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L-functions and theta correspondence for classical groups

Shunsuke Yamana

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Abstract Using the doubling method of Piatetski-Shapiro and Rallis, we develop a theory of local factors of representations of classical groups and apply it to give a necessary and sufficient condition for nonvanishing of global theta liftings in terms of analytic properties of the L-functions and local theta correspondence.

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S. Yamana (✉)
Faculty of Mathematics, Kyushu University, Motooka, Nishi-ku, Fukuoka 819-0395, Japan
e-mail: yamana@math.kyushu-u.ac.jp

1 Introduction

One of the fundamental problems in the theory of automorphic representations is the definition of L and epsilon factors. To this problem, Piatetski-Shapiro and Rallis [43, 44] have discovered a general type of zeta integrals which in one fell swoop generalized the Godement-Jacquet zeta integral from $\mathrm{GL}(n)$ to an arbitrary simple classical group. Their construction, known as the doubling method, includes the standard L-factors (twisted by Hecke characters) of all irreducible admissible representations of all simple classical groups. Though they presented general principles applicable to a wider class of groups as well, their results left ample room for elaboration.

In this paper we develop a theory of these zeta integrals. Our goal is twofold. First, we construct the standard L-factors of irreducible admissible representations of classical groups properly. Second, we show that nonvanishing of global theta liftings is characterized in terms of the analytic properties of the complete L-functions and the occurrence in the local theta correspondence.

It was the starting point of the work of Piatetski-Shapiro and Rallis that a special value of the global zeta integral is the Petersson inner product of relevant theta liftings. Such an identity is called the Rallis inner product formula and indicates that the theory of theta liftings is inextricably linked with the theory of automorphic L-functions. After the works of Waldspurger [60, 62] and Piatetski-Shapiro [42], Rallis proved the tower property of theta liftings in [47], which was to have a decisive influence on the direction of subsequent development of the theory of theta liftings, and initiated a project for determining the nonvanishing of theta liftings in [48]. His program can be divided into four steps: to produce a local theory of the standard L factor; to deduce the basic analytic properties of the standard L-functions from those of the Eisenstein series appearing in the integral; to generalize the Rallis inner product formula by extending the Siegel-Weil formula; to analyze local obstructions to the nonvanishing of the theta liftings.

Kudla and Rallis gave fairly complete information about the constituents of the degenerate principal series representations in [25, 27], prescribed exactly the location of the possible poles of the standard L-functions in [26], established the regularized Siegel-Weil formula and gave a sufficient condition for nonvanishing of theta liftings in terms of poles of the partial L-functions in [28]. However, their conclusion, eliminating sufficiently many finite number of Euler factors, is not clear-cut.

The first half of this paper produces a local theory of the standard L-factor with this application in view. The gamma factors have been given by Lapid and Rallis in [32] essentially as the proportionality constants for the local zeta integrals, with respect to a suitably normalized intertwining operator. The issue of the correct L-factor is more delicate. On the one hand, it has been defined as a "g.c.d." of the local zeta integrals for all good sections by Piatetski-Shapiro and Rallis (see [13, 43, 45]). On the other hand, it was alternatively defined from the gamma factors by Lapid and Rallis in such a way that it satisfies expected properties, several of which determine it uniquely. The two definitions have complementary strengths. We will prove that the two L-factors agree. Granted the L and gamma factors, epsilon factors are defined.

The local integral divided by the L-factor gives rise to a nonzero intertwining map from the degenerate principal series representation. We may reduce the global nonvanishing problem to the nonvanishing of a certain value or residue of the L-function and the local problem of the nonvanishing of the restriction of the map to a certain submodule of the degenerate principal series representation at all places. In light of the conservation relation the local nonvanishing can be controlled by the local theta correspondence, which yields the most practicable way of expressing the relationship between the global theory of theta liftings and the local theory.

To describe our problems and results, let us consider the case of orthogonal groups in this introduction. Let F be a number field with adele ring $\mathbb{A}$, $\psi=\bigotimes_v\psi_v$ a nontrivial additive character of $\mathbb{A}/F$, $\pi=\bigotimes_v\pi_v$ an irreducible automorphic cuspidal representation of the orthogonal group G of an n dimensional quadratic space W over F and V_π a realization of π in the space of cusp forms. Let $G^\square$ denote the orthogonal group of $W^\square=W\oplus(-W)$, where $-W$ is the space W with the negative of its given quadratic form. The Eisenstein series which occurs in the integral representation of the standard L-function of π is constructed from a meromorphic section $f^{(s)}$ of the degenerate principal series representations $I(s)$ for $G^\square$. If $f^{(s)}$ and $\xi\in\bar{V}_\pi\boxtimes V_\pi$ are factorizable, then the global integral $Z(\xi,f^{(s)})$ can be unfolded and written as a product of the local integrals $Z_v(\xi_v,f_v^{(s)})$ of $\xi_v\in\pi_v^\vee\boxtimes\pi_v$ and a local section $f_v^{(s)}$. The unramified local integral is the standard Langlands L-factor times a product of certain abelian L-factors, so that $Z(\xi,f^{(s)})$ interpolates the standard L-function of π.

Fix a place v of F. We adapt the same notation adding a subscript v for objects associated to F_v. Let Π be an irreducible admissible representation of G_v. Lapid and Rallis normalize the intertwining operator $M_v(s):I_v(s)\to I_v(-s)$ so that

$$Z_v\big(\xi_v,M^\dagger_{\psi_v}(s)f_v^{(s)}\big)=\Pi(-1)\varepsilon_{W_v,\psi_v}\gamma(s,\Pi,\psi_v)Z_v\big(\xi_v,f_v^{(s)}\big),$$

where ε_{W_v,ψ_v} is defined in the text. The result of [32] alluded to above is that the gamma factor $\gamma(s,\Pi,\psi_v)$ satisfies the ten properties, called the Ten Commandments, which determine it uniquely.

Recall the equation $M^\dagger_{\psi_v^{-1}}(-s)\circ M^\dagger_{\psi_v}(s)=\mathrm{Id}$. The family of good sections of $I_v(s)$ is defined as the smallest family of sections containing all holomorphic sections and closed under the normalized intertwining operator $M^\dagger_{\psi_v}(s)$. The reason for this definition will be clear from the form of the functional equation above. Another useful characterization is that $f_v^{(s)}$ is a good section if and only if both $f_v^{(s)}$ and $M^\dagger_{\psi_v^{-1}}(-s)f_v^{(-s)}$ are holomorphic in the half-plane $\Re s \geq 0$. To avoid excessive details, we assume that F_v is a nonarchimedean local field with residue field of order q_v. The L-factor $L(s,\Pi)$ is the function of the form $Q_\Pi(q_v^{-s+1/2})^{-1}$, with polynomial Q_Π satisfying $Q_\Pi(0)=1$, which is minimal in terms of degree with respect to the property that the products $Q_\Pi(q_v^{-s})Z_v(\xi_v, f_v^{(s)})$ are entire for all $\xi_v \in \Pi^\vee \boxtimes \Pi$ and all good sections $f_v^{(s)}$. The epsilon factor is defined by the formula

$$\varepsilon(s,\Pi,\psi_v)=\gamma(s,\Pi,\psi_v)L(s,\Pi)/L(1-s,\Pi).$$

The following result is a special case of Theorems 7.1 and 7.2.

Theorem 1 *If Π is the Langlands quotient of a standard module* $\mathrm{Ind}_Q^{G_v}(\sigma_1 \boxtimes \cdots \boxtimes \sigma_k \boxtimes \sigma_0)$, *where each σ_j $(1\leq j\leq k)$ is an essentially tempered representation of some general linear group and σ_0 is a tempered representation of an orthogonal group of lower rank*, *then*

$$L(s,\Pi)=L(s,\sigma_0)\prod_{j=1}^{k}L^{GJ}(s,\sigma_j)L^{GJ}\big(s,\sigma_j^\vee\big),$$

$$\varepsilon(s,\Pi,\psi_v)=\varepsilon(s,\sigma_0,\psi_v)\prod_{j=1}^{k}\varepsilon^{GJ}(s,\sigma_j,\psi_v)\varepsilon^{GJ}\big(s,\sigma_j^\vee,\psi_v\big),$$

where L^{GJ} and ε^{GJ} denote the Godement-Jacquet L and epsilon factors.

Jacquet has demonstrated the analogous property of the Godement-Jacquet local factors in [19]. The proof of Theorem 1 is broken into three parts: We first show that the L-factor is compatible with parabolic induction in Theorem 6.1. This part is relatively straightforward. Next we prove the following comparison result in Lemma 7.5:

$$L(s,\sigma_j)=L^{GJ}(s,\sigma_j)L^{GJ}\big(s,\sigma_j^\vee\big)\quad (j=1,2,\dots,k).$$

A final conclusion is Theorem 7.2 in the case of general linear groups. The L-factor in the left hand side is defined by applying the doubling construction to the general linear groups. Since $L(s,\sigma_j)$ and $L(1-s,\sigma_j^\vee)$ can have a common pole unlike the tempered case, this relation cannot be deduced directly from the analogous property of the gamma factor. To circumvent this problem, we introduce a zeta integral in two variables. With these results in hand, we know that the poles of $L(s,\Pi)$ are contained in the poles of the Lapid-Rallis L-factor, even with multiplicity. It is the most technically difficult part to establish that the Lapid-Rallis L-factor cannot have extraneous poles. Our approach is quite close to that of Jacquet [19], with additional complications.

The complete L-function is defined by the convergent Euler product $L(s,\pi)=\prod_v L(s,\pi_v)$ in the half-plane $\Re s>\frac{n}{2}$. We set $\varepsilon(s,\pi)=\prod_v \varepsilon(s,\pi_v,\psi_v)$. In view of Theorem 1 we retrieve the standard Langlands L-factor and $\varepsilon(s,\pi_v,\psi_v)=1$ if π_v and ψ_v are unramified. Using the integral representation of $L(s,\pi)$, we see that $L(s,\pi)$ continues to a meromorphic function in $\mathbb{C}$ which has at most simple poles and satisfies the functional equation:

$$L(s,\pi)=\varepsilon(s,\pi)L(1-s,\pi).$$

The result of [26] is that the poles can only occur at $s=j+1-\frac{n}{2}$ for $j\in\{0,1,2,\dots,n-1\}\setminus\{\frac{n-1}{2}\}$. The explanation for these poles is intimately tied up with the theory of theta liftings.

Let $Sp(V_j)$ be the symplectic group of a $2j$ dimensional vector space V_j equipped with a nondegenerate alternating form. The groups G and $Sp(V_j)$ comprise a dual reductive pair inside $Sp(\mathbb{W})$, where $\mathbb{W}=V_j\otimes_F W$ is the symplectic space whose alternating form is the tensor product of the forms on V_j and W. Fix a place v again. The symplectic group $Sp(\mathbb{W})_v$ has a unique nonlinear double cover $\mathrm{Mp}^{(2)}(\mathbb{W})_v$ unless $F_v=\mathbb{C}$. Let $\overline{Sp(V_j)}_v$ denote $Sp(V_j)_v$ in the case of even parity, or $\mathrm{Mp}^{(2)}(V_j)_v$ in the case of odd parity. We may pullback the Weil representation of $\mathrm{Mp}^{(2)}(\mathbb{W})_v$ to $G_v\times\overline{Sp(V_j)}_v$. The resulting representation is denoted by $\omega_{\psi_v,j}$. We say that π_v occurs in the correspondence for $G_v\times\overline{Sp(V_j)}_v$ if the space $\mathrm{Hom}_{G_v}(\omega_{\psi_v,j},\pi_v)$ is nonzero. If $F=\mathbb{R}$ or $\mathbb{C}$, then $\omega_{\psi_v,j}$ and π_v are actually Harish-Chandra modules and the homomorphism space consists of homomorphisms of Harish-Chandra modules.

We define the representation $\omega_{\psi,j}$ by pulling the global Weil representation $\omega_\psi=\bigotimes_v\omega_{\psi_v}$ back to $G(\mathbb{A})\times\overline{Sp(V_j)}_{\mathbb{A}}$. As is well-known, there is a natural intertwining map Θ^ψ of ω_ψ to the space of automorphic forms on $\mathrm{Mp}^{(2)}(\mathbb{W})_{\mathbb{A}}$. The theta lifting is defined in terms of the theta kernel above by the formula

$$\xi\mapsto\theta^\psi_{j,\phi}(\xi)(h)=\int_{G(F)\backslash G(\mathbb{A})}\overline{\xi(g)}\Theta^\psi(\phi)(gh)dg$$

for $\xi\in V_\pi$ and $\phi\in\omega_{\psi,j}$. Then $\theta^\psi_{j,\phi}$ is an automorphic form on $\overline{Sp(V_j)}_{\mathbb{A}}$. We denote by $\theta^\psi_j(\pi)$ the space of the automorphic representation generated by all $\theta^\psi_{j,\phi}(\xi)$, as ϕ and ξ vary.

Our objective is to determine whether $\theta_j^\psi(\pi)$ is zero or not. This was carried out by Waldspurger [60, 62] in the specific case for the dual pair $\overline{\mathrm{SL}(2)} \times \mathrm{PGL}(2)$. What is striking about his results is that one requires both global and local data to get a nonvanishing condition for the theta lifting. These two pieces of data detect the nonvanishing of $\theta_j^\psi(\pi)$ completely in general.

Theorem 2 *Let G be the orthogonal group of an n dimensional quadratic space. Let π be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$. Assume that $\theta_{j-1}^\psi(\pi)$ is zero.*

(1) *Assume that $j \le \frac{n}{2} - 1$. Then $\theta_j^\psi(\pi)$ is nonvanishing if and only if π_v occurs in the correspondence for $G_v \times \overline{Sp(V_j)}_v$ for all v and $L(s,\pi)$ has a pole at $s = j + 1 - \frac{n}{2}$.*
(2) *Assume that $j \ge \frac{n-1}{2}$. Then $\theta_j^\psi(\pi)$ is nonvanishing if and only if π_v occurs in the correspondence for $G_v \times \overline{Sp(V_j)}_v$ for all v and $L(s,\pi)$ is holomorphic and nonzero at $s = j + 1 - \frac{n}{2}$.*

The proof of this theorem distinguishes the cases $j \le \frac{n-1}{2}$ and $j \ge \frac{n}{2}$, which we refer to as the first term range and the second term range, respectively. For each place v we write sgn_v for the sign character of G_v. For a set T of even number of places we define a character sgn_T of $G(\mathbb{A})$ trivial on $G(F)$ by $\mathrm{sgn}_T(g) = \prod_{v \in T} \mathrm{sgn}_v(g_v)$.

In the first term range, one can prove that if $L(s,\pi)$ has a pole at $s = j + 1 - \frac{n}{2}$, then there is a set T such that $\theta_j(\pi \otimes \mathrm{sgn}_T)$ is nonvanishing. Kudla and Rallis [28] have proven an analogous statement for symplectic groups. The set T is uniquely determined by the local theta correspondence. Moreover, if $L(s,\pi)$ has a pole, if $j_0 + 1 - \frac{n}{2}$ is the leftmost pole of $L(s,\pi)$, if $\theta_{j_0}^\psi(\pi \otimes \mathrm{sgn}_{T_0})$ is nonzero and if $j \le n - 1 - j_0$, then $\theta_j^\psi(\pi \otimes \mathrm{sgn}_{T'})$ is vanishing unless $T' = T_0$, in which case $\theta_k^\psi(\pi \otimes \mathrm{sgn}_{T_0})$ is zero for $k < j_0$ and is nonzero for $k \ge j_0$. We have $0 \le j_0 \le \frac{n}{2} - 1$ because of the basic analytic properties of $L(s,\pi)$ alluded to above. These results are extended to all classical groups in Theorems 10.1 and 10.2. In the second term range, the theorem is valid if $\theta_{n-1-j}(\pi)$ is zero. Indeed, Corollary 10.1 shows that if $\theta_{n-1-j}(\pi)$ is zero, then $\theta_j(\pi)$ is square integrable, so that the Rallis inner product formula can apply. The result in the second term range is extended to the quaternion case (see Theorem 10.3).

Besides the work of Kudla and Rallis [28], a lot of works has been done on the nonvanishing of theta liftings. Among them, we mention the following sample: [8, 9, 36, 49, 58]. All these works consider partial standard L-functions and handle sufficient conditions of the nonvanishing of theta liftings. It is essential to include the missing L-factors in order to prove the other direction.

The key ingredient is the extended Rallis inner product formula, which tells us that the Petersson inner product of $\theta^{\psi}_{j,\phi_1}(\xi_1)$ and $\theta^{\psi}_{j,\phi_2}(\xi_2)$ is just the value at $s = t_j$ of the meromorphic function $Z(\overline{\xi_1} \boxtimes \xi_2, f^{(s)}_{\Phi})$, where $t_j = j - \frac{n-1}{2}$, $\Phi = \phi_1 \otimes \overline{\phi_2}$, and $f^{(s)}_{\Phi}$ is a section constructed from Φ. We write $\Theta^{\psi}_{j}(\mathbb{1})$ for the submodule of $I(t_j)$ spanned by functions $f^{(t_j)}_{\Phi}$. Note that $Z(\overline{\xi_1} \boxtimes \xi_2, f^{(s)}_{\Phi})$ splits into a product of two terms. The first term is the L-function $L(s+\frac{1}{2}, \pi)$, and the global obstruction is the vanishing of its pole or its special value. The second term is a finite product of the normalized local zeta integrals.

When $j < \frac{n-1}{2}$, it is convenient to consider the residue at $s = -t_j$ of a zeta integral which is tied to the special value $Z(\overline{\xi_1} \boxtimes \xi_2, f^{(s)}_{\Phi})|_{s=t_j}$ via the functional equation. Since the residue of the Eisenstein series factors through $I(-t_j) \to \bigoplus_T (\Theta^{\psi}_{j}(\mathbb{1}) \otimes \mathrm{sgn}_T)$, so does the residue of the global zeta integral. Therefore, there is a twist $\pi \otimes \mathrm{sgn}_T$ having a nonzero theta lift, provided that $L(s, \pi)$ has a pole at $s = t_j + \frac{1}{2}$. We can infer the uniqueness of the set T from the fact that if π_v occurs as a quotient of $\omega_{\psi_v, j}$ and if $\pi_v \otimes \mathrm{sgn}_v$ occurs as a quotient of $\omega_{\psi_v, j'}$, then $j + j' \geq n$.

As far as the poles or the central value of the L-function is concerned, the Rallis inner product formula is well understood in the general case of type (I) dual pairs. The argument outlined above goes through with no or little adjustment in general. Since the Rallis inner product formula is fully generalized by W.T. Gan and Takeda [8] in the orthogonal case, we can prove the general criterion stated in Theorem 2(2). Employing the Siegel-Weil formula extended by the author [69], we extend the inner product formula and Theorem 2 to quaternion dual pairs in Sects. 10.5, 10.6 and 10.7.

W.T. Gan, Qiu and Takeda [6] recently established the second term identity of the Siegel-Weil formula and generalized Theorem 2 for the other groups. Though we do not have sufficient control of the nonvanishing of the normalized zeta integral in certain archimedean cases, the substantial part of Rallis's project has been completed.

The second section sets up the notation. We include all cases to be dealt with in the framework of the doubling method, and try to give a uniform exposition.

In Sects. 3–8 we develop the local theory. We discuss the notion of good sections in Sect. 3. Section 4 records some known results on the doubling method. We construct the standard L and epsilon factors by the doubling method in Sect. 5. Sections 6 and 7 are devoted to proving the compatibility of the L factor with the Langlands classification. The main ingredient here is the theory of the gamma factors, as completed by Lapid and Rallis in [32]. Their arguments, which streamline the discussion about multiplicativity of the zeta integrals and the normalization of intertwining operators, have been transplanted to the proof. The remainder of Sect. 7 discusses some consequences of results of Sects. 5 and 6 on basic properties of the local factors. Section 8 reviews relevant aspects of the theory of local theta correspondence. The required nonvanishing of the bad local integral is related to the local theta correspondence.

After reviewing the integral representation of the standard L-function, we examine its analytic properties in Sect. 9. The material of this section is a refinement of that of [26, 44]. In Sect. 10 we first review the known extensions of the Siegel-Weil formula and derive the generalizations of the Rallis inner product formula, and then collect together the local results of Sects. 7 and 8 to deduce the criterion for the nonvanishing.

Appendix A discusses the local theory of the doubling method over an archimedean local field. In Appendix B we list some of main results proven by Lapid and Rallis [32].

2 Classical groups

We list our principal notations.

2.1 Notation

Besides the standard symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}_+^\times$ the group of strictly positive real numbers, by $\mathbb{C}^1$ the group of complex numbers of absolute value 1 and by μ_n the group of nth roots of unity. For an associative ring R with identity element, we denote by $R^\times$ the group of all its invertible elements, and by $\mathrm{M}_{mn}(R)$ the module of all $m \times n$ matrices with entries in R. We put $\mathrm{M}_n(R) = \mathrm{M}_{nn}(R)$ and $\mathrm{GL}_n(R) = \mathrm{M}_n(R)^\times$ particularly when we view the set as a ring. We denote by $\mathbf{1}_n$ and $\mathbf{0}_n$ or simply by 1 and 0 the identity and zero elements of the ring $\mathrm{M}_n(R)$. If G is a locally compact topological group, then a character of G is a continuous representation of G into $\mathbb{C}^\times$. In particular a character is not necessarily unitary unless we specify it. When C is a division algebra central and of finite dimension over a local field F, we identify characters of $F^\times$ with those of $\mathrm{GL}_n(C)$, using the reduced norm $\nu : \mathrm{GL}_n(C) \to F^\times$.

When X is a totally disconnected locally compact topological space, let $C_c^\infty(X)$ or $\mathcal{S}(X)$ denote the space of locally constant compactly supported functions on X. When X is a C^∞ manifold, let $C_c^\infty(X)$ be the space of compactly supported C^∞ functions on X and $\mathcal{S}(X)$ the space of C^∞ functions on X which have an appropriate rapid decrease property relative to derivatives.

To indicate that a union $X = \bigcup_{i \in I} Y_i$ is disjoint, we write $X = \bigsqcup_{i \in I} Y_i$. We understand that $\prod_{i=\alpha}^{\beta} = 1$ and $\sum_{i=\alpha}^{\beta} = 0$ if $\alpha > \beta$. For a nonzero real number x, put $\mathrm{sgn}(x) = x/|x|$. We denote the determinant characters of orthogonal groups also by sgn.

2.2 Algebras with involutions

The symbol F will be used to denote a local field or global field of characteristic zero. If F is a number field, then we denote by $\mathbb{A} = \mathbb{A}_F$ the adele ring of F, by C_F the idele classes of F and by ψ a nontrivial additive character of $\mathbb{A}/F$. If F is local, then let $C_F = F^\times$ and fix a nontrivial additive character ψ of F.

By an involution of an algebra D whose center E contains F, we mean an arbitrary anti-automorphism ρ of D of order two under which F is the fixed subfield of E. We denote the restriction of ρ to E also by ρ. We take a couple (D, ρ) belonging to the following five types:

(a) $D = E = F$ and ρ is the identity map;
(b) D is a division quaternion algebra over $E = F$ and ρ is the main involution of D;
(c) D is a division algebra central over a quadratic extension E of F and ρ generates $\mathrm{Gal}(E/F)$;
(d) $D = \mathrm{M}_2(E)$, $E = F$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\rho = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$;
(e) $D = \mathbf{D} \oplus \mathbf{D}^{\mathrm{op}}$, $E = F \oplus F$ and $(x, y)^\rho = (y, x)$, where $\mathbf{D}$ is a division algebra central over F and $\mathbf{D}^{\mathrm{op}}$ is its opposite algebra.

The rank of D as a module over E is a square of a natural number which will be denoted by δ. We will assume D to be division if F is a number field, so that D of type (d) (resp. (e)) will appear in our later discussion as a localization of a global D of type (b) (resp. (c)). Let ϵ be either 1 or -1. We fix once and for all the triple (D, ρ, ϵ). Put $D_\epsilon = \{a \in D \mid a^\rho = \epsilon a\}$ and

$$d_0 = \dim_F D, \qquad d_\epsilon = \dim_F D_\epsilon, \qquad \varepsilon = d_\epsilon / d_0.$$

When E is a field, we denote by $\alpha_E(x) = |x|_E$ the standard norm of C_E. A character of C_E is called principal if it is of the form α_E^s for some $s \in \mathbb{C}$. When $E = F \oplus F$, we set $C_E = C_F \times C_F$ and $\alpha_E(x, y) = |xy|_F$.

For a matrix $x = (x_{ij})$ with entries in D we denote by ${}^t x$ the transpose of x, and put $x^\rho = (x_{ij}^\rho)$ and $x^* = {}^t x^\rho$. Then $x \mapsto x^*$ gives an involution of $\mathrm{M}_n(D)$. If D is of type (e), then the involution interchanges the factors and induces an anti-isomorphism of $\mathrm{M}_n(\mathbf{D})$ onto $\mathrm{M}_n(\mathbf{D}^{\mathrm{op}})$. If x is a square matrix, then $\nu(x) \in E$ and $\tau(x) \in E$ stand for its reduced norm and reduced trace to the center E of D.

2.3 ϵ-skew hermitian forms and unitary groups

Let W be a free left D-module of rank n. By an ϵ-skew hermitian space we mean a structure $\mathcal{W} = (W, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is a ϵ-skew hermitian form on

W, that is, an F-bilinear map $\langle\ ,\ \rangle : W \times W \to D$ such that

$$\langle x, y\rangle^\rho = -\epsilon\langle y, x\rangle, \qquad \langle ax, by\rangle = a\langle x, y\rangle b^\rho \quad (a, b \in D;\ x, y \in W).$$

Such a form is called nondegenerate if $\langle x, W\rangle = 0$ implies that $x = 0$. We will assume that $\langle\ ,\ \rangle$ is nondegenerate. We denote the ring of all D-linear endomorphisms of W by $\mathrm{End}(W, D)$ and set $GL(W, D) = \mathrm{End}(W, D)^\times$. Note that $GL(W, D)$ acts on W on the right. When $D = F$, we omit D and write simply $GL(W)$ for ease of notations. Let

$$G = \big\{g \in GL(W, D) \mid \langle xg, yg\rangle = \langle x, y\rangle \text{ for all } x, y \in W\big\}$$

be the unitary group of $(W, \langle\ ,\ \rangle)$, which is a (possibly disconnected) reductive algebraic group defined over F. It is important to realize that G always comes together with a space W and a form $\langle\ ,\ \rangle$. We usually just speak of G and let the data $\mathcal{W} = (W, \langle\ ,\ \rangle)$ be implicitly understood. When the dependence of G on W needs to be stressed, we write $G = G(\mathcal{W})$. If F is local, then we shall deal with the representations of the group of F-rational points of G, while if F is global, then we consider the localization and adelization of G.

If D is of type (e), then the nondegenerate form $\langle\ ,\ \rangle$ identifies the free D-module W with the sum $\mathbf{W} + \mathbf{W}^\vee$, where $\mathbf{W}$ is a vector space over $\mathbf{D}$ and $\mathbf{W}^\vee$ is its dual. In this case G is isomorphic to $GL(\mathbf{W}, \mathbf{D})$. If $D = F$, then W is called symplectic or symmetric according as $\epsilon = 1$ or $\epsilon = -1$, and G is called a symplectic group or an orthogonal group accordingly.

When W is a symplectic space, we denote by $\mathrm{Mp}(W)$ the metaplectic extension of the symplectic group $G = Sp(W)$, which is nontrivial except when $F = \mathbb{C}$ and which we take to be the extension

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}(W) \to Sp(W) \to 1$$

obtained from the usual two-fold covering via the inclusion of μ_2 into $\mathbb{C}^1$. If $F = \mathbb{C}$, we write $\mathrm{Mp}(W)$ for $Sp(W) \times \mathbb{C}^1$. The group $\mathrm{Mp}(W)$ has a character $(g, z) \mapsto z^2$, whose kernel $\mathrm{Mp}^{(2)}(W)$ is precisely the metaplectic two-fold cover of $G = Sp(W)$. We include the case $\bar{G} = \mathrm{Mp}^{(2)}(W)$. In the other cases we put $\bar{G} = G$ to unify notation. For any subgroup J of G we write $\bar{J}$ for the pull-back of J in $\bar{G}$.

Throughout this paper we put

$$\rho_n = n + 2\varepsilon - 1.$$

We refer to the case in which $E = F \oplus F$ as Case (II) and refer to the other cases as Case (I). In Case (I) there are five cases to be dealt with, which we refer to as Cases (I_0)–(I_4):

(I_0) $D = E = F$, $\epsilon = -1$, $\varepsilon = 0$;

(I_1) D is a quaternion algebra over $E = F$, $\epsilon = 1$, $\varepsilon = 1/4$;
(I_2) E is a quadratic extension of F, $\epsilon = \pm 1$, $\varepsilon = 1/2$;
(I_3) D is a quaternion algebra over $E = F$, $\epsilon = -1$, $\varepsilon = 3/4$;
(I_4) $D = E = F$, $\epsilon = 1$, $\varepsilon = 1$.

We sometimes refer to Cases (I_2) and (II) (resp. (I_1) and (I_3)) collectively as the unitary (resp. quaternion) case. In the unitary case E is an étale quadratic F-algebra, and we can identify hermitian and skew hermitian forms by multiplying $\langle\ ,\ \rangle$ by an appropriate constant, and avail ourselves of this noncanonical identification to reduce the problem from one case to the other. We refer to the cases of symplectic and metaplectic groups as (I_4^s) and (I_4^m) respectively when they require a separate treatment. We distinguish the even dimensional orthogonal case (I_0^e) and the odd dimensional orthogonal case (I_0^o) if necessary.

For a character χ of C_E we denote the restriction of χ to C_F by χ^0. For $\lambda \in F^\times$ let χ_λ be a quadratic character of C_F corresponding to the extension $F(\sqrt{\lambda})$. We fix a character χ_W of C_E in the following way: Taking $A = (\langle x_i, x_j\rangle)$ for any basis $x_1, \dots, x_n$ of W, we define $\Delta(W) \in F^\times/F^{\times 2}$ by

$$\Delta(W) = (-1)^{n(n-1)/2} 2^{-n} \det A, \tag{I_0}$$

$$\Delta(W) = (-1)^n \nu(A). \tag{I_1}$$

In Cases (I_0) and (I_1) we set $\chi_W = \chi_{\Delta(W)}$. In the unitary case we write $\epsilon_{E/F}$ for the quadratic character of C_F whose kernel is equal to the image of the norm map $N_{E/F} : C_E \to C_F$. We set $\chi_W = \chi_n$, where χ_n is a fixed character of C_E such that $\chi_n^0 = \epsilon_{E/F}^n$. In Cases ($\mathrm{I}_3$) and ($\mathrm{I}_4$) let χ_W be the trivial character of C_F.

We formally include the case $n = 0$. We take χ_W to be the trivial character. Though there is a slight abuse of notation, it will be convenient to distinguish Case (I) from Case (II) even when $n = 0$ (cf. Remark 3.1(4)).

When F is local, we associate with an ϵ-skew hermitian space W an invariant $\eta(W)$, which is either 1 or -1, according to the type of W as follows: In Case (I_0) let $\eta(W)$ be the normalized Hasse invariant of W. This invariant assigns 1 to a hyperbolic form. We refer to [51, 54] for its definition. In the unitary case we define $\Delta(W) \in F^\times/N_{E/F}(E^\times)$ by $\Delta(W) = (-1)^{n(n-1)/2} \det A$ and put $\eta(W) = \epsilon_{E/F}(\Delta(W))$, taking $A = (\langle x_i, x_j\rangle)$ for any basis $x_1, \dots, x_n$ of W and assuming that $\epsilon = 1$. Set $\eta(W) = 1$ in all the other cases.

2.4 Doubling variables

Put $W^\square = W \oplus W$. We sometimes write $W^\square = W_+ \oplus W_-$ to distinguish the copies of W in $W^\square$. Define an ϵ-skew hermitian form $\langle\ ,\ \rangle^\square$ on $W^\square$ by

$$\left\langle x + y, x' + y'\right\rangle^\square = \left\langle x, x'\right\rangle - \left\langle y, y'\right\rangle \quad \left(x, x' \in W_+;\ y, y' \in W_-\right).$$

Let $G^\square$ denote the unitary group of $(W^\square, \langle\ ,\ \rangle^\square)$. Let

$$W^\nabla = \{(x, -x) \in W^\square \mid x \in W\}$$

be the graph of minus the identity map from W to W, and

$$W^\Delta = \{(x, x) \in W^\square \mid x \in W\}$$

the graph of the identity map. Then $W^\square = W^\nabla + W^\Delta$. Therefore $W^\square$ is split in Case (I), i.e., admits a complete polarization. We identify $G \times G$ with the subgroup of $G^\square$ preserving W_+ and W_-.

If $g \in GL(W, D)$ preserves a D-submodule Y of W, then we define $\alpha_Y(g) \in GL(W/Y, D)$ by $\alpha_Y(g)(x) = xg \pmod{Y}$ for $x \in W/Y$. When Y is a totally isotropic free D-submodule, let $P(Y)$ be the stabilizer of Y in G. The unipotent radical $N(Y)$ of $P(Y)$ consists of the elements in G which act as the identity on both Y and $Y^\perp/Y$, where

$$Y^\perp = \{x \in W \mid \langle x, y\rangle = 0 \text{ for all } y \in Y\}.$$

If $p \in P(Y)$ in Case (I), then p preserves $Y^\perp$ as well and $\alpha_{Y^\perp}(p)$ is the inverse of the ρ-dual of the action of p on Y, and the Levi part $M(Y)$ of $P(Y)$ is canonically isomorphic to $GL(Y, D) \times G(Y^\perp/Y)$. Put

$$\Delta_Y^W(p) = \nu\big(\alpha_{Y^\perp}(p)\big) = \big(\nu(p|_Y)^{-1}\big)^\rho \in E^\times.$$

For simplicity of notation we will denote $P(W^\Delta)$ and $\Delta_{W^\Delta}^{W^\square}$ by $P = P_{\mathcal{W}}$ and $\Delta = \Delta^{\mathcal{W}}$.

Suppose that $D = F$ is a local field. We consider the set $\overline{GL}(Y) \simeq GL(Y) \times \mu_2$ with multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = \big(g_1 g_2, t_1 t_2 (\det g_1, \det g_2)_F\big),$$

where $(\ ,\ )_F : F^\times \times F^\times \to \mu_2$ is the Hilbert symbol. We can define a genuine character of $\overline{\mathrm{GL}}(1)$ by

$$\gamma_\psi\big((\lambda, t)\big) = t\gamma(\psi)/\gamma(\psi_\lambda)$$

for $\lambda \in F^\times$ and $t \in \mu_2$, where $\psi_\lambda(x) = \psi(\lambda x)$ and $\gamma(\psi)$ is a Weil constant associated to ψ (see [46]). Using this character, we obtain a bijection, which depends on the choice of the additive character ψ, between the set of equivalence classes of admissible representations of $GL(Y)$ and that of genuine admissible representations of $\overline{GL}(Y)$ via $\sigma \mapsto \sigma_\psi$, where

$$\sigma_\psi\big((g, t)\big) = \gamma_\psi\big((\det g, t)\big)\sigma(g).$$

When W is a symplectic form and $\bar{G} = \mathrm{Mp}^{(2)}(W)$, we can regard $N(Y)$ canonically as a subgroup of $\bar{G}$ and the preimage $\bar{P}(Y)$ of $P(Y)$ in $\bar{G}$ is of the form $\bar{P}(Y) = \bar{M}(Y) \cdot N(Y)$. The restriction of the extension $\bar{G}$ to $GL(Y)$ is precisely $\overline{GL}(Y)$, and the preimage $\bar{M}(Y)$ of $M(Y)$ in $\bar{G}$ is isomorphic to $\overline{GL}(Y) \times_{\mu_2} \mathrm{Mp}^{(2)}(Y^{\perp}/Y)$.

3 Degenerate principal series representations

3.1 Holomorphic sections

We hereafter assume F to be a local field of characteristic zero and, we continue with this assumption until the end of Sect. 8. In the p-adic case F contains a ring $\mathfrak{o}$ of integers having a single prime ideal $\mathfrak{p}$ whose norm is denoted by q. We call ψ unramified if it is trivial on $\mathfrak{o}$ and nontrivial on $\mathfrak{p}^{-1}$. For any algebraic group G defined over F we will denote the group of F-rational points of G also by G.

There are two sets of notation that we shall be using in the local setting. For the moment G denotes the group of isometries of $\mathcal{W} = (W, \langle\ ,\ \rangle)$. Fix a minimal parabolic subgroup P_0 of G with a Levi decomposition $P_0 = M_0 U_0$. We choose a maximal compact subgroup K of G which is in a good position with respect to M_0. In particular, $G = PK$ and $M \cap K$ is a maximal compact subgroup of M for each parabolic subgroup P of G containing P_0 and each Levi subgroup M of G containing M_0. Similarly, we choose a suitable maximal compact subgroup $K^{\square}$ of $G^{\square}$. We assume $K^{\square}$ to contain $K \times K$ in the archimedean case.

Unless otherwise mentioned $P = P_{\mathcal{W}}$ hereafter denotes the parabolic subgroup of $G^{\square}$ stabilizing W^{Δ}. For $s \in \mathbb{C}$ and a character χ of $E^{\times}$, we set $\chi_s = \chi \cdot \alpha_E^s$ or $\chi_s = \chi_{\psi,s} = \chi_{\psi} \cdot \alpha_E^s$ according as $\bar{G}^{\square} = G^{\square}$ or not. We consider the normalized induced representation

$$I(s, \chi) = I_{\psi}(s, \chi) = \mathrm{Ind}_{\bar{P}}^{\bar{G}^{\square}}(\chi_s \circ \Delta).$$

It is understood that all relevant objects implicitly depend on the choice of ψ in the metaplectic case.

We call a function $(s, g) \mapsto f^{(s)}(g)$ on $\mathbb{C} \times \bar{G}^{\square}$ (or $f^{(s)}$ in short) a holomorphic section of $I(s, \chi)$ if the following three conditions are satisfied:

- for each $s \in \mathbb{C}$, $f^{(s)}(g)$ belongs to $I(s, \chi)$ as a function of $g \in \bar{G}^{\square}$;
- for each $g \in \bar{G}^{\square}$, $f^{(s)}(g)$ is holomorphic in s;
- $f^{(s)}(g)$ is right $\bar{K}^{\square}$-finite.

We call a holomorphic section $f^{(s)}$ a standard section if its restriction to $\mathbb{C} \times \bar{K}^{\square}$ does not depend on s. We call a function $f^{(s)}$ on $\mathbb{C} \times \bar{G}^{\square}$ a meromorphic section of $I(s,\chi)$ if there is an entire function β which is not identically zero and such that $\beta(s)f^{(s)}$ is a holomorphic section.

When F is a nonarchimedean local field of odd residual characteristic and ψ is unramified, the metaplectic covering $\bar{G}^{\square}$ is known to split canonically over $K^{\square}$, and we regard $K^{\square}$ as a subgroup of $\bar{G}^{\square}$. When χ is principal, we extend $\chi_s \circ \Delta$ to a right $K^{\square}$-invariant function on $\bar{G}^{\square}$, which we still denote by $\chi_s \circ \Delta$. The function $f_0^{(s)} = \chi_{s+\delta\rho_n/2} \circ \Delta$ is a unique section of $I(s,\chi)$ that is identically 1 on $K^{\square}$. Note that the modulus function of P is given by $\alpha_E^{\delta\rho_n/2} \circ \Delta$.

3.2 The cases of $D = \mathrm{M}_2(F)$ and $D = \mathbf{D} \oplus \mathbf{D}^{\mathrm{op}}$

We will need to consider the case in which D is not division. First we discuss the case in which D is of type (d). Put $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in D \simeq \mathrm{M}_2(F)$. Set $W_e = eW$ and let $\langle\ ,\ \rangle_e$ be the restriction of $\langle\ ,\ \rangle$ on W_e. Then $\langle\ ,\ \rangle_e$ is an F-bilinear mapping with value in the one dimensional F-vector space eDe^{ρ}, and it is nondegenerate and has the opposite symmetry as $\langle\ ,\ \rangle$ under interchange of the two variables (see [51, pp. 361–362]). The restriction $g \mapsto g|_{W_e}$ gives an isomorphism of G onto the group of all F-automorphisms of W_e that preserve $\langle\ ,\ \rangle_e$. We will fix once and for all an isomorphism, identify these two groups, and reduce Case (I_1) (resp. (I_3)) to Case (I_0) (resp. (I_4)), provided that D is a split quaternion algebra.

Next suppose that D is of type (e). Put $e_1 = (1,0) \in E$. The restriction $g \mapsto g|_{e_1 W}$ gives an isomorphism of G onto $GL(W_1, \mathbf{D})$. Under this identification, we arrive at $G^{\square} = \mathrm{GL}_{2n}(\mathbf{D})$ and the maximal parabolic subgroup $P = MN$ given by

$$M = \left\{ m(a,d) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \,\middle|\, a, d \in \mathrm{GL}_n(\mathbf{D}) \right\},$$

$$N = \left\{ n(b) = \left(\begin{smallmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{smallmatrix}\right) \,\middle|\, b \in \mathrm{M}_n(\mathbf{D}) \right\}.$$

When we write $\chi = (\chi_1, \chi_2)$, the induced representation $\mathrm{Ind}_P^{G^{\square}}(\chi_s \circ \Delta)$ is translated to

$$I(s,\chi) = \mathrm{Ind}_P^{\mathrm{GL}_{2n}(\mathbf{D})}\left(\left(\chi_1 \alpha_F^s\right) \circ \nu \boxtimes \left(\chi_2 \alpha_F^s\right)^{-1} \circ \nu \right).$$

It will often be convenient to consider the multiplicative group of a simple algebra instead of the group coming from the algebra with the involution exchanging the two summands. To apply an inductive argument, we have to deal with the case $\langle\ ,\ \rangle = 0$. Such a case is reduced to the cases just discussed: if $\langle\ ,\ \rangle = 0$ and D is of type (d) (resp. (e)), then $G \simeq \mathrm{GL}_{2n}(F)$ (resp. $G \simeq \mathrm{GL}_n(\mathbf{D}) \times \mathrm{GL}_n(\mathbf{D}^{\mathrm{op}})$).

3.3 Some change of notation

In what follows we differ slightly from our previous notation. Let C be a division algebra over F, and U a left C-vector space of dimension n. We take an F-bilinear form $\varphi : U \times U \to C$. We will consider the following three cases:

- C is an arbitrary division algebra with center F and $\varphi = 0$;
- C is one of the first two types referred to in Sect. 2.2 and φ is a nondegenerate ϵ-skew hermitian form on U;
- $C = E$ is a quadratic extension of F and φ is a nondegenerate ϵ-skew hermitian form on U.

We will continue to use this notation until the end of Sect. 7. Let δ^2 be the rank of C over its center. Unless otherwise mentioned we will always denote by $G = G(\mathcal{U})$ the group of isometries of the pair $\mathcal{U} = (U, \varphi)$, which is said to be of type (I) or (II) according as φ is nondegenerate or $\varphi = 0$.

Let $\mathcal{U}^\square$ be the space $U^\square = U \oplus U$ equipped with the ϵ-skew hermitian form $\varphi^\square = \varphi \oplus (-\varphi)$. We shall denote $G(\mathcal{U}^\square)$ simply by $G^\square$. For any subspace I of U we write

$$I^\square = I \oplus I, \qquad I_+ = I \oplus \{0\}, \qquad I_- = \{0\} \oplus I,$$
$$I^\Delta = \bigl\{(x, x) \in U^\square \bigm| x \in I\bigr\}, \qquad I^\nabla = \bigl\{(x, -x) \in U^\square \bigm| x \in I\bigr\}.$$

Let $P = P_{\mathcal{U}}$ be the stabilizer of U^Δ in $G^\square$, $N = N_{\mathcal{U}}$ its unipotent radical and $I(s, \chi) = I^{\mathcal{U}}(s, \chi)$ the representation of $\bar{G}^\square$ defined as before.

In Case (II) $E = F \oplus F$, χ is a pair (χ_1, χ_2) of characters of $F^\times$ and the representation $I^{\mathcal{U}}(s, \chi)$ is defined to be $\mathrm{Ind}_P^{G^\square}(\chi_s \circ \Delta)$, where $\Delta = (\Delta_1, \Delta_2) : P \to E^\times$ is defined by

$$\Delta_1(p) = \nu\bigl(\alpha_{U^\Delta}(p)\bigr), \qquad \Delta_2(p) = \nu(p|_{U^\Delta})^{-1}.$$

3.4 The compound case

We call a subspace Y of U totally isotropic if $\varphi(x, y) = 0$ for all $x, y \in Y$. In particular, any subspace of U is totally isotropic in Case (II). Let $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_k$ be a flag of totally isotropic subspaces of U. We write $P(\mathcal{Y})$ for its stabilizer in G. For reasons on induction we consider a formal sum $\mathcal{Y} = \boxplus_{i=1}^{k+1} \mathcal{Y}_i$ and the ordered product $G(\mathcal{Y}) = \prod_{i=1}^{k+1} G(\mathcal{Y}_i)$, where

$$\mathcal{Y}_{k+1} = \bigl(Y_k^\perp / Y_k, \varphi\bigr), \qquad \mathcal{Y}_i = (Y_i / Y_{i-1}, 0) \quad (i = 1, 2, \ldots, k).$$

In the metaplectic case we set

$$\bar{G}(\mathcal{Y}) = \bar{G}(\mathcal{Y}_1) \times_{\mu_2} \cdots \times_{\mu_2} \bar{G}(\mathcal{Y}_{k+1})$$

and note that an irreducible admissible genuine representation of $\bar{G}(\mathcal{Y})$ is of the form $\sigma_{1,\psi} \boxtimes \sigma_{2,\psi} \boxtimes \cdots \boxtimes \sigma_{k,\psi} \boxtimes \sigma_0$, where σ_0 is an irreducible admissible genuine representation of $\mathrm{Mp}^{(2)}(Y_k^{\perp}/Y_k)$ and σ_i is an irreducible admissible representation of $GL(Y_i/Y_{i-1})$ for $i = 1, 2, \ldots, k$. We call $\mathcal{Y}$ compound if $k > 0$ and simple if $k = 0$. We naturally identify the Levi parts of $P(\mathcal{Y})$ (resp. $\bar{P}(\mathcal{Y})$) with $G(\mathcal{Y})$ (resp. $\bar{G}(\mathcal{Y})$). The notations $\mathcal{U}^{\square}$, $\mathcal{U}^{\Delta}$ and $P_{\mathcal{U}}$ extend in an obvious way to the compound case. When G is of type (I), we define the homomorphism $\Delta^{\mathcal{Y}_i} : P_{\mathcal{Y}_i} \to E^{\times}$ by

$$\Delta^{\mathcal{Y}_i}(p) = \Delta_1^{\mathcal{Y}_i}(p)\Delta_2^{\mathcal{Y}_i}(p)^{\rho} \quad (i = 1, 2, \ldots, k).$$

We set $\Delta^{\mathcal{Y}} = \prod_{i=1}^{k+1} \Delta^{\mathcal{Y}_i}$ and

$$I^{\mathcal{Y}}(s, \chi) = \mathrm{Ind}_{\bar{P}_{\mathcal{Y}}}^{\bar{G}(\mathcal{Y}^{\square})}\big(\chi_s \circ \Delta^{\mathcal{Y}}\big) \simeq \mathop{\boxtimes}_{i=1}^{k+1} \mathrm{Ind}_{\bar{P}_{\mathcal{Y}_i}}^{\bar{G}(\mathcal{Y}_i^{\square})}\big(\chi_s \circ \Delta^{\mathcal{Y}_i}\big).$$

The map

$$f^{(s)} \mapsto f_{\psi}^{(s)}\big((g, t)\big) = \gamma_{\psi}\big((\det g, t)\big) f^{(s)}(g)$$

defines a $\bar{G}(\mathcal{Y}_i)$-equivariant isomorphism of

$$\big(\mathrm{Ind}_{P_{\mathcal{Y}_i}}^{G(\mathcal{Y}_i^{\square})}\big(\chi\alpha_F^s\big) \circ \Delta^{\mathcal{Y}_i}\big)_{\psi} \simeq \mathrm{Ind}_{\bar{P}_{\mathcal{Y}_i}}^{\bar{G}(\mathcal{Y}_i^{\square})}\big(\chi_{\psi}\alpha_F^s\big) \circ \Delta^{\mathcal{Y}_i} \quad (i = 1, 2, \ldots, k).$$

We will consider only the simple case in the proofs below. The extension to the compound case is straightforward.

3.5 Good sections

Let χ be a character of $E^{\times}$. We will frequently write $\rho(\chi)$ for the character defined by $\rho(\chi)(x) = \chi(x^{\rho})$ and write χ^0 for the restriction of χ to $F^{\times}$. Note that $\chi^0 = \chi_1\chi_2$ in Case (II). We define the intertwining operator $M(s, \chi) = M_{\mathcal{U}}(s, \chi) : I^{\mathcal{U}}(s, \chi) \to I^{\mathcal{U}}(-s, \rho(\chi)^{-1})$ for $\Re s \gg 0$ by the integral

$$\big[M(s, \chi) f^{(s)}\big](g) = \int_N f^{(s)}(wug)du, \quad w = (1, -1) \in G \times G$$

and by meromorphic continuation otherwise. When η is a character of $F^{\times}$, the local abelian L-factor is denoted by $L(s, \eta)$. If $x \in \mathbb{R}$, then $[x]$ will denote the biggest integer inferior or equal to x. Put $n' = \delta n$ and

$$a(s,\chi) = \prod_{j=1}^{[n'/2]} L\big(2s - n' + 2j, \chi^2\big),$$
$$b(s,\chi) = \prod_{j=1}^{[n'/2]} L\big(2s + n' + 1 - 2j, \chi^2\big), \qquad (\mathrm{I}_0, \mathrm{I}_1)$$

$$a(s,\chi) = \prod_{j=1}^{n'} L\big(2s - j + 1, \chi^0 \cdot \epsilon_{E/F}^{n'+j}\big),$$
$$b(s,\chi) = \prod_{j=1}^{n'} L\big(2s + j, \chi^0 \cdot \epsilon_{E/F}^{n'+j}\big), \qquad (\mathrm{I}_2, \mathrm{II})$$

$$a(s,\chi) = L\left(s - \frac{n'-1}{2}, \chi\right) \prod_{j=1}^{n'/2} L\big(2s - 2j + 2, \chi^2\big),$$
$$b(s,\chi) = L\left(s + \frac{n'+1}{2}, \chi\right) \prod_{j=1}^{n'/2} L\big(2s + 2j - 1, \chi^2\big), \qquad (\mathrm{I}_3, \mathrm{I}_4^s)$$

$$a(s,\chi) = \prod_{j=1}^{n'/2} L\big(2s - 2j + 1, \chi^2\big),$$
$$b(s,\chi) = \prod_{j=1}^{n'/2} L\big(2s + 2j, \chi^2\big). \qquad (\mathrm{I}_4^m)$$

It is well-known that if F is a p-adic field, $N(\mathfrak{o})$ has volume 1, χ is unramified, ψ is unramified and C is either F or an unramified quadratic extension of F, then

$$M(s,\chi) f_0^{(s)} = \frac{a(s,\chi)}{b(s,\chi)} f_0^{(-s)}. \tag{3.1}$$

We can prove (3.1) by the standard Gindikin–Karpelevich argument. It can also be deduced from Shimura's computation [52, Proposition 15.4] and [53, Theorem 16.2].

Lemma 3.1 *The operator $a(s,\chi)^{-1} M(s,\chi)$ is entire.*

We remark that this result is known in almost all cases. This is proven in [16, 45] in Case (I_4^s), and in [30, 70] in the unitary case. When F is a p-adic field, this is proven in [57] in Case (I_4^m), and in [67] in Cases (I_0), (I_1) and (I_3). So as not to interrupt the flow of the body of the section, we prove the remaining cases in Appendix B. Note that the operator $a(s,\chi)^{-1}M(s,\chi)$ can have a zero when G is not quasisplit.

Choose an element A of the Lie algebra $\mathfrak{n}$ of N as in Appendix B. A normalization $M^\dagger(s,\chi,A,\psi)$ of the operator $M(s,\chi)$ is defined in Definition B.1. Henceforth, we abbreviate $M^\dagger(s,\chi,A,\psi)$ to $M^\dagger(s,\chi)$.

Definition 3.1 Assume that χ^0 is unitary. Let $s' \in \mathbb{C}$ and $f^{(s)}$ a meromorphic section of $I(s,\chi)$. When $\Re s' > -\frac{1}{2}$, we say that $f^{(s)}$ is good at $s = s'$ if it is holomorphic at $s = s'$. When $\Re s' < 0$, we say that $f^{(s)}$ is good at $s = s'$ if $M^\dagger(s,\chi)f^{(s)}$ is holomorphic at $s = s'$. We call $f^{(s)}$ a good section if it is good at every point $s' \in \mathbb{C}$.

The definition extends in an obvious way to the compound case.

Remark 3.1

(1) The assumption on χ^0 does not restrict the generality of Definition 3.1. An arbitrary quasicharacter of $E^\times$ has the form $\chi\alpha_E^t$, where $t \in \mathbb{C}^\times$ and χ is a character of $E^\times$ whose restriction χ^0 is unitary. Since $I(s+t,\chi) = I(s,\chi\alpha_E^t)$, we can demand that χ^0 is unitary by a shift in s. We say that a meromorphic section $f^{(s)}$ of $I(s,\chi\alpha_E^t)$ is a good section if $h^{(s)} = f^{(s-t)}$ is.

(2) In view of Lemma B.1 there is an entire function $\alpha(s,\chi,A,\psi)$ having no zeros such that

$$M^\dagger(s,\chi,A,\psi) = \alpha(s,\chi,A,\psi)\frac{b(-s,\rho(\chi)^{-1})}{a(s,\chi)}M(s,\chi).$$

Thus Definition 3.1 is independent of the choice of A and ψ.

(3) By Proposition 3.1(2) below the two definitions coincide in the strip $-\frac{1}{2} < \Re s' < 0$.

(4) We allow here the specific case $n = 0$ so that $G^\square$ is a trivial group, $I(s,\chi)$ is a trivial representation and

$$a(s,\chi) = b(s,\chi) = 1, \qquad (\mathrm{I}_0\text{–}\mathrm{I}_2, \mathrm{I}_4^m, \mathrm{II})$$

$$a(s,\chi) = b(s,\chi) = L(s+1/2,\chi). \qquad (\mathrm{I}_3, \mathrm{I}_4^s)$$

Good sections are $L(s+1/2,\chi)$ times entire functions in Cases (I_3) and (I_4^s), and they are entire functions in the other cases.

The Mittag-Leffler theorem may be combined with the Weierstrass factorization theorem to give the following result:

Lemma 3.2 ([50, Theorem 15.13]) *Suppose that $B \subset \mathbb{C}$ has no limit point, and to each $\lambda \in B$ there are associated a nonnegative integer $m(\lambda)$ and complex numbers $w_{j,\lambda}$ $(0 \le j \le m(\lambda))$. Then there exists an entire function whose jth derivative at $s = \lambda$ is equal to $w_{j,\lambda}$ for each $\lambda \in B$ and $0 \le j \le m(\lambda)$.*

Proposition 3.1

(1) *Holomorphic sections are good sections.*
(2) *If $f^{(s)}$ is a good section of $I(s,\chi)$, then $b(s,\chi)^{-1} f^{(s)}$ is a holomorphic section.*
(3) *If $f^{(s)}$ is a meromorphic section which is good at $s = s'$, then there is a good section $F^{(s)}$ such that $f^{(s)} - F^{(s)}$ has a zero of any prescribed order at $s = s'$.*
(4) *For a meromorphic section $f^{(s)}$ of $I(s,\chi)$ the following conditions are equivalent*:
    (a) *$f^{(s)}$ is a good section*;
    (b) *$h^{(s)} = M^\dagger(-s,\chi) f^{(-s)}$ is a good section of $I(s, \rho(\chi)^{-1})$*;
    (c) *there are holomorphic sections $f_1^{(s)}$ of $I(s,\chi)$ and $f_2^{(-s)}$ of $I(-s, \rho(\chi)^{-1})$ such that $f^{(s)} = f_1^{(s)} + M^\dagger(-s, \rho(\chi)^{-1}) f_2^{(-s)}$.*
(5) *If $\delta = 1$, F is a p-adic field, E/F is unramified, χ is unramified and ψ is unramified, then*

$$b(s,\chi) M^\dagger(s,\chi) f_0^{(s)} = b\big(-s, \rho(\chi)^{-1}\big) f_0^{(-s)}.$$

In particular, $b(s,\chi) f_0^{(s)}$ is a good section.

Proof There is no harm in assuming that χ^0 is unitary. Then $b(s,\chi)$ has no poles in the right half-plane $\Re s > -\frac{1}{2}$. Lemma 3.1 and Remark 3.1(2) prove the first assertion, and the identity (B.3) below proves the second assertion and the equivalence of (a) and (b).

Let A_f be the set of all poles of $f^{(s)}$, and for each $\lambda \in A_f$, let $m_f(\lambda)$ be the order of the pole of $f^{(s)}$ at $s = \lambda$. Let β be an entire function which has a zero of order $m_f(\lambda)$ at each $\lambda \in A_f \setminus \{s'\}$ and such that $\beta(s) - 1$ has the necessary order at $s = s'$. Lemma 3.2 ensures the existence of such β. Then $\beta(s) f^{(s)}$ has removable singularities at the points of $A_f \setminus \{s'\}$ and has the same principal part as that of $f^{(s)}$ at $s = s'$. Hence the third assertion follows.

Part (c) implies (a), owing to (1). We now derive (c) from (a). Since A_f and $\{-s' \mid s' \in A_h\}$ have an empty intersection, Lemma 3.2 gives an entire function β satisfying the following conditions:

(i) $\beta(-s)$ has a zero of order $m_h(s')$ at each $s' \in A_h$;
(ii) $\beta(s) - 1$ has a zero of order $m_f(s')$ at each $s' \in A_f$.

Put $f_2^{(s)} = \beta(-s)h^{(s)}$. Then $f_2^{(s)}$ is a holomorphic section by (i). The singularities of $f^{(s)} - M^\dagger(-s, \rho(\chi)^{-1}, A, \psi^{-1}) f_2^{(-s)} = (1 - \beta(s)) f^{(s)}$ at the points of A_f are removable by (ii), hence we can extend it to a holomorphic section. Thus (a) implies (c) as claimed.

Noting that $\alpha(s, \chi, A, \psi) = 1$ in the notation of Remark 3.1(2), we get the last part of the proposition by (3.1). □

4 The local doubling zeta integrals

4.1 The basic results

Let Z_G stand for the center of G. Let π be an admissible representation of $\bar{G}$ on V_π. We assume π to be genuine in the metaplectic case. We occasionally identify the space V_π with π itself when there is no danger of confusion. In the p-adic case we say that π admits a central character if there is a character χ_π of $\bar{Z}_G$ such that $\pi(a) = \chi_\pi(a)\mathrm{Id}_{V_\pi}$ for $a \in \bar{Z}_G$. In the archimedean case we denote by $\mathfrak{g}$ the complexified Lie algebra of the real Lie group G and consider only admissible representations of the pair $(\mathfrak{g}, \bar{K})$ although we will often allow ourselves to speak of a representation of $\bar{G}$. One can define its contragredient representation $\pi^\vee$, its central characters χ_π, and its matrix coefficients, even though π is not a representation of the group $\bar{G}$.

For any continuous homomorphism $\mu : G \to \mathbb{C}^\times$ we define a representation $\pi \otimes \mu$ on the same space V_π by $(\pi \otimes \mu)(g) = \mu(g)\pi(g)$. To simplify notation, we frequently regard a character χ of $E^\times$ (or $F^\times$ in Case (II)) as defining a character of G via $g \mapsto \chi(\nu(g))$ and abbreviate $\pi \otimes (\chi \circ \nu)$ to $\pi \otimes \chi$. In Cases (I_1), (I_3) and (I_4) $\pi \otimes \chi = \pi$. We write $\pi^\vee$ for its admissible dual and denote by $\mathcal{P} = \mathcal{P}_{\pi^\vee} : \pi^\vee \boxtimes \pi \to \mathbb{C}$ the standard pairing.

We return to the notation of the earlier section on $\mathcal{U} = (U, \varphi)$ and use the doubling method to give the definition of the local factors of representations of $\bar{G}$. We identify $G \times G$ with the subgroup of $G^\Box$ preserving U_+ and U_-. For any subgroup J of G, put

$$J^\diamondsuit = \{(g, g) \in G^\Box \mid g \in J\}.$$

For $\xi \in \pi^\vee \boxtimes \pi$ let $H_\xi(g) = \mathcal{P}((\pi^\vee \boxtimes \pi)(g, e)\xi)$ be the corresponding matrix coefficient of $\pi^\vee$.

For any section $f^{(s)}$ of $I(s, \chi)$ we define the zeta integral by

$$Z\big(\xi, f^{(s)}\big) = Z^{\mathcal{U}}\big(\xi, f^{(s)}\big) = \int_{G^{\diamond}\backslash G\times G} \kappa_\chi(g)^{-1}\mathcal{P}\big(\big(\pi^\vee \boxtimes \pi\big)(g)\xi\big) f^{(s)}(g)dg$$
$$= \int_G H_\xi(g) f^{(s)}\big((g,e)\big)dg,$$

where we define the character $\kappa_\chi = \kappa_\chi^{\mathcal{U}}$ of $G \times G$ by

$$\kappa_\chi(g_1, g_2) = 1, \qquad (\mathrm{I}_1, \mathrm{I}_3, \mathrm{I}_4)$$
$$\kappa_\chi(g_1, g_2) = \chi\big(\nu(g_2)\big), \qquad (\mathrm{I}_0, \mathrm{I}_2)$$
$$\kappa_\chi(g_1, g_2) = \chi_1\big(\nu(g_2)\big)\chi_2\big(\nu(g_2)\big)^{-1}. \qquad (\mathrm{II})$$

Note that since the integrand is a product of two genuine functions on the metaplectic cover in Case (I_4^m), it is defined on G. We will sometimes denote $Z(\xi, f^{(s)})$ by $Z(H_\xi, f^{(s)})$. The integral is convergent for s in some open set and is to be interpreted by meromorphic continuation for the remaining values of s.

Let $\mathrm{Irr}(\bar{G})$ be the set of equivalence classes of irreducible admissible representations of $\bar{G}$ which are required to be genuine in the metaplectic case. The following basic theorem is stated in [43, 44] and demonstrated in [4, 32], provided that C is commutative. The argument can be easily modified to deal with the general case.

Theorem 4.1 (Cf.[4, 32, 43, 44]) *Suppose that* $\pi \in \mathrm{Irr}(\bar{G})$.

(1) *There is* $\sigma \in \mathbb{R}$ *such that for all* $\xi \in \pi^\vee \boxtimes \pi$ *and holomorphic sections* $f^{(s)}$ *of* $I(s,\chi)$ *the integrals defining* $Z(\xi, f^{(s)})$ *converge absolutely in the right half-plane* $\Re s > \sigma$.
(2) *The integral extends to a meromorphic function in* s, *which represents a rational function of* q^{-s} *if* F *is a* p*-adic field and* $f^{(s)}$ *is a standard section.*
(3) *There is a meromorphic function* $\Gamma^{\mathcal{U}}(s, \pi \times \chi, A, \psi)$ *such that for all* $\xi \in \pi^\vee \boxtimes \pi$ *and meromorphic sections* $f^{(s)}$ *of* $I(s,\chi)$

$$Z\big(\xi, M_{\mathcal{U}}^\dagger(s,\chi,A,\psi) f^{(s)}\big) = \Gamma^{\mathcal{U}}(s, \pi \times \chi, A, \psi) Z\big(\xi, f^{(s)}\big).$$

It is useful to translate the above notions in terms of equivariant maps. Fix any holomorphic section $f^{(s)}$ of $I(s,\chi)$. Since $f^{(s)}$ is right $\bar{K} \times \bar{K}$-finite, uniformly in s, for s in the realm of absolute convergence of the integrals, the map $\xi \mapsto Z(\xi, f^{(s)})$ gives a $\bar{K} \times \bar{K}$-finite bilinear form on $\pi^\vee \times \pi$ which varies holomorphically with s, or equivalently an element $Z(s,\pi,\chi) f^{(s)} \in \pi \boxtimes \pi^\vee$, depending holomorphically on s, such that for all $\xi \in \pi^\vee \boxtimes \pi$

$$\mathcal{P}_{\pi^\vee\boxtimes\pi}\big(\xi \otimes Z(s,\pi,\chi) f^{(s)}\big) = Z\big(\xi, f^{(s)}\big).$$

We have seen that the map $Z(s,\pi,\chi): I(s,\chi) \to (\pi \boxtimes \pi^\vee) \otimes \kappa_\chi$ can be continued as a meromorphic function in s which satisfies

$$Z\big(-s,\pi,\rho(\chi)^{-1}\big) \circ M^\dagger(s,\chi,A,\psi) = \Gamma(s,\pi\times\chi,A,\psi)Z(s,\pi,\chi),$$

the meaning of this assertion being clear since $Z(s,\pi,\chi)f^{(s)}$ takes values in a finite dimensional subspace of $\pi \boxtimes \pi^\vee$, uniformly in s.

4.2 Representations induced from parabolic subgroups

Let $Q = LN$ be a parabolic subgroup of G, δ_Q the modulus function of Q, σ an admissible representation of $\bar{L}$ on V_σ and proj_L the canonical projection from Q onto L. We construct, out of its pull-back to $\bar{Q}$, an admissible representation $\pi = \mathrm{Ind}_{\bar{Q}}^{\bar{G}} \sigma$ of $\bar{G}$ by right translations on the space of smooth functions $f : \bar{G} \to V_\sigma$ satisfying

$$f(pg) = \delta_Q(p)^{1/2}\sigma(p)f(g) \quad (p \in \bar{Q},\ g \in \bar{G}).$$

For $\xi \in \mathrm{Ind}_{\bar{Q}\times\bar{Q}}^{\bar{G}\times\bar{G}}(\sigma^\vee \otimes \sigma)$ the function $H : \bar{G}\times\bar{G} \to \mathbb{C}$ defined by $H(g) = \mathcal{P}_{\sigma^\vee}(\xi(g))$ transforms on the left according to

$$H(pg) = \delta_Q(p)H(g) \quad \big(p \in \bar{Q}^\diamondsuit,\ g \in \bar{G}\times\bar{G}\big).$$

Denote by $H \mapsto \int_{Q^\diamondsuit\backslash G^\diamondsuit} H(g)dg$ a positive right invariant form on the space of continuous functions on $G^\diamondsuit$ satisfying $H(pg) = \delta_Q(p)H(g)$ for $p \in Q^\diamondsuit$. We may set

$$\mathcal{P}_{\pi^\vee}(\xi) = \int_{Q^\diamondsuit\backslash G^\diamondsuit} \mathcal{P}_{\sigma^\vee}\big(\xi(g)\big)dg.$$

This invariant pairing allows us to identify $\pi^\vee$ with $\mathrm{Ind}_{\bar{Q}}^{\bar{G}} \sigma^\vee$. In particular, $\mathrm{Ind}_{\bar{Q}}^{\bar{G}} \sigma$ is unitary whenever σ is unitary.

Let $\pi \in \mathrm{Irr}(\bar{G})$. We say that π is supercuspidal if the matrix coefficients of π are compactly supported modulo the center of $\bar{G}$. We say that π is square integrable if χ_π is unitary and the matrix coefficients of π are square integrable modulo the center of $\bar{G}$. We say that π is tempered if it is a component of a representation $\mathrm{Ind}_{\bar{Q}}^{\bar{G}} \sigma$ induced from a square integrable representation σ of $\bar{Q}/N$ to $\bar{G}$. If G is of type (II), then we write $e(\pi)$ for the unique real number such that $\pi \otimes \alpha_F^{-e(\pi)}$ has unitary central character. We call π essentially tempered if $\pi \otimes \alpha_F^{-e(\pi)}$ is tempered. If G is of type (I), then we put $e(\pi) = 0$.

Let $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_k$ be a flag of totally isotropic subspaces of U. Let $\sigma = \sigma_1 \boxtimes \cdots \boxtimes \sigma_k \boxtimes \sigma_0$ be an irreducible admissible essentially tempered representation of $\bar{G}(\mathcal{Y})$ which satisfies $e(\sigma_1) < e(\sigma_2) < \cdots < e(\sigma_k) < e(\sigma_0)$. Then $\mathrm{Ind}_{\bar{P}(\mathcal{Y})}^{\bar{G}} \sigma$ is referred to as a standard module and has a unique irreducible quotient, which we call the Langlands quotient. Bear in mind that G acts on U on the right. It should be remarked that when the Langlands quotient is considered, most papers take a left action of the group and then, standard modules have the exponents in a decreasing order. A general irreducible admissible representation of $\bar{G}$ can be written essentially uniquely as a Langlands quotient of a standard module.

4.3 Multiplicativity

Assume now that Q is a maximal parabolic subgroup, i.e., $Q = P(Y)$ for a totally isotropic subspace Y of U of dimension a over C. We define the homomorphism $|\Delta_{\mathcal{Y};\mathcal{U}}| : P(Y^{\Box}) \to \mathbb{R}_{+}^{\times}$ via

$$\left|\Delta_{\mathcal{Y};\mathcal{U}}(p)\right| = \left|\nu(p|_{Y^{\Box}})\right|_{E}^{\delta(n-a)/2}, \tag{I}$$

$$\left|\Delta_{\mathcal{Y};\mathcal{U}}(p)\right| = \left|\nu\big(\alpha_{Y^{\Box}}(p)\big)\right|_{F}^{-\delta a/2}\left|\nu(p|_{Y^{\Box}})\right|_{F}^{\delta(n-a)/2}. \tag{II}$$

We remind the reader that G acts on U on the right, whereas it acts on the left in [32]. To begin, we record necessary results included in [32] (see also [4]). Though Lapid and Rallis [32] restrict their attention to the case when C is commutative and $\chi_1 = \chi_2$, their argument adapts without essential change to the general case. Recall that $P = P(U^{\Delta})$.

Lemma 4.1 *Notation being as above, we put $S = P \cap P(Y^{\Box})$.*

(1) *The injection $N(Y)^{\Diamond}\backslash N(Y) \times N(Y) \to N(Y^{\Box}) \cap P\backslash N(Y^{\Box})$ is an isomorphism.*
(2) *The image of S under $\mathrm{proj}_{M(Y^{\Box})}$ is the parabolic subgroup $P_{\mathcal{Y}}$ of $G(\mathcal{Y}^{\Box})$. Its unipotent radical is the image of $N \cap P(Y^{\Box})$.*
(3) *The restriction of $\Delta_{\mathcal{U}}$ to S is $\Delta_{\mathcal{Y}} \circ \mathrm{proj}_{M(Y^{\Box})}$.*
(4) *Let $\delta_{S;N(Y^{\Box})\cap P\backslash N(Y^{\Box})}$ denote the modulus character of S on $N(Y^{\Box}) \cap P\backslash N(Y^{\Box})$. Then for $m \in S$*

$$\delta_P(m)^{1/2} \cdot \delta_{S;N(Y^{\Box})\cap P\backslash N(Y^{\Box})}(m) = \delta_{P_{\mathcal{Y}}}(m)^{1/2}\left|\Delta_{\mathcal{Y};\mathcal{U}}(m)\right|\delta_{P(Y^{\Box})}(m)^{1/2}.$$

(5) *The restriction of $|\Delta_{\mathcal{Y};\mathcal{U}}|\delta_{P(Y^{\Box})}^{1/2}$ to $P(Y) \times P(Y)$ is $\delta_{P(Y)\times P(Y)}^{1/2}$.*

Proof The proof is a straightforward generalization of the arguments in Sect. 4 of [32]. □

For $\xi \in \mathrm{Ind}_{\bar{P}(Y)\times\bar{P}(Y)}^{\bar{G}\times\bar{G}}(\sigma^\vee \boxtimes \sigma)$, we get

$$\begin{aligned} Z^{\mathcal{U}}\big(\xi, f^{(s)}\big) &= \int_{G^\diamondsuit\backslash G\times G} \kappa_\chi(g)^{-1} f^{(s)}(g) \int_{P(Y)^\diamondsuit\backslash G^\diamondsuit} \mathcal{P}_{\sigma^\vee}\big(\xi(xg)\big)dxdg \\ &= \int_{P(Y)^\diamondsuit\backslash G\times G} \kappa_\chi(g)^{-1} f^{(s)}(g)\mathcal{P}_{\sigma^\vee}\big(\xi(g)\big)dg \\ &= \int_{P(Y)^\times P(Y)\backslash G\times G} \int_{P(Y)^\diamondsuit\backslash P(Y)^\times P(Y)} \delta_{P(Y)\times P(Y)}(p)^{-1/2} \\ &\quad \times \kappa_\chi(pg)^{-1}\mathcal{P}_{\sigma^\vee}\big(\big(\sigma^\vee \boxtimes \sigma\big)(p)\xi(g)\big) f^{(s)}(pg)dpdg. \end{aligned}$$

Writing $dp = dmdu$, we get

$$\begin{aligned} &\int_{P(Y)^\times P(Y)\backslash G\times G} \int_{M(Y)^\diamondsuit\backslash M(Y)^\times M(Y)} \delta_{P(Y)\times P(Y)}(m)^{-1/2} \\ &\quad \times \kappa_\chi^{\mathcal{Y}}(m)^{-1}\kappa_\chi^{\mathcal{U}}(g)^{-1}\mathcal{P}_{\sigma^\vee}\big(\big(\sigma^\vee \boxtimes \sigma\big)(m)\xi(g)\big)\big[J^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\big](mg)dmdg, \end{aligned}$$

where

$$\big[J^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\big](g) = \int_{N(Y)^\diamondsuit\backslash N(Y)\times N(Y)} f^{(s)}(ug)du, \quad g \in \bar{G}^\square.$$

From Lemma 4.1(1)

$$\big[J^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\big](g) = \int_{N(Y^\square)\cap P\backslash N(Y^\square)} f^{(s)}(ug)du.$$

Lemma 4.1(2), (3), (4) gives the last part of the following proposition:

Proposition 4.1 *The integral defining $J^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}$ converges for $\Re s \gg 0$ and has a meromorphic continuation to the whole s-plane. Moreover, the map $f^{(s)} \mapsto [\Psi(s,\chi)f^{(s)}]$, where*

$$\big[\Psi(s,\chi)f^{(s)}\big](g) : p \mapsto \big|\Delta_{\mathcal{Y};\mathcal{U}}(p)\big|^{-1}\delta_{P(Y^\square)}(p)^{-1/2}\big[J^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\big](pg),$$

defines an intertwining map

$$\Psi^{\mathcal{U},\mathcal{Y}}(s,\chi) = \Psi(s,\chi) : I^{\mathcal{U}}(s,\chi) \to \mathrm{Ind}_{\bar{P}(Y^\square)}^{\bar{G}^\square}\big(I^{\mathcal{Y}}(s,\chi) \otimes |\Delta_{\mathcal{Y};\mathcal{U}}|\big).$$

Proof This is nothing but Proposition 1 of [32]. □

Taking Lemma 4.1(5) into account, we finally get

$$Z^{\mathcal{U}}\big(\xi, f^{(s)}\big) = \int_{P(Y)\times P(Y)\backslash G\times G} \kappa_\chi^{\mathcal{U}}(g)^{-1} Z^{\mathcal{Y}}\big(\xi(g), \big[\Psi(s,\chi)f^{(s)}\big](g)\big)dg$$

$$= \int_{P(Y)\times P(Y)\backslash G\times G} \kappa_\chi^{\mathcal{U}}(g)^{-1}\mathcal{P}_{\sigma^\vee\boxtimes\sigma}\big(\xi(g), Z^{\mathcal{Y}}(s,\sigma,\chi)$$
$$\times\big[\Psi(s,\chi)f^{(s)}\big](g)\big)dg,$$

where one can easily check that for all s with $\Re s$ large enough

$$\kappa_\chi^{\mathcal{U}}(g)^{-1}Z^{\mathcal{Y}}(s,\sigma,\chi)\big[\Psi^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\big](g) \in \operatorname{Ind}_{\bar{P}(Y)\times\bar{P}(Y)}^{\bar{G}\times\bar{G}}\big(\sigma\boxtimes\sigma^\vee\big).$$

In summary we obtain the following:

Proposition 4.2 *If σ is an admissible representation of $\bar{M}(Y)$ and $\pi = \operatorname{Ind}_{\bar{P}(Y)}^{\bar{G}}\sigma$, then*

$$Z^{\mathcal{U}}(s,\pi,\chi) = \kappa_\chi^{-1}\cdot \operatorname{Ind}_{\bar{P}(Y)\times\bar{P}(Y)}^{\bar{G}\times\bar{G}} Z^{\mathcal{Y}}(s,\sigma,\chi)\circ\Psi^{\mathcal{U},\mathcal{Y}}(s,\chi).$$

This is merely a restatement of Proposition 2 of [32].

We hereafter suppose that A is maximally split. If $A\in\mathfrak{n}$ satisfies $A(Y^\Box)\subset Y^\Delta$, namely, $Y^\Box/Y^\Delta$ is a totally isotropic subspace of $(W^\Box/W^\Delta, A)$, then $A(Y^{\perp\Box})\subset Y^{\perp\Delta}$, and it gives rise to an element B in the Lie algebra of $N_{\mathcal{Y}}$. In our later discussion we will let A be maximally split, and as such, we can always assume that A preserves any given totally isotropic subspace of U by changing A to a congruent form.

Lemma 4.2 *Notation and assumption being as above, we have*

$$\Psi\big(-s,\rho(\chi)^{-1}\big)\circ M_{\mathcal{U}}^\dagger(s,\chi,A,\psi) = \operatorname{Ind}_{\bar{P}(Y^\Box)}^{\bar{G}^\Box} M_{\mathcal{Y}}^\dagger(s,\chi,B,\psi)\circ\Psi(s,\chi).$$

Proof This is substantially the same as [32, Lemma 9]. The normalized intertwining operator is defined in Definition B.1 slightly differently than [32], but the normalization factor pertaining to A is multiplicative in the obvious sense. Note that if G is of type (I), then

$$\nu_U(A) = \nu_Y(A|_{Y^\Box})\nu_{Y^\perp/Y}\big(\alpha_{Y^\Box}(A|_{Y^{\perp\Box}})\big)\nu_{U/Y^\perp}\big(\alpha_{Y^{\perp\Box}}(A)\big)$$
$$= (-1)^{\delta a}\nu_Y(A|_{Y^\Box})\nu_Y(A|_{Y^\Box})^\rho\nu_{Y^\perp/Y}\big(\alpha_{Y^\Box}(A|_{Y^{\perp\Box}})\big)$$

since the $\varphi^\Box$-dual of $Y^\Box/Y^\Delta$ is $(U/Y^\perp)^\Delta$. □

5 *L* and epsilon factors

5.1 The local factors of Godement and Jacquet

The local factors for the groups of type (II) is of interest in its own right, and is indispensable even in considering Case (I) as they sit inside G as factors of Levi subgroups of parabolic subgroups. Let $G = \mathrm{GL}_n(C)$ in this section. In the p-adic case we write $\mathcal{O}$ for the maximal compact subring of C and put $K = \mathrm{GL}_n(\mathcal{O})$. In the archimedean case we set $K = \{g \in \mathrm{GL}_n(C) \mid g^*g = \mathbf{1}_n\}$. Similarly, we define a maximal compact subgroup $K^\square$ of $G^\square$.

Let $\mathcal{S}(\mathrm{M}_{ab}(C))$ be the space of Schwartz functions on $\mathrm{M}_{ab}(C)$. The Fourier transform $\hat{\phi} \in \mathcal{S}(\mathrm{M}_{ba}(C))$ of $\phi \in \mathcal{S}(\mathrm{M}_{ab}(C))$ is defined by

$$\hat{\phi}(x) = \int_{\mathrm{M}_{ab}(C)} \phi(y)\psi\big(\tau(xy)\big)dy,$$

where the Haar measure dy is so chosen that $\int_{\mathrm{M}_{ab}(C)} \hat{\phi}({}^t y)dy = \phi(0)$. Let $S(\mathrm{M}_n(C))$ be the subspace of $\mathcal{S}(\mathrm{M}_n(C))$ as defined on p. 115 of [11] in the archimedean case. That subspace is dense in $\mathcal{S}(\mathrm{M}_n(C))$, invariant under K acting on the right and left, invariant by the universal enveloping algebra of $\mathfrak{g}$ and also by the Fourier transform. We set $S(\mathrm{M}_n(C)) = \mathcal{S}(\mathrm{M}_n(C))$ in the p-adic case.

Let π be an irreducible admissible representation of G. For $s \in \mathbb{C}$, $\phi \in \mathcal{S}(\mathrm{M}_n(C))$ and a matrix coefficient H of π we set

$$Z^{GJ}(s,\phi,H) = \int_G H(g)\phi(g)\big|\nu(g)\big|_F^{s+\delta n/2} d^\times g.$$

These integrals converge in some half-plane and can be continued meromorphically to the whole s-plane. We associate to each $\phi \in S(\mathrm{M}_n(C))$ a $\mathbb{C}$-linear endomorphism $Z^{GJ}(s,\pi,\phi)$ on π by

$$\mathcal{P}_\pi\big(Z^{GJ}(s,\pi,\phi)\xi\big) = Z^{GJ}(s,\phi,H_\xi), \quad \xi \in \pi \boxtimes \pi^\vee,$$

where $H_\xi(g) = \mathcal{P}((\pi \boxtimes \pi^\vee)(g,e)\xi)$. The L-factor $L(s,\pi)$ is a minimal Euler factor such that the ratios $Z^{GJ}(s,\pi,\phi)/L(s+1/2,\pi)$ are entire for all $\phi \in S(\mathrm{M}_n(C))$. In the archimedean case the local integral divided by $L(s+1/2,\pi)$ is continuous on $\mathcal{S}(\mathrm{M}_n(C))$ (see [19]), and hence $L(s,\pi)$ is precisely the Euler factor determined by the poles of the family of local integrals using functions in $S(\mathrm{M}_n(C))$ or in $\mathcal{S}(\mathrm{M}_n(C))$, or even in $C_c^\infty(\mathrm{M}_n(C))$. The local integrals satisfy the functional equation

$$\frac{Z^{GJ}(-s,\hat{\phi},\check{H})}{L^{GJ}(\frac{1}{2}-s,\pi^\vee)} = (-1)^{n(\delta-1)}\varepsilon^{GJ}\left(s+\frac{1}{2},\pi,\psi\right)\frac{Z^{GJ}(s,\phi,H)}{L^{GJ}(s+\frac{1}{2},\pi)},$$

where $\check{H}(g) = H(g^{-1})$ (see [11]). The gamma factor is defined via the formula

$$\gamma^{GJ}(s,\pi,\psi) = \varepsilon^{GJ}(s,\pi,\psi)L^{GJ}\big(1-s,\pi^\vee\big)\big/L^{GJ}(s,\pi).$$

The functional equation above now reads

$$Z^{GJ}\left(-s, \pi^{\vee}, \hat{\phi}\right) = (-1)^{n(\delta-1)} \gamma^{GJ}\left(s + \frac{1}{2}, \pi, \psi\right) {}^{t}Z^{GJ}(s, \pi, \phi).$$

Proposition 5.1 ([19, Remark 3.2.4]) *If* $\pi \in \mathrm{Irr}(G)$ *is tempered, then the poles of* $L^{GJ}(s, \pi)$ *are contained in* $\Re s \leq 0$.

Theorem 5.1 (Jacquet [19]) *If* π *is the Langlands quotient of a standard module* $\mathrm{Ind}_Q^G(\sigma_1 \boxtimes \cdots \boxtimes \sigma_t)$, *then*

$$L^{GJ}(s, \pi) = \prod_{j=1}^{t} L^{GJ}(s, \sigma_j), \qquad L^{GJ}\left(s, \pi^{\vee}\right) = \prod_{j=1}^{t} L^{GJ}\left(s, \sigma_j^{\vee}\right),$$

$$\varepsilon^{GJ}(s, \pi, \psi) = \prod_{j=1}^{t} \varepsilon^{GJ}(s, \sigma_j, \psi).$$

These results are proven in [19] when $\delta = 1$. The same proof applies to the case $\delta > 1$.

5.2 An inductive structure of good sections

We return to the general setting of Sect. 3.3. We begin by reviewing a criterion for convergence of intertwining operators. Fix a maximal split torus $T^{\square}$ in $G^{\square}$. Let $M_0^{\square}$ be its centralizer in $G^{\square}$, which is a minimal Levi subgroup, $\mathrm{N}(T^{\square})$ its normalizer in $G^{\square}$ and $X^*(T^{\square})$ the group of its rational characters. Let $W(T^{\square}) = \mathrm{N}(T^{\square})/M_0^{\square}$ be the Weyl group of $G^{\square}$. For $w \in W(T^{\square})$ and $x \in X^*(T^{\square})$ we define $x^w \in X^*(T^{\square})$ by $x^w(t) = x(\tilde{w}^{-1} t \tilde{w})$ for $t \in T^{\square}$, where $\tilde{w}$ is a representative of w in $\mathrm{N}(T^{\square})$. For a minimal parabolic subgroup $P_0^{\square}$ of $G^{\square}$ containing $M_0^{\square}$ we denote by $U_0^{\square}$ its unipotent radical, by $U_0^{\square-}$ the unipotent radical of the minimal parabolic subgroup opposite to $P_0^{\square}$ and by $\mathcal{R}(P_0^{\square})$ the set of positive F-roots of $T^{\square}$ in $G^{\square}$ determined by $U_0^{\square}$, that is, the Lie algebra of $U_0^{\square}$ is the direct sum of root subspaces for roots in this subset.

For a (genuine) character μ of $\bar{M}_0^{\square}$, let $\mathrm{Ind}_{\bar{P}_0^{\square}}^{\bar{G}^{\square}} \mu$ denote the induced representation consisting of right $\bar{K}^{\square}$-finite functions $f : \bar{G}^{\square} \to \mathbb{C}$ which satisfy $f(bg) = \mu(b)\delta_{P_0^{\square}}(b)^{1/2} f(g)$ for $b \in \bar{P}_0^{\square}$ and $g \in \bar{G}^{\square}$. For another minimal parabolic subgroup $Q_0^{\square} = M_0^{\square} N_0^{\square}$ of $G^{\square}$ we set

$$\left[J_{Q_0^{\square}|P_0^{\square}}(\mu) f\right](g) = \int_{U_0^{\square} \cap N_0^{\square} \backslash N_0^{\square}} f(ug)\, du.$$

Let $\langle\ ,\ \rangle$ be a $W(T^\square)$-invariant product on $X^*(T^\square)\otimes_{\mathbb{Z}}\mathbb{R}$.

Proposition 5.2 *If $\langle \Re\mu, \beta\rangle > 0$ for all $\beta \in \mathcal{R}(P_0^\square)\setminus\mathcal{R}(Q_0^\square)$, then the integral defining $J_{Q_0^\square|P_0^\square}(\mu)f$ converges absolutely.*

For the proof of this fact we refer to [21, Proposition 7.8] for the archimedean case and to [63, Proposition IV.2.1] for the p-adic case.

Lemma 5.1 *If χ^0 is unitary, then the integral defining $J^{\mathcal{U},\mathcal{Y}}(s,\chi)$ converges absolutely uniformly for s in a compact set in the right half-plane $\Re s > -\delta/2$.*

Proof We first argue in Case (I). Take a totally isotropic subspace X of U so that X and Y are nondegenerately paired by φ. Let Z be the orthogonal complement of $X+Y$ in U. Choose a D-basis of $U^\square$ compatible with the decomposition

$$U^\square = X_+ \oplus Z^\nabla \oplus Y_- \oplus Y^\Delta \oplus Z^\Delta \oplus X^\Delta$$

such that the form $\varphi^\square$ is expressed in the matrix form as follows:

$$\varphi^\square(x,y) = x w_n y^*, \qquad w_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\epsilon\mathbf{1}_n & 0 \end{pmatrix}$$

for $x, y \in U^\square$. Put

$$T^\square = \left\{ m(t) = \operatorname{diag}\left[t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1}\right] \,\middle|\, t_i \in F^\times \right\}.$$

Let x_i be the rational characters of $T^\square$ defined by $x_i : m(t) \mapsto t_i$ for $i = 1,2,\dots,n$. The group $X^*(T^\square)$ is isomorphic to $\mathbb{Z}^n$ with $\{x_i\}$ its canonical basis. We can easily check that

$$N\bigl(Y^\square\bigr) = \bigl(N\bigl(Y^\square\bigr)\cap P\bigr)\cdot\bigl(N\bigl(Y^\square\bigr)\cap N\bigl(X_+\oplus Z^\nabla\oplus Y_-\bigr)\bigr)$$

(cf. [32, Lemma 6]). It follows from Lemma 4.1(1) that

$$\bigl[J^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\bigr](g) = \int_{N(Y^\square)\cap N(X_+\oplus Z^\nabla\oplus Y_-)} f^{(s)}(ug)\,du.$$

Take a minimal parabolic subgroup $P_0^\square$ of $G^\square$ so that

$$\mathcal{R}\bigl(P_0^\square\bigr) = \{x_i + x_j \mid i \le j\} \cup \{x_k - x_l \mid k < l\}. \tag{I$_1$–I$_4$}$$

Recall that $a = \dim_C Y$. Note that the Lie algebra of $N(Y^\square)\cap N(X_+\oplus Z^\nabla\oplus Y_-)$ is the sum of root subspaces corresponding to the roots

$$\Psi = \{-x_i - x_j \mid i \le j,\ a+1 \le i \le n,\ n-a+1 \le j \le n\}. \tag{I$_1$–I$_4$}$$

In Case (I_0) $i \le j$ must be replaced by $i < j$. Note that the restriction of Δ to $T^\square$ is given by $\delta^{-1}d_0 \cdot \mathbb{1}$, where $\mathbb{1} = (1, 1, \dots, 1)$. We can regard $I(s, \chi)$ as a subrepresentation of $\mathrm{Ind}_{\bar{P}_0^\square}^{\bar{G}^\square} \mu_{s,\chi}$, where $\delta_{P_0^\square}$ and the real part of $\mu_{s,\chi}$, when restricted to $T^\square$, are easily seen to be

$$\delta_{P_0^\square} = 2d_0(n-1, n-2, \dots, 0) + 2d_\epsilon \cdot \mathbb{1},$$

$$\Re\mu_{s,\chi} = d_0(1, 2, \dots, n) + d_0\left(\delta^{-1}\Re s - \frac{n+1}{2}\right) \cdot \mathbb{1}.$$

There is an element w_a of $N(T^\square)$ such that

$$x_i^{w_a} = \begin{cases} x_i & \text{if } 1 \le i \le a, \\ x_{i+a} & \text{if } a+1 \le i \le n-a, \\ -x_{n+a+1-i} & \text{if } n-a+1 \le i \le n. \end{cases}$$

Let us set $Q_0^\square = w_a^{-1} P_0^\square w_a$. Note that $\mathcal{R}(Q_0^\square) = \mathcal{R}(P_0^\square)^{w_a^{-1}}$. Since $\Psi = \{-x \mid x \in \mathcal{R}(P_0^\square) \setminus \mathcal{R}(Q_0^\square)\}$, we have

$$N\big(Y^\square\big) \cap N\big(X_+ \oplus Z^\triangledown \oplus Y_-\big) = N_0^\square \cap U_0^{\square-}.$$

Therefore the map $f^{(s)} \mapsto J^{\mathcal{U},\mathcal{Y}}(s, \chi) f^{(s)}$ coincides with the restriction to $I(s, \chi)$ of the intertwining operator $J_{Q_0^\square | P_0^\square}(\mu_{s,\chi})$. It is easy to see that if $\Re s > -\delta/2$, then $\langle \Re\mu_{s,\chi}, \beta\rangle > 0$ for all $\beta \in \mathcal{R}(P_0^\square) \setminus \mathcal{R}(Q_0^\square)$.

We prove a stronger result in Case (II) in the next subsection (see Lemma 5.2). $\square$

Next we reveal an inductive structure of good sections.

Proposition 5.3 *Assume that $f^{(s)}$ is a good section of $I^{\mathcal{U}}(s, \chi)$. Then $[\Psi(s, \chi) f^{(s)}](g)$ is a good section of $I^{\mathcal{Y}}(s, \chi)$ for each $g \in \bar{G}^\square$.*

Proof Fix $s' \in \mathbb{C}$. If $\Re s' > -\frac{1}{2}$, then $[\Psi(s, \chi) f^{(s)}](g)$ is holomorphic at $s = s'$ by Lemma 5.1, and hence it is good at $s = s'$. If $\Re s' < 0$, then $M_{\mathcal{U}}^\dagger(s, \chi) f^{(s)}(g)$ is holomorphic at $s = s'$ by assumption, and so by Lemmas 4.2 and 5.1, $M_{\mathcal{Y}}^\dagger(s, \chi)[\Psi(s, \chi) f^{(s)}](g)$ is holomorphic at $s = s'$, which concludes that $[\Psi(s, \chi) f^{(s)}](g)$ is good at $s = s'$ as required. $\square$

5.3 The zeta integral in two variables

The groups are of type (II) in this section, namely, $\varphi = 0$, $G = \mathrm{GL}_n(C)$ and $G^\square = \mathrm{GL}_{2n}(C)$. In the archimedean case let $I^\infty(s, \chi)$ be the smooth induced representation with its Fréchet topology (see Appendix A). In the p-adic case we set $I^\infty(s,\chi)=I(s,\chi)$.

From now on we fix a pair (χ_1,χ_2) of unitary characters of $F^\times$ and consider the holomorphic family of representations $I^\infty(s_1,s_2)=I^\infty(s_1,s_2,\chi_1,\chi_2)$ defined by setting $I^\infty(s_1,s_2)=I^\infty(0,(\chi_1\alpha_F^{s_1},\chi_2\alpha_F^{s_2}))$ for $(s_1,s_2)\in\mathbb{C}^2$. The space $I^\infty(s_1,s_2)$ consists of smooth functions $f^{(s_1,s_2)}:G^\square\to\mathbb{C}$ which satisfy

$$f^{(s_1,s_2)}(pg)=\chi_1\big(\nu(a)\big)\chi_2\big(\nu(d)\big)^{-1}\big|\nu(a)\big|_F^{s_1+\delta n/2}\big|\nu(d)\big|_F^{-s_2-\delta n/2}f^{(s_1,s_2)}(g)$$

for all $g\in G^\square$ and $p=\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\in P$.

For any section $f^{(s)}$ of $I^\infty(s,\chi)$ we define a function on $\mathbb{C}^2\times G^\square$ by

$$(s_1,s_2,g)\mapsto f^{(s_1,s_2)}(g)=\big|\nu(g)\big|_F^{(s_1-s_2)/2}f^{((s_1+s_2)/2)}(g)$$

for $s_1,s_2\in\mathbb{C}$ and $g\in G^\square$. Then for each $(s_1,s_2)\in\mathbb{C}^2$, $f^{(s_1,s_2)}\in I^\infty(s_1,s_2)$ as a function of $g\in G^\square$. We call $f^{(s_1,s_2)}$ a holomorphic (resp. standard, good) section if $f^{(s)}$ is. With these definitions, the obvious adaptation of the results stated in Sects. 3.5 and 5.2 still holds for good sections of $I^\infty(s_1,s_2)$. We use the same notations for objects associated to $I(s,\chi)$ as in the previous section with (s,χ) replaced by (s_1,s_2).

Lemma 5.2 *Assume that G is of type (II). Suppose that $\Re s_1>-\delta/2$ and $\Re s_2>-\delta/2$. Then the integral defining $J^{\mathcal{U},\mathcal{Y}}(s_1,s_2)$ is absolutely convergent.*

Proof Take a subspace X of U so that $U=X\oplus Y$. Choose a D-basis of $U^\square$ compatible with the decomposition $U^\square=X^\triangledown\oplus Y^\triangledown\oplus X^\vartriangle\oplus Y^\vartriangle$. Let $T^\square$ be the group of diagonal matrices in $\mathrm{GL}_{2n}(F)\subset G^\square$. Define $x_i\in X^*(T^\square)$ via $x_i(\mathrm{diag}[t_1,\dots,t_{2n}])=t_i$ for $i=1,2,\dots,2n$. Take a minimal parabolic subgroup $P_0^\square$ of $G^\square$ with a Levi decomposition $P_0^\square=M_0^\square U_0^\square$ so that

$$\mathcal{R}\big(P_0^\square\big)=\{x_i-x_j\mid 1\le i<j\le 2n\}.$$

Observe that the Lie algebra of $N(Y^\square)\cap N(X^\triangledown\oplus Y^\triangledown)$ is the sum of root subspaces corresponding to the roots

$$\Psi=\{x_i-x_j\mid n+1\le i\le 2n-a,\ n-a+1\le j\le n\}.$$

We can regard $I^\infty(s_1,s_2)$ as a subrepresentation of $\mathrm{Ind}_{\bar{P}_0^\square}^{\bar{G}^\square}\mu_{s_1,s_2}$, where $\delta_{P_0^\square}$ and the real part of μ_{s_1,s_2}, if restricted to $T^\square$, are easily seen to be

$$\delta_{P_0^\square}=d_0(2n-1,2n-3,\dots,1-2n),$$

$$\Re\mu_{s_1,s_2} = \delta \sum_{j=1}^{n} \left[\left\{ \Re s_1 + \delta\left(j - \frac{n+1}{2} \right) \right\} x_j - \left\{ \Re s_2 - \delta\left(j - \frac{n+1}{2} \right) \right\} x_{j+n} \right].$$

There is an element w_a of $N(T^\Box)$ such that

$$w_a : \begin{cases} x_i \mapsto x_i & \text{if } 1 \le i \le n-a, \\ x_i \mapsto x_{i+n-a} & \text{if } n-a+1 \le i \le n, \\ x_i \mapsto x_{i-a} & \text{if } n+1 \le i \le 2n-a, \\ x_i \mapsto x_i & \text{if } 2n-a+1 \le i \le 2n. \end{cases}$$

Set $Q_0^\Box = w_a^{-1} P_0^\Box w_a$. Since $\Psi = \{-x \mid x \in \mathcal{R}(P_0^\Box),\ x^{w_a} \notin \mathcal{R}(P_0^\Box)\}$,

$$N\left(Y^\Box\right) \cap N\left(X^\triangledown \oplus Y^\triangledown\right) = N_0^\Box \cap U_0^{\Box-}.$$

The remaining part of the proof continues as in that in Case (I). $\Box$

When F is a p-adic field, we put $\mathfrak{R} = \mathbb{C}[q^{-s}, q^s]$. When $F = \mathbb{R}$ or $\mathbb{C}$, we let $\mathfrak{R}$ be the ring of entire functions on the complex plane.

Lemma 5.3 *Assume that G is of type (II). Let $\pi \in \mathrm{Irr}(G)$. For each good section $f^{(s_1,s_2)}$ of $I^\infty(s_1, s_2)$ the integral $Z(\xi, f^{(s_1,s_2)})$ converges absolutely if both $\Re s_1$ and $\Re s_2$ are sufficiently large. It continues to a meromorphic function on $\mathbb{C}^2$. Moreover, there are functions $\alpha_1, \alpha_2 \in \mathfrak{R}$ which are not identically zero and such that $\alpha_1(s_1)\alpha_2(s_2)Z(\xi, f^{(s_1,s_2)})$ is holomorphic everywhere on $\mathbb{C}^2$ for all $\xi \in \pi^\vee \boxtimes \pi$ and $f^{(s_1,s_2)}$. Furthermore,*

$$\frac{Z(\xi, M^\dagger(s_1, s_2) f^{(s_1,s_2)})}{L^{GJ}(\frac{1}{2} - s_1, \pi \otimes \chi_1^{-1}) L^{GJ}(\frac{1}{2} - s_2, \pi^\vee \otimes \chi_2^{-1})} = z(\pi) \frac{\varepsilon^{GJ}(s_1 + \frac{1}{2}, \pi^\vee \otimes \chi_1, \psi) \varepsilon^{GJ}(s_2 + \frac{1}{2}, \pi \otimes \chi_2, \psi)}{L^{GJ}(s_1 + \frac{1}{2}, \pi^\vee \otimes \chi_1) L^{GJ}(s_2 + \frac{1}{2}, \pi \otimes \chi_2)} Z\left(\xi, f^{(s_1,s_2)}\right).$$

Proof Theorem 4.1(3) together with Proposition B.1(4) gives rise to the last identity. Once we prove all the statements true for all holomorphic sections, it may be deduced from Proposition 3.1(4) and this identity that they are true for all good sections.

Let N_+^U (resp. N_-^U) be the unipotent radical of the parabolic subgroup of $G^\Box$ which stabilizes U_+ (resp. U_-) and $I_1^\infty(s_1, s_2)$ (resp. $I_2^\infty(s_1, s_2)$) the subspace of $I^\infty(s_1, s_2)$ consisting of all functions that vanish (with all of their derivatives in the archimedean case) outside PN_+^U (resp. PN_-^U). For $\phi_1 \in C_c^\infty(N_+^U)$ we can define a section $f_{\phi_1}^{(s_1,s_2)}$ of $I_1^\infty(s_1,s_2)$ by requiring that $f_{\phi_1}^{(s_1,s_2)}(x) = \phi_1(x+\mathbf{1}_n)$ for $x \in N_+^U$, where we view N_+^U as $\mathrm{M}_n(C)$ in a natural way. Since

$$\big((x,0)+(y,y)\big)g = (xg+yg-y,0)+(y,y) \quad (x,\ y \in U,\ g \in G \times e),$$

it follows that

$$f_{\phi_1}^{(s_1,s_2)}\big((g,e)\big) = \chi_1\big(\nu(g)\big)\big|\nu(g)\big|_F^{s_1+\delta n/2}\phi_1(g)$$

for $g \in G$. Similarly, for $\phi_2 \in C_c^\infty(N_-^U)$, we can find a section $f_{\phi_2}^{(s_1,s_2)}$ of $I_2^\infty(s_1,s_2)$ such that

$$f_{\phi_2}^{(s_1,s_2)}\big((g,e)\big) = \chi_2\big(\nu(g)\big)^{-1}\big|\nu(g)\big|_F^{-s_2-\delta n/2}\phi_2\big(g^{-1}\big)$$

for $g \in G$. Put $\pi_1 = \pi^\vee$ and $\pi_2 = \pi$. For $\xi \in \pi^\vee \boxtimes \pi$ and $g \in G$ we put

$$H_1(g) = \chi_1\big(\nu(g)\big)H_\xi(g), \qquad H_2(g) = \chi_2\big(\nu(g)\big)H_\xi\big(g^{-1}\big).$$

Note that H_i is a matrix coefficient of $\pi_i \otimes \chi_i$. The integrals become

$$Z\big(\xi, f_{\phi_i}^{(s_1,s_2)}\big) = Z^{GJ}(s_i,\phi_i,H_i) \quad (i=1,2). \tag{5.1}$$

Propositions 4.2 and 5.3 coupled with the subrepresentation theorem reduce the proof of Lemma 5.3 to the case where π is supercuspidal. First assume that $n=1$. Since $G^\square$ is the overlapping union of open subsets PN_+^U and PN_-^U, a function in $I^\infty(s_1,s_2)$ may be expressed as a sum of functions in the spaces $I_1^\infty(s_1,s_2)$ and $I_2^\infty(s_1,s_2)$, which allows us to take $\alpha_i(s) = L^{GJ}(s+\frac{1}{2},\pi_i \otimes \chi_i)^{-1}$ for $i=1,2$.

Having proven Lemma 5.3 for $n=1$, we prove it for general n. Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(C)$ with $n \geq 2$. Thus F is a p-adic field. Take a compact open subgroup K' of G so that H_ξ and $f^{(s)}$ are invariant on the right under K'. There is a finite subset $\mathcal{B}$ of G such that the support of H_ξ is contained in a finite disjoint union $\bigsqcup_{g\in\mathcal{B}} ZgK'$. Define an embedding $\varrho : \mathrm{GL}_2(F) \to G^\square$ and sections $f_g^{(s_1',s_2')}$ of some induced representation $I(s_1',s_2')$ of $\mathrm{GL}_2(F)$ by

$$\varrho(h) = \begin{pmatrix} a\mathbf{1}_n & b\mathbf{1}_n \\ c\mathbf{1}_n & d\mathbf{1}_n \end{pmatrix}, \qquad f_g^{(s_1',s_2')}(h) = f^{(s_1,s_2)}\big(\varrho(h)g\big)$$

for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$. Then

$$Z\big(\xi, f^{(s_1,s_2)}\big) = \sum_{g \in \mathcal{B}} H_\xi(g) Z\big(\omega_\pi^{-1}, f_g^{(s_1', s_2')}\big),$$

which completes the proof by reducing it to the case $n = 1$. □

5.4 Definition of L and epsilon factors

The central sign $z(\pi)$ of π is $\chi_\pi(-1)$ except in the metaplectic case. In the metaplectic case $z(\pi) = z_\psi(\pi) = \chi_\pi(-1)/\gamma_\psi(\underline{-1})$, where this quotient is independent of the choice of the preimage in $\bar{Z}_G$ of $-1 \in Z_G$. Put

$$\varepsilon_{W,\psi} = \eta(W), \qquad (\mathrm{I}_0^o)$$

$$\varepsilon_{W,\psi} = \varepsilon(1/2, \chi_W, \psi)^{-1}, \qquad (\mathrm{I}_0^e, \mathrm{I}_1)$$

$$\varepsilon_{W,\psi} = \eta(W)^{n+1}, \qquad (\mathrm{I}_2)$$

$$\varepsilon_{W,\psi} = 1. \qquad (\mathrm{I}_3, \mathrm{I}_4, \mathrm{II})$$

In the following theorem by "local Euler factor" in the p-adic case we mean a function of the form $Q(q^{-s})^{-1}$, where Q is a polynomial satisfying $Q(0) = 1$, and in the archimedean case we mean a finite product of Tate's local Euler factors for GL(1).

Theorem 5.2 *For any $\pi \in \mathrm{Irr}(\bar{G})$ and any character χ of $E^\times$ there exists a local Euler factor $L_\psi^{\mathcal{U}}(s, \pi^\vee \times \chi)$ with the following properties*:

- *the quotients $Z(\xi, f^{(s)})/L_\psi^{\mathcal{U}}(s + \frac{1}{2}, \pi^\vee \times \chi)$ are entire for all $\xi \in \pi^\vee \boxtimes \pi$ and all good sections $f^{(s)}$ of $I(s, \chi)$;*
- *for any fixed $s' \in \mathbb{C}$ there is a choice of $\xi \in \pi^\vee \boxtimes \pi$ and a good section $f^{(s)}$ of $I(s, \chi)$ such that $Z(\xi, f^{(s)})/L_\psi^{\mathcal{U}}(s + \frac{1}{2}, \pi^\vee \times \chi)$ does not have a zero at $s = s'$;*
- *there is a nowhere vanishing entire function $\varepsilon^{\mathcal{U}}(s, \pi^\vee \times \chi, \psi)$ such that for all ξ and $f^{(s)}$*

$$\frac{Z(\xi, M^\dagger(s, \chi) f^{(s)})}{L_{\psi^{-1}}^{\mathcal{U}}(\frac{1}{2} - s, \pi \times \chi^{-1})} = z(\pi)\varepsilon_{W,\psi}\varepsilon^{\mathcal{U}}\left(s + \frac{1}{2}, \pi^\vee \times \chi, \psi\right) \frac{Z(\xi, f^{(s)})}{L_\psi^{\mathcal{U}}(s + \frac{1}{2}, \pi^\vee \times \chi)}.$$

In the archimedean case these factors are defined only up to entire functions which have no zeros.

The gamma factor $\gamma^{\mathcal{U}}(s, \pi \times \chi, \psi)$ is extensively studied in [4, 32]. Properties of the gamma factors are recalled or derived in Appendix B. We remind the reader of the relation

$$\varepsilon^{\mathcal{U}}(s, \pi \times \chi, \psi) = \gamma^{\mathcal{U}}(s, \pi \times \chi, \psi) \frac{L^{\mathcal{U}}_{\psi}(s, \pi \times \chi)}{L^{\mathcal{U}}_{\psi^{-1}}(1-s, \pi^\vee \times \chi^{-1})}.$$

When $\mathcal{U}$ is understood, we will use the notation $L_\psi(s, \pi \times \chi)$. When χ is trivial, we often write $L_\psi(s, \pi)$ instead of $L_\psi(s, \pi \times \chi)$.

Theorem 5.2 is stated in [43].

Proof The last assertion follows from the first two assertions and Proposition 5.4 below. Indeed, the functional equation is merely a restatement of Theorem 4.1(3). The ratios on both sides prove to be holomorphic and nowhere vanishing, and hence so is $\varepsilon(s, \pi \times \chi, \psi)$.

Note that the local integrals do not have a common zero. Kudla and Rallis [26, 28] have proven this result in Case (I_4^s). Their proof is valid in the general case (see Lemmas A.2 and A.3 below). We have only to find a nonzero function $\alpha \in \mathfrak{R}$ which cancels all poles of the local integrals $Z(\xi, f^{(s)})$. The minimal Euler factor whose reciprocal cancels all the poles of the local integrals has the required properties. In Case (II) Lemma 5.3 produces such α. In view of Propositions 4.1, 4.2 and 5.3 coupled with the subrepresentation theorem we may let π be an irreducible supercuspidal representation of the group $\bar{G}$ of type (I). Then the integrand is compactly supported, and so the poles of $Z(\xi, f^{(s)})$ must arise from the poles of the section $f^{(s)}$, and hence $\alpha(s) = b(s, \chi)^{-1}$ works by Proposition 3.1(2). □

We could formulate Theorems 4.1 and 5.2 for the pair $(\pi, \pi^\vee)$ rather than π, the situation being symmetric in π and $\pi^\vee$.

Proposition 5.4 *Let* $\pi \in \mathrm{Irr}(\bar{G})$. *In the metaplectic case*

$$L_\psi\big(s, \pi^\vee \times \chi\big) = L_{\psi^{-1}}(s, \pi \times \chi), \qquad \varepsilon\big(s, \pi^\vee \times \chi, \psi\big) = \varepsilon\big(s, \pi \times \chi, \psi^{-1}\big).$$

In all other cases

$$L\big(s, \pi^\vee \times \chi\big) = L\big(s, \pi \times \rho(\chi)\big), \qquad \varepsilon\big(s, \pi^\vee \times \chi, \psi\big) = \varepsilon\big(s, \pi \times \rho(\chi), \psi\big).$$

Proof The argument is similar to that of Theorem 4(5) of [32]. Let $c : U^\square \to U^\square$ be the anti-isometry induced by switching U_+ and U_-. It acts on $G^\square$ by conjugation. The resulting automorphism of $G^\square$, which is denoted by sw, stabilizes $G \times G$ and acts by interchanging the factors. Note that sw stabilizes $P(U^\triangle)$ and $P(U^\triangledown)$, acts trivially on their intersection. In the metaplectic case it has a unique lift to $\mathrm{Mp}(U^{\square})$ which restricts to complex conjugation on the central $\mathbb{C}^1$. We denote this lift also by sw. We define a function $h^{(s)}: \bar{G}^{\square} \to \mathbb{C}$ by $h^{(s)}(g) = f^{(s)}(sw(g))$. Then $h^{(s)} \in I_{\psi^{-1}}(s, \chi)$. Put $\chi'' = \rho(\chi)$ in Case (I), and $\chi'' = \chi_2\chi_1^{-1}$ in Case (II). Since $\kappa_\chi^{-1} \circ sw = \chi'' \cdot \kappa_{\rho(\chi)}^{-1}$,

$$Z\big(\xi^\vee \boxtimes \xi, f^{(s)}\big) = Z\big(\xi \boxtimes \xi^\vee, \chi'' \cdot h^{(s)}\big) \quad \big(\xi \in \pi,\ \xi^\vee \in \pi^\vee\big).$$

This together with Remark 5.1(3) below proves the first equality. Using Proposition B.1(2), we get the second equality from the first. $\square$

Remark 5.1

(1) Let $\mathrm{BC}(\pi) = \mathrm{BC}_\psi(\pi)$ stand for the (hypothetical) transfer of π to $\mathrm{GL}_N(E)$, where $N = n-1$ in Case (I_0^o), $N = \delta n$ in Cases (I_0^e), (I_1), (I_2), (I_4^m), (II), and $N = \delta n + 1$ in Cases (I_3), (I_4^s). It is expected that

$$L_\psi(s, \pi \times \chi) = L^{GJ}\big(s, \mathrm{BC}(\pi) \otimes \chi\big),$$
$$\varepsilon(s, \pi \times \chi, \psi) = \varepsilon^{GJ}\big(s, \mathrm{BC}(\pi) \otimes \chi, \psi\big).$$

(2) In the proofs below we will often implicitly assume that χ^0 is unitary. Since $L_\psi(s, \pi \times \chi\alpha_E^{s'}) = L_\psi(s + s', \pi \times \chi)$, this assumption entails no loss of generality.

(3) The L-factor is independent of the choice of ψ except in the metaplectic case, in which case since $I_{\psi_\lambda}(s, \chi) = I_\psi(s, \chi\chi_\lambda)$,

$$L_{\psi_\lambda}(s, \pi \times \chi) = L_\psi(s, \pi \times \chi\chi_\lambda), \quad \lambda \in F^\times.$$

(4) One can deduce from (B.1) that

$$\varepsilon(s, \pi \times \chi, \psi_\lambda) = \big(\chi\alpha_E^{s-1/2}\big)^N(\lambda)\varepsilon(s, \pi \times \chi, \psi), \qquad (\mathrm{I}_0^o, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4^s, \mathrm{II})$$
$$\varepsilon(s, \pi \times \chi, \psi_\lambda) = \big(\chi\alpha_F^{s-1/2}\big)^N(\lambda)\chi_U(\lambda)\varepsilon(s, \pi \times \chi, \psi), \qquad (\mathrm{I}_0^e, \mathrm{I}_1)$$
$$\varepsilon(s, \pi \times \chi, \psi_\lambda) = \big(\chi\alpha_F^{s-1/2}\big)^N(\lambda)\varepsilon(s, \pi \times \chi\chi_\lambda, \psi). \qquad (\mathrm{I}_4^m)$$

(5) The functional equation

$$\varepsilon(s, \pi \times \chi, \psi)\varepsilon\big(1 - s, \pi^\vee \times \chi^{-1}, \psi^{-1}\big) = 1$$

follows from Proposition B.1(3).

We normalize the zeta integral by

$$\mathcal{Z}\big(\xi, f^{(s)}\big) = Z\big(\xi, f^{(s)}\big) / L_\psi\left(s + \frac{1}{2}, \pi^\vee \times \chi\right).$$

The limit $\lim_{s\to s'} \mathcal{Z}(\xi, f^{(s)})$ depends only on $f^{(s')}$. Indeed, let $h^{(s)}$ be a holomorphic section satisfying $h^{(s')} = f^{(s')}$. Put $F^{(s)} = (f^{(s)} - h^{(s)})/(s - s')$. Since $F^{(s)}$ is a holomorphic section,

$$\mathcal{Z}(\xi, F^{(s)}) = (\mathcal{Z}(\xi, f^{(s)}) - \mathcal{Z}(\xi, h^{(s)}))/(s - s')$$

is holomorphic at $s = s'$, and hence $\lim_{s\to s'} \mathcal{Z}(\xi, f^{(s)}) = \lim_{s\to s'} \mathcal{Z}(\xi, h^{(s)})$. If $\Re s' > -\frac{1}{2}$, then the bilinear form defined by $\xi, f^{(s')} \mapsto \lim_{s\to s'} \mathcal{Z}(\xi, f^{(s)})$ gives rise to a nonzero element

$$\mathcal{Z}(s', \pi, \chi) \in \operatorname{Hom}_{\bar{G}\times\bar{G}}(I(s', \chi), (\pi \boxtimes \pi^\vee) \otimes \kappa_\chi).$$

5.5 The local factors of Harris, Kudla and Sweet

Another useful way of thinking of the local L-factors is the following (cf. [13]). Let F be a p-adic field. Theorem 4.1 tells us that if $f^{(s)}$ is a standard section of $I(s, \chi)$ multiplied by an element of $\mathbb{C}[q^{-s}, q^s]$ or a section obtained by applying the normalized intertwining operator to such a section of $I(-s, \rho(\chi)^{-1})$, then $Z(\xi, f^{(s)})$ is a rational function of q^{-s}. Let $\mathcal{I}(\pi, \chi)$ be the subspace of $\mathbb{C}(q^{-s})$ spanned by these local integrals. Theorem 5.2 asserts that each such rational function can be written with a common denominator which depends only on π (and ψ in the metaplectic case) and that $\mathcal{I}(\pi, \chi)$ contains the constants. Thus $\mathcal{I}(\pi, \chi)$ is a fractional $\mathbb{C}[q^{-s}, q^s]$-ideal which has a single generator of the form $Q_{\pi,\chi}(q^{-s})^{-1}$, where the polynomial $Q_{\pi,\chi}$ satisfies $Q_{\pi,\chi}(0) = 1$.

Proposition 5.5 *Notation being as above, we have*

$$L_\psi(s, \pi^\vee \times \chi) = Q_{\pi,\chi}(q^{-s+1/2})^{-1}.$$

Proof It is evident that $L_\psi(s, \pi^\vee \times \chi)^{-1}$ is divisible by $Q_{\pi,\chi}(q^{-s+1/2})$. Since any holomorphic section can be expressed as a linear combination of standard sections with coefficients holomorphic functions of s, Proposition 3.1(4) shows that the family $Z(\xi, f^{(s)})$ for good sections cannot have a higher order pole at any point than $Q_{\pi,\chi}(q^{-s})^{-1}$. □

To prove an analogous statement in the archimedean case, we need to define a more refined notion of good sections. We do not discuss these aspects here, but refer the reader to [18] for further analysis.

6 Compatibility with parabolic induction

Theorem 6.1 *Let $\sigma = \boxtimes_j \sigma_j$ be an admissible representation of $\bar{G}(\mathcal{Y})$ which is not necessarily irreducible but has a central character. If the assertions of*

Theorem 5.2 are true for σ and χ, then they are true for $\pi = \mathrm{Ind}_{\bar{P}(\mathcal{Y})}^{\bar{G}} \sigma$ and χ. More precisely,

$$L_\psi(s, \pi \times \chi) = L_\psi^{\mathcal{Y}}(s, \sigma \times \chi),$$
$$\varepsilon(s, \pi \times \chi, \psi) = \varepsilon^{\mathcal{Y}}(s, \sigma \times \chi, \psi),$$

where we set

$$L_\psi^{\mathcal{Y}}(s, \sigma \times \chi) = \prod_j L_\psi(s, \sigma_j \times \chi),$$
$$\varepsilon^{\mathcal{Y}}(s, \sigma \times \chi, \psi) = \prod_j \varepsilon(s, \sigma_j \times \chi, \psi).$$

It is now easy to prove the first part of this theorem. For all $\xi \in \pi^\vee \otimes \pi$ and meromorphic sections $f^{(s)}$ of $I(s, \chi)$ Propositions 4.1 and 4.2 deduce the functional equation

$$Z\big(\xi, M^\dagger(s, \chi) f^{(s)}\big) = z(\sigma)\varepsilon_{\mathcal{Y},\psi}\gamma^{\mathcal{Y}}\left(s + \frac{1}{2}, \sigma^\vee \times \chi, \psi\right) Z\big(\xi, f^{(s)}\big)$$

from the functional equation of $Z^{\mathcal{Y}}(s, \sigma, \chi)$, where

$$z(\sigma) = \prod_j z(\sigma_j), \qquad \varepsilon_{\mathcal{Y},\psi} = \prod_j \varepsilon_{\mathcal{Y}_j,\psi},$$
$$\gamma^{\mathcal{Y}}(s, \sigma \times \chi, \psi) = \prod_j \gamma(s, \sigma_j \times \chi, \psi).$$

Recall that the gamma factors are compatible, in a strong way, with parabolic induction (see Proposition B.1(1)). Propositions 4.1, 4.2 and 5.3 show by induction that the ratio $Z(\xi, f^{(s)})/L_\psi^{\mathcal{Y}}(s + \frac{1}{2}, \sigma^\vee \times \chi)$ is entire for all $\xi \in \pi^\vee \otimes \pi$ and good sections $f^{(s)}$ of $I(s, \chi)$. We can take $L_\psi(s + \frac{1}{2}, \pi^\vee \times \chi)$ to be the minimal Euler factor having the same poles as the family $Z(\xi, f^{(s)})$ for good sections. Then $L_\psi^{\mathcal{Y}}(s, \sigma^\vee \times \chi)^{-1}$ is divisible by $L_\psi(s, \pi^\vee \times \chi)^{-1}$.

The following lemma is used not in the proof of Theorem 6.1 but in the next section.

Lemma 6.1 *Let $\sigma = \boxtimes_j \sigma_j$ be an irreducible admissible representation of $\bar{G}(\mathcal{Y})$ and π an irreducible subquotient of $\mathrm{Ind}_{\bar{P}(\mathcal{Y})}^{\bar{G}} \sigma$.*

(1) *$L_\psi^{\mathcal{U}}(s, \pi \times \chi)/L_\psi^{\mathcal{Y}}(s, \sigma \times \chi)$ is entire for every character χ of $E^\times$.*

(2) *If for any unitary character* χ *of* $E^\times$ *and any fixed* $s' \in \mathbb{C}$ *with* $\Re s' \geq 0$ *there are* $\xi \in \pi^\vee \boxtimes \pi$ *and a holomorphic section* $f^{(s)}$ *of* $I(s,\chi)$ *such that* $Z(\xi, f^{(s)})/L_\psi^{\mathcal{Y}}(s+\frac{1}{2}, \sigma^\vee \times \chi)$ *does not vanish at* $s = s'$, *then* $L_\psi^{\mathcal{U}}(s, \pi \times \chi) = L_\psi^{\mathcal{Y}}(s, \sigma \times \chi)$ *for all unitary characters* χ *of* $E^\times$.

Proof Since $\pi^\vee$ is a subquotient of $\mathrm{Ind}_{\bar{P}(\mathcal{Y})}^{\bar{G}} \sigma^\vee$, any matrix coefficient of $\pi^\vee$ is that of $\mathrm{Ind}_{\bar{P}(\mathcal{Y})}^{\bar{G}} \sigma^\vee$. Hence the first assertion is trivially deduced from the observation above.

To prove (2), we have to show that $L_\psi^{\mathcal{Y}}(s+\frac{1}{2}, \sigma^\vee \times \chi)$ not only cancels all poles of the local integrals for $\pi^\vee$, but also dividing by it introduces no extraneous zeros. There are no extraneous zeros in the right half-plane $\Re s \geq 0$ by assumption. Fix $s' \in \mathbb{C}$ with $\Re s' < 0$. Note that $L_\psi^{\mathcal{Y}}(s, \sigma^\vee \times \rho(\chi)^{-1}) = L_{\psi^{-1}}^{\mathcal{Y}}(s, \sigma \times \chi^{-1})$ by Proposition 5.4. By assumption we can find $\xi \in \pi^\vee \boxtimes \pi$ and a holomorphic section $f^{(s)}$ of $I(s, \rho(\chi)^{-1})$ such that the ratio $Z(\xi, f^{(s)})/L_{\psi^{-1}}^{\mathcal{Y}}(s+\frac{1}{2}, \sigma \times \chi^{-1})$ does not have a zero at $s = -s'$. Note that $h^{(s)} = M^\dagger(-s, \chi) f^{(-s)}$ is a good section of $I(s, \chi)$. It follows from the functional equation

$$\frac{Z(\xi, h^{(-s)})}{L_\psi^{\mathcal{Y}}(\frac{1}{2}-s, \sigma^\vee \times \chi)} = z(\sigma)\varepsilon_{\mathcal{Y},\psi}\varepsilon^{\mathcal{Y}}\left(s+\frac{1}{2}, \sigma \times \chi^{-1}, \psi^{-1}\right)\frac{Z(\xi, f^{(s)})}{L_{\psi^{-1}}^{\mathcal{Y}}(s+\frac{1}{2}, \sigma \times \chi^{-1})}$$

that $Z(\xi, h^{(s)})/L_\psi^{\mathcal{Y}}(s+\frac{1}{2}, \sigma^\vee \times \chi)$ does not vanish at $s = s'$. □

The proof of the equalities of Theorem 6.1 will occupy the rest of this section. The reader whose main interest is in Theorem 7.1 can skip the rest of this section and continue reading from Sect. 7.1 onwards. Recall that if $f^{(s)}$ is a good section, then so is $[\Psi(s,\chi) f^{(s)}](e)$ by Proposition 5.3. Conversely, any good section of $I^{\mathcal{Y}}(s,\chi)$ comes from a good section of $I^{\mathcal{U}}(s,\chi)$, as indicated in the following lemma. We here discuss the p-adic case. We shall indicate modifications which are needed in the archimedean case in Appendix A.

Lemma 6.2 *If* $h^{(s)}$ *is a good section of* $I^{\mathcal{Y}}(s,\chi)$, *then there is a good section* $f^{(s)}$ *of* $I^{\mathcal{U}}(s,\chi)$ *such that* $[\Psi(s,\chi) f^{(s)}](e) = h^{(s)}$.

Proof We focus our attention on Case (I). The argument below can be adapted to Case (II). We begin by recalling the orbit structure of $P\backslash G^\square$ for the action of $P(Y^\square)$. We identify $P\backslash G^\square$ with the variety $\Omega(U^\square)$ of n-dimensional totally isotropic subspaces of $U^\square$ via $Pg \mapsto U^\triangle g$. For such subspaces L and

L', there exists $p \in P(Y^\square)$ such that $L = L'p$ if and only if the intersections $L \cap Y^\square$ and $L' \cap Y^\square$ have the same dimension (see Proposition 4.1 and (4.12) of [10]). Therefore $G^\square$ is the disjoint union of the double cosets $\Omega_j = P\delta_j P(Y^\square)$ $(0 \le j \le 2a)$, where the image of Ω_j in $\Omega(U^\square)$ consists of subspaces L whose intersection $L \cap Y^\square$ has dimension j over C. The closure of an orbit Ω_j is given by $\bigsqcup_{i \ge j} \Omega_i$. Let

$$\mathrm{St}_j = \delta_j^{-1} P \delta_j \cap P\big(Y^\square\big)$$

be the stabilizer in $P(Y^\square)$ of the point $U^\Delta\delta_j \in \Omega(U^\square)$. The stratification of $G^\square$ by Ω_j gives rise to a $M(Y^\square)$-stable filtration

$$I(s,\chi) = I^{(2a)}(s,\chi) \supset \cdots \supset I^{(j)}(s,\chi) \supset \cdots \supset I^{(0)}(s,\chi),$$

where $I^{(j)}(s,\chi)$ is the set of functions in $I(s,\chi)$ whose restriction to the closure of Ω_{j+1} vanishes. The successive quotients are given by

$$I^{(j)}(s,\chi)/I^{(j-1)}(s,\chi) \simeq \text{c-ind}_{\mathrm{St}_j}^{P(Y^\square)} \xi_j^{(s)},$$

where $\xi_j^{(s)}$ is the character of St_j which takes g to $\chi(\nu(c))|\nu(c)|_E^{s+\delta\rho_n/2}$ if $\delta_j g \delta_j^{-1} = \mathrm{diag}[c,(c^{-1})^*]u \in P$.

We take $\delta_a = 1$. Then $\mathrm{St}_a = (N(Y^\square) \cap P) \cdot P_{\mathcal{Y}}$. For $\Re s$ sufficiently large the intertwining operator $\Psi(s,\chi)$ followed by restriction yields an $N(Y^\square)$-invariant map from $I(s,\chi)$ to $I^{\mathcal{Y}}(s,\chi)$. Observe that if $f^{(s)} \in I^{(a)}(s,\chi)$, then the integral $[\Psi(s,\chi)f^{(s)}](e)$ is absolutely convergent for all s. Moreover, it induces an isomorphism of the Jacquet module $(\text{c-ind}_{\mathrm{St}_a}^{P(Y^\square)} \xi_a^{(s)})_{N(Y^\square)}$ onto $I^{\mathcal{Y}}(s,\chi)$ by Proposition 6.2.1 of [2].

Proposition 3.1(4), which is easily extended to the compound case, enables us to take holomorphic sections $h_1^{(s)}$ and $h_2^{(s)}$ so that

$$h^{(s)} = h_1^{(s)} + M_{\mathcal{Y}}^\dagger\big(-s,\rho(\chi)^{-1}\big)h_2^{(-s)}.$$

By what we have just seen there are a holomorphic section $f_1^{(s)}$ of $I^{(a)}(s,\chi)$ and that of $I^{(a)}(s,\rho(\chi)^{-1})$ such that

$$\big[\Psi(s,\chi)f_i^{(s)}\big](e) = h_i^{(s)} \quad (i = 1,2).$$

From Lemma 4.2

$$\big[\Psi(s,\chi)M_{\mathcal{U}}^\dagger\big(-s,\rho(\chi)^{-1}\big)f_2^{(-s)}\big](e) = M_{\mathcal{Y}}^\dagger\big(-s,\rho(\chi)^{-1}\big)h_2^{(-s)}.$$

It suffices to take for $f^{(s)}$ the section $f_1^{(s)} + M_{\mathcal{U}}^\dagger\big(-s,\rho(\chi)^{-1}\big)f_2^{(-s)}$. $\square$

We finish the proof of Theorem 6.1. It suffices to prove the equality relative to $L^{\mathcal{U}}_{\psi}(s,\pi\times\chi)$ as the second equality follows from the multiplicativity relation of the gamma factor stated in Proposition B.1(1). We have already seen that $L_{\psi}(s,\pi\times\chi)/L^{\mathcal{Y}}_{\psi}(s,\sigma\times\chi)$ is entire. To complete our picture, we check that $L^{\mathcal{Y}}_{\psi}(s,\sigma\times\chi)$ cannot have extraneous poles. For each $\eta\in\sigma^{\vee}\boxtimes\sigma$ there is an element $\xi_{\eta}\in\pi^{\vee}\boxtimes\pi$ such that

$$\mathcal{P}_{\pi^{\vee}\boxtimes\pi}(\xi_{\eta}\boxtimes\xi)=\mathcal{P}_{\sigma^{\vee}\boxtimes\sigma}\big(\eta\boxtimes\xi(e)\big)$$

for all $\xi\in\pi\boxtimes\pi^{\vee}$. It follows that

$$\begin{aligned}Z^{\mathcal{U}}\big(\xi_{\eta},f^{(s)}\big)&=\mathcal{P}_{\pi^{\vee}\boxtimes\pi}\big(\xi_{\eta}\boxtimes\kappa_{\chi}^{-1}\operatorname{Ind}^{\bar{G}\times\bar{G}}_{\bar{P}(Y)\times\bar{P}(Y)}Z^{\mathcal{Y}}(s,\sigma,\chi)\big[\Psi(s,\chi)f^{(s)}\big]\big)\\&=\mathcal{P}_{\sigma^{\vee}\boxtimes\sigma}\big(\eta\boxtimes Z^{\mathcal{Y}}(s,\sigma,\chi)\big[\Psi(s,\chi)f^{(s)}\big](e)\big)\\&=Z^{\mathcal{Y}}\big(\eta,\big[\Psi^{\mathcal{U},\mathcal{Y}}(s,\chi)f^{(s)}\big](e)\big),\end{aligned}$$

which tells us that the quotient is not identically zero for each s.

Remark 6.1 Theorem 7.1 is proven independently of Lemma 6.2 and reproves Theorem 6.1 for standard modules.

7 Computation of the L-factor

7.1 Twisting by characters

If G is of type (I_2), then let E^1 be the kernel of the norm map $N_{E/F}:E^{\times}\to F^{\times}$ and μ a character of $E^{\times}$ trivial on $F^{\times}$. We can define a character $\mu':E^1\to\mathbb{C}^1$ by $\mu'(x/x^{\rho})=\mu(x)$. The character $\tilde{\mu}:G\to\mathbb{C}^1$ is defined as composition of this character with determinant.

Except in the unitary case, for given $g\in G^{\square}$, we define a quantity $x(g)$ of $F^{\times}/F^{\times 2}$ via the relative Bruhat decomposition of $G^{\square}$ with respect to P as in [23]. In Cases (I_0) or (I_1) the map $g\mapsto x(g)$ from $G^{\square}$ to $F^{\times}/F^{\times 2}$ is a homomorphism (cf. [67, Proposition 6.5]), so if μ is a quadratic character μ of $F^{\times}$, then composing μ by x gives a character $\tilde{\mu}$ of $G^{\square}$. We denote by $\tilde{\mu}$ as well a character of G obtained by restricting $\tilde{\mu}$ to $G\times e$. The restriction of the homomorphism x is a spinor norm of G in Case (I_0).

Lemma 7.1 *Let χ be a character of $E^{\times}$ and π an irreducible representation of G.*

(1) *If G is of type* (I_0), *then*

$$L\big(s,(\pi\otimes\mathrm{sgn})\times\chi\big)=L(s,\pi\times\chi),$$

$$\varepsilon\big(s, (\pi \otimes \mathrm{sgn}) \times \chi, \psi\big) = \varepsilon(s, \pi \times \chi, \psi).$$

(2) *If μ is a quadratic character of $F^\times$ in Cases* (I_0) *and* (I_1) *or if μ is a character of $E^\times$ trivial on $F^\times$ in Case* (I_2), *then*

$$\begin{aligned} L\big(s, (\pi \otimes \tilde{\mu}) \times \chi\big) &= L\big(s, \pi \times (\chi\mu)\big), \\ \varepsilon\big(s, (\pi \otimes \tilde{\mu}) \times \chi, \psi\big) &= \varepsilon\big(s, \pi \times (\chi\mu), \psi\big). \end{aligned} \tag{I_0–I_2}$$

(3) *If G is of type* (II) *and η is a character of $F^\times$, then*

$$\begin{aligned} L\big(s, (\pi \otimes \eta) \times \chi\big) &= L\big(s, \pi \times \big(\chi_1\eta, \chi_2\eta^{-1}\big)\big), \\ \varepsilon\big(s, (\pi \otimes \eta) \times \chi, \psi\big) &= \varepsilon\big(s, \pi \times \big(\chi_1\eta, \chi_2\eta^{-1}\big), \psi\big), \end{aligned}$$

where $\chi = (\chi_1, \chi_2)$.

(4) *If G is of type* (II) *and $\delta = 1$, then*

$$L_\psi(s, \pi_\psi \times \chi) = L(s, \pi \times \chi), \qquad \varepsilon(s, \pi_\psi \times \chi, \psi) = \varepsilon(s, \pi \times \chi, \psi).$$

Proof Let $\tilde{\Omega}$ be a character of $G^\square$ which we assume coincides with $\Omega \circ \nu$ on M for some character Ω of $F^\times$. We denote its restriction to $G \times e$ also by $\tilde{\Omega}$. Clearly, $f^{(s)}$ is a good section of $I(s, \chi)$ if and only if $\tilde{\Omega} \cdot f^{(s)}$ is that of $I(s, \chi\Omega)$. The identity

$$Z\big(H_\xi, \tilde{\Omega} \cdot f^{(s)}\big) = Z\big(\tilde{\Omega} \cdot H_\xi, f^{(s)}\big), \tag{7.1}$$

is easy to see, where the integral on the left hand side is with respect to $\pi^\vee$ and $\chi\Omega$, and on the right hand side with respect to $\pi^\vee \otimes \tilde{\Omega}$ and χ. Taking $\tilde{\Omega} = \mathrm{sgn}$ or $\tilde{\Omega} = \tilde{\mu}$, we obtain the first identity. Since twisting by characters plays no role on the integral defining l_{ψ_A},

$$\tilde{\Omega} \cdot \big(M^\dagger(s, \chi) f^{(s)}\big) = M^\dagger(s, \chi)\big(\tilde{\Omega} \cdot f^{(s)}\big),$$

which implies the second identity. The proof of (3) and (4) is similar. □

7.2 Tempered representations

Lemma 7.2 *If π is tempered and χ is unitary, then the integral defining $Z(\xi, f^{(s)})$ converges absolutely in the right half-plane $\Re s > -\delta/2$. In particular, $L_\psi(s, \pi \times \chi)$ is holomorphic for $\Re s > 0$.*

Remark 7.1 Lemma 7.2 sharpens and extends [32, Lemma 2, Proposition 5]. In the square integrable case W.T. Gan and Ichino prove a stronger result in Cases (I_0), (I_2) and (I_4) in [5]: if π is square integrable, then $Z(\xi, f^{(s)})$ still converges absolutely on the line $\Re s = -\frac{1}{2}$.

Proof Lemma 7.3 below includes the first statement in Case (II). The proof in Case (I) is similar. The second statement follows from the first. □

7.3 Essentially tempered representations

The group G is of type (II) in this section. Both χ_1 and χ_2 are assumed to be unitary in Lemmas 7.3 and 7.4.

Lemma 7.3 *If π is tempered and if $\Re s_1 > -\delta/2$ and $\Re s_2 > -\delta/2$, then the integral defining $Z(\xi, f^{(s_1,s_2)})$ is absolutely convergent for $\xi \in \pi^\vee \boxtimes \pi$.*

Proof Proposition 4.2 and Lemma 5.2 reduce the statement to the square integrable case. The statement in the square integrable case is proven in the appendix of [70]. □

Lemma 7.4 *If π is tempered, then for any $\xi \in \pi^\vee \boxtimes \pi$ and any good section $f^{(s_1,s_2)}$ of $I^\infty(s_1, s_2)$*

$$L^{GJ}\left(s_1 + \frac{1}{2}, \pi^\vee \otimes \chi_1\right)^{-1} L^{GJ}\left(s_2 + \frac{1}{2}, \pi \otimes \chi_2\right)^{-1} Z\big(\xi, f^{(s_1,s_2)}\big)$$

is holomorphic on the whole of $\mathbb{C}^2$.

Proof We write $\Upsilon(s_1, s_2)$ for the function above. Proposition 5.1 and Lemma 7.3 show that $\Upsilon(s_1, s_2)$ is holomorphic in the domain

$$\left\{(s_1, s_2) \,\middle|\, \Re s_1 > -\frac{1}{2},\ \Re s_2 > -\frac{1}{2}\right\},$$

and so by the functional equation of Lemma 5.3, it is holomorphic also in the domain

$$\left\{(s_1, s_2) \,\middle|\, \Re s_1 < \frac{1}{2},\ \Re s_2 < \frac{1}{2}\right\}.$$

It follows that the function $s_1 \mapsto \Upsilon(s_1, s_2)$ is entire if s_2 lies in the strip $-\frac{1}{2} < \Re s_2 < \frac{1}{2}$, and similarly for the function $s_2 \mapsto \Upsilon(s_1, s_2)$. By Lemma 5.3 there are nonzero elements $\alpha_1, \alpha_2 \in \mathfrak{R}$ such that $\Upsilon'(s_1, s_2) = \alpha_1(s_1)\alpha_2(s_2)\Upsilon(s_1, s_2)$ is a holomorphic function on the whole of $\mathbb{C}^2$. Assume that α_1 and α_2 are minimal with respect to this property. If s' is a zero of α_1, then since $\Upsilon'(s', s_2)$ is zero for any point in the strip $-\frac{1}{2} < \Re s_2 < \frac{1}{2}$, it is identically zero, so that we may replace $\alpha_1(s_1)$ by $\alpha_1(s_1)/(s_1 - s')$ or $\alpha_1(s_1)/(1 - q^{s'-s_1})$ according as F is archimedean or not. Since this contradicts our assumption on α_1, it turns out that α_1 and α_2 must be invertible. □

Lemma 7.5 *Let $\chi=(\chi_1,\chi_2)$ be a (not necessarily unitary) character of $E^\times$. If π is essentially tempered, then*

$$L(s,\pi\times\chi)=L^{GJ}(s,\pi\otimes\chi_1)L^{GJ}\big(s,\pi^\vee\otimes\chi_2\big).$$

Proof Taking Lemma 7.1(3) into account, we may assume that π is tempered. Lemma 7.4 gives elements α and $\alpha^\vee$ of $\mathfrak{R}$ such that

$$L(s,\pi\times\chi)=\alpha(s)L^{GJ}(s,\pi\otimes\chi_1)L^{GJ}\big(s,\pi^\vee\otimes\chi_2\big),$$
$$L\big(s,\pi^\vee\times\chi^{-1}\big)=\alpha^\vee(s)L^{GJ}\big(s,\pi^\vee\otimes\chi_1^{-1}\big)L^{GJ}\big(s,\pi\otimes\chi_2^{-1}\big).$$

Since (5.1) implies that $L^{GJ}(s,\pi^\vee\otimes\chi_2)/L(s,\pi\times\chi)$ is entire, so is $\alpha(s)^{-1}L^{GJ}(s,\pi\otimes\chi_1)^{-1}$. The same reasoning applied to $\pi^\vee$ and χ^{-1} shows that $\alpha^\vee(s)^{-1}L^{GJ}(s,\pi^\vee\otimes\chi_1^{-1})^{-1}$ is entire as well. Since $L^{GJ}(s,\pi\otimes\chi_1)^{-1}$ and $L^{GJ}(1-s,\pi^\vee\otimes\chi_1^{-1})^{-1}$ are coprime, so are $\alpha(s)$ and $\alpha^\vee(1-s)$. Proposition B.1(4) shows that $\varepsilon(s,\pi\times\chi,\psi)$ equals

$$\varepsilon^{GJ}(s,\pi\otimes\chi_1,\psi)\varepsilon^{GJ}\big(s,\pi^\vee\otimes\chi_2,\psi\big)\alpha(s)/\alpha^\vee(1-s).$$

We conclude that both $\alpha(s)$ and $\alpha^\vee(s)$ are invertible functions. They are necessarily equal to 1 in the nonarchimedean case. In the archimedean case the L-factors are defined only up to multiplication by invertible functions, and they can be taken to be as above. □

7.4 The Langland classification

Theorem 7.1 *If π is the Langlands quotient of a standard module* $\mathrm{Ind}_{\bar{P}(\mathcal{Y})}^{\bar{G}}\sigma$, *where $\sigma=\sigma_1\boxtimes\cdots\boxtimes\sigma_k\boxtimes\sigma_0$, then*

$$L_\psi(s,\pi\times\chi)=\prod_{j=0}^{k}L_\psi(s,\sigma_j\times\chi),$$

$$\varepsilon(s,\pi\times\chi,\psi)=\prod_{j=0}^{k}\varepsilon(s,\sigma_j\times\chi,\psi).$$

Remark 7.2 Since the archimedean L-factor is defined only up to multiplication by invertible functions, it would be more correct to say in the archimedean case that one can take $L_\psi(s,\pi\times\chi)$ to be as above.

The proof is based on a series of reduction steps. In view of Lemma 7.1(3) and Remark 5.1(2) we may assume that χ is unitary. We put $l=k$ in Case (I). If G is of type (II) and if $e(\sigma_1)\ge 0$ or $e(\sigma_0)\le 0$, then the proof is much simpler, i.e., Theorem 7.1 can be deduced from Lemma 6.1(2) and (5.1). In Case (II) we put

$$l = \max\{j \mid 1 \le j \le k,\ e(\sigma_j) \le 0\},$$

assuming that $e(\sigma_1) < 0$ and $e(\sigma_0) > 0$. Put $Y = Y_l$ and $G_1 = GL(Y, C)$. Let Q be the parabolic subgroup of G_1 stabilizing the flag $Y_1 \subset Y_2 \subset \cdots \subset Y_{l-1}$ and ϱ_1 the Langlands quotient of the standard module $\mathrm{Ind}_{\bar{Q}}^{\bar{G}_1}(\sigma_1 \boxtimes \cdots \boxtimes \sigma_l)$. In Case (I) we put $\varrho = \varrho_1 \boxtimes \sigma_0$ and take a totally isotropic subspace X of U which is nondegenerately paired with Y, and let Z be the orthogonal complement of $X + Y$ in U. In Case (II) we let $X = Z$ be a complement of Y in U and put $\varrho = \varrho_1 \boxtimes \varrho_0$, where ϱ_0 is the Langlands quotient of the standard module

$$\mathrm{Ind}_{Q'}^{GL(Z,C)}(\sigma_{l+1} \boxtimes \sigma_{l+2} \boxtimes \cdots \boxtimes \sigma_k \boxtimes \sigma_0)$$

and where Q' is the parabolic subgroup of $GL(Z, C)$ stabilizing the flag

$$Z \cap Y_{l+1} \subset Z \cap Y_{l+2} \subset \cdots \subset Z \cap Y_k.$$

We may take for a Levi subgroup $M(Y)$ the intersection $P(X) \cap P(Y)$ and identify it with $G_1 \times G(Z)$, and similarly for $M(Y^\square)$. Then π is a quotient of $\mathrm{Ind}_{\bar{P}(Y)}^{\bar{G}} \varrho$ and $\pi^\vee$ is a quotient of $\mathrm{Ind}_{\bar{P}(X)}^{\bar{G}} \varrho^\vee$. Note that the quotient map $\mathrm{Ind}_{\bar{P}(Y)}^{\bar{G}} \varrho \to \pi$ is given by

$$f \mapsto f'(g) = \int_{N(X)} f(ug)du.$$

The integral is absolutely convergent (cf. [21, 63]). Therefore, for every element $\xi \in \mathrm{Ind}_{\bar{P}(X) \times \bar{P}(Y)}^{\bar{G} \times \bar{G}}(\varrho^\vee \boxtimes \varrho)$,

$$\begin{aligned} H_\xi(g) &= \int_{P(X)^\diamond \backslash G^\diamond} \int_{N(X)} \mathcal{P}_{\varrho^\vee}\big(\xi\big((e, v)xg\big)\big)dvdx \\ &= \int_{M(Y)^\diamond \backslash G^\diamond} \mathcal{P}_{\varrho^\vee}\big(\xi(xg)\big)dx \end{aligned}$$

is a matrix coefficient of $\pi^\vee$.

We may compute the zeta integral as

$$\begin{aligned} &\int_{G^\diamond \backslash G \times G} \kappa_\chi^{\mathcal{U}}(g)^{-1} f^{(s)}(g) \int_{M(Y)^\diamond \backslash G^\diamond} \mathcal{P}_{\varrho^\vee}\big(\xi(xg)\big)dxdg \\ &\quad = \int_{M(Y)^\diamond \backslash G \times G} \kappa_\chi^{\mathcal{U}}(g)^{-1} f^{(s)}(g) \mathcal{P}_{\varrho^\vee}\big(\xi(g)\big)dg. \end{aligned}$$

Integrating first over $M(Y)^{\Diamond}\backslash P(X) \times P(Y)$, we obtain

$$\begin{aligned}
&\int_{P(X)\times P(Y)\backslash G\times G} \int_{M(Y)} \int_{N(X)} \int_{N(Y)} \kappa_\chi^{\mathcal{U}}(g)^{-1} f^{(s)}\big((mu, v)g\big) \\
&\qquad \times \delta_{P(X)}(m)^{1/2} H_{\xi(g)}(m)\, dv\, du\, dm\, dg \\
&\quad = \int_{P(X)\times P(Y)\backslash G\times G} \int_{M(Y)} \int_{N(X)} H_{\xi(g)}(m) \\
&\qquad \times \kappa_\chi^{\mathcal{U}}(g)^{-1} \big[\Psi^{\mathcal{U},\mathcal{Y}}(s,\chi) f^{(s)}\big]\big((u,e)g\big)\big((m,e)\big)\, du\, dm\, dg.
\end{aligned}$$

Lemma 7.6 (Cf. [19, 49]) *Notation being as above, we fix $\eta \in \varrho^\vee \boxtimes \varrho$ and a holomorphic section $f^{(s)}$ of $I(s,\chi)$.*

(1) *The integral*

$$A\big(\eta, f^{(s)}\big) = \int_{M(Y)} \int_{N(X)} H_\eta(m) \big[\Psi^{\mathcal{U},\mathcal{Y}}(s,\chi) f^{(s)}\big]\big((u,e)\big)\big((m,e)\big)\, du\, dm$$

converges absolutely in a right half-plane.

(2) *There is an element $\xi \in \mathrm{Ind}_{\bar{P}(X)\times \bar{P}(Y)}^{\bar{G}\times\bar{G}}(\varrho^\vee \boxtimes \varrho)$ such that*

$$Z\big(H_\xi, f^{(s)}\big) = A\big(\eta, f^{(s)}\big),$$

provided that $\Re s$ is sufficiently large.

Proof The explanation for (1) is the same as Remark 3.6.6 of [19]. The proof for (2) is completely analogous to the argument on p. 73 of [19]. The reader is referred to [49] for more detail. □

Recall that $N_\pm^Y$ is the subgroup of $G_1^\Box = GL(Y^\Box, C)$ consisting of the elements which act as the identity on both $Y_\pm$ and $Y^\Box/Y_\pm$. Note that $N_\pm^Y$ acts as the identity on both $X_\mp$ and $X^\Box/X_\mp$ in Case (I). Put $G' = G(Z_+)$ in Case (I), and put $G' = N_+^Z$ in Case (II).

Lemma 7.7 *Put $\Omega = PN(Y_-)N_-^Y G' N(X_+)$. Then Ω is an open set in $P\backslash G^\Box$. If F is nonarchimedean (resp. archimedean), then the map $(v, w, m_0, u) \mapsto Pvwm_0u$ is a homeomorphism (diffeomorphism) of $N(Y_-) \times N_-^Y \times G' \times N(X_+)$ onto Ω.*

Proof We focus on Case (I) as the proof in Case (II) is simpler. To prove the injectivity, we fix $u \in N(X_+)$, $v \in N(Y_-)$, $w \in N_-^Y$ and $m_0 \in G(Z_+)$. Put $m = vwm_0u$. Our task is to show that the space $U^\Delta m$ determines u, v, w and m_0, uniquely. Put $\Delta(t) = (t,t)$ for $t \in U$. We denote by pr_X and pr_Z the projections on the first and second factors of $U = X \oplus Z \oplus Y$, and by pr_+ and pr_- the projections on the first and second factors of $U^\square = U_+ \oplus U_-$. Since

$$\mathrm{pr}_X \circ \mathrm{pr}_-\big(\Delta(t)m\big) = \mathrm{pr}_X(t), \qquad \mathrm{pr}_Z \circ \mathrm{pr}_-\big(\Delta(t)m\big) = \mathrm{pr}_Z(tv),$$

it follows that

$$\big\{p \in U^\Delta m \mid \mathrm{pr}_X \circ \mathrm{pr}_-(p) = 0,\ \mathrm{pr}_Z \circ \mathrm{pr}_-(p) = 0\big\} = \Delta(Y)m.$$

For $y \in Y$ we can put $\Delta(y)w = ((0,0,y),(0,0,y + y\mu_w))$ with $\mu_w \in \mathrm{End}(Y, C)$. Then

$$\Delta(Y)m = \big\{\big(yu, (0,0,y + y\mu_w)\big) \bigm| y \in Y\big\},$$

from which we know u and w. Since

$$\mathrm{pr}_Z \circ \mathrm{pr}_+\big(\Delta(t)m\big) = zm_0 + y\gamma_u, \qquad \mathrm{pr}_Z \circ \mathrm{pr}_-\big(\Delta(t)m\big) = z$$

for $t = (0, z, y) \in U$, we know m_0. Observing that

$$\big\{p \in U^\Delta m \bigm| \mathrm{pr}_Y \circ \mathrm{pr}_+(p) = 0,\ \mathrm{pr}_Z \circ \mathrm{pr}_+(p) = 0\big\} = \Delta(X)m$$

and $\mathrm{pr}_-(\Delta(x)m) = xv$ for $x \in X$, we know v. Since

$$\dim N(X) + \dim N(Y) + \dim N_-^Y + \dim G' = \dim G = \dim\big(P\backslash G^\square\big),$$

Ω is an open set. The remaining part is easy to prove. $\square$

Now we prove Theorem 7.1 in the p-adic case. We are going to explain what kind of modifications are necessary in the archimedean case in Appendix A. We discuss only Case (I) as the proof in Case (II) is similar. Theorem 5.1 and Lemma 7.5 show that

$$L_\psi^{\mathcal{Y}}\big(s, \sigma^\vee \times \chi\big) = L_\psi\big(s, \sigma_0^\vee \times \chi\big) L^{GJ}\big(s, \varrho_1^\vee \otimes \chi\big) L^{GJ}\big(s, \varrho_1 \otimes \rho(\chi)\big).$$

The poles of $L_\psi(s, \sigma_0^\vee \times \chi)$ are contained in $\Re s \le 0$ by Lemma 7.2. Since $e(\sigma_j) < 0$ for $j = 1, 2, \dots, k$, Proposition 5.1 and Theorem 5.1 show that the poles of $L^{GJ}(s, \varrho_1^\vee \otimes \chi)$ are contained in $\Re s < 0$. Taking Lemma 6.1(2) into account, we have only to show that $Z(s, \pi, \chi)/L^{GJ}(s + \frac{1}{2}, \varrho_1 \otimes \rho(\chi))$ does not have a zero in $\Re s \ge 0$.

Fix s' with $\Re s' \ge 0$. Take $\phi_1 \in C_c^\infty(N_-^Y)$ and $\phi_0 \in C_c^\infty(G')$. Let $\phi' \in C_c^\infty(N(Y_-) \times N(X_+))$ have total integral 1 over $N(Y_-) \times N(X_+)$. Lemma 7.7 allows us to define a section $f^{(s)}$ of $I^\infty(s, \chi)$ by imposing the following conditions:

– $\operatorname{supp}(f^{(s)}) \subset \Omega$;
– for $u \in N(X_+)$, $v \in N(Y_-)$, $w \in N_-^Y$ and $m_0 \in G'$

$$f^{(s)}(vwm_0u) = \phi'(v,u)\phi_1(w)\phi_0(m_0).$$

Observe that

$$\int_{N(X)} \big[\Psi(s,\chi)f^{(s)}\big]\big((u,e)\big)\big((m_1m_0,e)\big)du = f_{\phi_1}^{(s)}(m_1)f_{\phi_0}^{(s)}(m_0)$$

for $m_1 \in G_1$ and $m_0 \in G(Z)$, where $f_{\phi_1}^{(s)}$ and $f_{\phi_0}^{(s)}$ are defined in Lemmas 5.3 and A.2. For $\eta_1 \in \varrho_1^\vee \boxtimes \varrho_1$ and $\eta_0 \in \sigma_0^\vee \boxtimes \sigma_0$, Lemma 7.6(2) gives an element $\xi \in \operatorname{Ind}_{\bar{P}(Y)\times\bar{P}(Y)}^{\bar{G}\times\bar{G}}(\varrho^\vee \boxtimes \varrho)$ such that

$$Z\big(\xi, f^{(s)}\big) = Z\big(\eta_1, f_{\phi_1}^{(s)}\big)Z\big(\eta_0, f_{\phi_0}^{(s)}\big).$$

Employing Lemma A.2 below, we can take the integral $Z(\eta_0, f_{\phi_0}^{(s)})$ to be 1 for a suitable choice of η_0 and ϕ_0. Our proof is complete by (5.1).

7.5 Unramified representations

Suppose for a moment that F is a nonarchimedean local field of odd residual characteristic and ψ is unramified. We call U unramified either if $\varphi = 0$ and $C = F$ or if $C = E$, χ_U^0 is unramified and $\eta(U) = 1$. If U is unramified, then G is defined over $\mathfrak{o}$, has a Borel subgroup B defined over F and is split over an unramified extension of F, and $K = G(\mathfrak{o})$ is a hyperspecial maximal compact subgroup of G, and we call $\pi \in \operatorname{Irr}(\bar{G})$ unramified if it contains a nonzero K-fixed vector. If U, π and ψ are unramified, then the transfer $\mathrm{BC}(\pi)$ to an unramified principal series representation of $\mathrm{GL}_N(E)$ is well-defined. Note that $\mathrm{BC}(\pi) = \pi \boxtimes \pi^\vee$ in Case (II) and that the parametrization depends on the choice of ψ in Case (I_4^m).

The following result is indispensable for the global theory.

Proposition 7.1 *If U, π, χ, ψ are unramified, then*

$$L_\psi(s, \pi \times \chi) = L^{GJ}\big(s, \mathrm{BC}_\psi(\pi) \otimes \chi\big), \qquad \varepsilon(s, \pi \times \chi, \psi) = 1.$$

Proof At this stage, we can infer this fact from Theorem 7.1. We can also argue directly as follows. Lemma 6.1(1) together with Lemma 7.5 shows that

$$L_\psi(s, \pi \times \chi) = Q\big(q^{-s}\big)L^{GJ}\big(s, \mathrm{BC}_\psi(\pi) \otimes \chi\big),$$

where Q is a polynomial of q^{-s}. In the metaplectic case we invoke Lemma 7.1(4). If $\xi_0 \in \pi \boxtimes \pi^\vee$ is a $K \times K$-fixed vector, then

$$Z\bigl(\xi_0, f_0^{(s)}\bigr) = L^{GJ}\left(s + \frac{1}{2}, \mathrm{BC}_\psi(\pi) \otimes \chi\right)\mathcal{P}(\xi_0)/b(s,\chi) \tag{7.2}$$

(see [32, 34, 44]). Since $b(s,\chi) f_0^{(s)}$ is a good section by Proposition 3.1(5), $L_\psi(s, \pi \times \chi)^{-1}$ is divisible by $L^{GJ}(s, \mathrm{BC}_\psi(\pi) \otimes \chi)^{-1}$, and hence Q is a unit, necessarily equal to 1. The second identity follows from (7.2) and Propositions 3.1(5). □

Remark 7.3 Arguing as in Sect. 5 of [29], one can prove Proposition 7.1 without recourse to Lemma 6.1(1).

7.6 The doubling zeta integrals for the unit group of a simple algebra

Theorem 7.2 *If G is of type (II), then for any $\pi \in \mathrm{Irr}(G)$ and any character χ of $E^\times$*

$$L(s, \pi \times \chi) = L^{GJ}(s, \pi \otimes \chi_1) L^{GJ}\bigl(s, \pi^\vee \otimes \chi_2\bigr),$$
$$\varepsilon(s, \pi \times \chi, \psi) = \varepsilon^{GJ}(s, \pi \otimes \chi_1, \psi)\varepsilon^{GJ}\bigl(s, \pi^\vee \otimes \chi_2, \psi\bigr).$$

Proof Write π as a Langlands quotient of a standard module $\mathrm{Ind}_Q^G(\sigma_1 \boxtimes \cdots \boxtimes \sigma_t)$. Then

$$L(s, \pi \times \chi) = \prod_{j=1}^t L(s, \sigma_j \times \chi) = \prod_{j=1}^t L^{GJ}(s, \sigma_j \otimes \chi_1) L^{GJ}\bigl(s, \sigma_j^\vee \otimes \chi_2\bigr)$$

by Theorem 7.1 and Lemma 7.5. The last product is equal to $L^{GJ}(s, \pi \otimes \chi_1) L^{GJ}(s, \pi^\vee \otimes \chi_2)$ by Theorem 5.1. One can derive the second identity by combining this with Proposition B.1(4). □

7.7 The local factors of Lapid and Rallis

Lapid and Rallis [32] use the gamma factors to define L and epsilon factors (cf. Appendix B). To reconcile the two definitions, we briefly recall how this is done. What is special about tempered representations is that $\gamma(s, \pi \times \chi, \psi)$ determines $L_\psi(s, \pi \times \chi)$ and $\varepsilon(s, \pi \times \chi, \psi)$ at the same time. From Lemma 7.2 the poles of $L_\psi(s, \pi \times \chi)$ are contained in $\Re s \le 0$, while those of $L_{\psi^{-1}}(1 - s, \pi^\vee \times \chi^{-1})$ are contained in $\Re s \ge 1$. Consequently, there is no cancellation and $L_\psi(s, \pi \times \chi)$ is equal to the inverse of a function which has the same zeros as $\gamma(s, \pi \times \chi, \psi)$. The local factors of general irreducible admissible representations are then defined by using the Langlands classification, multiplicativity, and the comparison result in Theorem 7.2.

8 Local theta correspondence

8.1 Splitting of reductive dual pairs

We give a brief discussion of the basic setup and results of local theta correspondence. Let F be a local field. Fix once and for all a right D vector space V of dimension m and equip it with a nondegenerate F-bilinear form $(\ ,\ ) : V \times V \to D$ such that

$$(x,y)^\rho = \epsilon(y,x), \qquad (xa,yb) = a^\rho(x,y)b \quad (a,b \in D;\ x,y \in V).$$

Put

$$H = \bigl\{h \in GL(V,D) \mid (hx,hy) = (x,y) \text{ for all } x,y \in V\bigr\}.$$

In the archimedean case let $\mathfrak{h}$ be the complexified Lie algebra of H.

Throughout this paper we put

$$s_m = \delta(m - \rho_n)/2.$$

We have a natural homomorphism $\iota_{V,W} : G \times H \to Sp(\mathbb{W})$, where $\mathbb{W} = V \otimes_D W$, viewed as an F-vector space of dimension dmn and equipped with the symplectic form

$$\langle\!\langle\ ,\ \rangle\!\rangle = \mathrm{Tr}_{E/F}\bigl(\tau\bigl((\ ,\ ) \otimes \langle\ ,\ \rangle^\rho\bigr)\bigr).$$

The metaplectic extension $\mathrm{Mp}(\mathbb{W})$ splits over G and H in all cases except when a symplectic group is paired with an odd orthogonal group, in which case $\mathrm{Mp}(\mathbb{W})$ splits over the orthogonal group but not over the symplectic group. For the proof of this fact we refer to [23, 37]. In the case of symplectic groups, we put $\bar{G} = \mathrm{Mp}^{(2)}(W)$ if m is odd, and we put $\bar{G} = Sp(W)$ if m is even. When G is not a symplectic group, we set $\bar{G} = G$ to make our exposition uniform, and similarly for H.

Having chosen the pair of characters $\chi = (\chi_V, \chi_W)$ as in Sect. 2.3, we fix a lift $\tilde{\iota}_{V,W} = \tilde{\iota}_{V,W,\chi,\psi}$ of the homomorphism $\iota_{V,W}$ as in [13]. We now explain its construction briefly. Recall that W_- is the space W with the ϵ-skew hermitian form $-\langle\ ,\ \rangle$. Put

$$\mathbb{W}^\square = V \otimes_D W^\square, \qquad \mathbb{W}_+ = V \otimes_D W_+, \qquad \mathbb{W}_- = V \otimes_D W_-.$$

The identity map on W gives an anti-isometry of W onto W_-, by which we identity $G(W)$ and $G(W_-)$, and similarly for $\mathbb{W}_-$ and $Sp(\mathbb{W}_-)$, as opposed to using the isomorphism induced by an isometry $\mathbb{W} \simeq \mathbb{W}_-$ which is the outer automorphism given by an element of $GSp(\mathbb{W})$ with similitude factor -1. Kudla [23] gave an explicit homomorphism $\tilde{\iota}^\square_{V,W} = \tilde{\iota}^\square_{V,W,\chi_V,\psi} : \bar{G}^\square \times H \to$

$\mathrm{Mp}(\mathbb{W}^\square)$ lifting the natural homomorphism $\iota_{V,W^\square}: G^\square \times H \to Sp(\mathbb{W}^\square)$. Since this lift depends on the choice of the character $\chi_V = \chi_m$ in the unitary case, so do relevant objects, but we suppress the dependence on χ_m in our notation. Let

$$i : Sp(\mathbb{W}) \times Sp(\mathbb{W}) \to Sp\big(\mathbb{W}^\square\big)$$

be the natural embedding. This map yields a homomorphism

$$\tilde{\iota} : \mathrm{Mp}(\mathbb{W}) \times \mathrm{Mp}(\mathbb{W}) \to \mathrm{Mp}\big(\mathbb{W}^\square\big),$$

which is unique if we demand that

$$\tilde{\iota} : \mathbb{C}^1 \times \mathbb{C}^1 \to \mathbb{C}^1, \quad \tilde{\iota}(z_1, z_2) = z_1 z_2^{-1}.$$

Since the inverse image of $Sp(\mathbb{W}) \times e$ in $\mathrm{Mp}(\mathbb{W}^\square)$ is isomorphic to $\mathrm{Mp}(\mathbb{W})$, the restriction of $\tilde{\iota}^\square_{V,W}$ to $\bar{G} \times e$ yields a homomorphism $\bar{G} \to \mathrm{Mp}(\mathbb{W})$, which we denote by $\tilde{\iota}_V$. Similarly, the restriction of $\tilde{\iota}^\square_{V,W}$ to $e \times \bar{G}$ yields a homomorphism $\tilde{\iota}_{V,-} : \bar{G} \to \mathrm{Mp}(\mathbb{W})$. It is important to note that the diagram

$$\begin{array}{ccc} \bar{G}^\square & \xrightarrow{\tilde{\iota}^\square_{V,W}} & \mathrm{Mp}(\mathbb{W}^\square) \\ \uparrow & & \uparrow \tilde{\iota} \\ \bar{G} \times \bar{G} & \xrightarrow{\tilde{\iota}_V \times \tilde{\iota}_{V,-}} & \mathrm{Mp}(\mathbb{W}) \times \mathrm{Mp}(\mathbb{W}) \end{array} \tag{8.1}$$

commutes by definition.

Lemma 8.1 $\tilde{\iota}_{V,-} = \chi_V^{-1} \cdot \tilde{\iota}_V$.

Proof The proof is the same as that for [13, Lemma 1.1]. □

We can construct a lifting homomorphism $\tilde{\iota}_W : \bar{H} \to \mathrm{Mp}(\mathbb{W})$ with the roles of W and V reversed. The homomorphism $\tilde{\iota}_{V,W} = (\tilde{\iota}_V, \tilde{\iota}_W) : \bar{G} \times \bar{H} \to \mathrm{Mp}(\mathbb{W})$ is the lift we want.

8.2 Local Howe conjecture

The group $\mathrm{Mp}(\mathbb{W})$ has a representation ω_ψ associated to ψ and called the Weil representation. Using the homomorphism $\tilde{\iota}_{W,V}$, we obtain the Weil representation

$$\omega_{\psi,V} = \omega_{\psi,V,W,\chi} = \omega_\psi \circ \tilde{\iota}_{V,W}$$

of $\bar{G} \times \bar{H}$ associated to the auxiliary data above.

The representation ω_ψ has various concrete realizations. If $\mathbb{W} = \mathbb{X} + \mathbb{Y}$ is a complete polarization, then $\mathrm{Mp}(\mathbb{W})$ can be identified with a certain group of unitary operators on $\mathcal{S}(\mathbb{X})$, which determines an isomorphism $\mathrm{Mp}(\mathbb{W}) \simeq Sp(\mathbb{W}) \times \mathbb{C}^1$ as sets. When F is a p-adic field, we put $S(\mathbb{X}) = \mathcal{S}(\mathbb{X})$. When $F = \mathbb{R}$ or $\mathbb{C}$, we work with the Harish-Chandra module associated to $\omega_{\psi,V}$ on the subspace $S(\mathbb{X})$ of $\mathcal{S}(\mathbb{X})$, which may be identified with the space of polynomials in a Fock model compatible with $\bar{K}$ and some maximal compact subgroup $\bar{K}_H$ of $\bar{H}$.

If $\pi \in \mathrm{Irr}(\bar{G})$, then the maximal quotient of $\omega_{\psi,V}$ on which $\bar{G}$ acts as a multiple of π is of the form $\pi \boxtimes \Theta^{\psi}_{V,W}(\pi)$, where $\Theta^{\psi}_{V,W}(\pi)$ is a representation of $\bar{H}$. We write $\Theta^{\psi}_{V,W}(\pi) = 0$ if π does not occur as a quotient of $\omega_{\psi,V}$. Let $\theta^{\psi}_{V,W}(\pi)$ be the maximal semisimple quotient of $\Theta^{\psi}_{V,W}(\pi)$. The following is the Howe duality conjecture:

- $\theta^{\psi}_{V,W}(\pi)$ is either zero or irreducible;
- the correspondence $\pi \mapsto \theta^{\psi}_{V,W}(\pi)$ defines a bijection between the subsets of $\mathrm{Irr}(\bar{G})$ and $\mathrm{Irr}(\bar{H})$ consisting of representations which occur in the correspondence.

This conjecture is known to hold except when F is an extension of $\mathbb{Q}_2$. For the proof we refer to [15] in the archimedean case, and to [61] in the p-adic case. It is expected to hold with no restriction. We will make use of only the following result:

Proposition 8.1 *For $\pi \in \mathrm{Irr}(\bar{G})$ the representation $\Theta^{\psi}_{V,W}(\pi)$ is either zero or of finite length.*

There is no restriction on the residual characteristic here. This result is proven in Théorème on p. 69 of [37], via the methods in [22, 47].

Bear in mind that m is even and $V = V^+$ is split whenever G is of type (I_0). To conserve notation, we set $\eta(V^-) = -1$ and take $\Theta^{\psi}_{V^-,W}(\pi)$ to be $\Theta^{\psi}_{V^+,W}(\pi \otimes \mathrm{sgn})$. It will be convenient to artificially view V^- as a nonsplit symplectic space and distinguish it from the split symplectic space V^+. With this convention, we insert here the classification of ϵ-hermitian spaces.

Proposition 8.2 (Cf. [51]) *When $F = \mathbb{R}$, we exclude Cases* (I_1), (I_2) *and* (I_4). *Two ϵ-hermitian spaces V and U are isometric if and only if* $\dim V = \dim U$, *$\chi_V = \chi_U$ and $\eta(V) = \eta(U)$.*

Remark 8.1

(1) Recall that χ_U depends only on $\dim U$ in Case (I_2).
(2) If F is a p-adic field in Cases (I_1), or if $F = \mathbb{C}$ in Case (I_4), or if G is of either type (I_3) or type (II), then there is at most one equivalence class of nondegenerate ϵ-hermitian spaces with dimension and character fixed.

8.3 Submodules associated to ϵ-hermitian forms

Let $\omega_\psi^\square$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}^\square)$ associated to ψ. We consider the standard Schrödinger model associated to the complete polarization

$$\mathbb{W}^\square = \big(V \otimes_D W^\triangledown\big) \oplus \big(V \otimes_D W^\vartriangle\big),$$

and so we take $V_{\omega_\psi^\square} = \mathcal{S}(V \otimes_D W^\triangledown)$. The pullback $\omega_{\psi,V}^\square = \omega_\psi^\square \circ \tilde{\iota}_{V,W}^\square$ of the representation $\omega_\psi^\square$ to $\bar{G}^\square \times H$ yields a representation of this product group on the space $\mathcal{S}(V \otimes_D W^\triangledown)$. It is well-known that

$$\omega_\psi^\square \circ \tilde{\iota} \simeq \omega_\psi \boxtimes \omega_\psi^\vee.$$

Lemma 8.1 and (8.1) show that

$$\omega_{\psi,V}^\square \simeq \big(\omega_{\psi,V} \boxtimes \omega_{\psi,V}^\vee\big) \otimes \kappa_{\chi_V} \tag{8.2}$$

as a representation of $\bar{G} \times \bar{G}$, where the anti-linear action of $\mathbb{C}^1$ for $\omega_\psi^\vee$ inverts the χ_V^{-1}. We remark that κ_{χ_V} is trivial except in the unitary case. We require that $h \in H$ acts by

$$\omega_{\psi,V}^\square(h)\Phi(w) = \Phi\big(h^{-1}w\big) \quad \big(\Phi \in \mathcal{S}\big(V \otimes_D W^\triangledown\big),\ w \in V \otimes_D W^\triangledown\big).$$

The choice of polarization of $\mathbb{W}$ induces a polarization $\mathbb{W}^\square = \mathbb{X}^\square + \mathbb{Y}^\square$ of $\mathbb{W}^\square$, where $\mathbb{X}^\square = \mathbb{X} \oplus \mathbb{X}$ and $\mathbb{Y}^\square = \mathbb{Y} \oplus \mathbb{Y}$. We can realize the representation $\omega_\psi^\square$ on $\mathcal{S}(\mathbb{X}^\square) = \mathcal{S}(\mathbb{X}) \otimes \mathcal{S}(\mathbb{X})$. Note that the integral $\phi \mapsto \int_{\mathbb{X}} \phi(u,u)du$ defines an invariant pairing $\mathcal{P}_{\omega_\psi} : \omega_\psi \otimes \omega_\psi^\vee \to \mathbb{C}$. There is an intertwining map $\sigma : \mathcal{S}(\mathbb{X}^\square) \to \mathcal{S}(V \otimes_D W^\triangledown)$ such that

$$\sigma(\phi)(0) = \mathcal{P}_{\omega_\psi}(\phi).$$

We briefly describe the representation $\Theta_{W^\square,V}^\psi(\mathbb{1})$ associated to the trivial representation of H via the local theta correspondence. In the p-adic case $\Theta_{W^\square,V}^\psi(\mathbb{1})$ is the maximal quotient of $S(V \otimes_D W^\triangledown)$ on which H acts trivially. By the explicit formula for $\omega_{\psi,V}^\square$ described in [23], there is an intertwining map

$$S\big(V \otimes_D W^\triangledown\big) \to I(s_m, \chi_V), \quad \Phi \mapsto f_\Phi^{(s_m)}(g) = \omega_{\psi,V}^\square(g)\Phi(0).$$

This map factors through $\Theta_{W^\square,V}^\psi(\mathbb{1})$ and, by the invariant distribution theorem of Rallis [37, 47], induces an injection $\Theta_{W^\square,V}^\psi(\mathbb{1}) \hookrightarrow I(s_m, \chi_V)$, which says that $\Theta^{\psi}_{W^{\square},V}(\mathbb{1})$ is nonvanishing. In the archimedean case let $\Theta^{\psi}_{W^{\square},V}(\mathbb{1})$ be the maximal quotient of $S(V \otimes_D W^{\triangledown})$ on which $(\mathfrak{h}, K_H)$ acts trivially, and the analogous statement still holds. Let us take $\Theta^{\psi}_{W^{\square},V^-}(\mathbb{1})$ to be $\Theta^{\psi}_{W^{\square},V^+}(\mathbb{1}) \otimes \mathrm{sgn}$ in Case (I_0) to simplify notation.

Using the Iwasawa decomposition, we extend $f_{\Phi}^{(s_m)}$ to the standard section $f_{\Phi}^{(s)}$ of $I(s, \chi_V)$. Put

$$F_{\Phi}^{(s)} = b(s, \chi_V) f_{\Phi}^{(s)}, \qquad h_{\Phi}^{(-s)} = M^{\dagger}(s, \chi_V) F_{\Phi}^{(s)}.$$

Lemma 8.2 *For any $\Phi \in S(V \otimes_D W^{\triangledown})$, the section $F_{\Phi}^{(s)}$ is good at $s = s_m$. In particular, if $m \le \rho_n$, then $h_{\Phi}^{(s)}$ is holomorphic at $s = -s_m$.*

Proof When $m \ge \rho_n$, there is nothing to prove. When $m < \rho_n$, this lemma is equivalent to [68, Proposition 3.1], [69, Proposition 4.11] and [70, Proposition 1.4]. □

Note that if we put $M^*(s, \chi_V) = b(-s, \chi_V)^{-1} M^{\dagger}(s, \chi_V)$, then the following identity follows from (B.3):

$$M^*(s, \chi_V) h_{\Phi}^{(s)} = f_{\Phi}^{(-s)}. \tag{8.3}$$

Lemma 8.3 *If $\Re s > -\delta/2$, then the function $g \mapsto f^{(s)}((g, e))$ belongs to $L^2(G)$.*

Proof We consider Case (I) as Case (II) is settled in the appendix of [70]. Observe first that when $g \in \bar{G}$ and $\Phi = \sigma(\phi_1 \otimes \overline{\phi_2})$ for $\phi_i \in S(\mathbb{X})$,

$$f_{\Phi}^{(s_m)}\big((g, e)\big) = \mathcal{P}_{\omega_{\psi}}\big(\omega_{\psi,V}(g)\phi_1 \otimes \overline{\phi_2}\big),$$

is a matrix coefficient of $\omega_{\psi,V}$.

It is known that if V is split and $\chi_V = 1$, then $\Theta^{\psi}_{W^{\square},V}(\mathbb{1})$ contains $f_0^{(s_m)}$. Recall that the section $f_0^{(s)}$ is defined in Sect. 3.1. We refer to [12, Lemma A.1] in Case (I_2). The author [67] proved this fact in the quaternion case. Let $\mathcal{H}$ be a hyperbolic plane. There is an element $\Phi \in S(\mathcal{H} \otimes_D W^{\triangledown})$ such that $f_{\Phi}^{(s_2)} = f_0^{(s_2)}$, where $s_2 = \delta(1 - \frac{\rho_n}{2})$, and hence $f_0^{(s_2)}$ is a linear combination of matrix coefficients of the Weil representation $\omega_{\psi,\mathcal{H}}$ of $\bar{G}$. Therefore all we have to do is to show that if $t > \rho_n - 1$, then for all $\phi_1, \phi_2 \in S(\mathbb{X})$ the integral

$$\int_G \big|\mathcal{P}_{\omega_{\psi}}\big(\omega_{\psi,\mathcal{H}}(g)\phi_1 \otimes \overline{\phi_2}\big)\big|^t dg$$

converges absolutely, which is nothing but [33, Theorem 3.2]. □

Lemma 8.4 *If π and χ are unitary, then $L_\psi(s, \pi \times \chi)$ is holomorphic in the half-plane $\Re s > \frac{1}{2}\{\delta(\rho_n - 2) + 1\}$.*

Proof Since any matrix coefficient of $\pi^\vee$ is bounded, the integrand of $Z(\xi, f^{(s)})$ is majorized by a multiple of $|f_0^{(s)}((g, e))|$, from which one can derive Lemma 8.4 from Lemma 8.3. $\square$

The subject of the structure of degenerate principal series representations is vast. Some relevant works are [25, 27, 30, 57, 67]. We here record only the relevant knowledge that we need.

Proposition 8.3

(1) *If $0 \le m < \rho_n$, then the maximal semisimple quotient of the representation $I(-s_m, \chi_V)$ is isomorphic to*

$$\bigoplus_{\dim U = m,\ \chi_U = \chi_V} \Theta^\psi_{W^\square, U}(\mathbb{1}) \subset I(s_m, \chi_V),$$

where U runs over all equivalence classes of ϵ-hermitian spaces over D such that $\dim U = m$ and $\chi_U = \chi_V$. Moreover, the quotient map is realized as the intertwining operator $M^*(-s_m, \chi_V)$.

(2) *If $m = \rho_n$, then*

$$I(0, \chi_V) = \bigoplus_{\dim U = \rho_n,\ \chi_U = \chi_V} \Theta^\psi_{W^\square, U}(\mathbb{1}).$$

8.4 First occurrence

Two ϵ-hermitian spaces are said to lie in the same Witt tower if they are isomorphic after the addition of a suitable number of hyperbolic planes.

Lemma 8.5 *Let $\pi \in \mathrm{Irr}(\bar{G})$. We exclude Case* (II). *When $F = \mathbb{C}$, we exclude Case* (I_4). *Let U and U' be two ϵ-hermitian spaces such that $\chi_U = \chi_{U'}$. If both $\Theta^\psi_{U,W}(\pi)$ and $\Theta^\psi_{U',W}(\pi)$ are nonzero, then either*

- *U and U' lie in the same Witt tower*; *or*
- $\dim U + \dim U' > 2\rho_n$.

Proof Notice that $\dim U$ and $\dim U'$ have the same parity except in the quaternion case. When F is a p-adic field, we refer to [29, Theorem 3.8] in Case (I_4), to [12] in Case (I_2), to [5] in Case (I_0) and to [55] in the quaternion case. When $F = \mathbb{R}$, this was proven in [1, Lemma 1.8] in Cases (I_0) and (I_4) (cf. [41, Proposition 22]), in [35, Proposition 3.38] in Case (I_1) and in [40, Proposition 3.2] in Case (I_2). Finally, we assume $F = \mathbb{R}$ and consider

Case (I_3). Given a positive integer m, there is a unique equivalence class of nondegenerate skew hermitian spaces of dimension m over D in this case. The assumption $\chi_U = \chi_{U'}$ implies that $\dim U - \dim U'$ is even, from which U and U' must lie in the same Witt tower. $\square$

Remark 8.2

(1) Lemma 8.5 is a part of the conservation relation which was conjectured by Kudla and Rallis [29] and has recently been proven by Sun and Zhu [55].
(2) Lemma 8.5 follows readily from the special case in which π is the trivial representation: if U_0 is a nonsplit ϵ-hermitian space of dimension $2\rho_n$ with trivial character, then $\Theta^{\psi}_{U_0,W}(\mathbb{1})$ is zero.

Corollary 8.1 *If $\pi \in \mathrm{Irr}(\bar{G})$ and $m \leq \rho_n + \frac{1}{2}$, then at most one of the representations $\Theta^{\psi}_{U,W}(\pi)$ is nonvanishing, where U runs over all equivalence classes of ϵ-hermitian spaces of dimension m with character χ_V.*

Remark 8.3 It is known that when $m = \rho_n$, precisely one of the representations $\Theta^{\psi}_{U,W}(\pi)$ is nonvanishing (cf. [1, 5, 7, 13, 39, 71]). The root number $\varepsilon(1/2, \pi \times \chi_V, \psi)$ is known to determine which representation is nonzero except when $F = \mathbb{R}$ and G is of type (I_1), (I_2) or (I_4). In these exceptional cases the root number only determines for which of two families of groups the lift is nonzero.

Recall that $s_m = \delta(m - \rho_n)/2$.

Lemma 8.6 *Let $\pi \in \mathrm{Irr}(\bar{G})$. Assume that $m \geq \rho_n$.*

(1) *If the restriction of $\mathcal{Z}(s_m, \pi, \chi_V)$ to $\Theta^{\psi}_{W^{\square},V}(\mathbb{1})$ is nonzero, then $\Theta^{\psi}_{V,W}(\pi)$ is nonvanishing.*
(2) *Suppose either that $I(s_m, \chi_V) = \Theta^{\psi}_{W^{\square},V}(\mathbb{1})$ or that*

$$I(s_m, \chi_V)/\Theta^{\psi}_{W^{\square},V}(\mathbb{1}) \simeq \Theta^{\psi}_{W^{\square},V'_0}(\mathbb{1}),$$

where V'_0 is an ϵ-hermitian space of dimension $2\rho_n - m$ with character χ_V such that V and V'_0 do not lie in the same Witt tower. Then the converse of (1) holds.

Remark 8.4

(1) The assumption of (2) is fulfilled except when $F = \mathbb{R}$ and G is of type (I_1), (I_2) or (I_4). In the exceptional cases the module structure of $I(s_m, \chi_V)$ is much more involved.

(2) Assume either G is of type (II); or $F = \mathbb{C}$ and G is of type (I_4); or $F = \mathbb{R}$ and G is of type (I_3). Then $I(s_m, \chi_V) = \Theta^{\psi}_{W^\square, V}(\mathbb{1})$, so that $\Theta^{\psi}_{V,W}(\pi)$ is nonzero and there are $\xi \in \pi^\vee \boxtimes \pi$ and $\Phi \in S(V \otimes_D W^\triangledown)$ satisfying $\mathcal{Z}(\xi, f^{(s_m)}_{\Phi}) \neq 0$.

Proof Since

$$\begin{aligned} &\mathrm{Hom}_{\bar{G}\times\bar{G}}\big(\Theta^{\psi}_{W^\square,V}(\mathbb{1}), \big(\pi \boxtimes \pi^\vee\big) \otimes \kappa_{\chi_V}\big) \\ &\quad\simeq \mathrm{Hom}_{\bar{H}}\big(\Theta^{\psi}_{V,W}(\pi) \otimes \Theta^{\psi}_{-V,W}\big(\pi^\vee\big), \mathbb{1}\big) \end{aligned}$$

by the local seesaw identity, the first part is evident. We consider the second part. Since there is nothing to prove if $I(s_m, \chi_V) = \Theta^{\psi}_{W^\square,U}(\mathbb{1})$, we assume that this is not the case. If $\mathcal{Z}(s_m, \pi, \chi_V)$ is identically zero on $\Theta^{\psi}_{W^\square,V}(\mathbb{1})$, then it gives rise to a nonzero element in the space

$$\mathrm{Hom}_{\bar{G}\times\bar{G}}\big(\Theta^{\psi}_{W^\square,V_0'}(\mathbb{1}), \big(\pi \boxtimes \pi^\vee\big) \otimes \kappa_{\chi_V}\big).$$

By the observation above $\Theta^{\psi}_{V_0',W}(\pi)$ is nonvanishing, from which Lemma 8.5 forces $\Theta^{\psi}_{V,W}(\pi)$ to be zero. □

9 The integral representation of the standard L-function

9.1 Eisenstein series

We take this opportunity to fix some notation that will be used in the global setting. Let D be a division algebra over a number field E of the first three types referred to in Sect. 2.2. We take ϵ, $\mathcal{W} = (W, \langle\ ,\ \rangle)$, $G = G(\mathcal{W})$, $G^\square = G(\mathcal{W}^\square)$ as in Sect. 2. Note that G is an algebraic group of type (I) defined over the fixed subfield F of E under ρ. We tacitly exclude the case in which W is a split binary quadratic space.

We denote by $\mathbf{a}$ and $\mathbf{h}$ the sets of archimedean and nonarchimedean places of F and, in the quaternion case, by $\mathbf{b}$ and $\mathbf{d}$ the sets of all archimedean and nonarchimedean places ramified in D. Put $\mathbf{v} = \mathbf{a} \cup \mathbf{h}$. For each place $v \in \mathbf{v}$, let F_v be the v-completion of F. We write

$$\begin{aligned} E_v &= E \otimes_F F_v, & D_v &= D \otimes_F F_v, \\ W_v &= W \otimes_F F_v, & G_v &= G(F_v) \end{aligned}$$

for simplicity. The D_v-module W_v is equipped with an ϵ-hermitian form by extension of scalars. We write $G(\mathbb{A})$ for the adelization of G. We fix a maximal order $\mathcal{O}$ of D. For $v \in \mathbf{h}$ let $\mathcal{O}_v$ be a maximal compact subring of D_v determined by $\mathcal{O}$. Fix a maximal compact subgroup K of $G(\mathbb{A})$ such that $K = \prod_v K_v$, where K_v is still a maximal compact subgroup of G_v and where $K_v = G(\mathfrak{o}_v)$ for almost all finite places. Fix a maximal compact subgroup $K^\square$ of $G^\square(\mathbb{A})$ such that $K^\square = \prod_v K_v^\square$, where if $v \in \mathbf{h}$, then $K_v^\square = G_v^\square \cap \mathrm{GL}_{2n}(\mathcal{O}_v)$. Note that the Iwasawa decomposition $G^\square(\mathbb{A}) = P(\mathbb{A})K^\square$ holds.

In Case (I_4^m) let $\bar{G}_{\mathbb{A}}$ denote the two-fold metaplectic cover of $G(\mathbb{A}) = Sp(W, \mathbb{A})$. Recall that $\bar{G}_{\mathbb{A}}$ splits over the subgroup of rational points $G(F)$ in $G(\mathbb{A})$. To simplify notation, we write $\bar{G}_{\mathbb{A}} = G(\mathbb{A})$ in all other cases. Let $\mathcal{A}(\bar{G}_{\mathbb{A}})$ stand for the space of automorphic forms on $\bar{G}_{\mathbb{A}}$.

For a fixed unitary character χ of C_E we define the global induced representation $I(s,\chi) = \bigotimes_v I_v(s,\chi_v)$ and the global intertwining operator $M(s,\chi) = \bigotimes_v M_v(s,\chi_v)$ in analogy with Sect. 3, where χ_v denotes the character of $E_v^\times$ induced on $E_v^\times$ by χ. We define the functions $a(s,\chi)$ and $b(s,\chi)$ by taking the complete Hecke L-functions in place of the local L-factors in the definition of $a_v(s,\chi_v)$ and $b_v(s,\chi_v)$.

We define standard sections, holomorphic sections and meromorphic sections similarly. We call a meromorphic section of $I(s,\chi)$ a good section if it is a finite sum of factorizable elements $\bigotimes_v f_v^{(s)}$ such that $f_v^{(s)}$ is a good section for all v and such that $f_v^{(s)} = b_v(s,\chi_v) f_{0,v}^{(s)}$ for almost all v. Observe that the product $\bigotimes_v f_v^{(s)}$ is absolutely convergent for $\Re s > 0$ and can be continued meromorphically to the whole s-plane.

Lemma 9.1 *If $f^{(s)} = \bigotimes_v f_v^{(s)}$ is a good section, then so is $M(s,\chi)f^{(s)}$, and we have $M(s,\chi)f^{(s)} = \bigotimes_v M_v^\dagger(s,\chi_v) f_v^{(s)}$.*

Proof One can deduce Lemma 9.1 from Proposition 3.1(5) and (B.2). □

For any holomorphic section $f^{(s)}$ of $I(s,\chi)$, we form the associated Eisenstein series $E(f^{(s)})$ on $G^\square(F)\backslash \bar{G}_{\mathbb{A}}^\square$ by

$$E\big(f^{(s)}\big)(g) = \sum_{\gamma \in P(F)\backslash G^\square(F)} f^{(s)}(\gamma g).$$

Such a series converges absolutely for $\Re s > \delta\rho_n/2$ (cf. Théorème 3 of [66] or Sect. A3 of [52]). By the theory of Eisenstein series, it can be continued to a meromorphic function in s on all of $\mathbb{C}$ satisfying a functional equation:

$$E\big(f^{(s)}\big) = E\big(M(s,\chi)f^{(s)}\big).$$

When $\chi = \rho(\chi)^{-1}$, we set

$$\Upsilon_{n,\chi} = \{0, 2, 4, \dots, 2(n+\epsilon)\}, \tag{$\mathrm{I}_0, \mathrm{I}_4^s$}$$

$$\Upsilon_{n,\chi} = \{j = 0, 1, 2, \dots, 2n \mid \chi^0 = \epsilon_{E/F}^{\delta j}\}, \tag{I_2}$$

$$\Upsilon_{n,\chi} = \{0, 1, 2, \dots, 2n - \epsilon\}, \tag{$\mathrm{I}_1, \mathrm{I}_3$}$$

$$\Upsilon_{n,\chi} = \{1, 3, 5, \dots, 2n+1\}. \tag{I_4^m}$$

Proposition 9.1 ([16, 17, 26, 59, 69, 70]) *If $f^{(s)}$ is a good section of $I(s,\chi)$, then the poles of $E(f^{(s)})$ are at most simple. If $\chi\rho(\chi)$ is not principal, then $E(f^{(s)})$ is entire. If $\chi = \rho(\chi)^{-1}$, then the poles can occur in the following set*:

$$\{s_j = \delta(j - \rho_n)/2 \mid j \in \Upsilon_{n,\chi}\} \setminus \{0\}.$$

Remark 9.1 More precisely, if $j < \rho_n$, then $E(f^{(s)})$ has a pole at $s = \delta(j - \rho_n)/2$ for some good section $f^{(s)}$ of $I(s, \chi_V)$ if and only if there is an ϵ-hermitian space U of dimension j and character $\chi_U = \chi_V$.

Proof For proofs of this result we refer to [16, 26] for symplectic and orthogonal groups, to [17] for metaplectic groups, to [59, 70] for unitary groups, and to [69] for quaternionic unitary groups. Observe that the normalization factors $b_v(s, \chi_v)$ knock out infinitely many unwanted poles of the Eisenstein series. □

9.2 Poles of the standard L-functions

Let π be an irreducible cuspidal automorphic representation of $\bar{G}_{\mathbb{A}}$ realized on a space $V_\pi \subset L^2(G(F)\backslash \bar{G}_{\mathbb{A}})$, where we fix an embedding $\pi \hookrightarrow V_\pi \subset \mathcal{A}(\bar{G}_{\mathbb{A}})$. The contragredient representation $\pi^\vee$ is realized on the complex conjugate $\overline{V_\pi}$ of V_π. The Petersson pairing $\mathcal{P} = \mathcal{P}_{\pi^\vee} : \overline{V_\pi} \boxtimes V_\pi \to \mathbb{C}$ is defined by

$$\mathcal{P}_{\pi^\vee}(\xi_1 \boxtimes \xi_2) = \int_{G(F)\backslash G(\mathbb{A})} \xi_1(g)\xi_2(g)dg.$$

We fix isomorphisms $V_\pi \simeq \bigotimes_v \pi_v$ and $\overline{V_\pi} \simeq \bigotimes_v \pi_v^\vee$ and choose standard local pairings $\mathcal{P}_{\pi_v^\vee} : \pi_v^\vee \boxtimes \pi_v \to \mathbb{C}$ in order that $\mathcal{P}_{\pi^\vee}(\xi) = \prod_v \mathcal{P}_{\pi_v^\vee}(\xi_v)$ for all $\xi = \bigotimes_v \xi_v \in \overline{V_\pi} \boxtimes V_\pi$, where $\mathcal{P}_{\pi_v^\vee}(\xi_{0,v}) = 1$ for almost all the $\bar{K}_v \times \bar{K}_v$-invariant vectors $\xi_{0,v} \in \pi_v^\vee \boxtimes \pi_v$ used to define the restricted tensor products.

We can at least formally define

$$L_\psi(s, \pi \times \chi) = \prod_v L_{\psi_v}(s, \pi_v \times \chi_v).$$

Proposition 7.1 shows that the product

$$\varepsilon_\psi(s,\pi\times\chi)=\prod_v \varepsilon(s,\pi_v\times\chi_v,\psi_v)$$

is in fact a finite product. The fact that $\varepsilon_\psi(s,\pi\times\chi)$ is independent of the choice of ψ except in Case (I_4^m) can be checked by analyzing how the epsilon factors vary as you vary ψ, as is done in Remark 5.1(4).

For an algebraic group G over F, we occasionally write $[G]$ in place of $G(F)\backslash G(\mathbb{A})$ in order to save space. For each pair of cusp forms $\xi_1\in\overline{V_\pi}$ and $\xi_2\in V_\pi$ and each section $f^{(s)}$ of $I(s,\chi)$ we consider the global zeta integral defined by

$$Z\big(\xi_1\boxtimes\xi_2,f^{(s)}\big)=\int_{[G\times G]}\chi\big(\nu(g_2)\big)^{-1}\xi_1(g_1)\xi_2(g_2)E\big(f^{(s)}\big)\big((g_1,g_2)\big)dg_1dg_2.$$

Since the two cusp forms are rapidly decreasing on $G(F)\backslash\bar{G}_{\mathbb{A}}$ and the Eisenstein series is only of moderate growth, we see that the integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic in s. Unfolding the Eisenstein series as in [44], we get

$$Z\big(\xi_1\boxtimes\xi_2,f^{(s)}\big)=\int_{G(\mathbb{A})}\mathcal{P}\big(\pi(g)\xi_1\boxtimes\xi_2\big)f^{(s)}\big((g,e)\big)dg.$$

If $\xi_i=\bigotimes_v\xi_{i,v}$ and $f^{(s)}=\bigotimes_v f_v^{(s)}$ are factorisable, then this integral factors into a product of local integrals

$$Z\big(\xi_1\boxtimes\xi_2,f^{(s)}\big)=\prod_v Z_v\big(\xi_{1,v}\boxtimes\xi_{2,v},f_v^{(s)}\big).$$

Theorem 9.1 ([26, 44]) *The infinite product $L_\psi(s,\pi\times\chi)$ converges absolutely for $\Re s>(\delta\rho_n+1)/2$, can be meromorphically continued to the whole $\mathbb{C}$ and satisfies the functional equation*:

$$L_\psi(s,\pi\times\chi)=\varepsilon_\psi(s,\pi\times\chi)L_{\psi^{-1}}\big(1-s,\pi^\vee\times\chi^{-1}\big).$$

If $\chi\rho(\chi)$ is not principal, then $L_\psi(s,\pi\times\chi)$ is entire. If $\chi=\rho(\chi)^{-1}$, then $L_\psi(s,\pi\times\chi)$ has at most simple poles and these can only occur at the points in $\{s_j+\frac{1}{2}\mid j\in\Upsilon_{n,\chi}\}\backslash\{\frac{1}{2}\}$.

Proof Lemma 8.4 implies that $L_{\psi_v}(s,\pi_v\times\chi_v)$ is holomorphic in $\Re s>(\delta\rho_n-1)/2$ for all v, which guarantees the convergence of $L_\psi(s,\pi\times\chi)$ for $\Re s>(\delta\rho_n+1)/2$. Let S be a large enough set of places of F containing all the archimedean places such that for all $v\notin S$, W_v, π_v, χ_v, ψ_v are unramified, $\xi_v = \xi_{0,v}$ and $f_v^{(s)} = b_v(s, \chi_v) f_{0,v}^{(s)}$. Proposition 7.1 and (7.2) give

$$Z\bigl(\xi, f^{(s)}\bigr) = L_\psi\left(s + \frac{1}{2}, \pi^\vee \times \chi\right) \prod_{v \in S} \mathcal{Z}_v\bigl(\xi_v, f_v^{(s)}\bigr). \tag{9.1}$$

We next turn to the functional equation. The functional equation

$$Z\bigl(\xi, f^{(s)}\bigr) = Z\bigl(\xi, M(s,\chi) f^{(s)}\bigr)$$

follows from the functional equation of the Eisenstein series. The right hand side also unfolds as

$$L_{\psi^{-1}}\left(-s + \frac{1}{2}, \pi \times \chi^{-1}\right) \prod_{v \in S} \mathcal{Z}_v\bigl(\xi_v, M_v^\dagger(s, \chi_v) f_v^{(s)}\bigr)$$

by Proposition 5.4 and Lemma 9.1 with convergence for $\Re s \ll 0$. By the local functional equation and the product formula $\prod_v z(\pi_v)\varepsilon_{W_v,\psi_v} = 1$, we get the global functional equation as stated.

By definition, each $\mathcal{Z}_v(\xi_v, f_v^{(s)})$ is entire, and for any $s' \in \mathbb{C}$ and any place v there is a choice of ξ_v and $f_v^{(s)}$ such that $\mathcal{Z}_v(\xi_v, f_v^{(s)})|_{s=s'} \neq 0$. So as we vary ξ and $f^{(s)}$ at the places $v \in S$ we see that division by these factors can introduce no extraneous poles or zeros in $L_\psi(s, \pi^\vee \times \chi)$. Thus the poles of $L_\psi(s + \frac{1}{2}, \pi^\vee \times \chi)$ are among those of the Eisenstein series. Proposition 9.1 now gives the expected set of possible poles. $\square$

Remark 9.2 The set of possible poles is independent of π, but the actual set of poles may be more restricted. For example, since the Eisenstein series has a constant residue (possibly zero) at $s = \delta\rho_n/2$, we can rule out the possibility that $j = 0$ or $2\rho_n$ unless W is anisotropic. See also [28, Theorem 7.2.5(ii)].

10 Global theta correspondence

10.1 Tower property

Once and for all we fix a global ϵ-hermitian space $\mathcal{V} = (V, (\ ,\ ))$ of dimension m whose group of isometries is denoted by H. Put $\mathbb{W} = V \otimes_D W$ and take a complete polarization $\mathbb{W} = \mathbb{X} + \mathbb{Y}$. Let ω_ψ be the Weil representation of $\mathrm{Mp}(\mathbb{W})_\mathbb{A}$ associated to ψ which can be realized on the restricted tensor product $S(\mathbb{X}(\mathbb{A})) = \bigotimes_v S(\mathbb{X}_v)$. Here the metaplectic extension $\mathrm{Mp}(\mathbb{W})_\mathbb{A}$ of $Sp(\mathbb{W}, \mathbb{A})$ is a certain group of unitary operators which preserves the space $\mathcal{S}(\mathbb{X}(\mathbb{A}))$ and is compatible with the local metaplectic groups. Let $\Theta : S(\mathbb{X}(\mathbb{A})) \to \mathbb{C}$ be the functional $\phi \mapsto \Theta(\phi) = \sum_{\xi \in \mathbb{X}(F)} \phi(\xi)$.

The splitting $Sp(\mathbb{W}, F) \to \mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ is determined by the condition that $\Theta(\omega_\psi(\gamma)\phi) = \Theta(\phi)$ for all $\gamma \in Sp(\mathbb{W}, F)$ and $\phi \in S(\mathbb{X}(\mathbb{A}))$. This functional gives an intertwining operator from the space of the Weil representation to the space of automorphic forms on $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$.

Using the local homomorphism constructed in Sect. 8, we get a splitting $\tilde{\iota}_{V,W} : \bar{G}_{\mathbb{A}} \times \bar{H}_{\mathbb{A}} \to \mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ and let $\omega_{\psi,V} = \omega_\psi \circ \tilde{\iota}_{V,W}$ be the pullback of the Weil representation of $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ to the product group $\bar{H}_{\mathbb{A}} \times \bar{G}_{\mathbb{A}}$. For each $\phi \in S(\mathbb{X}(\mathbb{A}))$ we consider the theta function

$$\Theta(g, h; \phi) = \Theta\big(\omega_{\psi,V}(g, h)\phi\big) \quad (g \in \bar{G}_{\mathbb{A}},\ h \in \bar{H}_{\mathbb{A}}).$$

Let (π, V_π) be an irreducible cuspidal automorphic representation of $\bar{G}_{\mathbb{A}}$. We consider a space of functions on $\bar{H}_{\mathbb{A}}$ given by the integrals

$$\theta^{\psi}_{V,W,\phi}(\xi)(h) = \int_{G(F)\backslash G(\mathbb{A})} \overline{\xi(g)}\Theta(g, h; \phi)dg,$$

where $\xi \in V_\pi$ and $\phi \in S(\mathbb{X}(\mathbb{A}))$. Since the theta functions are known to be slowly increasing functions on $G(F)\backslash\bar{G}_{\mathbb{A}} \times H(F)\backslash\bar{H}_{\mathbb{A}}$ and since ξ is rapidly decreasing on $G(F)\backslash\bar{G}_{\mathbb{A}}$, each of these integrals converges absolutely, and defines an automorphic form on $H(F)\backslash\bar{H}_{\mathbb{A}}$. Let $\theta^{\psi}_{V,W}(\pi)$ be the space of functions $\theta^{\psi}_{V,W,\phi}(\xi)$ so generated on $H(F)\backslash\bar{H}_{\mathbb{A}}$. We have compatibility with local theta correspondence in the following sense: if $\theta^{\psi}_{V,W}(\pi)$ is nonvanishing and $\sigma = \bigotimes_v \sigma_v$ is an irreducible constituent of $\theta^{\psi}_{V,W}(\pi)$, then σ_v is an irreducible constituent of $\Theta^{\psi_v}_{V_v,W_v}(\pi_v)$ for each v (cf. Proposition 7.1.2 of [28]). In particular, for $\theta^{\psi}_{V,W}(\pi)$ to be nonvanishing, $\Theta^{\psi_v}_{V_v,W_v}(\pi_v)$ is necessarily nonvanishing for each v.

In Cases (I_0) or (I_1) we need to introduce a bit more notation. Let T be a finite subset of places of F. We impose the following conditions:

- $\#T$ is even in Case (I_0);
- T and $\mathbf{b} \cup \mathbf{d}$ have empty intersection in Case (I_1).

Recalling that sgn_v denotes the sign character of the orthogonal group G_v, we can define a character $\mathrm{sgn}_T : G(F)\backslash G(\mathbb{A}) \to \mu_2$ by $\mathrm{sgn}_T(g) = \prod_{v\in T} \mathrm{sgn}_v(g_v)$. For $\xi \in V_\pi$ and $\phi \in S(\mathbb{X}(\mathbb{A}))$ we set

$$\theta^{\psi}_{V^T,W,\phi}(\xi) = \theta^{\psi}_{V,W,\phi}(\mathrm{sgn}_T \cdot \xi), \qquad \theta^{\psi}_{V^T,W}(\pi) = \theta^{\psi}_{V,W}(\pi \otimes \mathrm{sgn}_T)$$

to make our exposition simpler. The symbol V^T should be viewed, formally, as a global ϵ-hermitian space with localizations $V_v^{\epsilon_v(T)}$, where $\epsilon_v(T)$ is $+$ or $-$ according as $v \notin T$ or $v \in T$.

For each integer j, whenever it exists, we denote by $V[j]$ the ϵ-hermitian space obtained from V by addition of j hyperbolic planes or by deletion of $-j$ hyperbolic planes according as $j \geq 0$ or $j < 0$.

Proposition 10.1 (Rallis [47]) *Fix an ϵ-hermitian space U of dimension m over D. Let π be an irreducible cuspidal automorphic representation of $\bar{G}_{\mathbb{A}}$ and j_0 the smallest value of j for which $\theta^{\psi}_{U[j],W}(\pi)$ is nonvanishing.*

(1) *$\theta^{\psi}_{U[j_0],W}(\pi)$ is in the space of cusp forms.*
(2) *If $j \geq j_0$, then $\theta^{\psi}_{U[j],W}(\pi)$ is nonvanishing.*
(3) *If $j \geq n$, then $\theta^{\psi}_{U[j],W}(\pi)$ is nonvanishing.*
(4) *If $m + j + j_0 > \rho_n$, then $\theta^{\psi}_{U[j],W}(\pi)$ lies in the space of square integrable automorphic forms.*

Proof Let I be a totally isotropic subspace of $U[j]$. Fix an isotropic complement I^* to $I^{\perp}$ in $U[j]$ and let I_0 be the orthogonal complement of $I \oplus I^*$ in $U[j]$. Let $P(I)$ be the maximal parabolic subgroup stabilizing I and $N(I)$ the unipotent radical of $P(I)$. Set

$$\mathbb{I} = I^* \otimes_D W, \qquad \mathbb{W}_0 = I_0 \otimes_D W, \qquad \mathbb{I}^* = I \otimes_D W.$$

Let $(\omega_{0,\psi}, S_0)$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}_0)_{\mathbb{A}}$ and $S(\mathbb{I}(\mathbb{A}); S_0)$ the space of S_0-valued Schwartz functions on $\mathbb{I}(\mathbb{A})$. We may realize $\omega_{\psi,U[j]}$ on $S(\mathbb{I}(\mathbb{A}); S_0)$. The constant term of $\theta^{\psi}_{U[j],W,\phi}(\xi)$ along $P(I)$ has a relatively simple formula: for $\phi \in S(\mathbb{I}(\mathbb{A}); S_0)$ and $\xi \in V_\pi$

$$\theta^{\psi}_{U[j],W,\phi}(\xi)_{N(I)} = \theta^{\psi}_{I_0,W,\phi(0)}(\xi).$$

This formula explains the first two parts. Rallis [47] has proven this formula in Cases (I_0) and (I_4). The argument can easily be modified to deal with the general case. Note that the local analogues were proven in [22, 37] in all cases by a parallel computation.

Rallis [47] proved (3) in Cases (I_0) and (I_4). The proof is valid word for word in the general case.

To prove (4), we may assume that $j > j_0$. Since $\omega_{\psi,U[j]}(a)\phi(0) = \gamma(a)|\nu(a)|^{\delta n/2}\phi(0)$ for $a \in GL(I, D)_{\mathbb{A}}$, where $\gamma(a)$ is some fourth root of unity, the real parts of exponents of $\theta^{\psi}_{U[j],W,\phi}(\xi)$ are of the form

$$d_0\left(\frac{n-m}{2} - j + \varepsilon, \frac{n-m}{2} - j + 1 + \varepsilon, \dots, \frac{n-m}{2} - j + k - 1 + \varepsilon\right)$$

for some k with $1 \leq k \leq j - j_0$. The square integrability criterion proves (4) (see Lemma I.4.11 of [38]). □

The following corollary readily follows from Proposition 10.1(4).

Corollary 10.1 *Let r be the Witt index of V. Let $\xi \in V_\pi$ and $\phi \in \omega_{\psi,V}$.*

(1) *If $m - r > \rho_n$, then $\theta^{\psi}_{V,W,\phi}(\xi)$ is square integrable.*
(2) *Put $\rho'_n = [\rho_n]$. Assume that $m - r \le \rho_n < m$. Let V_0 be an ϵ-hermitian space of dimension $2\rho'_n - m$ such that V and V_0 belong to the same Witt tower. If $\theta^{\psi}_{V_0,W}(\pi)$ is zero, then $\theta^{\psi}_{V,W,\phi}(\xi)$ is square integrable.*

10.2 Nonvanishing of theta liftings

If $\chi = \rho(\chi)^{-1}$, then since

$$L_\psi(s, \pi \times \chi) = L_{\psi^{-1}}\big(s, \pi^\vee \times \chi^{-1}\big),$$

the poles and zeros of $L_\psi(s, \pi \times \chi)$ must occur symmetrically across the line of symmetry $\Re s = 1/2$, via the functional equation involving an exponential factor of proportionality $\varepsilon(s, \pi \times \chi, \psi)$. In view of Lemma 7.1(2) and Remark 9.1 there is no harm in assuming $\chi = \chi_V$ to characterize those π for which these poles occur.

Let π be an irreducible cuspidal automorphic representation of $\bar{G}_{\mathbb{A}}$. In the first term range we ultimately derive the following theorems in Sect. 10.6.

Theorem 10.1 *We exclude the quaternion case. Assume either*

– *$L_{\psi^{-1}}(s, \pi \times \chi_V^{-1})$ has a pole, and m is the smallest integer such that $L_{\psi^{-1}}(s, \pi \times \chi_V^{-1})$ has a pole at $s = \frac{1}{2}(\delta(m - \rho_n) + 1)$; or*
– *$L_{\psi^{-1}}(s, \pi \times \chi_V^{-1})$ is entire, $m = \rho_n$ and $L_{\psi^{-1}}(1/2, \pi \times \chi_V^{-1}) \ne 0$,*

where $m = \dim V$. Then the following assertions hold.

(1) *$\theta^{\psi}_{V,W}(\pi)$ is nonvanishing if and only if $\Theta^{\psi_v}_{V_v,W_v}(\pi_v)$ is nonvanishing for each v.*
(2) *There is a unique equivalence class of ϵ-hermitian spaces U of dimension m with character χ_V such that $\theta^{\psi}_{U,W}(\pi)$ is nonzero.*
(3) *Let U' be an ϵ-hermitian space with character χ_V of dimension less than or equal to $2\rho_n - m$. Then $\theta^{\psi}_{U',W}(\pi)$ is zero unless U and U' lie in the same Witt tower and $\dim U' \ge m$.*

Next we discuss the quaternion case. We put $s_D = \#(\mathbf{b} \cup \mathbf{d})$ and $\ell = 2^{s_D - 2}$ in Case (I_3). We set $\ell = 1$ in Case (I_1). Two ϵ-hermitian spaces V and U over D are called locally isometric if V_v is isometric to U_v for every v. It is known that there are exactly ℓ global equivalence classes of ϵ-hermitian forms locally isometric to V, say $V^{(1)}, \dots, V^{(\ell)}$. Namely, the Hasse principle holds in Case (I_1), but fails in Case (I_3).

Theorem 10.2 *Assume that G is of type* (I_1) *or* (I_3). *Let V be an ϵ-hermitian space of dimension m. If $L(s,\pi\times\chi_V)$ has a pole and if m is the smallest integer such that $L(s,\pi\times\chi_V)$ has a pole at $s=m-\rho_n+\frac{1}{2}$, then the following assertions hold*:

(1) *$L(s,\pi\times\chi_V)$ is holomorphic at $s=j-\rho_n+\frac{1}{2}$ for all integers $j<\rho_n$ such that j and m have opposite parity.*
(2) *$\theta^{\psi}_{V^{(j)},W}(\pi)$ is nonvanishing for some j if and only if $\Theta^{\psi_v}_{V_v,W_v}(\pi_v)$ is nonvanishing for each v.*
(3) *There is a unique local equivalence class of ϵ-hermitian spaces U of dimension m with character χ_V such that $\theta^{\psi}_{U^{(j)},W}(\pi)$ is nonvanishing for some space $U^{(j)}$ locally isometric to U.*
(4) *If U' is an ϵ-hermitian space with character χ_V and if $\theta^{\psi}_{U',W}(\pi)$ is nonvanishing, then either $\dim U'>2\rho_n-m$ or U_v and U'_v lie in the same Witt tower for all v.*

Remark 10.1 In Case (I_3) we do not know how to determine which of the theta lifts $\theta^{\psi}_{U^{(j)},W}(\pi)$ are nonvanishing.

10.3 The generalized Siegel-Weil formula

Using the local splitting constructed by Kudla in [23], we obtain the lifting homomorphism $\tilde{\iota}^{\Box}_{V,W}:\bar{G}^{\Box}_{\mathbb{A}}\to\mathrm{Mp}(\mathbb{W}^{\Box})_{\mathbb{A}}$. We consider the Weil representation $\omega^{\Box}_{\psi,V}=\omega^{\Box}_{\psi}\circ\tilde{\iota}^{\Box}_{V,W}$ of $\bar{G}^{\Box}_{\mathbb{A}}$ realized on $S((V\otimes_D W^{\nabla})(\mathbb{A}))$. For each $\Phi\in\omega^{\Box}_{\psi,V}$ we define sections $f^{(s)}_{\Phi}$, $F^{(s)}_{\Phi}$ and $h^{(s)}_{\Phi}$ of $I(s,\chi_V)$ in analogy with Sect. 8. Lemma 8.2 says that $F^{(s)}_{\Phi}$ and $h^{(-s)}_{\Phi}$ are good at $s=s_m$.

If $m<\rho_n$, then since $b(s,\chi_V)$ has a simple pole at $s=s_m$ and since $E(h^{(s)}_{\Phi})$ has at most a simple pole at $s=-s_m$, we see that $E(f^{(s)}_{\Phi})$ is holomorphic at $s=s_m$ and

$$\mathrm{Res}_{s=-s_m}E\big(h^{(s)}_{\Phi}\big)=-\mathrm{Res}_{s=s_m}b(s,\chi_V)\cdot E\big(f^{(s)}_{\Phi}\big)\big|_{s=s_m} \tag{10.1}$$

by evaluating the functional equation at $s=s_m$, and then using (8.3).

Remark 10.2 The discussion above is not entirely accurate when V is a split binary quadratic space. The reader can consult [68] for a complete account of the Siegel-Weil formula in this case. Corollary 2 to Theorem I.2.1 of [47] says that theta lifts to the orthogonal group of a split binary quadratic space vanish. Therefore we may exclude the split binary quadratic case. However, the method applies to this case with some modifications, which eventually concludes that $L(s,\pi)$ is holomorphic at $s=n/2$ in Case (I^s_4).

Next we consider the theta function

$$\Theta(g,h;\Phi)=\sum_{x\in(V\otimes_D W^{\triangledown})(F)}\omega^{\square}_{\psi,V}(g)\Phi\bigl(h^{-1}x\bigr)\quad\bigl(g\in\bar{G}^{\square}_{\mathbb{A}},\ h\in\bar{H}_{\mathbb{A}}\bigr).$$

Fix a Haar measure dh giving $H(F)\backslash H(\mathbb{A})$ volume 1. The integral

$$\theta^{\psi}_{W^{\square},V,\Phi}(1)(g)=\int_{H(F)\backslash H(\mathbb{A})}\Theta(g,h;\Phi)dh$$

gives a $\bar{G}^{\square}_{\mathbb{A}}$-intertwining and $H(\mathbb{A})$-invariant map from $\omega^{\square}_{\psi,V}$ to $\mathcal{A}(\bar{G}^{\square}_{\mathbb{A}})$. This integral must be defined by regularization if it diverges. The procedure outlined in [28, 69], using a certain differential operator or an element of the Bernstein center to kill support, can be applied at least when $m\le\rho_n$. We omit details.

If $m>2\rho_n$, then for all $\Phi\in\omega^{\square}_{\psi,V}$

$$E\bigl(f^{(s_m)}_{\Phi}\bigr)=\theta^{\psi}_{W^{\square},V,\Phi}(1),$$

where both sides converge absolutely. This is a classic result of Siegel and Weil (see [66]). Kudla and Rallis extended this identity beyond the range of absolute convergence, culminating in a regularized Siegel-Weil formula. The following proposition is a simple reformulation of their result.

Proposition 10.2 ([28, 68–70]) *If $m\le\rho_n$ and V is not a split binary quadratic space, then for all $\Phi\in\omega^{\square}_{\psi,V}$,*

$$E\bigl(f^{(s)}_{\Phi}\bigr)\big|_{s=s_m}=\varkappa\theta^{\psi}_{W^{\square},V,\Phi}(1),\qquad (\mathrm{I}_0\text{–}\mathrm{I}_2,\ \mathrm{I}_4)$$

$$E\bigl(f^{(s)}_{\Phi}\bigr)\big|_{s=s_m}=\ell^{-1}\sum_{j=1}^{\ell}\theta^{\psi}_{W^{\square},V^{(j)},\Phi}(1),\qquad (\mathrm{I}_3)$$

where $\varkappa=1$ if G is of type (I_1) *or if $m=1$ in Case* (I_4), *and $\varkappa=2$ in the other cases.*

When $m<\rho_n$, the regularized Siegel-Weil formula proven by Kudla and Rallis in [28] is an identity between the integral $\theta^{\psi}_{W^{\square},V,\Phi}(1)$ and the residue $\mathrm{Res}_{s=-s_m}E(h^{(s)}_{\Phi})$. Proposition 10.2 follows upon combining this identity with (10.1). For a detailed proof we refer to [68–70].

If $m>\rho_n$ and the theta integral diverges, then the Eisenstein series can have a pole at $s=s_m$ and the problem of regularization and extending the Siegel-Weil formula is rather delicate. However, the author [69] found that these difficulties are absent in the quaternion case.

Proposition 10.3 ([69]) *If $m > \rho_n$ and G is of type* (I$_1$) *or* (I$_3$), *then $E(f_\Phi^{(s)})$ is holomorphic at $s = s_m$ and*

$$E\big(f_\Phi^{(s)}\big)\big|_{s=s_m} = \theta^{\psi}_{W^\square, V^{(j)}, \Phi}(1) \quad \big(\Phi \in \omega^\square_{\psi, V},\ j = 1, 2, \ldots, \ell\big).$$

W.T. Gan and Takeda [8] derive an extension of the Siegel-Weil formula in Case (I$_0$) for those Φ in the span of the spherical Schwartz function. We do not give a precise statement but remark that the inner product formula stated in Lemma 10.1 below is an application of this extension and holds without the restriction on Φ.

10.4 Residues of the Eisenstein series

Let $\mathcal{C} = \{C_v\}$ be a collection of local ϵ-hermitian spaces of dimension m with characters χ_{V_v} and such that C_v is isometric to V_v for almost all v. We form the restricted tensor product $\Theta^{\psi}_{W^\square, \mathcal{C}}(\mathbb{1}) = \bigotimes_v \Theta^{\psi_v}_{W_v^\square, C_v}(\mathbb{1})$, which we can regard as a subrepresentation of $I(s_m, \chi_V)$. When $\mathcal{C} = \{V_v\}$, we will write $\Theta^{\psi}_{W^\square, V}(\mathbb{1})$ instead of $\Theta^{\psi}_{W^\square, \mathcal{C}}(\mathbb{1})$.

The classification of ϵ-hermitian forms over a number field was done by Minkowski and Hasse in Case (I$_4$), by Landherr in Case (I$_2$), and by Bartels in Case (I$_3$) (cf. [51, 69]). We formulate the results in the form that suits our purposes.

Proposition 10.4 *Notation being as above, we have the following*:

(1) *Suppose that G is of type* (I$_0$), (I$_2$) *or* (I$_4$). *Then $\mathcal{C}$ is the set of localizations of a global space if and only if $\prod_v \eta(C_v) = 1$.*
(2) *Every $\mathcal{C}$ is the set of localizations of a global space in the quaternion case.*

From now on we suppose that $m \le \rho_n$, i.e., $s_m \le 0$. We define a submodule R^+_{m,χ_V} of $I(s_m, \chi_V)$ by

$$R^+_{m,\chi_V} = \bigoplus_{\dim U = m,\ \chi_U = \chi_V} \Theta^{\psi}_{W^\square, U}(\mathbb{1}),$$

where U extends over all equivalence classes of ϵ-hermitian spaces of dimension m with character χ_V. Next we define a submodule R^-_{m,χ_V} of $I(s_m, \chi_V)$ by

$$R^-_{m,\chi_V} = \bigoplus_{\mathcal{C} = \{C_v\}:\ \prod_v \eta(C_v) = -1} \Theta^{\psi}_{W^\square, \mathcal{C}}(\mathbb{1}), \qquad (\mathrm{I}_0, \mathrm{I}_2, \mathrm{I}_4)$$

$$R^-_{m,\chi_V} = 0. \qquad (\mathrm{I}_1, \mathrm{I}_3)$$

As a corollary to Proposition 9.1, we can define an intertwining map $A(-s_m,\chi_V)$ from the degenerate principal series representation $I(-s_m,\chi_V)$ to the space of automorphic forms on $\bar{G}^\square_{\mathbb{A}}$ by

$$f^{(s_m)} \mapsto \operatorname{Res}_{s=-s_m} E\big(f^{(s)}\big) \quad (m<\rho_n),$$
$$f^{(s_m)} \mapsto E\big(f^{(s)}\big)\big|_{s=0} \quad (m=\rho_n).$$

Put

$$M^*(s,\chi_V)=b(-s,\chi_V)^{-1}M(s,\chi_V).$$

Since $M^*(s,\chi_V)$ is equal to $a(s,\chi_V)^{-1}M(s,\chi_V)$ up to multiplication by invertible functions, Lemma 3.1 combined with (3.1) shows that $M^*(s,\chi_V)$ is holomorphic in $\Re s>-\frac{1}{2}$.

Proposition 10.5 ([8, 28, 69, 70]) *Suppose that $m\le\rho_n$.*

(1) *$M^*(-s_m,\chi_V)$ maps $I(-s_m,\chi_V)$ surjectively onto $R^+_{m,\chi_V}\oplus R^-_{m,\chi_V}$.*
(2) *$A(-s_m,\chi_V)$ factors through the quotient $I(-s_m,\chi_V)\to R^+_{m,\chi_V}$.*

Proof The first statement follows from Propositions 8.3 and 10.4. For proofs of the second statement, we refer to [28] for symplectic groups, to [8] for orthogonal groups, to [69] for quaternion unitary groups, and to [70] for unitary groups. □

10.5 The extended Rallis inner product formula

The complete polarization $\mathbb{W}^\square=\mathbb{X}^\square+\mathbb{Y}^\square$ gives rise to a second model of the Weil representation on $\mathcal{S}(\mathbb{X}^\square(\mathbb{A}))$. The two models are related by the partial Fourier transform $\sigma:\mathcal{S}(\mathbb{X}^\square(\mathbb{A}))\to\mathcal{S}((V\otimes_D W^\triangledown)(\mathbb{A}))$ defined in the local setting in Sect. 8. For $g_1,g_2\in\bar{G}_{\mathbb{A}}$, $h\in\bar{H}_{\mathbb{A}}$ and $\phi_1,\phi_2\in S(\mathbb{X}(\mathbb{A}))$ it is easy to see that

$$\sigma(\phi_1\otimes\overline{\phi_2})\big(h^{-1}w\big)=\sigma\big(\omega_{\psi,V}(h)\phi_1\otimes\overline{\omega_{\psi,V}(h)\phi_2}\big)(w)$$

for $w\in(V\otimes_D W^\triangledown)(\mathbb{A})$. From (8.2) we find that

$$\Theta\big((g_1,g_2),h;\sigma(\phi_1\otimes\overline{\phi_2})\big)=\chi_V\big(\nu(g_2)\big)\Theta(g_1,h;\phi_1)\overline{\Theta(g_2,h;\phi_2)}. \quad (10.2)$$

This is the reason for our choice of $(\tilde{\iota}_V,\tilde{\iota}_{V,-})$.

Lemma 10.1 *Let r be the Witt index of V. Assume either V is anisotropic; or $m-r>\rho_n$; or V is isotropic and $\theta^\psi_{V[-1],W}(\pi)=0$; or $m-r\le\rho_n<m$ and $\theta^\psi_{V_0,W}(\pi)=0$, where V_0 is defined in Corollary 10.1. Let $\xi_1,\xi_2\in V_\pi$ and $\phi_1,\phi_2\in\omega_{\psi,V}$.*

(1) $Z(\overline{\xi_1}\boxtimes\xi_2, f_\Phi^{(s)})$ *is holomorphic at* $s=s_m$ *for all* $\Phi\in\omega_{\psi,V}^\Box$.
(2) *Assume that* $m\le\rho_n$ *or* $m-r>\rho_n$ *in Cases* (I_2) *and* (I_4). *We exclude Case* (I_3) *when* $m<\rho_n$. *Put* $\Phi=\sigma(\phi_1\otimes\overline{\phi_2})$. *Then*

$$Z\big(\overline{\xi_1}\boxtimes\xi_2, f_\Phi^{(s)}\big)\big|_{s=s_m}=\varkappa\mathcal{P}\Big(\theta^\psi_{V,W,\phi_1}(\xi_1)\boxtimes\overline{\theta^\psi_{V,W,\phi_2}(\xi_2)}\Big).$$

Proof We first remark that $\theta^\psi_{V,W}(\pi)$ is square integrable on account of Proposition 10.1. W.T. Gan and Takeda established the Rallis inner product formula in full generality in Case (I_0). The formula just presented is a reformulation of Theorems 6.4, 6.6 and 6.7 of [8]. One can see that Theorem 6.4 of [8] holds under the weaker assumption $\theta_{2(m-r-1)}(\pi)=0$ by an examination of the proof of Proposition 6.3 of [8].

If $m\le\rho_n$ or if $m-r>\rho_n$ or if G is of type (I_1) or (I_3), then since $E(f_\Phi^{(s)})$ is holomorphic at $s=s_m$, so is $Z(\overline{\xi_1}\boxtimes\xi_2, f_\Phi^{(s)})$. Exclude the quaternion case and assume that $m-r\le\rho_n<m$. By (10.1) there exists $\Phi'\in\omega^\Box_{\psi,V_0}$ such that the residue of $E(f_\Phi^{(s)})$ at $s=s_m$ is equal to $E(f_{\Phi'}^{(s)})|_{s=-s_m}$. Since $\theta^\psi_{V_0,W}(\pi)=0$, Proposition 10.2 and the computation below show that $Z(\overline{\xi_1}\boxtimes\xi_2, f_{\Phi'}^{(s)})$ has a zero at $s=-s_m$, from which $Z(\overline{\xi_1}\boxtimes\xi_2, f_\Phi^{(s)})$ is holomorphic at $s=s_m$.

Applying the Siegel-Weil formula, we obtain the identity

$$\begin{aligned}Z\big(\overline{\xi_1}\boxtimes\xi_2, f_\Phi^{(s)}\big)\big|_{s=s_m}&=\int_{[G\times G]}\kappa_{\chi_V}(g)^{-1}(\overline{\xi_1}\boxtimes\xi_2)(g)E\big(f_\Phi^{(s)}\big)(g)\big|_{s=s_m}dg\\&=\varkappa\int_{[G\times G]}\kappa_{\chi_V}(g)^{-1}(\overline{\xi_1}\boxtimes\xi_2)(g)\theta^\psi_{W^\Box,V,\Phi}(1)(g)dg.\end{aligned}$$

We want to interpret the expression on the right hand side. By virtue of (10.2) the integral in the last line is equal to

$$\begin{aligned}&\int_{[G\times G]}\chi_V\big(\nu(g_2)\big)^{-1}\overline{\xi_1(g_1)}\xi_2(g_2)\\&\qquad\times\int_{[H]}\Theta\big((g_1,g_2),h;z_0\cdot\sigma(\phi_1\otimes\overline{\phi_2})\big)dg_1dg_2\\&=\int_{[H]}\bigg(\int_{[G\times G]}\overline{\xi_1(g_1)}\Theta(g_1,h;\phi_1)\xi_2(g_2)\overline{\Theta(g_2,h;\phi_2)}\bigg)*z_0dg_1dg_2dh\\&=\int_{[H]}\theta^\psi_{V,W,\phi_1}(\xi_1)(h)\overline{\theta^\psi_{V,W,\phi_2}(\xi_2)(h)}*z_0dh\\&=\int_{[H]}\theta^\psi_{V,W,\phi_1}(\xi_1)(h)\overline{\theta^\psi_{V,W,\phi_2}(\xi_2)(h)}dh.\end{aligned}$$

Here, the operator z_0 is to be interpreted as identity operator in the convergent case, and z_0 is used to regularize the theta integral in the divergent case. Since $\theta^{\psi}_{V,W,\phi_1}(\xi_1)\overline{\theta^{\psi}_{V,W,\phi_2}(\xi_2)}$ is integrable on $H(F)\backslash H(\mathbb{A})$, it has dropped out. □

10.6 Proof of Theorems 10.1 and 10.2

Lemma 10.2 *Assume that $\theta^{\psi}_{U',W}(\pi)$ is zero for every ϵ-hermitian space U' of dimension m' with character χ_V such that $m' < m$ and $m' \equiv m \pmod 2$. If $m < \rho_n$, then the following conditions are equivalent*:

- *$L_{\psi}(s, \pi \times \chi_V)$ is holomorphic at $s = s_m + \frac{1}{2}$;*
- *$\theta^{\psi}_{U,W}(\pi)$ is zero for every ϵ-hermitian space U of dimension m with character χ_V.*

When one replaces the first condition with $L_{\psi}(1/2, \pi \times \chi_V) = 0$, the equivalence still holds in the case $m = \rho_n$.

Proof Suppose that $m < \rho_n$. As explained in the proof of Theorem 9.1, the first condition is equivalent to that $Z(\overline{\xi_1} \boxtimes \xi_2, f^{(s)})$ is holomorphic at $s = -s_m$ for every $\xi_1, \xi_2 \in V_{\pi}$ and every holomorphic section $f^{(s)}$ of $I(s, \chi_V)$. It follows from (10.1) that

$$\mathrm{Res}_{s=-s_m} Z\big(\overline{\xi_1} \boxtimes \xi_2, h^{(s)}_{\Phi}\big) = -\mathrm{Res}_{s=s_m} b(s, \chi_V) \cdot Z\big(\overline{\xi_1} \boxtimes \xi_2, f^{(s)}_{\Phi}\big)\big|_{s=s_m}$$

for $\Phi \in \omega^{\square}_{\psi,U}$. Proposition 10.5 implies that the residue of $Z(\overline{\xi_1} \boxtimes \xi_2, f^{(s)})$ at $s = -s_m$ is a finite linear combination of the quantities $Z(\overline{\xi_1} \boxtimes \xi_2, f^{(s)}_{\Phi})|_{s=s_m}$. Lemma 10.1 says that every such quantity can be obtained as the Petersson inner product of theta lifts of π to the isometry group of some U. We here utilize (7.1) in Cases (I_0) and (I_1). Thus the first condition is equivalent to that all these theta lifts are zero. In the case $m = \rho_n$, nonvanishing of the theta liftings can be related to nonvanishing of the central value of the L-function by the same reasoning. □

Corollary 10.2 (Cf. [14, 31]) *Assume that $\bar{G}$ is of type* (I^o_0), (I_2) *or* (I^m_4). *In Case* (I_2) *let χ be a character of C_E such that $\chi^0 = \epsilon^{\delta n}_{E/F}$. Let χ be the trivial character in the other cases. If π is an irreducible cuspidal automorphic representation of $\bar{G}_{\mathbb{A}}$ such that π_v is tempered for every v, then $L_{\psi}(s, \pi \times \chi)$ is entire and $L^S_{\psi}(1/2, \pi \times \chi) \geq 0$, where S is a large enough set of places of F containing all the archimedean places.*

Proof Since Lemma 7.2 shows that $L_{\psi}(s, \pi \times \chi)$ is holomorphic in $\Re s > 1$, it is entire by Theorem 9.1. The nonnegativity is proven in the same way as in [14] where the unitary case is discussed. □

For cuspidal generic representations of odd orthogonal groups this result has been proven by Lapid and Rallis [31] whose proof is entirely different. Now we finish the proof of Theorem 10.1.

Proof of Theorem 10.1 By Lemma 10.2 there is an ϵ-hermitian space U of dimension m with character χ_V such that $\theta^{\psi}_{U,W}(\pi)$ is nonzero. Since $m \leq \rho_n$ and since $\Theta^{\psi_v}_{U_v,W_v}(\pi_v)$ is nonvanishing for each v, the equivalence class of U is uniquely determined owing to Corollary 8.1. To prove the last assertion, we suppose that $\theta^{\psi}_{U',W}(\pi)$ is nonvanishing. Lemma 10.2 shows that $\dim U' \geq m$. If $m + \dim U' \leq 2\rho_n$, then Lemma 8.5 forces U_v and U'_v to lie in the same Witt tower for each v, so that U and U' must lie in the same Witt tower. $\square$

We explain what kind of modifications are necessary in the quaternion case. For $\xi_1, \xi_2 \in V_\pi$ and $\phi_j, \phi'_j \in \omega_{\psi,V^{(j)}}$ we can show by the same calculation that if

$$\Phi = \sigma\big(\phi_1 \otimes \overline{\phi'_1}\big) = \cdots = \sigma\big(\phi_\ell \otimes \overline{\phi'_\ell}\big),$$

then

$$Z\big(\overline{\xi_1} \boxtimes \xi_2, f^{(s)}_\Phi\big)\big|_{s=s_m} = \ell^{-1} \sum_{j=1}^{\ell} \mathcal{P}\Big(\theta^{\psi}_{V^{(j)},W,\phi_j}(\xi_1) \boxtimes \overline{\theta^{\psi}_{V^{(j)},W,\phi'_j}(\xi_2)}\Big).$$

Appealing to this formula, we can prove Theorem 10.2 by arguing exactly as in the proof of Theorem 10.1. We will focus on the first assertion. Suppose that $L(s, \pi \times \chi_V)$ has a pole at $s = j - \rho_n + \frac{1}{2}$ for $j < \rho_n$. Lemma 10.2 gives an ϵ-hermitian spaces U' of dimension j with character χ_V such that $\theta^{\psi}_{U',W}(\pi)$ is nonzero. Note that since $m < \rho_n$, we have $j < 2\rho_n - m$. Lemma 8.5 forces U_v and U'_v to lie in the same Witt tower for all v. In particular, $j - m$ must be even.

Remark 10.3 Notation being as above, Proposition 10.3 shows that if $m > \rho_n$, then the inner products

$$\mathcal{P}\Big(\theta^{\psi}_{V^{(1)},W,\phi_1}(\xi_1) \boxtimes \overline{\theta^{\psi}_{V^{(1)},W,\phi'_1}(\xi_2)}\Big), \dots, \mathcal{P}\Big(\theta^{\psi}_{V^{(\ell)},W,\phi_\ell}(\xi_1) \boxtimes \overline{\theta^{\psi}_{V^{(\ell)},W,\phi'_\ell}(\xi_2)}\Big)$$

are equal. One can prove a stronger result. Let U and U' be two skew hermitian spaces of dimension m with character χ_V. Let $\phi_1, \phi_2 \in \omega_{\psi,U}$ and

$\phi_1', \phi_2' \in \omega_{\psi,U'}$. If $f^{(s_m)}_{\sigma(\phi_1\otimes\overline{\phi_2})} = f^{(s_m)}_{\sigma(\phi_1'\otimes\overline{\phi_2'})}$, then

$$\mathcal{P}\Big(\theta^{\psi}_{U,W,\phi_1}(\xi_1)\boxtimes\overline{\theta^{\psi}_{U,W,\phi_2}(\xi_2)}\Big)=\mathcal{P}\Big(\theta^{\psi}_{U',W,\phi_1'}(\xi_1)\boxtimes\overline{\theta^{\psi}_{U',W,\phi_2'}(\xi_2)}\Big).$$

This is a kind of the matching phenomenon discussed by Kudla in [24].

10.7 End of the proof

We will finish the proof of Theorem 2 presented in the introduction, indicating the additional facts which are needed when $j > \frac{n-1}{2}$. Notice that Theorem 10.1 and Lemma 10.2 include Theorem 2(1).

The restriction of Theorems 10.1 and 10.2 to the case $m \le \rho_n$ arises from (i) the use of an extension of the Rallis inner product formula and (ii) the need to control the nonvanishing of the normalized local zeta integrals. In the case $m > \rho_n$ the inner product formula stated in Lemma 10.1 can apply to Cases (I_0), (I_1) and (I_3). As for (ii) Lemma 8.6 gives sufficient control of the local integral except in the real cases listed in Remark 8.4(1).

Theorem 10.3 *Assume that G is of type* (I_0), (I_1) *or* (I_3). *In Case* (I_1) *we assume that* $\mathbf{b}$ *is empty. In Case* (I_3) *we assume that there is no real archimedean place at which D splits. Assume that $\rho_n < m$. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. When $m - r \le \rho_n$, we suppose that $\theta^{\psi}_{V_0,W}(\pi)$ is zero, where V_0 is as in Corollary 10.1. Then $\theta^{\psi}_{V,W}(\pi)$ is nonvanishing if and only if the following conditions hold*:

- *$L(s, \pi \times \chi_V)$ is holomorphic and does not have a zero at $s = \frac{1}{2}(\delta(m - \rho_n) + 1)$;*
- *$\Theta^{\psi_v}_{V_v,W_v}(\pi_v)$ is nonvanishing for each v.*

The integral $Z(\xi, f^{(s)}_{\Phi})$ is holomorphic at $s = s_m$ for all $\xi \in \overline{V_\pi} \boxtimes V_\pi$ and $\Phi \in \omega^{\square}_{\psi,V}$ by Lemma 10.1(1). By Lemma 8.6, if $\Theta^{\psi_v}_{V_v,W_v}(\pi_v)$ is nonvanishing, then there is a choice of ξ_v and Φ_v such that $\mathcal{Z}_v(\xi_v, f^{(s)}_{\Phi_v})$ is nonzero at $s = s_m$. These combined with (9.1) show that if $\theta^{\psi}_{V,W}(\pi)$ is nonvanishing, then $L(s, \pi^\vee \times \chi_V)$ must be holomorphic at $s = s_m + \frac{1}{2} = \frac{1}{2}(\delta(m-\rho_n)+1)$. Note that $L(s, \pi^\vee \times \chi_V) = L(s, \pi \times \chi_V)$ by Proposition 5.4. If $\overline{\xi_1} \boxtimes \xi_2 = \bigotimes_v \xi_v$ and $\sigma(\phi_1 \otimes \overline{\phi_2}) = \bigotimes_v \Phi_v$, then Lemma 10.1(2) together with (9.1) gives

$$\mathcal{P}\Big(\theta^{\psi}_{V,W,\phi_1}(\xi_1)\boxtimes\overline{\theta^{\psi}_{V,W,\phi_2}(\xi_2)}\Big)=\frac{L(s_m+\frac{1}{2},\pi\times\chi_V)}{b(s_m,\chi_V)}\prod_{v\in S}\mathcal{Z}_v\big(\xi_v, F^{(s_m)}_{\Phi_v}\big).$$

This combined with Lemma 8.6 completes the proof.

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Appendix A: The archimedean theory

In this appendix we will let F be an archimedean local field except in the statement of Lemma A.2. Since the sections constructed in the proofs of Lemma 6.2 and Theorem 7.1 may not be $\bar{K}^\square$-finite, a complication arises in the archimedean case. To get around this problem, we redefine the L-factor by using general smooth sections rather than $\bar{K}^\square$-finite sections. To show that the two definitions agree, we shall show that the ratio of the local integral divided by this L-factor is continuous in the appropriate topology. Kudla and Rallis [26] have carried this out for symplectic and orthogonal groups. The subtle convergence arguments can be simplified considerably by applying the inductive argument developed in Sect. 5 to smooth sections.

A character χ of $E^\times$ determines a holomorphic family of characters $\chi_{\psi,s}\circ\Delta$ of $\bar{P}$ and $I^\infty(s,\chi)$ stands for the smooth degenerate principal series representation. Since $P\backslash G^\square = M\cap K^\square\backslash K^\square$, all representations $I^\infty(s,\chi)$ can be realized on the same space $I^\infty(\chi)$ of smooth functions f on $\bar{K}^\square$ which satisfy $f(pg)=\chi'(\Delta(p))f(g)$ for $g\in\bar{K}^\square$ and $p\in\bar{M}\cap\bar{K}^\square$. Here $\chi'=\chi_\psi$ in Case (I_4^m) and $\chi'=\chi$ in the other cases. Note that $I^\infty(\chi)$ is a Fréchet space with respect to the seminorms $\|Xf\|$, where $\|f\|=\max_{k\in\bar{K}^\square}|f(k)|$ and X ranges over the universal enveloping algebra $U(\mathfrak{k}^\square)$ of the complexified Lie algebra $\mathfrak{k}^\square$ of $K^\square$. In this way we can view $I^\infty(s,\chi)$ as a holomorphic family of representations.

A holomorphic section $f^{(s)}$ of $I^\infty(s,\chi)$ is a holomorphic function on $\mathbb{C}$ with values in $I^\infty(\chi)$. For each $g\in\bar{G}^\square$ we define $f^{(s)}(g)$ by writing $g=pk$ with $p\in\bar{P}$ and $k\in\bar{K}^\square$, and taking

$$f^{(s)}(g)=\chi_{\psi,s+\delta\rho_n/2}\big(\Delta(p)\big)f^{(s)}(k).$$

As in the $\bar{K}^\square$-finite setting, we have the concept of good sections in the smooth setting. The obvious adaptation of the results stated in Sects. 3.5 and 5.2 still holds for good sections of $I^\infty(s,\chi)$. However, it should be remarked that the proof of Lemma 6.2 is more involved in the archimedean case. Theorem 4.1 of [3] states that principal series representations admit a Bruhat filtration, and it is possible to extend this work to $I^\infty(s,\chi)$, which allows us to argue in the archimedean case in the same way as in the p-adic case.

Lemma A.1 *For any $\pi \in \mathrm{Irr}(\bar{G})$ and any character χ of $E^\times$ there exists a local Euler factor $L(s)$ such that for any $\xi \in \pi^\vee \boxtimes \pi$ and any good section $f^{(s)}$ of $I^\infty(s,\chi)$ the quotient $Z(\xi, f^{(s)})/L(s+1/2)$ is entire.*

Proof We can use Propositions 4.2 and 5.3 to reduce the statement to the case where φ is anisotropic or the case where $n=1$ and $\varphi=0$. The former case is evident. In the latter case $G=\mathrm{GL}_1(C)$, $G^\square=\mathrm{GL}_2(C)$ and the zeta integral is easy to compute (cf. [32]). The embedding of $G\times G$ into $G^\square$ is given by $(g_1,g_2)\mapsto w_1\left(\begin{smallmatrix} g_1 & 0\\ 0 & g_2\end{smallmatrix}\right)w_1^{-1}$, where $w_1=\left(\begin{smallmatrix} 2^{-1} & -2^{-1}\\ 1 & 1\end{smallmatrix}\right)$. Then P is the group of matrices whose lower left $n\times n$ blocks are zero. Put $C^1=\{z\in C^\times \mid |z|=1\}$. Then $C=\mathbb{R}_+^\times\cdot C^1$. By twisting π by an unramified character we may assume that χ_π is trivial on $\mathbb{R}_+^\times$. Let α_i be complex numbers such that $\chi_i(a)=a^{\alpha_i}$ for $a\in\mathbb{R}_+^\times$ and $i=1,2$. The zeta integral is

$$\begin{aligned}\int_{C^\times} H_\xi(g) f^{(s)}\left(w_1\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\right)dg\\ =\int_{C^1} H_\xi(c)\int_{\mathbb{R}_+^\times} f^{(s)}\left(w_1\begin{pmatrix} ac & 0\\ 0 & 1\end{pmatrix}\right)d^\times a\,dc.\end{aligned}$$

The Iwasawa decomposition for $w_1\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{R})$ is

$$w_1\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix} a(a^2+1)^{-1/2} & *\\ 0 & (a^2+1)^{1/2}\end{pmatrix}\kappa_\theta,$$

where

$$\kappa_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\qquad \tan\theta=a\quad (0<\theta<\pi/2).$$

If we write the translate of $f^{(s)}$ by $\left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right)$ for $f_c^{(s)}$, then the inner integral is

$$\begin{aligned}\int_{\mathbb{R}_+^\times}\left(\frac{a}{a^2+1}\right)^{s+\delta/2} a^{\alpha_1}\left(a^2+1\right)^{-(\alpha_1+\alpha_2)/2} f_c^{(s)}(\kappa_\theta)d^\times a\\ =\int_0^{\pi/2}(\sin\theta)^{s+\alpha_1-1+\delta/2}(\cos\theta)^{s+\alpha_2-1+\delta/2} f_c^{(s)}(\kappa_\theta)d\theta.\end{aligned}$$

Put $s_i = s + \alpha_i - 1 + \delta/2$. This integral converges, provided that $\Re s > -\min\{\Re\alpha_1, \Re\alpha_2\} - \delta/2$. We split the domain of integration into two parts. Integration by parts gives us

$$\int_0^{\pi/4} (\sin\theta)^{s_1}(\cos\theta)^{s_2} f_c^{(s)}(\kappa_\theta)d\theta$$
$$= \frac{1}{s_1+1}\left(2^{-(s_1+s_2)/2} f_c^{(s)}(\kappa_{\pi/4}) - \int_0^{\pi/4}(\sin\theta)^{s_1+1}h(s,\theta,c)d\theta\right),$$

where $h(s,\theta,c) = \frac{\partial}{\partial\theta}\{(\cos\theta)^{s_2-1} f_c^{(s)}(\kappa_\theta)\}$. Likewise for the integral over $\pi/4 \le \theta \le \pi/2$. This allows us to extends $Z(\xi, f^{(s)})$ meromorphically to the region $\Re s > -\min\{\Re\alpha_1, \Re\alpha_2\} - \delta/2 - 1$. We can again integrate by parts and eventually we see that it continues to a meromorphic function in s on the whole plane. Moreover, the poles of all the local integrals are contained in the poles of $\Gamma(s+\alpha_1+\delta/2)\Gamma(s+\alpha_2+\delta/2)$ with multiplicity. □

For $\phi \in C_c^\infty(\bar{G})$ we can define a section $f_\phi^{(s)}$ of $I^\infty(s,\chi)$ by requiring that $\mathrm{supp}(f_\phi^{(s)}) \subset \bar{P}\cdot(\bar{G}\times\bar{G})$ and $f_\phi^{(s)}((g,e)) = \phi(g)$ for $g \in \bar{G}$. The following lemma can be proven by choosing ϕ to be supported in a small neighborhood.

Lemma A.2 ([26, 49]) *For given $\pi \in \mathrm{Irr}(\bar{G})$ there is a choice of $\xi \in \pi^\vee \boxtimes \pi$ and $\phi \in C_c^\infty(\bar{G})$ such that $Z(\xi, f_\phi^{(s)}) = 1$.*

Now we choose a meromorphic function $L(s)$ in such a way that it not only cancels all poles of the local integrals for good sections of $I^\infty(s,\chi)$, but also dividing by it introduces no extraneous zeros. From Lemma A.2 $L(s)$ vanishes nowhere. The following lemma concludes that $L(s)$ is determined by the poles of the family of local integrals even using good sections of $I(s,\chi)$.

Lemma A.3 *$L(s,\pi^\vee\times\chi)/L(s)$ is an invertible function.*

Proof Fix $s' \in \mathbb{C}$. Our task is to show that $Z(\xi, f^{(s)})/L(s+1/2)$ does not have a zero at $s = s'$ for a suitable choice of $\xi \in \pi^\vee \boxtimes \pi$ and a good section $f^{(s)}$ of $I(s,\chi)$. For each $h \in I^\infty(\chi)$ and $s \in \mathbb{C}$ we define a function $h_s : \bar{G}^\square \to \mathbb{C}$ by

$$h_s(pk) = \chi_{\psi,s+\delta\rho_n/2}\big(\Delta(p)\big)h(k) \quad \big(p\in\bar{P},\ k\in\bar{K}^\square\big).$$

By the argument of [65], for any $X \in U(\mathfrak{g}^\square)$ and for s in a compact set, there is a seminorm μ such that $\|Xh_s\| \le \mu(h)$ for all $h \in I^\infty(\chi)$.

As we observed in the course of the proof of Lemma A.1, if $n = 1$ and $\varphi = 0$, then for $\xi \in \pi^\vee\boxtimes\pi$ and for s in a compact set, there exist a polynomial

$Q(s)$ and a seminorm μ such that for all $h \in I^\infty(\chi)$

$$\left|Q(s)Z(\xi, h_s)\right| \le \mu(h). \tag{A.1}$$

Lapid and Rallis [32] proved the functional equation of Theorem 4.1(3) for smooth but not necessarily $\bar{K}^\square$-finite sections. The functional equation reads as in Theorem 5.2 but with $L_\psi^{\mathcal{U}}(s, \pi^\vee \times \chi)$ replaced by $L(s)$. Since the constant of proportionality is an exponential factor in view of the definition of $L(s)$, we may suppose that $\Re s' > -\frac{1}{2}$. In this region the intertwining operator $\Psi(s, \chi)$ appearing in Proposition 4.2 is absolutely convergent by Lemma 5.1, and [65, Lemma 10.1.11] states that $\Psi(s, \chi)$ is a bounded operator with respect to $\|\cdot\|$, independently of s. We thus obtain an estimate similar to (A.1) in general. Let l be the order of the pole of $L(s + 1/2)$ at $s = s'$. Using the Cauchy's integral formula, we can find a seminorm μ such that for all $h \in I^\infty(\chi)$

$$\left|\lim_{s \to s'} (s - s')^l Z(\xi, h_s)\right| \le \mu(h).$$

Take a holomorphic section $h^{(s)}$ of $I^\infty(s, \chi)$ so that the limit $\lim_{s \to s'}(s - s')^l Z(\xi, h^{(s)}) \ne 0$. Since this limit depends only on $h^{(s')}$, we may assume that $h^{(s)}$ is of the form h_s for some $h \in I^\infty(\chi)$. Lemma A.3 now follows from the fact $\bar{K}^\square$-finite vectors are dense in $I^\infty(\chi)$. $\square$

We conclude this appendix by indicating how we must modify the arguments in the proofs of Theorems 6.1 and 7.1 in the archimedean case. Our explanation focuses on Theorem 7.1 as the proof of Theorem 6.1 can be modified in an easier manner.

Take $\eta_1 \in \varrho_1^\vee \boxtimes \varrho_1$ and $\phi_1 \in C_c^\infty(N_-^Y)$ in such a way that the quotient $Z(\eta_1, f_{\phi_1}^{(s)})/L^{GJ}(s + \frac{1}{2}, \varrho_1 \otimes \rho(\chi))$ does not have $s = s'$. As observed in the proof, one can always find $\eta \in \varrho^\vee \boxtimes \varrho$ and a holomorphic section $f^{(s)}$ of $I^\infty(s, \chi)$ which satisfy

$$A\bigl(\eta, f^{(s)}\bigr) = Z\bigl(\eta_1, f_{\phi_1}^{(s)}\bigr).$$

However, $f^{(s)}$ may not be $\bar{K} \times \bar{K}$ right finite, and hence we cannot directly apply Lemma 7.6(2). One can check that $A(\eta, h_s)$ can be meromorphically continued to the region $\Re s > -\frac{1}{2}$ and that the map

$$A_\eta : h \mapsto \lim_{s \to s'} A(\eta, h_s)/L^{GJ}\left(s + \frac{1}{2}, \varrho_1 \otimes \rho(\chi)\right)$$

is continuous on $I^\infty(\chi)$ by a careful analysis based on the technique of [26]. See the explanation of this point in Sect. 6 of [49]. Choose a $\bar{K}^\square$-finite element h in $I^{\infty}(\chi)$ such that $A_\eta(h) \neq 0$. Choosing $\xi \in \pi^\vee \boxtimes \pi$ so that $Z(\xi, h_s) = A(\eta, h_s)$, we have completed our proof.

Appendix B: Local factors for classical groups

We recall basic properties of the gamma factors which we need. We refer the reader to [32] for a complete description of the gamma factors and their characterization (cf. [4], where the metaplectic case is discussed).

B.1 Normalization of the intertwining operator

We begin by recalling the normalization of the intertwining operator. Let F be a local field of characteristic zero. We adopt the notation of Sect. 2. We remind the reader that the algebra D is a direct sum of mutually opposite division algebras in Case (II). We denote by N^- the unipotent radical of the maximal parabolic subgroup of $G^\square$ stabilizing W^∇. Let $\mathfrak{g}^\square$, $\mathfrak{n}$, $\mathfrak{n}^-$ be the Lie algebras of $G^\square$, N, N^-, respectively. Note that

$$\begin{aligned} \mathfrak{g}^\square &= \left\{X \in \mathrm{End}\left(W^\square, D\right) \mid \langle xX, y\rangle^\square + \langle x, yX\rangle^\square = 0 \text{ for } x, y \in W^\square\right\}, \\ \mathfrak{n} &= \left\{X \in \mathfrak{g}^\square \mid \mathrm{Ker} X \supset W^\Delta \supset \mathrm{Im} X\right\}. \end{aligned}$$

We can identify $\mathfrak{n}$ with the space of ϵ-hermitian forms on $W^\square / W^\Delta$ via $(x, y) \mapsto \langle xX, y\rangle^\square$. We sometimes identify the group N^- with its Lie algebra $\mathfrak{n}^-$ via the isomorphism $u \mapsto u - I$. For $A \in \mathfrak{n}$ we define a unitary character ψ_A of N^- by $\psi_A(X) = \psi(\tau(XA))$. Recall that $\tau : D \to E$ is the reduced trace. Observe that $\tau(XA) \in F$ in all cases. The map $A \mapsto \psi_A$ defines an isomorphism of $\mathfrak{n}$ onto the Pontryagin dual of N^-.

For simplicity we exclude the odd orthogonal case and the metaplectic case. The reader can consult [4, 5, 32] for a complete account of modifications required in these cases. We identify W with W^Δ via $x \mapsto (x, x)$, and $W^\square / W^\Delta$ with W via $(x, y) \mapsto x - y$. With these identifications the reduced norm $\nu_W(A) \in E$ is well-defined. Fix $A \in \mathfrak{n}$. Denote by $\mathbb{C}_{\psi_A}$ the one dimensional representation of N^- with action given by ψ_A. For a section $f^{(s)}$ of $I(s, \chi)$ the integral

$$l_{\psi_A}\left(f^{(s)}\right) = \int_{N^-} f^{(s)}(u)\psi_A(u)du$$

converges absolutely for $\Re s \gg 0$ and defines an N^- equivariant map from $I(s, \chi)$ to $\mathbb{C}_{\psi_A^{-1}}$. When $\nu_W(A) \in E^\times$ and F is a p-adic field, Karel [20] has proven that $l_{\psi_A}(f^{(s)})$ admits an entire analytic continuation to the whole s-plane and satisfies a functional equation

$$l_{\psi_A} \circ M(s, \chi) = \chi_s\big(\nu_W(A)\big)^{-1} c(s, \chi, A, \psi) l_{\psi_A}$$

for some meromorphic function $c(s, \chi, A, \psi)$. Analogous results are proven in the archimedean case in [64]. As is proven in [32, Lemma 10], the factor $c(s, \chi, A, \psi)$ depends only on the homothety class of A, and for $\lambda \in F^\times$

$$c(s, \chi, A, \psi_\lambda) = \chi_s(\lambda)^{-\delta n} c(s, \chi, A, \psi). \tag{B.1}$$

Let us put

$$C(s, \chi, A, \psi) = c(s, \chi, A, \psi), \qquad (\mathrm{I}_0^e, \mathrm{I}_1, \mathrm{I}_2)$$

$$C(s, \chi, A, \psi) = c(s, \chi, A, \psi)\gamma(s + 1/2, \chi \cdot \chi_A, \psi)^{-1}, \qquad (\mathrm{I}_3, \mathrm{I}_4^s)$$

$$C(s, \chi, A, \psi) = \chi_2(-1)^{\delta n} c(s, \chi, A, \psi). \qquad (\mathrm{II})$$

Lemma B.1 *Let $C(s, \chi, A, \psi)$ be defined as above. Then it is equal to $a(s, \chi)/b(-s, \rho(\chi)^{-1})$ up to multiplication by invertible functions.*

Proof We can find a number field $\mathbb{F}$, a quadratic or trivial extension $\mathbb{E}$ of $\mathbb{F}$, a division algebra $\mathbb{D}$ over $\mathbb{E}$, a place v_0 of $\mathbb{F}$, a nontrivial additive character ψ' of $\mathbb{A}_\mathbb{F}/\mathbb{F}$, a character χ' of $C_\mathbb{F}$ and an ϵ-hermitian matrix $A' \in \mathfrak{n}(\mathbb{F})$ so that $\mathbb{F}_{v_0} = F$, $\mathbb{E}_{v_0} = E$, $\mathbb{D}_{v_0} = D$, $\psi'_{v_0} = \psi$, $\chi'_{v_0} = \chi$ and A' is equivalent to A over $\mathbb{F}_{v_0}$. Let S be a finite set of places of $\mathbb{F}$, containing v_0, outside of which $\mathbb{E}$, $\mathbb{D}$, ψ', χ' and A are all unramified. It follows from (3.1) and [32, (24)] that

$$\frac{a_S(s, \chi')}{b_S(s, \chi')} \cdot \frac{\beta_S(-s, \rho(\chi')^{-1})}{b_S(-s, \rho(\chi')^{-1})} \cdot \prod_{v \in S} c_v\big(s, \chi'_v, A', \psi'_v\big) = \frac{\beta_S(s, \chi')}{b_S(s, \chi')},$$

where $\beta_v(s, \chi'_v) = L(s + 1/2, \chi'_v)$ in Cases (I_3) and (I_4^s), and $\beta_v(s, \chi'_v) = 1$ in all other cases. We can rewrite this as

$$\prod_{v \in S} C_v\big(s, \chi'_v, A', \psi'_v\big) = b_S\big(-s, \rho\big(\chi'\big)^{-1}\big)/a_S\big(s, \chi'\big). \tag{B.2}$$

This equality translates informally to "$\prod_v C_v(s, \chi'_v, A', \psi'_v) = 1$".

We first consider Case (I_4). When F is a p-adic field, the factor $c(s, \chi, A, \psi)$ is calculated explicitly by Sweet in [56, 57] (the formula in [56] can be extended to arbitrary characters of $F^\times$, as was noted in [57]). Applying (B.2) with $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{Q}(\sqrt{-1})$ and $v_0 = \infty$, we see that his formula remains true for $F = \mathbb{R}$ or $F = \mathbb{C}$. Lemma B.1 is a simple consequence of this formula.

In the p-adic case an explicit formula for the factor $c(s,\chi,A,\psi)$ is obtained in [30] in Case (I_2), and in [67] in Cases (I_0) and (I_1). By the same reasoning, we can check that these formulas are still true in the archimedean case, from which Lemma B.1 follows in Cases (I_0)–(I_2).

Finally, we consider Case (I_3). We can find a number field $\mathbb{F}$ which has two places v_0 and v_1 such that $\mathbb{F}_{v_1}=\mathbb{F}_{v_2}=F$. Let $\mathbb{D}$ be a quaternion algebra with center $\mathbb{F}$ ramified precisely at v_0 and v_1. Le us choose ψ', χ' and A' so that $\psi'_{v_i}=\psi$, $\chi'_{v_i}=\chi$ and A' is equivalent to A over $\mathbb{F}_{v_i}$ for $i=1,2$. Proposition 10.4(2) allows us to assume that $\prod_{v\neq v_1,v_2}\eta_v(A'_v)=1$. Fix a quadratic form T of dimension $2n$ over F satisfying $\chi_T=\chi_A$. Proposition 10.4(1) gives a global quadratic form which is equivalent to T over $\mathbb{F}_{v_i}$ for $i=1,2$ and whose localizations are A'_v outside the two places. We deduce from (B.2) that

$$c(s,\chi,A,\psi)^2=c(s,\chi,T,\psi)^2.$$

This finishes the list of remaining cases needed to be considered. □

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1 The proof mimics the argument of the proof of Theorem on p.106 of [45]. An important ingredient of the proof of [45] is the exact location of the poles of $c(s,\chi,A,\psi)$. This is carried out in Lemma B.1, and we can proceed in exactly the same way. □

Definition B.1 The normalization of $M(s,\chi)$ is defined by setting

$$M^\dagger_{\mathcal{W}}(s,\chi,A,\psi)=C(s,\chi,A,\psi)^{-1}M(s,\chi).$$

We can easily see that

$$M^\dagger_{\mathcal{W}}\big(-s,\rho(\chi)^{-1},A,\psi^{-1}\big)\circ M^\dagger_{\mathcal{W}}(s,\chi,A,\psi)=\mathrm{Id}. \tag{B.3}$$

B.2 Local theory via gamma factors

In the ensuing discussion we assume A to be maximally split, i.e., the form $(x,y)\mapsto\langle xA,y\rangle^\square$ on $W^\square/W^\Delta$ admits a totally isotropic subspace of dimension $[n/2]$. For $\pi\in\mathrm{Irr}(\bar{G})$ we set

$$\gamma^{\mathcal{W}}(s,\pi\times\chi,\psi)=z(\pi)\varepsilon_{W,\psi}^{-1}\Gamma^{\mathcal{W}}\left(s-\frac{1}{2},\pi^\vee\times\chi,A,\psi\right).$$

Recall that the factor $\Gamma^{\mathcal{W}}(s,\pi\times\chi,A,\psi)$ is defined as the proportionality constant of the functional equation in Theorem 4.1(3). See [4, 5, 32] for the definition and description of the gamma factor in the odd orthogonal and metaplectic cases. The gamma factor is independent of the choice of A except in the case G is of type (I_3) and n is odd, in which case it still depends on the choice of maximally split A, but we will suppress the dependence on A from the notation.

The calculation of the gamma factor is done in the archimedean case and minimal cases in [4, 32] except in the quaternion case. These cases are yet to be done. The gamma factor may not be quite correct in these cases, but in this paper we require only that it satisfies the formal properties described in following proposition.

Proposition B.1 (Cf. [4, 32]) *The factors $\gamma^{\mathcal{W}}(s, \pi \times \chi, \psi)$ satisfy the following properties.*

(1) *If π is a subquotient of $\mathrm{Ind}_{P(Y)}^{G} \sigma$, then*

$$\gamma^{\mathcal{W}}(s, \pi \times \chi, \psi) = \gamma^{\mathcal{Y}}(s, \sigma \times \chi, \psi).$$

(2) $\gamma^{\mathcal{W}}(s, \pi^\vee \times \chi, \psi) = \gamma^{\mathcal{W}}(s, \pi \times \rho(\chi), \psi)$ *except in the metaplectic case, in which case* $\gamma^{\mathcal{W}}(s, \pi^\vee \times \chi, \psi) = \gamma^{\mathcal{W}}(s, \pi \times \chi, \psi^{-1})$.

(3) $\gamma^{\mathcal{W}}(s, \pi \times \chi, \psi)\gamma^{\mathcal{W}}(1 - s, \pi^\vee \times \chi^{-1}, \psi^{-1}) = 1$.

(4) *In Case* (II) *we have*

$$\gamma^{\mathcal{W}}(s, \pi \times \chi, \psi) = \gamma^{GJ}(s, \pi \otimes \chi_1, \psi)\gamma^{GJ}\big(s, \pi^\vee \otimes \chi_2, \psi\big).$$

Proof These results are proven in [4, 32] when $\delta = 1$. Propositions 4.1, 4.2 and Lemma 4.2 prove the first assertion. We can prove the second statement in the same way as in [32] (see [4], where the metaplectic case is discussed, and see also the proof of Proposition 5.4). The third statement follows from (B.3). The last statement is proven in the appendix of [70]. □

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