What is a crystalline representation?

$\mathrm{Gal}(\overline{K}/K)$

$K/\mathbf{Q}_p$ finite, $\rho : \mathrm{Gal}_K \to G(\overline{\mathbf{Q}}_l)$.

"Crystalline" is the analogue for $l = p$ of "unramified" for $l \neq p$.

Crystalline representations form a Tannakian category.

The étale cohomology groups of smooth projective varieties over $\mathcal{O}_K$ are crystalline. $H^i(X_{\overline{K}}, \mathbb{Q}_p) \circlearrowleft \mathrm{Gal}_K$

The p-adic Galois representations associated to automorphic representations unramified at p are (conjecturally) crystalline.

e.g. for a newform f of level N, $\rho_f|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is crystalline if and only if $p \nmid N$.

The actual definition is technical.

Example: 1-dimensional representations of $\mathrm{Gal}_{\mathbf{Q}_p}$

e.g. unramified representations $\lambda_a : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{Gal}_{\mathbf{F}_p} \to \overline{\mathbf{Q}}_p^\times$, $\lambda_a(\mathrm{Frob}_p) = a$.

$H^2_{\text{ét}}(\mathbf{P}^1)$ gives the cyclotomic character $\varepsilon : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p) \to \mathbf{Z}_p^\times$, $g(\zeta_{p^\infty}) = \zeta_{p^\infty}^{\varepsilon(g)}$.

The crystalline characters of $\mathrm{Gal}_{\mathbf{Q}_p}$ are exactly the characters $\lambda_a \varepsilon^i$, $i \in \mathbf{Z}$.

Non-example: 1-dimensional representations of $\mathrm{Gal}_{\mathbf{Q}_p}$

Ramified finite order characters.

"ε^x" for $x \notin \mathbf{Z}$.

$\varepsilon : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times \xrightarrow{\theta} \mathbb{Z}_p^\times$

$\theta \in \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ $\quad \mathbb{Z}_p^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p)$

$\boxed{p > 2}$ $\quad \underset{\times}{1 + p\mathbb{Z}_p} \underset{\log}{\simeq} \underset{+}{\mathbb{Z}_p}$

The classification of crystalline characters, general K

Determined by $a \in \overline{\mathbf{Q}}_p^\times$, $(h_\sigma) \in (\mathbf{Z})^{\mathrm{Hom}_{\mathbf{Q}_p}(K, \overline{\mathbf{Q}}_p)}$, the Hodge–Tate weights.

e.g. $\lambda_a \varepsilon^i$ has all Hodge–Tate weights $-i$.

e.g. in talk B3: unramified = "crystalline of Hodge–Tate weight 0."

$GL_n \rightsquigarrow$ HT wts $0, 1, \dots, n-1$.

$[0, -1, \dots, 1-n.]$

Ordinary elliptic curves

$E/\mathbf{Q}_p$ elliptic curve with good ordinary reduction.

Associated crystalline $\mathrm{Gal}_{\mathbf{Q}_p}$-representation

$$\begin{pmatrix} \lambda_{1/a_p}\varepsilon & * \\ 0 & \lambda_{a_p} \end{pmatrix}$$

$H^1 \quad H^0 \quad H^2$

$a_p = 1 + p - \# E(\mathbb{F}_p)$

Conversely if $a \neq \pm 1$, every representation

HT wts $0, -1$.

$$\begin{pmatrix} \lambda_{1/a}\varepsilon & * \\ 0 & \lambda_{a} \end{pmatrix}$$

is crystalline.

(Not true for $a = \pm 1$, e.g. semistable elliptic curves.)

Ordinary crystalline deformations

Fix $\pm 1 \neq \overline{a} \in \mathbf{F}_p^\times$, $\overline{\rho} : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F}_p)$,

$$\overline{\rho} = \begin{pmatrix} \lambda_{1/\overline{a}}\overline{\varepsilon} & * \\ 0 & \lambda_{\overline{a}} \end{pmatrix}$$

non-split, so $\mathrm{End}_{\mathbf{F}_p[\mathrm{Gal}_{\mathbf{Q}_p}]}(\overline{\rho}) = \mathbf{F}_p$.

(Unique such $\overline{\rho}$ up to isomorphism by Tate local duality.)

$E[p]$ ↺ $\mathrm{Gal}_{\mathbb{Q}_p}$

+ Euler char formula.

Deformation problem: if A is Artin local with residue field $\mathbf{F}_p$, ask that $\rho_A : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(A)$ lifts $\overline{\rho}$ and has

$$\rho_A \cong \begin{pmatrix} \lambda_{1/\tilde{a}}\varepsilon & * \\ 0 & \lambda_{\tilde{a}} \end{pmatrix}$$

Ordinary crystalline deformations

Exercise: there is a universal deformation ρ to $\mathbf{Z}_p[[x]]$,

$$\rho \sim \begin{pmatrix} \lambda_{1/([a]+x)^{\varepsilon}} & * \\ 0 & \lambda_{[a]+x} \end{pmatrix}$$

(Again, uniqueness of such an extension is by Tate local duality.)

Fact: every "deformation of $\overline{\rho}$ to $\mathbf{Q}_p$" which is crystalline with Hodge–Tate weights $0, -1$ is of this form.

"Barsotti–Tate"

$\rho : G_{\mathbb{Q}_p} \to GL_2(\mathbb{Q}_p)$

$\overline{\rho}$

Ordinary crystalline deformations, general G

$K = \mathbb{Q}_p$

G is connected reductive split over $\mathbf{Z}_p$ with Borel B and maximal torus $T = B/U$, and $\overline{\rho} : \mathrm{Gal}_{\mathbf{Q}_p} \to B(\mathbf{F}_p)$.

Fix a dominant cocharacter $\lambda : \mathbf{G}_m \to T$ such that $\lambda \circ \varepsilon$ lifts $\overline{\rho}|_{I_K}$ (mod U).

Exercise: under a mild genericity condition, the corresponding weight λ ordinary deformation ring has dimension $\dim Z(G) + \dim G/B$.

$2 > 1 > 0$

$$\begin{pmatrix} \varepsilon^2\lambda_c & * & * \\ & \varepsilon\lambda_b & * \\ & & \lambda_a \end{pmatrix}$$

$\bar{a} \neq \bar{b} \neq \bar{c}$

$\bar{a} \neq \bar{c}$

$p > 3$

Crystalline representations which aren't ordinary

If $E/\mathbf{Q}_p$ is an elliptic curve with good supersingular reduction, the corresponding p-adic representation is irreducible, and crystalline with Hodge–Tate weights $0, -1$.

If $a_p = 0$ then it is the induction of a crystalline character of $\mathrm{Gal}_{\mathbf{Q}_{p^2}}$.

$\mathrm{Ind}_{\mathrm{Gal}_{\mathbb{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbb{Q}_p}} \chi$

If $a_p \neq 0$ I don't know how to describe it explicitly.

We would still like to define a crystalline deformation ring with Hodge–Tate weights $0, -1$.

Fontaine–Laffaille theory in Hodge–Tate weights $0, -1$

Assume $p > 2$. Let $\mathrm{MF}^1_{\mathrm{tor}}$ be the abelian category consisting of:

- A finite, torsion $\mathbf{Z}_p$-module M.
- A submodule $M^1 \subseteq M$.
- $\mathbf{Z}_p$-linear maps $\varphi : M \to M$, $\varphi^1 : M^1 \to M$, such that:
  - $\varphi|_{M^1} = p\varphi^1$.
  - $\varphi(M) + \varphi^1(M^1) = M$.

Theorem

(Fontaine–Laffaille) There is an exact, fully faithful functor from $\mathrm{MF}^1_{\mathrm{tor}}$ to the category of finite torsion $\mathbf{Z}_p$-modules with a continuous $\mathrm{Gal}_{\mathbf{Q}_p}$-action.
The essential image is the quotients T/T' of lattices in crystalline representations of Hodge–Tate weights $0, -1$.

Fontaine–Laffaille theory in Hodge–Tate weights $0, -1$

A submodule $M^1 \subseteq M$.

$\mathbf{Z}_p$-linear maps $\varphi : M \to M$, $\varphi^1 : M^1 \to M$, such that:

$\varphi|_{M^1} = p\varphi^1$.
$\varphi(M) + \varphi^1(M^1) = M$.

Examples over $\mathbf{F}_p$: λ_a corresponds to $M^1 = M$, $\varphi^1 = a$, and $\overline{\epsilon}$ is $M^1 = 0$, $\varphi = 1$.

In rank 2: $M = \mathbf{F}_p^2 = \mathbf{F}_p e_1 \oplus \mathbf{F}_p e_2$, $M^1 = \mathbf{F}_p e_1$,

$\varphi^1(e_1) = ae_1 + ce_2$, $\varphi(e_2) = be_1 + de_2$, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Exercise: classify the ordinary (i.e. reducible) modules.

Fontaine–Laffaille theory in Hodge–Tate weights $0, -1$

$M = \mathbf{F}_p^2 = \mathbf{F}_p e_1 \oplus \mathbf{F}_p e_2$, $M^1 = \mathbf{F}_p e_1$, $\varphi^1(e_1) = ae_1 + ce_2$, $\varphi(e_2) = be_1 + de_2$, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Supersingular (non-ordinary) case: replace $ae_1 + ce_2$ by e_2, so $a = 0$ and $c = 1$.

Claim: $d = 0$. Otherwise $e_2 + (b/d)e_1$ spans a submodule.

If we fix determinants, b is also determined, as we expect if we know the classification of irreducible $\mathbf{F}_p$-representations of $\mathrm{Gal}_{\mathbf{Q}_p}$.

Fontaine–Laffaille deformation rings

If A is Artin local with residue field $\mathbf{F}_p$, ask that $\rho_A : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(A)$ lifts $\overline{\rho}$ and is in the essential image of the Fontaine–Laffaille functor.

Exercise: in the ordinary case, this recovers what we had above.

Supersingular case: $M = \mathbf{F}_p^2 = \mathbf{F}_p e_1 \oplus \mathbf{F}_p e_2$, $M^1 = \mathbf{F}_p e_1$, $\varphi^1(e_1) = e_2$, $\varphi(e_2) = e_1$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Exercise (use Nakayama's lemma): if we fix determinants, deformations correspond to $M_A = A^2 = Ae_1 \oplus Ae_2$, $M_A^1 = Ae_1$, $\varphi^1(e_1) = e_2$, $\varphi(e_2) = e_1 + xe_2$, $x \in \mathfrak{m}_A$.

So again we have a universal deformation to $\mathbf{Z}_p[[x]]$.

Fontaine–Laffaille theory in general

$K = W(k)[1/p]$

Fontaine–Laffaille theory works for $K/\mathbf{Q}_p$ unramified, Hodge–Tate weights in any range of length at most $p-2$.

Replace $\mathbf{Z}_p$-modules with $W(k)$-modules, φ-semilinear, and filtration of length (at most) $p-2$.

Not hard to show for $G = \mathrm{GL}_n$ that the corresponding deformation ring is formally smooth of the expected dimension $\dim G/B$.

$\mathbb{Z}_p[[x_1, \ldots, x_{\dim G/B}]]$

$G = GL_n$.

Fontaine–Laffaille theory for general groups G?

Using the Tannakian perspective, can get a version of Fontaine–Laffaille theory by choosing a faithful representation, but this will often be very restrictive.

Booher–Levin: if G^{der} simply connected and $p \nmid \#\pi_1(G^{\mathrm{ad}})$, still have formally smooth crystalline deformation rings if $K/\mathbf{Q}_p$ is unramified and the pairings $\langle \mu, \alpha \rangle$ between the Hodge–Tate cocharacter μ and the roots α of G are all at most $p-2$.

(Proof of this does not use Fontaine–Laffaille theory, rather Kisin's theory.)

The derived enhancement

$\mathcal{X}$ "$= \mathrm{Spf}\,\mathcal{R}$" derived unrestricted deformation space for $\overline{\rho} : G_K \to \mathsf{GL}_n(\mathbf{F}_p)$.

$\mathcal{X}^{\mathrm{cl}} = \mathrm{Spf}\,\pi_0\mathcal{R}$ underlying classical space.

$\mathcal{X}^{\mathrm{cl}}_{\mathrm{FL}} \subseteq \mathcal{X}^{\mathrm{cl}}$ Fontaine–Laffaille subspace.

We define the derived Fontaine–Laffaille subspace $\mathcal{X}_{FL} \subseteq \mathcal{X}$ by $\mathcal{X}_{\mathrm{FL}} = \mathcal{X}^{\mathrm{cl}}_{\mathrm{FL}}$.

Why is this reasonable?

We expect that $\mathcal{X} = \mathcal{X}^{\mathsf{cl}}$ (i.e. that $\mathcal{X}^{\mathsf{cl}}$ is lci of the expected dimension), and we know that $\mathcal{X}_{\mathsf{FL}}$ is formally smooth (of the expected dimension).

This means that $\mathcal{X}_{\mathsf{FL}}^{\mathsf{global}} = \mathcal{X}^{\mathsf{global}} \times_{\mathcal{X}} \mathcal{X}_{\mathsf{FL}}$ has the expected tangent complex, i.e. Selmer group H^1_f.

(Should probably only need lci, but only the formally smooth case is considered in [GV].)

Beyond the Fontaine–Laffaille case?

In general, can define crystalline deformation rings via flat closure of the generic fibre.

They have the expected dimensions dim G/B, but in general little else is known about them (and they are usually not formally smooth).

It is unclear how to define derived enhancements ("equivalently" how to define an integral version without taking a flat closure).

The numerical coincidence

$F/\mathbf{Q}$ a number field, $\overline{\rho} : \mathrm{Gal}_F \to \hat{G}(\mathbf{F})$, G split/$\mathbf{Z}$ with dual $\hat{G}$. Fix regular Hodge–Tate weights at each $v|p$.

What is the "expected dimension" over $\mathbf{Z}_p$ of the deformation ring for representations which are regular of these Hodge–Tate weights?

The unrestricted local and global stacks have expected dimensions/$\mathbf{Z}_p$ given by the (negative) Euler characteristics χ_{F_v} and χ_F of the adjoint representation $\hat{\mathfrak{g}}$.

By the Euler characteristic formulas, for $v|p$,

$$\chi_{F_v} = [F_v : \mathbf{Q}_p] \dim G,$$
$$\chi_F = [F : \mathbf{Q}] \dim G - \sum_{v|\infty} \dim \hat{\mathfrak{g}}^{\mathrm{Gal}(\overline{F_v}/F_v)}.$$

The local (regular weight) crystalline stacks at $v|p$ have dimension $[F_v : \mathbf{Q}_p] \dim(\hat{G}/\hat{B})$.

The numerical coincidence

So the expected dimension of the global crystalline stack is

$$[F : \mathbf{Q}] \dim(\hat{G}/\hat{B}) - \sum_{v|\infty} \dim \hat{\mathfrak{g}}^{\mathrm{Gal}(\overline{F_v}/F_v)}.$$

For each $v|\infty$, this is at most $-(\mathrm{rank}(G_v) - \mathrm{rank}(K_v))$, with equality if and only if c_v is odd.

(e.g. for each complex place we get a contribution of $2\dim(\hat{G}/\hat{B}) - \dim \hat{G} = -\dim \hat{T}$.)

To get the expected dimension for the deformation ring, we have to add back on $\dim Z(\hat{G})$.

The numerical coincidence

In general the expected dimension is $-l_0$, where $l_0 = \mathrm{rank}(G_\infty) - \mathrm{rank}(A_\infty K_\infty)$.

(This is the “numerical coincidence”.)

In particular for F of signature (r_1, r_2) and $G = \mathrm{GL}_n$ we obtain $-l_0$ where $l_0 = r_1\lfloor(n+1)/2\rfloor + r_2 n - 1$.