EULER SUBGROUPS

I. I. PJATECKIJ-ŠAPIRO

Let $\mathscr{G}$ be a reductive algebraic group over a global field $\mathfrak{k}$ and G be an algebraic subgroup of $\mathscr{G}$ which is also defined over $\mathfrak{k}$. Suppose that there exists a nontrivial morphism $\mu: G \to \mathfrak{k}^*$.

The integral

$$\int_{G_\mathfrak{k} \setminus G_A} f(g)\,|\mu(g)|^s\,dg \tag{1}$$

where $f(g)$ is an automorphic form on $\mathscr{G}_A$, may be used for defining the zeta function only if it may be transformed into a product taken over all places of $\mathfrak{k}$. For instance, we are in such situation in the case $\mathscr{G} = GL(2, \mathfrak{k})$, $G = \left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}, \alpha \in \mathfrak{k}^*\right\}$.

More precisely, if $f(g)$ is a parabolic form on $GL(2, \mathfrak{k}) \setminus GL(2, A)$ we have an identity

$$\int_{\mathfrak{k}^* \setminus I_\mathfrak{k}} f\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} |\alpha|^s\,d\alpha = \int_{I_k} f_\varkappa \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} |\alpha|^s\,d\alpha \tag{2}$$

where

$$f_\chi(g) = \int_{Z_\mathfrak{k} \setminus Z_A} f(zg)\,\overline{\chi(z)}\,dz \tag{3}$$

$Z = \left\{\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right\}$ and χ is a nontrivial character of $Z_\mathfrak{k} \setminus Z_A$.

The aim of this paper is to give examples of similar identities for groups different from $GL(2)$.

For any place $\mathfrak{p}$ of $\mathfrak{k}$ we denote by $W_\chi^\mathfrak{p}$ the space of all functions $f: \mathscr{G}_\mathfrak{p} \to \mathbf{C}$ such that $f(zg) = \chi(z) f(g)$, where Z is an algebraic subgroup of $\mathscr{G}$ defined over $\mathfrak{k}$, χ is a character $Z_\mathfrak{p} \to \mathbf{C}$. The group $\mathscr{G}_\mathfrak{p}$ acts on $W_\chi^\mathfrak{p}$ by right translations.

We shall introduce the following property of $W_\chi^\mathfrak{p} \cdot (U^\mathfrak{p})$. For any irreducible admissible representation π of $\mathscr{G}_\mathfrak{p}$ $\dim \mathrm{Hom}_{\mathscr{G}_\mathfrak{p}}(\pi, W_\chi^\mathfrak{p}) \leq 1$.

The fact that the right-hand side of (2) can be represented as an Euler product is based on the property $(U^\mathfrak{p})$ [1].

The groups $\mathscr{G} = GL(2, \mathfrak{k})$, $Z = \left\{\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right\}$ give us an example where $(U^\mathfrak{p})$ is satisfied for all $\mathfrak{p}$ [1].

The results of Gelfand and Kajdan show that $(U^\mathfrak{p})$ holds if $\mathscr{G}$ is a Chevalley group, Z is its maximal unipotent subgroup, χ is a nondegenerate character of Z and $\mathfrak{p}$ is a finite place of $\mathfrak{k}$ [2].

The well-known method of description of the representations of principal series of Chevalley groups shows that if $(U^{\mathfrak{p}})$ holds for all $\mathfrak{p}$ then

$$\dim Z \geqq \dim_{\mathbf{C}} (\text{maximal unipotent subgroup of } \mathscr{G}_{\mathbf{C}}). \tag{4}$$

An algebraic subgroup G of $\mathscr{G}$ defined over $\mathfrak{k}$ is called an *Euler subgroup* if

1) there exists a nontrivial morphism $\mu: G \to \mathfrak{k}^*$

2)

$$\int_{G_{\mathfrak{k}} \diagdown G_A} f(g)|\mu(g)|^s \, dg = \int_{B_A} f_\chi(b)|\mu(b)|^s \, db \tag{5}$$

where B is a homogeneous space corresponding to a subgroup of G, Z is an algebraic subgroup of $\mathscr{G}$ defined over $\mathfrak{k}$ and satisfying (4), χ is a character of Z_A which is trivial on $Z_{\mathfrak{k}}$ and $f_\chi(g)$ is defined by a formula similar to (3).

In Section 1 I show that for $\mathscr{G} = GL(n)$ the subgroup $G = GL(n-1)$ is an Euler subgroup. In section 2 I give examples for Euler subgroups of $\mathscr{G}$ different from $GL(n)$. I give a sufficient condition for a subgroup G to be an Euler subgroup. It generalizes some results of Rankin [4], Selberg [5], [6], Ogg [7], and Andrianov [9], and gives a clue for consideration of further examples. I want to mention that the language used by these authors was quite different from that of this paper.

Let $\mathfrak{k}$ be a global field of positive characteristic. Using the results of [2] I prove in Section 1 the following.

Multiplicity-one theorem. The multiplicity of an irreducible representation of $GL(n, A)$ in $L_0^2(GL(n, \mathfrak{k}) \diagdown GL(n, A))$ is not greater than one.

It seems very plausible that a statement similar to the Gelfand—Kajdan theorem is true in the Archimedean case. If it were proved we should have a multiplicity-one theorem for $GL(n)$ over a number field.*

I do not know whether the multiplicity-one theorem is valid for groups different from $GL(n)$.

Section 3 is devoted to computation of local factors in this case: $\mathscr{G} = \mathrm{Sp}(4, \mathfrak{k})$. The main idea of this computation is contained in Andrianov's paper [9].

Now, from the point of view of the theory of zeta-functions, only in case $GL(2)$ is the theory more or less complete. However even in this case many important problems are unsolved. I want to mention one of them.

Let $\mathfrak{k}$ be a global field and K a Galois extension of $\mathfrak{k}$. Let $\pi = \otimes \pi_{\mathfrak{p}}$ be an irreducible representation of $GL(2, A_{\mathfrak{k}})$ belonging to $L_0^2(GL(2, \mathfrak{k}) \diagdown GL(2, A_{\mathfrak{k}}))$.

Does there exist an irreducible representation

$$\tilde{\pi} = \otimes \tilde{\pi}_{\mathfrak{p}} \in L_0^2(GL(2, K) \diagdown GL(2, A_K)) \quad \text{of} \quad GL(2, A_K)$$

* Now it has been proved by Shalika, *Bull. Amer. Math. Soc.* 79:2 (1973).

such that for each place $\mathfrak{p}$ of $\mathfrak{k}$ and each $\tilde{\mathfrak{p}}$ of K lying over $\mathfrak{p}$ the representations π and $\tilde{\pi}_{\mathfrak{p}}$ naturally correspond to each other, i.e. the representation of the Weil—Šafarevič group corresponding to $\tilde{\pi}_{\tilde{\mathfrak{p}}}$ is the restriction of the representation of the Weil—Šafarevič group corresponding to $\pi_{\mathfrak{p}}$? A positive answer to this question implies that zeta functions of arithmetic curves satisfy the Hasse—Weil conjecture. Here by arithmetic curve I mean a curve obtained as a factor of the upper half plane by an arithmetic group. Hitherto the Hasse—Weil conjecture was proved by Shimura only for some well-chosen models of arithmetic curves. Of course it is not the only example for relations between zeta functions of infinite-dimensional representations of adele groups.

The remarkable results of A. Weil, G. Shimura and P. Deligne give the hope that zeta functions of algebraic varieties may be expressed in terms of zeta functions of infinite-dimensional representations of adele groups of reductive groups.

Now I am going to say a few words about two problems. The first is the problem of the analytic continuation of integral (1) to the whole complex plane. In section 2 I give two methods of analytic continuation of integrals of this kind. The first method I call the Hecke method and the second one I call the Rankin—Selberg method. Moreover there exist examples of Euler subgroups (for instance Example 3 of § 2) where both of these methods fail. I think that integral (1) can always be extended to the whole complex plane as a meromorphic function. I want to mention that there are examples where the analytic continuation of the integral (1) is a meromorphic function with an infinite set of poles. More precisely this is the case in any application of the Rankin—Selberg method.

The second problem refers to finite-dimensional representations. Let $\mathscr{G}$ be a Chevalley group over a finite field. We call an irreducible representation of $\mathscr{G}$ in a finite-dimensional space V over $\mathbf{C}$ *hyperanalytic* if there exist two proper subgroups G and Z and an irreducible representation of Z in V_0 such that

$$V = \sum_{g \in G} g V_0. \tag{6}$$

It is known that every analytic representation of $GL(n)$ is hyperanalytic (with $G = GL(n-1)$ and Z maximal unipotent subgroup of $GL(n)$) [12]. This is closely related to the fact that $GL(n-1)$ is Euler subgroup of $GL(n)$.

The Andrianov theory for Sp (4) corresponds to the fact that Sp (4) has a hyperanalytic representation. This is the analytic representation of Sp (4) of minimal dimension. Gelfand proved (not published) that this representation is hyperanalytic in our sense.

Apparently there is a close connection between the examples of Euler subgroups and the examples of decomposition (6). I expect that to each example of a hyperanalytic representation of $\mathscr{G}$ over a finite field there corresponds an example of an Euler subgroup of $\mathscr{G}$ over a global field.

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§ 1. Representations of $GL(n)$

We denote by $\mathfrak{k}$ any global field (i.e. a number field or a field of algebraic functions of one variable over a finite field). If $\mathfrak{p}$ is a place of $\mathfrak{k}$ we denote by $\mathfrak{k}_\mathfrak{p}$ the corresponding completion of $\mathfrak{k}$.

Denote by $\mathscr{G}$ a reductive group over $\mathfrak{k}$.

The group $\mathscr{G}_A$ operates by right translations on the Hilbert space $L^2(\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A)$ of all measurable functions $\varphi(g)$ such that

$$\varphi(\gamma g) = \varphi(g) \quad \text{for every} \quad \gamma\in\mathscr{G}_\mathfrak{k} \tag{1}$$

and

$$\int\limits_{\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A} |\varphi(g)|^2\, dg < \infty. \tag{2}$$

Denote by $L_0^2=L_0^2(\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A)$ the closed invariant subspace of $L^2(\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A)$ consisting of all functions φ such that

$$\int\limits_{R_\mathfrak{k}\diagdown R_A} \varphi(rg)\, dr = 0 \quad (\text{almost all } g\in\mathscr{G}_A) \tag{3}$$

for any cuspidal subgroup R of G (a cuspidal subgroup is the unipotent radical of a parabolic subgroup).

Let C be the centre of $\mathscr{G}$ and let ψ be a character of $\mathscr{G}_A$ which is trivial on $\mathscr{G}_\mathfrak{k}$. Denote by $L^2_{0,\psi}=L^2_{0,\psi}(\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A)$ the Hilbert space which consists of all functions on $\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A$ which satisfy (3) and the following conditions

$$\varphi(cg) = \psi(c)\,\varphi(g) \quad \text{for every} \quad c\in C_A \tag{4}$$

$$\int\limits_{\mathscr{G}_\mathfrak{k} C_A\diagdown\mathscr{G}_A} |\varphi(g)|^2\, dg < \infty. \tag{5}$$

We can (by means of Fourier transforms) decompose L_0^2 into a continuous sum of unitary irreducible representations of $L^2_{0,\psi}$. It is well known that $L^2_{0,\psi}$ is a discrete sum of unitary representations of $\mathscr{G}_A$ each occurring with finite multiplicity [3]. We call a function $\varphi(g)$ on $\mathscr{G}_\mathfrak{k}\diagdown\mathscr{G}_A$ elementary if *1)* the space generated by the right translates of $\varphi(g)$ by the elements of $\mathscr{G}_A$, is isomorphic to an admissible irreducible representation $H=\otimes H_\mathfrak{p}$ of $\mathscr{G}_A$ *2)* under this isomorphism φ corresponds to a decomposable vector $\xi=\otimes\xi_\mathfrak{p}\in H$ *3)* there exists a maximal compact subgroup $U\subset\mathscr{G}_A$ such that the space generated by the right translates of $\varphi(g)$ by elements of U is finite-dimensional.

We shall denote by $A(\mathscr{G})$ the set of all elementary functions. Of course, $A(\mathscr{G})$ is not a linear space over $\mathbf{C}$. Denote by $S(\mathscr{G})$ the linear space consisting of all C^∞ slowly increasing functions on $\mathscr{G}_{\mathfrak{k}}\diagdown\mathscr{G}_A$. Of course, the linear span $\tilde{A}(\mathscr{G})$ of $A(\mathscr{G})$ belongs to $S(\mathscr{G})$.

From now on we shall consider in this section only the case $\mathscr{G}=GL(n)$, $G=GL(n-1)$. Denote by Z a maximal unipotent subgroup of $\mathscr{G}$. Let θ be a character of Z_A which is trivial on $Z_{\mathfrak{k}}$. We shall suppose that θ is nondegenerate, i.e. θ is nontrivial on every subgroup corresponding to a simple root. For instance, if Z is the upper triangle subgroup we can put

$$\theta(z) = \nu\left(\sum_{k=1}^{n-1} z_{kk+1}\right), \tag{6}$$

where ν is a character of $\mathfrak{k}\diagdown A$.

For any $f(g)\in S(\mathscr{G})$ we can define

$$f_\theta(g) = \int_{Z_{\mathfrak{k}}\diagdown Z_A} \overline{\theta(z)} f(zg)\, dz. \tag{7}$$

For $GL(2)$, Jacquet and Langlands have found a very simple explicit formula which shows that any $f(g)\in\tilde{A}(\mathscr{G})$ can be uniquely defined by means of $f_\theta(g)$. It says

$$f(g) = \sum f_\theta(\delta g), \tag{8}$$

where δ runs over the set of all matrices of the form

$$\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix}, \quad \delta\in\mathfrak{k}^*. \tag{9}$$

They also proved two important corollaries to (8).

Corollary 1. *The multiplicity-one theorem. The multiplicity of an irreducible representation H of $GL(2, A)$ in L_0^2 is not greater than one.*

Remark. An important particular case of this theorem was considered by Hecke. He proved that $SL(2, \mathbf{Z})$-modular forms are defined uniquely up to a scalar factor by the eigenvalues of all Hecke operators. It is equivalent to the Jacquet—Langlands theorem under the additional assumption that all non-Archimedean components $H_{\mathfrak{p}}$ of H are of class one.

Corollary 2. If $f(g)\in A(\mathscr{G})$, then the integral

$$\mathscr{I}(s) = \int_{\mathfrak{k}^*\diagdown I_{\mathfrak{k}}} f\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} |a|^s\, da \tag{10}$$

can be decomposed into an Euler product.

Remark. It is very simple to prove that *1)* the integral (10) converges absolutely in the half plane $\operatorname{Re} s > c$ for sufficiently large c, *2)* $\mathscr{I}(s)$ can be extended to the whole complex plane, as an entire function, *3)* $\mathscr{I}(s)$ has a "nice" functional equation.

The purpose of this section is to generalize formula (8) and its corollaries to $GL(n)$, for arbitrary n.

Theorem I.* *For* $f(g) \in \tilde{A}(\mathscr{G})$ *we have*

$$f(g) = \sum_{\delta \in (Z \cap G)_{\mathfrak{f}} \backslash G_{\mathfrak{f}}} f_\theta\left(\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} g\right) \tag{11}$$

where the series converges absolutely and uniformly on any compact subset of $\mathscr{G}_A$.

Proof. For the sake of simplicity we suppose that Z is the group of upper triangular matrices and $\theta = \nu\left(\sum_{k=1}^{n-1} Z_{k,k+1}\right)$ where ν is a character of $\mathfrak{k}\backslash A$.

Denote by P^n the subgroup of $GL(n)$, consisting of all matrices of the form

$$\begin{pmatrix} * & * \dots & * \\ \vdots & & \\ * & * \dots & * \\ 0 & 0 \dots 0 & 1 \end{pmatrix}.$$

By a standard cuspidal subgroup of P^n we mean a subgroup of P^n consisting of all matrices of the form

$$\begin{pmatrix} E_{k_1} & \cdots & * \\ & \ddots & \vdots \\ 0 & & E_{k_r} \end{pmatrix}, \quad k_1 + \dots + k_r = n.$$

Denote by S_n the set of all C^∞-functions (i.e. locally constant with respect to non-Archimedean variables and infinitely differentiable with respect to Archimedean variables) on P^n_A such that

$$\varphi(\gamma p) = \varphi(p) \quad \text{for every} \quad \gamma \in P^n_{\mathfrak{k}}, \tag{12}$$

$\int\limits_{R_{\mathfrak{f}} \backslash R_A} \varphi(rp)\, dr = 0$ for every standard cuspidal subgroup R.

It is easy to verify that $f(g) \in \tilde{A}(\mathscr{G})$ implies that $\varphi(p) = f(pg_0)$ belongs to S_n for every g_0. This shows that it is enough to prove the following

Lemma 1. For any $\varphi(p) \in S_n$ we have

$$\varphi(p) = \sum_{\delta \in Z_{\mathfrak{f}} \backslash P^n_{\mathfrak{f}}} \varphi_\theta(\delta p), \quad \varphi_\theta(p) = \int\limits_{Z_{\mathfrak{f}} \backslash Z_A} \varphi(zp) \overline{\theta(z)}\, dz \tag{13}$$

where the series converges absolutely and uniformly on any compact subset of P^n_A.

* This formula was independently obtained by Shalika.

Proof. First we introduce the subgroup $Y \subset P^n$ which consists of all matrices of the form

$$\begin{pmatrix} E & y \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

We define

$$\varphi_1(p_1) = \int_{Y_{\mathfrak{k}} \setminus Y_A} \varphi(yp)\, \nu(y_{n-1})\, dy. \tag{14}$$

It is not difficult to verify that

$$\varphi_1\left(\begin{pmatrix} \gamma & \beta \\ 0 & 1 \end{pmatrix} p\right) = \varphi_1(p) \quad \text{for any} \quad \gamma \in P_{\mathfrak{k}}^{n-1}, \quad \beta \subset \mathfrak{k}^{n-1}. \tag{15}$$

Denote by P^n the stationary subgroup of $\varphi_1(p)$. If χ is a nontrivial character of Y_A which is trivial on $Y_{\mathfrak{k}}$ we can find $\delta \in G_{\mathfrak{k}} = GL(n-1, \mathfrak{k})$ such that

$$\varphi_1\left(\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} p\right) = \int_{Y_{\mathfrak{k}} \setminus Y_A} \varphi(yp)\, \chi(y)\, dy. \tag{16}$$

Moreover Y is a standard cuspidal subgroup and hence

$$\int_{Y_{\mathfrak{k}} \setminus Y_A} \varphi(yp)\, dy = 0. \tag{17}$$

By making use of the ordinary Fourier expansion, we obtain from (15), (16) and (17) that

$$\varphi(p) = \sum_{\delta \in \tilde{P}_{\mathfrak{k}} \setminus P_{\mathfrak{k}}^n} \varphi_1(\delta p) \tag{18}$$

implies that the series (18) converges absolutely and uniformly on any compact subset of P_A^n.

Consider the function ψ on P_A^{n-1}, defined by

$$\psi(p) = \varphi_1\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} p_0\right), \quad \varphi(p) \in S_n \tag{19}$$

where p_0 is a fixed element of P_A^n.

It is clear that $\psi(p)$ is a C^∞-function on P_A^{n-1} and $\psi(\gamma p) = \psi(p)$ for any $\gamma \in P_{\mathfrak{k}}^{n-1}$. Let R be a standard cuspidal subgroup of P^{n-1}. We introduce the subgroup $R(k)$, $1 \leqq k \leqq n-2$ of P^n such that

$$R(k) = \left\{ \begin{pmatrix} r & y^k \\ 0 & 1 \end{pmatrix} \middle| \, y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

It is obvious that there exists K_0, $1 \leqq K_0 \leqq n-2$ such that $R^0 = R(K_0)$ is a standard cuspidal subgroup of P^n. We put $\tilde{R} = RY$. Since $R \subset P^{n-1} \subset \tilde{P}^n$, we have $\chi(ryr^{-1}) =$

$=\chi(y)$ for $r \in R_A$. Thus there exists an extension $\tilde{\chi}(r)$ of $\chi(y)=\nu(y_{n-1})$ to $\tilde{R}_A$ such that $\tilde{\chi}(r)=1$ for $r \in R_A$.

It is clear that R^0 is a normal subgroup of $\tilde{R}$ and $\tilde{\chi}(r)=1$ for $r \in R_A^0$. This shows that

$$\int\limits_{R_{\mathfrak{k}} \setminus R_A} \psi(rp)\,dr = \int\limits_{\tilde{R}_{\mathfrak{k}} \setminus R_A} \varphi(rp)\chi(r)\,dr = \int\limits_{\tilde{R}_{\mathfrak{k}}\tilde{R}_A^0 \setminus \tilde{R}_A} \chi(r_1)\,dr_1 \int\limits_{\tilde{R}_{\mathfrak{k}}^0 \setminus R_A^0} \varphi(r_0 r_1 p)\,dr = 0 \quad (20)$$

so we have proved that $\psi(p) \subset S_{n-1}$.

Now we shall prove our lemma by induction on n. For $n=2$ formula (18) is equivalent to the lemma. Suppose $n>2$ and suppose our lemma is true for $n-1$. Then (18) and the expansion for $\psi(p)$ (which follows from the inductive assumption) yield the statement of the lemma for n.

Now we can prove the theorem. Let $f(g) \in \tilde{A}(\mathscr{G})$. For $p \in P_A^n$ and g_0 fixed, we put $\varphi(p)=f(pg_0)$.

It is obvious that $\varphi(p) \in S_n$ and hence the expansion (11) holds for $f(g)$.

From this the statement of the theorem follows for $f(g)$.

Now I have to give some definitions. Let θ be a character of $Z_\mathfrak{p}$. Consider the space $S_\theta(\mathscr{G}_\mathfrak{p})$ of all measurable locally integrable functions on $\mathscr{G}_\mathfrak{p}$ for which

$$f(zg) = \theta(z) f(g) \quad \text{for any} \quad z \subset Z_\mathfrak{p}, \quad g \in \mathscr{G}_\mathfrak{p}. \quad (21)$$

Let H be an irreducible admissible representation of $\mathscr{G}_\mathfrak{p}$. Denote by Hom $(H, S_\theta(\mathscr{G}_\mathfrak{p}))$ the set of all mappings $\delta: H \to S_\theta(\mathscr{G}_\mathfrak{p})$ which satisfy the following conditions:

1) If $h_n \to h_0$, $f_n = \delta(hn)$, then

$$\int\limits_K f_n(g)\,dg \to \int\limits_K f_0(g)\,dg \quad (22)$$

for every compact $K \subset \mathscr{G}_\mathfrak{p}$;

2) If $\delta(h)=f(g)$, then for any $g_0 \in \mathscr{G}_\mathfrak{p}$ we have $\delta(T_{g_0}h)=f(gg_0)$ where T_{g_0} is the operator on H corresponding to g_0.

It is obvious that Hom $(H, S_\theta(\mathscr{G}_\mathfrak{p}))$ is a linear space over **C**.

Gelfand and Kajdan obtained that $\dim_\mathbf{C}$ Hom $(H, S_\theta(\mathscr{G}_\mathfrak{p})) \leqq 1$ for any non-Archimedean local field $\mathfrak{k}_\mathfrak{p}$ [2]. A similar statement is likely to hold in the Archimedean case as well.* This is the reason why we have to formulate our theorem in the following weak form.

Theorem 2.** *For every irreducible unitary representation H of $\mathscr{G}_A$ we have*

$$\dim \operatorname{Hom}(H, L^2_{0,\psi}) \leqq \prod_\mathfrak{p} \dim (H_\mathfrak{p}, S_\theta(\mathscr{G}_\mathfrak{p})), \quad (23)$$

where $\mathfrak{p}$ runs over all Archimedean places of $\mathfrak{k}$.

* Now it has been proved by Shalika.

** This theorem has been independently proved by Shalika.

Proof. Let θ be a character of Z_A which is equal to 1 on $Z_{\mathfrak{k}}$. We introduce the space $S_\theta(\mathscr{G}_A)$ consisting of all measurable locally integrable functions on $\mathscr{G}_A$ such that

$$f(zg) = \theta(z)f(g) \quad \text{for any} \quad z \in Z_A, \quad g \in \mathscr{G}_A. \tag{24}$$

As in the local case we can define $\operatorname{Hom}(H, S_\theta(\mathscr{G}_A))$. Consider the mapping: $\mu: L^2_{0,\psi} \to S_\theta(\mathscr{G}_A)$

$$\mu: f(g) \to f_\theta(g) = \int_{Z_{\mathfrak{k}} \diagdown Z_A} f(zg)\theta(z)\,dz. \tag{25}$$

Theorem 1 shows that the kernel of μ is zero. Hence for any unitary irreducible representation H, we have

$$\dim_{\mathbf{C}} \operatorname{Hom}(H, L^2_{0,\psi}) \leqq \dim_{\mathbf{C}} \operatorname{Hom}(H, S_\theta(\mathscr{G}_A)). \tag{26}$$

Following the pattern given in [1] we obtain from (26) the statement of our theorem.

I. M. Gelfand and D. A. Kajdan introduced a very important function which depends on two representations: one of $GL(n, \mathfrak{k}_\mathfrak{p})$ and one of $GL(n-1, \mathfrak{k}_\mathfrak{p})$. (This function also depends on a character of $\mathfrak{k}^+$ which will be fixed and will not appear in our notation.)

I. M. Gelfand and D. A. Kajdan conjectured [2] that there exists a zeta function $\zeta(\varrho, \sigma)$ which depends on two irreducible representations (ϱ, σ): one of $GL(n, A)$ and one of $GL(n-1, A)$ where $\varrho \in L^2_0(GL(n, \mathfrak{k}) \diagdown GL(n, A))$, $\sigma \in S(GL(n-1))$. (I should like to point out that this conjecture has stimulated me very much.) More precisely, they have suggested:

1) the local Γ-function can be represented in the form

$$\Gamma(\varrho_\mathfrak{p}, \sigma_\mathfrak{p}) = \frac{\zeta(\varrho_\mathfrak{p}, \sigma_\mathfrak{p})}{\zeta(\hat{\varrho}_\mathfrak{p}, \hat{\sigma}_\mathfrak{p})} \varepsilon(\varrho_\mathfrak{p}, \sigma_\mathfrak{p})$$

where ^ means the contragredient representation.

2) $\zeta(\varrho, \sigma) = \prod \zeta(\varrho_\mathfrak{p}, \sigma_\mathfrak{p})$ is a function holomorphic on the set of pairs of representations (ϱ, σ) which has been described above and satisfies the functional equation

$$\zeta(\hat{\varrho}_\mathfrak{p}, \hat{\sigma}_\mathfrak{p}) = \zeta(\varrho_\mathfrak{p}, \sigma_\mathfrak{p})\varepsilon(\varrho_\mathfrak{p}, \sigma_\mathfrak{p}). \tag{27}$$

They have also suggested that (27) implies

$$\varrho \in L^2_0(GL(n, \mathfrak{k}) \diagdown GL(n, A)).$$

The following does not contain the definition of the desired zeta function. We are going to consider some integral which might be useful for it. As for this integral, we prove that it has properties analogous to those of the similar integral for $GL(2)$.

Let $f \in \tilde{A}(\mathscr{G})$, $\varphi \in S(G)$. We put

$$\mathscr{I}(f, \varphi, s) = \int_{G_{\mathfrak{k}} \setminus G_A} f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h) |h|^s \, dh \tag{28}$$

where $|h| = |\det h|$.

Theorem 3. *For any* $f \in A(\mathscr{G})$, $\varphi \in S(\mathscr{G})$

1. $\mathscr{I}(f, \varphi, s)$ *converges absolutely in the half plane* $\operatorname{Re} s > c_1$ *for some* c_1;
2. $\mathscr{I}(f, \varphi, s)$ *is an entire function bounded in vertical strips;*
3. *if the right translations of* φ *by* G_A *generate an irreducible representation then* $\mathscr{I}(f, \varphi, s)$ *decomposes into an Euler product.*

Proof. First we have to prove the following

Lemma. If $f(g) \in A(\mathscr{G})$ then $\check{f}(g) = f(g'^{-1}) \in A(\mathscr{G})$.

Proof. First for any $\gamma \in \mathscr{G}_{\mathfrak{k}}$, $g \in \mathscr{G}_A$ we have

$$\check{f}(\gamma g) = f(\gamma'^{-1} g'^{-1}) = f(g'^{-1}) = \check{f}(g)$$

By making use of the equality $dg'^{-1} = dg$, we obtain

$$\int_{C_A \mathscr{G}_{\mathfrak{k}} \setminus \mathscr{G}_A} |\check{f}(g)|^2 \, dg = \int_{C_A \mathscr{G}_{\mathfrak{k}} \setminus \mathscr{G}_A} |f(g)|^2 \, dg. \tag{29}$$

If Z is a cuspidal subgroup the Z'^{-1} is also cuspidal. Consequently

$$\int_{Z_{\mathfrak{k}} \setminus Z_A} \check{f}(zg) \, dz = \int_{Z_{\mathfrak{k}}'^{-1} \setminus Z_A'^{-1}} f(zg'^{-1}) \, dz = 0. \tag{30}$$

Now the mapping $\delta : f \to \check{f}$ transforms an admissible irreducible representation into one with the same properties. That completes our proof.

Now we can prove our theorem.

It is well known that if $f(g) \in \tilde{A}(\mathscr{G})$ then f rapidly decreases at ∞. That shows that $\mathscr{I}(f, \varphi, s)$ converges absolutely in the half plane $\operatorname{Re} s > c_0$ for some c_0.

In order to prove that $\mathscr{I}(f, \varphi, s)$ is an entire function we have to use the usual trick.

We write

$$\mathscr{I} = \int_{|g| \geq 1} + \int_{|g| < 1}.$$

It is obvious that the first integral is an entire function. Consider the second integral

$$\int_{|g|<1} f\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \varphi(g) |g|^s \, dg = \int_{|g|>1} \check{f}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \tilde{\varphi}(g) |g|^{-s} \, dg$$

Using the lemma we obtain that $\check{f} \in A(\mathscr{G})$ and hence this integral is also an entire function. To prove the existence of the Euler product for $\mathscr{I}(f, \varphi, s)$ we have to prove the following formula:

$$\mathscr{I}(f, \varphi, s) = \int_{Z'_A \backslash G_A} f_\theta \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi_{\theta^{-1}}(h) |h|^s \, dh \tag{31}$$

where Z' is the upper triangle subgroup of G. Using (11), we have

$$\mathscr{I}(f, \varphi, s) = \int_{Z'_{\mathfrak{k}} \backslash G_A} f_\theta \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h) |h|^s \, dh = \int_{Z_A \backslash G_A} f_\theta \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi_{\theta^{-1}}(h) |h|^s \, dh$$

where

$$\varphi_{\theta^{-1}}(g) = \int_{Z'_{\mathfrak{k}} \backslash Z'_A} \theta(z) \varphi(zg) \, dz.$$

By making use of the Gelfand—Kajdan theorem we obtain for $\varphi \in S(G), f \in A(\mathscr{G})$ satisfying assumptions 3) the decompositions

$$f_\theta(g) = f^\infty(g^\infty) \prod_{\mathfrak{p}} f_{\theta_\mathfrak{p}}(g_\mathfrak{p})$$

$$\varphi_{\theta^{-1}}(g) = \varphi^\infty(g^\infty) \prod \varphi^{\mathfrak{p}}_{\theta_\mathfrak{p}^{-1}}(g_\mathfrak{p})$$

where the products are taken over all non-Archimedean places.

This gives

$$\mathscr{I}(f, \varphi, s) = \mathscr{I}_\infty \prod_{\mathfrak{p}n} \mathscr{I}_\mathfrak{p}$$

where $\mathfrak{p}$ runs over all finite places and

$$\mathscr{I}_\infty = \int_{Z_\infty \backslash G_\infty} f^\infty(g^\infty) \varphi^\infty(g^\infty) \, dg^\infty$$

$$\mathscr{I}_\mathfrak{p} = \int_{Z_\mathfrak{p} \backslash G_\mathfrak{p}} f^\mathfrak{p}_{\theta_\mathfrak{p}} \varphi_{\theta_\mathfrak{p}^{-1}} \, dg_\mathfrak{p}.$$

Here $G_\infty = \prod_{\mathfrak{p}} G_\mathfrak{p}$ (the product is taken over all Archimedean $\mathfrak{p}$).

§ 2. Some examples of Euler subgroups

In this section Theorem 1 gives some sufficient conditions for a subgroup $G \subset \mathscr{G}$ to be an Euler subgroup and Theorems 2 and 3 give sufficient conditions for the existence of a meromorphic continuation of the integral $\mathscr{I}(f, G, S)$.

Some examples of Euler subgroups will also be given. Example 1 for the case $K = \mathfrak{k} \oplus \mathfrak{k}$ and $\mathfrak{k} = \mathbf{Q}$ was considered by Rankin [4], Selberg [5], Ogg [7]. The case of arbitrary $\mathfrak{k}$ was considered from the point of view of the theory of representations by H. Jacquet (R. Godement's oral communication).

Andrianov was the first who considered Example 4. It seems to me that Examples 2 and 3 were not studied by anyone yet. Theorems 2 and 3 guarantee the possibility of a meromorphic continuation of $\mathscr{I}(f, G, s)$ to whole complex plane in Examples 1, 2, 4.

I do not know if a meromorphic continuation of $\mathscr{I}(f, G, s)$ to the whole complex plane exists in Example 3.

Of course, it is not difficult to give many other applications of Theorem 1.

Before formulating Theorem 1 let us agree upon the following notation. Let G be an algebraic group over $\mathfrak{k}$ and X and Y two of its algebraic subgroups defined over $\mathfrak{k}$. We shall write $G=XY$, if each element g of $G_{\mathfrak{k}}(G_A)$ can be uniquely represented in the form $g=xy$, where $x \in X_{\mathfrak{k}}(X_A)$, $y \in Y_{\mathfrak{k}}(Y_A)$.

Theorem 1. *Let $\mathscr{G}$ be a reductive algebraic group over $\mathfrak{k}$ and $H, N, S, G=NH, P=GS, Z=NS$ algebraic subgroups of $\mathscr{G}$ defined over $\mathfrak{k}$ with the following properties:*

1. *N has no nontrivial algebraic characters: $N \to \mathfrak{k}^*$,*
2. *the left invariant measure on G_A can be represented in the form*

$$\theta(h)\, dn\, dh \tag{1}$$

where $\theta(h)$ is a character of H_A, dn and dh are left invariant measures on N_A and H_A, respectively.

3. *S is a commutative unipotent normal subgroup of P.*
4. *There exists a nontrivial character $\chi_0 : S_A \to \mathbf{C}$ with the following properties:*

a) $\chi_0(x)=1$ *for* $x \in S_k$,

b) $\chi_0(nxn^{-1})=\chi_0(x)$ *for* $x \in S_A$, $n \in N_A$.

This shows that χ_0 can be extended to a character χ_0 of Z_A such that $\chi_0(n)=1$ for $n \in N_A$ (we use the same symbol for the character and its extension).

5. *$H_{\mathfrak{k}}$ acts on $S_{\mathfrak{k}}-0$ simply transitively. Hence each nontrivial character χ of $S_{\mathfrak{k}} \backslash S_A$ can be uniquely represented as $\chi_0(hxh^{-1})$ with $h \in H_{\mathfrak{k}}$.*
6. *P has a normal subgroup R which is a cuspidal subgroup of $\mathscr{G}$.*

For every function $f \in A(\mathscr{G})$ and any algebraic character $\mu : G \to \mathfrak{k}^*$ the identity

$$\int_{G_{\mathfrak{k}} \backslash G_A} f(g)\nu(\mu(g))\, dg = \int_{H_A} f_{\chi_0}(h)\,\nu(\mu(h))\theta(h)\, dh \tag{2}$$

holds, where

$$f_{\chi_0}(g) = \int_{Z_{\mathfrak{k}} \backslash Z_A} \chi_0(z) f(zg)\, dz, \tag{3}$$

ν is any character of $\mathfrak{k}^* \backslash I_{\mathfrak{k}}$ with sufficiently large real part.

Proof. First of all, Z has no nontrivial algebraic character and hence $Z_{\mathfrak{k}} \backslash Z_A$ has a finite volume [10], [11]. It shows that the integral (3) is correctly defined.

We assume that both integrals in (2) converge absolutely.

We put

$$\psi_0(g) = \int_{S_{\mathfrak{k}} \backslash S_A} \chi_0(s) f(sg)\, ds, \quad \psi_1(g) = \int_{S_{\mathfrak{k}} \backslash S_A} f(sg)\, ds. \tag{4}$$

The group S is commutative and hence the usual theory of Fourier series and assumption *5)* give

$$f(g) = \psi_1(g) + \sum_{h \in H_{\mathfrak{k}}} \psi_0(hg) \quad \text{for any} \quad f(g) \in A(\mathscr{G}). \tag{5}$$

Hence

$$\int_{G_{\mathfrak{k}} \backslash G_A} f(g)\nu(\mu(g))\,dg = \int_{G_{\mathfrak{k}} \backslash G_A} \psi_1(g)\nu(\mu(g))\,dg + \int_{G_{\mathfrak{k}} \backslash G_A} \Big(\sum_h \psi_0(hg)\,\nu(\mu(g))\Big)\,dg. \tag{6}$$

We shall show that the first term in the right-hand side of (6) equals zero. It follows from *3)* that μ can be extended to $\tilde{\mu}: P \to \mathfrak{k}^*$ such that $\tilde{\mu}(s)=1$. Obviously, any unipotent subgroup lies in the kernel of $\tilde{\mu}$ and hence $\tilde{\mu}(R)=1$, where R is defined in *6)*. Using this and the definition of ψ_1 in (4), we get

$$\int_{G_{\mathfrak{k}} \backslash G_A} \psi_1(g)\nu(\mu(g))\,dg = \int_{P_{\mathfrak{k}} \backslash P_A} f(g)\nu(\tilde{\mu}(g))\,dg = 0 \tag{7}$$

because the cuspidal subgroup R is a normal subgroup of P, $f \in A(\mathscr{G})$ and hence for any $p \in P_A$

$$\int_{R_{\mathfrak{k}} \backslash R_A} f(rp)\nu(\tilde{\mu}(rp))\,dr = \nu(\tilde{\mu}(p)) \int_{R_{\mathfrak{k}} \backslash R_A} f(rp)\,dr = 0.$$

Now we shall consider the second term of (6). Denote by F a fundamental domain in G_A with respect to the left translations by $G_{\mathfrak{k}}$. From $G=NH$ it follows that $F_1 = \bigcup_{h \in H_{\mathfrak{k}}} hF$ is a fundamental domain in G_A with respect to the left translations by $N_{\mathfrak{k}}$. Using this fact, we obtain

$$\int_F \Big(\sum_{h \in H_{\mathfrak{k}}} \psi_0(hg)\Big)\nu(\mu(g))\,dg = \sum_{h \in H_{\mathfrak{k}}} \int_F \psi_0(hg)\nu(\mu(g))\,dg =$$

$$= \sum_{h \in H_{\mathfrak{k}}} \nu(\mu(h^{-1})) \int_{hF} \psi_0(g)\nu(\mu(g))\,dg = \int_{N_{\mathfrak{k}} \backslash G_A} \psi_0(g)\nu(\mu(g))\,dg \tag{8}$$

because ν is trivial on $\mathfrak{k}^*$ and hence $\nu(\mu(\gamma))=1$ for $\gamma \in G_{\mathfrak{k}}$.

Applying $G=NH$ we get that $N_{\mathfrak{k}} \backslash G_A = N_{\mathfrak{k}} \backslash N_A H_A$. By making use of $\mu(N)=1$ and of *1)* and *2)* we obtain

$$\int_{N_{\mathfrak{k}} \backslash G_A} \psi_0(g)\nu(\mu(g))\,dg = \int_{H_A} f_{\chi_0}(h)\nu(\mu(h))\,\theta(h)\,dh \tag{9}$$

where

$$f_{\chi_0}(g) = \int_{Z_{\mathfrak{k}} \backslash Z_A} \chi_0(z) f(zg)\,dz = \int_{N_{\mathfrak{k}} \backslash N_A} \psi_0(ng)\,dn. \tag{10}$$

Remark. It is possible to prove a similar theorem under more general assumptions. For instance, it is not necessary to require the simple transitivity in *5)*.

Before considering some examples of applications of our theorem we shall discuss the problem of the analytic continuation of the integral (2). We have

$v(x)=v_0(x)|x|^s$, $x\in I_{\mathfrak{k}}$. Let s be a complex variable and denote by $\mathscr{I}(s)=\mathscr{I}(f, G, s)$ the integral defined in the left-hand side of (2). It is clear that $\mathscr{I}(s)$ is an analytic function in the half-plane $\operatorname{Re} s>c$ for sufficiently large c.

Of course, there does not necessarily exist an analytic continuation of $\mathscr{I}(s)$ to the whole complex plane as an entire function or as a function with a finite number of poles. But it is plausible that there exists a meromorphic continuation to the whole complex domain. From now on we assume that $G\cap C=\{1\}$, where C is the center of $\mathscr{G}$ and $G^0=\operatorname{Ker}\mu$ has no nontrivial algebraic character.

Now we describe two methods of analytic continuation. I shall call the first one the Hecke method and the second one the Rankin—Selberg method.

Theorem 2. *(The Hecke method.) If there exists an automorphism $\sigma: g\to g^\sigma$ of $\mathscr{G}$ such that*

(i) *$\sigma(G)=G$,*

(ii) *$\mu(g^\sigma)=\mu^{-1}(g)$ for every $g\in G$,*

then for any $f\in A(\mathscr{G})$ the integral $\mathscr{I}(f, G, s)$ is an entire function.

Proof. First we shall prove that if $f(g)\in A(\mathscr{G})$ then $\tilde{f}(g)=f(g^\sigma)\in A(\mathscr{G})$. It is clear that *1)* if $f(\gamma g)=f(g)$ for every $\gamma\in\mathscr{G}_{\mathfrak{k}}$ then

$$\tilde{f}(\gamma g)=f(\gamma^\sigma g^\sigma)=f(g^\sigma)=\tilde{f}(g);$$

2) if R is a cuspidal subgroup then $\sigma(R)$ is also a cuspidal subgroup; *3)* if the space generated by the right translates of $f(g)$ by $\mathscr{G}_A$ is irreducible and admissible then the same is true for $f(g)$. Hence $\tilde{f}(g)\in A(\mathscr{G})$.

To prove our theorem we use the usual trick. Denote by G^0, G^1, G^{-1} the subsets in G_A where $|\mu(g)|=1$. $|\mu(g)|>1$, $|\mu(g)|<1$ respectively. We have

$$\mathscr{I}(s)=\int\limits_{G_{\mathfrak{k}}\setminus G^{-1}}+\int\limits_{G_{\mathfrak{k}}\setminus G^{1}}+\int\limits_{G_{\mathfrak{k}}\setminus G^{0}}. \tag{11}$$

By making use of the fact that under our assumptions $f(g)$ rapidly decreases at ∞, we obtain that the first and the second terms are holomorphic in the whole complex plane. The substitution $g\to g^\sigma$ shows that the third term is also holomorphic.

Theorem 3. *(The Rankin—Selberg method.) Let $\tilde{G}$ be a reductive group over $\mathfrak{k}$ and G its parabolic subgroup. For any $f\in S(\tilde{G})$ the integral $\mathscr{I}(f, G, s)$ can be represented in the form*

$$\mathscr{I}(f, G, s)=\int\limits_{\tilde{G}_{\mathfrak{k}}\setminus\tilde{G}_A} f_0(g)E(g, s)\,dg \tag{12}$$

where $f_0(g)$ is contained in the space generated by the right translates of $f(g)$ by the elements of a maximal compact subgroup $U\subset\tilde{G}_A$; $E(g, s)$ is an Eisenstein series.

Proof. Let us fix a maximal compact subgroup $U \subset \tilde{G}_A$. We introduce

$$f_0(g) = \int_W f(gw)\, v_0(\mu(w)), \quad W = U \cap G_A \tag{14}$$

where the measure dw on W is chosen so that

$$\int_W dw = 1.$$

It is clear that

$$\mathscr{I}(f, G, s) = \mathscr{I}(f_0, G, s) \tag{15}$$

and

$$f_0(gw) = v_0^{-1}(\mu(w)) f_0(g) \quad \text{for} \quad w \in W. \tag{16}$$

It is obvious that the space L_0 generated by the right translates $f_0(gu)$, $u \in U$ is finite-dimensional. Let us consider a basis

$$f_1(g), f_2(g), \dots, f_n(g) \quad \text{of} \quad L_0$$

such that

$$f_1(g) = f_0(g),\ f_k(g) = f_0(g^{u_k}) \quad \text{for some} \quad u_k \in U, \quad 2 \leqq k \leqq n. \tag{17}$$

Denote by $F(g)$ the row vector the components of which are taken from (17) (in the natural order).

We have

$$F(gu) = F(g) A(u), \quad g \in \tilde{G}_A, \quad u \in U \tag{18}$$

where $A(u)$ is a matrix function on U. From (16), (18) and the definition of F it follows that

$$a_{11}(w) = v_0^{-1}(\mu(w)),\ a_{1i}(w) = 0, \quad 2 \leqq i \leqq k. \tag{19}$$

We introduce the column-vector

$$\Phi(g) = \begin{pmatrix} v(\mu(g)) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad g \in G_A. \tag{20}$$

For $\tilde{g} \in \tilde{G}_A$ we define

$$\Phi(\tilde{g}) = \Phi(gu) = A^{-1}(u)\Phi(g), \quad g \in G_A, \quad u \in U. \tag{21}$$

(19) shows that $\Phi(\tilde{g})$ is correctly defined. From (18) and (21) it follows that $F(g)\Phi(g)$ is stable under the action of U by right translations. We note that for almost all g, one has $g^{-1} U g \cap \tilde{G}_\mathfrak{k} = U \cap C_\mathfrak{k}$, where C is the center of $\tilde{G}$. From this we get

$$\int_{C_A G_\mathfrak{k} \setminus \tilde{G}_A} \psi(g)\, dg = c \int_{C_A G_\mathfrak{k} \setminus G_A} \psi(g)\, dg \tag{22}$$

where c is a scalar and $\psi(g)$ is any function on $C_A G_\mathfrak{k} \setminus \tilde{G}_A$ satisfying the following assumptions: *1)* $\psi(g)$ is stable under the right action of U, *2)* the integral on the right-hand side of (22) converges absolutely.

Applying (22) to $\psi(g)=F(g)\Phi(g)$ we get

$$\int_{C_A G_{\mathfrak{k}}\backslash G_A} F(g)\Phi(g)\,dg = \int_{C_A G_{\mathfrak{k}}\backslash G_A} F(g)\Phi(g)\,dg = \int_{C_A G_{\mathfrak{k}}\backslash G_A} f_0(g)\nu(\mu(g))\,dg. \tag{23}$$

Using the definition of $F(g)$ we obtain a function $\theta(g,s)$ on $\tilde{G}_A$ such that

$$\int_{C_A G_{\mathfrak{k}}/\tilde{G}_A} F(g)\Phi(g)\,dg = \int_{C_A G_{\mathfrak{k}}\backslash \tilde{G}_A} f_0(g)\theta(g,s)\,dg \tag{24}$$

and $\theta(g,s)$ is a holomorphic function of s stable under the left action of $G_{\mathfrak{k}}$.

We introduce the Eisenstein series

$$E(g,s) = \sum \theta(\gamma g, s). \tag{25}$$

We have

$$\int_{C_A G_{\mathfrak{k}}\backslash \tilde{G}_A} f_0(g)\theta(g,s)\,dg = \int_{C_A \tilde{G}_{\mathfrak{k}}\backslash \tilde{G}_A} f_0(g)E(g,s)\,dg. \tag{26}$$

Finally, combining (15), (23), (24) and (26) we get (12).

Now we are going to give some examples.

1) Let K be a two-dimensional algebra over $\mathfrak{k}$ which is isomorphic to $\mathfrak{k}\oplus\mathfrak{k}$ or a quadratic extension of $\mathfrak{k}$. Denote by i an imbedding: $\mathfrak{k}\to K$ which is the diagonal imbedding if $K=\mathfrak{k}\oplus\mathfrak{k}$ and the natural imbedding if K is a quadratic extension of $\mathfrak{k}$. Denote by σ the nontrivial automorphism of K which is trivial on $\mathfrak{k}$. The imbedding $i:\mathfrak{k}\to K$ defines an imbedding: $GL(2,\mathfrak{k})\to GL(2,K)$. We put $\mathcal{G}=GL(2,K)$, and set

$$H=\left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix},\ \alpha\in\mathfrak{k}\right\}$$

$$N=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ x\in\mathfrak{k}\right\}$$

$$S=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ x\in K,\ \sigma(x)=-x\right\}.$$

Furthermore, let $G=NH$, $P=GS$ and

$$R=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ x\in K\right\},\quad \chi_0(x)=\nu\ (\operatorname{tr}\alpha x)$$

where $\alpha\in K$, $\alpha^{\sigma}=-\alpha$ and ν is a nontrivial character on $\mathfrak{k}\backslash A_{\mathfrak{k}}$.

It is easy to show that assumptions (1—8) are satisfied for these subgroups.

2) Let $\mathfrak{A}$ be a division algebra over $\mathfrak{k}$. Denote by $n(x)$, $x\in\mathfrak{A}$, the norm on $\mathfrak{A}$. Let $\mathcal{G}=GL(2,\mathfrak{A})$.

Put

$$H=\left\{\begin{pmatrix}\alpha & 0\\ 0 & e\end{pmatrix},\ \alpha\in\mathfrak{A}^*\right\}$$

$$N=\left\{\begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix},\ \beta\in\mathfrak{A}^*,\ n(\beta)=1\right\}$$

$$S=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ x\in\mathfrak{A}\right\}.$$

Now, let $G=NH$, $P=GS$, $R=S$ and $\chi_0=\nu(\operatorname{tr}x)$ where ν is a nontrivial character of $\mathfrak{k}\diagdown A_{\mathfrak{k}}$. It is not too difficult to verify that the assumptions (1—8) are satisfied.

3) We shall give an example in which no intermediate reductive subgroup exists, in other words there is no reductive subgroup $\tilde{G}$ such that $\mathscr{G}\supset\tilde{G}\supset G$. Let $\mathscr{G}=GL(2,\mathfrak{k})\times\ldots\times GL(2,\mathfrak{k})$ (n times), and let

$$H=\left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix},\ldots\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right\}$$

$$N=\left\{\begin{pmatrix}1 & x_1\\ 0 & 1\end{pmatrix},\ldots\begin{pmatrix}1 & x_n\\ 0 & 1\end{pmatrix},\ \sum_{i=1}^{n}x_i=0\right\}$$

$$S=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ldots\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right\}.$$

We put $\chi_0(x)=\nu(x)$ for $x\in S$ (ν is a nontrivial character of $\mathfrak{k}\diagdown A_{\mathfrak{k}}$), and $R=NS$. It is obvious that if $n>2$, then no intermediate reductive group exists.

4) Let K be a quadratic extension of $\mathfrak{k}$. Denote by $\operatorname{End}_{\mathfrak{k}}K$ the ring of all $\mathfrak{k}$-linear endomorphisms of K which is considered as a two-dimensional space over $\mathfrak{k}$. Denote by $r(x)$ the result of the application $r\in\operatorname{End}_{\mathfrak{k}}K$ to $x\in K$. Denote by $\operatorname{tr}(r)$ and $n(r)$ the trace and the determinant of r, respectively. By means of the natural embedding $K\hookrightarrow\operatorname{End}_{\mathfrak{k}}K$ one can define $\operatorname{tr}(x)$ and $n(x)$ for $x\in K$. It is obvious that $\operatorname{tr}x=x+x^{\sigma}$ and $n(x)=xx^{\sigma}$, where σ is a nontrivial automorphism of K, coinciding with the identity on $\mathfrak{k}$.

Let $V=K^2$ and let $\alpha(v_1,v_2)$ be a nontrivial skew-form on V with values in K. Set $\varrho(v_1,v_2)=\operatorname{tr}\alpha(v_1,v_2)$. It is obvious that ϱ is a nondegenerate skew-form with values in $\mathfrak{k}$. Each element $g\in\operatorname{End}_{\mathfrak{k}}V$ can be written in the form

$$g=\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\tag{27}$$

where $g_{ij}\in\operatorname{End}_{\mathfrak{k}}K$.

In the following we shall assume that $\mathscr{G}$ is a factor group of the group $\tilde{\mathscr{G}}$ consisting of all $g\in\operatorname{End}_{\mathfrak{k}}V$ for which

$$\varrho(gx,gy)=\alpha\varrho(x,y),\quad \alpha\in\mathfrak{k}^*,\quad x,y\in\mathfrak{k}^4.\tag{28}$$

holds.

Now we introduce the subgroup R consisting of all $g\in\mathscr{G}$ having the form

$$g=\begin{pmatrix}e & r\\ 0 & e\end{pmatrix},\quad r\in\operatorname{End}_{\mathfrak{k}}K.\tag{29}$$

It is easy to verify that g of form (29) belongs to $\mathscr{G}$ if and only if $r\in\operatorname{End}_{\mathfrak{k}}K$ is symmetric, i.e.

$$\operatorname{tr}(r(x)y)=\operatorname{tr}(xr(y))\tag{30}$$

for all $x,y\in K$. Denote by $\operatorname{End}_{\mathfrak{k}}^{S}K$ the set of all $r\in\operatorname{End}_{\mathfrak{k}}K$ satisfying (30).

It is easy to verify that the image of K under the natural imbedding $K \hookrightarrow \mathrm{End}_{\mathfrak{k}} K$ belongs to $\mathrm{End}_{\mathfrak{k}}^{S} K$. Set

$$\tilde{H} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & e \end{pmatrix}, \ \alpha \in \mathfrak{k}^* \right\}$$

$$\tilde{S} = \left\{ \begin{pmatrix} e & x \\ 0 & e \end{pmatrix}, \ x \in \mathrm{End}_{\mathfrak{k}}^{S} K, \ \mathrm{tr}\, xa = 0 \ \text{for all} \ a \in K \right\}$$

$$\tilde{N} = \left\{ \begin{pmatrix} \delta & x \\ 0 & \delta^{\sigma} \end{pmatrix}, \ \sigma \in K^*, \ x \in K \right\}.$$

We define H, S, N as the images of $\tilde{H}$, $\tilde{S}$, $\tilde{N}$ in $\mathscr{G}$ under the natural homomorphism: $\tilde{\mathscr{G}} \to \mathscr{G}$.

We put $\chi_0(x) = v(\mathrm{tr}\,(\hat{\sigma}x))$, where $\hat{\sigma} \in \mathrm{End}_{\mathfrak{k}} K$ is defined by the relation $\hat{\sigma}(x) = x^{\sigma}$ for all $x \in K$, and v is a nontrivial character. It is not difficult to show that all assumptions (1—6) are satisfied.

§ 3. Computation of the local factors for a symplectic group of rank two

The idea of the proof of the key lemma (Lemma 1) is taken from Andrianov's paper [9].

Let $\mathfrak{k}$ be a local non-Archimedean field; denote by $\mathcal{O}$ the ring of integers, by $\mathfrak{p}$ the prime ideal of $\mathcal{O}$, by ω a generator of $\mathfrak{p}$ and by v the discrete valuation of $\mathfrak{k}$ such that $v(\omega) = 1$. We assume that the characteristic of the residue field of $\mathfrak{k}$ is not equal to 2.

We shall introduce the following skew symmetric matrix

$$\mathscr{I} = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Denote by $\mathscr{G}$ the set of all $g \in M(4, \mathfrak{k})$ such that

$$g' \mathscr{I} g = \alpha \mathscr{I}, \quad \alpha \in \mathfrak{k}^*. \tag{2}$$

Here and in the following g' means the transpose of g.

Let T denote the subgroup consisting of all $g \in \mathscr{G}$ which are of the form

$$\begin{pmatrix} e & t \\ 0 & e \end{pmatrix}, \quad t' = t. \tag{3}$$

The characters of T can be written in form

$$\chi(t) = v\ (\mathrm{tr}\,(\beta t)), \quad \beta \in M(2, \mathfrak{k}), \ \beta' = \beta \tag{4}$$

where v is the standard character on $\mathfrak{k}^+$.

The normalizer $N(T)$ of T equals PT, where $P \cong GL(2, \mathfrak{k}) \times \mathfrak{k}^*$ consists of the following matrices

$$P = \begin{pmatrix} r & 0 \\ 0 & \alpha r'^{-1} \end{pmatrix} \quad r \in GL(2, \mathfrak{k}), \ \alpha \in \mathfrak{k}^*. \tag{5}$$

Let $\chi(ptp^{-1})=\nu(\operatorname{tr}(\beta_1 t))$ where x is of form (4). Then

$$\beta_1 = \alpha^{-1}r'\beta r. \tag{6}$$

We call the character (4) nondegenerate if $\det \beta \neq 0$. Let χ be a nondegenerate character of T. Denote by P^χ the stabilizer of χ in P. Let $Z^\chi = P^\chi T$. It is clear that χ can be extended to the character of Z^χ such that $\chi(p)=1$ for each $p \in P$. It is not difficult to prove that $P \cong \hat{K}$, where K is a semisimple two-dimensional algebra over $\mathfrak{k}$, σ its canonical automorphism and $\hat{K}$ is the semidirect product of K^* and $\{1, \sigma\}$.

The following types of K can occur: $K=\mathfrak{k}\oplus\mathfrak{k}$ or $K=\mathfrak{k}(\sqrt{\delta})$, where $\sqrt{\delta}\notin\mathfrak{k}$.

To a representation of K one can attach a character $\chi(t)=\nu(\operatorname{tr}(\hat{\sigma}t))$ where $\hat{\sigma}$ satisfies the relation $\hat{\sigma}^{-1}a'\hat{\sigma}=a^\sigma$. It is easy to check that P^χ contains a subgroup of index two

$$\begin{pmatrix} a & 0 \\ 0 & \alpha a'^{-1} \end{pmatrix}, \quad a \in K, \ \alpha = \det a. \tag{7}$$

Denote by W_χ the set of all functions $f(g)$ on $\mathscr{G}$ such that

$$f(zg) = \chi(z)f(g) \tag{8}$$

for every $z \in Z^\chi$.

Let π be an irreducible representation of $\mathscr{G}$ containing a vector stable under the action of a maximal compact subgroup of $\mathscr{G}$. It is well known that such a vector is unique up to a scalar factor. Denote this vector by ξ_0.

Set

$$V_n = \{M \in M(4, \mathcal{O}) \mid M'\mathscr{I}M = \alpha\mathscr{I}, v(\alpha) = n\}. \tag{9}$$

It is well known that

$$V_n = \sum_{i=1}^{r} M_i \mathscr{G}_{\mathcal{O}}.$$

It follows that $\sum_{i=1}^{r} T_{M_i}\xi_0$ is $\mathscr{G}_{\mathcal{O}}$-stable and hence

$$\sum_{i=1}^{r} T_{M_i}\xi_0 = \lambda_n(\pi)\xi_0 \tag{11}$$

where $\lambda_n(\pi)\in\mathbf{C}$.

For any irreducible admissible representation π of $\mathscr{G}$ denote by $W_\chi(\pi)$ some subrepresentation of W_χ which is isomorphic to π. Assume that π contains the vector ξ_0 defined above. Denote by $f_0(g)$ the corresponding function in $W_\chi(\pi)$.

Collecting together all we know about $f_0(g)$ we have

$$f_0(zg) = \chi(z)f_0(g), \quad z \in Z^\chi \tag{12}$$

$$f_0(gu) = f_0(g), \qquad u \in \mathscr{G}_{\mathcal{O}} \tag{13}$$

$$\sum_{i=1}^{r} f_0(gM_i) = \lambda_n(\omega)f_0(g). \tag{14}$$

Applying the Iwasawa decomposition we see that M_i can be taken to the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a, b, d \in M(2, \mathcal{O}), \; ab' = ba', \; ad' = \omega^n e. \tag{15}$$

Denote by $V_n(a)$ the collection of all $M \subset V_n$ which are of the form (15) with a given a.

Denote by A_n the set of all $a \in M(2, \mathcal{O})$ such that $\omega^n a^{-1} \in M(2, \mathcal{O})$. We shall call $a_1, a_2 \in A_n$ *equivalent* if $a_1 a_2^{-1} \in GL(2, \mathcal{O}_{\mathfrak{p}})$. Denote by $B(a)$ the set of all $b \in M(2, \mathcal{O})$ such that $a^{-1}b = b'a'^{-1}$. We shall call $b_1, b_2 \in B(a)$ *congruent* if $a^{-1}(b_1 - b_2) \in M(2, \mathcal{O})$.

We put

$$\psi_\chi(a) = \sum \chi(\omega^{-n} ba') \tag{16}$$

where b runs over all incongruent elements of $B(a)$. (14) and (15) show that

$$\sum f_0 \begin{pmatrix} a & 0 \\ 0 & \omega^n a'^{-1} \end{pmatrix} \psi_\chi(a) = \lambda_n(\pi) f_0(E) \tag{17}$$

where a runs over all inequivalent elements of A_n.

Lemma 1. If $\psi_\chi(a) \neq 0$, $a \in A_n$, then a is equivalent to $\tilde{a} \in K$. More precisely, let $\chi = v(\operatorname{tr} \hat{\sigma} t)$, where v is the standard character on $\mathfrak{k}^*$.

We have *1)* if $K = \mathfrak{k} \oplus \mathfrak{k}$

$$K = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \; x_i \in \mathfrak{k} \right\}, \; \hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{18}$$

then

$$\tilde{a} = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^r \end{pmatrix}, \; \max(k, r) = n, \; \psi_\chi(a) = q^{n + 2\min(k,r)}, \tag{19}$$

2) if $\mathfrak{k}(\sqrt{\delta}) = K$ is a quadratic extension of k,

$$K = \left\{ \begin{pmatrix} x_0 & x_1 \\ x_1 \delta & x_0 \end{pmatrix}, \; x_0, x_1 \in \mathfrak{k} \right\}, \; \hat{\sigma} = \begin{pmatrix} \delta & 0 \\ 0 & -1 \end{pmatrix}, \tag{20}$$

then

$$\hat{a} = \begin{cases} \omega^n e, & v(\delta) = 0 \\ \omega^n e \;\text{ or }\; \begin{pmatrix} 0 & \omega^{n-1} \\ \omega^{n-1}\delta & 0 \end{pmatrix}, & v(\delta) = 1, \end{cases} \tag{21}$$

$$\psi_\chi(\tilde{a}) = \begin{cases} q^{3n}, & v(\delta) = 0 \\ q^{3n} \;\text{ or }\; q^{3n-2}, & v(\delta) = 1. \end{cases} \tag{22}$$

Proof. It is clear that if $\psi_\chi(a) \neq 0$, then $\chi(\omega^{-n} ba') = 1$ for all $b \in B(a)$, i.e.

$$\operatorname{tr}(\omega^{-n} \hat{\sigma} ba') \in \mathcal{O}. \tag{23}$$

Each $a \in A_n$ is equivalent to $\tilde{a}$ which is of the form

$$\tilde{a} = \begin{pmatrix} \omega^k & c \\ 0 & \omega^r \end{pmatrix}. \tag{24}$$

It is easy to show that $\bar{a} \in A_n$ if and only if

$$n \geqq k \geqq 0,\, n \geqq r \geqq 0,\, k > v(c) \geqq \max(0, k+r-n) \quad \text{or} \quad c = 0. \tag{25}$$

Moreover $b=(b_{ij}) \in B(\bar{a})$ if and only if

$$b_{ij} \in \mathcal{O} \tag{26}$$

$$b_{12}\omega^r = b_{21}\omega^k + b_{22}c \tag{27}$$

First we consider the case $K=\mathfrak{k}\oplus\mathfrak{k}$. We have

$$\operatorname{tr}(\omega^{-n}\hat{\sigma}ba') = 2b_{12}\omega^{r-n}. \tag{28}$$

(27), (23) and (28) show that $\max(k, r)=n$.

If $r=n$ it follows from (25) that $c=0$ and hence $a \in K$. If $r<n$, then $k=n$. (23), (27) and (28) imply that $v(c) \geqq 0$ and hence, according to (25), $c=0$. This means $\bar{a} \in K$. It is clear that $\psi_\chi(a)$ equals the number of incongruent elements in $B(a)$. We get $\psi_\chi(\bar{a})=q^{n+2\min(k,r)}$.

Now we consider the case $K=\mathfrak{k}(\sqrt{\delta})$ and $\sqrt{\delta} \notin \mathfrak{k}$. Suppose that the representation K is of the form (20).

It follows from (23) that

$$\delta(b_{11}\omega^k + b_{12}c) - b_{22}\omega^r \in \omega^n\mathcal{O}. \tag{29}$$

First we take b which is of the form

$$\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{30}$$

From (29) it follows that $k+v(\delta) \geqq n$. If $v(\delta)=0$ then $k=n$. Now we take

$$b = \begin{pmatrix} 0 & b_{12} \\ 0 & b_{22} \end{pmatrix}. \tag{31}$$

(25) shows that $v(c) \geqq r$ and hence (27) gives

$$b_{12} = c\omega^{-r}b_{22}. \tag{32}$$

(32) and (29) show that

$$v(\delta c^2 - 1) \geqq n - r. \tag{33}$$

Using the fact that $\sqrt{\delta} \notin \mathfrak{k}$, we obtain from (33) that $n=r$. Thus it follows from (25) that $c=0$, $\bar{a}=\omega^n e$. Now we consider the case $v(\delta)=1$. Suppose first that $k=n$. As in the previous case, it follows from (25) that $v(c) \geqq r$. If b is of the form (31) then (33) holds again and shows that $n=r$. Suppose now that $k=n-1$. Let

$$b = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}. \tag{34}$$

The substitution of b into (27) gives

$$b_{12}\omega^r = b_{21}\omega^{n-1} \tag{35}$$

We want to show that $r=n$ again. Suppose that $r<n$, then $v(b_{12}) \geqq n-1-r$ and there exists $\hat{b} \in B(a)$ of the form (34) for which $v(b_{12})=n-1-r$.

Substituting $\hat{b}$ into (29) shows $v(c) \geqq r$. For b of form (31) it follows from (27) that

$$b_{12} = b_{22} c\omega^{-r}.$$

If we substitute b of form (31) into (29) we obtain

$$v(\delta c^2 \omega^{-2r} - 1) \geqq n-r,$$

and this is a contradiction since $v(\delta c^2 \omega^{-2r}) > 0$. So the equality $r=n$ is proved. Application of (25) shows that $c=0$. Finally, we obtain

$$\bar{a} = \begin{pmatrix} \omega^{n-1} & 0 \\ 0 & \omega^n \end{pmatrix} = \begin{pmatrix} 0 & \omega^{n-1} \\ \omega^{n-1}\delta & 0 \end{pmatrix} u, \quad u \in GL(2, \mathcal{O}).$$

$\psi_\chi(a)$ can be computed in the same way. Hence our lemma is proved.

We define a formal power series as

$$\Phi(t) = \sum_{n=0}^{\infty} f_0 \begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix} t^n$$

$$\Lambda(t) = \sum_{n=0}^{\infty} \lambda_n(\pi) t^n.$$

Lemma 2.

$$\Phi(tq^3) = f_0(E)\varrho_K(t)\Lambda(t), \tag{36}$$

where

$$\varrho_K(t) = \begin{cases} \dfrac{1-qt}{1+qt} & \text{if} \quad K = \mathfrak{k} \oplus \mathfrak{k}, \\ (1+qt)^{-1} & \text{if} \quad K \text{ is a ramified extension of } \mathfrak{k}, \\ 1 & \text{if} \quad K \text{ is an unramified extension of } \mathfrak{k}. \end{cases}$$

Proof. First we consider the case $K=\mathfrak{k} \oplus \mathfrak{k}$. Formula (17) and Lemma 1 imply

$$\sum_{r=0}^{n-1} \left(f_0 \begin{pmatrix} a_r & 0 \\ 0 & \omega^n a_r^{-1} \end{pmatrix} + f_0 \begin{pmatrix} b_r & 0 \\ 0 & \omega^n b_r^{-1} \end{pmatrix} \right) q^{n+2r} + f_0 \begin{pmatrix} a_n & 0 \\ 0 & \omega^n a_n^{-1} \end{pmatrix} q^{3n} = \lambda_n(\pi) f_0(E) \tag{37}$$

where

$$a_r = \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^n \end{pmatrix}, \; b_r = \begin{pmatrix} \omega^n & 0 \\ 0 & \omega^r \end{pmatrix}. \tag{38}$$

Moreover

$$\begin{pmatrix} a_r & 0 \\ 0 & \omega^n a_r^{-1} \end{pmatrix} \begin{pmatrix} c_r & 0 \\ 0 & c_r^\sigma \end{pmatrix} = \begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix} \tag{39}$$

where

$$c_r = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{r-n} \end{pmatrix} \in K$$

and hence

$$\begin{pmatrix} c_r & 0 \\ 0 & c_r^\sigma \end{pmatrix} \in P^\varkappa$$

(P^χ is the stabilizer of χ_σ).

It shows that

$$f_0\begin{pmatrix} a_r & 0 \\ 0 & \omega^n a_r^{-1} \end{pmatrix} = f_0\begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix}. \tag{40}$$

Analogously, we have

$$f_0\begin{pmatrix} b_r & 0 \\ 0 & \omega^n b_r^{-1} \end{pmatrix} = f_0\begin{pmatrix} \omega^r e & 0 \\ 0 & e \end{pmatrix}. \tag{41}$$

A substitution of (40), (41) in (37) gives

$$2\sum_{r=0}^{n-1} f_0\begin{pmatrix} \omega^r e & 0 \\ 0 & e \end{pmatrix} q^{n+2r} + f_0\begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix} q^{3n} = \lambda_n(\pi) f_0(E). \tag{42}$$

Analogously, we can prove that

$$f_0\begin{pmatrix} \omega^{n-1} e & 0 \\ 0 & e \end{pmatrix} q^{3n-2} + f_0\begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix} q^{3n} = \lambda_n(\pi) f_0(E) \tag{43}$$

if K is a ramified extension of $\mathfrak{k}$,

$$f_0\begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix} q^{3n} = f_0(E)\, \lambda_n(\pi) \tag{44}$$

if K is an unramified extension of $\mathfrak{k}$.

By formal computation the statement of our lemma can be easily deduced from (42), (43) and (44).

Denote by H the following subgroup

$$H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha \in \mathfrak{k} \right\}. \tag{45}$$

Lemma 3.

$$\int_{H_\mathfrak{p}} f_0(h) |h|^s \, d^* h = f_0(E)\, \varrho_K(q^{-s-3})\, \Lambda(q^{-s-3})\, \mu(H_{E_\mathfrak{p}}) \tag{46}$$

where $E_\mathfrak{p}$ is a group of units of $\mathfrak{k}$.

Proof. $f_0(h)$ is invariant under the action of $H_{E_\mathfrak{p}}$. Thus it is enough to find $f_0(h_n)$, where

$$h_n = \begin{pmatrix} \omega^n e & 0 \\ 0 & e \end{pmatrix}. \tag{47}$$

By making use of (12) it is not too difficult to prove that $f_0(h_n) = 0$ for $n < 0$. Combining this fact with the result of Lemma 4 we get the desired equality.

Remark. Lemma 2 shows that the restriction of $f_0(g)$ to H is uniquely defined by the representation π. Of course it does not guarantee the uniqueness of the space $W_\chi(\pi)$. Perhaps, it can be proved by using the same method or that of paper [2].

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