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NILPOTENT CONNECTIONS  
AND THE MONODROMY THEOREM :  
APPLICATIONS OF A RESULT OF TURRITTIN

*by NICHOLAS M. KATZ*

INTRODUCTION

**(0.0)** Let  $\bar{S}/\mathbf{C}$  be a projective non-singular connected curve, and

$$S = \bar{S} - \{p_1, \dots, p_r\}$$

a Zariski-open subset of  $\bar{S}$ . Suppose that

$$(0.0.0) \quad \pi : X \rightarrow S$$

is a proper and smooth morphism. From the  $C^\infty$  viewpoint,  $\pi$  is a locally trivial fibre space, so that, for  $s \in S$  variable and  $i \geq 0$  a fixed integer, the  $\mathbf{C}$ -vector spaces “complex cohomology of the fibre”

$$(0.0.1) \quad H^i(X_s, \mathbf{C})$$

form a *local system* on  $S^{\text{anal}}$ .

This local system may be constructed in a purely algebraic manner, by using the algebraic de Rham cohomology sheaves  $H_{\text{DR}}^i(X/S)$ . For each  $i \geq 0$ ,  $H_{\text{DR}}^i(X/S)$  is a locally free coherent algebraic sheaf on  $S$ , whose “fibre” at each point  $s \in S$  is the  $\mathbf{C}$ -vector space  $H^i(X_s, \mathbf{C})$ , and has an *integrable connection*  $\nabla$ , the “Gauss-Manin connection”. From this data, the local system of  $H^i(X_s, \mathbf{C})$  may be recovered as the sheaf of germs of *horizontal sections* of the associated coherent analytic sheaf on  $S^{\text{anal}}$

$$(0.0.2) \quad H_{\text{DR}}^i(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_S^{\text{anal}}.$$

**(0.1)** Now, in down-to-earth terms,  $H_{\text{DR}}^i(X/S)$  is an algebraic differential equation on  $S$  (classically called the Picard-Fuchs equations), and the *local system* of  $H^i(X_s, \mathbf{C})$  is the local system of germs of *solutions* of that equation.

**(0.2)** The Griffiths-Landman-Grothendieck “Local Monodromy Theorem” asserts that if we restrict the local system of the  $H^i(X_s, \mathbf{C})$  to a small punctured disc  $D^*$  around one of the “missing” points  $p \in \bar{S} - S$ , then picking a base point  $s_0 \in D^*$ , the automorphism  $T$  of  $H^i(X_{s_0}, \mathbf{C})$  induced by the canonical *generator* of  $\pi_1(D^*, s_0)$  (the

generator being “ turning once around  $p$  counterclockwise ”) has a very special Jordan decomposition:

$$(o.2.0) \quad T = D \cdot U = U \cdot D$$

where

(o.2.1)  $D$  is semisimple of finite order (i.e., its eigenvalues are roots of unity) and

(o.2.2)  $U$  is *unipotent*, and  $(I - U)^{i+1} = 0$  (i.e., the local monodromy has *exponent of nilpotence*  $\leq i + 1$ ).

(o.3) We can interpret the “ Local Monodromy Theorem ” as a statement about the local monodromy of the Picard-Fuchs equations around the singular point  $p$ . Griffiths, by estimating the rate of growth of the periods as we approach the singular point  $p$ , was able to prove that the Picard-Fuchs equations have a “ regular singular point ” (in the sense of Fuchs) at  $p$ .

Given that the Picard-Fuchs equations have a regular singular point at  $p$ , the statement that the eigenvalues of its local monodromy are roots of unity is precisely the statement that the *exponents* of the Picard-Fuchs equation at  $p$  are rational numbers. (In fact, Brieskorn [2] has recently given a marvelous proof of the rationality of the exponents via Hilbert’s 7th Problem.)

(o.4) The purpose of this paper is to give an *arithmetic* proof that the Picard-Fuchs equations have only regular singular points, rational exponents, and exponent of nilpotence  $i + 1$  (for  $H^i$ ).

(o.5) The method is first to “ thicken ”

$$(o.5.0) \quad X \xrightarrow{\pi} S \rightarrow \text{Spec}(\mathbf{C})$$

to a family

$$(o.5.1) \quad \mathbf{X} \xrightarrow{\pi} \mathbf{S} \rightarrow \text{Spec}(\mathbf{R})$$

where  $\mathbf{R}$  is a subring of  $\mathbf{C}$ , finitely generated over  $\mathbf{Z}$ ,  $\mathbf{S}/\text{Spec}(\mathbf{R})$  is a smooth connected curve which “gives back”  $S/\mathbf{C}$  after extension of scalars  $\mathbf{R} \hookrightarrow \mathbf{C}$ , and  $\pi: \mathbf{X} \rightarrow \mathbf{S}$  is a proper and smooth morphism which “gives back”  $\pi: X \rightarrow S$  after the base change  $S \rightarrow \mathbf{S}$ .

For instance, the Legendre family of elliptic curves, given in homogeneous coordinates by

$$(o.5.2) \quad Y^2Z - X(X - Z)(X - \lambda Z) \quad \text{in} \quad \text{Spec}\left(\mathbf{C}\left[\lambda, \frac{1}{\lambda(1-\lambda)}\right]\right) \times \mathbf{P}^2$$

is (projective and) smooth over  $\text{Spec}\left(\mathbf{C}\left[\lambda, \frac{1}{\lambda(1-\lambda)}\right]\right) = \mathbf{A}^1 - \{0, 1\}$ . A natural thickening is just to keep the equation (o.5.2), but replace  $\mathbf{C}\left[\lambda, \frac{1}{\lambda(1-\lambda)}\right]$  by  $\mathbf{Z}\left[\lambda, \frac{1}{2\lambda(1-\lambda)}\right]$ , and replace  $\mathbf{C}$  by  $\mathbf{Z}[1/2]$ .

The thickening completed, we look at  $H_{\text{DR}}^i(\mathbf{X}/\mathbf{S})$ ; replacing  $\mathbf{S}$  by a Zariski open-set, we can suppose

(o.5.3)  $\mathbf{S}$  is affine, say  $\mathbf{S} = \text{Spec}(\mathbf{B})$ , and is étale over  $\mathbf{A}_T^1$  (i.e.,  $\mathbf{B}$  is étale over  $\mathbf{R}[\lambda]$ ).

(o.5.4)  $M = H_{\text{DR}}^i(\mathbf{X}/\mathbf{S})$  is a free  $\mathbf{B}$ -module of finite rank.

The datum of the Gauss-Manin connection is that of an  $\mathbf{R}$ -linear mapping

$$(o.5.5) \quad \nabla\left(\frac{d}{d\lambda}\right) : M \rightarrow M$$

which satisfies, for  $f \in \mathbf{B}$ ,  $m \in M$

$$(o.5.6) \quad \nabla\left(\frac{d}{d\lambda}\right)(fm) = \frac{df}{d\lambda} \cdot m + f \cdot \nabla\left(\frac{d}{d\lambda}\right)(m).$$

The next step is to prove that this connection is *globally nilpotent on  $\mathbf{B}$  of exponent  $i+1$* , which by definition means that for every prime number  $p$ , the  $\mathbf{R}$ -linear operation

$$(o.5.7) \quad \left(\nabla\left(\frac{d}{d\lambda}\right)\right)^{p(i+1)} : M \rightarrow M$$

induces the zero mapping of  $M/pM$ .

To prove this, we use the fact that,  $M = H_{\text{DR}}^i(\mathbf{X}/\mathbf{S})$  being free, we have

$$(o.5.8) \quad M/pM \simeq H_{\text{DR}}^i(\mathbf{X} \otimes \mathbf{F}_p / \mathbf{S} \otimes \mathbf{F}_p)$$

(the right hand side being an  $\mathbf{B}/p\mathbf{B}$  module). The problem is then to prove the nilpotence of the Gauss-Manin connection in characteristic  $p$ ; this is done in Section 5.

(o.5.9) The final step is to deduce, from the global nilpotence of exponent  $i+1$ , that the Picard-Fuchs equations have only regular singular points, and rational exponents, and that the exponent of nilpotence of the local monodromy is  $\leq i+1$ .

This deduction (13.0) is made possible by the fantastic Theorem (11.10) of Turrittin, which allows us to really see what keeps a *singular* point of a differential equation from being a regular singular point.

(o.6) The first sections (1-4) review the formalism of connections. They represent joint work with Oda, and nearly all of the results are either contained in or implicit in [31], which unfortunately was not cast in sufficient generality for the present applications.

Sections 5-6 take up nilpotent connections in characteristic  $p > 0$ . The notion of a nilpotent connection is due to Berthelot (cf. [1]). We would like to call attention to the beautiful formula (5.3.0) of Deligne. The main result (5.10) is that, in characteristic  $p$ , the Gauss-Manin connection on  $H_{\text{DR}}^i(\mathbf{X}/\mathbf{S})$  is nilpotent of exponent  $\leq i+1$  (or  $\leq 2n-i+1$ , if  $i > n = \dim(\mathbf{X}/\mathbf{S})$ ).

Section 7 is entirely due to Deligne. He had the idea of using the Cartier operation to lower the exponent of nilpotence of  $H_{\text{DR}}^i(\mathbf{X}/\mathbf{S})$  from  $i+1$  to the number of pairs  $(p, q)$

of integers with  $h^{p,q}(X/S) = H^q(X/S, \Omega_{X/S}^p) \neq 0$ , and  $p+q=i$ , thus relating the exponent of nilpotence to the Hodge structure.

Section 8 is a review of standard base-changing theorems, and Section 9 precises the notion of global nilpotence. Section 10 combines the results of Sections 7, 8 and 9 to show that  $H_{\text{DR}}^i(X/S)$  is globally nilpotent of exponent  $i+1$ , or (by Deligne), the number of non-zero terms in the Hodge decomposition of  $H^i(X_s, \mathbf{C})$ ,  $s$  being any  $\mathbf{C}$ -valued point of  $S$ .

Section 11 reviews the classical theory of regular singular points, and proves Turrittin's theorem. I am grateful to E. Brieskorn for having made me aware of the paper of D. Lutz [24], from which I learned of the existence of Turrittin's Theorem.

Section 12 recalls the classical theory of the local monodromy around a regular singular point. It is a pleasure to be able to refer to the elegant paper [25] of Manin for the main result (12.0).

In Section 13 we establish that global nilpotence of a differential equation implies that all of its singular points are regular singular points, with rational exponents (13.0). This theorem was originally conjectured by Grothendieck (and proved by him for a rank-one equation on  $\mathbf{P}^1!$ ). Needless to say, that conjecture was the starting point of the work presented here.

In Section 14, we "tie everything together", and give the final statement of the Local Monodromy Theorem (14.1), with Deligne's improvement on the exponent of nilpotence in terms of the Hodge structure. We also give Deligne's extension of the theorem (14.3) for *non-proper* smooth families, proved via the systematic use of Hironaka's resolution of singularities and Deligne's technique of systematically working with differentials having only logarithmic singularities along the divisor at  $\infty$ .

It is a pleasure to acknowledge the overwhelming influence of Grothendieck and Deligne on this work.

(1.0) Let  $T$  be a scheme,  $f: S \rightarrow T$  a smooth  $T$ -scheme, and  $\mathcal{E}$  a quasi-coherent sheaf of  $\mathcal{O}_S$ -modules. A  $T$ -connection on  $\mathcal{E}$  is a homomorphism  $\nabla$  of abelian sheaves

$$(1.0.0) \quad \nabla: \mathcal{E} \rightarrow \Omega_{S/T}^1 \otimes_{\mathcal{O}_S} \mathcal{E}$$

such that

$$(1.0.1) \quad \nabla(ge) = g\nabla(e) + dg \otimes e$$

where  $g$  and  $e$  are sections of  $\mathcal{O}_S$  and  $\mathcal{E}$  respectively over an open subset of  $S$ , and  $dg$  denotes the image of  $g$  under the canonical exterior differentiation  $d: \mathcal{O}_S \rightarrow \Omega_{S/T}^1$ . The kernel of  $\nabla$ , noted  $\mathcal{E}^\nabla$ , is the sheaf of germs of *horizontal sections* of  $(\mathcal{E}, \nabla)$ .

A  $T$ -connection  $\nabla$  may be extended to a homomorphism of abelian sheaves

$$\nabla_i: \Omega_{S/T}^i \otimes_{\mathcal{O}_S} \mathcal{E} \rightarrow \Omega_{S/T}^{i+1} \otimes_{\mathcal{O}_S} \mathcal{E}$$

by

$$(1.0.2) \quad \nabla_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla(e)$$

where  $\omega$  and  $e$  are sections of  $\Omega_{S/T}^i$  and  $\mathcal{E}$  respectively over an open subset of  $S$ , and where  $\omega \wedge \nabla(e)$  denotes the image of  $\omega \otimes \nabla(e)$  under the canonical map

$$\Omega_{S/T}^i \otimes_{\mathcal{O}_S} (\Omega_{S/T}^1 \otimes_{\mathcal{O}_S} \mathcal{E}) \rightarrow \Omega_{S/T}^{i+1} \otimes_{\mathcal{O}_S} \mathcal{E}$$

which sends  $\omega \otimes \tau \otimes e$  to  $(\omega \wedge \tau) \otimes e$ .

The curvature  $K = K(\mathcal{E}, \nabla)$  of the T-connection  $\nabla$  is the  $\mathcal{O}_S$ -linear map

$$K = \nabla_1 \circ \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_S} \Omega_{S/T}^2.$$

One easily verifies that

$$(\nabla_{i+1} \circ \nabla_i)(\omega \otimes e) = \omega \wedge K(e)$$

where  $\omega$  and  $e$  are sections of  $\Omega_{S/T}^i$  and  $\mathcal{E}$  over an open subset of  $S$ .

The T-connection  $\nabla$  is called *integrable* if  $K = 0$ . An integrable T-connection  $\nabla$  on  $\mathcal{E}$  thus gives rise to a complex (the de Rham complex of  $(\mathcal{E}, \nabla)$ )

$$(1.0.3) \quad 0 \rightarrow \mathcal{E} \xrightarrow{\nabla} \Omega_{S/T}^1 \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla_1} \Omega_{S/T}^2 \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla_2} \dots$$

which we denote simply by  $\Omega_{S/T}^\bullet \otimes_{\mathcal{O}_S} \mathcal{E}$  when the integrable T-connection  $\nabla$  is understood.

Let  $Der(S/T)$  denote the sheaf of germs of T-derivations of  $\mathcal{O}_S$  into itself. We note that  $Der(S/T)$  is naturally a sheaf of  $f^{-1}(\mathcal{O}_T)$ -Lie algebras, while, as  $\mathcal{O}_S$ -module, it is isomorphic to  $\text{Hom}_{\mathcal{O}_S}(\Omega_{S/T}^1, \mathcal{O}_S)$ .

Let  $End_T(\mathcal{E})$  denote the sheaf of germs of  $f^{-1}(\mathcal{O}_T)$ -linear endomorphisms of  $\mathcal{E}$ . We note that  $End_T(\mathcal{E})$  is naturally a sheaf of  $f^{-1}(\mathcal{O}_T)$ -Lie algebras.

Now fix a T-connection  $\nabla$  on  $\mathcal{E}$ ;  $\nabla$  gives rise to an  $\mathcal{O}_S$ -linear mapping

$$\nabla : Der(S/T) \rightarrow End_T(\mathcal{E})$$

sending  $D$  to  $\nabla(D)$ , where  $\nabla(D)$  is the composite

$$\mathcal{E} \xrightarrow{\nabla} \Omega_{S/T}^1 \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes 1} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \simeq \mathcal{E}.$$

We have

$$(1.0.4) \quad \nabla(D)(fe) = D(f)e + f\nabla(D)(e)$$

whenever  $D, f$  and  $e$  are sections of  $Der(S/T)$ ,  $\mathcal{O}_S$  and  $\mathcal{E}$  respectively over an open subset of  $S$ . Conversely, because  $S/T$  is smooth, any  $\mathcal{O}_S$ -linear mapping

$$Der(S/T) \rightarrow End_T(\mathcal{E})$$

satisfying (1.0.4) arises from a unique T-connection  $\nabla$ .

The T-connection  $\nabla$  is integrable precisely when the mapping  $Der(S/T) \rightarrow End_T(\mathcal{E})$  is also a Lie-algebra homomorphism. This is seen by using the well-known fact that for  $D_1$  and  $D_2$  sections of  $Der(S/T)$  over an open subset of  $S$ , we have

$$(1.0.5) \quad [\nabla(D_1), \nabla(D_2)] - \nabla([D_1, D_2]) = (D_1 \wedge D_2)(K)$$

where the right-hand side is the composite mapping

$$\mathcal{E} \xrightarrow{K} \Omega_{S/T}^2 \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D_1 \wedge D_2} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \simeq \mathcal{E}.$$

(1.1) Let  $(\mathcal{E}, \nabla)$  and  $(\mathcal{F}, \nabla')$  be quasi-coherent  $\mathcal{O}_S$ -modules with  $T$ -connections. An  $\mathcal{O}_S$ -linear mapping

$$\Phi : \mathcal{E} \rightarrow \mathcal{F}$$

is called *horizontal* if

$$(1.1.0) \quad \Phi(\nabla(D)(e)) = \nabla'(D)(\Phi(e))$$

whenever  $D$  and  $e$  are sections of  $Der(S/T)$  and  $\mathcal{E}$  respectively over an open subset of  $S$ .

We denote by  $MC(S/T)$  the abelian category whose objects are pairs  $(\mathcal{E}, \nabla)$  as above, and whose morphisms are the horizontal ones ( $MC$ =modules with connection). The category  $MC(S/T)$  has a tensor product, constructed as follows :

$$(\mathcal{E}, \nabla) \otimes (\mathcal{F}, \nabla') = (\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{F}, \nabla'''),$$

$\nabla'''$  defined by the formula

$$(1.1.1) \quad \nabla'''(D)(e \otimes f) = \nabla(D)(e) \otimes f + e \otimes \nabla'(D)(f)$$

where  $D$ ,  $e$ , and  $f$  are sections of  $Der(S/T)$ ,  $\mathcal{E}$ , and  $\mathcal{F}$  respectively over an open subset of  $S$ .

Each object  $(\mathcal{E}, \nabla)$  whose underlying module  $\mathcal{E}$  is locally of finite presentation defines an internal *Hom* functor

$$Hom_{\mathcal{O}_S}((\mathcal{E}, \nabla), ?) : MC(S/T) \rightarrow MC(S/T)$$

as follows :

$$Hom_{\mathcal{O}_S}((\mathcal{E}, \nabla), (\mathcal{F}, \nabla')) = (Hom_{\mathcal{O}_S}(\mathcal{E}, \mathcal{F}), \nabla''),$$

$\nabla''$  being defined by the formula

$$(1.1.2) \quad (\nabla''(D)(\Phi))(e) = \nabla'(D)(\Phi(e)) - \Phi(\nabla(D)(e))$$

where  $D$ ,  $\Phi$  and  $e$  are sections of  $Der(S/T)$ ,  $Hom_{\mathcal{O}_S}(\mathcal{E}, \mathcal{F})$  and  $\mathcal{E}$  respectively over an open subset of  $S$ .

Allowing ourselves a moderate abuse of language, we will say that  $MC(S/T)$  has an internal *Hom* which is not everywhere defined.

We denote by  $MIC(S/T)$  the full (abelian) subcategory of  $MC(S/T)$  consisting of sheaves of quasi-coherent  $\mathcal{O}_S$ -modules with *integrable* connections. This subcategory is *stable* under the internal *Hom* (when defined, cf. (1.1.2)) and tensor product of  $MC(S/T)$ .

We remark that the categories  $MC(S/T)$  and  $MIC(S/T)$  have an evident functoriality in the smooth morphism  $f : S \rightarrow T$ . Explicitly, if  $f' : S' \rightarrow T'$  is a smooth morphism, and

$$(1.1.3) \quad \begin{array}{ccc} S' & \xrightarrow{g} & S \\ \downarrow & & \downarrow f \\ T' & \xrightarrow{h} & T \end{array}$$

is a commutative diagram, there is an “ inverse image ” functor

$$(1.1.4) \quad (g, h)^* : \mathbf{MC}(S/T) \rightarrow \mathbf{MC}(S'/T')$$

(which maps  $\mathbf{MIC}(S/T)$  to  $\mathbf{MIC}(S'/T')$ ), as follows. Let  $(\mathcal{E}, \nabla)$  be an object of  $\mathbf{MC}(S/T)$ . Taking the usual inverse image by  $(g, h)$  of the mapping

$$(1.1.5) \quad \nabla : \mathcal{E} \rightarrow \Omega_{S/T}^1 \otimes_{\mathcal{O}_S} \mathcal{E}$$

gives a mapping

$$(1.1.6) \quad g^*(\mathcal{E}) \rightarrow (g, h)^* \Omega_{S/T}^1 \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}).$$

The canonical mapping

$$(1.1.7) \quad (g, h)^* \Omega_{S/T}^1 \rightarrow \Omega_{S'/T'}^1,$$

tensorized by  $g^*(\mathcal{E})$ , gives a map

$$(1.1.8) \quad (g, h)^* \Omega_{S/T}^1 \otimes_{\mathcal{O}_S} g^*(\mathcal{E}) \rightarrow \Omega_{S'/T'}^1 \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}).$$

The composition of (1.1.6) and (1.1.8) is thus a mapping  $(g, h)^*(\nabla)$

$$(1.1.9) \quad (g, h)^*(\nabla) : g^*(\mathcal{E}) \rightarrow \Omega_{S'/T'}^1 \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E})$$

which is easily seen to be a  $T'$ -connection on  $g^*(\mathcal{E})$ . The inverse image  $(g, h)^*(\mathcal{E}, \nabla)$  is, by definition,  $(g^*(\mathcal{E}), (g, h)^*(\nabla))$ .

One checks immediately that the curvature element

$$K(g^*(\mathcal{E}), (g, h)^*(\nabla)) \in \text{Hom}_{S'}(g^*(\mathcal{E}), \Omega_{S'/T'}^2 \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}))$$

is the *inverse image* of  $K(\mathcal{E}, \nabla) \in \text{Hom}_S(\mathcal{E}, \Omega_{S/T}^2 \otimes_{\mathcal{O}_S} \mathcal{E})$ .

**(1.2)** We remark that the category  $\mathbf{MIC}(S/T)$  has enough injectives, being (tautologically) equivalent to the category of quasicoherent modules over an appropriate sheaf of enveloping algebras (the sheaf P-D Diff. of Berthelot [1], or, equivalently, the enveloping algebra of Kostant, Rosenberg and Hochschild [19]).

**(2.0)** We define the de Rham cohomology sheaves on  $T$  of an object  $(\mathcal{E}, \nabla)$  in  $\mathbf{MIC}(S/T)$  by

$$(2.0.1) \quad H_{DR}^q(S/T, \mathcal{E}, \nabla) = \mathbf{R}^q f_*(\Omega_{S/T}^* \otimes_{\mathcal{O}_S} \mathcal{E})$$

where  $\Omega_{S/T}^* \otimes_{\mathcal{O}_S} \mathcal{E}$  is the de Rham complex of  $(\mathcal{E}, \nabla)$ , cf. (1.3), and  $\mathbf{R}^q f_*$  are the hyper-derived functors of  $\mathbf{R}^0 f_*$ . In particular,  $H_{DR}^0(S/T, (\mathcal{E}, \nabla)) = f_*(\mathcal{E}^\nabla)$ . As is proved in [17] and also in [19], the functors  $H_{DR}^q(S/T, ?)$  are the right derived functors of the left exact functor

$$(2.0.2) \quad H_{DR}^0(S/T, ?) : \mathbf{MIC}(S/T) \rightarrow \mathbf{MIC}(T/T) = (\text{quasicoherent sheaves on } T).$$

(3.0) Suppose now  $\pi: X \rightarrow S$  is a smooth morphism. The natural forgetful functor

$$(3.0.1) \quad \begin{aligned} \mathbf{MIC}(X/T) &\rightarrow \mathbf{MIC}(X/S) \\ (\mathcal{E}, \nabla) &\mapsto (\mathcal{E}, \nabla | Der(X/S)) \end{aligned}$$

allows us to define the de Rham complex of  $(\mathcal{E}, \nabla | Der(X/S))$ , which we will denote simply by  $\Omega_{X/S}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E}$ . Further abusing notation, we write

$$(3.0.2) \quad H_{DR}^q(X/S, (\mathcal{E}, \nabla)) = \mathbf{R}^q \pi_*(\Omega_{X/S}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Exactly as in [31], we may construct a canonical T-connection  $\nabla$  on the quasi-coherent  $\mathcal{O}_S$ -module  $H_{DR}^q(X/S, (\mathcal{E}, \nabla))$ , the “Gauss-Manin connection”, so that the functors  $H_{DR}^q(X/S, ?)$  may be interpreted as an exact connected sequence of cohomological functors

$$\mathbf{MIC}(X/T) \rightarrow \mathbf{MIC}(S/T).$$

*Remark (3.1).* — There is no difficulty in checking that these functors are none other than the right derived functors of

$$H_{DR}^0(X/S, ?) : \mathbf{MIC}(X/T) \rightarrow \mathbf{MIC}(S/T)$$

where the T-connection on  $H_{DR}^0(X/S, (\mathcal{E}, \nabla | Der(X/S))) = \pi_*(\mathcal{E}^{\nabla | Der(X/S)})$  is defined by using the exactness of the sequence of sheaves on X

$$(3.1.0) \quad 0 \rightarrow Der(X/S) \rightarrow Der(X/T) \rightarrow \pi^* Der(S/T) \rightarrow 0.$$

(3.2) For computational purposes, however, we recall the construction given in [31], of the entire de Rham complex  $\Omega_{S/T}^{\bullet} \otimes_{\mathcal{O}_S} H_{DR}^q(X/S, (\mathcal{E}, \nabla))$ . Consider the canonical filtration of  $\Omega_{X/T}^{\bullet}$  by locally free subsheaves

$$(3.2.0) \quad \Omega_{X/T}^{\bullet} = F^0(\Omega_{X/T}^{\bullet}) \supset F^1(\Omega_{X/T}^{\bullet}) \supset \dots$$

given by

$$(3.2.1) \quad F^i(\Omega_{X/T}^{\bullet}) = \text{image of } (\pi^*(\Omega_{S/T}^i) \otimes_{\mathcal{O}_X} \Omega_{X/T}^{*-i} \rightarrow \Omega_{X/T}^{\bullet}).$$

By smoothness, the associated graded objects  $gr^i = F^i/F^{i+1}$  are given the (locally free) sheaves

$$(3.2.2) \quad gr^i(\Omega_{X/T}^{\bullet}) = \pi^*(\Omega_{S/T}^i) \otimes_{\mathcal{O}_X} \Omega_{X/S}^{*-i}.$$

We filter the de Rham complex  $\Omega_{X/T}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E}$  by the subcomplexes

$$(3.2.3) \quad F^i(\Omega_{X/T}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E}) = F^i(\Omega_{X/T}^{\bullet}) \otimes_{\mathcal{O}_X} \mathcal{E};$$

the associated graded objects are the  $f^{-1}(\mathcal{O}_S)$ -linear complexes

$$(3.2.4) \quad gr^i(\Omega_{X/T}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E}) \simeq \pi^*(\Omega_{S/T}^i) \otimes_{\mathcal{O}_X} (\Omega_{X/S}^{*-i} \otimes_{\mathcal{O}_X} \mathcal{E})$$

(the differential in this complex is  $1 \otimes$  (the differential of  $\Omega_{X/S}^{*-i} \otimes_{\mathcal{O}_X} \mathcal{E}$ )).

Consider the functor  $\mathbf{R}^0 \pi_*$  from the category of complexes of abelian sheaves on X to the category of abelian sheaves on S. Applying the spectral sequence of a

finitely filtered object, we obtain a spectral sequence abutting to (the associated graded object with respect to the filtration of)  $\mathbf{R}^\bullet \pi_*(\Omega_{X/T}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E})$ , while

$$\begin{aligned} (3.2.5) \quad E_1^{p,q} &= \mathbf{R}^{p+q} \pi_*(\text{gr}^p) = \mathbf{R}^{p+q} \pi_*(\pi^*(\Omega_{S/T}^p) \otimes_{\mathcal{O}_X} (\Omega_{X/S}^{-p} \otimes_{\mathcal{O}_X} \mathcal{E})) \\ &= \Omega_{S/T}^p \otimes_{\mathcal{O}_S} \mathbf{R}^q \pi_*(\Omega_{X/S}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E}) \\ &= \Omega_{S/T}^p \otimes_{\mathcal{O}_S} H_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla)). \end{aligned}$$

The de Rham complex of  $H_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla))$  is then the complex  $(E_1^{\bullet, q}, d_1^{\bullet, q})$ , the  $q$ -th row of  $E_1$  terms of the above spectral sequence.

*Remark (3.3).* — The zealous reader who wishes to construct the “Leray spectral sequence” of de Rham cohomology for  $X \xrightarrow{\pi} S \xrightarrow{f} T$

$$(3.3.0) \quad E_2^{p,q} = H_{\text{DR}}^p(S/T, (H_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla)), \text{兄})) \Rightarrow H_{\text{DR}}^{p+q}(X/T, (\mathcal{E}, \nabla))$$

without availing himself of the previous remark (whose truth reduces the question to one of the usual composite functor spectral sequence) may employ the following trick, due to Deligne.

Let  $\mathcal{B}$  and  $\mathcal{C}$  be abelian categories,  $\mathcal{B}$  having enough injectives, and let  $N : \mathcal{B} \rightarrow \mathcal{C}$  be a left exact additive functor. Let  $K^\bullet$  be a complex ( $K^i = 0$  for  $i < 0$ ) over  $\mathcal{B}$ . By a C-E resolution with respect to  $N$  of  $K^\bullet$  we mean an augmented first quadrant bicomplex

$$K^\bullet \rightarrow M^\bullet$$

such that, for each  $i \geq 0$ , the complex  $M^{\bullet, i}$  is a resolution of  $K^i$  by  $N$ -acyclic objects, and such that for each  $p \geq 0$ , the complex

$$H^p(M^0, \cdot) \rightarrow H^p(M^1, \cdot) \rightarrow H^p(M^2, \cdot) \rightarrow \dots$$

is a resolution of  $H^p(K^\bullet)$  by  $N$ -acyclic objects.

If  $K^\bullet$  is a finitely filtered complex over  $\mathcal{B}$

$$K^\bullet = F^0(K^\bullet) \supset F^1(K^\bullet) \supset \dots,$$

then by a filtered C-E resolution of  $K^\bullet$  with respect to  $N$  we mean an augmented first quadrant finitely filtered bicomplex

$$M^{\bullet\bullet} = F^0(M^{\bullet\bullet}) \supset F^1(M^{\bullet\bullet}) \supset \dots$$

such that, for  $i \geq 0$ ,

$$F^i(K^\bullet) \rightarrow F^i(M^{\bullet\bullet})$$

and

$$\text{gr}^i(K^\bullet) \rightarrow \text{gr}^i(M^{\bullet\bullet})$$

are C-E resolutions with respect to  $N$  of  $F^i(K^\bullet)$  and  $\text{gr}^i(K^\bullet)$  respectively.

*Proposition (3.3.1).* — Let  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  be three abelian categories,  $\mathcal{A}$  and  $\mathcal{B}$  with enough injectives, and let

$$L : \mathcal{A} \rightarrow \mathcal{B}, \quad N : \mathcal{B} \rightarrow \mathcal{C}$$

be left exact additive functors, such that the image of an injective by  $L$  is  $T$ -acyclic. Suppose further that every finitely filtered complex over  $\mathcal{B}$  admits a filtered C-E resolution with respect to  $N$ .

Let  $A \supset F^1(A) \supset F^2(A) \supset \dots$  be a finitely filtered object of  $\mathcal{A}$ . The spectral sequence of a finitely filtered object for the functor  $L$  gives a spectral sequence

$$E_1^{p,q}(A) = R^{p+q}(L)(\text{gr}^p A) \Rightarrow R^{p+q}(L)(A).$$

For each  $q$ , we denote by  $E_1^{\bullet,q}(A)$  the complex

$$(E_1^{\bullet,q}(A), d_1^{\bullet,q}).$$

Then there is a spectral sequence

$$(3.3.2) \quad E_2^{p,q} = R^p(N)(E_1^{\bullet,q}(A)) \Rightarrow R^{p+q}(NL)(A).$$

*Remark (3.3.3).* — If  $\mathcal{B}$  is the category of abelian sheaves on a topological space  $S$ ,  $\mathcal{C}$  the category of abelian sheaves on a topological space  $T$ , and  $N$  the functor  $f_*$ , where  $f: S \rightarrow T$  is a continuous map, then taking “the canonical flasque resolution” componentwise *functorially* provides every finitely filtered complex over  $B$  with a finitely filtered C-E resolution with respect to  $N$ .

To apply the proposition, we take

$$\begin{cases} \mathcal{A} = \text{complexes of abelian sheaves on } X \\ \mathcal{B} = \text{abelian sheaves on } S \\ \mathcal{C} = \text{abelian sheaves on } T \\ L = R^0 \pi_* \\ N = f_* \\ A = \Omega_{X/T}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E} \text{ with the filtration (3.2.3).} \end{cases}$$

*Outline of proof.* — Take a finitely filtered injective resolution  $I^\bullet$  of  $A$ , so that, for each  $i \geq 0$ ,  $F^i I^\bullet$  and  $\text{gr}^i(I^\bullet)$  are injective resolutions of  $F^i(A)$  and  $\text{gr}^i(A)$  respectively. Put  $K^\bullet = L(I^\bullet)$ ,  $F^i(K^\bullet) = L(F^i(I^\bullet))$ . Let  $M^\bullet$  be a filtered C-E resolution with respect to  $N$  of  $K^\bullet$ , and define a *new* filtration  $\tilde{F}$  on  $M^\bullet$  by defining

$$\tilde{F}^i(M^{p,q}) = F^{i-p}(M^{p,q}).$$

Now let  $P^\bullet = N(M^\bullet)$ , filtered by  $\tilde{F}^i(P^{p,q}) = N(\tilde{F}^i M^{p,q}) = N(F^{i-p} M^{p,q})$ . The desired spectral sequence is that of the “totalized” complex of  $P^\bullet$ , with the filtration  $\tilde{F}$ .

(3.4) We now recall from [31] the explicit calculation of the Gauss-Manin connection. The question being local on  $S$ , we will suppose that  $S$  is affine.

Choose a finite covering of  $X$  by affine open sets  $\{U_\alpha\}$  such that each  $U_\alpha$  is *étale* over  $\mathbf{A}_S^n$ , so that, on  $U_\alpha$ , the sheaf  $\Omega_{X/S}^1$  is a free  $\mathcal{O}_X$ -module, with base  $\{dx_1^\alpha, \dots, dx_n^\alpha\}$ .

For any object  $(\mathcal{E}, \nabla)$  of  $\text{MIC}(X/T)$ , the  $S$ -modules  $R^i \pi_*(\Omega_{X/S}^1 \otimes_{\mathcal{O}_X} \mathcal{E})$  may be calculated as the total homology of the bicomplex of  $\mathcal{O}_S$ -modules

$$C^{p,q}(\mathcal{E}) \underset{(def)}{=} C^p(\{U_\alpha\}, \Omega_{X/S}^q \otimes_{\mathcal{O}_X} \mathcal{E})$$

of alternating Čech cochains on the nerve of the covering  $\{U_\alpha\}$ . We will now describe a  $T$ -connection (in general *not* integrable) on the totalized complex associated to the

bicomplex  $C^p(\{U_\alpha\}, \Omega_{X/S}^q \otimes_{\mathcal{O}_X} \mathcal{E})$ , which upon passage to homology yields the Gauss-Manin connection.

Let  $D$  be any  $T$ -derivation of the coordinate ring of  $S$ . For each index  $\alpha$ , let  $D_\alpha \in \text{Der}_T(\mathcal{O}_{U_\alpha}, \mathcal{O}_{U_\alpha})$  be the unique *extension* of  $D$  which kills  $dx_1^\alpha, \dots, dx_n^\alpha$ .  $D_\alpha$  induces a  $T$ -linear endomorphism of sheaves (a “Lie derivative”)

$$(3.4.0) \quad D_\alpha : \Omega_{U_\alpha/S}^q \rightarrow \Omega_{U_\alpha/S}^q$$

by

$$(3.4.1) \quad D_\alpha(h dx_{i_1}^\alpha \wedge \dots \wedge dx_{i_q}^\alpha) = D_\alpha(h) dx_{i_1}^\alpha \wedge \dots \wedge dx_{i_q}^\alpha$$

where  $h$  is a section of  $\mathcal{O}_X$  over an open subset of  $U_\alpha$ . Similarly,  $D_\alpha$  induces a  $T$ -linear endomorphism

$$(3.4.2) \quad D_\alpha : \Omega_{U_\alpha/S}^q \otimes_{\mathcal{O}_{U_\alpha}} \mathcal{E} \rightarrow \Omega_{U_\alpha/S}^q \otimes_{\mathcal{O}_{U_\alpha}} \mathcal{E}$$

by

$$(3.4.3) \quad D_\alpha(\omega \otimes e) = D_\alpha(\omega) \otimes e + \omega \otimes \nabla(D_\alpha)(e)$$

where  $\omega$  and  $e$  are sections of  $\Omega_{X/S}^q$  and  $\mathcal{E}$  respectively over an open subset of  $U_\alpha$ .

Choose a total ordering on the indexing set of the covering  $\{U_\alpha\}$ . We define a  $T$ -linear endomorphism  $\tilde{D}$  of bidegree  $(0, 0)$  of the bigraded  $\mathcal{O}_S$ -module  $C^{p,q}(\mathcal{E}) = C^p(\{U_\alpha\}, \Omega_{X/S}^q \otimes_{\mathcal{O}_X} \mathcal{E})$  by setting

$$(3.4.4) \quad \tilde{D}|_{\Gamma(U_{\alpha_0} \cap \dots \cap U_{\alpha_p}, \Omega_{X/S}^q \otimes_{\mathcal{O}_X} \mathcal{E})} = D_{\alpha_0}$$

if  $\alpha_0 < \alpha_1 < \dots < \alpha_p$ .

For each pair  $\alpha, \beta$  of indices, we define an  $\mathcal{O}_X$ -linear mapping of sheaves (the interior product with  $D_\alpha - D_\beta$ )

$$(3.4.5) \quad \lambda(D)_{\alpha, \beta} : \Omega_{X/S}^q|_{(U_\alpha \cap U_\beta)} \rightarrow \Omega_{X/S}^{q-1}|_{(U_\alpha \cap U_\beta)}$$

by

$$(3.4.6) \quad \lambda(D)_{\alpha, \beta}(h dx_1 \wedge \dots \wedge dx_q) = h \sum_i (-1)^i (D_\alpha - D_\beta)(x_i) dx_1 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_q$$

where  $h, x_1, \dots, x_q$  are sections of  $\mathcal{O}_X$  over an open subset of  $U_\alpha \cap U_\beta$ . (We put  $\lambda(D)_{\alpha, \beta} = 0$  on  $\mathcal{O}_X$ .) Similarly, we define an  $\mathcal{O}_X$ -linear mapping

$$(3.4.7) \quad \lambda(D)_{\alpha, \beta} : \Omega_{X/S}^q \otimes_{\mathcal{O}_X} \mathcal{E}|_{(U_\alpha \cap U_\beta)} \rightarrow \Omega_{X/S}^{q-1} \otimes_{\mathcal{O}_X} \mathcal{E}|_{(U_\alpha \cap U_\beta)}$$

by

$$(3.4.8) \quad \lambda(D)_{\alpha, \beta}(\omega \otimes e) = \lambda(D)_{\alpha, \beta}(\omega) \otimes e.$$

We define an  $\mathcal{O}_X$ -linear endomorphism  $\lambda(D)$  of bidegree  $(1, -1)$  of the bigraded  $\mathcal{O}_S$ -module  $C^{*,*}(\mathcal{E})$

$$(3.4.9) \quad \lambda(D) : C^p(\{U_\alpha\}, \Omega_{X/S}^q \otimes \mathcal{E}) \rightarrow C^{p+1}(\{U_\alpha\}, \Omega_{X/S}^{q-1} \otimes \mathcal{E})$$

by

$$(3.4.10) \quad (\lambda(D)(\sigma))_{\alpha_0, \dots, \alpha_{p+1}} = (-1)^q \lambda(D)_{\alpha_0, \alpha_1}(\sigma_{\alpha_1, \dots, \alpha_{p+1}}) \quad \text{if } \alpha_0 < \dots < \alpha_{p+1}$$

where  $\sigma$  is the alternating  $p$ -cochain whose value on  $U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$ ,  $\alpha_0 < \dots < \alpha_p$ , is  $\sigma_{\alpha_0, \dots, \alpha_p}$ .

Notice that  $\{D_\alpha - D_\beta\}$  is a 1-cocycle on the nerve of the covering  $\{U_\alpha\}$  with values in  $Der(X/S)$ , whose cohomology class in  $H^1(X, Der(X/S))$  is the value at  $D \in Der_T(\mathcal{O}_S, \mathcal{O}_S)$  of the *Kodaira-Spencer map*

$$(3.4.11) \quad \rho_{X/S} : Der_T(\mathcal{O}_S, \mathcal{O}_S) \rightarrow H^1(X, Der(X/S)).$$

The cochain map  $\lambda(D)$  is just the *cup-product* with the representative cocycle  $\{D_\alpha - D_\beta\}$ .

“The” Gauss-Manin connection on the bicomplex  $C^\bullet(\mathcal{E})$  is given by

$$(3.4.12) \quad \chi : D \in Der_T(\mathcal{O}_S, \mathcal{O}_S) \mapsto \chi(D) = \tilde{D} + \lambda(D).$$

This explicit formula has a number of immediate consequences, which we will now record.

*Theorem (3.5).*

(3.5.1)  $\chi(D)$  is compatible with the “Zariski” filtration  $F_{zar}$  of  $C^\bullet(\mathcal{E})$ ,  $F_{zar}^i = \sum_{p \geq i} C^{p,q}(\mathcal{E})$ , hence acts on the associated spectral sequence

$$(3.5.1.0) \quad E_1^{p,q} = C^p(\{U_\alpha\}, \mathcal{H}_{DR}^q(X/S, (\mathcal{E}, \nabla))) \Rightarrow H_{DR}^{p+q}(X/S, (\mathcal{E}, \nabla))$$

where  $\mathcal{H}_{DR}^q(X/S, (\mathcal{E}, \nabla))$  denotes the presheaf on  $X$  with values in  $MIC(S/T)$

$$U \mapsto H_{DR}^q(U/S, (\mathcal{E}, \nabla) | U).$$

(3.5.2)  $\chi(D)$  is not compatible with the “Hodge” filtration  $F_{Hodge}$  of  $C^\bullet(\mathcal{E})$ ,  $F_{Hodge}^i = \sum_{q \geq i} C^{p,q}(\mathcal{E})$ , and does not act on the associated spectral sequence

$$(3.5.2.0) \quad E_1^{p,q} = H^q(X, \Omega_{X/S}^p \otimes \mathcal{E}) \Rightarrow H_{DR}^{p+q}(X/S, (\mathcal{E}, \nabla)).$$

However,  $\chi(D)$  does respect the Hodge filtration on  $H_{DR}(X/S, (\mathcal{E}, \nabla))$  to a shift of one, i.e.,

$$\chi(D) F_{Hodge}^i \subset F_{Hodge}^{i-1}$$

(“Griffith’s transversality theorem”) and so induces, by passage to quotients, an  $S$ -linear mapping

$$(3.5.2.1) \quad \chi(D) : gr_{Hodge}^p H_{DR}^{p+q}(X/S, (\mathcal{E}, \nabla)) \rightarrow gr_{Hodge}^{p-1} H_{DR}^{p+q}(X/S, (\mathcal{E}, \nabla)).$$

In particular, if the spectral sequence (3.5.2.0) degenerates ( $E_1 = E_\infty$ ) (this is the case for example, if  $X$  is proper and smooth over  $S$ ,  $S$  is of characteristic zero, and  $\mathcal{E} = \mathcal{O}_X$  with the standard connection (cf. (8.7))), this induced mapping (3.5.2.1)

$$\chi(D) : H^q(X, \Omega_{X/S}^p \otimes \mathcal{E}) \rightarrow H^{q+1}(X, \Omega_{X/S}^{p-1} \otimes \mathcal{E})$$

is none other than the cup-product with the Kodaira-Spencer class  $\rho_{X/S}(D) \in H^1(X, Der(X/S))$ .

(3.5.3) If  $X$  is itself étale over  $A_S^n$ , with  $\Omega_{X/S}^1$  free with base  $\{dx_1, \dots, dx_n\}$ , then

$$H_{DR}^q(X/S, (\mathcal{E}, \nabla)) = H^q(\Gamma(X, \Omega_{X/S}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{E})),$$

and, for  $D \in \text{Der}_T(\mathcal{O}_S, \mathcal{O}_S)$ , the action of  $\tilde{\Gamma}(D)$  on  $H_{\text{dR}}^q(X/S, (\mathcal{E}, \nabla))$  is that induced from the  $T$ -endomorphism  $\tilde{D}$  of  $\Gamma(X, \Omega_{X/S}^* \otimes \mathcal{E})$ :

$$\tilde{D}(\omega \otimes e) = D_0(\omega) \otimes e + \omega \otimes \nabla(D_0)(e)$$

where  $D_0 \in \text{Der}_T(\mathcal{O}_X, \mathcal{O}_X)$  is the unique extension of  $D$  which kills  $dx_1, \dots, dx_n$ , and where  $\omega$  and  $e$  are sections of  $\Omega_{X/S}^*$  and  $\mathcal{E}$  respectively over  $X$ .

#### 4. Connections having logarithmic singularities.

(4.0) Let  $\pi : X \rightarrow S$  be a smooth morphism, and let  $i : Y \hookrightarrow X$  be the inclusion of a divisor with normal crossings relative to  $S$ ,  $j : X - Y \hookrightarrow X$  the inclusion of its complement. “Normal crossings” means that  $X$  may be covered by affine open sets  $U$  such that

(4.0.1)  $U$  is étale over  $A_S^n$ , via “coordinates”  $x_1, \dots, x_n$ .

(4.0.2)  $Y|U$  is defined by an equation  $x_1 \dots x_v = 0$  (i.e.,  $Y$  is the inverse image of the union of the first  $v \leq n$  of the coordinate hyperplanes in  $A_S^n$ ).

(4.1) We define a locally free  $\mathcal{O}_X$ -Module  $\Omega_{X/S}^1(\log Y)$ , by giving, as base over an open set  $U$  as above, the elements  $\frac{dx_1}{x_1}, \dots, \frac{dx_v}{x_v}, dx_{v+1}, \dots, dx_n$ . We define  $\Omega_{X/S}^i(\log Y) = \wedge_{\mathcal{O}_X}^i(\Omega_{X/S}^1(\log Y))$ . Viewing  $\Omega_{X/S}^*(\log Y)$  as a subsheaf of  $j_*(\Omega_{(X-Y)/S}^*)$ , we see that the usual exterior differentiation in  $j_*(\Omega_{(X-Y)/S}^*)$  preserves  $\Omega_{X/S}^*(\log Y)$ , which is thus (given the structure of) a complex (“the de Rham complex of  $X/S$  with logarithmic singularities along  $Y$ ”).

Now let  $\mathcal{M}$  be a quasicoherent  $\mathcal{O}_X$ -Module. An  $S$ -connection on  $\mathcal{M}$ , with logarithmic singularities along  $Y$ , is a homomorphism of abelian sheaves

$$(4.1.0) \quad \nabla : \mathcal{M} \rightarrow \Omega_{X/S}^1(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M}$$

such that

$$(4.1.1) \quad \nabla(gm) = g\nabla(m) + dg \otimes m$$

where  $g$  and  $m$  are sections of  $\mathcal{O}_X$  and  $\mathcal{M}$  respectively over an open subset of  $X$ . We denote by  $\mathcal{M}^\nabla$  the kernel of  $\nabla$ ;  $\mathcal{M}^\nabla$  is the sheaf of germs of horizontal sections.

(4.2) Just as for “ordinary” connections, we say that  $\nabla$  is *integrable* if the canonical extensions (1.0.2) of  $\nabla$  to maps

$$(4.2.0) \quad \nabla_i : \Omega_{X/S}^i(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \Omega_{X/S}^{i+1}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M}$$

make  $\Omega_{X/S}^*(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M}$  into a *complex* (“the de Rham complex of  $(\mathcal{M}, \nabla)$  with logarithmic singularities along  $Y$ ”).

Let  $\text{Der}_Y(X/S)$  be the sheaf on  $X$  defined by

$$(4.2.1) \quad \text{Der}_Y(X/S) = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}^1(\log Y), \mathcal{O}_X).$$

Over an open  $U$  as above,  $\text{Der}_Y(X/S)$  is  $\mathcal{O}_X$ -free on  $x_1 \frac{\partial}{\partial x_1}, \dots, x_v \frac{\partial}{\partial x_v}, \frac{\partial}{\partial x_{v+1}}, \dots, \frac{\partial}{\partial x_n}$ .

$\text{Der}_Y(X/S)$  is a sheaf of  $f^{-1}(\mathcal{O}_S)$ -Lie algebras, and an *integrable S-connection* in  $\mathcal{M}$  with logarithmic singularities along  $Y$  is nothing other than an  $\mathcal{O}_X$ -linear mapping

$$(4.2.2) \quad \nabla : \text{Der}_Y(X/S) \rightarrow \text{End}_{\mathcal{O}_S}(\mathcal{M})$$

which is compatible with brackets, and such that

$$(4.2.3) \quad \nabla(D)(gm) = D(g)m + g\nabla(D)(m)$$

where  $D, g$  and  $m$  are sections of  $\text{Der}_Y(X/S)$ ,  $\mathcal{O}_X$  and  $\mathcal{M}$  respectively over an open subset of  $X$ .

(4.3) We denote by  $\text{MIC}(X/S(\log Y))$  the abelian category of pairs  $(\mathcal{M}, \nabla)$ ,  $\mathcal{M}$  a quasicoherent  $\mathcal{O}_X$ -Module and  $\nabla$  an integrable S-connection on  $\mathcal{M}$  with logarithmic singularities along  $Y$ . (The morphisms are the horizontal ones.) Just as before (cf. (1.2)),  $\text{MIC}(X/S(\log Y))$  has enough injectives, and has a (not everywhere defined, cf. (1.1.2)) internal *Hom* and a tensor product.

(4.4) The de Rham cohomology sheaves on  $S$  of an object  $(\mathcal{M}, \nabla)$  in  $\text{MIC}(X/S(\log Y))$  are defined by

$$(4.4.0) \quad H_{\text{DR}}^q(X/S(\log Y), (\mathcal{M}, \nabla)) = \mathbf{R}^q \pi_*(\Omega_{X/S}^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M}).$$

Thus

$$(4.4.1) \quad H_{\text{DR}}^0(X/S(\log Y), (\mathcal{M}, \nabla)) = \pi_*(\mathcal{M}^{\nabla})$$

and the arguments of [17] or [19] show that the  $H_{\text{DR}}^q$  are the right derived functors of  $H_{\text{DR}}^0$ .

(4.5) Suppose now that  $f: S \rightarrow T$  is a smooth morphism. Then  $f \circ \pi: X \rightarrow T$  is a smooth morphism, and  $i: Y \hookrightarrow X$  is a divisor with normal crossings relative to  $T$ . As in (3.0) there is a natural forgetful functor

$$(4.5.0) \quad \begin{aligned} \text{MIC}(X/T(\log Y)) &\rightarrow \text{MIC}(X/S(\log Y)) \\ (\mathcal{E}, \nabla) &\mapsto (\mathcal{E}, \nabla)|_{\text{Der}_Y(X/S)} \end{aligned}$$

so that, just as in (3.0), we may define an exact connected sequence of cohomological functors

$$(4.5.1) \quad H_{\text{DR}}^q(X/S(\log Y), ?) : \text{MIC}(X/T(\log Y)) \rightarrow \text{MIC}(S/T)$$

by putting, for  $(\mathcal{E}, \nabla)$  an object of  $\text{MIC}(X/T(\log Y))$

$$(4.5.2) \quad H_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla)) = \mathbf{R}^q \pi_*(\Omega_{X/S}^{\bullet}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E}).$$

(4.6) The Gauss-Manin connection is constructed as before, using the canonical filtration of  $\Omega_{X/T}^{\bullet}(\log Y)$  by the subcomplexes

$$(4.6.0) \quad F^i(\Omega_{X/T}^{\bullet}(\log Y)) = (\text{image } \pi^*(\Omega_{S/T}^i) \otimes_{\mathcal{O}_X} \Omega_{X/T}^{*-i}(\log Y))$$

whose associated graded complexes are

$$(4.6.1) \quad \text{gr}^i(\Omega_{X/T}^{\bullet}(\log Y)) = \pi^*(\Omega_{S/T}^i) \otimes_{\mathcal{O}_X} \Omega_{X/T}^{*-i}(\log Y).$$

We then filter the de Rham complex of  $(\mathcal{E}, \nabla)$  with logarithmic singularities along  $Y$  by the subcomplexes

$$(4.6.2) \quad F^i(\Omega_{X/T}^\bullet(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E}) = F^i(\Omega_{X/T}^\bullet(\log Y)) \otimes_{\mathcal{O}_X} \mathcal{E}$$

whose associated graded objects are given by

$$(4.6.3) \quad \text{gr}^i(\Omega_{X/T}^\bullet(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E}) = \pi^*(\Omega_{S/T}^i) \otimes_{\mathcal{O}_X} (\Omega_{X/S}^{\bullet-i}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Then the de Rham complex  $\Omega_{S/T}^\bullet \otimes_{\mathcal{O}_S} H_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla))$  of the Gauss-Manin connection on  $H_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla))$  is the complex  $(E_1^{*,q}, d_1^{*,q})$  of  $E_1$  terms of the spectral sequence of the filtered (as above) object  $\Omega_{X/T}^\bullet(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E}$  and the functor  $\mathbf{R}^0\pi_*$ .

When  $S$  is affine, the Gauss-Manin connection can be “lifted” to a connection on the Čech bicomplex

$$(4.6.4) \quad C^p(\{U_\alpha\}, \Omega_{X/S}^q(\log Y) \otimes_{\mathcal{O}_X} \mathcal{E})$$

by exactly the same formulas as before, provided that:

(4.6.5) We use a covering of  $X$  by  $U$ 's as in (4.0.1), and use coordinates  $x_1, \dots, x_n$  on  $U$  so that  $Y$  is defined by  $x_1 \dots x_n = 0$ .

(4.6.6) We lift  $D \in \text{Der}(S/T)$  to the derivation of  $\mathcal{O}_X$  which extends it and kills  $dx_1, \dots, dx_n$  (so that, in particular, the lifting is *tangent* to  $Y$ ).

The Gauss-Manin connection acts on the spectral sequence over an affine  $S$  associated to a covering of  $X$  by affine open sets  $U_i$  verifying (4.0.1) and (4.0.2) (by using the Zariski filtration (3.5.1) of the Čech bicomplex)

$$(4.6.7) \quad E_1^{p,q} = C^p(\{U_i\}, \mathcal{H}_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla))) \Rightarrow H_{\text{DR}}^{p+q}(X/S(\log Y), (\mathcal{E}, \nabla))$$

where  $\mathcal{H}_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla))$  is the presheaf on  $X$  with values in  $\text{MIC}(S/T)$  given by

$$(4.6.8) \quad U \mapsto H_{\text{DR}}^q(U/S(\log Y), (\mathcal{E}, \nabla)|U).$$

## 5. Connections in Characteristic $p (> 0)$ .

(5.0) In this section, we suppose the base scheme  $T$  to be of characteristic  $p > 0$ , i.e., that  $p\mathcal{O}_T = 0$ . As before, let  $f: S \rightarrow T$  be a smooth  $T$ -scheme. Recall the Leibniz rule

$$(5.0.1) \quad D^n(gh) = \sum_{i=0}^n \binom{n}{i} D^i(g) D^{n-i}(h)$$

where  $D$ ,  $g$  and  $h$  are sections of  $\text{Der}(S/T)$ ,  $\mathcal{O}_S$  and  $\mathcal{O}_S$  respectively over an open subset of  $S$ . Putting  $n=p$ , we find (being in characteristic  $p$ ) that

$$(5.0.2) \quad D^p(gh) = D^p(g) \cdot h + gD^p(h)$$

i.e., that the  $p$ -th iterate of a derivation is a derivation, so that  $\text{Der}(S/T)$  is a sheaf of restricted  $p$ -Lie algebras.

Let  $(\mathcal{E}, \nabla)$  be an object of  $\text{MIC}(S/T)$ . Since  $\text{End}_T(\mathcal{E})$  is also a sheaf of restricted

$p$ -Lie algebras (taking the  $p$ -th iterate of a T-endomorphism), it is natural to ask whether or not the homomorphism

$$\nabla : \text{Der}(S/T) \rightarrow \text{End}_T(\mathcal{E})$$

is compatible with the  $p$ -structures, i.e., whether or not it is the case that

$$\nabla(D^p) = (\nabla(D))^p$$

whenever D is a section of  $\text{Der}(S/T)$  over an open subset of S.

With this question in mind, we define the “ $p$ -curvature”  $\psi$  of the connection  $\nabla$  as a mapping of sheaves

$$(5.0.3) \quad \psi : \text{Der}(S/T) \rightarrow \text{End}_T(\mathcal{E})$$

by setting

$$(5.0.4) \quad \psi(D) = (\nabla(D))^p - \nabla(D^p).$$

We remark that  $\psi$  “is” actually a mapping

$$(5.0.5) \quad \psi : \text{Der}(S/T) \rightarrow \text{End}_S(\mathcal{E})$$

(i.e., that  $\psi(D)$  is S-linear). To see this, we use the Leibniz rule

$$(5.0.6) \quad (\nabla(D))^m(ge) = \sum_{i=0}^m \binom{m}{i} D^i(g)(\nabla(D))^{m-i}(e)$$

where D, g and e are sections of  $\text{Der}(S/T)$ ,  $\mathcal{O}_S$  and  $\mathcal{E}$  over an open subset of S. Putting  $m=p$ , we get

$$(5.0.7) \quad (\nabla(D))^p(ge) = D^p(g)e + g(\nabla(D))^p e.$$

Since we have also the “connection-rule”

$$(5.0.8) \quad \nabla(D^p)(ge) = D^p(g)e + g\nabla(D^p)(e),$$

subtracting (5.0.8) from (5.0.7) gives the desired formula

$$(5.0.9) \quad \psi(D)(ge) = g\psi(D)(e).$$

We recall that having “ $p$ -curvature zero” means having enough horizontal sections. More precisely:

*Theorem (5.1)* (Cartier). — Let  $f: S \rightarrow T$  be a smooth T-scheme of characteristic  $p$ .

(5.1.0) Let  $F_{\text{abs}}: T \rightarrow T$  be the absolute Frobenius (i.e., the  $p$ -th power mapping on  $\mathcal{O}_T$ ), and

(5.1.1)  $S^{(p)} = S \times_{T, F_{\text{abs}}} T$ , the fibre product of  $F_{\text{abs}}: T \rightarrow T$  and  $f: S \rightarrow T$ . Let  $F: S \rightarrow S^{(p)}$  be the relative Frobenius (i.e., elevation of vertical coordinates to the  $p$ -th power).

There is an equivalence of categories between the category of quasi-coherent sheaves on  $S^{(p)}$  and the full subcategory of  $\text{MIC}(S/T)$  consisting of objects  $(\mathcal{E}, \nabla)$  whose  $p$ -curvature is zero. This equivalence may be given explicitly as follows :

Let  $\mathcal{F}$  be a quasicoherent sheaf on  $S^{(p)}$ . Then there is a unique  $T$ -connection  $\nabla_{\text{can}}$ , integrable and of  $p$ -curvature zero, on  $F^*(\mathcal{F})$ , such that

$$\mathcal{F} \xrightarrow{\sim} (F^*(\mathcal{F}))^{\nabla_{\text{can}}}.$$

The desired functor is  $\mathcal{F} \mapsto (F^*(\mathcal{F}), \nabla_{\text{can}})$ .

Given an object  $(\mathcal{E}, \nabla)$  of  $\mathbf{MIC}(S/T)$  of  $p$ -curvature zero, we form  $\mathcal{E}^\nabla$ , which is in a natural way a quasicoherent sheaf on  $S^{(p)}$ . The desired inverse functor is

$$(\mathcal{E}, \nabla) \mapsto \mathcal{E}^\nabla.$$

*Proof.* — The only point requiring proof is that, if an object  $(\mathcal{E}, \nabla)$  of  $\mathbf{MIC}(S/T)$  has  $p$ -curvature zero, then the canonical mapping of  $\mathcal{O}_S$ -modules

$$(5.1.1.0) \quad F^*(\mathcal{E}^\nabla) \rightarrow \mathcal{E}$$

is an isomorphism. The question being local on  $S$ , we may suppose  $S$  is affine, and étale over  $A_T^r$ , with  $\Omega_{S/T}^1$  free on  $\{ds_1, \dots, ds_r\}$ . Consider the  $F^{-1}(\mathcal{O}_{S^{(p)}})$ -linear endomorphism  $P$  of  $\mathcal{E}$ , given by

$$(5.1.2) \quad P = \sum_w \prod_{i=1}^r \left( \frac{(-s_i)^{w_i}}{w_i!} \right) \prod_{i=1}^r \nabla \left( \frac{\partial}{\partial s_i} \right)^{w_i}$$

the sum taken over all  $r$ -tuples  $(w_1, \dots, w_r)$  of integers satisfying  $0 \leq w_i \leq p-1$ .

One immediately verifies that

$$(5.1.3) \quad P(\mathcal{E}) \subset \mathcal{E}^\nabla$$

$$(5.1.4) \quad P|_{\mathcal{E}^\nabla} = \text{id}$$

$$(5.1.5) \quad P^2 = P \text{ is a projection onto } \mathcal{E}^\nabla$$

$$(5.1.6) \quad \bigcap_w \text{Kernel of } P \cdot \prod_{i=1}^r \nabla \left( \frac{\partial}{\partial s_i} \right)^{w_i} = \{0\},$$

the intersection extended to all  $r$ -tuples of integers  $(w_1, \dots, w_r)$  with  $0 \leq w_i \leq p-1$ .

It follows that the mapping inverse to (5.1.1.0) is given explicitly by

$$(5.1.7) \quad \begin{aligned} \text{Taylor} : \mathcal{E} &\rightarrow F^*(\mathcal{E}^\nabla) \\ \text{Taylor}(e) &= \sum_w \prod_{i=1}^r \frac{s_i^{w_i}}{w_i!} P \cdot \prod_{i=1}^r \left( \nabla \left( \frac{\partial}{\partial s_i} \right) \right)^{w_i} (e). \end{aligned}$$

We now develop the basic properties of the  $p$ -curvature.

*Proposition (5.2).*

(5.2.0) The mapping  $\psi : \text{Der}(S/T) \rightarrow \text{End}_S(\mathcal{E})$  is  $p$ -linear, i.e., it is additive, and  $\psi(gD) = g^p \psi(D)$  whenever  $g$  and  $D$  are sections of  $\mathcal{O}_S$  and  $\text{Der}(S/T)$  over an open subset of  $S$ .

(5.2.1) If  $D$  is a section of  $\text{Der}(S/T)$  over an open subset  $U$  of  $S$ , the three  $T$ -endomorphisms of  $\mathcal{E}|_U$

$$\nabla(D), \nabla(D^p), \psi(D)$$

mutually commute.

(5.2.2) If  $D$  and  $D'$  are any two sections of  $\text{Der}(S/T)$  over an open subset of  $S$ , then  $\psi(D)$  and  $\psi(D')$  commute.

(5.2.3)  $\psi$  takes values in the sheaf of germs of horizontal  $S$ -endomorphisms of  $\mathcal{E}$ .

*Proof.* — To prove that  $\psi$  is additive, we use the Jacobson formula. If  $a$  and  $b$  are elements of an associative ring  $R$  of characteristic  $p$ , then

$$(5.2.4) \quad (a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)$$

where the  $s_i(a, b)$  are the universal Lie polynomials obtained by passing to  $R[t]$ ,  $t$  an indeterminate, and writing

$$(5.2.5) \quad (\text{ad}(ta+b))^{p-1}(a) = \sum_{i=1}^{p-1} i s_i(a, b) t^{i-1}.$$

Now let  $D, D' \in \text{Der}(S/T)$ . Then by (5.2.4)

$$(5.2.6) \quad (D+D')^p = D^p + D'^p + \sum_{i=1}^{p-1} s_i(D, D')$$

and, as the connection  $\nabla$  is integrable and the  $s_i$  are Lie polynomials, we have

$$(5.2.7) \quad \nabla((D+D')^p) = \nabla(D^p) + \nabla(D'^p) + \sum_{i=1}^{p-1} s_i(\nabla(D), \nabla(D')).$$

Again applying (5.2.4), we have

$$(5.2.8) \quad (\nabla(D) + \nabla(D'))^p = (\nabla(D))^p + (\nabla(D'))^p + \sum_{i=1}^{p-1} s_i(\nabla(D), \nabla(D')).$$

Subtracting, we find that  $\psi(D+D') = \psi(D) + \psi(D')$ .

We next prove that  $\psi$  is  $p$ -linear. For this we use Deligne's identity for  $(gD)^p$ .

*Proposition (5.3)* (Deligne). — Let  $A$  be an associative ring of characteristic  $p$ ,  $g$  and  $D$  two elements of  $A$ . For each integer  $n \geq 0$ , put  $g^{(n)} = (\text{ad}(D))^n(g) = [\underbrace{D, [D, \dots [D}_n g] \dots]$ . Suppose that the elements  $g^{(n)}$ , for  $n \geq 0$  mutually commute. Then

$$(5.3.0) \quad (gD)^p = g^p D^p + g(g^{p-1})^{(p-1)} D.$$

*Proof.* — Reducing to the “universal” case, we may suppose that  $g$  is invertible; let  $h = g^{-1}$ . By induction, it is easily seen that for each positive integer  $n$ , we have

$$(5.3.1) \quad (*_n) \quad (gD)^n = (h^{-1}D)^n = h^{-2n} \sum_{\mathbf{m}} A_{\mathbf{m}} (\prod_i h^{(m_i)} D^{n-\sum m_i})$$

the sum being over all  $n$ -tuples of integers  $\mathbf{m} = (m_1, \dots, m_n)$  having  $0 \leq m_1 \leq m_2 \leq \dots \leq m_n$  and  $\sum_i m_i < n$ , with the  $A_{\mathbf{m}} \in \mathbf{F}_p$ .

Consider now the special case of the ring of additive endomorphisms of the field

$\mathbf{F}_p(X_0, \dots, X_{p-2}, T)$ , and let  $D = \frac{d}{dT}$ ,  $h = \sum_{i=0}^{p-2} X_i T^i / i!$ . Then  $D^p = 0$ , and because  $h = D(k)$  with  $k = \sum_{i=0}^{p-2} X_i \frac{T^{i+1}}{(i+1)!}$  we have  $h^{(p-1)} = D^{p-1}(h) = 0$  and  $(h^{-1}D)^p = 0$  (since  $h^{-1}D = \frac{d}{dk}$ ). On the other hand,  $D, D^2, \dots, D^{p-1}$  are linearly independent over  $K$ .

Putting  $n=p$  in 5.3.1, we thus find that for each integer  $j$ ,  $0 < j < p$ ,

$$(5.3.2) \quad 0 = \sum_{\mathbf{m}} A_{\mathbf{m}} \left( \prod_{i=1}^n h^{(m_i)} \right)$$

the sum being over those  $\mathbf{m} = (m_1, \dots, m_p)$ ,  $0 \leq m_1 \leq \dots \leq m_p$  with  $\sum_i m_i = j$ . As  $g^{(0)}, \dots, g^{(p-2)}$  are algebraically independent over  $\mathbf{F}_p$  (their values at  $T=0$  being  $X_0, \dots, X_{p-2}$ ), we have

$$(5.3.3) \quad A_{\mathbf{m}} = 0 \text{ if } \mathbf{m} = (m_1, \dots, m_p), \quad 0 \leq m_1 \leq \dots \leq m_p, \quad \sum_i m_i < p \text{ and each } m_i < p-1.$$

The only possibly non-zero  $A_{\mathbf{m}}$  in  $(*)_p$  are thus  $A_{(0, \dots, 0)}$  and  $A_{(0, \dots, 0, p-1)}$ . Returning to  $(*_n)$ , it is immediately verified by induction on  $n$  that  $A_{(0, \dots, 0)} = 1$  and  $A_{(0, \dots, 0, n-1)} = 1$ , so that (5.3.1) with  $n=p$  becomes the desired formula:

$$(5.3.4) \quad \begin{aligned} (gD)^p &= (h^{-1}D)^p = h^{-2p} (h^p D^p + h^{p-1} h^{(p-1)} D) \\ &= g^p D^p + g(g^{p-1})^{(p-1)} D. \end{aligned}$$

(5.4) We now return to the proof of (5.2). Applying (5.3.0) to  $g$  and  $D$  in  $\text{End}_T(\mathcal{O}_S)$ , we have

$$(5.4.0) \quad \begin{aligned} (gD)^p &= g^p D^p + g(\text{ad}(D))^{p-1} (g^{p-1}) \cdot D \\ &= g^p D^p + g D^{p-1} (g^{p-1}) \cdot D \end{aligned}$$

whence

$$(5.4.1) \quad \nabla((gD)^p) = g^p \nabla(D^p) + g D^{p-1} (g^{p-1}) \nabla(D).$$

Applying (5.3.0) to  $g$  and  $\nabla(D)$  in  $\text{End}_T(\mathcal{E})$ , we have

$$(5.4.2) \quad \begin{aligned} (\nabla(gD))^p &= (g\nabla(D))^p = g^p (\nabla(D))^p + g(\text{ad}(\nabla(D)))^{p-1} (g^{p-1}) \nabla(D) \\ &= g^p \nabla(D)^p + g D^{p-1} (g^{p-1}) \nabla(D). \end{aligned}$$

Subtracting (5.4.1) from (5.4.2) gives the desired  $p$ -linearity.

To prove (5.2.1), we remark that  $D$  and  $D^p$  commute, thus,  $\nabla$  being integrable, so do  $\nabla(D)$  and  $\nabla(D^p)$ , whence  $\psi(D) = (\nabla(D))^p - \nabla(D^p)$  commutes with  $\nabla(D)$  and  $\nabla(D^p)$ .

We now prove (5.2.2) and (5.2.3). The question being local on  $S$ , we may suppose that  $S$  is affine, and is étale over  $\mathbf{A}_T^r$ , so that  $\Omega_{S/T}^1$  is free, with base  $\{ds_1, \dots, ds_r\}$ .

We denote  $\left\{ \frac{\partial}{\partial s_1}, \dots, \frac{\partial}{\partial s_r} \right\}$  the dual base of  $\text{Der}(S/T)$ . Let

$$(5.4.3) \quad D = \sum_i a_i \frac{\partial}{\partial s_i}, \quad D' = \sum_i b_i \frac{\partial}{\partial s_i};$$

we must prove that

$$(5.4.4) \quad [\psi(D), \psi(D')] = o = [\psi(D), \nabla(D')].$$

But

$$\psi(D) = \sum_i a_i^p \psi\left(\frac{\partial}{\partial s_i}\right) = \sum_i a_i^p \left(\nabla\left(\frac{\partial}{\partial s_i}\right)\right)^p,$$

$$\psi(D') = \sum_i b_i^p \left(\nabla\left(\frac{\partial}{\partial s_i}\right)\right)^p, \quad \text{and} \quad \nabla(D') = \sum_i b_i \nabla\left(\frac{\partial}{\partial s_i}\right),$$

so that

$$[\psi(D), \psi(D')] = \sum_{i,j} a_i^p b_j^p \left[ \left(\nabla\left(\frac{\partial}{\partial s_i}\right)\right)^p, \left(\nabla\left(\frac{\partial}{\partial s_j}\right)\right)^p \right] = o,$$

and

$$[\psi(D), \nabla(D')] = \sum_{i,j} a_i^p b_j \left[ \left(\nabla\left(\frac{\partial}{\partial s_i}\right)\right)^p, \nabla\left(\frac{\partial}{\partial s_j}\right) \right] = o.$$

*Corollary (5.5). — Let  $f : S \rightarrow T$  be a smooth  $T$ -scheme of characteristic  $p$ ,  $(\mathcal{E}, \nabla)$  an object of  $\mathbf{MIC}(S/T)$ , and  $n \geq 1$  an integer. The following conditions are equivalent.*

(5.5.0) *There exists a filtration of  $(\mathcal{E}, \nabla)$  of length  $\leq n$  (i.e.,  $F^0 = \text{all}$ ,  $F^n = \{o\}$ ) whose associated graded object has  $p$ -curvature zero.*

(5.5.1) *Wherever  $D_1, \dots, D_n$  are sections of  $\text{Der}(S/T)$  over an open subset of  $S$ ,  $\psi(D_1)\psi(D_2)\dots\psi(D_n) = o$ .*

(5.5.2) *There exists a covering of  $S$  by affine open subsets  $U$ , and on each  $U$  “coordinates”  $u_1, \dots, u_r$  (i.e., sections of  $\mathcal{O}_S$  over  $U$  such that  $\Omega_{U/T}^1$  is free on  $du_1, \dots, du_r$ ) such that for every  $r$ -tuple  $(w_1, \dots, w_r)$  of integers with  $\sum_i w_i = n$ ,*

$$\left(\nabla\left(\frac{\partial}{\partial u_1}\right)\right)^{pw_1} \cdots \left(\nabla\left(\frac{\partial}{\partial u_r}\right)\right)^{pw_r} = o.$$

*Proof.* — (5.5.0)  $\Leftrightarrow$  (5.5.1) is clear.

$$(5.5.1) \Rightarrow (5.5.2) \text{ because } \psi\left(\frac{\partial}{\partial u_i}\right) = \left(\nabla\left(\frac{\partial}{\partial u_i}\right)\right)^p.$$

(5.5.2)  $\Rightarrow$  (5.5.1) by the  $p$ -linearity of  $\psi$ ; for, covering any open set by its intersection with the covering of (3), we are immediately reduced to the case in which  $D_1, \dots, D_n \in \text{Der}(U/T)$ . We expand each  $D_i$  using the given coordinates on  $U$

$$D_i = \sum_j a_{ij} \frac{\partial}{\partial u_j}$$

whence

$$\psi(D_i) = \sum_j a_{ij}^p \psi\left(\frac{\partial}{\partial u_j}\right) = \sum_j a_{ij} \left(\nabla\left(\frac{\partial}{\partial u_j}\right)\right)^p,$$

and the assertion is clear.

*Definition (5.6). — We say that  $(\mathcal{E}, \nabla)$  is nilpotent of exponent  $\leq n$  when one of the equivalent conditions of (5.5) is verified. We say that  $(\mathcal{E}, \nabla)$  is nilpotent if there exists*

a positive integer  $n$  such that it is nilpotent of exponent  $\leq n$ . We denote by  $\text{Nilp}(S/T)$  the full subcategory of  $\text{MIC}(S/T)$  of objects  $(\mathcal{E}, \nabla)$  which are nilpotent, by  $\text{Nilp}^n(S/T)$  those which are nilpotent of exponent  $\leq n$ .  $\text{Nilp}^1$  consists of those of  $p$ -curvature zero.

We record for future reference:

*Proposition (5.7).*

(5.7.0)  $\text{Nilp}(S/T)$  is an exact abelian subcategory of  $\text{MIC}(S/T)$ .

(5.7.1) Each  $\text{Nilp}^n(S/T)$  is stable under the operations of taking sub-objects and quotient objects.

(5.7.2)  $\text{Nilp}(S/T)$  is stable under the operations of internal Hom (when it is defined, cf. (1.1.2)) and internal tensor product, and if  $A$  and  $B$  are objects of  $\text{Nilp}^n(S/T)$  and  $\text{Nilp}^m(S/T)$  respectively, then  $A \otimes B$  and  $\text{Hom}(A, B)$ , if defined, are in  $\text{Nilp}^{n+m-1}(S/T)$ .

*Proposition (5.8).* —  $(\mathcal{E}, \nabla)$  is nilpotent if and only if for any section  $D$  of  $\text{Der}(S/T)$  (over an open subset  $U$  of  $S$ ) which, as a  $T$ -endomorphism of  $\mathcal{O}_U$ , is nilpotent, the corresponding  $T$ -endomorphism  $\nabla(D)$  of  $\mathcal{E}|_U$  is nilpotent.

*Proof.* — ( $\Rightarrow$ ) If  $D^p = 0$  in  $\text{Der}(U/T)$ ,  $\psi(D) = (\nabla(D))^p$  is nilpotent by assumption. By induction on the integer  $v$  such that  $D^{p^v} = 0$  in  $\text{Der}(U/T)$ , we may suppose already proven the nilpotence of  $\nabla(D^p)$  (since  $(D^p)^{p^{v-1}} = 0$ ). But  $(\nabla(D))^p = \psi(D) + \nabla(D^p)$ , a sum of commuting (5.2.1) nilpotents.

( $\Leftarrow$ ) take a finite covering of  $S$  by affine open sets  $U$  which are étale over  $\mathbf{A}_T^r$ . On each  $U$ , choose “coordinates”  $u_1, \dots, u_r$  (i.e., sections of  $\mathcal{O}_U$  which define an étale morphism  $U \rightarrow \mathbf{A}_T^r$ ). Then each  $\left(\frac{\partial}{\partial u_i}\right)^p = 0$  in  $\text{Der}(U/T)$ . Let  $n_U$  be an integer such that, for each  $i$ ,  $\left(\nabla\left(\frac{\partial}{\partial u_i}\right)\right)^{p^{n_U}} = 0$  in  $\text{End}_T(\mathcal{E}/U)$ ; and take  $n = \sup_{\{U\}} n_U$ . Then  $(\mathcal{E}, \nabla)$  is nilpotent of exponent  $\leq n^2$ .

*Theorem (5.9).* — Let  $f: S \rightarrow T$  and  $f': S' \rightarrow T'$  be smooth morphisms, and

$$(5.9.0) \quad \begin{array}{ccc} S' & \xrightarrow{g} & S \\ f' \downarrow & & \downarrow f \\ T' & \xrightarrow{h} & T \end{array}$$

a commutative diagram. Suppose  $T$  is of characteristic  $p$ . Then under the inverse image functor

$$(5.9.1) \quad (g, h)^*: \text{MIC}(S/T) \rightarrow \text{MIC}(S'/T')$$

we have, for every integer  $n \geq 1$ ,

$$(5.9.2) \quad (g, h)^*(\text{Nilp}^n(S/T)) \subset \text{Nilp}^n(S'/T').$$

*Proof.* — The proof is by induction on  $n$ , the exponent of nilpotence. Suppose first the theorem proven for  $v=1, \dots, n-1$ , and take an object  $(\mathcal{E}, \nabla)$  in  $\text{Nilp}^n(S/T)$ . By definition there is an exact sequence in  $\text{MIC}(S/T)$

$$0 \rightarrow (\mathcal{E}', \nabla') \rightarrow (\mathcal{E}, \nabla) \rightarrow (\mathcal{E}'', \nabla'') \rightarrow 0$$

with  $(\mathcal{E}', \nabla') \in \mathbf{Nilp}^1(S/T)$ ,  $(\mathcal{E}'', \nabla'') \in \mathbf{Nilp}^{n-1}(S/T)$ . Pulling back gives an exact sequence in  $\mathbf{MIC}(S'/T')$

$$(g, h)^*(\mathcal{E}', \nabla') \rightarrow (g, h)^*(\mathcal{E}, \nabla) \rightarrow (g, h)^*(\mathcal{E}'', \nabla'') \rightarrow 0$$

and by hypothesis,  $(g, h)^*(\mathcal{E}', \nabla') \in \mathbf{Nilp}^1(S'/T')$ , and  $(g, h)^*(\mathcal{E}'', \nabla'') \in \mathbf{Nilp}^{n-1}(S'/T')$ , whence, by definition,  $(g, h)^*(\mathcal{E}, \nabla) \in \mathbf{Nilp}^n(S'/T')$ .

Thus it suffices to prove that  $\mathbf{Nilp}^1$  is stable under inverse image. To do this, we make use of the fibre product to reduce to checking two cases.

*Case 1.* —  $S = S \times_T T'$ ,  $g = \text{pr}_1$ ,  $f' = \text{pr}_2$ . The question is local on  $S$ , which we suppose is affine, and étale over  $\mathbf{A}_T^r$ , with  $\Omega_{S/T}^1$  free on  $ds_1, \dots, ds_r$ . Then  $S'$  is étale over  $\mathbf{A}_{T'}^r$ , with  $\Omega_{S'/T'}^1$  free on  $ds'_1, \dots, ds'_r$ , where  $s'_i = g^*(s_i)$ . By the  $p$ -linearity of  $\psi$ , it suffices to check that  $\psi\left(\frac{\partial}{\partial s'_i}\right) = \left(\nabla\left(\frac{\partial}{\partial s'_i}\right)\right)^p = 0$  in  $\text{End}_{S'}(g^*(\mathcal{E}))$ . But  $\nabla\left(\frac{\partial}{\partial s'_i}\right) \in \text{End}_{T'}(g^*(\mathcal{E}))$  is the  $T'$ -linear endomorphism of  $g^*(\mathcal{E}) \simeq \mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$  deduced from the  $T$ -linear endomorphism  $\nabla\left(\frac{\partial}{\partial s_i}\right)$  of  $\mathcal{E}$  by extension of scalars  $\mathcal{O}_T \rightarrow \mathcal{O}_{T'}$ .

*Case 2.* —  $T' = T$ ,  $h = \text{id}$ . We have the commutative diagram of  $T$ -schemes (cf. (5.1.1))

$$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ \downarrow F' & & \downarrow F \\ S'^{(p)} & \xrightarrow{g^{(p)}} & S^{(p)} \end{array}$$

By Cartier's theorem (5.1), any object  $(\mathcal{E}, \nabla) \in \mathbf{MIC}(S/T)$  with  $p$ -curvature zero is isomorphic to  $(F^*(\mathcal{F}), \nabla_{\text{can}})$ , where  $\mathcal{F}$  is a quasicoherent  $S^{(p)}$ -Module (namely  $\mathcal{E}^\nabla$ ). Clearly we have

$$(g, \text{id})^*(F^*(\mathcal{F}), \nabla_{\text{can}}) = (F'^*(g^{(p)*}(\mathcal{F})), \nabla_{\text{can}}),$$

an object of  $p$ -curvature zero.

We now prove the stability of nilpotence under higher direct images.

*Theorem (5.10).* — Let  $\pi : X \rightarrow S$  and  $f : S \rightarrow T$  be smooth morphisms, with  $T$  a scheme of characteristic  $p$ . Let  $n$  be the relative dimension of  $X/S$ , supposed constant. Suppose  $S$  is affine, and consider the spectral sequence (3.5.1.0) associated to a covering  $\{U_\alpha\}$  of  $X$  by open subsets étale over  $\mathbf{A}_S^n$  and an object  $(\mathcal{E}, \nabla) \in \mathbf{Nilp}^n(X/T)$ :

$$(5.10.0) \quad E_1^{p,q} = G^p(\{U_\alpha\}, \mathcal{H}_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla))) \Rightarrow H_{\text{DR}}^{p+q}(X/S, (\mathcal{E}, \nabla)),$$

on which  $\text{Der}(S/T)$  acts through the Gauss-Manin connection.

(5.10.1) Each term  $E_r^{p,q} \in \mathbf{Nilp}^n(S/T)$ .

(5.10.2) For each integer  $i \geq 0$  we put

$$\tau(i) = \text{the number of integers } p \text{ with } E_\infty^{p,i-p} \neq 0.$$

Then

$$H_{\text{DR}}^i(X/S, (\mathcal{E}, \nabla)) \in \mathbf{Nilp}^{n-\tau(i)}(S/T).$$

$$(5.10.3) \quad \tau(i) \leq i+1, \quad \text{and} \quad \tau(i) \leq 2n-i+1.$$

*Proof.* — To prove (5.10.1), it suffices to show each  $E_1^{p,q} \in \mathbf{Nilp}^v(S/T)$ . But

$$(5.10.4) \quad E_1^{p,q} = \prod_{i_0 < \dots < i_p} H_{\text{DR}}^q((U_{i_0} \cap \dots \cap U_p)/S, (\mathcal{E}, \nabla)|_{(U_{i_0} \cap \dots \cap U_{i_p})})$$

so that we must prove that if  $(\mathcal{E}, \nabla) \in \mathbf{Nilp}^v(S/T)$  and  $X$  is étale over  $A_S^n$ , then  $H_{\text{DR}}^i(X/S, (\mathcal{E}, \nabla)) \in \mathbf{Nilp}^v(S/T)$ .

Let us remark first that, if  $X$  is étale over  $A_S^n$ , the Gauss-Manin connection  $\alpha$  on  $H_{\text{DR}}(X/S, (\mathcal{E}, \nabla)) = R\pi_*(\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$  is deduced from an integrable  $T$ -connection on the complex  $\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}$  (cf. (3.5.3)) which may be described explicitly as follows. Let  $\Omega_{X/S}^1$  be free on  $dx_1, \dots, dx_n$ , and for each  $D \in \text{Der}(S/T)$  denote by  $D_0 \in \text{Der}(X/T)$  the unique extension of  $D$  which kills  $dx_1, \dots, dx_n$ . Then the Gauss-Manin connection is deduced from the integrable connection

$$\alpha : \text{Der}(S/T) \rightarrow \text{End}_T(\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$$

given by  $\alpha(D)((dx_{i_1} \wedge \dots \wedge dx_{i_r}) \otimes e) = (dx_{i_1} \wedge \dots \wedge dx_{i_r}) \otimes \nabla(D_0)(e)$ .

Clearly we have

$$(\alpha(D))^p((dx_{i_1} \wedge \dots \wedge dx_{i_r}) \otimes e) = (dx_{i_1} \wedge \dots \wedge dx_{i_r}) \otimes (\nabla(D_0))^p(e),$$

thus the hypothesis  $(\mathcal{E}, \nabla) \in \mathbf{Nilp}^v(X/T)$  implies that, for any  $D^{(1)}, \dots, D^{(v)} \in \text{Der}(S/T)$ , the endomorphism  $\psi(D_0^{(1)}) \dots \psi(D_0^{(v)})$  of  $\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}$  is zero, and hence it is zero on  $H_{\text{DR}}(X/S, (\mathcal{E}, \nabla))$ , which concludes the proof of (5.10.1).

To prove (5.10.2), notice that

$$(5.10.5) \quad E_\infty^{p,q} = \text{gr}_{\text{zar}}^p H_{\text{DR}}^{p+q}(X/S, (\mathcal{E}, \nabla)),$$

so that  $H_{\text{DR}}^i(X/S, (\mathcal{E}, \nabla))$  has a horizontal filtration with  $\tau(i)$  non-zero quotients, each quotient in  $\mathbf{Nilp}^v(S/T)$ .

To prove (5.10.3), we observe first that  $E_1^{p,q} = 0$ , unless  $p \geq 0$  and

$$0 \leq p \leq n = \text{rel. dim}(X/S).$$

To conclude the proof, it suffices to show that  $E_2^{p,q} = 0$  (and hence  $E_\infty^{p,q} = 0$ ) if  $p > n$ . (The problem is that, while  $\pi : X \rightarrow S$  has cohomological dimension  $\leq n$  for sheaves, our  $E_2^{p,q}$  terms are, *a priori*, only the Čech cohomology of certain presheaves. But being in characteristic  $p$  will allow us to circumvent these difficulties by using an idea of Deligne.)

Let  $S \xrightarrow{F_{\text{abs}}} S$  be the absolute Frobenius,  $X^{(p)}$  the fibre product of  $\pi : X \rightarrow S$  and  $F_{\text{abs}} : S \rightarrow S$ , and  $F : X \rightarrow X^{(p)}$  the relative Frobenius (cf. (5.1.0)).

The complex  $\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}$  is linear over  $\pi^{-1}(\mathcal{O}_S)$  and over  $(\mathcal{O}_X)^p$ ; in other words,  $F_*(\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$  is an  $\mathcal{O}_{X^{(p)}}^p$ -linear complex of quasicoherent  $\mathcal{O}_{X^{(p)}}$ -Modules. Thus the cohomology presheaves of this complex,  $\mathcal{H}^i(F_*(\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}))$ , are sheaves of quasicoherent  $\mathcal{O}_{X^{(p)}}$ -Modules. As we have an isomorphism of presheaves on  $X^{(p)}$

$$(5.10.6) \quad F_* \mathcal{H}_{\text{DR}}^i(X/S, (\mathcal{E}, \nabla)) \xrightarrow{\sim} \mathcal{H}^i(F_*(\Omega_{X/S}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}))$$

it follows that the presheaves  $\mathcal{H}_{\text{DR}}^i(X/S, (\mathcal{E}, \nabla))$  are in fact *sheaves*. Furthermore, we have

(5.10.7)

$$\begin{aligned} E_2^{p,q} &= \check{H}^p(X, \{U_\alpha\}, \mathcal{H}_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla))) = \check{H}^p(X^{(p)}, \{F(U_\alpha)\}, F_*\mathcal{H}_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla))) \\ &= H^p(X^{(p)}, F_*\mathcal{H}_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla))) \end{aligned}$$

the last equality because  $F_*\mathcal{H}_{\text{DR}}^q(X/S, (\mathcal{E}, \nabla))$  is quasicoherent on  $X^{(p)}$ , and  $\{F(U_\alpha)\}$  is a covering of  $X^{(p)}$  by *affine* open sets. As  $X^{(p)}/S$  is of cohomological dimension  $\leq n$ , we have  $E_2^{p,q} = 0$  if  $p > n$ , which concludes the proof of (5.10.3).

*Remark (5.10.8).* — The interpretation (5.10.7) of the  $E_2^{p,q}$  term of the spectral sequence (5.10.0) shows that the Zariski filtration it defines on the  $H_{\text{DR}}^i(X/S, (\mathcal{E}, \nabla))$  is *independent* of the choice of covering of  $X$  by affine open sets étale over  $A_S^n$ . Indeed, it shows that the entire spectral sequence, from  $E_2$  on, is independent of that choice. We do not know if this is true when  $S$  is no longer of characteristic  $p$ .

## 6. Connections in characteristic $p > 0$ having logarithmic singularities.

(6.0) Let  $\pi: X \rightarrow S$  be a smooth morphism,  $i: Y \hookrightarrow X$  the inclusion of a divisor with normal crossings relative to  $S$ , and  $f: S \rightarrow T$  a smooth morphism, with  $T$  of characteristic  $p$ .

We define  $\text{Nilp}^\nu(X/T(\log Y))$  to be the full subcategory of  $\text{MIC}(X/T(\log Y))$  consisting of objects admitting a filtration which has at most  $\nu$  non-zero quotients, each of  $p$ -curvature zero. In this context, the  $p$ -curvature of an object  $(\mathcal{E}, \nabla)$  in  $\text{MIC}(X/T(\log Y))$  is the  $p$ -linear mapping

$$(6.0.1) \quad \begin{aligned} \psi: \text{Der}_Y(X/T) &\rightarrow \text{End}_T(\mathcal{E}) \\ \psi(D) &= (\nabla(D))^p - \nabla(D^p). \end{aligned}$$

The proof of (5.10) carries over *mutatis mutandis* to give

*Theorem (6.1)=(5.10) bis.* — *Assumptions as above, suppose that  $S$  is affine, and let  $n = \text{rel. dim}(X/S)$ . Let  $(\mathcal{E}, \nabla)$  be an object of  $\text{MIC}(X/T(\log Y))$ . Consider the spectral sequence (4.6.7),*

$$(6.1.0) \quad E_1^{p,q} = G^p(\{U_\alpha\}, \mathcal{H}_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla))) \Rightarrow H_{\text{DR}}^{p+q}(X/S(\log Y), (\mathcal{E}, \nabla))$$

which by (5.10.7), has

$$(6.1.1) \quad E_2^{p,q} = H^p(X^{(p)}, F_*\mathcal{H}_{\text{DR}}^q(X/S(\log Y), (\mathcal{E}, \nabla)))$$

and, from  $E_2$  on, is independent of choice of covering.  $\text{Der}(S/T)$  acts on this spectral sequence through the Gauss-Manin connection. Suppose  $(\mathcal{E}, \nabla) \in \text{Nilp}^\nu(X/T(\log Y))$ . Then

(6.1.2) Each term  $E_r^{p,q} \in \text{Nilp}^\nu(S/T)$ .

(6.1.3) For each integer  $i \geq 0$ , put

$$\tau(i) = \text{the number of integers } p \text{ with } E_\infty^{p,i-p} \neq 0.$$

Then

$$H_{\text{DR}}^i(X/S(\log Y), (\mathcal{E}, \nabla)) \in \mathbf{Nilp}^{\vee, \tau(i)}(S/T).$$

$$(6.1.4) \quad \tau(i) \leq i+1 \quad \text{and} \quad \tau(i) \leq 2n-i+1.$$

## 7. Ordinary de Rham cohomology in characteristic $p$ .

(7.0) Let  $\pi : X \rightarrow S$  be a smooth morphism,  $i : Y \hookrightarrow X$  the inclusion of a divisor with normal crossings relative to  $S$ . The structural sheaf  $\mathcal{O}_X$ , with the integrable  $S$ -connection “exterior differentiation”

$$(7.0.1) \quad d_{X/S} : \mathcal{O}_X \rightarrow \Omega_{X/S}^1$$

defines an object in  $\mathbf{MIC}(X/S)$ .

We denote the de Rham cohomology sheaves on  $S$  of this object simply  $H_{\text{DR}}(X/S)$ , i.e. by definition

$$(7.0.2) \quad H_{\text{DR}}^q(X/S) = \mathbf{R}^q \pi_*(\Omega_{X/S}^\bullet).$$

Similarly, by composing (7.0.2) with the canonical inclusion  $\Omega_{X/S}^1 \hookrightarrow \Omega_{X/S}^1(\log Y)$ , we obtain an object in  $\mathbf{MIC}(X/S(\log Y))$  whose de Rham cohomology sheaves on  $S$  are denoted  $H_{\text{DR}}(X/S(\log Y))$ , i.e., by definition

$$(7.0.3) \quad H_{\text{DR}}^q(X/S(\log Y)) = \mathbf{R}^q \pi_*(\Omega_{X/S}^\bullet(\log Y)).$$

For any smooth morphism  $f : S \rightarrow T$ , the objects of  $\mathbf{MIC}(X/S)$  and  $\mathbf{MIC}(X/S(\log Y))$  defined by  $(\mathcal{O}_X, d_{X/S})$  come via (3.0.1) and (4.5.0) from the objects  $\mathbf{MIC}(X/T)$  and  $\mathbf{MIC}(X/T(\log Y))$  defined by  $(\mathcal{O}_X, d_{X/T})$ . Thus the sheaves  $H_{\text{DR}}^q(X/S)$  and  $H_{\text{DR}}^q(X/S(\log Y))$  are provided with a canonical integrable  $T$ -connection whenever  $f : S \rightarrow T$  is a smooth morphism.

(7.1) Suppose now that  $S$  is of characteristic  $p$ . As before (5.1.0) we denote by  $X^{(p)}$  the scheme which makes the following diagram cartesian

$$(7.1.0) \quad \begin{array}{ccc} X^{(p)} & \xrightarrow{w} & X \\ \downarrow \pi^{(p)} & & \downarrow \pi \\ S & \xrightarrow{F_{\text{abs}}} & S \end{array}$$

(i.e.,  $X^{(p)}$  is the fibre product of  $\pi : X \rightarrow S$  and  $F_{\text{abs}} : S \rightarrow S$ ) and we denote by  $F : X \rightarrow X^{(p)}$  the relative Frobenius. The diagram

$$(7.1.1) \quad \begin{array}{ccccc} X^{(p)} & \xrightarrow{w} & X & \xrightarrow{F} & X^{(p)} \\ \downarrow \pi^{(p)} & & \downarrow \pi & & \searrow \pi^{(p)} \\ S & \xrightarrow{F_{\text{abs}}} & S & & \end{array}$$

is commutative, and  $W \circ F$  is the absolute Frobenius endomorphism of  $X$ ,  $F \circ W$  the absolute Frobenius of  $X^{(p)}$ . We denote by  $Y^{(p)}$  the fibre product of  $Y \hookrightarrow X$  and  $X^{(p)} \xrightarrow{W} X$ ;  $Y^{(p)}$  is a divisor in  $X^{(p)}$  with normal crossings relative (via  $\pi^{(p)}$ ) to  $S$ . The spectral sequences of ordinary de Rham cohomology of  $X/S$  may be written (writing  $\mathcal{H}^q$  for cohomology sheaf)

$$(7.1.2) \quad E_2^{p,q} = R^p \pi_*^{(p)}(\mathcal{H}^q(F_*(\Omega_{X/S}^{\bullet}))) \Rightarrow \mathbf{R}^{p+q} \pi_*(\Omega_{X/S}^{\bullet}) = H_{\text{DR}}^{p+q}(X/S)$$

and that for de Rham cohomology of  $X/S(\log Y)$  may be similarly written

$$(7.1.3) \quad \begin{aligned} E_2^{p,q}(\log Y) &= R^p \pi_*^{(p)}(\mathcal{H}^q(F_*(\Omega_{X/S}^{\bullet}(\log Y)))) \\ &\Rightarrow \mathbf{R}^{p+q} \pi_*(\Omega_{X/S}^{\bullet}(\log Y)) = H_{\text{DR}}^{p+q}(X/S(\log Y)). \end{aligned}$$

The  $E_2$  terms have a remarkably simple interpretation due to Deligne, via the Cartier operation.

*Theorem (7.2)* (Cartier). — *There is a unique isomorphism of  $\mathcal{O}_{X^{(p)}}$ -modules*

$$(7.2.0) \quad C^{-1} : \Omega_{X^{(p)}/S}^i \xrightarrow{\sim} \mathcal{H}^i(F_* \Omega_{X/S}^{\bullet})$$

which verifies

$$(7.2.1) \quad C^{-1}(1) = 1$$

$$(7.2.2) \quad C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$$

$$(7.2.3) \quad C^{-1}(d(W^{-1}(f))) = \text{the class of } f^{p-1} df \text{ in } \mathcal{H}^1(F_* \Omega_{X/S}^{\bullet}).$$

Furthermore,  $C^{-1}$  induces an isomorphism of  $\mathcal{O}_{X^{(p)}}$ -modules (by restricting  $C^{-1}$  to

$$\begin{aligned} \Omega_{X^{(p)}/S}^i(\log Y^{(p)}) &\subset \Omega_{(X^{(p)} - Y^{(p)})/S}^i \\ (7.2.4) \quad C^{-1} : \Omega_{X^{(p)}/S}^i(\log Y^{(p)}) &\rightarrow \mathcal{H}^i(F_* \Omega_{X/S}^{\bullet}(\log Y)). \end{aligned}$$

*Proof.* — First, we construct  $C^{-1}$ , following a method of Deligne. We need only construct  $C^{-1}$  for  $i=1$ , for then the asserted multiplicativity (2) will determine it uniquely for  $i \geq 1$ , and for  $i=0$  the condition (1) and  $\mathcal{O}_{X^{(p)}}$ -linearity suffice.

An  $\mathcal{O}_{X^{(p)}}$ -linear mapping

$$(7.2.5) \quad C^{-1} : \Omega_{X^{(p)}/S}^1 \rightarrow \mathcal{H}^1(F_* \Omega_{X/S}^{\bullet})$$

is nothing other than a  $(\pi^{(p)})^{-1}(\mathcal{O}_S)$ -linear derivation of  $\mathcal{O}_{X^{(p)}}$  into  $\mathcal{H}^1(F_* \Omega_{X/S}^{\bullet})$ . Making explicit use of the definition of  $X^{(p)}$  as a fibre product, we have

$$(7.2.6) \quad \mathcal{O}_{X^{(p)}} = \mathcal{O}_X \otimes_{\pi^{-1}(\mathcal{O}_S)} \pi^{-1}(\mathcal{O}_S) \quad (\text{where } \pi^{-1}(\mathcal{O}_S) \text{ is a module over itself by } F_{\text{abs}})$$

so that such a derivation is a mapping of sheaves

$$(7.2.7) \quad \begin{aligned} \delta : \mathcal{O}_X \times \pi^{-1}(\mathcal{O}_S) &\rightarrow \mathcal{H}^1(\Omega_{X/S}^{\bullet}) \\ (f, s) &\mapsto \delta(f, s) \end{aligned}$$

which is biadditive and verifies

$$(7.2.8) \quad \begin{cases} \delta(fs, s') = \delta(f, s^p s') \\ \delta(gf, s) = g^p \delta(f, s) + f^p \delta(g, s) \\ \delta(f, 1) = \text{the class of } f^{p-1} df. \end{cases}$$

We define

$$(7.2.9) \quad \delta(f, s) = \text{the class of } sf^{p-1}df.$$

The properties (7.2.8) are obvious; as for biadditivity, we calculate

$$(7.2.10)$$

$$\begin{aligned} \delta(f+g, s) - \delta(f, s) - \delta(g, s) &= s((f+g)^{p-1}(df+dg) - f^{p-1}df - g^{p-1}dg) \\ &= d\left(s \cdot \left(\frac{(f+g)^p - f^p - g^p}{p}\right)\right). \end{aligned}$$

Having defined  $C^{-1}$ , we must now prove it is an isomorphism. The question being local on  $X$ , we may suppose  $X$  is étale over  $\mathbf{A}_s^n$  via coordinates  $x_1, \dots, x_n$ , such that the divisor  $Y$  is defined by the equation  $x_1 \dots x_v = 0$ . Then  $F_*(\Omega_{X/S}^\bullet)$  is the  $\mathcal{O}_{X(p)}$ -linear complex

$$(7.2.11) \quad \mathcal{O}_{X(p)} \otimes_{\mathbf{F}_p} K^\bullet(n)$$

where for every integer  $n \geq 1$ ,  $K^\bullet(n)$  is the complex of  $\mathbf{F}_p$ -vector spaces with basis the differential forms

$$\begin{aligned} &x_1^{w_1} \dots x_n^{w_n} dx_{\alpha_1} \wedge \dots \wedge dx_{\alpha_j} \\ &\left\{ \begin{array}{l} 0 \leq w_i \leq p-1 \quad \text{for } i=1, \dots, n \\ 1 \leq \alpha_1 < \dots < \alpha_j \leq n \end{array} \right. \end{aligned}$$

and differential the usual exterior derivative in  $n$  variables. Thus

$$(7.2.12) \quad H^i(F_* \Omega_{X/S}^\bullet) \simeq \mathcal{O}_{X(p)} \otimes_{\mathbf{F}_p} H^i(K^\bullet(n)).$$

Similarly,  $F_*(\Omega_{X/S}^\bullet(\log Y))$  is the  $\mathcal{O}_{X(p)}$ -linear complex

$$(7.2.13) \quad \mathcal{O}_{X(p)} \otimes_{\mathbf{F}_p} L^\bullet(n, v)$$

where for each pair of integers  $i \leq v \leq n$ ,  $L^\bullet(n, v)$  is the complex of  $\mathbf{F}_p$ -vector spaces with basis the differential forms

$$\begin{aligned} &x_1^{w_1} \dots x_n^{w_n} \omega_{\alpha_1} \wedge \dots \wedge \omega_{\alpha_j} \\ &\left\{ \begin{array}{l} 1 \leq w_i < p-1 \quad \text{for } i=1, \dots, n \\ 1 \leq \alpha_1 < \dots < \alpha_j \leq n \\ \omega_i = \begin{cases} dx_i/x_i & i=1, \dots, v \\ dx_i & i=v+1, \dots, n \end{cases} \end{array} \right. \end{aligned}$$

and differential the usual exterior derivative in  $n$  variables. We have

$$(7.2.14) \quad H^i(F_*(\Omega_{X/S}^\bullet(\log Y))) \simeq \mathcal{O}_{X(p)} \otimes_{\mathbf{F}_p} H^i(L^\bullet(n, v)).$$

What must be proved, then, is that

- a)  $H^0(L^\bullet(n, v)) = H^0(K^\bullet(n)) = \mathbf{F}_p$ .
- b)  $H^1(K^\bullet(n))$  has as base the classes  $x_i^{p-1} dx_i, i=1, \dots, n$  and  $H^1(L^\bullet(n, v))$  has as base the classes  $\begin{cases} \omega_i & i=1, \dots, v \\ x_i^{p-1} \omega_i & i=v+1, \dots, n. \end{cases}$
- c)  $H^i(K^\bullet(n)) = \wedge^i H^1(K^\bullet(n))$  and  $H^i(L^\bullet(n, v)) = \wedge^i H^1(L^\bullet(n, v))$ .

To see this, we observe that the complexes  $K^\bullet(n)$  and  $L^\bullet(n, v)$  are easily expressed as tensor products of “ $i$ -variable” complexes. Namely

$$K^\bullet(n) = \underbrace{K^\bullet(1) \otimes_{\mathbf{F}_p} K^\bullet(1) \otimes_{\mathbf{F}_p} \dots \otimes_{\mathbf{F}_p} K^\bullet(1)}_{n \text{ times}}$$

$$L^\bullet(n, v) = \underbrace{L^\bullet(1, 1) \otimes_{\mathbf{F}_p} \dots \otimes_{\mathbf{F}_p} L^\bullet(1, 1)}_{v \text{ times}} \otimes_{\mathbf{F}_p} \underbrace{K^\bullet(1) \otimes_{\mathbf{F}_p} \dots \otimes_{\mathbf{F}_p} K^\bullet(1)}_{n-v \text{ times}}.$$

By Künneth, it suffices to show

$$H^i(K^\bullet(1)) = \begin{cases} \mathbf{F}_p \cdot 1 & i=0 \\ \mathbf{F}_p \cdot (\text{class of } x^{p-1} dx), & i=1 \\ 0 & i \geq 2 \end{cases}$$

and

$$H_i(L^\bullet(1, 1)) = \begin{cases} \mathbf{F}_p \cdot 1 & i=0 \\ \mathbf{F}_p \cdot (\text{class of } \omega = dx/x) & i=1 \\ 0 & i \geq 2 \end{cases}$$

which is clear.

This concludes the proof of theorem (7.2).

(7.3) Thus the spectral sequences (7.1.2) and (7.1.3) may be written

$$(7.3.0) \quad E_2^{p, q} = R^p \pi_*^{(p)}(\Omega_{X/(p)/S}^q) \Rightarrow R^{p+q} \pi_*(\Omega_{X/S}^*) = H_{DR}^{p+q}(X/S)$$

and

$$(7.3.1)$$

$$E_2^{p, q}(\log Y) = R^p \pi_*^{(p)}(\Omega_{X/(p)/S}^q(\log Y^{(p)})) \Rightarrow R^{p+q} \pi_*(\Omega_{X/S}^*(\log Y)) = H_{DR}^{p+q}(X/S(\log Y)).$$

Suppose now that either of the two following conditions is true.

(7.3.2) For each pair of integers  $p, q \geq 0$ , the formation of the sheaves  $R^p \pi_*(\Omega_{X/S}^q)$  and  $R^p \pi_*(\Omega_{X/S}^q(\log Y))$  commutes with arbitrary base change.

(7.3.3) The morphism “absolute Frobenius”  $F_{abs}: S \rightarrow S$  is flat; this is the case, for example, if  $S$  is smooth over a field, or locally admits a “ $p$ -base” (cf. 4).

Since the diagram (7.1.0)

$$(7.3.4)$$

$$\begin{array}{ccc} X^{(p)} & \xrightarrow{W} & X \\ \downarrow \pi^{(p)} & & \downarrow \pi \\ S & \xrightarrow{F_{abs}} & S \end{array}$$

is cartesian, and

$$(7.3.5) \quad \begin{cases} \Omega_{X(p)/S}^q = W^*(\Omega_{X/S}^q) \\ \Omega_{X(p)/S}^q(\log Y^{(p)}) = W^*(\Omega_{X/S}^q(\log Y)) \end{cases}$$

either of the assumptions (7.3.2) or (7.3.3) implies that, for all  $p, q \geq 0$ , we have isomorphisms

$$(7.3.6) \quad E_2^{p,q} = R^p \pi_*^{(p)}(\Omega_{X(p)/S}^q) = F_{abs}^*(R^q \pi_*(\Omega_{X/S}^q))$$

$$(7.3.7) \quad E_2^{p,q}(\log Y) = R^p \pi_*^{(p)}(\Omega_{X(p)/S}^q(\log Y^{(p)})) = F_{abs}^*(R^p \pi_*(\Omega_{X/S}^q(\log Y))).$$

*Remark.* — When (7.3.2) or (7.3.3) holds, the above isomorphisms provide (via Cartier's theorem (5.1)) an *a priori* construction of the Gauss-Manin connection on the  $E_2$  terms of (7.1.2) and (7.1.3).

We summarize our findings.

*Theorem (7.4).* — Let  $\pi: X \rightarrow S$  be a smooth morphism,  $i: Y \hookrightarrow X$  the inclusion of a divisor with normal crossings. Suppose that  $S$  is a scheme of characteristic  $p$ , and that either

(7.4.0) for each pair of integers  $p, q \geq 0$ , the formation of the sheaves  $R^p \pi_*(\Omega_{X/S}^q)$  and  $R^p \pi_*(\Omega_{X/S}^q(\log Y))$  commutes with arbitrary base change; or

(7.4.1) the morphism “absolute Frobenius”  $F_{abs}: S \rightarrow S$  is flat.

Then the spectral sequences (7.1.2) and (7.1.3) may be rewritten

$$(7.4.2) \quad E_2^{p,q} = F_{abs}^*(R^p \pi_*(\Omega_{X/S}^q)) \Rightarrow H_{DR}^{p+q}(X/S)$$

$$(7.4.3) \quad E_2^{p,q}(\log Y) = F_{abs}^*(R^q \pi_*(\Omega_{X/S}^q(\log Y))) \Rightarrow H_{DR}^{p+q}(X/S(\log Y)).$$

For any smooth morphism  $f: S \rightarrow T$ , these spectral sequences are endowed with a canonical integrable  $T$ -connection, that of Gauss-Manin, which has  $p$ -curvature zero on the terms  $E_r^{p,q}$ ,  $r \geq 2$ .

*Corollary (7.5)* (Deligne). — For each integer  $i \geq 0$ , let  $h(i)$  (respectively  $h_Y(i)$ ) be the number of integers  $p$  for which  $E_2^{p,i-p}$  (resp.  $E_2^{p,i-p}(\log Y)$ ) is non-zero. (Clearly  $h(i)$  and  $h_Y(i)$  are  $\leq \sup(i+1, 2 \dim(X/S) + 1 - i)$ .) Then for any smooth morphism  $f: S \rightarrow T$ , we have

$$(7.5.0) \quad H_{DR}^i(X/S) \in \mathbf{Nilp}^{h(i)}(S/T)$$

$$(7.5.1) \quad H_{DR}^i(X/S(\log Y)) \in \mathbf{Nilp}^{h_Y(i)}(S/T).$$

## 8. Base-changing the de Rham and Hodge cohomology.

We first recall a rather crude version of the “base-changing” theorems (cf. EGA [14], Mumford [29], and Deligne [6]).

*Theorem (8.0).* — Let  $S$  be a noetherian scheme, and  $\pi: X \rightarrow S$  a proper morphism. Let  $K^\bullet$  be a complex of abelian sheaves on  $S$ , such that

(8.0.0)  $K^i = 0$  if  $i < 0$  and for  $i \gg 0$ .

(8.0.1) Each  $K^i$  is a coherent  $\mathcal{O}_X$ -Module, flat over  $S$ .

(8.0.2) The differential of the complex  $K^\bullet$  is  $\pi^{-1}(\mathcal{O}_S)$ -linear.

Then the following conditions are equivalent:

- (8.0.3) For every integer  $n \geq 0$ , the coherent  $\mathcal{O}_S$ -Modules  $\mathbf{R}^n\pi_*(K^\bullet)$  are locally free.  
 (8.0.4) For every morphism  $g : S' \rightarrow S$ , we form the (cartesian) diagram

$$(8.0.4.0) \quad \begin{array}{ccc} S' \times_S X & \xrightarrow{\text{pr}_2} & X \\ \text{pr}_1 \downarrow & & \downarrow \pi \\ S' & \xrightarrow{g} & S \end{array}$$

The canonical morphism of base-change,

$$(8.0.4.1) \quad g^*\mathbf{R}^n\pi_*(K^\bullet) \rightarrow \mathbf{R}^n\text{pr}_{1*}(\text{pr}_2^*(K^\bullet))$$

is an isomorphism for every integer  $n \geq 0$ .

(8.0.5) Same as (8.0.4) for every morphism  $g : S' \rightarrow S$  which is the inclusion of a point of  $S$ .

Furthermore, there is a non-empty open subset  $U$  of  $S$  such that, for the morphism  $\pi_U : \pi^{-1}(U) \rightarrow U$  and the complex  $K^\bullet|_U$ , each of these equivalent conditions is satisfied.

When these conditions are satisfied, we say that the formation of the  $\mathbf{R}^n\pi_*(K^\bullet)$  commutes with base change.

*Corollary (8.1).* — Let  $\pi : X \rightarrow S$  be a proper and smooth morphism, and suppose  $S$  is noetherian. There is a non-empty open subset  $U \subset S$ , such that each of the coherent sheaves on  $S$

$$(8.1.0) \quad \mathbf{R}^q\pi_*(\Omega_{X/S}^p), \quad p, q \geq 0$$

$$(8.1.1) \quad H_{\text{DR}}^n(X/S) = \mathbf{R}^n\pi_*(\Omega_{X/S}^*), \quad n \geq 0$$

is locally free over  $U$ .

(8.2) Let us define the Hodge cohomology of  $X/S$  by

$$(8.2.0) \quad H_{\text{Hodge}}^n(X/S) = \prod_{p+q=n} \mathbf{R}^q\pi_*(\Omega_{X/S}^p);$$

it is bigraded :

$$(8.2.1) \quad H_{\text{Hodge}}^{p,q}(X/S) = \mathbf{R}^q\pi_*(\Omega_{X/S}^p).$$

*Corollary (8.3).* — Let  $\pi : X \rightarrow S$  be a proper and smooth morphism, and suppose  $S$  is noetherian. Suppose that each of the coherent  $\mathcal{O}_S$ -modules

$$(8.3.0) \quad H_{\text{Hodge}}^{p,q}(X/S), \quad p, q \geq 0$$

$$(8.3.1) \quad H_{\text{DR}}^n(X/S), \quad n \geq 0$$

is locally free.

Then for any change of base  $g : S' \rightarrow S$ , the canonical morphisms of sheaves on  $S'$

$$(8.3.2) \quad g^*H_{\text{Hodge}}^{p,q}(X/S) \rightarrow H_{\text{Hodge}}^{p,q}((S' \times_S X)/S')$$

$$(8.3.3) \quad g^*H_{\text{DR}}^n(X/S) \rightarrow H_{\text{DR}}^n((S' \times_S X)/S')$$

are isomorphisms for all values of  $p$ ,  $q$  and  $n$ . In particular, the Hodge and de Rham cohomology sheaves of  $(S' \times_S X)/S'$  are locally free sheaves on  $S'$ .

*Corollary (8.4).* — Under the assumptions of (8.1), let

$$i : Y \hookrightarrow X$$

be the inclusion of a divisor with normal crossings relative to  $S$ . Then there is a non-empty open subset  $U \subset S$  such that each of the coherent sheaves on  $S$

$$(8.4.0) \quad R^q \pi_*(\Omega_{X/S}^p(\log Y)) \quad p, q \geq 0$$

$$(8.4.1) \quad H_{DR}^n(X/S(\log Y)) = R^n \pi_*(\Omega_{X/S}^{\bullet}(\log Y))$$

is locally free over  $U$ .

**(8.5)** Let us define the Hodge cohomology of  $X/S(\log Y)$  by

$$(8.5.0) \quad H_{Hodge}^n(X/S(\log Y)) = \prod_{p+q=n} R^q \pi_*(\Omega_{X/S}^p(\log Y));$$

it is again bigraded :

$$(8.5.1) \quad H_{Hodge}^{p,q}(X/S(\log Y)) = R^q \pi_*(\Omega_{X/S}^p(\log Y)).$$

*Corollary (8.6).* — Assumptions being as in (8.4), suppose that each of the coherent sheaves on  $S$

$$(8.6.0) \quad H_{Hodge}^{p,q}(X/S(\log Y)) \quad p, q \geq 0$$

$$(8.6.1) \quad H_{DR}^n(X/S(\log Y)) \quad n \geq 0$$

is locally free.

Then for any change of base  $g : S' \rightarrow S$ , denoting by  $Y'$  the fibre product of  $i : Y \hookrightarrow X$  and  $\text{pr}_2 : S' \times_S X \rightarrow X$  (cf. (8.0.4.0)), which is a divisor in  $S' \times_S X$  with normal crossings relative to  $S'$ , the canonical morphisms of sheaves on  $S'$

$$(8.6.2) \quad g^* H_{Hodge}^{p,q}(X/S(\log Y)) \rightarrow H_{Hodge}^{p,q}((S' \times_S X)/S'(\log Y'))$$

$$(8.6.3) \quad g^* H_{DR}^n(X/S(\log Y)) \rightarrow H_{DR}^n((S' \times_S X)/S'(\log Y'))$$

are isomorphisms for all values of  $p$ ,  $q$  and  $n$ . In particular, the Hodge and de Rham cohomology sheaves of  $(S' \times_S X)/S'(\log Y')$  are locally free sheaves on  $S'$ .

**(8.7)** It is proven in [6] that, if  $S$  is of characteristic zero, then the open subset of Corollary (8.1) is all of  $S$ , and the spectral sequence of sheaves on  $S$

$$(8.7.1) \quad E_1^{p,q} = H_{Hodge}^{p,q}(X/S) \Rightarrow H_{DR}^{p+q}(X/S)$$

degenerates at  $E_1$ .

Similar arguments, using Deligne's theory of "mixed Hodge structures" (unpublished) allow one to prove that, if  $S$  is of characteristic zero, then the open subset of Corollary (8.4) is all of  $S$ , and that the spectral sequence of sheaves on  $S$

$$(8.7.2) \quad E_1^{p,q} = H_{Hodge}^{p,q}(X/S(\log Y)) \Rightarrow H_{DR}^{p+q}(X/S(\log Y))$$

degenerates at  $E_1$ .

There is an elementary proof of the fact that the  $H_{DR}(X/S)$  are locally free on  $S$ , if  $X/S$  is proper and smooth, and  $S$  is smooth over a field  $k$  at characteristic zero. It is based only on the fact that the  $H_{DR}(X/S)$  are coherent sheaves on  $S$ , with an integrable  $k$ -connection (that of Gauss-Manin!).

*Proposition (8.8).* — Let  $S$  be smooth over a field  $k$  of characteristic zero, and let  $(\mathcal{M}, \nabla)$  be an object of  $MIC(S/k)$ , such that  $\mathcal{M}$  is coherent. Then  $\mathcal{M}$  is a locally free sheaf on  $S$ .

*Proof.* — The question being local on  $S$ , it suffices to prove that, for every closed point  $s \in S$ , the module  $\mathcal{M}_s$  over  $\mathcal{O}_{S,s}$  is free. As  $\mathcal{M}_s$  is finitely generated over  $\mathcal{O}_{S,s}$  by hypothesis, it suffices to prove that  $\hat{\mathcal{M}}_s = \mathcal{M}_s \otimes_{\mathcal{O}_{S,s}} \hat{\mathcal{O}}_{S,s}$ , the completion of  $\mathcal{M}_s$  for the topology defined by powers of the maximal ideal of  $\mathcal{O}_{S,s}$ , is free over  $\hat{\mathcal{O}}_{S,s}$ . Thus it suffices to prove an analogue of Cartier's theorem (5.1) :

*Proposition (8.9).* — Let  $K$  be a field of characteristic zero,  $K[[t_1, \dots, t_n]]$  the ring of formal power series over  $K$  in  $n$  variables. Let  $M$  be a finitely generated module over  $K[[t_1, \dots, t_n]]$ , given with an integrable connection  $\nabla$  (for the continuous  $K$ -derivations of  $K[[t_1, \dots, t_n]]$  to itself). Then  $M^\nabla$ , the  $K$ -space of horizontal elements of  $M$ , is finite-dimensional over  $K$ , and the pair  $(M, \nabla)$  is isomorphic to the pair  $(M^\nabla \otimes_K K[[t_1, \dots, t_n]], I \otimes d)$  where  $d$  denotes the “identical” connection on  $K[[t_1, \dots, t_n]]$ .

*Proof.* — We begin by constructing an additive endomorphism of  $M$ . For  $i=1, \dots, n$ , let

$$(8.9.0) \quad D_i = \nabla \left( \frac{\partial}{\partial t_i} \right)$$

and for each integer  $j \geq 0$ , let

$$(8.9.1) \quad D_i^{(j)} = \frac{I}{j!} \left( \nabla \left( \frac{\partial}{\partial t_i} \right) \right)^j; \quad D_i^{(0)} = I.$$

For an  $n$ -tuple  $J = (j_1, \dots, j_n)$  of non-negative integers, we put

$$(8.9.2) \quad D^{(J)} = \prod_{i=1}^n D_i^{(j_i)}$$

$$(8.9.3) \quad t^J = \prod_{i=1}^n (t_i)^{j_i}$$

$$(8.9.4) \quad (-I)^J = \prod_{i=1}^n (-I)^{j_i}.$$

We then define a  $K$ -linear endomorphism  $P$  of  $M$

$$(8.9.5) \quad P : M \rightarrow M$$

$$P = \sum_J (-I)^J t^J D^{(J)}.$$

One successively verifies

$$(8.9.6) \quad P(fm) = f(0)P(m) \quad \text{for} \quad f \in K[[t_1, \dots, t_n]] \text{ and } m \in M,$$

by Leibniz's rule, so that  $\text{Kernel}(\mathbf{P}) \supset (t_1, \dots, t_n) \mathbf{M}$

$$(8.9.7) \quad \begin{aligned} \mathbf{P}(m) &\equiv m \text{ modulo } (t_1, \dots, t_n) \mathbf{M}, \quad \text{so that} \\ \text{Kernel}(\mathbf{P}) &= (t_1, \dots, t_n) \cdot \mathbf{M} \end{aligned}$$

$$(8.9.8) \quad \mathbf{P}|_{\mathbf{M}^\nabla} = \text{id.}$$

$$(8.9.9) \quad \begin{aligned} \mathbf{P}(\mathbf{M}) &\subset \mathbf{M}^\nabla \quad (\text{by a "telescoping"}, \text{ so}) \\ \mathbf{P}^2 &= \mathbf{P} \quad \text{is a projection onto } \mathbf{M}^\nabla \end{aligned}$$

$$(8.9.10) \quad \begin{aligned} \mathbf{P} &\text{ induces an isomorphism of K-vector spaces.} \\ \mathbf{P} : \mathbf{M}/(t_1, \dots, t_n) \mathbf{M} &\xrightarrow{\sim} \mathbf{M}^\nabla. \end{aligned}$$

This shows that  $\mathbf{M}^\nabla$  is finite dimensional, and that (by Nakayama), the canonical mapping

$$(8.9.11) \quad \mathbf{M}^\nabla \otimes_K K[[t_1, \dots, t_n]] \rightarrow \mathbf{M}$$

is *surjective*. To show it is an isomorphism, we must show that if  $m_1, \dots, m_r$  are  $K$ -linearly independent elements of  $\mathbf{M}^\nabla$ , then there can be no non-trivial relation

$$(8.9.12) \quad \sum_i f_i m_i = \mathbf{o} \quad \text{in } \mathbf{M}.$$

Supposing the contrary, assume  $f_1 \neq \mathbf{o}$ . Then for some  $J = (j_1, \dots, j_n)$ , we have

$$(8.9.13) \quad \left( \prod_{v=1}^n \frac{1}{j_v!} \left( \frac{\partial}{\partial t_v} \right)^{j_v} (f_1) \right) (\mathbf{o}) \neq \mathbf{o}.$$

Since the  $m_i$  are *horizontal*, applying  $D^{(J)}$  to (8.9.12) gives

$$(8.9.14) \quad \mathbf{o} = D^{(J)} \left( \sum_i f_i m_i \right) = \sum_i \prod_{v=1}^n \frac{1}{j_v!} \left( \frac{\partial}{\partial t_v} \right)^{j_v} (f_i) \cdot m_i$$

a relation of the form

$$(8.9.15) \quad \sum_i g_i m_i = \mathbf{o}, \quad g_1(\mathbf{o}) \neq \mathbf{o}.$$

Applying  $\mathbf{P}$  to (7.11.14) gives a relation

$$\sum_i g_i(\mathbf{o}) m_i = \mathbf{o}$$

which is impossible. Q.E.D.

*Remark (8.9.16).* — Heuristically,  $\mathbf{P}(m)(t_1, \dots, t_n) = m(t_1 - t_1, \dots, t_n - t_n)$  expanded in Taylor series. In fact, the proof of (7.11) is just a concrete spelling-out of the formal descent theory (with a section, no less), cf. [17].

*Remark (8.10)* (an afterthought). — Of course when  $S = \text{Spec}(\mathbf{C})$ ,  $X/\mathbf{C}$  proper and smooth,  $Y \hookrightarrow X$  a divisor with normal crossings, we have isomorphisms

$$(8.10.1) \quad \begin{aligned} H_{\text{DR}}^i(X/\mathbf{C}) &\simeq H^i(X^{\text{anal}}, \mathbf{C}) \\ H_{\text{DR}}^i(X/\mathbf{C}(\log Y)) &\simeq H_{\text{DR}}^i((X - Y)/\mathbf{C}) \simeq H^i(X^{\text{anal}} - Y^{\text{anal}}, \mathbf{C}). \end{aligned}$$

For  $S$  any scheme of characteristic zero, if  $\pi : X \rightarrow S$  is proper and smooth,  $i : Y \hookrightarrow X$  the inclusion of a divisor with normal crossings relative to  $S$ ,  $j : X - Y \hookrightarrow X$  the inclusion of its complement, then the canonical morphism of complexes of sheaves on  $X$

$$(8.10.2) \quad \Omega_{X/S}^\bullet(\log Y) \rightarrow j_* \Omega_{(X-Y)/S}^\bullet$$

is a quasi-isomorphism (i.e., an *isomorphism* on cohomology sheaves) (cf. Atiyah-Hodge [o]), from which it follows that the maps deduced by applying the  $\mathbf{R}^i\pi_*$  to (7.12.2)

$$(8.10.3) \quad H_{\text{DR}}^i(X/S(\log Y)) \rightarrow H_{\text{DR}}^i((X-Y)/S)$$

are *isomorphisms* of sheaves on  $S$ .

### 9. Nilpotence over a global base.

(9.0) Let  $R$  be an integral domain which is finitely generated (as a ring) over  $\mathbf{Z}$ , and whose field of fractions has characteristic zero. Let  $T = \text{Spec}(R)$ ; we call  $T$  a “*global affine variety*”. Let  $f : S \rightarrow T$  be a smooth morphism.

Let  $p$  be prime number which is *not* invertible on  $S$ . This excludes a finite set of primes. Put

$$(9.0.1) \quad T \otimes \mathbf{F}_p = \text{Spec}(R/pR) = \text{Spec}(R \otimes_{\mathbf{Z}} \mathbf{F}_p)$$

and

$$(9.0.2) \quad S \otimes \mathbf{F}_p = S \times_{\mathbf{Z}} \mathbf{F}_p.$$

We have the diagram (in which all squares are cartesian)

$$(9.0.3)$$

$$\begin{array}{ccc} S \otimes \mathbf{F}_p & \searrow & S \\ \downarrow & & \downarrow \\ T \otimes \mathbf{F}_p & \searrow & T \\ \downarrow & & \downarrow \\ \text{Spec}(\mathbf{F}_p) & \searrow & \text{Spec}(\mathbf{Z}) \end{array}$$

(9.1) Let  $(\mathcal{M}, \nabla)$  be an object of  $\mathbf{MIC}(S/T)$ , with  $\mathcal{M}$  locally free of finite rank on  $S$ . Taking its inverse image (cf. (1.1.3)) in  $\mathbf{MIC}(S \otimes \mathbf{F}_p/T \otimes \mathbf{F}_p)$ , which we denote  $(\mathcal{M} \otimes \mathbf{F}_p, \nabla \otimes \mathbf{F}_p)$ , we may ask whether or not it is nilpotent, and, if nilpotent, then nilpotent of what exponent?

We will say that  $(\mathcal{M}, \nabla)$  is *globally nilpotent* on  $S/T$  if, for every prime  $p$  which is not invertible on  $S$ , we have

$$(9.1.0) \quad (\mathcal{M} \otimes \mathbf{F}_p, \nabla \otimes \mathbf{F}_p) \in \text{Nilp}(S \otimes \mathbf{F}_p / T \otimes \mathbf{F}_p).$$

Let  $v$  be an integer,  $v \geq 1$ . We will say that  $(\mathcal{M}, \nabla)$  is *globally nilpotent of exponent v* on  $S/T$ , if, for every prime  $p$  which is not invertible on  $S$ , we have

$$(9.1.1) \quad (\mathcal{M} \otimes \mathbf{F}_p, \nabla \otimes \mathbf{F}_p) \in \text{Nilp}^v(S \otimes \mathbf{F}_p / T \otimes \mathbf{F}_p).$$

Clearly we have

*Proposition (9.2). — Let  $f: S \rightarrow T$  and  $f': S' \rightarrow T'$  be smooth morphisms, with  $T$  and  $T'$  global affine varieties (cf. (9.0)), and suppose given a commutative diagram of morphisms*

$$(9.2.0) \quad \begin{array}{ccc} S' & \xrightarrow{g} & S \\ \downarrow f' & & \downarrow f \\ T' & \xrightarrow{h} & T \end{array}$$

*Let  $(\mathcal{M}, \nabla)$  be an object of  $\text{MIC}(S/T)$ , with  $\mathcal{M}$  locally free of finite rank on  $S$ . Then*

**(9.2.1)** *If  $(\mathcal{M}, \nabla)$  is globally nilpotent on  $S/T$ , then its inverse image  $(g, h)^*(\mathcal{M}, \nabla)$  is globally nilpotent on  $S'/T'$ .*

**(9.2.2)** *If  $(\mathcal{M}, \nabla)$  is globally nilpotent of exponent  $v$  on  $S/T$ , then its inverse image  $(g, h)^*(\mathcal{M}, \nabla)$  is globally nilpotent of exponent  $v$  on  $S'/T'$ .*

We also have the self evident

*Proposition (9.3). — Let  $T$  be a global affine variety,  $f: S \rightarrow T$  a smooth morphism, and  $g: S' \rightarrow S$  a proper étale morphism. Let  $(\mathcal{M}, \nabla)$  be an object of  $\text{MIC}(S'/T)$ , with  $\mathcal{M}$  locally free of finite rank on  $S'$ . Then*

**(9.3.0)**  *$(\mathcal{M}, \nabla)$  is globally nilpotent on  $S'/T$  if and only if  $(g_* \mathcal{M}, \nabla)$  is globally nilpotent on  $S/T$ .*

**(9.3.1)**  *$(\mathcal{M}, \nabla)$  is globally nilpotent of exponent  $v$  on  $S'/T$  if and only if  $(g_* \mathcal{M}, \nabla)$  is globally nilpotent of exponent  $v$  on  $S/T$ .*

## 10. Global nilpotence of de Rham cohomology.

Putting together sections 7, 8 and 9, we find

*Theorem (10.0). — Let  $T$  be a global affine variety (cf. (9.0)),  $f: S \rightarrow T$  a smooth morphism, with  $S$  connected, and  $\pi: X \rightarrow S$  a proper and smooth morphism.*

*Suppose that each of the coherent sheaves on  $S$  (cf. (8.2))*

$$\begin{aligned} H_{\text{Hodge}}^{p,q}(X/S) &\quad p, q \geq 0 \\ H_{\text{DR}}^n(X/S) &\quad n \geq 0 \end{aligned}$$

is locally free on  $S$  (a hypothesis which is always verified over a non-empty open subset of  $S$  cf. (8.4)).

For each integer  $i \geq 0$ , let  $h(i)$  be the number of integers  $a$  such that  $H_{\text{Hodge}}^{a, i-a}(X/S)$  is non-zero. (Thus  $h(i)$  is the number of non-zero groups  $H^{i-a}(X_s, \Omega_{X_s/\mathbb{C}}^a)$  which occur in the Hodge decomposition of the  $i$ -th cohomology group of the fibre  $X_s$  of  $\pi$  over any  $\mathbb{C}$ -valued point of  $S$ .)

Then for each integer  $i \geq 0$ , the locally free sheaf  $H_{\text{DR}}^i(X/S)$ , with the Gauss-Manin connection, is globally nilpotent of exponent  $h(i)$  on  $S/T$ .

**Theorem (10.0) ( $\log Y$ )** — Let  $T$  be a global affine variety,  $f: S \rightarrow T$  a smooth morphism, with  $S$  connected,  $\pi: X \rightarrow S$  a proper and smooth morphism, and  $i: Y \hookrightarrow X$  the inclusion of a divisor with normal crossings relative to  $S$ . Suppose that each of the coherent sheaves (cf. (8.5))

$$H_{\text{Hodge}}^{p, q}(X/S(\log Y)) \quad p, q \geq 0$$

$$H_{\text{DR}}^n(X/S(\log Y)) \quad n \geq 0$$

is locally free on  $S$  (a hypothesis always verified on a non-empty open subset of  $S$ ).

For each integer  $i \geq 0$ , let  $h_Y(i)$  be the number of integers  $a$  with  $H_{\text{Hodge}}^{a, i-a}(X/S(\log Y))$  non-zero. Then for each integer  $i \geq 0$ , the locally free sheaf  $H_{\text{DR}}^i(X/S(\log Y))$ , with the Gauss-Manin connection, is globally nilpotent of exponent  $h_Y(i)$  on  $S/T$ .

### 11. Classical theory of regular singular points.

(11.0) Let  $k$  be a field of characteristic zero, and  $K$  a field of functions in one variable over  $k$ , i.e.,  $K$  is the function field of a projective, smooth, absolutely irreducible curve over  $k$ .

Let  $W$  be a finite-dimensional vector space over  $K$ . A  $k$ -connection  $\nabla$  on  $W$  is an additive mapping

$$(11.0.1) \quad \nabla: W \rightarrow \Omega_{K/k}^1 \otimes_K W$$

which satisfies

$$(11.0.2) \quad \nabla(fw) = df \otimes w + f\nabla(w)$$

for  $f \in K$ ,  $w \in W$ . Equivalently,  $\nabla$  "is" a  $K$ -linear mapping

$$(11.0.3) \quad \nabla: \text{Der}(K/k) \rightarrow \text{End}_k(W)$$

such that

$$(11.0.4) \quad (\nabla(D))(fw) = D(f)w + f(\nabla(D))(w)$$

for  $D \in \text{Der}(K/k)$ ,  $f \in K$ , and  $w \in W$ .

The connection is necessarily integrable, i.e., compatible with brackets, since  $\Omega_{K/k}^2 = 0$ .

If  $(W, \nabla)$  and  $(W', \nabla')$  are two such objects, a horizontal morphism  $\varphi$  from  $(W, \nabla)$  to  $(W', \nabla')$  is a  $K$ -linear mapping of  $W$  to  $W'$  which is compatible with the connections, i.e.,

$$(III.0.5) \quad \varphi(\nabla(D)(w)) = \nabla'(D')(\varphi(w)).$$

The objects  $(W, \nabla)$  as above, with morphisms the horizontal ones, form an abelian category  $\mathbf{MC}(K/k)$ . (N.B. — This notation is slightly misleading, since, unlike what was required in the geometrical case of a smooth morphism  $S \rightarrow k$ , we are requiring that  $W$  be finite dimensional over  $K$  (i.e., *coherent*), rather than just quasicoherent.) Just as in (I.1),  $\mathbf{MC}(K/k)$  has an internal *Hom* and a tensor product.

(III.1) Let  $p$  be a place of  $K/k$  (i.e., a closed point of the projective and smooth curve over  $k$  whose function field is  $K$ ),  $\mathcal{O}_p$  its local ring,  $\mathfrak{m}_p$  its maximal ideal,  $\text{ord}_p : K \rightarrow \mathbf{Z} \cup \{\infty\}$  the associated valuation “order of zero at  $p$ ”. Thus

$$(III.1.0) \quad \mathcal{O}_p = \{f \in K \mid \text{ord}_p(f) \geq 0\}$$

$$(III.1.1) \quad \mathfrak{m}_p = \{f \in K \mid \text{ord}_p(f) \geq 1\}.$$

Let  $\text{Der}_p(K/k)$  denote the  $\mathcal{O}_p$ -submodule of  $\text{Der}(K/k)$

$$(III.1.2) \quad \text{Der}_p(K/k) = \{D \in \text{Der}(K/k) \mid D(\mathfrak{m}_p) \subset \mathfrak{m}_p\}.$$

In terms of a uniformizing parameter  $h$  at  $p$ ,  $\text{Der}_p(K/k)$  is the free  $\mathcal{O}_p$ -module with basis  $h \frac{d}{dh}$ . In fact, for any function  $y \in K^*$ , which is not a unit at  $p$ ,  $y \frac{d}{dy}$  is an  $\mathcal{O}_p$ -basis for  $\text{Der}_p(K/k)$ .

(III.2) Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$ . We say that  $(W, \nabla)$  has a *regular singular point* at  $p$  if there exists an  $\mathcal{O}_p$ -lattice  $W_p$  of  $W$  (i.e. a subgroup of  $W$  which is a free  $\mathcal{O}_p$ -module of rank equal to  $\dim_K(W)$ ) such that

$$(III.2.0) \quad \text{Der}_p(K/k)(W_p) \subset W_p.$$

In more concrete terms, we ask if there is a base  $\mathbf{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$  of  $W$  as  $K$ -space, such that

$$(III.2.1) \quad \nabla \left( h \frac{d}{dh} \right) \mathbf{e} = B \mathbf{e} \quad \text{with } B \in \mathbf{M}_n(\mathcal{O}_p)$$

for some (and hence for any) uniformizing parameter  $h$  at  $p$ .

*Remark (III.2.2).* — If  $p$  is a regular singular point of  $(W, \nabla)$ , there is no unicity in the lattice  $W_p$  which “works” in (III.2.0). We will return to this question later (cf. especially (I2.0) and (I2.5)).

*Proposition (III.3).* — Suppose

$$(III.3.0) \quad 0 \rightarrow (V, \nabla') \rightarrow (W, \nabla) \rightarrow (U, \nabla'') \rightarrow 0$$

is an exact sequence in  $\mathbf{MC}(K/k)$ . Then  $(W, \nabla)$  has a regular singular point at  $p$  if and only if both  $(V, \nabla')$  and  $(U, \nabla'')$  have a regular singular point at  $p$ .

*Proof.* — Suppose first that  $(V, \nabla')$  and  $(U, \nabla'')$  have regular singular points at  $p$ . This means we can choose a  $K$ -base of  $W$  of the form

$$(III.3.1) \quad \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_{n_1} \\ f_1 \\ \vdots \\ f_{n_2} \end{pmatrix}$$

of  $W$  so that  $\mathbf{e}$  is a base of  $V$ , and  $\mathbf{f}$  projects to a base of  $U$ , in terms of which the connection is expressed

$$(III.3.2) \quad \nabla \left( h \frac{d}{dh} \right) \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} A & \mathbf{O} \\ B & C \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

with  $A \in \mathbf{M}_{n_1}(\mathcal{O}_p)$ ,  $C \in \mathbf{M}_{n_2}(\mathcal{O}_p)$ . The problem is that  $B$  may not be holomorphic at  $p$ . But for any integer  $v$ , we readily compute

$$(III.3.3) \quad \nabla \left( h \frac{d}{dh} \right) \begin{pmatrix} \mathbf{e} \\ h^v \mathbf{f} \end{pmatrix} \begin{pmatrix} A & \mathbf{O} \\ h^v B & C + v \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ h^v \mathbf{f} \end{pmatrix},$$

and for  $v \gg 0$ , we have  $h^v B$  holomorphic at  $p$ .

Conversely, suppose that  $(W, \nabla)$  has a regular singular point at  $p$ , and let  $W_p$  be an  $\mathcal{O}_p$ -lattice in  $W$  which “works” for (III.2.0). Then  $V \cap W_p$  is an  $\mathcal{O}_p$ -lattice in  $V$  (*elementary divisors*) which works for (III.2.0). Similarly,  $U \cap (\text{image of } W_p \text{ in } U)$  is an  $\mathcal{O}_p$ -lattice in  $U$  which works for (III.2.0).

*Remark (III.3.4).* — The full abelian subcategory of  $\mathbf{MC}(K/k)$  consisting of objects with a regular singular point at  $p$  is *stable* under the internal *Hom* and tensor product of  $\mathbf{MC}(K/k)$ .

**(III.4)** Let us say that  $(W, \nabla)$  is *cyclic* if there is a vector  $w \in W$ , such that for some (and hence for any) non-zero derivation  $D \in \text{Der}(K/k)$ , the vectors  $w, \nabla(D)(w), (\nabla(D))^2(w), (\nabla(D))^3(w), \dots$  span  $W$  over  $K$ . (We should remark that for  $w \in W$ , the  $K$ -span of the vectors  $w, \nabla(D)(w), (\nabla(D))^2(w), \dots$  is independent of the choice of non-zero  $D \in \text{Der}(K/k)$ , and is thus a  $\text{Der}(K/k)$ -stable subspace of  $W$ .)

*Corollary (III.5).* — Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$ . Then  $(W, \nabla)$  has a regular singular point at  $p$  if and only if every cyclic subobject of  $(W, \nabla)$  has a regular singular point at  $p$ .

*Proof.* — “Only if” by (III.3), “if” because  $(W, \nabla)$  is a quotient of a direct sum of finitely many of its cyclic subobjects (so apply (III.3) again).

**(III.6)** Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$ , and  $W_p$  and  $\mathcal{O}_p$ -lattice in  $W$ . We say that  $(W, \nabla)$  satisfies “*Jurkat’s Estimate*” (J) at  $p$  for the lattice  $W_p$  if there is an

integer  $\mu$ , such that, for every integer  $j \geq 1$  and every  $j$ -tuple  $D_1, \dots, D_j \in \text{Der}_p(K/k)$ , we have (denoting by  $h$  a uniformizing parameter at  $p$ )

$$(11.6.0) \quad \nabla(D_1) \cdot \nabla(D_2) \cdots \nabla(D_j)(W_p) \subset h^\mu(W_p).$$

Let us reformulate this condition. Let  $D_0$  be an  $\mathcal{O}_p$ -base of  $\text{Der}_p(K/k)$ . One quickly checks by induction that for any  $D_1, \dots, D_j \in \text{Der}_p(K/k)$ , one has

$$\nabla(D_1) \cdot \nabla(D_2) \cdots \nabla(D_j) = \sum_{v=0}^j a_v(\nabla(D_0))^v$$

with  $a_0, \dots, a_j \in \mathcal{O}_p$ . Thus (11.6.0) holds for all  $j \geq 1$  if and only if, for some  $\mathcal{O}_p$ -base  $D_0$  of  $\text{Der}_p(K/k)$ , one has

$$(11.6.0 \text{ bis}) \quad (\nabla(D_0))^j(W_p) \subset h^\mu(W_p) \quad \text{for all } j \geq 1.$$

In terms of an  $\mathcal{O}_p$ -base  $\mathbf{e}$  of  $W_p$ , the condition (11.6.0 bis) may be expressed as follows. For each  $j \geq 1$ , define a matrix  $B_j \in \mathbf{M}_n(K)$  by

$$(11.6.1) \quad (\nabla(D_0))^j \mathbf{e} = B_j \mathbf{e}.$$

Then (11.6.0 bis) is equivalent to

$$(11.6.2) \quad \text{ord}_p(B_j) \geq \mu \quad \text{for all } j \geq 1.$$

In applications we will speak of a  $K$ -base  $\mathbf{e}$  of  $W$  as satisfying (J) at  $p$ , rather than of the lattice given by its  $\mathcal{O}_p$ -span. Also, we will usually take as  $\mathcal{O}_p$ -base of  $\text{Der}_p(K/k)$  a derivation  $h \frac{d}{dh}$ ,  $h$  a uniformizing parameter at  $p$ , although we may occasionally use  $y \frac{d}{dy}$  as base, for a non-zero  $y$  which is a *non-unit* at  $p$ .

*Proposition (11.6.3).* — *If  $(W, \nabla)$  satisfies (J) at  $p$  for one base, it satisfies it for every base.*

*Proof.* — Let  $\mathbf{e}$  be a base of  $W$  and  $\mu$  an integer such that, for all  $j \geq 1$

$$(11.6.3.0) \quad \left( \nabla \left( h \frac{d}{dh} \right) \right)^j \mathbf{e} = B_j \mathbf{e}, \quad \text{ord}_p(B_j) \geq \mu.$$

Let  $\mathbf{f}$  be another base of  $W$ , so that

$$(11.6.3.1) \quad \mathbf{f} = A \mathbf{e}, \quad \mathbf{e} = A^{-1} \mathbf{f}, \quad A \in \mathbf{GL}_n(K).$$

We define the sequence of matrices  $C_j$  by

$$(11.6.3.2) \quad \left( \nabla \left( h \frac{d}{dh} \right) \right)^j \mathbf{f} = C_j \mathbf{f}.$$

We easily calculate the  $C_j$  in terms of the  $B_j$ , by using Leibniz's rule :

$$(III.6.3.3) \quad \begin{aligned} \left( \nabla \left( h \frac{d}{dh} \right) \right)^j \mathbf{f} &= \left( \nabla \left( h \frac{d}{dh} \right) \right)^j (A \cdot \mathbf{e}) \\ &= \sum_{i=0}^j \binom{j}{i} \left( \left( h \frac{d}{dh} \right)^{j-i} (A) \right) \cdot B_i \mathbf{e} \end{aligned}$$

whence

$$(III.6.3.4) \quad C_j = \sum_{i=0}^j \binom{j}{i} \left( \left( h \frac{d}{dh} \right)^{j-i} (A) \right) \cdot B_i A^{-1}.$$

Since for any element  $f \in K$  we have

$$(III.6.3.5) \quad \text{ord}_p \left( h \frac{df}{dh} \right) \geq \text{ord}_p(f),$$

(III.6.3.4) gives immediately

$$(III.6.3.6) \quad \text{ord}_p(C_j) \geq \min_{0 \leq i \leq j} (\text{ord}_p(A) + \text{ord}_p(B_i) + \text{ord}_p(A^{-1}))$$

i.e.,

$$(III.6.3.7) \quad \text{ord}_p(C_j) \geq \mu + \text{ord}_p(A) + \text{ord}_p(A^{-1}).$$

*Proposition (III.7).* — If  $(W, \nabla)$  has a regular singular point at  $p$ , then it satisfies (J) at  $p$ .

*Proof.* — Indeed in a suitable base  $\mathbf{e}$ , we have

$$(III.7.0) \quad \nabla \left( h \frac{d}{dh} \right) \mathbf{e} = B \mathbf{e}, \quad B \in \mathbf{M}_n(\mathcal{O}_p).$$

As the  $B_j$  are formed successively according to the rule

$$(III.7.1) \quad B_{j+1} = h \frac{d}{dh} (B_j) + B_j B$$

we see that each  $B_j$  is holomorphic at  $p$ , i.e.,  $\text{ord}_p(B_j) \geq 0$ .

**(III.8)** Let  $a$  be a positive integer. In the extension field  $K(h^{1/a})$  of  $K$ , there is a unique place  $p^{1/a}$  which extends  $p$ , and  $h^{1/a}$  is a uniformizing parameter there.

*Proposition (III.8.1).* — Let  $(W, \Delta)$  be an object of  $\mathbf{MC}(K/k)$ . Then  $(W, \nabla)$  satisfies (J) at  $p$  if and only if its inverse image in  $\mathbf{MC}(K(h^{1/a})/k)$  satisfies (J) at  $p^{1/a}$ .

*Proof.* — Calculate the matrices  $B_j$  of (III.6.1), using a  $K$ -base  $\mathbf{e}$  of  $W$ , and  $h \frac{d}{dh}$  as  $\mathcal{O}_p$ -base, for both  $(W, \nabla)$  and its inverse image. They are the same matrices.

*Theorem (III.9)* (Fuchs [8], Turrin [34], Lutz [24]). — Let  $(W, \nabla)$  be a cyclic object of  $\mathbf{MC}(K/k)$ ,  $w \in W$  a cyclic vector,  $p$  a place of  $K/k$ ,  $h$  a uniformizing parameter at  $p$ ,  $n = \dim_K(W)$ . The following conditions are equivalent.

(III.9.1)  $(W, \nabla)$  does not have a regular singular point at  $p$ .

(III.9.2) In terms of the base

$$\mathbf{e} = \begin{pmatrix} w \\ \nabla \left( h \frac{d}{dh} \right) (w) \\ \vdots \\ \left( \nabla \left( h \frac{d}{dh} \right) \right)^{n-1} (w) \end{pmatrix}$$

of  $W$ , the connection is expressed as

$$(III.9.2.0) \quad \nabla \left( h \frac{d}{dh} \right) \mathbf{e} = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \ddots & 0 \\ 0 & & & \ddots & I \\ & & & \dots & 0 \\ f_0 & f_1 & \dots & & f_{n-1} \end{pmatrix} \mathbf{e}$$

and, for some value of  $i$ , we have  $\text{ord}_p(f_i) < 0$ .

(III.9.3) For every multiple  $a$  of  $n!$ , the inverse image of  $(W, \nabla)$  in  $\mathbf{MC}(K(h^{1/a})/k)$  admits a base  $\mathbf{f}$  in terms of which the connection is expressed (putting  $t = h^{1/a}$ ) as

$$(III.9.3.0) \quad \nabla \left( t \frac{d}{dt} \right) \mathbf{f} = B \mathbf{f}$$

such that, for an integer  $v \geq 1$ , we have

$$(III.9.3.1) \quad B = t^{-v} B_{-v}, \quad B_{-v} \in \mathbf{M}_n(\mathcal{O}_{p^{1/a}}),$$

and the image of  $B_{-v}$  in  $\mathbf{M}_n(k(p))$  (i.e., the value of  $B_{-v}$  at  $p^{1/a}$ ) is not nilpotent.

(III.9.4) For every multiple  $a$  of  $n!$ , the inverse image of  $(W, \nabla)$  in  $\mathbf{MC}(K(h^{1/a})/k)$  does not satisfy (J) at  $p^{1/a}$  (using  $h^{1/a}$  as parameter).

(III.9.5)  $(W, \nabla)$  does not satisfy (J) at  $p$ .

*Proof.* — (III.9.1)  $\Rightarrow$  (III.9.2) by definition of a regular singular point.

(III.9.2)  $\Rightarrow$  (III.9.3). After the base change  $K \rightarrow K(t)$ ,  $t^a = h$ , we have, in terms of the given base  $\mathbf{e}$ ,

$$(III.9.6) \quad \frac{1}{a} \nabla \left( t \frac{d}{dt} \right) \mathbf{e} = \begin{pmatrix} 0 & I & & & \\ 0 & 0 & \ddots & & \\ 0 & & \ddots & \ddots & 0 \\ & & & \ddots & I \\ f_0 & & \dots & & f_{n-1} \end{pmatrix} \mathbf{e} = C \mathbf{e}.$$

By assumption, we have  $\text{ord}_{p^{1/a}}(f_i) < 0$  for at least one value of  $i$ , while for every value of  $j$ , the integer  $\text{ord}_{p^{1/a}}(f_j)$  is divisible by  $n!$ . Consider the strictly positive integer  $v$  defined by

$$(III.9.7) \quad v = \max_{0 \leq j \leq n-1} (-\text{ord}_{p^{1/a}}(f_j)/(n-j)).$$

Consider the basis  $\mathbf{f}$  of  $W \otimes_K K(t)$ , with  $t = h^{1/a}$ , given by

$$(III.9.8) \quad \mathbf{f} = \begin{pmatrix} I \\ t^v & O \\ & t^{2v} \\ O & \ddots & \ddots \\ & & t^{(n-1)v} \end{pmatrix} \mathbf{e} = A\mathbf{e}.$$

We readily calculate (cf. (III.9.6))

$$\begin{aligned} (III.9.9) \quad \nabla \left( t \frac{d}{dt} \right) \mathbf{f} &= \nabla \left( t \frac{d}{dt} \right) (A\mathbf{e}) = \left( \left( t \frac{d}{dt} \right) (A) \right) \cdot \mathbf{e} + A \cdot \nabla \left( t \frac{d}{dt} \right) (\mathbf{e}) \\ &= \left( \left( t \frac{d}{dt} \right) (A) \right) A^{-1} + ACA^{-1} \mathbf{f} = B\mathbf{f} \end{aligned}$$

and an immediate computation then yields

$$(III.9.10) \quad \frac{I}{a} \nabla \left( t \frac{d}{dt} \right) \mathbf{f} = B\mathbf{f}$$

with

$$B = \begin{pmatrix} 0 & & & & & & & \\ v & & & & & & & \\ & 2v & & & & & & \\ & & \ddots & & & & & \\ & & & (n-1)v & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix} + \begin{pmatrix} 0 & t^{-v} & & & & & & \\ \vdots & \ddots & \ddots & & & & & \\ 0 & & \ddots & & & & & \\ \vdots & & & t^{-v} & & & & \\ \vdots & & & & \ddots & & & \\ 0 & 0 & & & & \ddots & & \\ t^{(n-1)v} f_0 & t^{(n-2)v} f_1 & \dots & t^{(n-i-1)v} f_i & \dots & f_{n-1} & & t^{-v} \end{pmatrix}.$$

By definition of  $v$  (III.9.7), we have

$$(III.9.11) \quad \text{ord}_{p^{1/a}}(t^{(n-j-1)v} f_j) = (n-j-1)v + \text{ord}_{p^{1/a}}(f_j) \geq -v$$

and, for at least one value of  $i$ , we have

$$\text{ord}_{p^{1/a}}(t^{(n-i-1)v} f_i) = -v.$$

Thus we may write  $B = t^{-v} B_{-v}$ , with  $B_{-v} \in \mathbf{M}_n(\mathcal{O}_{\mathfrak{p}^{1/a}})$ . In fact, putting

$$(11.9.12) \quad g_j = t^{n-1-j} f_j, \quad \text{for } j=0, \dots, n-1$$

so that

$$(11.9.13) \quad \begin{cases} g_j \in \mathcal{O}_{\mathfrak{p}^{1/a}} & \text{for all } j \\ g_i \notin \mathfrak{m}_{\mathfrak{p}^{1/a}} & \text{for some } i \end{cases}$$

we have

$$(11.9.14) \quad B_{-v} \equiv \begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & \ddots & \ddots & O \\ O & \vdots & O & I \\ 0 & 0 & \cdots & 0 & I \\ g_0 & g_1 & \cdots & & g_{n-1} \end{pmatrix} \pmod{\mathfrak{p}^{1/a}}$$

which is *not* nilpotent modulo  $\mathfrak{p}^{1/a}$ . Indeed, we have

$$(11.9.15) \quad \det(TI_n - B_{-v}) \equiv T^n - \sum_{i=0}^{n-1} g_i T^i \pmod{\mathfrak{p}^{1/a}}$$

so that  $B_{-v}$  is nilpotent modulo  $\mathfrak{p}^{1/a}$  if and only if each  $g_j \in \mathfrak{m}_{\mathfrak{p}^{1/a}}$ , which by (11.9.13) is not the case. This concludes the proof that (11.9.2)  $\Rightarrow$  (11.9.3).

(11.9.3)  $\Rightarrow$  (11.9.4). We use the base  $\mathbf{f}$  to test the estimate. Writing

$$(11.9.16) \quad \left( \nabla \left( t \frac{d}{dt} \right) \right)^j \mathbf{f} = B_j \mathbf{f}$$

we have

$$(11.9.17) \quad B_{j+1} = \left( t \frac{d}{dt} \right) B_j + B_j B$$

and one checks immediately (despite the confusing notation) that

$$(11.9.18) \quad B_j = t^{-vj} B_{-vj} \quad \text{with} \quad B_{-vj} \in \mathbf{M}_n(\mathcal{O}_{\mathfrak{p}^{1/a}})$$

$$\text{and} \quad B_{-vj} \equiv (B_{-v})^j \pmod{\mathfrak{p}^{1/a}}$$

so that

$$(11.9.19) \quad \text{ord}_{\mathfrak{p}^{1/a}}(B_j) = -vj$$

so that (J) is not satisfied.

To conclude the proof of (11.9), we note that (11.9.4)  $\Rightarrow$  (11.9.5) by (11.8.1), and (11.9.5)  $\Rightarrow$  (11.9.1) by (11.7).

*Corollary (11.9.20)* (Manin [25]). — Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$ ,  $\mathfrak{p}$  a place of  $K/k$ . Then  $(W, \nabla)$  has a regular singular point at  $\mathfrak{p}$  if and only if, for every  $w \in W$ , the smallest  $\mathcal{O}_{\mathfrak{p}}$ -module stable under  $\text{Der}_{\mathfrak{p}}(K/k)$  (cf. (11.1.2)) and containing  $w$  (if  $h$  is a uniformizing

parameter at  $\mathfrak{p}$ , this is the  $\mathcal{O}_{\mathfrak{p}}$ -span of  $w$ ,  $\nabla\left(h \frac{d}{dh}\right)(w), \dots, \left(\nabla\left(h \frac{d}{dh}\right)\right)^i(w), \dots$  is of finite type over  $\mathcal{O}_{\mathfrak{p}}$ .

*Proof.* — If  $(W, \nabla)$  has a regular singular point at  $\mathfrak{p}$ , then, for any element  $w \in W$ , the K-span of the elements  $\left(\nabla\left(h \frac{d}{dh}\right)\right)^i(w)$ ,  $i \geq 0$ , “is” a cyclic object of  $\mathbf{MC}(K/k)$  having a regular singular point at  $\mathfrak{p}$ . Letting  $n_1$  be the K-dimension of this span, we see by (11.9) that the  $\mathcal{O}_{\mathfrak{p}}$ -span of the elements  $\left(\nabla\left(h \frac{d}{dh}\right)\right)^i(w)$ , for  $i \geq 0$ , is free of rank  $n_1$  over  $\mathcal{O}_{\mathfrak{p}}$ . (In fact, the elements  $\left(\nabla\left(h \frac{d}{dh}\right)\right)^i(w)$ , for  $i = 0, \dots, n_1 - 1$ , form an  $\mathcal{O}_{\mathfrak{p}}$ -base.)

Conversely, suppose that for every  $w \in W$ , the  $\mathcal{O}_{\mathfrak{p}}$ -span of the elements  $\left(\nabla\left(h \frac{d}{dh}\right)\right)^i(w)$ , for  $i \geq 0$ , is of finite type. This means that every  $w \in W$  is annihilated by a monic polynomial in  $\nabla\left(h \frac{d}{dh}\right)$  whose coefficients are in  $\mathcal{O}_{\mathfrak{p}}$ , hence that the K-span of the elements  $\left(\nabla\left(h \frac{d}{dh}\right)\right)^i(w)$ , for  $i \geq 0$ , is a quotient in  $\mathbf{MC}(K/k)$  of an object with a regular singular point at  $\mathfrak{p}$ . We conclude, by (11.5), that  $(W, \nabla)$  has a regular singular point at  $\mathfrak{p}$ .

**Theorem (11.10)** (Turrittin). — *Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$ ,  $\mathfrak{p}$  a place of  $K/k$ ,  $h$  a uniformizing parameter at  $\mathfrak{p}$ ,  $n = \dim_K(W)$ . The following conditions are equivalent.*

(11.10.1)  $(W, \nabla)$  does not have a regular singular point at  $\mathfrak{p}$ .

(11.10.2) For every integer  $a$  multiple of  $n!$ , there exists a base  $\mathbf{f}$  of  $W \otimes_K K(h^{1/a})$  in terms of which the connection is expressed (putting  $t = h^{1/a}$ ) as

$$(11.10.2.0) \quad \begin{cases} \nabla\left(t \frac{d}{dt}\right) \mathbf{f} = B\mathbf{f} \\ B = t^{-\nu} B_{-\nu}, \text{ with } \nu \text{ an integer } \geq 1, \text{ and} \\ B_{-\nu} \in \mathbf{M}_n(\mathcal{O}_{\mathfrak{p}^{1/a}}) \text{ has non-nilpotent image in } \mathbf{M}_n(k(\mathfrak{p})). \end{cases}$$

(11.10.3) The estimate (J) is not satisfied at  $\mathfrak{p}$ .

*Proof.* — The implications  $(11.10.2) \Rightarrow (11.10.3)$  and  $(11.10.3) \Rightarrow (11.10.1)$  are obvious, using (11.9.16-19) and (11.8.1) for the first, and (11.7) for the second. We now turn to the serious part of the proof.

$(11.10.1) \Rightarrow (11.10.2)$ . We proceed by induction. (If  $n=1$ , we are in the cyclic case.) If  $(W, \nabla)$  has no proper non-zero subobjects, it is necessarily cyclic. If  $(W, \nabla)$  has a non-trivial subobject  $(V, \nabla')$ , we have a short exact sequence in  $\mathbf{MC}(K/k)$

$$0 \rightarrow (V, \nabla') \rightarrow (W, \nabla) \rightarrow (U, \nabla'') \rightarrow 0$$

with  $n_1 = \dim_K(V) < n$ ,  $n_2 = \dim_K(U) < n$ .

By (11.3.0), either  $(V, \nabla')$  or  $(U, \nabla'')$  does not have a regular singular point at  $\mathfrak{p}$ .

So, by induction, there exists a basis of  $W \otimes_K K(h^{1/a})$  of the form  $\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$ , where  $\mathbf{e}$  is a basis of  $V \otimes_K K(h^{1/a})$ , and where  $\mathbf{f}$  projects to a basis of  $U \otimes_K K(h^{1/a})$ , in terms of which the connection is expressed (putting  $t = h^{1/a}$ ) as

$$(III.10.4) \quad \nabla \left( t \frac{d}{dt} \right) \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} A & \mathbf{O} \\ B & C \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

such that

$$(III.10.5) \quad \begin{aligned} A &= t^{-v} A_{-v} \quad \text{for an integer } v \geq 0 \\ A_{-v} &\in \mathbf{M}_{n_1}(\mathcal{O}_{\mathfrak{p}^{1/a}}) \end{aligned}$$

if  $v > 0$ ,  $A_{-v}$  has *non-nilpotent* image in  $\mathbf{M}_{n_1}(k(\mathfrak{p}))$  and

$$(III.10.6) \quad \begin{aligned} C &= t^{-\tau} C_{-\tau} \quad \text{for an integer } \tau \geq 0 \\ C_{-\tau} &\in \mathbf{M}_{n_2}(\mathcal{O}_{\mathfrak{p}^{1/a}}) \end{aligned}$$

if  $\tau > 0$ , then  $C_{-\tau}$  has *non-nilpotent* image in  $\mathbf{M}_{n_2}(k(\mathfrak{p}))$ , and finally

$$(III.10.7) \quad v + \tau > 0.$$

Replacing the basis  $\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$  by the basis  $\begin{pmatrix} \mathbf{e} \\ t^N \mathbf{f} \end{pmatrix}$  for  $N$  large, we obtain the new connection matrix

$$(III.10.8) \quad \nabla \left( t \frac{d}{dt} \right) \begin{pmatrix} \mathbf{e} \\ t^N \mathbf{f} \end{pmatrix} = \begin{pmatrix} A & \mathbf{O} \\ t^N B & C + NI \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ t^N \mathbf{f} \end{pmatrix}$$

with  $A, B, C$  as before (but now  $t^N B$  is holomorphic at  $\mathfrak{p}^{1/a}$ ). Clearly this connection matrix (III.10.8) has a pole at  $\mathfrak{p}^{1/a}$  of order  $\sup(v, \tau)$ , and

$$t^{\sup(v, \tau)} \begin{pmatrix} A & \mathbf{O} \\ t^N B & C + NI \end{pmatrix}$$

has *non-nilpotent* image in  $\mathbf{M}_n(k(\mathfrak{p}))$ . This concludes the proof of Turrittin's theorem.

*Proposition (III.11).* — Let

$$F \hookrightarrow K \hookrightarrow L$$

be a tower of function fields in one variable over a field  $k$  of characteristic zero, with  $\deg(K/F) < \infty$  and  $\deg(L/K) < \infty$ . Let  $\mathfrak{p}$  be a place of  $L/k$ ,  $\mathfrak{p}'$  the induced place of  $K/k$ , and  $\mathfrak{p}''$  the induced place of  $F/k$ . Let  $\mathfrak{p}'_1, \dots, \mathfrak{p}'_r$  be all the places of  $K/k$  which lie over the place  $\mathfrak{p}''$  of  $F/k$ .

Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$ . Then:

(III.11.1)  $(W, \nabla)$  has a regular singular point at  $\mathfrak{p}'$  if and only if the inverse image  $(W \otimes_K L, \nabla_L)$  of  $(W, \nabla)$  in  $\mathbf{MC}(L/k)$  has a regular singular point at  $\mathfrak{p}$ .

(III.11.2) The “direct image” ( $W$  as  $F$ -space,  $\nabla|_{\text{Der}(F/k)}$ ) of  $(W, \nabla)$  in  $\mathbf{MC}(F/k)$  has a regular singular point at  $\mathfrak{p}''$ , if and only if  $(W, \nabla)$  has a regular singular point at each place  $\mathfrak{p}'_i$  of  $K/k$  which lies over  $\mathfrak{p}''$ .

*Proof.* — We have (cf. (11.1.2))

$$(11.11.3) \quad \text{Der}_{\mathcal{P}}(L/k) \xleftarrow{\sim} \text{Der}_{\mathcal{P}'}(K/k) \otimes_{\mathcal{O}_{\mathcal{P}'}} \mathcal{O}_{\mathcal{P}}$$

and

$$(11.11.4) \quad \text{Der}_{\mathcal{P}_i}(K/k) \xleftarrow{\sim} \text{Der}_{\mathcal{P}''}(F/k) \otimes_{\mathcal{O}_{\mathcal{P}''}} \mathcal{O}_{\mathcal{P}_i}, \quad i=1, \dots, r.$$

To prove (11.11.1), observe that if  $W_{\mathcal{P}'}$  is an  $\mathcal{O}_{\mathcal{P}'}$ -lattice in  $W$ , stable under  $\text{Der}_{\mathcal{P}'}(K/k)$ , then  $W_{\mathcal{P}'} \otimes_{\mathcal{O}_{\mathcal{P}'}} \mathcal{O}_{\mathcal{P}}$  is an  $\mathcal{O}_{\mathcal{P}}$ -lattice in  $W \otimes_K L$ , stable under  $\text{Der}_{\mathcal{P}}(L/k)$ . To prove (11.11.2)  $\Rightarrow$  (11.11.1), observe that if  $(W \otimes_K L)_{\mathcal{P}}$  is an  $\mathcal{O}_{\mathcal{P}}$ -lattice in  $W \otimes_K L$ , stable under  $\text{Der}_{\mathcal{P}}(L/k)$ , then  $W \cap (W \otimes_K L)_{\mathcal{P}}$  is an  $\mathcal{O}_{\mathcal{P}'}$ -lattice in  $W$  which is stable under  $\text{Der}_{\mathcal{P}'}(K/k)$ .

Similarly, to prove (11.11.3), note that if for  $i=1, \dots, r$ ,  $W_{\mathcal{P}_i}$  is an  $\mathcal{O}_{\mathcal{P}_i}$ -lattice in  $W$ , stable under  $\text{Der}_{\mathcal{P}_i}(K/k)$ , then  $\bigoplus_i W_{\mathcal{P}_i}$  is an  $\mathcal{O}_{\mathcal{P}''}$ -lattice in  $W$ , stable under  $\text{Der}_{\mathcal{P}''}(F/k)$ . To prove the converse we simply apply the criterion (11.9.20) of Manin.

*Corollary (11.12).* — Let  $K/k$  be a function field in one variable over a field  $k$  of characteristic zero,  $\mathfrak{p}$  a place of  $K/k$ ,  $\bar{k}$  an algebraic closure of  $k$ ,  $\bar{\mathfrak{p}}$  the induced place of  $K\bar{k}/\bar{k}$ ,  $(W, \nabla)$  an object of  $\mathbf{MC}(K/k)$ , and  $(W_{\bar{k}}, \nabla_{\bar{k}})$  its inverse image in  $\mathbf{MC}(K\bar{k}/\bar{k})$ . Then  $(W, \nabla)$  has a regular singular point at  $\mathfrak{p}$  if and only if  $(W_{\bar{k}}, \nabla_{\bar{k}})$  has a regular singular point at  $\bar{\mathfrak{p}}$ .

*Proof.* — Use the equivalence (11.10.1)  $\Leftrightarrow$  (11.10.3), calculating with a  $K$ -base of  $W$ , and a parameter at  $\mathfrak{p}$ .

## 12. The Monodromy around a Regular Singular Point.

We refer to the elegant paper [25] of Manin for a proof of the following theorem, which ought to be well-known.

*Theorem (12.0).* — Let  $K/k$  be a function field in one variable, with  $k$  of characteristic zero. Let  $\mathfrak{p}$  be a place of  $K/k$  which is rational, i.e.  $k(\mathfrak{p})=k$ . Suppose that  $(W, \nabla)$  is an object of  $\mathbf{MC}(K/k)$  which has, at  $\mathfrak{p}$ , a regular singular point. In terms of a uniformizing parameter  $t$  at  $\mathfrak{p}$ , and a basis  $\mathbf{e}$  of an  $\mathcal{O}_{\mathfrak{p}}$ -lattice  $W_{\mathfrak{p}}$  of  $W$  which is stable under  $\nabla\left(t \frac{d}{dt}\right)$ , we express the connection as

$$(12.0.1) \quad \nabla\left(t \frac{d}{dt}\right)\mathbf{e} = B\mathbf{e}, \quad B \in \mathbf{M}_n(\mathcal{O}_{\mathfrak{p}}).$$

Suppose that the matrix  $B(\mathfrak{p}) \in \mathbf{M}_n(k)$  (the value of  $B$  at  $\mathfrak{p}$ , whose conjugacy class depends only on the lattice  $W_{\mathfrak{p}}$ , not on the particular choice of a base of  $W_{\mathfrak{p}}$  or on the choice of the uniformizing parameter  $t$ ) has all of its proper values in  $k$ . Then :

(12.0.2) The set of images in the additive group  $k^+/\mathbf{Z}$  of the proper values of  $B(\mathfrak{p})$  (the exponents of  $(W, \nabla)$  at  $\mathfrak{p}$ ) is independent of the choice of the  $\nabla\left(t \frac{d}{dt}\right)$ -stable  $\mathcal{O}_{\mathfrak{p}}$ -lattice  $W_{\mathfrak{p}}$  in  $W$ .

(12.0.3) Fix a set-theoretic section  $\phi : k^+/\mathbf{Z} \rightarrow k^+$  of the projection mapping  $k^+ \rightarrow k^+/\mathbf{Z}$ .

(For instance, if  $k = \mathbf{C}$ , we might require  $0 \leq \operatorname{Re}(\varphi) < 1$ .) There exists a unique  $\mathcal{O}_{\mathfrak{p}}$ -lattice  $W'_{\mathfrak{p}}$  of  $W$ , stable under  $\nabla \left( t \frac{d}{dt} \right)$ , in terms of a base  $\mathbf{e}'$  of which the connection is expressed as

$$(12.0.3.1) \quad \nabla \left( t \frac{d}{dt} \right) \mathbf{e}' = C \mathbf{e}', \quad \text{with } C \in \mathbf{M}_n(\mathcal{O}_{\mathfrak{p}})$$

and such that the proper values of  $C(\mathfrak{p}) \in \mathbf{M}_n(k)$  are all fixed by the composition  $k^+ \xrightarrow{\text{proj}} k^+ / \mathbf{Z} \xrightarrow{\varphi} k^+$ . (The point is that non-equal proper values of  $C(\mathfrak{p})$  do not differ by integers.)

(12.0.4) The completion  $\hat{W}'_{\mathfrak{p}}$  of the  $\mathcal{O}_{\mathfrak{p}}$ -lattice  $W'_{\mathfrak{p}}$  of  $W$  above admits a base  $\hat{\mathbf{e}}$  in terms of which the connection is simply

$$\nabla \left( t \frac{d}{dt} \right) \hat{\mathbf{e}} = C(\mathfrak{p}) \cdot \hat{\mathbf{e}}.$$

*Remark (12.1).* — If we require of  $\varphi$  that  $\varphi(\mathbf{Z}) = \{0\}$ , and if  $B(\mathfrak{p})$  has all its proper values in  $\mathbf{Z}$ , then the matrix  $C(\mathfrak{p})$  is nilpotent.

*Remark (12.2).* — In general, let  $C(\mathfrak{p}) = D + N$ , with  $[D, N] = 0$ , be the Jordan decomposition of  $C(\mathfrak{p})$  as a sum of a semi-simple matrix  $D$  and a nilpotent matrix  $N$ . Then the conjugacy class of  $N$  is independent of the choice of  $\varphi$ . (And the eigenvalues of  $D$  are, modulo  $\mathbf{Z}$ , the exponents (cf. (12.0.2)) at  $\mathfrak{p}$ .)

*Remark (12.3).* — Suppose  $k \subset \mathbf{C}$ , and let  $\mathcal{O}_{\mathfrak{p}}^{\text{anal}}$  be the local ring of germs of analytic functions at  $\mathfrak{p}$ . Then the base  $\hat{\mathbf{e}}$  of  $\hat{W}'_{\mathfrak{p}}$  comes by extension of scalars  $\mathcal{O}_{\mathfrak{p}}^{\text{anal}} \rightarrow \hat{\mathcal{O}}_{\mathfrak{p}}$  from a base  $\mathbf{e}^{\text{anal}}$  of  $W'_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}}^{\text{anal}}$ . In terms of this base  $\mathbf{e}^{\text{anal}}$ , a multivalued “ fundamental matrix of horizontal sections ” over a small punctured disc around  $\mathfrak{p}$  is given by

$$(12.3.1) \quad t^{-C(\mathfrak{p})} = \exp(-C(\mathfrak{p}) \log t).$$

Thus, when “  $t$  turns once around  $\mathfrak{p}$  counterclockwise ”,  $\log(t)$  becomes  $\log(t) + 2\pi i$ , and the fundamental matrix

$$(12.3.2) \quad t^{-C(\mathfrak{p})}$$

becomes

$$(12.3.3) \quad \exp(-2\pi i C(\mathfrak{p})) t^{-C(\mathfrak{p})}$$

or, what is the same,

$$(12.3.4) \quad \exp(-2\pi i D) \exp(-2\pi i N) t^{-D-N}.$$

In particular, the proper values of the monodromy substitution for “  $t$  turning once around  $\mathfrak{p}$  counterclockwise ” are the numbers  $\exp(-2\pi i \sigma_1), \dots, \exp(-2\pi i \sigma_n)$ , where  $\sigma_1, \dots, \sigma_n$  are the exponents at  $\mathfrak{p}$ .

*Definition (12.4).* — Let  $K/k$  be a function field in one variable, with  $k$  a field of characteristic zero. Let  $\mathfrak{p}$  be a place of  $K/k$  which is rational,  $(W, \nabla)$  an object of  $\mathbf{MC}(K/k)$  which has a regular singular point at  $\mathfrak{p}$ . We say that the local monodromy

at  $\mathfrak{p}$  is *quasi-unipotent* if the *exponents* at  $\mathfrak{p}$  are rational numbers. If the local monodromy at  $\mathfrak{p}$  is quasi-unipotent, we say that its *exponent of nilpotence* is  $\leq v$  if, in the notation of (12.2), we have  $N^v = 0$ .

*Definition (12.4 bis).* — If  $\mathfrak{p}$  is any place of  $K/k$  (*not necessarily rational*), at which  $(W, \nabla)$  has a regular singular point, we say that the local monodromy at  $\mathfrak{p}$  is quasi-unipotent (resp. quasi-unipotent with exponent of nilpotence  $\leq v$ ) if this *becomes* true after the change of base  $k \rightarrow \bar{k} =$  an algebraic closure of  $k$ , at the induced place  $\bar{\mathfrak{p}}$  of  $K \cdot \bar{k}/\bar{k}$ .

**(12.5)** *An example.* — Let  $k = \mathbf{C}$ ,  $K = \mathbf{C}(z)$ ,  $(W, \nabla)$  the object of  $\mathbf{MC}(K/k)$  given by

**(12.5.1)**  $W$ , a  $K$ -space of dimension 2, with basis  $e_1, e_2$ . In terms of this base, the connection

$$(12.5.2) \quad \nabla \left( z \frac{d}{dz} \right) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = B \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -z \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Thus  $(W, \nabla)$  has a regular singular point at the place  $\mathfrak{p} : z = 0$ , and its *exponents* there (the proper values, mod  $\mathbf{Z}$ , of  $B(0)$ ) are *integers*.

Although  $\exp(2\pi i B(0)) = I$ , the *monodromy* of local horizontal sections in a punctured disc around zero is *non-trivial*. Indeed, a basis of these (multivalued) horizontal sections is

$$(12.5.3) \quad \begin{cases} v_1 = ze_2 \\ v_2 = \frac{1}{2\pi i} (e_1 + z \log(z) \cdot e_2). \end{cases}$$

After a counterclockwise turn around  $z = 0$ ,

$$(12.5.4) \quad \begin{cases} v_1 \rightarrow v_1 \\ v_2 \rightarrow v_2 + v_1. \end{cases}$$

In terms of a section  $\varphi : \mathbf{C}/\mathbf{Z} \rightarrow \mathbf{C}$  which maps  $\mathbf{Z}$  to 0, the unique  $\mathcal{O}_{\mathfrak{p}}$ -lattice of (12.0.3) is the  $\mathcal{O}_{\mathfrak{p}}$ -span of the vectors

$$(12.5.5) \quad \begin{cases} e'_1 = e_1 \\ e'_2 = -ze_2 \end{cases}$$

in terms of which the connection is expressed (cf. (12.0.3)) as

$$\nabla \left( z \frac{d}{dz} \right) \begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} = C \begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix}.$$

*Remark (12.6).* — Returning to the “abstract” case, suppose that  $\mathfrak{p}$  is a rational place of  $K/k$  at which  $(W, \nabla)$  has a regular singular point, and  $W_{\mathfrak{p}}$  is an  $\mathcal{O}_{\mathfrak{p}}$ -lattice of  $W$

which is stable under  $\nabla\left(t\frac{d}{dt}\right)$ . Suppose the completion  $\hat{W}_p$  of  $W_p$  admits a base  $\hat{\mathbf{e}}$  in terms of which the connection is expressed as

$$(12.6.1) \quad \nabla\left(t\frac{d}{dt}\right)\hat{\mathbf{e}} = C\hat{\mathbf{e}} \quad \text{with} \quad C \in \mathbf{M}_n(k).$$

If the proper values of  $C$  all lie in  $k$ , then we may *rechoose* the base  $\hat{\mathbf{e}}$  of  $\hat{W}_p$  so that the connection is expressed as

$$(12.6.2) \quad \nabla\left(t\frac{d}{dt}\right)\hat{\mathbf{e}} = C\hat{\mathbf{e}}, \quad C \in \mathbf{M}_n(k)$$

and such that  $C$  is in the form

$$(12.6.3) \quad \begin{pmatrix} C_1 & & \\ & \ddots & \mathbf{O} \\ \mathbf{O} & \ddots & \\ & & C_r \end{pmatrix}$$

where each  $C_i$  is a square matrix of size  $v_i$  whose *only* proper value is  $\lambda_i$ . In terms of the section  $\varphi$  of  $k^+ \rightarrow k^+/\mathbf{Z}$ , we put  $n_i = \varphi(\lambda_i) - \lambda_i$ . Then we replace the lattice  $W_p$  by the lattice  $W'_p$  whose *completion* admits as base

$$(12.6.4) \quad \hat{\mathbf{e}}' = \begin{pmatrix} t^{n_1} I_{v_1} & & \\ & \ddots & \mathbf{O} \\ \mathbf{O} & \ddots & \\ & & t^{n_r} I_{v_r} \end{pmatrix} \hat{\mathbf{e}}.$$

In terms of the base  $\hat{\mathbf{e}'}$ , the connection is expressed as

$$(12.6.5) \quad \nabla\left(t\frac{d}{dt}\right)\hat{\mathbf{e}'} = \begin{pmatrix} C_1 + n_1 I_{v_1} & & \\ & \ddots & \mathbf{O} \\ \mathbf{O} & \ddots & \\ & & C_r + n_r I_{v_r} \end{pmatrix} \hat{\mathbf{e}'} = C' \hat{\mathbf{e}'}.$$

It follows (the proper values of  $C'$  being fixed by  $\varphi$ ) that  $W'_p$  is the unique lattice specified in (12.0.3) by the choice of  $\varphi$ . Noting that  $C$  and  $C'$  have the *same* nilpotent parts in their Jordan decomposition, we have

*Proposition (12.6.6).* — *Suppose  $(W, \nabla)$  has a regular singular point at the rational place  $p$  of  $K/k$ , and there exists an  $\mathcal{O}_p$ -lattice  $W_p$  whose completion admits a base  $\hat{\mathbf{e}}$  in terms of which the connection is expressed as*

$$\nabla\left(t\frac{d}{dt}\right)\hat{\mathbf{e}} = C\hat{\mathbf{e}}, \quad \text{with} \quad C \in \mathbf{M}_n(k).$$

*Then the local monodromy of  $(W, \nabla)$  at  $p$  is quasi-unipotent of exponent of nilpotence  $\leq v$  if and only if in the Jordan decomposition of  $C$ ,*

$$C = D + N, \quad [D, N] = 0$$

with  $D$  semisimple and  $N$  nilpotent, the proper values of  $D$  are rational numbers, and  $N^r = 0$ .

**Proposition (12.7).** — Let  $F \hookrightarrow K \hookrightarrow L$  be a tower of function fields in one variable over a field  $k$  of characteristic zero. Let  $\mathfrak{p}$  be a place of  $L/k$ ,  $\mathfrak{p}'$  the induced place of  $K/k$ , and  $\mathfrak{p}''$  the induced place of  $F/k$ . Let  $\mathfrak{p}'_1, \dots, \mathfrak{p}'_r$  be all the places of  $K/k$  which lie over the place  $\mathfrak{p}''$  of  $F/k$ . Let  $(W, \nabla)$  be an object of  $\mathbf{MC}(K/k)$  which has regular singular points at each place  $\mathfrak{p}'_1, \dots, \mathfrak{p}'_r$ , and  $v \geq 1$  an integer. Then:

(12.7.1) The inverse image  $(W \otimes_K L, \nabla_L)$  of  $(W, \nabla)$  in  $\mathbf{MC}(L/k)$ , which has a regular singular point at  $\mathfrak{p}$  by (11.12), has quasi-unipotent local monodromy at  $\mathfrak{p}$  of exponent of nilpotence  $\leq v$ , if and only if  $(W, \nabla)$  has quasi-unipotent local monodromy at  $\mathfrak{p}'$ , of exponent of nilpotence  $\leq v$ .

(12.7.2) The direct image  $(W \text{ as } F\text{-space}, \nabla|_{\text{Der}(F/k)})$  of  $(W, \nabla)$  in  $\mathbf{MC}(F/k)$  has quasi-unipotent local monodromy at  $\mathfrak{p}''$  of exponent of nilpotence  $\leq v$  if and only if  $(W, \nabla)$  has quasi-unipotent local monodromy of exponent of nilpotence  $\leq v$  at each place  $\mathfrak{p}'_i$  of  $K/k$  lying over  $\mathfrak{p}''$ .

*Proof.* — By making the base-change  $k \rightarrow \bar{k}$  = an algebraic closure of  $k$ , we are immediately reduced to the case of  $k$  algebraically closed. Let  $t$  be a uniformizing parameter at  $\mathfrak{p}''$ .

To prove (12.7.1), we choose an  $\mathcal{O}_{\mathfrak{p}'}\text{-lattice } W_{\mathfrak{p}'}$  in  $W$ , stable under  $\nabla\left(t \frac{d}{dt}\right)$ , whose completion  $\hat{W}_{\mathfrak{p}'}$  admits a base  $\hat{\mathbf{e}}$  in terms of which the connection is expressed (putting  $\varepsilon(\mathfrak{p}'/\mathfrak{p}'') =$  the ramification index) as

$$(12.7.3) \quad \varepsilon(\mathfrak{p}'/\mathfrak{p}'') \nabla \left( t \frac{d}{dt} \right) \hat{\mathbf{e}} = C \hat{\mathbf{e}}, \quad C \in \mathbf{M}_n(k).$$

Consider the lattice  $W_{\mathfrak{p}'} \otimes_{\mathcal{O}_{\mathfrak{p}'}} \mathcal{O}_{\mathfrak{p}}$  in  $W \otimes_K L$ ; its completion admits the “ same ” base  $\hat{\mathbf{e}}$ , and the connection is

$$(12.7.4) \quad \varepsilon(\mathfrak{p}/\mathfrak{p}'') \nabla \left( t \frac{d}{dt} \right) \hat{\mathbf{e}} = \varepsilon(\mathfrak{p}/\mathfrak{p}') C \hat{\mathbf{e}}.$$

We conclude the proof of (12.7.1) by applying the criterion (12.6.6) to the matrices  $C$  and  $\varepsilon(\mathfrak{p}/\mathfrak{p}')C$ .

Now let us prove (12.7.2). For each point  $\mathfrak{p}'_i$  lying over  $\mathfrak{p}''$ , we choose a lattice  $W_{\mathfrak{p}'_i}$  in terms of a base  $\hat{\mathbf{e}}_i$  of whose completion the connection is expressed (writing  $\varepsilon_i = \varepsilon(\mathfrak{p}'_i/\mathfrak{p}'')$ ) as

$$(12.7.5) \quad \varepsilon_i \nabla \left( t \frac{d}{dt} \right) (\hat{\mathbf{e}}_i) = C(\mathfrak{p}'_i) \hat{\mathbf{e}}_i, \quad C(\mathfrak{p}'_i) \in \mathbf{M}_n(k).$$

Consider the  $\mathcal{O}_{\mathfrak{p}''}\text{-lattice } \bigoplus_i W_{\mathfrak{p}'_i}$  in  $W$  considered as  $F$ -space. In the natural basis of its completion, consisting of the blocks of vectors

$$(12.7.6) \quad (t)^{a/\varepsilon_i} \hat{\mathbf{e}}_i, \quad a = 0, 1, \dots, \varepsilon_i - 1, \quad i = 1, \dots, r$$

the connection, *stable* on the span of each block  $t^{a/\varepsilon_i} \hat{\mathbf{e}}_i$ , is expressed on each block as

$$(12.7.7) \quad \nabla \left( t \frac{d}{dt} \right) (t^{a/\varepsilon_i} \hat{\mathbf{e}}_i) = \frac{1}{\varepsilon_i} (C(\mathbf{p}_i) + a(t^{a/\varepsilon_i} \hat{\mathbf{e}}_i)).$$

Again, we conclude by using condition (12.6.6), which is satisfied by each of the matrices  $C(\mathbf{p}_i)$  if and only if it is satisfied by each of the matrices  $\frac{1}{\varepsilon_i} (C(\mathbf{p}_i) + a)$ , with  $a = 0, 1, \dots, \varepsilon_i - 1$ .

### 13. Consequences of Tjurittin's Theorem.

We are now in a position to apply Tjurittin's theorem (11.10) to the study of globally nilpotent connections.

**Theorem (13.0).** — Let  $T$  be a global affine variety (cf. (9.0)) and  $f: S \rightarrow T$  a smooth morphism of relative dimension one, whose generic fibre is geometrically connected. Let  $(\mathcal{M}, \nabla)$  be an object of  $\mathbf{MIC}(S/T)$ , with  $\mathcal{M}$  locally free of finite rank on  $S$ . Let  $k$  denote the function field of  $T$ ,  $K$  the function field of  $S$ . Thus  $K$  is a field of functions in one variable over a field  $k$  of characteristic zero.

(13.0.1) Suppose that  $(\mathcal{M}, \nabla)$  is globally nilpotent on  $S/T$  (cf. (9.1)). Then the inverse image of  $(\mathcal{M}, \nabla)$  in  $\mathbf{MC}(K/k)$  has a regular singular point at every place  $\mathfrak{p}$  of  $K/k$ , and has quasi-unipotent local monodromy at every place  $\mathfrak{p}$  of  $K/k$ .

(13.0.2) Suppose that  $(\mathcal{M}, \nabla)$  is globally nilpotent of exponent  $v$  on  $S/T$ . Then at every place  $\mathfrak{p}$  of  $K/k$ , the local monodromy of the inverse image of  $(\mathcal{M}, \nabla)$  in  $\mathbf{MC}(K/k)$  is quasi-unipotent of exponent  $\leq v$ .

*Proof.* — Using (9.3.1) and (9.2), and (11.12.2) and (12.7.2), we are immediately reduced to the case:

(13.0.3)  $S$  is a principal open subset of  $\mathbf{A}_T^1$ , i.e.,  $T = \text{Spec}(R)$ , and  $S = \text{Spec}\left(R[t] \left[\frac{1}{g(t)}\right]\right)$  with  $g(t) \in R[t]$ .

(13.0.4) We wish to check at the place of  $K = k(t)$  defined by  $t = 0$ .

(13.0.5)  $M$  is a free  $R[t] \left[\frac{1}{g(t)}\right]$ -module.

(13.0.6)  $g(t) = t^j h(t)$ , with  $h(t) \in R[t]$  and  $j \geq 1$  (otherwise there is no singularity at  $t = 0$ ).

And  $h(0)$  an invertible element of  $R$  (at the expense of localizing  $R$  at  $h(0)$ ).

Suppose that  $(M, \nabla)$  is globally nilpotent, but that  $t = 0$  is not a regular singular point of its restriction to  $\mathbf{MC}(k(t)/k)$ . Let  $n$  be the rank of  $M$ . Let us make the base change (putting  $z = t^{1/n!}$ )

$$(13.0.7) \quad R[t] \left[\frac{1}{g(t)}\right] \rightarrow R[z] \left[\frac{1}{g(z)}\right].$$

By (9.2), the inverse image of  $(M, \nabla)$  on  $R[z] \left[ \frac{I}{g(z)} \right]$  is still globally nilpotent, but by Turrin's theorem (11.10), there exists a basis  $\mathbf{m}$  of  $M$  (over an open subset of  $\text{Spec} \left( R[z] \left[ \frac{I}{g(z)} \right] \right)$ , which, by "enlarging"  $g$ , we may suppose to be all of  $\text{Spec} \left( R[z] \left[ \frac{I}{g(z)} \right] \right)$ , in terms of which the connection is expressed as

$$(13.0.8) \quad \nabla \left( z \frac{d}{dz} \right) \cdot \mathbf{m} = z^{-\mu} (A + zB) \mathbf{m}, \quad \mu \geq 1$$

$$(13.0.9) \quad \begin{cases} A \in \mathbf{M}_n(R) \text{ non-nilpotent} \\ B \in \mathbf{M}_n \left( R[z] \left[ \frac{I}{h(z)} \right] \right) \end{cases}$$

$$(13.0.10) \quad \begin{cases} A \in \mathbf{M}_n(R) \text{ non-nilpotent} \\ B \in \mathbf{M}_n \left( R[z] \left[ \frac{I}{h(z)} \right] \right) \text{ (and } h(0) \text{ invertible in } R\text{).} \end{cases}$$

An immediate calculation then shows that, for each integer  $j \geq 1$ , we have

$$(13.0.11) \quad \left( \nabla \left( z \frac{d}{dz} \right) \right)^j \mathbf{m} = z^{-\mu j} (A^j + zB_j) \mathbf{m} \quad \text{with} \quad B_j \in \mathbf{M}_n \left( R[z] \left[ \frac{I}{h(z)} \right] \right).$$

Now let  $p$  be a prime number. Recall that in  $\text{Der}(\mathbf{F}_p[z]/\mathbf{F}_p)$  we have  $\left( z \frac{d}{dz} \right)^p = z \frac{d}{dz}$ .

Thus the hypothesis of global nilpotence is that, for every prime number  $p$ , there is an integer  $\alpha(p)$  such that

$$(13.0.12) \quad \left( \left( \nabla \left( z \frac{d}{dz} \right) \right)^p - \nabla \left( z \frac{d}{dz} \right) \right)^{\alpha(p)} M \subset pM$$

or, equivalently, using (5.0.9), that, for every prime  $p$

$$(13.0.12 \text{ bis}) \quad (z^{-\mu p} (A^p + zB_p) - z^{-\mu} (A + zB))^{\alpha(p)} \in p \mathbf{M}_n \left( R[z] \left[ \frac{I}{g(z)} \right] \right).$$

Hence looking at the most polar term, we conclude

$$(13.0.13) \quad A^{p \cdot \alpha(p)} \in p \mathbf{M}_n(R) \quad \text{for every prime } p.$$

Now look at the characteristic polynomial of  $A$ ,  $\det(XI_n - A)$ . According to (13.0.13), its value at every closed point of  $T = \text{Spec}(R)$  is  $X^n$ , and hence

$$(13.0.14) \quad \det(XI_n - A) = X^n$$

which implies that  $A$  is nilpotent, a contradiction. This proves that  $t=0$  was a regular singular point of the inverse image of  $(M, \nabla)$  in  $\mathbf{MC}(k(t)/k)$ .

We now turn to proving quasi-unipotence of the local monodromy at  $t=0$ . By definition of a regular singular point, there exists a basis  $\mathbf{m}$  of  $M$  (over an open subset

of  $\text{Spec } R[t] \left[ \frac{I}{g(t)} \right]$ , which by “enlarging”  $g$ , we may suppose to be *all* of  $\text{Spec} \left( R[t] \left[ \frac{I}{g(t)} \right] \right)$  in terms of which the connection is expressed as

$$(13.0.15) \quad \nabla \left( t \frac{d}{dt} \right) \mathbf{m} = (A + tB) \mathbf{m}$$

with  $A \in \mathbf{M}_n(R)$ ,  $B \in \mathbf{M}_n \left( R[t] \left[ \frac{I}{h(t)} \right] \right)$ , and  $h(0)$  invertible in  $R$ .

By adjoining to the ring  $R$  the proper values of  $A$ , and perhaps localizing the resulting ring a bit, we can assume that the Jordan decomposition is defined over  $R$ , i.e. that

$$(13.0.16) \quad A = D + N, \quad [D, N] = 0$$

$$\begin{cases} D \in \mathbf{M}_n(R) & \text{diagonal} \\ N \in \mathbf{M}_n(R) & \text{nilpotent super-triangular.} \end{cases}$$

Suppose that  $(M, \nabla)$  is *globally nilpotent*. For each prime number  $p$ , we thus have

$$(13.0.17) \quad \left( \left( \nabla \left( t \frac{d}{dt} \right) \right)^p - \nabla \left( z \frac{d}{dz} \right) \right)^{\alpha(p)} M \subset pM.$$

As before (13.0.11), an immediate calculation shows that, for each integer  $j \geq 1$

$$(13.0.18) \quad \left( \nabla \left( t \frac{d}{dt} \right) \right)^j \mathbf{m} = (A^j + tB_j) \mathbf{m}$$

with  $B_j \in \mathbf{M}_n \left( R[t] \left[ \frac{I}{h(t)} \right] \right)$ ,  $h(0)$  invertible in  $R$ .

Now using (5.0.9) and looking at the *constant* term of the matricial expression of (13.0.12), we find

$$(13.0.19) \quad (A^p - A)^{\alpha(p)} \in p\mathbf{M}_n(R) \quad \text{for every prime } p.$$

Writing  $A = D + N$  (cf. 13.0.15), we have (because  $[D, N] = 0$ )

$$(13.0.20) \quad 0 \equiv (A^p - A)^{\alpha(p)} \equiv (D^p - D + N^p - N)^{\alpha(p)} \quad \text{modulo } p\mathbf{M}_n(R)$$

and, looking at the diagonal terms, we find

$$(13.0.21) \quad (D^p - D)^{\alpha(p)} \in p\mathbf{M}_n(R).$$

Let  $d$  be a proper value of  $D$ ; then  $d$  is a quantity in an integral domain  $R$  of finite type over  $\mathbf{Z}$ , whose quotient field is of characteristic zero, such that at every closed point  $\mathfrak{p}$  of  $\text{Spec}(R)$ , the image of  $d$  in the residue field  $R/\mathfrak{p}$  at  $\mathfrak{p}$  lies in the *prime field*. As is well-

known, this implies that  $d \in R \cap Q$ . This proves the quasi-unipotence of the local monodromy.

Now we must estimate the exponent of nilpotence of the local monodromy, assuming  $(M, \nabla)$  globally nilpotent of exponent  $v$ . At a closed point  $p$  of  $R$  of residue characteristic  $p$ , we have ( $D$  being diagonal)

$$(13.0.22) \quad D^p \equiv D \pmod{p}$$

so that (13.0.20) gives (since we may take  $\alpha(p) = v$  for all  $p$ )

$$(13.0.23) \quad (N^p - N)^v \equiv 0 \pmod{p}.$$

But  $N$  is *nilpotent*; let us write

$$(13.0.24) \quad (N^p - N)^v = (-1)^v N^v (1 - N^{p-1})^v$$

and notice that  $(1 - N^{p-1})^v$  is *invertible* in  $M_n(R)$ , so (13.0.23) is equivalent to

$$(13.0.25) \quad N^v \equiv 0 \pmod{p} \quad \text{for every closed point } p$$

which implies that  $N^v = 0$  in  $M_n(R)$ . Q.E.D.

**(13.1)** *A counter-example* (d'après Deligne). — Let  $\pi : S \rightarrow T$  be a smooth morphism. There is a bijective correspondence between  $T$ -connections  $\nabla$  on  $\mathcal{O}_S$  and global sections of  $\Omega_{S/T}^1$ . Namely, to a  $T$ -connection  $\nabla$  on  $\mathcal{O}_S$

$$(13.1.0) \quad \nabla : \mathcal{O}_S \rightarrow \Omega_S^1$$

corresponds the global section of  $\Omega_{S/T}^1$

$$(13.1.1) \quad \omega = \nabla(1).$$

Conversely, to a global section  $\omega$  of  $\Omega_{S/T}^1$  corresponds the  $T$ -connection  $\nabla_\omega$  on  $\mathcal{O}_S$ , defined by

$$(13.1.2) \quad \nabla_\omega(f) = df + f\omega.$$

The *curvature*  $K_\omega$  of the connection  $\nabla_\omega$  is

$$(13.1.3) \quad \begin{cases} K_\omega : \mathcal{O}_S \rightarrow \Omega_{S/T}^2 \\ K_\omega(f) = f \cdot d\omega. \end{cases}$$

Thus  $\nabla_\omega$  is *integrable* precisely when  $\omega$  is *closed*.

Suppose that  $T$  (and hence  $S$ ) is a *reduced* scheme of characteristic  $p$ , and let  $\omega$  be a *closed* global section of  $\Omega_{S/T}^1$ . What does it mean that the connection  $\nabla_\omega$  be *nilpotent*? First, since  $\mathcal{O}_S$  is free of rank one,  $S$  is reduced, and the  $p$ -curvature  $\psi_\omega(D)$  of a local section of  $\text{Der}(S/T)$  is a nilpotent  $\mathcal{O}_S$ -linear endomorphism of  $\mathcal{O}_S$ , it means that  $\nabla_\omega$  has *p-curvature zero*. By Cartier's theorem (5.1), the  $\mathcal{O}_S$ -span of the horizontal (for  $\nabla_\omega$ ) sections of  $\mathcal{O}_S$  is all of  $\mathcal{O}_S$ , and hence there exists an open covering  $\{U_i\}$  of  $S$ , and sections  $f_i$  of  $\mathcal{O}_S^*$  over  $U_i$  such that  $f_i$  is *horizontal* for  $\nabla_\omega$ , i.e.

$$(13.1.4) \quad \omega = -df_i/f_i \quad \text{on } U_i.$$

Thus, if  $T = \text{Spec}(\mathbf{F}_p)$ , and  $S$  is an elliptic curve  $E$  over  $\mathbf{F}_p$ , and  $\omega$  is a (non-zero) differential of the first kind on  $E$ , then  $\nabla_\omega$  is *nilpotent* if and only if the “Hasse invariant” of  $E$  is 1, i.e. if and only if

$$(13.1.5) \quad \text{Card}(E(\mathbf{F}_p)) \equiv 0 \pmod{p}$$

where  $E(\mathbf{F}_p)$  denote the group of *rational points* of  $E$ . By the Riemann Hypothesis for elliptic curves,

$$(13.1.6) \quad \sqrt{p}-1 \leq \sqrt{\text{Card}(E(\mathbf{F}_p))} \leq \sqrt{p}+1.$$

Thus if  $p \geq 7$ , and if  $E(\mathbf{F}_p)$  has a non-trivial element of order *two* (so that  $\text{Card}(E(\mathbf{F}_p))$  is *even*), (13.1.5) and (13.1.6) are incompatible, and so  $\nabla_\omega$  is not nilpotent. Thus may we construct counter-examples to the converse of (13.0).

*Example (13.2).* — Let  $a, b \in \mathbf{Z}$ , with  $a^2 \neq 4b$ . Consider the projective and smooth elliptic curve  $\mathbf{E}$  over  $\text{Spec}\left(\mathbf{Z}\left[\frac{1}{30(a^2-4b)}\right]\right)$  given in homogeneous coordinates  $X, Y, Z$  by the equation

$$(13.2.1) \quad Y^2 = X(X^2 + aXZ + bZ^2).$$

Then the connection in  $\mathcal{O}_{\mathbf{E}}$  given by

$$f \mapsto df + f\omega$$

(where  $\omega = d(X/Z)/(Y/Z)$  is the differential of the first kind on  $\mathbf{E}$ ) gives a connection on the function field of  $\mathbf{E}_\mathbf{Q}$  for which every place is a regular singular point (indeed *not* a singular point at all) and has quasi-unipotent monodromy (namely none at all). However, the connection, far from being globally nilpotent, induces on the structure sheaf of the fibre over *every* closed point of the base a *non-nilpotent* connection.

*Remark (13.3).* — If we project this example to the  $x$ -axis, we get a rank-two counter-example over an open subset of  $\mathbf{A}_x^1$ , whose inverse image on  $\mathbf{Q}(x)$  has singular points precisely at 0,  $\infty$ , and the roots of  $x^2 + ax + b$ . (These are the points over which the  $x$ -coordinate is not étale; compare with (12.7.6-7).)

**(13.4)** In the “positive” direction, Messing (unpublished) has shown that, if  $a, b, c \in \mathbf{Q}$ , then the rank two module over

$$\mathbf{Z}\left[x, \frac{1}{n \cdot x(x-1)}\right] \quad \left(n \in \mathbf{Z} \text{ so chosen that } a, b, c \in \mathbf{Z}\left[\frac{1}{n}\right]\right)$$

corresponding to the *hypergeometric* differential equation with parameters  $\{a, b, c\}$ , is globally nilpotent. Of course here there are only *three* singular points,  $x=0, 1$ , or  $\infty$ .

#### 14. Application to the Local Monodromy Theorem.

**(14.0)** Let  $S/\mathbf{C}$  be a smooth connected curve, and let  $\pi: X \rightarrow S$  be a proper and smooth morphism. Clearly there exist:

(14.0.1) A subring  $R$  of  $\mathbf{C}$  which is finitely generated over  $\mathbf{Z}$ .

(14.0.2) A smooth connected curve  $S/R$ , which “gives back”  $S/\mathbf{C}$  after the base change  $R \hookrightarrow \mathbf{C}$ .

(14.0.3) A proper and smooth morphism  $\pi : X \rightarrow S$  which “gives back”  $\pi : X \rightarrow S$  after the base change  $S \rightarrow \mathbf{C}$ .

Combining (10.0) and (13.0), we find

**Theorem (14.1)** (the Local Monodromy Theorem). — Let  $S/\mathbf{C}$  be a smooth connected curve,  $K/\mathbf{C}$  its function field,  $\pi : X \rightarrow S$  a proper and smooth morphism,  $X_K/K$  the generic fibre of  $\pi$ .

For each integer  $i \geq 0$ , let  $h(i)$  (cf. (10.0)) be the number of pairs  $(p, q)$  of integers with  $p + q = i$  and  $H^{p,q}(X_K/K) = \dim_K H^q(X_K, \Omega_{X_K/K}^p) = \text{rank}_{\mathcal{O}_S} R^q \pi_*(\Omega_{X/S}^p)$  non-zero. Then the inverse image of  $H_{\text{DR}}^i(X/S)$ , with the Gauss-Manin connection, in  $\mathbf{MC}(K/\mathbf{C})$  (or what is the same, the  $K$ -space  $H_{\text{DR}}^i(X_K/K)$  with the Gauss-Manin connection) has regular singular points at every place of  $K/\mathbf{C}$  (indeed has no singularity at any place in  $S$ ) and quasi-unipotent local monodromy, whose exponent of nilpotence is  $\leq h(i)$ .

(14.2) Let  $K/\mathbf{C}$  be the function field of a smooth connected curve  $S/C$ , and let

$$(14.2.1) \quad \pi : U \rightarrow \text{Spec}(K)$$

be a smooth morphism (not necessarily proper).

By Hironaka [18], there exists a finite extension  $L/K$ , a proper and smooth morphism  $\rho : X \rightarrow \text{Spec}(L)$ , and a divisor,  $i : Y \hookrightarrow X$ , with normal crossings relative to  $\text{Spec}(L)$ , such that the morphism

$$(14.2.2) \quad \pi_L : U_L = U \times_K L \rightarrow \text{Spec}(L)$$

is the morphism

$$(14.2.3) \quad \rho|_{(X-Y)} : X - Y \rightarrow \text{Spec}(L).$$

Clearly there exist

(14.2.4) A subring  $R$  of  $\mathbf{C}$ , finitely generated over  $\mathbf{Z}$ .

(14.2.5) A smooth connected curve  $S/R$ , the generic point of whose fibre over the given point  $\text{Spec}(C) \rightarrow \text{Spec}(R)$  is  $L$ .

(14.2.5) A proper and smooth morphism  $\rho : X \rightarrow S$ , and a divisor  $i : Y \hookrightarrow X$  with normal crossings relative to  $S$ , whose fibres over the given point  $\text{Spec}(L) \rightarrow S$  are  $\rho : X \rightarrow L$  and  $i : Y \hookrightarrow X$  respectively.

Applying (10.0) ( $\log Y$ ), (13.0), (8.10), the fact that

$$H_{\text{DR}}^i((X-Y)/L) = H_{\text{DR}}^i(U \times_K L/L) = H_{\text{DR}}^i(U/K) \otimes_K L,$$

(11.12.1) and (12.7.1), we find

**Theorem (14.3)** (Deligne) (The “Open” Local Monodromy Theorem). — Assumptions and notations being as in (14.2.1-3), let  $\pi : U \rightarrow \text{Spec}(K)$  be a smooth morphism. For each

integer  $i \geq 0$ , let  $h_Y(i)$  (cf. (10.0)(log Y)) be the number of pairs  $(p, q)$  of integers with  $p+q=i$  and  $\dim_L H^q(X, \Omega_{X/L}^p(\log Y))$  non-zero. Then the object of  $MC(K/\mathbf{C})$  given by  $H_{\text{DR}}^i(U/K)$  with the Gauss-Manin connection, has regular singular points at every place of  $K/\mathbf{C}$ , and at each the local monodromy is quasi-unipotent, of exponent of nilpotence  $\leq h_Y(i)$ .

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