Spherical varieties and L-functions

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Outline

1. Introduction
   - Branching laws
   - Motivation: images of Langlands transfer
   - Motivation: period integrals and L-functions
2. Spherical varieties
   - Dual group
   - Classification
   - Degeneration
3. More Ichino-Ikeda
   - Generalized global Ichino-Ikeda conjectures

Part 1: Orgins

Notions

Local story (main):

1. $F = \mathbb{Q}_p$ or $F = \mathbb{F}_q((t))$ is a local field.
2. G is a (split) reductive group over F.
3. π is an irreducible (unitary) smooth $\mathbb{C}$-coefficient representation of $G(F)$.
4. W_F is the Weil-Deligne group of F.

Global story :

1. $k = \mathbb{Q}$ or $k = \mathbb{F}_q(C)$ is a global field.
2. Still denote by G a (split) reductive group over k.
3. Still denote by π an irreducible (cuspidal) automorphic representation of $[G] = G(\mathbb{A})/G(k)$.

Branching laws

π an irr rep of $G(F)$, $H \subseteq G$ a nice ("spherical") subgroup.
A central problem in representation theory: when $\mathrm{Hom}_H(\pi, 1) \neq 0$?
How to produce elements in $\mathrm{Hom}_H(\pi, 1)$? $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, 1) =?$
Global analog for automorphic representations?

Frobenius reciprocity

$\mathrm{Hom}_H(\pi, 1) \neq 0$ iff $\pi \hookrightarrow C^\infty(G(F)/H(F))$.
This motivates the spectral study of function spaces on F-points of the spherical variety $X = G/H$.
A subtly is that $G(F)/H(F) \neq (G/H)(F)$, we will ignore the nature of inner forms and L-packets in this talk.

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Today:

- Why spherical X? Langlands transfer, period integrals and L-functions...
- How to think about spherical varieties? Examples, classifications...
- What is Ichino-Ikeda type conjecture? Relations between period integral and central L-values, relation between local and global program...

We also ignore convergence issues or derived branching laws, so things will be supercuspidal (local) and cuspidal (global).

Motivation: images of Langlands transfer

Local Langlands correspondence and functoriality (informal)

$$\{\text{irr smooth } \mathbb{C}\text{-reps of } G(F)\} \overset{\text{finite-to-one}}{\to} \{\text{Galois reps } \phi : W_F \to G^\vee(\mathbb{C})\}$$

Given good $f : G_1^\vee \to G_2^\vee$, composition by f on RHS gives a transfer map from irr reps of G_1 to irr reps of G_2 on LHS.

Example: quadratic base change, parabolic induction.

Question (local/global)

Choose a map $G_X^\vee \to G^\vee$. For a rep π of $G(F)$, when does $\phi_\pi : W_F \to G^\vee(\mathbb{C})$ factor through $G_X^\vee$? For example, when is π a transfer from G_1?

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Periods and L-values

Lots of examples $\rightsquigarrow$ we can detect this by

- (global) poles or nonvanishing of certain L-function at $s = s_0$ (center or nearly center points);
- (global) nonvanishing of certain automorphic period integrals;
- (local) certain branching laws $\mathrm{Hom}_H(\pi, 1) \neq 0$.

So another question: relation between period integrals and L-functions in general?

Reminder on L-functions

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_n \frac{1}{n^s}.$$

Given a rep π of $G(F)$, and $\rho : G^\vee \to \mathrm{GL}(V)$, the associated local L-function ($s \in \mathbb{C}$) is

$$L(\pi, \rho, s) = \det(1 - q_F^{-s} \rho \circ \phi_\pi(\mathrm{Frob}_F)|_{V^I})^{-1} \in \mathbb{C}[q^s, q^{-s}].$$

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This is defined unconditionally for unramified π by Satake isomorphism. Given a rep π of $[G]$, and $\rho : G^\vee \to \mathrm{GL}(V)$, the associated (incomplete) global L-function is

$$L(\pi, \rho, s) = \prod_v L(\pi_v, \rho_v, s)$$

where the product is over finite places v of k where π_v is unramified.

Reminder on global period integrals

Let $H \subseteq G$ be a "nice" subgroup. The "niceness" is encoded in $X = G/H$, e.g X is affine iff H is reductive. Below we ignore important convergence issues.

Automorphic period integrals

(global) For $\phi \in \pi$ on $[G]$, $P_X(\phi) := \int_{[H]} \phi(h)dh$.

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- L-functions: $\int_{[H]} \phi(h)dh = (*)L(?, ?, s_0)$.
- Branching laws: note $P_X \in \mathrm{Hom}_H(\pi, 1)$, so $P_X \neq 0$ implies $\mathrm{Hom}_H(\pi, 1) \neq 0$ i.e π is H-distinguished.
- Langlands transfer: functoriality shall also be realized by integration along certain kernel functions (geometrization).

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Local period integrals?

In practice, people study or construct L-functions by relating it to some (period) integrals e.g to show analytic continuation. See Tate thesis.
$\rightsquigarrow$ how about local decomposition of global period integrals?
$[G]$ **is not a direct product,** $\phi_v \in \pi_v$ **is not naturally a function on some spaces**.

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$[G]$ **is not a direct product,** $\phi_v \in \pi_v$ **is not naturally a function on some spaces**.
Ichino-Ikeda type conjecture: global $|P_H|^2$ can be decomposed into local pairings $(v_1, v_2) \mapsto \int_H \langle h.v_1, v_2\rangle\, dh$, $v_1 \in \pi, v_2 \in \pi^\vee$, after some important normalizations. This will relate central L-values to period integrals, in a precise way.

Examples

We give examples for previous two questions.

- Dirichlet L-function $L(\chi, s)$ has a pole at $s_0 = 1$ iff $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is trivial i.e its Langlands parameter factors through the trivial subgroup.
- (Hecke period) For any normalized cusp form f $(a_1 = 1)$, $L(f, s) = \int_0^\infty f(it)t^s \frac{dt}{t}$. $H = \mathbb{G}_m = \mathrm{diag}\{*, 1\} \hookrightarrow G = \mathrm{PGL}_2$. In automorphic language, $L(\pi, s) = \int_{[H]} f(h)|h|^{s-1/2}dh$ (center $s = 1/2$).
- (Waldspurger period) $G = \mathrm{PGL}_2$, $H' = (\mathrm{Res}_{k'/k}\mathbb{G}_m)/\mathbb{G}_m$ a non-split torus. Then $|\int_{[H']} \phi|^2 = \frac{L(\pi,1/2)L(\pi\otimes\eta_{E/F},1/2)}{L(\pi,\mathrm{Ad},1)}$.
- (Whittaker period) Fourier coefficients are also integrals. $X = (G/N, \psi)$.

Examples

- (Rankin-Selberg) $L(\pi_1 \times \pi_2, s) = \int_{[GL_2]} f_1(g) f_2(g) E(g,s) dg$.
  $G = GL_2 \times GL_2$, $X = \mathbb{A}^2 \times GL_2$.
- (Tate thesis) $L(\chi_p, s) = \int_{GL_1(F)} 1_{\mathrm{Mat}_{1\times 1}(O)}(x) \chi_p(x) |\det(x)|^s d^\times x$ for unramified χ_p. $G = \mathrm{GL}_1$, $X = \mathbb{A}^1$.
- (Godement-Jacquet) $G = \mathrm{GL}_n \times \mathrm{GL}_n$, $X = \mathrm{Mat}_{n\times n}$:
  $L(\pi_p, \mathrm{Std}, s) = \int_{GL_{n\times n}(F)} 1_{\mathrm{Mat}_{n\times n}(O)}(x) \langle \phi_1(x), \phi_2 \rangle |\det(x)|^s d^\times x$ for unramified π_p.

You see more examples beyond homogeneous $X = G/H$. **The main player is the G-variety X** (H will be the stabilizer of the open G-orbit).

Program of Sakellaridis–Venkatesh (local)

Slogan: For any "nice" (quasi-affine spherical) G-variety X, one can construct a local X-period integral, and a Langlands dual group $G_X^\vee$ over $\mathbb{C}$ with a distinguished map $\iota : G_X^\vee \to G^\vee$ (see Part 2).

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Conjecture

- There is a local period integral $|P_X|_\pi^2$, such that $|P_X|_\pi^2 \neq 0$ iff there is a functorial lifting ϕ_σ of π to $G_X^\vee$.
- There exists an (graded) algebraic rep $\rho_X : G_X^\vee \to GL(V_X)$ such that $|P_X|_\pi^2 = (*)L(\sigma, \rho_X, s_0) = (*)L_X(\pi_v)$.

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$\rightsquigarrow$ images of local Langlands transfer is related to local branching laws i.e what G-reps will occur in $L^2(X)$.
$\rightsquigarrow$ local period integrals "=" local L-values.
Also, there shall exist precise relative character identities relating relative characters π and σ as distributions.

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If $X = G/H$, $|P_X|^2$ is the natural pairing $(v_1, v_2) \mapsto \int_H \langle h.v_1, v_2 \rangle \, dh$, $v_1 \in \pi, v_2 \in \pi^\vee$. But ρ_X is mysterious.
To get $|P_X|^2$ and $L(-, \rho_X, s)$ in general, the idea is to study Plancherel decomposition of $L^2(X(F))$ (or $C_c^\infty(X(F))$) under the action of $G(F)$.

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Conjecture

$$L^2(X) \cong \int_{\widehat{G}_X} \iota_*(\sigma)^{\oplus m(\sigma)} d\mu_{G_X}(\sigma),$$

where $\mu_{G_X}(\sigma)$ denotes the Plancherel measure of G_X and $m(\sigma)$ is a multiplicity space.

The unramified spectrum $C_c^\infty(X(F))^{G(O)}$ is already interesting (related to relative Satake).
The IC function "$1_{X(O)}$" $\rightsquigarrow$ local unramified L-function, hence global (incomplete) L-function by products.

Part 2: spherical varieties

Spherical varieties

Now G is a reductive group with a Borel B over a field k_0, X is a normal G-variety over k_0. In practice, F is a local field with residue field $k_0 = \mathbb{F}_q$. For simplicity, now $k_0 = \mathbb{C}$, $F = k_0((t))$.

X is spherical if X has an open dense B-orbit X°.

We denote by $X^\bullet$ the G-orbit containing X°. $X^\bullet \cong H\backslash G$ for some H.

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- Toric varieties for $G = T$ a torus. $G = \mathbb{G}_m, X = \mathbb{A}^1$.
- Flag variety G/B
- (Whittaker) $X = G/U$.
- **Fundamental example**: $X = H$, $G = H \times H$ (group case).
- More generally, symmetric spaces $X = G/K$, $K = G^\theta$.
- $G = \mathrm{SL}_2$ on $X = \mathbb{A}^2$, $X^\bullet = \mathbb{A}^2 \backslash \{0\} = G/U$.
- (GGP) $G = SO_n \times SO_{n+1}$, $H = SO_n$.

Why?

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Geometry: many things on G can be generalized to spherical X: root lattice Δ_X, weight lattice Λ_X, Weyl group W_X, dual group $G_X^\vee$, Chevalley isomorphism...

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Rep theory: you can do geometric haramonic analysis on $X(F)$: study of $L^2(X(F))$, Fourier transform, asymptotics (or nearby cycles), Satake isomorphism...

- If X is affine, $k_0[X]$ is a multiplicity-free G-module.
- X has only finitely many B-orbits.
- In practice (wavefront condition), $\dim \mathrm{Hom}_{G(F)}(\pi, C^\infty(X(F))) < +\infty$: uniqueness of Whittaker model for GL_n, $\dim_{\mathbb{C}} \mathrm{Hom}_{SO_{n-1}}(\pi_{SO_n}, 1) \leq 1$..

Root datum: Borel action is the key

Classically, the action of $H \times H$ on $k[H]$ encodes $\mathrm{Rep}(H)$ hence everything. The $B_H \times B_H$-action will encode root datum. As we don't assume X is affine, it's better to work with fraction field $k_0(X) = k_0(X^\circ)$.

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- Weight lattice: Let $k_0(X)^{(B)}$ be the B-eigenfunctions in $k_0(X)$. The weight lattice $\Lambda_X \subseteq \Lambda_G$ consists of B-eigencharacters in $k_0(X)^{(B)}$.
- Cartan torus of X: $T_X = \operatorname{Spec} \Lambda_X$. $T_X = T_G/\mathrm{Im}(B \cap H)$ acts freely on X°.
- The cone $\mathcal{V}$ generated by anti-dominant weights in $\Lambda^\vee_{X,\mathbb{Q}}$: let $\mathcal{V}$ denote the set of G-invariant valuations $k_0(X)^\times \to \mathbb{Q}$. Evaluating a valuation on B-eigenfunctions gives an injective map $\mathcal{V} \to \Lambda^\vee_{X,\mathbb{Q}}$.

The advantage of considering valuations is to generalize the notion of prime divisors birationally.

Root datum: Borel action is the key

- **Weight lattice** Λ_X: $k_0(X)^{(B)}$ = the B-eigenfunctions in $k_0(X)$. $\Lambda_X \subseteq \Lambda_G$ consists of B-eigencharacters in $k_0(X)^{(B)}$. Similarly, define Λ_X^{++} for $k[X]^{(B)}$ if X is quasi-affine, then $\Lambda_X = \mathbb{Z}[\Lambda_X^{++}]$. $\Lambda_X = k_0(X)^{(B),\times}/k_0^\times$ (mult one).
- Cartan torus: $T_X = \operatorname{Spec} \Lambda_X$.
- The polyhedral cone $\mathcal{V} \subseteq \Lambda_{X,\mathbb{Q}}^\vee$: the set of G-invariant valuations $k_0(X)^\times \to \mathbb{Q}$.
- Spherical roots $\Sigma_X \subseteq \Lambda_X$: generators of (extremal rays of $-\mathcal{V}^\vee) \cap \Lambda_X$.
- Normalized spherical root Δ_X: integral issues, $\Sigma_X = \Delta_X$ in many cases e.g Hecke, GGP..
- Dual group $G_X^\vee$ over $\mathbb{C}$: given by the root datum $(\Lambda_X, \Lambda_X^\vee, \Delta_X, \Delta_X^\vee)$.

Fact: up to $\{1, 2, 1/2\}$, any spherical root of X is sum of two roots of G.

Some generalizations

Classically, $W_G = N(T)/T$. W_X is defined as the group generated by the reflections about the codimension-1-faces of the valuation cone $\mathcal{V}(X)$. It's the Weyl group of $G_X^\vee$.
Chevalley restriction theorem:

$$\mathfrak{g}^*//G \cong \mathfrak{a}_G^*//W_G.$$

Example: $\mathrm{Mat}_{n\times n}//\mathrm{GL}_n = \mathbb{A}^n//S_n$.
Spherical variety version:

$$\mathfrak{g}_X^*//G \cong \mathfrak{a}_X^*//W_X.$$

Cartan decomposition

$G(O)$-orbits in $X^\bullet(F)$ is in bijection to $\Lambda_X^\vee/W_X$.

Colors

The dual group does not determine X, because it only depends on $X^\bullet$.
Fact: in many interesting cases (e.g X is strongly tempered),
$G_X^\vee = G^\vee$.
To classify X, we need colors of X. In a dual way, you can think a B-eigenfunction f via the B-stable divisor $\mathrm{div}(f)$.

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To classify X, we need colors of X. In a dual way, you can think a B-eigenfunction f via the B-stable divisor $\mathrm{div}(f)$.

- $\mathcal{D}(X)$ is the finite set of all B-stable prime divisors of X.
- **A color of** X is a B-stable but not G-stable prime divisor of X, and $\mathcal{D} = \mathcal{D}(H\backslash G)$ is the set of colors.

Colors as coweights

- $\rho_X : \mathcal{D}(X) \to \Lambda_X^\vee : D \mapsto v_D$: any $D \in \mathcal{D}(X)$ gives a valuation on $k(X)^\times$ hence on Λ_X. ρ_X is similar to $\mathcal{V} \to \Lambda_{X,\mathbb{Q}}^\vee$, but may not be injective.
- The rational cone $\mathcal{C}_0 = \mathcal{C}_0(X) \subseteq \Lambda_{X,\mathbb{Q}}^\vee$ generated by $\rho_X(\mathcal{D}(X))$. $\mathrm{Hom}(\Lambda_X^{++}, \mathbb{Z}_{\geq 0}) = \mathcal{C}_0 \cap \Lambda_X^\vee$.

Examples

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In Hecke case $G = \mathrm{PGL}_2, H = \mathbb{G}_m = \mathrm{diag}\{*, 1\}, X = H\backslash G$, we have $G_X^\vee = G^\vee = SL_2$.
To see this, B-orbits on X is the same as $H = \mathbb{G}_m$-orbits on $G/B = \mathbb{P}^1$, so three orbits $\mathbb{G}_m, 0, \infty$, and two colors $D^+(0)$, $D^-(\infty)$.

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To see this, B-orbits on X is the same as $H = \mathbb{G}_m$-orbits on $G/B = \mathbb{P}^1$, so three orbits $\mathbb{G}_m, 0, \infty$, and two colors $D^+(0)$, $D^-(\infty)$.
Note $k_0[\mathbb{G}_m\backslash SL_2]^{(B)} = k_0[\mathbb{G}_m\backslash(\mathbb{A}^2 - 0)]^{(\mathbb{G}_m)}$.
$k_0[\mathbb{G}_m\backslash(\mathbb{A}^2 - 0)] = k_0[xy]$, you see the T-eigenvalues for $\mathbb{G}_m\backslash SL_2$ are generated by $t \mapsto t^2$, the simple root of SL_2.
For $X = \mathbb{G}_m\backslash\mathrm{PGL}_2$, things are dual, so $\Lambda_X^\vee = \Lambda_G^\vee = \mathbb{Z}\frac{1}{2}\alpha^\vee$, where $\alpha : t \mapsto t$ is the simple root of PGL_2. And $v_{D^+} = v_{D^-} = \frac{1}{2}\alpha^\vee$.

Rank 1 cases

The rank of spherical roots is easy to compute, and is called the rank of X.
One can classify all rank 1 cases. Beyond the group case, there are more examples.
For example, $X = SO_{2n-1} \backslash SO_{2n}$ has $G_X^\vee = \mathrm{PGL}_2$. $SO_2 \backslash SO_3$ is the Hecke case as before.
$X \cong \{x \in V_{2n} | (x, x) = 1\}$ is a "sphere" (maybe a motivation for the name "spherical varieties").
$C_c^\infty(V_{2n}) \to C_c^\infty(X(F))$ is surjective, so one can use Weil representation and theta correspondence tools.

Classification of homogeneous spherical $H\backslash G$

Homogeneous spherical varieties $H\backslash G$ are classified by combinatorial invariants called homogeneous spherical datum.
For a simple root α of G, let $B \subseteq P_\alpha$ denote the corresponding sub-minimal parabolic of G.
For any B-orbit closure $Y \subseteq X$, we say that α moves Y if $P_\alpha Y \neq Y$.
Let $\mathcal{D}(\alpha)$ denote the set of colors in $\mathcal{D}$ such that α moves D.
Using this, one can describe those spherical roots of X that come from G.

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$\#\mathcal{D}(\alpha) \leq 2$

Theorem [Lun97, 3.2 and 3.4]

$\#\mathcal{D}(\alpha) \leq 2$. There are 4 cases:

- $\mathcal{D}(\alpha) = \emptyset$. Equivalently, α is among the simple roots associated to the stabilizer $P(X^\circ) \subseteq G$ of X°.
- (Type U, SL_2/U) $\mathcal{D}(\alpha) = \{D\}$, and no multiple of α is in X. In this case $v_D = \alpha^\vee|_{\Lambda_X}$.
- (Type N, PGL_2/O_2) $\mathcal{D}(\alpha) = \{D\}$, and some non-trivial multiple of α is in X. In this case $v_D = \frac{1}{2}\alpha^\vee|_{\Lambda_X}$, and $2\alpha \in \Sigma$.
- (Type T, $\mathrm{PGL}_2/\mathbb{G}_m$) $\mathcal{D}(\alpha) = \{D_\alpha^+, D_\alpha^-\}$. Equivalently, $\alpha \in \Sigma_X$, and $v_{D_\alpha^+} + v_{D_\alpha^-} = \alpha^\vee|_{\Lambda_X}$.

Homogeneous spherical datum

Roughly speaking, the homogeneous spherical datum associated to $X = H\backslash G$ consists of the lattice X; the colors $D(\alpha)$ for $\alpha \in \Sigma \cap \Delta_G$ i.e in case 4; the set Σ_X, and the set of simple roots moving no color.

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Next step: Then one needs to classify all spherical embeddings $X^\bullet = H\backslash G \hookrightarrow X$ for fixed $X^\bullet$.

Origin: classification of toric varieties by families of cones: firstly do affine toric varieties, then glue.

Fact (using normality): any spherical variety X is covered by quasi-affine G-stable open subsets.

Luna-Vust theory of spherical embeddings

Assume $H\backslash G$ is quasi-affine. Assume X is affine spherical, so X has an unique closed G-orbit Y.
Let $\mathcal{C}(X) \subseteq \mathcal{C}_0(X)$ be the cone in $\Lambda^{\vee}_{X,\mathbb{Q}}$ generated by the valuations v_D for all B-stable divisors $D \in \mathcal{D}(X)$ containing Y. Then

Luna-Vust theory of spherical embeddings

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[Kno91, Theorems 3.1 and 6.7]

$X \mapsto \mathcal{C}(X)$ gives a bijection (up to iso) between affine spherical embeddings of $X^\bullet$ and admissible rational polyhedral cones in $\Lambda^\vee_{X,\mathbb{Q}}$.

In short, the colors $\mathcal{D} = \mathcal{D}(H\backslash G)$ plus the cone $C(X)$ (**adimissile colored cone**) give a complete understanding of all quasi-affine spherical varieties X. Then the full classification follows by gluing.

Rankin-Selberg example

$n > 1$, $G = \mathrm{GL}_n \times \mathrm{GL}_n$, $H = \begin{pmatrix} \mathrm{GL}_{n-1} & * \\ 0 & 1 \end{pmatrix}$. $H\backslash G = \mathrm{GL}_n \times (\mathbb{A}^n - 0)$ quasi-affine but not affine.
There are $(3n-3)$-colors and the dual group $G_X^\vee = G^\vee$. for a simple root of GL_n, the set $D(\alpha_i; 0) \cup D(0, \alpha_i)$ has cardinality 3 and there are no other overlaps.
Let $H\backslash G \hookrightarrow X = \mathrm{GL}_n \times \mathbb{A}^n$ be the canonical affine embedding. The cone $\mathcal{C}(X) \cap \mathcal{V} \subset \Lambda_{X,\mathbb{Q}}^\vee = \mathbb{Q}^n \times \mathbb{Q}^n$ corresponds to $-\mathbb{Q}_{\geq 0}$ diagonally embedded inside $\mathbb{Q}^n \times \mathbb{Q}^n$.

Degeneration

To do harmonic analysis on $X(F)$, a trick is to degenerate X to simple spherical variety $X_\emptyset$.
A G-variety $X_\emptyset$ is horospherical if for each $x \in X_\emptyset$, its stabilizer subgroup in G contains the unipotent radical of a Borel subgroup of G.
Fact: If $X_\emptyset$ is horospherical and spherical, then its dual group is always the dual torus $T_{X_\emptyset}$.

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[SV, 2.5]

There exists a principal degeneration $\mathcal{X} \to \mathbb{A}^1$ degenerating X to a horospherical variety $X_\emptyset$.

Idea: deformation to the normal cone.

Part 3: Ichino-Ikeda

Global branching laws and global GGP

Globally, we have a candidate P_H. If $P_H \neq 0$, it's necessary that all local spaces $\mathrm{Hom}_{H_v}(\pi_v, 1) \neq 0$.
Consider $H = SO_n \hookrightarrow G = SO_n \times SO_{n+1}$.

Gan-Gross-Prasad conjecture

- (local) Whether $\mathrm{Hom}_H(\pi, 1) \neq 0$ can be understood by ϵ-factors/ genericity of σ.
- (global) under local non-vanishing assumptions, globally we have $L(\pi, 1/2) \neq 0 \Leftrightarrow \exists \phi \in \pi, \int_{[H]} \phi \neq 0$.

Hecke periods

Examples

$G = \mathrm{PGL}_2, H = \mathbb{G}_m$. For a cusp eigenform f, its central L-value satisfies $(\int_{[N]} f(n)\psi(n)dn)L(\pi, 1/2) = \int_{[H]} f(h)dh$. Rankin-Selberg gives $|\int_{[N]} f(n)\psi(n)dn|^2 = \prod_v \int_{N(F_v)} \langle \pi(h)f, f\rangle \psi(n)dn$.

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$$|\int_{[H]} f(h)dh|^2 = (*)\prod_v |P_v(\phi_v)|^2$$

where $|P_v(\phi_v)|^2 = \frac{L(\pi_v, 1/2)L(\overline{\pi}_v, 1/2)}{L(\pi_v, \mathrm{Ad}, 1)}$ for unramified places.

The square absolute value of the period integral shall have a precise formula, related to central L-values of π. This is Ichino-Ikeda conjecture, generalizing Waldspurger's formula or Hecke's formula $(n = 1)$.

Global Ichino-Ikeda type conjectures

For a good pair (G, H), $\dim \mathrm{Hom}_{H_v}(\pi_v, 1) \le 1$. The global period integral decompose to tensor product of local linear functionals, uniquely up to scalar.
The global period integral gives a global pairing
$P^{Aut} : \pi \otimes \bar{\pi} \to \mathbb{C}, P^{Aut}(\phi_1, \phi_2) := \int_{[H]} \phi_1 dh \int_{[H]} \overline{\phi_2} dh = P_X(\phi_1)\overline{P_X(\phi_2)}$.
The local Plancherel decomposation gives a local pairing
$P_{X,\pi_v} : \pi_v \otimes \overline{\pi_v} \to \mathbb{C}, P_v^{Planch}(u_1, u_2) = \int_{H(k_v)} \langle \pi_v(h) u_1, u_2 \rangle \, du,$

Ichino-Ikeda conjecture (imprecise)

$$P^{Aut} = (*) \prod_v{}' P_v^{Planch}$$

with a formula for $(*)$.

$$|P_{X,\pi}(\phi)|^2 = (*) \prod_v |P_{X,\pi_v}(\phi_v)|^2.$$

Normalizations

Normalization is needed for the convergence of Euler products, we can normalize unramified local term to be 1.
More precisely, one computes P_v^{Planch} for spherical unit vectors, it's $(*)\frac{L_X(\pi_v,1/2)}{L(\pi_v,\mathrm{Ad},1)}$.

$$P_v^{Planch,*} := ((*)\frac{L_X(\pi_v,1/2)}{L(\pi_v,\mathrm{Ad},1)})^{-1}P_v^{Planch}.$$

Program of Sakellaridis–Venkatesh (global)

$[SV]$ gives an conjectural generalization of local Plancherel formula, and the Ichino-Ikeda conjecture: for $\phi = \otimes_v \phi_v \in \pi = \otimes_v \pi_v$,

$$\left|P_{X,\pi}(\phi)\right|^2 = c(\pi) \cdot \frac{L_X(\pi, 1/2)}{L(\pi, \mathrm{Ad}, 1)} \cdot \prod_v \left|P^*_{X,\pi_v}(\phi_v)\right|^2$$

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Examples: original Ichino-Ikeda in the GGP case, Rallis inner product formula..

- The incomplete global L-values $\frac{L_X(\pi,1/2)}{L(\pi,\mathrm{Ad},1)}$ is defined by analytic continuation. **So the local and global normalization by central L-values don't cancel trivially**.
- **The adjoint L-values occur, as the normalization is based on Petersson inner products**.
- $c(\pi)$ = products of some measure normalization constants and a power of 2 (size of Vogan L-packet).
- We ignore multiplicity > 1 issues.

What do we know?

For $H = U_n \hookrightarrow G = U_n \times U_{n+1}$, $L_X(\pi, 1/2) = L(\pi, \mathrm{Std}, \frac{1}{2})$, it is proved using relative trace formula after the work of many: Jacquet-Rallis, Z. Yun, W. Zhang...
For $H = SO_n \hookrightarrow G = SO_n \times SO_{n+1}$...

Main references

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Thank you!