PART I

Variation of Hodge Structures

§1. Hodge bundles

In this section we adopt the algebraic (and analytic) definition of the De Rham cohomology sheaves from Illusie's notes [**Ill**]. The naïve filtration on the complex of relative differential forms yields the Hodge to De Rham spectral sequence and defines the Hodge filtration on the limit, the relative cohomology. In the case of a family of complex projective manifolds this is the Hodge filtration [**Dem**], a filtration by holomorphic sub-bundles of the bundle of relative cohomology groups. The language here is that of hypercohomology [**Ill**].

Let us fix the notions used in the sequel: a scheme is a scheme of finite type over an algebraically closed field k of characteristic zero. One may suppose $k = \mathbb{C}$ if one wishes. For positive characteristic we refer to [**Ill**]. When $k = \mathbb{C}$, we pass from the scheme-structure to the structure of the associated analytic space without explicit mentioning. Likewise, if X is a smooth scheme, we'll pass to the underlying C^∞-structure without mentioning this.

A sheaf will be a sheaf of $\mathcal{O}_X$-modules or an abelian sheaf if one only considers the C^∞-structure. Cohomology, an indispensable tool for manipulating families, is cohomology of coefficients in a sheaf (see the book [**God**] for example).

We shall also use the language of *hypercohomology*. Let $\Omega^\bullet$ be a complex (bounded from below) and $\Omega^\bullet \to I^\bullet$ an injective (or flasque) resolution, i.e. one which induces an isomorphism on the level of cohomology sheaves. Then $H^\bullet(X, \Omega^\bullet)$ by definition is the graded object $h^\bullet(\Gamma(X, I^\bullet))$; likewise, if $f : X \to S$ is a continuous application, morphism of schemes, etc., $\mathbb{R}^\bullet f_*(\Omega^\bullet) = h^\bullet(f_*(I^\bullet))$ is the graded object formed by the higher direct images with coefficients in the complex $\Omega^\bullet$.

Consider for example the cohomology of a smooth manifold, with constant coefficients $\mathbb{C}$. This, by definition is $H^i(X, \mathbb{C})$, $\mathbb{C}$ = the constant sheaf. The De Rham complex $\mathcal{A}^\bullet_X$ of smooth differential forms with complex coefficients is a resolution of the constant sheaf $\mathbb{C}$ (this is the *Poincaré lemma*), thus the classical fact:

$$H^i(X, \mathbb{C}) = H^i(X, \mathcal{A}^\bullet_X) = H^i(\Gamma(X, \mathcal{A}^\bullet_X)).$$

If X is a complex manifold, the sheaf Ω^p_X being the sheaf of holomorphic p-forms, the holomorphic De Rham complex $\Omega^\bullet_X$ is a resolution of $\mathbb{C}$ (*holomorphic Poincaré lemma*), from which we get

$$H^i(X, \mathbb{C}) = H^i(X, \Omega^\bullet_X) \text{ (hypercohomology).}$$

If X is a smooth (non singular) scheme, one can consider the De Rham complex $\Omega^\bullet_{X/k}$ of algebraic differential forms. The vector spaces $H^i(X, \Omega^\bullet_{X/k})$ are by definition, the (algebraic) De Rham cohomology groups of X. If $k = \mathbb{C}$, and if X is projective, it results from Serre's comparison theorems (GAGA) [**Se**] that $H^i(X, \Omega^\bullet_X)$ is the same whether $\Omega^\bullet_X$ is the algebraic, or the complex holomorphic De Rham complex. Thus if $k = \mathbb{C}$, the cohomology of X for the transcendental topology, can be computed using algebraic differential forms.

Let us pass to the *relative situation*. Let $f : X \to S$ be a morphism of schemes; we assume that f is proper. One defines ([**Ill**]) the complex of relative Kähler forms $\Omega^\bullet_{X/S}$, which is a complex of $\mathcal{O}_X$-modules of finite type (f is of finite type), for which the derivative $d_{X/S}$ is an $f^{-1}(\mathcal{O}_S)$-linear operator. The formation of $\Omega^\bullet_{X/S}$ is compatible with base change. If S, X and f are nonsingular, and if $k = \mathbb{C}$, the analytic analogue $\Omega^{\bullet\text{an}}_{X/S}$ (resp. C^∞, $\mathcal{A}^\bullet_{X/S}$) is compatible with base change. One defines the De Rham (algebraic) cohomology sheaves by ([**Ill**])

$$\mathcal{H}^k(X/S) := \mathbb{R}^k f_*(\Omega^\bullet_{X/S}).$$

Intuitively, the fiber above $s \in S$ of $\mathcal{H}^k(X/S)$ is $H^k(X_s, \Omega^\bullet_{X_s/k})$, where $X_s = f^{-1}(s)$. In addition one has the *Hodge sheaves* $R^q f_*(\Omega^p_{X/S})$, i.e. $H^q(X, \Omega^p_{X/k})$ if S is reduced to a point.

If $k = \mathbb{C}$, and if one replaces the relative algebraic differential forms $\Omega^\bullet_{X/S}$ by relative holomorphic forms $\Omega^{\bullet\text{an}}_{X/S}$, again the comparison theorems insure that the result is the same.

Let us filter the complex $\Omega^\bullet_{X/S}$ (algebraic, holomorphic,...) by the *Hodge (or naïve) filtration*

$$F^p(\Omega^\bullet_{X/S}) = (\Omega^\bullet_{X/S})^{\geq p}$$

which is the complex which has the same term in degree $i \geq p$, and which is zero in degree $< p$. Then, the spectral sequence associated to this finite filtration and to the functor f_*, is the Hodge to De Rham spectral sequence:

$$\text{(HDR)} \qquad E_1^{pq} = R^q f_*(\Omega^p_{X/S}) \Longrightarrow \mathcal{H}^{p+q}(X/S) = \mathbb{R}^{p+q} f_*(\Omega^\bullet_{X/S})$$

There results a filtration on the limit $F^p\mathcal{H}^k(X/S)$, the Hodge filtration on the De Rham cohomology whose associated gradeds are

$$E_\infty^{p,q} = \mathrm{Gr}^p\left(\mathcal{H}^{p+q}(X/S)\right).$$

The essential assumption is:

1.1. Assumption. *The (relative) Hodge to De Rham spectral sequence degenerates at E_1.*

This means

$$E_1 = E_2 = \cdots = E_\infty$$

and in particular

$$E_1^{pq} = R^q f_*(\Omega^p_{X/S}) = \frac{F^p\mathcal{H}^{p+q}(X/S)}{F^{p+1}\mathcal{H}^{p+q}(X/S)}.$$

This assumption is discussed in [**Ill**]. Let us only mention the following statement which indicates the most important consequences for Hodge sheaves (see [**Del3**], th. 5.5).

1.2. Theorem. *Let S be a scheme of characteristic 0; assume that the morphism $f : X \to S$ is proper and smooth. Then*

(i) *The sheaves $R^p f_*(\Omega^p_{X/S})$ are locally free of finite type and they are compatible with base change.*

(ii) *The spectral sequence* (HDR) *degenerates at* E_1.

(iii) *The sheaves* $\mathcal{F}^p\mathcal{H}^{p+q}(X/S)$ *are locally free of finite rank compatible with base change.*

(iv) *The spectral sequence* (HDR) *restricted to the fiber above* $s \in S$ *yields the spectral sequence corresponding to* X_s.

1.3. REMARKS. If S is smooth and connected, the transcendental theory [**Dem**] says that Hodge numbers $h^{p,q}(s) = \dim H^{p,q}(X_s)$ are constant. Then one can deduce that $R^p f_*(\Omega^p_{X/S})_s = H^q(X_s, \Omega^p_{X_s})$. In the general case, the degeneration of the Hodge spectral sequence at a point $s \in S$ leads to (i) to (iv) by general arguments due to Grothendieck ("lemme d'échange" [**Del3**], th. 5.5).

Assume $k = \mathbb{C}$, and as always $f : X \to S$ proper and smooth. By the relative holomorphic Poincaré lemma, the complex $\Omega^{\bullet\mathrm{an}}_{X/S}$ is a resolution of the sheaf $f^{-1}(\mathcal{O}_S)$ (inverse sheaf-theoretical image). (It suffices to restrict to the fiber $X_s = f^{-1}(s)$). Then

$$\begin{aligned}\mathcal{H}^k(X/S) &= R^k f_*(f^{-1}(\mathcal{O}_S))\\ &\cong \mathcal{O}_S \otimes_{\mathbb{C}} R^k f_*\mathbb{C}.\end{aligned}$$

To justify this identification, recall the base change theorem in cohomology (with proper supports) (see Iversen [**Iv**]). Consider a Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{p} & S \end{array}$$

with f proper and X, S, T, Y locally compact spaces. For any abelian sheaf F on X, one has a canonical isomorphism

$$p^* R^k f_* F \xrightarrow{\sim} R^k g_* q^* F.$$

Using the projection formula (loc. cit.) one gets the above identification. In this way one has (if $k = \mathbb{C}$) a relation between De Rham cohomology and cohomology with constant coefficients, which we make precise in §2.A.

In the sequel we use the following convention:

1.4. DEFINITION. A *family of projective manifolds* consists of a smooth morphism $f : X \to S$ with connected fibers such that for some closed immersion $i : X \to \mathbb{P}^N \times S$ we have $f = \mathrm{pr}_S \circ i$.

The morphism f is thus proper. Let us assume that the Hodge spectral sequence degenerates. The decreasing filtration $\mathcal{F}^p(\mathcal{H}^k(X/S))$, is let us recall, the *Hodge filtration*. The bundles $\mathcal{F}^p$ are sub-bundles of $\mathcal{H}^k(X/S)$, and $\mathcal{F}^p/\mathcal{F}^{p+1} \cong R^q f_*(\Omega^p_{X/S})$.

§2. Gauss-Manin connection

To study how the De Rham cohomology classes vary in a family, it is essential to be able to differentiate these classes with respect to local coordinates on the base S. The goal of this section is to make this precise by explaining in detail the constructions of Katz and Oda ([**Ka**], [**K-O**]). These use a "connection" on a resolution of the De Rham complex which, after passage to cohomology, leads to the Gauss-Manin connection. In this section the framework is that of schemes.

§2.A. Local Systems.

Let S be a topological space. A locally constant sheaf $\mathcal{V}$ (of sets, of groups, of vector spaces etc.) is called a *local system*. Thus there exists an open covering of S such that $\mathcal{V}$ is constant on the open sets of this covering. One sees easily that a locally constant sheaf is constant on a simply connected space and thus the pull back of $\mathcal{V}$ to the universal covering is constant with fiber V say; then one gets $\mathcal{V}$ as quotient of $\tilde{S} \times V$ by the fundamental group of S which acts on $\tilde{S} \times V$ in a natural way: $\gamma((\tilde{s}), f) = (\tilde{s} \cdot \gamma, \gamma^{-1} f)$, where $\gamma \in \pi_1(S)$ acts from the right on $\tilde{S}$ and from the left on V. This action leads to a representation of $\pi_1(S)$ on V, the *monodromy representation*.

Assume that S is a scheme (resp. analytic space, C^∞ manifold) with structure sheaf $\mathcal{O}_S$. The sheaf of $\mathcal{O}_S$-modules $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_S$ is called the *sheaf associated* to the local system $\mathcal{V}$. It is locally free, i.e. a vector bundle on S. Such a vector bundle is characterized by the fact that there exists a trivializing open covering such that on the intersection of two of these open sets, the transition matrix has constant coefficients. Recall [**Dem**] that a *connection* on $\mathcal{F}$, is a k-linear operator $\nabla : \mathcal{F} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{F}$ (resp. $\Omega^{1,\mathrm{an}}_S \otimes \mathcal{F}, \mathcal{A}^1_S \otimes \mathcal{F}$), which satisfies Leibnitz' rule

$$\nabla(ae) = da \otimes e + a\nabla e.$$

One can extend ∇ to a k-linear map $\nabla : \Omega^p_S \otimes_{\mathcal{O}_S} \mathcal{F} \to \Omega^{p+q}_S \otimes_{\mathcal{O}_S} \mathcal{F}$ by forcing the rule $\nabla(\alpha \otimes e) = d\alpha \otimes e + (-1)^p \alpha \wedge \nabla e$ ($\alpha = p$-form). Then the operator $R = \nabla\nabla$ is $\mathcal{O}_S$-linear; one has

$$R \in \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \Omega^2_S \otimes_{\mathcal{O}_S} \mathcal{F}) = \Omega^2_S(\mathrm{End}(\mathcal{F})),$$

the curvature operator associated to ∇. The connection is called *integrable* (or *flat*) if $R = 0$, thus if

$$0 \longrightarrow \mathcal{F} \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{F} \xrightarrow{\nabla} \Omega^2_S \otimes \mathcal{F} \longrightarrow \cdots$$

is a complex. One calls it the *associated De Rham complex*, in view of the particular case $\mathcal{F} = \mathcal{O}_S$, $\nabla = d$. Recall that for any vector field v on X (or on a open subset of X), the k-linear operator ∇_v (contraction of ∇ with v) is called the covariant derivative in the direction of v. The integrability condition reads

$$(1) \qquad [\nabla_v, \nabla_w] = \nabla_{[v,w]} \ (v,\ w \text{ vector fields on } X),$$

$[v, w]$ being the bracket of the vector fields v and w. If $k = \mathbb{C}$, and if ∇ is an integrable connection on the vector bundle $\mathcal{F}$ (locally free sheaf of rank n), the existence of local solutions for the linear differential equation $\nabla e = 0$ implies that $\mathcal{V} = \ker(\nabla) \subset \mathcal{F}$ is a locally constant sub-sheaf and that $\mathcal{F} = \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_S$ is the bundle associated to $\mathcal{V}$. Then $\nabla = 1 \otimes d$, which means that if one chooses locally a basis $\{e_i\}$ of F, composed of flat sections (i.e., sections of $\mathcal{V}$) one has $\nabla(\sum_i a_i \otimes e_i) = \sum_i da_i \otimes e_i$.

Conversely, for a locally constant sheaf $\mathcal{V}$ on S, $\nabla = 1 \otimes d$ is a flat connection on the associated bundle $\mathcal{F} = \mathcal{V} \otimes \mathcal{O}_S$, with $\ker \nabla = \mathcal{V}$. Thus, there is an equivalence of categories between flat bundles, i.e. pairs $(\mathcal{F}, \nabla)$ with $R_\nabla = 0$) and locally constant sheaves of $\mathbb{C}$ vector spaces, the morphisms of bundles being the horizontal morphisms. Let us come back to the geometrical situation of a family of projective manifolds $f : X \to S$. We have seen [**Dem**], §10, that a family is locally trivial from the differentiable point of view and thus the abelian sheaves $R^k f_*\mathbb{C}$, $R^k f_*\mathbb{R}$, $R^k f_*\mathbb{Z}$ are local systems. In this way $\mathcal{H}^k(X/S) = R^k f_*(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_S$ and the *Gauss-Manin connection* on the cohomology bundle $\mathcal{H}^k(X/S)$ is the unique (holomorphic) connection with horizontal (or flat) sections, the sections of $R^k f_*(\mathbb{C})$, i.e.

$$\nabla_{\mathrm{GM}}(e) = 0 \iff e \in R^k f_*(\mathbb{C}).$$

§**2.B. The Kodaira-Spencer map.**

Now k is arbitrary, and S is a smooth scheme of finite type over k. The smoothness of f yields an exact sequence

$$0 \longrightarrow f^*(\Omega^1_S) \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0\ . \tag{2}$$

This extension, in general non trivial, is given by a class $c \in \mathrm{Ext}^1(\Omega^1_{X/S}, f^*(\Omega^1_S))$, and as $\Omega^1_{X/S}$ is locally free, one has

$$\mathrm{Ext}^1(\Omega^1_{X/S}, f^*(\Omega^1_S)) \cong H^1(X, \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, f^*(\Omega^1_S))\ .$$

The image of c by the canonical map

$$\begin{array}{ccc} H^1(X, \underbrace{\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, f^*(\Omega^1 S))}_{= T_{X/S} \otimes f^*(\Omega^1_S)}) & \longrightarrow & H^0(X, R^1 f_*(T_{X/S} \otimes f^*(\Omega^1_S)) \\ & & \| \\ & & H^0(X, \Omega^1_S \otimes R^1 f_*(T_{X/S})) \end{array}$$

is called the *Kodaira-Spencer class* of X/S ; one can see this class as a morphism, *the Kodaira-Spencer morphism* $\rho_{X/S} : T_S \longrightarrow R^1 f_*(T_{X/S})$. The fiber $(\rho_{X/S})_s = \rho_s : T_{S,s} \longrightarrow H^1(X_s, T_{X_s})$ is *the Kodaira-Spencer map* at $s \in S$.

Recall ([**Ill**]) that if $c \in H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G}))$ is the class of an extension $0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0$, the boundary morphism $\partial : H^q(X, \mathcal{F}) \to H^{q+1}(X, \mathcal{G})$ can be identified as cup product with c.

The Kodaira-Spencer map at s measures how X_s deforms in the family X/S in the neighborhood of s, at least infinitesimally.

We shall come back to the Kodaira-Spencer map in §3.C.

§**2.C. Algebraicity of the Gauss-Manin connection.**

The sheaves $\mathcal{H}^k(X/S)$, $\mathcal{F}^p\mathcal{H}^k$ have an algebraic definition, via the algebraic De Rham cohomology; we shall see that this is also true for the Gauss-Manin connection. We shall pass to the De Rham complex $\Omega^\bullet_{X/k} = \Lambda^\bullet \Omega^1_{X/k}$. This complex is not in general "multiplicative" with respect to the two extremes. The Koszul filtration on $\Omega^\bullet_{X/k}$ measures this deviation. The definition works for any extension $0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0$ (of locally free $\mathcal{O}_X$-modules). Put

$$F^p \Lambda^\bullet \mathcal{H} = \mathrm{image}\big(\Lambda^p \mathcal{G} \otimes \Lambda^\bullet \mathcal{F}[-p] \longrightarrow \Lambda^\bullet \mathcal{H}\big).$$

One has clearly $\mathrm{Gr}^p = F^p/F^{p+1} \cong \Lambda^p \mathcal{G} \otimes \Lambda^\bullet \mathcal{F}[-p]$, $[-p]$ means that there is a shift of $-p$ in the degree. Consider the exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}^1 & \longrightarrow & F^0/F^2 & \longrightarrow & \mathrm{Gr}^0 & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathcal{G} \otimes \Lambda^\bullet \mathcal{F}[-1] & & & & \Lambda^\bullet \mathcal{F} & & \end{array}$$

which in degree k, leads to the extension

$$0 \longrightarrow \mathcal{G} \otimes \Lambda^{k-1}\mathcal{F} \longrightarrow (F^0/F^2)^k \longrightarrow \Lambda^k\mathcal{F} \longrightarrow 0\ .$$

An easy verification shows that the class $c_k \in H^1(X, \mathrm{Hom}(\Lambda^k\mathcal{F}, \mathcal{G} \otimes \Lambda^{k-1}\mathcal{F})$ is derived from c by means of interior product

$$I : \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\Lambda^k\mathcal{F}, \mathcal{G} \otimes \Lambda^{k-1}\mathcal{F})$$

where $I(\lambda)(f_1 \wedge \cdots \wedge f_p) = \sum_{i=1}^{p}(-1)^{i-1}\lambda(f_i) \otimes f_1 \wedge \cdots \wedge \hat{f_i} \wedge \cdots \wedge f_p$.

We shall return to the geometrical situation. The Kodaira-Spencer map can be derived from the extension class (2); we shall see that the Gauss-Manin connection can be derived from the extension class of the complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}^1 & \longrightarrow & F^0/F^2 & \longrightarrow & \mathrm{Gr}^0 & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & f^*(\Omega^1_S) \otimes \Omega^\bullet_{X/S}[-1] & & & & \Omega^\bullet_{X/S} & & \end{array} \tag{3}$$

(we refer to **[Ill]** for a precise definition).

Consider the boundary morphism in hypercohomology

$$\partial : R^k f_*(\mathrm{Gr}^0) \longrightarrow R^{k+1} f_*(\mathrm{Gr}^1)$$

which after identification becomes

$$\begin{array}{ccc} \partial : \mathcal{H}^k(X/S) & \longrightarrow & R^{k+1} f_*(f^*(\Omega^1_S) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[-1]) \\ & & \| \\ & & \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{H}^k(X/S) \end{array} .$$

We arrive at the main result

2.1. THEOREM.
1. *∂ is an integrable connection on the De Rham cohomology bundle $\mathcal{H}^k(X/S)$.*
2. *The associated De Rham complex $(\mathcal{H}^k(X/S) \otimes \Omega^\bullet_S, \partial)$ can be identified with the complex $E_1^{\bullet,k}$ derived from the spectral sequence of $\Omega^\bullet_X$ filtered by the Koszul filtration relative to the functor f_*.*
3. *If $k = \mathbb{C}$, after identification of the sheaves, ∂ coincides with ∇.*

Before giving the details of the proof, let us indicate that (1) and (2) are obtained easily if one takes into account the compatibility of the Koszul filtration with respect to the exterior product $F^i \wedge F^j \subset F^{i+j}$. So one can define a pairing on the spectral sequence

$$E_1^{pq} \times E_1^{p'q'} \longrightarrow E_1^{p+p',q+q'},\ (e, e') \longmapsto ee'$$

such that $e'e = (-1)^{(p+q)(p'+q')}ee'$ and $d_1(ee') = (d_1e)e' + (-1)^{p+q}e \cdot d_1(e')$. It is maybe more convincing to give an explicit formula for ∂ from which the integrability will then be an easy consequence. This is the procedure that we shall make precise in several steps.

Step 1

The problem being local on S, one can suppose that S is affine (or Stein, in the analytic framework). We assume from the start on that $X = S \times T$ is a product, without supposing that T is projective; one can even suppose that X is étale over $\mathbb{A}^n \times S$. With this assumption that the family is trivial, the exact sequence (2) splits, and $\Omega^\bullet_X = p_1^*(\Omega^\bullet_S) \otimes p_2^*(\Omega^\bullet_T)$ (tensor product of complexes). One can identify $p_2^*(\Omega^\bullet_T)$ and $\Omega^\bullet_{X/S}$, and then the total derivative d_X decomposes as $d_X = d_S + d_{X/S}$. Let us note that the tensor product is over $\mathcal{O}_X$, and hence the derivative is only k-linear. Locally, one can describe the situation as follows. With $S = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(B)$ we have $X = \mathrm{Spec}(A \otimes_k B)$. Consider $\Omega^\bullet_S = \Lambda^\bullet \Omega^1_{A/k}$, $\Omega^\bullet_T = \Lambda^\bullet \Omega^1_{B/k}$, which are graded algebras over A (resp. B). Then one has $\Omega^1_{X/k} = \Omega^1_S \otimes_k B \oplus A \otimes_k \Omega^1_T$ and $\Omega^\bullet_X = \Omega^\bullet_S \otimes_k \Omega^\bullet_T$ with the natural structure of an $A \otimes_k B$ graded algebra. The derivative is $d_X = d_S + d_T$, with the usual meaning $d_X(\alpha \otimes \beta) = d_S(\alpha) \otimes \beta + (-1)^p \alpha \otimes d_T(\beta)$, if $\alpha \in \Omega^p_S$, $\beta \in \Omega^q_T$. Observe that $\Omega^\bullet_{X/S} = A \otimes_k \Omega^\bullet_T$, $d_{X/S} = 1 \otimes d_T$. The quotient morphism $\pi : \Omega^\bullet_X \to \Omega^\bullet_{X/S}$ admits a natural section φ (of groups), such that, with an abuse of notation

$$\varphi(h d_{X/S} f_1 \wedge \cdots \wedge d_{X/S} f_p) = h d_{X/S} f_1 \wedge \cdots \wedge d_{X/S} f_p,$$

where on the right $d_{X/S}$ is the partial derivative from the decomposition $d_X = d_S + d_{X/S}$. One has

$$F^p = \bigoplus_{i \geq p} \Omega^i_S \otimes \Omega^j_T \text{ et } \Omega^\bullet_X = F^1 \bigoplus \Omega^\bullet_{X/S}.$$

2.2. Lemma. *There exists a derivation I (total interior product) of the algebra $\Omega^\bullet_X$, i.e. $I(\alpha \wedge \beta) = I(\alpha) \wedge \beta + \alpha \wedge I(\beta)$, such that $I(dg) = d_S g$. Moreover, for any form $\omega \in \Omega^\bullet_X$ one has*

$$\varphi\pi(\omega) - \omega \equiv -I(\omega) \pmod{F^2\Omega^\bullet_X}.$$

Proof. With the description $\Omega^\bullet_X = \Omega^\bullet_S \otimes_k \Omega^\bullet_T$, one takes for I, the "derivation",

$$d(\alpha \wedge \beta) = p\alpha \wedge \beta \text{ if } \alpha \text{ has degree } p\ .$$

Observe that I is $\mathcal{O}_X$ $(= A \otimes_k B)$ linear.

For the second assertion, one can suppose $g = a \otimes b$, $(a \in A,\ b \in B)$, then $d_X g = d_S a \otimes b + a \otimes d_T b$ with $d_S g = d_S a \otimes b$ and $d_{X/S} g = a \otimes d_T b$. One has $I(d_X g) = d_S a \otimes b = d_S g$. This yields more generally $I(g dg_1 \wedge \cdots \wedge dg_p) = \sum_{i=1}^n g dg_1 \wedge \cdots \wedge d_S g_i \wedge \cdots \wedge dg_p$.

For the last property, assume $\omega = g dg_1 \wedge \cdots \wedge d_S g_i \wedge \cdots \wedge dg_p$, then

$$\begin{aligned}
\varphi\pi(\omega) &= g d_{X/S} g_1 \wedge \cdots \wedge d_{X/S} g_p \\
&= g(dg_1 - d_S g_1) \wedge \cdots \wedge (g_p - d_S g_p) \\
&= \omega - I(\omega) \pmod{F^2\Omega^\bullet_X}\ .
\end{aligned}$$

□

Step 2

Assume that S is affine, and choose a finite open covering $X = \bigcup_{\alpha=1}^m U_\alpha$, where U_α is supposed to be étale over $\mathbb{A}^n \times S$. With such a trivialization (see the appendix) one can, as indicated in step 1, decompose the De Rham complex $\Omega^\bullet_{U_\alpha/k}$ as a tensor product $\Omega^\bullet_S \otimes_{\mathcal{O}_X} \Omega^\bullet_{U_\alpha/S}$, and then write the derivative d_X on U_α as $d_{X/U_\alpha} = d^\alpha_S + d^\alpha_{X/S}$. Let φ_α be the section of $\pi : \Omega^\bullet_{U_\alpha} \to \Omega^\bullet_{U_\alpha/S}$ which results from this decomposition, and let I_α the corresponding interior product.

The Gauss-Manin connection describes in a cohomological way this (local) decomposition of $\Omega_X^\bullet$ as a tensor product $\Omega_S^\bullet \otimes \Omega_{X/S}^\bullet$. Consider $\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega_X^\bullet)$ (resp. $\check{\mathcal{C}}^\bullet(\mathcal{U}, F^p)$, etc.), the Čech complex with coefficients in $\Omega_X^\bullet$ (resp. ...) ([**Ill**]). One knows (loc. cit.) that the canonical morphism $\Omega_X^\bullet \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega_X^\bullet)$ is a quasi-isomorphism. Note also, that since the open set U_α is affine, the functor $K^\bullet \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, K^\bullet)$ is exact, and hence

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Gr}^p(\Omega_X^\bullet)) = \check{\mathcal{C}}^\bullet(\mathcal{U}, F^p)/\check{\mathcal{C}}^\bullet(\mathcal{U}, F^{p+1}).$$

The derivative of the Čech complex is denoted $d + \delta$ where d is the derivative on the level of forms, and δ the Čech-derivative

$$(\delta\beta)(i_0, \dots, i_q) = (-1)^p \sum_{j=0}^{q} (-1)^j \beta(i_0, \dots, \hat{i}_j, \dots, i_q)$$

(if $\beta \in \mathcal{C}^{p,q} = \mathcal{C}^q(\mathcal{U}, \Omega_X^p)$).

For any index α with $h_\alpha = d_S^\alpha \circ \varphi_\alpha$ viewed as morphism of complexes, one has $h_\alpha : \mathrm{Gr}^0(\Omega_X^\bullet)_{|U_\alpha} \longrightarrow \mathrm{Gr}^1(\Omega_X^\bullet)[1]_{|U_\alpha}$ (immediate verification). Let $\psi_{\alpha\beta} = \varphi_\beta - \varphi_\alpha$ $(\mathrm{mod}\, F^2)$ so that

$$\psi_{\alpha\beta} : \mathrm{Gr}^0(\Omega_X^\bullet)_{|U_\alpha \cap U_\beta} \longrightarrow \mathrm{Gr}^1(\Omega_X^\bullet)_{|U_\alpha \cap U_\beta}.$$

We have $\psi_{\alpha\beta} + \psi_{\beta\alpha} = \psi_{\alpha\gamma}$ (for any (α, β, γ)), and $(h_\beta - h_\alpha)_{|U_\alpha \cap U_\beta} = d\psi_{\alpha\beta}$, which implies

$$d_S^\alpha \varphi_\alpha - d_S^\beta \varphi_\beta = (d_S^\alpha + d_{X/S}^\alpha)\varphi_\alpha - (d_S^\beta + d_{X/S}^\beta)\varphi_\beta - (\varphi_\alpha - \varphi_\beta) d_{X/S}$$

defining thus a morphism of complexes

$$h : \mathrm{Gr}^0 \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Gr}^1[1])$$

which induces (in the derived category) a morphism

$$\nabla : \mathrm{Gr}^0 \longrightarrow \mathrm{Gr}^1[1].$$

The reader should compare this with the proof of lemma (5.4) in [**Ill**].

If one passes to cohomology, ∇ induces the Gauss-Manin connection. We shall make a more precise construction, and deduce from it the integrability of the Gauss-Manin connection.

Step 3

Let $\beta \in \mathcal{C}^q(\mathcal{U}, \Omega_X^p)$, set

$$\mathcal{L}(\beta)(i_0, \dots, t_q) = d_S^{i_0}(\beta(i_0, \dots, i_q)) \text{ (total Lie derivative)}$$

then

$$I(\beta)(i_0, \dots, i_{q+1}) = (-1)^p (I_{i_0} - I_{i_1})(\beta(i_1, \dots, i_{q+1}) \text{ (total interior product)}$$

and

$$\varphi(\beta)(i_0, \dots, i_p) = \varphi_{i_0}(\beta(i_0, \dots, i_p)) .$$

Note that $\mathcal{L}$ is of bi-degree $(1, 0)$, I of bi-degree $(0, 1)$.

An elementary computation leads to

2.3. LEMMA. *$\nabla = \mathcal{L} + I$ is a morphism of complexes*

$$\mathcal{L} + I \in \operatorname{Hom}\big(\check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega_X^\bullet), \check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega_X^\bullet[1])\big).$$

Let us note that by construction, $\nabla(\check{\mathcal{C}}(\mathcal{U}, F^i)) \subseteq \check{\mathcal{C}}(\mathcal{U}, F^{i+1})$. To ensure that on the level of cohomology, the morphism induced by $\nabla = \mathcal{L} + I : \mathrm{Gr}^0 \to \mathrm{Gr}^1[1]$ is indeed the Gauss-Manin connection, the following property is exactly what is needed:

2.4. LEMMA.

$$(d_X + \delta)\varphi - \varphi(d_{X/S} + \delta) \equiv (\mathcal{L} + I) \circ \varphi \;(\operatorname{mod} \check{\mathcal{C}}^\bullet(F^2)).$$

PROOF. Let $\beta \in \mathcal{C}^q(\mathcal{U}, \Omega^p_{X/S})$ and

$$(d_X\varphi - \varphi d_{X/S})(\beta)(i_0, \dots, i_q) = d_S^{i_0}\varphi_{i_0}(\beta(i_0, \dots, i_q)) \;.$$

An easy computation shows that

$$(\delta\varphi - \varphi\delta)(\beta)(i_0, \dots, i_{q+1}) = (-1)^{p+1}(\varphi_{i_0} - \varphi_{i_1})(\beta(i_1, \dots, i_{q+1})).$$

It suffices then to verify that

$$(-1)^{p+1}(\varphi_{i_0} - \varphi_{i_1})(\omega) \equiv (-1)^p I^{i_0}\varphi_{i_1}(\omega) \;(\operatorname{mod} F^2)$$

for any form $\omega \in \Gamma(U_{i_0} \cap U_{i_1}, \Omega_{X/S})$. If one puts $\varphi_{i_1}(\omega) = \alpha$, this congruence is equivalent to $\varphi_{i_0}(\pi(\alpha)) - \alpha \equiv -I_{i_0}(\alpha) \;(\operatorname{mod} F^2)$, which results then from lemma 2. □

The integrability of the Gauss-Manin connection follows immediately from the formula of lemma 4. Indeed

$$\nabla((d_X + \delta)\varphi - \varphi(d_{X/S} + \delta)) \equiv \nabla^2 \circ \varphi \;(\operatorname{mod} \check{\mathcal{C}}^\bullet(F^3))$$

because ∇ is of degree 1 for the Koszul filtration and since ∇ and $d_X + \delta$ commute,

$$(d_X + \delta)(\nabla \circ \varphi) - (\nabla \circ \varphi)(d_{X/S} + \delta) \equiv \nabla^2 \circ \varphi (\operatorname{mod} F^3).$$

Thus ∇^2 induces the zero morphism from Gr^0 to $\mathrm{Gr}^2[2]$ (in the sense of derived categories).

Before concluding, it is useful to stress the following point. What has been constructed in the steps 1 to 3 (lemma 2.3) is a connection ("the Gauss-Manin connection") on the level of the De Rham complex (the Čech-De Rham complex). This (not necessarily integrable) connection induces the (integrable) Gauss-Manin connection on the level of cohomology.

To finish the proof of the theorem, it remains to verify that the above construction does not depend on the choice of $\mathcal{U} = \{U_\alpha\}$. This is completely standard. If $k = \mathbb{C}$, it is also immediate that the above construction can be applied with S and U_α Stein, and the result is identical. To convince oneself that this construction of ∇ coincides with the purely topological definition, one observes that the decomposition of the De Rham complex, which is local on X in the algebraic and analytic case, becomes local on S with the complex of C^∞ forms. Thus one can suppose that $X \to S$ is a fibration which is trivial in the C^∞ sense. In this case, one can construct the morphisms ∇ on the level of the C^∞ De Rham complexes, say $\nabla : \mathcal{A}^\bullet_X \to \mathcal{A}^\bullet_X[1]$, and $\nabla = d_S$ (relative to some decomposition). Then, it is almost obvious that ∇ induces in cohomology $d_S \otimes 1$

$$\nabla = d_S \otimes 1 : \mathbb{R}^q f_*(\mathcal{A}^\bullet_{X/S}) \longrightarrow \mathbb{R}^{q+1} f_*(\mathrm{Gr}^1[1]) = \mathcal{A}^1_S \otimes \mathbb{R}^q f_*(\mathcal{A}^\bullet_{X/S}).$$

and that $\mathbb{R}^q f_*(\mathcal{A}^\bullet_{X/S}) = \mathbb{R}^q f_*(\Omega^{\bullet\mathrm{an}}_{X/S})$ is the constant sheaf $H^q(T, \mathbb{C})$ on S (if $X = T \times S$). □

§2.D. Transversality of ∇.

For any complex $\Omega^\bullet$, the Hodge filtration of $\Omega^\bullet$ is the naïve filtration $\Omega^{\bullet\geq}$. After shifting degrees one has

$$(\Omega^\bullet[n])^{\geq p} = (\Omega^{\bullet \geq p+n})[n].$$

Consider the exact sequence of complexes (cf. (3))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}^1 & \longrightarrow & F^0/F^2 & \longrightarrow & \mathrm{Gr}^0 & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & f^*(\Omega^1_S) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[-1] & & & & \Omega^\bullet_{X/S} & \longrightarrow & 0 \end{array}$$

and pass to the i-th level of the Hodge filtration. One has the exact sequence

$$0 \longrightarrow f^*(\Omega^1_S) \otimes \Omega^{\bullet \geq i-1}_{X/S}[-1] \longrightarrow (F^0/F^2)^{\geq i} \longrightarrow \Omega^{\bullet\geq i}_{X/S} \longrightarrow 0$$

and in cohomology one gets a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^k f_*(\Omega^\bullet_{X/S}) & \xrightarrow{\partial} & \mathbb{R}^{k+1} f_*(\mathrm{Gr}^1[1]) = \Omega^1_S \otimes_{\mathcal{O}_S} \mathbb{R}^k f_*(\Omega^\bullet_{X/S}) \\ \uparrow & & \uparrow \\ \mathbb{R}^k f_*(\Omega^{\bullet\geq i}_{X/S}) & \xrightarrow{\partial} & \mathbb{R}^{k+1} f_*(\mathrm{Gr}^1[1]^{\geq i}) = \mathbb{R}^{k+1} f_*(f^*(\Omega^1_S) \otimes (\Omega^{\bullet\geq i-1}_{X/S}[-1]) \cdot \\ & & \| \\ & & \Omega^1_S \otimes_{\mathcal{O}_S} \mathbb{R}^k f_*(\Omega^{\bullet\geq i-1}_{X/S}) \end{array}$$

The images of the vertical maps are resp. $\mathcal{F}^i\mathcal{H}^k(X/S)$ and $\Omega^1_S \otimes \mathcal{F}^{i-1}\mathcal{H}^k(X/S)$, and $\partial = \nabla$, thus one has the *transversality property* for the Gauss-Manin connection:

$$\nabla(\mathcal{F}^i\mathcal{H}^k(X/S)) \subseteq \Omega^1_S \otimes \mathcal{F}^{i-1}\mathcal{H}^k(X/S) .$$

Assume that the Hodge to De Rham spectral sequence degenerates (for example if $k = \mathbb{C}$). Then $E_1^{pq} = R^q f_*(\Omega^p_{X/S}) = F^p\mathcal{H}^{p+q}(X/S)/F^{p+1}\mathcal{H}^{p+q}(X/S)$.

Moreover, it is clear that passing to the associated graded of the Hodge filtration on $\mathcal{H}^k(X/S)$, ∇ induces an $\mathcal{O}_S$-linear map

$$\overline{\nabla} : R^q f_*(\Omega^p_{X/S}) \longrightarrow \Omega^1_S \otimes R^{q+1} f_*(\Omega^{p-1}_{X/S}) .$$

Then $\overline{\nabla}$ is the cup product with the Kodaira-Spencer map

$$\rho_{X/S} \in H^0(S, \Omega^1_S \otimes \mathbb{R}^1 f_*(T_{X/S})).$$

Hence finally:

2.5. THEOREM. *With respect to the Hodge filtration $F^\bullet\mathcal{H}^k(X/S)$, the Gauss-Manin connection satisfies Griffiths' transversality property*

$$\nabla(F^i\mathcal{H}^k) \subseteq \Omega^1_S \otimes F^{i-1}(\mathcal{H}^k) \ .$$

If the Hodge to De Rham spectral sequence degenerates (if for example $k = \mathbb{C}$), the $\mathcal{O}_S$-linear map which the derivation ∇ induces on the associated Hodge bundles coincides with cup product with the Kodaira-Spencer class.

□

2.6. NOTES.

1. All of what has been done in §1 and §2, admits an almost immediate translation in the logarithmic framework. In this set-up $D \subset X$ is a divisor which is a union of nonsingular divisors relative to the base S, and D has normal crossings relative to S. In this context, one can define (see [**Ill**], §7) the complex $\Omega^\bullet_{X/S}(\log D)$ of the *differential forms, regular on $X \setminus D$ having logarithmic poles along D*. For the easier case of a smooth hypersurface see §8 and for the general case see [**Ka**]. We get a Hodge to De Rham spectral sequence

$$E_1^{pq} = R^q f_*(\Omega^p_{X/S}(\log D)) \Longrightarrow \mathbb{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D))$$

and a Gauss-Manin connection

$$\nabla : R^q f_*(\Omega^\bullet_{X/S}(\log D)) \longrightarrow \Omega^1_S \otimes_{\mathcal{O}_S} \mathbb{R}^q f_*(\Omega^\bullet_{X/S}(\log D))$$

which satisfies the property of Griffiths' transversality with respect to the Hodge filtration $F^\bullet$, at least if one assumes that the above spectral sequence degenerates at E_1 ($k = \mathbb{C}$). The reader should consult the fundamental work of N. Katz [**Ka**] for details.

2. Recently Hinich and Schechtman [**H-S**] have introduced a higher order Kodaira-Spencer map, which can be applied to differential operators and not only to derivations.

Appendix: "local coordinates" in algebraic geometry.

To take away any doubt concerning the local computations in §2, recall how one works with local coordinates in algebraic geometry.

Let X/k be a scheme of finite type, smooth over the field k. Then the $\mathcal{O}_X$-module $\Omega^1_{X/k}$ is locally free of finite rank $n = \dim X$ and in a neighborhood of any point $x \in X$, one can find regular sections $s_1, \ldots, s_n \in \Gamma(U, \mathcal{O}_X)$ such that $\{ds_1, \ldots, ds_n\}$ is a basis of $\Omega^1_{X/h}$ over U.

2.7. DEFINITION. One calls $s_1, \ldots, s_n$ a system of uniformizing coordinates (or local parameters) on U.

On can then define the partial derivative $\frac{\partial}{\partial s_i}$ by means of the formula ($\alpha \in \Gamma(U, \mathcal{O}_X)$)

$$d\alpha = \sum_{i=1}^{n} \frac{\partial \alpha}{\partial s_i} ds_i \ .$$

The relation $d^2 = 0$ is equivalent to $\frac{\partial^2}{\partial s_i \partial s_j} = \frac{\partial^2}{\partial s_j \partial s_i}$ $(\forall(i,j))$. If $f : X \to Y$ is an étale morphism, the canonical morphism $df : f^*(\Omega^1_{Y/k}) \to \Omega^1_{X/k}$ is an isomorphism. So, if $s_1, \ldots, s_n$ are uniformizing coordinates on $V \subset Y$, $t_1 = f^*(s_1), \ldots, t_n =$

$f^*(s_n)$ is a system of uniformizing coordinates on $U = f^{-1}(V)$. One has by construction

$$\frac{\partial}{\partial t_i}(f^*(\alpha)) = f^*\big(\frac{\partial \alpha}{\partial s_i}\big) \ (\alpha \in \Gamma(V, \mathcal{O}_Y)) \ .$$

If now U is an open subset of the scheme X, U is étale over $\mathbb{A}^n \times S$, then $s_1, \dots, s_n$ are natural coordinates on $\mathbb{A}^n$. The restrictions to U of these $n+m$ coordinates define local coordinates on U.

§3. Variation of Hodge structures

In this section we introduce the notions of variation of Hodge structures, of period domain and of infinitesimal variation of Hodge structures.

§3.A. Introduction to Variation of Hodge structures.

Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ a smooth projective manifold of dimension n. $H_{\mathbb{R}} = H^k(X, \mathbb{R})$ carries a so-called *real Hodge structure of weight* k given by one of the following equivalent data:

i) A (Hodge) decomposition

$$H_{\mathbb{C}} := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

with $H^{p,q} = \overline{H}^{q,p}$.

ii) A Hodge filtration $F^p = \bigoplus_{i \geq p} H^{i,j}$, such that $H^{p,q} = F^p \cap \overline{F}^q$, $(p+q=k)$ and $H_{\mathbb{C}} = F^p \oplus \overline{F}^{q+1}$.

If H and H' are the real vector spaces which carry a (real) Hodge structure of weight k, resp. k', then it is easy to see that on H^*, $H \otimes H'$ and $\mathrm{Hom}(H, H')$ there is a natural Hodge structure of weight $-k$, $k+k'$ and $k'-k$. In particular $\mathrm{Hom}(H, H)$ has a Hodge structure of weight 0, and

$$\mathrm{Hom}(H,H)^{(a,b)} = \big\{\lambda : H \to H, \ \lambda(H^{p,q}) \subseteq H^{p+a,q+b}\big\}.$$

One can interpret a Hodge structure as a real representation of the real algebraic group

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) = \mathrm{Spec}\,\mathbb{R}\big[x, y, (x^2+y^2)^{-1}\big]$$

(restriction à la Weil of the algebraic group $\mathbb{C}^*$ of $\mathbb{C}$ to $\mathbb{R}$). This explains why one can perform the operations of duality and $\otimes$ on Hodge structures.

In the geometrical case, when X is Kähler, recall that on $H_{\mathbb{C}} = H^k(X, \mathbb{C})$ there is a bilinear form of parity $(-1)^k$, the *Hodge-Riemann form* ([**Dem**])

$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{n-k} \qquad (\dim X = n)$$

(Q is symmetrical if k is even, skew if k is odd). When the real $(1,1)$ form ω is integral, thus if X is projective algebraic, and ω comes from the class of a hyperplane section, Q is then integral on the lattice $H^k(X, \mathbb{Z})/(\text{torsion})$. Recall the *Hodge-Riemann bilinear relations*, the first of which reads:

(R1) $\qquad Q(H^{p,q}, H^{p',q'}) = 0$ except if $(p', q') = (q, p)$.

It is equivalent to say that the space Q-orthogonal to F^p is F^{k-p+1}. In fact, $F^p = \bigoplus_{i\geq p} H^{i,j}$, $F^{k-p+1} = \bigoplus_{i\geq k-p+1} H^{i,j}$. So, if $i \geq p$, one has $j = k-i \leq k-p$, hence $Q(F^\ell, F^{k-p+1}) = 0$. But

$$\dim F^p = \sum_{i\geq p} h^{i,j} = \sum_{i\geq p} h^{j,i} = \sum_{j\leq k-p} h^{j,i} = \operatorname{codim}(F^{k-p+1}) \ .$$

Thus $F^{k-p+1} = (F^p)^\perp$.

The second relation is

(R2) If $0 \neq \xi \in \mathrm{Prim}^{p,q}$, $\sqrt{-1}^{p-q} Q(\xi, \overline{\xi}) > 0$.

If one introduces the *Weil operator*, which is the real operator $C \in \mathrm{Aut}_{\mathbb{C}}(H_{\mathbb{C}})$, such that

$$C_{\big| H^{p,q}} = \sqrt{-1}^{p-q},$$

the form $\langle \alpha, \beta\rangle := Q(C\alpha, \bar{\beta})$ is a hermitian form called the *Hodge form*. The Hodge form is positive on the primitive part $\mathrm{Prim}^k(X, \mathbb{C})$ of $H^k(X, \mathbb{C})$.

A *polarized* Hodge structure of weight k consists of a real Hodge structure of weight k ($H_{\mathbb{R}}, H_{\mathbb{R}} \otimes \mathbb{C} = \oplus H^{p,q}$, $H^{p,q} = \overline{H}^{p,q}$) and of a *polarization*, i.e. a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$ equipped with a non-degenerate bilinear form of parity $(-1)^k$ on $H_{\mathbb{R}}$, which is integral on the lattice (but not necessarily unimodular)

$$Q(H_{\mathbb{Z}} \times H_{\mathbb{Z}}) \subseteq \mathbb{Z}.$$

One demands that the two Riemann conditions (R1) and (R2) hold. In particular

$$\langle \alpha, \alpha\rangle = Q(C\alpha, \bar{\alpha}) > 0 \quad (\alpha \neq 0)$$

with $C =$ the Weil operator.

An isomorphism of polarized Hodge structures is an isomorphism which preserves the polarizations, i.e. the integral structures and the bilinear forms.

Let now $f : X \to S$ be a family of projective manifolds. So $X \subset \mathbb{P}^N \times S$ and f is the restriction to X of the projection on S. Let $X_s = f^{-1}(s) = X \cap \mathbb{P}^N_{\mathbb{C}} \times \{s\}$. As explained in [**Dem**], X_s carries a real Hodge structure on each $H^k(X_s, \mathbb{C})$ and $\mathrm{Prim}^k(X_s, \mathbb{C})$ carries in addition a polarization defined by the form $\omega_s \in H^{1,1}(X_s, \mathbb{Z})$, deduced from the embedding $X_s \subset \mathbb{P}^N$.

These real (resp. polarized) Hodge structures define a family (or variation) of Hodge structures. One has in fact the following objects:

1. A local system of free abelian groups of finite (constant rank),

$$\mathcal{H}^k_{\mathbb{Z}} = R^k f_*(\mathbb{Z})/(\text{torsion}),$$

   idem with $\mathcal{H}^k_{\mathbb{R}}$, $\mathcal{H}^k_{\mathbb{C}}$).
2. A vector bundle (locally free $\mathcal{O}_S$ module) $\mathcal{H}^k = \mathbb{R}^k f_*(\Omega^\bullet_{X/S})$ ($\Omega^\bullet_{X/S}$ = algebraic, or holomorphic forms).
3. A decreasing filtration on $\mathcal{H}^k$ by holomorphic subbundles $\{\mathcal{F}^p\}_{p=0,\dots,k}$ (Hodge filtration) (if one passes to the fiber in $s \in S$, $\mathcal{H}^k_s = H^k(X_s, \mathbb{C})$ and $\mathcal{F}^p_s$ is exactly the Hodge filtration on $H^k(X_s, \mathbb{C})$). One has $\mathcal{F}^p \cap \overline{\mathcal{F}}^{q+1} = 0$, $(p+q=k)$.

   Let $\omega \in H^0(S, R^2 f_*(\mathbb{Z}))$ the image of the class of the relative hyperplane section (a locally constant section). The section ω induces at each $s \in S$, $\omega_s \in H^2(X_s, \mathbb{Z})$, the integral form $(1,1)$ which polarizes X_s. One has then

4. A locally constant non-degenerate bilinear form $Q : \mathcal{H}^k_{\mathbb{Z}} \otimes \mathcal{H}^k_{\mathbb{Z}} \longrightarrow \mathcal{H}^{2n}_{\mathbb{Z}} = \mathbb{Z}$ ("the Hodge-Riemann form").
5. An (integrable) connection $\nabla : \mathcal{H}^k \to \Omega^1_S \otimes \mathcal{H}^k$: the Gauss-Manin connection, such that the local system of its horizontal sections is $\mathcal{H}^k_{\mathbb{C}}$.
6. Griffiths' transversality property
$$\nabla(\mathcal{F}^p) \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}.$$
7. The Lefschetz operator L admits a global form; L is the cup product with ω. Observe that L is a horizontal operator and one defines $\mathcal{H}^k_{\text{prim}}$ as the kernel of
$$L^{n-k+1} : \mathcal{H}^k \longrightarrow \mathcal{H}^{2n-k+2}.$$
The fiber above $s \in S$ of $\mathcal{H}^k_{\text{prim}}$ is $\text{Prim}^k(X_s, \mathbb{C})$.

One gathers all of these data into the following definition of (polarized) variation of Hodge structures ("VHS" in short). The following definition has been formulated by Griffiths ([**Grif1**]).

3.1. Definition. A family of (real) Hodge structures of weight k, on S, consists of

1. A locally constant sheaf of real vector spaces $\mathcal{H}_{\mathbb{R}}$ on S.
2. A finite filtration $\{\mathcal{F}^p\}$ on the vector bundle $\mathcal{H} := \mathcal{H}_{\mathbb{R}} \otimes \mathcal{O}_S$ ($\mathcal{F}^p$ is a holomorphic subbundle).

With the conditions

(VHS-1) The natural connection $\nabla = 1 \otimes d_S$ on $\mathcal{H}$ is such that $\nabla \mathcal{F}^p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$

(VHS-2) For any point $s \in S$, $\{\mathcal{F}^p_s\}$ defines a (real) Hodge structure of weight k on $\mathcal{H}_{\mathbb{R}})_s$.

A polarization consists in addition, of a locally constant sheaf $\mathcal{H}_{\mathbb{Z}} \subseteq \mathcal{H}_{\mathbb{R}}$, of free $\mathbb{Z}$-modules of finite rank, with $\mathcal{H}_{\mathbb{R}} \equiv \mathcal{H}_{\mathbb{Z}} \otimes \mathbb{R}$, and a locally constant non-degenerate bilinear form
$$Q : \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \longrightarrow \mathbb{Z}$$
which for any $s \in S$ induces a polarization on $(\mathcal{H}_{\mathbb{R}})_s$.

We shall only consider polarized Hodge structures.

3.2. Definition. Let $\{H_{\mathbb{Z}}, \{\mathcal{F}^p\}, \nabla, Q\}$ be a variation of polarized Hodge structures on S. The monodromy group of the locally constant sheaf $H_{\mathbb{Z}}$ is called the monodromy group of the variation (VHS).

To define this group, one fixes $s_0 \in S$ and following the prescription of §2.A one considers the monodromy representation
$$T : \pi_1(S, s_0) \longrightarrow \text{Aut}_{\mathbb{Z}}((H_{\mathbb{Z}})_{s_0}).$$
From the fact that the form Q is (locally) constant, the image of T (i.e. the monodromy group) is included in the orthogonal group $G_{\mathbb{Z}} := \text{Aut}_{\mathbb{Z}}((H_{\mathbb{Z}})_{s_0}, Q)$.

Recall that the locally constant sheaf $\mathcal{H}_{\mathbb{Z}}$ is obtained as
$$\mathcal{H}_{\mathbb{Z}} = \widetilde{S} \times H_{\mathbb{Z}} / \pi_1(S, s_0) \ (H_{\mathbb{Z}} = (\mathcal{H}_{\mathbb{Z}})_{s_0})$$
where $\pi_1(S, s_0)$ acts as $\gamma(t, \alpha) = (t\gamma, T(\gamma)^{-1}\alpha)$. The monodromy representation describes how a local section of $\mathcal{H}_{\mathbb{Z}}$ changes under analytic continuation along a loop. As one assumes S to be connected, all the fibers of $\mathcal{H}_{\mathbb{Z}}$ are isomorphic to $H_{\mathbb{Z}} = (\mathcal{H}_{\mathbb{Z}})_{s_0}$ but not canonically.

§3.B. Griffiths' period domain.

It is natural to describe "the set of Hodge structures" on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$ polarized by the form Q on $H_{\mathbb{Z}}$ and with fixed Hodge numbers $h^{p,q}$. One fixes thus $H_{\mathbb{Z}} \cong \mathbb{Z}^n$, an abelian group, Q a form (skew or symmetrical according to the parity of the weight k) which is integral and non-degenerate on $H_{\mathbb{Z}}$. Further, one fixes Hodge numbers $h^{p,q}$ $(= h^{q,p})$, with $\sum\limits_{p+q=k} h^{p,q} = n$. Note that then $\dim F^p = \sum\limits_{i \geq p, i+j=k} h^{i,j}$ is fixed $(= f^p)$. One denotes by $\mathrm{Gr}(k, H_{\mathbb{C}})$ the Grassmannian of subspaces of dimension k of $H_{\mathbb{C}}$. Recall the Riemann relations:

1. The subspace of $H_{\mathbb{C}}$ Q-orthogonal to F^p is F^{k-p+1} and
2. If C is the Weil operator with $C(\xi) = \sqrt{-1}^{p-q}\xi$ for $\xi \in H^{p,q}$, one has $Q(C\xi, \bar{\xi}) > 0$ if $0 \neq \xi$.

3.3. Notation.

- $\check{D} = \{$filtrations $F^\bullet = \{F^p\}_{p=0,\dots,h}$ of $H = H_{\mathbb{C}}$, such that $\dim F^p = f^p$, and for any p, $Q(F^p, F^{k-p+1}) = 0\}$. (Then F^{k-p+1} is the space Q-orthogonal to F^p).
- D denotes the subset of $\check{D}$ consisting of Hodge structures, i.e. satisfying condition 2 above.
- Let $G_{\mathbb{R}}$ be the orthogonal group of $(H_{\mathbb{R}}, Q)$ and $G_{\mathbb{C}}$ its complexification, $G_{\mathbb{C}} = O(H_{\mathbb{C}}, Q)$.

3.4. Proposition.

1. *$\check{D}$ is a non singular submanifold of $\prod\limits_p \mathrm{Gr}(f^p, H)$, which is in fact a homogeneous space under the complex Lie group $G_{\mathbb{C}}$.*

$$\check{D} = G_{\mathbb{C}}/B, \ (B \text{ parabolic subgroup})$$

2. *D is open in $\check{D}$, an orbit of the real Lie group $G_{\mathbb{R}}$:*

$$D = G_{\mathbb{R}}/V \ (V = G_{\mathbb{R}} \cap B).$$

Proof. The proof not is not too difficult, it is an exercise in linear algebra (use Witt's theorem for example). Note that D gets the structure of complex manifold (open in $\check{D}$). It is often convenient to fix an initial Hodge structure $\{H_0^{p,q}\}$, and then one can identify $\check{D}$ with $G_{\mathbb{C}}/B$, where B is the stabilizer of $\{F_0^p\}$, and V the stabilizer of $\{H_0^{p,q}\}$ in $G_{\mathbb{R}}$. □

It is important to describe the tangent bundle to $\check{D}$ as well as the universal sub bundles $\mathcal{F}^p$ of the trivial bundle $H \otimes \mathcal{O}_{\check{D}}$ as homogeneous vector bundles on these homogeneous spaces.

Recall that if $F \subset H$ is a subspace of dimension d, the tangent space of $\mathrm{Gr}(d, H)$ at $[F]$ can be canonically identified with $\mathrm{Hom}(F, H/F)$. Hence, a tangent vector of $\check{D}$ at the point $F^\bullet = \{F^p\}$ may be identified with a collection of linear maps

$$\xi_p : F^p \longrightarrow H/F^p$$

fitting into commutative diagrams

$$\begin{array}{ccc} F^p & \xrightarrow{\xi_p} & H/F^p \\ \uparrow & & \uparrow \\ F^{p+1} & \xrightarrow{\xi_{p+1}} & H/F^{p+1} \end{array}$$

such that moreover

$$Q(\xi_p(\alpha), \beta) + Q(\alpha, \xi_{k-p+1}(\beta)) = 0 \qquad (\alpha \in F^p,\ \beta \in F^{k-p+1})$$

[infinitesimal version of the first bilinear relation]. Recall that we have chosen a base point $\{H_0^{p,q}\} \in D$. Let

$$\mathfrak{g} = \operatorname{Lie}(G_{\mathbb{C}}) = \{X \in \operatorname{End}(H), Q(X(\alpha), \beta) + Q(\alpha, X(\beta)) = 0\}.$$

One has a Hodge structure of weight 0 on $\mathfrak{g}$, with

$$\mathfrak{g}^{r,s} = \{X \in \mathfrak{g},\ X(H^{p,q}) \subseteq H^{p+q,q+s}\}.$$

Then $\mathfrak{b} = \operatorname{Lie}(B) = \bigoplus_{r \geq 0} \mathfrak{g}^{r,-r}$, and the tangent bundle $T_{\check{D}}$ is the homogeneous bundle $G_{\mathbb{C}} \times_B \mathfrak{g}/\mathfrak{b}$, B acting by the adjoint representation. Let $\mathfrak{g}_0 = \operatorname{Lie}(G_{\mathbb{R}})$, then $\mathfrak{v} = \operatorname{Lie}(V) = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. Observe that $\mathfrak{g}_0/\mathfrak{v} \cong \mathfrak{g}/\mathfrak{b}$, which corresponds to the fact that the open subset $D \subset \check{D}$ is an orbit of $G_{\mathbb{R}}$.

3.5. Definition. [**Grif1**] The subbundle $G_{\mathbb{C}} \times_B \mathfrak{g}^{-1,1}/\mathfrak{b}$ of $T_{\check{D}}$ is called the *horizontal subbundle* (notation $T_{\mathrm{hor}}(\check{D})$). A tangent vector $\xi = \{\xi_p\}$ is *horizontal* if $\xi_p(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1}/\mathcal{F}^p$.

To describe the universal bundle $\mathcal{F}^p \subset H \otimes \mathcal{O}_{\check{D}}$ in terms of bundles associated to principal bundles, remark that the trivial bundle $H \otimes \mathcal{O}_{\check{D}}$ (with fiber H and base $\check{D}$) is the bundle $G_{\mathbb{C}} \times_B H$, and $\mathcal{F}^p = G_{\mathbb{C}} \times_B \mathcal{F}_0^p$ (by definition B is the stabilizer of the filtration $\{\mathcal{F}_0^p\}$.

3.6. Example (Siegel's upper half space). Let $k = 1$ (weight a). The Hodge filtration reduces to $H = F^0 \supset F^1 \supset 0$, with $(F^1)^\perp = F^1$ (for the skew form Q). Then $\check{D}$ is the Lagrangian Grassmannian of (H, Q), and $\check{D} = \operatorname{Sp}(2g, \mathbb{C})/B$ if $\dim H = 2g$. It is a classical fact (and easy to check) that $D = \operatorname{Sp}(2g, \mathbb{R})/U(g, \mathbb{C})$ can be identified with Siegel's upper half space $\{\tau \in M_g(\mathbb{C}), {}^t\tau = \tau \text{ and } \operatorname{Im}\tau > 0\}$.

Let there be given a VHS $\{\mathcal{H}_{\mathbb{Z}}, \mathcal{F}^p, \nabla, Q\}$. It does not directly give a morphism $S \to D$, because $\mathcal{H}_{\mathbb{Z}}$ is only locally constant. However, locally on an open subset $\mathcal{U}$ of S one can trivialize the vector bundle $\mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_S$, by means of flat sections (of $\mathcal{H}_{\mathbb{Z}}$) and then the filtration induced by the $\mathcal{F}^p$ yields a (holomorphic) morphism

$$\Phi : \mathcal{U} \to D \subset \check{D}.$$

Globally, one can transport the VHS to the universal covering $\widetilde{S}$ of S, and the choice of a trivialization of the local system $\mathcal{H}_{\mathbb{Z}}$ on $\widetilde{S}$, leads to a morphism $\tilde{\Phi} : \widetilde{S} \to D \subset \check{D}$.

One sees immediately that the (global) monodromy group Γ acts properly discontinuously on D. One gets the *period map* after taking the quotient by Γ:

$$\Phi : S \longrightarrow \Gamma \backslash D.$$

An important property of $\tilde{\Phi}$ is that it is horizontal, which is the translation of the condition of Griffiths' transversality for ∇: for any $s \in S$

$$d\Phi_s(T_{S,0}) \subseteq T_{\mathrm{hor},\Phi(s)}(D). \tag{4}$$

To make this property clear, let us make it explicit on a neighborhood of $s_0 \in S$. Trivialize the vector bundle $\mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_S$ by means of flat sections $\{\tau_i\}$. One may suppose that $H_0^{p,q}$ $(= H^{p,q}(s_0))$ has a basis $(\tau_i)_{f_{p+1}<i\le f_p}$, and then a basis for $\mathcal{F}_0^p$ is $\{\tau_i\}_{i\le f_p}$. One can find a local trivialization $\{e_i(s)\}$ of $\mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_S$ with $e_i(s_0) = \tau_i$, and such that locally $(e_1, \dots, e_{f_p})$ is a basis of $\mathcal{F}^p$. Let $e_i = \sum A_{ji}\tau_j$ be the matrix of change of frames (we abbreviate $s_0 = 0$). The section τ_i being flat, we have $\nabla e_i = \sum dA_{ji}\tau_j = \sum_k c_{ki}e_k$, with $c = A^{-1}dA$. Transversality implies that $c_{ji} = 0$ for $i \le f_p$, $j > f_{p+1}$. The linear map ξ_α associated to $d\Phi_{s_0}(\partial/\partial s_\alpha)$ is such that for $i \le f_p$ one has

$$\begin{aligned}\xi_\alpha(\tau_i) &= \frac{\partial e_i}{\partial s_\alpha}(0) \pmod{\mathcal{F}_0^p}\\ &= \sum_{f_{p+1}<j} \frac{\partial A_{ji}}{\partial s_\alpha}(0)\tau_j.\end{aligned}$$

As

$$\frac{\partial A_{ji}(0)}{\partial s_\alpha} = \left\langle c_{ji}(0), \frac{\partial}{\partial s_\alpha}\right\rangle,$$

one has indeed (4).

For later use, let us indicate the following property, consequence of the vanishing of the curvature of ∇.

3.7. Proposition. *If* $\partial_1, \partial_2 \in T_{S,s}$, $\xi_i = d\Phi_s(\partial_i) \in \mathfrak{g}^{-1,1}$, *then* $[\xi_1, \xi_2] = 0$ (*bracket in* $\mathfrak{g}$).

Proof. Recall the formula (1) of section 2. It implies that it suffices to show that $[\partial_1, \partial_2] = 0$ seen as endomorphism of $H_0^{p,q}$. Thus one may suppose $\xi_1 = \partial/\partial s_\alpha$, $\xi_2 = \partial/\partial s_\beta$. Set $c_{ji}^\alpha = \langle c_{ji}, \partial/\partial s_\alpha\rangle$, then $c_{ji}^\alpha(0) = \partial A_{ji}/\partial s_\alpha\big|_0$. It suffices to show that $\partial c_{ji}^\alpha/\partial s_\beta - \partial c_{ji}^\beta/\partial s_\beta$ is annihilated in 0 for $f_{p+1} < i, j \le f_p$. We have seen that

$$c_{ji}^\alpha = \sum_k A_{jk}^{-1}\frac{\partial A_{ki}}{\partial s_\alpha} = 0 \text{ if } i \le f_p,\ j > f_{p+1}\ .$$

Differentiating with respect to s_β, and then evaluating at 0, one gets for i, j in the interval (f_{p+1}, f_p):

$$\begin{aligned}\frac{\partial^2 A_{ji}}{\partial s_\beta \partial s_\alpha}(0) &= \sum_k \frac{\partial A_{jk}}{\partial s_\alpha}(0)\frac{\partial A_{ki}}{\partial s_\beta}(0)\\ &= \sum_k \frac{\partial A_{jk}}{\partial s_\beta}(0)\frac{\partial A_{ki}}{\partial s_\alpha}(0)\end{aligned}$$

which yields the result. □

§3.C. Deformations and IVHS (Infinitesimal variations of Hodge structures).

A natural question at this stage is whether the variation of complex structure is determined by its variation of Hodge structures (*Torelli problem*). It is clear that this is false in general: take the product family. Thus one restricts to families for which the complex structure varies truly, at least infinitesimally. The infinitesimal complex variation being measured by the Kodaira-Spencer map, one only considers families having the property that the Kodaira-Spencer map is everywhere injective. These are the *effective families.* The variation of Hodge structures at the infinitesimal level is described by the derivative of the period map. An infinitesimal version of the Torelli problem is whether this derivative is injective for effective families. See [**P-S**] for this circle of ideas. Let us complete the discussion in making precise the notion of *universal* or *versal deformation.* Here one fixes a manifold X_o, one works with families on a pointed base manifold (S, o), considered as germ, such that the fiber above o is the fixed manifold X_o. These types of families are called *deformations of* X_o. Such a deformation $f : X \to S$ is *complete* if any other deformation $g : Y \to T$ of X_o can be obtained from $f : X \to S$ by a change of basis (at the level of germs)

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{p} & S \end{array}$$

where $p : (T, o) \to (S, o)$, and the above square is Cartesian. If the morphism p is unique, X/S is called *universal.* In general this not the case, but often the derivative $dp(o)$ is unique and in this case the deformation X/S is called *versal.* For example, if X/S is complete, to obtain a versal deformation, one may restrict the family to a suitable submanifold which passes through o. Warning: a deformation can very well be (uni)versal at $o \in S$ but can fail to be so at other points of S. Kodaira and Spencer have shown [**K-S2**] that f is complete if and only if the Kodaira-Spencer map is a surjection.

3.8. CONSEQUENCE. *$f : X \to S$ is versal at o if and only if the Kodaira-Spencer map*

$$\rho : T_{S,o} \to H^1(T_{X_o})$$

is an isomorphism.

In this situation, $\dim S$ is considered to be the number of parameters for the complex structure).

Although this result has been ameliorated by Kuranishi, Kodaira's result suffices for many examples, notably for hypersurfaces in projective space, as in the following example. The result of Kuranishi says that there is always a versal family, provided that one allows analytic spaces as possible base space S (it is essential to allow nilpotent elements in the structure sheaf of S). In this framework, a family $f : X \to S$ is a holomorphic and proper map such that, if one replaces S by a smaller set, locally X is a product $U_i \times S$ and $f|U_i \to S$ coincides with the projection on the second factor. See [**Ku**] for details.

3.9. EXAMPLE. If $f : X \to S$ is the tautological family of hypersurfaces of degree d in $\mathbb{P}^{n+1}$, the Kodaira-Spencer map is a surjection if $n \geq 2$ or $n = 2$ and $d \neq 4$. To obtain a versal deformation one has to restrict this family to a small disk transversal to a $\mathrm{PGL}(n+1, \mathbb{C})$-orbit of a fixed hypersurface. For details see [**K-S**], §14(C).

Let us come back to the derivative of the period map for a versal family $f : X \to S$. Fix $o \in S$ and consider the fiber X_o above the point o and the derivative

$\delta = d\Phi(o)$ at o. This is a linear map

$$\delta : T_{S,o} \longrightarrow \mathfrak{g}^{-1,1} \subset \bigoplus_p \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$$

with the following properties (3.7):
(1) *For any tangent vector t in o, $\delta(t) \in \mathfrak{g} = \operatorname{Lie}(G_{\mathbb{C}})$.*
(2) *If $t_1, t_2 \in T_{S,o}$, the endomorphisms $\delta(t_1)$ and $\delta(t_2)$ commute.*

3.10. DEFINITION. Let there be given a (real) Hodge structure on H. The (linear algebra) data (T, H, δ, Q)

$$\delta : T \longrightarrow \mathfrak{g}^{-1,1}$$

which satisfies (1) and (2) is called (by Griffiths and Harris [**C-G-G-H**]), an *infinitesimal variation of Hodge structures* (IVHS).

For a geometrical IVHS, the linear map induced by

$$\delta(t) : H^{p,q} \to H^{p-1,q+1}$$

is cup product with the image of t under the Kodaira-Spencer map $T_{S,o} \to H^1(X_0, T_{X_0})$ (cf. §2B and §2C).

Starting with a IVHS, one can perform linear algebra operations. For example, if $t_1, \dots, t_k \in T_{S,o} = T$, one can compose the maps $\delta(t_1), \dots, \delta(t_k)$, which yields a map

$$H^{k,0} \xrightarrow{\delta(t_1)} H^{k-1,1} \xrightarrow{\delta(t_2)} H^{k-2,2} \longrightarrow \cdots \xrightarrow{\delta(t_k)} H^{0,k}.$$

Denote the result by $\delta(t_1, \dots, t_k) \in \operatorname{Hom}(H^{k,0}, H^{0,k})$. Recall that $H^{0,k}$ and $H^{k,0}$ are each others dual under Q. Then the properties (1) and (2) lead easily to the following results:

1. $\delta(t_1, \dots, t_k)$ is a symmetric bilinear form on $H^{k,0}$
2. $\delta(t_1, \dots, t_k)$ is symmetric in the arguments $t_1, \dots, t_k$.

Hence a linear map

$$\delta : \operatorname{Sym}^k(T) \longrightarrow \operatorname{Hom}_{\operatorname{Sym}}(H^{k,0}, H^{0,k}) \cong \operatorname{Sym}^2(H^{k,0}) \tag{5}$$

which, as one can expect, contains significant information about X_0.

§4. Degenerations

In this section we introduce the notion of a mixed Hodge structure. Next we consider families with base a punctured disk, deduced from a proper morphism over the disk by deleting the fiber above of the origin. Such situation is called a degeneration, because the fiber above the origin can be singular. In this situation, turning once around the origin induces in cohomology the Picard-Lefschetz or local monodromy operator. This map is a quasi-unipotent, a fundamental property which is discussed briefly in §4.B. Finally, in §4.C we define the nearby and vanishing cycles, notions which we need to understand the recent developments concerning local monodromy.

§4.A. Mixed Hodge structures.

Let $H_{\mathbb{Q}}$ be a $\mathbb{Q}$-space vector of finite dimension equipped with an increasing filtration $W_\bullet$

$$\cdots W_k \subset W_{k+1} \subset W_{k+2} \cdots$$

Let us assume that there is a decreasing filtration $F^\bullet$

$$\cdots F^k \subset F^{k-1} \subset F^{k-2} \cdots$$

on $H = H_{\mathbb{Q}} \otimes \mathbb{C}$. These two filtrations define *a mixed Hodge structure* if the filtration induced by $F^\bullet$ on $\mathrm{Gr}_\ell^W = W_\ell/W_{\ell-1}$ is a pure Hodge structure of weight ℓ. The induced filtration is $F^p(\mathrm{Gr}_\ell^W) = W_\ell \cap F^p/W_{\ell-1} \cap F^p$. The *Hodge numbers* are the Hodge numbers of Gr_ℓ^W. Thus $h^{p,q} = h^{q,p}$ but in general there are non zero Hodge numbers for different values of $p+q$. If one can find a bigrading $H = \bigoplus H^{p,q}$ such that $W_\ell \otimes \mathbb{R} = \sum_{r+s\leq\ell} H^{r,s}$ and $F^p = \sum_{r\geq p} H^{r,s}$ one says that the mixed structure is *split*. Deligne has found (see [**C-K-S1**]) a canonical splitting

$$I^{a,b} = F^p \cap W_{a+b} \cap ((\bar{F}^b \cap W_{a+b}) + \overline{G}^{b-1}_{a+b-2}), \quad \text{with } G^p_q := \sum_{j\geq 0} F^{p-j} \cap W_{q-j}$$

and thus

$$W_\ell = \bigoplus_{a+b\leq\ell} I^{a,b}, \; F^p = \bigoplus_{a\geq p} I^{a,b}.$$

Warning: although $h^{a,b} = h^{b,a}$, in general it is not the case that $I^{b,a} = \overline{I}^{a,b}$, but if this symmetry property holds, we say that the splitting is defined over $\mathbb{R}$. One can always "deform" a mixed structure defined over $\mathbb{R}$ (i.e. in the definition of mixed structure one starts with an $\mathbb{R}$-space vector) into a real split mixed Hodge structure. In this case $I^{a,b} = F^a \cap \bar{F}^b \cap W_{a+b}$. In [**Dem**] it is shown that the cohomology group $H^w(X,\mathbb{Z})$ of a compact Kähler manifold X carries a pure Hodge structure of weight w. In particular this applies to complex projective manifolds.

Deligne has proved [**Del4**], [**Del5**], that $H^w(X,\mathbb{Z})$ carries a mixed Hodge structure which depends functorially on X for any quasi-projective, possibly singular variety X, in fact for any scheme of finite type over $\mathbb{C}$. The Hodge numbers of this structure can be shown to be non-zero at most in the range $0 \leq p,q \leq w$, and in the more restricted range $w-n \leq p,q \leq n$ if $w \geq n = \dim X$. If X is smooth, one only has weights $\geq w$ ($h^{p,q} = 0$ if $p+q < w$); however for X proper, there are only weights $\leq w$. Of course if X is smooth and projective, the mixed Hodge structure on $H^w(X,\mathbb{Z})$ reduces to the classical pure Hodge structure of weight w.

Let us indicate where the mixed Hodge structure comes from when X is a smooth quasi-projective $\mathbb{C}$-scheme (see [**Del4**] for the details). One first compactifies X, i.e. realizes X as the complement $X = \overline{X} \setminus D$ of a normal crossings divisor ([**Ill**], §7). Such a compactification exists; to simplify the discussion, we assume that D is the union of nonsingular divisors which cross transversally. We work in the analytic framework; the forms are thus holomorphic. The ordinary De Rham theorem, which says that $H^w(X,\mathbb{C}) = \mathbb{H}^w(X,\Omega^\bullet_X)$ does not suffice. If $j : X \hookrightarrow \overline{X}$ is the inclusion note that $H^w(X,\mathbb{C}) = \mathbb{H}^w(\overline{X}, j_*\Omega^\bullet_X)$. Let $\Omega^\bullet_{\overline{X}}(\log D)$ be the subcomplex of $j_*\Omega^\bullet_X$, whose the sections are the meromorphic forms on $\overline{X}$ that are holomorphic on X and which have logarithmic poles along D. Recall [**Ill**] that a section of $\Omega^1_{\overline{X}}(\log D)$ defined at $x \in D$, is a linear combination $\{\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n\}$ if $(z_1,\ldots,z_n)$ is a system of local coordinates around x such that $z_1 \cdot \ldots \cdot z_k = 0$ is a local equation of D. Set $\Omega^p_{\overline{X}}(\log D) = \wedge^p \Omega^1_{\overline{X}}(\log D)$. One may check that $\Omega^\bullet_{\overline{X}}$ is the smallest subcomplex of $j_*\Omega^\bullet_X$ containing $\Omega^\bullet_{\overline{X}}$ and the logarithmic differentials $\frac{df}{f}$ for any local section which is meromorphic along D. The logarithmic version of the De Rham theorem then says that the two complexes $\Omega^\bullet_{\overline{X}}(\log D)$ and $j_*\Omega^\bullet_X$ are quasi-isomorphic and hence

$$H^w(X,\mathbb{C}) = \mathbb{H}^w(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D).$$

For D is a smooth hypersurface the reader can find a proof in §8.

As in §1, the Hodge filtration $F^\bullet$ leads to a spectral sequence

$$E_1^{p,q} = H^q(\overline{X}, \Omega^p_{\overline{X}}(\log D)) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

and to a Hodge filtration $F^p H^w(X, \mathbb{C})$ on the limit. Note that the coherent algebraic sheaves $\Omega^p_{\overline{X}}(\log D)$ can be substituted for their analytic analogs ([**Ill**], §7).

The complex $\Omega^\bullet_{\overline{X}}(\log D)$ admits a second filtration, the weight filtration $W_\bullet$ (an increasing filtration); W_m is the image of the exterior product map

$$\Omega^m_{\overline{X}}(\log D) \otimes \Omega^\bullet_{\overline{X}}[-m] \longrightarrow \Omega^\bullet_{\overline{X}}(\log D).$$

If one sets $W^m = W_{-m}$ (making it into a decreasing filtration), one can thus consider the spectral sequence associated to the filtered complex $(\Omega^\bullet_{\overline{X}}(\log D), W^\bullet)$, let

$$E_1^{-n,w+n} = \mathbb{H}^w(\overline{X}, \mathrm{Gr}^w_n(\Omega^\bullet_{\overline{X}}(\log D))) \Longrightarrow H^w(X, \mathbb{C}).$$

Assume that $D_1, \ldots, D_r$ are the components of D. It is not hard to see that the Poincaré "residue" operation furnishes an isomorphism (see §8)

$$\mathrm{Gr}^W_n(\Omega^\bullet_{\overline{X}}(\log D)) = \begin{cases} 0 & \text{if } n < 0 \\ \Omega^\bullet_{\overline{X}} & if \text{n=0} \\ \oplus_{1 \le i_1 \le \cdots \le i_n \le r} \Omega^\bullet_{D_{i_1} \cap \cdots \cap D_{i_n}}[-n] & \text{if } n \ge 1 \end{cases}$$

On $H^w(X, \mathbb{C})$ one has thus two filtrations $W_\bullet$ and $F^\bullet$. We then study how these filtrations live together. The E_1 term of the spectral sequence reads

$$E_1^{-n,w+n} = \bigoplus_{1 \le i_1 \le \cdots \le i_n \le r} H^{w-n}(D_{i_1} \cap \cdots \cap D_{i_n}, \mathbb{C}).$$

The Hodge filtration induces on this term a pure Hodge structure of weight $w+n$, which is derived from the one of weight $w-n$ on each group $H^{w-n}(D_{i_1} \cap \cdots \cap D_{i_n}, \mathbb{C})$ by means of a shift. Then one can show inductively by a rather delicate analysis, that the derivative d_r is zero for $r \ge 2$, in particular

$$E_2^{p,q} = \cdots = E_\infty^{p,q} = \mathrm{Gr}^W_p(H^{p+q}(X, \mathbb{C})).$$

We may then easily conclude that the filtration $F^\bullet$ on $W_n/W_{n-1} = \mathrm{Gr}^W_n(H^w(X, \mathbb{C}))$ yields a pure Hodge structure of weight $w+n$. Then the shifted filtration $W_\bullet[w]$ together with the filtration $F^\bullet$ define on $H^w(X)$ a mixed Hodge structure. The reader can find in [**Del4**] an explanation why $W_\bullet$ is in fact defined over $\mathbb{Q}$, and also why the result is independent of the compactification. As an example let us regard the case of a smooth hypersurface $D \subset \overline{X}$ and $X = \overline{X} \setminus D$. Keeping account of the shift, the weight filtration on $H^w(X, \mathbb{C})$ is $0 \subset W_w \subset W_{w+1} = H^w(X, \mathbb{C})$. One has $W_w = \mathrm{Im}[H^w(\overline{X}, \mathbb{C}) \to H^w(X, \mathbb{C})]$. To interpret the quotient W_{w+1}/W_w, consider the derivative d_1 of the spectral sequence

$$\begin{array}{ccccccc} 0 & \to & E_1^{-1,w+1} & \xrightarrow{d_1} & E_1^{0,w-1} & \to & 0 \\ & & \| & & \| & & \\ & & H^{w-1}(D) & & H^{w+1}(X) & & \end{array}$$

Degeneration at E_2 is here clear. We have

$$E_2^{-1,w+1} = \ker(d_1) = \cdots = E_\infty^{-1,w+1} = W_{w+1}/W_w.$$

We shall see in §7 that the derivative d_1 can be interpreted by means of the Gysin exact sequence

$$\cdots H^w(\overline{X}) \to H^w(X) \to H^{w-1}(D) \xrightarrow{\partial} H^{w+1}(\overline{X}) \to \dots$$

Here, $\partial = d_1$ being of bidegree $(1,1)$, the proof of Deligne's theorem is immediate.

Let us specify this example even more, by by taking for $\overline{X}$ a complete smooth curve of genus g and for D a set of n points. The above Gysin sequence shows that $W_1 \cong H^1(\overline{X})$ carries a pure Hodge structure of weight 1 and that $W_2/W_1 \cong \ker(H^0(D) \to H^2(\overline{X}))$ is of rank $n-1$ with a pure Hodge structure of weight 2, with only one term of type $(1,1)$ (just as for $H^2(\overline{X})$), which corresponds to the fact that $b_1(X) = g + n - 1$.

The method of Deligne yields also a mixed structure on the cohomology of Kähler manifolds admitting Kähler compactifications. In another direction, cohomology with compact support as well as Borel-Moore homology (of a separated scheme over $\mathbb{C}$ or of a Kähler manifold admitting a Kähler compactification) carry also mixed Hodge structures.

§4.B. Limit Structures.

Consider the situation of a degeneration $f : X \to \Delta$, i.e. a proper and holomorphic map of a complex manifold$X$ to the disk Δ such that f is smooth outside the origin. Let

$$\mathfrak{h} \to \Delta^*, \quad \tau \mapsto s = \exp(2\pi \mathbf{i} \tau)$$

be the universal covering of the punctured disk. Let $\tilde{X} = X \times_{\Delta^*} \mathfrak{h}$ be the product bundle and let $k : \tilde{X} \to X$ be the natural map. The map $h : \tilde{X} \to \tilde{X}$, $h(x,\tau) = (x, \tau+1)$ induces the monodromy operation T on the cohomology groups $H^k(X_s)$ ($s = \exp(2\pi\mathbf{i}\tau)$ and $X_s = f^{-1}(s)$). In the case of a geometrical VHS, $H_{\mathbb{Z}} = R^k f_*(\mathbb{Z})_{s_0} = H^k(X_{s_0}, \mathbb{Z})/\text{torsion}$, T is the *Picard-Lefschetz transformation.* A fundamental property of T is

4.1 THEOREM. *(*[**La**]*) The map T is quasi-unipotent, i.e.. $(T^\ell - \mathbf{1})$ is nilpotent for suitable $\ell \in \mathbb{N}$; in fact, the index of nilpotency is $\leq k+1$ so that $(T^\ell - \mathbf{1})^{k+1} = 0$* (Local Monodromy Theorem)

For abstract variations this theorem has been proved by Schmid in [**S**]. The local monodromy theorem without the bound on the index of nilpotency results (according to an idea of Borel) from curvature properties of the period domain. We sketch the argument given in [**S**].

Recall the notion of *sectional curvature* of a hermitian metric h on a complex manifold M. Let F_h be the curvature (see [**Dem**], §1) of the metric connection on the holomorphic tangent bundle $T(M)$. The *sectional curvature* is the function $\kappa : T(M) \setminus \{\text{zero-section}\} \to \mathbb{C}$ given by

$$\kappa(v) = \frac{h\left(F_h(v, \overline{v})v, v\right)}{h(v,v)^2}.$$

4.2 EXAMPLE. Assume that $\dim M = 1$. Then $F_h = \overline{\partial}\partial \log(h)$, where $\omega = \frac{\mathbf{i}}{2} h \; dz \wedge d\overline{z}$ is the form associated to the metric. We check easily that $\kappa(\partial/\partial z)$ is the Gaussian curvature $K_h = -h^{-1} \cdot \partial^2/\partial z \partial \overline{z}(\log h)$. This result can be written as follows:

$$\frac{1}{\mathbf{i}} \text{ curvature } h = - \text{ Gaussian curvature of the metric } h.$$

Particular cases:

1. Let Δ be the unit disk with the Poincaré metric
$$h = \frac{1}{(1-|z|^2)^2} dz \otimes d\overline{z}.$$
We find
$$\kappa(\partial/\partial z) \equiv -1.$$
2. The upper half plane $\mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, with the metric
$$h = \frac{1}{|\operatorname{Im} z|^2} dz \otimes d\overline{z}.$$
The Gaussian curvature is equal to -1.
3. The punctured disk $\Delta^* = \{\zeta \in \mathbb{C} : |\zeta| < 1,\ \zeta \neq 0\}$ admit $\mathfrak{h}$ as universal covering. The Poincaré metric is invariant by translation and thus induces a metric on Δ^*
$$\frac{1}{|\xi|^2(\log|\xi|^2)^2} d\xi \otimes d\overline{\xi}$$
with Gaussian curvature -1.

LEMMA (AHLFORS-SCHWARZ). *A holomorphic map $f : \Delta \to M$ of the unit disk to a complex manifold equipped with a hermitian metric h having the property that $f(\Delta)$ is tangent to the directions in which the curvature κ satisfies $\kappa \leq -1$, then, with ω_h the form associated to h and ω_Δ the form associated to the Poincaré metric one has*
$$f^*\omega_h \leq \omega_\Delta,$$
i.e. f is distance decreasing.

PROOF. The assumption of the lemma says that the sectional curvature calculated in the direction of $f(\Delta)$ is estimated from above by -1. We know that the curvature decreases in subbundles (see [**Grif4**], Chapt. II) and thus the sectional curvature calculated with the metric induced on T_M is bounded by -1. Recall that the *Ricci form* of a metric with form ω_N on a manifold N of dimension 1 is given by
$$\operatorname{Ric} \omega_N = \frac{1}{2} \mathbf{i} \partial\overline{\partial} \log h = -K\omega_N,$$
where K denotes the Gaussian curvature. Thus $f^*\omega_h \leq \operatorname{Ric} f^*\omega_h$ and it suffices to show that $\operatorname{Ric} f^*\omega_h \leq \omega_\Delta$.

Consider a smaller disk of radius r and let
$$\eta_r = \frac{\mathbf{i} \cdot r^2 dz \wedge d\overline{z}}{(r^2 - |z|^2)^2}$$
be the Poincaré metric on this disk. Introduce
$$\Psi := f^*\omega_h = u\eta_r.$$
Since Ψ remains bounded on each disk of radius $r < 1$, while η_r tends to infinity when one approaches the circle $|z| = r$, the function u remains bounded on this disk and thus takes on an interior maximum, say at the point z_0. At this point one has
$$0 \geq \mathbf{i}\partial\overline{\partial} \log u = \operatorname{Ric} \Psi - \operatorname{Ric} \eta_r.$$

The Gaussian curvature of η_r is equal to -1, where:

$$\operatorname{Ric}\Psi \le \operatorname{Ric}\eta_r = \eta_r$$

and one gets the inequality $u(z_0) \le 1$. But u takes on its maximum at z_0 and thus $\operatorname{Ric}\Psi \le \eta_r$. Taking the limit, one gets indeed $\operatorname{Ric}\Psi \le \eta_\Delta$. □

One can see the upper half plane as a special case of a period domain. Here the curvature is -1. For an arbitrary period domain the curvature of the invariant metric in general won't be negative, but it will be negative along the horizontal directions. More precisely, the holomorphic sectional curvature of the horizontal subbundle $T_{\mathrm{hor}}(D)$ is bounded by a (uniform) negative constant

$$\kappa(\xi) \le -1, \ \forall x \in T_{\mathrm{hor}}(D)$$

(so as to normalize the metric). For the original proof see **[G-S1]**.

Let there now be given a VHS on Δ^*, with monodromy transformation T, as indicated. Lift the period map to

$$\tilde{\Phi} : \mathfrak{h} \to D \subset \check{D}.$$

Recall that the map $\tilde{\Phi}$ is horizontal and then the Ahlfors-Schwarz lemma implies that $\tilde{\Phi}$ is distance decreasing (on $\mathfrak{h}$ one puts the hyperbolic metric with curvature -1)

$$\tilde{\Phi}^*(ds_D^2) \le ds_h^2$$

hence $\tilde{\Phi}$ is decreasing with respect to the associated Riemannian distances:

$$d_D(\tilde{\Phi}(p), \tilde{\Phi}(q)) \le d_h(p,q) \ .$$

Note that if $x, r \in \mathbb{R}_+$, $d(\mathbf{i}r, \mathbf{i}r + x) = \frac{x}{r}$. Then since $\tilde{\Phi}(\tau+1) = T\tilde{\Phi}(\tau)$ one has

$$d_D(\tilde{\Phi}(\mathbf{i}n), T\tilde{\Phi}(\mathbf{i}n)) \le \frac{1}{n}.$$

Fix a base point $v \in D$, and one identifies D with the orbit $G_{\mathbb{R}}/V$ of v. The map $G_{\mathbb{R}} \to D$ is proper because V is compact. Let $\tilde{\Phi}(\mathbf{i}n) = g_n v$. One has $d_D(g_n v, T g_n v) = d_D(v, g_n^{-1} T g_n v) \le 1/n$ because d_D is $G_{\mathbb{R}}$-invariant. It follows that $g_n^{-1} T g_n v \to v$. Passing to a subsequence of $\{g_n\}$, one may suppose that $g_n^{-1} T g_n$ converges to an element g of V. Since V is compact (it is a subgroup of a unitary group), the eigenvalues of g, and thus those of T are complex numbers of norm 1. The element T is in fact in $G_{\mathbb{Z}}$. Then, if λ is an eigenvalue of T, the same is true for any complex conjugate, and a classical fact (due to Kronecker) implies that λ must be a root of unity. □

Let us next explain a fundamental result **[S]** of W. Schmid: "The nilpotent orbit theorem" for a VHS on Δ^*. Here Δ is a disk with parameter s, and $\Delta^* = \Delta \setminus \{0\}$. Recall that the universal covering of Δ^* is given by

$$\mathfrak{h} \to \Delta^*, \quad \tau \mapsto s = \exp(2\pi\mathbf{i}\tau).$$

The monodromy of the locally constant sheaf $\mathcal{H}_{\mathbb{Z}}$ is described by analytic continuation along a circle traversed counterclockwise and is thus an operator T on $(\mathcal{H}_{\mathbb{Z}})_{s_0} = H_{\mathbb{Z}}$ (one fixes a base point $s_0 \in \Delta^*$). Recall that this means that the inverse image of $\mathcal{H}_{\mathbb{Z}}$ on $\mathfrak{h}$ is the constant sheaf $\mathfrak{h} \times H_{\mathbb{Z}}$, with the operation $\sigma : (\tau, \alpha) \to (\tau + 1, T^{-1}\alpha)$, and that $\mathcal{H}_{\mathbb{Z}} = \mathfrak{h} \times H_{\mathbb{Z}}/\{\sigma\}$.

The holomorphic vector bundle $\mathcal{H} := \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_{\Delta^*}$, on Δ^* is trivial for general reasons: a holomorphic bundle on a non compact Riemann surface is trivializable ([**Fo**], §30). In the present situation one can choose a privileged trivialization, using the fact that the monodromy is quasi-unipotent.

Assume that T is in fact a unipotent operator $(T-1)^m = 0$, to simplify a bit. Then $N = \log(T) = (T-1) - \frac{1}{2}(T-1)^2 + \cdots + (-1)^{m-2}\frac{1}{m-1}(T-1)^{m-1}$ is defined, and $N \in \mathfrak{g}_{\mathbb{Q}}$, i.e. N is a rational element of the Lie algebra of the group $G_{\mathbb{C}} = \mathrm{Aut}(H_{\mathbb{C}}, Q)$. Observe that for any $t \in \mathbb{C}$, $\exp(tN) \in G_{\mathbb{C}}$.

Trivializing the vector bundle $\mathcal{H} = \mathcal{H}(X/S)$ on Δ^* according to the description

$$\mathcal{H} = \mathfrak{h} \times H/\{\sigma\}$$

(H is equipped with the complex topology) is the same as trivializing the class $\{T\}$ in $H^1(\mathbb{Z}, G_{\mathbb{C}})$.

It suffices to remark that

$$\exp((\tau+1)\cdot N)\cdot \exp(\tau N)^{-1} = T\ .$$

In other words, if $\theta(\tau, \alpha) = (\tau, \exp(\tau N)\alpha)$ is a "change of coordinates" on $\mathfrak{h} \times H$, the action of $\pi_1(\Delta^*)$ in the new coordinates becomes

$$(\theta\sigma\theta^{-1})(\tau, \alpha) = (\tau + 1, \alpha).$$

This leads to a privileged trivialization of $\mathcal{H}$ over Δ^*. In this trivialization, the horizontal sections (those that extend over $s = 0$), are the sections $\alpha(s) = (s, \alpha)$ in the new coordinates. In the old ones, this means that if $\alpha \in H = (\mathcal{H})_{s_0}$, the analytic continuation of α defines a multi-valued section of the locally constant sheaf $\mathcal{H}_{\mathbb{C}}$, and $s \mapsto \exp\big(-\frac{\log s}{2\pi \mathbf{i}}N\big)\alpha(s)$ defines a holomorphic section of the vector bundle, which is horizontal relative to the privileged trivialization. By definition, the section

$$\alpha^*(s) = \exp\Big(-\frac{\log s}{2\pi \mathbf{i}}N\Big)\alpha(s)$$

is defined at $s = 0$. These are the horizontal sections of the vector bundle $\overline{\mathcal{H}}(X/S)$, extended to the whole disk Δ.

The fundamental result of W. Schmid is the following [**S**]:

4.3. THEOREM. *The Hodge bundles $\mathcal{F}^p \subset \mathcal{H}(X/S)$ can be extended to subbundles of the bundle $\overline{\mathcal{H}}(X/S)$. In particular in $s = 0$ one has a limit filtration $\mathcal{F}^\bullet(0) \in \check{D}$ (in general $\mathcal{F}^\bullet(0) \notin D$).*

The theorem can be stated in another way: let

$$\Phi : \mathfrak{h} \longrightarrow D \subset \check{D}\ .$$

Then if $\widetilde{\psi}(\tau) = \exp(-\tau N)\widetilde{\Phi}(\tau)$, one has $\widetilde{\psi}(\tau+1) = \widetilde{\psi}(\tau)$, consequently $\widetilde{\psi}$ defines a holomorphic function $\psi : \Delta^* \to \check{D}$ by $\psi(s) = \widetilde{\psi}\big(\frac{\log s}{2\pi\mathbf{i}}\big)$. The result is that ψ extends holomorphically to a map $\psi : \Delta \to \check{D}$. Recall that $H = H^k(X_{t_0}, \mathbb{Z})/\text{torsion}$. Consider $\psi(0)$ as a filtration

$$0 \subset F^k_\infty \subset F^{k-1}_\infty \subset \cdots \subset F^0_\infty$$

on $H \otimes \mathbb{C}$; this is the limit filtration.

The second part of Schmid's theorem – which we shall not use – concerns the nilpotent orbit

$$N(\tau) = \exp(\tau N)[\mathcal{F}^\bullet(0)]\ .$$

For $\operatorname{Im}\tau \gg 0$, $N(\tau) \in D$, and N is horizontal; thus for $\operatorname{Im}\tau$ big enough $N(\tau)$ defines a variation of Hodge structures, for which one can prove that it furnishes an approximation of the initial variation (in a sense one can make precise).

Introduce the *(monodromy) weight filtration*. It results from the following construction [**S**]: let V be a vector space over a field K of characteristic zero, and let $N \in \operatorname{End}(V)$, such that $N^{k+1} = 0$. Then there is a unique filtration

$$W_{-1} = \{0\} \subset W_0 \subset W_1 \subset \cdots \subset W_{2k} = V$$

such that $N(W_\alpha) \subseteq W_{\alpha-2}$, and such that N^ℓ induces an isomorphism $\mathrm{Gr}^W_{k+\ell} \xrightarrow{\sim} \mathrm{Gr}^W_{k-\ell}$, where $\mathrm{Gr}^W_\alpha = W_\alpha/W_{\alpha-1}$. So on H one disposes of two filtrations $F^\bullet_\infty$ and $W^\bullet$, the weight filtration $W_\bullet$ being defined on $\mathbb{Q}$. The most important result is the following part of Schmid's nilpotent orbit theorem.

4.4. Theorem([**S**]). *The filtrations $F^\bullet_\infty$ (the limit Hodge filtration) and $W_\bullet$ define a mixed Hodge structure on H.*

There is a generalization which is much more delicate (the Sl(2)-Orbit theorem of in n variables) with applications to degenerations with n parameters. See [**C-K-S1**].
For the case of a degeneration of one parameter, Steenbrink [**St1**] and Clemens-Schmid [**Cl**] have constructed this mixed Hodge structure in a geometrical way and from it they draw important consequences, for example

4.5. Local Invariant Cycle Theorem. *A class in $H^k(X_s, \mathbb{Q})$ is invariant if and only if it is the restriction of a global class on $H^k(X, \mathbb{Q})$.*

This assertion, although intuitively clear, is false in the non-Kähler setting!

§4.C. Nearby and vanishing Cycles.

It is useful to recall here the constructions of the sheaves of nearby and vanishing cycles associated to a degeneration $f : X \to \Delta$ (or more generally to a function $f : X \to \mathbb{C}$). These sheaves have support contained in $X_0 = f^{-1}(0)$.

The construction uses *the Milnor fiber at $x \in X$* which is the intersection of a small sphere around x of maximal real dimension in X with X_t, t close to 0. It can shown that the homotopy type of the Milnor fiber is independent of t and of the radius of the sphere, provided that these be carefully chosen (see [**Mil**]). Consider the cohomology groups resp. the reduced cohomology groups. For x variable, these groups form sheaves and one can construct two complexes which "calculate" these two cohomology groups, *the complex of the nearby cycles* resp. *the complex of the vanishing cycles*. To define the complex $\psi_f(\mathbb{C}_X)$ of the nearby cycles, take an injective resolution of the constant sheaf on $\tilde{X}$. Then restrict the direct image by $k : \tilde{X} \to X$ to X_0 (in other words $\psi_f(\mathbb{C}_X) = i^*Rk_*\mathbb{C}_{\tilde{X}}$, the inverse image under $i : X_0 \to X$ of the direct image by k of the constant sheaf on $\tilde{X}$). The complex of the vanishing cycles $\phi_f(\mathbb{C}_X)$ is defined as the cone of the natural morphism $\mathbb{C}_{X_0} \to \psi_f(\mathbb{C}_X)$ coming from $\mathbb{C}_X \to Rk_*k^*\mathbb{C}_X$. Here we recall that the cone $C(f)^\bullet$ of a morphism $f^\bullet : A^\bullet \to B^\bullet$ between complexes is defined by $C^p(f) = A^{p+1} \oplus B^p$ with derivation given by $\begin{pmatrix} -d_A^{p+1} & 0 \\ f^p & d_B^p \end{pmatrix}$. There is a short exact sequence $0 \to B^\bullet \to C(f)^\bullet \to A^\bullet[1] \to 0$ and applying this sequence, one finds that indeed for $j > 0$, $\mathcal{H}^j(\phi_f(\mathbb{C}_X)) = \mathcal{H}^j(\psi_f(\mathbb{C}_X))$ computes the j-th cohomology group of the Milnor fiber. For $j = 0$ there is a difference: $\mathcal{H}^0(\phi_f(\mathbb{C}_X))$ computes the cohomology but $\mathcal{H}^0(\psi_f(\mathbb{C}_X))$ computes the reduced cohomology.

The advantage of this description can be seen from the fact that $H^w(X_t, \mathbb{Q}) = H^w(\tilde{X}, \mathbb{Q}) = H^w(\psi_f^{\mathrm{uni}}\mathbb{Q}_X)$ where the superscript "uni" means that one takes the maximal subcomplex of $\psi_f\mathbb{Q}_X$ on which the natural monodromy action is unipotent. Thus the mixed Hodge structure can be constructed on the level of this complex. This is what Steenbrink ([**St1**]) in fact does in the case where the monodromy acts unipotently and X_0 is a divisor with normal crossings with a structure of algebraic variety. Navarro-Aznar ([**NA**]) has generalized this construction. See [**St2**] for applications to isolated singularities. See also [**D-S**] where a nice supplement to the monodromy theorem can be found in the case of an isolated singularity: if T admits a Jordan block of maximal size $n = \dim X$ – and necessarily for an eigenvalue different from 1 –, then it will have also a block of size $n - 1$ with eigenvalue 1.

Let us finally observe that the above description suggests that $\mathbb{C}_{\tilde{X}} = k^*\mathbb{C}_X$ can be replaced by $k^*\mathcal{K}^\bullet$, where $\mathcal{K}^\bullet$ is an arbitrary bounded complex of sheaves on X. This plays an important role Saito's works (see below).

5. Higgs bundles

The goal of this section is to give some details of Simpson's work on the construction of variation of Hodge structures. In particular we shall briefly explain how his results lead to restrictions on the possible fundamental groups of a Kähler manifold. These results can be found in [**Si3**]. They depend on [**Si1**] and [**Si4**]. The reader should also consult [**Si2**]. For other results on the fundamental groups which depend on the theory of the Higgs bundles, see [**A1**], [**A2**], [**Z1**], [**Z2**].

In §3 we have introduced the notion of a *variation of of polarized (VHS)* of weight w on a base manifold S which is supposed to be projective, smooth and defined over $\mathbb{C}$. Briefly, a such structure consists of a quadruple $\{\mathcal{H}, \nabla, Q, \{\mathcal{H}^{r,s}\}\}$ where $\mathcal{H}$ is a holomorphic bundle (equipped with a real structure), ∇ a flat connection, Q a bilinear form, $(-1)^w$-symmetric and ∇-parallel, $\mathcal{H} = \bigoplus_{r+s=w} \mathcal{H}^{r,s}$ a decomposition into differentiable subbundles $\mathcal{H}^{r,s}$ with the property that $\mathcal{H}^{r,s}$ is the complex conjugate of $\mathcal{H}^{s,r}$ (a Hodge decomposition). In addition, one demands that the Hodge bundles $\mathcal{F}^p = \bigoplus_{r \geq p} \mathcal{H}^{r,s}$ be holomorphic and that ∇ send $\mathcal{F}^p$ to $\mathcal{F}^{p-1} \bigotimes \Omega_S^1$ (Griffiths' transversality). Finally one demands that the Hodge decomposition be h-orthogonal with respect to the hermitian and ∇-parallel form, defined by $h(x, y) = (-i)^w Q(x, \overline{y})$, and that $(-1)^r h$ be positive on $\mathcal{H}^{r,s}$. So one could also start from $\{\mathcal{H}, \nabla, h, \{\mathcal{H}^{r,s}\}\}$.

Forgetting the real structure, and dropping the condition $\mathcal{H}^{r,s} = \overline{\mathcal{H}^{s,r}}$, we obtain the notion of a *complex variation of Hodge structures* (cVHS), provided Griffiths' transversality is interpreted correctly: not only it is required that $\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega_S^1$ but so that the bundles $\overline{\mathcal{F}^q} \stackrel{\mathrm{def}}{=} \bigoplus_{s \geq q} \mathcal{H}^{r,s}$ carry an anti-holomorphic structure on which ∇ acts by sending $\overline{\mathcal{F}^q}$ to $\overline{\mathcal{F}^{q-1}} \bigotimes_{\overline{\mathcal{O}_S}} \overline{\Omega_S^1}$.

A cVHS yields a particular example of a *Higgs bundle*, i.e. a holomorphic bundle $\mathcal{H}$ with a homomorphism $\theta : \mathcal{H} \to \mathcal{H} \otimes \Omega_S^1$ satisfying the integrability property $\theta \wedge \theta = 0$. Here $\mathcal{H} = \bigoplus_p \mathcal{F}^p/\mathcal{F}^{p+1}$ and θ is the direct sum of the $\mathcal{O}_S$-linear homomorphisms $\mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{F}^{p-1}/\mathcal{F}^p \bigotimes \Omega_S^1$ induced by ∇.

The Higgs bundle coming from a cVHS in addition is stable under the action of $\mathbb{C}^*$ given by $t\cdot(\mathcal{H},\theta)=(\mathcal{H},t\theta)$. More precisely, $\phi_t:\mathcal{H}\to\mathcal{H}$ given by $\mathcal{H}^{r,s}\ni x\mapsto t^r x$ induces an isomorphism $(\mathcal{H},\theta)\to(\mathcal{H},t\theta)$.

One can show ([**Si3**], Theorem 4.2) that if S is compact (and thus projective), each local system (of complex vector spaces) yields a Higgs bundle and if the system is semi-simple, the Higgs bundle $(\mathcal{H},\theta)$ comes from a cVHS if and only if the isomorphism class of $(\mathcal{H},\theta)$ is $\mathbb{C}^*$-invariant. In other words: among the representations of the fundamental group of S in $\mathbb{C}^d$, those which carry a cVHS are the semi-simple ones which are fixed by the action of $\mathbb{C}^*$. Another theorem of Simpson ([**Si3**], Theorem 3) says a representation of $\pi_1(S)$ can always be deformed into such a representation. In particular the Zariski closure G of the monodromy group (in the group $\mathrm{Gl}(d,\mathbb{R})$) must be very special, called *of Hodge type*, i.e. the rank of G must be equal to the rank of the maximal compact subgroup of G. For example the groups $\mathrm{Sl}(n,\mathbb{R})$ for $n\geq 3$ not are not of Hodge type.

It follows from these results that a lattice (discrete subgroup with quotient of finite volume) Γ in $\mathrm{Sl}(n,\mathbb{R})$ (for example $\mathrm{Sl}(n,\mathbb{Z})$) not can not figure as the fundamental group of a Kähler manifold.

To give a short indication of the proof, recall that a representation $\rho:\pi\to\mathrm{Gl}(d,\mathbb{C})$ is called rigid if the $\mathrm{Gl}(d,\mathbb{C})$-orbit of ρ under conjugation on $\mathrm{Hom}(\pi,\mathrm{Gl}(d,\mathbb{C}))$ is open. Thus this orbit is a connected component, because $\mathrm{Gl}(d,\mathbb{C}))$ is reductive. By Simpson's last theorem this component contains a cVHS and thus the Zariski closure of the monodromy group is of Hodge type.

On the other hand, a result of Margulis implies that the natural representation of the lattice Γ is rigid and thus, if Γ would be the fundamental group of a Kähler manifold, $\mathrm{Sl}(d,\mathbb{R})$ would be of Hodge type, leading to a contradiction.

The reader can find the details, as well as many other examples in [**Si3**].

6. Hodge modules

The goal of this section is to give an introduction to Morihiko Saito work on Hodge modules. One of the main applications is to intersection cohomology treated briefly in §6.A.

§6.A. Intersection and L_2-cohomology.

Recently, the intersection cohomology groups $IH^w(X)$ (for X complex and quasi-projective) have been introduced by Goresky and MacPherson ([**G-M**]). This cohomology is better adapted to singular manifolds than ordinary cohomology. For example there is a version of Poincaré duality of and the strong and weak Lefschetz theorems are valid. Cheeger, Goresky and MacPherson ([**C-G-M**]) have stated the conjecture that $IH^w(X)$ should carry a pure Hodge structure of weight w if X is projective. Saito [**Sa1,2**] has proved this with his theory of Hodge modules (see below).

One can ask whether classical Hodge theory (valid for compact Kähler manifolds) can be generalized for example to quasi-projective manifolds using a suitable Kähler metric so that the intersection cohomology can be calculated in terms of harmonic forms on the smooth part. Indeed, a Hodge decomposition theorem can be proved for complete Kähler metrics using forms which are locally L_2 with respect to the metric. In this case, as in the classical case, the decomposition into harmonic forms in components of pure bidegree leads to a Hodge decomposition for the cohomology groups $H_2^w(X,\mathbb{C})$ provided that this group has finite rank. See [**B-Z**] §3. Thus, if there would exist an identification between $H_2^w(X,\mathbb{C})$ and $IH^w(X,\mathbb{C})$, one would have a Hodge structure on intersection cohomology. There always is a natural map $H_2^w(X,\mathbb{C}) \to IH^w(X,\mathbb{C})$ which conjecturally is an isomorphism. This conjecture is true for isolated conical singularities [**Ch1**], [**Ch2**].

Observe that the conjecture of Cheeger, Goresky and MacPherson is more precise than the mere existence of a Hodge structure on $IH^w(X)$; one requires that

1. $IH^w(X)$ be canonically isomorphic to the group $H_2^w(X \setminus \mathrm{Sing}X)$, the cohomology group computed using forms which are locally L_2 with respect to the *Fubini-Study metric*,
2. the Hodge structure is induced by this isomorphism.

It is this more refined conjecture which has been proved in the case of isolated conical singularities, but it has not been proved yet in general. See [**B-Z**], §3 for a detailed discussion.

Deligne has generalized the Hodge decomposition theorem by replacing $\mathbb{C}$ by a variation of Hodge structures H_X on a compact Kähler manifold X. The same argument works in the L_2-framework with X quasi-projective admitting a complete Kähler metric, provided that the group $H_2^w(X,H_X)$ has finite dimension. It has been shown in this case that $H_2^w(X,H_X)$ admits a pure Hodge structure of weight $w+v$ where v is the weight of H_X (see [**Zu**]).

Next, Cattani, Kaplan, Schmid [**C-K-S2**] and Kashiwara and Kawai [**K-K**] have shown that if $\overline{X}$ is a smooth compactification of X such that $\overline{X} \setminus X$ is a divisor with normal crossings, $IH^w(\overline{X},H_X)$ is isomorphic to $H_2^w(X,H_X)$ and thus carries a Hodge structure of weight $w+v$.

§6.B. Saito's work.

Let S be a complex manifold and H_S a local system of real vector spaces. The holomorphic bundle associated $\mathcal{H}_S = H_S \otimes \mathcal{O}_S$ admits a flat connection $\nabla = 1 \otimes d$. Thus $\mathbf{D}_S$, the sheaf of differentials operators on S acts on $\mathcal{H}_S$ (the action of a holomorphic vector field ξ is given by $s \mapsto \nabla_\xi s$) giving $\mathcal{H}_S$ the structure of a $\mathbf{D}_S$-module. In fact, such a $\mathbf{D}_S$-module is a *coherent* and even a *holonomic* $\mathbf{D}_S$-module. The definitions of these notions can be found in [**Bo**], where also the details can be found of the following discussion.

In the framework of algebraic geometry, we often encounter the situation where S is a Zariski open subset of projective manifold X and $D := X \setminus S$ is a divisor. In this framework, the notion of a connection with *regular singularities* along D makes sense and it is known that ∇ admits such singularities. It can even be shown that $(\mathcal{H}_S,\nabla) \mapsto H_S$ establishes an equivalence between the category of holomorphic bundles on S equipped with a connection having regular singularities (along D) and the category of the local systems of complex vector spaces (*'Riemann-Hilbert correspondence'*).

The notion of regularity can be extended to holonomic $\mathbf{D}_S$-modules and in this framework one also has a 'Riemann-Hilbert' correspondence. To explain this, the notion of *perverse sheaf* is needed. So let us take a $\mathbf{D}_S$-module $\mathcal{M}$ and we begin by observing that the $\mathbf{D}_S$-module structure allows one to define a complex, called *De Rham complex* $\mathrm{DR}(\mathcal{M}) := \Omega_S^\bullet \otimes \mathcal{M}$. Consider this complex in a suitable derived category where, let us recall, two complexes get identified when a morphism between them exist inducing an isomorphism between the cohomology sheaves [**Ill**]; the complexes are said to be *quasi-isomorphic*. In the case of a $\mathbf{D}_S$-module coming from a local system, there only is cohomology in dimension zero: the local system itself (by the holomorphic Poincaré lemma). And thus in this case $\mathrm{DR}(\mathcal{H}_S)$ is quasi-isomorphic to H_S.

An important construction in this category is that of *Verdier duality*. We do not give the details here; it suffices to know that the dual complex of $\mathrm{DR}(\mathcal{H}_S)$ in the sense of Verdier is represented by the complex $\mathrm{DR}(\mathcal{H}_S^\vee)$ and thus in the derived category the dual of H_S is $H_S^\vee$.

A complex $\mathcal{K}^\bullet$ of sheaves of $\mathbb{C}$- vector spaces is said to be *perverse* if the cohomology sheaf in dimension j of $\mathcal{K}^\bullet$ as well of its Verdier dual is constructible and supported in dimension at most $-j$. A word of explanation: the convention is such that the De Rham complex of a $\mathbf{D}_S$-module starts in degree $-n$ and thus a local system H_S is perverse because the support of H_S as well as of its dual is S and thus has dimension n. More generally, a local system H_S on a Zariski open dense subset S of an algebraic manifold X can be extended in a minimal way to a perverse sheaf $IC(H_S)$ on X. In this case, if X is compact one has $IH^w(X, H_S) = H^w(X, IC(H_S))$.

So a perverse sheaf can be viewed as a generalization of a local system; the correspondence which associates to a holonomic $\mathbf{D}_S$-module its De Rham complex induces an equivalence of categories between the category of holonomic $\mathbf{D}_S$-modules with regular singularities and the category of the perverse sheaves of $\mathbb{C}$-spaces vector (Riemann-Hilbert correspondence of [**Kas**], [**Me1, Me2**]). One can convince oneself that this new framework is a consequent generalization of §2.

Now assume that the local system H_S carries a variation of Hodge structures of weight w. The Hodge filtration induces a filtration called *good* $\mathcal{M}_p := \mathcal{F}^{-p}$, i.e. the action of the operators of order ≤ 1 send $\mathcal{M}_p$ to $\mathcal{M}_{p+1}$ (translation of Griffiths' transversality). Such a filtered $\mathbf{D}_S$-module is an example of a *Hodge Module of weight* w. The definition of these objects is rather indirect, as we shall see, and it is a difficult theorem that a variation of Hodge structures is indeed a Hodge Module. See [**Sa**] for a proof as well as that for of the details of the discussion which follows.

Saito defines Hodge modules by induction. Begin with those which have their supported in a point $s \in S$: this are simply the (real) Hodge structures with an increasing Hodge filtration ($F_p := F^{-p}$). By taking the direct image under the inclusion $s \to S$ one gets a constructible sheaf on S considered as a perverse sheaf and thus as a $\mathbf{D}_S$-module. The Hodge filtration in fact gives it the structure of a filtered $\mathbf{D}_S$-module and this is an object in the category $MF_h(\mathbf{D}_S)$ of filtered $\mathbf{D}_S$-modules. Finally, to obtain a real structure, the fiber product with the category of real perverse sheaves should be taken.

In §4.C, we recalled the definition of the *nearby and vanishing cycles* relative to the zeroes of a non constant holomorphic function $g : S \to \mathbb{C}$: $\psi_g(\mathbb{C}_S) = i^*Rk_*\mathbb{C}_{\tilde{S}} = i^*Rk_*k^*\mathbb{C}_S$ and $\phi_g(\mathbb{C}_S)$ being the cone over $\{\mathbb{C}_{S_0} = i^*\mathbb{C}_S \to i^*Rk_*k^*\mathbb{C}_S\}$. Replacing $\mathbb{C}_S$ by a bounded complex $\mathcal{K}^\bullet$ on S, one arrives at $\psi_g(\mathcal{K}^\bullet)$ resp. $\phi_g(\mathcal{K}^\bullet)$. We have seen that the monodromy acts on these complexes and induces the the weight filtration.

Gabber (see [**Bry**]) has shown that for $\mathcal{K}^\bullet$ perverse, these complexes (shifted by $[-1]$) are perverse sheaves on S_0 and Saito has proposed a construction of the functors ϕ and ψ at the level of filtered holonomic $\mathbf{D}_S$-modules. In particular, the resulting nearby and vanishing modules admit weight filtrations $W_\bullet$. Saito now completes the inductive definition of his Hodge Modules in two steps: first restrict to a full sub-category of $MF_h(\mathbf{D}_S)$ such that its objects possess good properties with respect to the functors ϕ and ψ and next, declare a module $\mathcal{M}$ in this subcategory to be a Hodge Module if and only if it is so for the W-graded modules $\psi_g(\mathcal{M})$ and $\phi_g(\mathcal{M})$ for any function $g : S \to \mathbb{C}$ (more precisely: one has to restricts to the maximal submodule on which T acts unipotently). Since these modules have support on the fiber S_0 of g over 0, they are supported in strictly smaller dimension and since one knows inductively what Hodge modules supported in these dimensions are, the definition is complete.

The best known application of the fact that a variation of Hodge structures H_S of weight v parametrized by a complex manifold S is a Hodge Module of weight v is the following theorem that we already have announced: for any S Zariski open in a compact Kähler manifold X, the group $IH^w(X, H_S)$ carries a polarized Hodge structure of weight $v + w$. Indeed, we have seen that $IH^w(X, H_S) = H^w(X, IC(H_S))$ and that H_S and thus so $IC(H_S)$ are Hodge modules of weight v. Thus $H^w(X, IC(H_S))$ as a Hodge module supported in a point, carries a Hodge structure (of weight $v + w$)).

In particular, the polarized structure on $IH^w(X, \mathbb{Q})$ is a direct factor of the pure structure on $IH^w(\tilde{X}, \mathbb{Q})$ where $\tilde{X}$ is a resolution of the singularities of X.