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Abstract. We study various moduli spaces of local Shtukas in the setting of Fargues' program for GL_n. In certain cases, this gives us an explicit description of the spectral action which was recently introduced by Fargues and Scholze. This description sheds light to the categorical local Langlands program for GL_n and allows us to construct Hecke eigensheaves associated to certain ℓ-adic Weil representations of rank n and to prove some parts of Fargues' conjecture. Moreover, by using this description, we can prove new cases of the Harris-Viehmann conjecture for non-basic Rapoport-Zink spaces and compute some parts of the cohomology of the Igusa varieties associated to GL_n.

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Contents

1. Introduction

Let $\ell \neq p$ be primes and fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ with associated Weil group $W_{\mathbb{Q}_p}$. In [Far16], Fargues formulated his program to geometrize the local Langlands correspondence for $\mathbb{Q}_p$ via the stack $\mathrm{Bun_G}$ of G-bundles on the Fargues-Fontaine curve, with G a reductive group over $\mathbb{Q}_p$.

Fargues' program has been put forward in the seminal recent work [FS21]. The aims of Fargues' program is to relate sheaves on the stack of Langlands parameters, "the spectral side", to sheaves on $\mathrm{Bun_G}$, "the geometric side". On the spectral side, one considers $\mathrm{Perf}^{\mathrm{qc}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}])$ (resp. $\mathrm{D}^{b,\mathrm{qc}}_{\mathrm{coh}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}])$), the derived category of perfect complexes on the stack of L-parameters $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}]$ with quasi-compact support (resp. bounded derived category of sheaves with coherent cohomology with quasi-compact support), as in [DHKM20, Zhu20] and [FS21, Section VIII.I]. On the geometric side, one considers $\mathrm{D_{lis}}(\mathrm{Bun_G}, \overline{\mathbb{Q}}_\ell)$, the category of lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves and $\mathrm{D_{lis}}(\mathrm{Bun_G}, \overline{\mathbb{Q}}_\ell)^\omega$ the sub-category of compact objects, as defined in [FS21, Section VII.7]. Fargues and Scholze constructed an action (called the spectral action) of the category $\mathrm{Perf}^{\mathrm{qc}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}])$ on the category $\mathrm{D_{lis}}(\mathrm{Bun_G}, \overline{\mathbb{Q}}_\ell)$ and expect this action relates these categories in a precise way. Fix a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ with unipotent radical U and a generic character $\psi : \mathrm{U}(\mathbb{Q}_p) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. Let $\mathcal{W}_\varphi$

be the Whittaker sheaf, which is the sheaf concentrated on the stratum $\mathrm{Bun}_{\mathrm{G}}^1$ corresponding to the representation $c\text{-}\mathrm{Ind}_{\mathrm{U}}^{\mathrm{G}}\,\psi$ of $\mathrm{G}(\mathbb{Q}_p)$. One can rephrase their conjecture by saying that the "non-abelian Fourier transform"

$$\begin{aligned} \mathrm{Perf}^{\mathrm{qc}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}] &\longrightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell) \\ \mathrm{M} &\longmapsto \mathrm{M} \star \mathcal{W}_\psi \end{aligned}$$

is fully faithful and extends to an equivalence of $\overline{\mathbb{Q}}_\ell$-linear small stable ∞-categories

$$\mathrm{D}^{b,\mathrm{qc}}_{\mathrm{coh}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}] \longrightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)^\omega,$$

where $\star$ denotes the spectral action and $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)^\omega$ denotes the full subcategory of compact objects.

This program is expected to capture in a geometric way everything we know about the local Langlands conjectures such as the internal structure of L-packets, Jacquet-Langlands correspondences, functoriality transfers and even some form of the local-global compatibility. We consider the case $\mathrm{G} = \mathrm{GL}_n$, the main goal of the manuscript is to compute explicitly the action of $\mathrm{Perf}^{\mathrm{qc}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n])$ on π_b where $b \in B(\mathrm{GL}_n)$ and π_b is a smooth irreducible representation of $\mathrm{G}_b(\mathbb{Q}_p)$ whose corresponding L-parameter satisfies some extra conditions. Besides many geometric and cohomological applications, these computations also shed light to the relation between the categorical and the usual local Langlands conjectures.

We consider an ($\widehat{\mathrm{GL}}_n$-conjugacy class of) L-parameter ϕ of GL_n which has a decomposition

$$\phi = \phi_1 \oplus \ldots \oplus \phi_r,$$

where the ϕ_i's are irreducible and for $1 \leq i \neq j \leq r$, there does not exist unramified character χ such that $\phi_i \simeq \phi_j \otimes \chi$. Let π be the irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ corresponding to ϕ via the local Langlands correspondence. In this case we know that π is isomorphic to the normalized parabolic induction $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n}(\pi_1 \otimes \ldots \otimes \pi_r)$ where $\mathrm{P} = \mathrm{MN}$ is some standard parabolic subgroup whose Levi factor is given by $\mathrm{M} = \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$ and the π_i's are supercuspidal representations.

We have a map $\theta : [Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n] \longrightarrow Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)//\widehat{\mathrm{GL}}_n$ where $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)//\widehat{\mathrm{GL}}_n$ is the categorical quotient. Since ϕ is semi-simple, its orbit in $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)$ by the action of GL_n is closed. Let x be the $\overline{\mathbb{Q}}_\ell$-point in $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)//\widehat{\mathrm{GL}}_n$ corresponding to this orbit. We can show that $\theta^{-1}(x)$ is closed and consists of exactly the point in $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$ corresponding to ϕ (proposition 3.4). We denote by $[C_\phi]$ the connected component of $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$ containing ϕ. Recall that by the computation in [KMSW14, pg. 83], we have an explicit description of the centralizer $S_\phi := \mathrm{Cent}(\phi) = \prod_{i=1}^{r} \mathbb{G}_m$ and by the results in [DHKM20], we can show (proposition 3.6) that $[C_\phi] \simeq [\mathbb{G}_m^r/\mathbb{G}_m^r]$ where the quotient is taken with respect to the trivial action of $\mathbb{G}_m^r$.

Let $C \in \mathrm{Perf}^{\mathrm{qc}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n])$ be a perfect complex such that its support does not intersect with $\theta^{-1}(x)$. We then (by [Ham21, lemma 3.8]) know that $C \star \mathcal{F}_\pi = 0$ where $\mathcal{F}_\pi$ is the sheaf over $\mathrm{Bun}_{\mathrm{n}}$ concentrated on Bun_n^1 associated with π. More generally, if the support of C does not intersect $[C_\phi]$ and π' is any finitely generated representation of $\mathrm{G}_b(\mathbb{Q}_p)$ regarded as sheaf on $\mathrm{Bun}_{\mathrm{n}}$ whose (semi-simplified) L-parameter of any irreducible constituent is given by an L-parameter in $[C_\phi]$ then $C \star \mathcal{F}_{\pi'} = 0$. The study of the spectral action on $\mathcal{F}_\pi$ is therefore reduced to the description of the complex $C \star \mathcal{F}_\pi$ for C perfect complexes supported on the connected component $[C_\phi]$. We denote by $\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{n}}, \overline{\mathbb{Q}}_\ell)^\omega$ the full sub-category of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{n}}, \overline{\mathbb{Q}}_\ell)^\omega$ such that the Schur-irreducible objects in this subcategory all have Fargues-Scholze parameter given by an L-parameter in $[C_\phi]$.

1.1. **Hecke operators and cohomology of local Shimura varieties.**

1.1.1. *Hecke operators and Hecke eigensheaves.*

We fix a maximal split torus T and a Borel subgroup B of GL_n and fix an L-parameter ϕ satisfying some conditions as above. In particular we have $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ and $S_\phi = \prod_{i=1}^{r} \mathbb{G}_m$. Suppose that $\dim \phi_i = n_i$, thus $\sum_{i=1}^{r} n_i = n$.

In this manuscript, we compute explicitly the action of a perfect complex $\mathbb{L} \in \mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n])^{\mathrm{qc}}$ on $\mathcal{F}_{\pi'}$. Since the category $\mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n])^{\mathrm{qc}}$ is generated by cones and retracts of Hecke operators, the first step is to compute the Hecke actions acting on $\mathcal{F}_{\pi'}$ where π' is irreducible. Since we have an explicit description of $[C_\phi]$, the category $\mathrm{Perf}([C_\phi])$ is not too hard to understand. In particular, we have a monoidal embedding of categories

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi) \longrightarrow \mathrm{Perf}([C_\phi])$$

where the image of an irreducible character χ is the vector bundle on $[C_\phi]$ corresponding to the structural sheaf on $\mathbb{G}_m^r$ together with the $\mathbb{G}_m^r$-action defined by χ. We denote by C_χ this perfect sheaf and we want to concretely describe the $C_\chi \star \mathcal{F}_\pi$ where π is the irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ with L-parameter given by ϕ.

Remark that S_ϕ is commutative then the set $\mathrm{Irr}(S_\phi)$ of its characters forms a group under the tensor product operator. In this specific case that group is isomorphic to $\prod_{i=1}^{r} \mathbb{Z}$. Then a set of generators of $\mathrm{Irr}(S_\phi)$ is given by the characters χ_i corresponding to $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 is in the i^{th}-position), $1 \leq i \leq r$.

Let us describe the first main result describing the action of the C_χ's and the action of the Hecke operators. For each character $\chi = (d_1, \ldots, d_r) \in \prod_{i=1}^{r} \mathbb{Z}$, we define an element $b_\chi \in B(\mathrm{GL}_n)$, an irreducible representation π_χ of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ and a sheaf $\mathcal{F}_\chi$ on $\mathrm{Bun_n}$ as follow:

- b_χ is the unique element in $B(\mathrm{GL}_n)$ such that its corresponding vector bundle is isormorphic to $\mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_r)^{m_r}$ where $\mathcal{O}(\lambda_i)$ is the stable vector bundle of slope $\lambda_i = d_i/n_i$ and $m_i = \gcd(d_i, n_i)$.
- Consider the group G_{b_χ}, it is an inner form of a standard Levi subgroups of GL_n. Let $\mathrm{G}^*_{b_\chi}$ be the split inner form of G_{b_χ}. For each i, let $\mathrm{G}_i := \mathrm{GL}_{m_i}(D_{-\lambda_i})$ where $D_{-\lambda_i}$ is the division algebra whose invariant is $-\lambda_i$, thus $\mathrm{G}^*_i = \mathrm{GL}_{n_i}$. We have a map $\phi_i : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}^*_i}(\overline{\mathbb{Q}}_\ell)$ and the direct sum $\phi_1 \oplus \ldots \oplus \phi_r$ gives us a map $W_{\mathbb{Q}_p} \longrightarrow \prod_{i=1}^{r} \widehat{\mathrm{G}^*_i}(\overline{\mathbb{Q}}_\ell)$ whose post-composition with the natural embeddings $\prod_{i=1}^{r} \widehat{\mathrm{G}^*_i}(\overline{\mathbb{Q}}_\ell) \hookrightarrow \widehat{\mathrm{G}^*_{b_\chi}}(\overline{\mathbb{Q}}_\ell)$ defines an L-parameter ϕ_χ of G_{b_χ}. Moreover, the post-composition of ϕ_χ with $\widehat{\mathrm{G}^*_{b_\chi}}(\overline{\mathbb{Q}}_\ell) \hookrightarrow \widehat{\mathrm{GL}}_n(\overline{\mathbb{Q}}_\ell)$ is the ($\widehat{\mathrm{GL}}_n$-conjugacy class of) ϕ. Finally, π_χ is the representation of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ whose L-parameter is given by ϕ_χ via the local Langlands correspondence for general linear groups.
- We can suppose that $\mathrm{G}^*_{b_\chi}$ is standard. By (3), there exists a unique standard parabolic subgroup P of GL_n with Levi factor given by $\mathrm{G}^*_{b_\chi}$ such that $\nu_{b_\chi} \in X_*(\mathrm{P})^+$. Let δ_{P} be the modulus character with respect to P and denote $\delta_{\mathrm{P}|\mathrm{G}^*_{b_\chi}}$ its restriction to $\mathrm{G}^*_{b_\chi}$. Then we denote by δ_{b_χ} the character of G_{b_χ} whose L-parameter is the same as that of $\delta_{\mathrm{P}|\mathrm{G}^*_{b_\chi}}$. We consider the embedding $i_{b_\chi} : \mathrm{Bun}_n^{b_\chi} \longrightarrow \mathrm{Bun_n}$ and define $\mathcal{F}_\chi := i_{b_\chi !}(\delta_{b_\chi}^{-1/2} \otimes \pi_\chi)[-d_\chi]$ where $d_\chi = \langle 2\rho, \nu_{b_\chi} \rangle$ and ρ is the half sum of all positive roots of GL_n. In particular, if we denote by Id the identity element in $\mathrm{Irr}(S_\phi)$ then $\mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\pi$.

For avoiding confusion, we recall that if $\mathrm{P} = \mathrm{MN}$ where M is the Levi subgroup and N is the unipotent radical then the modulus character is defined by $\delta_{\mathrm{P}}(mn) := |\det(\mathrm{ad}(m); \mathrm{LieN})|$ where $|\cdot|$ denotes the normalized absolute value of the field $\overline{\mathbb{Q}}_\ell$.

Remark 1.1. Our characters δ_b is the same as the character δ_b defined by Hamann and Imai in [HI23]. We recall that for $b \in B(\mathrm{GL}_n)$, we have $\nu_b = (-\nu_{\mathcal{E}_b})_{\mathrm{dom}}$ then the parabolic that preserves the Harder-Narasimhan filtration of $\mathcal{E}_{b_\chi}$ is the opposite parabolic subgroup of P above.

The following result is theorem 4.5.

Theorem 1.2. *Let ϕ be an L-parameter satisfying the above conditions and let $\chi = (d_1, \dots, d_r)$ be an element in $\mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^{r} \mathbb{Z}$. Then we have*

$$C_\chi \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_\chi$$

where Id *is the identity element in* $\mathrm{Irr}(S_\phi)$.

Remark 1.3. (1) Let G be an inner form of a Levi subgroup of GL_n. Then $\mathrm{G}(\mathbb{Q}_p)$ has an irreducible representation with L-parameter given by ϕ if and only if $\mathrm{G}(\mathbb{Q}_p) \simeq \mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ for some $\chi \in \mathrm{Irr}(S_\phi)$. Therefore the vector bundles C_χ realize all the possible transfers of π to the inner forms of the Levi subgroups of GL_n by acting on $\mathcal{F}_\pi \simeq \mathcal{F}_{\mathrm{Id}}$. It could be seen as geometric local Jacquet-Langlands and Langlands-Jacquet correspondences. Thus the categorical local Langlands program contains rich information about the classical Langlands correspondences as expected.
(2) The modulus character $\delta_{b_\chi}^{-1/2}$ in the above formula is closely related to the twist appeared in [FS21, Corollary IX.7.3]. More precisely, there are several Fargues-Scholze parameters associated to the $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$-representation π_χ : the one constructed by using excursion operators on Bun_n and the one constructed by using $\mathrm{Bun}_{\mathrm{G}_{b_\chi}}$. These parameters differ by a twist as explained in [FS21, Corollary IX.7.3]. Thus, in fact $\delta_{b_\chi}^{-1/2} \otimes \pi_\chi$ has the same Fargues-Scholze parameter (with respect to Bun_n) with π_{Id} and it is in line with [Ham21, Prop. 3.14].
(3) The case $r = 1$ is treated in [ALB21, theorem 1.2], [Han20, theorem 1.5]. The above theorem could also be seen as non-supercuspidal analogue of some of the results in [BMHN22, §4].

Let us describe roughly the strategy of the proof. It is enough to prove the theorem for $\chi = (d_1, \dots, d_r)$ where the d_i's are non-negative. We denote by $\mathrm{Irr}(S_\phi)^+$ the set of all such characters and moreover denote by $\mu = (1, 0^{(n-1)})$ the simplest cocharacter of GL_n. Then we have a Hecke operator T_μ corresponding to μ. Remark that we can compute the action of C_{χ_i} by using the Hecke operator T_μ and the latter is closely related to the cohomology of Rapoport-Zink spaces [RZ96], or more generally to the cohomology of Scholze's local Shimura varieties $\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu)$ where b_1, b_2 are not necessarily basic [SW20].

Indeed, we recall that $\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu)$ denotes the moduli space of modifications of type μ from $\mathcal{E}_{b_1}$ to $\mathcal{E}_{b_2}$. As $S_\phi \times W_{\mathbb{Q}_p}$-representations we have

$$r_\mu \circ \phi = \bigoplus_{i=1}^{r} \chi_i \boxtimes \phi_i$$

where r_μ is the highest weight representation associated to μ. In this case it is the standard representation. Thus by [FS21, Coro. VIII.4.3, Theo. X.1.1], we deduce that for any irreducible sheaf $\mathcal{F}_{\pi'}$ on $\mathrm{Bun_n}$ whose (semi-simple) L-parameter is ϕ we have

$$\mathrm{T}_\mu(\mathcal{F}_{\pi'}) = \bigoplus_{i=1}^{r} C_{\chi_i} \star \mathcal{F}_{\pi'} \boxtimes \phi_i$$

as sheaves on $\mathrm{Bun_n}$ with $W_{\mathbb{Q}_p}$-action (see also proposition 3.9). Thus by [FS21, §IX.3] the description of $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu), \overline{\mathbb{Q}}_\ell)$ as $W_{\mathbb{Q}_p} \times \mathrm{G}_{b_1}(\mathbb{Q}_p) \times \mathrm{G}_{b_2}(\mathbb{Q}_p)$-modules allows us to compute the action of C_{χ_i} by identifying the $[\phi_i]$-isotypical parts (see also lemma 2.10).

We proceed by induction on r. First we can deduce the case $r = 1$ by [Far20] [Zou22], [ALB21, theorem 1.2], [Han20, theorem 1.5]. Suppose that the theorem is true for $1, \dots, r-1$, we show that it is true for r. Now we proceed by induction on $D = \sum_{i=1}^{r} d_i$. For $D = 1$, one could use known cases of the Harris-Viehmann's conjecture and [ALB21, theorem 1.2], [Han20, theorem 1.5] to compute $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu), \overline{\mathbb{Q}}_\ell)$ where $b_1 = 1$ and then get a description of $C_{\chi_i} \star \mathcal{F}_{\mathrm{Id}}$ by identifying the $W_{\mathbb{Q}_p}$-action on both sides.

In order to prove the induction step from $D = s-1$ to $D = s$, we will study the cohomology of the local Shimura varieties $\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu)$ for various b_1 and b_2 such that $\kappa(b_1) = 1 - s$ and $\kappa(b_2) = -s$. An important ingredient of this step is an analogue of Boyer's trick for $\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu)$ when b_1, b_2 are non-basic (proposition 5.1, corollary 5.2 and lemmas 5.5, 5.6), generalizing classical Boyer's trick [Boy99], [HT01], [Man08], [She14], [Han21], [Hon18], [GI19]. This result allows us to relate $\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu)$ with local Shimura varieties associated to GL_m where $m < n$ and we can then apply the induction hypothesis on r. Let us give more details on the strategy of this induction step.

Let $\chi = (d_1, \dots, d_r)$ be a character of $\mathrm{Irr}(S_\phi)^+$ such that $\sum_{i=1}^{r} d_i = s$. We would like to compute $C_\chi \star \mathcal{F}_{\mathrm{Id}}$. First we want to compute the restriction of the complex $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ to the stratum corresponding to b_χ. We can choose a character $\chi' = \chi \otimes \chi_i^{-1}$ for some well chosen i such that χ' belongs to $\mathrm{Irr}(S_\phi)^+$ and the Newton polygons of b_χ and $b_{\chi'}$ are sufficiently close so that we can use the analogue of Boyer's trick to the triple $(b_{\chi'}, b_\chi, \mu)$. In this case, by monoidal property of spectral action we have

$$i^*_{b_\chi} C_\chi \star \mathcal{F}_{\mathrm{Id}} = i^*_{b_\chi} C_{\chi_i} \star (C_{\chi'} \star \mathcal{F}_{\mathrm{Id}}).$$

However, by the induction hypothesis on $D = s-1$, we see that $C_{\chi'} \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_{\chi'}$. Since the complex $\mathcal{F}_{\chi'}$ is supported only on the stratum corresponding to $b_{\chi'}$, we can use $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b_\chi, \mu))[\pi_{\chi'}]$ to compute $i^*_{b_\chi} C_{\chi_i} \star \mathcal{F}_{\chi'}$ as above (by identifying the appropriate $W_{\mathbb{Q}_p}$-action). The analogue of Boyer's trick (proposition 5.1, lemma 5.5) applying to $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b_\chi, \mu))[\pi_{\chi'}]$ allows us to use the induction hypothesis on $t < r$ to compute $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b_\chi, \mu))[\pi_{\chi'}]$ and deduce that $i^*_{b_\chi} C_\chi \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_\chi$.

We remark that more generally for an arbitrary b, we can use $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b, \mu))[\pi_{\chi'}]$ to compute the restriction $i^*_b C_{\chi_i} \star \mathcal{F}_{\chi'}$ to any stratum corresponding to b. In particular, we deduce that $C_\chi \star \mathcal{F}_{\mathrm{Id}} = C_{\chi_i} \star \mathcal{F}_{\chi'}$ is only supported on the strata b such that there exists a modification of type μ from $\mathcal{E}_{b_{\chi'}}$ to $\mathcal{E}_b$.

The next step is to show that the restrictions of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ to the strata different from b_χ vanish. It is the most technical part of the manuscript. We know that the irreducible constituents appearing in the cohomology of $i^*_b C_\chi \star \mathcal{F}_{\mathrm{Id}}$ are irreducible representations of $\mathrm{G}_b(\mathbb{Q}_p)$ whose (semi-simplified) L-parameters are given by ϕ after post-composing with the natural L-embedding $\widehat{\mathrm{G}_b^*} \hookrightarrow \widehat{\mathrm{GL}}_n$. By proposition 4.3, we deduce that if $i^*_b C_\chi \star \mathcal{F}_{\mathrm{Id}}$ does not vanish then b is of the form b_ξ for some $\xi \in \mathrm{Irr}(S_\phi)$. It is a generalization of the more familiar fact that the support of an irreducible sheaf $\mathcal{F}$ on $\mathrm{Bun_n}$ with cuspidal L-parameter is contained in the semi-stable locus. Some simpler forms of this property were already used in [CS17, CS19, Kos21] to study cohomology of Shimura varieties with torsion coefficients. This property relies ultimately on the classification of irreducible representations by L-parameters and the compatibility up to semi-simplification between Fargues-Scholze L-parameters and the usual L-parameters for inner forms of $\mathrm{GL}_n(\mathbb{Q}_p)$ [FS21, IX.7.3] [HKW21, Theo. 6.6.1].

Thus we need to describe the elements of the form b_ξ that can be obtained by some modifications of type μ of $b_{\chi'}$ and then study the cohomology of the associated moduli spaces $\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b_\xi, \mu)$. Remark that in general, for a non semi-stable vector bundle $\mathcal{E}$ of rank n, the set of elements b in $B(\mathrm{GL}_n)$ such that there exists a modification $\mathcal{E} \longrightarrow \mathcal{E}_b$ of type μ is unknown. In order to get some information on the b_ξ's that could appear, we notice that the modification of type μ

$$f : \mathcal{E}_{b_{\chi'}|X\setminus\infty} \longrightarrow \mathcal{E}_{b_\xi|X\setminus\infty}$$

induces a bijection from the set of P-reductions of $\mathcal{E}_{b_{\chi'}}$ to the set of P-reductions of $\mathcal{E}_{b_\xi}$ where P is a standard parabolic subgroup of GL_n (where X denotes the Fargues-Fontaine curve associated to the algebraically closed perfectoid field $C^\flat$ and ∞ is some Cartier divisor associated to some untilt C of $C^\flat$). Therefore, by choosing an appropriate P-reduction $\mathcal{E}^{\mathrm{P}}_{b_{\chi'}}$ of $\mathcal{E}_{b_{\chi'}}$ corresponding to a P-reduction of $b_{\chi'}$, we get a P-reduction $\mathcal{E}^{\mathrm{P}}_{b_\xi}$ of $\mathcal{E}_{b_\xi}$. In our case, a P-reduction is no other than a filtration of vector bundle. Then by the theory of extensions and of Harder-Narasimhan filtration of vector bundles on the Fargues-Fontaine curve, we see that the information on $\mathcal{E}^{\mathrm{P}}_{b_\xi}$ yields some controls on b_ξ. One important fact is that we can use [CFS21, lemma 2.6] and [Vie, lemma 3.11] to compute $\mathcal{E}^{\mathrm{P}}_{b_\xi}$. More precisely, the modification f above is encoded by an element x in the Schubert cell $\mathrm{Gr}^{\mu}_n(C)$ associated to μ inside the B^+_{dR}-Grassmannian $\mathrm{Gr}_n(C)$. If M denotes the Levi factor of P then the restriction to M of $\mathcal{E}^{\mathrm{P}}_{b_\xi}$ is the image of a modification f^{M} of the restriction to M of $\mathcal{E}^{\mathrm{P}}_{b_{\chi'}}$. That modification f^{M} is encoded by the element $y := \mathrm{Pr}_{\mathrm{M}}(x)$ in $\mathrm{Gr}_{\mathrm{M}}(C)$ by using the Iwasawa decomposition. In particular, the type of the modification f^{M} is closely related to μ and can be computed explicitly. Since $\mu = (1, 0^{(n-1)})$, the above information is traceable. We study the above process case by case depending on the combinatoric description of $\mathcal{E}^{\mathrm{P}}_{b_{\chi'}}$.

The upshot is that if $b_\xi \neq b_\chi$ then we can show that in all cases b_ξ and b_χ are close enough so that we can apply the analogue of Boyer's trick to reduce the computation of $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b_\xi, \mu))[\pi_{\chi'}]$ to some computations involving only local Shimura varieties associated to GL_m with $m < n$ and then use the induction hypothesis. We emphasis that we do not need to compute the whole cohomology $R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_n, b_{\chi'}, b_\xi, \mu))[\pi_{\chi'}]$; we only need to compute its $[\phi_i]$-isotypical part since we only need to determine the action of C_{χ_i}. This makes the computation process simpler since $C_{\chi_i} \star (C_{\chi_i^{-1}} \star (-)) = C_{\mathrm{Id}} \star (-)$ is the identity functor of $\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$. More precisely, it allows us to use the topological properties of Bun_n via the semi-orthogonal decomposition of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ into the $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}^b_n, \overline{\mathbb{Q}}_\ell)$'s where b varies in $B(\mathrm{GL}_n)$ (lemma 2.6) and to use various adjunction formulas.

The above theorem could lead to some interesting applications. First of all, since the irreducible representations of S_ϕ are all of dimension 1, we deduce the following description of the regular representation $V_{\mathrm{reg}} = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi$. Thus the sheaf $\mathcal{G}_\phi := \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \mathcal{F}_\chi$ is non trivial and by using the above theorem, we can compute $\mathrm{T}_V(\mathcal{G}_\phi)$ for every algebraic representation V of GL_n and then prove the following result.

Theorem 1.4. *The sheaf $\mathcal{G}_\phi$ is a non trivial Hecke eigensheaf corresponding to the L-parameter ϕ.*

1.1.2. *Cohomology of local Shimura varieties and Harris-Viehmann's conjecture.*

As we know, the theory of Rapoport-Zink spaces and local Shimura varieties play a crucial role in the Langlands program. There are two important conjectures in the study of the cohomology of these spaces.

(1) The Kottwitz conjecture which concerns the discrete part of the cohomology of the basic local Shimura varieties.
(2) The Harris-Viehmann conjecture which gives an inductive formula of the cohomology of the non-basic local Shimura varieties.

By combining these two conjectures, one should be able to *describe* the cohomology of these spaces.

The Kottwitz conjecture was proven in many cases. The Lubin-Tate case was proven by [Boy99, HT01], the case of basic EL unramified spaces was proven in [Shi12, Far04], the case of basic PEL odd-unitary groups was treated in [Ngu19, BMN21] and the case of basic PEL type associated with GSp_4 was proven in [Ham21]. However, the Harris-Viehmann conjecture was only known in some cases involving the Hodge-Newton reducible conditions and classical Boyer's trick.

In [BMHN22, §4.1.2] we prove the Kottwitz conjecture for non-minuscule cocharacters for unitary similitude groups by exploiting the explicit description of the spectral action on the supercuspidal L-packets. Now we know how the Hecke operatos act on the representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ whose L-parameter satisfies our starting hypothesis. Thus by the same argument, we can *compute* some parts of the cohomology of non-basic local Shtukas spaces and then deduce new cases of the Harris-Viehmann's conjecture for GL_n. Let $b \in B(\mathrm{GL}_n)$ and suppose that $\mathcal{E}_b \simeq \mathcal{E}(\lambda_1) \oplus \ldots \oplus \mathcal{E}(\lambda_k)$ for $\lambda_1 > \ldots > \lambda_k$ and where $\mathcal{E}(\lambda_j)$ is a semi-stable vector bundle of slope λ_j. Let π_b be a representation of $\mathrm{G}_b(\mathbb{Q}_p)$ with the corresponding L-parameter ϕ^b. We denote by M the standard Levi subgroup of GL_n that is the split inner form of G_b and denote by P the standard parabolic subgroup of GL_n with Levi factor M. Then $\mathrm{M} = \mathrm{GL}_{m_1} \times \ldots \times \mathrm{GL}_{m_k}$ where $m_j := \operatorname{rank} \mathcal{E}(\lambda_j)$ and the natural embedding ${}^L\mathrm{G}_b(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ induces a morphism of L-groups $\eta : {}^L\mathrm{M}(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. We define an L-parameter of GL_n by $\phi := \eta \circ \phi^b$ and suppose that ϕ satisfies the conditions in theorem 4.5.

Let μ be an arbitrary cocharacter of GL_n and suppose that as $S_\phi \times W_{\mathbb{Q}_p}$-modules we have

$$r_\mu \circ \phi = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi \boxtimes \sigma_\chi.$$

Then we have (up to some shifts and twists)

$$\mathrm{T}_\mu(i_{b!}\pi_b) \simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} C_\chi \star i_{b!}\pi_b \boxtimes \sigma_\chi.$$

Therefore theorem 4.5 allows us to describe the Hecke action $\mathrm{T}_\mu(i_{b!}\pi_b)$ and compute the cohomology of moduli of local Shtukas. The Levi subgroup M is a product of general linear groups then the same arguments allow us to compute the cohomology of moduli of local Shtukas associated to M. Therefore, the (generalized) Harris-Viehmann conjecture follows from this description of the cohomologies. For simplicity, let us state the result up to some shifts and twists and only in the minuscule case where $\mu = (1^{(a)}, 0^{(n-a)})$ for some integer $1 \leq a \leq n$.

Theorem 1.5. *With the above notations, if μ is minuscule then we have the following identity of $\mathrm{GL}_n(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-representations*

$$R\Gamma_c(\mathrm{GL}_n, b, \mu)[\pi_b] \simeq \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n} R\Gamma_c(\mathrm{M}, b_{\mathrm{M}}, \mu_{\mathrm{M}})[\pi_b].$$

where b_{M} is the reduction of b to M *and $\mu_{\mathrm{M}} = \mu_1 \times \ldots \times \mu_k$ and where $\mu_j = (1^{(\deg \mathcal{E}(\lambda_j))}, 0^{(\operatorname{rank} \mathcal{E}(\lambda_j) - \deg \mathcal{E}(\lambda_j))})$.*

1.2. **Spectral action.**

We now want to compute the action of a perfect complex $\mathbb{L} \in \mathrm{Perf}([C_\phi])$ on $\mathcal{F}_\pi$. The connected component $[C_\phi]$ containing ϕ gives rise to a direct summand

$$\mathrm{Perf}([C_\phi]) \hookrightarrow \mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]).$$

Therefore the spectral action gives rise to a corresponding direct summand

$$\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega.$$

We want to understand the structure of the category $\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$ together with the action of $\mathrm{Perf}([C_\phi])$. However we have

$$\mathrm{Perf}([C_\phi]) \simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \mathrm{Perf}(\mathcal{O}(\mathbb{G}^r_m)),$$

where $\mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^{r} \mathbb{Z}$ is the set of characters of S_ϕ. From the theory of Bernstein decomposition, we know that the category of $\mathcal{O}(\mathbb{G}_m^r)$-modules is equivalent to the Bernstein block containing a representation π of $\mathrm{G}(\mathbb{Q}_p)$ whose L-parameter is given by ϕ where G is any inner form of a Levi sub-group of GL_n. Thus $\mathrm{Perf}(\mathcal{O}(\mathbb{G}_m^r))$ is equivalent to the full sub-category of compact objects of the derived category of these Bernstein blocks. Therefore, it is expected that $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$ is the direct sum of all of these categories, indexed by the elements in $\mathrm{Irr}(S_\phi)$.

For each $\chi \in \mathrm{Irr}(S_\phi)$, we define a full subcategory $\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega$ of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$ as follow. Let $\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))$ be the derived category of the Bernstein block of the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{G}_{b_\chi}(\mathbb{Q}_p))$ containing the irreducible representation π_χ. We denote by $\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega$ its full sub-category of compact objects. Denote by $\mathcal{W}_{\mathfrak{s}_\phi}$ the projection of $\mathcal{W}_\psi$ on $\mathrm{Rep}(\mathfrak{s}_\phi(\mathrm{Id})))^\omega$. The following results are theorems 9.3, 9.5 and 9.7.

Theorem 1.6. *Let* $\mathbb{L}$ *be a perfect complex in* $\mathrm{Perf}([C_\phi])$*. Then*

(1) *We can compute explicitly* $\mathbb{L} \star \mathcal{W}_{\mathfrak{s}_\phi}$*,*
(2) *We have a decomposition of categories*

$$\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega \simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\chi)} \mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega.$$

Let us describe briefly the strategy of the proof. The first step is to understand $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$ and we proceed by induction on r as in the proof of theorem 4.5. We deduce the case $r = 1$ by using [ALB21, theorem 1.2], [Han20, theorem 1.5] and Bernstein's result on the equivalence between $\mathrm{Rep}(\mathfrak{s}_\phi(\mathrm{Id})))$ and the category of $\overline{\mathbb{Q}}_\ell[X, X^{-1}]$-modules. Then we argue as in the proof of theorem 4.5 to deduce the general case. Note that it is less complicated since we can use theorem 4.5 to simplify the arguments.

By the construction of the spectral action, we know that if we have morphisms, cones or retracts of perfect complexes in $\mathrm{Perf}([C_\phi])$, then by acting on an object $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$, it induces corresponding morphisms, cones or retracts in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$. Moreover, the category $\mathrm{Perf}([C_\phi])$ is generated under cones and retracts by $\mathrm{Rep}(S_\phi)$ then by tracing back the construction, we are able to compute $\mathbb{L} \star \mathcal{W}_{\mathfrak{s}_\phi}$. One important ingredient is the explicit description of the morphisms

$$\Psi_{\mathrm{GL}_n} : \mathcal{O}([C_\phi]) \longrightarrow \mathcal{Z}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))$$

between the factor $\mathcal{O}([C_\phi])$ of the spectral Bernstein center and the Bernstein center of the blocks $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$'s. This description relies ultimately on the compatibility between Fargues-Scholze L-parameters and the usual L-parameters of inner forms of GL_n.

In the final part, we would like to verify some cases of the local-global compatibility conjecture. The rough idea is that by using the comparison between the fibers of the Hodge-Tate period map with the Igusa varieties [CS17, Thm. 1.15], one reduces the local-global compatibility to the computation of the cohomology of the Igusa varieties. Then we can use Mantovan's formula and the description of the Hecke operators to understand Igusa varieties.

Remark 1.7. (1) In [Hel23, Zhu20], Eugen Hellmann and Xinwen Zhu stated some conjectures relating the derived category of smooth representations of a p-adic split reductive group with the derived category of (quasi-)coherent sheaves on a stack of L-parameters. Hellmann and also proved his conjectures for GL_2. However it is not clear how to incorporate the spectral actions into their conjectures.

(2) By studying geometric Eisenstein series [Ham22], Linus Hamann also obtained the description of Hecke eigensheaves associated with L-parameters that factor through some maximal torus of an arbitrary reductive group. A construction of Hecke eigensheaves associated with "generous" L-parameters and a proof of Harris-Viehmann's conjecture for an arbitrary reductive group as well as various foundational results would appeared in the forthcoming work of Hamann, Hansen and Scholze [HHS].

(3) Some forms of the main results in the case $r = 1$ were also obtained by Konrad Zou and Chenji-Fu, even for integral coefficients.

Organization of the paper.

In section 2, we recollect some well-known results on vector bundles over the Fargues-Fontaine curve and the stack of these bundles as well as the construction of Fargues-Scholze L-parameters. In section 3, we recall the definition of the stack of L-parameters and study some of its basic geometric properties. We also study perfect complexes on this stack and deduce some consequences on the Hecke operators. In section 4, we state theorem 4.5, our first main results. In section 5, we prove a generalization of Boyer's trick for moduli spaces of local Shtukas $\mathrm{Sht}(\mathrm{GL}_n, b_1, b_2, \mu)$ where we allow both b_1, b_2 to be non basic. Then we also deduce some computations that are important for the proof of theorem 4.5 that occupies the whole section 6. Then in the following sections, we give some applications of our theorem 4.5. In section 7, we construct Hecke eigensheaves associated to the L-parameter ϕ and in sections 8, we prove new cases of the Harris-Viehmann's conjecture for non-basic Rapoport-Zink spaces associated to GL_n. In section 9, we describe some parts of the map Ψ_{GL_n} between spectral Bernstein center and the usual Bernstein center constructed recently by Fargues and Scholze. Then we use all these results to compute the spectral action of $\mathrm{Perf}([C_\phi])$ on $\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$. In the final section 10, we compute parts of the cohomology of the Igusa varieties and deduce some weak form of the local-global compatibility.

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Notation

We use the following notation:

- $\breve{\mathbb{Q}}_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius σ.
- G is a connected reductive group over $\mathbb{Q}_p$. Let H be a quasi-split inner form of G and fix an inner twisting $\mathrm{G}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} \mathrm{H}_{\breve{\mathbb{Q}}_p}$.
- $A \subseteq T \subseteq B$ where A is a maximal split torus, $T = Z_H(A)$ is the centralizer of A in T and B is a Borel subgroup in H. Let U be its unipotent radical.
- $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is the absolute root datum of G with positive roots Φ^+ and simple roots Δ with respect to the choice of B.
- ρ is the half sum of the positive roots.
- Further, $(X^*(A), \Phi_0, X_*(A), \Phi_0^\vee)$ denotes the relative root datum, with positive roots Φ_0^+ and simple roots Δ_0.
- On $X_*(A)_{\mathbb{Q}}$ resp. $X_*(T)_{\mathbb{Q}}$ we consider the partial order given by $\nu \leq \nu'$ if $\nu' - \nu$ is a non-negative rational linear combination of positive resp. relative coroots.
- Let P be a parabolic subgroup of G, then $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$, resp. $\mathrm{ind}_{\mathrm{P}}^{\mathrm{G}}$, denotes the normalized, resp. un-normalized parabolic induction.
- Let $C|\overline{\mathbb{Q}}_p$ be an algebraically closed complete field. Let C° resp. $C^{\flat,\circ}$ be the subring of power-bounded elements of C resp. $C^\flat$ and let ξ be a generator of the kernel of the surjective map $W(C^{\flat,\circ}) \to C^\circ$. Let $B^+_{\mathrm{dR}} := B^+_{\mathrm{dR}}(C)$ be the ξ-adic completion of $W(C^{\flat,\circ})[1/p]$ and $B_{\mathrm{dR}} = B_{\mathrm{dR}}(C) = B^+_{\mathrm{dR}}[\xi^{-1}]$. Then $B^+_{\mathrm{dR}} \cong C[[\xi]]$ and $B_{\mathrm{dR}} \cong C((\xi))$.
- Let X be the schematic Fargues-Fontaine curve over $C^\flat$. The untilt C of $C^\flat$ corresponds to a point $\infty \in |X|$ with residue field C and $\widehat{\mathcal{O}}_{X,\infty} \cong B^+_{\mathrm{dR}}$.

- For $\lambda \in \mathbb{Q}$, we denote the stable vector bundle on X whose slope is λ by $\mathcal{O}(\lambda)$. If $\lambda = 0$, we simply write $\mathcal{O}$ for $\mathcal{O}(0)$.
- Let $B(\mathrm{G})$ be the set of $\mathrm{G}(\breve{\mathbb{Q}}_p)$-$\sigma$-conjugacy classes of elements of $\mathrm{G}(\breve{\mathbb{Q}}_p)$. For each elements $[b] \in B(\mathrm{G})$, we denote by G_b the σ-centralizer of b. By work of Kottwitz, elements $[b]$ are classified by their Kottwitz point $\kappa_{\mathrm{G}}(b) \in \pi_1(\mathrm{G})_\Gamma$ and their Newton point $\nu_b \in X_*(A)_{\mathbb{Q},\mathrm{dom}}$.
- For a G-bundle $\mathcal{E}$ on X let $\nu_{\mathcal{E}} \in X_*(A)_{\mathbb{Q},\mathrm{dom}}$ be the corresponding Newton polygon.

2. Generalities on Bun_n

2.1. **Vector bundles on the Fargues-Fontaine curve.** Let $\mathrm{Bun}(X)$ denote the category of vector bundles on the Fargues-Fontaine curve and recall from [FF18] that the curve X is complete in the sense that if $f \in k(X)$ is any nonzero rational function on X, then the divisor of f has degree zero. Thus if $\mathcal{E}$ is a rank n vector bundle on the curve X then we can define its degree by $\deg(\mathcal{E}) := \deg \Lambda^n \mathcal{E}$ and its slope by $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$.

Recall that a vector bundle $\mathcal{E}$ on X is stable if it has no proper, non-zero subbundles $\mathcal{F} \longrightarrow \mathcal{E}$ with $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$. We say that $\mathcal{E}$ is semi-stable if it has no proper, non-zero subbundles $\mathcal{F} \longrightarrow \mathcal{E}$ with $\mu(\mathcal{F}) > \mu(\mathcal{E})$.

Definition 2.1. *A Harder-Narasimhan (HN) filtration of a vector bundle $\mathcal{E}$ is a filtration of $\mathcal{E}$ by subbundles $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \mathcal{E}_m = \mathcal{E}$ such that the quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are semi-stable with strictly decreasing slopes $\mu_1 > \mu_2 > \ldots > \mu_m$. The Harder-Narasimhan (HN) polygon of $\mathcal{E}$ is the upper convex hull of the points $(\operatorname{rank} \mathcal{E}_i, \deg \mathcal{E}_i)$.*

We have the following results about vector bundles on the Fargues-Fontaine curve.

Theorem 2.2. *(Fargues-Fontaine, Kedlaya)* [FF18, Ked08] *Vector bundles on X satisfy the following properties:*

(1) *The set of isomorphism classes of rank n vector bundles over X is classified by the Kottwitz set $B(\mathrm{GL}_n)$.*
(2) *Every vector bundle $\mathcal{E}$ admits a canonical Harder-Narasimhan filtration.*
(3) *For every λ in $\mathbb{Q}$, there is a unique stable bundle of slope λ on X, which is denoted by $\mathcal{O}(\lambda)$. Writing $\lambda = r/s$ in lowest terms, then the bundle $\mathcal{O}(\lambda)$ has rank s and degree r.*
(4) *Any semistable bundle of slope λ is a finite direct sum $\mathcal{O}^d(\lambda)$, and tensor products of semi-stable bundles are semi-stable.*
(5) *For any $\lambda \in \mathbb{Q}$, we have*
$$H^0(\mathcal{O}^d(\lambda)) = 0 \quad \textit{if and only if} \quad \lambda < 0$$
and
$$H^1(\mathcal{O}^d(\lambda)) = 0 \quad \textit{if and only if} \quad \lambda \geq 0$$
In particular, any vector bundle $\mathcal{E}$ admits a splitting $\mathcal{E} = \bigoplus_i \mathcal{O}(\lambda_i)$ of its Harder-Narasimhan filtration.

Recall that X is the Fargues-Fontaine curve over $C^\flat$, which comes equipped with a point ∞ corresponding to C. By Beauville-Laszlo's gluing theorem [BL95] we have a bijective correspondence between vector bundles $\mathcal{E}$ on X and triples $(\mathcal{E}^e, \mathcal{E}_{B^+_{\mathrm{dR}}}, \iota)$ where $\mathcal{E}^e$ is a vector bundle over $X \backslash \{\infty\}$, where $\mathcal{E}_{B^+_{\mathrm{dR}}}$ is a vector bundle on $\operatorname{Spec}(B^+_{\mathrm{dR}})$ and where $\iota : \mathcal{E}^e \otimes_{B_e} B_{\mathrm{dR}} \to \mathcal{E}_{B^+_{\mathrm{dR}}} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$ is an isomorphism. Here, the triple corresponding to some $\mathcal{E}$ is given by the respective base changes of $\mathcal{E}$ together with the induced isomorphism.

The pullback of a vector bundle $\mathcal{E}_b$ via $\operatorname{Spec}(B^+_{\mathrm{dR}}) \longrightarrow X$ is trivial. Indeed, the inclusion $\mathbb{Q}_p \hookrightarrow C \hookrightarrow B^+_{\mathrm{dR}}$ extends to an embedding of an algebraic closure $\overline{\mathbb{Q}}_p$ into B^+_{dR}. By Lang's theorem there is a $g \in \mathrm{G}(\overline{\mathbb{Q}}_p)$ with $gb\sigma(g^{-1}) = 1$, which induces the desired trivialization. It is well-defined up to the action of $\mathrm{G}(\mathbb{Q}_p)$. The Beauville-Laszlo uniformization depends on the choice of such a trivialization. From now on we consider $\mathcal{E}_b$ together with a trivialization of $\mathcal{E}_{b,B^+_{\mathrm{dR}}}$, without explicitly mentioning it. If $b = 1$, we choose the natural trivialization. In all cases, the trivialization of $\mathcal{E}_{b,B_{\mathrm{dR}}^+}$ induces a trivialization of $\mathcal{E}_{b,B_{\mathrm{dR}}^+}\otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$ (i.e., an identification with ${B_{\mathrm{dR}}}^n$ where n is the rank) identifying $\mathcal{E}_{b,B_{\mathrm{dR}}^+}$ with the standard lattice $(B_{\mathrm{dR}}^+)^n$ in ${B_{\mathrm{dR}}}^n$.

In this context, a modification f from $\mathcal{E}_1 = (\mathcal{E}_1^e, \mathcal{E}_{1,B_{\mathrm{dR}}^+}, \iota_1)$ to $\mathcal{E}_2 = (\mathcal{E}_2^e, \mathcal{E}_{2,B_{\mathrm{dR}}^+}, \iota_2)$ is an isomorphism $f : \mathcal{E}_1^e \longrightarrow \mathcal{E}_2^e$. It induces an isomorphism

$$\overline{f} := \iota_2^{-1}\circ f\circ \iota_1^{-1} : \mathcal{E}_{1,B_{\mathrm{dR}}^+}\otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \longrightarrow \mathcal{E}_{2,B_{\mathrm{dR}}^+}\otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}.$$

The type of this modification is the relative position of $(B_{\mathrm{dR}}^+)^n$ with respect to $\overline{f}\Big((B_{\mathrm{dR}}^+)^n\Big)$. In particular, if the type is given by the tuple $(k_1,\dots,k_n)$ where $k_1\geq \dots k_n\geq 0$ then $\overline{f}\Big((B_{\mathrm{dR}}^+)^n\Big)\subset (B_{\mathrm{dR}}^+)^n$. Hence the couple $(f, \overline{f}_{|(B_{\mathrm{dR}}^+)^n})$ gives us an injective map from $\mathcal{E}_1$ to $\mathcal{E}_2$.

For each $x\in \mathrm{Gr}_n(C)$ one can construct a modification $\mathcal{E}_{b,x}$ of $\mathcal{E}_b$ as follows. Using the trivialization of $\mathcal{E}_{b,B_{\mathrm{dR}}^+}$, we can write the triple corresponding to $\mathcal{E}_b$ as $(\mathcal{E}_{b|X\setminus\infty}, \mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+}, \iota)$ where $\mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+}$ is the trivial bundle of rank n on $\mathrm{Spec}(B_{\mathrm{dR}}^+)$. Then $\mathcal{E}_{b,x}$ is given as the vector bundle corresponding to the triple $(\mathcal{E}_{b|X\setminus\infty}, \mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+}, \iota_x)$ where the isomorphism ι_x is given by the commutative diagram

$$\begin{tikzcd}
\mathcal{E}_b^e\otimes_{B_e} B_{\mathrm{dR}} \arrow[r, "\iota"] \arrow[d, "\iota_x"] & \mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+}\otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \\
\mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+}\otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}. \arrow[ur, "x"'] &
\end{tikzcd}$$

Here, $B_e = H^0(X\setminus\infty, \mathcal{O}_X)$ and the map x in the diagram is multiplication by a representative of x on ${B_{\mathrm{dR}}}^n$. The isomorphism class of the triple only depends on the lattice $x(\mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+})\subset {B_{\mathrm{dR}}}^n$ and is in particular independent of the choice of the representative.

Write $\Lambda_x := x(\mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+})$ and $\mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}} := \mathcal{E}^{n,\mathrm{tri}}_{B_{\mathrm{dR}}^+}\otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$. The type of the induced modification $f_x : \mathcal{E}_b\longrightarrow \mathcal{E}_{b,x}$ is the relative position of Λ_x with respect to $(B_{\mathrm{dR}}^+)^n$. By the Cartan's decomposition

$$\mathrm{Gr}_{\mathrm{GL}_n}(C) = \coprod_{\mu} \mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)\mu^{-1}(\xi)\mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)/\mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)$$

where the union is over all conjugacy classes of cocharacters of GL_n. We have the following decomposition of locally spatial diamonds ([SW20, 19.2])

$$\mathrm{Gr}_{\mathrm{GL}_n} = \coprod_{\mu\in X_*(T)_{\mathrm{dom}}} \mathrm{Gr}_{\mathrm{GL}_n,\mu}$$

The element x is type μ if and only if $x\in \mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)\mu^{-1}(\xi)\mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)/\mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)$. Recall that the Iwasawa's decomposition induces a decomposition

$$\mathrm{Gr}_{\mathrm{GL}_n}(C) = \coprod_{\lambda\in X_*(T)} \mathrm{U}(B_{\mathrm{dR}})(C)\lambda(\xi)\mathrm{GL}_n(B_{\mathrm{dR}}^+)(C)/\mathrm{GL}_n(B_{\mathrm{dR}}^+)(C),$$

and we can define the semi-infinite orbits S_λ associated to λ as in [Vie, Definition 2.7].

Let $\mathcal{E},\mathcal{E}'$ be vector bundles on X such that $\mathcal{E}'\simeq \mathcal{E}_x$ for some $x\in \mathrm{Gr}_{\mathrm{GL}_n}(C)$. Recall from [CFS21, lemma 2.4] that the isomorphism between $\mathcal{E}_{|X\setminus\infty}$ and $\mathcal{E}'_{|X\setminus\infty}$ induces for every parabolic subgroup P a bijection between

$$\{\text{reduction of } \mathcal{E} \text{ to } P\}\longrightarrow \{\text{reduction of } \mathcal{E}' \text{ to } P\}.$$

By [CFS21, lemma 2.6] and [Vie, lemma 3.10], we can compute explicit this bijection in certain cases by using the information from the intersections $S_\lambda\cap \mathrm{Gr}_{\mathrm{GL}_n,\mu}(C)$. Moreover, in some cases it also allows us to study modifications of vector bundles.

Example 2.3. Let $\mathcal{E}$ be the rank n vector bundle $\mathcal{O}(1/n')\oplus\mathcal{O}^{n-n'}$. Then there exists a modification

$$f:\mathcal{E}'\longrightarrow\mathcal{E}$$

of type $\mu=(1,0,\dots,0)$ if and only if $\mathcal{E}'\in S:=\{\mathcal{O}^n,\mathcal{O}(1/n')\oplus\mathcal{O}^m\oplus\mathcal{O}(-1/m')\mid n'+m+m'=n\}$.

It is clear that if $\mathcal{E}'\in S$ then there exists a modification of type μ from $\mathcal{E}'$ to $\mathcal{E}$. We are going to show the inverse inclusion. Indeed, we consider the canonical Harder-Narasimhan filtration $\mathcal{E}_1=\mathcal{O}(1/n')\subset\mathcal{E}$. Then it induces a filtration $\mathcal{E}'_1\subset\mathcal{E}'$. We want to compute $\mathcal{E}'_1$ and $\mathcal{E}'/\mathcal{E}'_1$. The modification f induces two modifications

$$f_1:\mathcal{E}'_1\longrightarrow\mathcal{E}_1\qquad f_2:\mathcal{E}'/\mathcal{E}'_1\longrightarrow\mathcal{O}^{n-m}$$

of type μ_1 and μ_2 respectively. By [CFS21, lemma 2.6] and [Vie, lemma 3.10], we can compute explicitly their types. There are only two cases.

Case 1: $\mu_1=(1,0^{(m-1)})$ and $\mu_2=(0^{(n-m)})$. We see that $\mathcal{E}'/\mathcal{E}'_1\simeq\mathcal{O}^{n-m}$ and $\mathcal{O}^{n'}$ and thus $\mathcal{E}'\simeq\mathcal{O}^n$.

Case 2: $\mu_1=(0^{(m)})$ and $\mu_2=(1,0^{(n-m-1)})$. We see that $\mathcal{E}'_1\simeq\mathcal{O}(1/n')$ and $\mathcal{E}'/\mathcal{E}'_1\in\{\mathcal{O}^m\oplus\mathcal{O}(-1/m')\mid m+m'=n-n'\}$. Since $H^1(\mathcal{O}^d(\lambda))=0$ if $\lambda\geq 0$, we deduce that $\mathcal{E}'\simeq\mathcal{O}(1/n')\oplus\mathcal{O}^m\oplus\mathcal{O}(-1/m')$. Therefore $\mathcal{E}'$ belongs to the set S.

2.2. **Stack of vector bundles on the Fargues-Fontaine curve.** Let $\mathrm{G}/\mathbb{Q}_p$ be a connected reductive group. We let Perf denote the category of perfectoid spaces over $\overline{\mathbb{F}}_p$. We write $*:=\mathrm{Spd}(\overline{\mathbb{F}}_p)$ for the natural base. The key object of study is the moduli pre-stack $\mathrm{Bun}_{\mathrm{G}}$ sending $S\in\mathrm{Perf}$ to the groupoid of G-bundles on the relative Fargues–Fontaine curve X_S. The following theorem gives a geometric description of $\mathrm{Bun}_{\mathrm{G}}$.

Theorem 2.4. [FS21, Theorem III.0.2] *The pre-stack* $\mathrm{Bun}_{\mathrm{G}}$ *satisfies the following properties:*

(1) *The prestack* $\mathrm{Bun}_{\mathrm{G}}$ *is a small v-stack.*
(2) *The points* $|\mathrm{Bun}_{\mathrm{G}}|$ *are naturally in bijection with Kottwitz' set* $B(\mathrm{G})$ *of* G*-isocrystals.*
(3) *The map*
$$\nu:|\mathrm{Bun}_{\mathrm{G}}|\longrightarrow B(\mathrm{G})\longrightarrow (X_*(T)_{\mathbb{Q},\mathrm{dom}})^{\Gamma}$$
is semi-continuous and
$$\kappa:|\mathrm{Bun}_{\mathrm{G}}|\longrightarrow B(\mathrm{G})\longrightarrow \pi_1(\mathrm{G}_{\overline{\mathbb{Q}}_p})_{\Gamma}$$
is locally constant. Moreover, the map $|\mathrm{Bun}_{\mathrm{G}}|\longrightarrow B(\mathrm{G})$ *is a homeomorphism when* $B(\mathrm{G})$ *is equipped with the order topology* [Vie, Han17].
(4) *The semistable locus* $\mathrm{Bun}_{\mathrm{G}}^{ss}\subset\mathrm{Bun}_{\mathrm{G}}$ *is open, and given by*
$$\mathrm{Bun}_{\mathrm{G}}^{ss}\simeq\coprod_{b\in B(\mathrm{G})_{\mathrm{basic}}}[\bullet/\underline{\mathrm{G}_b(\mathbb{Q}_p)}].$$
(5) *For any* $b\in B(\mathrm{G})$*, the corresponding subfunctor*
$$i^b:\mathrm{Bun}_{\mathrm{G}}^b\subset\mathrm{Bun}_{\mathrm{G}}$$
is locally closed, and isomorphic to $[\bullet/\widetilde{\mathrm{G}}_b]$ *where* $\widetilde{\mathrm{G}}_b$ *is a v-sheaf of groups such that* $\widetilde{\mathrm{G}}_b\longrightarrow *$ *is representable in locally spatial diamonds with* $\pi_0\widetilde{\mathrm{G}}_b=\mathrm{G}_b(\mathbb{Q}_p)$*. The connected component* $\widetilde{\mathrm{G}}_b^0\subset\widetilde{\mathrm{G}}_b$ *of the identity is cohomologically smooth of dimension* $\langle 2\rho,\nu_b\rangle$*.*

For $\mathrm{G}=\mathrm{GL}_n$, we have $X_*(\mathrm{T})\simeq\mathbb{Z}^n$ and the target of the map ν is the set of nonincreasing sequences of rational numbers, which are the slopes of the Newton polygon of the corresponding isocrystal. Moreover, $\pi_1(\mathrm{G}_{\overline{\mathbb{Q}}_p})_{\Gamma}=X_*(T)/(\text{coroot lattice})$ is naturally isomorphic to $\mathbb{Z}$, and in this case $\kappa(b)$ is the endpoint of the Newton polygon. We can make the root data of GL_n explicit. The positive roots of GL_n (corresponding to the Borel subgroup given by the upper triangular matrices) are

$$\Phi^+=\{e_k-e_h\mid k,h\in\{1,2,\dots,n\},k<h\},$$

and the simple roots are

$$\Delta=\{e_i-e_{i+1}\mid i\in\{1,2,\dots,n-1\}\}.$$

Thus if $\nu_b = (\underbrace{\lambda_1, \dots, \lambda_1}_{m_1}, \dots, \underbrace{\lambda_r, \dots, \lambda_r}_{m_r})$ then $\langle 2\rho, \nu_b\rangle = \sum_{i<j} m_i m_j(\lambda_i - \lambda_j)$.

2.3. **Overview of the Fargues-Scholze local Langlands correspondence.** We recall that, for any Artin v-stack X, Fargues–Scholze define a triangulated category $\mathrm{D}_{\blacksquare}(X, \overline{\mathbb{Q}}_\ell)$ of solid $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Section VII.1] and isolate a nice full subcategory $\mathrm{D}_{\mathrm{lis}}(X, \overline{\mathbb{Q}}_\ell) \subset \mathrm{D}_{\blacksquare}(X, \overline{\mathbb{Q}}_\ell)$ of lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Section VII.6.]. We will be interested in the derived category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)$. The key point is that objects in this category are manifestly related to smooth admissible representations of $\mathrm{G}(\mathbb{Q}_p)$. The strata of $\mathrm{Bun}_{\mathrm{G}}$ admit a natural map $\mathrm{Bun}_{\mathrm{G}}^b \to [*/\mathrm{G}_b(\mathbb{Q}_p)]$ to the classifying stack defined by the $\mathbb{Q}_p$-points of the σ-centralizer G_b, and, by [FS21, Proposition VII.7.1], pullback along this map induces an equivalence

$$\mathrm{D}(\mathrm{G}_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{D}_{\mathrm{lis}}([*/\mathrm{G}_b(\mathbb{Q}_p)], \overline{\mathbb{Q}}_\ell) \xrightarrow{\simeq} \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}^b, \overline{\mathbb{Q}}_\ell),$$

where $\mathrm{D}(\mathrm{G}_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ is the unbounded derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{G}_b(\mathbb{Q}_p)$.

We denote by $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$ the stable ∞-category of compact objects of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_n}, \overline{\mathbb{Q}}_\ell)$. For any $b \in B(\mathrm{GL}_n)$, there is a local chart

$$\mathrm{Ch_b} : \mathcal{M}_{\mathrm{b}} \longrightarrow \mathrm{Bun_n}$$

that is representable in locally spatial diamonds, partially proper and cohomologically smooth [FS21, Theo. V.3.7]. The image of $\mathrm{Ch_b}$ is the open sub-stack $\mathrm{Bun}_n^{\leq b}$ consisting of the strata smaller than b with respect to the usual partial order in $B(\mathrm{GL}_n)$ [Han17, Theo. 1.1] [Vie, Theo. 6.7].

Remark 2.5. For $b, b' \in B(\mathrm{GL}_n)$, we have $\nu_b = (-\nu_{\mathcal{E}_b})_{\mathrm{dom}}$ and $\nu_{b'} = (-\nu_{\mathcal{E}_{b'}})_{\mathrm{dom}}$. Thus b is smaller than b' with respect to the usual partial order in $B(\mathrm{GL}_n)$ if and only if $\nu_{\mathcal{E}_b}$ is smaller than $\nu_{\mathcal{E}_{b'}}$ with respect to the usual partial order in $X_*(\mathrm{T})_{\mathbb{Q}}$ where T is a maximal split torus of GL_n.

Recall that via excision triangles, there is an infinite semi-orthogonal decomposition of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_n}, \overline{\mathbb{Q}}_\ell)$ into the various $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell)$ for $b \in B(\mathrm{GL}_n)$ [FS21, Chap. VII].

Lemma 2.6. *Let b be an element in $B(\mathrm{GL}_n)$ and let $\mathcal{F}, \mathcal{G}$ be in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_n}, \overline{\mathbb{Q}}_\ell)$ such that $\mathcal{F}$ is supported on $\mathrm{Bun}_n^{\leq b}$ and the intersection of the support of $\mathcal{G}$ with $\mathrm{Bun}_n^{\leq b}$ is empty. Then there is no non-trivial map in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_n}, \overline{\mathbb{Q}}_\ell)$ from $\mathcal{F}$ to $\mathcal{G}$.*

Proof. The local chart $\mathrm{Ch_b} : \mathcal{M}_{\mathrm{b}} \longrightarrow \mathrm{Bun_n}$ is cohomologically smooth by [FS21, Theo. V.3.7] and moreover the proof of this theorem actually show that the map $\mathrm{Ch}'_{\mathrm{b}} : \mathcal{M}_{\mathrm{b}} \longrightarrow \mathrm{Bun}_{\mathrm{n}}^{\leq \mathrm{b}}$ is also cohomologically smooth. Therefore by [FS21, Def. IV.1.11], the open immersion $i : \mathrm{Bun}_n^{\leq b} \hookrightarrow \mathrm{Bun}_n$ is cohomologically smooth. Denote by $\mathcal{C}$ (resp. $\mathcal{C}'$) the category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_n}, \overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_n}^{\leq b}, \overline{\mathbb{Q}}_\ell)$) for short. We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathcal{C}}(i_! i^* \mathcal{F}, \mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{C}'}(i^* \mathcal{F}, i^! \mathcal{G}) \quad (i^! \text{ is the right adjoint of } i_!).\end{aligned}$$

However, since i is cohomologically smooth, the map $i^!$ is given by taking derived tensor product of the dualizing object with i^*. We see that $i^*\mathcal{G} \simeq 0$ because the intersection of the support of $\mathcal{G}$ with $\mathrm{Bun}_n^{\leq b}$ is empty. Hence $i^!\mathcal{G} \simeq 0$ and

$$\mathrm{Hom}_{\mathcal{C}'}(i^* \mathcal{F}, i^! \mathcal{G}) = 0.$$

□

For a smooth irreducible representation $\pi \in \Pi_{\overline{\mathbb{Q}}_\ell}(\mathrm{G}_b(\mathbb{Q}_p))$, we get an object $\rho \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}^b, \overline{\mathbb{Q}}_\ell) \subset \mathrm{D}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)$ by extension by zero along the locally closed embedding $i_b : \mathrm{Bun}_{\mathrm{G}}^b \hookrightarrow \mathrm{Bun}_{\mathrm{G}}$, and the Fargues–Scholze parameter comes from acting on this representation by endofunctors of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)$ called Hecke operators. To introduce this, for a finite index set I, we let $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L\mathrm{G}^I)$ denote the category of algebraic representations of I-copies of the Langlands dual group, and we let Div^I be the product of I-copies of the diamond

$\mathrm{Div}^1 = \mathrm{Spd}(\breve{\mathbb{Q}}_p)/\mathrm{Frob}^{\mathbb{Z}}$. The diamond Div^1 parametrizes, for $S \in \mathrm{Perf}$, characteristic 0 untilts of S, which in particular give rise to Cartier divisors in X_S. We then have the Hecke stack

$$\begin{tikzcd} & \mathrm{Hck} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}\times supp"] & \\ \mathrm{Bun}_{\mathrm{G}} & & \mathrm{Bun}_{\mathrm{G}} \times \mathrm{Div}^I \end{tikzcd}$$

defined as the functor parametrizing, for $S \in \mathrm{Perf}$ together with a map $S \to \mathrm{Div}^I$ corresponding to characteristic 0 untilts $S_i^{\sharp}$ defining Cartier divisors in X_S for $i \in I$, a pair of G-torsors $\mathcal{E}_1$, $\mathcal{E}_2$ together with an isomorphism

$$\beta : \mathcal{E}_1|_{X_S \setminus \bigcup_{i\in I} S_i^{\sharp}} \xrightarrow{\simeq} \mathcal{E}_2|_{X_S \setminus \bigcup_{i\in I} S_i^{\sharp}},$$

where $h^{\leftarrow}((\mathcal{E}_1, \mathcal{E}_2, i, (S_i^{\sharp})_{i\in I})) = \mathcal{E}_1$ and $h^{\rightarrow} \times supp((\mathcal{E}_1, \mathcal{E}_2, \beta, (S_i^{\sharp})_{i\in I})) = (\mathcal{E}_2, (S_i^{\sharp})_{i\in I})$. For each element $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L\mathrm{G}^I)$, the geometric Satake correspondence of Fargues–Scholze [FS21, Chapter VI] furnishes a solid $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{S}_W$ on Hck. This allows us to define Hecke operators.

Definition 2.7. *For each $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L\mathrm{G}^I)$, we define the Hecke operator*

$$\mathrm{T}_W : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\blacksquare}(\mathrm{Bun}_{\mathrm{G}} \times X^I, \overline{\mathbb{Q}}_\ell)$$

$$A \mapsto R(h^{\rightarrow} \times supp)_{\natural}(h^{\leftarrow *}(A) \otimes^{\mathbb{L}} \mathcal{S}_W),$$

where $\mathcal{S}_W$ is a solid $\overline{\mathbb{Q}}_\ell$-sheaf and the functor $R(h^{\rightarrow} \times supp)_{\natural}$ is the natural push-forward (i.e the left adjoint to the restriction functor in the category of solid $\overline{\mathbb{Q}}_\ell$-sheaves [FS21, Proposition VII.3.1]*).*

It follows by [FS21, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3] that, if E denotes the reflex field of W, this induces a functor

$$\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)^{BW^I_{\mathbb{Q}_p}}$$

which we will also denote by T_W. The Hecke operators are natural in I and W and compatible with exterior tensor products. For a finite set I, a representation $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L\mathrm{G}^I)$, maps $\alpha : \overline{\mathbb{Q}}_\ell \to \Delta^* W$ and $\beta : \Delta^* W \to \overline{\mathbb{Q}}_\ell$, and elements $(\gamma_i)_{i\in I} \in W^I_{\mathbb{Q}_p}$ for $i \in I$, one defines the excursion operator on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)$ to be the natural transformation of the identity functor given by the composition:

$$\mathrm{id} = \mathrm{T}_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\alpha} \mathrm{T}_{\Delta^* W} = \mathrm{T}_W \xrightarrow{(\gamma_i)_{i\in I}} \mathrm{T}_W = \mathrm{T}_{\Delta^* W} \xrightarrow{\beta} \mathrm{T}_{\overline{\mathbb{Q}}_\ell} = \mathrm{id}.$$

In particular, for all such data, we get an endomorphism of a smooth irreducible $\pi \in \mathrm{D}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}^1_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell) \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)$ which, by Schur's lemma will give us a scalar in $\overline{\mathbb{Q}}_\ell$. In other words, to the datum $(I, W, (\gamma_i)_{i\in I}, \alpha, \beta)$ we assign a scalar. The natural compatibilities between Hecke operators will give rise to natural relationships between these scalars. These scalars and the relations they satisfy can be used, via Lafforgue's reconstruction theorem [Laf18, Proposition 11.7], to construct a unique continuous semisimple map

$$\phi_\pi^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^L\mathrm{G}(\overline{\mathbb{Q}}_\ell),$$

which is the Fargues–Scholze parameter of π. It is characterized by the property that for all I, W, α, β and $(\gamma_i)_{i\in I} \in W^I_{\mathbb{Q}_p}$, the corresponding endomorphism of π defined above is given by multiplication by the scalar that results from the composite

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi_\pi(\gamma_i))_{i\in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell.$$

Fargues and Scholze show that their correspondence has various good properties which we will invoke throughout.

Theorem 2.8. [FS21, Theorem I.9.6] *The mapping defined above*

$$\pi \mapsto \phi_\pi^{\mathrm{FS}}$$

enjoys the following properties:

(1) *(Compatibility with Local Class Field Theory) If* $\mathrm{G} = T$ *is a torus, then* $\pi \mapsto \phi_\pi$ *is the usual local Langlands correspondence*
(2) *The correspondence is compatible with character twists, passage to contragredients, and central characters.*
(3) *(Compatibility with products) Given two irreducible representations* π_1 *and* π_2 *of two connected reductive groups* G_1 *and* G_2 *over* $\mathbb{Q}_p$*, respectively, we have*

$$\pi_1 \boxtimes \pi_2 \mapsto \phi_{\pi_1}^{\mathrm{FS}} \times \phi_{\pi_2}^{\mathrm{FS}}$$

under the Fargues–Scholze local Langlands correspondence for $\mathrm{G}_1 \times \mathrm{G}_2$.
(4) *(Compatibility with parabolic induction) Given a parabolic subgroup* $P \subset \mathrm{G}$ *with Levi factor* M *and a representation* π_M *of* M*, then the Weil parameter corresponding to any sub-quotient of* $I_P^{\mathrm{G}}(\pi_M)$ *the (normalized) parabolic induction is the composition*

$$W_{\mathbb{Q}_p} \xrightarrow{\phi_{\pi_M}^{\mathrm{FS}}} {}^L M(\overline{\mathbb{Q}}_\ell) \to^L \mathrm{G}(\overline{\mathbb{Q}}_\ell)$$

where the map ${}^L M(\overline{\mathbb{Q}}_\ell) \to {}^L \mathrm{G}(\overline{\mathbb{Q}}_\ell)$ *is the natural embedding.*
(5) *(Compatibility with Harris–Taylor/Henniart LLC) For* $\mathrm{G} = \mathrm{GL}_n$ *or an inner form of* G *the Weil parameter associated to* π *is the (semi-simplified) parameter* ϕ_π *associated to* π *by Harris–Taylor/Henniart.* [Han20]

Let G be a reductive group and let b, b' be elements in $B(\mathrm{G})$. Given a geometric dominant cocharacter μ of G with reflex field E, we call the quadruple (G, b, b', μ) a local shtuka datum. Attached to it, we define the shtuka space

$$\mathrm{Sht}(\mathrm{G}, b, b', \mu) \longrightarrow \mathrm{Spd}(\breve{E}),$$

as in [SW20] where $\breve{E} := \breve{\mathbb{Q}}_p E$, to be the space parametrizing modifications

$$\mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$$

of G-bundles on the Fargues–Fontaine curve X with type bounded by μ.

Remark 2.9. We note that our definition of $\mathrm{Sht}(\mathrm{G}, b, b', \mu)$ coincides with $\mathrm{Sht}(\mathrm{G}, b, b', -\mu)$ in the notation of [SW20], where $-\mu$ is the dominant inverse of μ. This convention limits the appearance of duals when studying the cohomology of these spaces.

This has commuting actions of $\mathrm{G}_b(\mathbb{Q}_p)$ and $\mathrm{G}_{b'}(\mathbb{Q}_p)$ coming from automorphisms of $\mathcal{E}_b$ and $\mathcal{E}_{b'}$, respectively. Moreover the equlities $bb^\sigma(b^{-1})^\sigma = b$ and $(b')^{-1}b'(b')^\sigma = (b')^\sigma$ induces the isomorphisms $t_b : \mathcal{E}_{b^\sigma} \simeq \mathcal{E}_b$ and $t_{b'} : \mathcal{E}_{b'} \simeq \mathcal{E}_{(b')^\sigma}$. Thus we have a Weil descent datum of $\mathrm{Sht}(\mathrm{G}, b, b', \mu)$ defined by

$$\begin{aligned} \mathrm{Sht}(\mathrm{G}, b, b', \mu) &\longrightarrow \mathrm{Sht}(\mathrm{G}, b^\sigma, (b')^\sigma, \mu) \\ f \quad &\longmapsto t_{b'} \circ f \circ t_b. \end{aligned}$$

We define the tower

$$\mathrm{Sht}(\mathrm{G}, b, b', \mu)_K := \mathrm{Sht}(\mathrm{G}, b, b', \mu)/\underline{K} \longrightarrow \mathrm{Spd}(\breve{E})$$

of locally spatial diamonds [SW20, Theorem 23.1.4] for varying open compact subgroups $K \subset \mathrm{G}_{b'}(\mathbb{Q}_p)$. We can define the cohomology $R\Gamma_c(\mathrm{Sht}(\mathrm{G}, b, b', \mu)_K, \mathcal{S}_\mu)$ as in [Ima19, section 3].

When b is trivial, we denote the moduli space $\mathrm{Sht}(\mathrm{G}, b, b', \mu)$ by $\mathrm{Sht}(\mathrm{G}, b', \mu)$. There is a natural map

$$\mathsf{p} : \mathrm{Sht}(\mathrm{G}, b, b', \mu)_\infty \longrightarrow \mathrm{Hck}$$

mapping to the locus of modifications with type bounded by μ. Attached to the geometric cocharacter μ, consider the highest weight representation $V_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L\mathrm{G})$. The associated $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{S}_\mu$ on Hck (by the geometric Satake isomorphism) considered above will be supported on this locus, and we abusively denote $\mathcal{S}_\mu$ for the pullback of this sheaf along p. Since p factors through the quotient of $\mathrm{Sht}(\mathrm{G}, b, b', \mu)_\infty$ by the simultaneous group action of $\mathrm{G}_b(\mathbb{Q}_p) \times \mathrm{G}_{b'}(\mathbb{Q}_p)$, this sheaf will be equivariant with respect to these actions. This allows us to define the complex

$$R\Gamma_c(\mathrm{G}, b, b', \mu) := \mathrm{colim}_{K \to \{1\}} R\Gamma_c(\mathrm{Sht}(\mathrm{G}, b, b', \mu)_{K, \mathbb{C}_p}, \mathcal{S}_\mu)$$

which will be a complex of smooth admissible $\mathrm{G}_b(\mathbb{Q}_p) \times \mathrm{G}_{b'}(\mathbb{Q}_p) \times W_{E_\mu}$-modules, where $\mathrm{Sht}(\mathrm{G}, b, b', \mu)_{K, \mathbb{C}_p}$ is the base change of $\mathrm{Sht}(\mathrm{G}, b, b', \mu)_K$ to $\mathbb{C}_p$. Remark that if μ is minuscule then $\mathcal{S}_\mu \simeq \overline{\mathbb{Q}}_\ell[d_\mu](\frac{d_\mu}{2})$ where $d_\mu = \langle 2\rho, \mu \rangle$. For $\pi_b \in \Pi_{\overline{\mathbb{Q}}_\ell}(\mathrm{G}_b)$, this allows us to define the following complexes

$$R\Gamma_c^\flat(\mathrm{G}, b, b', \mu)[\pi_b] := R\mathcal{H}om_{\mathrm{G}_b(\mathbb{Q}_p)}(R\Gamma_c(\mathrm{G}, b, b', \mu), \pi_b) \tag{1}$$

and

$$R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_b] := R\Gamma_c(\mathrm{G}, b, b', \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(\mathrm{G}_b)} \pi_b \tag{2}$$

where $\mathcal{H}(\mathrm{G}_b)$ is the smooth Hecke algebra. Analogously, for $\pi_{b'} \in \Pi_{\overline{\mathbb{Q}}_\ell}(\mathrm{G}_{b'})$, we can define $R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_{b'}]$ and $R\Gamma_c^\flat(\mathrm{G}, b, b', \mu)[\pi_{b'}]$. It follows by [FS21, Corollary I.7.3] and [FS21, Page 317] that these will be valued in smooth admissible representations of finite length.

Recall that for each $b \in B(\mathrm{GL}_n)$, we can define a character $\kappa_b : \mathrm{G}_b(\mathbb{Q}_p) \longrightarrow \overline{\mathbb{Q}}_\ell$ and we have

$$R\Gamma_c(\widetilde{\mathrm{G}}_b^0, \overline{\mathbb{Q}}_\ell) = \kappa_b[-2d_b],$$

as in [GI19], lemma 4.18 and as in [Ham22], page 91, before lemma 11.1. We now relate the above complexes to Hecke operators on Bun_G. In particular, we have the following result.

Lemma 2.10. [FS21, Section IX.3] *Given a local shtuka datum* (G, b, b', μ) *as above and* $\pi_{b'}$ *(resp.* π_b*) a finitely generated smooth representation of* $\mathrm{G}_{b'}(\mathbb{Q}_p)$ *(resp.* $\mathrm{G}_b(\mathbb{Q}_p)$*), we can consider the associated sheaves* $\pi_b \in \mathrm{D}(\mathrm{G}_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathrm{G}^b)$ *and* $\pi_{b'} \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathrm{G}^{b'}) \simeq \mathrm{D}(\mathrm{G}_{b'}(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ *on the HN-strata* $i_b : \mathrm{Bun}_\mathrm{G}^b \hookrightarrow \mathrm{Bun}_\mathrm{G}$ *and* $i_{b'} : \mathrm{Bun}_\mathrm{G}^{b'} \hookrightarrow \mathrm{Bun}_\mathrm{G}$. *There then exists isomorphisms*

$$R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_b \otimes \kappa_b^{-1}][2d_b] \simeq i_{b'}^* \mathrm{T}_\mu i_{b!}(\pi_b),$$

$$R\Gamma_c^\flat(\mathrm{G}, b, b', \mu)[\pi_b][-2d_{b'}] \simeq i_{b'}^! \mathrm{T}_\mu R i_{b*}(\pi_b) \otimes \kappa_b^{-1},$$

of complexes of $\mathrm{G}_{b'}(\mathbb{Q}_p) \times W_{E_\mu}$*-modules and isomorphisms*

$$R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_{b'} \otimes \kappa_{b'}^{-1}][2d_{b'}] \simeq i_b^* \mathrm{T}_{-\mu} i_{b'!}(\pi_{b'}),$$

$$R\Gamma_c^\flat(\mathrm{G}, b, b', \mu)[\pi_{b'}][-2d_b] \simeq i_b^! \mathrm{T}_{-\mu} R i_{b'*}(\pi_{b'}) \otimes \kappa_{b'}^{-1},$$

of complexes of $\mathrm{G}_b(\mathbb{Q}_p) \times W_{E_\mu}$*-modules, where* $-\mu$ *is the dominant cocharacter conjugate to the inverse of* μ *and where* $d_b = \langle 2\rho, \nu_b \rangle$ *and* $d_{b'} = \langle 2\rho, \nu_{b'} \rangle$.

Proof. We sketch the proof for the first 2 formulas, the other 2 are similar. We can argue as in [Far16], pages 38-40 to deduce that

$$R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_b \otimes \kappa_b^{-1}][2d_b] \simeq i_{b'}^* \mathrm{T}_\mu i_{b!}(\pi_b).$$

Note that the Hodge-Tate period map π_{HT} in [Far16] is a $\widetilde{\mathrm{G}}_b$-torsor over the Newton stratum corresponding to $[b']$. Thus we need to take into account the cohomology of the locally spatial diamond $R\Gamma_c(\widetilde{\mathrm{G}}_b^0, \overline{\mathbb{Q}}_\ell) \simeq \kappa_b[-2d_b]$, hence the shift $[2d_b]$ and the character κ_b^{-1} appear in the formula.

By the arguments as in [FS21, section IX.3], we deduce that

$$R\Gamma_c^\flat(\mathrm{G}, b, b', \mu)[\pi_b][-2d_{b'}] \simeq i_{b'}^! \mathrm{T}_\mu R i_{b*}(\pi_b) \otimes \kappa_b^{-1}.$$

Note that the functor $i_{b'}^!$ appears since in general, it is the right adjoint to $i_{b'!}$ and if b' is basic then $i_{b'}^! = i_{b'}^*$.

If π_b is admissible then $\mathbb{D}(\mathbb{D}(\pi_b)) \simeq \pi_b$ and therefore we can pull through Verdier duality and using derived Hom-Tensor duality to prove the second formula from the first formula as in [FS21, section. IX.3, page 317]. In more details we have

$$\mathbb{D}_{\mathrm{Bun}_n^{b'}}\big(R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_b \otimes \kappa_b^{-1}][2d_b]\big) \simeq \mathbb{D}_{\mathrm{Bun}_n^{b'}}\big(i_{b'}^* \mathrm{T}_\mu i_{b!}(\pi_b)\big).$$

From [Ima19, Lemmas 3.6, 3.7], we see that $i_{b'}^{!}\circ\mathbb{D} = \mathbb{D}\circ i_{b'}^{*}$. Combining it with the computation of dualizing object of $\mathrm{Bun}_n^{b''}$ for an arbitrary stratum ([Ham22, §11.2]), we have

$$\begin{aligned}\big(R\Gamma_c(\mathrm{G}, b, b', \mu)[\pi_b \otimes \kappa_b^{-1}]\big)^{\vee}[-2d_b - 2d_{b'}] &\simeq i_{b'}^{!}\mathbb{D}\big(\mathrm{T}_{\mu} i_{b!}(\pi_b)\big)\\ &\simeq i_{b'}^{!}\mathrm{T}_{\mu}\mathbb{D}\big(i_{b!}(\pi_b)\big)\\ &\simeq i_{b'}^{!}\mathrm{T}_{\mu} R i_{b*}\mathbb{D}_{\mathrm{Bun}_n^b}(\pi_b)\\ &\simeq i_{b'}^{!}\mathrm{T}_{\mu} R i_{b*}(\pi_b^{\vee})[-2d_b]\end{aligned}$$

where $\vee$ denotes the contragradient representation. Now by applying the Hom-Tensor duality we get the second formula. $\square$

3. The stack of L-parameters

We fix a prime $\ell \neq p$. Let G be a reductive group over $\mathbb{Q}_p$ and then we get the dual group $\widehat{G}/\mathbb{Z}_\ell$, which we endow with its usual algebraic action of the Weil group $W_{\mathbb{Q}_p}$. We will be mainly interested in the case $\mathrm{G} = \mathrm{GL}_n$ in this section.

3.1. **The geometry of the stack of L-parameters.** In this paragraph we recall the definition of the stack of L-parameters and recollect some of its geometric properties such as characterization of some smooth points and connected components.

3.1.1. *Smoothness.* Let A be any $\mathbb{Z}_\ell$-algebra, regarded as a condensed $\mathbb{Z}_\ell$-algebra.

Definition 3.1. *An L-parameter for* G *with coefficients in A is a section*

$$\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{G}(A) \rtimes W_{\mathbb{Q}_p}$$

of the natural map of condensed groups $\widehat{G}(A) \rtimes W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p}$. Equivalently, an L-parameter for G *with coefficients in A is a condensed 1-cocycle $\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}}(A)$ for the given $W_{\mathbb{Q}_p}$ action. More concretely, if A is endowed with the topology given by writing $A = colim_{A' \subset A} A'$ where A' is finitely generated $\mathbb{Z}_\ell$-algebra with its ℓ-adic topology then an L-parameter with values in A is a 1-cocycle $\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}}(A)$ such that if $\widehat{\mathrm{G}} \hookrightarrow \mathrm{GL}_n$ the associated map $\phi : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_n(A)$ is continuous.*

Remark 3.2. In constrast to definition 4.1, we do not require that ϕ sends semi-simple elements to semi-simple elements in the above definition.

With this definition of L-parameter over any $\mathbb{Z}_\ell$-algebra A, we can define a moduli space, denoted by $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})_{\mathbb{Z}_\ell}$, over $\mathbb{Z}_\ell$, whose A-points are the continuous 1-cocycles $\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}}$ with respect to the natural action of $W_{\mathbb{Q}_p}$ on $\widehat{\mathrm{G}}(A)$. This defines the scheme considered in, [FS21, DHKM20, Zhu20].

Any condensed 1-cocycle $\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}}(A)$ is trivial on an open subgroup of the wild inertia subgroup $\mathrm{P}_{\mathbb{Q}_p}$; note also that $\mathrm{P}_{\mathbb{Q}_p}$ acts on $\widehat{\mathrm{G}}$ through a finite quotient. Thus by a discretization process, Fargues and Scholze show [FS21, Theorem I.9.1] that $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})_{\mathbb{Z}_\ell}$ can be written as a union of open and closed affine subschemes $Z^1(W_{\mathbb{Q}_p}/\mathrm{P}, \widehat{\mathrm{G}})_{\mathbb{Z}_\ell}$ as P runs through subgroups of the wild inertia $\mathrm{P}_{\mathbb{Q}_p}$ of $W_{\mathbb{Q}_p}$, where $Z^1(W_{\mathbb{Q}_p}/\mathrm{P}, \widehat{\mathrm{G}})_{\mathbb{Z}_\ell}$ parametrizes condensed 1-cocycles that are trivial on P. Each $Z^1(W_{\mathbb{Q}_p}/\mathrm{P}, \widehat{\mathrm{G}})_{\mathbb{Z}_\ell}$ is a flat local complete intersection over $\mathbb{Z}_\ell$ of dimension $\dim(\mathrm{G})$.

This allows us to consider the Artin stack quotient $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}]_{\mathbb{Z}_\ell}$, where $\widehat{\mathrm{G}}$ acts via conjugation. We then consider the base change to $\mathbb{Q}_\ell$, denoted by $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}]$ and referred to it as the stack of Langlands parameters, as well as the category $\mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}])$ of perfect complexes of coherent sheaves on this space.

To study the geometric properties of the stack of L-parameters, we also need to consider its coarse moduli quotient in the category of scheme

$$Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})//\widehat{\mathrm{G}}$$

of $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})$ by the action of $\widehat{\mathrm{G}}$ via conjugation. Concretely, for every connected component $\operatorname{Spec} R \subset Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})$, we get a corresponding connected component $\operatorname{Spec} R^{\widehat{\mathrm{G}}} \subset Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})//\widehat{\mathrm{G}}$. We recall the following definition of semi-simple L-parameter.

Definition 3.3. *Let K be an algebraically closed field over $\mathbb{Z}_\ell$. An L-parameter $\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}}(K) \rtimes W_{\mathbb{Q}_p}$ is semi-simple if whenever the image of ϕ is contained in a parabolic subgroup of $\widehat{\mathrm{G}} \rtimes W_{\mathbb{Q}_p}$ then it is contained in the Levi subgroup of this parabolic subgroup.*

Given an L-parameter $\phi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}} \rtimes W_{\mathbb{Q}_p}$, by [FS21, Proposition VIII.3.2] and [DHKM20, Proposition 4.13] the $\widehat{\mathrm{G}}$-orbit of ϕ defines a closed $\overline{\mathbb{Q}}_\ell$-point of $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})//\widehat{\mathrm{G}}$ if and only if ϕ is a semi-simple parameter. The natural map

$$\theta : [Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})/\widehat{\mathrm{G}}] \longrightarrow Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{G}})//\widehat{\mathrm{G}}$$

evaluated on a $\overline{\mathbb{Q}}_\ell$-point in the stack quotient defined by an L-parameter ϕ defines a $\overline{\mathbb{Q}}_\ell$-point in the coarse moduli space given by its semi-simplification ϕ^{ss}.

From now on we suppose $\mathrm{G} = \mathrm{GL}_n$. In particular the action of $W_{\mathbb{Q}_p}$ on $\widehat{\mathrm{G}}$ is trivial. Let $\phi : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an L-parameter such that ϕ sends semi-simple elements to semi-simple elements. In particular, ϕ is semi-simple and Frobenius semi-simple. Thus it defines a geometric point x in $Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)//\mathrm{GL}_n$ and a point $\overline{x}$ in the stack $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$. The geometric properties of $\theta^{-1}(x)$ will be essentially important since they reveal interesting information on the perfect complexes over $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$. Since ϕ is semi-simple, there is a decomposition $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ of ϕ into irreducible representations.

Proposition 3.4. *Let $|\cdot|_p$ be the norm character $W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^\times \longrightarrow \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Suppose that there do not exist $1 \leq i \neq j \leq r$ such that $\phi_i \simeq \phi_j$ or $\phi_i \simeq \phi_j \otimes |\cdot|_p$ then $\theta^{-1}(x)$ has only 1 geometric point $\overline{x}$. In particular, $\overline{x}$ is a closed point in $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$.*

Proof. We need to show that there is no other L-parameter $\varphi : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that its semi-simplification φ^{ss} is isomorphic to ϕ. Suppose that there exists such a φ. Recall that φ and ϕ_i are trivial on an open sub-group $\mathrm{P}'_{\mathbb{Q}_p}$ of the wild inertia subgroup $\mathrm{P}_{\mathbb{Q}_p}$. Let $W_{\mathbb{Q}_p} \longrightarrow W'_{\mathbb{Q}_p}$ be the quotient of $W_{\mathbb{Q}_p}$ by $\mathrm{P}'_{\mathbb{Q}_p}$. We look at the discrete dense subgroup W of $W'_{\mathbb{Q}_p}$ generated by $\mathrm{P}_{\mathbb{Q}_p}$, a choice of Frobenius σ and a choice of generator of the tame inertia τ. Thus we have an exact sequence

$$0 \longrightarrow \mathrm{I} \longrightarrow W \longrightarrow \sigma^{\mathbb{Z}} \longrightarrow 0$$

where I is some quotient of the inertia subgroup. We also have another exact sequence

$$0 \longrightarrow \mathrm{P} \longrightarrow \mathrm{I} \longrightarrow \tau^{\mathbb{Z}[\frac{1}{p}]} \longrightarrow 0$$

where P is a finite p-group. Moreover, in W/P, the elements τ and σ satisfy the commutation $\sigma^{-1}\tau\sigma = \tau^p$. In particular, the eigenvalues of τ are roots of unity of order $p-1$. By the proof of theorem $VIII.1.3$ in [FS21], the representations φ and ϕ_i are completely determined by its restrictions to W (since $W \subset W'_{\mathbb{Q}_p}$ is dense and $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is quasi-separated).

Since P is a finite group, the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}\mathrm{P}$ is semi-simple. Hence $\varphi_{|\mathrm{P}}$ is a sum of irreducible P-representations. Moreover $\varphi_{|\mathrm{P}} = \phi_{1|\mathrm{P}} \oplus \ldots \oplus \phi_{r|\mathrm{P}}$. Let us consider the Jordan normal form of elements $\varphi(\sigma)$ and $\varphi(\tau)$ in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Since $\overline{\mathbb{Q}}_\ell$ is algebraically closed, their Jordan blocs are of the form

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

for some eigenvalues λ.

Let V be the n-dimensional $\overline{\mathbb{Q}}_\ell$-vector space underlying the $W_{\mathbb{Q}_p}$-representation φ and let $V_1, \ldots, V_r$ the underlying $\overline{\mathbb{Q}}_\ell$-vector space of V corresponding to $\phi_1, \ldots, \phi_r$ respectively. Notice that the ϕ_i's send semi-simple elements to semi-simple elements. Thus the semi-simplification $\varphi^{ss} \simeq \phi$ is exactly the P-representation V together with the action of the semi-simple part τ^{ss} respectively σ^{ss} of τ respectively σ.

Since φ is not semi-simple, we can suppose that $\sigma(V_1 \oplus \ldots \oplus V_i) = V_1 \oplus \ldots \oplus V_i$ for all $1 \leq i \leq r$ and moreover that $\varphi(V_2) \not\subset V_2$. Suppose further that $\varphi(\tau)$ is not semi-simple and write its matrix in the base (V_1, V_2) as

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}.$$

By using the fact that τ acts on the finite p-group P by $g \longmapsto \tau g \tau^{-1}$, we deduce that Y induces an morphism from V_2 to V_1 as P-representations.

By the definition of taking semi-simplification, we deduce that the semi-simplification of $\varphi(\tau)$ is $\mathrm{diag}(X, Z)$ and then $\varphi^{ss}(\tau)_{|V_2}$ has a basic consisting of eigenvectors of $\varphi(\tau)$. Let $t \in V_2$ be an eigenvector of eigenvalue λ of $\phi_1(\tau)$, by direct computation and by using the form of the Jordan normal blocs of $\varphi(\tau)$, we deduce that Yt is either zero or an eigenvector of eigenvalue λ of $\phi_2(\tau)$. Hence Y induces a morphism of I-representations from V_2 to V_1. We denote by V_2' the co-image and by V_1' the image of Y. Thus Y is an isomorphism of I-representations from V_2' to V_1'. Finally, by using the equality $\sigma^{-1}\tau\sigma = \tau^p$ as well as the fact that τ^p is equal to $\mathrm{diag}(S_1, \ldots, S_m)$ where S_i is a matrix of the following form (note that $\lambda_i^p = \lambda_i$)

$$\begin{pmatrix} \lambda_i & p \\ 0 & \lambda_i \end{pmatrix},$$

we deduce that Y induces an isomorphism between V_1' and $V_2' \otimes |\cdot|_p$. In particular V_1' is a sub $W_{\mathbb{Q}_p}$-representation of V_1, hence $V_1' = V_1$. We deduce that Y induces an isomorphism between ϕ_1 and $\phi_2|\cdot|_p$. It is a contradiction.

If $\varphi(\tau)$ is semi-simple then $\varphi(\sigma)$ is not semi-simple, the same argument shows that ϕ_1 and ϕ_2 are isomorphic and it is again a contradiction. We conclude that $\theta^{-1}(x)$ has only 1 geometric point. □

The above proposition implies that $\theta^{-1}(x) \simeq [\bullet/S_\phi] \simeq [\bullet/\prod_{i=1}^{r}\mathbb{G}_m]$ and the immersion $[\bullet/S_\phi] \hookrightarrow [Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$ is closed. The following proposition tells us that the immersion is in fact closed and regular.

Proposition 3.5. *Let $|\cdot|_p$ be the norm character $W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^\times \longrightarrow \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Suppose that there do not exist $1 \leq i \neq j \leq r$ such that $\phi_i \simeq \phi_j$ or $\phi_i \simeq \phi_j \otimes |\cdot|_p$ then ϕ defines a closed and smooth geometric point of $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$.*

Proof. Denote by x the point determined by ϕ and denote by $\widehat{g}$ the Lie algebra of $\widehat{\mathrm{G}}$. Since ϕ is semi-simple, x is a closed point. Then, using the notation in [FS21, section VIII.2], we need to show that $x^*\mathrm{Sing}_{[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]_{\mathbb{Z}_\ell}} = H^0(W_{\mathbb{Q}_p}, \widehat{g}^* \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell(1))$ is trivial where the action of $W_{\mathbb{Q}_p}$ on $\widehat{g}^* \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell(1)$ is the (adjoint) action defining the L-group twisted by ϕ and the cyclotomic character. Let $v \in \widehat{g} \simeq \widehat{g}^*$. By [FS21, Proposition VIII.2.11], we know that v belongs to the nilpotent cone. However, our group is split then the action of $W_{\mathbb{Q}_p}$ acting on $\widehat{g}^* \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell(1)$ is determined by ϕ. In particular, $\phi(g)v\phi(g)^{-1} = v$ for all g in the inertia subgroup I of $W_{\mathbb{Q}_p}$ and $\phi(\sigma)v\phi(\sigma)^{-1} = p \cdot v$. Since there do not exist $1 \leq i \neq j \leq r$ such that $\phi_i \simeq \phi_j \otimes |\cdot|_p$, the latter commutative condition implies that v is a diagonal matrix. Therefore v is trivial. □

3.1.2. *Connected components.* We recall the description of the connected components of $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$, following [DHKM20]. We fix a lift σ of the arithmetic Frobenius to $W_{\mathbb{Q}_p}/\mathrm{P}_{\mathbb{Q}_p}$ and τ a topological generator of $\mathrm{I}_{\mathbb{Q}_p}/\mathrm{P}_{\mathbb{Q}_p}$. Let $(W/\mathrm{P}_{\mathbb{Q}_p})^0$ be the subgroup of $W_{\mathbb{Q}_p}/\mathrm{P}_{\mathbb{Q}_p}$ generated by τ, σ and $\mathrm{P}_{\mathbb{Q}_p}$, regarded as discrete group and let W^0 be the inverse image of $(W/\mathrm{P}_{\mathbb{Q}_p})^0$ in $W_{\mathbb{Q}_p}$. The functor that sends R to $Z^1(W^0_{\mathbb{Q}_p}, \mathrm{GL}_n(R))$ is representable by an affine scheme denoted by $Z^1(W^0_{\mathbb{Q}_p}, \mathrm{GL}_n)$. We study the connected components of the stack $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$ by a discretization process. Thus we choose a decreasing sequence $(\mathrm{P}^e)_{e\in\mathbb{N}}$ of open normal subgroups of the wild inertia $\mathrm{P}_{\mathbb{Q}_p}$ whose intersection is $\{1\}$. We know that $Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\mathbb{Z}_\ell}$ can be written as a union of open and closed affine sub-schemes $Z^1(W^0_{\mathbb{Q}_p}/\mathrm{P}^e, \mathrm{GL}_n)$ for $e \in \mathbb{N}$.

For each $e \in \mathbb{N}$, by a reduction to tame parameters argument [DHKM20, Theorem 3.1, equation 4.2], there exists a finite set Φ_e^{adm} consisting of 1-cocycle $\varphi : \mathrm{P}_{\mathbb{Q}_p}/\mathrm{P}^e \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that we have a decomposition

$$[Z^1(W^0_{\mathbb{Q}_p}/\mathrm{P}^e, \mathrm{GL}_n)/\mathrm{GL}_n] = \coprod_{\varphi\in\Phi_e^{\mathrm{adm}}} [Z^1(W^0_{\mathbb{Q}_p}, \mathrm{GL}_n)_\varphi/C(\varphi)]$$

where $C(\varphi)$ is the centralizer of φ and $Z^1(W^0_{\mathbb{Q}_p}, \mathrm{GL}_n)_\varphi$ is the (non empty) sub-scheme of $Z^1(W^0_{\mathbb{Q}_p}, \mathrm{GL}_n)$ parametrizing the 1-cocycles that extend φ. There is also a similar decomposition in the situation where we consider the categorical quotient.

We are now interested in the situation where $\varphi : \mathrm{P}_{\mathbb{Q}_p}/\mathrm{P}^e \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is multiplicity free. For each $e \in \mathbb{N}$, let I^e be the open sub-group of the inertia $\mathrm{I}_{\mathbb{Q}_p}$ as in [DHKM20, Corollary 4.16]. In particular, every semi-simple map $f : W^0/\mathrm{P}^e \longrightarrow \mathrm{GL}_n(L)$ is trivial on $\mathrm{I}^e/\mathrm{P}^e$ and then extends canonically to $W_{\mathbb{Q}_p}/\mathrm{P}^e$, where L is any algebraically closed field of characteristic 0 or $\ell \neq p$. With the group I^e, we can consider the GL_n-stable closed sub-scheme $Z^1(W_{\mathbb{Q}_p}/\mathrm{I}^e, \mathrm{GL}_n)$ of $Z^1(W^0_{\mathbb{Q}_p}/\mathrm{P}^e, \mathrm{GL}_n)$ parametrizing the maps that are trivial on I^e. Since φ is multiplicity free, the semi-simplifications of its extensions to $W_{\mathbb{Q}_p}$ satisfy conditions of proposition 3.4 and therefore all of its extensions are semi-simple. Hence the opened and closed sub-scheme $Z^1(W^0_{\mathbb{Q}_p}, \mathrm{GL}_n)_\varphi$ is an open and closed sub-scheme of $Z^1(W_{\mathbb{Q}_p}/\mathrm{I}^e, \mathrm{GL}_n)$. Thus, it is enough to study this scheme.

We only consider the stack of L-parameters defined over $\overline{\mathbb{Q}}_\ell$ for some prime $\ell \neq p$ or defined over $\overline{\mathbb{Z}}_\ell$ for some prime ℓ that is invertible in $\mathbb{Z}[\frac{1}{p\mathrm{N}_{\mathrm{GL}_n}}]$ where the constant $\mathrm{N}_{\mathrm{GL}_n}$ is as in [DHKM20, Proposition 6.4]. The advantage of this assumption is that there exists a finite set $\widetilde{\Phi}_e^{\mathrm{adm}}$ consisting of 1-cocycle $\widetilde{\varphi} : \mathrm{I}_{\mathbb{Q}_p}/\mathrm{I}^e \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that we have a decomposition similar to the above tamely discretization process

$$[Z^1(W_{\mathbb{Q}_p}/\mathrm{I}^e, \mathrm{GL}_n)/\mathrm{GL}_n] = \coprod_{\widetilde{\varphi}\in\widetilde{\Phi}_e^{\mathrm{adm}}} [Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi}}/C(\widetilde{\varphi})]$$

where $C(\widetilde{\varphi})$ is the centralizer of $\widetilde{\varphi}$ and $Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi}}$ is the (non empty) sub-scheme of $Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)$ parametrizing the 1-cocycles ϕ such that $\phi_{|\mathrm{I}_{\mathbb{Q}_p}} = \widetilde{\varphi}$.

Since all the extension of φ to $W_{\mathbb{Q}_p}$ is multiplicity free, it is enough to consider the case $\widetilde{\varphi}$ is multiplicity free. Thus we can write $\widetilde{\varphi} = \widetilde{\varphi}_1 \oplus \ldots \oplus \widetilde{\varphi}_k$ for some k and the centralizer group $C(\widetilde{\varphi})$ is isomorphic to $\mathbb{G}_m^k$.

Proposition 3.6. *Suppose that φ is multiplicity free then we have an isomorphism*

$$[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi}}/C(\widetilde{\varphi})] \simeq [\mathbb{G}_m^r/\mathbb{G}_m^r],$$

the quotient stack of $\mathbb{G}_m^r$ by the trivial action of $\mathbb{G}_m^r$ for some natural number r.

Proof. Fix an element $\phi \in Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi}}$ then the conjugation action of $W_{\mathbb{Q}_p}$ on $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ by ϕ stabilizes $C(\widetilde{\varphi})$ and the restricted action on this group factors over $W_{\mathbb{Q}_p}/\mathrm{I}_{\mathbb{Q}_p}$. Denoting by Ad_ϕ this action and denoting by $Z^1_{\mathrm{Ad}_\phi}(W_{\mathbb{Q}_p}/\mathrm{I}_{\mathbb{Q}_p}, C(\widetilde{\varphi}))_{\overline{\mathbb{Q}}_\ell}$ the affine scheme of 1-cocycles $f : W_{\mathbb{Q}_p}/\mathrm{I}_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_n$. Note that for any $w \in W_{\mathbb{Q}_p}$ and any element $\phi' \in Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell))_{\widetilde{\varphi}}$ we can write $\phi'(w) = \eta(w)\phi(w)$ and by Schur's lemma, we can deduce that $\eta(w)$ belongs to $C(\widetilde{\varphi})(\overline{\mathbb{Q}}_\ell)$. Hence, as in [DHKM20, page 20], the map $\eta \longmapsto \eta \cdot \phi$ sets up an isomorphism of $\overline{\mathbb{Q}}_\ell$-schemes

$$Z^1_{\mathrm{Ad}_\phi}(W_{\mathbb{Q}_p}/\mathrm{I}_{\mathbb{Q}_p}, C(\widetilde{\varphi}))_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\simeq} Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi},\overline{\mathbb{Q}}_\ell}.$$

The scheme $Z^1_{\mathrm{Ad}_\phi}(W_{\mathbb{Q}_p}/\mathrm{I}_{\mathbb{Q}_p}, C(\widetilde{\varphi}))_{\overline{\mathbb{Q}}_\ell}$ is in fact isomorphic to $C(\widetilde{\varphi})$. We analyze the action of $C(\widetilde{\varphi}) \simeq \mathbb{G}_m^k$ on $Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi},\overline{\mathbb{Q}}_\ell} \simeq \mathbb{G}_m^k$.

Suppose first that ϕ is irreducible and $\phi_{|I_{\mathbb{Q}_p}} \simeq \widetilde{\varphi} \simeq \widetilde{\varphi}_1 \oplus \ldots \oplus \widetilde{\varphi}_k$. If V is a vector space together with a morphism $f : \mathrm{I}_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}(V)$ then we denote by V^σ the representation of $\mathrm{I}_{\mathbb{Q}_p}$ whose underlying vector space is V and where $i \in \mathrm{I}_{\mathbb{Q}_p}$ acts by $f(\sigma i \sigma^{-1})$. Denote by V_i the underlying vector space of $\widetilde{\varphi}_i$ and $V = \bigoplus_{i=1}^{k} V_i$. Since the action of the Frobenius gives an isomorphism between the $\mathrm{I}_{\mathbb{Q}_p}$-representations V and V^σ, we deduce that up to permutation, there are isomorphisms of $\mathrm{I}_{\mathbb{Q}_p}$-representations $V_1 \simeq V_2^\sigma \simeq V_3^{\sigma^2} \simeq \ldots \simeq V_k^{\sigma^{k-1}}$ and moreover $V_1 \simeq V_1^{\sigma^k}$. Thus, we can suppose that in the basic $V_1 \oplus V_2 \oplus \ldots \oplus V_k$, the $\phi(\sigma)$ is given by

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and therefore the action of $t = (t_1, \ldots, t_k) \in C(\widetilde{\varphi}) \simeq \mathbb{G}_m^r$ acting on $Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi},\overline{\mathbb{Q}}_\ell} \simeq \mathbb{G}_m^k$ is given by $t \cdot (x_1, \ldots, x_k) = (\frac{t_1}{t_2}x_1, \ldots, \frac{t_k}{t_1}x_k)$. By using the map $\mathbb{G}_m^r \longrightarrow \mathbb{G}_m$ that send $(t_1, \ldots, t_k)$ to $\prod_{i=1}^{k} t_i$, we see that

$$[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi}}/C(\widetilde{\varphi})] \simeq [\mathbb{G}_m/\mathbb{G}_m]$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$. Moreover, for $t \in \overline{\mathbb{Q}}_\ell^\times$ if we denote by χ_t the unramified character of $W_{\mathbb{Q}_p}$ such that $\chi_t(\sigma) = t$ then the conjugacy class of the L-parameter $\phi \otimes \chi_t$ corresponds to the closed immersion $[\bullet/\mathbb{G}_m] \hookrightarrow [\mathbb{G}_m/\mathbb{G}_m]$ whose image is the closed point t in $\mathbb{G}_m$.

In general, if $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ where the ϕ_i's are irreducible and $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free then the same arguments implies that

$$[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)_{\widetilde{\varphi}}/C(\widetilde{\varphi})] \simeq [\mathbb{G}_m^r/\mathbb{G}_m^r]$$

where $\mathbb{G}_m^r$ acts trivially on $\mathbb{G}_m^r$ and the closed point $t = (t_1, \ldots, t_r) \in \mathbb{G}_m^r$ corresponds to the conjugacy class of the L-parameter $\phi_1 \otimes \chi_{t_1} \oplus \ldots \oplus \phi_r \otimes \chi_{t_r}$. □

Finally we have the following lemma characterizing the conjugacy classes of L-parameters ϕ such that $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free.

Lemma 3.7. *Let ϕ be an L-parameter of GL_n, then $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free if and only if $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ where the ϕ_i's are irreducible and if $i \neq j$ then there does not exists unramified character χ such that $\phi_i \otimes \chi \simeq \phi_j$.*

Proof. It is clear that if $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free then $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ where the ϕ_i's are irreducible and if $\dim \phi_i = \dim \phi_j$ for $i \neq j$ then there does not exists unramified character χ such that $\phi_i \otimes \chi \simeq \phi_j$.

We prove the inverse direction. By considering the action of the Frobenius σ as before we see that for $1 \leq i \leq r$, $\phi_{i|\mathrm{I}_{\mathbb{Q}_p}} \simeq V_i \oplus V_i^\sigma \oplus \ldots \oplus V_i^{\sigma^{k_i - 1}}$ where k_i is the smallest integer such that $V_i^{\sigma^{k_i}} \simeq V_i$. Therefore $\phi_{i|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free. Moreover if $V_i \simeq V_j^{\sigma^h}$ then $V_i \oplus V_i^\sigma \oplus \ldots \oplus V_i^{\sigma^{k_i-1}} \simeq V_j^{\sigma^h} \oplus V_j^{\sigma^{h+1}} \oplus \ldots \oplus V_j^{\sigma^{h+k_j-1}}$. Since ϕ_i and ϕ_j are irreducible, we deduce that $i = j$. Therefore $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free. □

Remark 3.8. By the same argument using the topological generator τ of $\mathrm{I}_{\mathbb{Q}_p}/\mathrm{P}_{\mathbb{Q}_p}$ instead of the Frobenius σ, one can show that $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free if and only if $\phi_{|\mathrm{P}_{\mathbb{Q}_p}}$ is multiplicity free.

3.2. **Perfect complexes on the stack of L-parameters.** In this subsection we talk about the spectral action defined in [FS21]. Denote by $\mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])$ the derived category of perfect complexes on $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$. We write $\mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])^{BW^{\mathrm{I}}_{\mathbb{Q}_p}}$ for the derived category of objects with a continuous $W^{\mathrm{I}}_{\mathbb{Q}_p}$ action for a finite index set I, and $\mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$ for the triangulated sub-category of compact objects in $\mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)$. By [FS21, Corollary X.I.3], there exists a $\overline{\mathbb{Q}}_\ell$-linear action

$$\mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])^{BW^I_{\mathbb{Q}_p}} \to \mathrm{End}(\mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega)^{BW^{\mathrm{I}}_{\mathbb{Q}_p}}$$
$$C \mapsto \{A \mapsto C \star A\}$$

which, extending by colimits, gives rise to an action

$$\mathrm{IndPerf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])^{BW^I_{\mathbb{Q}_p}} \to \mathrm{End}(\mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega)^{BW^{\mathrm{I}}_{\mathbb{Q}_p}}$$

where $\mathrm{IndPerf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])$ is the triangulated category of Ind-Perfect complexes, and this action is uniquely characterized by some complicated properties. For our purposes, we will need the following:

(1) For $V = \boxtimes_{i\in I} V_i \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L\mathrm{GL}_n^I)$, there is an attached vector bundle $C_V \in \mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])^{BW^I_{\mathbb{Q}_p}}$ whose evaluation at a $\overline{\mathbb{Q}}_\ell$-point of $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$ corresponding to a (not necessarily semi-simple) L-parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is the vector space V with $W^I_{\mathbb{Q}_p}$-action given by $\boxtimes_{i\in I} r_{V_i} \circ \tilde{\phi}$. Then the Hecke operator T_V defined in the previous sections, is given by the endomorphism

$$C_V \star (-) : \mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell) \to \mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^{BW^I_{\mathbb{Q}_p}}.$$

by compatibility between Hecke operators and spectral action [FS21, Theorem X.1.1].

(2) The action is symmetric monoidal in the sense that given $C_1, C_2 \in \mathrm{IndPerf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n])$, we have a natural equivalence of endofunctors:

$$(C_1 \otimes^{\mathbb{L}} C_2) \star (-) \simeq C_1 \star (C_2 \star (-)).$$

Let $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ be a semi-simple and Frobenius semi-simple L-parameter as in the previous sub-section, more precisely the ϕ_i's are irreducible and if $i \neq j$ then there does not exists unramified character ξ such that $\phi_i \otimes \xi \simeq \phi_j$. Let S_ϕ be the centralizer of our ϕ and let $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)$ be the category of finite-dimensional algebraic $\overline{\mathbb{Q}}_\ell$-representations of the group S_ϕ. The parameter ϕ defines a closed (by the semi-simplicity of ϕ) $\overline{\mathbb{Q}}_\ell$-point inside the moduli stack $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$, giving rise to a closed embedding

$$[\bullet/S_\phi] \hookrightarrow [Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$$

which is regularly immersed (by 3.5) inside the connected component containing ϕ which we denote by $[C_\phi]$. By proposition 3.6, $[C_\phi]$ consists of the L-parameters of the form $\phi_1 \otimes \xi_1 \oplus \ldots \oplus \phi_r \otimes \xi_r$ where the ξ_i's are unramified characters. This connected component gives rise to a direct summand

$$\mathrm{Perf}([C_\phi]) \hookrightarrow \mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]).$$

Therefore the spectral action gives rise to a corresponding direct summand

$$\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega \subset \mathrm{D_{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$$

such that the Schur-irreducible objects in this subcategory all have Fargues-Scholze parameter given by an L-parameter in $[C_\phi]$.

Now we consider the vector bundles on the connected component $[C_\phi]$. By proposition 3.6 we have $[C_\phi] \simeq [\mathbb{G}_m^r/\mathbb{G}_m^r]$ where $\mathbb{G}_m^r$ acts trivially. Let $\mathrm{Irr}(S_\phi)$ be the set of irreducible algebraic representations of S_ϕ. For each $\chi \in \mathrm{Irr}(S_\phi) = \mathrm{Irr}(\mathbb{G}_m^r)$ we have an associated vector bundle C_χ on $[\mathbb{G}_m^r/\mathbb{G}_m^r]$. More precisely the trivial line bundle on $\mathbb{G}_m^r$ together with the action of $\mathbb{G}_m^r$ defined by χ at every fibers gives rise to a $\mathbb{G}_m^r$-equivariant vector bundle on $\mathbb{G}_m^r$ and this vector bundle corresponds to C_χ on $[\mathbb{G}_m^r/\mathbb{G}_m^r]$. Since $\mathbb{G}_m^r$ is affine, we see that C_χ is projective. Therefore it is clear that for $\chi, \chi' \in \mathrm{Irr}(S_\phi)$ we have $C_{\chi\otimes\chi'} \simeq C_\chi \otimes C_{\chi'} \simeq C_\chi \otimes^{\mathbb{L}} C_{\chi'}$. In particular, $C_\chi \otimes C_{\chi^{-1}} \simeq C_{\chi^{-1}} \otimes C_\chi \simeq C_{\mathrm{Id}}$ is the identity functor of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$. Hence for an arbitrary $\chi \in \mathrm{Irr}(S_\phi)$, C_χ is an auto-equivalence of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$.

Let $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n)$, by the arguments in pages 339–340 and theorem $X.1.1$ in [FS21], we can express the action of the Hecke operator T_V on $\mathrm{D}_{\mathrm{lis}}^{\phi}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$ in terms of the spectral action of the C_χ's for $\chi \in \mathrm{Irr}(S_\phi)$. Recall that the action of T_V is given by the vector bundle C_V defined above. By [Ham21, lemma 3.8], it is enough to consider the restriction of C_V to $[\mathbb{G}_m^r/\mathbb{G}_m^r]$ and by simplicity we also use C_V to denote the restriction of C_V to $[\mathbb{G}_m^r/\mathbb{G}_m^r]$.

The restriction of V to S_ϕ admits a commuting $W_{\mathbb{Q}_p}$-action given by ϕ. This defines a monoidal functor

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n) \longrightarrow \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}.$$

Suppose that as a $S_\phi \times W_{\mathbb{Q}_p}$-representation, we can decompose V into a direct sum

$$V \simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi \boxtimes \sigma_\chi,$$

where σ_χ is the $W_{\mathbb{Q}_p}$-representation $\mathrm{Hom}_{S_\phi}(\chi, V)$. If we forget the $W_{\mathbb{Q}_p}$-action, we see that

$$\mathrm{T}_V = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} C_\chi^{\dim \sigma_\chi}.$$

However, T_V has an action of $W_{\mathbb{Q}_p}$ and we want to understand this action. Recall that $W_{\mathbb{Q}_p}$ acts on C_V and this action gives rise to the action of $W_{\mathbb{Q}_p}$ on T_V. By the concrete description of $W_{\mathbb{Q}_p}$ on the fibers of C_V and remark that $\phi'_{|\mathrm{I}_{\mathbb{Q}_p}} \simeq \phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ for every ϕ' in $[C_\phi]$, we deduce that the action of $\mathrm{I}_{\mathbb{Q}_p}$ on T_V can be expressed as $\mathrm{T}_V = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} C_\chi \boxtimes \sigma_{\chi|\mathrm{I}_{\mathbb{Q}_p}}$. In other words, the following diagram of monoidal functors commutes

$$\begin{tikzcd}
\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n) \arrow[r] \arrow[d] & \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathrm{D}_{\mathrm{lis},\lambda}^{\phi}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega)^{B\mathrm{I}_{\mathbb{Q}_p}} \\
\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{B\mathrm{I}_{\mathbb{Q}_p}} \arrow[ur] &
\end{tikzcd}$$

If we apply the above diagram to a Schur-irreducible sheaf $\mathcal{F}$ on Bun_n whose L-parameter is given by ϕ then we can get precise information of the action of the Frobenius.

Proposition 3.9. *Suppose that as a $S_\phi \times W_{\mathbb{Q}_p}$-representation, we can decompose V into a direct sum*

$$V \simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi \boxtimes \sigma_\chi,$$

where σ_χ is the $W_{\mathbb{Q}_p}$-representation $\mathrm{Hom}_{S_\phi}(\chi, V)$. Then for $\mathcal{F}$ a Schur-irreducible sheaf on Bun_n whose L-parameter is given by ϕ, we have

$$\mathrm{T}_V(\mathcal{F}) = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} C_\chi \star \mathcal{F} \boxtimes \sigma_\chi.$$

as object in $\mathrm{D}_{\mathrm{lis}}^{\phi}((\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega)^{BW_{\mathbb{Q}_p}}$.

Proof. By the definition of the vector bundle C_V, we see that the Frobenius σ gives rise to an element $\alpha = (\alpha_\chi)_{\dim \sigma_\chi > 0}$ of $\prod_{\chi \in \mathrm{Irr}(S_\phi); \dim \sigma_\chi > 0} \mathrm{Aut}(C_\chi^{\dim \sigma_\chi} \star \mathcal{F})$.

Since $\mathcal{F}$ is Schur-irreducible and $C_{\chi^{-1}}\star C_\chi = C_{\mathrm{Id}}$ is the identity functor, we deduce that $C_\chi\star\mathcal{F}$ is also Schur-irreducible. Thus we have a decomposition as objects in $\mathrm{D}^{\phi}_{\mathrm{lis}}((\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^{\omega})^{BW_{\mathbb{Q}_p}}$

$$\mathrm{T}_V(\mathcal{F}) = \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} C_\chi\star\mathcal{F}\boxtimes\sigma'_\chi.$$

where $\sigma'_{\chi|\mathrm{I}_{\mathbb{Q}_p}} \simeq \sigma_{\chi|\mathrm{I}_{\mathbb{Q}_p}}$ and $\sigma'_\chi(\sigma)$ acts by α_χ. In order to show that $\sigma_\chi \simeq \sigma'_\chi$, we use the construction of the Fargues-Scholze's L-parameter of $\mathcal{F}$.

Suppose first that $V \simeq \mathrm{std}$ is the standard representation. We choose the excursion data $(\{1,2\}, \mathrm{std}\boxtimes\mathrm{std}^\vee, x, y, (\gamma,\gamma))$ where x and y are the tautological maps $x : 1 \longrightarrow \mathrm{std}\boxtimes\mathrm{std}^\vee$ and $y : \mathrm{std}\boxtimes\mathrm{std}^\vee \longrightarrow 1$ and where γ is any element of $W_{\mathbb{Q}_p}$.

Now by evaluating the excursion operators at $\mathcal{F}$ when γ varies, we see that the $W_{\mathbb{Q}_p}$-representations $\bigoplus_{\chi\in\mathrm{Irr}(S_\phi)}\sigma'_\chi$ and $\bigoplus_{\chi\in\mathrm{Irr}(S_\phi)}\sigma_\chi$ have the same traces. Therefore they are isomorphic up to semi-simplification. Since they are already semi-simple and $\sigma'_{\chi|\mathrm{I}_{\mathbb{Q}_p}} \simeq \sigma_{\chi|\mathrm{I}_{\mathbb{Q}_p}}$, we deduce that $\sigma_\chi \simeq \sigma'_\chi$.

In general, remark that any irreducible algebraic representation of GL_n is a sub-representation of $\mathrm{std}^m\otimes(\mathrm{std}^\vee)^{m'}$ for some $m, m'\in\mathbb{N}$. The result follows from the monoidal property of the Hecke operators. □

4. Hecke operators on $\mathrm{Bun_n}$

4.1. **Combinatoric description of the Hecke operators.**

4.1.1. *Local Langlands correspondence for inner forms of* GL_n.

Let F be a field of characteristic zero, $\overline{F}$ be an algebraic closure, and Γ the Galois group of $\overline{F}/F$. Let G be a connected reductive group defined over F. An inner form of G is a connected reductive group G_1 defined over F for which there exists an isomorphism $\xi : \mathrm{G}\times F \longrightarrow \mathrm{G}_1\times F$ such that for all $\sigma\in\Gamma$, the automorphism $\xi^{-1}\sigma(\xi) = \xi^{-1}\circ\sigma\circ\xi\circ\sigma^{-1}$ is an inner automorphism of G. If $\xi_1 : \mathrm{G}\longrightarrow\mathrm{G}_1$ and $\xi_2 : \mathrm{G}\longrightarrow\mathrm{G}_2$ are two inner twists, then an isomorphism $\xi_1\longrightarrow\xi_2$ consists of an isomorphism $f : \mathrm{G}_1\longrightarrow\mathrm{G}_2$ defined over F and having the property that $\xi_2^{-1}\circ f\circ\xi_1$ is an inner automorphism of G. The map $\xi\longmapsto\xi^{-1}\sigma(\xi)$ sets up a bijection from the set of isomorphism classes of inner twists of G to the set set $H^1(\Gamma, \mathrm{G}_{ad})$.

When $\mathrm{G} = \mathrm{GL}_n$ then $H^1(\Gamma, \mathrm{G}_{ad}) = H^1(\Gamma, \mathrm{PGL}_n) \simeq \mathbb{Z}/n\mathbb{Z}$ and all the inner forms of GL_n are the groups $\mathrm{Res}_{D/F}\mathrm{GL}_m$, where D is any division algebra over F of degree g^2, where g is a natural number such that $gm = n$.

We consider a local field $\mathbb{Q}_v$ for v any place of $\mathbb{Q}$. The local Langlands group is defined by $\mathcal{L}_{\mathbb{Q}_v} := W_{\mathbb{R}}$ if $v=\infty$ and by $W_{\mathbb{Q}_p}\times\mathrm{SL}_2(\mathbb{C})$ if $v = p$ is a prime. For a connected reductive group G, we also set ${}^L\mathrm{G} = \widehat{\mathrm{G}}\rtimes W_{\mathbb{Q}_v}$ as a topological group where $\widehat{\mathrm{G}}$ is the Langlands dual group of G. In our case we see that ${}^L\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})\times W_{\mathbb{Q}_v}$ as GL_n is a split group and then $W_{\mathbb{Q}_v}$ acts trivially on $\mathrm{GL}_n(\mathbb{C})$.

Definition 4.1. *A local L-parameter for a connected reductive group* G *defined over $\mathbb{Q}_v$ is a continuous morphism $\phi : \mathcal{L}_{\mathbb{Q}_v}\longrightarrow {}^L\mathrm{G}$ which commutes with the canonical projections of $\mathcal{L}_{\mathbb{Q}_v}$ and ${}^L\mathrm{G}$ to $W_{\mathbb{Q}_v}$ and such that ϕ sends semisimple elements to semisimple elements.*

We denote by $\Phi(\mathrm{G})$ the set of $\widehat{\mathrm{G}}$-conjugacy classes of L-parameters. An L-parameter ϕ is called bounded (resp. discrete) if its image in ${}^L\mathrm{G}$ projects to a relatively compact subset of $\widehat{\mathrm{G}}$ (resp. if its image is not contained in any proper parabolic subgroup of ${}^L\mathrm{G}$). We denote by $\Phi_{\mathrm{bdd}}(\mathrm{G})$, (resp. $\Phi_2(\mathrm{G})$) the subset of $\Phi(\mathrm{G})$ consisting of bounded (resp. discrete) L-parameters.

We denote the set of isomorphism classes of irreducible admissible representations of a connected reductive group G by $\Pi(\mathrm{G})$. We denote the set of tempered, essentially square integrable, and unitary representations by $\Pi_{\mathrm{temp}}(\mathrm{G})$, $\Pi_2(\mathrm{G})$, and $\Pi_{\mathrm{unit}}(\mathrm{G})$ respectively. We denote $\Pi_{\mathrm{temp}}(\mathrm{G})\cap\Pi_2(\mathrm{G})$ by $\Pi_{2,\mathrm{temp}}(\mathrm{G})$.

For each L-parameter ϕ, we define the centralizer group as below, which plays an important role in the theory

$$S_\phi := \mathrm{Cent}(\mathrm{Im}\phi, \widehat{\mathrm{G}}).$$

Every $\phi \in \Phi(\mathrm{GL}_n)$ can be written as $\phi = k_1\phi_1 \oplus k_2\phi_2 \oplus \ldots \oplus k_r\phi_r$ with $\phi_i \in \Phi_2(\mathrm{GL}_{n_i})$ and $k_1n_1 + k_2n_2 + \ldots + k_rn_r = n$. Thus we have

$$S_\phi \simeq \mathrm{GL}_{k_1} \times \ldots \times \mathrm{GL}_{k_r}.$$

Each $\phi_i \in \Phi_2(\mathrm{GL}_{n_i})$ is of the form $\nu_i \otimes \mathrm{Sym}^{m_i-1}$ where ν_i is an irreducible representation of $W_{\mathbb{Q}_v}$ of dimension t_i and Sym^{m_i-1} is the representation of dimension m_i of $\mathrm{SL}_2(\mathbb{C})$ such that $t_i \cdot m_i = n_i$. Moreover if for each i, the restriction $\phi_{i|\mathrm{SL}_2(\mathbb{C})}$ is trivial then ϕ is semi-simple.

Now let $\mathrm{G} = \mathrm{Res}_{D/F}\mathrm{GL}_m$ be an inner form of GL_n, where D is a division algebra over $\mathbb{Q}_v$ of degree g^2. Then the parameter ϕ is relevant for G if and only if g divides n_i for all $1 \leq i \leq r$. We can now recall the results of [Bad08, BR10] on the local Langlands correspondence for G.

Theorem 4.2. *Let $\phi = k_1\phi_1 \oplus k_2\phi_2 \oplus \ldots \oplus k_r\phi_r \in \Phi(\mathrm{GL}_n)$ where $\phi_i \in \Phi_2(\mathrm{GL}_{n_i})$ be a decomposition of an L-parameter into simple constituents. If ϕ is relevant for* G *then there exists a unique irreducible unitary representation π_ϕ of $\mathrm{G}(\mathbb{Q}_v)$ corresponding to ϕ and characterised by traces identities. Moreover, as ϕ runs over $\Phi(\mathrm{GL}_n)$, the representations π_ϕ are different and exhaust* $\Pi(\mathrm{G})$.

4.1.2. *Combinatoric description of the spectral action.*

Next we prove that there is a bijection between the irreducible representations of the centralizer S_ϕ of some L-parameters ϕ and the irreducible sheaves over $\mathrm{Bun_n}$ whose L-parameter is ϕ. The bijection is similar to the one described in the introduction. It is of combinatoric nature and could be proved independently with the theory of spectral action. Thus we do not need to impose too much assumption on the L-parameter ϕ.

We fix a maximal split torus T and a Borel subgroup B of GL_n. Let $\overline{C}$ denote the closed Weyl chamber in $X_*(\mathrm{T})_\mathbb{R}$ associated to B and let $\overline{C}_\mathbb{Q}$ denote its intersection with $X_*(\mathrm{T})_\mathbb{Q}$. For any standard parabolic subgroup P with Levi decomposition P = MN such that T ⊂ M (i.e., M is a standard Levi subgroup), we put

$$X_*(\mathrm{P})^+ := \{\mu \in X_*(\mathrm{M}) \mid \langle \mu, \alpha\rangle > 0 \text{ for any root of T in N}\}.$$

Then we have a decomposition

$$\overline{C} = \coprod_{\mathrm{P}} X_*(\mathrm{P})^+$$

where the index is the set of standard parabolic subgroups of GL_n. We define the subset $B(\mathrm{GL}_n)_\mathrm{P}$ of $B(\mathrm{G})$ to be the pre-image of $X_*(\mathrm{P})^+$ under the Newton map. This gives a decomposition

$$B(\mathrm{GL}_n) = \coprod_{\mathrm{P}} B(\mathrm{GL}_n)_\mathrm{P} \tag{3}$$

where the index is the set of standard parabolic subgroups of GL_n. For a general standard parabolic P = MN, $B(\mathrm{GL}_n)_\mathrm{P}$ has the following description. By noting that the image of the Newton map ν_M for M lies in $X_*(\mathrm{M})$, we define $B(\mathrm{M})^+_{\mathrm{bas}}$ by

$$B(\mathrm{M})^+_{\mathrm{bas}} := \{b \in B(\mathrm{M})_{\mathrm{bas}} \mid \nu_\mathrm{M}(b) \in X_*(\mathrm{P})^+\}.$$

Then the canonical map $B(\mathrm{M}) \longrightarrow B(\mathrm{G})$ induces a bijection $B(\mathrm{M})^+_{\mathrm{bas}} \simeq B(\mathrm{GL}_n)_\mathrm{P}$ (see [Kot97, §5.1]).

Let us recall the formula given in the introduction. We could consider the case where ϕ has a decomposition

$$\phi = \phi_1 \oplus \ldots \oplus \phi_r$$

where the ϕ_i's are irreducible of dimension n_i and if $n_i = n_j$ then there does not exists unramified character ξ such that $\phi_i \simeq \phi_j \otimes \xi$. In this case we know that $\pi = \mathrm{Ind}_\mathrm{P}^{\mathrm{GL}_n}(\pi_1 \otimes \ldots \otimes \pi_r)$ for some standard parabolic subgroup P and supercuspidal representations π_i. Thus by the computation in [KMSW14, page 83], we can see that $S_\phi := \mathrm{Cent}(\phi) = \prod_{i=1}^{r} \mathbb{G}_m$ and the set $\mathrm{Irr}(S_\phi)$ of irreducible representations of S_ϕ is isomorphic to the abelian group $\prod_{i=1}^{r} \mathbb{Z}$. For each character $\chi = (d_1, \dots, d_r) \in \prod_{i=1}^{r} \mathbb{Z}$, we define an element $b_\chi \in B(\mathrm{GL}_n)$, an irreducible representation π_χ of $\mathrm{G}_\chi(\mathbb{Q}_p) := \mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ and a sheaf $\mathcal{F}_\chi$ as follow:

- b_χ is the unique element in $B(\mathrm{GL}_n)$ such that $\mathcal{E}_{b_\chi} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \dots \oplus \mathcal{O}(\lambda_r)^{m_r}$ where $\mathcal{O}(\lambda_i)$ is the stable vector bundle of slope $\lambda_i = d_i/n_i$ and $m_i = (d_i, n_i)$.
- Consider the group G_{b_χ}, it is an inner form of a standard Levi subgroups of GL_n. Let $\mathrm{G}^*_{b_\chi}$ be the split inner form of G_{b_χ}. For each i, denote $\mathrm{G}_i := \mathrm{GL}_{m_i}(D_{-\lambda_i})$ where $D_{-\lambda_i}$ is the division algebra whose invariant is $-\lambda_i$, thus $\mathrm{G}^*_i = \mathrm{GL}_{n_i}$. Thus we have a map $\phi_i : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}^*_i}(\overline{\mathbb{Q}}_\ell)$ and the direct sum $\phi_1 \oplus \dots \oplus \phi_r$ gives us a map $W_{\mathbb{Q}_p} \longrightarrow \prod_{i=1}^{r} \widehat{\mathrm{G}^*_i}(\overline{\mathbb{Q}}_\ell)$ whose post-composition with the natural embeddings $\prod_{i=1}^{r} \widehat{\mathrm{G}^*_i}(\overline{\mathbb{Q}}_\ell) \hookrightarrow \widehat{\mathrm{G}^*_\chi}(\overline{\mathbb{Q}}_\ell)$ defines an L-parameter ϕ_χ of G_{b_χ}. Moreover, the post-composition of ϕ_χ with $\widehat{\mathrm{G}^*_{b_\chi}}(\overline{\mathbb{Q}}_\ell) \hookrightarrow \widehat{\mathrm{GL}}_n(\overline{\mathbb{Q}}_\ell)$ is the ($\widehat{\mathrm{GL}}_n$-conjugacy class of) ϕ. Finally, π_χ is the representation of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ whose L-parameter is given by ϕ_χ via the local Langlands correspondence for general linear groups.
- We can suppose that $\mathrm{G}^*_{b_\chi}$ is standard. By (3), there exists a unique standard parabolic subgroup P of GL_n with Levi factor given by $\mathrm{G}^*_{b_\chi}$ such that $\nu_{b_\chi} \in X_*(\mathrm{P})^+$. Let δ_P be the modulus character with respect to P. Then we denote by δ_{b_χ} (or δ_χ) the character of G_{b_χ} whose L-parameter is the same as that of $\delta_{\mathrm{P}|\mathrm{G}^*_{b_\chi}}$. We consider the embedding $i_{b_\chi} : \mathrm{Bun}_n^{b_\chi} \longrightarrow \mathrm{Bun_n}$ and define $\mathcal{F}_\chi := i_{b_\chi !}(\delta_{b_\chi}^{-1/2} \otimes \pi_\chi)[-d_\chi]$ where $d_\chi = \langle 2\rho, \nu_{b_\chi} \rangle$.

  We recall that if $\mathrm{P} = \mathrm{MN}$ where M is the Levi subgroup and N is the unipotent radical then the modulus character is defined by $\delta_\mathrm{P}(mn) := |\det(\mathrm{ad}(m); \mathrm{LieN})|$ where $|\cdot|$ denotes the normalized absolute value of the field $\overline{\mathbb{Q}}_\ell$.

  Our characters δ_b is the same as the character δ_b defined by Hamann and Imai in [HI23]. We recall also that for $b \in B(\mathrm{GL}_n)$, we have $\nu_b = (-\nu_{\mathcal{E}_b})_{\mathrm{dom}}$ then the parabolic that preserves the Harder-Narasimhan filtration of $\mathcal{E}_{b_\chi}$ is the opposite parabolic subgroup of P above.

We show that the map $\chi \longmapsto (b_\chi, \pi_\chi)$ is bijective. First suppose that there exists $\chi_1 = (d_{1,1}, \dots, d_{1,r})$ and $\chi_2 = (d_{2,1}, \dots, d_{2,r})$ such that $(b_{\chi_1}, \pi_{\chi_1}) = (b_{\chi_2}, \pi_{\chi_2})$.

Thus we have $\mathcal{E}_{b_{\chi_1}} \simeq \mathcal{E}_{b_{\chi_2}} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \dots \oplus \mathcal{O}(\lambda_k)^{m_k}$ where $\lambda_1 > \lambda_2 > \dots > \lambda_k$. By the construction of b_{χ_1} and b_{χ_2}, for each λ_i there exists subsets $\mathrm{I}(\lambda_i)$ and $\mathrm{J}(\lambda_i)$ of $\{1, 2, \dots, r\}$ such that $d_{1,x}/n_x = \lambda_i$ if and only if $x \in \mathrm{I}(\lambda_i)$ and $d_{2,y}/n_y = \lambda_i$ if and only if $y \in \mathrm{J}(\lambda_i)$. More over $\mathrm{G}_{\chi_1}(\mathbb{Q}_p) = \mathrm{G}_{\chi_2}(\mathbb{Q}_p) = \prod_{i=1}^{k} \mathrm{GL}_{m_i}(D_{-\lambda_i})$ where $\mathrm{GL}_{m_i}(D_{-\lambda_i})$ is isomorphic to the group of automorphisms of the semi-stable vector bundle $\mathcal{O}(\lambda_i)^{m_i}$. By the construction of π_{χ_1} and π_{χ_2}, for each $1 \le i \le k$ one can construct representations π_i^1 and π_i^2 of $\mathrm{GL}_{m_i}(D_{-\lambda_i})$ such that $\pi_{\chi_1} = \boxtimes_{i=1}^k \pi_i^1$ and $\pi_{\chi_2} = \boxtimes_{i=1}^k \pi_i^2$. Moreover the L-parameter of π_i^1 is given by $\bigoplus_{j \in \mathrm{I}(i)} \phi_j$ and the L-parameter of π_i^2 is given by $\bigoplus_{j \in \mathrm{J}(i)} \phi_j$. Thus $\pi_{\chi_1} \simeq \pi_{\chi_2}$ if and only if $\pi_i^1 \simeq \pi_i^2$ as representations of $\mathrm{GL}_{m_i}(D_{-\lambda_i})$ for all $1 \le i \le k$.

Thus for each $1 \leq i \leq k$ we have $\bigoplus_{j\in \mathrm{I}(i)} \phi_j = \bigoplus_{j \in \mathrm{J}(i)} \phi_j$. We deduce that $\mathrm{I}(i) = \mathrm{J}(i)$ and more over $\lambda_i = d_{2,t}/n_t = d_{1,t}/n_t$ for every $t \in \mathrm{I}(i)$. We deduce that $d_{2,t} = d_{1,t}$ for all $t \in \mathrm{I}(i)$ and for all $1 \leq i \leq k$. In other words, $\chi_1 = \chi_2$.

We now show that the map $\chi \longmapsto (b_\chi, \pi_\chi)$ is surjective. Consider a pair (b, π_b) consisting of an element $b \in B(\mathrm{GL}_n)$ and an irreducible representation π_b of $\mathrm{G}_b(\mathbb{Q}_p)$ whose L-parameter is ϕ (after post-composing with $\widehat{\mathrm{G}_b^*} \hookrightarrow \widehat{\mathrm{GL}}_n$).

We can decompose $\mathcal{E}_b \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_k)^{m_k}$ where λ_i and $\lambda_1 > \lambda_2 > \ldots > \lambda_k$. As before $\mathrm{G}_b(\mathbb{Q}_p) = \prod_{i=1}^{d} \mathrm{GL}_{m_i}(D_{-\lambda_i})$. Thus for each $1 \leq i \leq k$, there exists a subset $\mathrm{I}(i)$ of $\{1, 2, \ldots, r\}$ such that $\bigoplus_{j\in\mathrm{I}(i)} \phi_j$ is an L-parameter of $\mathrm{GL}_{m_i}(D_{-\lambda_i})$ and the direct sum $\bigoplus_{1\leq i\leq k} \bigoplus_{j\in\mathrm{I}(i)} \phi_j$ gives us ϕ.

Now for each $t \in \mathrm{I}(i)$, by the explicit description of Levi subgroups of $\mathrm{GL}_{m_i}(D_{-\lambda_i})$, there exists a unique integer d_t such that $\lambda_i = d_t/n_t$. If we take $\chi = (d_1, \ldots, d_r)$ then we can see that $(b_\chi, \pi_\chi) = (b, \pi_b)$. Thus the map $\chi \longmapsto (b_\chi, \pi_\chi)$ is surjective.

From the above discussion and the compatibility between Fargues-Scholze L-parameters with the usual L-parameters for GL_n and its inner forms up to semi-simplification, we deduce the following result.

Proposition 4.3. *If $\mathcal{F}$ is a non-zero irreducible sheaf on Bun_n supported on a stratum corresponding to $b \in B(\mathrm{GL}_n)$ whose L-parameter is ϕ then there exists $\chi \in \mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^{r} \mathbb{Z}$ such that $b = b_\chi$.*

Remark 4.4. In [FS21, Remark I.10.3], Fargues and Scholze note that Fargues' conjecture should also include a comparison of t-structures. Then the equivalence would also yield a bijection between irreducible objects in the abelian hearts. On one side, these irreducible objects would then be enumerated by pairs (b, π_b) of an element $b \in B(\mathrm{G})$ and an irreducible smooth representation π_b, by using intermediate extensions. On the other side, they would likely correspond to a Frobenius-semisimple L-parameter $\varphi : W_{\mathbb{Q}_p} \longrightarrow \widehat{\mathrm{G}}(\overline{\mathbb{Q}}_\ell)$ together with an irreducible representation of the centralizer S_φ of φ. The above combinatoric bijection $\chi \longmapsto (b_\chi, \pi_\chi)$ could be a good candidate for Fargues-Scholze's predicted bijection for GL_n. In [BMO22], Bertoloni Meli and Oi proposed a good candidate for the bijection between irreducible objects in the abelian hearts for an arbitrary reductive group G and L-parameters φ satisfying some conditions similar to ours. The readers could also see [Han23] for more conjectures.

4.2. **Statement of the first main theorem.** Let $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$ be the stack of L-parameters of $\mathrm{GL}_{n,\mathbb{Q}_p}$ and let $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)//\widehat{\mathrm{GL}}_n$ be the categorical quotient. We have a map $\theta : [Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n] \longrightarrow Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)//\widehat{\mathrm{GL}}_n$ given by the semi-simplification. Let ϕ be an L-parameter of GL_n that sends semi-simple elements to semi-simple elements and let $|\cdot|_p$ be the norm character $W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p}^{\mathrm{ab}} \simeq \mathbb{Q}_p^\times \longrightarrow \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. We suppose that ϕ satisfies the following condition:

(A1) We have a decomposition $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ where the ϕ_i's are pairwise disjoint irreducible representations of dimension n_i and if $i \neq j$ then there does not exist unramified character χ such that $\phi_i \simeq \phi_j \otimes \chi$.

By the computation in [KMSW14, page 83], we can see that $S_\phi := \mathrm{Cent}(\phi) = \prod_{i=1}^{r} \mathbb{G}_m$.

By proposition 3.4, we see that ϕ defines a closed point in $Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)//\widehat{\mathrm{GL}}_n$ and $\theta^{-1}(x)$ is the image of the closed embedding $[\bullet/S_\phi] \longrightarrow [Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$. Denote by $[C_\phi]$ the connected component of $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$ containing the image of ι. By proposition 3.6, we know that $[C_\phi] \simeq [\mathbb{G}_m^r/\mathbb{G}_m^r]$ where the quotient is taken with respect to the trivial action of $\mathbb{G}_m^r$. Let $C \in \mathrm{Perf}^{\mathrm{qc}}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n])$ be a sheaf such that its support does not intersect with $[C_\phi]$. Then by [Ham21, lemma 3.8], we know that $C \star \mathcal{F} = 0$ where $\mathcal{F}$ is any Schur irreducible sheaf on $\mathrm{Bun_n}$ whose L-parameter belongs to $[C_\phi]$. The study of the spectral action on those sheaves $\mathcal{F}$ is therefore reduced to the description of the $C \star \mathcal{F}$ for C the vector bundles on $[C_\phi]$.

Denote by $\mathrm{Perf}([C_\phi])$ the category of perfect complex supported on $[C_\phi]$. We have a monoidal embedding of categories

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi) \longrightarrow \mathrm{Perf}([C_\phi]) \longrightarrow \mathrm{Perf}([Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n])$$

where the image of an irreducible character χ is the vector bundle on $[C_\phi]$ corresponding to the structural sheaf on $\mathbb{G}_m^r$ together with the $\mathbb{G}_m^r$-action defined by χ.

Remark that S_ϕ is commutative then the set $\mathrm{Irr}(S_\phi)$ of its algebraic characters forms a group under the tensor product operator. In this specific case that group is isomorphic to $\prod_{i=1}^{r} \mathbb{Z}$. Let $\chi = (d_1, \dots, d_r) \in \prod_{i=1}^{r} \mathbb{Z}$ be the character of S_ϕ such that $\chi(t_1, \dots, t_r) = \prod_{i=1}^{r} t_i^{d_i}$. Then we denote by C_χ the corresponding vector bundle on $[C_\phi]$. We can also define a triple $(b_\chi, \pi_\chi, \mathcal{F}_\chi)$ as in the previous section. Recall that $\mathcal{F}_\chi := i_{b_\chi !}(\delta_{b_\chi}^{-1/2} \otimes \pi_\chi)[-d_\chi]$ where $d_\chi = \langle 2\rho, \nu_{b_\chi}\rangle$.

Theorem 4.5. *Let ϕ be an L-parameter satisfying the conditions in the beginning of subsection 4.2. Let $\chi = (d_1, \dots, d_r)$ be an element in $\prod_{i=1}^{r} \mathbb{Z}$, then we have*

$$C_\chi \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_\chi.$$

where Id *is the identity of* $\prod_{i=1}^{r} \mathbb{Z}$.

Remark 4.6. While preparing this manuscript, the author was informed that Linus Hamann and David Hansen formulated a conjecture describing the Hecke operators for an arbitrary reductive group G and L-parameters satisfying similar conditions as ours [Han23]. Hamann and Hansen call the L-parameters satisfying their specific conditions "generous L-parameters" since it is closely related to the vanishing of the cohomology of Shimura varieties with mod ℓ-coefficient.

5. An analogue of Boyer's trick

In this section, we prove an analogue of theorems 1.7 and 4.13 in [Han21] based on the results developed in loc.cit. and derive some applications.

5.1. **An analogue of Boyer's trick.** For each $\lambda \in \mathbb{Q}$, let $\mathcal{O}(\lambda)$ be the stable vector bundle of slope λ. Consider $b, b' \in B(\mathrm{GL}_n)$ and suppose that we have $\mathcal{E}_b = \mathcal{O}(\lambda_1) \oplus \ldots \oplus \mathcal{O}(\lambda_s)$ and $\mathcal{E}_{b'} = \mathcal{O}(\lambda'_1) \oplus \ldots \oplus \mathcal{O}(\lambda'_t)$ where $\lambda_i \geq \lambda_{i+1}$ and $\lambda'_i \geq \lambda'_{i+1}$. Let $\mu = (k_1, k_2, \dots, k_n)$ be a minuscule dominant cochatacter of GL_n and let $\deg(\mu) := \sum_{i=1}^{n} k_i = \deg(\mathcal{E}_{b'}) - \deg(\mathcal{E}_b)$.

Recall that for each $b \in B(\mathrm{GL}_n)$, we can define a character $\kappa_b : \mathrm{G}_b(\mathbb{Q}_p) \longrightarrow \overline{\mathbb{Q}}_\ell$ as in [GI19], lemma 4.18 and as in [Ham22], page 91, before lemma 11.1.

Proposition 5.1. *Suppose that we have decompositions $\mathcal{E}_b = \mathcal{E}_{b_1} \oplus \mathcal{E}_{b_2}$ and $\mathcal{E}_{b'} = \mathcal{E}_{b'_1} \oplus \mathcal{E}_{b'_2}$ where $\mathcal{E}_{b_1} = \mathcal{O}(\lambda_1) \oplus \ldots \oplus \mathcal{O}(\lambda_{s'})$; $\mathcal{E}_{b_2} = \mathcal{O}(\lambda_{s'+1}) \oplus \ldots \oplus \mathcal{O}(\lambda_s)$ and $\mathcal{E}_{b'_1} = \mathcal{O}(\lambda'_1) \oplus \ldots \oplus \mathcal{O}(\lambda'_{t'})$; $\mathcal{E}_{b'_2} = \mathcal{O}(\lambda'_{t'+1}) \oplus \ldots \oplus \mathcal{O}(\lambda'_t)$ such that* $\mathrm{rank}(\mathcal{E}_{b_1}) = \mathrm{rank}(\mathcal{E}_{b'_1}) = m < n$ *and* $\deg(\mathcal{E}_{b'_1}) = \deg(\mathcal{E}_{b_1}) + \sum_{i=1}^{m} k_i$ *as well as $\lambda'_{t'} > \lambda'_{t'+1}$. Suppose that we have a filtration $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell \subseteq \mathcal{E}_{b_1} \subset \mathcal{F}_{\ell+1} \subset \ldots \subset \mathcal{E}_b$ where $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell \subset \mathcal{F}_{\ell+1} \subset \ldots \subset \mathcal{E}_b$ is the canonical Harder-Narasimhan filtration of*

$\mathcal{E}_b$. *This filtration defines a parabolic subgroup* P_b *of* G_b. *We define* $\mu_1 := (k_1, k_2, \ldots, k_m)$ *and* $\mu_2 := (k_{m+1}, k_{m+2}, \ldots, k_n)$ *then we have the following equality.*

$$\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu) = \Big(\mathrm{Sht}(\mathrm{GL}_m, b_1, b_1', \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b_2', \mu_2) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathcal{J}^U\Big) \times^{\underline{\mathrm{P}_b}} \underline{\mathrm{G}}_b.$$

where $\mathcal{J}^U$ *is the unipotent diamond in group* $\widetilde{\mathrm{G}}^0_{b'}/\widetilde{\mathrm{G}}^0_{b_1'} \times \widetilde{\mathrm{G}}^0_{b_2'}$. *Moreover there is a natural isomorphism of* $\mathrm{G}_b(\mathbb{Q}_p) \times \mathrm{G}_{b'}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$*-modules*

$$H^*_c(\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{ind}_{\mathrm{P}_b}^{\mathrm{G}_b}\Big(H_c^{*-2d}(\mathrm{Sht}(\mathrm{GL}_m, b_1, b_1', \mu_1) \times \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b_2', \mu_2), \overline{\mathbb{Q}}_\ell) \otimes \kappa\Big)$$

where $\mathrm{ind}_{\mathrm{P}_b}^{\mathrm{G}_b}$ *denotes the un-normalized parabolic induction;* $d = \langle 2\rho, \nu_{b'}\rangle - \langle 2\rho_m, \nu_{b_1'}\rangle - \langle 2\rho_{n-m}, \nu_{b_2'}\rangle$ *is the dimension of* $\mathcal{J}^U$ *and* ρ, ρ_m, ρ_{n-m}*, respectively, are half of the sum of positive roots of* GL_n, GL_m, GL_{n-m} *respectively and where* $\kappa = \kappa_{b'} \otimes (\kappa_{b_1'}^{-1} \times \kappa_{b_2'}^{-1})$.

Corollary 5.2. *Suppose that* $\mu = (k_1, \ldots, k_n)$ *with* $k_{n-m+1} = \ldots = k_n = 0$ *and we have a decomposition* $\mathcal{E}_b = \mathcal{E}_{b_1} \oplus \mathcal{E}_{b_2}$ *and* $\mathcal{E}_{b'} = \mathcal{E}_{b_1'} \oplus \mathcal{E}_{b_2'}$ *where* $\mathcal{E}_{b_1'} \simeq \mathcal{E}_{b_1} = \mathcal{O}(\lambda_1) \oplus \ldots \oplus \mathcal{O}(\lambda_{s'})$; $\mathcal{E}_{b_2} = \mathcal{O}(\lambda_{s'+1}) \oplus \ldots \oplus \mathcal{O}(\lambda_s)$ *and* $\mathcal{E}_{b_2'} = \mathcal{O}(\lambda'_{s'+1}) \oplus \ldots \oplus \mathcal{O}(\lambda'_t)$ *such that* $\mathrm{rank}(\mathcal{E}_{b_1}) = \mathrm{rank}(\mathcal{E}_{b_1'}) = m < n$ *as well as* $\lambda_{s'} > \lambda_{s'+1}$. *Suppose that we have a filtration* $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell \subseteq \mathcal{E}_{b_1'} \subset \mathcal{F}_{\ell+1} \subset \ldots \subset \mathcal{E}_{b'}$ *where* $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\ell \subset \mathcal{F}_{\ell+1} \subset \ldots \subset \mathcal{E}_{b'}$ *is the canonical Harder-Narasimhan filtration of* $\mathcal{E}_{b'}$. *This filtration defines a parabolic subgroup* $\mathrm{P}_{b'}$ *of* $\mathrm{G}_{b'}$. *Denote* $\mu_1 = (0, 0, \ldots, 0)$ *and* $\mu_2 = (k_1, k_2, \ldots, k_{n-m})$ *then we have the following equality.*

$$\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu) = \Big(\mathrm{Sht}(\mathrm{GL}_m, b_1, b_1', \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b_2', \mu_2) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathcal{J}^U\Big) \times^{\underline{\mathrm{P}_{b'}}} \underline{\mathrm{G}}_{b'}.$$

where $\mathcal{J}^U$ *is the unipotent diamond in group* $\widetilde{\mathrm{G}}^0_{b}/\widetilde{\mathrm{G}}^0_{b_1} \times \widetilde{\mathrm{G}}^0_{b_2}$. *Moreover there is a natural isomorphism of* $\mathrm{G}_b(\mathbb{Q}_p) \times \mathrm{G}_{b'}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$*-modules*

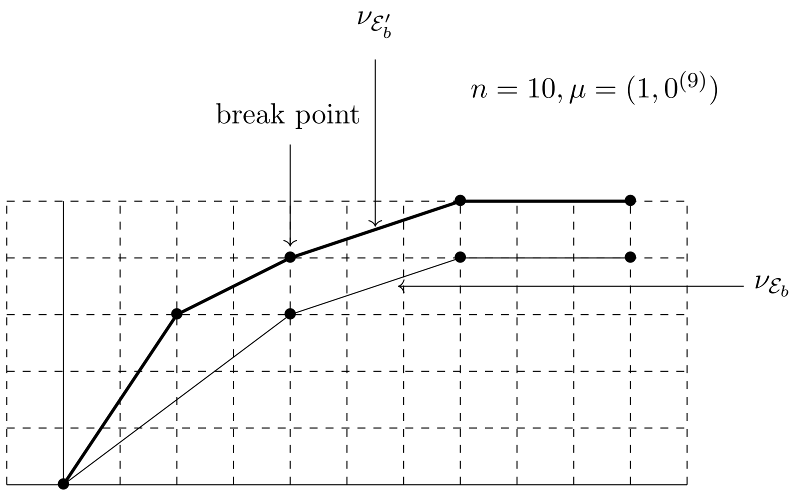
$$H^*_c(\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{ind}_{\mathrm{P}_{b'}}^{\mathrm{G}_{b'}}\Big(H_c^{*-2d}(\mathrm{Sht}(\mathrm{GL}_m, b_1, b_1', \mu_1) \times \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b_2', \mu_2), \overline{\mathbb{Q}}_\ell) \otimes \kappa\Big)$$

where $d = \langle 2\rho, \nu_b\rangle - \langle 2\rho_m, \nu_{b_1}\rangle - \langle 2\rho_{n-m}, \nu_{b_2}\rangle$ *is the dimension of* $\mathcal{J}^U$ *and* ρ, ρ_m, ρ_{n-m}*, respectively, are half of the sum of positive roots of* GL_n, GL_m, GL_{n-m} *respectively and* $\kappa = \kappa_b \otimes (\kappa_{b_1}^{-1} \times \kappa_{b_2}^{-1})$.

Remark 5.3. By (3), there exists a unique standard parabolic subgroup P such that the restriction of $\mathcal{E}_b$ to the Levi factor of P is semi-stable and the Newton point ν_b is in $X_*(\mathrm{P})^+_{\mathbb{Q}}$. Moreover the parabolic subgroup of GL_n that preserves the Harder-Narasimhan filtration of $\mathcal{E}_b$ is the opposite parabolic subgroup of P since $\nu_b = -(\nu_{\mathcal{E}_b})_{\mathrm{dom}}$. The same remark also valid for $\mathcal{E}_{b'}$ and P', in particular the opposite parabolic subgroups of P_b, $\mathrm{P}_{b'}$ above are standard.

For a proof of the corollary, we just need to apply the theorem to the moduli space of shtukas $\mathrm{Sht}(\mathrm{GL}_n, b', b, -\mu)$ where $-\mu = (-k_n, -k_{n-1}, \ldots, -k_1)$ is the dominant cocharacter corresponding to the inverse of μ.

Below are some illustrating pictures.



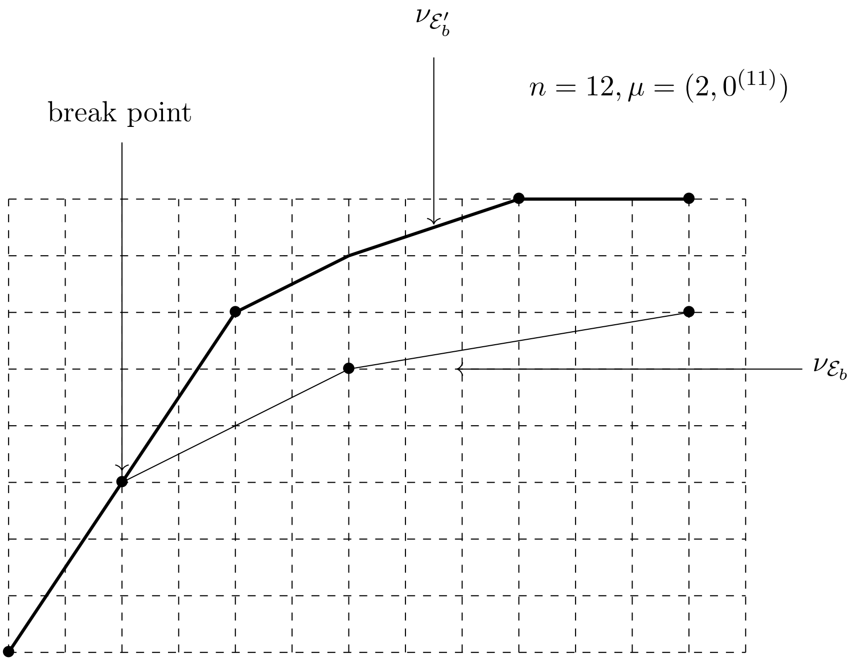
In this case we have $\mu = (1, 0^{(9)})$ and

$$\mathcal{E}_b = \mathcal{O}(3/4) \oplus \mathcal{O}(1/3) \oplus \mathcal{O}^3 \qquad \mathcal{E}_{b'} = \mathcal{O}(3/2) \oplus \mathcal{O}(1/2) \oplus \mathcal{O}(1/3) \oplus \mathcal{O}^3.$$

Thus $\mathcal{E}_{b_1} = \mathcal{O}(3/4)$; $\mathcal{E}_{b'_1} = \mathcal{O}(3/2) \oplus \mathcal{O}(1/2)$ and $\mathcal{E}_{b_2} \simeq \mathcal{E}_{b'_2} \simeq \mathcal{O}(1/3) \oplus \mathcal{O}^3$ and $\mu_1 = (1, 0^{(3)})$; $\mu_2 = (0^{(6)})$ as well as

$$P_b(\mathbb{Q}_p) = \mathrm{G}_b(\mathbb{Q}_p) \simeq D^{\times}_{-3/4}(\mathbb{Q}_p) \times D^{\times}_{-1/3}(\mathbb{Q}_p) \times \mathrm{GL}_3(\mathbb{Q}_p)$$

Moreover we have $\langle \rho_4, \nu_{b'_1} \rangle = 4$; $\langle \rho_6, \nu_{b'_2} \rangle = 3$ and $\langle \rho_{10}, \nu_{b'} \rangle = 26$, hence $d = 19$.



In this case we see that $\mu = (2, 0^{(11)})$ and

$$\mathcal{E}_b \simeq \mathcal{O}(3/2) \oplus \mathcal{O}(1/2)^2 \oplus (1/6) \qquad \mathcal{E}_{b'} \simeq \mathcal{O}(3/2)^2 \oplus \mathcal{O}(1/2) \oplus \mathcal{O}(1/3) \oplus \mathcal{O}^3.$$

Thus $\mathcal{E}_{b'_1} \simeq \mathcal{E}_{b_1} \simeq \mathcal{O}(3/2)$ and $\mathcal{E}_{b_2} \simeq \mathcal{O}(1/2)^2 \oplus (1/6)$; $\mathcal{E}_{b'_2} \simeq \mathcal{O}(3/2) \oplus \mathcal{O}(1/2) \oplus \mathcal{O}(1/3) \oplus \mathcal{O}^3$ and $\mu_1 = (0^{(3)})$; $\mu_2 = (2, 0^{(8)})$. Moreover

$$P_{b'}(\mathbb{Q}_p) = D^{\times}_{-3/2}(\mathbb{Q}_p) \times D^{\times}_{-3/2}(\mathbb{Q}_p) \times D^{\times}_{-1/2}(\mathbb{Q}_p) \times D^{\times}_{-1/3}(\mathbb{Q}_p) \times \mathrm{GL}_3(\mathbb{Q}_p)$$

and

$$G_{b'}(\mathbb{Q}_p) = \mathrm{GL}_2(D_{-3/2}(\mathbb{Q}_p)) \times D^{\times}_{-1/2}(\mathbb{Q}_p) \times D^{\times}_{-1/3}(\mathbb{Q}_p) \times \mathrm{GL}_3(\mathbb{Q}_p).$$

Proof. Notice that if $f : \mathcal{O}(x_1)\oplus\ldots\oplus\mathcal{O}(x_r) \longrightarrow \mathcal{E}$ is a modification of type $\mu_{\mathrm{cent}} = (1,\ldots,1)$ then $\mathcal{E} \simeq \mathcal{O}(x_1+1)\oplus\ldots\oplus\mathcal{O}(x_r+1)$. Thus by the same arguments in [Ngu20, Theorem 6.3], we can replace the moduli space of Shtukas $\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu)$ by $\mathrm{Sht}(\mathrm{GL}_n, b, b'\cdot b_{\mathrm{cent}}, \mu\cdot\mu_{\mathrm{cent}})$ where b_{cent} is the unique basic element in $B(\mathrm{GL}_n, \mu_{\mathrm{cent}})$. Therefore we can suppose that $\mu = (k_1,\ldots,k_n)$ with $k_n \geq 0$.

We use the same strategy as in [Han21]. First we prove that any modification $f : \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$ gives rise to a couple of modifications $f_1 : \mathcal{E}_{b_1} \longrightarrow \mathcal{E}_{b'_1}$ and $f_2 : \mathcal{E}_{b_2} \longrightarrow \mathcal{E}_{b'_2}$ of type μ_1 and μ_2 respectively. Then we prove the first statement of the proposition by examining the action of $\widetilde{\mathrm{Aut}}(\mathcal{E}_{b'_1}\oplus\mathcal{E}_{b'_2})$ on the moduli spaces of Shtukas. Finally we deduce the cohomological consequence by using the results concerning $\mathcal{J}^U$ proved in section 4.3 of [Han21].

Step 1 First we prove that any modification of type μ from $\mathcal{E}_b$ to $\mathcal{E}_{b'}$ gives rise to modifications from $\mathcal{E}_{b_1}$ to $\mathcal{E}_{b'_1}$ of type μ_1 and from $\mathcal{E}_{b_2}$ to $\mathcal{E}_{b'_2}$ of type μ_2.

The case of a point

Let C be a complete algebraically closed field over $\mathbb{Q}_p$ and let X be the Fargues-Fontaine curve associated to $(C, C^\flat)$. We denote by ∞ the Cartier divisor in X corresponding to the untilt C of $C^\flat$. Consider a modification

$$f : \mathcal{E}_{b|X\setminus\infty} \longrightarrow \mathcal{E}_{b'|X\setminus\infty}$$

of type μ between vector bundles on X.

We choose a Borel subgroup B of GL_n. The decomposition $\mathcal{E}_{b'} = \mathcal{E}_{b'_1}\oplus\mathcal{E}_{b'_2}$ is compatible with the canonical reduction of $\mathcal{E}_{b'}$. Thus there exists a standard parabolic subgroup P whose Levi component M is isormophic to $\mathrm{GL}_m\times\mathrm{GL}_{n-m}$ and a reduction $\mathcal{E}_{b',P}$ of $\mathcal{E}_{b'}$ to P corresponding to the natural filtration coming from the above decomposition. By [CFS21, lemma 2.4], the modification f and the reduction $\mathcal{E}_{b',P}$ induces a reduction $\mathcal{E}_{b,P}$ of $\mathcal{E}_b$ to P. We denote by $\mathcal{E}_{b,M} = \mathcal{E}_1\times\mathcal{E}_2$ the reduction to M of $\mathcal{E}_{b,P}$. Notice that $\mu = (k_1,\ldots,k_n)$ with $k_n \geq 0$ then f can be extended to an injection from $\mathcal{E}_b$ to $\mathcal{E}_{b'}$. Indeed, we view a vector bundle $\mathcal{E}$ as a triple $(\mathcal{E}_{|X\setminus\infty}, \mathcal{E}^{\mathrm{tri}}_{B^+_{\mathrm{dR}}}, \iota)$. Thus we have an isomorphism

$$f : \mathcal{E}_{b|X\setminus\infty} \longrightarrow \mathcal{E}_{b'|X\setminus\infty},$$

and by using $\iota_b, \iota_{b'}$, it can be extended to an isomorphism

$$\overline{f} : \mathcal{E}^{\mathrm{tri}}_{b,B^+_{\mathrm{dR}}}\otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}} \longrightarrow \mathcal{E}^{\mathrm{tri}}_{b'B^+_{\mathrm{dR}}}\otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$$

Since $k_n \geq 0$, we see that $\overline{f}(\mathcal{E}^{\mathrm{tri}}_{b,B^+_{\mathrm{dR}}}) \subset \mathcal{E}^{\mathrm{tri}}_{b',B^+_{\mathrm{dR}}}$. Thus it induces a map $f' : \mathcal{E}^{\mathrm{tri}}_{b,B^+_{\mathrm{dR}}} \longrightarrow \mathcal{E}^{\mathrm{tri}}_{b',B^+_{\mathrm{dR}}}$. The couple (f, f') is an injective map from $\mathcal{E}_b$ to $\mathcal{E}_{b'}$.

In particular we see that $\mathcal{E}_1$ is exactly the intersection $\mathcal{E}_b\cap\mathcal{E}_{b'_1}$.

We will show that $\mathcal{E}_1 \simeq \mathcal{E}_{b_1}$ and $\mathcal{E}_2 \simeq \mathcal{E}_{b_2}$.

Indeed, remark first that $\deg(\mathcal{E}_1) \leq \deg(\mathcal{E}_{b_1})$ by the property of the Harder-Narasimhan filtration of $\mathcal{E}_b$ [NV, Theorem A.5]. By [CFS21, lemma 2.6] [Vie, lemma 3.10], the modification f induces the modifications

$$f_1 : \mathcal{E}_{1|X\setminus\infty} \longrightarrow \mathcal{E}_{b'_1|X\setminus\infty} \quad \text{and} \quad f_2 : \mathcal{E}_{2|X\setminus\infty} \longrightarrow \mathcal{E}_{b'_2|X\setminus\infty}$$

of type μ'_1 resp. μ'_2 such that $(\mu'_1\times\mu'_2)_{\mathrm{dom}} \leq \mu = (k_1,\ldots,k_n)$. In particular we have the equality

$$\deg(\mathcal{E}_1) + \deg(\mu'_1) = \deg(\mathcal{E}_{b'_1}).$$

Moreover the condition $(\mu'_1\times\mu'_2)_{\mathrm{dom}} \leq \mu$ implies that $\deg(\mu'_1) \leq \sum_{i=1}^{m} k_i$. Thus we have

$$\deg(\mathcal{E}_{b'_1}) = \deg(\mathcal{E}_1) + \deg(\mu'_1) \leq \deg(\mathcal{E}_{b_1}) + \sum_{i=1}^{m} k_i = \deg(\mathcal{E}_{b'_1}).$$

Therefore we have $\deg(\mathcal{E}_1) = \deg(\mathcal{E}_{b_1})$ and $\deg(\mu'_1) = \sum_{i=1}^{m} k_i$.

The Harder-Narasimhan filtration $\mathcal{F}_1 \subset \dots \mathcal{F}_\ell = \mathcal{E}_1$ of $\mathcal{E}_1$ induces a filtration $\mathcal{F}_1 \subset \dots \mathcal{F}_\ell = \mathcal{E}_1 \subset \mathcal{E}_b$. Therefore, by the property of Herder-Narasimhan filtration, we see that $\nu_{\mathcal{E}_1}(\mathrm{rank}(\mathcal{F}_i)) \leq \nu_{\mathcal{E}_b}(\mathrm{rank}(\mathcal{F}_i))$ for $1 \leq i \leq \ell$ where $\nu_{\mathcal{E}_1}$, resp $\nu_{\mathcal{E}_b}$ are the Harder-Narasimhan polygons of $\mathcal{E}_1$, resp. $\mathcal{E}_b$. Thus we deduce that $\nu_{\mathcal{E}_1}$ lies below $\nu_{\mathcal{E}_{b_1}}$. The same argument together with the equality $\deg(\mathcal{E}_1) = \deg(\mathcal{E}_{b_1})$ implies that $\nu_{\mathcal{E}_2}$ lies below $\nu_{\mathcal{E}_{b_2}}$. However, by [Che21, corollary 2.9], the polygon ν forming from the slopes of $\nu_{\mathcal{E}_1}$ and $\nu_{\mathcal{E}_2}$ lies above the Newton polygon ν_b. Therefore, by the uniqueness of Harder-Narasimhan filtration, we deduce that $\mathcal{E}_1 = \mathcal{E}_{b_1}$ and $\mathcal{E}_2 \simeq \mathcal{E}_{b_2}$. Now we view a vector bundle $\mathcal{E}$ as a triple $(\mathcal{E}_{|X\backslash\infty}, \mathcal{E}^{n,\mathrm{tri}}_{B^+_{\mathrm{dR}}}, \iota)$ and notice that in this case B^+_{dR} is a discrete valuation ring. Then by applying [Han21, lemma 3.2] to the modifications f, f_1, f_2; we deduce that μ is a direct summand of μ_1 and μ_2. It implies that f_1, respectively, f_2 are of type μ_1 and μ_2 respectively.

The general case

Let S be a perfectoid space over $\mathbb{Q}_p$ and consider the vector bundles $\mathcal{E}_{b'} = \mathcal{E}_{b'_1} \oplus \mathcal{E}_{b'_2}$ over the Fargues-Fontaine curve associated to S. Then $\mathcal{E}_{b'_1}$ is a saturated sub-bundle of $\mathcal{E}_{b'}$. Let $f : \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$ be a modification of type μ. Since $\mu = (k_1, \dots, k_n)$ with $k_n \geq 0$, we deduce that that f can be extended to an injection from $\mathcal{E}_b$ to $\mathcal{E}_{b'}$. Consider the intersection $\mathcal{E}_1 = \mathcal{E}_{b'_1} \cap \mathcal{E}_b$. By proceeding as in [Han21, §3.2] and by using the B-pairs and also proposition 2.4 in [Han21], we can show that $\mathcal{E}_1 \simeq \mathcal{E}_{b_1}$ and the quotient $\mathcal{E}_b/\mathcal{E}_{b_1}$ is isormorphic to $\mathcal{E}_{b_2}$. Moreover the modification f induces modifications

$$f_1 : \mathcal{E}_{b_1} \longrightarrow \mathcal{E}_{b'_1}, \qquad f_2 : \mathcal{E}_{b_2} \longrightarrow \mathcal{E}_{b'_2},$$

of type μ_1 respectively μ_2.

Step 2 Analyzing the effect of the actions of $\widetilde{\mathrm{G}}^0_{b'}/\widetilde{\mathrm{G}}^0_{b'_1} \times \widetilde{\mathrm{G}}^0_{b'_2}$ and of $\underline{\mathrm{G}_b(\mathbb{Q}_p)}$.

We consider an intermediate space $\mathrm{Sht}(P_b, b, b', \mu)$ of P_b-shtukas as in [Han21, section 1.5]. More precisely, the points of $\mathrm{Sht}(P_b, b, b'\mu)$ are triples $(\mathcal{F}, x, \alpha^P)$ where $\mathcal{F}$ is a vector bundle isomorphic to $\mathcal{E}_b$ and $x \in \mathrm{Gr}^{\mu}_{\mathrm{GL}_n}$ is a point that giving rise to a type μ modification

$$f : \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$$

As in step 1, the decomposition $\mathcal{E}_{b'} = \mathcal{E}_{b'_1} \oplus \mathcal{E}_{b'_2}$ induces a filtration $0 \subset \mathcal{E}_1 \subset \mathcal{E}_b$. Then $\alpha^P : \mathcal{E}_b \longrightarrow \mathcal{E}_b$ is an isomorphism of vector bundle matching the filtration $0 \subset \mathcal{E}_1 \subset \mathcal{E}_b$ with the filtration $0 \subset \mathcal{E}_{b_1} \subset \mathcal{E}_b$. If $\lambda_{s'} = \lambda_{s'+1}$ then $P_b(\mathbb{Q}_p)$ is a proper parabolic subgroup of $\mathrm{G}_b(\mathbb{Q}_p)$. However, if $\lambda_{s'} > \lambda_{s'+1}$ then $P_b(\mathbb{Q}_p) \simeq \mathrm{G}_b(\mathbb{Q}_p)$ and then by the uniqueness of Harder-Narasimhan filtration, $\mathrm{Sht}(P_b, b, b'\mu) \simeq \mathrm{Sht}(\mathrm{GL}_n, b, b'\mu)$.

The same analyse in [Han21] goes through in our situation and we have an isomorphism commuting with all additional structures

$$\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu) = \mathrm{Sht}(P_b, b, b', \mu) \times^{\underline{P_b}} \underline{\mathrm{G}}_b.$$

Now we show that

$$\mathrm{Sht}(P_b, b, b', \mu) \simeq \Big(\mathrm{Sht}(\mathrm{GL}_m, b_1, b'_1, \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b'_2, \mu_2) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathcal{J}^U\Big).$$

Let $f : \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$ be a modification of type μ. Thus by step 1, it gives rise to modifications

$$f_1 : \mathcal{E}_{b_1} \longrightarrow \mathcal{E}_{b'_1}, \qquad f_2 : \mathcal{E}_{b_2} \longrightarrow \mathcal{E}_{b'_2},$$

of type μ_1 respectively μ_2. Moreover, the direct sum $f' = f_1 \oplus f_2$ is a modification from $\mathcal{E}_b$ to $\mathcal{E}_{b'}$ of type μ. We show that there exists $u \in \mathcal{J}^U \simeq \widetilde{\mathrm{G}}^0_{b'}/\widetilde{\mathrm{G}}^0_{b'_1} \times \widetilde{\mathrm{G}}^0_{b'_2}$ such that $f = u \circ f'$.

We know that $f, f' \in \mathrm{Hom}(\mathcal{E}_b, \mathcal{E}_{b'})$. If we write $\mathcal{E}_b \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{E}_{b'} = \mathcal{E}_{b'_1} \oplus \mathcal{E}_{b'_2}$ then f can be written by

$$\begin{pmatrix} f_1 & t \\ 0 & f_2 \end{pmatrix},$$

where $t = (t_e, t')$ is a map from $\mathcal{E}_2$ to $\mathcal{E}_{b'_1}$. We show that there exists a map $h = (h_e, h')$: $\mathcal{E}_{b'_2} \longrightarrow \mathcal{E}_{b'_1}$ such that $t = h \circ f_2$. We use again the description of a vector bundle $\mathcal{E}$ by a triple $(\mathcal{E}_{|X\setminus\infty}, \mathcal{E}^{\mathrm{tri}}_{B^+_{\mathrm{dR}}}, \iota)$. Since $f_2 : \mathcal{E}_{2|X\setminus\infty} \simeq \mathcal{E}_{b'_2|X\setminus\infty}$ is an isomorphism, it induces a map $h_e : \mathcal{E}_{b'_2|X\setminus\infty} \longrightarrow \mathcal{E}_{b'_1|X\setminus\infty}$ such that $t_e = h_e \circ f_2$. By using the map $\iota_{b'_2}$ and $\iota_{b'_1}$, we have an isomorphism

$$\overline{h}_e := \iota_{b'_1} \circ h_e \circ \iota^{-1}_{b'_2} : \mathcal{E}^{\mathrm{tri}}_{b'_2, B^+_{\mathrm{dR}}} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}} \longrightarrow \mathcal{E}^{\mathrm{tri}}_{b'_1 B^+_{\mathrm{dR}}} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}.$$

Similarly, we can consider the maps $\overline{t}_e$ and $\overline{f}_2$ and we have the identity $\overline{t}_e = \overline{h}_e \circ \overline{f}_2$ such that the restriction to $\mathcal{E}^{\mathrm{tri}}_{2,B^+_{\mathrm{dR}}}$ gives rise to an identity:

$$t' : \mathcal{E}^{\mathrm{tri}}_{2,B^+_{\mathrm{dR}}} \xrightarrow{f'_2} f'_2(\mathcal{E}^{\mathrm{tri}}_{2,B^+_{\mathrm{dR}}}) \xrightarrow{\overline{h}^*_e} \mathcal{E}^{\mathrm{tri}}_{b'_1,B^+_{\mathrm{dR}}},$$

where $\overline{h}^*_e$ is the restriction of $\overline{h}_e$ to $f'_2(\mathcal{E}^{\mathrm{tri}}_{2,B^+_{\mathrm{dR}}})$. Note that f_2 is a modification of type $(k_{m+1}, \dots, k_n)$ with $k_n \geq 0$ then $\xi^{k_{m+1}} \mathcal{E}^{\mathrm{tri}}_{b'_2,B^+_{\mathrm{dR}}} \subset f'_2(\mathcal{E}^{\mathrm{tri}}_{2,B^+_{\mathrm{dR}}}) \subset \mathcal{E}^{\mathrm{tri}}_{b'_2,B^+_{\mathrm{dR}}}$. Note that $\mu_1 = (k_1, \dots, k_m)$ then we have $\overline{h}^*_e(f'_2(\mathcal{E}^{\mathrm{tri}}_{2,B^+_{\mathrm{dR}}})) \subset \xi^{k_m} \mathcal{E}^{\mathrm{tri}}_{b'_1,B^+_{\mathrm{dR}}}$. Since $k_m \geq k_{m+1}$, we deduce that $\overline{h}^*_e(\mathcal{E}^{\mathrm{tri}}_{b'_2,B^+_{\mathrm{dR}}}) \subset \mathcal{E}^{\mathrm{tri}}_{b'_1,B^+_{\mathrm{dR}}}$. Therefore the couple $(h_e, \overline{h}_e)$ gives rise to a map $h : \mathcal{E}_{b'_2} \longrightarrow \mathcal{E}_{b'_1}$ such that $t = h \circ f_2$. Now if we choose the map u of the form

$$\begin{pmatrix} \mathrm{Id}_{\mathcal{E}_{b'_1}} & h \\ 0 & \mathrm{Id}_{\mathcal{E}_{b'_2}} \end{pmatrix},$$

with respect to $\mathcal{E}_{b'} = \mathcal{E}_{b'_1} \oplus \mathcal{E}_{b'_2}$ then we have $f = u \circ f'$. We conclude that there is an isomorphism commuting with all additional structures

$$\mathrm{Sht}(\mathrm{P}_b, b, b', \mu) \simeq \Big(\mathrm{Sht}(\mathrm{GL}_m, b_1, b'_1, \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b'_2, \mu_2) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathcal{J}^U\Big).$$

Finally, we deduce the cohomological consequences by arguing as in the proof of [HI23, Proposition 4.11] and by using the computation of the cohomology of $\mathcal{J}^U$ (page 91, before lemma 11.1 in [Ham22]). ☐

Lemma 5.4. *For $b \in B(\mathrm{GL}_n)$, we have $\kappa_b = \delta_b^{-1}$.*

Proof. In [HI23], Hamann and Imai will show a more general result. We only give a sketch of the proof in our cases at hand. It is based on the facts that Fargues-Scholze's construction of L-parameters is compatible with twisting by a central character and commute with Hecke operators in the sense of [Ham21, Proposition 3.14, Corollary 3.15].

We prove the lemma by induction on the number of distinct sloves of $\nu_{\mathcal{E}_b}$. By modifications of type $\mu_{\mathrm{cent}} = (a, a, \dots, a)$ for some integer a, it is enough to treat the case where all the slopes of $\nu_{\mathcal{E}_b}$ are positive. Thus we have a decomposition $\mathcal{E}_b = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_k)^{m_k}$ with $\lambda_1 > \ldots \lambda_k \geq 0$.

If $k = 1$ then there is nothing to prove. We suppose $k \geq 2$. We use induction on $\deg(\mathcal{E}_b)$. If $\deg(\mathcal{E}_b) = 1$ then $\mathcal{E}_b = \mathcal{O}(\lambda) \oplus \mathcal{O}^{m_1}$ for some λ of the form $1/m$. We fix a maximal torus and a Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{GL}_n$. Let π be an irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ such that its L-parameter is of the form $\phi_1 \oplus \phi_2$ where the ϕ_i's are irreducible, does not belong to the same Bernstein component and $\dim(\phi_1) = m$; $\dim(\phi_2) = m_1$. Note that $\mathrm{G}_b(\mathbb{Q}_p) \simeq D^\times_{-1/m}(\mathbb{Q}_p) \times \mathrm{GL}_{m_1}(\mathbb{Q}_p)$ and denote by $\pi_{1,b}$ irreducible representation of $\mathrm{G}_b(\mathbb{Q}_p)$ whose L-parameter (with respect to $\mathrm{G}_b(\mathbb{Q}_p)$) is $\phi_1 \times \phi_2$. Remember that Fargues and Scholze also associate to $\pi_{1,b}$ an L-parameter ϕ' with respect to $\mathrm{GL}_n(\mathbb{Q}_p)$. By the compatibility of the Fargues-Scholze's construction with G_b ([FS21, Chapter IX.7.1]) we see that $\phi' = \phi \otimes \chi_b^{1/2}$ where χ_b corresponds to δ_b via the local Langlands correspondence for tori. If $m = m_1$ then we denote by $\pi_{2,b}$ irreducible representation of $\mathrm{G}_b(\mathbb{Q}_p)$ whose L-parameter is $\phi_2 \times \phi_1$. Similar remark also applies to $\pi_{2,b}$.

We will analyse the L-parameters of the constituents of $i_b^* \mathrm{T}_\mu i_{1!}(\pi)$ where $\mu = (1, 0, \dots, 0)$. By [Ham21, Proposition 3.14, Corollary 3.15], the irreducible constituents of $i_b^* \mathrm{T}_\mu i_{1!}(\pi)$ belong to the set $\{\pi_{1,b} \otimes \delta_b^{-1/2}, \pi_{2,b} \otimes \delta_b^{-1/2}\}$ since their Fargues-Scholze L-parameter with respect to GL_n is given by ϕ.

By lemma 2.10 we have

$$R\Gamma_c(\mathrm{G}, 1, b, \mu)[\pi] \simeq i_b^* \mathrm{T}_\mu i_{1!}(\pi),$$

and by using Hom-Tensor duality, second adjointness theorem combining with proposition 5.1, we see that

$$\begin{aligned}(R\Gamma_c(\mathrm{G}, 1, b, \mu)[\pi] \otimes \kappa_b^{-1})^\vee &\simeq \mathrm{RHom}_{\overline{\mathbb{Q}}_\ell}\big((\mathrm{ind}_{\mathrm{P}_1}^{\mathrm{GL}_n} X) \otimes_{\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p))} \pi, \overline{\mathbb{Q}}_\ell\big) \\ &\simeq \mathrm{RHom}_{\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p))}\big(\mathrm{ind}_{\mathrm{P}_1}^{\mathrm{GL}_n} X, \pi^\vee\big) \\ &\simeq \mathrm{RHom}_{\mathcal{H}(\mathrm{M}(\mathbb{Q}_p))}\big(\delta_{\mathrm{P}_1}^{-1} X, (r_{\mathrm{M}}^{\mathrm{GL}_n} \pi)^\vee\big)\end{aligned}$$

where $X = R\Gamma_c(\mathrm{Sht}(\mathrm{GL}_m, 1, b_1, (1, 0^{(m-1)})) \times \mathrm{Sht}(\mathrm{GL}_{m_1}, 1, 1, \mathrm{Id}))$, $\mathcal{E}_{b_1} \simeq \mathcal{O}(\lambda)$ and M is the Levi factor of P_1, $r_{\mathrm{M}}^{\mathrm{GL}_n}$ denotes the normalized Jacquet functor.

Note that the opposite parabolic group P_1^{op} is standard then δ_b is the character whose L-parameter is the same as that of the character $\delta_{\mathrm{P}_1}^{-1}$. Therefore, the irreducible constituents of $R\Gamma_c(\mathrm{G}, 1, b, \mu)[\pi]$ belong to the set $\{\pi_{1,b} \otimes \kappa_b \otimes \delta_b^{1/2}, \pi_{2,b} \otimes \kappa_b \otimes \delta_b^{1/2}\}$. Hence we deduce that $\kappa_b = \delta_b^{-1}$.

Now we suppose that $\deg(\mathcal{E}_b) > 1$. By the same argument, we can treat the case $\lambda_k = 0$. If $\lambda_k > 0$ then we choose $b' \in B(\mathrm{GL}_n)$ such that $\mathcal{E}_{b'} = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}} \oplus \mathcal{E}$ where $\mathcal{E}$ is the semi-stable vector bundle such that $\dim(\mathcal{E}) = \dim(\mathcal{O}(\lambda_k)^{m_k})$ and $\deg(\mathcal{E}) = \deg(\mathcal{O}(\lambda_k)^{m_k}) - 1$. In particular, we know that $\kappa_{b'} = \delta_{b'}^{-1}$.

In order to compute κ_b, we choose an appropriate irreducible representation π_b of $\mathrm{G}_b(\mathbb{Q}_p)$ and analyse the L-parameter (with respect to GL_n) of the constituents of

$$i_{b'}^* \mathrm{T}_\mu i_{b!}(\pi_b)$$

by using lemma 2.10, corollary 5.2 and [FS21, IX.7.1]. $\square$

5.2. **Some preliminary computations.**

Let $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ be an L-parameter satisfying the conditions of theorem 4.5. We know that $\mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^r \mathbb{Z}$. Let $\chi = (d_1, \ldots, d_r)$ be a character in $\mathrm{Irr}(S_\phi)^+ := \{(t_1, \ldots, t_r) \in \mathrm{Irr}(S_\phi) \mid t_i \geq 0 \ \forall\ 1 \leq i \leq r\}$. Suppose that $\mathcal{E}_{b_\chi} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_k)^{m_k}$ where $\lambda_1 > \ldots > \lambda_k$. For each $1 \leq \ell \leq k$ we denote by $\mathrm{I}(\ell) \subset \{1, 2, \ldots, r\}$ the set of index i such that $d_i/n_i = \lambda_\ell$ where $n_i := \dim \phi_i$ and $\sum_{i=1}^r n_i = n$.

Let $b \in B(\mathrm{GL}_n)$ be an element such that $\mathcal{E}_b \simeq \mathcal{G} \oplus \mathcal{O}(\lambda_i)^{m_i'} \oplus \mathcal{O}(\lambda_{i+1})^{m_{i+1}} \oplus \ldots \oplus \mathcal{O}(\lambda_k)^{m_k}$ where $1 \leq i \leq k$ and $0 < m_i' \leq m_i$ be integers and $\mathcal{G}$ is a vector bundle such that the slopes of its Newton polygon are strictly bigger than the smallest slope of $\mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{i-1})^{m_{i-1}} \oplus \mathcal{O}(\lambda_i)^{m_i - m_i'}$.

Let $b' \in B(\mathrm{GL}_n)$ be an element such that $\mathcal{E}_{b'} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{i-1})^{m_{i-1}} \oplus \mathcal{O}(\lambda_i)^{m_i} \oplus \mathcal{H}$ where $1 \leq i \leq k$ be integers and $\mathcal{H}$ is a vector bundle such that the biggest slope of its Newton polygon is smaller than λ_i and strictly bigger than λ_{i+1}.

We define $D := \deg(\mathcal{E}_{b_\chi})$ and $D_1 := \deg(\mathcal{E}_b)$ and $D_2 := \deg(\mathcal{E}_{b'})$. Suppose that $D < D_1$ and $D < D_2$. Let $\chi_i \in \mathrm{Irr}(S_\phi)$ be the element $(d_1, \ldots, d_r)$ where $d_i = 1$ and $d_j = 0$ for $1 \leq j \neq i \leq r$. In the following lemmas, we analyse the complexes $i_b^* C_{\chi_i} \star \mathcal{F}_\chi$ and $i_{b'}^* C_{\chi_i} \star \mathcal{F}_\chi$ where $\mathcal{F}_\chi$ is defined as in §4.1.

Lemma 5.5. *Let $\chi' = (a_1, \ldots, a_r) \in \mathrm{Irr}(S_\phi)^+$ be a character such that $\sum_{j=1}^r a_j = D_1 - D = 1$. We define $S_+ := \{1 \leq j \leq r \mid a_j > 0\}$ and $S_0 := \{1, 2, \ldots, k\} \setminus S_+$. Suppose that theorem 4.5 is true for every L-parameters $\phi' = \bigoplus_{j \in J \subsetneq \{1, \ldots, r\}} \phi_j$, then:*

(1) *If* $S_+ \subset \big(\bigcup_{i<j\leq k} \mathrm{I}(j)\big)$ *then*

$$i_b^* C_{\chi'} \star (i_{b_\chi !}\pi_\chi) \simeq 0$$

where π_χ *is the irreducible representation of* $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ *whose L-parameter is given by* ϕ_χ *as in §4.1 (whose post-composition with* $\widehat{\mathrm{G}^*_{b_\chi}} \hookrightarrow \widehat{\mathrm{GL}}_n$ *is given by* ϕ*).*

(2) *For* $S_+ = \{c\} \subset \big(\bigcup_{1\leq j\leq i} \mathrm{I}(j)\big)$ *we have*

$$i_b^* C_{\chi'} \star \mathcal{F}_\chi \simeq \begin{cases} \mathcal{F}_{\chi'\otimes\chi} & \text{if } b = b_{\chi'\otimes\chi} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly we have

Lemma 5.6. *Let* $\chi' = (a_1, \dots, a_r) \in \mathrm{Irr}(S_\phi)^+$ *be a character such that* $\sum_{j=1}^r a_j = D_2 - D = 1$. *We define* $S_+ := \{1 \leq j \leq r \mid a_j > 0\}$ *and* $S_0 := \{1, 2, \dots, k\} \setminus S_+$. *Suppose that theorem 4.5 is true for every L-parameters* $\phi' = \bigoplus_{j\in J\subsetneq\{1,\dots,r\}} \phi_j$*, then:*

(1) *If* $S_+ \subset \big(\bigcup_{1\leq j<i} \mathrm{I}(j)\big)$ *then*

$$i_b^* C_{\chi'} \star (i_{b_\chi !}\pi_\chi) \simeq 0$$

where π_χ *is the irreducible representation of* $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ *whose L-parameter is given by* ϕ_χ *as in §4.1 (whose post-composition with* $\widehat{\mathrm{G}^*_{b_\chi}} \hookrightarrow \widehat{\mathrm{GL}}_n$ *is given by* ϕ*).*

(2) *For* $S_+ = \{c\} \subset \big(\bigcup_{i\leq j\leq k} \mathrm{I}(j)\big)$ *we have*

$$i_{b'}^* C_{\chi'} \star \mathcal{F}_\chi \simeq \begin{cases} \mathcal{F}_{\chi'\otimes\chi} & \text{if } b' = b_{\chi'\otimes\chi} \\ 0 & \text{otherwise.} \end{cases}$$

Remark that part (2) of the lemmas does not completely describe $C_{\chi'} \star \mathcal{F}_\chi$ since we assume that b satisfies some constraints related to b_χ. If we suppose $D > D_1$ and $D > D_2$ as well as all the a_j's are non-positive then we also get similar results.

Proof. We only prove lemma 5.5, the proof for lemma 5.6 is similar and less complicated.

We suppose first that $m'_i = m_i$ and denote by μ the cocharacter $(1, 0^{(n-1)})$. Thus we have $r_\mu = \mathrm{StdGL}_n$ and since all the a_j's are non negative, we see that $\mathrm{Hom}_{S_\phi}(\chi', r_\mu \circ \phi)$ is non trivial. Note that in this case $i > 1$ and the triple (b_χ, b, μ) satisfies the condition in proposition 5.1. Therefore we have

$$\mathrm{Sht}(\mathrm{GL}_n, b_\chi, b, \mu) = \mathrm{Sht}(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathcal{J}^U$$

where

- $\mathcal{E}_{\widetilde{b}} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \dots \oplus \mathcal{O}(\lambda_{i-1})^{m_{i-1}}$; $\mathcal{E}_{b_2} \simeq \mathcal{O}(\lambda_i)^{m_i} \oplus \dots \oplus \mathcal{O}(\lambda_k)^{m_k}$ and $\mathcal{E}_{b_1} \simeq \mathcal{G}$. In particular $\mathcal{E}_{b_\chi} \simeq \mathcal{E}_{\widetilde{b}} \oplus \mathcal{E}_{b_2}$ and $\mathcal{E}_b \simeq \mathcal{E}_{b_1} \oplus \mathcal{E}_{b_2}$.
- $\mu_1 = (1, 0^{(m-1)})$ and $\mu_2 = (0^{(n-m)})$.
- $\mathcal{J}^U$ is the unipotent diamond in group $\widetilde{\mathrm{G}}^0_b/(\widetilde{\mathrm{G}}^0_{b_1} \times \widetilde{\mathrm{G}}^0_{b_2})$.

Recall that for χ, we have an associated representation π_χ of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p) = \prod_{1\leq j\leq k} \mathrm{Aut}_{\mathcal{O}^{m_j}(\lambda_j)}(\mathbb{Q}_p)$ whose L-parameter is given by $\prod_{1\leq j\leq k} \bigoplus_{h\in \mathrm{I}(j)} \phi_h$. If we write $\pi_\chi = \widetilde{\pi} \times \pi_2$ as representation of

$\mathrm{G}_{b_\chi}(\mathbb{Q}_p) = \mathrm{G}_{\widetilde{b}}(\mathbb{Q}_p) \times \mathrm{G}_{b_2}(\mathbb{Q}_p)$, then we have an isomorphism of $\mathrm{G}_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-modules

$$\begin{aligned} & R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\delta_\chi^{-1/2} \otimes \pi_\chi] \otimes \kappa^{-1} \\ \simeq\ & R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\delta_{\chi|\mathrm{G}_{\widetilde{b}}}^{-1/2} \otimes \widetilde{\pi}] \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\delta_{\chi|\mathrm{G}_{b_2}}^{-1/2} \otimes \pi_2][h-2d](\tfrac{h}{2}) \end{aligned} \tag{4}$$

where

- $h = \langle \mu, 2\rho \rangle - \langle \mu_1, 2\rho_m \rangle$ and the Tate twist $(\frac{h}{2})$ as well as the shift $[h]$ come from the difference in the normalization of the sheaves $\mathcal{S}_\mu$ and $\mathcal{S}_{\mu_1}$ in the definition of $R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)$ and $R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)$.
- $d = d_b - d_{b_1} - d_{b_2}$ and the shift $[-2d]$ comes from the cohomology of the automorphism group $\mathcal{J}^U$. Here $d_b = \langle \nu_b, 2\rho \rangle$, $d_{b_1} = \langle \nu_{b_1}, 2\rho_m \rangle$ and $d_{b_2} = \langle \nu_{b_2}, 2\rho_{n-m} \rangle$ and $\kappa = \kappa_b \otimes (\kappa_{b_1}^{-1} \times \kappa_{b_2}^{-1})$.

Besides, by lemma 2.10, we have

$$i_b^* \mathrm{T}_\mu(i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi) \simeq R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\delta_\chi^{-1/2} \otimes \pi \otimes \kappa_{b_\chi}^{-1}][2d_{b_\chi}]. \tag{5}$$

Since T_{Id} is the identity functor, lemma 2.10 implies that

$$R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\delta_{\chi|\mathrm{G}_{b_2}}^{-1/2} \otimes \pi_2] \simeq \delta_{\chi|\mathrm{G}_{b_2}}^{-1/2} \otimes \pi_2 \otimes \kappa_{b_2}[-2d_{b_2}]. \tag{6}$$

Suppose that as $S_\phi \times W_{\mathbb{Q}_p}$-representations, we have an identification

$$r_\mu \circ \phi = \bigoplus_{\xi \in \mathrm{Irr}(S_\phi)} \xi \boxtimes \sigma_\xi$$

where σ_ξ is the $W_{\mathbb{Q}_p}$-representations $\mathrm{Hom}_{S_\phi}(\xi, r_\mu \circ \phi)$. Since the Fargues-Scholze's parameter (with respect to Bun_n) of $i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi$ is given by ϕ, we have a decomposition of Hecke operator

$$\mathrm{T}_\mu(i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi) = \bigoplus_{\xi \in \mathrm{Irr}(S_\phi)} C_\xi \star (i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi) \boxtimes \sigma_\xi.$$

Remark that the condition on ϕ allows us to understand $C_\xi \star (i_{b!} \delta_\chi^{-1/2} \otimes \pi)$ by identifying the $[\sigma_\xi]$-isotypical part of $\mathrm{T}_\mu(i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi)$.

Note that the L-parameter of π_2 is given by $\bigoplus_{t \in \bigcup_{i \le j \le k} \mathrm{I}(j)} \phi_t$ and the L-parameter of $\widetilde{\pi}$ is given by $\bigoplus_{t \in \bigcup_{1 \le j < i} \mathrm{I}(j)} \phi_t$. The observation is that if we regard the cohomology groups $R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\delta_\chi^{-1/2} \otimes \pi]$ as $W_{\mathbb{Q}_p}$-module then by (4), its irreducible sub-quotients are of the form ϕ_t for $t \in \bigcup_{1 \le j < i} \mathrm{I}(j)$ since $W_{\mathbb{Q}_p}$ acts trivially on $R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\delta_{\chi|\mathrm{G}_{b_2}}^{-1/2} \otimes \pi_2]$. However if $S_+ = \{c\} \subset \big(\bigcup_{i \le j \le k} \mathrm{I}(j) \big)$ then $\sigma_\chi(\varphi) \simeq \phi_c$ Thus the condition on the irreducible factors of ϕ in the beginning of §4.2 allows us to conclude that the σ_χ-isotypical part of $R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\delta_\chi^{-1/2} \otimes \pi]$ vanishes. Therefore $i_b^* C_{\chi'} \star (i_{b_\chi !} \pi_\chi) \simeq 0$, hence the first point of the lemma.

We are going to prove the second point. We see that $r_\mu \circ \phi = \bigoplus_{1 \le j \le r} \phi_j$. Hence

$$\mathrm{T}_\mu(i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi) = \bigoplus_{j=1}^{r} C_{\chi_j} \star (i_{b_\chi !} \delta_\chi^{-1/2} \otimes \pi) \boxtimes \phi_j,$$

and we only need to consider the case $\chi' = \chi_c$ for some $c \in \big(\bigcup_{1 \le j < i} \mathrm{I}(j) \big)$. We need to compute the $[\phi_c]$-isotypical part of the $W_{\mathbb{Q}_p}$-module $R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\delta_{\chi|\mathrm{G}_{\widetilde{b}}}^{-1/2} \otimes \widetilde{\pi}]$.

Let $\widetilde{\phi}$ denote the L-parameter $\bigoplus_{t\in\bigcup_{1\leq j<i}\mathrm{I}(j)}\phi_t$ and let $\widetilde{\chi}$, respectively $\widetilde{\chi}_c$ be the element in $\mathrm{Irr}(S_{\widetilde{\phi}})\simeq\prod_{t\in\bigcup_{1\leq j<i}\mathrm{I}(j)}\mathbb{Z}$ obtained from χ, respectively χ_c by removing all the indexes corresponding to ϕ_t, where $t\in\bigcup_{i\leq j<k}\mathrm{I}(j)$. Thus we can check that $\widetilde{b}=b_{\widetilde{\chi}}$ and $\widetilde{\pi}=\pi_{\widetilde{\chi}}$. Hence by lemma 2.10, we have

$$i^*_{b_1}\mathrm{T}_{\mu_1}(i_{\widetilde{b}!}\delta^{-1/2}_{\chi|\mathrm{G}_{\widetilde{b}}}\otimes\pi_{\widetilde{\chi}}\otimes\kappa_{\widetilde{b}})\simeq R\Gamma_c(\mathrm{GL}_m,\widetilde{b},b_1,\mu_1)[\delta^{-1/2}_{\chi|\mathrm{G}_{\widetilde{b}}}\otimes\pi_{\widetilde{\chi}}][2d_{\widetilde{b}}], \tag{7}$$

where $d_{\widetilde{b}}=\langle\nu_{\widetilde{b}},2\rho_m\rangle$

By theorem 4.5, applying to the L-parameter $\widetilde{\phi}$ we deduce that

$$i^*_{b_1}C_{\widetilde{\chi}_c}\star(i_{b_{\widetilde{b}}!}\delta^{-1/2}_{\widetilde{b}}\otimes\pi_{\widetilde{\chi}})\simeq\begin{cases}\mathcal{F}_{\widetilde{\chi}_c\otimes\widetilde{\chi}}[d_{\widetilde{b}}] & \text{if } b_1=b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}\\ 0 & \text{otherwise.}\end{cases}$$

We can check from the constructions that $\mathcal{E}_{b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}}\oplus\mathcal{E}_{b_2}\simeq\mathcal{E}_{b_{\chi_c\otimes\chi}}$. We see that if $b\neq b_{\chi_c\otimes\chi}$ then $b_1\neq b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}$ and the $[\phi_c]$-isotypical part of $R\Gamma_c(\mathrm{GL}_m,\widetilde{b},b_1,\mu_1)[\delta^{-1/2}_{\chi|\mathrm{G}_{\widetilde{b}}}\otimes\widetilde{\pi}\otimes\kappa^{-1}_{\widetilde{b}}]$ is empty. Thus by equality (4), the $[\phi_c]$-isotypical part of $R\Gamma_c(\mathrm{GL}_n,b_\chi,b,\mu)[\delta^{-1/2}_\chi\otimes\pi]$ is empty and then $i^*_bC_{\chi_c}\star\mathcal{F}_\chi\simeq 0$.

Note that in this case by using the fact that all the slopes of the Newton polygon of $\mathcal{E}_{b_{\widetilde{\chi}}}$ are bigger than that of $\mathcal{E}_{b_2}$ we can see that the filtration $\mathcal{E}_{b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}}\subset\mathcal{E}_{b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}}\oplus\mathcal{E}_{b_2}\simeq\mathcal{E}_{b_{\chi_c\otimes\chi}}$ can be refined to the canonical Harder-Narasimhan filtration of $\mathcal{E}_{b_{\chi_c\otimes\chi}}$. However in the context of lemma 5.6, it is not always true. Nevertheless, it is still true under the assumption that $b'=b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}$.

Now we suppose that $b=b_{\chi_c\otimes\chi}$ then $b_1=b_{\widetilde{\chi}_c\otimes\widetilde{\chi}}$. Note that by the assumption, all the slopes of the Newton polygon of $\mathcal{E}_{b_1}\simeq\mathcal{G}$ are strictly bigger than λ_{i-1}, we deduce that $\mathrm{I}(i-1)=\{c\}$. By (7) we see that

$$R\Gamma_c(\mathrm{GL}_m,\widetilde{b},b_1,\mu_1)[\delta^{-1/2}_{\widetilde{b}}\otimes\widetilde{\pi}][2d_{\widetilde{b}}]\simeq\bigoplus_{b_{\widetilde{\chi}_{c'}\otimes\widetilde{\chi}}=b_1}\delta^{-1/2}_{b_1}\otimes\pi_{\widetilde{\chi}_{c'}\otimes\widetilde{\chi}}\otimes\kappa_{b_1}\boxtimes\phi_{c'}[d_{\widetilde{b}}-d_{b_1}]$$

where the sum is over all $c'\in\left(\bigcup_{1\leq j<i}\mathrm{I}(j)\right)$ such that $\widetilde{\chi}_{c'}\otimes\widetilde{\chi}=b_1$.

By combining with equations (4), (5), (6) we have:

$$\begin{aligned}
&i^*_b\mathrm{T}_\mu(i_{b_\chi!}\delta^{-1/2}_\chi\otimes\pi\otimes\kappa_{b_\chi})\\
&\simeq\kappa\otimes\Big(R\Gamma_c(\mathrm{GL}_m,\widetilde{b},b_1,\mu_1)[\delta^{-1/2}_{\chi|\mathrm{G}_{\widetilde{b}}}\otimes\widetilde{\pi}]\times R\Gamma_c(\mathrm{GL}_{n-m},b_2,b_2,\mu_2)[\delta^{-1/2}_{\chi|\mathrm{G}_{b_2}}\otimes\pi_2]\Big)[h-2d+2d_{b_\chi}](\tfrac{h}{2})\\
&\simeq\kappa_b\otimes(\kappa^{-1}_{b_1}\times 1)\otimes\Big(R\Gamma_c(\mathrm{GL}_m,\widetilde{b},b_1,\mu_1)[\delta^{-1/2}_{\chi|\mathrm{G}_{\widetilde{b}}}\otimes\widetilde{\pi}]\times\delta^{-1/2}_{\chi|\mathrm{G}_{b_2}}\pi_2\Big)[h-2d+2d_{b_\chi}-2d_{b_2}](\tfrac{h}{2})\\
&\simeq\bigoplus_{b_{\widetilde{\chi}_{c'}\otimes\widetilde{\chi}}=b_1}\kappa_b\otimes\big(\delta^{-1/2}_{\chi\otimes\chi_{c'}|\mathrm{G}_{b_1}}\pi_{\widetilde{\chi}_{c'}\otimes\widetilde{\chi}}\times\delta^{-1/2}_{\chi|\mathrm{G}_{b_2}}\pi_2\big)\boxtimes\phi_{c'}[h-2d+2d_{b_\chi}-2d_{b_2}-d_{\widetilde{b}}-d_{b_1}]
\end{aligned}$$

where in the last line, there is a Tate twist $(\frac{-h}{2})=(\frac{m-n}{2})$ coming from the difference between the action of $W_{\mathbb{Q}_p}$ in $i^*_{b_1}\mathrm{T}_{\mu_1}\big(i_{\widetilde{b}!}\delta^{-1/2}_{\chi|\mathrm{G}_{\widetilde{b}}}\otimes\pi_{\widetilde{\chi}}\big)$ and in $i^*_{b_1}\mathrm{T}_{\mu_1}\big(i_{\widetilde{b}!}\delta^{-1/2}_{\widetilde{b}}\otimes\pi_{\widetilde{\chi}}\big)$. Since $\mathcal{F}_\chi=(i_{b_\chi!}\delta^{-1/2}_\chi\otimes\pi)[-d_{b_\chi}]$ and by the condition $\mathrm{I}(i-1)=\{c\}$, we can verify that

$$d_b=d_{b_\chi}-d_{\widetilde{b}}+d_{b_1}+h,$$

moreover, we have the identity $d=d_b-d_{b_1}-d_{b_2}$. Thus, by using lemma 5.4 and by replacing π by $\pi\otimes\kappa^{-1}_{b_\chi}$ in the computation of $i^*_b\mathrm{T}_\mu(i_{b_\chi!}\delta^{-1/2}_\chi\otimes\pi\otimes\kappa_{b_\chi})$, we see that

$$i^*_b\mathrm{T}_\mu(i_{b_\chi!}\delta^{-1/2}_\chi\otimes\pi)\simeq\bigoplus_{b_{\widetilde{\chi}_{c'}\otimes\widetilde{\chi}}=b_1}\big(\delta^{-1/2}_{\chi\otimes\chi_{c'}}\pi_{\chi\otimes\chi_{c'}}\big)\boxtimes\phi_{c'}[-h+d_{\widetilde{b}}-d_{b_1}].$$

By identifying the $W_{\mathbb{Q}_p}$ action, we see that

$$i^*_bC_{\chi_c}\star(i_{b_\chi!}\delta^{-1/2}_\chi\otimes\pi)[-d_{b_\chi}]\simeq\delta^{-1/2}_{\chi\otimes\chi_c}\pi_{\chi\otimes\chi_c}[-h+d_{\widetilde{b}}-d_{b_\chi}-d_{b_1}]$$

and we deduce that

$$i_b^* C_{\chi'} \star \mathcal{F}_\chi \simeq \mathcal{F}_{\chi \otimes \chi_c}.$$

Now we treat the case $m'_i < m_i$. Thus again by proposition 5.1 we have

$$\mathrm{Sht}(\mathrm{GL}_n, b_\chi, b, \mu) = \Big(\mathrm{Sht}(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathcal{J}^U\Big) \times^{\underline{\mathrm{P}}} \underline{\mathrm{G}}$$

with the similar notations as before. The only difference is that now P is a proper parabolic subgroup of $\mathrm{G} := \mathrm{G}_{b_\chi}$. Unravelling the definition of these groups, we see that

$$\mathrm{G}_{b_\chi} = \mathrm{Aut}(\mathcal{O}(\lambda_1)^{m_1}) \times \ldots \times \mathrm{Aut}(\mathcal{O}(\lambda_i)^{m_i}) \times \ldots \times \mathrm{Aut}(\mathcal{O}(\lambda_k)^{m_k})$$

and P is the opposite parabolic subgroup whose Levi factor is

$$\mathrm{M} = \mathrm{Aut}(\mathcal{O}(\lambda_1)^{m_1}) \times \ldots \times \mathrm{Aut}(\mathcal{O}(\lambda_i)^{m_i - m'_i}) \times \mathrm{Aut}(\mathcal{O}(\lambda_i)^{m'_i}) \times \ldots \times \mathrm{Aut}(\mathcal{O}(\lambda_k)^{m_k}).$$

Thus

$$R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu) = \mathrm{ind}_{\mathrm{P}}^{\mathrm{G}}\Big(R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1) \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2) \times R\Gamma_c(\mathcal{J}^U)\Big)$$

and we want to compute $R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\delta_{b_\chi}^{-1/2} \otimes \pi_\chi \otimes \kappa_{b_\chi}^{-1}]$ where π_χ is a representation of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$ associated to the character χ. Thus we need to find a way to get rid of the parabolic induction $\mathrm{ind}_{\mathrm{P}}^{\mathrm{G}}$. Note that we do not need to deal with this issue in the context of lemma 5.6 since the parabolic induction in that case is from some parabolic subgroup of $\mathrm{G}_b(\mathbb{Q}_p)$.

Let π be an irreducible representation of $\mathrm{G}'(\mathbb{Q}_p)$ where $\mathrm{G}' = \mathrm{Aut}(\mathcal{O}^{m_i}(\lambda_i))$ whose L-parameter is given by $\bigoplus_{j \in \mathrm{I}(i)} \phi_j$ then by second adjointness theorem we see that

$$\mathrm{Hom}_{\mathrm{G}'}(\mathrm{Ind}_{\mathrm{P}'}^{\mathrm{G}'}(\rho), \pi) = \mathrm{Hom}_{\mathrm{M}'}(\rho, r_{\overline{\mathrm{P}'}}(\pi))$$

where $r_{\overline{\mathrm{P}'}}$ denotes the normalized Jacquet functor; $\mathrm{M}' = \mathrm{Aut}(\mathcal{O}(\lambda_i)^{m_i - m'_i}) \times \mathrm{Aut}(\mathcal{O}(\lambda_i)^{m'_i})$ and P' is the standard parabolic subgroup of G' whose Levi factor is M' and $\overline{\mathrm{P}'}$ is the opposite parabolic of P'. Thus the constituents of $r_{\overline{\mathrm{P}'}}(\pi)$ are the representations of $\mathrm{M}'(\mathbb{Q}_p)$ whose L-parameter post-compose with the natural embedding $\widehat{\mathrm{M}'} \longrightarrow \widehat{\mathrm{G}'}$ is given by $\bigoplus_{j \in \mathrm{I}(i)} \phi_j$. Note that the ϕ_j's are pairwise distinct then the L-parameters of these representations of $\mathrm{M}'(\mathbb{Q}_p)$ are pairwise distinct. Now since there is no non-trivial extension between tempered representations with different cuspidal supports, we deduce that

$$r_{\overline{\mathrm{P}'}}(\pi) = \bigoplus_{A \in \mathbb{A}} \pi_A$$

where $\mathbb{A}$ is the set of subsets A of $\mathrm{I}(i)$ such that $\sum_{j \in A} \dim \phi_j = \mathrm{rank}\, \mathcal{O}(\lambda_i)^{m_i - m'_i}$ and π_A is the representation $\pi_{1,A} \times \pi_{2,A}$ of $\mathrm{M}'(\mathbb{Q}_p)$ whose L-parameter is given by $\bigoplus_{j \in A} \phi_j \times \bigoplus_{j \in \mathrm{I}(i) \setminus A} \phi_{j'}$.

By using the Hom-Tensor duality we see that (up to some twists)

$$R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\pi_\chi] \otimes \kappa^{-1} = \bigoplus_{A \in \mathbb{A}} R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\pi_{1,\chi}^A] \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\pi_{2,\chi}^A],$$

where for each $A \in \mathbb{A}$, $\pi_{1,\chi}^A \times \pi_{2,\chi}^A$ is the representation of $\mathrm{M}(\mathbb{Q}_p)$ such that the $\mathrm{Aut}(\mathcal{O}^{m_j}\lambda_j)$-factor is given by π_A if $j = i$ and is given by the $\mathrm{Aut}(\mathcal{O}(\lambda_j)^{m_j})$-factor of π_χ if $i \neq j$.

Now we can use the same arguments as in the case $m_i = m'_i$. We can show the vanishing results in exactly the same manner. More precisely, if $c \in \big(\mathrm{I}(i) \setminus A\big) \cup \big(\bigcup_{i < j \le k} \mathrm{I}(j)\big)$ then the $[\phi_c]$-isotypical part of $R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\pi_{1,\chi}^A] \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\pi_{2,\chi}^A]$ is empty since $W_{\mathbb{Q}_p}$ acts trivially on $R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)$. In particular if $c \in \big(\bigcup_{i<j\le k} \mathrm{I}(j)\big)$ then the $[\phi_c]$-isotypical part of $R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\pi_\chi]$ is empty.

If $b \neq b_{\chi\otimes\chi_c}$ then $i_{b'}^* C_{\chi'} \star \mathcal{F}_\chi \simeq 0$ as before. Now we suppose that $c \in \left(\bigcup_{1<j\leq i} \mathrm{I}(j)\right)$ and $b = b_{\chi\otimes\chi_c}$ then we can show that $c \in \mathrm{I}(i)$ and $\dim \phi_c = \operatorname{rank} \mathcal{O}(\lambda_i)^{m_i - m'_i}$. In particular, if $c \in A$ then $A = \{c\}$. Thus when taking the $[\phi_c]$-isotypical part we have (up to some twists)

$$\begin{aligned} R\Gamma_c(\mathrm{GL}_n, b_\chi, b, \mu)[\pi_\chi][\phi_c] &= \bigoplus_{A\in\mathbb{A}} R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\pi_{1,\chi}^A][\phi_c] \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\pi_{2,\chi}^A] \\ &= \bigoplus_{A\in\mathbb{A},\ c\in A} R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\pi_{1,\chi}^A][\phi_c] \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\pi_{2,\chi}^A] \\ &= R\Gamma_c(\mathrm{GL}_m, \widetilde{b}, b_1, \mu_1)[\pi_{1,\chi}^{\{c\}}][\phi_c] \times R\Gamma_c(\mathrm{GL}_{n-m}, b_2, b_2, \mu_2)[\pi_{2,\chi}^{\{c\}}] \end{aligned}$$

There is actually at most 1 term in the last sum and by the same computations as before we can show that

$$i_b^* C_{\chi'} \star \mathcal{F}_\chi \simeq \mathcal{F}_{\chi\otimes\chi_c}.$$

and we are done. ☐

6. Proof of the first main theorem

This section is devoted to the proof of theorem 4.5. By [Ngu20, theorem 6.3], it is enough to prove the theorem for $\chi \in \mathrm{Irr}(S_\phi)^+ := \{(d_1, \dots, d_r) \in \mathrm{Irr}(S_\phi) \mid d_i \geq 0\ \forall\ 1 \leq i \leq r\}$. We will proceed by induction on r. One can deduce the case $r = 1$ by [ALB21, theorem 1.2], [Han20, theorem 1.5]. Thus we mainly focus on proving the case $r \geq 2$. The arguments are rather technical and complicated. However it is greatly simpler if we suppose further that ϕ is a direct sum of characters. In particular, under this assumption, the slopes of the Newton point ν_{b_χ} are all integer for $\chi \in \mathrm{Irr}(S_\phi)$. The readers who only want to grab the main ideas could impose that condition in the rest of this section.

Suppose that the theorem is true for $1, \dots, r-1$ then we show that it is true for r. Now we proceed by induction on $D = \sum_{i=1}^r d_i =: |\chi|$. For $D = 1$, one could use lemmas 5.5, 5.6 to verify the theorem. However, we will use the known cases of Harris-Viehmann's conjecture to show the base case. Then we suppose that the theorem is true for $D < s$ and prove the case $D = s$. By a direct application of proposition 5.1 and lemmas 5.5, 5.6, we can show that $i_{b_\chi}^* C_\chi \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_\chi$. The most technical part of the proof is to show that the restrictions of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ to other strata vanish.

In this section, we fix a maximal split torus and a Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{GL}_n$. For $b, b' \in B(\mathrm{GL}_n)$, we have $\nu_b = (-\nu_{\mathcal{E}_b})_{\mathrm{dom}}$ and $\nu_{b'} = (-\nu_{\mathcal{E}_{b'}})_{\mathrm{dom}}$. Thus b is smaller than b' with respect to the usual partial order in $B(\mathrm{GL}_n)$ if and only if $\nu_{\mathcal{E}_b}$ is smaller than $\nu_{\mathcal{E}_{b'}}$ with respect to the usual partial order in $X_*(\mathrm{T})_{\mathbb{Q}}$. We always use the partial order when we compare elements $b, b' \in B(\mathrm{GL}_n)$ and elements $\nu_{\mathcal{E}_b}$, $\nu_{\mathcal{E}_{b'}}$ in $X_*(\mathrm{T})_{\mathbb{Q}}$.

6.1. The base case.

We prove the case $D = 1$. For $1 \leq i \leq r$, let $\chi_i = (d_1, \dots, d_r)$ be the character such that $d_i = 1$ and $d_j = 0$ for $1 \leq j \neq i \leq r$. We will show that $C_{\chi_i^{-1}} \star \mathcal{F}_{\chi_i} = \mathcal{F}_{\mathrm{Id}}$. Then by the monoidal property and the fact that $C_{\mathrm{Id}} \star \mathcal{F}_\chi = \mathcal{F}_\chi$ for all χ, we deduce that $C_{\chi_i} \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_{\chi_i}$.

Let $\mu = (1, 0^{(n-1)})$. Fix $1 \leq i \leq r$, we have $\mathcal{E}_{b_{\chi_i}} \simeq \mathcal{O}(1/n_i) \oplus \mathcal{O}^{n-n_i}$. By the Harris-Viehmann conjecture (proposition 5.1) for $\mathrm{Sht}(\mathrm{GL}_n, b_{\chi_i}, \mu)$ and [HT01], [ALB21, theorem 1.2], [Han20, theorem 1.5], we see that

$$\mathrm{Sht}(\mathrm{GL}_n, b_{\chi_i}, \mu) = \left(\mathrm{Sht}(\mathrm{GL}_{n-n_i}, 1_{\mathrm{GL}_{n-n_i}}, \mathrm{Id}) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \mathrm{Sht}(\mathrm{GL}_{n_i}, b_1, \mu_1) \times_{\mathrm{Spd}\breve{\mathbb{Q}}_p} \widetilde{\mathrm{G}}_{b_{\chi_i}}^0\right) \times^{\underline{\mathrm{P}}} \underline{\mathrm{GL}_n}.$$

where $1_{\mathrm{GL}_{n-n_i}}$ is the trivial element in $\mathrm{GL}_{n-n_i}(\breve{\mathbb{Q}}_p)$, $\mathcal{E}_{b_1} \simeq \mathcal{O}(1/n_i)$, $\mu_1 = (1, 0^{(n_i-1)})$ is a cocharacter of GL_{n_i} and P is the opposite parabolic subgroup of GL_n whose Levi factor is

$\mathrm{GL}_{n_i} \times \mathrm{GL}_{n-n_i}$. By taking the cohomology and lemma 5.4, we have

$$
\begin{aligned}
& R\Gamma_c(\mathrm{GL}_n, b_{\chi_i}, \mu)[\delta_{b_{\chi_i}}^{-1/2} \otimes \pi_{\chi_i} \otimes \kappa_{b_{\chi_i}}^{-1}] \\
\simeq\ & R\Gamma_c(\mathrm{GL}_n, b_{\chi_i}, \mu)[\delta_{b_{\chi_i}}^{-1/2} \otimes \pi_{\chi_i}] \\
\simeq\ & \mathrm{ind}_{\mathrm{P}}^{\mathrm{GL}_n}\Big(\mathrm{Sht}(\mathrm{GL}_{n_i}, b_1, \mu_1)(\delta_{b_{\chi_i}|\mathrm{GL}_{n-n_i}}^{-1/2} \otimes \pi) \boxtimes \mathrm{Sht}(\mathrm{GL}_{n_i}, b_1, \mu_1)[\delta_{b_{\chi_i}|\mathrm{G}_{b_1}}^{-1/2} \otimes \pi_1^{b_1}]\Big)[(n-n_i) - 2(n-n_i)](\tfrac{n-n_i}{2}) \qquad (8) \\
\simeq\ & \mathrm{ind}_{\mathrm{P}}^{\mathrm{GL}_n}\Big(\delta_{\mathrm{P}}^{1/2} \otimes \big(\pi \boxtimes \pi_1\big)\Big) \boxtimes \phi_i^{\vee}[n_i - n] \\
\simeq\ & \pi_{\mathrm{Id}} \boxtimes \phi_i^{\vee}[n_i - n] \qquad (9)
\end{aligned}
$$

as complexes of $\mathrm{GL}_n(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-modules where π is the representation of $\mathrm{GL}_{n-n_i}(\mathbb{Q}_p)$ whose L-parameter is given by $\bigoplus_{1 \leq j \neq i \leq r} \phi_j$ and $\pi_1^{b_1}$ resp. π_1 are supercuspidal representations of $\mathrm{G}_{b_1}(\mathbb{Q}_p)$, resp. $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ whose L-parameter is given by ϕ_i and $\vee$ denotes the contragredient representation. In particular, we see that $\pi_{\chi_i} \simeq \pi \boxtimes \pi_1^{b_1}$. Moreover, since the opposite parabolic subgroup P^{op} is standard, we see that $\delta_{b_{\chi_i}}$ and δ_{P}^{-1} have the same L-parameters.

Note that in equation (8) the cohomology of the connected component of the automorphisms group $\widetilde{G}_{b_{\chi_i}}^0$ contributes a shift $[-2(n-n_i)]$, the normalization in the sheaves $\mathcal{S}_\mu$ and $\mathcal{S}_{\mu_1}$ contributes a twist $(\frac{n-n_i}{2})$ and a shift $[n-n_i]$. There is no Tate twist in the third line since the modulus character $\delta_{b_{\chi_i}|\mathrm{G}_{b_1}}^{-1/2}$ contributes a twist $(-\frac{n-n_i}{2})$ when taking the cohomology of local Shimura variety and then $\mathrm{Sht}(\mathrm{GL}_{n_i}, b_1, \mu_1)[\delta_{b_{\chi_i}|\mathrm{G}_{b_1}}^{-1/2} \otimes \pi_1^{b_1}] \simeq \delta_{\mathrm{P}|\mathrm{GL}_{n_i}}^{1/2} \otimes \pi_1 \boxtimes \phi_i^{\vee}(-\frac{n-n_i}{2})$.

However $r_{\mu^{-1}} \circ \phi = \bigoplus_{j=1}^{r} \chi_j^{-1} \boxtimes \phi_j^{\vee}$ and $\mathcal{F}_{\chi_i} := i_{b_{\chi_i}!}(\delta_{b_{\chi_i}}^{-1/2} \otimes \pi_{\chi_i})[n_i - n]$ then by lemma 2.10 we deduce that

$$
R\Gamma_c(\mathrm{GL}_n, b_{\mathrm{Id}}, b_{\chi_i}, \mu)[\delta_{b_{\chi_i}}^{-1/2} \otimes \pi_{\chi_i} \otimes \kappa_{b_{\chi_i}}^{-1}][n-n_i] \simeq i_1^* \mathrm{T}_{\mu^{-1}}(\mathcal{F}_{\chi_i}) \simeq \bigoplus_{j=1}^{r} i_1^* C_{\chi_j^{-1}} \star \mathcal{F}_{\chi_i} \boxtimes \phi_j^{\vee}.
$$

By using equation (9), we see that the left hand side is isomorphic to $\mathcal{F}_{\mathrm{Id}} \boxtimes \phi_i^{\vee}$. By identifying the $W_{\mathbb{Q}_p}$-action, we deduce that

$$
i_1^* C_{\chi_i^{-1}} \star \mathcal{F}_{\chi_i} \simeq \mathcal{F}_{\mathrm{Id}},
$$

and for $j \neq i$, we have

$$
i_1^* C_{\chi_j^{-1}} \star \mathcal{F}_{\chi_i} \simeq 0. \qquad (10)
$$

To show that $C_{\chi_i^{-1}} \star \mathcal{F}_{\chi_i} \simeq \mathcal{F}_{\mathrm{Id}}$, we need to show that the restrictions of $C_{\chi_i^{-1}} \star \mathcal{F}_{\chi_i}$ to other HN-strata are zero. Thus we need to compute the image of b_{χ_i} by some modifications of type μ^{-1}. The set of images is given by $\Big\{\mathcal{O}^n,\ \mathcal{E}_{b_{m,m'}} := \mathcal{O}(1/n_i) \oplus \mathcal{O}^m \oplus \mathcal{O}(-1/m') \mid n_i + m + m' = n\Big\}$ by example 2.3.

Thus we need to compute $R\Gamma_c(\mathrm{GL}_n, b_{\chi_i}, b_{m,m'}, \mu)$ for $m' \neq 0$. To do this, we use the same arguments as in lemmas 5.5, 5.6. Hence we have

$$
i_{b_{m,m'}}^* C_{\chi_i^{-1}} \star \mathcal{F}_{\chi_i} = 0.
$$

6.2. Induction step I : Restriction of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ to the stratum b_χ.

Suppose that the theorem is true for $D = 1, \ldots, s-1$. Now consider the case $D = s$. Let $\chi = (d_1, \ldots, d_r)$ be an element in $\mathrm{Irr}(S_\phi)$ such that the d_i's are non-negative and $\sum_{i=1}^{r} d_i = s$.

Suppose that we have $\mathcal{E}_{b_\chi} = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_k)^{m_k}$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_k \geq 0$ and $m_i > 0$ for $1 \leq i \leq k$. By the construction of b_χ, there exists for each $1 \leq i \leq k$, a non-empty subset $\mathrm{I}(i)$ of $\{1, 2, \ldots, r\}$ such that for $j \in \mathrm{I}(i)$, we have $d_j/n_j = \lambda_i$. We denote by $r(\chi)$ the number $\min\{\dim \phi_j \mid j \in \mathrm{I}(k)\}$.

We show that $i^*_{b_\chi} C_\chi \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_\chi$.

We suppose that $\lambda_k = 0$. The case $\lambda_k > 0$ is simpler and can be treated similarly.

In this case $\lambda_{k-1} > 0$. We consider a character $\chi' = (d'_1, \dots, d'_r)$ where for one index $j^* \in \mathrm{I}(k-1)$ we have $d'_{j^*} = d_{j^*} - 1$ and $d'_m = d_m$ for all other $r-1$ indexes. In particular the d_j's are non-negative and $\sum_{i=1}^{k} d'_i = s-1$. By construction $\mathcal{E}_{b_{\chi'}} = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}-t} \oplus \mathcal{O}(\lambda'_k)^{t'} \oplus \mathcal{O}(\lambda_k)^{m_k}$ where $\lambda'_k = \frac{d'_{j^*}}{n_{j^*}}$; $t = \gcd(d_{j^*}, n_{j^*})$ and $t' = \gcd(d'_{j^*}, n_{j^*})$. Moreover we have $\chi = \chi_{j^*} \otimes \chi'$. Thus by monoidal property we have

$$C_\chi \star \mathcal{F}_{\mathrm{Id}} = C_{\chi_{j^*}} \star (C_{\chi'} \star \mathcal{F}_{\mathrm{Id}}).$$

By the induction hypothesis (on D) for χ' we see that $C_{\chi'} \star \mathcal{F}_{\mathrm{Id}} = \mathcal{F}_{\chi'}$. Thus we need to compute $i^*_{b_\chi} C_{\chi_{j^*}} \star \mathcal{F}_{\chi'}$. We can do it by computing $R\Gamma_c(\mathrm{GL}_n, b_{\chi'}, b_\chi, \mu)[\delta^{1/2}_{b_{\chi'}} \otimes \pi_{\chi'}]$ where $\mu = (1, 0^{(n-1)})$. The triple $(b_{\chi'}, b_\chi, \mu)$ satisfies the hypothesis of corollary 5.2 and lemma 5.6 then we deduce that

$$i^*_{b_\chi} C_{\chi_{j^*}} \star \mathcal{F}_{\chi'} = \mathcal{F}_\chi.$$

Now we need to show that the restrictions of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ to other strata vanish.

6.3. **Induction step II : Vanishing of the restrictions of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ to other strata.**

From now on we suppose that $C_\chi \star \mathcal{F}_{\mathrm{Id}} \not\simeq \mathcal{F}_\chi$ and try to find a contradiction.

6.3.1. *The case where $\lambda_k > 0$.*

We suppose $\lambda_k > 0$ in this paragraph.

In the last subsection, we associated a natural number $r(\chi)$ to χ. Let C denotes the set of characters $\xi \in \mathrm{Irr}(S_\phi)^+$ such that $\mathcal{E}_{b_\xi}$ is a direct sum of semi-stable vector bundles of strictly positive slopes, $C_\xi \star \mathcal{F}_{\mathrm{Id}} \not\simeq \mathcal{F}_\xi$ and $|\xi| = s$. Since C is finite, we can assume that $r(\chi) = \min\{r(\xi) \mid \xi \in C\}$.

Consider a character $\chi' = (d'_1, \dots, d'_r)$ where $d'_{j^*} = d_{j^*} - 1$ for one index $j^* \in \mathrm{I}(k)$ such that $\dim \phi_{j^*} = r(\chi)$ and $d'_i = d_i$ for all other $r-1$ indexes. Hence $\mathcal{E}_{b_{\chi'}} = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}} \oplus \mathcal{O}(\lambda_k)^{m_k - t} \oplus \mathcal{O}(\lambda'_k)^{t'}$ where $t = \gcd(d_j, n_j)$; $t' = \gcd(d'_{j^*}, n_{j^*})$ and $\lambda'_k = \frac{d'_{j^*}}{n_{j^*}}$.

We have $\chi = \chi_{j^*} \otimes \chi'$ and then $C_\chi \star \mathcal{F}_{\mathrm{Id}} = C_{\chi_{j^*}} \star \mathcal{F}_{\chi'}$. Thus the set of strata where the restriction of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non null is a subset of the strata indexed by images of the modification of $\mathcal{E}_{b_{\chi'}}$ of type $\mu = (1, 0^{(n-1)})$. Moreover, by proposition 4.3 and the fact that the formalism of Fargues-Scholze parameter and the restrictions to Harder-Narasimhan strata commute [Ham21, Proppsition 3.14, Corollary 3.15], we deduce that these strata are of the form b_ξ for some $\xi \in \mathrm{Irr}(S_\phi)$.

Since $C_\chi \star \mathcal{F}_{\mathrm{Id}} \neq \mathcal{F}_\chi$, we can choose an element $b \neq b_\chi \in B(\mathrm{GL}_n)$ such that $i^*_b C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non trivial. Thus there is a modification of type μ

$$f : \mathcal{E}_{b_{\chi'}} \longrightarrow \mathcal{E}_b.$$

Remark that if the biggest slope of $\nu_{\mathcal{E}_b}$ equals to λ_1 then we can apply lemmas 5.5, 5.6 to the triple $\mathcal{E}_{b_{\chi'}}$, $\mathcal{E}_b$ and $\mu = (1, 0^{(n-1)})$ to compute $i^*_b C_\chi \star \mathcal{F}_{\mathrm{Id}}$ and check that if $b \neq b_\chi \in B(\mathrm{GL}_n)$ then the restriction $i^*_b C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is trivial, which is a contradiction. Hence we deduce that the biggest slope of $\nu_{\mathcal{E}_b}$ is not equal to λ_1.

Consider the filtration $\mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}} \oplus \mathcal{O}(\lambda_k)^{m_k-t} \subset \mathcal{E}_{b_{\chi'}}$. Denote by P the standard parabolic subgroup corresponding to this filtration as well as M its Levi factor. Hence $\mathrm{M} \simeq \mathrm{GL}_{n-n_{j^*}} \times \mathrm{GL}_{n_{j^*}}$. The modification μ induces a filtration $\mathcal{E} \subset \mathcal{E}_b$ of $\mathcal{E}_b$. The knowledge of this filtration and that of extension between vector bundles allows us to understand b.

The M-bundle corresponding to the filtration of $\mathcal{E}_{\chi'}$ is $\mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}} \oplus \mathcal{O}(\lambda_k)^{m_k-t} \times \mathcal{O}(\lambda'_k)^{t'}$. The modification f induces modifications

$$f_1 : \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}} \oplus \mathcal{O}(\lambda_k)^{m_k-t} \longrightarrow \mathcal{E}$$

and

$$f_2 : \mathcal{O}(\lambda'_k)^{t'} \longrightarrow \mathcal{E}_b/\mathcal{E} =: \mathcal{E}'.$$

of type μ_1 and μ_2 respectively. By [CFS21, lemma 2.6], [Vie, lemma 3.10], there are 2 possibilities:

Case 1: $\mu_1 = (1, 0^{(n-n_{j^*}-1)})$ and $\mu_2 = (0^{(n_{j^*})})$.

Thus $\mathcal{E}' \simeq \mathcal{O}(\lambda'_k)^{t'}$ and in particular, all the slopes of $\nu_{\mathcal{E}}$ are strictly bigger than λ'_k (even strictly bigger than λ_k if $|\mathrm{I}(k)| = 1$). Since $H^1(\mathcal{O}(\lambda)) = 0$ if $\lambda \geq 0$ we deduce that $\mathcal{E}_b = \mathcal{E} \oplus \mathcal{O}(\lambda'_k)^{t'}$.

Thus we can apply lemma 5.5 to the triple $\mathcal{E}_{b_{\chi'}}$, $\mathcal{E}_b$ and χ_{j^*} to deduce that if $b \neq b_\chi \in B(\mathrm{GL}_n)$ then

$$i_b^* C_{\chi_{j^*}} \star \mathcal{F}_{\chi'} = i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}} \simeq 0,$$

which is a contradiction as above.

Case 2: $\mu_1 = (0^{(n-n_{j^*})})$ and $\mu_2 = (1, 0^{(n_{j^*}-1)})$.

In particular $\mathcal{E} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}} \oplus \mathcal{O}(\lambda_k)^{m_k-t}$. Since $\mathcal{O}(\lambda'_k)^{t'}$ is semi-stable, we can compute $\mathcal{E}'$ explicitly. If $\mathcal{E}'$ is a semi-stable vector bundle then it is semi-stable of slope λ_k and therefore we find out that $b = b_\chi$ and we are done. If it is not the case we still see that $\mathcal{E}' = \mathcal{E}_x \oplus \mathcal{E}''$ where $\mathcal{E}''$ is a semi-stable vector bundle of slope λ such that $\lambda_k > \lambda \geq 0$. Moreover if h is a slope of $\nu_{\mathcal{E}_x}$ then $\lambda < h \leq \lambda'_k + 1 \leq \lambda + 1$.

Lemma 6.1. *We have $\lambda > 0$.*

Proof. Indeed, if $\lambda = 0$ then $\lambda'_k = 0$ and therefore $\mathcal{E}_b \simeq \mathcal{E}_y \oplus \mathcal{O}^{\ell_1}$ for some vector bundle $\mathcal{E}_y$ such that the slopes of $\nu_{\mathcal{E}_y}$ are strictly positive and $\ell_1 < n_{j^*}$. We also see that $\mathcal{O}^{n_{j^*}}$ is a direct factor of $\mathcal{E}_{\chi'}$. Therefore we can apply lemma 5.5 to triple $\mathcal{E}_{\chi'}$, $\mathcal{E}_b$ and χ_{j^*} to deduce that

$$i_b^* C_{\chi_{j*}} \star \mathcal{F}_{\chi'} \simeq 0,$$

a contradiction. Hence we have $\lambda > 0$. □

By the property of the Harder-Narasimhan filtration [NV, Theorem A.5], we see that $\nu_{\mathcal{E}_b}$ is bounded below by the concatenation $\nu_{\mathcal{E}} \oplus \nu_{\mathcal{E}'}$ of $\nu_{\mathcal{E}}$ and $\nu_{\mathcal{E}'}$. By [Che21, lemma 2.9] we also have $\nu_{\mathcal{E}_b} \leq \nu_{\mathcal{E} \oplus \mathcal{E}'}$. Therefore $\mathcal{E}_b \simeq \mathcal{E}_y \oplus \mathcal{E}''$ where $\mathcal{E}_y$ is a vector bundle such that if h is a slope of $\nu_{\mathcal{E}_y}$ then $h > \lambda$. Moreover we already know that if $h_{\max}$ is the biggest slope of $\nu_{\mathcal{E}_y}$ then $h_{\max} \neq \lambda_1$. Therefore $h_{\max} > \lambda_1$ and we deduce that b is not smaller than b_χ with respect to the usual order in $B(\mathrm{GL}_n)$.

Now we see that by proposition 4.3, there exists $\xi = (a_1, \ldots, a_r) \in \mathrm{Irr}(S_\phi)^+$ such that $b = b_\xi$, all the slopes of $\nu_{\mathcal{E}_{b_\xi}}$ are strictly positive and $r(\xi) < r(\chi)$ (since $\operatorname{rank} \mathcal{E}'' < r(\chi)$). Therefore

$$C_\xi \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\xi.$$

We know that the spectral action preserves compact objects and ULA objects then $i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is a ULA object and moreover all of its Schur-irreducible constituents have L-parameter given by ϕ (twisted by some map as in [FS21, Corollary IX.7.3]). Therefore, up to replacing ξ by another ξ' such that $b_\xi = b_{\xi'} = b$ and up to some shift, we can suppose that there is a non-trivial morphism

$$g_1 : i_{b!} i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}} \longrightarrow \mathcal{F}_\xi,$$

more precisely, we can construct g_1 from a quotient of the highest degree non-trivial cohomology group of $i_{b!} i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$.

Since the set $S_{\mathrm{supp}} \subset B(\mathrm{GL}_n)$ of strata where the restriction of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non trivial is finite, there exists maximal elements in S_{supp} with respect to the restriction of the usual order in $B(\mathrm{GL}_n)$. Since b is not smaller than b_χ with respect to that partial order, we can suppose that b is a maximal element in S_{supp}. Hence, by applying the excision exact triangle to the closed embedding $\mathrm{Bun}_n^{\geq b} \longrightarrow \mathrm{Bun}_n$, we deduce that there is a non trivial morphism

$$g_2 : C_\chi \star \mathcal{F}_{\mathrm{Id}} \longrightarrow i_{b!}i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}} \xrightarrow{g_1} \mathcal{F}_\xi \simeq C_\xi \star \mathcal{F}_{\mathrm{Id}}.$$

Recall that λ is the smallest slope of $\mathcal{E}_{b_\xi}$ and there exists an index $\widetilde{j}$ such that $\frac{a_{\widetilde{j}}}{n_{\widetilde{j}}} = \lambda$. Since $\lambda > 0$ we see that $\chi, \xi \in \mathrm{Irr}(S_\phi)^{>0} := \{(c_1, \ldots, c_r) \mid c_i > 0\}$. However for an arbitrary character $\theta \in \mathrm{Irr}(S_\phi)$, $C_\theta \star (C_{\theta^{-1}} \star (-)) = C_{\theta^{-1}} \star (C_\theta \star (-)) = C_{\mathrm{Id}} \star (-)$ is the identity functor of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$, we deduce that $C_{\theta^{-1}}(-)$ is an auto-equivalence of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$. Thus by applying the auto-equivalence $C_{\chi_{\widetilde{j}}^{-1}}$ to g_2 we get a non trivial morphism

$$g : C_{\chi \otimes \chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}} \longrightarrow C_{\xi \otimes \chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}}.$$

The characters $\chi \otimes \chi_{\widetilde{j}}^{-1}$ and $\xi \otimes \chi_{\widetilde{j}}^{-1}$ belong to $\mathrm{Irr}(S_\phi)^+$ then by the induction hypothesis (on D), we have $C_{\chi \otimes \chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_{\chi \otimes \chi_{\widetilde{j}}^{-1}}$ and $C_{\xi \otimes \chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_{\xi \otimes \chi_{\widetilde{j}}^{-1}}$. By the choice of $\widetilde{j}$, the biggest slope of $\mathcal{E}_{b_{\xi \otimes \chi_{\widetilde{j}}^{-1}}}$ is still $h_{\max}$ and the biggest slope of $\mathcal{E}_{b_{\chi \otimes \chi_{\widetilde{j}}^{-1}}}$ is not bigger than λ_1. In particular, $b_{\xi \otimes \chi_{\widetilde{j}}^{-1}}$ is not smaller than $b_{\chi \otimes \chi_{\widetilde{j}}^{-1}}$ with respect to the usual partial order of $B(\mathrm{GL}_n)$. Thus by lemma 2.6 there is no non-trivial morphism from $\mathcal{F}_{\chi \otimes \chi_{\widetilde{j}}^{-1}}$ to $\mathcal{F}_{\xi \otimes \chi_{\widetilde{j}}^{-1}}$. Hence a contradiction. In other words, the restriction $i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$ must be trivial.

6.3.2. *The case where $\lambda_k = 0$ and $r > 2$.*

We suppose that $\lambda_k = 0$ in this paragraph.

Consider a character $\chi' = (d'_1, \ldots, d'_r)$ where $d'_{j^*} = d_{j^*} - 1$ for one index $j^* \in \mathrm{I}(k-1)$ and $d'_i = d_i$ for all other $r-1$ indexes. Thus $\mathcal{E}_{b_{\chi'}} = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}-t} \oplus \mathcal{O}(\lambda'_k)^{t'} \oplus \mathcal{O}(\lambda_k)^{m_k}$ where $\lambda'_k = \frac{d'_{j^*}}{n_{j^*}}$; $t = \gcd(d_{j^*}, n_{j^*})$ and $t' = \gcd(d'_{j^*}, n_{j^*})$. In particular $\chi' \in \mathrm{Irr}(S_\phi)^+$ and $\chi = \chi_{j^*} \otimes \chi'$. Then by induction hypothesis we have $C_\chi \star \mathcal{F}_{\mathrm{Id}} = C_{\chi_{j^*}} \star \mathcal{F}_{\chi'}$.

As before we can choose an element $b \neq b_\chi \in B(\mathrm{GL}_n)$ such that $i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non trivial. Hence, there is a modification of type $\mu = (1, 0^{(n-1)})$

$$f : \mathcal{E}_{b_{\chi'}} \longrightarrow \mathcal{E}_b,$$

in particular $\mathcal{E}_b$ is a direct sum of semi-stable vector bundles with non-negative slopes. If the trivial line bundle $\mathcal{O}$ is a direct factor of $\mathcal{E}_b$ then we can apply lemma 5.5 to the triple $\mathcal{E}_{b_{\chi'}}$, $\mathcal{E}_b$ and μ to deduce that $\mathcal{E}_b \simeq \mathcal{E}_{b_\chi}$ and we are done.

Thus we suppose that $\mathcal{E}_b$ is a direct sum of semi-stable vector bundles whose slopes are strictly positive.

Consider the filtration $\mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}-t} \oplus \mathcal{O}(\lambda'_k)^{t'} \subset \mathcal{E}_{b_{\chi'}}$. Denote by P the standard parabolic subgroup corresponding to the filtration as well as M the Levi factor. Hence $\mathrm{M} \simeq \mathrm{GL}_{n-m} \times \mathrm{GL}_m$ where $m = \mathrm{rank}(\mathcal{O}(\lambda_k)^{m_k})$. The modification f induces a filtration $\mathcal{E} \subset \mathcal{E}_b$ of $\mathcal{E}_b$ and $\mathcal{E}_b/\mathcal{E} \simeq \mathcal{E}'$.

The M-bundle corresponding to the filtration of $\mathcal{E}_{b_{\chi'}}$ is $\mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda'_k)^{t'} \times \mathcal{O}(\lambda_k)^{m_k}$. The modification f induces modifications

$$f_1 : \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda'_k)^{t'} \longrightarrow \mathcal{E}$$

and

$$f_2 : \mathcal{O}(\lambda_k)^{m_k} \longrightarrow \mathcal{E}'$$

of type μ_1 and μ_2 respectively. As before we can compute the M-bundle $\mathcal{E} \times \mathcal{E}'$ by using [CFS21, lemma 2.6]. There are 2 possibilities:

Case 1: $\mu_1 = (1, 0^{(n-m-1)})$ and $\mu_2 = (0^{(m)})$.

Thus $\mathcal{E}' \simeq \mathcal{O}(\lambda_k)^{m_k}$ and all the slopes of $\nu_{\mathcal{E}}$ are strictly positive. Since $H^1(\mathcal{O}(\lambda)) = 0$ if $\lambda \geq 0$ we deduce that $\mathcal{E}_b = \mathcal{E} \oplus \mathcal{O}(\lambda_k)^{m_k}$. It can not happen since $\lambda_k = 0$ and we suppose that the trivial line bundle $\mathcal{O}$ is not a direct factor of $\mathcal{E}_b$.

Case 2: $\mu_1 = (0^{(n-m)})$ and $\mu_2 = (1, 0^{(m-1)})$ and in particular we have $\mathcal{E} \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda'_k)^{t'}$.

Since $\mathcal{O}(\lambda_k)^{m_k} \simeq \mathcal{O}^m$, we can compute $\mathcal{E}'$ explicitly.

If $\mathcal{E}'$ is not semi-stable then it is of the form $\mathcal{E}_z \oplus \mathcal{O}^a$ for some $a > 0$ and it can not happen as in case 1. Therefore we deduce that $\mathcal{E}'$ is semi-stable of slope $\lambda = 1/m$. Note that since $s > 1$ we see that $\deg \mathcal{E} \geq 1$.

Lemma 6.2. *We have*

$$\sum_{j=1}^{k-1} |\mathrm{I}(j)| > 1$$

Proof. Suppose the contrary that $\sum_{j=1}^{k-1} |\mathrm{I}(j)| = 1$. Thus we see that $k = 2$ and $\mathcal{E}$ is a semi-stable vector bundle of slope $\lambda'_1 > 0$. By the property of the Harder-Narasimhan filtration, we see that $\nu_{\mathcal{E}_b}$ is bounded below by the concatenation of $\nu_{\mathcal{E}}$ and $\nu_{\mathcal{E}'}$ that we denote by $\nu_{\mathcal{E}} \oplus \nu_{\mathcal{E}'}$. Moreover, by [Che21, lemma 2.9], $\nu_{\mathcal{E}_b}$ is bounded above by $\nu_{\mathcal{E} \oplus \mathcal{E}'}$.

If $\lambda < \lambda'_1$ then from the inequalities $\nu_{\mathcal{E}} \oplus \nu_{\mathcal{E}'} \leq \nu_{\mathcal{E}_b} \leq \nu_{\mathcal{E} \oplus \mathcal{E}'}$, we deduce that $\mathcal{E}_b = \mathcal{E} \oplus \mathcal{E}'$. In this case we can apply lemma 5.6 to the triple $\mathcal{E}_b$, $\mathcal{E}_{b_{\chi'}}$ and $C_{\chi_{j^*}}$ to deduce that $i_b^* C_{\chi_{j^*}} \star \mathcal{F}_{\chi'}$ vanishes if $b \neq b_\chi$.

Suppose that $\lambda \geq \lambda'_1$. Notice that by [Che21, Corollary 2.9] we see that if h is a slope of $\nu_{\mathcal{E}_b}$ then $\lambda \geq h > 0$ since all the slopes of $\nu_{\mathcal{E}}$ and of $\nu_{\mathcal{E}'}$ are strictly positive and smaller than λ.

Note that we have $|\mathrm{I}(2)| \geq 2$ since $r > 2$. By using proposition 4.3, we deduce that $\mathcal{E}_b$ has a slope not smaller than $1/n_i$ for every $i \in \mathrm{I}(2)$. However since $|\mathrm{I}(2)| \geq 2$, we deduce that $m = \operatorname{rank} \mathcal{O}(\lambda_2)^{m_2} > n_i$. It contradicts the fact that $\lambda = 1/m \geq 1/n_i$ and allows us to conclude that

$$\sum_{j=1}^{k-1} |\mathrm{I}(j)| > 1$$

□

Now by the lemma, we know that $\sum_{j=1}^{k-1} |\mathrm{I}(j)| > 1$, then the biggest slope of $\nu_{\mathcal{E}}$ is λ_1. Hence if $h_{\max}$ is the biggest slope of $\nu_{\mathcal{E}_b}$ then $h_{\max} \geq \lambda_1$.

If $h_{\max} = \lambda_1$ then again, by applying lemma 5.6 to $\mathcal{E}_b$, $\mathcal{E}_{\chi'}$ and $\mu = (1, 0^{(n-1)})$ we can conclude that $i_b^* C_{\chi_{j^*}} \star \mathcal{F}_{\chi'}$ vanishes if $b \neq b_\chi$ and it gives us a contradiction. Thus we deduce that $h_{\max} > \lambda_1$ and that b is not smaller than b_χ with respect to the usual order in $B(\mathrm{GL}_n)$.

By proposition 4.3, there exists $\xi = (a_1, \ldots, a_r) \in \mathrm{Irr}(S_\phi)^+$ such that $b = b_\xi$. Since $\mathcal{E}_b$ is a direct sum of semi-stable vector bundles with strictly positive slopes, by the previous paragraph we deduce that

$$C_\xi \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\xi.$$

We know that the spectral action preserves compact object then $i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is a compact object and moreover all of its Schur-irreducible constituents have L-parameter given by ϕ (twisted by some map as in [FS21, Corollary IX.7.3]). Therefore, up to replacing ξ by another ξ' such that $b_\xi = b_{\xi'} = b$ and up to some shift, we can suppose as before that there is a non-trivial morphism

$$g_1 : i_{b!} i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}} \longrightarrow \mathcal{F}_\xi.$$

Since the set $S_{\mathrm{supp}} \subset B(\mathrm{GL}_n)$ of strata where the restriction of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non trivial is finite, there exists maximal elements in S_{supp} with respect to the restriction of the usual order in $B(\mathrm{GL}_n)$. Since b is not smaller than b_χ with respect to that partial order, we can suppose that b is a maximal element in S_{supp} as before. Hence, by applying the excision exact triangle to the closed embedding $\mathrm{Bun}_n^{\geq b} \longrightarrow \mathrm{Bun}_n$, we deduce that there is a non trivial morphism

$$g_2 : C_\chi \star \mathcal{F}_{\mathrm{Id}} \longrightarrow i_{b!}i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}} \xrightarrow{g_1} \mathcal{F}_\xi \simeq C_\xi \star \mathcal{F}_{\mathrm{Id}}.$$

We define $\mathrm{I} := \{i \mid \frac{a_i}{n_i} = h_{\max}\}$ and since $\sum_{j=1}^{k-1} |\mathrm{I}(j)| > 1$, we can choose an index $\widetilde{j} \in \bigcup_{j=1}^{k-1} \mathrm{I}(j)$ such that $|\mathrm{I} \setminus \{\widetilde{j}\}| \geq 1$.

By applying the auto-equivalence $C_{\chi_{\widetilde{j}}^{-1}}$ to g_2 we get a non trivial morphism

$$g : C_{\chi\otimes\chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}} \longrightarrow C_{\xi\otimes\chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}}.$$

Note that by the choice of $\widetilde{j}$, we see that $\chi \otimes \chi_{\widetilde{j}}^{-1}, \xi \otimes \chi_{\widetilde{j}}^{-1} \in \mathrm{Irr}(S_\phi)^+$ then by the induction hypothesis on $D = s-1$, we deduce that $C_{\chi\otimes\chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}}$ is supported on $b_{\chi\otimes\chi_{\widetilde{j}}^{-1}}$ and $C_{\xi\otimes\chi_{\widetilde{j}}^{-1}} \star \mathcal{F}_{\mathrm{Id}}$ is supported on $b_{\xi\otimes\chi_{\widetilde{j}}^{-1}}$.

Again, by the choice of $\widetilde{j}$, the biggest slope of $\mathcal{E}_{b_{\xi\otimes\chi_{\widetilde{j}}^{-1}}}$ is still $h_{\max}$ and the biggest slope of $\mathcal{E}_{b_{\chi\otimes\chi_{\widetilde{j}}^{-1}}}$ is not bigger than λ_1. In particular, $b_{\xi\otimes\chi_{\widetilde{j}}^{-1}}$ is not smaller than $b_{\chi\otimes\chi_{\widetilde{j}}^{-1}}$ with respect to the usual partial order of $B(\mathrm{GL}_n)$. Thus by lemma 2.6 there is no non-trivial morphism from $\mathcal{F}_{\chi\otimes\chi_{\widetilde{j}}^{-1}}$ to $\mathcal{F}_{\xi\otimes\chi_{\widetilde{j}}^{-1}}$. Hence a contradiction. Therefore, the restriction $i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$ must be trivial.

6.3.3. *The case where $\lambda_k = 0$ and $r = 2$.*

At this point we know that $k = r = 2$ hence we can suppose $\mathrm{I}(1) = \{1\}$ and $\mathrm{I}(2) = \{2\}$. Thus $\mathcal{E}_\chi \simeq \mathcal{O}(\lambda_1)^{m_1} \oplus \mathcal{O}^m$ with $\chi = (x, 0)$ for some $x \in \mathbb{N}$. Then we choose $\chi' = (x-1, 0)$ and denote $b \neq b_\chi \in B(\mathrm{GL}_n)$ an element such that $i_b^* C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non trivial. Hence, there is a modification of type $\mu = (1, 0^{(n-1)})$

$$f : \mathcal{E}_{b_{\chi'}} \longrightarrow \mathcal{E}_b.$$

All the arguments in the last paragraph (the case $r > 2$) before lemma 6.2 still work. Therefore we deduce that $\mathcal{E}_b \simeq \mathcal{E}_\xi$ where $\xi = (x_1, x_2) \in \mathrm{Irr}(S_\phi)^{>0}$ and where $\operatorname{rank} \mathcal{O}(\lambda_1)^{m_1} = \dim \phi_1 =: n_1$; $m = \dim \phi_2 =: n_2$ and $x_1 + x_2 = x$. Moreover we can also deduce that there is an injection $\mathcal{O}(\lambda'_1) \hookrightarrow \mathcal{E}_b$ and the quotient is isomorphic to $\mathcal{O}(\lambda)$ where $\lambda'_1 = \frac{x_1+x_2-1}{n_1}$, $\lambda = \frac{1}{n_2}$. Remark that we also know (by the case $\lambda_k > 0$) that

$$C_\xi \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\xi.$$

By using lemma 5.6 to the triple $\mathcal{E}_b$, $\mathcal{E}_{b_{\chi'}}$ and $C_{\chi_{j^*}}$, we can deduce that $\lambda \geq \lambda'_1$ as before. In particular we see that $\frac{x_2}{n_2} > \frac{x_1}{n_1}$. We consider the following cases

Case 1 : $\frac{x_2}{n_2} > \lambda_1$.

Thus $\nu_{\mathcal{E}_b}$ is not smaller than $\nu_{\mathcal{E}_{b_\chi}}$ with respect to the usual partial order in $X_*(\mathrm{T})_{\mathbb{Q}}$. By choosing an element b maximal with these properties and using the excision exact triangles argument with respect to the closed embedding $\mathrm{Bun}_n^{\geq b} \hookrightarrow \mathrm{Bun}_n$ as above, we can obtain a contradiction.

Case 2 : $\lambda_1 \geq \frac{x_2}{n_2}$.

In this case $\nu_{\mathcal{E}_b}$ is smaller than $\nu_{\mathcal{E}_{b_\chi}}$ with respect to the usual partial order in $X_*(\mathrm{T})_{\mathbb{Q}}$. Note that the spectral action preserves ULA objects then $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is ULA and then $i_{b_\xi}^*(C_\chi \star \mathcal{F}_{\mathrm{Id}})$ is also ULA. Hence there is a non-trivial morphism

$$g : C_\xi \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\xi \longrightarrow i_{b_\xi}^*(C_\chi \star \mathcal{F}_{\mathrm{Id}}),$$

more precisely, we can construct g from a sub-module of the non-trivial cohomology group of $i_{b_\xi}^*(C_\chi \star \mathcal{F}_{\mathrm{Id}})$ which has lowest degree.

Now we can suppose that b is minimal with respect to the partial order among the strata where the restriction of $C_\chi \star \mathcal{F}_{\mathrm{Id}}$ is non-trivial. In particular b is smaller than and not equal to b_χ in

$B(\mathrm{GL}_n)$. By similar excision exact triangles argument for the open embedding $\mathrm{Bun}_n^{b\geq} \hookrightarrow \mathrm{Bun}_n$, we can suppose that up to some shift, there is a non-trivial morphism

$$g_1 : C_\xi \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\xi \longrightarrow C_\chi \star \mathcal{F}_{\mathrm{Id}}.$$

Now we apply the auto-equivalence $C_{\chi_1^{-1}}$ on g_1 to get a non-trivial morphism (where $\chi_1 = (1, 0)$)

$$g_2 : C_{\xi\otimes\chi_1^{-1}} \star \mathrm{Id} \simeq \mathcal{F}_{\xi\otimes\chi_1^{-1}} \longrightarrow C_{\chi\otimes\chi_1^{-1}} \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_{\chi\otimes\chi_1^{-1}}.$$

If $x_1 + x_2 > 2$ then we see that $b_{\chi\otimes\chi_1^{-1}}$ is not smaller than $b_{\xi\otimes\chi_1^{-1}}$ and lemma 2.6 gives rise to a contradiction.

If $x_1 + x_2 = 2$ then $x_1 = x_2 = 1$ and $\xi \otimes \chi_1^{-1} = (0, 1)$ and $\chi \otimes \chi_1^{-1} = (1, 0)$. Applying again the auto-equivalence χ_2^{-1} on g_2 to get a non-trivial morphism

$$g_3 : C_{\mathrm{Id}} \star \mathcal{F}_{\mathrm{Id}} \longrightarrow C_{\xi'} \star \mathcal{F}_{\mathrm{Id}},$$

where $\xi' = (1, -1)$. By lemma 2.6 the restriction $i_1^* C_{\xi'} \star \mathcal{F}_{\mathrm{Id}}$ is non trivial. However it contradicts equation (10).

It allows us to finally conclude that

$$C_\chi \star \mathcal{F}_{\mathrm{Id}} \simeq \mathcal{F}_\chi.$$

7. Hecke-eigensheaves for GL_n

The main goals of this subsection is to use theorem 4.5 to give an explicit description of the Hecke eigensheaves associated to some L-parameters ϕ.

Throughout this paragraph, we fix an L-parameter $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ satisfying the conditions of theorem 4.5. In particular the connected component $[C_\phi]$ of $[Z^1(W_{\mathbb{Q}_p}, \widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$ containing ϕ is isomorphic to $[\mathbb{G}_m^r/\mathbb{G}_m^r]$ with the trivial action of $\mathbb{G}_m^r$. Then $S_\phi = \prod_{i=1}^r \mathbb{G}_m$, $\mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^r \mathbb{Z}$ and all the algebraic irreducible representations of S_ϕ are of dimension 1. Moreover, the category $\mathrm{Rep}_{\mathbb{Q}_\ell}(S_\phi)$ is semi-simple, we see that the regular representation of S_ϕ has the following description

$$V_{\mathrm{reg}} = \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} \chi.$$

We deduce an explicit description of a Hecke eigensheaf associated to ϕ.

Theorem 7.1. *The sheaf*

$$\mathcal{G}_\phi := \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} \mathcal{F}_\chi$$

is a non trivial Hecke eigensheaf corresponding to the L-parameter ϕ.

Proof. By the description of the $\mathcal{F}_\chi$'s, it is clear that the stalk of $\mathcal{G}_\phi$ at the stratum Bun_n^1 is isomorphic to $\mathcal{F}_{\mathrm{Id}}$. Thus the sheaf $\mathcal{G}_\phi$ is non-zero.

Remark that for every $\chi' \in \mathrm{Irr}(S_\phi)$ we have $\chi' \otimes \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} \chi = \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} \chi$. Thus by theorem 4.5 and the monoidal property of the spectral action, we deduce that

$$C_{\chi'} \star \mathcal{G}_\phi = C_{\chi'} \star \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} C_\chi \star \mathcal{F}_{\mathrm{Id}} = \bigoplus_{\chi\in\mathrm{Irr}(S_\phi)} C_\chi \star \mathcal{F}_{\mathrm{Id}} = \mathcal{G}_\phi.$$

Let V be an algebraic representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ and let r_V resp. T_V be the corresponding highest weight representation resp. the corresponding Hecke operator. We show that $\mathrm{T}_V(\mathcal{G}_\phi) =$

$\mathcal{G}_\phi \boxtimes r_V \circ \phi$ as sheaf on $\mathrm{Bun_n}$ with $W_{\mathbb{Q}_p}$-action. Indeed, suppose that as $S_\phi \times W_{\mathbb{Q}_p}$-representation, we have

$$r_V \circ \phi \simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi \boxtimes \sigma_\chi,$$

where $\sigma_\chi := \mathrm{Hom}_{S_\phi}(\chi, r_V \circ \phi)$ is a $W_{\mathbb{Q}_p}$-representation.

In particular, since $\dim \chi = 1$ we deduce that $\bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \sigma_\chi = r_V \circ \phi$.

On the other hand, we have

$$\mathrm{T}_V(\mathcal{G}_\phi) = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} C_\chi \star \mathcal{G}_\phi \boxtimes \sigma_\chi = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \mathcal{G}_\phi \boxtimes \sigma_\chi = \mathcal{G}_\phi \boxtimes \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \sigma_\chi = \mathcal{G}_\phi \boxtimes r_V \circ \phi.$$

Therefore $\mathcal{G}_\phi$ is a non-zero Hecke eigensheaf of the L-parameter ϕ. □

Example 7.2. Let us describe the stalks of $\mathcal{G}_\phi$ in some special cases.

Suppose further that $\phi = \phi_1 \oplus \ldots \oplus \phi_n$ is a sum of n characters. Thus the Newton polygon of the vector bundle $\mathcal{E}_{b_\chi}$ are of the form $(d_1^{(n_1)}, \ldots, d_k^{(n_k)})$ for a decreasing chain $d_1 > d_2 > \ldots > d_k$ of integers (where $d_i^{(n_i)}$ indicates that d_i appears with multiplicity n_i).

Let $b = b_\xi$ for $\xi = (d_1^{(n_1)}, \ldots, d_k^{(n_k)}) \in \mathrm{Irr}(S_\phi)$. We see that $\mathrm{G}_b = \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_k}$ and for a character $\chi = (t_1, \ldots, t_n) \in \mathrm{Irr}(S_\phi)$ we have $b_\chi = b$ if and only if $(t_1, \ldots, t_n)$ is a permutation of $(d_1^{(n_1)}, \ldots, d_k^{(n_k)})$. The stabilizer of $(t_1, \ldots, t_n)$ under the action of the permutation group S_n is a subgroup isomorphic to $S_{n_1} \times S_{n_2} \times \ldots \times S_{n_k}$. Therefore the stalk of $\mathcal{G}_\phi$ at Bun_n^b is given by

$$\bigoplus_{w \in W_{\mathrm{GL}_n}/W_{\mathrm{G}_b}} \mathcal{F}_{\xi^w}$$

where W_{G} denotes the Weyl group of G. This description is compatible with the one given by L. Hamann in [Ham22, Theorem 1.14].

One expects that the restriction of $i_b^* \mathcal{G}_\phi$ to the stratum Bun_n^b is closely related to the functor Red_b defined by Shin [Shi12, §6.2] and Bertoloni-Meli [BM, Definition 5.6]. It is first observed by Hamann in [Ham22, §11].

8. Harris-Viehmann's conjecture

The main goals of this section is to compute part of the cohomology of the moduli spaces $\mathrm{Sht}(\mathrm{GL}_n, b, b', \mu)$ and deduce new cases of the Harris-Viehmann's conjecture for GL_n.

Throughout this section, we fix an L-parameter $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ satisfying the conditions of theorem 4.5. In particular we have $S_\phi = \prod_{i=1}^r \mathbb{G}_m$, $\mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^r \mathbb{Z}$ and $[C_\phi] \simeq [\mathbb{G}_m^r / \mathbb{G}_m^r]$, the quotient of $\mathbb{G}_m^r$ by the trivial action of $\mathbb{G}_m^r$.

Let $b, b' \in B(\mathrm{GL}_n)$ be such that ϕ is $\mathrm{G}_b(\mathbb{Q}_p)$-relevant and and let π_b be the irreducible representation of $\mathrm{G}_b(\mathbb{Q}_p)$ corresponding to ϕ under the local Langlands correspondence. Let $\lambda \in \mathrm{Irr}(S_\phi)$ be the character corresponding to the pair (b, π_b). Then by lemmas 2.10 and 5.4 we have

$$R\Gamma_c(\mathrm{GL}_n, b', b, \mu)[\delta_b^{1/2} \otimes \pi_b][d_b] \simeq i_{b'}^* \mathrm{T}_{\mu^{-1}} \mathcal{F}_\lambda,$$

where $d_b = \langle 2\rho, \nu_b \rangle$.

Suppose that as $S_\phi \times W_{\mathbb{Q}_p}$-representations, we have an identification

$$r_{\mu^{-1}} \circ \phi = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi \boxtimes \sigma_\chi$$

where σ_χ is the $W_{\mathbb{Q}_p}$-representations $\mathrm{Hom}_{S_\phi}(\chi, r_{\mu^{-1}} \circ \phi)$, then we have

$$\mathrm{T}_{\mu^{-1}} \mathcal{F}_\lambda = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} C_\chi \star \mathcal{F}_\lambda \boxtimes \sigma_\chi.$$

Hence we can use the explicit description of C_χ to compute $R\Gamma_c(\mathrm{GL}_n, b', b, \mu))[\delta_b^{1/2} \otimes \pi_b]$.

Theorem 8.1. *(some cases of the generalized Harris-Viehmann's conjecture) Let $b \in B(\mathrm{GL}_n)$ be an element such that ϕ is $\mathrm{G}_b(\mathbb{Q}_p)$-relevant. Let μ be an arbitrary cocharacter of GL_n and π_b be an irreducible representation of $\mathrm{G}_b(\mathbb{Q}_p)$ such that its corresponding L-parameter ϕ^b post-composed with the natural embedding ${}^L\mathrm{G}_b(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is ϕ. Thus we have*

$$R\Gamma_c(\mathrm{GL}_n, b, \mu)[\delta_b^{1/2} \otimes \pi_b] = \pi_1 \boxtimes \mathrm{Hom}_{S_\phi}(\chi_b^{-1}, r_{\mu^{-1}} \circ \phi)[-d] \tag{11}$$

where π_1 is the irreducibe representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ whose L-parameter is ϕ and χ_b is the unique character of S_ϕ corresponding to the couple (b, π_b) and where $d = \langle 2\rho, \nu_b\rangle$. In particular, the generalized Harris-Viehmann's conjecture is true in this case.

Proof. We recall that $\mathrm{Sht}(\mathrm{GL}_n, b, \mu)$ is the space parametrizing modifications $f : \mathcal{E}_1 \longrightarrow \mathcal{E}_b$ of type μ. As before, we have

$$R\Gamma_c(\mathrm{GL}_n, b, \mu)[\delta_b^{1/2} \otimes \pi_b] \simeq i_1^*\mathrm{T}_{\mu^{-1}}\mathcal{F}_{\chi_b}[-d].$$

Then we consider the $S_\phi \times W_{\mathbb{Q}_p}$-representation $r_{\mu^{-1}} \circ \phi = \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} \chi \boxtimes \sigma_\chi$ where σ_χ is the $W_{\mathbb{Q}_p}$-representation $\mathrm{Hom}_{S_\phi}(\chi, r_{\mu^{-1}} \circ \phi)$. Since the couple (b, π_b) corresponds to the character χ_b, we deduce that $i_1^*C_\chi \star \mathcal{F}_{\chi_b} \simeq 0$ if $\chi \neq \chi_b^{-1}$. Thus

$$\begin{aligned} R\Gamma_c(\mathrm{GL}_n, b, \mu)[\delta_b^{1/2} \otimes \pi_b] &\simeq \bigoplus_{\chi \in \mathrm{Irr}(S_\phi)} i_1^*C_\chi \star \mathcal{F}_{\chi_b} \boxtimes \sigma_\chi[-d] \\ &\simeq \quad i_1^*C_{\chi_b^{-1}} \star \mathcal{F}_{\chi_b} \boxtimes \sigma_{\chi_b^{-1}}[-d] \\ &\simeq \pi_1 \boxtimes \mathrm{Hom}_{S_\phi}(\chi_b^{-1}, r_{\mu^{-1}} \circ \phi)[-d]. \end{aligned}$$

Suppose that $\mathcal{E}_b \simeq \mathcal{E}(\lambda_1) \oplus \ldots \oplus \mathcal{E}(\lambda_k)$ where $\lambda_1 > \ldots > \lambda_k$ and for each j, $\mathcal{E}(\lambda_j)$ is a semi-stable vector bundle of slope λ_j. Denote by M the standard Levi subgroup of GL_n that is the split inner form of G_b. Then $\mathrm{M} \simeq \mathrm{GL}_{m_1} \times \ldots \times \mathrm{GL}_{m_k}$ where $m_j := \mathrm{rank}\, \mathcal{E}(\lambda_j)$ and the natural embedding ${}^L\mathrm{G}_b(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ induces a morphism of L-groups $\eta :\ {}^L\mathrm{M}(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that $\phi = \eta \circ \phi^b$. In particular we have $S_{\phi^b} \simeq S_\phi$ and from now on, for each $\chi \in \mathrm{Irr}(S_\phi)$, we denote by χ_M its corresponding character in $\mathrm{Irr}(S_{\phi^b})$. Now we consider $r_{\mu^{-1}}$ as an M-representation by η. Since the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}\mathrm{M}$ is semi-simple, there is a decomposition

$$r_{\mu^{-1}} = \bigoplus_{\mu_M \in X_*(M)} (r_{\mu_M})^{m_{\mu_M}} \tag{12}$$

for some multiplicity $m_{\mu_M} = \dim \mathrm{Hom}(r_{\mu_M}, r_{\mu|M})$. Moreover the decomposition (12) is compatible with $W_{\mathbb{Q}_p}$-action since $\phi = \eta \circ \phi^b$. Thus

$$\mathrm{Hom}_{S_\phi}(\chi_b^{-1}, r_\mu \circ \phi) = \bigoplus_{\mu_M \in X_*(M)} \mathrm{Hom}_{S_{\phi^b}}(\chi_{b,\mathrm{M}}^{-1}, r_{\mu_M} \circ \phi^b)^{m_{\mu_M}}.$$

Denote by b_M the reduction of b to M and π_1^M the unique irreducible representation of $\mathrm{M}(\mathbb{Q}_p)$ whose L-parameter is ϕ^b then $\pi_1 = \mathrm{Ind}_\mathrm{P}^{\mathrm{GL}_n}\pi_1^\mathrm{M}$ where P is the standard parabolic subgroup whose Levi factor is M (normalized parabolic induction). By using the formula (11) for M we deduce that

$$\begin{aligned} R\Gamma_c(\mathrm{GL}_n, b, \mu)[\delta_b^{1/2} \otimes \pi_b][d - d_\mathrm{M}] &\simeq \pi_1 \boxtimes \mathrm{Hom}_{S_\phi}(\chi_b^{-1}, r_\mu \circ \phi)[-d_\mathrm{M}] \\ &\simeq \pi_1 \boxtimes \bigoplus_{\mu_M \in X_*(M)} \mathrm{Hom}_{S_{\phi^b}}(\chi_{b,\mathrm{M}}^{-1}, r_{\mu_M} \circ \phi^b)^{m_{\mu_M}}[-d_\mathrm{M}] \\ &\simeq \bigoplus_{\mu_\mathrm{M} \in X_*(\mathrm{M})} \left(\mathrm{Ind}_\mathrm{P}^{\mathrm{GL}_n} R\Gamma_c(\mathrm{M}, b_\mathrm{M}, \mu_\mathrm{M})[\delta_{b_M}^{1/2} \otimes \pi_b]\right)^{m_{\mu_\mathrm{M}}}. \end{aligned}$$

where $d_\mathrm{M} = \langle 2\rho_\mathrm{M}, \nu_{b_\mathrm{M}}\rangle$ and $d = \langle 2\rho, \nu_b\rangle$.

The above formula is a particular case of the generalized Harris-Viehmann's conjecture formulated by Hansen-Scholze. Moreover, a proof of this conjecture for general reductive group G would appear in their forthcoming joint work with L. Hamann [HHS].

If $\mu = (1^{(a)}, 0^{(n-a)})$ is a minuscule cocharacter then $r_{\mu^{-1}} = (\Lambda^a \mathrm{Std})^\vee$. Hence if $\chi_b = (d_1, \ldots, d_r)$ for $d_1, \ldots, d_r \geq 0$ then

$$\mathrm{Hom}_{S_\phi}(\chi_b^{-1}, r_{\mu^{-1}} \circ \phi) = \bigotimes_{i=1}^{r} (\Lambda^{d_i} \phi_i)^\vee,$$

thus

$$R\Gamma_c(\mathrm{GL}_n, b, \mu)[\delta_b^{1/2} \otimes \pi_b][d - d_{\mathrm{M}}] \simeq \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n} R\Gamma_c(\mathrm{M}, b_{\mathrm{M}}, \mu_{\mathrm{M}})[\delta_{b_M}^{1/2} \otimes \pi_b]$$

where $\mu_{\mathrm{M}} = \mu_1 \times \ldots \times \mu_k$ and $\mu_i = (1^{(\deg \mathcal{E}(\lambda_i))}, 0^{(\mathrm{rank}\, \mathcal{E}(\lambda_i) - \deg \mathcal{E}(\lambda_i))})$. Hence we obtain the Harris-Viehmann's conjecture in this particular case [Har01, Conjecture 5.2] [RV14, Conjecture 8.5].

□

Example 8.2. We suppose that $b \in B(\mathrm{GL}_2)$ is an element such that $\mathcal{E}_b = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$; $\mu^{-1} = (3, 0)$ and $\phi = \phi_1 \oplus \phi_2$ is the sum of 2 characters. We see that $S_\phi \simeq \mathbb{G}_m \times \mathbb{G}_m$ and $\mathrm{Irr}(S_\phi) \simeq \mathbb{Z} \times \mathbb{Z}$. We denote by $\chi_{(x,y)}$ the character corresponding to (x, y). Since $r_{\mu^{-1}} = \mathrm{Sym}^3 \mathrm{Std} \mathrm{GL}_2$, we have the following identification of $S_\phi \times W_{\mathbb{Q}_p}$-representations

$$r_{\mu^{-1}} \circ \phi \simeq \chi_{(3,0)} \boxtimes \phi_1^3 \bigoplus \chi_{(2,1)} \boxtimes \phi_1^2 \otimes \phi_2 \bigoplus \chi_{(1,2)} \boxtimes \phi_1 \otimes \phi_2^2 \bigoplus \chi_{(0,3)} \boxtimes \phi_2^3.$$

Let π_b be the representation such that the couple (b, π_b) corresponds to the character $\chi_{(-1,-2)}$. Thus theorem 4.5 implies that $i_1^* C_{\chi_{(3,0)}} \star \mathcal{F}_{\chi_{(-1,-2)}} = i_1^* C_{\chi_{(2,1)}} \star \mathcal{F}_{\chi_{(-1,-2)}} = i_1^* C_{\chi_{(0,3)}} \star \mathcal{F}_{\chi_{(-1,-2)}} = 0$ and $i_1^* C_{\chi_{(1,2)}} \star \mathcal{F}_{\chi_{(-1,-2)}} = \pi_1$. Hence

$$R\Gamma_c(\mathrm{GL}_n, b, \mu))[\delta_b^{1/2} \otimes \pi_b] = \pi_1 \boxtimes \phi_1 \otimes \phi_2^2[-d],$$

where $d = \langle 2\rho, \nu_b \rangle = 1$.[1]

Example 8.3. We suppose that $\phi = \phi_1 \oplus \phi_2$ where ϕ_1 and ϕ_2 are irreducible representations whose dimension are given by natural numbers $n_1 > n_2$. Let $b \in B(\mathrm{GL}_n)$ be the element such that $\mathcal{E}_b = \mathcal{O}(-1/n_1) \oplus (-1/n_2)$. We are going to compute the cohomology of Rapoport-Zink spaces/local Shimura varieties $R\Gamma_c(\mathrm{GL}_n, b, \mu_i))[\pi_b]$ $(i = 1, 2)$ where $\mu_1^{-1} = (1^{(2)}, 0^{(n-2)})$; $\mu_2^{-1} = (2, 0^{(n-1)})$ and π_b is the irreducible representation of $\mathrm{G}_b(\mathbb{Q}_p)$ whose L-parameter is ϕ. Thus the couple (b, π_b) corresponds to the character $\chi_{(-1,-1)}$.

We see that $S_\phi \simeq \mathbb{G}_m \times \mathbb{G}_m$ and $\mathrm{Irr}(S_\phi) \simeq \mathbb{Z} \times \mathbb{Z}$. We denote by $\chi_{(x,y)}$ the character corresponding to (x, y). Since $r_{\mu_1^{-1}} = \Lambda^2 \mathrm{Std}$, we deduce the following identification of $S_\phi \times W_{\mathbb{Q}_p}$-representations

$$r_{\mu_1^{-1}} \circ \phi \simeq \chi_{(-2,0)} \boxtimes \Lambda^2 \phi_1 \bigoplus \chi_{(-1,-1)} \boxtimes \phi_1 \otimes \phi_2 \bigoplus \chi_{(0,-2)} \boxtimes \Lambda^2 \phi_2.$$

By theorem 4.5, we have $i_1^* C_{\chi_{(2,0)}} \star \mathcal{F}_{\chi_{(1,1)}} = i_1^* C_{\chi_{(0,2)}} \star \mathcal{F}_{\chi_{(1,1)}} = 0$ and $i_1^* C_{\chi_{(1,1)}} \star \mathcal{F}_{\chi_{(1,1)}} = \pi_1$ where π_1 is the irreducible representation of GL_n whose L-parameter is ϕ. Hence

$$R\Gamma_c(\mathrm{GL}_n, b, \mu_1)[\delta_b^{1/2} \otimes \pi_b] = \pi_1 \boxtimes \phi_1 \otimes \phi_2[-d],$$

where $d = \langle 2\rho, \nu_b \rangle = n_2 - n_1$.

Similarly, we have $r_{\mu_2^{-1}} = \mathrm{Sym}^2 \mathrm{Std}$, we deduce the following identification of $S_\phi \times W_{\mathbb{Q}_p}$-representations

$$r_{\mu_2^{-1}} = \chi_{(2,0)} \boxtimes \mathrm{Sym}^2 \phi_1 \bigoplus \chi_{(1,1)} \boxtimes \phi_1 \otimes \phi_2 \bigoplus \chi_{(0,2)} \boxtimes \mathrm{Sym}^2 \phi_2.$$

Therefore

$$R\Gamma_c(\mathrm{GL}_n, b, \mu_2))[\delta_b^{1/2} \otimes \pi_b] = \pi_1 \boxtimes \phi_1 \otimes \phi_2[-d],$$

where $d = \langle 2\rho, \nu_b \rangle = n_2 - n_1$.

[1]See also the computation in [Ima19, Example 8.10].

9. On the categorical form of Fargues' conjecture for GL_n

In this section, we describe the map Ψ_{GL_n} from the spectral Bernstein center of GL_n to its Bernstein center by using the compatibility between Fargues-Scholze L-parameters and the usual L-parameters for GL_n. Then we combine this description with theorem 4.5 to describe the action of $\mathrm{Perf}([C_\phi])$ on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$.

9.1. **Bernstein centers.**

Recall that the spectral Bernstein center of GL_n is $\mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) := \mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n))^{\mathrm{GL}_n}$, the ring of global functions on the stack/the coarse moduli space of L-parameters. The geometric Bernstein center of GL_n is $\mathcal{Z}^{\mathrm{geom}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) := \pi_0(\mathrm{End}(\mathrm{id}_{\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)}))$. There is a natural identification between $\mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell)$ and the algebra of excursion operators ([FS21, theorem VIII.5.1]). By [FS21, theo. VIII.4.1], there is a map $\mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{Z}^{\mathrm{geom}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell)$ which induces a map

$$\Psi_{\mathrm{GL}_n} : \mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{Z}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell)$$

by the inclusion $\mathrm{D}_{\mathrm{lis}}(\mathrm{GL}_n(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ and where $\mathcal{Z}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell)$ is the Bernstein center of GL_n.

Let $f \in \mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n))^{\mathrm{GL}_n}$ be an element and denote by $\mathcal{O}$ the structural sheaf of $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$. In particular the multiplication by f gives rise to a map of $\mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n))^{\mathrm{GL}_n}$-modules $\mathcal{O} \xrightarrow{\cdot f} \mathcal{O}$. Note that $\mathcal{O}$ is the vector bundle C_{tri} corresponding to the Hecke operator of the trivial representation of GL_n. Hence via the spectral action, the map $\mathcal{O} \xrightarrow{\cdot f} \mathcal{O}$ induce an element in $\mathrm{End}(\mathcal{O} \star A) \simeq \mathrm{End}(A)$ for every $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ and therefore it induces an element $\widetilde{f}$ in $\mathrm{End}(\mathrm{id}_{\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega})$. Now the excursion operator corresponding to f also gives rise to an element $\overline{f}$ in $\mathrm{End}(\mathrm{id}_{\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega})$. By the compatibility of the spectral action and the excursion operators [Zou22, Theorem 5.2.1], we have $\widetilde{f} = \overline{f}$. More concretely, let A be a Schur-irreducible element in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$ whose Fargues-Scholze L-parameter is given by ϕ. Then $\mathcal{O} \xrightarrow{\cdot f} \mathcal{O}$ acting on A gives us an endomorphism

$$\{\mathcal{O} \star A = A \longrightarrow \mathcal{O} \star A = A\} \in \mathrm{End}(A) = \overline{\mathbb{Q}}_\ell,$$

which will be precisely the scalar α given by evaluating A on the excursion datum corresponding to f (see also [Ham21, page 24]). By [FS21, Proposition VIII.3.8], ϕ corresponds to a surjective map

$$\mathrm{Ev}_\phi : \mathcal{O}(\mathrm{Z}^1(\mathrm{W}_{\mathbb{Q}_\mathrm{p}}, \mathrm{GL}_\mathrm{n}))^{\mathrm{GL}_\mathrm{n}} \longrightarrow \overline{\mathbb{Q}}_\ell$$

and the scalar α above obtained by evaluating ϕ on the excursion datum corresponding to f is exactly $\mathrm{Ev}_\phi(f)$.

Now let $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ be an L-parameter satisfying the condition of theorem 4.5 and let $[C_\phi]$ be the connected component of $[Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n)/\mathrm{GL}_n]$ containing ϕ. By proposition 3.6, $[C_\phi]$ is isomorphic to $[\mathbb{G}_m^r/\mathbb{G}_m^r]$ where $\mathbb{G}_m^r$ acts trivially. Therefore, the ring of global functions of $[C_\phi]$ is given by $\overline{\mathbb{Q}}_\ell[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}]$ and it is a direct factor of $\mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \mathrm{GL}_n))^{\mathrm{GL}_n}$. Now we want to give an explicit description of the restriction of the map Ψ_{GL_n} to the factor $\mathcal{O}([C_\phi]) \simeq \overline{\mathbb{Q}}_\ell[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}]$. As we will see, this description ultimately comes from the fact that the Fargues-Scholze L-parameter and the usual L-parameter are compatible up to semi-simplification for irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Let $\mathrm{B} \subset \mathrm{GL}_n$ be the standard upper triangular Borel subgroup and let ψ be a generic character of the unipotent radical $\mathrm{U} \subset \mathrm{B}$. This Whittaker datum yields the Whittaker representation $c\text{-}\mathrm{Ind}_{\mathrm{U}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \psi$. The so-called Whittaker sheaf is the sheaf $\mathcal{W}_\psi$ concentrated on Bun_n^1 corresponding to $c\text{-}\mathrm{Ind}_{\mathrm{U}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \psi$.

Let $\mathrm{M} := \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$ where $n_i = \dim \phi_i$ $(1 \leq i \leq r)$ be the Levi subgroup of GL_n corresponding to ϕ. Let π^{M} be the supercuspidal representation of M whose L-parameter is given by $\phi_1 \times \ldots \times \phi_r$. Thus $\mathfrak{s}_\phi := (\mathrm{M}, \pi^{\mathrm{M}})$ is a cuspidal pair of GL_n and we denote by $\mathrm{Rep}(\mathfrak{s}_\phi)$

the corresponding Bernstein block of the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}\mathrm{GL}_n(\mathbb{Q}_p)$ of smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. By [Ber84], $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}\mathrm{GL}_n(\mathbb{Q}_p)$ decomposes into a product of indecomposable Bernstein blocks. Let $\mathcal{W}_{\mathfrak{s}_\phi}$ be the Bernstein component of $c\text{-}\mathrm{Ind}_{\mathrm{U}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\psi$ in $\mathfrak{s}_\phi$ and let $\mathcal{Z}_{\mathfrak{s}_\phi}$ be the center of the block $\mathrm{Rep}(\mathfrak{s}_\phi)$. Let M^{un} be the variety of unramified characters of M. It is known by [Ber84], [BH03] that $\mathcal{Z}_{\mathfrak{s}_\phi}$ is isomorphic to the global functions of the variety M^{un}. In our case, M^{un} is isomorphic to $\mathbb{G}_m^r$. Then it is also known that $\mathcal{W}_{\mathfrak{s}_\phi}$ is a pro-generator of the block $\mathrm{Rep}(\mathfrak{s}_\phi)$ and

$$\mathcal{Z}_{\mathfrak{s}_\phi} \simeq \mathrm{End}(\mathcal{W}_{\mathfrak{s}_\phi}) \simeq \overline{\mathbb{Q}}_\ell[Y_1,\dots,Y_r,Y_1^{-1},\dots,Y_r^{-1}].$$

More precisely, if ϕ is a supercuspidal L-parameter then $\pi := \pi^{\mathrm{GL}_n}$ is supercuspidal. Moreover by [BH03, §9.2], we know that $\mathcal{W}_{\mathfrak{s}_\phi} = c\text{-}\mathrm{Ind}_{\mathrm{GL}_n^o}^{\mathrm{GL}_n}\pi'$ where $\mathrm{GL}_n^o := \{g \in \mathrm{GL}_n | \det g \in \mathbb{Z}_p^*\}$ is an open, dense, normal subgroup of GL_n and π' is the unique direct factor of the restriction $\pi_{|\mathrm{GL}_n^o}$ such that $\mathrm{Hom}_{\mathrm{GL}_n^o}(c\text{-}\mathrm{Ind}_{\mathrm{U}}^{\mathrm{GL}_n^o}\psi, \pi')$ is non trivial.

In general, $\mathfrak{s}_\phi = [\mathrm{M}, \pi^{\mathrm{M}}]$ is a cuspidal pair for GL_n and we also consider $\mathfrak{t}_\phi = [\mathrm{M}, \pi^{\mathrm{M}}]$ as a cuspidal pair of M. Hence, by [Sol22, §4] the representation $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n}(\mathcal{W}_{\mathfrak{t}_\phi})$ is a pro-generator of $\mathrm{Rep}(\mathfrak{s}_\phi)$ where P is the standard parabolic subgroup of GL_n whose Levi factor is M. By [BH03, §9.3] and [Sol22, Proposition 4.1] we deduce that $\mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n}(\mathcal{W}_{\mathfrak{t}_\phi})$ is a pro-generator of $\mathrm{Rep}(\mathfrak{s}_\phi)$ (note that for the Bernstein blocks we consider, the algebras $\mathrm{End}_{\mathrm{G}}(\mathrm{I}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{E}_{\mathrm{B}}))$ and $\mathrm{End}_{\mathrm{L}}(\mathrm{I}_{\mathrm{P}\cap\mathrm{L}}^{\mathrm{L}}(\mathrm{E}_{\mathrm{B}}))$ in [Sol22, Proposition 4.1] are isomorphic).

Moreover, the block $\mathrm{Rep}(\mathfrak{s}_\phi)$ is equivalent to the category of modules over $\mathcal{Z}_{\mathfrak{s}_\phi}$ via the functor

$$\begin{aligned} \mathrm{F} : \mathrm{Rep}(\mathfrak{s}_\phi) &\longrightarrow \mathcal{Z}_{\mathfrak{s}_\phi} - \mathrm{Mod} \\ \pi &\longmapsto \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{s}_\phi)}(\mathcal{W}_{\mathfrak{s}_\phi}, \pi). \end{aligned}$$

Let π_ϕ be the irreducible representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ whose L-parameter is ϕ. Without loss of generality, by replacing ϕ by an appropriate L-parameter in $[C_\phi]$, we can suppose that $\mathrm{F}(\pi_\phi)$ is isomorphic to $\overline{\mathbb{Q}}_\ell$ as $\mathcal{Z}_{\mathfrak{s}_\phi}$-module where $\mathcal{Z}_{\mathfrak{s}_\phi}$ acts via the character

$$\overline{\mathbb{Q}}_\ell[Y_1,\dots,Y_r,Y_1^{-1},\dots,Y_r^{-1}] \longrightarrow \overline{\mathbb{Q}}_\ell, \quad Y_i \longmapsto 1.$$

More generally, let $\xi = (\xi_1,\dots,\xi_r)$ be an unramified character of M, we denote by $\phi \otimes \xi$ the L-parameter $\bigoplus_{i=1}^r \phi_i \otimes \xi_i$ and by $\pi_{\phi\otimes\xi}$ the corresponding irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. Then $\mathrm{F}(\pi_{\phi\otimes\xi})$ is isomorphic to $\overline{\mathbb{Q}}_\ell$ as $\mathcal{Z}_{\mathfrak{s}_\phi}$-module where $\mathcal{Z}_{\mathfrak{s}_\phi}$ acts via the reduction

$$\overline{\mathbb{Q}}_\ell[Y_1,\dots,Y_r,Y_1^{-1},\dots,Y_r^{-1}] \longrightarrow \overline{\mathbb{Q}}_\ell[Y_1,\dots,Y_r,Y_1^{-1},\dots,Y_r^{-1}]/\mathfrak{m}_\xi \simeq \overline{\mathbb{Q}}_\ell$$

where $\mathfrak{m}_\xi$ is the maximal ideal corresponding to the closed point ξ in M^{un}.

Recall that we associate a pair (b_χ, π_χ) to each character χ of $S_\phi \simeq \mathbb{G}_m^r$ where $b_\chi \in B(\mathrm{GL}_n)$ and π_χ is an irreducible representation of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$. Let $\mathfrak{s}_\phi(\chi) = [\tau_\chi, \mathrm{M}_{b_\chi}]$, resp. $\mathfrak{t}_\phi(\chi) = [\tau_\chi, \mathrm{M}_{b_\chi}]$ be the cuspidal pair of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$, resp. of $\mathrm{M}_{b_\chi}(\mathbb{Q}_p)$ corresponding to π_χ, resp. τ_χ where $(\tau_\chi, \mathrm{M}_{b_\chi})$ is the cuspidal support of π_χ. Let $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$, resp. $\mathrm{Rep}(\mathfrak{t}_\phi(\chi))$ be the corresponding Bernstein block of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$, resp. $\mathrm{M}_{b_\chi}(\mathbb{Q}_p)$. As above we can describe a pro-generator of this category. Indeed, $\mathcal{W}_{\mathfrak{t}_\phi(\chi)} := c\text{-}\mathrm{Ind}_{\mathrm{M}_{b_\chi}^o}^{\mathrm{M}_{b_\chi}}(\tau_\chi)'$ is a pro-generator of $\mathrm{Rep}(\mathfrak{t}_\phi(\chi))$ where $\tau(\chi)'$ is a direct factor of the restriction $\tau_{\chi|\mathrm{M}_{b_\chi}^o}$. Similarly $\mathcal{W}_{\mathfrak{s}_\phi(\chi)} := \mathrm{Ind}_{\mathrm{P}_{b_\chi}}^{\mathrm{G}_{b_\chi}}(\mathcal{W}_{\mathfrak{t}_\phi(\chi)})$ is a pro-generator of $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$. It is also known that $\mathcal{Z}_{\mathfrak{s}_\phi(\chi)} = \mathrm{End}(\mathcal{W}_{\mathfrak{s}_\phi(\chi)}) \simeq \overline{\mathbb{Q}}_\ell[X_1,\dots,X_r,X_r^{-1},\dots,X_r^{-1}]$ and the block $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$ is equivalent to the category of modules over $\mathcal{Z}_{\mathfrak{s}_\phi(\chi)}$ via the functor

$$\begin{aligned} \mathrm{F} : \mathrm{Rep}(\mathfrak{s}_\phi(\chi)) &\longrightarrow \mathcal{Z}_{\mathfrak{s}_\phi(\chi)} - \mathrm{Mod} \\ \pi &\longmapsto \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{s}_\phi(\chi))}(\mathcal{W}_{\mathfrak{s}_\phi(\chi)}, \pi). \end{aligned}$$

We also denote by $\mathcal{F}_{\mathcal{W}(\chi)}$ the sheaf $i_{b_\chi !}(\delta_{b_\chi}^{-1/2} \otimes \mathcal{W}_{\mathfrak{s}_\phi(\chi)})[-d_\chi]$ supported on b_χ where $d_\chi = \langle 2\rho, \nu_{b_\chi}\rangle$ and where $i_{b_\chi} : \mathrm{Bun}_n^{b_\chi} \longrightarrow \mathrm{Bun}_n$ is the canonical immersion. By abuse of notations, we sometimes use $\mathcal{W}_{\mathfrak{s}_\phi}$ instead of $\mathcal{F}_{\mathcal{W}(\mathrm{Id})}$ to denote $i_{1!}(\mathcal{W}_{\mathfrak{s}_\phi})$.

We denote by R the ring $\mathcal{O}([C_\phi]) \simeq \overline{\mathbb{Q}}_\ell[X_1, \dots, X_r, X_1^{-1}, \dots, X_r^{-1}]$ and by R_χ the ring $\mathcal{Z}_{\mathfrak{s}_\phi(\chi)} \simeq \overline{\mathbb{Q}}_\ell[Y_1, \dots, Y_r, Y_1^{-1}, \dots, Y_r^{-1}]$ to lighten the notations. We have a natural identification between R and R_χ by the map that sends X_i to Y_i for $1 \leq i \leq r$. If $\chi = \mathrm{Id}$, we even denote R_χ by R'. Note that for each Bernstein block $\mathfrak{s}$ of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(\overline{\mathbb{Q}}_p))$, we have a canonical projection map $\mathrm{Pr}_{\mathfrak{s}} : \mathcal{Z}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{Z}_{\mathfrak{s}}$.

The following lemma is well known for the experts.

Lemma 9.1. *If $\mathfrak{s} \neq \mathfrak{s}_\phi$ then the map $\mathrm{Pr}_{\mathfrak{s}} \circ \Psi_{\mathrm{GL}_n|\mathcal{O}([C_\phi])} : \mathcal{O}([C_\phi]) \longrightarrow \mathcal{Z}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{Z}_{\mathfrak{s}}$ is the zero map and if $\mathfrak{s} = \mathfrak{s}_\phi$ then we have the following description*

$$\begin{aligned} \mathrm{Pr}_{\mathfrak{s}_\phi} \circ \Psi_{\mathrm{GL}_n|\mathcal{O}([C_\phi])} : \mathcal{O}([C_\phi]) &\longrightarrow \mathcal{Z}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{Z}_{\mathfrak{s}_\phi} \\ X_i &\longmapsto Y_i. \end{aligned}$$

Proof. Let π be any irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ that does not belong to the Bernstein block $\mathrm{Rep}(\mathfrak{s}_\phi)$. By the compatibility of the Fargues-Scholze L-parameter and the usual L-parameter, we see that the L-parameter φ of π does not belong to the connected component $[C_\phi]$. Thus $\Psi_{\mathrm{GL}_n|\mathcal{O}([C_\phi])}(X_i) \in \mathcal{Z}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell)$ acts trivially on π since the valuation of X_i at the closed point corresponding to φ^{ss} vanishes. Hence $\mathrm{Pr}_{\mathfrak{s}} \circ \Psi_{\mathrm{GL}_n|\mathcal{O}([C_\phi])}$ is the zero map.

Now let $\xi = (\xi_1, \dots, \xi_r)$ be an unramified character of M^{un} and π be an irreducible representation whose L-parameter is given by $\phi \otimes \xi$. We denote by $\mathfrak{m}_\xi$ (resp. $\mathfrak{m}'_\xi$) the maximal ideal of R (resp. R') corresponding to the point ξ. Then by the above description, if we let α be the image of X_i in the quotient $R/\mathfrak{m}_\xi$ then the excursion operator corresponding to X_i acts on π by $\alpha \in \overline{\mathbb{Q}}_\ell \simeq \mathrm{End}(\pi)$. That means the element $Y'_i := \mathrm{Pr}_{\mathfrak{s}_\phi} \circ \Psi_{\mathrm{GL}_n|\mathcal{O}([C_\phi])}(X_i)$ also acts on π by $\alpha \in \overline{\mathbb{Q}}_\ell \simeq \mathrm{End}(\pi)$. Since the usual L-parameter of π is the same as its Fargues-Scholze L-parameter, we deduce that the image of Y'_i in $R'/\mathfrak{m}'_\xi$ is also given by α. In other words, $Y'_i - Y_i$ is in the maximal ideal $\mathfrak{m}'_\xi$. This holds for an arbitrary point ξ, hence $Y'_i = Y_i$. □

In more general situation where we consider a character $\chi \in \mathrm{Irr}(S_\phi)$, we can embed the derived category of $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$ into $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ by the functor

$$\begin{aligned} i_\chi : \mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi))) &\longrightarrow \quad \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \\ \pi \qquad &\longmapsto \quad i_{b_\chi!}(\delta_{b_\chi}^{-1/2} \otimes \pi)[-d_\chi] \end{aligned}$$

where $d_\chi = \langle 2\rho, \nu_{b_\chi} \rangle$. Therefore we have an induced map

$$\Psi^\chi_{\mathrm{GL}_n} : \mathcal{Z}^{\mathrm{spec}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{Z}_{\mathfrak{s}_\phi(\chi)}.$$

We know that Fargues-Scholze L-parameter is also compatible with the usual one for inner forms of GL_n [HKW21, Theorem 6.6.1], then by the same arguments as in lemma 9.1, we can show the following result.

Lemma 9.2. *The restriction of the map $\Psi^\chi_{\mathrm{GL}_n}$ to $\mathcal{O}([C_\phi])$ is given explicitly by:*

$$\begin{aligned} \Psi^\chi_{\mathrm{GL}_n|\mathcal{O}([C_\phi])} : \mathcal{O}([C_\phi]) &\longrightarrow \mathcal{Z}_{\mathfrak{s}_\phi(\chi)} \\ X_i &\longmapsto Y_i. \end{aligned}$$

9.2. On spectral action on Bun_n.

The main goal of this paragraph is to study the full sub-category $\mathcal{C} := \mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega$ consisting of compact objects whose Schur-irreducible constituents have L-parameter in the connected component $[C_\phi]$ and also the action of $\mathrm{Perf}([C_\phi])$ on $\mathcal{C}$. First, we are going to show the identity $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq i_{b_\chi!}(\delta_{b_\chi}^{-1/2} \otimes \mathcal{W}_{\mathfrak{s}_\phi(\chi)})[-d_\chi]$ for $\chi \neq \mathrm{Id}$ (the case $\chi = \mathrm{Id}$ is trivial since C_{Id} is the identity functor of $\mathcal{C}$).

Theorem 9.3. *Let $\phi = \phi_1 \oplus \ldots \oplus \phi_r$ be an L-parameter satisfying the conditions of theorem 4.5. With the above notations, for each $\chi \in \mathrm{Irr}(S_\phi)$ we have*

$$C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_{\mathcal{W}(\chi)}.$$

Proof. We proceed by induction on r. Remark that if $i^*_{b_\chi}(C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi})$ is non trivial then there exists some $\xi \in \mathrm{Irr}(S_\phi)$ such that $b = b_\xi$.

Step 1: The case where ϕ is irreducible.

In this case $S_\phi \simeq \mathbb{G}_m$ and $\mathrm{Irr}(S_\phi) \simeq \mathbb{Z}$. Let $\xi \in \mathrm{Irr}(S_\phi)$ be character corresponding to (1). Since all the irreducible sub-quotients of $\mathcal{W}_{\mathfrak{s}_\phi}$ have supercuspidal L-parameter, then so does all the irreducible sub-quotients of $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$. Thus $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$ is supported only on semi-stable locus. By direct computation we see that C_{ξ^m} is a direct factor of $C_{\mathrm{Sym}^m \mathrm{GL}_n}$. Hence $C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}$ is supported on the connected component of Bun_n corresponding to $m \in \pi_1(\mathrm{GL}_n)_\Gamma \simeq \mathbb{Z}$. Therefore we deduce that $C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}$ is supported on the basic stratum corresponding to b_{ξ^m}.

We show that $C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}$ is concentrated in degree 0.

Suppose that it is not true. Since $\mathcal{W}_{\mathfrak{s}_\phi}$ is compact, so is $C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}$. In other words, $C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}$ is bounded and its cohomology groups are finitely generated smooth representations. Thus they corresponds to finitely generated $\mathcal{Z}_{\mathfrak{s}_\phi(\xi^m)}$-modules B by the equivalence between $\mathrm{Rep}(\mathfrak{s}_\phi(\xi^m))$ and the category of $\mathcal{Z}_{\mathfrak{s}_\phi(\xi^m)}$-modules. In our situation, $\mathcal{Z}_{\mathfrak{s}_\phi(\xi^m)} \simeq \overline{\mathbb{Q}}_\ell[Y, Y^{-1}]$ is a principal ideal domain. Since the homological dimension of $\overline{\mathbb{Q}}_\ell[Y, Y^{-1}]$ is 1, there exists a non-zero integer k and a supercuspidal representation π of $\mathrm{G}_{b_{\xi^m}}(\mathbb{Q}_p)$ whose L-parameter belongs to the connected component $[C_\phi]$ such that $\mathrm{Hom}_{\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega}(C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}, i_{b_{\xi^m}!}(\pi)[k]) \neq 0$. Note that $C_{\xi^m} \star C_{\xi^{-m}}$ is the identity functor, we deduce that

$$\mathrm{Hom}_{\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega}(\mathcal{W}_{\mathfrak{s}_\phi}, \pi'[k]) \neq 0$$

where π' is the supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ whose L-parameter is the same as that of π. It is a contradiction since $\mathcal{W}_{\mathfrak{s}_\phi}$ is projective and $k \neq 0$. Hence $C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}$ is concentrated in degree 0.

Thus we have

$$B \simeq \overline{\mathbb{Q}}_\ell[Y, Y^{-1}]^t \oplus \overline{\mathbb{Q}}_\ell[Y, Y^{-1}]/(f)$$

where f is some non-unit element of $\overline{\mathbb{Q}}_\ell[Y, Y^{-1}]$. If B is not torsion-free then there exists a supercuspidal representation τ of $\mathrm{G}_{b_{\xi^m}}(\mathbb{Q}_p)$ whose L-parameter is in the connected component $[C_\phi]$ such that $\mathrm{Hom}_{\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega}(i_{b_{\xi^m}!}(\tau), C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi}) \neq 0$. As before we deduce that

$$\mathrm{Hom}_{\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega}(i_{1!}(\tau'), \mathcal{W}_{\mathfrak{s}_\phi}) \neq 0$$

where τ' is the supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ whose L-parameter is the same as that of τ. It is a contradiction since it induces that there is a non-trivial map from $\overline{\mathbb{Q}}_\ell[Y, Y^{-1}]/(g)$ to $\overline{\mathbb{Q}}_\ell[Y, Y^{-1}]$ for some non-unit element g in $\overline{\mathbb{Q}}_\ell[Y, Y^{-1}]$. Therefore B is torsion-free and then $B \simeq \overline{\mathbb{Q}}_\ell[Y, Y^{-1}]^t$ for some positive integer t. Finally the condition $\mathrm{Hom}_{\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega}(\mathcal{W}_{\mathfrak{s}_\phi}, i_{1!}(\pi)) = \overline{\mathbb{Q}}_\ell$ for every irreducible representation in $\mathrm{Rep}(\mathfrak{s}_\phi)$ implies that $B \simeq \overline{\mathbb{Q}}_\ell[Y, Y^{-1}]$. In other words, we have

$$C_{\xi^m} \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_{\mathcal{W}(\xi^m)}.$$

By the monoidal property of spectral action, we see that $C_\xi \star \mathcal{F}_{\mathcal{W}(\xi^{m-1})} = \mathcal{F}_{\mathcal{W}(\xi^m)}$. Moreover $C_{V_{\mathrm{std}}} = C^n_\xi$ then if we forget the $W_{\mathbb{Q}_p}$-action, we have

$$\mathrm{T}_{V_{\mathrm{std}}}(\mathcal{F}_{\mathcal{W}(\xi^{m-1})}) = (\mathcal{F}_{\mathcal{W}(\xi^m)})^n.$$

However, by the discussion before proposition 3.9 we have

$$\mathrm{T}_{V_{\mathrm{std}}}(\mathcal{F}_{\mathcal{W}(\xi^{m-1})}) = \mathcal{F}_{\mathcal{W}(\xi^m)} \boxtimes \phi_{|\mathrm{I}_{\mathbb{Q}_p}} \tag{13}$$

as objects in $(\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun_n}, \overline{\mathbb{Q}}_\ell)^\omega)^{B\mathrm{I}_{\mathbb{Q}_p}}$.

Step 2: The general case.

For $\chi = (t_1, \dots, t_r)$ we denote by $|\chi|$ the sum $\sum_{i=1}^{r} t_i$. As in the proof of theorem 4.5, we only need to prove the statement for χ with non-negative entries and moreover, we will proceed by induction on $|\chi|$. Notice that by lemme 3.7, the restriction $\phi_{|\mathrm{I}_{\mathbb{Q}_p}}$ is multiplicity free. Thus by using the result for $r = 1$, equation (13) and by replacing the $W_{\mathbb{Q}_p}$-action by $\mathrm{I}_{\mathbb{Q}_p}$-action, we see that some arguments in the proof of theorem 4.5 still work in this situation. In particular, we can show that if $|\chi| = 1$ then

$$C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_{\mathcal{W}(\chi)}$$

and moreover if the theorem is true for every $|\xi|$ such that $1 \le |\xi| \le s-1$ then for every $|\chi|$ such that $|\chi| = s$, we have

$$i^*_{b_\chi} C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_{\mathcal{W}(\chi)}.$$

Thus we need to show that

$$i^*_b C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0$$

if $b \neq b_\chi$ in $B(\mathrm{GL}_n)$.

Lemma 9.4. *If* $\mathrm{Bun}_n^b \not\subset \mathrm{Bun}_n^{\le b_\chi}$ *then*

$$i^*_b C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0.$$

Proof. Remark that $\mathrm{Bun}_n^{\le b_\chi}$ (resp. $\mathrm{Bun}_n^{\ge b_\chi}$) is an open (resp. closed) substack of Bun_n. Since the spectral action preserve compact objects, we deduce that the support of $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$ is a finite union of strata. Suppose that the support of $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$ does not contained in $\mathrm{Bun}_n^{\le b_\chi}$. Then there exists $b \neq b_\chi \in B(\mathrm{GL}_n)$ such that $i^*_b C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$ is not trivial and for all $b < b'$ we also have $i^*_{b'} C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0$. Therefore by excision exact triangles, there is a non-trivial morphism

$$C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \longrightarrow i_{b!} i^*_b (C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}).$$

Since $i_{b!} i^*_b (C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi})$ is non trivial and compact then there exists an integer k and an irreducible representation π of $\mathrm{G}_b(\mathbb{Q}_p)$ whose L-parameter is given by a closed point of $[C_\phi]$ such that there exists a non trivial morphism $i_{b!} i^*_b (C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}) \longrightarrow i_{b!}(\delta_b^{-1/2} \otimes \pi)[k]$ (we can construct such a morphism from a quotient of the highest non-trivial cohomology group of $i_{b!} i^*_b (C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi})$). Hence, there is a non-trivial morphism

$$f : C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \longrightarrow i_{b!}(\delta_b^{-1/2} \otimes \pi)[k].$$

Let $\xi \in \mathrm{Irr}(S_\phi)$ be the element corresponding to (b, π). Since $C_\chi \star C_{\chi^{-1}}$ is the identity functor we deduce that f gives rise to a non-trivial element in $\mathrm{Hom}_{\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega}(\mathcal{W}_{\mathfrak{s}_\phi}, C_{\chi^{-1}} \star (i_{b!}(\delta_b^{-1/2} \otimes \pi)[k]))$. By theorem 4.5, $C_{\chi^{-1}} \star (i_{b!}(\delta_b^{-1/2} \otimes \pi))[k]$ is supported on the non-basic stratum $b_{\xi \otimes \chi^{-1}}$.

However, by lemma 2.6, there is no non-trivial morphism from a sheaf supported on Bun_n^1 to a sheaf supported on a non-basic stratum, a contradiction. Hence

$$i^*_b C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0.$$

□

Thus we only need to show that if $\mathrm{Bun}_n^b \subset \mathrm{Bun}_n^{< b_\chi}$ then $i^*_b C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0$. We will use again some arguments in the proof of theorem 4.5.

Suppose $\chi = (d_1, \dots, d_r)$ where the integers d_i's are non-negative and $\sum_{i=1}^{r} d_i = s$. Suppose that we have $\mathcal{E}_{b_\chi} = \mathcal{O}(\lambda_1)^{m_1} \oplus \dots \oplus \mathcal{O}(\lambda_k)^{m_k}$ with $\lambda_1 > \lambda_2 > \dots > \lambda_k \ge 0$ and $m_i > 0$ for $1 \le i \le k$. By the construction of b_χ, there exists for each $1 \le i \le k$, a non-empty subset $\mathrm{I}(i)$ of $\{1, 2, \dots, r\}$ such that for $j \in \mathrm{I}(i)$, we have $d_j / n_j = \lambda_i$.

If $k = 1$ then b_χ is basic and thus $\mathrm{Bun}_n^{< b_\chi}$ is empty and we have nothing to prove. We suppose $k \ge 2$ from now on. There are two cases.

Case 1: $k \ge 3$ or $k = 2$ and $\lambda_2 > 0$.

We only treat the case $k \geq 3$, the case $k = 2$ and $\lambda_2 > 0$ can be treated similarly.

In this case $\lambda_{k-1} > 0$. We consider a character $\chi' = (d'_1, \ldots, d'_r)$ where for one index $j^* \in \mathrm{I}(k-1)$ we have $d'_{j^*} = d_{j^*} - 1$ and $d'_m = d_m$ for all other $r-1$ indexes. In particular the d_j's are non-negative and $\sum_{i=1}^{k} d'_i = s - 1$. By construction $\mathcal{E}_{b_{\chi'}} = \mathcal{O}(\lambda_1)^{m_1} \oplus \ldots \oplus \mathcal{O}(\lambda_{k-1})^{m_{k-1}-t} \oplus \mathcal{O}(\lambda'_k)^{t'} \oplus \mathcal{O}(\lambda_k)^{m_k}$ where $\lambda'_k = \frac{d'_{j^*}}{n_{j^*}}$; $t = \gcd(d_{j^*}, n_{j^*})$ and $t' = \gcd(d'_{j^*}, n_{j^*})$. Moreover we have $\chi = \chi_{j^*} \otimes \chi'$. Thus by monoidal property we have

$$C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} = C_{\chi_{j^*}} \star (C_{\chi'} \star \mathcal{W}_{\mathfrak{s}_\phi}) \simeq C_{\chi_{j^*}} \star \mathcal{F}_{\mathcal{W}(\chi')}.$$

Let $b \in B(\mathrm{GL}_n)$ be an element such that $\mathrm{Bun}_n^b \subset \mathrm{Bun}_n^{<b_\chi}$ and $i_b^*(C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}) \neq 0$. Since $C_{\chi_{j^*}}$ is a direct factor of $C_{V_{\mathrm{std}}}$, we deduce as in the proof of theorem 4.5 that there is a modification of type $\mu = (1, 0^{(n-1)})$ from $\mathcal{E}_{b_{\chi'}}$ to $\mathcal{E}_b$. Since $b < b_\chi$, we deduce that $\mathcal{O}(\lambda_1)^{m_1}$ is a direct factor of both $\mathcal{E}_b$ and $\mathcal{E}_{b_{\chi'}}$. By replacing the action of $W_{\mathbb{Q}_p}$ by the action of $\mathrm{I}_{\mathbb{Q}_p}$, the same arguments as in the proof of lemmas 5.5 and 5.6 show that $i_b^* C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0$, contradiction.

Case 2: $k = 2$ and $\lambda_2 = 0$.

In this case $\lambda_1 > 0$. If $|\mathrm{I}(1)| > 1$ then the above arguments still work and we see that $\mathcal{O}(\lambda_1)^{m_1 - t}$ is a direct factor of both $\mathcal{E}_b$ and $\mathcal{E}_{b_{\chi'}}$. The same arguments as in lemmas 5.5 and 5.6 show that $i_b^* C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0$.

Now we suppose that $|\mathrm{I}(1)| = 1$. We consider a character $\chi' = (d'_1, \ldots, d'_r)$ where for the index $j^* \in \mathrm{I}(1)$ we have $d'_{j^*} = d_{j^*} - 1$ and $d'_m = d_m$ for all other $r-1$ indexes. In particular the d_j's are non-negative and $\sum_{i=1}^{k} d'_i = s - 1 \geq 1$. By construction $\mathcal{E}_{b_{\chi'}} = \mathcal{O}(\lambda'_1)^{m'_1} \oplus \mathcal{O}(\lambda_2)^{m_2}$ where $\lambda'_1 = \frac{d_{j^*} - 1}{n_{j^*}}$. Moreover we have $\chi = \chi_{j^*} \otimes \chi'$. Thus by monoidal property we have

$$C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} = C_{\chi_{j^*}} \star (C_{\chi'} \star \mathcal{W}_{\mathfrak{s}_\phi}) \simeq C_{\chi_{j^*}} \star \mathcal{F}_{\mathcal{W}(\chi')}.$$

Let $b \in B(\mathrm{GL}_n)$ be an element such that $\mathrm{Bun}_n^b \subset \mathrm{Bun}_n^{<b_\chi}$ and $i_b^*(C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}) \neq 0$. Since $C_{\chi_{j^*}}$ is a direct factor of $C_{V_{\mathrm{std}}}$, we deduce as in the proof of theorem 4.5 that there is a modification f of type $\mu = (1, 0^{(n-1)})$ from $\mathcal{E}_{b_{\chi'}}$ to $\mathcal{E}_b$. We consider the filtration $\mathcal{O}(\lambda'_1)^{m'_1} \subset \mathcal{E}_{b_{\chi'}}$. Thus the modification f induces a filtration $\mathcal{E}' \subset \mathcal{E}_b$ and then 2 modifications

$$f_1 : \mathcal{O}(\lambda'_1)^{m'_1} \longrightarrow \mathcal{E}'$$

and

$$f_2 : \mathcal{O}(\lambda_2)^{m_2} \longrightarrow \mathcal{E}_b/\mathcal{E}'.$$

As before, there are 2 possibilities. If f_1 is of type $(1, 0^{(t_1-1)})$ and f_2 is of type $(0^{(t_2)})$ where $t_1 := \operatorname{rank} \mathcal{O}(\lambda'_1)^{m'_1}$ and $t_2 := \operatorname{rank} \mathcal{O}(\lambda_2)^{m_2}$ then we can deduce that $\nu_{\mathcal{E}_b}$ is not smaller than $\nu_{\mathcal{E}_{b_\chi}}$, a contradiction.

We can then suppose that f_1 and f_2 are of type $(0^{(t_1)})$, resp. $(1, 0^{(t_2-1)})$. Since $\lambda_2 = 0$, we see that $\mathcal{E}_b/\mathcal{E}' \simeq \mathcal{O}(\lambda) \oplus \mathcal{O}^t$ for some t and λ. If $t > 0$ then $\mathcal{O}^t$ is a direct factor of $\mathcal{E}_b$. Therefore we can use the arguments in the proof of lemmas 5.5, 5.6 and an induction argument on r to deduce that $i_b^* C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0$.

If $t = 0$ then $\nu_{\mathcal{E}_b}$ is bounded by the Newton polygon of $\mathcal{O}(\lambda) \oplus \mathcal{O}(\lambda'_1)^{m'_1}$. Using the fact that $b = b_\xi$ for some $\xi \in \mathrm{Irr}(S_\phi)$, we deduce that $\mathrm{I}(2) = 1$ then $r = 2$, $\xi = (a_1 - 1, 1)$, $\chi = (a_1, 0)$ and $\mathcal{E}_b \simeq \mathcal{O}(\lambda) \oplus \mathcal{O}(\lambda'_1)^{m'_1}$ where $a_1 - 1 = \deg \mathcal{O}(\lambda'_1)^{m'_1}$. We can moreover suppose that $\lambda > \lambda_1$ (otherwise we can use the same arguments as in lemmas 5.5, 5.6 to conclude).

Since the spectral action preserves compact objects, we deduce that the cohomology groups of $i_b^* C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$ are finitely generated and belong to the category $\mathrm{Rep}(\mathfrak{s}_\phi(\xi))$. At this point we know that $C_\xi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_{\mathcal{W}(\xi)}$. Thus there are integers a, k and a non trivial morphism in the category $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$

$$g_1 : (\mathcal{F}_{\mathcal{W}(\xi)})^a[k] \longrightarrow i_b^* C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi},$$

more precisely, we can construct g_1 from a surjection of $(\mathcal{F}_{\mathcal{W}(\xi)})^a$ to the lowest non-trivial cohomology group of $i_b^*C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$.

By excision exact triangles, we see that there is a non trivial morphism in the category $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$

$$g_2 : i_b^*C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \longrightarrow C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}$$

thus there is a non-trivial morphism

$$g : (\mathcal{F}_{\mathcal{W}(\xi)})^a[k] \longrightarrow C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi}.$$

Note that $C_{\xi^{-1}}$ is an auto-equivalence of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$ then by applying $C_{\xi^{-1}}$ to g, we get a non trivial map

$$g' : (\mathcal{W}_{\mathfrak{s}_\phi})^a[k] \longrightarrow C_{\chi'} \star \mathcal{W}_{\mathfrak{s}_\phi}$$

where $\chi' = (1, -1)$. Since by lemma 2.6, there is no non-trivial map from a sheaf supported on Bun_n^1 to a sheaf supported on the non-semi-stable locus. Thus the restriction of $C_{\chi'} \star \mathcal{W}_{\mathfrak{s}_\phi}$ to Bun_n^1 is non trivial.

However, $\chi' = \chi_1 \otimes \chi_2^{-1}$ and we know that $C_{\chi_1} \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_{\mathcal{W}(\chi_1)}$. Then we can compute $i_1^*(C_{\chi_2^{-1}} \star \mathcal{F}_{\mathcal{W}(\chi_1)})$ by using cohomology of local Shtukas spaces. And by using the same arguments as in the base case of the theorem 4.5, we can show that

$$i_1^*(\chi_2^{-1} \star \mathcal{F}_{\mathcal{W}(\chi_1)}) \simeq 0$$

and it is a contradiction. Thus we conclude that

$$i_b^*C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq 0.$$

□

Let Id be the trivial character of S_ϕ and let C_{tri} be the vector bundle corresponding to the Hecke operator $\mathrm{T}_{\mathrm{tri}}$ of the trivial representation of GL_n. We know by construction that C_{Id} is the restriction of C_{tri} on the connected component $[C_\phi]$. By [Ham21, Lemma 3.8] we have the identity $C_{\mathrm{Id}} \star \mathcal{W} = C_{\mathrm{Id}} \star \mathcal{W}_{\mathfrak{s}_\phi} = \mathcal{W}_{\mathfrak{s}_\phi}$. Remark that we can identify R and R' by the map θ that sends X_i to Y_i. The next goal is to describe $\mathbb{L} \star \mathcal{W}_{\mathfrak{s}_\phi}$ where $\mathbb{L}$ is a perfect complex on $[C_\phi]$.

Theorem 9.5. *Let $\mathbb{L}$ be a perfect complex of R-module. For each $\chi \in \mathrm{Irr}(S_\phi)$, let $\mathbb{L}(\chi)$ be the perfect complex $\mathbb{L}$ together with the action of $\mathbb{G}_m^r$ acting by χ so that it gives rise to a perfect complex on $[C_\phi]$. We denote by $\pi_{\mathbb{L}}(\chi)$ the complex of smooth $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$-representations in the derived category of the Bernstein block $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$ corresponding to the R_χ-perfect complex $\mathbb{L}$ (by the natural identification between R and R_χ). Then*

$$\mathbb{L}(\chi) \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq i_{b_\chi !}(\delta_{b_\chi}^{-1/2} \otimes \pi_{\mathbb{L}}(\chi))[-d_\chi].$$

Proof. In general, one uses cones and retracts to construct the spectral action of a perfect complex from the Hecke operators. In the case we consider the perfect complex $\mathbb{L}(\chi)$ is generated by cones and retracts from the vector bundle C_χ. Hence one can compute $\mathbb{L}(\chi) \star \mathcal{W}_{\mathfrak{s}_\phi}$ by using the identity $C_\chi \star \mathcal{W}_{\mathfrak{s}_\phi} = i_{b_\chi !}(\delta_{b_\chi}^{-1/2} \otimes \mathcal{W}_{\mathfrak{s}_\phi(\chi)})[-d_\chi]$ and by tracing back the process of constructing $\mathbb{L}(\chi)$ from C_χ.

For simplicity we suppose $\chi = \mathrm{Id}$, the general case is similar. In the following, every perfect complex of R-modules is equipped with the trivial $\mathbb{G}_m^r$-action so that it gives rise to a perfect complex on $[C_\phi]$. In order to lighten the notation, we simply write $\mathbb{L}$ for $\mathbb{L}(\mathrm{Id})$. Since $\mathbb{L}$ is a perfect complex of R-modules, it has a finite resolution by finitely generated projective R-modules

$$0 \longrightarrow P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \dots \xrightarrow{f_1} P_0 \xrightarrow{f_0} \mathbb{L}.$$

Now, we trace back the construction of the spectral action of $\mathbb{L}$. We consider the cone $\mathrm{Cone}(f_m)$ of f_m. There is an exact triangle

$$P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{g_m} \mathrm{Cone}(f_m)$$

and there is an isomorphism $\mathrm{Cone}(f_m) \simeq P_{m-1}/f_m(P_m)$ in the derived category of the category of R-modules. By Quillen-Suslin's theorem, the R-modules P_i's are free. Thus $P_i \simeq R^{t_i}$ for $1 \leq i \leq m$ and $f_m \in \mathrm{Hom}(R^{t_m}, R^{t_{m-1}}) \simeq \mathrm{End}(R,R)^{t_m t_{m-1}}$. By applying the above exact triangle on $\mathcal{W}_{\mathfrak{s}_\phi}$, we get an exact triangle

$$\mathcal{W}_{\mathfrak{s}_\phi}^{t_m} \xrightarrow{\widetilde{f_m}} \mathcal{W}_{\mathfrak{s}_\phi}^{t_{m-1}} \xrightarrow{\widetilde{g_m}} \mathrm{Cone}(f_m) \star \mathcal{W}_{\mathfrak{s}_\phi},$$

and $\mathrm{Cone}(f_m) \star \mathcal{W}_{\mathfrak{s}_\phi}$ is the cone of $\widetilde{f_m}$. By the equivalence of $\mathrm{Rep}(\mathfrak{s}_\phi)$ and the category of R'-modules via the functor F explained above, we get an exact triangle in the derived category of $\mathrm{Rep}(\mathfrak{s}_\phi)$

$$(R')^{t_m} \xrightarrow{F(\widetilde{f_m})} (R')^{t_{m-1}} \xrightarrow{F(\widetilde{g_m})} F(\mathrm{Cone}(f_m) \star \mathcal{W}_{\mathfrak{s}_\phi}),$$

and therefore $F(\mathrm{Cone}(f_m) \star \mathcal{W}_{\mathfrak{s}_\phi})$ is the cone of $F(\widetilde{f_m})$. However, we can see that an element in $\mathrm{End}(R,R)$ corresponds to the multiplying by a function in R. We can identify R and R' via the natural map θ that sends X_i to Y_i for $1 \leq i \leq r$ and then we can extend θ to identify the category of R-modules with that of R'-modules. Thus by lemma 9.1, $F(\widetilde{f_m})$ is given by the image of $\theta(f_m)$. Therefore, by uniqueness of cones, $F(\mathrm{Cone}(f_m) \star \mathcal{W}_{\mathfrak{s}_\phi}) \simeq \theta(\mathrm{Cone}(f_m))$.

The morphism $P_{m-1} \xrightarrow{f_{m-1}} P_{m-2}$ induces a map g from $\mathrm{Cone}(f_m)$ to P_{m-2}. Similarly, the map $\theta(f_{m-1}) : \theta(P_{m-1}) \longrightarrow \theta(P_{m-2})$ induces a map $\theta(g)$ from $\theta(\mathrm{Cone}(f_m))$ to $\theta(P_{m-2})$ and we can show that $F(\widetilde{g}) = \theta(g)$ (up to some quasi-isomorphism from $\theta(\mathrm{Cone}(f_m))$ to $\theta(\mathrm{Cone}(f_m))$) where $\widetilde{g}$ is the morphism from $\mathrm{Cone}(f_m) \star \mathcal{W}_{\mathfrak{s}_\phi}$ to $P_{m-2} \star \mathcal{W}_{\mathfrak{s}_\phi}$ induced by the spectral action.

Now we can apply the above process to the chain

$$\mathrm{Cone}(f_m) \xrightarrow{g} P_{m-2} \xrightarrow{f_{m-2}} \ldots \xrightarrow{f_1} P_0 \xrightarrow{f_0} \mathbb{L}.$$

Therefore by induction argument we get the identity

$$\mathbb{L} \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq i_{1!}\pi_{\mathbb{L}}.$$

□

Recall that $S_\phi \simeq \prod_{i=1}^{r} \mathbb{G}_m$ and $\mathrm{Irr}(S_\phi) \simeq \prod_{i=1}^{r} \mathbb{Z}$. As before, for each $1 \leq j \leq r$ we denote by χ_j the element of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is in the j^{th}-position. Denote by $\mathfrak{m}_\phi$ the maximal ideal in R corresponding to ϕ. We can define an R-module $\mathbb{L}$ whose underlying $\overline{\mathbb{Q}}_\ell$-vector space is $\overline{\mathbb{Q}}_\ell$ and where R acts via the map $R \longrightarrow R/\mathfrak{m}_\phi \simeq \overline{\mathbb{Q}}_\ell$. For each character $\chi \in \mathrm{Irr}(S_\phi)$, we can define a perfect sheaf $A(\chi)$ on $[C_\phi]$ by letting $\mathbb{G}_m^r$ acts on the fiber of $\mathbb{L}$ via the character χ. It is a skyscraper sheaf on $[C_\phi]$ supported on the closed point defined by ϕ. Note that in general the derived tensor product $A(\chi) \otimes^{\mathbb{L}} A(\chi')$ is not isomorphic to $A(\chi \otimes \chi')$. Recall that for each ϕ and $\chi \in \mathrm{Irr}(S_\phi)$, we defined a sheaf $\mathcal{F}_\chi$ as in theorem 4.5.

Proposition 9.6. *For each $\chi \in \mathrm{Irr}(S_\phi)$ we have*

$$A(\chi) \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}_\chi.$$

Proof. Let V be any algebraic representation of GL_n. Since the restriction of C_V on $[C_\phi]$ is projective, we have $C_V \otimes^{\mathbb{L}} A(\chi) \simeq C_V \otimes A(\chi)$ for any $\chi \in \mathrm{Irr}(S_\phi)$.

Suppose that as $W_{\mathbb{Q}} \times S_\phi$-representation, $V \circ \phi$ decomposes as

$$V \circ \phi \simeq \bigoplus_{\xi \in \mathrm{Irr}(S_\phi)} \sigma_\xi \boxtimes \xi$$

where $\sigma_\xi = \mathrm{Hom}_{S_\phi}(\xi, V \circ \phi)$. Then by the description of the fibers of C_V, we can check that

$$C_V \otimes A(\chi) \simeq \bigoplus_{\xi \in \mathrm{Irr}(S_\phi)} \sigma_\xi \boxtimes A(\chi \otimes \xi).$$

As in the proof of theorem 4.5, it is enough to compute $A(\chi)\star\mathcal{W}_{\mathfrak{s}_\phi}$ for $\chi=(t_1,\dots,t_r)$ where $t_1,\dots,t_r\geq 0$ and we will show that by induction on $|\chi|=\sum_{i=1}^{r}t_i$. We proved the identity $A(\mathrm{Id})\star\mathcal{W}_{\mathfrak{s}_\phi}\simeq\mathcal{F}_{\mathrm{Id}}$ in theorem 9.5.

Suppose that $A(\chi)\star\mathcal{W}_{\mathfrak{s}_\phi}\simeq\mathcal{F}_\xi$ for all ξ such that $0\leq|\xi|<s$. Let χ be an element in $\mathrm{Irr}(S_\phi)^+$ such that $|\chi|=s$. We can find an index j and a character $\chi'\in\mathrm{Irr}(S_\phi)^+$ such that $\chi'\otimes\chi_j=\chi$. Hence we have $|\chi'|=s-1$. Therefore, by induction hypothesis we have

$$A(\chi')\star\mathcal{W}_{\mathfrak{s}_\phi}\simeq\mathcal{F}_{\chi'}.$$

Remark that as a $W_{\mathbb{Q}}\times S_\phi$-representation, $V_{\mathrm{std}}\circ\phi$ decomposes as

$$V_{\mathrm{std}}\circ\phi\simeq\bigoplus_{i=1}^{r}\phi_i\boxtimes\chi_i$$

where V_{std} is the standard representation of GL_n. By monoidal property of the spectral action we have

$$\begin{aligned}\mathrm{T}_{V_{\mathrm{std}}}(\mathcal{F}_{\chi'})&\simeq C_{V_{\mathrm{std}}}\star(A(\chi')\star\mathcal{W}_{\mathfrak{s}_\phi})\\&\simeq(C_{V_{\mathrm{std}}}\otimes A(\chi'))\star\mathcal{W}_{\mathfrak{s}_\phi}\\&\simeq\bigoplus_{i=1}^{r}\phi_i\boxtimes A(\chi'\otimes\chi_i)\star\mathcal{W}_{\mathfrak{s}_\phi}\end{aligned}$$

However, by the theorem 4.5, we have the identity

$$\mathrm{T}_{V_{\mathrm{std}}}(\mathcal{F}_{\chi'})\simeq\bigoplus_{i=1}^{r}\phi_i\boxtimes\mathcal{F}_{\chi'\otimes\chi_i}.$$

By identifying the $W_{\mathbb{Q}_p}$-action, we deduce that $A(\chi'\otimes\chi_i)\star\mathcal{W}_{\mathfrak{s}_\phi}\simeq\mathcal{F}_{\chi'\otimes\chi_i}$ for all $1\leq i\leq r$. In particular, we have

$$A(\chi)\star\mathcal{W}_{\mathfrak{s}_\phi}\simeq\mathcal{F}_\chi$$

and we are done. □

We now want to study the full sub-category $\mathcal{C}:=\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun_n},\overline{\mathbb{Q}}_\ell)^\omega\subset\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun_n},\overline{\mathbb{Q}}_\ell)^\omega$ consisting of compact objects whose Schur-irreducible constituents have L-parameter in the connected component $[C_\phi]$. For each χ, we regard the derived category $\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega$ as a full subcategory of $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun_n},\overline{\mathbb{Q}}_\ell)^\omega$ via the functor i_χ as before.

$$\begin{aligned}i_\chi:\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega&\longrightarrow\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)^\omega\\\pi&\longmapsto i_{b_\chi!}(\delta_{b_\chi}^{-1/2}\otimes\pi)[-d_\chi]\end{aligned}$$

Theorem 9.7. *We have an orthogonal decomposition*

$$\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)^\omega\simeq\bigoplus_{\chi\in\mathrm{Irr}(S_\chi)}\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega$$

Proof. Let $\mathcal{H}$ be a complex in $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)^\omega$ then by proposition 4.3, the support of $\mathcal{H}$ is a finite union of strata of the form b_χ where $\chi\in\mathrm{Irr}(S_\phi)$. Consider a restriction $i_{b_\chi}^*\mathcal{H}$ of $\mathcal{H}$ to such a stratum. Then all the irreducible constituents of the cohomology groups of $i_{b_\chi}^*\mathcal{H}$ have Fargues-Scholze L-parameters in the connected component $[C_\phi]$. Therefore, as smooth representations of $\mathrm{G}_{b_\chi}(\mathbb{Q}_p)$, these irreducible constituents belong to the direct sum

$$\bigoplus_{b_{\chi'}=b_\chi}\mathrm{Rep}(\mathfrak{s}_\phi(\chi'))^\omega$$

of Bernstein blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{G}_{b_\chi}(\mathbb{Q}_p))$. Hence $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)^\omega$ is generated by the full subcategories $\mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega$'s where $\chi\in\mathrm{Irr}(S_\phi)$.

To prove the direct decomposition now, it suffices to prove that if $\mathcal{F} \in \mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi)))^\omega$ and $\mathcal{G} \in \mathrm{D}(\mathrm{Rep}(\mathfrak{s}_\phi(\chi')))^\omega$ for $\chi \neq \chi'$ then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = 0$.

If $b_\chi = b_{\chi'} = b$ then it is clear that $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = 0$ since $\mathrm{Rep}(\mathfrak{s}_\phi(\chi))$ and $\mathrm{Rep}(\mathfrak{s}_\phi(\chi'))$ are two distinct Bernstein blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{G}_b(\mathbb{Q}_p))$.

We suppose that $b_\chi \neq b_{\chi'}$. Let $\mathbb{L}$ and $\mathbb{L}'$ be the complexes in $\mathrm{D}(\mathcal{Z}_{\mathfrak{s}_\phi(\chi)}\text{-Mod})$ and in $\mathrm{D}(\mathcal{Z}_{\mathfrak{s}_\phi(\chi')}\text{-Mod})$ corresponding respectively to $\mathcal{F}$ and $\mathcal{G}$. Since they are compact, the complexes $\mathbb{L}$ and $\mathbb{L}'$ are perfect. Then by theorems 9.3 and 9.5 we see that $\mathbb{L}(\chi) \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{F}$ and $\mathbb{L}'(\chi') \star \mathcal{W}_{\mathfrak{s}_\phi} \simeq \mathcal{G}$. Since $C_{\chi^{-1}} \star (-)$ is an auto-equivalence of $\mathrm{D}^{[C_\phi]}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$, we deduce that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathcal{C}}(\mathbb{L}(\chi) \star \mathcal{W}_{\mathfrak{s}_\phi}, \mathbb{L}'(\chi') \star \mathcal{W}_{\mathfrak{s}_\phi}) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathbb{L}(\mathrm{Id}) \star \mathcal{W}_{\mathfrak{s}_\phi}, \mathbb{L}'(\chi' \otimes \chi^{-1}) \star \mathcal{W}_{\mathfrak{s}_\phi}).\end{aligned}$$

Again by theorem 9.5 the complex $\mathbb{L}(\mathrm{Id}) \star \mathcal{W}_{\mathfrak{s}_\phi}$ is supported on Bun_n^1 and $\mathbb{L}'(\chi' \otimes \chi^{-1}) \star \mathcal{W}_{\mathfrak{s}_\phi}$ is supported on the complement of Bun_n^1. Therefore lemma 2.6 implies that

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{L}(\mathrm{Id}) \star \mathcal{W}_{\mathfrak{s}_\phi}, \mathbb{L}'(\chi' \otimes \chi^{-1}) \star \mathcal{W}_{\mathfrak{s}_\phi}) = 0$$

and it allows us to conclude. $\square$

10. Local-Global compatibility

In this section we prove some particular cases of Fargues' and Caraiani-Scholze's local-global conjecture [Far16, §7], [CS17, 1.18].

10.1. **Cohomology of Igusa varieties.**

10.1.1. *Shimura varieties and Mantovan's formula.*

We consider some simple Shimura varieties of type A as in [Kot92]. Assume that F is an imaginary quadratic field, p is split in F and that we have an unramified integral PEL datum of the form $(\mathrm{B}, \mathcal{O}_\mathrm{B}, *, \mathrm{V}, \Lambda_0, \langle,\rangle, \mathrm{h})$, where

- B is a division algebra with center F,
- $*$ is an involution of the second kind,
- $\mathrm{B}, \mathcal{O}_\mathrm{B}$ is a $\mathbb{Z}_{(p)}$ maximal order in B that is preserved by $*$ such that $\mathrm{B}, \mathcal{O}_\mathrm{B} \otimes_\mathbb{Z} \mathbb{Z}_\mathrm{p}$ is a maximal order in $\mathrm{B}_{\mathbb{Q}_\mathrm{p}}$,
- V is a simple B-module,
- $\langle,\rangle : V \times V \longrightarrow \mathbb{Q}$ is a $*$-Hermitian pairing with respect to the B-action,
- Λ_0 is a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ that is preserved by $\mathcal{O}_\mathrm{B}$ and self-dual for $\langle,\rangle$.
- $h : \mathbb{C} \longrightarrow \mathrm{End}_\mathbb{R}(V_\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism satisfying the equality $\langle h(z), w\rangle = \langle v, h(z^c)w\rangle, \forall v, w \in V_\mathbb{R}$ and $z \in \mathbb{C}$ and such that the bilinear pairing $(v, w) \longmapsto \langle v, h(\sqrt{-1})w\rangle$ is symmetric and positive definite.

We can define a similitude unitary group $\mathbf{G}$ over $\mathbb{Q}$ associated to this PEL datum and denote by $\mathbf{G}'$ the unitary group that is the kernel of the similitude factor, thus $\mathbf{G}'(\mathbb{R}) \simeq \mathrm{U}(q, n-q)$ for some $q \in \mathbb{N}$. We assume further that the p-adic group $\mathbf{G}'_{\mathbb{Q}_\mathrm{p}}$ is isomorphic to $\mathrm{GL}_{n,\mathbb{Q}_p}$ and hence $\mathbf{G}_{\mathbb{Q}_p} \simeq \mathrm{GL}_{n,\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p}$.

Let $\mu : \mathbb{G}_m \longrightarrow \mathbf{G}$ be the group homomorphism over $\mathbb{C}$ corresponding to h. Associated to the above PEL datum is a system of *projective* Shimura varieties $\{Sh_K\}$ defined over the reflex field E and where K runs over the set of sufficiently small open compact subgroups of $\mathbf{G}(\mathbb{A}^\infty)$. Moreover, the PEL datum also gives rise to a system of integral models $\mathcal{S}_p = \{\mathcal{S}_{K^p}\}$ defined over $\mathcal{O}_{E,(p)}$ such that the generic fiber of each $\mathcal{S}_{K^p}$ is naturally identified with $Sh_{K^p K_p^{\mathrm{hs}}}$ where K^p runs over the set of sufficiently small open compact subgroups of $\mathbf{G}(\mathbb{A}^{\infty,\mathrm{p}})$ and $K_p^{\mathrm{hs}} \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is a hyperspecial subgroup.

On the other hand, for each $b \in B(\mathbf{G}'_{\mathbb{Q}_\mathrm{p}}, -\mu)$, we can define an Igusa variety Ig_b which is closely related to the special fiber $\overline{\mathcal{S}}_p$ of $\mathcal{S}_p$. The Igusa variety is a projective system of smooth varieties $\mathrm{Ig}_b = \{\mathrm{Ig}_{b,m,K^p}\}$ where $m \in \mathbb{N}$ and K^p are sufficiently small open compact subgroups of $\mathbf{G}(\mathbb{A}^{\infty,\mathrm{p}})$ as before. Let $\mathcal{L}$ be an algebraic representation of $\mathbf{G}$ over $\overline{\mathbb{Q}}_\ell$. Then it gives rises to ℓ-adic sheaves on Sh and Ig_b, which will be denoted by $\mathcal{L}$ (by abuse of notation). Define

$$R\Gamma_c(Sh,\mathcal{L}) := \mathrm{colim}_{\overrightarrow{K}} R\Gamma_c(Sh_K,\mathcal{L}),$$

$$R\Gamma_c(\mathrm{Ig}_b,\mathcal{L}) := \mathrm{colim}_{\overrightarrow{K^p,m}} R\Gamma_c(\mathrm{Ig}_{b,m,K^p},\mathcal{L}).$$

These complexes are endowed with an action of $\mathbf{G}(\mathbb{A}^\infty)\times \mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E})$ and of $\mathbf{G}(\mathbb{A}^{\infty,p})\times \big(\mathrm{G}_b(\mathbb{Q}_p)\times\mathbb{Q}_p^\times\big)$ respectively.

For each K, the special fiber $\overline{\mathcal{S}}_K$ has a stratification by the Kottwitz set $B(\mathrm{GL}_n,-\mu)$. Thus we can compute the cohomology of Shimura varieties by the cohomology of its strata. We recall a formula of Mantovan that expresses a cohomological relation between $R\Gamma_c(Sh,\mathcal{L})$, $R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})$'s and the Hecke operator $\mathrm{T}_{-\mu}$. Denote by π an irreducible representation of $\mathbf{G}(\mathbb{Q}_p)$ and let Π be any $\mathcal{L}$-cohomological automorphic representation of $\mathbf{G}(\mathbb{A})$ globalizing π. We denote by $R\Gamma_c(Sh,\mathcal{L})[\Pi^{\infty,p}]$ and $R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]$ the $[\Pi^{\infty,p}]$-isotypic part of the cohomology of the Shimura variety, respectively the $[\Pi^{\infty,p}]$-isotypic part of the cohomology of the Igusa varieties.

Proposition 10.1. *(*[Man05, Theorem 22], [LS18, Theorems. 6.26, 6.32]*) We set $d := \dim Sh = \langle 2\rho_{\mathrm{GL}_n},\mu\rangle$ and for $b\in B(\mathrm{GL}_n,-\mu)$, we set $d_b := \dim \mathrm{Ig}_b = \langle 2\rho_{\mathrm{GL}_n},\nu_b\rangle$. Then the complex $R\Gamma_c(Sh,\mathcal{L})[\Pi^{\infty,p}]$ has a filtration as a complex of $\boldsymbol{G}(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$ representations with graded pieces isomorphic to $i_1^*\mathrm{T}_{-\mu}(i_{b!}\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}])[-d](-\frac{d}{2})$ where b runs in the Kottwitz set $B(\mathrm{GL}_n,-\mu)$. More precisely, the graded pieces are isomorphic to*

$$R\Gamma_c(\mathrm{GL}_n,b,\mu)\otimes^{\mathbb{L}}_{\mathcal{H}(\mathrm{G}_b)} R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}][2d_b-d](-\tfrac{d}{2})$$

as complexes of $\boldsymbol{G}(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$-modules.

Remark 10.2.
(1) In fact, Mantovan and Lan-Stroh proved the proposition with $R\operatorname{Hom}_{\mathcal{H}(\mathrm{G}_b)}$ instead of $\otimes^{\mathbb{L}}_{\mathcal{H}(\mathrm{G}_b)}$. However the cohomology groups of the complex $R\Gamma_c(Sh,\mathcal{L})[\Pi^{\infty,p}]$, respectively of the complex $R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]$ are admissible as $\mathbf{G}(\mathbb{Q}_p)$-representations, respectively as $\mathrm{G}_b(\mathbb{Q}_p)$-representations. Therefore the Hom-Tensor duality gives us the dual statement with $\otimes^{\mathbb{L}}_{\mathcal{H}(\mathrm{G}_b)}$ in place of $R\operatorname{Hom}_{\mathcal{H}(\mathrm{G}_b)}$. By using lemma 2.10, we deduce the expression in term of $\mathrm{T}_{-\mu}$.
(2) Koshikawa and Hamann-Lee proved [Kos21, Theorem. 7.1], [HL23, Theorem. 1.13] some analogue of this proposition for torsion coefficients and for more general Shimura varieties of type A and type C.

10.1.2. *Hodge-Tate period map and local-global compatibility.*

Recall that by [CS17, Theorem. 1.10], for each sufficiently small compact open subgroup $K^p\subset\mathbf{G}(\mathbb{A}^{\infty,\mathrm{p}})$ there is a perfectoid space Sh_{K^p} over E_p such that

$$Sh_{K^p}\sim \lim_{\overleftarrow{K_p}} Sh_{K_pK^p}\otimes_E E_p$$

and there is also a Hodge Tate period map

$$\pi_{\mathrm{HT}} : Sh_{K^p}\longrightarrow \mathcal{F}\ell_{\mathrm{GL}_n,\mu}.$$

Let ϕ be any local L-parameter of GL_n. Fargues and Caraiani-Scholze have conjectured that the perverse sheaves $R\pi_{\mathrm{HT}*}\overline{\mathbb{Q}}_\ell$ on $\mathcal{F}\ell_{\mathrm{GL}_n,\mu}$ are related to the conjectural Hecke eigensheaf on Bun_n associated with ϕ via the natural map $\overleftarrow{h} : \mathcal{F}\ell_{\mathrm{GL}_n,\mu}\longrightarrow \mathrm{Bun}_n$, by some form of local-global compatibility. We remark that Caraiani-Scholze also compared the fibers of the Hodge-Tate period map with Igusa varieties. Hence we will first compute the cohomology of Igusa varieties and then deduce some form of the local-global compatibility.

Let $\overline{\phi}=\overline{\phi}_1\oplus\ldots\oplus\overline{\phi}_r$ be an L-parameter of GL_n satisfying condition (A1). Then $S_{\overline{\phi}}\simeq\mathbb{G}_m^r$ and the connected component of $\overline{\phi}$ in $[Z^1(W_{\mathbb{Q}_p},\widehat{\mathrm{GL}}_n)/\widehat{\mathrm{GL}}_n]$ is isomorphic to $[\mathbb{G}_m^r/\mathbb{G}_m^r]$ as before. We consider an L-parameter $\phi=\phi_1\otimes\chi_{t_1}\oplus\ldots\oplus\phi_1\otimes\chi_{t_r}$ where $t=(t_1,\ldots,t_r)$ is a closed point in $\mathbb{G}_m^r$ and where χ_{t_i} is the unramified character of $W_{\mathbb{Q}_p}$ corresponding to t_i. We suppose that

(A2) The elements $p, t_1, \dots, t_r$ are linearly independent in the $\mathbb{Z}$-module $\overline{\mathbb{Q}}_\ell^\times$ and if $1 \leq i \neq j \leq r$ then $V_i \cap V_j = \{0\}$ where V_i is the $\mathbb{Z}$-module generated by the set $\{t_i \times \alpha \mid \alpha \text{ is an eigenvalue of the Frobenius of } \overline{\phi}_i\}$.

Note that the locus of the points t such that ϕ satisfies condition (A2) is an open dense subset of $\mathbb{G}_m^r$. Let $r_{-\mu}$ be the highest weight representation associated with $-\mu$. A consequence of the above condition is that if for $\chi \in \mathrm{Irr}(S_\phi)$ we denote by σ_χ the $W_{\mathbb{Q}_p}$-representation

$$\mathrm{Hom}_{S_\phi}(\chi, r_{-\mu} \circ \phi),$$

then $\mathrm{Hom}_{W_{\mathbb{Q}_p}}(\sigma_\chi, \sigma_{\chi'}) = 0$ if $\chi \neq \chi'$.

Denote by π the irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ whose L-parameter is given by ϕ and fix a character ω of $\mathbb{Q}_p^\times$. Let Π be any $\mathcal{L}$-cohomological automorphic representation of $\mathbf{G}(\mathbb{A})$ globalizing $\pi \times \omega$, in particular Π is cuspidal. For each $b \in B(\mathrm{GL}_n, -\mu)$, we are going to compute the $\Pi^{\infty,p}$-isotypic part

$$R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]$$

of the cohomology of the variety Ig_b as $\mathrm{G}_b(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ representation. The rough idea is that Mantovan's formula gives us a relation between $R\Gamma_c(Sh, \mathcal{L})$, $R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})$'s and the Hecke operator $\mathrm{T}_{-\mu}$. However, in our case, we fully described the Hecke operator $\mathrm{T}_{-\mu}$ and it leads to a transparent relation between $R\Gamma_c(Sh, \mathcal{L})$ and $R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})$'s. Finally, the knowledge of $R\Gamma_c(Sh, \mathcal{L})$ allows us to describe $R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})$'s.

Theorem 10.3. *Suppose that ϕ satisfies conditions (A1) and (A2). Then there exists a multiplicity $m \in \mathbb{N}$ such that for all $b \in B(\mathrm{GL}_n, -\mu)$, the complex $R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]$ concentrates in middle degree and we have an isomorphism of $\mathrm{G}_b(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$-representations*

$$H_c^{d_b}(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}] = m \bigoplus_{\substack{\chi \in \mathrm{Irr}(S_\phi) \\ [b_\chi]=[b]}} \delta_b^{1/2} \otimes \big(\pi_\chi \times \omega\big).$$

where b_χ and π_χ are constructed from χ as in section 4.

Proof. The first step is to compute the $[\Pi^{\infty,p}]$-isotypic component $R\Gamma_c(Sh, \mathcal{L})[\Pi^{\infty,p}]$ as complex of $\mathbf{G}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-modules. By the hypothesis imposed on Π, the complex $R\Gamma_c(Sh, \mathcal{L})[\Pi^{\infty,p}]$ is concentrated in degree d. Therefore, by the results of Kottwitz and Harris-Taylor we deduce that there exists a multiplicity $m \in \mathbb{N}$ such that

$$H_c^d(Sh, \mathcal{L})[\Pi^{\infty,p}] = m(\pi \times \omega) \boxtimes r_{-\mu} \circ \phi(-\tfrac{d}{2})$$

as $\mathbf{G}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-representations.

By Mantovan's product formula, there is a filtration of the $\mathbf{G}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-modules $R\Gamma_c(Sh, \mathcal{L})[\Pi^{\infty,p}]$ whose graded pieces are isomorphic to $i_1^*\mathrm{T}_{-\mu}(i_{b!}\delta_b^{-1} \otimes R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}])[-d](-\tfrac{d}{2})$ for $b \in B(\mathrm{GL}_n, -\mu)$. Thus, it induces a spectral sequence converging to the cohomology groups of $R\Gamma_c(Sh, \mathcal{L})[\Pi^{\infty,p}]$.

Since the Fargues-Scholze L-parameter of π is given by ϕ, we deduce that every irreducible subquotient of the $\mathrm{G}_b(\mathbb{Q}_p)$-modules $H_c^k(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]$ also has Fargues-Scholze L-parameter given by ϕ. However, the complex $R\Gamma_c(Sh, \mathcal{L})[\Pi^{\infty,p}]$ of $\mathrm{GL}_n(\mathbb{Q}_p)$-modules is admissible and the complexes $R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]$ of $\mathrm{G}_b(\mathbb{Q}_p)$-modules are also admissible. Therefore, by theorem 9.7, we deduce that for each $b \in B(\mathrm{GL}_n, -\mu)$, there is an orthogonal decomposition

$$i_{b!}\big(\delta_b^{-1} \otimes R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]\big) = \bigoplus_{\substack{\chi \in \mathrm{Irr}(S_\phi) \\ [b_\chi]=[b]}} i_{b!}\big(\delta_b^{-1} \otimes R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]\big)_\chi.$$

where $i_b : \mathrm{Bun}_b \longrightarrow \mathrm{Bun}_n$ is the usual embedding and $i_{b!}\big(\delta_b^{-1} \otimes R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]\big)_\chi$ belongs to the full subcategory of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ corresponding to χ and the connected component $[C_\phi]$ as in the previous sections. Thus it suffices to compute $i_{b!}\big(\delta_b^{-1} \otimes R\Gamma_c(\mathrm{Ig}_b, \mathcal{L})[\Pi^{\infty,p}]\big)_\chi$. Moreover we have

$$\begin{aligned}
&i_1^*\mathrm{T}_{-\mu}(i_{b!}\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}])[-d](-\tfrac{d}{2})\\
\simeq\ & i_1^*\mathrm{T}_{-\mu}\bigoplus_{\substack{\chi\in\mathrm{Irr}(S_\phi)\\ [b_\chi]=[b]}} i_{b!}\big(\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]\big)_\chi[-d](-\tfrac{d}{2})\\
\simeq\ & i_1^*\bigoplus_{\chi'\in\mathrm{Irr}(S_\phi)}\Big(C_{\chi'}\star\bigoplus_{\substack{\chi\in\mathrm{Irr}(S_\phi)\\ [b_\chi]=[b]}} i_{b!}\big(\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]\big)_\chi\Big)\boxtimes\sigma_{\chi'}[-d](-\tfrac{d}{2})\\
\simeq\ & \bigoplus_{\substack{\chi\in\mathrm{Irr}(S_\phi)\\ [b_\chi]=[b]}}\Big(C_{\chi^{-1}}\star i_{b!}\big(\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]\big)_\chi\Big)\boxtimes\sigma_{\chi^{-1}}[-d](-\tfrac{d}{2}), \qquad (14)
\end{aligned}$$

where the last isomorphism comes from the fact that $[b_\xi]\neq[1]$ if $\xi\neq\mathrm{Id}$ and the complex $C_{\chi'}\star\Big(i_{b!}\big(\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]\big)_\chi\Big)$ is supported on $[b_{\chi'\otimes\chi}]$.

Recall that there is a filtration of the $\mathbf{G}(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$-modules $R\Gamma_c(Sh,\mathcal{L})[\Pi^{\infty,p}]$ whose graded pieces are isomorphic to $i_1^*\mathrm{T}_{-\mu}(i_{b!}\delta_b^{-1}\otimes R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}])[-d](-\frac{d}{2})$ for $b\in B(\mathrm{GL}_n,-\mu)$ and then it induces a spectral sequence converging to the cohomology groups of $R\Gamma_c(Sh,\mathcal{L})[\Pi^{\infty,p}]$. However, by combining equation (14) and the fact that $\mathrm{Hom}_{W_{\mathbb{Q}_p}}(\sigma_\chi,\sigma_{\chi'})=0$ for $\chi\neq\chi'$, we deduce that the spectral sequence degenerates. Thus we have an isomorphism of complexes of $\mathbf{G}(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$-modules

$$\begin{aligned}
\bigoplus_{\chi\in\mathrm{Irr}(S_\phi)}\Big(C_{\chi^{-1}}\star i_{b_\chi!}\big(\delta_{b_\chi}^{-1}\otimes R\Gamma_c(\mathrm{Ig}_{b_\chi},\mathcal{L})[\Pi^{\infty,p}]\big)_\chi\Big)\boxtimes\sigma_{\chi^{-1}}[-d](-\tfrac{d}{2}) &\simeq R\Gamma_c(Sh,\mathcal{L})[\Pi^{\infty,p}]\\
&\simeq m\big(\pi\times\omega\big)\boxtimes r_{-\mu}\circ\phi[-d](-\tfrac{d}{2}).
\end{aligned}$$

By using the fact that $\mathrm{Hom}_{W_{\mathbb{Q}_p}}(\sigma_\chi,\sigma_{\chi'})=0$ for $\chi\neq\chi'$ and identifying the $W_{\mathbb{Q}_p}$-action in both sides we deduce that

$$C_{\chi^{-1}}\star\Big(i_{b_\chi!}\big(\delta_{b_\chi}^{-1}\otimes R\Gamma_c(\mathrm{Ig}_{b_\chi},\mathcal{L})[\Pi^{\infty,p}]\big)_\chi\Big)\simeq m\cdot m_{\chi^{-1}}\big(\pi\times\omega\big),$$

where $m_{\chi^{-1}}=1$ if $\sigma_{\chi^{-1}}\neq 0$ and $m_{\chi^{-1}}=0$ otherwise. Since μ is minuscule, we can check that $\sigma_{\chi^{-1}}\neq 0$ if and only if $b_\chi\in B(\mathrm{GL}_n,-\mu)$. Now by applying C_χ on both sides, we get

$$i_{b_\chi!}\big(\delta_{b_\chi}^{-1}\otimes R\Gamma_c(\mathrm{Ig}_{b_\chi},\mathcal{L})[\Pi^{\infty,p}]\big)_\chi\simeq m\cdot m_{\chi^{-1}}\big(\delta_{b_\chi}^{-1/2}\otimes\pi_\chi\times\omega\big)[-d_b].$$

Thus for each $b\in B(\mathrm{GL}_n,-\mu)$, by taking the sum over the χ's such that $[b_\chi]=[b]$, we obtain the following formula

$$R\Gamma_c(\mathrm{Ig}_b,\mathcal{L})[\Pi^{\infty,p}]=m\bigoplus_{\substack{\chi\in\mathrm{Irr}(S_\phi)\\ [b_\chi]=[b]}}\delta_b^{1/2}\otimes\big(\pi_\chi\times\omega\big)[-d_b].$$

□

We have completely described the Hecke eigensheaves associated with the L-parameters ϕ satisfying conditions (A1), (A2) and we have also computed some part of the cohomology of the Igusa varieties related to ϕ. Hence we can deduce some weak form of the local-global compatibility.

Corollary 10.4. *Let ϕ be an L-parameter satisfying conditions (A1) and (A2). Denote by $\mathcal{G}_\phi$ the Hecke eigensheaf on Bun_n corresponding to ϕ by theorem 7.1. Let x be a point of $\mathcal{F}\ell_{\mathrm{GL}_n,\mu}$ in the stratum corresponding to b. Then there exists a multiplicity $m\in\mathbb{N}$ such that we have an isomorphism*

$$\delta_b\otimes\big(R\pi_{\mathrm{HT}*}\overline{\mathbb{Q}}_\ell\big)_x[\Pi^{\infty,p}]\simeq m\overleftarrow{h}^*\big(i_b^*\mathcal{G}_\phi\big),$$

where $\overleftarrow{h}:\mathcal{F}\ell_{\mathrm{GL}_n,\mu}\longrightarrow\mathrm{Bun}_n$ is the natural map.

Proof. This is a consequence of the description of the stalks of $\mathcal{G}_\phi$ by theorem 7.1, the computation of the cohomology of Igusa varieties Ig_b by theorem 10.3 and the comparision of the fibers of π_{HT} with Igusa varieties [CS17, Theorem. 1.15], [HL23, Corollary 3.13]. □

Remark 10.5. (1) It is now clear that Caraiani-Scholze's comparison theorem [CS17, Theorem 1.15] and the local-global compatibility conjecture imply a strong relation between the cohomology of Igusa varieties and the stalks of the conjectural Hecke eigensheaves.

(2) Theorem 10.3 generalizes the $[\Pi^{\infty,p}]$-isotypic part of [Shi12, Theorem 6.7] and gives more precise information. However, the results in [Shi12] and [BMS] could give information on more general part of the cohomology of the Igusa varieties. Hence as mentioned above, it could reveal important information on the conjectural Hecke eigensheaves associated to more general L-parameters such as the ones corresponding to the Steinberg representations.

10.2. **Scope of generalization.**

We conclude with comments on the possibility of generalizing our work.

10.2.1. *Torsion coefficients.*

We only consider the category $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun_n}, \Lambda)^\omega$ where $\Lambda = \overline{\mathbb{Q}}_\ell$ in this paper but with extra efforts, one could run the same arguments in the case $\Lambda = \overline{\mathbb{F}}_\ell$, at least when ℓ is a very good prime. These questions will be studied in an upcoming project. Combining the expected results in the case $\Lambda = \overline{\mathbb{F}}_\ell$ with the method developed by Linus Hamann and Yi-Sing Lee [HL23] (which is based on the works [CS17, CS19, Kos21, Zha23]), one can obtain many new cases of the vanishing of the cohomology of Shimura varieties with torsion coefficients.

10.2.2. *Other groups.*

The ideas and arguments in this paper are primarily geometric and many of them could work in more general contexts other than GL_n. However, the paper still depends on a number of special aspects of GL_n such as the compatibility of the usual local L-parameter with the one constructed by Fargues-Scholze and the explicit combinatoric nature of the Kottwitz set $B(\mathrm{GL}_n)$. In particular, there is a cocharacter $\mu := (1, 0^{(n-1)})$ whose highest weight representation is simple and convenient for our analysis.

We expect that one can extend the main results of this paper to the groups SL_n, $\mathrm{Res}_{\mathbb{Q}_p}(\mathrm{GL}_{n,F})$, $(\mathrm{G})\mathrm{U}_n$ or even $(\mathrm{G})\mathrm{SO}_{2n+1}$ if the compatibility of L-parameters is available. The SL_n and $(\mathrm{G})\mathrm{U}_{2n}$ cases are particularly interesting since it could shed light to the functoriality between GL_n and SL_n as well as between GU_{2n} and U_{2n} in the context of categorical local Langlands program.

It is not clear whether one can apply the method to the groups $(\mathrm{G})\mathrm{Sp}_{2n}$ and $(\mathrm{G})\mathrm{SO}_{2n}$. In these cases, the compatibility of L-parameters seems to be out of reach and more importantly, the highest weight representations associated with minuscule cocharacters are the spin representations which are harder to analyse[2]. However, thanks to [Ham21], one can apply the method to the group $(\mathrm{G})\mathrm{Sp}_4$.

For a general reductive group G, it is reasonable to speculate that some form of the theorems 4.5, 9.5 and 9.7 are true, at least for L-parameters ϕ such that the corresponding connected component $[C_\phi]$ in the stack of L-parameters is isormorphic to $[\mathbb{G}_m^r/\mathbb{G}_m^r \times S]$ or $[\mathbb{G}_m^{r-1}/\mathbb{G}_m^r \times S]$ where r is a natural number and S is an abelian group and $\mathbb{G}_m^r \times S$ acts trivially. The point is that in this case we have an orthogonal decomposition

$$\mathrm{Perf}([C_\phi]) \simeq \bigoplus_{\chi \in \mathrm{Irr}(\mathbb{G}_m^r \times S)} \mathrm{Perf}(\mathbb{G}_m^r),$$

and the categorical local Langlands program would implied a similar orthogonal decomposition of the category $\mathrm{D}_{\mathrm{lis}}^{[C_\phi]}(\mathrm{Bun}_{\mathrm{G}}, \overline{\mathbb{Q}}_\ell)^\omega$.

It is interesting to use the method in this paper to study the case $\Lambda = \overline{\mathbb{F}}_\ell$ or $\Lambda = \overline{\mathbb{Z}}_\ell$. It is also reasonable to speculate that some versions of the main theorems still hold in this context.

[2] I thank Peter Scholze for pointing out the possible difficulties concerning complicated highest weight representations associated with minuscule cocharacters.

10.2.3. *The equal characteristic case.*

In [FS21], Fargues and Scholze treated the mixed characteristic and equal characteristic cases uniformly. One might ask to what extend the method in this paper could apply to the equal characteristic case. Some important results such as the classification of G-bundles over the "equal characteristic Fargues-Fontaine curve" and the compatibility between Fargues-Scholze's construction of L-parameters with that of Genestier-Lafforgues for GL_n were obtained in [Ans19, LH23]. However, one might need to establish many other important results in order to apply the arguments in this paper.

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