \in \mathbb{R}_+$. Assume that a single semi-norm p generates a complete set of seminorms under the action of G, i.e., the seminorms $p \circ g$, where g ranges over G, form a complete set. Such are, for example, "locally defined" spaces of functions on a homogeneous space X of G, e.g., $L^2_{loc}(X)$, C(X) etc. Then we define:

$$V_N := \{ v \in V | \sup_{g \in G} \frac{p(gv)}{\|g\|^N} < \infty \}.$$
 (B.2)

This subspace depends on the chosen height $\| \bullet \|$ on G but not on the choice of p, and for any other compatible height function $\| \bullet \|'$ with corresponding spaces V_N' there are positive constants c, C such that $V_{cN} \subset V_N' \subset V_{CN}$ for every N.

B.2.1. **Lemma.** The subspace V_N , endowed with the norm:

$$p_N(v) := \sup_{g \in G} \frac{p(gv)}{\|g\|^N},$$

is complete.

Proof. Suppose that $(v_n)_n$ is a Cauchy sequence with respect to this norm, and let v be its limit in V. For any $\epsilon > 0$ and every g we can choose an arbitrarily large n_g such that:

$$\frac{p(gv_{n_g} - gv)}{\|g\|^N} \le \epsilon.$$

In particular, we can assume that all $n_g \ge m$, where m is such that for all $n \ge m$ we have:

$$\sup_{g \in G} \frac{p(gv_n - gv_m)}{\|g\|^N} < \epsilon.$$

Then:

$$\sup_{g \in G} \frac{p(gv)}{\|g\|^N} \le \sup_{g \in G} \frac{p(gv_{n_g} - gv)}{\|g\|^N} + \sup_{g \in G} \frac{p(gv_{n_g})}{\|g\|^N} \le$$

$$\le \epsilon + \sup_{g \in G} \frac{p(gv_{n_g} - gv_m)}{\|g\|^N} + \sup_{g \in G} \frac{p(gv_m)}{\|g\|^N} \le 2\epsilon + p_N(v_m) < \infty,$$

hence $v \in V_N$.

Hence, V_N is a Banach representation, in particular an F-representation. We let:

$$V_{mg} = \bigcup_{N \in \mathbb{N}} V_N, \tag{B.3}$$

where "mg" stands for "moderate growth".

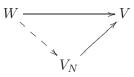
The spaces V_N^{∞} are SF-representations, and we set:

$$V_{mg}^{\infty} = \bigcup_{N} V_{N}^{\infty}.$$
 (B.4)

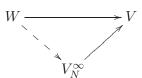
Clearly, as a subspace of V it does not depend on choices.

Since $V_N \hookrightarrow V_{N+1}$ is not closed, there is no good topology on V_{mg} (for instance, the finest topology making the inclusions of all V_N 's continuous is in general non-Hausdorff). By abuse of language, we will say that a map into V_{mg} (resp. V_{mg}^{∞}) is continuous if it factors as a continuous map through some V_N (resp. V_N^{∞}). For example, in this language the following lemma states that a certain map factors "continuously" through V_{mg} , resp. V_{mg}^{∞} .

B.2.2. **Lemma.** Let V be as above, and let W be an F-representation (resp. an SF-representation). Any morphism $T:W\to V$ factors (continuously) as:



resp.:



for some N.

Proof. Start by writing W as an inverse limit of Banach representations of G, and letting ν_i be the corresponding seminorms.

By continuity, there is an i and a positive constant C such that $\rho(Tw) \leq C\nu_i(w)$ for every $w \in W$. Now recall that Banach representations have the moderate growth property: there exist N and D > 0 such that $\nu_i(gw) \leq D\|g\|^N\nu_i(w)$ for any $w \in W$. Thus:

$$\rho(g(Tw)) = \rho(T(gw)) \le C\nu_i(gw) \le CD\nu_i(w)||g||^N,$$

which shows that T factors through a continuous map: $W \to V_N$.

The statement on SF-representations follows by applying the functor "smooth vectors" to the morphism: $W \to V_N$.

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