A NOTE ON INTEGRAL SATAKE ISOMORPHISMS

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ABSTRACT. We formulate a Satake isomorphism for the integral spherical Hecke algebra of an unramified p-adic group G and generalize the formulation to give a description of the Hecke algebra $H_G(V)$ of weight V, where V is a lattice in an irreducible algebraic representation of G.

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In this note, we formulate a canonical integral Satake isomorphism, by identifying the integral spherical Hecke algebra H_G of an unramified p-adic group G with a \mathbb{Z} -algebra associated to an affine monoid $V_{\hat{G},\rho_{\rm ad}}$ of the Langlands dual group \hat{G} . The precise statement can be found in Proposition 5. There are two motivations. First, the usual Satake isomorphism depends on a choice of $q^{1/2}$ and therefore only gives a description of $H_G \otimes \mathbb{Z}[q^{\pm 1/2}]$ in terms of the dual group \hat{G} (e.g. see [Gr96]). Using Deligne's modification of the Langlands dual group (e.g. see [BG11]), one can formulate a canonical Satake isomorphism for $H_G \otimes \mathbb{Z}[p^{-1}]$. On the other hand, there is the mod p Satake isomorphism, as discussed in [He11, HV15]. However, the Langland duality does not appear explicitly in these works. Therefore, it is desirable to extend the classical Satake isomorphism to \mathbb{Z} , which after mod p recovers the mod p Satake isomorphism. Another motivation comes from the recent work of R. Cass ([Ca19]) on the geometric Satake equivalence for perverse \mathbb{F}_p -sheaves on the affine Grassmannian. In his work, what controls the tensor category is not the Langlands dual group, but some affine monoid M_G . I hope $V_{\hat{G},\rho_{\rm ad}}$ and M_G are closely related 1.

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Our formulation generalizes easily to give a description of the Hecke algebra $H_G(V)$ of weight V in terms of the affine monoid $V_{\hat{G},\lambda_{\text{ad}}+\rho_{\text{ad}}}$, where V is a lattice in an irreducible algebraic representation of G of highest weight λ . See Proposition 14 for the precise formulation. It follows from our formulation that the ring of invariant functions on the Vinberg monoid specializes to all $H_G(V)$'s.

Supported by the National Science Foundation under agreement Nos. DMS-1902239.

¹However, as explained to me by Cass, the monoid appearing in his work is solvable and therefore cannot be isomorphic to the one appearing in this note.

1. The Satake isomorphism

1.1. **The** C-**group.** Let G be a connected reductive group over a field F. Let $(\hat{G}, \hat{B}, \hat{T}, \hat{e})$ be a pinned dual group of G over \mathbb{Z} . There is a natural action of the Galois group Γ_F of F on $(\hat{G}, \hat{B}, \hat{T}, \hat{e})$, induced by its action on the root datum of G. This action factors as $\Gamma_F \twoheadrightarrow \Gamma_{\widetilde{F}/F} \stackrel{\xi}{\hookrightarrow} \operatorname{Aut}(\hat{G}, \hat{B}, \hat{T}, \hat{e})$, where $\Gamma_{\widetilde{F}/F}$ is the Galois group of a finite Galois extension \widetilde{F}/F .

Let $2\rho: \mathbb{G}_m \to \hat{T}$ denote the cocharacter given by sum of positive coroots of \hat{G} (with respect to \hat{B}), and $2\rho_{\rm ad}$ its projection to the adjoint torus $\hat{T}_{\rm ad} \subset \hat{G}_{\rm ad} = \hat{G}/Z_{\hat{G}}$ of \hat{G} . It admits a unique square root $\rho_{\rm ad}: \mathbb{G}_m \to \hat{T}_{\rm ad}$. Define an action of \mathbb{G}_m on \hat{G} by

$$\operatorname{Ad}\rho_{\operatorname{ad}}: \mathbb{G}_m \stackrel{\rho_{\operatorname{ad}}}{\to} \hat{T}_{\operatorname{ad}} \stackrel{\operatorname{Ad}}{\to} \operatorname{Aut}(\hat{G}).$$

Note that $\mathrm{Ad}\rho_{\mathrm{ad}}$ still preserves (\hat{B},\hat{T}) , but not \hat{e} . The two actions $\mathrm{Ad}\rho_{\mathrm{ad}}$ and ξ on \hat{G} commute with each other. Following the terminology of Buzzard-Gee [BG11], we define the C-group of G as an affine reductive group scheme over \mathbb{Z} as

$${}^{C}G := \hat{G} \rtimes (\mathbb{G}_m \times \Gamma_{\widetilde{F}/F}).$$

This group scheme appears naturally from the geometric Satake equivalence, as to be reviewed in Theorem 16 below. Note that our definition is different from [BG11, Definition 5.3.2], but is equivalent to it (see below). We may write

$${}^{C}G \cong {}^{L}G \rtimes \mathbb{G}_{m} \cong \hat{G}^{T} \rtimes \Gamma_{\widetilde{F}/F},$$

where ${}^LG := \hat{G} \rtimes_{\xi} \Gamma_{\widetilde{F}/F}$ is the usual Langlands dual group, and where $\hat{G}^T := \hat{G} \rtimes_{\operatorname{Ad}\rho_{\operatorname{ad}}} \mathbb{G}_m$, which fits into the short exact sequence

$$(1.1) 1 \to \hat{G} \to \hat{G}^T \xrightarrow{d_{\rho_{ad}}} \mathbb{G}_m \to 1.$$

Note that there is an isomorphism

$$(1.2) \qquad (\hat{G} \times \mathbb{G}_m)/(2\rho \times \mathrm{id})(\mu_2) \cong \hat{G}^T, \quad (g,t) \mapsto (g2\rho(t)^{-1}, t^2),$$

and the left hand side is the more familiar form of Deligne's modification of the Langlands dual group (e.g. see [BG11, Proposition 5.3.3]).

Remark 1. We may regard \hat{G}^T as the dual group of a reductive group G^T , which is a central extension of G by \mathbb{G}_m over F, and then regard ${}^CG \cong {}^LG^T$ as the usual Langlands dual group of G^T . This is the definition given in [BG11, Definition 5.3.2].

We discuss a few examples.

Example 2. (1) If G = T is a torus, then $\hat{T}^T = \hat{T} \times \mathbb{G}_m$ and $^CT = (\hat{T} \rtimes \Gamma_{\tilde{E}/F}) \times \mathbb{G}_m$.

- (2) If G is an inner form of a split reductive group, then ${}^CG = \hat{G}^T$. For example, if $G = \operatorname{PGL}_2$, ${}^CG = \hat{G}^T = \operatorname{GL}_2$ and $d_{\rho_{\operatorname{ad}}}$ is the usual determinant map. In particular, \hat{G}^T is not isomorphic to $\hat{G} \times \mathbb{G}_m$ in general.
- (3) However, if ρ_{ad} lifts to a cocharacter $\tilde{\rho} \in \mathbb{X}_{\bullet}(\hat{T})$ (which does not necessarily satisfy $2\tilde{\rho} = 2\rho$), then $\tilde{\rho}$ induces an isomorphism

(1.3)
$$\hat{G}^T \simeq \hat{G} \times \mathbb{G}_m, \ (g,t) \mapsto (g\tilde{\rho}(t),t).$$

For example, if $G = \operatorname{GL}_n$ so $\hat{G} = \operatorname{GL}_n$, we can choose $\tilde{\rho}(t) = \operatorname{diag}\{t^{n-1}, \dots, t, 1\}$; If $G = \operatorname{PGL}_{2n+1}$ so $\hat{G} = \operatorname{SL}_{2n+1}$, we can choose $\tilde{\rho}(t) = \operatorname{diag}\{t^n, t^{n-1}, \dots, t^{-n}\}$.

- (4) In general, even if ρ_{ad} lifts to an element $\tilde{\rho} \in \mathbb{X}_{\bullet}(\hat{T})$, so $\hat{G}^T \cong \hat{G} \times \mathbb{G}_m$, CG may not be isomorphic to ${}^LG \times \mathbb{G}_m$, unless $\tilde{\rho}$ can be chosen to be $\Gamma_{\widetilde{F}/F}$ -invariant. For example, if $G = \mathrm{U}_{2n}$ is an even unitary group, associated to a quadratic extension \widetilde{F}/F , then $\hat{G} = \mathrm{GL}_{2n}$, and $\xi : \Gamma_{\widetilde{F}/F} = \{1, c\} \to \mathrm{Aut}(\mathrm{GL}_{2n})$ is defined by $\xi(c)(A) = J_{2n}(A^T)^{-1}J_{2n}^{-1}$, where J_{2n} is the anti-diagonal matrix with (i, 2n + 1 i)-entry $(-1)^i$. In this case, \hat{G}^T is isomorphic to $\hat{G} \times \mathbb{G}_m$, but CG is not isomorphic to ${}^LG \times \mathbb{G}_m$.
- 1.2. **Affine monoids.** We define an affine monoid scheme $V_{\hat{G},\rho_{ad}}$ equipped with a faithfully flat monoid morphism

$$(1.4) d_{\rho_{\rm ad}}: V_{\hat{G},\rho_{\rm ad}} \to \mathbb{A}^1$$

which extends the homomorphism $d_{\rho_{ad}}: \hat{G}^T \to \mathbb{G}_m$ from (1.1). It will be obtained as the pullback of a universal monoid (called the Vinberg monoid) associated to \hat{G} , whose definition we first briefly recall following the approach in [XZ19, §2, §3]. Note that the discussions of *loc. cit.* are given over a field. But as they work over any field, they actually work over the base \mathbb{Z} .

We will use notations from [XZ19] with $(\hat{G}, \hat{B}, \hat{T})$ in place of (G, B, T) in loc. cit. We identify $\mathbb{X}^{\bullet}(\hat{T}_{ad}) \subset \mathbb{X}^{\bullet}(\hat{T})$ with the root lattice inside the weight lattice. Let $\mathbb{X}^{\bullet}(\hat{T}_{ad})_{pos}$ be the submonoid generated by simple roots $\{\hat{\alpha}_1, \dots, \hat{\alpha}_r\}$ of \hat{G} with respect to (\hat{B}, \hat{T}) , let $\mathbb{X}^{\bullet}(\hat{T})^? \subset \mathbb{X}^{\bullet}(\hat{T})$ for ? = +, - be the submonoids of dominant and anti-dominant weights, and let $\mathbb{X}^{\bullet}(\hat{T})^+_{pos} \subset \mathbb{X}^{\bullet}(\hat{T})$ be the submonoid generated by $\mathbb{X}^{\bullet}(\hat{T}_{ad})_{pos}$ and $\mathbb{X}^{\bullet}(\hat{T})^+$. Let

$$\hat{T}_{\mathrm{ad}}^+ = \mathrm{Spec}\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T}_{\mathrm{ad}})_{\mathrm{pos}}],$$

which is an affine monoid containing \hat{T}_{ad} as the group of invertible elements. Note that every dominant coweight $\lambda: \mathbb{G}_m \to \hat{T}_{ad}$ of \hat{G}_{ad} can be extended to a monoid homomorphism

$$\lambda^{+}: \mathbb{A}^{1} \to \hat{T}_{ad}^{+}.$$

We equip $\mathbb{X}^{\bullet}(\hat{T})$ with the usual partial order \leq , i.e. $\hat{\lambda}_1 \leq \hat{\lambda}_2$ if $\hat{\lambda}_2 - \hat{\lambda}_1 \in \mathbb{X}^{\bullet}(\hat{T}_{ad})_{pos}$. For $\hat{\nu} \in \mathbb{X}^{\bullet}(\hat{T})$, let $\hat{\nu}^* := -w_0(\hat{\nu})$. Recall that the left and the right multiplication of \hat{G} on itself induce a natural $(\hat{G} \times \hat{G})$ -module structure on $\mathbb{Z}[\hat{G}]$, which admits a canonical multi-filtration

$$\mathbb{Z}[\hat{G}] = \sum_{\hat{\nu} \in \mathbb{X}^{\bullet}(\hat{T})^{+}_{\mathrm{pos}}} \operatorname{fil}_{\hat{\nu}} \mathbb{Z}[\hat{G}]$$

by $(\hat{G} \times \hat{G})$ -submodules indexed by $\mathbb{X}^{\bullet}(\hat{T})^+_{\text{pos}}$. Here $\text{fil}_{\hat{\nu}} \mathbb{Z}[\hat{G}]$ denotes the saturated \mathbb{Z} -module that is maximal among all $(\hat{G} \times \hat{G})$ -submodules $V \subset \mathbb{Z}[\hat{G}]$ with the following property: for any pair $(\hat{\mu}, \hat{\mu}') \in \mathbb{X}^{\bullet}(\hat{T} \times \hat{T})$, if the weight space $V(\hat{\mu}, \hat{\mu}') \neq 0$, then $\hat{\mu} \leq \hat{\nu}^*$ and $\hat{\mu}' \leq \hat{\nu}$. One knows that $\text{fil}_{\nu} \mathbb{Z}[\hat{G}]$ is finite free over \mathbb{Z} and that

$$\operatorname{gr} \mathbb{Z}[\hat{G}] = \bigoplus_{\hat{\nu} \in \mathbb{X}^{\bullet}(\hat{T})^{+}} S_{\hat{\nu}^{*}} \otimes S_{\hat{\nu}},$$

where $S_{\hat{\nu}}$ denotes the Schur module of highest weight $\hat{\nu}$ (i.e. the induced \hat{G} -module from the character $-\hat{\nu}$ of \hat{B}). See [XZ19, §2,§3] for detailed discussions, including the definition of associated graded of a multi-filtration. Now the Vinberg monoid $V_{\hat{G}}$ of \hat{G} is defined as

$$V_{\hat{G}} = \operatorname{Spec} R_{\mathbb{X}^{\bullet}(\hat{T})^{+}_{\operatorname{pos}}} \mathbb{Z}[\hat{G}],$$

where $R_{\mathbb{X}^{\bullet}(\hat{T})_{\text{pos}}^{+}}\mathbb{Z}[\hat{G}] := \bigoplus_{\hat{\nu} \in \mathbb{X}^{\bullet}(\hat{T})_{\text{pos}}^{+}} \operatorname{fil}_{\hat{\nu}} \mathbb{Z}[\hat{G}]$ denotes the Rees algebra associated to the above defined multi-filtration. It is an affine monoid \mathbb{Z} -scheme of finite type which admits a faithfully flat monoid morphism $d: V_{\hat{G}} \to \hat{T}_{\text{ad}}^{+}$ such that

- $d^{-1}(\hat{T}_{ad})$ contains the group of invertible elements of $V_{\hat{G}}$ and is isomorphic to the quotient of $\hat{G} \times \hat{T}$ by $Z_{\hat{G}}$ with respect to the diagonal action $z \cdot (g,t) = (gz,tz)$ (so in particular \hat{G} acts on $V_{\hat{G}}$ by left and right translations);
- $d^{-1}(0) \cong \operatorname{As}_{\hat{G}}$ as $(\hat{G} \times \hat{G})$ -schemes, where

$$\operatorname{As}_{\hat{G}} := \operatorname{Spec} \operatorname{gr} \mathbb{Z}[\hat{G}]$$

is called the asymptotic cone of \hat{G} .

In fact, the Vinberg monoid of \hat{G} can be characterized as the unique affine monoid scheme M equipped with a faithfully flat monoid morphism $d: M \to \hat{T}_{\mathrm{ad}}^+$ satisfying the above two properties.

Now, for a dominant coweight $\lambda: \mathbb{G}_m \to \hat{T}_{ad}$, let

$$(1.6) d_{\lambda}: V_{\hat{G},\lambda} := \mathbb{A}^{1} \times_{\lambda^{+},\hat{T}_{ad}} V_{G} \to \mathbb{A}^{1}$$

be the pullback of of the homomorphism $d: V_{\hat{G}} \to \hat{T}_{ad}^+$ along the map λ^+ from (1.5). The homomorphism $\hat{G} \times \mathbb{G}_m \xrightarrow{\mathrm{id} \times 2\rho} \hat{G} \times \hat{T}$ induces a homomorphism $\hat{G}^T \to \hat{G} \times^{Z(\hat{G})} \hat{T}$ by (1.2), and therefore we obtain desired map (1.4) by setting $\lambda = \rho_{ad}$ in (1.6).

Note that $V_{\hat{G}}$ and \hat{T}_{ad}^+ are acted by $\Gamma_{\widetilde{F}/F}$ (induced by its action ξ) and the homomorphism $d:V_{\hat{G}}\to\hat{T}_{\mathrm{ad}}^+$ is $\Gamma_{\widetilde{F}/F}$ -equivariant. If $\lambda:\mathbb{G}_m\to\hat{T}_{\mathrm{ad}}$ is a dominant coweight fixed by $\Gamma_{\widetilde{F}/F}$, the monoid $V_{\hat{G},\lambda}$ is also acted by $\Gamma_{\widetilde{F}/F}$ and the map d_{λ} extends to a homomorphism

$$\tilde{d}_{\lambda}: V_{\hat{G},\lambda} \rtimes \Gamma_{\widetilde{F}/F} \to \mathbb{A}^1 \times \Gamma_{\widetilde{F}/F}.$$

Note that $\rho_{\rm ad}$ is $\Gamma_{\widetilde{F}/F}$ -invariant. It follows that there is a natural isomorphism

(1.7)
$$\tilde{d}_{\rho_{ad}}^{-1}(\mathbb{G}_m \times \Gamma_{\widetilde{F}/F}) \cong {}^{C}G.$$

1.3. **Invariant theory.** We fix a finite order automorphism σ of $(\hat{G}, \hat{B}, \hat{T}, \hat{e})$ and a positive integer q. We consider the σ -twisted conjugation action of \hat{G} on $V_{\hat{G}}$ given by

$$c_{\sigma}(g)(x) = gx\sigma(g)^{-1}, \quad g \in \hat{G}, \ x \in V_{\hat{G}}.$$

Let $\lambda: \mathbb{G}_m \to \hat{T}_{ad}$ be a dominant coweight. Then the conjugation action of \hat{G} on $V_{\hat{G}} \rtimes \langle \sigma \rangle$ restricts to an action on $V_{\hat{G}}|_{d=\lambda(q)}\sigma$. Then we have the isomorphism $\mathbb{Z}[V_{\hat{G}}|_{d=\lambda(q)}\sigma]^{\hat{G}} = \mathbb{Z}[V_{\hat{G}}|_{d=\lambda(q)}]^{c_{\sigma}(\hat{G})}$.

Let $V_{\hat{T}}$ be the closure of $\hat{T} \times^{Z_{\hat{G}}} \hat{T} \subset \hat{G} \times^{Z_{\hat{G}}} \hat{T}$ inside $V_{\hat{G}}$. This is a commutative submonoid of $V_{\hat{G}}$. If we write the ring of regular functions of $\hat{T} \times^{Z_{\hat{G}}} \hat{T}$ by

$$\mathbb{Z}[\hat{T} \times^{Z_{\hat{G}}} \hat{T}] = \bigoplus_{(\hat{\lambda}, \hat{\nu})} \mathbb{Z}(e_1^{\hat{\lambda}} \otimes e_2^{\hat{\nu}}),$$

where the sum is taken over pairs of weights $(\hat{\lambda}, \hat{\nu})$ of \hat{T} such that $\hat{\nu} + \hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T}_{ad})^2$, and where $e_1^{\hat{\lambda}} \otimes e_2^{\hat{\nu}}$ denotes the corresponding regular function on $\hat{T} \times^{Z_{\hat{G}}} \hat{T}$, then the ring of regular functions on $V_{\hat{T}}$ is the subring

$$\mathbb{Z}[V_{\hat{T}}] = \bigoplus_{(\hat{\lambda}, \hat{\nu}), \hat{\nu} + \hat{\lambda}_- \in \mathbb{X}^{\bullet}(\hat{T}_{\mathrm{ad}})_{\mathrm{pos}}} \mathbb{Z}(e_1^{\hat{\lambda}} \otimes e_2^{\hat{\nu}}),$$

where $\hat{\lambda}_{-} \in \mathbb{X}^{\bullet}(\hat{T})^{-}$ denotes the anti-dominant weight in the Weyl group orbit of $\hat{\lambda}$.

²In [XZ19] the condition reads as $\hat{\nu} - \hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T}_{ad})$. The sign convention here is more suitable for our purpose.

For $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})$, let $e^{\hat{\lambda}}$ denote the corresponding regular function on \hat{T} , and for $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T}_{ad})_{pos}$, let $\bar{e}^{\hat{\lambda}}$ be the corresponding regular function on \hat{T}_{ad}^+ . The homomorphism $d: V_{\hat{T}} \to \hat{T}_{ad}^+$ is given by the ring map $\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T}_{ad})_{pos}] \to \mathbb{Z}[V_{\hat{T}}]$ sending $\bar{e}^{\hat{\lambda}}$ to $1 \otimes e_2^{\hat{\lambda}}$, which admits a section $\mathfrak{s}: \hat{T}_{ad}^+ \to V_{\hat{T}}$

$$\mathbb{Z}[V_{\hat{T}}] \to \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T}_{\mathrm{ad}})_{\mathrm{pos}}], \quad e_1^{\hat{\lambda}} \otimes e_2^{\hat{\nu}} \mapsto \bar{e}^{(\hat{\nu}+\hat{\lambda})}.$$

Note that $\mathfrak{s}|_{\hat{T}_{ad}}: \hat{T}_{ad} \to \hat{T} \times^{Z_{\hat{G}}} \hat{T}$ is induced by the diagonal embedding $\hat{T} \to \hat{T} \times \hat{T}$. We still denote by \mathfrak{s} the composed section $\hat{T}_{ad}^+ \to V_{\hat{G}}$. Its restriction to \hat{T}_{ad} induces a section $\hat{T}_{ad} \to \hat{G} \times^{Z_{\hat{G}}} \hat{T}$, and therefore an isomorphism $\hat{G} \times^{Z_{\hat{G}}} \hat{T} \cong \hat{G} \times \hat{T}_{ad}$, whose pullback along $\rho_{ad}: \mathbb{G}_m \to \hat{T}_{ad}$ gives the isomorphism (1.2).

There is a natural injective map $i_1: \hat{T} \to \hat{T} \times^{Z_{\hat{G}}} \hat{T} \to V_{\hat{T}}$, where the first map is the inclusion into the first factor. Then we obtain a map $(i_1, \mathfrak{s}): \hat{T} \times \hat{T}_{\mathrm{ad}}^+ \to V_{\hat{T}}$ over \hat{T}_{ad}^+ , which induces the ring map

$$\mathbb{Z}[V_{\hat{T}}] \to \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})] \otimes \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T}_{ad})_{pos}], \quad e_1^{\hat{\lambda}} \otimes e_2^{\hat{\nu}} \mapsto e^{\hat{\lambda}} \otimes \bar{e}^{(\hat{\nu}+\hat{\lambda})}.$$

It in turn induces an injective map

(1.8)
$$\mathbb{Z}[V_{\hat{T}}|_{d=\lambda(q)}] \subset \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})], \quad e_1^{\hat{\lambda}} \otimes e_2^{\hat{\nu}} \mapsto q^{\langle \lambda, \hat{\nu} + \hat{\lambda} \rangle} e^{\hat{\lambda}}.$$

Let $W = N_{\hat{G}}(\hat{T})/\hat{T}$ be the Weyl group of (\hat{G}, \hat{T}) and let $W_0 = W^{\sigma}$ be the subgroup of elements fixed by σ , which naturally acts on $\mathbb{X}^{\bullet}(\hat{T})^{\sigma}$. Let \hat{N}_0 be the preimage of W_0 in $N_{\hat{G}}(\hat{T})$. The σ -twisted conjugation c_{σ} of \hat{G} on $V_{\hat{G}}$ induces the σ -twisted action of \hat{N}_0 on $V_{\hat{T}}$. Recall that there is the Chevalley restriction isomorphism (see [XZ19, Proposition 4.2.3], which was denoted as $\operatorname{Res}_{+,1}^{\sigma}$)

Res:
$$\mathbb{Z}[V_{\hat{G}}]^{c_{\sigma}(\hat{G})} \cong \mathbb{Z}[V_{\hat{T}}]^{c_{\sigma}(\hat{N}_0)},$$

compatible with the $\mathbb{Z}[X^{\bullet}(\hat{T}_{ad})_{pos}]$ -structure on both sides. The same argument as in [XZ19, Lemma 4.2.1] implies that the isomorphism Res induces isomorphisms

$$(1.9) \quad \text{Res}: \mathbb{Z}[V_{\hat{G}}|_{d=\lambda(q)}]^{c_{\sigma}(\hat{G})} \cong \mathbb{Z}[V_{\hat{G}}]^{c_{\sigma}(\hat{G})} \otimes_{\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T}_{\text{ad}})_{\text{pos}}], \bar{e}^{\hat{\alpha}_{i}} \mapsto q^{(\lambda, \hat{\alpha}_{i})}} \mathbb{Z}$$

$$\cong \mathbb{Z}[V_{\hat{T}}]^{c_{\sigma}(\hat{N}_{0})} \otimes_{\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T}_{\text{ad}})_{\text{pos}}], \bar{e}^{\hat{\alpha}_{i}} \mapsto q^{(\lambda, \hat{\alpha}_{i})}} \mathbb{Z} \cong \mathbb{Z}[V_{\hat{T}}|_{d=\lambda(q)}]^{c_{\sigma}(\hat{N}_{0})}.$$

Remark 3. The c_{σ} -action of \hat{N}_0 on $V_{\hat{T}}|_{d=\lambda(q)}$ induces an action of \hat{N}_0 on $\mathbb{Z}[V_{\hat{T}}|_{d=\lambda(q)}]$. On the other hand, the c_{σ} -action of \hat{N}_0 on \hat{T} induces an action of \hat{N}_0 on $\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})]$. The inclusion (1.8) is not equivariant with respect to these two actions. Indeed, the base change of (1.8) to \mathbb{Q} becomes an isomorphism, under which the action of \hat{N}_0 on $\mathbb{Q}[V_{\hat{T}}|_{d=\lambda(q)}]$ induces an action of W_0 on $\mathbb{Q}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$ given by

$$(1.10) w \bullet_{\lambda} e^{\hat{\lambda}} = q^{\langle \lambda, w \hat{\lambda} - \hat{\lambda} \rangle} e^{w \hat{\lambda}}, \quad w \in W_0, \ \hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}.$$

The following lemma follows from (1.8).

Lemma 4. The image of the map

$$(1.11) \qquad \qquad \mathbb{Z}[V_{\hat{T}}|_{d=\lambda(q)}]^{c_{\sigma}(\hat{N}_{0})} \subset \mathbb{Z}[V_{\hat{T}}|_{d=\lambda(q)}] \xrightarrow{(1.8)} \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})]$$

is the subring of $\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$ with a \mathbb{Z} -basis given by elements of the form

$$\sum_{\hat{\lambda}' \in W_0 \hat{\lambda}} q^{\langle \lambda, \hat{\lambda}' - \hat{\lambda} \rangle} e^{\hat{\lambda}'}, \quad \hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma, -} := \mathbb{X}^{\bullet}(\hat{T})^{\sigma} \cap \mathbb{X}^{\bullet}(\hat{T})^{-}.$$

1.4. Classical Satake isomorphism over \mathbb{Z} . Now, we assume that F is a non-archimedean local field, with \mathcal{O} its ring of integers and $k \simeq \mathbb{F}_q$ the residue field. Let G be a connected reductive group over \mathcal{O} . In this case, \widetilde{F}/F is an unramified extension and $\Gamma_{\widetilde{F}/F} \cong \langle \sigma \rangle$ is generated by the geometric q-Frobenius σ of k^3 . We also fix a uniformizer $\varpi \in \mathcal{O}$. Then every $\hat{\lambda} \in \mathbb{X}_{\bullet}(T)^{\sigma}$ gives a well defined point $\hat{\lambda}(\varpi) \in T(F) \subset G(F)$.

Let $K = G(\mathcal{O})$, and let $H_G = C_c(K \setminus G(F)/K, \mathbb{Z})$ be the space of compactly supported bi-K-invariant \mathbb{Z} -valued functions on G(F). This is a natural algebra under convolution (with the chosen Haar measure on G(F) such that the volume of K is 1). Let T be the abstract Cartan of G, which is defined as the quotient of a Borel subgroup $B \subset G$ over \mathcal{O} by its unipotent radical $U \subset B$. (But T is canonically defined, independent of the choice of B, e.g. see [Zhu17b, 0.3.2]). For a choice of Borel subgroup $B \subset G$ over \mathcal{O} , we define the classical Satake transform

(1.12)
$$\operatorname{CT}^{\operatorname{cl}}: H_G \to \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}], \quad f \mapsto \operatorname{CT}^{\operatorname{cl}}(f) = \sum_{\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}} \left(\sum_{u \in U(F)/U(\mathcal{O})} f(\hat{\lambda}(\varpi)u)\right) e^{\hat{\lambda}}.$$

This map is independent of the choice of B and the uniformizer ϖ .

Proposition 5. There exists a unique isomorphism

$$\operatorname{Sat}^{\operatorname{cl}}: \mathbb{Z}[V_{\hat{G},\rho_{\operatorname{ad}}} \rtimes \langle \sigma \rangle |_{\tilde{d}_{\rho_{\operatorname{ad}}}=(q,\sigma)}]^{\hat{G}} \xrightarrow{\cong} H_{G},$$

which we call the Satake isomorphism, making the following diagram commutative

$$\begin{split} \mathbb{Z}[V_{\hat{G},\rho_{\mathrm{ad}}} \rtimes \langle \sigma \rangle |_{\tilde{d}_{\rho_{\mathrm{ad}}} = (q,\sigma)}]^{\hat{G}} & \xrightarrow{\mathrm{Sat^{cl}}} & H_{G} \\ & \underset{\mathrm{Res}}{\bigvee} \cong & \bigvee_{\mathrm{CT^{cl}}} \\ \mathbb{Z}[V_{\hat{T}}|_{d = \rho_{\mathrm{ad}}(q)}]^{c_{\sigma}(\hat{N}_{0})} & \xrightarrow{(1.11)} & \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]. \end{split}$$

In particular, H_G is finitely generated.

Proof. The uniqueness is clear. To prove the existence, by Lemma 4, it is enough to show that the Satake transform (1.12) induces an isomorphism

$$\operatorname{CT}^{\operatorname{cl}}: H_G \xrightarrow{\cong} \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}] \cap \mathbb{Q}[\mathbb{X}^{\bullet}(\hat{T}^{\sigma})^{W_0 \bullet_{\rho_{\operatorname{ad}}}},$$

where $W_0 \bullet_{\rho_{\rm ad}}$ denotes the action given in (1.10) (with $\lambda = \rho_{\rm ad}$). Indeed, this follows from the usual Satake isomorphism by noticing that (1.12) differs from the usual Satake transform (e.g. see [Gr96, (3.4)] in the split case) by a square root of the modular character.

Remark 6. The above isomorphism might look artificial as we identify both sides with a subring of $\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$. In the next section, we will deduce this isomorphism from the geometric Satake⁴. This alternative approach has the advantage that it is more natural and is independent of the usual Satake isomorphism, and is useful for some arithmetic applications.

Let us explain the relation of $\operatorname{Sat}^{\operatorname{cl}}$ with the classical Satake isomorphism (e.g see [Gr96, Proposition 3.6] in the split case) and the mod p Satake isomorphism as in [He11, HV15].

First by (1.7), $(V_{\hat{G},\rho_{\mathrm{ad}}}|_{\tilde{d}_{\rho_{\mathrm{ad}}}=(q,\sigma)})_{\mathbb{Z}[q^{-1}]} \cong ({}^{C}G|_{d_{\rho_{\mathrm{ad}}}=(q,\sigma)})_{\mathbb{Z}[q^{-1}]}$, so the above isomorphism induces a canonical isomorphism

$$H_G \otimes \mathbb{Z}[q^{-1}] \cong \mathbb{Z}[q^{-1}][{}^C G|_{d_{\rho_{\mathrm{ad}}}=(q,\sigma)}]^{\hat{G}}.$$

³Using the arithmetic Frobenius will lead to a different formulation by taking the dual.

⁴In fact, this is how the formulation given here was first discovered.

After choosing a square root $q^{1/2}$, there is a \hat{G} -equivariant isomorphism (comparing with (1.3))

$${}^{C}G|_{d_{\rho,d}=(q,\sigma)} \cong \hat{G}\sigma, \quad (g,(q,\sigma)) \in \hat{G} \rtimes (\mathbb{G}_m \times \langle \sigma \rangle) \mapsto g2\rho(q^{-\frac{1}{2}})\sigma \in {}^{L}G.$$

Note that the following diagram is commutative

$$\mathbb{Z}[q^{\pm\frac{1}{2}}][^{C}G|_{d_{\rho_{\mathrm{ad}}}=(q,\sigma)}]^{\hat{G}} \xrightarrow{\cong} \mathbb{Z}[q^{\pm\frac{1}{2}}][\hat{G}\sigma]^{\hat{G}}$$

$$(1.11)\circ\mathrm{Res} \downarrow \qquad \qquad \downarrow_{\mathrm{Res}}$$

$$\mathbb{Z}[q^{\pm\frac{1}{2}}][\mathbb{X}^{\bullet}(\hat{T})^{\sigma}] \xrightarrow{e^{\hat{\lambda}} \mapsto (q^{-\frac{1}{2}})^{(2\rho,\hat{\lambda})}e^{\hat{\lambda}}} \mathbb{Z}[q^{\pm\frac{1}{2}}][\mathbb{X}^{\bullet}(\hat{T})^{\sigma}],$$

where the right vertical map is the restriction map from functions on $\hat{G}\sigma$ to function on $\hat{T}\sigma$. The composition of (1.12) with the bottom map in the above diagram is the usual Satake transform. We thus obtain the usual classical Satake isomorphism

$$H_G \otimes \mathbb{Z}[q^{\pm \frac{1}{2}}] \cong \mathbb{Z}[q^{\pm \frac{1}{2}}][\hat{G}\sigma]^{\hat{G}}.$$

On the other hand, after mod p, (1.12) is the formula used in [He11, HV15] to define the mod p Satake isomorphism. In addition, $\mathbb{F}_p[V_{\hat{G}}|_{d=\rho_{\mathrm{ad}}(q)}] = \mathbb{F}_p[\mathrm{As}_{\hat{G}}]$.

Corollary 7. There is a canonical isomorphism $H_G \otimes \mathbb{F}_p \cong \mathbb{F}_p[As_{\hat{G}}]^{c_{\sigma}(\hat{G})}$.

This gives a natural description of the mod p Hecke algebra by Langlands duality. Note that by definition

$$\mathbb{F}_p[\mathrm{As}_{\hat{G}}]^{c_{\sigma}(\hat{G})} = \bigoplus_{\nu \in \mathbb{X}^{\bullet}(\hat{T})^+} (S_{\nu^*} \otimes S_{\nu})^{c_{\sigma}(\hat{G})} \cong \mathbb{F}_p[\mathbb{X}_{\bullet}(T)^{\sigma,-}].$$

Therefore, we recover the mod p Satake isomorphism [He11, HV15] (for trivial V in loc. cit.).

Example 8. Let $G = \operatorname{PGL}_2$, so that ${}^CG = \hat{G}^T = \operatorname{GL}_2$ and $V_{\hat{G},\rho_{\operatorname{ad}}} = V_{\hat{G}} = M_2$ is the monoid of 2×2 -matrices, and $d = \det: M_2 \to \mathbb{A}^1$ is the usual determinant map. Then $\mathbb{Z}[M_2|_{\det=q}]^{\operatorname{SL}_2} \cong \mathbb{Z}[\operatorname{tr}]$ is the polynomial ring generated by the trace function. On the other hand, $H_G = \mathbb{Z}[T_p]$ is a polynomial ring generated by the T_p -operator. Under the canonical Satake isomorphism T_p matches tr.

Remark 9. (1) Proposition 5 is compatible with the Weil restriction of G along unramified extensions. We leave the verification as an exercise.

- (2) As suggested by Bernstein, it is the C-group rather than the L-group that should be used in the formulation of the Langlands functoriality. Similarly, we expect the Vinberg monoid of \hat{G} might be useful to formulate the more subtle arithmetic aspect of the Langlands functoriality.
- 1.5. Satake isomorphism for the Hecke algebra of weight V. We retain notations from §1.4. Let Λ be a \mathbb{Z} -algebra, and let (V, π) be a Λ -module equipped with a Λ -linear action of $K = G(\mathcal{O})$.

We first briefly recall the general formalism of Hecke algebra $H_G(V)$ of "weight" V and the Satake transform. We refer to [HV15] for a more general treatment in a more abstract setting. First, we define

(1.13)

$$H_G(V) := \{ f : G(F) \to \operatorname{End}_{\Lambda}(V) \mid f(k'gk)(v) = k'(f(g)(kv)), \forall k, k' \in K, \quad \operatorname{Supp}(f) \text{ is compact} \},$$

with the ring structure given by convolution

$$(f_1 \star f_2)(g)(v) = \sum_{h \in G/K} f_1(h)(f_2(h^{-1}g)(v)), \quad g \in G(F), v \in V.$$

Example 10. Let G = T be a torus over \mathcal{O} . Let $\Lambda = \mathcal{O}_L$, where L is a non-archimedean local field over F. Let V be the rank one free module over Λ on which $T(\mathcal{O})$ acts through a continuous character $\chi: T(\mathcal{O}) \to \mathcal{O}_L^{\times}$. In this case, we write $H_T(V)$ as $H_T(\chi)$. There is an isomorphism

(1.14)
$$H_T(\chi) \cong \mathcal{O}_L[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}], \quad f \mapsto \sum_{\hat{\lambda} \in \mathbb{X}^{\bullet}(T)^{\sigma}} f(\hat{\lambda}(\varpi)) e^{\hat{\lambda}}.$$

If χ is non-trivial, this isomorphism depends on the choice of a uniformizer $\varpi \in F$.

Similar to (1.12), there is the Satake transform

$$(1.15) \quad \operatorname{CT}_V^{\operatorname{cl}}: H_G(V) \to H_T(V^{U(\mathcal{O})}), \quad f \mapsto \Big(\operatorname{CT}_V^{\operatorname{cl}}(f): t \in T(F) \mapsto \sum_{u \in U(F)/U(\mathcal{O})} f(tu)|_{V^{U(\mathcal{O})}}\Big).$$

To justify the definition, note that the sum $\sum_{u \in U(F)/U(\mathcal{O})} f(tu)|_{V^{U(\mathcal{O})}}$ is finite and that for $v \in \mathcal{O}$ $V^{U(\mathcal{O})}$, $\sum_{u \in U(F)/U(\mathcal{O})} f(tu)(v) \in V^{U(\mathcal{O})}$. In addition, one verifies that (1.15) is a homomorphism, either by a direct computation (e.g. see Step 2 in the proof of [He11, Theorem 1.2]), or by considering the action of $H_G(V)$ on the "principal series representation of weight V" (e.g. see [HV15, Section 2). By virtue of (1.14), (1.15) specializes to (1.12) when V=1 is the trivial representation.

Remark 11. In most literature, V is assumed to be a smooth K-module, i.e. the stabilizer of every element $v \in V$ in K is open. But this assumption is in fact not necessary in the above discussions.

In the above generality, there is very little one can say about (1.15). In the sequel, we specialize V to the following situation. Let L be a non-archimedean local field over F, with \mathcal{O}_L its ring of integers. Let V be a finite free \mathcal{O}_L -module arising from an algebraic representation of G over \mathcal{O}_L , such that $V^{U(\mathcal{O})}$ is free of rank one. This is the case if and only if $\dim_L V_L^{U_L} = 1$. In this case, $T(\mathcal{O}) = B(\mathcal{O})/U(\mathcal{O})$ acts on $V^{U(\mathcal{O})}$ via a dominant weight λ of T. It follows from (1.14) that $H_T(V^{U(\mathcal{O})}) = H_T(\lambda)$ is commutative.

Lemma 12. Under the above assumption, the map CT_V^{cl} is injective.

Proof. The proof given below follows the same strategy in [ST06, He11, HV15]. First,

Lemma 13. Fix a uniformizer $\varpi \in F$. For every $\hat{\mu} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma,-}$, There is a unique element $f_{\hat{\mu}} \in H_G(V)$ satisfying

- $f_{\hat{\mu}}$ is supported on $K\hat{\mu}(\varpi)K$; $f_{\hat{\mu}}(\hat{\mu}(\varpi))|_{V^{U(\mathcal{O})}} = \mathrm{id}: V^{U(\mathcal{O})} \to V^{U(\mathcal{O})}$.

In addition, The collection $\{f_{\hat{\mu}}\}_{\hat{\mu} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma,-}}$ form an \mathcal{O}_L -basis of $H_G(V)$.

Proof. Clearly, restricting a map $f:G(F)\to \operatorname{End}_{\mathcal{O}_L}(V)$ to $\hat{\mu}(\varpi)$ induces a bijection between elements in $H_G(V)$ satisfying the first condition and \mathcal{O}_L -linear maps $\varphi: V \to V$ satisfying

$$(1.16) \pi(\hat{\mu}(\varpi)k\hat{\mu}(\varpi)^{-1})\varphi(w) = \varphi(\pi(k)w), \forall k \in K \cap \hat{\mu}(\varpi)^{-1}K\hat{\mu}(\varpi), w \in V.$$

As $K \cap \hat{\mu}(\varpi)^{-1}K\hat{\mu}(\varpi)$ is Zariski dense in G(F), the rational map $\varphi_L: V_L \to V_L$ must be equal to $c\pi(\hat{\mu}(\varpi))$ for some $c \in L$. Then φ preserves the integral lattice V if and only if $c \in \varpi^{\langle \lambda, -\hat{\mu} \rangle} \mathcal{O}_L$.

It follows from the above considerations that $f_{\hat{\mu}}(\hat{\mu}(\varpi)) = \varpi^{\langle \lambda, \hat{\mu} \rangle} \pi(\hat{\mu}(\varpi))$ is the desired element as in the lemma. In addition $\{f_{\hat{\mu}}\}_{\hat{\mu}\in\mathbb{X}^{\bullet}(\hat{T})^{\sigma,-}}$ form an \mathcal{O}_L -basis of $H_G(K)$.

Note that the natural map $H_G(V) \otimes L \to H_G(V_L)$ is an isomorphism, and there is the following commutative diagram

(1.17)
$$H_{G}(\mathbf{1}) \otimes L \xrightarrow{\operatorname{CT}_{\mathbf{1}}^{\operatorname{cl}}} H_{T}(\mathbf{1}) \otimes L$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$H_{G}(V) \otimes L \xrightarrow{\operatorname{CT}_{V}^{\operatorname{cl}}} H_{T}(V^{U(\mathcal{O})}) \otimes L$$

where the left vertical map sends $f: G(F) \to L$ to $\tilde{f}: G(F) \to \operatorname{End}(V_L), \ g \mapsto f(g)\pi(g)$, and the right vertical map sends $f: T(F) \to L$ to $\tilde{f}: T(F) \to \operatorname{End}(V_L^{U_L}) = L, \ t \mapsto f(t)\lambda(t)$.

The left vertical map sends $1_{K\hat{\mu}(\varpi)K}$ to $\varpi^{\langle\lambda,\hat{\mu}\rangle}f_{\hat{\mu}}$, and therefore is an isomorphism. Similarly, the right vertical map is an isomorphism. As $\operatorname{CT}_1^{\operatorname{cl}}\otimes L$ is injective, we conclude Lemma 12.

In this sequel, we further assume that $F = \mathbb{Q}_p$, and choose the uniformizer $\varpi = p$. Then we write the Satake transform (1.15) as a homomorphism $\operatorname{CT}_V^{\operatorname{cl}}: H_G(V) \to \mathcal{O}_L[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$ using (1.14).

For a dominant weight λ of G, let $\lambda_{\rm ad}: \mathbb{G}_m \to \hat{T} \to \hat{T}_{\rm ad}$ be the corresponding cocharacter of $\hat{T}_{\rm ad}$.

Proposition 14. There exists a unique isomorphism

$$\operatorname{Sat}^{\operatorname{cl}}: \mathcal{O}_L[V_{\hat{G}}|_{d=(\lambda_{\operatorname{ad}}+\rho_{\operatorname{ad}})(p)}]^{c_\sigma(\hat{G})} \xrightarrow{\cong} H_G(V),$$

making the following diagram commutative

$$\mathcal{O}_{L}[V_{\hat{G}}|_{d=(\lambda_{\mathrm{ad}}+\rho_{\mathrm{ad}})(p)}]^{c_{\sigma}(\hat{G})} \xrightarrow{\mathrm{Sat^{cl}}} \mathcal{H}_{G}(V)$$

$$\underset{\mathrm{Res}}{\otimes} \downarrow \cong \qquad \qquad \downarrow \mathrm{CT^{cl}_{V}}$$

$$\mathcal{O}_{L}[V_{\hat{T}}|_{d=(\lambda_{\mathrm{ad}}+\rho_{\mathrm{ad}})(p)}]^{c_{\sigma}(\hat{N}_{0})} \xrightarrow{(1.11)} \mathcal{O}_{L}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}].$$

In particular, $H_G(V)$ is finitely generated.

Proof. As in the proof of Proposition 5, it is enough to prove

$$\operatorname{CT}_V^{\operatorname{cl}}: H_G(V) \cong \mathcal{O}_L[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}] \cap L[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]^{W_0 \bullet_{\lambda_{\operatorname{ad}} + \rho_{\operatorname{ad}}}}.$$

But this follows from the case $V = \mathbf{1}$ and the commutative diagram (1.17).

Remark 15. (1) It follows from (1.9) that the algebra $\mathcal{O}_L[V_{\hat{G}}]^{c_{\sigma}(\hat{G})}$ specializes all $H_G(V)$'s.

- (2) Taking the p-adic completion and inverting p allows us to recover some results of [ST06] on the Banach-Hecke algebra $\widehat{H}_G(V)[1/p]$.
- (3) Again, the way to identify $H_G(V)$ with $\mathcal{O}[V_{\hat{G}}|_{d=(\lambda_{\mathrm{ad}}+\rho_{\mathrm{ad}})(p)}]^{c_{\sigma}(\hat{G})}$ given above might look artificial. It would be interesting to have a geometric version of Proposition 14.

2. Compatibility with the geometric Satake

In this section, we deduce Proposition 5 from the geometric Satake equivalence.

2.1. The geometric Satake equivalence. We refer to [Zhu17a, Zhu17b] and references cited there for detailed discussions of the geometric Satake equivalence. We retain notations from §1.4. Let \tilde{F}/F be the splitting field of G. It is a finite unramified extension of F, with $\tilde{\mathcal{O}}$ its ring of integers and \tilde{k} the residue field. Let $r = [\tilde{F} : F] = [\tilde{k} : k]$. Let L^+G denote the positive loop group of G over k and let G denote its affine Grassmannian over K.

For $\hat{\mu} \in \mathbb{X}^{\bullet}(\hat{T})^+$, let $k_{\hat{\mu}} \subset \tilde{k}$ be its field of definition and $d_{\hat{\mu}} := [k_{\hat{\mu}} : k]$. Let $\operatorname{Gr}_{\leq \hat{\mu}}$ denote the Schubert variety corresponding to $\hat{\mu}$, which is a (perfect) projective scheme defined over $k_{\hat{\mu}}$. Let $\operatorname{Gr}_{\hat{\mu}}$ denote the open Schubert cell.

We fix $\ell \neq p$. Let $IC_{\hat{\mu}}$ be the intersection complex with \mathbb{Q}_{ℓ} -coefficient on $Gr_{\leq \hat{\mu}}$, so that

$$IC_{\hat{\mu}}|_{Gr_{\hat{\mu}}} = \mathbb{Q}_{\ell}[\langle 2\rho, \hat{\mu} \rangle].$$

If k'/k is an algebraic extension in \bar{k} , let $\operatorname{Sat}_{G,k',\ell}$ denote the category of $L^+G \otimes k'$ -equivariant perverse sheaves on $\operatorname{Gr} \otimes k'$ with \mathbb{Q}_{ℓ} -coefficients, which is a tensor abelian categories. Inside $\operatorname{Sat}_{G,\tilde{k},\ell}$, there is a full semisimple monoidal abelian subcategory $\operatorname{Sat}_{G,\tilde{k},\ell}^T$ as defined in $[\operatorname{Zhu}17b]^5$: it is the full semisimple tensor abelian category generated by all $\{\operatorname{IC}_{\hat{\mu}}(i), \hat{\mu} \in \mathbb{X}_{\bullet}(T)^+, i \in \mathbb{Z}\}$. Now we define $\operatorname{Sat}_{G,\ell}^T$ as the category of $\operatorname{Gal}(\tilde{k}/k)$ -equivariant objects in $\operatorname{Sat}_{G,\tilde{k},\ell}^T$. I.e., objects are pairs (\mathcal{F},γ) , where $\mathcal{F} \in \operatorname{Sat}_{G,\tilde{k},\ell}$ and $\gamma: \sigma^*\mathcal{F} \simeq \mathcal{F}$ is an isomorphism such that the induced isomorphism $\mathcal{F} = (\sigma^r)^*\mathcal{F} \xrightarrow{\gamma\sigma(\gamma)\cdots\sigma^{r-1}(\gamma)} \mathcal{F}$ is the identity map, and morphisms from (\mathcal{F},γ) to (\mathcal{F}',γ') are morphisms from \mathcal{F} to \mathcal{F}' in $\operatorname{Sat}_{G,\tilde{k},\ell}^T$ that are compatible with γ and γ' . This is still a semisimple tensor category.

For a (not necessarily connected) split reductive group H over a field E of characteristic zero, let $\operatorname{Rep}(H_E)$ denote the category of finite dimensional algebraic representations of H over E. Let $\sigma-\operatorname{Mod}_{\mathbb{Q}_\ell}$ denote the category of representations of σ on finite dimensional \mathbb{Q}_ℓ -vector spaces. Here is the version of the geometric Satake equivalence we need in the sequel.

Theorem 16. There is a natural equivalence of tensor categories $\operatorname{Sat}: \operatorname{Rep}(^{C}G_{\mathbb{Q}_{\ell}}) \cong \operatorname{Sat}_{G,\ell}^{T}$ such that the composition with the cohomology functor $\operatorname{H}^{*}(\operatorname{Gr}_{\bar{k}}, -): \operatorname{Sat}_{G,\ell}^{T} \to \sigma - \operatorname{Mod}_{\mathbb{Q}_{\ell}}$ is the restriction functor $\operatorname{Rep}(^{C}G_{\mathbb{Q}_{\ell}}) \to \sigma - \operatorname{Mod}_{\mathbb{Q}_{\ell}}$ induced by the inclusion $\sigma \mapsto (1, q, \sigma) \in \hat{G} \rtimes (\mathbb{G}_{m} \times \langle \sigma \rangle)$.

Proof. If G is split (so ${}^CG = \hat{G}^T$), this was stated in [Zhu17b, Lemma 5.5.14]. So we obtain a natural equivalence $\operatorname{Rep}(\hat{G}_{\mathbb{Q}_{\ell}}^T) \cong \operatorname{Sat}_{G,\tilde{k},\ell}^T$ satisfying the desired properties as in the theorem with σ replaced by σ^r . In addition, \hat{G}^T is equipped with a pinning (induced from the pinning of \hat{G} as described in [Zhu17b, Corollary 5.3.23]).

Now for $(\mathcal{F}, \gamma) \in \operatorname{Sat}_{G,\ell}^T$, we have a canonical isomorphism $\operatorname{H}^*(\operatorname{Gr}_{\bar{k}}, \mathcal{F}) \cong \operatorname{H}^*(\operatorname{Gr}_{\bar{k}}, \sigma^*\mathcal{F}) \cong \operatorname{H}^*(\operatorname{Gr}_{\bar{k}}, \mathcal{F})$. Using the formalism as in [RZ14, Lemma A.3] and an argument similar to [RZ14, Proposition A.6] (see also [Zhu17b, Lemma 5.5.5] and the paragraphs before it), we see that the above equivalence induces the pinned action of σ on $\hat{G}_{\mathbb{Q}_{\ell}}^T$, and the desired equivalence.

- Remark 17. (1) Without adding the \mathbb{G}_m factor, the Galois action of σ on \hat{G} obtained by the formalism [RZ14, Lemma A.3] does not preserve the pinning. The semidirect product of Γ_k with $\hat{G}(\mathbb{Q}_\ell)$ using this action was denoted by ${}^LG^{\text{geom}}$ in loc. cit. (and later denoted by ${}^LG^{\text{geo}}$ in [Ri14, §5] and in [Zhu17b, §5.5]). Theorem 16 is compatible with [Ri14, §5] (see also [Zhu17b, Theorem 5.5.12]), via restriction along the injective map ${}^LG^{\text{geo}} \to {}^CG(\mathbb{Q}_\ell)$.
- (2) One of the consequences of the above theorem is as follows. For a σ -invariant dominant weight $\hat{\mu}$, there is a unique (up to isomorphism) irreducible representation $V_{\hat{\mu}}$ of ${}^{C}G_{\mathbb{Q}_{\ell}}$ such that $V_{\hat{\mu}}|_{\hat{G}}$ is irreducible of highest weight $\hat{\mu}$ and that the action of $\mathbb{G}_{m} \times \langle \sigma \rangle$ on the lowest weight line of $V_{\hat{\mu}}$ (w.r.t. (\hat{G}, \hat{B})) is trivial. Namely, under the geometric Satake, $V_{\hat{\mu}}$ corresponds to $IC_{\hat{\mu}}$ equipped with the natural $Gal(\tilde{k}/k)$ -equivariant structure. Of course, this fact is well-known.

We will need the following properties of the geometric Satake equivalence.

First, let T be the abstract Cartan of G. We can define the Satake category $\operatorname{Sat}_{T,\ell}^T$ as a subcategory of perverse sheaves on Gr_T . For a choice a Borel subgroup $B \subset G$ over \mathcal{O} , we have the correspondence $\operatorname{Gr}_T \xleftarrow{r} \operatorname{Gr}_B \xrightarrow{i} \operatorname{Gr}_G$. Recall that for $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})$, we have the point $\hat{\lambda}(\varpi) \in \operatorname{Gr}_T(\tilde{k})$.

⁵Strictly speaking, only equal characteristic version was considered in [Zhu17b]. However its counterpart in mixed characteristic is obvious, using [Zhu17a]. The similar remark applies to the discussions in the sequel.

Let $\operatorname{Gr}_{B,\hat{\lambda}}=r^{-1}(\hat{\lambda}(\varpi))$ be the (geometrically) connected component of Gr_B given by $\hat{\lambda}$. We write $r_{\hat{\lambda}}$ (resp. $i_{\hat{\lambda}}$) be the restriction of r (resp. i) to $\operatorname{Gr}_{B,\hat{\lambda}}$. Then we define the Mirković-Vilonen's weight functor

$$\operatorname{CT}: \operatorname{Sat}_{G,\ell}^T \to \operatorname{Sat}_{T,\ell}^T, \quad \operatorname{CT}(\mathcal{F}) = \bigoplus_{\hat{\lambda}} r_{\hat{\lambda},!} i_{\hat{\lambda}}^* \mathcal{F}[(2\rho, \hat{\lambda})].$$

We note that $\mathrm{CT}(\mathcal{F})$ is a sheaf on $\mathrm{Gr}_{T,\tilde{k}}$ naturally equipped with a $\mathrm{Gal}(\tilde{k}/k)$ -equivariant structure so that the above definition makes sense. Then the geometric Satake fits into the following commutative diagram

(2.1)
$$\operatorname{Rep}({}^{C}G_{\mathbb{Q}_{\ell}}) \xrightarrow{\operatorname{Sat}} \operatorname{Sat}_{G,\ell}^{T}$$

$$\operatorname{Res} \downarrow \qquad \qquad \downarrow^{\operatorname{CT}}$$

$$\operatorname{Rep}({}^{C}T_{\mathbb{Q}_{\ell}}) \xrightarrow{\operatorname{Sat}} \operatorname{Sat}_{T,\ell}^{T}.$$

This follows from the usual compatibility between the geometric Satake and the restriction to the maximal torus, with the Galois action taking into account.

For $\mathcal{F} \in \operatorname{Sat}_{G,\ell}^T$, its Frobenius trace function

$$f_{\mathcal{F}} \in C_c(K \backslash G(F)/K, \mathbb{Q}_\ell)$$

makes sense as usual. The next fact we need is a description of $f_{\mathrm{Sat}(V)}$ for $V \in {}^C G_{\mathbb{Q}_\ell}$. For this purpose, we need to recall the so-called Brylinski-Kostant filtration. Let $\{\hat{e}, 2\rho, \hat{f}\}$ be the principal \mathfrak{sl}_2 -triple of \hat{G} containing $\hat{e} \in \mathrm{Lie}\hat{B}$ and $2\rho \in \mathbb{X}_{\bullet}(\hat{T}) \subset \mathrm{Lie}\hat{T}$. For a representation V of \hat{G} and $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^-$, we define the Brylinski-Kostant filtration on $V(\hat{\lambda})$ as

$$F_iV(\hat{\lambda}) := \ker(\hat{f}^{i+1} : V \to V) \cap V(\hat{\lambda}).$$

Let $\operatorname{gr}_{\bullet}^F V(\hat{\lambda})$ denote its associated graded. Note that if $\hat{\lambda}$ is σ -invariant, then $F_iV(\hat{\lambda})$ is σ -stable and therefore σ acts on $\operatorname{gr}_{\bullet}^F V(\hat{\lambda})$.

Proposition 18. Let V be a representation of ${}^{C}G_{\mathbb{Q}_{\ell}}$, and $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma,-}$. Then

$$f_{\mathrm{Sat}(V)}(\hat{\lambda}(\varpi)) = (-1)^{\langle 2\rho, \hat{\lambda} \rangle} \mathrm{tr}((1, q, \sigma) \mid \mathrm{gr}_i^F V(\hat{\lambda})) q^{-i}.$$

Proof. This follows [Zhu15, §5], by taking into account of the Galois action. Note that the proof of Lemma 5.8 of *loc. cit.* relies on the existence of "big open cell" of the affine Grassmannian, which is not known in mixed characteristic. However, one can easily replace the purity argument in *loc. cit.* by a parity argument, e.g. the argument in the middle of p. 452 of [Zhu17a].

Remark 19. Recall that for the representation $V = V_{\hat{\mu}}$ of ${}^{C}G_{\mathbb{Q}_{\ell}}$ as in Remark 17, and for $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}$, Jantzen's twisted character formula ([Ja73, Satz 9]) expresses $\operatorname{tr}((1,1,\sigma) \mid V_{\hat{\mu}}(\hat{\lambda}))$ as the dimension of a representation of a reductive group \hat{G}_{σ} whose weight lattice is $\mathbb{X}^{\bullet}(\hat{T})^{\sigma}$. It would be very interesting to have its q-analogue, expressing $\sum_{i} \operatorname{tr}(\sigma \mid \operatorname{gr}_{i}^{F} V_{\hat{\mu}}(\hat{\lambda})) q^{-i}$ in terms of representations of \hat{G}_{σ} .

We do not have such a formula at the moment. The following lemma is sufficient for our purpose.

Lemma 20. Let $V_{\hat{\mu}}$ be as above. Then for every $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}$, $\operatorname{tr}(\sigma \mid \operatorname{gr}_{i}^{F} V_{\hat{\mu}}(\hat{\lambda})) \in \mathbb{Z}$.

Proof. We may assume that $\hat{\lambda} \in \mathbb{X}^{\bullet}(T)^{\sigma,-}$. The root datum of G defines a reductive group \mathbb{G} over \mathbb{C} equipped with a \mathbb{C} -automorphism σ . Let $\mathrm{Gr}_{\mathbb{G}}$ be its affine Grassmannian, acted by σ . We have the geometric Satake for $\mathrm{Gr}_{\mathbb{G}}$, and the analogous statement of Proposition 18 in this setting is

$$\operatorname{tr}(\sigma \mid \operatorname{gr}_{i}^{F} V_{\hat{\mu}}(\hat{\lambda})) = \operatorname{tr}(\sigma \mid \mathcal{H}_{\hat{\lambda}(\varpi)}^{-2i + \langle 2\rho, \hat{\lambda} \rangle} \operatorname{IC}_{\hat{\mu}}),$$

where $\mathcal{H}_{\hat{\lambda}(\varpi)}^{-2i+\langle 2\rho,\hat{\lambda}\rangle}$ $IC_{\hat{\mu}}$ denotes the stalk cohomology of $IC_{\hat{\mu}}$ at $\hat{\lambda}(\varpi)$. Over \mathbb{C} , the sheaf $IC_{\hat{\mu}}$ has a natural \mathbb{Z} -structure preserved by the action of σ ([MV07, Proposition 8.1]). The lemma follows. \square

2.2. The representation ring. We generalize some well-known relations between the representation ring of a reductive group and its ring of invariant functions. Let E be a characteristic zero field. Let $\operatorname{Rep}^+(\hat{G}_E^T) \subset \operatorname{Rep}(\hat{G}_E^T)$ denote the subcategory consisting of those objects on which all weights of $\mathbb{G}_m \subset \hat{G} \rtimes_{\operatorname{Ad}\rho_{\operatorname{ad}}} \mathbb{G}_m = \hat{G}^T$ are ≥ 0 . Let $\operatorname{Rep}(V_{\hat{G},\rho_{\operatorname{ad}},E})$ denote the category of finite dimensional algebraic representations of $V_{\hat{G},\rho_{\operatorname{ad}},E}$.

Lemma 21. The inclusion $\hat{G}^T \subset V_{\hat{G},\rho_{\mathrm{ad}}}$ induces an equivalence of categories $\mathrm{Rep}^+(\hat{G}_E^T) \cong \mathrm{Rep}(V_{\hat{G},\rho_{\mathrm{ad}},E})$.

Proof. First, under the inclusion $\hat{G} \times^{Z_{\hat{G}}} \hat{T} \to V_{\hat{G}}$, an irreducible representation V of $\hat{G}_E \times \hat{T}_E$ can be extended to a representation of $V_{\hat{G},E}$ if and only if the following holds: if $V|_{\hat{G}_E}$ has the highest weight $\hat{\mu}$, and $V|_{\hat{T}_E}$ has the weight $\hat{\nu}$, then $\hat{\nu} + w_0(\hat{\mu})$ is a sum of nonnegative roots of \hat{G} . Therefore, an irreducible representation of $(\hat{G} \times \mathbb{G}_m)/(2\rho \times \mathrm{id})(\mu_2)$ can be extended to a representation of $V_{\hat{G},\rho_{\mathrm{ad}}}$ if and only if the following two conditions hold:

- (i) \mathbb{G}_m acts on V by some weight n; and
- (ii) If $\hat{\mu}$ is the highest weight of V as a \hat{G} -representation, then $\langle 2\rho, \hat{\mu} \rangle \leq n$ and $(-1)^{\langle 2\rho, \hat{\mu} \rangle} = (-1)^n$.

But under the isomorphism (1.2), this exactly corresponds to an object in Rep⁺(\hat{G}_{E}^{T}).

Now let σ be a finite order automorphism of $(\hat{G}, \hat{B}, \hat{T}, \hat{e})$ as before. We let $\operatorname{Rep}^{\operatorname{int}}(\mathbb{G}_m \times \langle \sigma \rangle)$ denote the full tensor subcategory of finite dimensional algebraic E-linear representations of $\mathbb{G}_m \times \langle \sigma \rangle$ consisting of W satisfying the following two properties

- all \mathbb{G}_m -weights in W are ≥ 0 ;
- the trace of σ on each \mathbb{G}_m -weight subspace of W is an integer.

This is equivalent to requiring that the trace of (q, σ) on W is an integer.

Let $\operatorname{Rep}^{\operatorname{int}}({}^{C}G_{E})$ be the full tensor subcategory of $\operatorname{Rep}({}^{C}G_{E})$ consisting of those representations V such that for every σ -invariant weight $\hat{\lambda}$ of \hat{T} and every $i \geq 0$, the space $\operatorname{gr}_{i}^{F}V(\hat{\lambda})$ as a representation of $\mathbb{G}_{m} \times \langle \sigma \rangle \subset \hat{G} \rtimes (\mathbb{G}_{m} \times \langle \sigma \rangle)$ belongs to $\operatorname{Rep}^{\operatorname{int}}(\mathbb{G}_{m} \times \langle \sigma \rangle)$. Note that the restriction of such a representation to \hat{G}^{T} belongs to $\operatorname{Rep}^{+}(\hat{G}_{E}^{T})$, and therefore we have the functor

$$\operatorname{Rep}^{\operatorname{int}}({}^{C}G_{E}) \to \operatorname{Rep}(V_{\hat{G},\rho_{\operatorname{ad}},E} \rtimes \langle \sigma \rangle).$$

When σ is trivial, we have $\operatorname{Rep}^{\operatorname{int}}({}^{\mathbb{C}}G_E) \cong \operatorname{Rep}^+(\hat{G}_E^T) \cong \operatorname{Rep}(V_{\hat{G},\rho_{\operatorname{ad}},E})$.

Lemma 22. There exists a unique homomorphism $\operatorname{tr}: K(\operatorname{Rep}^{\operatorname{int}}({}^{C}G_{E})) \to \mathbb{Z}[V_{\hat{G},\rho_{\operatorname{ad}}}|_{d_{\rho_{\operatorname{ad}}}=q}]^{c_{\sigma}(\hat{G})}$ making the following diagram commutative

$$(2.2) K(\operatorname{Rep}^{\operatorname{int}}({}^{C}G_{E})) \xrightarrow{\operatorname{tr}} \mathbb{Z}[V_{\hat{G},\rho_{\operatorname{ad}}}|d_{\rho_{\operatorname{ad}}=q}]^{c_{\sigma}(\hat{G})}$$

$$K(\operatorname{Res}) \downarrow \qquad \qquad \downarrow (1.11) \circ \operatorname{Res}$$

$$K(\operatorname{Rep}^{\operatorname{int}}({}^{C}T_{E})) \xrightarrow{[V] \mapsto \sum_{\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}} \operatorname{tr}((q,\sigma)|V(\hat{\lambda})) e^{\hat{\lambda}}} \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}].$$

Proof. We define the map

$$\operatorname{tr}: K(\operatorname{Rep}^{\operatorname{int}}({}^{C}G_{E})) \to K(\operatorname{Rep}(V_{\hat{G},\rho_{\operatorname{ad}},E} \rtimes \langle \sigma \rangle) \to \overline{E}[V_{\hat{G},\rho_{\operatorname{ad}}}|_{d_{\rho_{\operatorname{ad}}}=q}]^{c_{\sigma}(G)},$$

where the second arrow is induced by taking the trace of representations at elements in $V_{\hat{G}}|_{d=\rho_{ad}(q)}\sigma$. The diagram is clearly commutative when the right vertical map is tensored with \overline{E} . To see that tr

factors through $\mathbb{Z}[V_{\hat{G},\rho_{\mathrm{ad}}}|_{d=q}]^{c_{\sigma}(\hat{G})}$, it is enough to note that $\mathbb{Z}[V_{\hat{G},\rho_{\mathrm{ad}}}|_{d=q}]^{c_{\sigma}(\hat{G})} = \overline{E}[V_{\hat{G},\rho_{\mathrm{ad}}}|_{d=q}]^{c_{\sigma}(\hat{G})} \cap \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$ inside $\overline{E}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$ which follows easily from Lemma 4. The uniqueness is clear as the right vertical map is injective.

Lemma 23. The homomorphism in Lemma 22 is surjective.

Proof. For a σ -invariant dominant weight $\hat{\mu}$, let $V_{\hat{\mu}}$ be the unique (up to isomorphism) irreducible representation of ${}^{C}G_{E}$ as in Remark 17 (2). We claim that $V_{\hat{\mu}} \in \operatorname{Rep}^{\operatorname{int}}({}^{C}G_{E})$. Then the lemma follows from the explicit description of $\mathbb{Z}[V_{\hat{G},\rho_{\operatorname{ad}}}|_{d_{\rho_{\operatorname{ad}}}=q}]^{c_{\sigma}(\hat{G})}$ inside $\mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}]$, as in Lemma 4.

We need to show that $\operatorname{tr}((1,q,\sigma)\mid\operatorname{gr}_F^i V_{\hat{\mu}}(\hat{\lambda}))\in\mathbb{Z}$ for σ -invariant weight $\hat{\lambda}$. Note that under the map $\hat{G}\times\mathbb{G}_m\to\hat{G}^T$ from the isomorphism (1.2), the second factor \mathbb{G}_m acts on $V_{\hat{\mu}}$ by a fixed weight (namely $\langle 2\rho,\hat{\mu}\rangle$). Therefore, $(1,q)\in\hat{G}\rtimes\mathbb{G}_m=\hat{G}^T$ acts on $V_{\hat{\mu}}(\hat{\lambda})$ by $q^{\langle\rho_{\mathrm{ad}},\hat{\lambda}-w_0\hat{\mu}\rangle}$. Therefore, the lemma follows by noticing that the trace of $(1,1,\sigma)$ on $V_{\hat{\mu}}(\hat{\lambda})$ is an integer, by Lemma 20.

2.3. From Sat to Sat^{cl}. Let Sat^{T,int} be the full subcategory of Sat^T_{G,ℓ} consisting of those \mathcal{F} such that $f_{\mathcal{F}} \in H_G$, i.e. $f_{\mathcal{F}}(\hat{\lambda}(\varpi)) \in \mathbb{Z}$ for every $\hat{\lambda} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}$. By Proposition 18, we have the following statement.

Lemma 24. The geometric Satake induces an equivalence $\operatorname{Rep}^{\operatorname{int}}({}^{\mathbb{C}}G_{\mathbb{E}}) \cong \operatorname{Sat}_{G,\ell}^{T,\operatorname{int}}$.

Lemma 25. Taking the trace Frobenius function induces a surjective ring homomorphism

$$\operatorname{tr}: K(\operatorname{Sat}_{G,\ell}^{T,\operatorname{int}}) \to H_G.$$

Proof. For dominant $\hat{\mu} \in \mathbb{X}^{\bullet}(\hat{T})^{\sigma}$, by Proposition 18 and Lemma 23,

$$(2.3) f_{\mathrm{IC}_{\hat{\mu}}} = (-1)^{\langle 2\rho, \hat{\mu} \rangle} 1_{K\hat{\mu}(\varpi)K} + \sum_{\hat{\mu}' < \hat{\mu}} a_{\hat{\mu}\hat{\mu}'} 1_{K\hat{\mu}'(\varpi)K}, \quad a_{\hat{\mu}\hat{\mu}'} \in \mathbb{Z},$$

where $1_{K\hat{\mu}'(\varpi)K}$ denotes the characteristic function on $K\hat{\mu}'(\varpi)K$ and "<" denotes the Bruhat order. Then the lemma follows since these $1_{K\hat{\mu}(\varpi)K}$'s form a \mathbb{Z} -basis of H_G .

Note that the Grothendieck-Lefschetz trace formula implies that

$$\operatorname{tr} \circ K(\operatorname{CT}) = \operatorname{CT}^{\operatorname{cl}} \circ \operatorname{tr} : K(\operatorname{Sat}_{G,\ell}^{T,\operatorname{int}}) \to \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}].$$

Together with (2.1) and (2.2), we obtain the following commutative diagram

$$K(\operatorname{Rep}^{\operatorname{int}}({}^{C}G_{\mathbb{Q}_{\ell}})) \xrightarrow{\operatorname{tr}} \mathbb{Z}[V_{\hat{G}}|_{d=\rho_{\operatorname{ad}}(q)}]^{c_{\sigma}(G)} \xrightarrow{\operatorname{Res}} \mathbb{Z}[V_{\hat{T}}|_{d=\rho_{\operatorname{ad}}(q)}]^{c_{\sigma}(\hat{N}_{0})}$$

$$\downarrow K(\operatorname{Sat}) \downarrow \cong \qquad \qquad \downarrow (1.11)$$

$$K(\operatorname{Sat}_{G,\ell}^{T,\operatorname{int}}) \xrightarrow{\operatorname{tr}} H_{G} \xrightarrow{\operatorname{CT}^{\operatorname{cl}}} \mathbb{Z}[\mathbb{X}^{\bullet}(\hat{T})^{\sigma}].$$

Both trace maps tr in the above diagram are surjective, by Lemma 23 and 25. Therefore, we can complete the diagram by adding an isomorphism Sat^{cl} in the middle column, as desired.

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