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Green Polynomials and Singularities of Unipotent Classes

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Let X be an irreducible algebraic variety of dimension d over an algebraically closed field.

Deligne [2] has associated to X a complex ${}^{\pi}\mathbb{Q}_l$ of l-adic sheaves (canonical up to quasi-isomorphism) which has constructible cohomology sheaves $\mathscr{H}^i(X)$, which is self-dual in the derived category, which is equivalent to the complex reduced to constant sheaf $\mathbb{Q}_l$ in degree 0 over the smooth part of X, and which has the property: $\mathscr{H}^i(X) = 0$ for $i < 0$, $\mathscr{H}^i(X)$ has support of dimension $\leqslant d - i - 1$ if $i > 0$.

His construction, which is sketched in [8, Sect. 3] is an algebraic analogue of the Goresky and Macpherson middle intersection cohomology theory [3, 4]. We shall call $\mathscr{H}^i(X)$ the DGM sheaves of X.

The purpose of this paper is to describe an application of this theory to the study of irreducible characters of the finite group $GL_n(\mathbb{F}_q)$. Let k be an algebraic closure of $\mathbb{F}_q$. Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\ (\geqslant 0))$ be a partition of n: $n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. We associate to λ the unipotent class $X_\lambda \subset GL_n(k)$ consisting of the unipotent elements which have Jordan blocks of size $\lambda_1, \lambda_2, \ldots, \lambda_n$. We also associate to λ the irreducible unipotent representation E_λ of $GL_n(\mathbb{F}_q)$: it is the "biggest" component of the representation induced by the identity representation of the stabilizer of a flag of subspaces of dimensions $\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3, \ldots$, in $\mathbb{F}_q^n$. Consider the DGM sheaves $\mathscr{H}^i(\bar{X}_\lambda)$ of the closure of X_λ. In the following theorem the sheaves $\mathscr{H}^i(\bar{X}_\lambda)$ will be regarded as sheaves on the whole variety of unipotent elements in $GL_n(k)$, equal to zero on the complement of $\bar{X}_\lambda$.

THEOREM 1. *Let $u \in GL_n(F_q)$ be a unipotent element. Let $n(\lambda)$ be*

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defined by (1.2). *We have*

$$\operatorname{Tr}(u, E_\lambda) = q^{n(\lambda)} \sum_{i \geqslant 0} q^i \dim \mathscr{H}_u^{2i}(\bar{X}_\lambda).$$

Moreover, $\mathscr{H}^i(\bar{X}_\lambda) = 0$ *if* i *is odd.*

The proof will be given in Sections 1 and 2. The idea of the proof is to show that the singularities of the closure of the unipotent class X_λ are a special case of the singularities of a certain generalized Schubert variety (in the sense of [8, Sect. 5]) associated to elements in an affine Weyl group and then to use the results of [8, Sect. 5]. The algebraic formalism we shall use is that of the Hall algebra; an excellent exposition can be found in Macdonald's book [12].

1. The Space of Lattices

Let V be a fixed vector space of dimension n over the field $k((t))$. A $k[[t]]$-submodule of rank n of V is said to be a lattice. Let $\mathscr{L}$ be the set of all lattices in V, and let L_0 be a fixed latice in V. The lattices L such that $L \subset L_0$ form a subset $\mathscr{L}^+$ of $\mathscr{L}$. It can be regarded as a countable disjoint union of (finite dimensional) projective varieties $\mathscr{L}_i^+$ $(i \geqslant 0)$ over k, where $\mathscr{L}_i^+$ is the set of lattices $L \subset L_0$ such that $\dim(L_0/L) = i$. If $L \in \mathscr{L}_i^+$, then $t^{ni}L_0 \subset L$, hence $\mathscr{L}_i^+$ may be identified with the set of all codimension i subspaces of the $n^2 i$-dimensional k-vector space $L_0/t^{ni}L_0$, which are stable under the nilpotent endomorphism t of $L_0/t^{ni}L_0$.

The group $GL(L_0)$ of automorphisms of L_0 acts in a natural way on $\mathscr{L}^+$; its orbits are in 1–1 correspondence with the elements of the set

$$\mathscr{P}_n = \{\lambda \in \mathbb{N}^n \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\};$$

the orbit $\mathscr{O}_\lambda \subset \mathscr{L}^+$ corresponding to $\lambda \in \mathscr{P}_n$ consists of the lattices $L \subset L_0$ such that the nilpotent transformation t of L_0/L has Jordan blocks of sizes $\lambda_1, \lambda_2, \dots, \lambda_n$. (In particular, $\mathscr{O}_\lambda \subset \mathscr{L}_{|\lambda|}^+$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.) The closure of $\mathscr{O}_\lambda$ is the union of the orbits $\mathscr{O}_\mu$ $(\mu \leqslant \lambda)$, where, for $\mu, \lambda \in \mathscr{P}_n$, $\mu \leqslant \lambda$ means that $|\mu| = |\lambda|$ and $\mu_1 \leqslant \lambda_1$, $\mu_1 + \mu_2 \leqslant \lambda_1 + \lambda_2$, $\mu_1 + \mu_2 + \mu_3 \leqslant \lambda_1 + \lambda_2 + \lambda_3$, etc.

Let $\mu \leqslant \lambda$ be two elements of $\mathscr{P}_n$. We wish to compute

$$\Pi_{\mu,\lambda} = \sum_{i \geqslant 0} q^{i/2} \dim \mathscr{H}_x^i(\bar{\mathscr{O}}_\lambda),$$

where $\mathscr{H}_x^i(\bar{\mathscr{O}}_\lambda)$ are the stalks of the DGM sheaves of the variety $\bar{\mathscr{O}}_\lambda$ at a point $x \in \mathscr{O}_\mu$, and $q^{1/2}$ is an indeterminate. Let Λ_n be the algebra of symmetric polynomials in the variables $X=(X_1, X_2, \ldots, X_n)$ with coefficients in $Q(q^{1/2})$. For $\lambda \in \mathscr{P}_n$, we denote by $P_\lambda(X, q) \in \Lambda_n$ the Hall–Littlewood polynomial corresponding to λ [12, III(2.1)]. Then $s_\lambda(X)=P_\lambda(X, 0)$ are the Schur functions [12, I(3.1)]. There are well-defined polynomials (in q) $K_{\lambda,\mu}(q)$ defined for all $\mu \leqslant \lambda$ in $\mathscr{P}_n$ such that

$$s_\lambda(X)=\sum_{\mu \leqslant \lambda} K_{\lambda,\mu}(q)\, P_\mu(X, q) \tag{1.1}$$

for all $\lambda \in \mathscr{P}_n$ [12, III(2.6)]. For each $\lambda \in \mathscr{P}_n$, we set

$$n(\lambda)=\sum_{i \geqslant 1} (i-1)\, \lambda_i; \tag{1.2}$$

cf. [12, I(1.5)].

THEOREM 2. *If $\mu \leqslant \lambda$ are two elements in $\mathscr{P}_n$, we have*

$$\Pi_{\mu,\lambda}=q^{n(\mu)-n(\lambda)} K_{\lambda,\mu}(q^{-1}).$$

Let $\mathscr{B}$ be the set of all sequences $L^0 \supsetneqq L^1 \supsetneqq L^2 \supsetneqq \cdots \supsetneqq L^n=tL^0$, where L^i are lattices in V, and let $L_0 \supsetneqq L_1 \supsetneqq L_2 \supsetneqq \cdots \supsetneqq L_n=tL_0$ be such a fixed sequence (with L_0 as above).

Let I be the stabilizer of this sequence in $GL(L_0)$. It is known (see, for example [8, Sect. 5]) that $\mathscr{B}$ is in a natural way an infinite dimensional algebraic variety; more precisely it is an increasing union of (finite dimensional) projective varieties. Each orbit of I on $\mathscr{B}$ is isomorphic to an affine space. There are n distinguished orbits $s_1, s_2, \ldots, s_n$ of dimension 1: if $1 \leqslant i \leqslant n-1$, s_i consists of all sequences $L_0 \supsetneqq L_1 \supsetneqq \cdots \supsetneqq L_{i-1} \supsetneqq L^i \supsetneqq L_{i+1} \supset \cdots \supset L_n=tL_0$, $L^i \neq L_i$; if $i=n$, s_n consists of all sequences $L^0 \supsetneqq L_1 \supsetneqq \cdots \supsetneqq L_{n-1} \supsetneqq L^n=tL^0$, $L^0 \neq L_0$. There is also a distinguished orbit τ of dimension 0: it consists of $L_1 \supsetneqq L_2 \supsetneqq \cdots \supsetneqq L_{n-1} \supsetneqq tL_0 \supsetneqq tL_1$. The set $\tilde{W}$ of orbits of I on $\mathscr{B}$ has a natural group structure; it has generators $s_1, s_2, \ldots, s_n, \tau$, where $s_1, s_2, \ldots, s_n$ are the standard generators of an affine Weyl group and τ is an element of infinite order satisfying $\tau s_i=s_{i+1}\tau$ (here i is taken as an integer modulo n). Let $l\colon \tilde{W} \to \mathbb{N}$ be the length function: $l(w)$ is the dimension of the corresponding orbit in $\mathscr{B}$.

Let H_n' be the Hecke algebra corresponding to $\tilde{W}$: it is an algebra over $Q(q^{1/2})$ with basis T_w ($w \in \tilde{W}$) and multiplication defined by

$$\begin{aligned} T_w T_{w'} &= T_{ww'}, && \text{if} \quad l(ww')=l(w)+l(w'), \\ (T_{s_i}+1)(T_{s_i}-q) &= 0, && \text{if} \quad i=1, 2, \ldots, n. \end{aligned}$$

There is a canonical involution of the ring H_n' (see [7, Sect. 1]). It is the unique ring homomorphism $h \to \bar{h}$ of H'_n into itself such that $\overline{q^{1/2}} = q^{-1/2}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$.

If y, w are elements of $\tilde{W}$ such that $y \leqslant w$ (i.e., the orbit defined by y is in the closure of the orbit defined by w) then there is a well-defined polynomial $P_{y,w}$ in q of degree $\leqslant \frac{1}{2}(l(w) - l(y) - 1)$ (if $y < w$) and such that $P_{w,w} = 1$; it is characterized by the identity

$$\overline{q^{-l(w)/2} \sum_{y \leqslant w} P_{y,w} T_y} = q^{-l(w)/2} \sum_{y \leqslant w} P_{y,w} T_y \qquad (\forall\, w \in \tilde{W}); \tag{1.3}$$

see [7, (1.1.6)]. The polynomial $P_{y,w}$ has the interpretation [8, Sect. 5]

$$P_{y,w} = \sum_{i \geqslant 0} q^{i/2} \dim \mathscr{H}^i_\alpha(\bar{w}),$$

where $\mathscr{H}^i_\alpha(w)$ are the stalks of the DGM sheaves of the closure of the orbit defined w at a point α in the orbit defined by y.

Let $\mathscr{B}^+$ be the subset of $\mathscr{B}$ consisting of sequences $L^0 \supsetneqq L^1 \supsetneqq \cdots \supsetneqq L^n = tL^0$ such that $L^0 \subset L_0$. There is a natural map $\mathscr{B}^+ \to^\pi \mathscr{L}^+$ taking $L^0 \supset L^1 \supset \cdots \supset L^n$ to L^0. Its fibres are isomorphic to the flag manifold of an n dimensional vector space over k. The inverse image of $\mathscr{O}_\lambda \subset \mathscr{L}^+$ under this map is a finite union of orbits w of I on $\mathscr{B}^+$; these w form a single double coset with respect to the subgroup $W \subset \tilde{W}$ generated by $s_1, \ldots, s_{n-1}$ ($W \approx \mathfrak{S}_r$). Among these orbits w, there is a unique one of maximal dimension, say, w_λ. If $\mu \leqslant \lambda$, so that $\mathscr{O}_\mu \subset \bar{\mathscr{O}}_\lambda$, then it is clear that $w_\mu \leqslant w_\lambda$ and that the stalk of $\mathscr{H}^i(\bar{\mathscr{O}}_\lambda)$ at a point of $\mathscr{O}_\mu$ is isomorphic to the stalk of $\mathscr{H}^i(\bar{w}_\lambda)$ at a point of w_μ. In particular, we have

$$\Pi_{\mu,\lambda} = P_{w_\mu, w_\lambda}, \tag{1.4}$$

so that $\mathscr{H}^i(\bar{\mathscr{O}}_\lambda) = 0$ for odd i (i.e., $\Pi_{\mu,\lambda}$ is a polynomial in q).

Next, we note that the elements

$$u_\lambda = \frac{1}{\Phi(q)} \sum_{\substack{w \subset \mathscr{B}^+ \\ \pi(w) \subset \mathscr{O}_\lambda}} T_w \in H'_n \qquad (\lambda \in \mathscr{P}_n), \tag{1.5}$$

where $\Phi(q) = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1})$, span a subspace H_n of H'_n which is closed under multiplication and has identity element u_0. Indeed, H_n could be characterized as the set of elements h in H such that $hu_0 = u_0 h = h$ and such that h is a linear combination of elements T_w ($w \subset \mathscr{B}^+$) (see [1, 2.10]), and hence H_n is an algebra. An argument in [12, V(2.6)] shows that

$$u_\mu \cdot u_\nu = \sum_\lambda g^\lambda_{\mu\nu}(q)\, u_\lambda,$$

where $g^{\lambda}_{\mu\nu}(q)$ are the Hall polynomials (see [12, II, 2]); note that $g^{\lambda}_{\mu\nu}=0$ unless $|\lambda|=|\mu|+|\mu|$.

The involution $h \to \bar{h}$ of H'_n keeps u_0 fixed:

$$\bar{u}_0=\frac{1}{\Phi(q^{-1})}\sum_{w\in W} T^{-1}_{w^{-1}}=\frac{q^{-n(n-1)/2}}{\Phi(q^{-1})}\sum_{w\in W} T_w=\frac{1}{\Phi(q)}\sum_{w\in W} T_w;$$

hence it leaves H_n stable. (From $hu_0=u_0h=h$, it follows that $\bar{h}u_0=u_0\bar{h}=\bar{h}$.)

From (1.3), (1.4), (1.5) and the identity $P_{zw_\mu z',w_\lambda}=P_{w_\mu,w_\lambda}$ ($\forall z, z' \in W$) (see [7, (2.3.g)]), it follows immediately that the identity

$$\overline{q^{-d(\lambda)/2}\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}u_\mu}=q^{-d(\lambda)/2}\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}u_\mu \qquad (\forall \lambda\in\mathscr{P}_n) \tag{1.6}$$

holds in H_n; here $d(\lambda)=l(w_\lambda)-\nu$ is the dimension of $\mathscr{O}_\lambda$.

Next we note that for $1\leqslant r\leqslant n$, the orbit $\mathscr{O}_{(1^r)}$ is just the set of all lattices $L\subset L_0$ of codimension r such that $t=0$ on L_0/L. These are in 1–1 correspondence with the codimension r subspaces of L_0/tL_0, hence form a Grassmanian of dimension $r(n-r)$. In particular, this is a closed orbit. It follows that for $\lambda=(1^r)$, we have $\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}u_\mu=u_\lambda$; hence (1.6) becomes:

$$\overline{q^{-r(n-r)/2}u_{(1^r)}}=q^{-r(n-r)/2}u_{(1^r)}. \tag{1.7}$$

It is known [12, III(3.4)] that there is a unique isomorphism $\Psi: H_n \to \Lambda_n$ of $Q(q^{1/2})$ algebras such that $\Psi(u_\lambda)=q^{-n(\lambda)}P_\lambda(X,q^{-1})$ ($\forall \lambda\in\mathscr{P}_n$).

In particular, $\Psi(u_{1^r})=q^{-r(r-1)/2}e_r$, where $e_r\in\Lambda_n$ is the rth elementary symmetric function. Transporting the involution $h\to\bar{h}$ of H_n to Λ_n, via Ψ, we get a ring involution $p\to\bar{p}$ of Λ_n such that $\overline{q^{1/2}}=q^{-1/2}$ and $\overline{q^{-r(r-1)/2}e_r}=q^{-r(n-r)}q^{-r(r-1)/2}e_r$, i.e.,

$$\overline{e_r}=q^{-r(n-1)}e_r \qquad (1\leqslant r\leqslant n). \tag{1.8}$$

Since $P_\lambda(X,q^{-1})$ has total degree of homogeneity $|\lambda|$ in X, it is a linear combination of products $e_{r_1}e_{r_2}\cdots e_{r_s}$, $r_1+\cdots+r_s=|\lambda|$, with coefficients in $\mathbb{Z}[q^{-1}]$, and it follows that

$$\overline{P_\lambda(X,q^{-1})}=q^{-|\lambda|(n-1)}P_\lambda(X,q). \tag{1.9}$$

Thus, applying Ψ to both sides of (1.6) we see that the identity

$$q^{d(\lambda)/2}\sum_{\mu\leqslant\lambda}\bar{\Pi}_{\mu,\lambda}q^{n(\mu)}\cdot q^{-|\mu|(n-1)}P_\mu(X,q)$$
$$=q^{-d(\lambda)/2}\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}q^{-n(\mu)}P_\mu(X,q^{-1}) \qquad (\forall \lambda\in\mathscr{P}_n)$$

holds in Λ_n. According to [12, V(2.9)] we have

$$n(\lambda) = \frac{|\lambda|(n-1)}{2} - \frac{d(\lambda)}{2}$$

hence

$$\sum_{\mu \leqslant \lambda} q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda} P_\mu(X, q)$$

$$= \sum_{\mu \leqslant \lambda} q^{-n(\mu)+n(\lambda)} \Pi_{\mu,\lambda} P_\mu(X, q^{-1}) \qquad (\forall\, \lambda \in \mathscr{P}_n). \tag{1.10}$$

Now, if $\mu < \lambda$, $\Pi_{\mu,\lambda}$ is a polynomial in q of degree $\leqslant \frac{1}{2}(d(\lambda) - d(\mu) - 1) = n(\mu) - n(\lambda) - \frac{1}{2}$; hence $q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda}$ is a polynomial in q without constant term. Thus, the left-hand side of (1.10) is a polynomial in X and in q. This polynomial is invariant under the substitution $q \to q^{-1}$, hence it does not involve q. Hence the left-hand side of (1.10) is equal to its value for $q = 0$. Thus

$$\sum_{\mu \leqslant \lambda} q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda} P_\mu(X, q) = P_\lambda(X, 0) = s_\lambda(X) \qquad (\forall\, \lambda \in \mathscr{P}_n).$$

Comparing with (1.2) it follows that $q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda} = K_{\lambda,\mu}$ ($\forall\, \mu \leqslant \lambda$) and Theorem 2 is proved.

COROLLARY 3. *The image of* $c_\lambda = q^{-d(\lambda)/2} \sum_{\mu \leqslant \lambda} \Pi_{\mu,\lambda} u_\mu$ *under the isomorphism*

$$\Psi: H_n \overset{\approx}{\to} \Lambda_n$$

is

$$q^{-|\lambda|(n-1)/2} s_\lambda(X).$$

In particular, if $\mu, \nu \in \mathscr{P}_n$, *the product* $c_\mu \cdot c_\nu$ *is a combination with constant coefficients of elements* c_λ. *The multiplication constants are those which give the product of the corresponding Schur functions.*

COROLLARY 4 (see [9; 12, III(6.5)].) *The polynomial* $K_{\lambda,\mu}(q)$ *has* $\geqslant 0$ *coefficients* ($\mu \leqslant \lambda$).

2. A COMPACTIFICATION OF THE VARIETY OF UNIPOTENT ELEMENTS IN $GL_n(k)$

Let $\bar{V}$ be an n-dimensional k-vector space. Define $E = \bar{V} \oplus \cdots \oplus \bar{V}$ (n copies) and let $t: E \to E$ be defined by $t(v_1, ..., v_n) = (0, v_1, v_2, ..., v_{n-1})$. Let

Y be the variety of all n-dimensional t-stable subspaces of E and let Y_0 be the open subvariety of Y consisting of those subspaces in Y which are transversal to

$$\underbrace{\bar{V} \oplus \cdots \oplus \bar{V}}_{n-1} \oplus 0.$$

To give a subspace $E' \in Y_0$ is the same as to give linear maps $f_1, f_2,..., f_{n-1} : \bar{V} \to \bar{V}$ such that

$$E' = \{(f_{n-1}(v), f_{n-2}(v),..., f_1(v), v) \mid v \in \bar{V}\}.$$

The condition for E' to be t-stable is that $(0, f_{n-1}(v), f_{n-2}(v),..., f_1(v)) \in E'$ for all $v \in \bar{V}$, i.e., that $f_2(v) = f_1^2(v)$, $f_3(v) = f_2 f_1(v),..., f_{n-1} = f_{n-2}(f_1(v))$, $0 = f_{n-1}(f_1(v))$ or equivalently that $f_i = f_1^i$ $(1 \leqslant i \leqslant n-1)$ and $f_1^n = 0$. Thus, the correspondence

$$f_1 \to E' = \{(f_1^{n-1}(v), f_1^{n-2}(v),..., f_1(v), v) \mid v \in V\}$$

gives an isomorphism between the variety X' of nilpotent endomorphisms of $\bar{V}$ and the open subvariety Y_0 of Y.

We shall now identify Y with the variety $\mathscr{L}_n^+$ defined in the previous section. Note that $\mathscr{L}_n^+$ can be identified with the set of t-stable codimension n subspaces of the n^2-dimensional k-vector space $L_0/t^n L_0$, hence with the set of t-stable n dimensional subspaces of the dual space $(L_0/t^n L_0)^*$. But this last space is isomorphic to $E = \bar{V} \oplus \cdots \oplus \bar{V}$ as a $k[t]$-module, and thus we may identify Y and $\mathscr{L}_n^+$. In this way, the variety X of unipotent elements in $GL(\bar{V})$ (which is canonically isomorphic to X') appears as an open subset of $\mathscr{L}_n^+$. Thus, $\mathscr{L}_n^+$ may be regarded as a compactification of X. The imbedding of X into $\mathscr{L}_n^+$ has the property that the unipotent class $X_\lambda \subset X$ is equal to $\mathscr{O}_\lambda \cap X$ for all $\lambda \in \mathscr{P}_n$, $|\lambda| = n$. In particular for such λ, the DGM sheaf $\mathscr{H}^i(\bar{X}_\lambda)$ of $\bar{X}_\lambda$ (closure in X) is just the restriction of the DGM sheaf $\mathscr{H}^i(\bar{\mathscr{O}}_\lambda)$ of $\bar{\mathscr{O}}_\lambda$ to $\bar{X}_\lambda$. It then follows from Theorem 2 that $\mathscr{H}^i(\bar{X}_\lambda) = 0$ for i odd and that

$$\sum_{i \geqslant 0} q^i \dim \mathscr{H}_x^{2i}(\bar{X}_\lambda) = q^{n(\mu) - n(\lambda)} K_{\lambda,\mu}(q^{-1}), \ (x \in X_\mu) \tag{2.1}$$

for all $\mu \leqslant \lambda$ in $\mathscr{P}_n$, $|\lambda| = |\mu| = n$. If we specialize q to be a prime power, and if $u \in GL_n(F_q)$ is a unipotent element with Jordan blocks of sizes $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$, then, by the results of Green [5],

$$q^{n(\mu)} K_{\lambda,\mu}(q^{-1}) = \mathrm{Tr}(u, E_\lambda) \tag{2.2}$$

for all $\mu \leqslant \lambda$ in $\mathscr{P}_n$, $|\lambda| = |\mu| = n$; moreover, $\mathrm{Tr}(u, E_\lambda) = 0$ if $\mu \nleqslant \lambda$.

Now Theorem 1 follows from (2.1) and (2.2).

3. Some Problems

Let $\mathscr{E}$ be a locally constant l-adic sheaf on an open smooth subset U of an irreducible variety X. Deligne's definition of the complex ${}^{\pi}\mathbb{Q}_l$ is applicable without change to $\mathscr{E}$ instead of the constant sheaf $\mathbb{Q}_l$ and leads to a complex ${}^{\pi}\mathscr{E}$ of l-adic sheaves on X which on U is equivalent to the complex reduced to $\mathscr{E}$ in degree zero. (This construction was used recently by Vogan and by Beilinson and Bernstein in connection with the question of describing the characters of a real semisimple Lie group. This section was influenced by their work.) Let $\mathscr{H}^i({}^{\pi}\mathscr{E})$ denote the cohomology sheaves of ${}^{\pi}\mathscr{E}$. If we are given a finite group W of automorphisms of $\mathscr{E}$ (inducing identity on X) then, by functoriality, W will act on each of the sheaves $\mathscr{H}^i({}^{\pi}\mathscr{E})$, inducing identity on X.

Consider, for example, the case where $X = G$, a reductive connected algebraic group over k. Let U be the open subset of G consisting of all regular semisimple elements in G. There is a canonical principal bundle $\tilde{U} \to^{p} U$ with group W (where W is the Weyl group of G): $\tilde{U}$ is the set of pairs (s, B), where $s \in U$ and B is a Borel subgroup of G containing s. Let $\mathscr{E} = p_*(\mathbb{Q}_l)$. This is a locally constant sheaf on V, with a natural action of W. (W acts on each stalk by the regular representation.) It follows that W acts naturally on each of the sheaves $\mathscr{H}^i({}^{\pi}\mathscr{E})$. If $g \in G$, the stalk $\mathscr{H}^i_g({}^{\pi}\mathscr{E})$ is naturally isomorphic to $H^i(\mathscr{B}_g, \mathbb{Q}_l)$, where $\mathscr{B}_g$ is the variety of Borel subgroups containing g. Indeed, let $\tilde{G}$ be the set of pairs (g', B), where $g' \in G$ and $B \in \mathscr{B}_g$; and let $p_1 : \tilde{G} \to G$ be the projection $(g', b) \to g'$; then, one can show that ${}^{\pi}\mathscr{E} = (p_1)_* (\mathbb{Q}_l)$. This follows by a general argument of Goresky and Macpherson [4, 4.2] as soon as one checks that there exists a finite partition of G into locally closed, irreducible subsets $G_0, G_1, \dots, G_n$ of G such that $G_0 = U$ and $\dim p_1^{-1}(g) \leqslant \frac{1}{2}$ (codim $G_i - 1$) for all $g \in G_i$ ($i = 1, 2, \dots, n$). The existence of such a partition follows from the finiteness of the number of unipotent classes in a reductive group and from the known inequality $\dim \pi^{-1}(g) \leqslant \frac{1}{2} (\dim Z(g) - \operatorname{rank}(G))$. Thus, we see that there is a natural action of W on $H^i(\mathscr{B}_g, \mathbb{Q}_l)$, for any $g \in G$. (In the case where g is unipotent, and the characteristic of k is not too small this can be identified with Springer's representation [15]; however, our definition seems to be closer to Slodowy's approach [13] to Springer's representation.)

Assume now that G is defined over $\mathbb{F}_q$; let $F: G \to G$ be the corresponding Frobenius map. Then, for any $g \in G$, there is a natural map $F: \mathscr{H}^i_g({}^{\pi}\mathscr{E}) \to \mathscr{H}^i_{F(g)}({}^{\pi}\mathscr{E})$.

Let $w \in W$, and let $\mathscr{B}_{(w)}$ be the variety of Borel subgroup $B \subset G$ such that B, FB are in relative position w; G^F acts on $\mathscr{B}_{(w)}$ by conjugation. In view of [6, 15], it seems natural to state

Conjecture 1.

$$\sum_i (-1)^i \operatorname{Tr}(Fw, \mathscr{H}^i_g({}^{\pi}\mathscr{E})) = \sum_i (-1)^i \operatorname{Tr}(g, H^i(\mathscr{B}_{(w)}, \mathbb{Q}_l)), \textit{for all } g \in G^F.$$

Now let ρ be an irreducible representation of W (over $\mathbb{Q}_l$) and let ρ' be its dual; we define the sheaves $\mathscr{H}^i({}^\pi\mathscr{E})_\rho$ to be $(\mathscr{H}^i({}^\pi\mathscr{E}) \otimes \rho')^W$. We wish to describe the restriction of the sheaf $\mathscr{H}^i({}^\pi\mathscr{E})_\rho$ to the variety $\mathscr{U}$ of unipotent elements in G.

CONJECTURE 2. *Given ρ as above, there is a well-defined unipotent class $C \subset G$ and a locally constant, l-adic sheaf $\mathscr{E}$ on C associated to an irreducible representation of the group of components of the centralizer of an element $u \in C$ with the following property: The sheaf $\mathscr{H}^i({}^\pi\mathscr{E})[-b_u]$ on $\bar{C}$, extended by zero on $\mathscr{U} - \bar{C}$ is isomorphic to the restriction of $\mathscr{H}^i({}^\pi\mathscr{E})_\rho$ to $\mathscr{U}$ (where $b_u = \dim \mathscr{B}_u$).*

This conjecture, which would make [15, 6.10] more precise, is supported by Theorem 1.

Remarks. (a) According to an unpublished theorem of Deligne, the variety of all unipotent elements in G is rationally smooth (in the sense of [7, 4.1]), at least in sufficiently large characteristic. It follows that, if $\mathscr{E}$ is the constant sheaf $\mathbb{Q}_l$ on the regular unipotent class, then $\mathscr{H}^i({}^\pi\mathscr{E})$ is the constant sheaf $\mathbb{Q}_l$ for $i = 0$ and it is zero if $i > 0$.

(b) Assume that G is simple and split over $\mathbb{F}_q$. There is a unique unipotent class $C \subset G$ of dimension $2(h-1)$, where h is the Coxeter number. When all root lengths are the same this is the minimal unipotent class not containing the neutral element e; according to an unpublished theorem of Kostant, the number of $\mathbb{F}_q$-rational points of C is given by

$$(q^h - 1)(q^{e_1-1} + q^{e_2-1} + \cdots + q^{e_l-1}), \tag{3.1}$$

where $e_1, \ldots, e_l$ are the exponents of G.

When there are roots of different lengths, C is no longer the minimal unipotent class not containing e. However, one can check, using a case by case analysis, that $\bar{C} - \{e\}$ is rationally smooth (in the sense of [7, A1]) and that the number of $\mathbb{F}_q$-rational points of $\bar{C} - \{e\}$ is again given by (3.1).

Let $\mathscr{E}$ be the constant sheaf $\mathbb{Q}_l$ on C; we consider the corresponding sheaves $\mathscr{H}^i({}^\pi\mathscr{E})$ on $\bar{C}$. Using the method of ([7, Appendix]) it follows that, in general, its stalks at e are described by

$$\sum_{i>0} q^{i/2} \dim \mathscr{H}^i_e({}^\pi\mathscr{E}) = \sum_{i=1}^{l} q^{e_i-1}. \tag{3.2}$$

(c) Let ρ be a special representation of W (in the sense of [10, 11]), and let C_ρ be the unipotent class in G corresponding to ρ by Conjecture 2. Such a unipotent class is said to be special. This concept was introduced in [10] in a slightly different way, which is however, equivalent to the present definition.

Another definition for special unipotent classes was proposed by Spaltenstein [14]. Following Spaltenstein, we associate to a special unipotent class, C, the subset $\tilde{C} \subset U$ consisting of all elements g in the closure of C which are not in the closure of any special unipotent class $C' \neq C$, $C' \subset \bar{C}$. For example, if C is as in (b), then C is the minimal special unipotent class other than $\{e\}$ and we have $\tilde{C} = \bar{C} - \{e\}$. The results in (b) suggest

CONJECTURE 3. *If C_ρ is any special unipotent class in G, then $\tilde{C}_\rho$ is rationally smooth. If, moreover, G is simple and split over F_q, then the number of F_q-rational points of $\tilde{C}$ is given by a polynomial in q which depends only on ρ (i.e., it is independent of characteristic and is the same for types B_n and C_n).*

REFERENCES

1. C. W. CURTIS, N. IWAHORI, AND R. KILMOYER, Hecke algebras and characters of parabolic type of finite groups with *BN* pairs, *Inst. Hautes. Études Sci. Publ. Math.* **40** (1972), 81–116.
2. P. DELIGNE, letter to D. Kazhdan and G. Lusztig, 20 April 1979.
3. M. GORESKY AND R. MACPHERSON, Intersection homology theory. I. Topology.
4. M. GORESKY AND R. MACPHERSON, Intersection homology theory, II, preliminary version.
5. J. A. GREEN, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.
6. D. KAZHDAN, Proof of Springer's hypothesis, *Israel J. Math.* **28** (1977), 272–286.
7. D. KAZHDAN AND G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
8. D. KAZHDAN AND G. LUSZTIG, "Schubert Vareties and Poincaré Duality," Proc. Symp. Pure Math. Vol. 36, Amer. Math. Soc., Providence, R.I., 1980.
9. A. LASCOUX AND M. P. SCHUTZENBERGER, Sur une conjecture de H. O. Foulkes, *C. R. Acad. Sci. Paris Ser. A* **285** (1978), 323–324.
10. G. LUSZTIG, A class of irreducible representations of a Weyl group, *Proc. Kon. Ned. Akad. Wetensch. Ser. A* **82**, No. 3 (1979), 323–335.
11. G. LUSZTIG, Some problems in the representation theory of a finite Chevalley group, *in* "Proceedings, Symposium Pure Mathematics," Vol. 37, Amer. Math. Soc., Providence, R.I., 1980.
12. I. G. MACDONALD, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press (Clarendon), Oxford, 1979.
13. P. SLODOWY, "Four Lectures on Simple Groups and Singularities," Communications of the Mathematical Institute, Utrecht, 1980.
14. N. SPALTENSTEIN, forthcoming book on unipotent classes.
15. T. A. SPRINGER, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, *Invent. Math.* **36** (1976), 173–207.