# $Bun_G$ minicourse: introduction RAMpAGe seminar

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# The main theorem of [FS]

Fargues-Scholze gives a purely geometric construction of the automorphic-to-Galois direction of local Langlands:

#### Theorem (Fargues-Scholze)

Let  $F/\mathbb{Q}_p$  be a finite extension, and let G/F be a reductive group. To every irreducible smooth representation  $\pi$  of G(F) (with coefficients in  $\overline{\mathbb{Q}}_\ell$ ) there is an associated semisimple L-parameter  $\varphi \colon W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ .

The construction mirrors that of V. Lafforgue in the function field setting. The goal of this talk is to review the ideas leading up to this theorem.

## Local class field theory

The Langlands program begins with class field theory.

For a global field K, this refers to the reciprocity map  $\mathbf{A}_K^{\times}/K^{\times} \to \operatorname{Gal}(\overline{K}/K)^{\operatorname{ab}}$ .

For a p-adic field F, this refers to the isomorphism  $F^{\times} \to W_F^{\mathrm{ab}}$ .

The latter can be constructed via a Lubin-Tate formal group  $H/\mathcal{O}_{\breve{F}}$ : this is the unique connected p-divisible  $\mathcal{O}_F$ -module of height 1 and dimension 1. Adjoining the torsion of H to  $\breve{F}$  produces its maximal abelian extension.

(If  $F=\mathbb{Q}_p$ , then  $H=\mu_{p^\infty}$  and we recover the local Kronecker-Weber theorem.)

# Local Langlands for GL<sub>n</sub>

## Theorem (Harris-Taylor 2002)

There is a bijection  $\pi \mapsto \varphi_{\pi}$  from smooth irreducible representations of  $GL_n(F)$  to Frobenius-semisimple Weil-Deligne representations, compatible with L- and  $\varepsilon$ -factors of pairs.

A Weil-Deligne representation is a continuous homomorphism  $W_F \to \operatorname{GL}_n(\mathbb{C})$  together with a monodromy operator; these can be packaged together as an L-parameter  $W_F \times \operatorname{SL}_2(\mathbb{C}) \to \operatorname{GL}_n(\mathbb{C})$ .

The proof uses geometry: one studies the deformation space M of a connected p-divisible  $\mathcal{O}_F$ -module of dimension 1 and height n, together with some crucial global input from unitary Shimura varieties.

M is Lubin-Tate space; it is an example of a Rapoport-Zink space.

# Local Langlands for general G

Let G/F be a reductive group. Let  $\hat{G}$  be the (complex) dual group: this has the dual root datum as G.

#### Conjecture

There is a finite-to-one surjective map  $\pi \mapsto \varphi_{\pi}$  from irreducible admissible representations of G(F) to L-parameters  $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ , satisfying various desiderata.

The conjecture is known in many cases (see Tasho's talk), but one may still ask for a geometric construction following Harris-Taylor which is uniform in G. It is not immediately clear what the analogue of Lubin-Tate space should be in general.

## Interlude: Geometric Langlands

Let X be a curve over a finite field k, with function field K. (Level 1) automorphic forms on K are functions on

$$\prod_{\nu} \operatorname{GL}_n(\mathcal{O}_{K_{\nu}}) \backslash \operatorname{GL}_n(\mathbf{A}_K) / \operatorname{GL}_n(K),$$

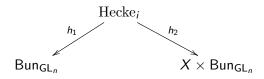
but this set has a meaning: it classifies rank n vector bundles on X.

This suggests we can *geometrize* the theory by replacing the set with the Artin stack  $Bun_{GL_n}$ , which assigns to a scheme S/k the groupoid of rank n vector bundles on  $X \times_k S$ .

In this geometrization, automorphic forms become complexes of sheaves on  $Bun_{GL_n}$ , that is, objects of  $D(Bun_{GL_n})$ .

## Hecke operators

We can also geometrize Hecke operators. For each  $i=1,\ldots,n$  we have a correspondence



classifying modifications  $\mathcal{E} \to \mathcal{E}'$  whose cokernel is a skyscraper sheaf supported at a point  $x \in X$  and isomorphic to  $\mathcal{O}_x^{\oplus i}$ . Then  $h_2$  is a  $\operatorname{Grass}(n,i)$ -bundle.

A Hecke eigensheaf is an object  $A \in D(Bun_{GL_n})$  satisfying

$$Rh_{2*}h_1^*A\cong \bigwedge^i\varphi\boxtimes A[i(n-i)]$$

for i = 1, ..., n. Here the eigenvalue  $\varphi$  is a local system of rank n.

# Geometric Langlands for GL<sub>n</sub>

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for  $i=1,\ldots,n$ . Here the eigenvalue  $\varphi$  is a local system of rank n.

#### Theorem (Frenkel-Gaitsgory-Vilonen 2001)

Given an irreducible rank n local system  $\varphi$  on X, there exists a perverse sheaf A on  $\mathsf{Bun}_{\mathsf{GL}_n}$  which is a Hecke eigensheaf with eigenvalue  $\varphi$ .

This is a geometrization of the result of L. Lafforgue (the global Langlands correspondence between automorphic representations  $\pi$  and Galois representations  $\varphi$ ). The Hecke eigensheaf property corresponds to the equality of L-factors

$$L_{\nu}(\pi,s)=L_{\nu}(\varphi,s)$$

for each place  $v \in |X|$ .

## More general groups, and the Satake isomorphism

The whole discussion can be generalized to the setting of a (split) reductive group G. Unramified automorphic forms are functions on

$$\prod_{\nu} G(\mathcal{O}_{K_{\nu}}) \backslash G(\mathbf{A}_{K}) / G(K).$$

For each  $v \in |X|$  the spherical Hecke algebra

$$\mathcal{H}_{\nu} = C_{c}(G(\mathcal{O}_{K_{\nu}}) \backslash G(K_{\nu}) / G(\mathcal{O}_{K_{\nu}}))$$

acts on automorphic forms; the eigenvectors for these correspond to automorphic representations of G.

#### Theorem (Satake isomorphism)

There is an isomorphism

$$\mathcal{H}_{\mathsf{v}} \to \operatorname{Rep} \hat{\mathsf{G}}$$

onto the representation ring of  $\hat{G}$ .

# More general groups, and the Satake isomorphism

#### Theorem (Satake isomorphism)

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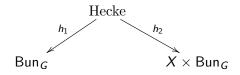
$$\mathcal{H}_{\mathsf{v}} \to \operatorname{Rep} \hat{\mathsf{G}}$$

onto the representation ring of  $\hat{G}$ .

Example for GL<sub>2</sub>: the indicator functions for the double cosets of  $\operatorname{diag}(\pi_{\nu},1)$  and  $\operatorname{diag}(\pi_{\nu},\pi_{\nu})$  correspond (up to some scalar factors) to the standard representation of GL<sub>2</sub> and its determinant, respectively.

#### Geometric Satake

Let  $\operatorname{Bun}_{\mathcal{G}}$  classify  $\mathcal{G}$ -torsors on  $\mathcal{X}$ . Let us define one (big) Hecke correspondence



classifying all modifications  $\mathcal{E} \to \mathcal{E}'$  at  $x \in X$ . Then the fiber of  $h_2$  over each  $v \in |X|$  is  $G(K_v)/G(\mathcal{O}_{K_v})$ . This is the set of k-points of the affine Grassmannian  $\mathrm{Gr}_G = LG/L^+G$ , an ind-scheme.

#### Theorem (Geometric Satake, Mirkovic-Vilonen)

There is an equivalence  $V \mapsto \mathcal{S}_V$  of symmetric monoidal categories between:

- Representations of  $\hat{G}$ , and
- $L^+G$ -equivariant perverse sheaves on  $Gr_G$ .

#### Geometric Satake

#### Theorem (Geometric Satake, Mirkovic-Vilonen)

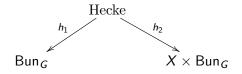
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- Representations of  $\hat{G}$ , and
- $L^+G$ -equivariant perverse sheaves on  $Gr_G$ .

We might even think of this theorem as a canonical model for  $\hat{G}$ : The symmetric monoidal category of  $L^+G$ -equivariant perverse sheaves on  $\mathrm{Gr}_G$  determines under the Tannakian formalism an algebraic group, which is exactly our  $\hat{G}$ .

# The geometric Langlands conjecture

Let X be a curve, let G be a split reductive group, and consider



#### Conjecture

Let  $\varphi$  be an irreducible  $\hat{G}$ -local system on X (that is,  $\varphi \colon \pi_1(X) \to \hat{G}$ ). There exists a nonzero object  $A_{\varphi}$  of  $D(\mathsf{Bun}_G)$  satisfying the eigensheaf property:

$$Rh_{2*}(h_1^*A_{\varphi}\otimes \mathcal{S}_V)\cong (r\circ \varphi)\boxtimes A_{\varphi}$$

for each representation  $r: \hat{G} \to \operatorname{Aut} V$ .

## Back to the p-adic world

How to carry these ideas from a curve X to a p-adic field F?

We could perhaps set X= "Spa  $\breve{F}/\phi$ ", where  $\breve{F}$  is the completion of the max unramified extension of F, and  $\phi$  is the Frobenius. The idea here is that representations of  $\pi_1(X)=W_F$  appear in local Langlands (rather than  $\operatorname{Gal}(\overline{F}/F)$ ).

Vector bundles on this X would be  $\check{F}$ -vector spaces equipped with a Frobenius-linear automorphism; ie, *isocrystals*. Already some of these (those with slopes in [0,1]) correspond to p-divisible  $\mathcal{O}_F$ -modules over  $\overline{k}$  (k=residue field of F).

# The object $X = \operatorname{\mathsf{Spa}} reve{\mathsf{F}}/\phi$

We want to define a stack  $\operatorname{Bun}_G$  which associates to a test object S the groupoid of G-torsors on  $X \times S$ . But what sort of object is our S, and what does  $X \times S$  mean?

Answer: S is drawn from the category Perf of perfectoid spaces over  $\overline{\mathbf{F}}_q$ .

When  $F = \mathbf{F}_q((t))$ , we can have  $S = \operatorname{Spa}(R, R^+)$  be an adic space over  $\overline{\mathbf{F}}_q$ , and then  $X \times S$  gets interpreted literally: it the quotient of the open disc over S modulo the Frobenius automorphism of S (cf. Hartl-Pink).

When F is p-adic, " $X \times S$ " no longer makes literal sense as a fiber product. Instead, it must be interpreted as the F-argues-F-ontaine curve.

## The Fargues-Fontaine curve

Let F be a p-adic field with uniformizer  $\pi$ , and let  $S = \operatorname{Spa}(R, R^+)$  be a perfectoid space over  $\overline{\mathbf{F}}_q$ , with pseudo-uniformizer  $\varpi$ . Let

"Spa 
$$R^+ imes \mathsf{Spa}\, \mathcal{O}_{\mathit{F}}$$
" := Spa  $W_{\mathcal{O}_{\mathit{F}}}(R^+)$ 

and let

$$Y_S = "S imes \mathsf{Spa}\,F" \subset "\mathsf{Spa}\,R^+ imes \mathsf{Spa}\,\mathcal{O}_F"$$

be the open subset defined by  $|\pi[\varpi]| \neq 0$ . The Fargues-Fontaine curve is the adic space

$$X_S = Y_S/\phi_S$$

where  $\phi_S$  is the Frobenius automorphism on S.

Warning: There is no morphism  $X_S \rightarrow S$ .

# The absolute Fargues-Fontaine curve

$$X_S = "S imes \operatorname{Spa} F"/\varphi_S$$

When  $S=\operatorname{Spa} C$  for C algebraically closed, the curve  $X_S=X_C$  has a schematic counterpart  $X_C^{\operatorname{sch}}$  which (despite not being finite type over a field) is very nice:  $X_C^{\operatorname{sch}}$  less one point is the spectrum of a PIDs, and total degrees of meromorphic functions are 0. There is a morphism  $X_C \to X_C^{\operatorname{sch}}$  satisfying a GAGA theorem (Fargues).

Closed points of  $X_C^{\rm sch}$  (="classical points of  $X_C$ ") correspond to Frobenius-classes of untilts of C to  $C^{\sharp}/F$ .

#### Vector bundles on the absolute curve

$$X_C=$$
 "Spa  $C imes$  Spa  $F$ "  $/arphi_S$ 

Let  $(V, \sigma_V)$  be an F-isocrystal: this means a  $\check{F}$ -vector space together with a Frobenius-linear automorphism  $\sigma_V$ .

Since  $\mathcal{O}_{\breve{F}} \subset W_{\mathcal{O}_F}(\mathcal{O}_C)$ , there is a morphism "Spa  $C \times$  Spa F"  $\to$  Spa  $\breve{F}$ , so we can use V to construct a (trivial) vector bundle on  $Y_C$ , which we can then descend to  $X_C$  using  $\sigma_V$ .

Thus we can talk about vector bundles  $\mathcal{O}(q)$  for any  $q \in \mathbb{Q}$ .

## Theorem (Fargues-Fontaine, Fargues)

Every vector bundle on  $X_C$  is isomorphic to a direct sum of  $\mathcal{O}(q)s$ . More generally, isomorphism classes of G-torsors on  $X_C$  are in bijection with the Kottwitz set

$$B(G) = G(\check{F})/\sigma$$
-conjugacy

# The relative Fargues-Fontaine curve, and $Bun_G$

More generally, if  $S \in \text{Perf}$ , we have the relative curve  $X_S$ . Degree 1 divisors on  $X_S$  correspond to Frobenius-classes of untilts of S over F.

#### **Definition**

 $Bun_G$  is the stack assigning to S the groupoid of G-torsors on  $X_S$ .

#### Theorem (FS)

 $Bun_G$  has the following properties.

- **1** Bun<sub>G</sub> is a smooth Artin v-stack of dimension 0.
- $|\mathsf{Bun}_G| \cong B(G)$ . (Viehmann  $\implies$  homeomorphism.)
- ullet The semistable locus  $\operatorname{Bun}_G^{\operatorname{ss}}$  is dense and open, and given by

$$\mathsf{Bun}_G^{\mathrm{ss}} = \coprod_{b \in B(G)_{\mathrm{ss}}} [*/\underline{G_b(F)}].$$

## Bun<sub>G</sub> and local Langlands

Ultimately, [FS] uses the geometry of  $\operatorname{Bun}_{\mathcal{G}}$  to construct  $\pi \mapsto \varphi_{\pi}$ .

Leaving the construction itself to the later talks, we will now discuss two directions related to local Langlands:

- A (very general) Kottwitz conjecture describing the cohomology of spaces generalizing the Rapoport-Zink spaces,
- ② Fargues' conjecture, asserting the existence of an eigensheaf  $A_{\varphi}$  on Bun $_{G}$  for each L-parameter  $\varphi$ .

# Example with $F = \mathbb{Q}_p$ and $\mathsf{GL}_2$

Every rank 2 vector bundle on  $X_C$  is either  $\mathcal{O}(m) \oplus \mathcal{O}(n)$  or else  $\mathcal{O}(k/2)$  for k odd.

There is one basic vector bundle in each degree: the  $\mathcal{O}(m)^{\oplus 2}$  and the  $\mathcal{O}(k/2)$ . Their automorphism groups are  $\mathrm{GL}_2(\mathbb{Q}_p)$  and  $D^{\times}$  respectively. D =quaternion algebra).

Thus the degree 0 part of  $\mathsf{Bun}_{\mathcal{G}}$  has dense open  $[*/\underline{\mathsf{GL}_2(F)}]$  corresponding to the trivial vector bundle,  $\mathcal{O}^{\oplus 2}$  which can degenerate to  $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ ,  $\mathcal{O}(2) \oplus \mathcal{O}(-2)$ , etc.

The degree 1 part of  $\mathsf{Bun}_G$  has dense open subset  $[*/\underline{D}^\times]$  corresponding to  $\mathcal{O}(1/2)$ , which can degenerate to  $\mathcal{O}(1) \oplus \mathcal{O}$ , etc.

# Contact with *p*-divisible groups

Continuing with this example, for  $S = \text{Spa}(R, R^+)$  we have

$$H^0(X_S, \mathcal{O}(1/2)) \cong B^+_{\mathrm{cris}}(R)^{\phi^2=p} \cong \tilde{H}_0(R^\circ)$$

where  $H_0/\overline{\mathbf{F}}_q$  is the formal group of height 1/2 and dimension 1 (Fontaine), and  $\tilde{H}_0$  means  $\varprojlim_p H_0$ .

If  $S^{\sharp}$  is an untilt of S, corresponding to a degree 1 divisor  $D_{S^{\sharp}} \subset X_{S}$ , and H is a deformation of  $H_{0}$  to  $R^{\sharp \circ}$  then we get an isomorphism

$$\tilde{H}(R^{\sharp \circ}) \cong \tilde{H}_0(R^{\circ}).$$

The logarithm exact sequence

$$0 o V_{
ho} H o ilde{H}(R^{\sharp \circ}) \overset{\mathsf{log}}{ o} R^{\sharp} o 0$$

is the result of applying  $H^0$  to

$$0 \to \textit{V}_{\textit{p}}\textit{H} \otimes \mathcal{O} \to \mathcal{O}(1/2) \to \textit{i}_{\textit{D}_{\textit{S}\sharp}*}\mathcal{O}_{\textit{S}\sharp} \to 0.$$

# Contact with *p*-divisible groups

So each time we have a deformation of  $H_0$  to H, we get an exact sequence

$$0 \to V_p H \otimes \mathcal{O} \to \mathcal{O}(1/2) \to i_{D_{S\sharp}*} \mathcal{O}_{S\sharp} \to 0.$$

#### Theorem (Scholze-W.)

Let  $H_0/\mathbf{F}_q$  be a p-divisible group, and let  $\mathcal{O}(H_0)$  be the vector bundle associated to its isocrystal. Given a perfectoid space S in characteristic 0, the following categories are equivalent:

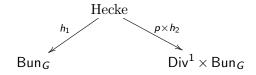
- Deformations of H<sub>0</sub> to S up to isogeny,
- Modifications on  $X_{S^{\flat}}$  at the divisor  $D_S$ :

$$0 \to \mathcal{T} \to \mathcal{O}(H_0) \to i_{D_S*} \mathrm{Lie}(H_0) \to 0$$

where  $\mathcal{T}$  is semistable of slope 0 (ie, trivial over every geometric point).

## Generalization to spaces of shtukas

The Hecke correspondence in the context of  $Bun_G$  is:



It parametrizes modifications of G-torsors along a divisor on the curve; those are parametrized by  $\mathrm{Div}^1 = \mathrm{Spd}\, \check{F}/\varphi$ .

Let  $b \in G(F)_{\text{bas}}$ . Let  $\mathsf{Sht}_{G,b} \to \mathsf{Div}^1$  be the pullback through  $h_1 \times h_2$  of the map  $* \to \mathsf{Bun}_G \times \mathsf{Bun}_G$  corresponding to  $(\mathcal{E}_1, \mathcal{E}_b)$ .

Thus if  $S/\breve{F}$  is a perfectoid space, then  $\operatorname{Sht}_{G,b}(S)$  parametrizes modifications  $\mathcal{E}_1 \dashrightarrow \mathcal{E}_b$  along  $D_S$ . Then  $\operatorname{Sht}_{G,b}$  has an action of  $G(F) \times G_b(F)$ .

## The Kottwitz conjecture for shtuka spaces

The cohomology of  $\mathsf{Sht}_{G,b}$  suggests a means of transfering representations from  $G_b(F)$  to  $G(F) \times W_F$ .

Let  $V \in \text{Rep } \hat{G}$ , and let  $S_V$  be the corresponding  $\mathbb{Z}_{\ell}$ -sheaf on  $\text{Sht}_{G,b}$ : this is equivariant for all actions.

#### **Definition**

Let  $\rho$  be a smooth irreducible representation of  $G_b(F)$  with coefficients in  $\overline{\mathbb{Q}}_{\ell}$ . Define

$$R\Gamma(G,b,V)[\rho] = \varinjlim_{K \subset G(F)} R\mathrm{Hom}_{G_b(F)} (R\Gamma_c(\mathsf{Sht}_{G,b}/K,\mathcal{S}_V) \otimes \overline{\mathbb{Q}}_\ell,\rho),$$

a derived representation of G(F) carrying an action of  $W_F$ .

#### Theorem (FS)

 $R\Gamma(G, b, V)[\rho]$  is admissible as a G(F)-module.

## The Kottwitz conjecture for shtuka spaces

Given G, b, V and a smooth irreducible representation  $\rho$  of  $G_b(F)$ , we have constructed a G(F)-admissible representation  $R\Gamma(G, b, V)[\rho]$  carrying an action of  $W_F$ . How does this interact with the (conjectural) Langlands correspondence?

#### Conjecture

Assume the local Langlands conjecture (as Tasho will describe it). Suppose  $\rho$  has supercuspidal L-parameter  $\varphi$ . Let  $S_{\varphi}$  be the centralizer of  $\varphi$  in  $\hat{G}$ . As classes in the Grothendieck group of  $G(F) \times W_F$ , we have

$$R\Gamma(G, b, V)[\rho] = \sum_{\pi \in \Pi_{\varphi}(G)} \pi \boxtimes \operatorname{Hom}_{S_{\varphi}}(\delta_{\pi, \rho}, r_{V} \circ \varphi).$$

Here  $\Pi_{\varphi}(G)$  is the L-packet of representations of G belonging to  $\varphi$ , and  $\delta_{\pi,\rho}$  is the relative position.

# The Kottwitz conjecture for shtuka spaces

#### Conjecture

For  $\rho$  in a supercuspidal L-packet:

$$R\Gamma(G, b, V)[\rho] = \sum_{\pi \in \Pi_{\varphi}(G)} \pi \boxtimes \operatorname{Hom}_{S_{\varphi}}(\delta_{\pi, \rho}, r_{V} \circ \varphi).$$

#### Theorem (Hansen-Kaletha-W.)

Assume the local Langlands conjecture. The equality in the conjecture is true, up to ignoring the  $W_F$ -action.

The proof will be explained in the 3rd and 4th talks. It reduces everything to a Lefschetz trace formula.

# $D(Bun_G)$ and Fargues' conjecture

Fargues' original article on geometrization put forward a conjecture about  $Bun_G$  which is exactly in line with geometric Langlands:

## Conjecture (Fargues)

(Choose a Whittaker datum.) Let  $\varphi$  be a discrete L-parameter. There exists an object  $A_{\varphi} \in D(\operatorname{Bun}_G, \overline{\mathbb{Q}}_{\ell})$  carrying an action of  $S_{\varphi}$  (=centralizer of  $\varphi$  in  $\hat{G}$ ) which is a Hecke eigensheaf with eigenvalue  $\varphi$ . The restriction of  $A_{\varphi}$  to  $[*/\underline{G_b(F)}]$  decomposes as a direct sum of  $\pi_b$  over the L-packet  $\Pi_{G_b}(\varphi)$ .

Stating this conjecture precisely requires defining  $D(\mathsf{Bun}_G, \overline{\mathbb{Q}}_\ell)$  correctly, which is highly nontrivial (and requires condensed mathematics).

Actually [FS] constructs a candidate for  $A_{\varphi}$  using the *spectral action*, but it is not clear that  $A_{\varphi} \neq 0$ !

# $D(Bun_G)$ and Fargues' conjecture

#### Conjecture (Fargues)

(Choose a Whittaker datum.) Let  $\varphi$  be a discrete L-parameter. There exists an object  $A_{\varphi} \in D(\operatorname{Bun}_G, \overline{\mathbb{Q}}_{\ell})$  carrying an action of  $S_{\varphi}$  (=centralizer of  $\varphi$  in  $\hat{G}$ ) which is a Hecke eigensheaf with eigenvalue  $\varphi$ . The restriction of  $A_{\varphi}$  to  $[*/\underline{G_b(F)}]$  decomposes as a direct sum of  $\pi_b$  over the L-packet  $\Pi_{G_b}(\varphi)$ .

#### Theorem (Anschütz-le Bras)

The conjecture is true for  $G = GL_n$  and irreducible  $\varphi$ .