MAZUR'S DEFORMATION RINGS

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This talk is about deformation theory à la Mazur.

Fix a prime p and k a finite field of characteristic p, and let $\mathcal{O} = W(k)$ denote its ring of Witt vectors.

Definition 0.0.1. We let $\widehat{\mathcal{C}}_{\mathcal{O}}$ denote the category whose objects are pairs consisting of a complete local Noetherian ring R and a fixed isomorphism $R/\mathfrak{m}_R \xrightarrow{\sim} k$, and whose morphisms are local homomorphisms which respect the isomorphism to k. We let $\mathcal{C}_{\mathcal{O}}$ denote the full subcategory of $\widehat{\mathcal{C}}_{\mathcal{O}}$ of Artinian rings.

Definition 0.0.2. A functor $F : \mathcal{C}_{\mathcal{O}} \to \mathrm{Set}$ is called *pro-representable* if there exists $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$ such that $F \cong \mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(A, -)$.

Remark 0.0.3.

- If a functor $F : \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Set}$ is representable by $A \in \mathcal{C}_{\mathcal{O}}$ then $A = \varprojlim_n A/\mathfrak{m}_A^n$, so F is uniquely determined by $F|_{\mathcal{C}_{\mathcal{O}}}$ because if $B \in \widehat{\mathcal{C}}_{\mathcal{O}}$ then
$$F(B) = \mathrm{Hom}_{\widehat{\mathcal{C}}_{\mathcal{O}}}(A, B) = \mathrm{Hom}_{\widehat{\mathcal{C}}_{\mathcal{O}}}(A, \varprojlim_n B/\mathfrak{m}_B^n) = \varprojlim_n \mathrm{Hom}_{\widehat{\mathcal{C}}_{\mathcal{O}}}(A, B/\mathfrak{m}_B^n)$$
and $B/\mathfrak{m}_n^B \in \mathcal{C}_{\mathcal{O}}$. This is why in practice it is convenient to consider functors on $\mathcal{C}_{\mathcal{O}}$ only.
- $\mathcal{C}_{\mathcal{O}}$ is closed under fiber products.

Example 0.0.4. The ring of dual numbers $k[\epsilon] = k[x]/x^2$ is in $\mathcal{C}_{\mathcal{O}}$, and plays an important role in the theory. We note now that
$$\begin{aligned} k[\epsilon] \times_k k[\epsilon] &= \{(\lambda_0 + \lambda_1\epsilon, \mu_0 + \mu_1\epsilon) \in k[\epsilon] \times k[\epsilon] \mid \lambda_0 = \mu_0\} \\ &= k[x, y]/(x^2, y^2, xy) \end{aligned}$$

Definition 0.0.5. The *Zariski tangent space* of F is $F(k[\epsilon])$.

Given a diagram $A \to C \leftarrow B$ in $\mathcal{C}_{\mathcal{O}}$, then we get a map of sets $h_{A,B,C} : F(A \times_C B) \to F(A) \times_{F(C)} F(B)$.

We want conditions for a functor $F : \mathcal{C}_{\mathcal{O}} \to \mathrm{Set}$ to be pro-representable.

Theorem 0.0.6 (Grothendieck). *Suppose $F(k) = \{\bullet\}$. Then F is pro-representable if and only if for all $A \to C \leftarrow B$ in $\mathcal{C}_{\mathcal{O}}$, the map $h_{A,B,C}$ is bijective and $F(k[\epsilon])$ is a finite dimensional k-vector space.*

[1] notes taken by Ashwin Iyengar

Ok, we haven't yet said why $F(k[\epsilon])$ is a k-vector space, but there is a completely natural structure that one can define when $h_{k[\epsilon],k[\epsilon],k}$ is bijective.

Grothendieck's condition turns out to be difficult to check in practice, but it can be significantly weakened as follows.

Definition 0.0.7. If $\alpha : A \to B$ in $\mathcal{C}_{\mathcal{O}}$ is a morphism, it is *small* if it is surjective such that the kernel is principal and killed by $\mathfrak{m}_A$.

Theorem 0.0.8 (Schlessinger)**.** *If $F(k) = \{\bullet\}$, then F is pro-representable if and only if the following conditions hold (letting $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ denote an arbitrary diagram in $\mathcal{C}_{\mathcal{O}}$ in each condition below)*

1. *If $\alpha : A \twoheadrightarrow C$ is a small surjection, then $h_{A,B,C}$ is surjective.*
2. *If $\alpha : k[\epsilon] \to k$ is the map killing ϵ, then $h_{k[\epsilon],B,k}$ is bijective.*
3. $\dim_k F(k[\epsilon]) < \infty$ *(by condition (2) there is a vector space structure, as above)*
4. *If $\alpha = \beta$ are both small then $h_{A,B,C}$ is bijective.*

Definition 0.0.9. A profinite group G satisfies the "p-finiteness condition" if

$$\dim_{\mathbb{F}_p} \mathrm{Hom}^{\mathrm{cts}}(H, \mathbb{F}_p) < \infty$$

for any open subgroup $H \leq G$.

Example 0.0.10. If $K/\mathbb{Q}_\ell$ is a finite extension, then $G = G_K = \mathrm{Gal}(K/\mathbb{Q}_p)$ is p-finite. If $F/\mathbb{Q}$ is a finite extension and S is a finite set of primes, then $G = G_{F,S} = \mathrm{Gal}(F_S/F)$ does as well. Here F_S is the maximal algebraic extension of F unramified at primes outside S.

Definition 0.0.11.

1. A *representation* if G of dimension n is a continuous group homomorphism $\rho : G \to \mathrm{GL}_n(A)$ for $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$.
2. If $\rho_0 : G \to \mathrm{GL}_n(A_0)$ and $\varphi : A \to A_0$ is a map in $\mathcal{C}_{\mathcal{O}}$ then we say that a *lifting* of ρ_0 to A is a representation $\rho : G \to \mathrm{GL}_n(A)$ such that $\mathrm{GL}_n(\varphi) \circ \rho = \rho_0$.
3. Two liftings of ρ_0 to A are called *strictly equivalent* if they are conjugate by an element in the kernel of $\mathrm{GL}_n(\varphi)$.
4. A *deformation* of ρ_0 to A is a strict equivalence class of liftings.

We can now define deformation functors of representations of profinite groups.

Definition 0.0.12. Fix a representation $\overline{\rho} : G \to \mathrm{GL}_n(k)$. Then we let

$$\begin{aligned} D^{\square}_{\overline{\rho}} : \mathcal{C}_{\mathcal{O}} &\to \mathsf{Set} \\ A &\mapsto \{\text{liftings of } \overline{\rho} \text{ to } A\} \end{aligned}$$

and

$$\begin{aligned} D_{\overline{\rho}} : \mathcal{C}_{\mathcal{O}} &\to \mathsf{Set} \\ A &\mapsto \{\text{deformations of } \overline{\rho} \text{ to } A\} \end{aligned}$$

Fact 0.0.13. *$D^{\square}_{\overline{\rho}}$ is pro-representable by its universal lifting ring $R^{\square}_{\overline{\rho}} \in \widehat{\mathcal{C}_{\mathcal{O}}}$, which comes with a universal lifting $\rho^{\square} : G \to \mathrm{GL}_n(R^{\square}_{\overline{\rho}})$. If $\mathrm{End}_G(\overline{\rho}) = k$, then $D_{\overline{\rho}}$ is also pro-representable by its universal deformation ring $R_{\overline{\rho}}$ which comes with a universal deformation $\rho^{\mathrm{univ}} : G \to \mathrm{GL}_n(R_{\overline{\rho}})$.*

Example 0.0.14.
- If $n = 1$, so that $\overline{\rho} : G \to k^\times$ is a character, then $D^\square_{\overline{\rho}} = D_{\overline{\rho}}$ is represented by $\mathcal{O}\left[\!\left[G^{\mathrm{ab},p}\right]\!\right]$, where ab denotes the abelianization and p the pro-p-completion.

  For instance if $F/\mathbb{Q}_p$ is a finite extension, then $R_{\overline{\rho}} = \mathcal{O}[\mu_{p^\infty}(F)]\left[\!\left[X_1, \dots, X_{[F:\mathbb{Q}]}\right]\!\right]$.
- If instead F_m is the free pro-p group on m generators and $\overline{\rho} = 1 \oplus \cdots \oplus 1$ (n times), then
$$R^\square_{\overline{\rho}} = \mathcal{O}\left[\!\left[X^{(k)}_{i,j} \mid 1 \le k \le m, 1 \le i, j \le n\right]\!\right]$$
  and $\rho^\square$ takes γ_k to $1 + (X^{(k)}_{ij})$.
- Here is a non-example. If $G = F_1$ as above and $\overline{\rho} = 1 \oplus 1$, then let
$$D_{\mathrm{ord}}(A) := \{\text{liftings of } \overline{\rho} \text{ fixing a flag}\}$$
  This is *not* pro-representable because the first condition in Theorem 0.0.8 is not satisfied because
$$D^\square_{\mathrm{ord}}(k[\epsilon] \times_k k[\epsilon]) \to D^\square_{\mathrm{ord}}(k[\epsilon]) \times D^\square_{\mathrm{ord}}(k[\epsilon])$$
  is not surjective. For instance two liftings are given by $\gamma \mapsto \begin{pmatrix} 1 & \psi(g)\epsilon \\ 0 & 1 \end{pmatrix}$ and $\gamma \mapsto \begin{pmatrix} 1 & 0 \\ \psi(g)\epsilon & 1 \end{pmatrix}$ but one can check that they don't lift to something ordinary.

From now on assume $\mathrm{End}_G(\overline{\rho}) = k$ so that $R_{\overline{\rho}}$ exists. Define $\mathrm{ad}\,\overline{\rho} = \mathrm{End}_k(\overline{\rho})$ with G acting via ρ composed with conjugation.

Lemma 0.0.15. $\mathrm{Hom}_k(\mathfrak{m}_{R_{\overline{\rho}}}/(\mathfrak{m}^2_{R_{\overline{\rho}}}, p), k) = \mathrm{Hom}_{\widehat{\mathcal{C}_\mathcal{O}}}(R_{\overline{\rho}}, k[\epsilon]) = D_{\overline{\rho}}(k[\epsilon]) \xrightarrow{\sim} H^1_{\mathrm{cts}}(G, \mathrm{ad}\,\overline{\rho})$.

Proof. A lifting $\rho : G \to \mathrm{GL}_n(k[\epsilon]) \in D^\square_{\overline{\rho}}(k[\epsilon])$ must be of the form $(1 + \epsilon c(g))\overline{\rho}(g)$ where $c : G \to \mathrm{ad}\,\overline{\rho}$ and then one shows that the fact that ρ is a continuous group homomorphisms translates into the fact that $c \in Z^1(G, \mathrm{ad}\,\overline{\rho})$. So then $D^\square_{\overline{\rho}}(k[\epsilon]) \xrightarrow{\sim} Z^1(G, \mathrm{ad}\,\overline{\rho})$ and in fact taking strict equivalence classes corresponds to killing coboundaries. □

Now take $\rho_0 : G \to \mathrm{GL}_n(A_0) \in D_{\overline{\rho}}(A_0)$. Take $\varphi : A \twoheadrightarrow A_0$ in $\mathcal{C}_\mathcal{O}$ and take $I = \ker \varphi$ such that $\mathfrak{m}_A I = 0$. Then we want a class $\mathcal{O}(\rho_0) \in H^2(G, \mathrm{ad}\,\overline{\rho}) \otimes I$ which vanishes if and only if ρ_0 lifts along φ. For this take a set theoretic lifting $\gamma : G \to \mathrm{GL}_n(A_1)$ of ρ_0 and define
$$(g_1, g_2) \mapsto \gamma(g_1 g_2)\gamma)(g_2)^{-1}\gamma(g_1)^{-1}.$$
One can show (this is very annoying to show) that this is a 2-cocycle, and that $[c] \in H^2$ does not depend on γ.

Now take $h^i := \dim_k H^i(G, \mathrm{ad}\,\overline{\rho})$. Then

Proposition 0.0.16. *There is a (non-canonical) isomorphism*
$$R_{\overline{\rho}} \cong \mathcal{O}[\![x_1, \dots, x_{h^1}]\!]/(f_1, \dots, f_{h^2})$$
and in particular $\dim R_{\overline{\rho}} \ge 1 + h^1 - h^2$.

Proof. We can find a surjection $\pi : S = \mathcal{O}[\![X_1, \dots, X_{h^1}]\!] \twoheadrightarrow R_{\overline{\rho}}$ which is an isomorphism on tangent spaces (basically by lifting a basis of the tangent space), and we take $J = \ker \pi$. But there is an exact sequence
$$0 \to J/\mathfrak{m}_S J \to S/\mathfrak{m}_S J \to R_{\overline{\rho}} \to 0$$
so we want to bound the number of generators $J/\mathfrak{m}_S J$. But one can show that the map $\mathrm{Hom}_k(J/\mathfrak{m}_S J, k) \to H^2(G, \mathrm{ad}\,\overline{\rho})$ taking
$$f \mapsto (1 \otimes f)(\mathcal{O}(\overline{\rho}^{\mathrm{univ}}))$$
is injective, and then we're done. □

Remark 0.0.17. By this presentation, if we know that $H^2(G, \operatorname{ad}\overline{\rho}) = 0$ then $R_{\overline{\rho}}$ is formally smooth over $\mathcal{O}$ of dimension h^1, and if $\dim R_{\overline{\rho}} = 1 + h^1 - h^2$ then $R_{\overline{\rho}}$ is a complete intersection.

Example 0.0.18. If $F/\mathbb{Q}_p$ is finite, and $G = G_F$, then local Tate duality says that

$$H^2(G, \operatorname{ad}\overline{\rho}) \cong H^0(G, \operatorname{ad}\overline{\rho}^*(1)) \cong \operatorname{Hom}_G(1, \operatorname{ad}\overline{\rho}^*(1)) = \operatorname{Hom}_G(\overline{\rho}, \overline{\rho}(1)).$$

Therefore, if $\operatorname{Hom}_G(\overline{\rho}, \overline{\rho}(1)) = 0$ then $R_{\overline{\rho}}$ is formally smooth over $\mathcal{O}$ of relative dimension h^1. But in fact we can compute the relative dimension using the Euler characteristic: $h^0 - h^1 + h^2 = -[F : \mathbb{Q}_p] \dim_k \operatorname{ad}\overline{\rho}$. In particular $R_{\overline{\rho}}$ is formally smooth over $\mathcal{O}$ of relative dimension $h^1 = [F : \mathbb{Q}_p]n^2 + \dim(\operatorname{ad}\overline{\rho})^G = [F : \mathbb{Q}_p]n^2 + 1$.

References