SPRINGER FIBERS: BASIC PROPERTIES AND APPLICATIONS TO CATEGORIFICATION. TALK 1

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1. Basics of Springer Fibers

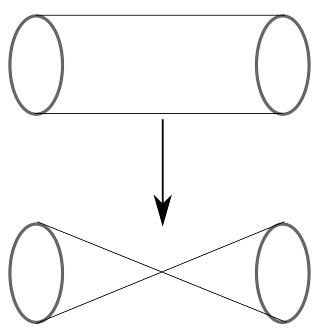
Let $\mathfrak{g}$ be a semi-simple complex Lie algebra. Inside $\mathfrak{g}$ we have the cone of nilpotent elements $\mathcal{N}$.

Example 1.1. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \;\middle|\; a^2 + bc = 0 \right\}$$

The equation $a^2 + bc = 0$ ensures the matrix is nilpotent. It also defines a type A_1 singularity which is the singularity formed on the quotient $\mathbb{C}^2 / \mathbb{Z}_2$.

The variety $\mathcal{N}$ is singular at the origin. It is resolved by $T^*\mathbb{P}^1$. We call this *the Springer resolution* in type A_1. If we draw just the real points, we have a double cone resolved by a cylinder.



We can identify

$$\begin{aligned} &T^*\mathbb{P}^1 \longleftrightarrow \left\{(L,A) \in \mathbb{P}^1 \times \mathcal{N} \mid A(\mathbb{C}^2) \subseteq L, A(L) \subseteq 0\right\} \\ &(L, f) \mapsto (L, A_f) \end{aligned}$$

as follows. A point L in $\mathbb{P}^1$ defines a one-dimensional subspace $L \subseteq \mathbb{C}^2$. Identifying $\mathbb{C}^2/L$ with $T_L\mathbb{P}^1$ we use $f : T_L\mathbb{P}^1 \to \mathbb{C}$ to define A_f via $A_f(y) = f(y)L$ and $A_f(L) = 0$. With respect to this identification the Springer resolution is

$$\mu : (L, A) \mapsto A.$$

In general, we identify

$$T^*(G/B) = \{(\mathfrak{b}, x) \mid \mathfrak{b} \subset \mathfrak{g} \text{ borel}, x \in \mathfrak{b} \text{ nilpotent}\}$$

and then define *the Springer resolution* as

$$\begin{aligned} \mu : T^*(G/B) &\longrightarrow \mathcal{N} \\ (\mathfrak{b}, x) &\longmapsto x. \end{aligned}$$

In type A as in the example for $\mathfrak{sl}_2$, we can describe $T^*(G/B)$ in terms of flags

$$T^*(SL_n/B) = \{(F_\bullet, x) \mid 0 = F_0 \subset F_1 \subset \dots F_n = \mathbb{C}^n, \dim(F_i/F_{i-1}) = 1, x(F_i) \subset F_{i-1}\}$$

Remark 1.2. When $\mathfrak{g} = \mathfrak{so}$ or $\mathfrak{sp}$ we can give similar descriptions of $T^*(G/B)$ in terms of flags. In this case $\mathbb{C}^m$ for $m = 2n$ or $2n+1$ has a bilinear form and we use flags $F_\bullet$ such that $F_i = F_{m-i}^\perp$.

Definition 1.3. The Springer fiber is

$$\mathcal{B}_x = \mu^{-1}(x).$$

Note that the Springer fiber only depends on which G-orbit x is in. Thus, in type A, for example, $\mathcal{B}_x$ is specified by a Jordan type.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_n$. Let $x = 0$. Then $\mathcal{B}_x = G/B$.

Example 1.5. Let x be a regular nilpotent element

$$\begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix},$$

then $\mathcal{B}_x = \{\mathrm{pt}\}$.

Example 1.6. Let $\mathfrak{g} = \mathfrak{sl}_3$. Choose $\{e_1, e_2, f\}$ a basis. Then let x be

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To compute $\mathcal{B}_x$ we need to find flags $F_1 \subset F_2 \subset \mathbb{C}^3$ stabilized by x. Two such families are

$$\begin{aligned} C_1^\circ &= \{\langle e_1 \rangle \subset \langle e_1, e_2 + bf \rangle \subset \mathbb{C}^3 \mid b \in \mathbb{C}\}, \\ C_2^\circ &= \{\langle e_1 + bf \rangle \subset \langle e_1 + bf, e_2 \rangle \subset \mathbb{C}^3 \mid b \in \mathbb{C}\}. \end{aligned}$$

The closures of these are

$$\begin{aligned} C_1 = \overline{C_1^\circ} &= \{\langle e_1 \rangle \subset \langle e_1, Ae_2 + Bf \rangle \subset \mathbb{C}^3 \mid [A:B] \in \mathbb{P}^1\}, \\ C_2 = \overline{C_2^\circ} &= \{\langle Ae_1 + Bf \rangle \subset \langle Ae_1 + Bf, e_2 \rangle \subset \mathbb{C}^3 \mid [A:B] \in \mathbb{P}^1\}. \end{aligned}$$

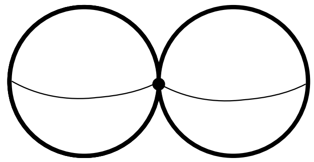
These are each isomorphic to $\mathbb{P}^1$. Their intersection

$$C_1 \cap C_2 = \{\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \mathbb{C}^3 \mid [A:B] \in \mathbb{P}^1\}$$

is just one point. We *claim* that the union is $C_1 \cup C_2 = \mathcal{B}_x$. If we assume the claim, then we have shown

$$\mathcal{B} = \mathbb{P}^1 \bigwedge \mathbb{P}^1.$$

as illustrated below.



This is an example of an A_2 singularity.

Remark 1.7. The action of the torus T on G/B does not descend to $\mathcal{B}_x$. Only those elements in

$$C_T(x) = \{t \in T \mid tx = xt\}$$

act on $\mathcal{B}_x$.

Example 1.8. In Example 1.6, we have

$$C_T(x) = \left\{ \left(\begin{array}{ccc} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{array} \right) \;\middle|\; t \in \mathbb{C}^* \right\}$$

In type A, if x has Jordan type λ, i.e. λ is a partition of n, then the $C_T(x)$-fixed points of $\mathcal{B}$ are in one-to-one correspondence with S_n/S_λ. This correspondence is given by considering the flags generated by adding coordinate vectors $\{e_1, \dots e_n\}$ one at a time which are preserved by x.

Definition 1.9. We define the *intersection graph* of $\mathcal{B}_x$ to have a vertex for each irreducible component C_i and an edge between C_i and C_j if their intersection is nonempty. The *advanced intersection graph* has the same vertex set, but is directed with $m_{ij} = \dim(H^*(C_i \cap C_J))$ arrows from C_i to C_j.

Example 1.10. In Example 1.6, the intersection graph is

$$\begin{tikzcd} C_1 \arrow[r, dash] & C_2 \end{tikzcd}$$

The advanced intersection graph is

$$\begin{tikzcd} C_1 \arrow[loop left] \arrow[r, bend left] & C_2 \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

By Spaltenstein, the irreducible components of $\mathcal{B}_x$ all have the same dimension.

We can define a *convolution algebra* structure on

$$A_{\text{conv}} = \bigoplus_{(C_i, C_j)} H^*(C_i \cap C_j)$$

using the usual push-pull convolution product on

$$\begin{tikzcd} & C_i \cap C_j \cap C_k \arrow[dl] \arrow[d] \arrow[dr] & \\ C_i \cap C_j & C_i \cap C_k & C_j \cap C_k. \end{tikzcd}$$

This behaves well if the intersections $C_i \cap C_j$ are smooth.

Special Cases:

(1) If $x = 0$ then $A_{\text{conv}} = H^*(G/B)$ leads to Soergel bimodules.
(2) In type A with x of Jordan type $(n-1, 1)$, the algebra A_{conv} is the Khovanov-Seidel algebra. (It arises from the Fukaya category of the Milnor fiber.)

(3) The principal block of $\mathcal{U}_q(\mathfrak{sl}_2)$ when $q \neq \pm 1$ is a root of unity (Anderson-Tubbenhauer, Arkhipov-Bezrukavnikov-Ginzburg).
(4) The principal block of $\mathcal{O}$ of finite dimensional representations of $\mathfrak{gl}(1|1)$.

Seven Ideas From an Old Theorem

Theorem 1.11 (S.-Webster)**.** *Let $x \in \mathfrak{sl}_n$ with 1 or 2 Jordan blocks. Then:*

(1) Let Z be a nilpotent slice and $i : \mathcal{B}_x \hookrightarrow Z$. Then

$$A_{\text{conv}} \cong \bigoplus_{(C_1,C_2)} \operatorname{Ext}^\bullet_{\operatorname{Coh}(Z)}\left(i_*\Omega^{1/2}_{C_1}, i_*\Omega^{1/2}_{C_2}\right).$$

(2) Let $\mathfrak{p}_x$ be the parabolic corresponding to the Jordan type of x. Let $\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n)$ be the principal block of parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$. Then there exists a fully faithful functor

$$\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n) \longrightarrow A_{\text{conv}}.$$

(3) Let p be a fixed point of the $C_T(x)$ action on $\mathcal{B}_x$. Define C_p to be the closure of the attracting cell in $\mathcal{B}_x$ to p. If we fix a cocharacter $\mathbb{C}^* \hookrightarrow T$, then we can write

$$C_p = \{x \mid \lim_{t\to 0} tx = p\}$$

Summing over pairs of such fixed points define

$$A'_{\text{conv}} = \bigoplus_{(p,p')} H^*(C_p \cap C'_p).$$

Then

$$\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n) \cong A'_{\text{conv}}.$$

Seven Ideas:

Idea 1. Think of A_{conv} as an algebraic version of the Fukaya category.
Idea 2. The $i_*\Omega^{1/2}_{C_i}$ are simple objects in an “exotic” t-structure.
Idea 3. Think of

$$\mathrm{KZ} : \mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n) \longrightarrow A_{\text{conv}}\text{—mod}.$$

Idea 4. Use this to give a geometric construction of category $\mathcal{O}$.
Idea 5. Consider $\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n)$ as a wrapped Fukaya category.
Idea 6. The functor

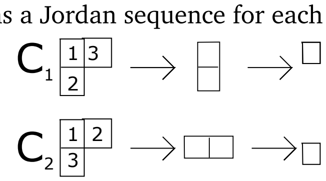
$$A_{\text{conv}}\text{—mod} \longrightarrow A'_{\text{conv}}\text{—mod}$$

is *Schurification* (in the sense of Roquier).
Idea 7. There is a homological grading on

$$H^*(C_p \cap C_{p'}).$$

Classification of Irreducible Components of $\mathcal{B}_x$ in Types ABCD.

In Example 1.6, there was a Jordan sequence for each component as such



given by starting with the partition corresponding to the nilpotent orbit of x and removing a box at each step corresponding to how x interacts with the F_i.

Theorem 1.12 (Vargos-Spaltenstein-Springer,Van leeuwen)**.** *Let λ be the partition corresponding to a nilpotent orbit. Then there is a one-to-one correspondence (which in type A is)*

$$\{\textit{components of } \mathcal{B}_x\} \longleftrightarrow \{\textit{ standard tableau of shape } \lambda\}$$
$$\{F_\bullet \mid F_\bullet \textit{ has Jordan sequence } T\} \leftarrow\!\shortmid T.$$

In types BCD, we would need to use signed domino tableau, but a similar correspondence holds.

We can describe the components very well in two special cases.

[Ehrig-S.] For λ in type ABCD with 2 rows, we have an explicit description of all components. Note that pairwise intersections of components are always iterated $\mathbb{P}^1$-bundles. The diagramatics in this case leads to Khovanov algebras.

[Sarbin] For λ a hook partition, we get iterated partial flag variety bundles.

[Fresse-Melnikov] The partition λ is 2-row or hook in type A if and only if all the components are smooth.

Final Observations.

By Springer, there is an action of W, the Weyl group, on $H^*(\mathcal{B}_x)$.

Lusztig showed in type A,

$$H^*(\mathcal{B}_\lambda) \cong \operatorname{Ind}_{S_\lambda}^{S_n}(\text{triv}).$$

By Schur-Weyl duality, if V is the natural $\mathfrak{sl}_2$-module, $V^{\otimes d}$ carries actions of $\mathfrak{sl}_2$ and S_d. The representations which occur are labeled by 2-row partitions. The weight spaces are permutation modules. When we quanitze, we get the Jones polynomial from knot theory.

Similarly if V is the natural representation of $\mathfrak{gl}(1|1)$, then $V^{\otimes d}$ carries actions of $\mathfrak{gl}(1|1)$ and S_d. The representations which occur are labeled by hook partitions. Quantization gives the Alexander polynomial from knot theory.