TENSOR CATEGORIES (AFTER P. DELIGNE)

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Abstract. This is my talk at the MIT Lie Groups Seminar. I give an exposition of a recent paper by P. Deligne "Catégories tensorielles".

1. Introduction

In a recent preprint [1] P. Deligne proved that any tensor (=rigid symmetric monoidal abelian) category over an algebraically closed field of characteristic 0 satisfying certain very mild conditions comes from representations of an affine super group (see exact statement below). This result is interesting by itself and also has applications in the theory of Hopf algebras (see [4] and especially [5]) since it allows to classify completely for example finite-dimensional triangular Hopf algebras.

In this note I give an exposition of Deligne's proof oriented on representation theorists. So I tried to be as elementary as possible. It is assumed that the reader knows basic notions of the category theory and is familiar with representations of the symmetric groups over the complex numbers. This paper does not contain anything original (except, possibly, mistakes) and can not be considered as a substitute for Deligne's paper. In many cases I leave proofs (and even definitions) to the reader, in all such cases the reader is referred to Deligne's exellent exposition.

2. Main Theorem

2.1. **Tensor categories.** Let k be an algebraically closed field of characteristic 0.

Definition. A tensor category $\mathcal{A}$ is a small abelian $k-$linear category endowed with a biexact $k-$linear functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, associativity, commutativity constraints, unit object $\mathbf{1}$ such that $(\mathcal{A}, \otimes)$ is a symmetric monoidal category (see e.g. [6]). Furthermore, the category $\mathcal{A}$ is assumed to be rigid, that is for any object $X \in \mathcal{A}$ there exists object $X^\vee$ and morphisms $coev : \mathbf{1} \to X \otimes X^\vee$ and $ev : X^\vee \otimes X \to \mathbf{1}$ such that the compositions $X \to X \otimes X^\vee \otimes X \to X$ and $X^\vee \to X^\vee \otimes X \otimes X^\vee$ are identity morphisms. Finally it is assumed that $\mathrm{End}(\mathbf{1}) = k$ (or equivalently $\mathbf{1}$ is a simple object of $\mathcal{A}$, see [2]).

One says that a category $\mathcal{A}$ is finitely $\otimes-$generated if there is an object $X \in \mathcal{A}$ such that any object of $\mathcal{A}$ is isomorphic to a subquotient of an object which is a direct sum of objects $X^{\otimes n}$. Such an object X is called a $\otimes-$generator of $\mathcal{A}$.

Examples. (0) The category Vec of finite dimensional vector spaces over k is a tensor category. This category is obviously finitely $\otimes-$generated.

(1) The category $\mathrm{Rep}(G)$ of finite dimensional representations of an affine algebraic group G over k. Any faithful representation X of G is $\otimes-$generator of this category.

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(2) The category $\mathrm{Rep}(G)$ of finite dimensional discrete (= factorizing through a finite quotient) representations of a profinite group G. This category is finitely $\otimes-$generated if and only if G is finite.

(3) The category $s\mathrm{Vec}$ of finite dimensional super vector spaces. Recall that the objects of $s\mathrm{Vec}$ are $\mathbb{Z}/2\mathbb{Z}-$graded spaces $V_0 \oplus V_1$ and commutativity morphism is given by $x \otimes y \mapsto (-1)^{deg(x)deg(y)} y \otimes x$ where $x \in V_{deg(x)}$ and $y \in V_{deg(y)}$.

(4) Let $\mathcal{O}(G)$ be a supercommutative super Hopf algebra finitely generated as an algebra (e.g. the exterior algebra $\wedge^\bullet(V)$ of a vector space V with comultiplication $x \mapsto x \otimes 1 + 1 \otimes x$ is a super Hopf algebra, but not a usual Hopf algebra). One considers $\mathcal{O}(G)$ as the functions algebra on an affine "super group" G. Then the category $\mathrm{Rep}(G)$ of finite dimensional (super) comodules over $\mathcal{O}(G)$ is a finitely $\otimes-$generated tensor category.

(5) Let G be a super group and let $\varepsilon \in G(k)$ (that is ε is a homomorphism $\mathcal{O}(G) \to k$) such that $\varepsilon^2 = 1$ (that is the map $\mathcal{O}(G) \to k$ $h \mapsto \varepsilon(h_{(1)})\varepsilon(h_{(2)})$ coincides with the counit ϵ; here $h \mapsto h_{(1)} \otimes h_{(2)}$ is the comultiplication) and inner automorphism of G induced by ε is just the parity automorphism (that is automorphism of $\mathcal{O}(G)$ $h \mapsto \varepsilon(h_{(1)})h_{(2)}\varepsilon(S(h_{(3)}))$ coincides with $h \mapsto (-1)^{deg(h)}h$; here $h \mapsto h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ is twice iterated comultiplication and $h \mapsto S(h)$ is the antipode). Consider the category $\mathrm{Rep}(G, \varepsilon)$ consisting of objects V of $\mathrm{Rep}(G)$ such that ε acts on V by the parity automorphism. The category $\mathrm{Rep}(G)$ is a tensor category. It is finitely $\otimes-$generated if and only if $\mathcal{O}(G)$ is finitely generated algebra.

Exercise. Show that examples (0), (1), (3), (4) above are special cases of example (5). (Hint: for example (4) consider semidirect product of G and of $\mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts on G via parity automorphism).

2.2. **Schur functors.** For any object $X \in \mathcal{A}$ we have a natural action of the symmetric group S_n on the object $X^{\otimes n}$ induced by commutativity isomorphisms. For any partition λ of n let V_λ denote the corresponding irreducible representation of S_n. We have the diagonal action of S_n on $V_\lambda \otimes X^{\otimes n}$.

Definition. The *Schur functor* S_λ is defined by the formula

$$S_\lambda(X) = (V_\lambda \otimes X^{\otimes n})^{S_n} := (\sum_{g \in S_n} g))(V_\lambda \otimes X^{\otimes n}).$$

For example if $\lambda = n$ then $S_\lambda(X) = Sym^n(X)$ is just symmetric power and if $\lambda = 1 + 1 + \cdots + 1$ then $S_\lambda(X) = \wedge^n(X)$ is the exterior power.

Exercise. Let $V = V_0 \oplus V_1$ be a superspace of dimension $p|q$, that is $\dim V_0 = p$ and $\dim V_1 = q$. Prove that $S_\lambda(V) = 0$ if and only if there is $i > p$ such that $\lambda_i > q$ (in other words $S_\lambda(V) \neq 0$ if and only if the Young diagram of λ lies in the union of two strips of width p in "λ_i-direction" and of width q in "$i-$direction"). Hint: One has the following formula:

$$S_\lambda(X \oplus Y) = \bigoplus_{\mu,\nu} (S_\mu(X) \otimes S_\nu(Y))^{a^\lambda_{\mu,\nu}}$$

where the summation runs over partitions μ, ν such that $|\mu| + |\nu| = |\lambda|$ and $a^\lambda_{\mu,\nu}$ are Littlewood-Richardson coefficients (additional exercise: state a similar formula for $S_\lambda(X \otimes Y)$).

In particular for any object X of the category $\mathrm{Rep}(G, \varepsilon)$ there exists λ such that $S_\lambda(X) = 0$.

2.3. **Main Theorem.** Here is the main result:

Theorem. Let $\mathcal{A}$ be a finitely $\otimes-$generated tensor category such that for any $X \in \mathcal{A}$ there is λ with $S_\lambda(X) = 0$. Then $\mathcal{A}$ is equivalent as tensor category to $\mathrm{Rep}(G, \varepsilon)$ for some supergroup G.

Remarks. (i) The set of objects $X \in \mathcal{A}$ annihilated by some (depending on X) Schur functor is stable under direct sums, tensor products, taking dual, subquotients, extensions; any such object has finite length. We leave this as an exercise to the reader.

(ii) The condition that an object X is annihilated by some Schur functor is equivalent to the existence of N such that $length(X^{\otimes n}) \leq N^n$ for all $n \geq 0$ (in one direction this is a consequence of the Theorem and in other direction the decomposition $X^{\otimes n} = \oplus_\lambda V_\lambda \otimes S_\lambda(X)$ shows that if $S_\lambda(X) \neq 0$ for all λ then $length(X^{\otimes n}) \geq \sum_\lambda \dim V_\lambda \geq (\sum_\lambda (\dim V_\lambda)^2)^{1/2} = \sqrt{n!}$). This condition is automatically satisfied if category $\mathcal{A}$ has only finitely many simple objects (for example if $\mathcal{A}$ is the category of representations of a finite dimensional (weak) Hopf algebra).

As an immediate consequence one gets

Corollary. Assume that $\mathcal{A}$ is semisimple with finitely many simple objects. Then $\mathcal{A}$ is equivalent to category $\mathrm{Rep}(G, \varepsilon)$ where G is a finite group and $\varepsilon \in G$ is a central element of order at most 2.

2.4. **Strategy of the proof.** We begin with the following

Definition. Let $\mathcal{A}$ and $\mathcal{A}'$ be two tensor categories. A *tensor functor* from $\mathcal{A}$ to $\mathcal{A}'$ is a functor $F : \mathcal{A} \to \mathcal{A}'$ endowed with an isomorphism $\mathbf{1} \to F(\mathbf{1})$ and functorial isomorphisms $F(X) \otimes F(Y) \to F(X \otimes Y)$ compatible with associativity, commutativity and unit constraints.

The main difficulty of Deligne's Theorem is the following: the tensor category $\mathrm{Rep}(G, \varepsilon)$ has an additional structure, the super fiber functor (that is tensor functor to the category $s\mathrm{Vec}$). Conversely it is not very hard (and is a standard exercise in Tannakian formalism) to prove that if a category $\mathcal{A}$ admits a super fiber functor then it is equivalent to the category $\mathrm{Rep}(G, \varepsilon)$, see [2] for the case of usual (not super) fiber functors. So we are reduced to showing that the category $\mathcal{A}$ admits a super fiber functor. For this one generalizes the notion of super fiber functor in the following way: for a supercommutative algebra R one defines $R-$fiber functor to be exact tensor functor to the category of $R-$modules with tensor product over R as a tensor product (note that strictly speaking category of $R-$modules is not a tensor category in our sense since $\mathrm{End}(\mathbf{1}) = R \neq k$). Then one shows (this is a key result) that category $\mathcal{A}$ admits $R-$fiber functor for some (very big) algebra R. Then using standard technique from algebraic geometry one deduces that category $\mathcal{A}$ admits super fiber functor over k.

2.5. **Some counterexamples.** It is not trivial to construct an example of tensor category with object V such that $S_\lambda(V) \neq 0$ for any λ. Here we present two such examples.

Orthosymplectic example. ([3]) Let t be an indeterminate. Consider the following category $OSP_{\mathbb{Q}(t)}$:

Objects: finite sets; tensor product: disjoint union; morphisms: $\mathrm{Hom}(X, Y)$ is free $\mathbb{Q}(t)-$module generated by bordisms from X to Y (= 1-dimensional manifolds with boundary $X \sqcup Y$) modulo the relation [bordism $\sqcup$ circle] = t[bordism]; the composition of morphisms is induced by the composition of bordisms.

Now we define the category OSP_t to be the Karoubian envelope of the category $OSP_{\mathbb{Q}(t)}$.

The category GL_t. ([3, 2]) This example is completely analogous to the previous example except that we consider oriented finite sets (that is finite sets X together with map $\varepsilon : X \to \{\pm 1\}$) and oriented bordisms. The resulting category is denoted GL_t.

The categories GL_t and OSP_t are abelian semisimple categories; the simple objects of these categories are absolutely simple.

Exercise. Let V be the object of GL_t or OSP_t corresponding to a finite set with one element. Show that $S_\lambda(V) \neq 0$ for any λ.

3. Existence of a super fiber functor

The main point of Deligne's proof is the possibility to imitate affine algebraic geometry in the category $\mathcal{A}$. In other words the notion of an algebra (in what follows "algebra" means a nonzero associative commutative algebra with unit) makes sense in the category $\mathcal{A}$: an algebra in $\mathcal{A}$ is an ind-object A of $\mathcal{A}$ endowed with multiplication morphism $A \otimes A \to A$ satisfying certain axioms (we leave as an exercise for the reader to state precisely these axioms). For example for any object $X \in \mathcal{A}$ one defines the symmetric algebra $Sym^*(X)$ (again we leave details to the reader). For an algebra A one easily defines the notions of $A-$modules, homomorphisms and tensor products over A.

3.1. **Key Lemma.** Let A be an algebra in $\mathcal{A}$. The notion of a rigid $A-$module is defined exactly as before. Note that a direct summand of a rigid $A-$module is rigid. Also symmetric powers of modules over A are defined.

An $A-$algebra B is an algebra B in $\mathcal{A}$ together with a homomorphism $A \to B$. For any $A-$module M one defines its *extension of scalars* to be $B-$module $M_B := M \otimes_A B$. In particular $\mathbf{1}_B$ is B itself considered as $B-$module. Clearly, extension of scalars is a functor. It is obvious that extension of scalars of a rigid module is again rigid module. The Schur functors are defined over an algebra A and commute with extension of scalars.

Key Lemma. Let M be a rigid $A-$module. The existence of $A-$algebra B such that M_B has $\mathbf{1}_B$ as a direct summand is equivalent to the condition $Sym^n_A(M) \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. One direction is trivial since the natural map $Sym^n_A(X) \to Sym^n_A(X \oplus Y)$ is injective for any X and Y.

Let us prove another direction. We are looking for an $A-$algebra B and two maps $\alpha : \mathbf{1}_B \to M_B$ and $\beta : M_B \to \mathbf{1}_B$ such that $\beta\alpha = \mathrm{Id}$. Now idea is very simple: let us try to find universal algebra with such properties. For this we translate our conditions into the language of algebra B. First the map of $B-$modules $\beta : M_B \to \mathbf{1}_B$ is the same as the map of $A-$modules $M \to \mathbf{1}_B$ or $v : M \to B$ which is equivalent to the map of $A-$algebras $v_{alg} : Sym^*_A(M) \to B$. Similarly, to give the map $\alpha : \mathbf{1}_B \to M_B$ is the same as to give the map of $A-$modules $u : M^\vee \to B$ or the map of $A-$algebras $u_{alg} : Sym^*_A(M^\vee) \to B$. The equation $\beta\alpha = \mathrm{Id}$ is translated to the following condition: the map

$$\mathbf{1}_A \xrightarrow{coev} M \otimes M^\vee \xrightarrow{v \otimes u} B \otimes B \xrightarrow{multiplication} B \qquad (*)$$

coincides with the map $A \to B$ coming from the fact that B is $A-$algebra.

Summarizing we can say that universal algebra B can be described by "generators" $M \oplus M^\vee$ and "relation": the map $(*)$ coincides with the unit morphism $A \to B$ (so algebra B is a quotient of algebra $Sym_A^*(M \oplus M^\vee) = Sym_A^*(M) \otimes_A Sym_A^*(M^\vee)$ by the "ideal" generated by the morphism $1 - \delta : \mathbf{1}_A \to Sym_A^*(M \oplus M^\vee)$ where $1 : \mathbf{1}_A \to Sym_A^0(M) \otimes Sym_A^0(M^\vee)$ is the unit morphism and $\delta : \mathbf{1}_A \to Sym_A^1(M) \otimes Sym_A^1(M^\vee)$ is the coevaluation morphism). The only problem now is to show that algebra described by such generators and relations is *nonzero*.

For this it is enough to show that 1 does not lie in the ideal generated by $1 - \delta$. Assume converse, that is $1 = (1-\delta)x$ (here x is a morphism $\mathbf{1}_A \to Sym_A^*(M \oplus M^\vee)$). Algebra $Sym_A^*(M) \otimes_A Sym_A^*(M^\vee)$ has natural grading by the group $\mathbb{Z} \oplus \mathbb{Z}$; 1 lies in $(0,0)-$graded component and δ lies in $(1,1)-$graded component. Decompose x in the sum of graded component $x = x_{0,0} + x_{1,0} + x_{0,1} + x_{1,1} + \ldots$. Clearly, we can assume that $x_{p,q} = 0$ for $p \neq q$ (since if x is solution of $1 = (1-\delta)x$ then $x' = x_{0,0} + x_{1,1} + \ldots$ is a solution too). Now the equation $1 = (1-\delta)x$ is equivalent to the following graded equations:

$$x_{0,0} = 1;\ x_{1,1} - \delta x_{0,0} = 0;\ x_{2,2} - \delta x_{1,1} = 0; \ldots$$

This means that $x_{p,p} = \delta^p$ and $\delta^n = 0$ for large enough n since the sum $x = x_{0,0} + x_{1,1} + \ldots$ is finite. Conversely, if $\delta^n = 0$ then $1 = (1-\delta)(1 + \delta + \ldots + \delta^{n-1})$.

So the universal algebra B is nontrivial if and only if $\delta^n \neq 0$ for all n. Now $\delta^n : \mathbf{1}_A \to Sym_A^n(M) \otimes_A Sym_A^n(M^\vee)$ equals to the coevaluation map $\mathbf{1}_A \to Sym_A^n(M) \otimes_A Sym_A^n(M)^\vee$ and is zero if and only if $Sym_A^n(M) = 0$. The Lemma is proved. □

3.2. **Local properties.** One says that a system of objects and morphisms has some property *locally* if this property holds after some extension of scalars (here the word "locally" refers to the topology fppf — fidelement plat de presentation finie). For example two objects X and Y of the category $\mathcal{A}$ are locally isomorphic if there exists a (nonzero) algebra A such that $X \otimes A$ is isomorphic to $Y \otimes A$ as $A-$modules.

Exercise. Let G be an affine algebraic group. Two objects $X, Y \in \mathrm{Rep}(G)$ are locally isomorphic if and only if $\dim X = \dim Y$. Hint: consider algebra of functions on the affine variety $Isom(X,Y)$ — open subset of the vector space $X^* \otimes Y$ consisting of isomorphisms; this variety has natural $G-$action, so algebra of functions has natural structure of algebra in $\mathrm{Rep}(G)$.

Example. Any short exact sequence in the category $\mathcal{A}$ locally splits. Indeed first we reduce ourselves to exact sequence of the form $0 \to X \to Y \xrightarrow{b} \mathbf{1} \to 0$ by the standard argument: splitting of the sequence $0 \to M \to N \to P \to 0$ is equivalent to the splitting of the sequence $0 \to M \otimes P^\vee \to E \to \mathbf{1} \to 0$ where E is the preimage of $\mathbf{1} \in P \otimes P^\vee$ under the map $N \otimes P^\vee \to P \otimes P^\vee$. Now one proceeds similarly (but easier) to the Key Lemma: it is enough to show that algebra $Sym^*(Y^\vee)/(b^t - 1)$ (where $b^t : \mathbf{1} \to Y^\vee$ is the morphism dual to $b : Y \to \mathbf{1}$) is nonzero; as before this is equivalent to nonnilpotency of b^t which is obvious.

Now suppose that category $\mathcal{A}$ contains an object $\bar{\mathbf{1}}$ such that $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}$ is isomorphic to $\mathbf{1}$ and the commutativity morphism $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}} \to \bar{\mathbf{1}} \otimes \bar{\mathbf{1}}$ is the multiplication by (-1). Such an object allows to define tensor functor $F : s\mathrm{Vec} \to \mathcal{A}$ by the formula $F(V) = V^0 \otimes \mathbf{1} \oplus V^1 \otimes \bar{\mathbf{1}}$ which is an equivalence of the category $s\mathrm{Vec}$ and the subcategory $< \mathbf{1}, \bar{\mathbf{1}} >$ of $\mathcal{A}$ consisting of direct sums of $\mathbf{1}$ and $\bar{\mathbf{1}}$.

Proposition. For an object $X \in \mathcal{A}$ the following conditions are equivalent:

(i) There exist p and q such that X is locally isomorphic to $\mathbf{1}^p \oplus \bar{\mathbf{1}}^q$.

(ii) There exists the Schur functor S_λ such that $S_\lambda(X) = 0$.

Proof. (i)⇒(ii) is trivial. Let us prove that (ii) ⇒ (i). Assume that after some extension of scalars we have $X_A = \mathbf{1}_A^r \oplus \bar{\mathbf{1}}_A^s \oplus S$ for some $A-$module S. Consider three cases:

(a) $Sym_A^n(S) \neq 0$ for all n. Then using Key Lemma we can find $A-$algebra B such that S_B has $\mathbf{1}_B$ as a direct summand and we get decomposition $X_B = \mathbf{1}_B^{r+1} \oplus \bar{\mathbf{1}}_B^s \oplus S'$.

(b) $Sym_A^n(\bar{\mathbf{1}} \otimes S) = \bar{\mathbf{1}}^{\otimes n} \otimes \wedge_A^n(S) \neq 0$ for all n. Then again using Key Lemma one finds $A-$algebra B such that $X_B = \mathbf{1}_B^r \oplus \bar{\mathbf{1}}_B^{s+1} \oplus S'$.

(c) Neither (a) nor (b) is true, that is there are n and m such that $Sym_A^{n+1}(S) = \wedge_A^{m+1}(S) = 0$. Let k be any integer greater than mn and let λ be a partition of k. Then it is easy to see that $S_\lambda(S) = 0$ (since partition λ contains either row of length greater than n or column of length greater than m; in other words any representation of S_k contains either trivial representation of S_n or sign representation of S_m). Hence $S^{\otimes k} = \bigoplus_\lambda V_\lambda \otimes S_\lambda(S) = 0$. Hence $S = 0$ (S is a direct summand of rigid module, so is rigid; for a rigid module S the equality $S \otimes_A M = 0$ implies $M = 0$).

Now we apply iteratively constructions (a) and (b) beginning from the case $A = \mathbf{1}, r = s = 0, S = X$. If this process never stops then X locally has a direct summand $\mathbf{1}^p \oplus \bar{\mathbf{1}}^q$ with $p + q$ arbitrarily large. But this contradicts to the condition $S_\lambda(X) = 0$ for some λ (see Exercise 2.2). So at some moment we arrive at (c) and get that X is locally isomorphic to $\mathbf{1}^p \oplus \bar{\mathbf{1}}^q$. □

3.3. **Super fiber functor over big ring.** Now we can prove that the super fiber functor exists over sufficiently big (infinitely generated) $k-$algebra.

Proposition. Assume that any object of the category $\mathcal{A}$ is annihilated by some (depending on object) Schur functor. Then there exists a nonzero supercommutative $k-$algebra R and the $R-$fiber functor of the category $\mathcal{A}$.

Proof. We can (and will) assume that the category $\mathcal{A}$ contains object $\bar{\mathbf{1}}$ with the properties above (otherwise consider category $\mathcal{A}_1 = \mathcal{A} \boxtimes s\mathrm{Vec}$, since $\mathcal{A}_1$ contains $\mathcal{A}$ the existence of fiber functor for $\mathcal{A}_1$ implies existence of fiber functor for $\mathcal{A}$).

By 2.2 we know that for any object X of $\mathcal{A}$ there exists an algebra B such that $X_B = \mathbf{1}_B^r \oplus \bar{\mathbf{1}}_B^s$ and for any short exact sequence in $\mathcal{A}$ there exists algebra B such that this sequence splits after extension of scalars to B. Let A be the tensor product of all such algebras (so we need to consider the tensor product of infinitely many algebras; this is just inductive limit of tensor products of finitely many algebras). Then after extension of scalars to A any short exact sequence in $\mathcal{A}$ splits and for any $X \in \mathcal{A}$ one has $X_A = \mathbf{1}_A^r \oplus \bar{\mathbf{1}}_A^s$ (in particular this means that a superdimension of objects of $\mathcal{A}$ is well defined).

Consider the functor $\rho : \mathcal{A} \to s\mathrm{Vec}$ defined by $\rho(X)_0 = \mathrm{Hom}(\mathbf{1}, X)$ and $\rho(X)_1 = \mathrm{Hom}(\bar{\mathbf{1}}, X)$ (this functor is not exact in general). Clearly $\rho(A)$ is a supercommutative algebra and if M is $A-$module then $\rho(M)$ is $\rho(A)-$module (since $< \mathbf{1}, \bar{\mathbf{1}} >$ is tensor subcategory of $\mathcal{A}$). Moreover for two $A-$modules M, N the canonical morphism $can : \rho(M) \otimes_{\rho(A)} \rho(N) \to \rho(M \otimes_A N)$ is defined. Note that if M has the form $A \otimes M_0$ with $M_0 \in < \mathbf{1}, \bar{\mathbf{1}} >$ then $\rho(M) = \rho(A) \otimes M_0$ (so for any $X \in \mathcal{A}$ the $\rho(A)-$module $\rho(X_A)$ is free); if N also has the form $A \otimes N_0$ with $N_0 \in < \mathbf{1}, \bar{\mathbf{1}} >$ then $M \otimes_A N = A \otimes (M_0 \otimes N_0)$ and the morphism $can : \rho(M) \otimes_{\rho(A)} \rho(N) \to \rho(M \otimes_A N)$ is isomorphism.

Now set $R := \rho(A)$ and define the functor $\omega : \mathcal{A} \to R - mod$ by the formula $\omega(X) := \rho(X_A) = \rho(X \otimes A)$. The remarks above show that this functor has a natural structure of tensor functor. Moreover since any short exact sequence in $\mathcal{A}$ splits after extension of scalars to A this functor is exact. The Proposition is proved. $\Box$

3.4. **From $R-$fiber functor to a super fiber functor.** In this section we explain how to get a super fiber functor F from an $R-$fiber functor ω. Very roughly the idea is the following: let X be a $\otimes-$generator of $\mathcal{A}$. Since the superdimension of X is well defined (see previous section) the superspace $F(X)$ is uniquely determined. Now one needs to define sufficiently many maps between spaces $F(X)^{\otimes n}$ to ensure that $F(Y)$ is defined for every $Y \in \mathcal{A}$ and axioms of super fiber functor are satisfied (recall that any object of $\mathcal{A}$ is a subquotient of a direct sum of objects of the form $X^{\otimes n}$). This is a problem with countably many variables and countably many equations and it has solution in some algebra R; this implies that this problem has a solution over a field k (see [1] for precise statements and proofs).

We restrict ourselves to the case when category $\mathcal{A}$ is semisimple and has only finitely many simple objects $X_1, \ldots, X_n$. Again the super spaces $F(X_i)$ are uniquely determined and the only problem is to define tensor structure on the functor F. For this let us introduce variables which describe all possible isomorphisms $F(X_i) \otimes F(X_j) \to F(X_i \otimes X_j)$ (where in the RHS we use decomposition of $X_i \otimes X_j$ into the sum of simple objects); these variables (there are only finitely many of them) should satisfy finitely many equations which express the fact that isomorphisms above commute with associativity, commutativity and unit constraints. From the existence of $R-$fiber functor we know that this problem has a solution with values in R, so it has solution with values in some finitely generated algebra (since there are only finitely many variables), hence it has solution with values in k (since by Hilbert Nullstellensatz any finitely generated algebra over k admits a homomorphism to k). Hence the super fiber functor exists.

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