INVERSE SATAKE TRANSFORMS

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ABSTRACT. Let H be a split reductive group over a local non-archimedean field, and let \hat{H} denote its Langlands dual group. We present an explicit formula for the generating function of an unramified L-function associated to a highest weight representation of the dual group, considered as a series of elements in the Hecke algebra of H. This offers an alternative approach to a solution of the same problem by Wen-Wei Li. Moreover, we generalize the notion of "Satake transform" and perform the analogous calculation for a large class of spherical varieties.

CONTENTS

1.	Introduction	1
2.	Assumptions and Cartan decomposition	3
3.	Definition: Satake transform	6
4.	Boundary degenerations and asymptotics	9
5.	Range of validity of asymptotics	11
6.	Asymptotics of the basic function	13
7.	Inverse Satake transforms	19
8.	The group case; relation to the formula of Wen-Wei Li	24
References		27

1. Introduction

The purpose of this paper is to discuss the notion of "Satake transform" for a spherical variety X under a reductive group G over a p-adic field F, generalizing the corresponding notion for the case X= a reductive group, and to present an explicit formula for its inversion, based on the theory of asymptotics of smooth G-representations on X. In particular, in the group case this gives an alternative approach to the one discovered by Wen-Wei Li in [Li].

Let H be a reductive group over a p-adic field F, and let us assume for simplicity that H is split. The Satake transform establishes an isomorphism between the unramified Hecke algebra of H (with respect to some hyperspecial open compact subgroup) and the algebra of invariant polynomials on the complex dual group \check{H} .

While the inversion of this transform is known to be given by Kazhdan-Lusztig polynomials, one would like an "efficient" method of inversion, especially when invariant polynomials on \hat{H} are replaced by certain rational functions on H (which, on the other side of the Satake isomorphism, should correspond to series of elements in the Hecke algebra). In particular, one is motivated by the paper of B.C. Ngô [Ngô], which suggests a relationship between "basic functions" on reductive monoids and local unramified automorphic L-functions. The prototype of this is the characteristic function of $n \times n$ integers, which was famously used by Godement and Jacquet [GJ72] to study the standard L-function for GL_n . Ngô's discovery shows that, at least in the local, unramified setting, the Godement-Jacquet approach is part of a more general story, where all automorphic L-functions appear. In order, however, to study global problems, one would need to generalize Fourier transforms and the Poisson summation formula. Part of the motivation of studying the inverse Satake transform has to do with the hope of finding an explicit (non-spectral) description of a Fourier transform in this setting.

In the paper [Li], Wen-Wei Li showed that the inversion problem for elements of the Hecke algebra corresponding to L-functions can be efficiently solved, circumventing the tedious, algorithmic process of decomposing symmetric powers of a representation of \check{H} and then using an infinite number of Kazhdan-Lusztig polynomials. His Theorem 5.3.1 offers a complete resolution to this problem. The goal of the present paper is to offer an alternative approach, which automatically applies to spherical varieties as well.

Indeed, as was shown in [Sak12], the Godement-Jacquet method and its generalizations proposed by Ngô are themselves part of a broader framework which includes the Rankin-Selberg method. The basic object for this generalization is that of an *affine spherical variety*. While we do not yet know the precise relationship with *L*-functions in the most general case, it seems to be confirmed by all known examples [BFGM02, BNS], and it may not be long before it is established. Thus, for the same reasons as above it will be interesting to have some understanding of inverse spectral transforms in this generality and, in any case, as we will see, the theory of asymptotics on spherical varieties provides a very straightforward approach to the problem of inversion, including in the group case.

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2. ASSUMPTIONS AND CARTAN DECOMPOSITION

We will assume throughout that G is split over F, in order to use results that are not yet available in the general case of unramified groups. In particular, we may fix the Chevalley model for G over the ring of integers $\mathfrak o$ of F, and denote by K its hyperspecial maximal compact subgroup $G(\mathfrak o)$. Whenever there is no confusion, we will be denoting G(F) simply by G, X(F) by X etc.

We let *X* be a homogeneous spherical variety which satisfies the following conditions:

- It is quasi-affine, admits a *G*-eigenmeasure, and its open Borel orbit admits a *B*-invariant measure. These assumptions cause no harm to generality, since every homogeneous variety admits a torus bundle whose total space satisfies them [Sak08, §3.8].
- It is *wavefront* [SV, §2.1]. This property, whose definition will be recalled below, applies to almost all spherical varieties, and certainly all symmetric ones (such as: reductive groups themselves). The reason for imposing it is to be able to take advantage of the theory of asymptotics, which is for now missing in the non-wavefront case.
- It satisfies the conclusions of [Sak12, Theorems 2.3.8, 2.3.10] on orbits of a hyperspecial and an Iwahori subgroup. These conditions are satisfied at almost every place when G, X are defined over a global field, and will also be recalled below.

To formulate the above conditions, we introduce the notion of a *Cartan* subtorus $A_X \subset X$, pointing the reader to [SV, $\S 2$] for more details (where the notation A_X was also used, but not the term "Cartan subtorus"). At first reading I would recommend to the reader to skip the definitions, and just accept the existence of a "root system" formalism allowing for a generalization of the Cartan decomposition to an arbitrary spherical variety.

A Cartan subtorus of X is a subvariety obtained by fixing a triple (x_0, B, f) , where $x_0 \in X$, B is a Borel subgroup such that x_0 is contained in its open orbit X° , and f is a B-eigenfunction whose set-theoretic zero locus is $X \setminus X^{\circ}$. If we pull the function f back to a function on G via the action map $G \ni g \mapsto x_0g \in G$, its differential becomes an element of the coadjoint representation whose centralizer we denote by L(X). Then the image of the action map:

$$L(X) \ni \ell \mapsto x_0 \cdot \ell \in X$$

will be called a Cartan subtorus A_X of X; through the action map, it is isomorphic to a torus quotient of L(X), and the resulting group structure will be considered as part of the data of a Cartan subtorus, as will the resulting identification with the *universal Cartan* (which we will be denoting by the same letter) $A_X \simeq X^\circ/N$ (where N is the unipotent radical of B).

In the group case, X=H under the $G=H\times H$ action (defined throughout as a right action, i.e. $x\cdot (h_1,h_2):=h_1^{-1}xh_2$), the choice of a Cartan subtorus amounts to a choice of a pair of Borel subgroups B_1,B_2 of H, together with a point x_0 in the corresponding open Bruhat cell. The stabilizer of the point in $B_1\times B_2$ is then a torus, whose centralizer is a Cartan subgroup, whose orbit containing x_0 is the "Cartan subtorus".

We will denote by A the universal Cartan of G, A = B/N, so we have a quotient map of algebraic tori: $A woheadrightarrow A_X$, which may not be surjective on F-points. The group L(X) constructed above is a Levi subgroup of the parabolic stabilizing the open Borel orbit X° ,

$$P(X) := \{ g \in G | X^{\circ} g = X^{\circ} \} \supset B.$$

This parabolic plays an important role in harmonic analysis, since it gives rise to an "Arthur SL_2 "-parameter – a deviation from temperedness (when $P(X) \neq B$).

The vector space $\mathfrak{a}_X^* := \operatorname{Hom}(A_X, \mathbb{G}_m) \otimes \mathbb{Q}$ carries an almost canonical based root system. "Almost canonical" refers to the fact that there are different choices in the literature about the length of roots; for a specific choice, we get the root system of the dual group \check{G}_X of X defined in [SV]. We will return to this root system in section 6, but until then we only need its Weyl group W_X (the "little Weyl group of X") and its canonical antidominant Weyl chamber $\mathfrak{a}_X^+ \subset \mathfrak{a}_X$. We have maps:

$$\mathfrak{a}_X^+ \subset \mathfrak{a}_X := \operatorname{Hom}(\mathbb{G}_m, A_X) \otimes \mathbb{Q} \leftarrow \operatorname{Hom}(\mathbb{G}_m, B) \otimes \mathbb{Q} =: \mathfrak{a} \supset \mathfrak{a}^+,$$

where a^+ denotes the antidominant Weyl chamber corresponding to the universal Cartan of G.

The wavefront condition is the condition that the map: $\mathfrak{a} \to \mathfrak{a}_X$ sends \mathfrak{a}^+ onto \mathfrak{a}_X^+ . This technical condition is satisfied for all symmetric varieties and almost all spherical varieties; it is being imposed because this is the case in which the theory of asymptotics of [SV, §5] has been completed. (Though, conjecturally, the wavefront condition shouldn't be necessary.)

The other two conditions state that there is a Cartan subtorus $A_X \subset X$, such that the following hold:

(1) The natural map:

$$\tilde{\Lambda}_X^+ := A_X(F)^+ / A(\mathfrak{o}) \to X(F) / K \tag{2.1}$$

is surjective. Elements of $\tilde{\Lambda}_X^+$ which map to distinct elements of $\Lambda_X^+ = A_X(F)^+/A_X(\mathfrak{o})$ correspond to distinct K-orbits on X. We also assume that X has a model over \mathfrak{o} (compatible with that of G), and that $X(\mathfrak{o})$ consists of the orbits which map to $0 \in \Lambda_X^+$ under the map $\tilde{\Lambda}_X^+ \to \Lambda_X^+$.

Here we denote by $A_X(F)^+ = A_X^+$ the set of antidominant elements of the torus A_X with respect to the above based root system; that is, the elements of $A_X(F)^+$ are those elements a that satisfy:

 $|e^{\gamma}(a)| \ge 1$ for every positive root γ . (In order to use additive notation on \mathfrak{a}_X^* , we use exponentials to denote actual characters of A_X .)

(2) There is an Iwahori subgroup J relative to a Borel B used to define A_X such that for every $x \in A_X(F)^+$ we have:

$$xJ = x(J \cap B).$$

From now on we fix such a Cartan subtorus A_X .

This is the best decomposition that one can hope for in general, and it holds at almost all places if G and X are defined over a global field, as explained in [Sak12]. During the first reading it is advisable to restrict to the case that $A \to A_X$ is surjective on F-points, where A = B/N. In that case we have:

$$\tilde{\Lambda}_X = A_X(F)/A(\mathfrak{o}) = A_X(F)/A_X(\mathfrak{o}) =: \Lambda_X,$$

which can also be identified with the cocharacter group of A_X via the map:

$$\check{\lambda} \mapsto \check{\lambda}(\varpi) A_X(\mathfrak{o})$$

for any cocharacter λ into A_X . However, such a clean description is in general impossible, as demonstrated by the following example:

2.1. *Example.* Let $X = T \setminus \operatorname{SL}_2$. As a variety, it can also be identified with the quotient of PGL_2 by a torus. Thus, we have a surjection: $X(F)/\operatorname{SL}_2(\mathfrak{o}) \twoheadrightarrow X(F)/\operatorname{PGL}_2(\mathfrak{o})$. One can easily see that $A_X = A_{\operatorname{PGL}_2}$, hence:

$$X(F)/\operatorname{PGL}_2(\mathfrak{o}) \leftrightarrow A_{\operatorname{PGL}_2}(F)^+/A_{\operatorname{PGL}_2}(\mathfrak{o}) \leftrightarrow \mathbb{N}.$$

Choose a good, smooth model over $\mathfrak o$ (e.g.: T being the special orthogonal group of an integral, residually nondegenerate quadratic form). Then, under the above parametrization, "zero" corresponds to $X(\mathfrak o)$.

The fibers of the map:

$$\tilde{\Lambda}_X = A_X(F)^+ / A_{\mathrm{SL}_2}(\mathfrak{o}) \to A_{\mathrm{PGL}_2}(F) / A_{\mathrm{PGL}_2}(\mathfrak{o}) = \Lambda_X$$

correspond to square classes in \mathfrak{o}^{\times} . However, it is easy to see that $X(\mathfrak{o})$ forms a unique $\mathrm{SL}_2(\mathfrak{o})$ -orbit. Hence, the map (2.1) is not injective in that case.

On the other hand, we claim that for sufficiently large elements of $\tilde{\Lambda}_X$ the map is injective. Indeed, the theory of asymptotics that we will recall below states that on "very antidominant" elements of $A_X(F)$ the map (2.1) has to be injective; more precisely, there is a bijection of "very large" elements of X(F)/K and "very large" elements of Y(F)/K, where Y is the boundary degeneration that we will encounter. In this case, $Y(F) = N^- \backslash \operatorname{PGL}_2$, where N^- is unipotent, and clearly $Y(F)/\operatorname{SL}_2(\mathfrak{o}) \leftrightarrow A_X(F)/A_{\operatorname{SL}_2}(\mathfrak{o})$.

2.2. Example. In the group case, X = H, $G = H \times H$, we have $\Lambda_X =$ the quotient of $\Lambda_H \times \Lambda_H$ (where Λ_H is the coweight lattice of the universal Cartan of H) by coweights of the form: $(\check{\lambda}, -\check{\lambda}^{\vee})$, where for a given coweight $\check{\lambda}$ of H, $\check{\lambda}^{\vee}$ denotes the "dual" weight, $\check{\lambda}^{\vee} = -^{w_0}\check{\lambda}$, $w_0 =$ the longest Weyl

group element. Thus, Λ_X can be identified with Λ_H , but one needs to specify whether the identification is on the "left" or on the "right" copy – the two differ by the operation $\check{\lambda} \mapsto \check{\lambda}^\vee$. In either case, the set Λ_X^+ is the set of antidominant elements of Λ_H . We remark that the "left" choice gives the *opposite* of the "obvious" Cartan decomposition for H, i.e. an antidominant cocharacter $\check{\lambda}$ is associated to the coset $K_H\check{\lambda}(\varpi)^{-1}K_H$, which is the coset of $K_H\check{\lambda}^\vee(\varpi)K_H$, due to the way that multiplication on the left is defined as a right action.

In sections 3-5 we will present a general method for reducing the problem of inverse Satake transforms to a (much easier) problem on horospherical "boundary degenerations" of X. Then, from section 6 on, we will impose additional, strong assumptions on X that allow us to explicitly perform this calculation based on the results of [Sak13]. These additional assumptions *contain* the following:

- (1) $\Lambda_X = \Lambda_X$; in other words, the F-points of a Borel subgroup act with a unique open orbit on X(F).
- (2) X is affine homogeneous, or Whittaker-induced from an affine homogeneous spherical variety of a Levi subgroup in the sense of [SV, $\S 2.6$].

There are *more* assumptions needed, in order to ensure the validity of a theorem of *loc.cit*. which we recall as Theorem 6.1; these conditions are of combinatorial nature, can be checked relatively easily in each case, and they are *expected* to be no stronger than the above two; however, I do not know how to prove this. They hold in the group case, of course.

3. Definition: Satake transform

The canonical map of universal Cartans:

$$A \to A_X$$

gives rise to a map with finite kernel between their complex dual tori:

$$\check{A}_X \to \check{A}.$$
 (3.1)

Recall that (the complex points of) $A_X = \operatorname{Hom}(\Lambda_X, \mathbb{C}^\times)$ (and similarly for \check{A}); we identify its elements in the standard way with unramified characters of $A_X(F)$, via the identification $A_X(F)/A_X(\mathfrak{o}) \simeq \Lambda_X$ that we discussed previously.

The map (3.1) is an embedding if and only if $\Lambda_X = \Lambda_X$, i.e. if and only if the map $A \to A_X$ is surjective on F-points. In any case, the image of this map will be denoted by $\check{A}_{X,GN} \subset \check{A}$; it is the Cartan of the *Gaitsgory-Nadler dual group* of X.

The map (3.1) can be used to study the unramified spectrum of X, but it requires a correction which takes into account the deviation from temperedness. (For example, for X = a point we have $\check{A}_X = 1$, but the trivial representation does not have trivial Langlands parameter.) For a fixed

Borel B we consider $\delta_{(X)}^{\frac{1}{2}}:=$ the square root of the modular character (defined as the quotient of right by left Haar measure) of the group $B\cap L(X)$, considered as an unramified character of B and hence as an element of \check{A} . It is stable under the action of W_X , and we consider the W_X -equivariant morphism:

$$\check{A}_X \ni \tilde{\chi} \mapsto \chi \delta_{(X)}^{\frac{1}{2}} \in \check{A},\tag{3.2}$$

where χ is the image of $\tilde{\chi}$ under $\check{A}_X \to \check{A}_{X,GN}$.

3.1. Remark. One can replace every occurrence of $\delta_{(X)}$ in this paper by $\delta_{(X)}^{-1}$ without introducing any errors; indeed, the two elements are conjugate under the Weyl group of L(X), which acts trivially on \check{A}_X , and therefore whether one uses $\delta_{(X)}$ or its inverse plays no role in the restriction of W-invariant functions on \check{A} , which is the only setting where this character will appear.

In order to not get confused between the maps (3.1) and (3.2), we will be writing $\delta_{(X)}^{\frac{1}{2}}\check{A}_X$ to emphasize that certain restriction maps are taken with respect to (3.2). When (3.1) is injective, of course, $\delta_{(X)}^{\frac{1}{2}}\check{A}_X$ can be identified with the subvariety $\delta_{(X)}^{\frac{1}{2}}\check{A}_{X,GN}$ of \check{A} .

Now consider the unramified Hecke algebra $\mathcal{H}(G,K)$ of G, which via the Satake isomorphism is canonically isomorphic to $\mathbb{C}[\check{G}]^{\check{G}} \simeq \mathbb{C}[\check{A}]^W$. By restriction to the image of (3.2) we get a morphism of algebras:

$$\mathcal{H}(G,K) \simeq \mathbb{C}[\check{G}]^{\check{G}} = \mathbb{C}[\check{A}]^W \xrightarrow{(*)} \mathbb{C}[\delta_{(X)}^{\frac{1}{2}} \check{A}_{X,GN}]^{W_X} =: \mathcal{H}_X. \tag{3.3}$$

We set $\mathcal{S}(X)=C_c^\infty(X)$ and $\Phi^0=$ the characteristic function of $X(\mathfrak{o})$ (which, recall, we have assumed to consist of the K-orbits that map to $0\in\Lambda_X^+$ under $\tilde{\Lambda}_X^+\to\Lambda_X^+$). These definitions are the "correct" ones only when X is affine, which is the case we will eventually focus on. Then:

3.2. **Theorem** ([Sak08, Theorem 6.2.1]). The Hecke algebra $\mathcal{H}(G, K)$ acts on $\mathcal{S}(X)^K$ via its quotient (*), and $\mathcal{S}(X)^K$ is torsion-free as a module for the image of $\mathcal{H}(G, K)$ under (*).

In many cases it is known, and in general it is expected, that the action of $\mathcal{H}(G,K)$ extends "naturally" to an action of \mathcal{H}_X on $\mathcal{S}(X)^K$ (and this action is also to be denoted as a convolution: $(h,\Phi)\mapsto h\star\Phi$). When the image of (*) generates \mathcal{H}_X rationally (i.e. generates its field of fractions), such an extension is necessarily unique by the above torsion-freeness statement. Since this covers most of the interesting cases, I will remind of the property characterizing this "natural" extension in the general case in the proof of Proposition 7.4 (see also [Sak08, Conjecture 6.3].

There are several related questions that one might ask in order to enrich the upcoming notion of inverse Satake transforms, for example: whether the action of \mathcal{H}_X on $\mathcal{S}(X)^K$ extends further to an action of $\mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]^{W_X}$. In some cases, the answer is no, at least not in the same "natural" way that we alluded to before. The cases where I know that this fails are the cases where the dual group \check{G}_X cannot be defined (due to "reflections of type N", s. [SV]).

In any case, in the present paper I ignore such questions. I will restrict to the case when the extension of the action to \mathcal{H}_X is known, and when it is not known the reader should replace \mathcal{H}_X in everything that follows by $\mathcal{H}_X' =$ the image of (*). Until section 7, the exact nature of the extension to \mathcal{H}_X does not matter for the statements.

We will generally distinguish notationally between an element h of \mathcal{H}_X considered as an operator on $\mathcal{S}(X)^K$ (or on $C^\infty(X)^K$), and its "Satake transform" $\hat{h} \in \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_{X,GN}]^{W_X}$. For an element $h \in \mathcal{H}_X$, we will denote by h^\vee the "dual" element, characterized by:

$$\widehat{h^{\vee}}(\delta_{(X)}^{\frac{1}{2}}\chi) := \widehat{h}(\delta_{(X)}^{\frac{1}{2}}\chi^{-1})$$

(as polynomials on $\delta_{(X)}^{\frac{1}{2}}\check{A}_{X,GN}$). This is compatible under the above map with the involution on the Hecke algebra $\mathcal{H}(G,K)$ to be denoted by the same symbol:

$$h^{\vee}(g) := h(g^{-1}),$$

because the latter corresponds to inverting the Satake parameter and, as we noticed in a remark above, $\delta_{(X)}$ is W-conjugate to $\delta_{(X)}^{-1}$.

3.3. **Definition.** The *inverse Satake transform* is the map:

$$\operatorname{Sat}^{-1}: \mathbb{C}[\delta_{(X)}^{\frac{1}{2}} \check{A}_{X,GN}]^{W_X} \ni \hat{h} \mapsto h^{\vee} \star \Phi^0 \in \mathcal{S}(X)^K$$
(3.4)

The map is injective by the torsion-freeness statement of Theorem 3.2. The *Satake transform* Sat is the inverse of this map, defined, of course, only on its image.

3.4. *Remark.* When $\tilde{\Lambda}_X = \Lambda_X \Leftrightarrow \check{A}_X = \check{A}_{X,GN}$, the ring \mathcal{H}_X can also be identified with the ring of invariant polynomials on the *dual group* \check{G}_X of X described in [SV]. We will be writing again:

$$\mathcal{H}_X \simeq \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{G}_X]^{\check{G}_X},$$

to remind of the shift when we restrict invariant functions on \check{G} . Notice that by [SV, §3.3] the dual group of X and the element $\delta_{(X)}^{\frac{1}{2}}$ commute in \check{G} (where $\delta_{(X)}^{\frac{1}{2}} = e^{2\rho_{L(X)}}(q^{-\frac{1}{2}})$ in the notation of *loc. cit.*), therefore \check{G} -invariants will indeed restrict to \check{G}_X -invariants on $\delta_{(X)}^{\frac{1}{2}}\check{G}_X$.

3.5. Example. In the group case we have $\check{G}_X \simeq \check{H}$, but one must decide whether it is embedded as: $\check{H} \ni z \mapsto (z,z^c) \in \check{H} \times \check{H}$ or $\check{H} \ni z \mapsto (z^c,z) \in \check{H}$, where the exponent c denotes the Chevalley involution fixing the canonical pinning. This choice has to be done in accordance with the identification $\Lambda_X \simeq \Lambda_H$ as explained in Example 2.2. In the first case, our Satake transform is *dual* to the usual one (i.e. differs by the involution $h \mapsto h^\vee$ on the Hecke algebra), while in the second it is equal to the usual Satake transform.

4. BOUNDARY DEGENERATIONS AND ASYMPTOTICS

To each X we can associate a horospherical G-variety Y, denoted X_{\emptyset} in [SV], called its (most degenerate) "boundary degeneration". We will take it to be homogeneous, in which case it is characterized by the following properties:

- *Y* is homogeneous and horospherical (i.e. stabilizers contain maximal unipotent subgroups);
- P(X) = P(Y); notice that P(Y) is maximal such that the stabilizer of a point of Y contains the commutator [P, P], where P is a parabolic subgroup *opposite* to P(Y);
- $\Lambda_X = \Lambda_Y$ and $\tilde{\Lambda}_X = \tilde{\Lambda}_Y$.

The Cartan-Iwasawa decomposition for *Y* states:

$$Y/K \leftrightarrow \tilde{\Lambda}_Y = \tilde{\Lambda}_X. \tag{4.1}$$

Evidently, such a bijection can be shifted by any element of the G-automorphism group of Y (hence, by any element of $\tilde{\Lambda}_Y$), but we fix it once and for all in order to state the following theorems; there is a more "geometric" realization of Y as an open orbit in a normal bundle, which leads to a rigidification of this decomposition relative to the Cartan decomposition for X (s. the proof of Theorem 4.3).

4.1. *Example.* In the group case, X = H, $G = H \times H$, the boundary degeneration Y is isomorphic to:

$$A^{\operatorname{diag}}(N\backslash H\times N^-\backslash H),$$

where B = AN, $B^- = AN^-$ are two opposite Borel subgroups of H. (There is, of course, no obvious reason here to present it like that since B and B^- are conjugate; however, this is the presentation that generalizes to the intermediate boundary degenerations, which will not be used in this paper.)

4.2. Remark. The "universal Cartan" $A_Y = A_X$ of Y acts on Y "on the left". We clarify the conventions, which can be a source of confusion. The variety Y is isomorphic to $U^-S\backslash G$, where U^- is the unipotent radical of a parabolic in the class of parabolics *opposite* to P(X) and S is a subgroup of the corresponding Levi L(X) which contains the commutator of the Levi. In

this presentation, the universal Cartan of *X* is:

$$A_X = L(X)/S \leftarrow P(X)/U[L(X), L(X)] \leftarrow B/N = A,$$

where U is the unipotent radical of P(X) and A is the universal Cartan of G.

This shows what the natural definition for the action of A_X is, namely, lifting an element $a \in A_X$ to an element $\tilde{a} \in L(X)$ we have:

$$a \cdot U^- Sx := U^- S \tilde{a} x \in U^- S \backslash G = Y.$$

For example, if we have a presentation $Y \simeq N \backslash G$ for some maximal unipotent subgroup N, we should not identify $A_Y = A_X = A$ with the quotient B/N, where B is the normalizer of N, and let it act in the obvious way via this identification. Instead, if B is our fixed Borel then we should present Y as $N^- \backslash G$ for some unipotent radical N^- of a parabolic B^- opposite to B, identify A = B/N with the intersection of B and B^- , and let it act "on the left" as a subgroup of B^- . The two actions differ by the action of the longest Weyl element on A.

We will return to the A_X -action on Y in section 7.

The basic theorem of asymptotics, restricted to K-invariants, is:

4.3. **Theorem.** There is a unique $\mathcal{H}(G,K)$ -equivariant morphism:

$$Asymp: C^{\infty}(X)^K \to C^{\infty}(Y)^K$$

with the property that, for any $\check{\lambda}$ "deep enough" in $\tilde{\Lambda}_X^+$, we have:

$$\Phi(x_{\check{\lambda}}K) = \operatorname{Asymp}(y_{\check{\lambda}}K),$$

where we denote $\check{\lambda} \mapsto x_{\check{\lambda}} K$, resp. $y_{\check{\lambda}} K$, the Cartan decomposition for X (resp. Y).

"Deep enough" or "large" will be used invariantly to signify that the given elemens of a commutative monoid are sufficiently far from its "walls".

Proof. This is [SV, Theorem 5.1.2], where this map is denoted by e_{\emptyset}^* , up to showing that the isomorphism:

$$\tilde{\Lambda}_X \simeq \tilde{\Lambda}_Y$$

can be chosen so that the association induced by the Cartan decomposition:

$$x_{\check{\lambda}}K \mapsto y_{\check{\lambda}}K \ (\check{\lambda} \in \tilde{\Lambda}_X)$$

is compatible with the "exponential map" in the sense of loc.cit., §4.3.

Let X be a smooth toroidal embedding of X, and let Z be any G-orbit in \bar{X} whose normal bundle contains a subvariety isomorphic to Y (necessarily as its open G-orbit). By the local structure theorem of Brion-Luna-Vust (s. *loc.cit*. Theorem 2.3.4), there is a P(X)-stable open subset $S \subset \bar{X}$, meeting every G-orbit, which is P(X)-equivariantly isomorphic to $\overline{A_X} \times U_{P(X)}$,

where $\overline{A_X}$ denotes the closure of A_X in S. Thus, $\overline{A_X}$ is a smooth toric variety, from which it is easy to see that there is a P(X)-equivariant open embedding:

$$A_X \times U_{P(X)} \hookrightarrow N_{S \cap Z} S$$
 (4.2)

(normal bundle to $S \cap Z$ in S) and a p-adic analytic map:

$$\begin{array}{ccc}
N_{S \cap Z} S & \xrightarrow{\varphi} & S \\
\downarrow & & \downarrow \\
A_X \times U_{P(X)} & \longrightarrow & A_X \times U_{P(X)}
\end{array}$$

which is the identity on $S \cap Z$ and on its normal bundle, and the identity on the lower horizontal arrow of the above diagram.

We can now identify Y with the open G-orbit in $N_Z\bar{X}$ and the subvariety A_X of (4.2) with a Cartan subtorus of Y, and under this identification we have:

$$\varphi(y_{\check{\lambda}}A(\mathfrak{o})) = x_{\check{\lambda}}A(\mathfrak{o}),$$

in particular the association:

$$x_{\check{\Lambda}}K \mapsto y_{\check{\Lambda}}K$$

is compatible with the exponential map of *loc.cit*. §4.3.

It is easily seen from the defining property that Asymp is dual to a morphism:

$$Asymp^*: \mathcal{M}(Y) \to \mathcal{M}(X), \tag{4.3}$$

where $\mathcal{M}(\bullet)$ denotes spaces of *compactly supported smooth measures*, with the property that $1_{y_{\tilde{\lambda}}K} \mapsto 1_{x_{\tilde{\lambda}}K}$ for large $\check{\lambda} \in \tilde{\Lambda}_X^+$ (where 1_S denotes the characteristic measure of an open compact subset S).

4.4. Remark. In [SV] this map (denoted e_{\emptyset}) was defined between spaces of functions, but here it's more convenient to define it on spaces of measures, thus avoiding some factors in the formulas that follow as well as the need to fix a G-eigenmeasure. We point the reader's attention to the fact that $1_{x_{\bar{\lambda}}K}$ etc. denote *characteristic measures*, not functions.)

5. RANGE OF VALIDITY OF ASYMPTOTICS

We remain, for now, in the general setting where $\tilde{\Lambda}_X$ is not necessarily equal to Λ_X ; more precisely, Λ_X is the quotient of $\tilde{\Lambda}_X$ by its torsion subgroup. Hence, the complexified dual:

$$\tilde{A}_X := \operatorname{Hom}(\tilde{\Lambda}_X, \mathbb{C}^\times)$$

has the natural structure of a complex algebraic group, whose identity component is the torus $\check{A}_X = \operatorname{Hom}(\Lambda_X, \mathbb{C}^\times)$. We have natural morphisms:

$$\check{A}_X \hookrightarrow \tilde{A}_X \stackrel{(**)}{\twoheadrightarrow} \check{A}_{X,GN} \hookrightarrow \check{A},$$
 (5.1)

where the arrow in the middle is obtained by restricting a character to the image of A(F). We let $\tilde{\chi} \mapsto \chi$ denote the map (**), and we let f denote the kernel of (**); it is the finite group of characters of $A_X(F)$ trivial on the image of A(F).

The following result is proven in [Sak13] under the assumptions of §2.

- 5.1. **Theorem.** There is a rational family of $\mathcal{H}(G,K)$ -eigenfunctions $\tilde{A}_X \ni \tilde{\chi} \mapsto \Omega_{\tilde{\chi}}$ on X, with the following properties:
 - (1) In terms of the Cartan decomposition, $\Omega_{\tilde{\chi}}$ has the form:

$$\Omega_{\tilde{\chi}}(x_{\check{\lambda}}) = q^{\left\langle \rho_{P(X)}, \check{\lambda} \right\rangle} \sum_{w \in W_X} \sum_{\psi \in \mathfrak{f}} a_w^{\psi}(\tilde{\chi})(\psi \tilde{\chi})(e^{w\check{\lambda}}), \tag{5.2}$$

for certain rational coefficients a_w^{ψ} , where $\rho_{P(X)}$ is the half-sum of roots in the unipotent radical of P(X). (We use exponential notation when elements of $\tilde{\Lambda}_X$ are considered as homomorphisms: $\tilde{A}_X \to \mathbb{C}^{\times}$.)

- (2) $\mathcal{H}(G,K)$ acts on $\Omega_{\tilde{\chi}}$ via the character $\chi \delta_{(X)}^{\frac{1}{2}}$ (identified with its image in \check{A}/W).
- (3) The specializations of $\Omega_{\tilde{\chi}}$ at any Zariski dense subset of \tilde{A}_X where they are defined span a dense subspace of $(\mathcal{M}(X)^K)^*$; in other words, if $\langle \Omega_{\tilde{\chi}}, \mu \rangle = 0$ for $\tilde{\chi}$ in a Zariski dense subset, then $\mu \in \mathcal{M}(X)^K$ is zero.

Proof. This is [Sak13, Theorem 4.2.2] (notice that $\rho_{P(X)} = \rho$ on $\tilde{\Lambda}_X$), except for the density statement which is [Sak08, Theorem 6.1.1]

5.2. Remark. The notation here is slightly different from loc.cit., where $\tilde{\chi}$ is a character of a certain subgroup $R\subset A(\bar{F})$, namely the subgroup of elements which map to $A_X(F)$ under the quotient map: $A\to A_X$. The character $\tilde{\chi}$ in loc. cit. was varying over all characters of R which extend elements of $\delta^{\frac{1}{2}}_{(X)}\check{A}_{X,GN}$ on A(F). The above formula is derived from formula (4.2) of loc.cit. which involves the characters ${}^w\tilde{\chi}\delta^{-\frac{1}{2}}$ which do descend to characters of $A_X(F)$; more precisely, the character ${}^w\tilde{\chi}\delta^{-\frac{1}{2}}$ of loc. cit. is equal to what we presently denote by $\delta^{-\frac{1}{2}}_{P(X)}{}^w\tilde{\chi}$, which explains the passage from one formula to the other.

We are ready to draw our first conclusion:

5.3. **Proposition.** The morphism $\operatorname{Asymp}^*: \mathcal{M}(Y) \to \mathcal{M}(X)$, which a priori maps $1_{y_{\tilde{\lambda}}K}$ to $1_{x_{\tilde{\lambda}}K}$ only for "large" $\tilde{\lambda} \in \tilde{\Lambda}_X^+$, actually has this property for every $\tilde{\lambda} \in \tilde{\Lambda}_X^+$.

¹A "rational family" can be defined as an element of $\operatorname{Hom}(\mathcal{M}(X)^K, \mathbb{C}[\tilde{A}_X]) \otimes_{\mathbb{C}[\tilde{A}_X]} \mathbb{C}(\tilde{A}_X)$; equivalently, it is a $\mathbb{C}(\tilde{A}_X)$ -valued function on X/K, with only a *finite* number of poles.

5.4. *Remark.* Notice that different $\check{\lambda} \in \tilde{\Lambda}_X^+$ with the same image in Λ_X^+ can correspond to the same K-orbit on X, as we saw in example 2.1.

Proof. By the defining property of Asymp, Asymp($\Omega_{\tilde{\chi}}$) has to be an $\mathcal{H}(G,K)$ -eigenfunction on $C^{\infty}(Y)^{\tilde{K}}$ with the same eigencharacter, and given by the formula (5.2) for all large $\check{\lambda} \in \tilde{\Lambda}_Y = \tilde{\Lambda}_X$. The only such eigenfunction is given by the formula (5.2) for all $\check{\lambda} \in \tilde{\Lambda}_Y$.

By the density property, $\operatorname{Asymp}^*(1_{y_{\tilde{\lambda}}K})$ is characterized by the property that for (almost) all $\tilde{\chi} \in \tilde{A}_X$:

$$\langle \operatorname{Asymp}^*(1_{y_{\tilde{\lambda}}K}), \Omega_{\tilde{\chi}} \rangle = \langle 1_{y_{\tilde{\lambda}}K}, \operatorname{Asymp} \Omega_{\tilde{\chi}} \rangle.$$

But this formula holds for $1_{x_{\check{\lambda}}K}$ in place of $\operatorname{Asymp}^*(1_{y_{\check{\lambda}}K})$, for $\check{\lambda} \in \tilde{\Lambda}_X^+$, by (5.2).

5.5. **Corollary.** For any $\Phi \in C^{\infty}(X)^K$, we have $\Phi = \operatorname{Asymp}(\Phi)|_{\tilde{\Lambda}_X^+}$ as functions on $\tilde{\Lambda}_X^+$.

This is the key to computing explicitly the inverse Satake transforms of various functions, since it is much easier to compute the Hecke action on $C^{\infty}(Y)^K$, than on $C^{\infty}(X)^K$.

6. ASYMPTOTICS OF THE BASIC FUNCTION

From now on we assume that X is affine homogeneous or Whittaker-induced from an affine homogeneous spherical variety of a Levi subgroup in the sense of [SV, $\S 2.6$]. We also require that $\tilde{\Lambda}_X = \Lambda_X$ (equivalently, $\check{A}_X = \check{A}_{X,GN}$, and we will write χ instead of $\tilde{\chi}$ for a character of Λ_X). The formulas that follow will involve the coroot system of X (i.e. the root system of its dual group \check{G}_X), as normalized in [SV, $\S 3.1$]. The set of positive roots of \check{G}_X will be denoted by $\check{\Phi}_X^+$.

In the affine case, the characteristic function of $X(\mathfrak{o})$ (which under our present assumptions forms a single K-orbit, parametrized by $0 \in \Lambda_X^+$) will be denoted by Φ^0 .

I point the reader to [SV, §2.6] for the general formalism of Whittaker-induction, but the basic idea is very familiar; in our case, we start with an affine homogeneous variety $H \setminus L$ of a Levi subgroup L, and a *generic* character $\Psi: U_P(F) \to \mathbb{C}^\times$ of the unipotent radical of a parabolic with Levi L, such that Ψ is fixed by H (and hence extends to a character of HU_P). Then, instead of smooth functions on $X:=HU_P \setminus G$ one considers smooth sections of the induced character (which can be thought of as a complex line bundle \mathcal{L}_Ψ over the F-points of X). Everything that we have established so far extends to the Whittaker-induced case, with the dual group (and hence the set Λ_X^+ of anti-dominant weights) being *different* from that of X considered as a variety without that line bundle. In this case, the Cartan decomposition does not parametrize all K-orbits on X but only the "relevant" ones (i.e. those which can support K-invariant sections of the line bundle).

Of course, as in the usual case, we need the analogous assumptions of §2 to hold for the Cartan decomposition, and they do at almost every place if X is defined over a global field. If, in the presentation above, $HU_P \cdot 1 \in X$ is on the orbit represented by $0 \in \Lambda_X^+$, the role of the "basic function" here will be played by the section Φ^0 defined by:

$$\Phi^0(g) = \begin{cases} \Psi(h), & \text{if } g = hk, h \in HU_P, k \in K; \\ 0, & \text{otherwise.} \end{cases}$$

In either case, from now on we will require that the assumptions of [Sak13, Theorem 7.2.1] hold; as remarked in §2, this includes, and is expected to be equivalent to, the requirement that X is affine homogeneous or Whittaker-induced from such; however, one must for now check additional combinatorial conditions in each case. The case of X =a reductive group satisfies these conditions.

I will not repeat the conditions here (as they involve a long list of definitions), but they have to do with a set Θ^+ of triples $(\check{\theta}, \sigma_{\check{\theta}}, r_{\check{\theta}})$, where $\theta^+ \in \Lambda_X$, $\sigma_{\check{\theta}}$ is + or -1, and $r_{\check{\theta}}$ is a half-integer. This set is obtained from the combinatorial invariants of the spherical variety, and in particular the valuations induced by its *colors* (B-stable divisors). I refer the reader to [Sak13, §7.1] for the definitions. Roughly speaking, the conditions state that Θ^+ behaves like the set of positive roots of a root system with Weyl group W_X : it can be completed to a W_X -stable set (where W_X acts on such triples by acting just on $\check{\theta}$) by inverting the $\check{\theta}$'s, "loses" a prescribed subset of elements when acted upon by a simple reflection etc. We will see some examples below. By abuse of notation, we will sometimes write $\check{\theta} \in \Theta$, instead of the corresponding triples.

Notice that the condition $\tilde{\Lambda}_X = \Lambda_X$ implies that $X(\mathfrak{o})$ is a single K-orbit, $\check{A}_X = \check{A}_{X,GN}$, and that the Hecke eigenfunctions of Theorem 5.1 are now parametrized by $\chi \in \check{A}_X$ (with no finite group \mathfrak{f} entering in their formula).

We recall and reformulate the statement of [Sak13, Theorem 7.2.1] under our present assumptions (more precisely, its restriction to affine or Whittaker-induced from affine cases where, in the notation of the theorem, $\omega =$ a constant):

6.1. **Theorem.** There is a positive constant c such that the Hecke eigenfunctions Ω_{χ} of Theorem 5.1, normalized so that their value at $X(\mathfrak{o})$ is 1, are equal to:

$$\frac{\Omega_{\chi}(x_{\check{\lambda}})}{\Omega_{\chi}(x_0)} = c^{-1} q^{\langle \rho_{P(X)}, \check{\lambda} \rangle} \cdot P_{\check{\lambda}}(\chi),$$

where $P_{\check{\lambda}}$ is the W_X -invariant polynomial on \check{A}_X given by:

$$P_{\check{\lambda}} = \sum_{w \in W_{X}} \left(\frac{\prod_{\check{\theta} \in \Theta^{+}} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})}{\prod_{\check{\gamma} \in \check{\Phi}_{X}^{+}} (1 - e^{\check{\gamma}})} e^{\check{\lambda}} \right)^{w}$$
(6.1)

Proof. This is a restatement of *loc.cit*. (7.4); see the second formula of the proof for the reformulation that we have presented here. The fact that the $P_{\check{\lambda}}$ s are polynomials can easily be seen from (7.4), where they are expressed in terms of Schur polynomials.

Moreover:

6.2. **Proposition.** The polynomials $P_{\check{\lambda}}$, for $\check{\lambda}$ varying over the antidominant elements of Λ_X , form a basis for the W_X -symmetric polynomials on \check{A}_X .

Proof. This is included in the proof of [Sak13, Theorem 8.0.2]. \Box

6.3. *Example*. In the group case, X=H, we have $\Theta^+=\check{\Phi}_H^+$ (positive coroots of H), $\sigma_{\check{\theta}}=+1$ and $r_{\check{\theta}}=1$ for all $\check{\theta}$, so we get Macdonald's formula according to which:

$$P_{\check{\lambda}} = \sum_{w \in W_H} \left(\prod_{\check{\gamma} \in \check{\Phi}_H^+} \frac{1 - q^{-1} e^{\check{\gamma}}}{1 - e^{\check{\gamma}}} e^{\check{\lambda}} \right)^w.$$

6.4. *Example*. In the Whittaker case, $X = N \setminus G$, where N is a maximal uniponent subgroup endowed with a nondegenerate character Ψ , we have: $\check{G}_X = \check{G}, \, \Theta^+ = \emptyset$, and:

$$P_{\check{\lambda}} = \sum_{w \in W} \left(\prod_{\check{\gamma} \in \check{\Phi}_G^+} \frac{1}{1 - e^{\check{\gamma}}} e^{\check{\lambda}} \right)^w = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{\check{\rho}_B - w\check{\rho}_B + w\check{\lambda}}}{\prod_{\check{\gamma} \in \check{\Phi}_G^+} (1 - e^{\check{\gamma}})},$$

where $\ell(w)$ is the length of w and $\check{\rho}_B = \frac{1}{2} \sum_{\check{\gamma} \in \check{\Phi}_G^+} \check{\gamma}$. The right hand side is, of course, the character (Schur polynomial) of the irreducible representation of \check{G} with *lowest* weight $\check{\lambda}$.

6.5. Example. When $X = \operatorname{Sp}_{2n} \setminus \operatorname{GL}_{2n}$ we have P(X) = the standard parabolic with Levi of type $\operatorname{GL}_2 \times \operatorname{GL}_2 \times \cdots \times \operatorname{GL}_2$, and the dual group \check{G}_X is isomorphic to GL_n (embedded in $\check{G} = \operatorname{GL}_{2n}$ via the spherical roots $\alpha_1 + 2\alpha_2 + \alpha_3$, $\alpha_3 + 2\alpha_4 + \alpha_5$,...). We have:

$$P_{\check{\lambda}} = \sum_{w \in W_X} \left(\prod_{\check{\gamma} \in \check{\Phi}_X^+} \frac{1 - q^{-2} e^{\check{\gamma}}}{1 - e^{\check{\gamma}}} \right)^w.$$

It requires a long introduction to the structure of spherical varieties (and the definition of the set Θ^+) in order to explain how these are computed, but I will give a few hints: The calculation of P(X) is easy, and the spherical roots can be read off from the diagrams in Luna's paper [Lun01]. One can then compute the PGL_2 -spherical varieties corresponding to each simple root α of G: these are the varieties $X^{\circ}P_{\alpha}/\mathcal{R}(P_{\alpha})$, where P_{α} is the parabolic whose Levi has a single positive root α , $\mathcal{R}(P_{\alpha})$ is its radical, and X° is the open Borel orbit. One sees that for $\alpha_1, \alpha_3, \alpha_5$ etc. this PGL_2 -variety is a point

(which is why P(X) is the standard parabolic containing the negatives of those roots), while for $\alpha_2, \alpha_4, \ldots$ they are of the form $N \setminus \mathrm{PGL}_2$, where N is unipotent. This implies that $X^\circ P_{\alpha_{2i}}$ contains a *color*, a B-stable divisor, whose valuation gives rise to the element $\check{\theta} = \check{\alpha}_{2i}$ of Θ^+ . Notice that $\check{\alpha}_{2i}$ is here equal to the coroot corresponding to the root $\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}$. The rest of the triple $(\check{\theta}, \sigma_{\check{\theta}}, r_{\check{\theta}}) = (\check{\alpha}_{2i}, +1, 2)$, and the other elements of Θ^+ , can be computed from the definitions of [Sak13, §7.1].

Before we continue, we need to discuss how we will denote certain functions on the horospherical boundary degeneration Y (and on X) as rational functions on \check{A}_X/W_X .

We introduce a basis of $S(Y)^K$ indexed by Λ_X , where $\check{\lambda} \in \Lambda_X$ is associated to the function:

$$e^{\check{\lambda}} := q^{\langle \rho_{P(X)}, \check{\lambda} \rangle}$$
 times the characteristic function of $y_{\check{\lambda}}K$. (6.2)

We will be writing $\hat{\Phi}$ for the expression of an arbitrary element of $C^{\infty}(Y)^K$ as a series in the elements $e^{\check{\lambda}}$, and we will also use rational functions to denote the corresponding power series. Notice that a rational function does not correspond to a unique power series, unless extra conditions are given on the support of the power series, e.g.:

$$\frac{1}{1 - e^{\check{\alpha}}}$$

could correspond to both $\sum_{i\geq 0}e^{i\check{\alpha}}$ and $-\sum_{i\geq 1}e^{-i\check{\alpha}}$.

In what follows, we will fix a strictly convex cone \mathcal{C}_X in Λ_X (i.e. the intersection of Λ_X with a strictly convex, finitely generated cone in the \mathbb{Q} -vector space it spans) and will require throughout that all our power series have support in a translate of this cone, without the need to repeat this condition every time. (Later, we will also introduce a larger strictly convex cone \mathcal{C}_X' , depending on the function that we want to represent; notice that as long as the latter contains the former and is strictly convex, any rational function that can be expanded as a series in a translate of \mathcal{C}_X , also corresponds unambiguously to the same series if we want to expand it in a translate of \mathcal{C}_X' .) The cone \mathcal{C}_X is defined as follows: Recall that we assume that X is affine, and we have a decomposition of the coordinate ring:

$$F[X] = \bigoplus_{\chi} V_{\chi} \tag{6.3}$$

into a multiplicity-free direct sum of highest weight modules. The set of B-weights appearing in this decomposition is actually a saturated monoid of the weights of the quotient torus A_X , and we let \mathcal{C}_X denote the dual cone:

$$\mathcal{C}_X = \{\check{\lambda} \in \Lambda_X | \langle \chi, \check{\lambda} \rangle \geq 0 \text{ for all } \chi \text{ appearing in (6.3)} \}.$$

Since the χ 's appearing in (6.3) are all dominant, this cone contains the images of all positive coroots of G in Λ_X .

6.6. Example. When $G = \operatorname{SL}_2$ and $Y = N \setminus \operatorname{SL}_2$, where N is a maximal unipotent subgroup, and $\check{\alpha}$ is the positive coroot of its universal Cartan, we have that \mathcal{C}_X is spanned by the positive coroots (there is no other possibility in this one-dimensional case, no matter what X was) and Y and Y are the Borel subgroup. The expression:

$$\frac{1}{1 - q^{-1}e^{\check{\alpha}}}$$

stands for the characteristic function of $\mathfrak{o}^2 \setminus \{0\}$, under the identification of Y(F) with $F^2 \setminus \{0\}$.

Indeed, first of all we expand in a power series in positive powers of $e^{\check{\alpha}}$, since \mathcal{C}_X must contain positive multiples of $\check{\alpha}$. Secondly, we interpret $q^{-i}e^{i\check{\alpha}}=q^{-i\langle\rho,\check{\alpha}\rangle}e^{i\check{\alpha}}$ as the characteristic function of the coset $y_{i\check{\alpha}}K$. Finally, for the Iwasawa decomposition of Y we should fix a Borel AN^- opposite from the "standard" one, and use an isomorphism $Y\simeq N^-\backslash G$ to represent $y_{\check{\lambda}}$ by $\check{\lambda}(\varpi)\in A(F)$. Then we immediately see that under a suitable embedding of Y(F) in F^2 we have: $y_{i\check{\alpha}}K=$ the subset $(\mathfrak{p}^i)^2\smallsetminus(\mathfrak{p}^{i+1})^2$ of F^2 .

Now we are ready to describe the image of the basic function under the asymptotics map. Recall that $\Phi^0 \in \mathcal{S}(X)^K$ denotes the characteristic function of $X(\mathfrak{o})$. We have the following:

6.7. **Proposition.** The support of Asymp (Φ^0) , as a function on $Y/K = \Lambda_X$, belongs to a translate of the cone \mathcal{C}_X .

Proof. This is [SV, Proposition 5.4.5]; s. also its proof, where the affine embedding containing its support is specified as the horospherical "affine degeneration" of X, i.e. the affine embedding of Y whose coordinate ring, as a G-module, contains the same highest weight representations as F[X]. \square

This shows that for the calculations that follow we can unambiguously represent functions on Y/K as rational functions, as long as they have a power series expansion supported in a translate of \mathcal{C}_X . In the next section we will do the same with functions on X/K, by restricting those power series (functions on $\Lambda_X = \Lambda_Y$) to Λ_X^+ .

Our basic result, now, is the following:

6.8. **Theorem.** *In the notation above, we have:*

$$\operatorname{Asymp}(\Phi^{0}) = \frac{\prod_{\check{\gamma} \in \check{\Phi}_{X}^{+}} (1 - e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta^{+}} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})}.$$
(6.4)

6.9. *Remark*. It follows from the definition of the set Θ^+ in [Sak13, §7.1] that it belongs to the cone C_X .

Proof. We begin with an extension of Proposition 5.3:

6.10. **Proposition.** *For any* $\lambda \in \Lambda_X$, *let:*

$$q^{\left<\rho_{P(X)},\check{\lambda}\right>}P_{\check{\lambda}} = \sum_{\check{\mu}\in\Lambda_X^+} c_{\check{\lambda}}^{\check{\mu}} q^{\left<\rho_{P(X)},\check{\mu}\right>} P_{\check{\mu}}$$

be the decomposition into the basis of Proposition 6.2. Then:

$$\operatorname{Asymp}^*(1_{y_{\check{\lambda}}K}) = \sum_{\check{\mu} \in \Lambda_X^+} c_{\check{\lambda}}^{\check{\mu}} 1_{x_{\check{\mu}}K}.$$

The argument of the proof is an obvious extension of that of Proposition 5.3 and will be omitted. Thus, the polynomials $P_{\tilde{\lambda}}$, even when $\tilde{\lambda}$ is not antidominant, have a meaning of their own! They represent the "exponential" map Asymp*.

Going back to the proof of the theorem, it is now enough to show that the inner product of Φ^0 with $\operatorname{Asymp}^*(1_{y_{\tilde{\lambda}}K})$, that is: the coefficient $c_{\tilde{\lambda}}^0$, is equal to $q^{\left\langle \rho_{P(X)},\check{\lambda}\right\rangle}$ times the coefficient of $e^{\check{\lambda}}$ in the power series expansion of the right hand side of (6.4). That is, we need to show that the coefficient of $e^{\check{\lambda}}$ is equal to the constant $c_{\tilde{\lambda}}^0$ in the notation of the last proposition.

It is shown in [Sak13, §9] that the polynomials $P_{\check{\lambda}}$, with $\check{\lambda}$ antidominant, are orthogonal with respect to the inner product:

$$[P,Q] = \int_{\check{A}_X^1/W_X} P(\chi) \cdot \overline{Q(\chi)} \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}_X} (1 - e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})} (\chi) d\chi,$$

where \check{A}_X^1 denotes the maximal compact subgroup of \check{A}_X .

In particular, since P_0 is equal to the (positive) constant c of Theorem 6.1, for arbitrary $\check{\lambda} \in \Lambda_H$ we have:

$$\begin{split} [P_{\check{\lambda}}, P_0] &= c \cdot \int_{\check{A}_X^1/W_X} \sum_{w \in W_X} \left(\frac{\prod_{\check{\theta} \in \Theta^+} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})}{\prod_{\check{\gamma} \in \check{\Phi}_X^+} (1 - e^{\check{\gamma}})} e^{\check{\lambda}} \right)^w \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}_X} (1 - e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})} (\chi) d\chi \\ &= c \cdot \int_{\check{A}_X^1} \frac{\prod_{\check{\gamma} \in \check{\Phi}_X^+} (1 - e^{-\check{\gamma}})}{\prod_{\check{\theta} \in \Theta^+} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{-\check{\theta}})} e^{\check{\lambda}} (\chi) d\chi. \end{split}$$

By a complex analysis/contour shift argument one can see that for the probability measure on \check{A}_X^1 this integral is equal to the constant term of the power series that one gets by expanding the inverse of the denominator in the obvious way: $(1-\sigma_{\check{\theta}}q^{-r_{\check{\theta}}}e^{-\check{\theta}})^{-1}=\sum_{i\geq 0}(\sigma_{\check{\theta}}q^{-r_{\check{\theta}}}e^{-\check{\theta}})^i$.

Thus, the result of the calculation is c times the coefficient of $e^{-\check{\lambda}}$ in the power series expansion of $\frac{\prod_{\check{\gamma}\in\check{\Phi}_X^+}(1-e^{-\check{\gamma}})}{\prod_{\check{\theta}\in\Theta^+}(1-\sigma_{\check{\theta}}q^{-r_{\check{\theta}}}e^{-\check{\theta}})}$, or equivalently c times the coefficient of $e^{\check{\lambda}}$ on the right hand side of (6.4).

On the other hand, this argument shows that $[P_0, P_0] = c$, and hence:

$$c_{\check{\lambda}}^0 = \frac{[P_{\check{\lambda}}, P_0]}{[P_0, P_0]} =$$
the coefficient of $e^{\check{\lambda}}$ on the right hand side of (6.4).

6.11. *Example.* In the group case X = H we have:

$$\operatorname{Asymp}(\Phi^0) = \prod_{\check{\gamma} \in \check{\Phi}_H^+} \frac{1 - e^{\check{\gamma}}}{1 - q^{-1}e^{\check{\gamma}}}$$

(cf. Example 6.3).

7. INVERSE SATAKE TRANSFORMS

Now recall that "Hecke" ring \mathcal{H}_X acting on $\mathcal{S}(X)^K$. Recall that we distinguish notationally between an element h of this ring considered as an operator on $\mathcal{S}(X)^K$ (or on $C^\infty(X)^K$), and its "Satake transform" $\hat{h} \in \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]^{W_X}$. (Under our present assumptions: $\check{A}_X = \check{A}_{X,GN}$.)

We also have the torus A_X acting "on the left" on Y. The reader should necessarily read Remark 4.2, to avoid potential confusion about the A_X -action on Y as we discuss the Satake isomorphism.

Accordingly, the torus A_X acts on $C^{\infty}(Y)^K$; we normalize this action as:

$$a \cdot f(y) := \delta_{P(X)}^{\frac{1}{2}}(a)f(ay),$$

so that it is unitary on the subspace of L^2 -functions, and define the action of its Hecke algebra $\mathcal{H}(A_X,A_X(\mathfrak{o}))\simeq \mathbb{C}[\Lambda_X]\simeq \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]$ on $A_X(\mathfrak{o})$ -invariant functions accordingly:

$$h\star f(y):=\int_{A_X/A_X(\mathfrak{o})\simeq \Lambda_X} a\cdot f(y)h(y).$$

Notice the isomorphism $\mathbb{C}[\Lambda_X]\simeq \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]$, which is simply coming from the canonical isomorphism $\mathbb{C}[\Lambda_X]=\mathbb{C}[\check{A}_X]$ composed with the obvious identification (translation by $\delta_{(X)}^{\frac{1}{2}}$) between \check{A}_X and $\delta_{(X)}^{\frac{1}{2}}\check{A}_X$. We will insist on introducing this shift, as we did in (3.3), for compatibility reasons with the Satake isomorphism that we are about to discuss. Despite the fact that these isomorphisms are completely canonical, for an element $h\in\mathcal{H}(A_X,A_X(\mathfrak{o}))\simeq\mathbb{C}[\Lambda_X]$ we will write \hat{h} for its image in $\mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]$.

With this definition of the action, the characteristic measure of $\check{\lambda} \in \Lambda_X$ takes the function that we denoted above by $e^{\check{\mu}}$ to $e^{\check{\mu}-\check{\lambda}}$; this explains our choice of basis. We can formulate this in terms of h and \hat{h} :

7.1. **Lemma.** For $\Phi \in \mathcal{S}(Y)^K$, let $\hat{\Phi}$ be its expression in the basis $(e^{\check{\lambda}})_{\check{\lambda} \in \Lambda_X}$, thought of as an element of $\mathbb{C}[\check{A}_X]$.

Let $h \in \mathcal{H}(A_X, A_X(\mathfrak{o}))$, and denote as before by h^{\vee} the dual element: $h^{\vee}(a) = h(a^{-1})$. Then:

$$\widehat{h^{\vee} \star \Phi} = \hat{h} \cdot \hat{\Phi}.$$

The definition of this action is compatible with the action of $\mathcal{H}(G,K)$ under the usual Satake isomorphism:

$$S: \mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[\check{A}]^W$$

in the following sense:

7.2. **Lemma.** Let $h \in \mathcal{H}(G, K)$, $h' \in \mathbb{C}[\Lambda_X] = \mathbb{C}[\delta_{(X)}^{\frac{1}{2}} \check{A}_X]$ such that the Satake transform \hat{h} of h, when composed with the restriction map:

$$\mathcal{H}(G,K) \simeq \mathbb{C}[\check{G}]^{\check{G}} \xrightarrow{(*)} \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X],$$

(where (*) is as in (3.3)) is equal to $\widehat{h'} \in \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]$.

Then, for all $\Phi \in \mathcal{S}(Y)^K$, we have:

$$h \star \Phi = h' \star \Phi$$
,

where, obviously, the convolution on the left is with respect to the G-action and the convolution on the right is with respect to the A_X -action.

Proof. It is enough to show this for the special case $Y = N^- \backslash G$. For, any other Y' is a quotient of this of the form $S \backslash G$, where S lives between a parabolic $P(Y')^-$ containing N^- and its derived group, and the action of $\mathbb{C}[\check{A}]$ on $S(N^- \backslash G)$ descends to $S(S \backslash G)$ via the corresponding restriction map:

$$\mathbb{C}[\check{A}] \to \mathbb{C}[\delta^{\frac{1}{2}}_{(Y')}\check{A}_{Y'}],$$

where $\delta_{(Y')}$ is defined in complete analogy with $\delta_{(X)}$ earlier.

When $Y = N \setminus G = N \setminus G$, this is the setting of the original Satake transform. Following [Gro98, (3.4)], the Satake transform of an element $h \in \mathcal{H}(G,K)$ considered as a function (fixing the Haar measure dg on G which gives mass 1 to K), is defined as the following function on the universal Cartan A:

$$Sh(a) := \delta_B(a)^{\frac{1}{2}} \int_N h(an) dn, \ (a \in A)$$

where B is any Borel subgroup, A is identified with its reductive quotient, and the measure on N is such that $dg = \delta_B(t)^{-1} dn dt dk$ according to the Iwasawa decomposition G = NTK, where T is a Cartan subgroup of B and $dt(T(\mathfrak{o})) = 1$.

Let us say that B is chosen opposite to the subgroup N^- above, and let $T=B\cap B^-$, where B^- is the normalizer of N^- . The embedding $T\hookrightarrow B\twoheadrightarrow A$ identifies T with A. Let $w\in K$ be an element which belongs to the normalizer of T and corresponds to the longest element of the Weyl group, then by the invariance of Sh(t) under W we have, for $t\in T\simeq A$:

$$Sh(t) = Sh(wtw^{-1}) = \delta_B(t)^{-\frac{1}{2}} \int_N h(wtw^{-1}n) dn = \delta_B(t)^{-\frac{1}{2}} \int_{N^-} h(tn) dn = \delta_B(t)^{-\frac{1}{2}} \int_$$

$$= \delta_B(t)^{-\frac{1}{2}} h \star \Phi^0(N^- t^{-1}).$$

Hence, for $\check{\mu} \in \Lambda_Y$, the evaluation of Sh at the associated representative $y_{\check{\mu}} \in A$ is:

 $Sh(y_{\check{\mu}}) = \text{ the coefficient of } e^{-\check{\mu}} \text{ when we write } h \star \Phi^0 \text{ in the basis consisting of the elements } e^{\check{\lambda}} \text{ that we introduced above.}$

It follows from Lemma 7.1 that if $h' \in \mathcal{H}(A,A(\mathfrak{o}))$ is the element $\mathcal{S}h(t)dt$ (so that $\hat{h}=\hat{h'}\in\mathbb{C}[\check{A}]$ as required by the present lemma), then the coefficient of $e^{-\check{\mu}}$ in $\widehat{h'}\star\widehat{\Phi^0}$ is equal to its coefficient in $\widehat{h'^{\vee}}$, i.e. equal to the coefficient of $e^{\check{\mu}}$ in $\widehat{h'}$, i.e. equal to $\mathcal{S}h(y_{\check{\mu}})$. Hence:

$$\widehat{h' \star \Phi^0} = \widehat{h \star \Phi^0}.$$

The same has to hold if we replace Φ^0 by any element of $\mathcal{S}(Y)^K$, since it generates all of them under the action of $\mathcal{H}(A,A(\mathfrak{o}))$, and the actions of A and G commute.

Now we come to combining the theory of asymptotics with the explicit formulas of the previous section. Notice that under the assumptions of the present section we have:

7.3. **Proposition.** The whole ring $\mathcal{H}_X = \mathbb{C}[\delta_{(X)}\check{A}_X]^{W_X}$ acts on $\mathcal{S}(X)^K$, and the Satake transform of Definition 3.2 is an isomorphism:

$$\mathrm{Sat}: \mathcal{S}(X)^K \xrightarrow{\sim} \mathcal{H}_X.$$

Proof. This is [Sak13, Theorem 8.0.2].

On the other hand, we may identify \mathcal{H}_X as the subring of W_X -invariants in the ring $\mathbb{C}[\delta_{(X)}\check{A}_X]$ acting on $\mathcal{S}(Y)^K$. Then:

7.4. **Proposition.** The asymptotics map Asymp : $C^{\infty}(X)^K \to C^{\infty}(Y)^K$ is \mathcal{H}_X -equivariant.

Proof. This is easy to see if the image of the restriction map:

$$\mathbb{C}[\check{A}]^W \to \mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]^{W_X} \simeq \mathcal{H}_X$$

generates \mathcal{H}_X rationally (i.e. generates its field of fractions), or equivalently: when the map $\check{A}_X/W_X \to \check{A}/W$ is generically injective.

This is the case in most examples of wavefront spherical varieties, and certainly in the case of symmetric varieties (see the – stronger – notion of "generic injectivity" in [SV, §14.2], and Lemma 15 in Delorme [Del] which proves it in the symmetric case).

Indeed, since Asymp is equivariant under the action of $\mathcal{H}(G,K)\simeq\mathbb{C}[\check{A}]^W$, and by Lemma 7.2 the action of the latter is compatible with the action of $\mathbb{C}[\delta_{(X)}^{\frac{1}{2}}\check{A}_X]$ "on the left", it follows that the asymptotics map is equivariant with respect to the image of the restriction map, considered as a subring of

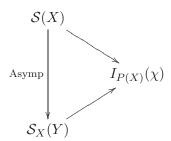
 \mathcal{H}_X . Since the modules are torsion-free, if the image generates \mathcal{H}_X rationally then it has to be equivariant with respect to the whole ring \mathcal{H}_X .

I sketch the proof in the general case: The action of \mathcal{H}_X on $\mathcal{S}(X)^K$ was characterized in [Sak13] by the requirement of being equivariant with respect to certain operators which in the literature (although not in *loc.cit*.) are sometimes called "Eisenstein integrals". These are a certain rational family of operators:

$$S_{\chi}: \mathcal{S}(X) \to I_{P(X)}(\chi),$$

where $I_{P(X)}(\chi)$ denotes the normalized principal series induced from the character χ of P(X), as χ varies in \check{A}_X . The operators (or rather, the functional obtained by composing with "evaluation at 1") are defined in some convergent region by an integral on the open P(X)-orbit, and extended rationally to all \check{A}_X . Again, our parametrization of characters is shifted by $\delta_{(X)}^{\frac{1}{2}}$ compared to that of [Sak13], cf. the remark following Theorem 5.1. The action of the ring \mathcal{H}_X on $I_{P(X)}(\chi)$ is defined to be by the scalar obtained by evaluation at $\delta_{(X)}^{\frac{1}{2}}\chi$.

The same operators can be defined on S(Y), and [SV, Proposition 5.4.6] states that the following diagram commutes:



where by $S_X(Y)$ we denote the image of S(X) in $C^{\infty}(Y)$.

This suffices to show equivariance of the asymptotics map under \mathcal{H}_X .

In particular:

7.5. **Proposition.** Let $h \in \mathcal{H}_X$ and $\Phi \in C^{\infty}(X)^K$. Then:

$$\widehat{\mathrm{Asymp}(h^{\vee}} \star \Phi) = \hat{h} \cdot \mathrm{Asymp}(\Phi),$$

both sides thought of as formal series in Λ_X (supported on a translate of the cone C_X).

In particular,

$$\operatorname{Asymp}(\operatorname{Sat}^{-1}(\hat{h})) = \hat{h} \cdot \operatorname{Asymp}(\Phi^0).$$

Moreover, considered as functions on Λ_X^+ , $\mathrm{Sat}^{-1}(\hat{h})$ and $\hat{h} \cdot \mathrm{Asymp}(\Phi^0)$ coincide.

Proof. This is a combination of what has been proven thus far, namely: Proposition 7.4, Lemma 7.1 (which is easily extended to functions represented by power series) and Corollary 5.5.

We would like to extend the above to suitable series of elements in \mathcal{H}_X , when $\Phi \in \mathcal{S}(X)^K$ (so that $\operatorname{Asymp}(\Phi)$ is supported on a translate of the cone \mathcal{C}_X). More precisely, let \mathcal{K}_X denote all series in the elements $e^{\check{\lambda}}$, $\check{\lambda} \in \Lambda_X$ which are W_X -invariant, and such that their support on every translate of the sublattice spanned by \mathcal{C}_X is compact. Via the identification $\mathcal{H}_X = \mathbb{C}[\Lambda_X]^{W_X}$, the ring \mathcal{H}_X is the subring of compactly supported elements of \mathcal{K}_X .

7.6. Remark. The assumption on support places a strong restriction on X, if \mathcal{K}_X is to contain elements of non-compact support. Namely, \mathcal{C}_X should not (rationally) span the whole lattice Λ_X , which implies that there is a non-trivial eigenfunction of G on the coordinate ring F[X]. This in essence leaves out varieties which, in the language of [SV], are not "factorizable", which should eventually be included in the theory.

Let Λ_X' =the quotient of Λ_X by the sublattice rationally spanned by \mathcal{C}_X , and let det denote the quotient map. We will use the same symbol to denote the map $X \to \Lambda_X'$ induced by the Cartan decomposition $X/K = \Lambda_X^+$. For $\hat{h} \in \mathcal{K}_X$ we have a decomposition:

$$\hat{h} = \sum_{\check{\delta} \in \Lambda'_Y} \hat{h}_{\delta},$$

where $\hat{h}_{\delta} \in \mathbb{C}[\Lambda_X]^{W_X} \simeq \mathcal{H}_X$ is supported in $\det^{-1}(\delta)$, and if h_{δ} is the operator on $\mathcal{S}(X)^K$ corresponding to \hat{h}_{δ} then, for any $\Phi \in \mathcal{S}(X \cap \det^{-1}(\delta_1))^K$ we have $h_{\delta} \star \Phi \in \mathcal{S}(X \cap \det^{-1}(\delta_1 + \delta))^K$. Hence, for every $\Phi \in \mathcal{S}(X)$ we have a well-defined element:

$$h \star \Phi \in C^{\infty}(X)^K$$

whose support on every subset of the form $\det^{-1}(\delta_1)$ is compact.

In particular, the inverse Satake transform extends to K_X :

$$\operatorname{Sat}^{-1}(\hat{h}) := h^{\vee} \star \Phi^0.$$

Now let $\check{\rho} \in \Lambda_X^+$ be outside of the rational span of \mathcal{C}_X ; in particular, the cone \mathcal{C}_X' spanned by $\check{\rho}$ and \mathcal{C}_X is strictly convex. We write $\hat{h} = L(\check{\rho}) \in \mathbb{C}(\delta_{(X)}^{\frac{1}{2}}\check{A}_X)^{W_X}$ for the rational function:

$$\delta_{(X)}^{\frac{1}{2}}\chi \mapsto \det\left(I - \chi|_{V_{\check{\rho}}}\right)^{-1},$$

where $V_{\check{\rho}}$ is the irreducible module of \check{G}_X of *lowest weight* $\check{\rho}$. We consider it as a power series with support in the cone \mathcal{C}_X' ; t satisfies the assumptions of the preceding discussion, i.e. it belongs to \mathcal{K}_X .

A combination of Proposition 7.5 and Theorem 6.8 now gives:

7.7. **Theorem.** Let $\hat{h} = L(\check{\rho}) \in \mathbb{C}(\delta_{(X)}^{\frac{1}{2}}\check{A}_X)^{W_X}$, then, as functions on $X/K = \Lambda_X^+$:

$$\operatorname{Sat}^{-1}(L(\check{\rho})) = h^{\vee} \star \Phi^{0} = L(\check{\rho}) \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}_{X}^{+}} (1 - e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta^{+}} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})} \bigg|_{\Lambda_{Y}^{+}}. \tag{7.1}$$

We remind that the term $e^{\check{\lambda}}$ on the right hand side should be interpreted as $q^{\langle \rho_{P(X)}, \check{\lambda} \rangle}$ times the characteristic function of $x_{\check{\lambda}}K$.

8. The group case; relation to the formula of Wen-Wei Li

We now restrict to the case X = H, $G = H \times H$, where H is a (split) reductive group over F. Here we have $\check{G}_X = \check{H}$ but, according to Example 3.5, there is a choice to be made in the identification. We make the choice to embed \check{H} in \check{G} as:

$$z \mapsto (z^c, z)$$

in order for our Satake transform to be compatible with the usual one. Recall from Example 2.2 that this choice is compatible with the "obvious" Cartan decomposition in terms of antidominant coweights, i.e. $\check{\lambda} \in \Lambda_H^+$ corresponds to the double K_H -coset of $\check{\lambda}(\varpi)$, where ϖ is a uniformizer of our field.

In the group case we have C_X = the cone spanned by the positive coroots. Assume that $\check{\rho} \in \Lambda_H^+$ is outside of the rational span of positive coroots, and let $L(\check{\rho})$, as above, be the rational function:

$$\det(I - \bullet|_{V_{\check{o}}})^{-1},$$

where $V_{\check{\rho}}$ is the irreducible representation of \check{H} of lowest weight $\check{\rho}$. Then we have:

8.1. **Corollary.** Let h be the series of elements in the Hecke algebra whose (usual) Satake transform is $\hat{h} = L(\check{\rho})$. Identify it with a function on $K_H \backslash H/K_H$ by fixing the Haar measure on H which is 1 on K_H .

Then the value of h on $\check{\lambda}(\varpi)$, $\check{\lambda} \in \Lambda_H^+$, is equal to $q^{\langle \rho_{B_H}, \check{\lambda} \rangle}$ (where ρ_{B_H} is the half-sum of positive roots on H) times the coefficient of $e^{\check{\lambda}}$ in the power series of:

$$L(\check{\rho}) \cdot \prod_{\check{\gamma} \in \check{\Phi}_H^+} \frac{1 - e^{\check{\gamma}}}{1 - q^{-1}e^{\check{\gamma}}}.$$
 (8.1)

(expanded in terms of the cone spanned by $\check{\rho}$ and the positive coroots).

Proof. Immediate combination of Theorem 7.7 and Example 6.3, using the fact that in this case, for $\check{\lambda} \in \Lambda_X = \Lambda_H$, $\langle \rho_B, \check{\lambda} \rangle = \langle \rho_{B_H}, \check{\lambda} \rangle$.

8.2. *Example* (Godement-Jacquet for GL_2). We choose a basis ϵ_1, ϵ_2 for the cocharacter lattice of $H = GL_2$ such that the positive coroot is $\check{\alpha} = \epsilon_1 - \epsilon_2$.

We set $z_i = e^{\epsilon_i}$. The lowest weight of the standard representation of \check{H} is ϵ_2 , and hence:

$$\hat{h} := L(\text{std}) = \frac{1}{(1 - z_1)(1 - z_2)}.$$

Thus:

$$h^{\vee} \star \Phi^{0} = \frac{1 - \frac{z_{1}}{z_{2}}}{1 - q^{-1} \frac{z_{1}}{z_{2}}} \cdot \frac{1}{(1 - z_{1})(1 - z_{2})} \bigg|_{\Lambda_{H}^{+}}.$$
 (8.2)

It is not immediately evident that this is related to the characteristic function of $\mathrm{Mat}_2(\mathfrak{o})$. However, notice that the above expression is equal to:

$$\frac{1}{(1-q^{-1}\frac{z_1}{z_2})(1-z_2)} - \frac{\frac{z_1}{z_2}}{(1-q^{-1}\frac{z_1}{z_2})(1-z_1)}.$$

The second summand can be discarded, as its support does not meet the set of antidominant coweights. (The set of antidominant coweights corresponds to monomials of the form $z_1^i z_2^j$ with $j \ge i$.)

sponds to monomials of the form $z_1^i z_2^j$ with $j \geq i$.) The support of the first summand intersects the antidominant coweights on the set of monomials $z_1^i z_2^j$ with $j \geq i \geq 0$, and the coefficient of such a monomial is: q^{-i} .

On the other hand, $\langle \rho, i\epsilon_1 + j\epsilon_2 \rangle = \frac{i-j}{2}$. Therefore, the characteristic function of the coset of the element $\begin{pmatrix} \varpi^i \\ \varpi^j \end{pmatrix}$ appears with coefficient:

$$q^{-i+\frac{i-j}{2}} = q^{-\frac{i+j}{2}} = |\det|^{\frac{1}{2}},$$

and the function of (8.2) is equal to $|\det|^{\frac{1}{2}}$ times the characteristic function of $\operatorname{Mat}_2(\mathfrak{o})$.

(We remind that for GL_n and $\hat{h} = L(std)$ one has: $h^{\vee} \star \Phi^0 = |\det|^{\frac{n-1}{2}}$ times the characteristic function of $Mat_n(\mathfrak{o})$.)

Finally, let us discuss the relationship of Corollary 8.1 with the result [Li, Theorem 5.3.1] of Wen-Wei Li. I am very grateful to Wen-Wei Li for explaining this relationship to me and allowing me to reproduce the arguments here.

As in the setting of Corollary 8.1, we fix an anti-dominant coweight $\check{\rho}$ which is not contained in the linear span of coroots. We assume that the character group of H is generated by a character "det" with the property that $\langle \det, \check{\rho} \rangle = 1$. (This is not a serious restriction, and one can recover Corollary 8.1 directly from the formulation of Wen-Wei Li, without this assumption.)

To introduce the result of [Li] we let Ψ denote the multiset arising as the *multiset union* of the set $\check{\Phi}_H^+$ of positive corrots of H and the multiset V of weights of the representation $\check{\rho}$ of the dual group. Consider the product:

$$\prod_{\check{\nu}\in\Psi}\frac{1}{1-qe^{-\check{\nu}}},$$

thought of as a series in $\operatorname{span}_{\mathbb{Z}_{\geq 0}}(\Psi)$ with coefficients in $\mathbb{Z}[q]$. For $\check{\mu} \in \Lambda_H$, let $\mathscr{P}_{\Psi}(\check{\mu},q) \in \mathbb{Z}[q]$ denote the coefficient of $e^{\check{\mu}}$ in the above power series, i.e. formally:

$$\sum_{\check{\mu}\in\Lambda_H}\mathscr{P}_{\Psi}(\check{\mu},q)e^{\check{\mu}}=\prod_{\check{\nu}\in\Psi}\frac{1}{1-qe^{-\check{\nu}}}.$$

The formula of [Li, Theorem 5.3.1] asserts that $\mathrm{Sat}^{-1}(L(\check{\rho}))$ equals the restriction to Λ_H^+ of the following series in our basis elements $e^{\check{\mu}}$:

$$\sum_{\check{\mu}\in\Lambda_{H},\langle\det,\check{\mu}\rangle\geq 0} c_{\check{\mu}}(q)e^{\check{\mu}},\tag{8.3}$$

with:

$$c_{\check{\mu}}(q) = \sum_{w \in W_H} (-1)^{\ell(w)} \mathscr{P}_{\Psi}(\check{\rho}_B - w\check{\rho}_B, -\check{\mu}; q^{-1}) q^{\langle \det, \check{\mu} \rangle},$$

where $\check{\rho}_B$ denotes half the sum of positive coroots of \check{H} and $\ell(w)$ is the length of w.

The coefficient $c_{\check{\mu}}(q)$ was defined only for $\check{\mu} \in \Lambda_H^+$ in [Li], but the same formula works in general.

To establish the equivalence between the two formulas, we first observe that the coefficients $\mathscr{P}_{\Psi}(\check{\mu},q)$ admit an alternative presentation where the roles of $\check{\Phi}_H^+$ and V in Ψ are distinguished, namely:

$$\sum_{\check{\mu}\in\Lambda_H}\mathscr{P}_{\Psi}(-\check{\mu},q^{-1})q^{\langle\det,\check{\mu}\rangle}e^{\check{\mu}}=\prod_{\check{\gamma}\in\check{\Phi}_H^+}\frac{1}{1-q^{-1}e^{\check{\gamma}}}\prod_{\check{\nu}\in V}\frac{1}{1-e^{\check{\nu}}}.$$

Indeed, for a certain $\mathbb{Z}_{\geq 0}$ -combination of elements of Ψ to be equal to $\check{\mu}$, we must have $k:=\langle \det,\check{\mu}\rangle \geq 0$, and the elements from V must be used exactly k times.

Combining this with the definition of $c_{\tilde{\mu}}$, changing $\check{\mu}$ to $\check{\mu} + \check{\rho}_B - w\check{\rho}_B$ and taking into account that $\langle \det, \check{\mu} + \check{\rho}_B - w\check{\rho}_B \rangle = \langle \det, \check{\mu} \rangle$, the series (8.3) can be written as:

$$\sum_{\check{\mu}\in\Lambda_{H},\langle\det,\check{\mu}\rangle\geq 0} \mathscr{P}_{\Psi}(-\check{\mu};q^{-1})q^{\langle\det,\check{\mu}\rangle} \sum_{w\in W_{H}} (-1)^{\ell(w)} e^{\check{\mu}+\check{\rho}_{B}-w\check{\rho}_{B}} =$$

$$= \prod_{\check{\gamma}\in\check{\Phi}_{H}^{+}} \frac{1}{1-q^{-1}e^{\check{\gamma}}} \prod_{\check{\nu}\in V} \frac{1}{1-e^{\check{\nu}}} \sum_{w\in W_{H}} (-1)^{\ell(w)} e^{\check{\rho}_{B}-w\check{\rho}_{B}}.$$

Finally, invoking the Weyl denominator formula:

$$\prod_{\check{\gamma} \in \check{\Phi}_H^+} (1 - e^{\check{\gamma}}) = \sum_{w \in W_H} (-1)^{\ell(w)} e^{\check{\rho}_B - w\check{\rho}_B}$$

we get:

$$\prod_{\check{\gamma}\in\check{\Phi}_H^+}\frac{1-e^{\check{\gamma}}}{1-q^{-1}e^{\check{\gamma}}}\prod_{\check{\nu}\in V}\frac{1}{1-e^{\check{\nu}}}=L(\check{\rho})\cdot\prod_{\check{\gamma}\in\check{\Phi}_H^+}\frac{1-e^{\check{\gamma}}}{1-q^{-1}e^{\check{\gamma}}},$$

which is equal to (8.1).

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