Prismatic F-gauges

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Preface

These are notes for a course given at Princeton University in Fall 2022. The goal of this course is to ultimately explain some work in progress of the author with Jacob Lurie on prismatic F-gauges. While some references are given, we have made no serious attempt to give exhaustive references.

Currently (in Fall 2022), these notes are being continuously updated and may change at any time. Please use with caution.

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Any original result is joint work with Jacob Lurie; all mistakes are due to the author alone.

Chapter 1

Motivation and goals

This course concerns algebraic geometry in mixed characteristic, so our basic geometric objects of study are bounded p-adic formal schemes[1]. In recent years, prismatic cohomology has provided a useful integral p-adic cohomology theory to study such spaces. The goal of this course is explain the first steps towards a reasonable theory of "coefficients" for prismatic cohomology and some consequences. To explain what this means as well as our motivation, let us recall two important examples of coefficient systems for more classical cohomology theories in algebraic geometry.

1.1 Hodge-theoretic coefficients

Say X is a complex variety (regarded as an analytic space). Then Saito attached to X the category $\mathrm{MHM}(X)$ of mixed Hodge modules on X. An object $E \in \mathrm{MHM}(X)$ consists a perverse sheaf $L \in \mathrm{Perv}(X, \mathbf{Q})$ and a filtration (in quasi-coherent $\mathcal{O}_X$-modules) on the associated $\mathcal{D}_X$-module $\mathrm{RH}(L_{\mathbf{C}})$ satisfying many conditions. An important example (for X smooth) is the Tate twist $\mathbf{Q}_X(i)[\dim(X)]$: take $L = \mathbf{Q}[\dim(X)]$ (with the shift ensuring perversity) with filtration concentrated in degree $-i$. For our purposes, the salient features of this construction are:

1. The value on a point: When $X = pt$, we can identify $\mathrm{MHM}(Y)$ with the category of polarizable mixed Hodge structures.
2. Pushforward stability: If $f : X \to Y$ is a map, then there is a naturally defined pushforward $Rf_* : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ compatible with pushforward on underlying sheaves. When $Y = pt$, this captures the fact that $H^i(X, \mathbf{Q})$ carries a natural polarizable mixed Hodge structure (Deligne).
3. Motivic nature: For any reasonable notion of "motives over X", there ought to be a realization functor towards $D^b(\mathrm{MHM}(X))$. In particular, the groups $H^i_{MHM}(X, \mathbf{Q}_X(j)) := \mathrm{Ext}^i_{\mathrm{MHM}(X)}(\mathbf{Q}_X, \mathbf{Q}_X(j))$ provides a Hodge-theoretic approximation to the motivic cohomology $H^i_{Mot}(X, \mathbf{Q}(j))$ of X and thus to algebraic cycles. For example, $H^2_{MHM}(X, \mathbf{Q}_X(1)) \simeq \mathrm{Pic}(X^{an})_{\mathbf{Q}}$. In general, the group $H^{2i}_{MHM}(X, \mathbf{Q}_X(i))$ is built from two pieces: the discrete

[1] A bounded p-adic formal scheme is a formal scheme which locally has the form $\mathrm{Spf}(R)$ for a p-adically complete ring R with bounded p^∞-torsion (e.g., any noetherian p-adically complete ring). Note that any scheme of characteristic p is a bounded p-adic formal scheme. For our purposes, the main examples of bounded p-adic formal schemes are those which are smooth over $\mathrm{Spf}(R)$, where R is either a perfect field of characteristic p or the ring of integers of a p-adic field (such as a local field or a perfectoid field). Studying these will often entail allow more "very ramified" examples.

part is the group of rationalized Hodge classes in codimension i, and the continuous part is an intermediate Jacobian.

1.2 Étale coefficients

Fix a field k with separable closure $\overline{k}$ and a prime ℓ invertible on k. To a variety X/k, Grothendieck and collaborators attached the constructible derived category $D^b_{cons}(X, \mathbf{Q}_\ell)$ of ℓ-adic sheaves on X. An important example of an object here is the invertible lisse ℓ-adic sheaf $\mathbf{Q}_\ell(1) := (\lim_n \mu_{\ell^n})[\frac{1}{\ell}]$. For our purposes, the salient features of this construction are:

1. The value on a point: When $X = \mathrm{Spec}(k)$, we can identify $D^b_{cons}(X, \mathbf{Q}_\ell)$ with the derived category of finite dimensional continuous ℓ-adic representations of $\mathrm{Gal}(\overline{k}/k)$.
2. Pushforward stability: If $f : X \to Y$ is a map of k-varieties, then there is a naturally defined pushforward $Rf_* : D^b_{cons}(X, \mathbf{Q}_\ell) \to D^b_{cons}(Y, \mathbf{Q}_\ell)$. For $Y = \mathrm{Spec}(k)$, this captures the fact that $H^i(X_{\overline{k}}, \mathbf{Q}_\ell)$ is naturally a continuous finite dimensional ℓ-adic representation of $\mathrm{Gal}(\overline{k}/k)$.
3. Motivic nature: For any reasonable notion of "motives over X", there ought to be a realization functor towards $D^b_{cons}(X, \mathbf{Q}_\ell)$. Thus, the groups $H^i(X, \mathbf{Q}_\ell(j)) = \mathrm{Ext}^i_{D^b_{cons}(X,\mathbf{Q}_\ell)}(\mathbf{Q}_\ell, \mathbf{Q}_\ell(j))$, often called "absolute étale cohomology", provide a Galois-theoretic approximation to motivic cohomology and thus algebraic cycles. For example, the Hochschild–Serre spectral sequence

$$H^i(\mathrm{Gal}(\overline{k}/k), H^j(X_{\overline{k}}, \mathbf{Q}_\ell(i))) \Rightarrow H^{i+j}(X, \mathbf{Q}_\ell(i))$$

   yields a filtration on $H^*(X, \mathbf{Q}_\ell(i))$. For k a number field, the resulting filtration on $H^{2i}(X, \mathbf{Q}_\ell(i))$ has a close conjectural relationship with the Bloch-Beilinson filtration on $\mathrm{CH}^i(X)$ under the cycle class map $\mathrm{CH}^i(X) \otimes \mathbf{Q}_\ell \to H^{2i}(X, \mathbf{Q}_\ell(i))$, see [Jan94].

1.3 Goals

Broadly speaking, the goal of these lectures is to introduce a partial analog of the stories in §1.1 and §1.2 for bounded p-adic formal schemes and prismatic cohomology. More precisely, given a p-adic formal scheme X satisfying mild conditions, we shall construct a symmetric monoidal stable ∞-category $\mathrm{F\text{-}Gauge}_{\Delta}(X)_{\mathrm{perf}}$ of perfect prismatic F-gauges on X; roughly, this will play the role $D^b(\mathrm{MHM}(X))$ or $D^b_{cons}(X, \mathbf{Q}_\ell)$ in the above discussion[2]. This category comes equipped with a line bundle $\mathcal{O}\{1\}$ (called the *Breuil-Kisin twist*) playing the role of the Tate twist. Moreover, there are "realization" functors towards more classical notions of coefficient systems (e.g., F-crystals on the special fibre $X_{p=0}$, local systems on the generic fibre X_η, filtered modules with integrable connection on X under certain assumptions, etc.). A large part of this course is dedicated to the study the F-gauge analogs of the 3 salient features mentioned in §1.1 and §1.2. In particular, we have pushforward stability under proper smooth maps, leading to many interesting examples of F-gauges. Moreover, similar to what happens in both Hodge theory and étale cohomology, the case of a point (i.e., $X = \mathrm{Spf}(\mathbf{Z}_p)$) is highly interesting and non-trivial. In fact, one of our target theorems concerns this case, and is the following:

[2] A disclaimer is already in order: perfect prismatic F-gauges capture only the lisse objects. The analogy with Hodge theory or étale cohomology breaks down outside the lisse case due to the phenomenon that integral p-adic cohomology theories, such as prismatic or crystalline cohomology, do not have good finiteness properties for, e.g., open varieties. Correspondingly, we only have pushforward stability of perfect prismatic F-gauges under proper smooth morphisms in general.

Theorem 1.3.1. *Let $X = \mathrm{Spf}(\mathbf{Z}_p)$. For any $E \in F\text{-}Gauge_{\Delta}(X)_{\mathrm{perf}}$, the complex* $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) := \mathrm{RHom}_{F\text{-}Gauge_{\Delta}(X)_{\mathrm{perf}}}(\mathcal{O}, E) \in D(\mathbf{Z}_p)$ *is called the* syntomic cohomology of $\mathbf{Z}_p$ with coefficients in E.

1. Finiteness: *For any $E \in F\text{-}Gauge_{\Delta}(X)_{\mathrm{perf}}$, the complex $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \in D(\mathbf{Z}_p)$ is perfect.*
2. Étale realization: *There is a natural symmetric monoidal "étale realization" functor*

$$T : F\text{-}Gauge_{\Delta}(X)_{\mathrm{perf}} \to \mathcal{D}^b_{cons}(\mathrm{Spec}(\mathbf{Q}_p), \mathbf{Z}_p) \simeq \mathcal{D}^b_{\mathrm{fd,cts}}(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), \mathbf{Z}_p)$$

with the property that each cohomology group of $T(E)[1/p]$ is a crystalline representation. Moreover, we have a natural isomorphism $T(\mathcal{O}\{1\}) = \mathbf{Z}_p(1)$.

3. Lagrangian refinement of Tate duality: *For any $E \in F\text{-}Gauge_{\Delta}(X)_{\mathrm{perf}}$, write E^* for the $\mathcal{O}$-linear dual. Then there is a natural fibre sequence*

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \to R\Gamma(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), T(E)) \to \mathrm{RHom}_{\mathbf{Z}_p}(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^*\{1\}[2]), \mathbf{Z}_p),$$

where the first map is induced by the functor T for E, while the second map is induced by T for E^* as well as the classical Tate duality isomorphism

$$R\Gamma(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), T(E)) \simeq \mathrm{RHom}_{\mathbf{Z}_p}(R\Gamma(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), T(E^*)(1)[2]), \mathbf{Z}_p).$$

The informal heuristic here is that prismatic F-gauges on $\mathrm{Spf}(\mathbf{Z}_p)$ give a reasonable definition of "crystalline objects"[3] of the derived category of Galois representations, and moreover that F-gauge cohomology $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)$ can be seen as the "crystalline part"[4] of the Galois cohomology $R\Gamma(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), T(E))$. To help demystify these objects, let us explain Theorem 1.3.1 (3) by writing out all the terms in in an example (assuming some blackboxes).

Example 1.3.2. Say $E = \mathcal{O}\{1\}$, whence $E^*\{1\} = \mathcal{O}$. As we shall later see, one has

$$R\Gamma(\mathcal{O}\{1\}) = R\Gamma_{fl}(\mathrm{Spf}(\mathbf{Z}_p), \mathbf{Z}_p(1)) = \lim_n R\Gamma_{fl}(\mathrm{Spf}(\mathbf{Z}_p), \mu_{p^n})$$

and

$$R\Gamma(\mathcal{O}) = R\Gamma_{et}(\mathrm{Spf}(\mathbf{Z}_p), \mathbf{Z}_p) \simeq R\Gamma_{et}(\mathrm{Spec}(\mathbf{F}_p), \mathbf{Z}_p),$$

both for general reasons (i.e., valid for any X). Specializing to our context, using Kummer theory, one can compute that

$$H^i(\mathcal{O}\{1\}) = H^i_{fl}(\mathrm{Spf}(\mathbf{Z}_p), \mathbf{Z}_p(1)) = \begin{cases} 0 & \text{if } i = 0 \\ \text{the } p\text{-adic completion } (\mathbf{Z}_p^*)^\wedge & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.3.1}$$

and

$$H^i(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), \mathbf{Z}_p(1)) \begin{cases} 0 & \text{if } i = 0 \\ \text{the } p\text{-adic completion } (\mathbf{Q}_p^*)^\wedge & \text{if } i = 1 \\ \mathbf{Z}_p (\text{via the fundamental class}) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.2}$$

[3] The étale realization functor T is not fully faithful, so this terminology is a perhaps a bit misleading: being "crystalline" is now additional structure.

[4] With rational coefficients, such a notion was defined at the level of cohomology groups by Bloch–Kato [BK07, §3]: for a crystalline $\mathbf{Q}_p$-representation V of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, they defined a subspace $H^1_f(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), V) \subset H^1(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), V)$ using the period ring B_{crys}, and proved an analog of Theorem 1.3.1 (3). Time permitting, we shall explain how this compares to $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)$ when $V = T(E)$.

Writing $\pi = \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)^\wedge$ for the pro-p-completion (which is isomorphic to $\mathbf{Z}_p$ as $\mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p) \simeq \widehat{\mathbf{Z}}$ via the Frobenius), one then computes that

$$H^i(\mathcal{O}) = H^i_{et}(\mathrm{Spec}(\mathbf{F}_p), \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p & \text{if } i = 0 \\ \pi^\vee := \mathrm{Hom}(\pi, \mathbf{Z}_p) & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.3.3}$$

The exact triangle in Theorem 1.3.1 (3) then induces an exact sequence in degree 1 given by

$$0 \to (\mathbf{Z}_p^*)^\wedge \to (\mathbf{Q}_p^*)^\wedge \to \pi \to 0.$$

This sequence is in fact simply isomorphic to the p-adic completion of the exact sequence defined by the valuation map $v : \mathbf{Q}_p^* \to \mathbf{Z}$. Moreover, thinking about the geometric meaning of the terms in this sequence helps explain the heuristic mentioned above in this example: indeed, the subspace $(\mathbf{Z}_p^*)^\wedge \subset (\mathbf{Q}_p^*)^\wedge$ exactly parametrizes those $\mathbf{Z}_p(1)$-torsors on $\mathrm{Spec}(\mathbf{Q}_p)$ that are crystalline (i.e., have good reduction or equivalently spread out to $\mathbf{Z}_p(1)$-torsors on $\mathrm{Spf}(\mathbf{Z}_p)$), while the dual $\pi^\vee$ of the cokernel $\pi \simeq (\mathbf{Q}_p^*/\mathbf{Z}_p^*)^\wedge$ can be viewed (via Tate duality) as the subspace of $H^1(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), \mathbf{Z}_p(1))^\vee \simeq H^1(\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p), \mathbf{Z}_p)$ parametrizing those $\mathbf{Z}_p$-torsors on $\mathrm{Spec}(\mathbf{Q}_p)$ that are crystalline in the same sense.

Besides the above, we hope to discuss certain other results from the perspective of F-gauges (and closely related notions such as prismatic crystals), including the following:

- Mazur's theorem on Newton-above-Hodge.
- Artin-Milne duality theorem for the p-adic Picard group of a surface over a finite field.
- Drinfeld's refinement of the Deligne–Illusie theorem

1.4 Strategy: geometrization[5]

Fix a p-adic formal scheme X which is quasi-syntomic[6]. We shall construct the category $\text{F-Gauge}_{\Delta}(X)_{\mathrm{perf}}$ via geometrization, i.e., as perfect complexes on a certain (close to algebraic) stack X^{Syn} attached to X. More precisely, we shall attach the following stacks and categories to X:

- X^{Δ} - the *prismatization of* X. The ∞-category of *prismatic crystals* is $\mathrm{Crys}_{\Delta}(X) := \mathcal{D}_{qc}(X^{\Delta})$.
- $X^{\mathcal{N}}$ - the *(Nygaard) filtered prismatization of* X. The ∞-category of *prismatic gauges* is $\mathrm{Gauge}_{\Delta}(X) := \mathcal{D}_{qc}(X^{\mathcal{N}})$.

The stack $X^{\mathcal{N}}$ contains two open substacks isomorphic to X^{Δ}. Consequently, we can define:

- X^{Syn} - the *syntomification of* X, obtained by glueing together the two copies of X^{Δ} inside $X^{\mathcal{N}}$. The ∞-category of *prismatic F-gauges* is defined as $\text{F-Gauge}_{\Delta}(X) := \mathcal{D}_{qc}(X^{\mathrm{Syn}})$.

[5]We shall use this word in this course to refer to the process of translating problems of interest into algebraic geometric terms, e.g., realizing a category of interest in terms of quasi-coherent sheaves on a variety or a stack.

[6]This is a condition on singularities of X ensuring that the cotangent complex has good properties. For noetherian X's, it amounts to the lci condition. While it is possible to develop the theory without this hypothesis at the expense of using derived algebraic geometry more fully, we shall avoid doing so in these lectures in the interest of simplicity.

The subcategory of *perfect prismatic F-gauges* inside $\text{F-Gauge}_{\Delta}(X)$ is defined as the subcategory of perfect complexes, i.e., $\text{F-Gauge}_{\Delta}(X)_{\text{perf}} := \text{Perf}(X^{\text{Syn}})$, and similary for prismatic gauges and crystals. The glueing description of X^{Syn} shows that

$$\text{F-Gauge}_{\Delta}(X) \simeq \text{Eq}(\text{Gauge}_{\Delta}(X) \rightrightarrows \text{Crys}_{\Delta}(X)) \quad \text{and} \quad \text{F-Gauge}_{\Delta}(X)_{\text{perf}} \simeq \text{Eq}(\text{Gauge}_{\Delta}(X)_{\text{perf}} \rightrightarrows \text{Crys}_{\Delta}(X)_{\text{perf}})$$

where the two maps are restriction to each copy of X^{Δ} inside $X^{\mathcal{N}}$.

Remark 1.4.1 (Relation to the prismatic site)**.** The stacks described above are closely related to the absolute prismatic site of X [BS19]. In fact, the ∞-categories described above often admit direct descriptions in terms of the prismatic site. For example, the category $\text{Crys}_{\Delta}(X)$ is identified [BL22b] with the ∞-category of crystals on absolute prismatic site of X; similarly, $\text{Gauge}_{\Delta}(X)$ can be identified with the ∞-category of filtered modules over the filtered Nygaard complex for certain "very ramified" X's (to be explained, [BL]). Nonetheless, the stacky perspective will be important to understanding their finer structure when X is not very ramified (e.g., for $X = \text{Spf}(\mathbf{Z}_p)$).

Remark 1.4.2 (Relative theory)**.** The constructions $X \mapsto X^{\Delta}, X^{\mathcal{N}}, X^{\text{Syn}}$ as well as the natural maps relating them are covariantly functorial in X. Consequently, if $f : X \to Y$ is a map of p-adic formal schemes, one obtains maps $f^{\text{Syn}} : X^{\text{Syn}} \to Y^{\text{Syn}}$, etc. These maps give pushforward/pullback stability for prismatic (F-)gauges and crystals, compatibly with the identifications in Remark 1.4.1.

Remark 1.4.3 (Relation to syntomic cohomology)**.** For each $i \in \mathbf{Z}$, the complex $R\Gamma_{\text{Syn}}(X, \mathbf{Z}_p(i)) := R\Gamma(X^{\text{Syn}}, \mathcal{O}\{i\})$ can be identified with the syntomic cohomology[7] of X; this is the reason X^{Syn} is called the syntomification of X. Syntomic cohomology is a form of p-adic étale motivic cohomology of X. For example, $R\Gamma_{\text{Syn}}(X, \mathbf{Z}_p(1)) \simeq R\Gamma_{fl}(X, \mathbf{G}_m)^\wedge[-1]$, where the completion is p-adic. More generally, there is a natural filtration on the étale K-theory of X with graded pieces given by syntomic cohomology, analogous to how algebraic K-theory is filtered by motivic cohomology (see [CMM21, Mat21, BMS19]).

The stacks discussed above will be quite close to algebraic and thus quite amenable to methods of algebraic geometry. To give a flavour of what we will study in more depth later, we describe these stacks in the simplest example.

Example 1.4.4 (The syntomification of a perfect field of characteristic p)**.** For $X = \text{Spec}(k)$ with k a perfect field of characteristic p, these objects are described as follows:

- $X^{\Delta} = \text{Spf}(W(k))$.
- $X^{\mathcal{N}} = \text{Spf}(W(k)[u,t]/(ut - p))/\mathbf{G}_m$. Note that the open substacks $X^{\Delta}_{\mathcal{N}, u\neq 0}$ and $X^{\Delta}_{\mathcal{N}, t\neq 0}$ can each identified with $\text{Spf}(W(k))$ via the projection $X^{\mathcal{N}} \to X^{\Delta}$.
- X^{Syn} is obtained by glueing the two copies of $X^{\Delta} \subset X^{\Delta}_{\mathcal{N}}$ found in the previous line using the Frobenius on $W(k)$. More precisely, X^{Syn} sits in a pushout square

$$\begin{tikzcd} X^{\Delta}_{\mathcal{N}, u\neq 0} \sqcup X^{\Delta}_{\mathcal{N}, t\neq 0} \arrow[r] \arrow[d, "\varphi^{-1}"', "\text{id}"] & X^{\mathcal{N}} \arrow[d] \\ X^{\Delta} \arrow[r] & X^{\text{Syn}}, \end{tikzcd}$$

[7]Syntomic cohomology was defined by Fontaine–Messing [FM87], in certain situations, for the purpose of proving p-adic comparison theorems. Following a large body of work, a reasonable definition for general X was recently obtained via prismatic cohomology [BMS19, BS19]; its specialization to characteristic p was already known to Kato [Kat19], and imporant special cases were also known via motivic methods [Sch94, Gei04, Sat07].

where the horizontal maps are open immersions, and the vertical map are surjective local isomorphisms. In particular, due to the Frobenius involved in the glueing procedure, the stack X^{Syn} is only $\mathbf{Z}_p$-linear and not $W(k)$-linear, unlike the rest of the terms in the pushout diagram. This reflects the fact that syntomic cohomology of X is only $\mathbf{Z}_p$-linear. In fact, as we shall see later and was hinted at in Example 1.3.2, we have $R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p) \simeq R\Gamma_{et}(X, \mathbf{Z}_p)$ for general reasons; it is a pleasant exercise to verify this by computing cohomology on the stack X^{Syn} described above.

In the lectures, we will first (slowly) explain the construction of X^{Syn} and cousins via their moduli interpretations when X has characteristic p, drawing motivation from known structures on de Rham cohomology. The eventual description will make the extension to mixed characteristic relatively straightforward. We shall then explain the relation to prismatic cohomology and the Nygaard filtration, and then zero in on $\mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}}) \simeq \text{F-Gauge}_{\Delta}(\mathbf{Z}_p)$.

1.5 Credits

While we have made no attempt to systematically give references, let us at least indicate some of the original sources for the material we will cover:

- Crystalline cohomology: [Gro68, Ber06]
- F-gauges in characteristic p: [FJ13]
- Syntomic cohomology:
  - Mixed characteristic, in low dimension and Hodge–Tate weights or up to isogeny: [FM87, K+87, KM92]
  - Characteristic p: [Kat19]
  - Mixed characteristic in general: [BMS19, BS19, BL22a, BM22]
- Prismatic cohomology: [BMS18, BS19]
- Prismatic F-crystals and F-gauges: [Dri20, Dri21a, BL22a, BL22b, BS21]

In particular, the filtered prismatization $X^{\mathcal{N}}$ and thus the syntomification X^{Syn} was first constructed by Drinfeld [Dri20] via a modular description. These lectures will rely on Drinfeld's paper as well as the forthcoming [BL].

Remark 1.5.1 (A comment on notation)**.** The paper [Dri20] studies the stacks X^{Δ}, $X^{\mathcal{N}}$ and X^{Syn} under the names X^{Δ}, $X^{\Delta'}$ and $X^{\Delta''}$ (as well as Σ, Σ' and Σ'' when $X = \mathrm{Spf}(\mathbf{Z}_p)$). The stack X^{Δ} was also studied independently in [BL22a, BL22b] under the name WCart_X. We apologize for introducing yet another naming scheme. The present choice of notation reflects our belief that the notation should be compact and yet reflect what the stacks are designed to capture: X^{Δ}, $X^{\mathcal{N}}$ and X^{Syn} capture the prismatic, Nygaard filtered prismatic, and syntomic cohomology of X respectively.

Chapter 2

Algebraic de Rham cohomology

Prismatic cohomology can be roughly regarded as a deformation of de Rham cohomology. Consequently, many structures on prismatic cohomology have counterparts for de Rham cohomology that are more classical and more explicit. In this chapter, we will explain how to interpret some of those via stacks.

2.1 Review

In this section, we review the theory of relative algebraic de Rham cohomology for a smooth morphism of schemes.

Definition 2.1.1 (Algebraic de Rham and Hodge complexes)**.** Let $f : X \to S$ be a morphism of schemes. The complex

$$\Omega^\bullet_{X/S} := \left(\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} ...\right)$$

of $f^{-1}\mathcal{O}_S$-modules on X is called the *algebraic de Rham complex* of X/S. Similarly, the complex

$$\Omega^H_{X/S} := \left(\mathcal{O}_X \xrightarrow{0} \Omega^1_{X/S} \xrightarrow{0} \Omega^2_{X/S} \xrightarrow{0} ...\right) = \bigoplus_i \Omega^i_{X/S}[-i]$$

is called the *Hodge complex* of X/S; this is a graded complex of $\mathcal{O}_X$-modules.

We shall be interested in $Rf_*\Omega^\bullet_{X/S}$ and $Rf_*\Omega^H_{X/S}$; assume from now that f is finitely presented, so these are quasi-coherent complexes on S, i.e., they lie in $\mathcal{D}_{qc}(S)$. Exterior product of differential forms turns $\Omega^\bullet_{X/S}$ into a commutative differential graded $f^{-1}\mathcal{O}_S$-algebra, while $\Omega^H_{X/S}$ is a graded $\mathcal{O}_X$-algebra via the same construction. As Rf_* is lax symmetric monoidal, it follows that $Rf_*\Omega^\bullet_{X/S}$ and $Rf_*\Omega^H_{X/S}$ give commutative algebra objects in $\mathcal{D}_{qc}(S)$. Let us recall some important features of this construction.

1. The Hodge filtration: Forgetting terms of the complex defines a complete descending $\mathbf{N}$-indexed filtration Fil^*_H on $\Omega^\bullet_{X/S}$ with associated graded $\Omega^H_{X/S}$. This filtration is multiplicative: the product carries $\mathrm{Fil}^i \otimes \mathrm{Fil}^j$ into Fil^{i+j}. Pushing down to S, we learn:

   Proposition 2.1.2. *The object $Rf_*\Omega^\bullet_{X/S} \in \mathcal{D}_{qc}(S)$ admits a natural complete descending multiplicative $\mathbf{N}$-indexed Hodge filtration Fil^*_H together with an isomorphism $\mathrm{gr}^*_H Rf_*\Omega^\bullet_{X/S} \simeq Rf_*\Omega^H_{X/S}$.*

2. Grothendieck's comparison [Gro66]: If $S = \mathrm{Spec}(\mathbf{C})$ and X is smooth over $\mathbf{C}$, then analytification yields an isomorphism

$$H^i(X, \Omega^\bullet_{X/\mathbf{C}}) \simeq H^i_{dR}(X^{an}; \mathbf{C}) \simeq H^i_{Sing}(X^{an}, \mathbf{C}).$$

In particular, each $H^i(X, \Omega^\bullet_{X/\mathbf{C}})$ is a finite dimensional $\mathbf{C}$-vector space; it vanishes for $i \notin [0, 2\dim(X)]$; and one has Poincare duality when X is additionally proper. Let us check this in an interesting example by hand:

Example 2.1.3 (The case of a torus)**.** Say $X = \mathbf{G}_m = \mathrm{Spec}(\mathbf{C}[t^{\pm 1}])$. Then

$$\begin{aligned} H^i(X, \Omega^\bullet_{X/\mathbf{C}}) &= H^i\left(\mathbf{C}[t^{\pm 1}] \xrightarrow{d} \mathbf{C}[t^{\pm 1}]\frac{dt}{t}\right) \\ &= H^i\left(\bigoplus_{i \in \mathbf{Z}} \mathbf{C}t^i \xrightarrow{t^i \mapsto it^i\frac{dt}{t}} \bigoplus_{i \in \mathbf{Z}} \mathbf{C}t^i\frac{dt}{t}\right) \\ &= H^i\left(\mathbf{C}t^0 \xrightarrow{0} \mathbf{C}t^0\frac{dt}{t}\right) \\ &= \begin{cases} \mathbf{C} & \text{if } i = 0 \text{ or } i = 1 \\ 0 & \text{otherwise.} \end{cases} \\ &= H^i_{Sing}(\mathbf{C} - \{0\}; \mathbf{C}), \end{aligned}$$

as wanted.

3. Poincaré duality (see [Sta18, Tag 0G8F], [Cla21]): If $f : X \to S$ is proper and smooth, then $Rf_*\Omega^i_{X/S} \in \mathrm{Perf}(S)$ for all i, whence $Rf_*\Omega^\bullet_{X/S} \in \mathrm{Perf}(S)$. Moreover, if f has relative dimension n, then there is a trace map $t_{X/S} : R^{2n}f_*\Omega^\bullet_{X/S} \to \mathcal{O}_S$ such that the induced pairing

$$Rf_*\Omega^\bullet_{X/S} \otimes Rf_*\Omega^\bullet_{X/S} \xrightarrow{\text{multiply}} Rf_*\Omega^\bullet_{X/S} \xrightarrow{\text{can}} R^{2n}f_*\Omega_{X/S}[-2n] \xrightarrow{t_{X/S}} \mathcal{O}_S[-2n]$$

is a perfect pairing, i.e., it induces an isomorphism

$$Rf_*\Omega^\bullet_{X/S} \simeq \underline{\mathrm{RHom}}_S(Rf_*\Omega^\bullet_{X/S}[2n], \mathcal{O}_S)$$

in $\mathrm{Perf}(S)$.

Remark 2.1.4 (Hodge filtered Poincaré duality)**.** For a filtered object E, write $E\{n\}$ for the result of shifting the filtration by n, i.e., $\mathrm{Fil}^i(E\{n\}) = \mathrm{Fil}^{i+n}(E)$. The construction of the trace map $t_{X/S}$ in [Sta18, Tag 0G8F], relying on [Sta18, Tag 0G8J], shows that the map

$$Rf_*\omega_{X/S}[-n] = \mathrm{Fil}^n_H Rf_*\Omega^\bullet_{X/S} \to Rf_*\Omega^\bullet_{X/S}$$

induces an isomorphism on $\mathcal{H}^{2n}$, and that the resulting map $\mathcal{H}^n(Rf_*\omega_{X/S}) \simeq \mathcal{H}^{2n}(Rf_*\Omega^\bullet_{X/S}) \to \mathcal{O}_S$ coincides with the trace map from Grothendieck duality up to a unit[8]. It follows that the same recipe used above refines to give a filtered isomorphism

$$Rf_*\Omega^\bullet_{X/S} \simeq \underline{\mathrm{RHom}}_S(Rf_*\Omega^\bullet_{X/S}\{n\}[2n], \mathcal{O}_S),$$

[8] It would be useful to pin down this ambiguity via the cycle class map

whose associated graded gives an isomorphism

$$Rf_*\Omega^H_{X/S} \simeq \underline{\mathrm{RHom}}_S(Rf_*\Omega^H_{X/S}\{n\}[2n], \mathcal{O}_S)$$

of graded objects (where the twist now indicates a shift of grading); in degree i, the above coincides (up to units) with the isomorphism

$$Rf_*\Omega^i_{X/S} \simeq \underline{\mathrm{RHom}}_S(Rf_*\Omega^{n-i}_{X/S}[n+i], \mathcal{O}_S),$$

induced by cup product of differential forms and Grothendieck duality for f.

4. Cartier theory: Say S has characteristic p, and $f : X \to S$ is smooth. Recall that we have following fundamental diagram defining relative Frobenius $F_{X/S} : X \to X^{(1)}$:

$$\begin{tikzcd}
X \arrow[dr, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "f"'] & & \\
& X^{(1)} \arrow[r] \arrow[d, "f^{(1)}"] & X \arrow[d, "f"] \\
& S \arrow[r, "F_S"] & S.
\end{tikzcd}$$

The map $F_{X/S} : X \to X^{(1)}$ is a universal homeomorphism, and the differential of the complex $F_{X/S,*}\Omega^\bullet_{X/S}$ is naturally $\mathcal{O}_{X^{(1)}}$-linear: for local sections $f, g \in \mathcal{O}_X$ and $s \in \mathcal{O}_S$, we have $d(f^p g) = f^p d(g)$ and $d(sg) = sd(g)$. Consequently, we can regard $F_{X/S,*}\Omega^\bullet_{X/S}$ as a coherent $\mathcal{O}_{X^{(1)}}$-complex. When regarded as such, there is a natural isomorphism $\mathcal{H}^*(F_{X/S,*}\Omega^\bullet_{X/S}) \simeq \Omega^*_{X^{(1)}/S}$ of graded rings; in particular, $F_{X/S,*}\Omega^\bullet_{X/S} \in \mathrm{Perf}(X^{(1)})$. Pushing forward down to S, we learn the following:

Proposition 2.1.5. *The complex $Rf_*\Omega^\bullet_{X/S} \in \mathcal{D}_{qc}(S)$ admits a natural multiplicative increasing exhaustive $\mathbf{N}$-indexed* conjugate filtration Fil^{conj}_* *together with an isomorphism* $\mathrm{gr}^{conj}_* Rf_*\Omega^\bullet_{X/S} \simeq \bigoplus_i Rf^{(1)}_*\Omega^i_{X^{(1)}/S}[-i]$.

Let us explain the conjugate filtration in a key example.

Example 2.1.6 (The case of a torus)**.** Say $S = \mathrm{Spec}(\mathbf{F}_p)$ and $X = \mathbf{G}_m = \mathrm{Spec}(\mathbf{F}_p[t^{\pm 1}])$. To avoid confusion, let $Y = \mathbf{G}_m = \mathrm{Spec}(\mathbf{F}_p[y^{\pm 1}])$, and let $f : X \to Y$ be determined by $y \mapsto t^p$, so f can be identified with the relative Frobenius for X/S. In this case, the Cartier isomorphism asserts that $H^i(\Omega^\bullet_{\mathbf{F}_p[t^{\pm 1}]/\mathbf{F}_p})$ is canonically identified with $\Omega^i_{\mathbf{F}_p[y^{\pm 1}]/\mathbf{F}_p}$.

$$\begin{aligned}
H^i(\Omega^\bullet_{\mathbf{F}_p[t^{\pm 1}]/\mathbf{F}_p}) &= H^i\left(\mathbf{F}_p[t^{\pm 1}] \xrightarrow{d} \mathbf{F}_p[t^{\pm 1}]\frac{dt}{t}\right) \\
&= H^i\left(\bigoplus_{i\in\mathbf{Z}} \mathbf{F}_p t^i \xrightarrow{t^i \mapsto it^i\frac{dt}{t}} \bigoplus_{i\in\mathbf{Z}} \mathbf{F}_p t^i \frac{dt}{t}\right) \\
&\simeq H^i\left(\bigoplus_{i\in p\mathbf{Z}} \mathbf{F}_p t^i \xrightarrow{0} \bigoplus_{i\in p\mathbf{Z}} \mathbf{F}_p t^i \frac{dt}{t}\right) \\
&\simeq H^i\left(\mathbf{F}_p[y^{\pm 1}] \xrightarrow{0} \text{“}\frac{1}{p}\text{”} \cdot \mathbf{F}_p[y^{\pm 1}]\frac{dy}{y}\right) \\
&\simeq \Omega^i_{\mathbf{F}_p[y^{\pm 1}]/\mathbf{F}_p},
\end{aligned}$$

as wanted.

Remark 2.1.7 (Conjugate filtered Poincare duality)**.** The analog of the assertion in Remark 2.1.4 also applies to the conjugate filtration and is in fact easier: the desired filtered trace map is defined via the composition $Rf_*\Omega^\bullet_{X/S} \to Rf_*\omega_{X^{(1)}/S}[-n] \to \mathcal{O}_S[-2n]$, where the first map is projecting to gr^{conj}_n.

5. Existence of crystalline cohomology: Say $f : X \to S$ is a finitely presented smooth map and S has characteristic p. Assume that S is given as the mod p reduction of a $\mathbf{Z}_p$-flat p-adic formal scheme $\widetilde{S}$. Then the formalism of crystalline cohomology [Ber06] shows the following:
   - The complex $Rf_*\Omega^\bullet_{X/S} \in \mathcal{D}_{qc}(S)$ has a functorial lift $E_{X/\widetilde{S}}$ to $\mathcal{D}_{qc}(\widetilde{S})$ given by the crystalline cohomology of $X/\widetilde{S}$. The formation of $E_{X/\widetilde{S}}$ is compatible with base change on $\widetilde{S}$.
   - If $f : X \to S$ admits a (necessarily smooth) lift $\widetilde{f} : \widetilde{X} \to \widetilde{S}$ to a map of $\mathbf{Z}_p$-flat p-adic formal schemes, then there is a natural identification $E_{X/\widetilde{S}} \simeq Rf_*\Omega^\bullet_{\widetilde{X}/\widetilde{S}}$. In particular, the latter is independent of choice of lift $\widetilde{f}$.

   We refer to [BdJ11] for a relatively quick proof of the comparison with de Rham cohomology. A typical use of this formalism is the following consequence:

   Corollary 2.1.8. *Say k is a perfect field of characteristic p and $X/W(k)$ is a proper smooth scheme. Then the object $M := R\Gamma(X, \Omega^\bullet_{X/W(k)}) \in \mathcal{D}(W(k))$ depends only on mod p fibre $Y := X \otimes_{W(k)} k$. Moreover, there is a natural map $\phi_{X/W(k)} : \phi^* M \to M$ (where $\phi : W(k) \to W(k)$ is the Frobenius) lifting the map $F_k^* R\Gamma(Y, \Omega^\bullet_{Y/k}) \simeq R\Gamma(Y^{(1)}, \Omega^\bullet_{Y^{(1)}/k}) \xrightarrow{F^*_{Y/k}} R\Gamma(Y, \Omega^\bullet_{Y/k})$.*

   We warn the reader that the lift $\phi_{X/W(k)}$ does *not* respect the Hodge filtration in general.

2.2 Linear algebra via stacks: filtrations and endomorphisms

In this section, we recall two geometric (stacky) perspectives of objects in linear algebra (namely, filtrations and endomorphisms).

2.2.1 Filtrations and $\mathbf{A}^1/\mathbf{G}_m$

Fix a commutative ring R. Our goal in this section is to review[9] a standard dictionary between filtrations on R-modules and the stack $\mathbf{A}^1/\mathbf{G}_m$. As a warmup, let us recall the analog of this dictionary for graded objects.

Construction 2.2.1 (Graded objects of the derived category)**.** The derived ∞-category of graded R-complexes is defined as $\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R))$, where $\mathbf{Z}$ is viewed as a discrete category. This has a symmetric monoidal structure given by Day convolution:

$$(F \otimes G)(n) = \operatorname*{colim}_{i+j=n} F(i) \otimes G(j) = \bigoplus_{i+j=n} F(i) \otimes G(j).$$

Recall that the stack $B\mathbf{G}_m$ classifies line bundles on R-schemes; in particular, it carries a tautological line bundle $\mathcal{O}_{B\mathbf{G}_m}(1)$. There is a symmetric monoidal R-linear equivalence

$$\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R)) \simeq \mathcal{D}_{qc}(B\mathbf{G}_m), \quad F \mapsto \bigoplus_{i \in \mathbf{Z}} F(i) \otimes_R \mathcal{O}(-i),$$

[9] For proofs, we refer [GP18, Mou21, MRT19, BMS19].

with inverse described as $M \mapsto (i \mapsto R\Gamma(B\mathbf{G}_{m,R}, M(i)))$ (so $\mathcal{O}(1)$ corresponds to the graded R-module R placed in grading degree -1). Under this dictionary, the functor of forgetting the grading on the left side (i.e., taking the colimit) corresponds[10] to pullback along the tautological map $\pi : \mathrm{Spec}(R) \to B\mathbf{G}_m$

Next, let us recall some basic formalism around the filtered derived category (parametrizing decreasing filtrations) mostly to introduce notation.

Construction 2.2.2 (The filtered derived ∞-category)**.** For a commutative ring R, we define the *filtered derived ∞-category* of R as $\mathcal{DF}(R) = \mathrm{Fun}(\mathbf{Z}^{op}, \mathcal{D}(R))$, where $\mathbf{Z}$ denotes the poset of integers with usual ordering. Let us name name some structures associated to this construction:

1. Underlying and assocaited graded objects: Given a filtered object $F : \mathbf{Z}^{op} \to \mathcal{D}(R)$, we can attach the following objects of $\mathcal{D}(R)$:
   - $\underline{F} = \mathrm{colim}_i F(i)$; this is called the *underlying R-complex.*
   - $\mathrm{Fil}^i \underline{F} = F(i)$ and $\mathrm{gr}^i_{\mathrm{Fil}} \underline{F} = \mathrm{Cone}(\mathrm{Fil}^{i+1}\underline{F} \to \mathrm{Fil}^i \underline{F})$. The construction
   $$F \mapsto \mathrm{gr}^*_{\mathrm{Fil}} \underline{F} := \bigoplus_i \mathrm{gr}^i_{\mathrm{Fil}} \underline{F}$$
   gives an exact colimit preserving functor to the ∞-category of graded objects in $\mathcal{D}(R)$.

   Thus, we have a diagram
   $$... \to \mathrm{Fil}^{i+1}\underline{F} \to \mathrm{Fil}^i\underline{F} \to \mathrm{Fil}^{i-1}\underline{F} \to ...$$
   with colimit $\underline{F}$, so we can roughly regard $\mathrm{Fil}^\bullet \underline{F}$ as a filtration on $\underline{F}$ with graded pieces $\mathrm{gr}^*_{\mathrm{Fil}}\underline{F}$.
2. Completeness: Given a filtered object $F : \mathbf{Z}^{op} \to \mathcal{D}(R)$, we say that F is *complete* if $\lim_i \mathrm{Fil}^i \underline{F} = 0$. For any filtered object F, there is a universal map $F \to \widehat{F}$ to a complete filtered object computed via the formula $\widehat{F} = \mathrm{Cone}(\mathrm{Const}(\lim_i \mathrm{Fil}^i \underline{F}) \to F)$, where $\mathrm{Const}(-)$ denotes the functor of regarding an R-complex M as a filtered R-complex with constant filtration. Write $\widehat{\mathcal{DF}}(R) \subset \mathcal{DF}(R)$ for the full subcategory of complete objects.

   Example 2.2.3 (The canonical filtration)**.** Any $K \in \mathcal{D}(R)$ has a preferred lift $\widetilde{K} \in \mathcal{DF}(R)$ defined by the canonical filtration: $\mathrm{Fil}^i \widetilde{K} = \tau^{\leq -i} K$. Note that $\mathrm{gr}^i \widetilde{K} = H^{-i}(K)$. The construction $K \mapsto \widetilde{K}$ gives a fully faithful embedding $\mathcal{D}(R) \to \mathcal{DF}(R)$ whose essential image is exactly those $F \in \mathcal{DF}(R)$ with the property that $\mathrm{gr}^i_{\mathrm{Fil}} \underline{F}$ is concentrated in cohomological degree $-i$ and such that F is complete; this statement can be proven using the formalism of t-structures.

   Example 2.2.4 (The stupid filtration)**.** Any actual chain complex $K^\bullet$ of R-modules has a naive filtration given by forgetting terms of the complex: $\mathrm{Fil}^i K^\bullet = K^{\geq i}$. Passing to the derived category, this yields a complete filtration on $K \in \mathcal{D}(R)$ with associated graded given by $\mathrm{gr}^i_{\mathrm{Fil}} \simeq K^i[-i]$. In fact, this construction yields a fully faithful functor $\mathrm{Ch}(R) \to \mathcal{DF}(R)$ from the *abelian* category of chain complexes of R-modules to $\mathcal{DF}(R)$ which is exact in the sense that it carries exact sequences to exact triangles. Moreover, the essential image is exactly those objects of $\mathcal{DF}(R)$ which are complete and have gr^i concentrated in cohomological degree i; this statement is one the basic features of the Beilinson t-structure (see (5) below).

[10] We caution the reader that the formation of limits in $\mathcal{D}_{qc}(B\mathbf{G}_m)$ does not commute with $\pi^* : \mathcal{D}_{qc}(B\mathbf{G}_m) \to \mathcal{D}(R)$; for instance, the natural map
$$\bigoplus_{i \in \mathbf{Z}} \mathcal{O}(i) \to \prod_{i \in \mathbf{Z}} \mathcal{O}(i)$$
is actually an equivalence on $B\mathbf{G}_m$. This will turn out to be a feature rather than a bug for our later purposes.

3. Symmetric monoidal structure: There is a natural symmetric monoidal structure on $\mathcal{DF}(R)$ defined via Day convolution from the commutative monoid structure on $\mathbf{Z}$. Explicitly,

$$(F \otimes G)(n) = \operatorname*{colim}_{i+j \geq n} F(i) \otimes G(j).$$

The associated graded functor gr^* mentioned in (1) i symmetric monoidal, i.e., there is a natural identification

$$\mathrm{gr}^*_{\mathrm{Fil}}\underline{F} \otimes \mathrm{gr}^*_{\mathrm{Fil}}\underline{G} \simeq \mathrm{gr}^*_{\mathrm{Fil}}\underline{F \otimes G}$$

of graded objects of $\mathcal{D}(R)$.

4. Shifting the filtration: Given a filtered object F and an integer n, one obtains a new filtered object $F\{n\}$ by shifting the filtration by n, normalized as follows: $\mathrm{Fil}^i\underline{F\{n\}} = \mathrm{Fil}^{i+n}\underline{F}$. Thus, for any filtered object F, there is a natural map $F\{1\} \to F$ with the property that $\mathrm{Fil}^0(F/F\{1\}) = \mathrm{gr}^0 F$.

5. t-structures: There are two t-structures on $\mathcal{DF}(R)$ that will be important for us:
   - The standard t-structure: This is the t-structure induced from the usual t-structure on $\mathcal{D}(R)$: an object $F \in \mathcal{DF}(R)$ is connective[11] (resp. coconnective) if each $\mathrm{Fil}^i\underline{F}$ is so. The heart of this t-structure is the abelian category $\mathrm{Fun}(\mathbf{Z}^{op}, \mathrm{Mod}_R)$ of $\mathbf{Z}^{op}$-indexed diagrams of R-modules.
   - The Beilinson t-structure: An object $F \in \mathcal{DF}(R)$ is connective for the Beilinson t-structure if $\mathrm{gr}^i_{\mathrm{Fil}}\underline{F} \in \mathcal{D}^{\leq i}$ for all i. It turns out that the the coconnective F's are exactly those with $\mathrm{Fil}^i\underline{F} \in D^{\geq i}$ for all i; if F is already complete, then this is equivalent to $\mathrm{gr}^i\underline{F} \in D^{\geq i}$ for all i. The heart of this t-structure is the abelian category of chain complexes via the embedding of Example 2.2.4. See [Bei87, Appendix A], [BMS19, §5], [BL22a, Appendix D] for more.

Our goal is to describe this ∞-category geometrically as quasi-coherent sheaves on the quotient stack $\mathbf{A}^1/\mathbf{G}_m$ (with respect to the standard action). To avoid ambiguity, let us specify the grading convention: we write $\mathbf{A}^1 = \mathrm{Spec}(R[t])$, with $\mathbf{G}_m$-action giving t grading degree 1. Thus, we can identify $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ with $\mathcal{D}_{\mathrm{graded}}(R[t])$. To proceed further, it will be useful to think about $\mathbf{A}^1/\mathbf{G}_m$ functorially and name some important structures it carries.

Construction 2.2.5 ($\mathbf{A}^1/\mathbf{G}_m$ and its tautological line bundle)**.** Given a scheme T, the groupoid $\mathbf{A}^1/\mathbf{G}_m(T)$ is identified with the groupoid of $\mathbf{G}_m$-torsors $T' \to T$ together with a $\mathbf{G}_m$-equivariant map $T' \to \mathbf{A}^1$. As a $\mathbf{G}_m$-torsor must uniquely have the form $\mathrm{Spec}(\bigoplus_{i \in \mathbf{Z}} L^{-i})$ for a line bundle L on T, the groupoid $\mathbf{A}^1/\mathbf{G}_m(T)$ identifies with the groupoid of virtual Cartier divisors on T, i.e., to the groupoid of $\mathcal{O}_T$-linear maps $L \to \mathcal{O}_T$ where $L \in \mathrm{Pic}(T)$. In particular, there is a universal such pair

$$t : \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(-1) \to \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}$$

over $\mathbf{A}^1/\mathbf{G}_m$; we refer to t (as well as all twists) as the tautological section. In terms of the description $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m) \simeq \mathcal{D}_{\mathrm{graded}}(R[t])$, the map above corresponds to the inclusion

$$tk[t] \subset k[t]$$

[11] The word "connective" is shorthand, borrowed from topologists, that means "cohomologically bounded above at 0" in the classical derived category literature on t-structures. Similarly, "coconnective" means " cohomologically bounded below at 0".

of graded $k[t]$-modules. The vanishing locus of t is the Cartier divisor $B\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$. Note that we have $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(-1)|_{B\mathbf{G}_m} = \mathcal{O}_{B\mathbf{G}_m}(-1)$ in the normalization from Construction 2.2.1: this amounts to observing that the graded $R[t]$-module $tR[t]/t^2R[t]$ is a copy of R in grading degree 1, and hence corresponds to the line bundle $\mathcal{O}_{B\mathbf{G}_m}(-1)$.

Our target theorem is the following; the idea here goes back to Simpson's work in non-abelian Hodge theory (see [Sim96, §5]), and a complete proof in modern language can be found in [Mou21].

Proposition 2.2.6 (Geometrization of the filtered derived category)**.** *Let R be a commutative ring. Consider the stack $\mathbf{A}^1/\mathbf{G}_m = \mathrm{Spec}(R[t])/\mathbf{G}_m$. The Rees module construction, informally described by carrying a filtered object $F \in \mathcal{DF}(R)$ to the graded $R[t]$-module*

$$\mathrm{Rees}(F) := \bigoplus_{i \in \mathbf{Z}} \mathrm{Fil}^i \underline{F} \cdot t^{-i},$$

defines a symmetric monoidal equivalence

$$\mathrm{Rees} : \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m) \simeq \mathcal{DF}(R).$$

This equivalence enjoys the following properties:

1. Rees *is t-exact for the standard t-structures.*
2. *Restriction to the open substack* $\mathrm{Spec}(R) = \mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ *corresponds to forgetting the filtration.*
3. *Restriction to the closed substack $i : B\mathbf{G}_m := \mathrm{Spec}(R)/\mathbf{G}_m \overset{t=0}{\subset} \mathbf{A}^1/\mathbf{G}_m$ corresponds to passage to the associated graded with a change of sign. More precisely, given $F \in \mathcal{DF}(R)$ and $i \in \mathbf{Z}$, we have*
$$\mathrm{gr}^i_{\mathrm{Fil}} \underline{F} \simeq R\Gamma(B\mathbf{G}_m, i^*\mathrm{Rees}(F)(-i)).$$
4. Rees *matches the completeness condition for objects in $\mathcal{DF}(R)$ with derived t-completeness for objects in $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$.*
5. *For an integer n, tensoring with the line bundle $\mathcal{O}(-n)$ on $\mathbf{A}^1/\mathbf{G}_m$ corresponds to the twist operation $F \mapsto F\{n\}$ on filtered objects.*

Sketch of inverse construction. Consider the tautological virtual Cartier divisor

$$t : \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(-1) \to \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}$$

over $\mathbf{A}^1/\mathbf{G}_m$. Taking powers of this section gives a $\mathbf{Z}$-indexed diagram

$$... \to \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(i-1) \to \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(i) \to \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(i+1) \to ...$$

in $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$. Given $M \in \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$, tensoring with the ($\mathbf{Z}^{op}$-indexed version of the) above diagram and taking global sections yields an object of $\mathcal{DF}(R)$, giving the inverse construction. □

Remark 2.2.7 (Completeness on $\mathbf{A}^1/\mathbf{G}_m$)**.** The stack $\mathbf{A}^1/\mathbf{G}_m$ behaves in many ways like a proper stack[12]. For our purposes, the following manifestation of properness will be important: any perfect complex $M \in \mathrm{Perf}(\mathbf{A}^1/\mathbf{G}_m)$ is derived t-complete, i.e., the inverse limit

$$\cdots M(-n) \xrightarrow{t} M(-n+1) \xrightarrow{t} .... \xrightarrow{t} M(-1) \xrightarrow{t} M$$

[12] For example, this stack is "formally proper" in the sense of [HLP14], i.e., its base change to $\mathrm{Spf}(R)$, where R is an adic noetherian ring, satisfies the conclusion of formal GAGA for pseudocoherent complexes.

vanishes[13]. By standard reductions, it suffices to show this claim for $M = \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}$. In this case, the claim essentially follows as the graded k-algebra $k[t]$ is t-adically complete in the graded sense, i.e., $k[t] \simeq \lim_n k[t]/(t^n)$, where everything is interpreted in graded vector spaces. One can repackage this argument as follows: show that the pushforward along the affine map $f : \mathbf{A}^1/\mathbf{G}_m \to B\mathbf{G}_m$ carries $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}$ to $\bigoplus_{i \leq 0} \mathcal{O}(i)$, then identify this sum with the corresponding product as in the footnote in Construction 2.2.1, and then directly compute that the relevant limit vanishes.

More generally, the preceding argument shows t-completeness of any $M \in \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ with the property that f_*M has bounded below $\mathbf{G}_m$-weights ($\Leftrightarrow \exists i_0$ such that $\mathrm{RHom}_{B\mathbf{G}_m}(\mathcal{O}(i), f_*M) = 0$ for all $i \geq i_0$); this applies to certain "big" objects, e.g., to $\mathcal{O}^{\oplus J}_{\mathbf{A}^1/\mathbf{G}_m}$ for any set J. Note that having bounded above weights is not enough (e.g., the graded $k[t]$-module $k[t^{\pm 1}]/k[t]$, viewed as an object in $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$, is not derived t-complete).

Remark 2.2.8 (Vector bundles on $\mathbf{A}^1/\mathbf{G}_m$)**.** The Rees equivalence in Proposition 2.2.6 identifies the category $\mathrm{Vect}(\mathbf{A}^1/\mathbf{G}_m)$ with the category of pairs (M, F^*), where M is a finite projective R-module, and F^* is a finite filtration on M (in the genuine sense: each F^i is a submodule of M) such that $\mathrm{gr}^i_F M$ is finite projective for all i. Let us explain the forward direction. If E is a vector bundle on $\mathbf{A}^1/\mathbf{G}_m$, then we obtain a vector bundle M on $\mathrm{Spec}(R)$ via $E|_{\mathbf{G}_m/\mathbf{G}_m}$. For any flat quasi-coherent sheaf (such as a bundle) E, the tautological maps $t : E(i) \to E(i+1)$ are injective; the inverse of the Rees construction sketched above then shows that M comes equipped with a genuine decreasing filtration $F^* = R\Gamma(\mathbf{A}^1/\mathbf{G}_m, E(-*))$. Moreover, identifying E with a finite projective graded $R[t]$-module, one learns that $R\Gamma(\mathbf{A}^1/\mathbf{G}_m, E(i))$ vanishes for $i \ll 0$ and stabilizes (via the tautological maps) for $i \gg 0$, ensuring that F^* is a finite filtration. Finally, as E is a vector bundle, the restriction $E|_{B\mathbf{G}_m}$ is also a vector bundle, whence each $\mathrm{gr}^i_F M$ is finite projective.

Remark 2.2.9 (Canonical filtrations)**.** As discussed in Example 2.2.3, sending an object $K \in \mathcal{D}(R)$ to its canonical filtration $\mathrm{Fil}^* = \tau^{\leq -*}K$ gives a fully faithful functor $\mathcal{D}(R) \to \mathcal{DF}(R)$ whose essential image is those objects which are complete and have gr^i concentrated in cohomological degree $-i$. Transporting via the Rees equivalence, we obtain a fully faithful functor $\mathcal{D}(R) \to \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ whose essential image is those objects $M \in \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ which are complete and have the feature that $\mathcal{H}^i(M)(i)$ is constant for all i, i.e., it is (necessarily uniquely) pulled back from $\mathrm{Spec}(R)$.

Remark 2.2.10 (Non-abelian filtrations)**.** Motivated by Proposition 2.2.6 and following Simpson, we shall view a morphism $f : \mathcal{X} \to \mathbf{A}^1/\mathbf{G}_m$ of stacks as a filtration on the stack $\underline{\mathcal{X}} := f^{-1}(\mathbf{G}_m/\mathbf{G}_m)$ and regard $\mathrm{Gr}(\mathcal{X}) := f^{-1}(B\mathbf{G}_m)$ as the associated graded. This terminology is justified by the following observation: assuming pushforward along f preserves quasi-coherence (e.g., f is representable qcqs), for any $M \in \mathcal{D}_{qc}(\mathcal{X})$, the pushforward $f_*M \in \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$, seen as an object of $\mathcal{DF}(R)$ via Proposition 2.2.6, is a filtration on $R\Gamma(\underline{\mathcal{X}}, M|_{\underline{\mathcal{X}}})$ with associated graded $R\Gamma(\mathrm{Gr}(\mathcal{X}), M|_{\mathrm{Gr}(\mathcal{X})})$. We shall thus sometimes refer to a morphism $f : \mathcal{X} \to \mathbf{A}^1/\mathbf{G}_m$ as a *filtered stack*.

2.2.2 Endomorphisms and $B\widehat{\mathbf{G}_a}$

Fix a commutative ring R, and work with R-schemes. Our goal in this review the variants and consequences of the following principle: when R has characteristic 0, specifying a representation of $\widehat{\mathbf{G}_a}$ on an R-module V is equivalent to specifying an endomorphism $t : V \to V$ (by differentiating the action), and this equivalence passes to derived categories.

[13] As the proof shows, it is critical that we work with $\mathbf{A}^1/\mathbf{G}_m$. The analogous statement on $\mathbf{A}^1$ is clearly false.

Notation 2.2.11 (Formal completions of vector bundle schemes)**.** Let $\widehat{\mathbf{G}_a} \subset \mathbf{G}_a$ be the formal completion at 0; its functor of points is described as $\widehat{\mathbf{G}_a}(S) = \mathrm{Nil}(S)$ for any R-algebra S. More generally, given a finite projective R-module E, write $\mathbf{V}(E) = \mathrm{Spec}(\mathrm{Sym}^*_R(E^\vee))$ for the associated vector bundle scheme over $\mathrm{Spec}(R)$, and let $\widehat{\mathbf{V}(E)}$ be its formal completion at the 0 section; thus, for an R-algebra S, we have $\widehat{\mathbf{V}(E)}(S) = E \otimes_R \mathrm{Nil}(S)$ as a subset of $\mathbf{V}(E)(S) = E \otimes_R S$.

Given a group scheme G/S, vector bundles on the stack BG are identified with representations of G on finite projective R-modules (essentially by definition); similarly for formal group schemes, such as the ones appearing above. Our goal in this section is to study representations of $\widehat{\mathbf{G}_a}$, their cohomology, as well as a general version that allows $\widehat{\mathbf{G}_a}$ to vary in a family. Before formulating the general result, let us explain the statements in the case of $\widehat{\mathbf{G}_a}$ itself.

Example 2.2.12 ($\widehat{\mathbf{G}_a}$-representations)**.** Let k be a field of characteristic 0. Writing t for a fixed generator of the tangent space $\mathrm{Lie}(\widehat{\mathbf{G}_a})$ of $\widehat{\mathbf{G}_a}$ at 0, Proposition 2.2.13 gives an equivalence[14]

$$\Phi : \mathcal{D}_{qc}(B\widehat{\mathbf{G}_a}) \simeq \mathcal{D}(k[t]).$$

The equivalence Φ has the following features:

- If $\pi : \mathrm{Spec}(k) \to B\widehat{\mathbf{G}_a}$ denotes the tautological map, then Φ intertwines π^* with the forgetful functor $\mathcal{D}(k[t]) \to \mathcal{D}(k)$. In other words, if one regards objects in $\mathcal{D}_{qc}(B\widehat{\mathbf{G}_a})$ as $\widehat{\mathbf{G}_a}$-representations on R-complexes, then the equivalence Φ is the identity functor on underlying R-complexes. Thus, the equivalence is observing that a $\widehat{\mathbf{G}_a}$-representation carries a functorial endomorphism t that complete determines the representation, and moreover that there are no additional constraints on the endomorphism t.

- $\Phi(\mathcal{O}_{B\widehat{\mathbf{G}_a}}) \simeq k[t]/(t) \simeq k$. Hence, for $M \in \mathcal{D}_{qc}(B\widehat{\mathbf{G}_a})$, we have a natural isomorphism

  $$R\Gamma(B\widehat{\mathbf{G}_a}, M) \simeq \mathrm{RHom}_{k[t]}(k, \Phi(M)) \simeq \mathrm{Fib}(\Phi(M) \xrightarrow{t} \Phi(M)).$$

  In particular, $R\Gamma(B\widehat{\mathbf{G}_a}, -)$ has cohomological dimension 1.

- Φ carries tensor products on the left hand side to convolution on the right hand side. In particular, given $M, N \in \mathrm{QCoh}(B\widehat{\mathbf{G}_a})$, we have $\Phi(M \otimes N) \simeq \Phi(M) \otimes_k \Phi(N)$ with $k[t]$-action determined by the formula $t_M \otimes 1 + 1 \otimes t_N$. Thus, we learn that

  $$\mathrm{CAlg}(\mathrm{QCoh}(B\widehat{\mathbf{G}_a})) \simeq \{(R, D : R \to R) \mid R \in \mathrm{CAlg}(k), D \text{ is a } k\text{-linear derivation.}\}$$

  In fact, as $\widehat{\mathbf{G}_a}$ is formal local, this description globalizes: specifying a $\widehat{\mathbf{G}_a}$-action on a k-scheme X is the same as specifying a k-linear derivation $D : \mathcal{O}_X \to \mathcal{O}_X$.

As a concrete example, observe that the inclusion $\widehat{\mathbf{G}_a} \subset \mathbf{G}_a$ gives rise to a $\widehat{\mathbf{G}_a}$-action on the scheme $\mathbf{G}_a$. By the last point above, this corresponds to a $k[t]$-module structure on $\mathcal{O}(\mathbf{G}_a) := k[X]$, where t acts as a derivation. One then computes that $t = \frac{d}{dX}$.

[14]Let us expliclity describe the formula for the inverse at the abelian level to demystify the equivalence. Write X for the co-ordinate on $\widehat{\mathbf{G}_a}$ that is $\mathbf{G}_m$-equivariantly dual to t. Given a $k[t]$-module M, the corresponding quasi-coherent sheaf on $B\widehat{\mathbf{G}_a}$, seen as a topological $\mathcal{O}(\widehat{\mathbf{G}_a})$-comodule, is simply M itself with coaction map

$$c : M \to M[\![X]\!], \quad m \mapsto \exp(tX)(m) := \sum_{i \geq 0} t^i(m) \frac{X^i}{i!}.$$

Note that the action of t on M can be recovered from the coaction map as the coefficient of X in the coaction map: we have $t(m) = (\frac{d}{dX} c(m))|_{X=0}$.

Let us now formulate the general statement where we allow the formal additive group to vary in a family.

Proposition 2.2.13 (Representations of formally completed vector bundles)**.** *Let R be a commutative $\mathbf{Q}$-algebra R. For a finite projective R-module E, there is a natural equivalence*

$$\mathcal{D}_{qc}(B\widehat{\mathbf{V}(E)}) \simeq \mathcal{D}_{qc}(\mathbf{V}(E^\vee))$$

of R-linear stable ∞-categories intertwining pullback along the tautological map $\pi : \mathrm{Spec}(R) \to B\widehat{\mathbf{V}(E)}$ with pushforward along the structure map $\mathbf{V}(E^\vee) \to \mathrm{Spec}(R)$. Moreover, if we endow the target with the convolution symmetric monoidal structure, then this equivalence is naturally symmetric monoidal.

Idea of proof. We only sketch the idea of the proof at the abelian level. For any commutative ring R and finite projective R-module E, the natural duality identification

$$\mathrm{Sym}^n_R(E)^\vee = \Gamma^n_R(E^\vee)$$

leads to an isomorphism

$$\mathrm{Hom}_R(\mathrm{Sym}^*_R(E), R) \simeq \widehat{\Gamma^*(E^\vee)}$$

of topological coalgebras, where the RHS denotes the completion of the divided power coalgebra $\Gamma^*(E^\vee)$ by the natural filtration by divided powers ideals. When R is a $\mathbf{Q}$-algebra, divided powers and symmetric powers coincide, so we can write

$$\mathrm{Hom}_R(\mathrm{Sym}^*_R(E), R) \simeq \widehat{\mathrm{Sym}^*_R(E^\vee)},$$

where the completion is with respect to the standard filtration $\{\mathrm{Sym}^{\geq n}_R(E^\vee)\}_{n\geq 0}$ of the symmetric algebra. Now quasi-coherent sheaves on $\mathbf{V}(E^\vee)$ are identified with $\mathrm{Sym}^*_R(E)$-modules, while quasi-coherent sheaves on $B\widehat{\mathbf{V}(E)}$ are identified[15] with R-modules equipped with a continuous $\widehat{\mathrm{Sym}^*_R(E^\vee)}$-comodule structure, i.e., a map $M \to M\widehat{\otimes}_R\mathrm{Sym}^*_R(E^\vee)$ satisfying suitable axioms. Using the above duality isomorphism, it is easy to see that these notions are equivalent to each other: a $\mathrm{Sym}^*_R(E)$-module structure on an R-module M yields a "continuous" $\mathrm{Sym}^*_R(E^\vee)$-comodule structure $M \to M\widehat{\otimes}_R\mathrm{Sym}^*_R(E^\vee)$ and vice versa. □

Remark 2.2.14 (Cohomology of $\widehat{\mathbf{V}(E)}$-representations)**.** Continue with notation as in Proposition 2.2.13. A given $M \in \mathcal{D}_{qc}(B\widehat{\mathbf{V}(E)})$ may be regarded as a representation of $\widehat{\mathbf{V}(E)}$ on the R-complex $\pi^*M \in \mathcal{D}(R)$. The proposition implies that the R-complex π^*M naturally admits an R-linearaction of the R-algebra $S = \mathrm{Sym}^*_R(E)$, and moreover that

$$R\Gamma(B\widehat{\mathbf{V}(E)}, M) := \mathrm{RHom}_{B\widehat{\mathbf{V}(E)}}(\mathcal{O}, M) \simeq \mathrm{RHom}_S(R, \pi^*M),$$

where R on the rightmost term is given the S-action where E acts trivially. Using the Koszul resolution

$$\big(... \to \wedge^2 E \otimes_R S \to E \otimes_R S \to S\big) \to R$$

[15] To understand this, recall that for any sheaf of groups G on $\mathrm{Spec}(R)$, one has a simplicial presentation

$$\Big(\cdots \;\substack{\to\\[-2pt]\to\\[-2pt]\to\\[-2pt]\to}\; G \times G \;\substack{\to\\[-2pt]\to\\[-2pt]\to}\; G \rightrightarrows \mathrm{Spec}(R) \Big) \to BG.$$

of BG. Using this presentation, one checks that when G is a formal affine scheme, quasi-coherent sheaves on BG are equivalent to R-modules M equipped with a "continuous" $\mathcal{O}(G)$-comodule structure.

to compute $\mathrm{RHom}_S(R, \pi^* M)$ then gives an explicit recipe for computing $R\Gamma(B\widehat{\mathbf{V}(E)}, M)$ in terms of the "underlying R-complex" of the representation M. For example, if E is a line bundle, then we learn that there is a fibre sequence

$$R\Gamma(B\widehat{\mathbf{V}(E)}, M) \to \pi^* M \to \pi^* M \otimes E^\vee$$

of R-complexes.

Remark 2.2.15 (Cohomology of actions of formally completed line bundles)**.** Say we are given a qcqs morphism $f : Y \to Z$ of characteristic 0 schemes, a line bundle L on Z, and a Z-linear action of $G := \widehat{\mathbf{V}(L)}$ on Y. Let us explain how to compute pushforwards along $f_G : Y/G \to Z$. Consider the cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi_Y"] & Z \arrow[d, "\pi_Z"] \\ Y/G \arrow[r, "\tilde{f}"] & BG. \end{tikzcd}$$

As the horizontal map is representable qcqs, pushforward along this map preserves quasi-coherence. Moreover, given $M \in \mathcal{D}_{qc}(Y/G)$, flat base change shows that $\pi_Z^* R\tilde{f}_* M \simeq Rf_* \pi_Y^* M$. Pushing forward $R\tilde{f}_* M$ along the structure map $g : BG \to Z$ and using Remark 2.2.14, we learn that $Rf_{G,*}M \simeq Rg_* R\tilde{f}_* M$ sits in a fibre sequence

$$Rf_{G,*}M \to Rf_* \pi_Y^* M \to Rf_* \pi_Y^* M \otimes L^{-1}.$$

In particular, the pushforward $Rf_{G,*}M$ is quasi-coherent. Changing perspective, if we regard $N = \pi_Y^* M$ as a G-equivariant sheaf on Y (with G-equivariant structure corresponding to the descent M to Y/G), then the fibre sequence gives a recipe for computing the equivariant cohomology $R\Gamma_G(Y, N) := R\Gamma(Y/G, M)$ of N in terms of its non-equivariant cohomology $R\Gamma(Y, N)$ together with some additional structure on the latter induced by the G-equivariant structure. One can formulate a similar statement if L is replaced by a higher rank bundle (left to the reader).

Variant 2.2.16. In characteristic 0, we can identify representations of $\mathbf{G}_a$ with nilpotent endomorphisms (more precisely: $\mathcal{D}_{qc}(B\mathbf{G}_a) \simeq \mathcal{D}_{qc}(\widehat{\mathbf{G}_a})$) by similar arguments.

We shall be working with quotients by $\widehat{\mathbf{G}_a}$-actions in the sequel. To control this effectively, the following variant of Serre vanishing will be useful, at least psychologically:

Lemma 2.2.17 (Higher cohomology of $\widehat{\mathbf{G}_a}$ vanishes on affines)**.** *Let R be any commutative ring. Then $R\Gamma_{et}(\mathrm{Spec}(R), \widehat{\mathbf{G}_a}) \simeq \mathrm{Nil}(R)[0]$.*

Proof. By general nonsense, it suffices to show that for any étale cover $R \to S$ with Cech nerve $R \to S^\bullet$, we have

$$\mathrm{Nil}(R) \simeq \lim \mathrm{Nil}(S^\bullet).$$

But $R \to S$ is étale, so $\mathrm{Nil}(R) \otimes_R S^\bullet \simeq \mathrm{Nil}(S^\bullet)$, whence $\lim \mathrm{Nil}(S^\bullet) \simeq \lim \mathrm{Nil}(R) \otimes_R S^\bullet \simeq \mathrm{Nil}(R)$ by fpqc descent for quasi-coherent sheaves. □

Remark 2.2.18 (Higher fppf cohomology of $\widehat{\mathbf{G}_a}$)**.** For completeness, let us record the behaviour of $R\Gamma_{fppf}(\mathrm{Spec}(R), \widehat{\mathbf{G}_a})$ when one works over a field.

- *The case of characteristic p:* As the nilradical of an $\mathbf{F}_p$-algebra is always annihilated by the map to the perfection, we obtain a short exact sequence

$$0 \to \widehat{\mathbf{G}_a} \to \mathbf{G}_a \to \mathbf{G}_{a,\mathrm{perf}} \to 0$$

  of fppf sheaves on $\mathbf{F}_p$-algebras, where $\mathbf{G}_{a,\mathrm{perf}}(R) = R_{\mathrm{perf}}$ is the colimit perfection of R, and the surjectivity on the right is due to the ability to extract p-th roots of functions fppf locally. Now both $\mathbf{G}_a$ and $\mathbf{G}_{a,\mathrm{perf}}$ are cyclic: this amounts to observing that if $R \to S^\bullet$ is the Cech nerve of an fppf cover, then $R \simeq \lim S^\bullet$ (by fppf descent), and $R_{\mathrm{perf}} \simeq \lim S^\bullet_{\mathrm{perf}}$ (by passage to the colimit from the previous statement). Taking global sections, we learn that

$$R\Gamma_{fppf}(\mathrm{Spec}(R), \widehat{\mathbf{G}_a}) \simeq \mathrm{Cone}(R \to R_{\mathrm{perf}})[-1]$$

  for any $\mathbf{F}_p$-algebra R. In particular, we have non-vanishing H^1 if R is not semiperfect.

- *The case of characteristic* 0 *(de Jong):* We claim that $R\Gamma_{fppf}(\mathrm{Spec}(R), \widehat{\mathbf{G}_a}) \simeq \mathrm{Nil}(R)[0]$ for a $\mathbf{Q}$-algebra R. In other words, we claim that $R \mapsto \mathrm{Nil}(R)$ is an fppf sheaf of complexes on $\mathbf{Q}$-algebras. As the class of étale covers is already of universal $\mathrm{Nil}(-)$ descent by (Lemma 2.2.17), it suffices by [Sta18, Tag 0DET] to prove the same for finite locally free covers, i.e., we must show that if $R \to S$ is a finite locally free extension with Cech nerve $S^\bullet$, then the natural map induces an equivalence $\mathrm{Nil}(R) \simeq \lim \mathrm{Nil}(S^\bullet)$ in $\mathcal{D}(\mathrm{Ab})$; we show this using a variant of the standard argument for faithfully flat descent. Consider the normalized trace map $t_{S/R} : S \to R$ given by $t_{S/R}(x) = \frac{1}{\deg(S/R)} \cdot \mathrm{Tr}_{S/R}(x)$. Then t is an R-linear map that splits the inclusion. Moreover, t carries $\mathrm{Nil}(S)$ into $\mathrm{Nil}(R)$: this can be detected after base change to fields, and then it is the classical statement that the trace of a nilpotent endomorphism of a finite dimensional vector space is 0. For each $i \in \Delta$, consider the induced maps

$$s : \mathrm{Nil}(S^i) \to \mathrm{Nil}(S^{i-1}) \quad \text{via} \quad s(x_0 \otimes x_1 \otimes .... \otimes x_i) = (x_0 \otimes x_1 \otimes ... \otimes x_{i-1}) t_{S/R}(x_i),$$

  so $s = t_{S^i/S^{i-1}}$, where the map $S^{i-1} \to S^i$ is determined by including the first i factors. One then checks that these maps yield a contracting homotopy of the augmented complex

$$\mathrm{Nil}(R) \to \mathrm{Nil}(S) \to \mathrm{Nil}(S \otimes_R S) \to ...,$$

  showing the desired acyclicity.

We do not know what happens in mixed characteristic.

2.3 de Rham cohomology in characteristic 0 via stacks

In this section, we work over a ground field k of characteristic 0. All quotients are interpreted as sheaves in the étale topology on the big site of all finitely presented k-schemes.

Our goal is to explain a geometrization procedure (due to Simpson) to recover algebraic de Rham cohomology of smooth k-schemes X as the $\mathcal{O}$-cohomology of an auxiliary geometric object X^{dR}. In fact, the procedure naturally accommodates coefficients too: vector bundles on X^{dR} identify with vector bundles equipped with a flat connection on X. To make the analogy with later constructions more transparent, we shall explain the construction $X \mapsto X^{dR}$ via the following intermediary:

Construction 2.3.1 (The ring stack $\mathbf{G}_a^{dR}$)**.** The scheme $\mathbf{G}_a$ is naturally ring scheme, and the subfunctor $\widehat{\mathbf{G}_a} \subset \mathbf{G}_a$ is an ideal scheme. Consequently, the quotient sheaf $\mathbf{G}_a^{dR} = \mathbf{G}_a/\widehat{\mathbf{G}_a}$ is a sheaf of rings. In fact, we have $\mathbf{G}_a^{dR}(R) = R_{red}$ for any k-algebra R by Lemma 2.2.17. We shall sometimes refer to $\mathbf{G}_a^{dR}$, which is a sheaf of sets, as a ring stack since closely related variants will be genuinely stacky.

Using the ring stack $\mathbf{G}_a^{dR}$, one formally obtains a "de Rhamification" operation on k-schemes:

Definition 2.3.2 (The de Rham space)**.** For any k-scheme X, write X^{dR} for the functor on finite type k-algebras given by

$$X^{dR}(R) = X(\mathbf{G}_a^{dR}(R)) = X(R_{red}).$$

We call this the *de Rham space of X*.

The functor $X \mapsto X^{dR}$ commutes with limits, and we have $(\mathbf{A}^1)^{dR} = \mathbf{G}_a^{dR}$ by definition, so the notation is consistent. In general, there is a natural map $X \to X^{dR}$ induced by the quotient map $\mathbf{G}_a \to \mathbf{G}_a^{dR}$. If X is smooth, then this map $X \to X^{dR}$ is a surjection of étale sheaves (or actually even presheaves on finite type k-algebras) by the infinitesimal lifting property of smoothness. Our goal is to show:

Theorem 2.3.3 (de Rham cohomology via X^{dR} (Simpson))**.** *For a smooth k-scheme X, there is a natural identification*

$$R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \simeq R\Gamma(X, \Omega^\bullet_{X/k}).$$

Under this isomorphism, pullback along $X \to X^{dR}$ corresponds to the projection $R\Gamma(X, \Omega^\bullet_{X/k}) \to \mathrm{gr}^0_H R\Gamma(X, \Omega^\bullet_{X/k}) \simeq R\Gamma(X, \mathcal{O}_X)$ to the displayed piece of the Hodge filtration.

One relatively standard way to prove this theorem is to identify the Cech nerve of the cover $X \to X^{dR}$ with the formal completion of the Cech nerve of $X \to \mathrm{Spec}(k)$ along the diagonally embedded copy of X; this gives a comparison with Grothendieck's infinitesimal cohomology [Gro68], which in turn is known to compute de Rham cohomology. With an eye towards introducing a simple model of more complicated later constructions, we shall sketch a different proof that also yields stronger filtered refinement recovering the Hodge filtration on $R\Gamma(X, \Omega^\bullet_{X/k})$ in terms of a filtration on $R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}})$. In fact, following Remark 2.2.10, we shall construct a filtration on X^{dR} itself. This relies on the following filtered variant of $\mathbf{G}_a^{dR}$ (Construction 2.3.1):

Construction 2.3.4 (The ring stack $\mathbf{G}_a^{dR,+}$)**.** Consider the universal effective Cartier divisor $t : \mathcal{O}(-1) \to \mathcal{O}$ on the stack $\mathbf{A}^1/\mathbf{G}_m$. Passing to the associated vector bundle schemes, we obtain a morphism

$$d : \mathbf{V}(\widehat{\mathcal{O}(-1)}) \xrightarrow{t} \mathbf{V}(\mathcal{O}) = \mathbf{G}_a$$

over $\mathbf{A}^1/\mathbf{G}_m$. This map has the following features: the target is a ring scheme, the source is a module over the target and the map is a linear map satisfying $xdy = ydx$ for local sections x, y of the source. In this situation, the stack quotient

$$\mathbf{G}_a^{dR,+} = \mathrm{Cone}(d : \mathbf{V}(\widehat{\mathcal{O}(-1)}) \xrightarrow{t} \mathbf{G}_a)$$

admits[16] a natural the structure of a 1-truncated animated $\mathbf{G}_a$-algebra stack over $\mathbf{A}^1/\mathbf{G}_m$. This object enjoys the following properties:

[16] The paper [Dri21b] gives multiple concrete descriptions of the ∞-category of 1-truncated animated rings; one of those is via "quasi-ideals", and the map d appearing provides an example. This style of construction will appear repeatedly in the sequel.

- $\mathbf{G}_a^{dR,+}(\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m) = \mathrm{Cone}(\mathrm{Nil}(R)(-1) \to R)$, where the twist and the map are defined via pullback from $\mathbf{A}^1/\mathbf{G}_m$. In other words, if the map $\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m$ is given by a pair $(L \in \mathrm{Pic}(R), L \to R)$, then $\mathbf{G}_a^{dR,+}(\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m) = \mathrm{Cone}(\mathrm{Nil}(R) \otimes_R L \to R)$.
- The restriction $\mathbf{G}_a^{dR,+}|_{\mathbf{G}_m/\mathbf{G}_m}$ is $\mathbf{G}_a^{dR}$.
- The restriction $\mathbf{G}_a^{dR,+}|_{B\mathbf{G}_m}$ is the split square-zero extension

$$\mathbf{G}_a^{Hodge} := \mathbf{G}_a \oplus \mathbf{V}(\widehat{\mathcal{O}(-1)})[1],$$

so $\mathbf{G}_a^{Hodge}(\mathrm{Spec}(R) \to B\mathbf{G}_m)$ is the split square-zero extension $R \bigoplus \mathrm{Nil}(R)(-1)[1]$.

Slightly informally, $\mathbf{G}_a^{dR,+}$ degenerates the cdga $\widehat{\mathbf{G}_a} \xrightarrow{d} \mathbf{G}_a$ to the underlying graded algebra by scaling the differential down to 0. Using this stack, we can mimic the construction of X^{dR} to obtain a filtered analog:

Definition 2.3.5 (The filtered de Rham space and the graded Hodge stack)**.** For a smooth k-scheme X, the *filtered de Rham space* is the map $X^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ whose functor of points is the following:

$$X^{dR,+}(\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m) = X(\mathbf{G}_a^{dR,+}(R)),$$

where the right side denotes the groupoid of maps $\mathrm{Spec}(\mathbf{G}_a^{dR,+}(R)) \to X$ computed in derived algebraic geometry[17]. The fibre

$$X^{Hodge} := X^{dR,+} \times_{\mathbf{A}^1/\mathbf{G}_m} B\mathbf{G}_m$$

is called the *Hodge stack* of X, so the functor X^{Hodge} on $B\mathbf{G}_m$-schemes is given by

$$X^{Hodge}(\mathrm{Spec}(R) \to B\mathbf{G}_m) = X(\mathbf{G}_a^{Hodge}(R)),$$

where again the right side denotes a mapping space in derived algebraic geometry.

Note that $(\mathbf{A}^1)^{dR,+} = \mathbf{G}_a^{dR,+}$ by construction. In general, the filtered stack $\pi : X^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ recovers X^{dR} over the open substack $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$, so we can regard it as a filtration on X^{dR} with associated graded X^{Hodge}, following the principle enunciated in Remark 2.2.10. Theorem 2.3.3 then follows from the following stronger assertion:

Theorem 2.3.6 (Hodge-filtered de Rham cohomology via $X^{dR,+}$ (Simpson))**.** *For X/k a smooth variety, let $\pi_X : X^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ be the structure map. Then $\mathcal{H}_{dR,+}(X) := R\pi_{X,*}\mathcal{O}_{X^{dR,+}}$ is quasi-coherent and complete. Moroever, the corresponding object of $\widehat{\mathcal{DF}}(k)$ identifies with the Hodge-filtered de Rham cohomology $\mathrm{Fil}_H^* R\Gamma(X, \Omega^\bullet_{X/k})$.*

In fact, the proof gives a more precise assertion: the construction carrying an open $U \subset X$ to the pushforward $\mathcal{H}_{dR,+}(U) \in \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ can be regarded as a Zariski sheaf $\mathcal{F}$ on X valued in $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$. The category $\mathrm{Shv}(X; \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m))$ of all such sheaves carries a Beilinson t-structure by transferring the eponymous t-structure on the filtered derived category via (a sheafified version of) the Rees equivalence (Proposition 2.2.6); the heart is equivalent to the abelian category of chain complexes of sheaves of k-modules on X. The proof below shows that $\mathcal{F}$ lies in the heart of this t-structure, and the correspond chain complex is exactly the de Rham complex $\Omega^\bullet_{X/k}$.

[17]This phrase is unambious as we are working in characteristic 0. But in fact even away from the characteristic 0 there is no ambiguity: as the derived rings appearing are 1-truncated, the mapping space can be computed in either derived algebraic geometry built from animated rings, or in spectral algebraic geometry built from E_∞-rings, without changing its meaning.

Proof. Let us make some preliminary remarks.

(a) The functor $X \mapsto X^{dR,+}$ from k-schemes to stacks over $\mathbf{A}^1/\mathbf{G}_m$ commutes with products and more generally Tor-independent finite limits. Indeed, observe that the formula defining $X^{dR,+}$ can be regarded as giving a functor from derived k-schemes to sheaves on derived schemes over $\mathbf{A}^1/\mathbf{G}_m$; when viewed as such, the functor commutes with all limits, essentially by definition.

(b) The map $X \times \mathbf{A}^1/\mathbf{G}_m \to X^{dR,+}$ is a cover for the étale topology by smoothness of X.

(c) If $f : U \to X$ is an étale map (resp. open immersion), then the diagram

$$\begin{tikzcd} U \times \mathbf{A}^1/\mathbf{G}_m \arrow[r] \arrow[d] & U^{dR,+} \arrow[d] \\ X \times \mathbf{A}^1/\mathbf{G}_m \arrow[r] & X^{dR,+} \end{tikzcd}$$

is cartesian (and thus, by (b), the right vertical map is a representable étale map (resp. open immersion)). Indeed, this follows from the infinitesimal lifting property for étale maps (in derived algebraic geometry).

(d) If X is written as the colimit of a finite diagram $U^\bullet$ of affine open subschemes of X, then $U^{\bullet,dR,+}$ forms a finite diagram of affine open subfunctors of $X^{dR,+}$ with colimit $X^{dR,+}$: this follows from the previous item as the horizontal maps in the square appearing there are surjections of étale sheaves (by smoothness).

Note that these remarks also apply to X^{Hodge} by base change. Let us now sketch to use these remarks to prove the theorem.

1. *Properties of the pushforward:* We claim that the pushforward $\mathcal{H}_{dR,+}(X) \in \mathcal{D}(\mathbf{A}^1/\mathbf{G}_m, \mathcal{O})$ is quasi-coherent and complete, and moreover that its restriction to $B\mathbf{G}_m$ (as a quasi-coherent complex) agrees with the corresponding pushforward along $X^{Hodge} \to B\mathbf{G}_m$ via the natural comparison map. By Remark (d), this can be checked when X admits an étale map $f : X \to \mathbf{A}^n$. Write $G = \mathbf{V}(\widehat{\mathcal{O}(-1)})^n$, regarded as a group scheme over $\mathbf{A}^1/\mathbf{G}_m$. Consider the commutative diagram

$$\begin{tikzcd} X \times \mathbf{A}^1/\mathbf{G}_m \arrow[r] \arrow[d] & X^{dR,+} \arrow[d] \\ \mathbf{A}^n \times \mathbf{A}^1/\mathbf{G}_m \arrow[r] & (\mathbf{A}^n)^{dR,+}. \end{tikzcd}$$

By Remark (c), the above diagram is Cartesian. Now the bottom horizontal map is a G-torsor: this reduces by Remark (a) to the case $n = 1$, where it is true by the definition of $(\mathbf{A}^1)^{dR,+} = \mathbf{G}_a^{dR,+}$ as a quotient. It follows that the top horizontal map is a G-torsor as well, so $X^{dR,+} = \left(X \times \mathbf{A}^1/\mathbf{G}_m\right)/G$. The recipe given in Remark 2.2.15 for computing pushforward along $(X \times \mathbf{A}^1/\mathbf{G}_m)/G \to \mathbf{A}^1/\mathbf{G}_m$ then gives quasi-coherence of the pushforward $\mathcal{H}_{dR,+}(X)$ and the base change compatibility. Moreover, the completeness also follows from this recipe using the observations in Remark 2.2.7.

2. *Identification of the associated Hodge stack*: The functor of points of X^{Hodge} over $B\mathbf{G}_m$ is given by

$$X^{Hodge}(\mathrm{Spec}(R) \xrightarrow{\eta} B\mathbf{G}_m) = X(R \oplus \mathrm{Nil}(R)(-1)[1]).$$

It follows by derived deformation theory[18] that we can identify X^{Hodge} with the classifying stack $B\mathbf{V}(\widehat{T_{X/k}(-1)})$ over $X \times B\mathbf{G}_m$, where $T_{X/k}(-1)$ is shorthand for $pr_1^* T_{X/k} \otimes pr_2^* \mathcal{O}_{B\mathbf{G}_m}(-1)$. Using Proposition 2.2.13 to compute cohomology, we learn that

$$\mathcal{H}_{Hodge}(X) := R\pi_{X,*}\mathcal{O}_{X^{Hodge}} \simeq \bigoplus_i R\Gamma(X, \Omega^i_{X/k}[-i])(i)$$

in $\mathcal{D}_{qc}(B\mathbf{G}_m)$.

3. *Identification of the consturction $X \mapsto \mathcal{H}_{dR,+}(X)$ as a complex of sheaves*: Consider the $\mathcal{DF}(k)$-valued presheaf $\mathcal{F}$ on X given by transporting the $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ valued presheaf $U \mapsto \mathcal{H}_{dR,+}(U)$ via the Rees equivalence in Theorem 2.2.6. Remark (d) above shows that the presheaf is a sheaf, so it can be regarded as an object of the filtered derived ∞-category $\mathcal{DF}(X,k)$ of $\mathcal{DF}(k)$-valued sheaves on X. Moreover, by items (1) and (2), the value of this sheaf on affine opens in X lies in the heart of the Beilinson t-structure on $\mathcal{DF}(k)$. Consequently, $\mathcal{F}$ itself lies in the heart of the (appropriately defined) Beilinson t-structure on $\mathcal{DF}(X,k)$. The identification of the heart of the Beilinson t-structure and the calculation in (2) then shows that $\mathcal{F}$ is given by a chain complex

$$\mathcal{O}_X \xrightarrow{\delta} \Omega^1_{X/k} \xrightarrow{\delta} \Omega^2_{X/k} \xrightarrow{\delta} ....$$

(for some currently unknown differential δ) equipped with the stupid filtration. To finish proving the theorem, it then suffices to identify δ with the de Rham differential. In fact, by multiplicativity and naturality, it suffices to check this when $X = \mathbf{A}^1$. In this case, we have $X^{dR,+} = \mathbf{G}_a^{dR,+}$, which is the quotient of $\mathbf{G}_a$ by the action of $\mathbf{V}(\widehat{\mathcal{O}(-1)})$ defined by the tautological section $t : \mathcal{O}(-1) \to \mathcal{O}$ on $\mathbf{A}^1/\mathbf{G}_m$. Using Remark 2.2.15, one then computes $\mathcal{H}_{dR,+}(X)$ is the graded $k[t]$-complex given by

$$k[t,x] \xrightarrow{t\frac{d}{dx}} k[t,x](1),$$

where the differential is computed by observing that it is a graded $k[t]$-linear derivation that gives $\frac{d}{dx}$ on underlying non-filtered objects via the calculation in Example 2.2.12 (especially in Footnote 14). Translating to filtered objects, this shows that $\delta = \frac{d}{dx}$, so we win. □

Remark 2.3.7 (Coefficients for de Rham cohomology)**.** For a smooth k-variety X, the stack $X^{dR,+}$ not only geometrizes (Hodge-filtered) de Rham cohomology, it also geometrizes the natural notion of coefficients for (Hodge-filtered) de Rham cohomology. Let us formulate the result.

Via pullback along the map $X \times \mathbf{A}^1/\mathbf{G}_m \to X^{dR,+}$ appearing above, the category $\mathrm{Vect}(X^{dR,+})$ of vector bundles on $X^{dR,+}$ can be identified with the category of triples (E, ∇, F^*), where E is a vector bundle on X, $\nabla : E \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} E$ is a flat connection, and F^* is a finite filtration of E by subbundles satisfying Griffiths transversality, i.e., $\nabla(F^i) \subset \Omega^1_{X/k} \otimes_{\mathcal{O}_X} F^{i-1}$. Similarly, by Proposition 2.2.13, pullback along $X \times B\mathbf{G}_m \to X^{Hodge}$ identifies the category $\mathrm{Vect}(X^{Hodge})$ identifies with the category of graded Higgs bundles, i.e., graded vector bundles $M = \oplus_i M_i$ together

[18] More precisely, we use the following assertion in derived algebraic geometry: given a finite type k-scheme X, an animated k-algebra R, a map $\eta : \mathrm{Spec}(R) \to X$ of derived k-schemes, and a square-zero extension $R' \to R$ in animated k-algebras of R by $N \in D^{\leq 0}(R)$, the fibre of the map $X(R') \to X(R)$ over $\eta \in X(R)$ is a torsor for $\mathrm{Der}_k(\mathcal{O}_X, \eta_* N) \simeq \mathrm{Map}_R(\eta^* L_{X/k}, N)$. When X is smooth and $N = L[1] \in \mathcal{D}^{\leq -1}(R)$, we have $\mathrm{Map}_R(\eta^* L_{X/k}, N) \simeq B(\eta^* T_{X/k} \otimes_R L)$. When $R' \to R$ comes equipped with a splitting, we conclude that the fibre of $X(R') \to X(R)$ is also a split torsor, and hence canonically identified with $B(\eta^* T_{X/k} \otimes_R L)$.

with a Higgs field[19] $\Theta : M \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} M$ carrying M_i into $\Omega^1_{X/k} \otimes_{\mathcal{O}_X} M_{i+1}$. Under these descriptions, the restriction functor $\mathrm{Vect}(X^{dR,+}) \to \mathrm{Vect}(X^{Hodge})$ carries (E, ∇, F^*) to the graded vector bundle $M = \oplus_i M_i$ where $M_i = \mathrm{gr}_F^{-i} E$ with the Higgs field Θ induced by ∇.

Remark 2.3.8 (Transmutation)**.** Definition 2.3.2 and Definition 2.3.5 both rely on a very general construction. Namely, given a ring R, a scheme B, and an animated R-algebra stack A on B-schemes, we obtain a functor from schemes over R to stacks over B by carrying an R-scheme X to the B-stack X^A given by $X^A(\mathrm{Spec}(S) \to B) = X(A(S))$; one can then contemplate the cohomology theory on R-schemes defined by, e.g., $X \mapsto R\Gamma(X^A, \mathcal{O}_{X^A})$. This process of producing stack valued invariants (such as $X \mapsto X^A$) or cohomology theories (such as $X \mapsto R\Gamma(X^A, \mathcal{O}_{X^A})$) out of a single ring stack shall appear repeatedly in the sequel, and we refer to it informally as *transmutation*[20].

This section can be summarized as follows: via the process of transmutation (Remark 2.3.8), the ring stack $\mathbf{G}_a^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ captures the theory of algebraic de Rham cohomology with its additional algebro-geometric structures in characteristic 0. In the sequel, we will give similar constructions for other cohomology theories.

2.4 Linear algebra via stacks: $B\mathbf{G}_a^\sharp$ and nilpotent endomorphisms

In §2.3, we gave a description of algebraic de Rham cohomology via stacks in characteristic 0. Beyond the key definitions, the main calculation powering this description was the calculation of the cohomology of $B\widehat{\mathbf{G}_a}$ in Proposition 2.2.13, which is only valid in characteristic 0. In this section, we give a (somewhat) analogous calculation in mixed/positive characteristic by replacing the additive group with its PD-version; this will be used later in describing de Rham cohomology via stacks in a p-adic setting.

Definition 2.4.1 (The group scheme $\mathbf{G}_a^\sharp$)**.** Let $\mathbf{G}_a^\sharp$ be the PD-hull of the origin in $\mathbf{G}_a$ over $\mathbf{Z}$. Explicitly, if $\mathbf{G}_a = \mathrm{Spec}(\mathbf{Z}[t])$, then $\mathbf{G}_a^\sharp = \mathrm{Spec}(\mathbf{Z}[t, \frac{t^2}{2!}, \frac{t^3}{3!}, ...])$, so $\mathbf{G}_a^\sharp(R)$ consists of elements of R equipped with a compatible system of divided powers. There is a natural map $\mathbf{G}_a^\sharp \to \mathbf{G}_a$ which is an isomorphism on rationalization. Moreover, the group law on $\mathbf{G}_a$ induces (a necessarily unique) one on $\mathbf{G}_a^\sharp$ (which amounts to observing that $\frac{(x+y)^n}{n!} = \sum_{i+j=n} \frac{x^i}{i!}\frac{y^j}{j!}$). The multiplicative $\mathbf{G}_a$-action on itself induces one on $\mathbf{G}_a^\sharp$ (which amounts to observing $\frac{(xy)^n}{n!} = \frac{x^n}{n!} \cdot y^n$) , turning the latter into a $\mathbf{G}_a$-module scheme, and the structure map $\mathbf{G}_a^\sharp \to \mathbf{G}_a$ into a quasi-ideal.

Example 2.4.2. If R is $\mathbf{Z}$-flat, then $\mathbf{G}_a^\sharp(R) \to \mathbf{G}_a(R) = R$ is injective, with image exactly those $t \in R$ such that $\frac{t^n}{n!} \in R$ for all $n \geq 0$. In particular, for $R = \mathbf{Z}_p$, an elementary argument with valuations shows that $\mathbf{G}_a^\sharp(\mathbf{Z}_p) = p\mathbf{Z}_p \subset \mathbf{Z}_p$. On the other hand, if R is an $\mathbf{F}_p$-algebra, then $\mathbf{G}_a^\sharp(R) \to \mathbf{G}_a(R)$ is injective if and only if R is reduced.

Variant 2.4.3 (PD-hulls of vector bundles)**.** If E is a vector bundle on a scheme X, we write $\mathbf{V}(E)^\sharp$ for the PD-hull of the 0 section in $\mathbf{V}(E)$. Then $\mathbf{V}(E)^\sharp$ is a $\mathbf{G}_a$-module scheme over X, and the map $\mathbf{V}(E)^\sharp \to \mathbf{V}(E)$ is a $\mathbf{G}_a$-module map.

[19]A Higgs structure on a quasi-coherent sheaf M is an $\mathcal{O}_X$-linear map $\Theta : M \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} M$ satisfying $\Theta \wedge \Theta = 0$. In other words, it is exactly the data that gives an action of $\mathrm{Sym}^*_{\mathcal{O}_X}(T_{X/k})$ on M, or equivalently the data needed to realize M as the pushforward to X of a quasi-coherent sheaf on the cotangent bundle T^*X.

[20]Our inspiration comes from the definition of "transmute" as "to change or alter in form, appearance, or nature and *especially to a higher form*", see `https://www.merriam-webster.com/dictionary/transmute`.

Proposition 2.4.4 (Representations of $\mathbf{G}_a^\sharp$)**.** *There is a natural equivalence*

$$\mathcal{D}_{qc}(B\mathbf{G}_a^\sharp) \simeq \mathcal{D}_{qc}(\widehat{\mathbf{G}_a}).$$

1. *This equivalence naturally intertwines pullback along* $\mathrm{Spec}(\mathbf{Z}) \to B\mathbf{G}_a^\sharp$ *on the LHS with the "local cohomology at 0" functor on the RHS.*
2. *If we endow the right side with the convolution product, then this equivalence is naturally symmetric monoidal.*
3. *The equivalence carries the standard t-structure on the LHS to the torsion t-structure*[21] *on the RHS.*

A (slightly more complex) variant of the above statement appears in [BL22a, §3.5]; see also [Dri21a, Appendix B] for the corresponding ring-theoretic assertion.

Sketch of construction. Recall that, as in Footnote 15, for any affine group scheme $G/\mathbf{Z}$, one can identify $\mathrm{QCoh}(BG)$ with comodules over the Hopf algebra $\mathcal{O}(G)$ via pullback along the tautological point $f : \mathrm{Spec}(\mathbf{Z}) \to BG$, with the comodule structure coming from the descent data. In this dictionary, the comodule given by $\mathcal{O}(G)$ acting on itself corresponds to $f_*\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})}$.

Specializing to our case, let $f : \mathrm{Spec}(\mathbf{Z}) \to B\mathbf{G}_a^\sharp$ be the tautological map. Thanks to the formula $\frac{d}{dt}(\frac{t^n}{n!}) = \frac{t^{n-1}}{(n-1)!}$ for $n \geq 1$, we have an evident exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}(\mathbf{G}_a^\sharp) \xrightarrow{\frac{d}{dt}} \mathcal{O}(\mathbf{G}_a^\sharp) \to 0.$$

One checks that this sequence $\mathbf{G}_a^\sharp$-equivariant for the standard action of $\mathbf{G}_a^\sharp$ on the second and third term, and the trivial action on the first. Under the dictionary relating $\mathcal{O}(\mathbf{G}_a^\sharp)$-comodules with $\mathrm{QCoh}(B\mathbf{G}_a^\sharp)$, this yields an exact sequence

$$0 \to \mathcal{O}_{B\mathbf{G}_a^\sharp} \to f_*\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})} \xrightarrow{N} f_*\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})} \to 0$$

on $B\mathbf{G}_a^\sharp$. As f_* and f^* commute with filtered colimits, it follows from the projection formula and the above sequence that $R\Gamma(B\mathbf{G}_a^\sharp, -)$ also commutes with filtered colimits. Given $V \in \mathcal{D}_{qc}(B\mathbf{G}_a^\sharp)$, this sequence also shows that $f^*V \simeq R\Gamma(B\mathbf{G}_a^\sharp, V \otimes f_*\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})})$ carries a canonical endomorphism N_V. As $\frac{d}{dt} : \mathcal{O}(\mathbf{G}_a^\sharp) \to \mathcal{O}(\mathbf{G}_a^\sharp)$ is locally nilpotent, the same holds true for N_V by the compatibility of $R\Gamma(B\mathbf{G}_a^\sharp, -)$ with filtered colimits, so the pair (f^*V, N_V) yields an object of $\mathcal{D}_{qc}(\widehat{\mathbf{G}_a})$, yielding one direction of the construction. It is also easy to see from the above sequence that this direction is fully faithful.

Conversely, given a $\mathbf{Z}$-module M with a locally nilpotent endomorphism $N : M \to M$, we obtain a $\mathcal{O}(\mathbf{G}_a^\sharp)$-comodule structure on M via

$$M \to M \otimes \mathcal{O}(\mathbf{G}_a^\sharp) = M \otimes_{\mathbf{Z}} \mathbf{Z}[t, \frac{t^2}{2!}, \frac{t^3}{3!}, ...], \quad m \mapsto \exp(Nt)(m) = \sum_{i \geq 0} N^i(m) \otimes \frac{t^i}{i!},$$

where the infinite sum makes sense by local nilpotence of N. We leave it to the reader to check that this yields an inverse to the preceding construction. ◻

[21] Given a commutative ring R with finitely generated ideal I, let $\mathcal{D}_{I-\mathrm{comp}}(R)$ and $\mathcal{D}_{I-\mathrm{tors}}(R)$ be the full subcategories of $\mathcal{D}(R)$ spanned by derived I-complete and I^∞-torsion R-complexes respectively. Then the functor of taking local cohomology $R\Gamma_I(-)$ and derived I-completion $(-)^\wedge_I$ give an equivalence $\mathcal{D}_{I-\mathrm{comp}}(R) \simeq \mathcal{D}_{I-\mathrm{tors}}(R)$, called the *complete-torsion equivalence*, see [Sta18, Tag 0A6X]. The standard t-structure on the torsion side then induces a t-structure on the complete side that we refer to as the "torsion t-structure". In the text above, this is applied for $R = \mathbf{Z}[\![t]\!]$ with $I = (t)$ (as $\widehat{\mathbf{G}_a} = \mathrm{Spf}(R)$).

This result has the following relative variant whose proof we omit:

Proposition 2.4.5. *Let X be a scheme. Let E be a vector bundle on X. Then there is a natural identification*

$$\mathcal{D}_{qc}(B\mathbf{V}(E)^\sharp) \simeq \mathcal{D}_{qc}(\widehat{\mathbf{V}(E^\vee)}),$$

enjoying compatibilities with respect to the forgetful functors, symmetric monoidal structures, and t-structures as in Proposition 2.4.4.

Remark 2.4.6 (Koszul description of cohomology on $BE^\sharp$)**.** Similar to what we saw in Remark 2.2.14 in charateristic 0, Proposition 2.4.5 has the following consequence. With notation as above, if $\pi : B\mathbf{V}(E)^\sharp \to X$ and $\nu : \widehat{\mathbf{V}(E^\vee)} \to X$ denote the projections and $i : X \to \widehat{\mathbf{V}(E^\vee)}$ is the 0-section, then we have a natural identification

$$R\pi_*\mathcal{O}_{B\mathbf{V}(E)^\sharp} \simeq R\nu_*\underline{\mathrm{RHom}}_{\widehat{\mathbf{V}(E^\vee)}}(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \simeq \bigoplus_i \wedge^i E^\vee[-i] \tag{2.4.1}$$

in $\mathcal{D}_{qc}(X)$; the first equality follows from Proposition 2.4.5, while the second can be calculated using the Koszul resolution of $i_*\mathcal{O}_X$. (One can even obtain this isomorphism at the level of commutative algebra objects in $\mathcal{D}_{qc}(X)$ using the natural $\mathbf{G}_m$-action on E.)

For our later use, we shall need the following vanishing theorem, analogous to Lemma 2.2.17.

Lemma 2.4.7 (Higher cohomology of $\mathbf{G}_a^\sharp$ vanishes in degrees ≥ 2)**.** *Let R be a p-nilpotent ring. Then the complex $R\Gamma_{fl}(\mathrm{Spec}(R), \mathbf{G}_a^\sharp)$ is concentrated*[22] *in degrees 0 and 1.*

Proof. We shall prove later (Remark 2.6.2) that there is a short exact sequence

$$0 \to \mathbf{G}_a^\sharp \to W \xrightarrow{F} W \to 0$$

of group schemes, where W denotes the ring scheme of Witt vectors, the map F is the Witt vector Frobenius, and exactness is measured in the flat topology. Granting this, taking cohomology, it suffices to show that $R\Gamma_{fl}(\mathrm{Spec}(R), W)$ is concentrated in degree 0. Now $W = \lim_n W_n = R\lim_n W_n$ as sheaves in the flat topology, so $R\Gamma(\mathrm{Spec}(R), W) \simeq R\lim_n R\Gamma(\mathrm{Spec}(R), W_n)$, whence it is enough to show that $R\Gamma_{fl}(\mathrm{Spec}(R), W_n) = W_n(R)[0]$ is concentrated in degree 0 and that the transition maps $W_n(R) \to W_{n-1}(R)$ are surjective for all n. The latter is clear by construction. For the former, we use that $W_n(-)$ admits a finite filtration with associated graded being copies of $\mathbf{G}_a$ (as abelian sheaves) to reduce to the assertion that $R\Gamma_{fl}(\mathrm{Spec}(R), \mathbf{G}_a) = R[0]$; this is a standard consequence of fpqc descent. ☐

Remark 2.4.8. Lemma 2.4.7 has the following consequence of interest to us. Say A is any abelian sheaf with vanishing higher cohomology on affines (e.g., $\mathbf{G}_a$). Fix a map $d : \mathbf{G}_a^\sharp \to A$ of abelian sheaves. Then $\mathbf{G}_a^\sharp$ acts (linearly) on A, and we may form the quotient stack $[A/\mathbf{G}_a^\sharp]$; this quotient can also be regarded as the 1-truncated connective object of the derived category given by $\mathrm{Cone}(d)$. The above vanishing ensures that $R\Gamma_{fl}(\mathrm{Spec}(R), [A/\mathbf{G}_a^\sharp]) \simeq [A/\mathbf{G}_a^\sharp](R)$ is connective.

[22]The notion of flat cohomology is delicate in general. To define it, one can, for example, use cutoff cardinals; but then one must worry whether the choice of the cardinal affects the answer. However, in the case at hand (and all cases relevant to us), these subtleties are irrelevant. Let us spell this out for $\mathbf{G}_a^\sharp$. First, for a p-nilpotent ring R, there is a perfectly reasonable notion of an fpqc $\mathbf{G}_a^\sharp$-torsor on $\mathrm{Spec}(R)$ devoid of set-theoretic subtleties: it consists of a faithfully flat affine R-scheme $X \to \mathrm{Spec}(R)$ equipped with an R-linear $\mathbf{G}_a^\sharp$-action with the property that for any R-algebra S, the action of $\mathbf{G}_a^\sharp(S)$ on $X(S)$ is simply transitive provided $X(S) \neq \emptyset$. Note that any such X is countably presented over R, and thus there are countably presented faithfully flat R-algebras where $X(S) \neq \emptyset$. It therefore makes sense to consider the groupoid $B\mathbf{G}_a^\sharp(R)$ of all fpqc $\mathbf{G}_a^\sharp$-torsors on $\mathrm{Spec}(R)$; this is naturally an object of $\mathcal{D}(\mathrm{Ab})$ (as $\mathbf{G}_a^\sharp$ is commtuative). The assertion in Lemma 2.4.7 can then be formulated as the statement that functor carrying a p-nilpotent ring R to $B\mathbf{G}_a^\sharp(R)$ is a $\mathcal{D}(\mathrm{Ab})$-valued sheaf for the fpqc topology.

2.5 de Rham cohomology of p-adic formal schemes via stacks

The goal of this section is to provide an analog of the constructions and results of §2.3 in the setting of p-adic formal schemes. Some relevant references for the material in this section are [Dri20, Dri18, LM21b, Mon22, BL22b].

Let V be a p-complete commutative ring with bounded p^∞-torsion; the primary cases of interest are rings of characteristic p or p-torsionfree rings. We shall work in the category of (bounded) p-adic formal schemes over V throughout unless otherwise specified. As a first step towards finding p-adic analogs of §2.3 for smooth p-adic formal V-schemes, we introduce the relevant ring stacks.

Definition 2.5.1 (The ring stack $\mathbf{G}_a^{dR}$ and variants)**.** Working over $\mathbf{A}^1/\mathbf{G}_m$, the natural map $t' : \mathbf{V}(\mathcal{O}(-1))^\sharp \to \mathbf{V}(\mathcal{O}) = \mathbf{G}_{a,\mathbf{A}^1/\mathbf{G}_m}$, induced by the tautological map $t : \mathcal{O}(-1) \to \mathcal{O}$, is a quasi-ideal, and hence we may form the quotient

$$\mathbf{G}_a^{dR,+} = \mathrm{Cone}(\mathbf{V}(\mathcal{O}(-1))^\sharp \xrightarrow{t'} \mathbf{G}_{a,\mathbf{A}^1/\mathbf{G}_m})$$

to obtain a 1-truncated animated $\mathbf{G}_a$-algebra stack on p-nilpotent V-algebras over $\mathbf{A}^1/\mathbf{G}_m$. Regarding $\mathbf{G}_a^{dR,+}$ as a filtered stack, we obtain two auxiliary stacks:

- $\mathbf{G}_a^{dR} = \mathbf{G}_a^{dR,+}|_{\mathbf{G}_m/\mathbf{G}_m} = \mathrm{Cone}(\mathbf{G}_a^\sharp \xrightarrow{can} \mathbf{G}_a)$ is the underlying non-filtered stack, and is a 1-truncated animated $\mathbf{G}_a$-algebra stack on p-nilpotent V-algebras. Concretely, we have

  $$\mathbf{G}_a^{dR}(R) = \mathrm{Cone}(B\mathbf{G}_a^\sharp(R)[-1] \xrightarrow{can} R),$$

  regarded as a 1-truncated animated R-algebra.

- $\mathbf{G}_a^{Hodge} := \mathbf{G}_a^{dR,+}|_{B\mathbf{G}_m} \to B\mathbf{G}_m$ is the associated graded stack, and is a 1-truncated animated $\mathbf{G}_a$-algebra stack on p-nilpotent V-algebras over $B\mathbf{G}_m$. Concretely, we can identify

  $$\mathbf{G}_a^{Hodge}(\mathrm{Spec}(R) \to B\mathbf{G}_m) = R \oplus B\mathbf{G}_a^\sharp(R)(-1)$$

  as a 1-truncated split square-zero extension of R in animated k-algebras, where the twist on the right is defined via pullback from $B\mathbf{G}_m$.

Remark 2.5.2. Let R be a p-nilpotent ring. By Remark 2.4.8, we have

$$\mathbf{G}_a^{dR}(R) \simeq R\Gamma_{fl}(\mathrm{Spec}(R), \mathbf{G}_a^{dR}),$$

where the RHS denotes the flat cohomology of $\mathrm{Spec}(R)$ with coefficients in the sheaf $\mathbf{G}_a^{dR}(-)$ of 1-truncated R-complexes on R-algebras; similarly for $\mathbf{G}_a^{dR,+}$ and $\mathbf{G}_a^{Hodge}$.

Thus, $\mathbf{G}_a^{dR}$ and $\mathbf{G}_a^{dR,+}$ are analogs similarly named constructions from §2.3: the former is an explicIty defined cdga, while the latter is a degeneration of the former to the underlying graded algebra obtained by rescaling the differential. Following the transmutation construction as in §2.3, we can use the ring stacks above to introduce the "de Rhamification" functors:

Definition 2.5.3 (The (Hodge filtered) de Rham stack)**.** Fix a smooth p-adic formal scheme X/V.

- The *Hodge filtered de Rham stack of* X/V is the stack $\pi_X : (X/V)^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ defined by

  $$(X/V)^{dR,+}(\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m) = X(\mathbf{G}_a^{dR,+}(R)) := \mathrm{Map}_V(\mathrm{Spec}(\mathbf{G}_a^{dR,+}(R)), X),$$

  where the mapping space on the right is computed in derived algebraic geometry over V.

- The fibre of π_X over $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ is labelled $(X/V)^{dR}$ and called the *de Rham stack of X/V*. Concretely, we have

$$(X/V)^{dR}(R) = \mathrm{Map}_V(\mathrm{Spec}(\mathbf{G}_a^{dR}(V)), X).$$

- The fibre of π_X over $B\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ is called $(X/V)^{Hodge}$ and called the *Hodge stack of X/V*. Concretely, we have

$$(X/V)^{Hodge}(\mathrm{Spec}(R) \to B\mathbf{G}_m) = \mathrm{Map}_V(\mathrm{Spec}(\mathbf{G}_a^{Hodge}(R)), X).$$

Remark 2.5.4 (Group schemes vs formal group schemes)**.** Observe a slight difference from the characteristic 0 story in §2.3: the role of the formal group scheme $\widehat{\mathbf{G}_a}$ in the latter story is now played by the group scheme $\mathbf{G}_a^\sharp$. As the latter is a genuine scheme instead of a formal scheme, certain technical arguments become easier (e.g., Remark 2.5.5, Remark 2.5.7).

Remark 2.5.5 (Quasi-coherence of pushforwards)**.** The following shall be repeatedly used in the sequel: if Y is a scheme, G/Y is a flat affine group scheme, and $f : X \to Y$ is a BG-torsor for the flat topology, then $Rf_*\mathcal{O}_X \in \mathcal{D}(Y, \mathcal{O}_Y)$ lies in $\mathcal{D}_{qc}(Y)$ and the formation of the latter commutes with flat base change. Indeed, this is a general fact about suitable "fpqc-algebraic" stacks. In our case, one can argue explicitly as follows. As the assertion can be checked locally on Y, so we may assume $Y = \mathrm{Spec}(R)$ and that $X \to Y$ is a trivial BG-torsor. Choose a point $Y = X^0 \to X$ giving a trivialization, and let $X^\bullet \to X$ be the Cech nerve of the $X^0 \to X$. Then each X^n is a flat affine Y-scheme, and $R\Gamma(X, \mathcal{O}_X) \simeq \lim R\Gamma(X^\bullet, \mathcal{O}_{X^\bullet})$ by descent. Moreover, the same picture holds true after base change on R. The claim then follows from the fact that the formation of totalization of cosimplicial objects in $\mathcal{D}^{\geq 0}(R)$ commutes with flat base change on R for any commutative ring R.

With these definitions in place, the picture is quite analogous to characteristic 0

Theorem 2.5.6 (Hodge-filtered de Rham cohomology via stacks)**.** *Let X/V be a smooth qcqs p-adic formal scheme. Let $\pi_X : X^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ be the filtered de Rham stack of X/V. Then $\mathcal{H}_{dR,+}(X) := R\pi_{X,*}\mathcal{O}_{X^{+,dR}}$ is quasi-coherent and complete. Moreover, the corresponding object of $\widehat{\mathcal{DF}}_{p-comp}(V)$ identifies with the Hodge-filtered de Rham complex $\mathrm{Fil}^*_H R\Gamma(X, \Omega^\bullet_{X/V})$.*

Sketch of proof. The proof is entirely analogous to that of Theorem 2.3.6, except we need to justify certain arguments that were obvious in characteristic 0. We only sketch the differences.

Before starting, we make some general remarks. Observe that p-nilpotent rings R that are $\mathbf{G}_a^\sharp$-acyclic (i.e., all $\mathbf{G}_a^\sharp$-torsors are trivial) form a basis of the flat topology: as $\mathbf{G}_a^\sharp$ is countably presented, this follows by a small object argument. When R is $\mathbf{G}_a^\sharp$-acyclic, the map $R \to \mathbf{G}_a^{dR}(R)$ is surjective on π_0 with a locally nilpotent kernel: the surjectivity is immediate from acyclicity, while the assertion about the kernel follows from existence of divided powers of kernel elements together with the p-nilpotence of R. In particular, the map $R \to \mathbf{G}_a^{dR}(R)$ induces an isomorphism on étale sites. Moreover, if a $\mathbf{G}_a^\sharp$-acyclic p-nilpotent ring R comes equipped with a map $\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m$, then the same analysis shows that $R \to \mathbf{G}_a^{dR,+}(R)$ is also surjective on π_0 with locally nilpotent kernel, and hence induces an isomorphism on étale sites.

Next, let us explain how to justify the analogs of Remarks (a) through (d) in the proof of Theorem 2.3.6. For Remark (a), the same proof applies. The analog of Remark (c) also follows similarly as the assertion can be tested on $\mathbf{G}_a^\sharp$-acyclic test rings. For Remark (b), we have the following flat analog: the maps $X \to (X/V)^{dR}$ and $X \times \mathbf{A}^1/\mathbf{G}_m \to (X/V)^{dR,+}$ are covers for the flat topology. Indeed, the smoothness of X/V shows that if $R \to S$ is any map of p-nilpotent animated V-algebras which is surjective on π_0 with locally nilpotent kernel, then $X(R) \to X(S)$ is surjective on π_0, so the desired flat analog follows from the observations on $\mathbf{G}_a^\sharp$-acyclic rings recorded above. The preceding flat analog of Remark (b) then also gives the analog of Remark (d).

The rest of the proof of Theorem 2.3.6 then adapts to our present context, with the calculation from Proposition 2.4.4 (especially the variant in Remark 2.4.6) replacing that from Proposition 2.2.13 in determining the Hodge stack. $\square$

Remark 2.5.7 (Local models for $(X/V)^{dR}$)**.** The proof of Theorem 2.5.6 implicitly gives local models for the stack $(X/V)^{dR}$, so let us record them explicitly here. Assume we have an étale map $X \to Y = \widehat{\mathbf{A}^n}$ of p-adic formal schemes, corresponding to $f_1, ..., f_n \in \mathcal{O}(X)$. Then the action of the group scheme $G = (\mathbf{G}_a^\sharp)^n$ on $\widehat{\mathbf{A}^n}$ lifts naturally to X — the action of the i-th copy of $\mathbf{G}_a^\sharp$ corresponds to the derivation $\frac{d}{df_i}$ under the dictionary in Proposition 2.4.4 — and we have $(X/V)^{dR} \simeq X/G$. In particular, the stack $(X/V)^{dR}$ is the quotient of a flat p-adic formal V-scheme by a flat p-adic formal group scheme, and its formation commutes with base change on V. In the special case where V has characteristic p (or is merely p-nilpotent), we learn that $(X/V)^{dR}$ is quite close to being an Artin stack: it is locally the quotient of a smooth V-scheme by a countably presented flat group V-scheme. In contrast, the analogous object over a characteristic 0 ring k was locally the quotient of a smooth k-scheme by a flat *formal* group scheme over k (see part (1) of proof of Theorem 2.3.6). This distinction is a variant of the fact that infinitesimal site in characteristic 0 is defined using nilpotent thickenings, while the (non-PD-nilpotent) crystalline site in characteristic p is defined using PD-thickenings without any nilpotence assumption.

Remark 2.5.8 (Vector bundles on $(X/V)^{dR,+}$)**.** Parallel to Remark 2.3.7, it can be shown via quasisyntomic descent that vector bundles on $(X/V)^{dR,+}$ give rise to the natural coefficient systems for Hodge-filtered de Rham cohomology: they correspond to triples $(E, \nabla, \mathrm{Fil}^\bullet)$, where E is a vector bundle on X, $\mathrm{Fil}^\bullet$ is a filtration of E by subbundles, and $\nabla : E \to \Omega^1_{X/V} \otimes_{\mathcal{O}_X} E$ is a flat connection that satisfies Griffiths transversality with respect to $\mathrm{Fil}^\bullet$ and such that ∇ has nilpotent p-curvature modulo p. We do not explain the proof here except to observe that this can either be shown via quasi-syntomic descent, or by identifying the Cech nerve of the fpqc cover $X \times \mathbf{A}^1/\mathbf{G}_m \to (X/V)^{dR,+}$ (coming via transmutation from the canonical surjection $\mathbf{G}_a \to \mathbf{G}_a^{dR,+}$) with the Rees stack of the PD-envelope $D_{I^\bullet_\Delta}(X^\bullet)$ of small diagonal in the Cech nerve $X^\bullet$ of $X \to \mathrm{Spec}(V)$, where $\mathcal{O}_{D_{\Delta^\bullet}(X^\bullet)}$ is given the PD-filtration.

Remark 2.5.9 (Vector bundles on $(X/V)^{Hodge}$)**.** The proof of Theorem 2.5.6 shows that $(X/V)^{Hodge}$ is the classifying stack $B\mathbf{V}(T_X^\sharp(-1))$ over $X \times B\mathbf{G}_m$. Consequently, vector bundles on $(X/V)^{Hodge}$ can be identified with graded Higgs bundles on X where the Higgs field decreases degree 1 and is nilpotent[23]; this follows from Proposition 2.4.5. Under the explicit description of $\mathrm{Vect}((X/V)^{dR,+})$ mentioned in Remark 2.5.8, pullback along $(X/V)^{Hodge} \to (X/V)^{dR,+}$ simply corresponds to taking the associated graded.

We end this section by observing that the stacky perspective on de Rham cohomology in mixed characteristic gives a clean conceptual explanation of the "crystalline miracle", i.e., the phenomenon that de Rham cohomology of a smooth projective scheme over $\mathbf{Z}_p$ only depends on the mod p fibre.

Corollary 2.5.10 (Crystalline miracle)**.** *Let X/V be a smooth qcqs p-adic formal scheme. Assume V is p-torsionfree. Then $R\Gamma(X, \Omega^\bullet_{X/V}) \in \mathcal{D}_{p-comp}(V)$ functorially depends on $X_{p=0}$.*

[23]In fact, the nilpotence of the Higgs field is automatic by grading considerations as we are working with vector bundles. But we prefer to emphasize it with an eye towards generalizations.

Proof. Passing to underlying stacks in Theorem 2.5.6, we learn that $R\Gamma((X/V)^{dR}, \mathcal{O}) \simeq R\Gamma(X, \Omega^\bullet_{X/V})$. It is therefore enough to show that the stack $(X/V)^{dR}$ itself only depends on $X_{p=0}$. Now for any p-nilpotent V-algebra R, we have

$$(X/V)^{dR}(R) = \mathrm{Map}_V(\mathrm{Spec}(\mathbf{G}_a^{dR}(R)), X).$$

We claim that the animated V-algebra $\mathbf{G}_a^{dR}(R)$ admits a functorial V/p-algebra structure. Granting this, the above can be rewritten as

$$(X/V)^{dR}(R) = \mathrm{Map}_V(\mathrm{Spec}(\mathbf{G}_a^{dR}(R)), X) \simeq \mathrm{Map}_{V/p}(\mathrm{Spec}(\mathbf{G}_a^{dR}(R)), X_{p=0}),$$

which clearly implies the corollary. To prove the claim[24], it suffices to show that $G(\mathbf{Z}_p) = \mathrm{Cone}(\mathbf{G}_a^\sharp(\mathbf{Z}_p) \to \mathbf{Z}_p)$ admits an animated $\mathbf{F}_p$-algebra structure: there is a natural map $G(\mathbf{Z}_p) \to \mathbf{G}_a^{dR}(R)$ of animated rings for any p-nilpotent V-algebra R. But $\mathbf{Z}_p$ is p-torsionfree, so the map $\mathbf{G}_a^\sharp(\mathbf{Z}_p) \to \mathbf{Z}_p$ is injective, whence $G(\mathbf{Z}_p)$ is actually a discrete commutative ring. Since $p \in \mathbf{Z}_p$ admits (unique) divided powers, we learn that $G(\mathbf{Z}_p)$ is a $\mathbf{F}_p$-algebra, as wanted. (In fact, one can check that $G(\mathbf{Z}_p) = \mathbf{F}_p$.) □

Remark 2.5.11 (Crystalline Frobenius)**.** In the context of Corollary 2.5.10, assume that $V = W(k)$ for a perfect field k of characteristic p, and let $\phi : W(k) \to W(k)$ be the Frobenius automorphism. Then the functoriality asserted in the corollary implies that the de Rham complex $M := R\Gamma(X, \Omega^\bullet_{X/W(k)}) \in \mathcal{D}_{p-comp}(W(k))$ comes equipped with a natural map $\phi_M : \phi^* M \to M$, induced by functoriality from the relative Frobenius $X_{p=0} \to (X_{p=0})^{(1)} \simeq (\phi^* X)_{p=0}$ over k.

Remark 2.5.12 (The crystallization functor)**.** Set $k = V/p$. The proof of Corollary 2.5.10 actually yields a more precise assertion: there is a functor $(-/V)^{crys}$ from smooth k-schemes to sheaves on p-nilpotent V-algebras as well as a functorial isomorphism $(X/V)^{dR} \simeq (X_{p=0}/V)^{crys}$ for smooth p-adic formal schemes V. Indeed, for a smooth k-scheme Y, we may simply take

$$(Y/V)^{crys}(R) := \mathrm{Map}_{V/p}(\mathrm{Spec}(\mathbf{G}_a^{dR}(R)), Y).$$

Moreover, the discussion in the proof of Theorem 2.5.6 as well as Remark 2.5.7 extends to this setting: the functor $(-/V)^{\mathrm{crys}}$ preserves étale maps (resp. covers), commutes with Tor independent finite limits, and satisfies $(\mathbf{A}^n_k/V)^{\mathrm{crys}} \simeq (\mathbf{A}^n_V/V)^{dR} \simeq \mathbf{A}^n_V/(\mathbf{G}_a^\sharp)^n$. In particular, if $Y \to \mathbf{A}^n_k$ is an étale map, we learn that $(Y/V)^{\mathrm{crys}} \simeq X/(\mathbf{G}_a^\sharp)^n$, where X/V is the unique smooth V-lift of Y determined by the given étale co-ordinates. Thus, $(Y/V)^{\mathrm{crys}}$ admits a Zariski open cover by quotients of a flat affine $\mathrm{Spf}(V)$-schemes by flat affine group $\mathrm{Spf}(V)$-schemes. It follows from these properties that $R\Gamma((Y/V)^{\mathrm{crys}}, \mathcal{O}) \in \mathrm{CAlg}(\mathcal{D}(V))$ is a p-complete commutative algebra in $\mathcal{D}(V)$ lifting the de Rham cohomology algebra $R\Gamma((Y/k)^{dR}, \mathcal{O}) \simeq R\Gamma(Y, \Omega^\bullet_{Y/k}) \in \mathrm{CAlg}(\mathcal{D}(k))$. One can in fact show that there is a natural identification $R\Gamma((Y/V)^{\mathrm{crys}}, \mathcal{O}) \simeq R\Gamma_{\mathrm{crys}}(Y/V)$ with the crystalline cohomology of Y/V, but we do not explain this comparison in these notes. Instead, we shall simply take $R\Gamma((Y/V)^{\mathrm{crys}}, \mathcal{O})$ as a definition of crystalline cohomology.

2.6 The group scheme $\mathbf{G}_a^\sharp$ via the Witt vectors

In this section, we describe the group scheme $\mathbf{G}_a^\sharp$ via the Witt vectors (as used in the proof of Lemma 2.4.7), and deduce several consequences about the stack $\mathbf{G}_a^{dR}$. In the next section, these

[24]See Corollary 2.6.8 for a more conceptual reason for the V/p-algebra structure.

consequences will give (via transmutation) a stacky description of the conjugate filtration on de Rham cohomology in characteristic p.

We write W (resp. W_n) for the ring scheme of p-typical Witt vectors (resp. n-truncated p-typical Witt vectors) over $\mathbf{Z}_{(p)}$ (unless otherwise specified). As schemes, we have $W_n = \prod_{i=0}^n \mathbf{A}^1$ via the Witt components, and $W = \lim_n W_n \simeq \prod_{i\geq 0} \mathbf{A}^1$, so each W_n is smooth over $\mathbf{Z}_{(p)}$ and W itself is pro-smooth (and hence flat) over $\mathbf{Z}_{(p)}$. The restriction, Frobenius, and Verschiebung maps on the Witt vectors give the following maps of schemes

- The projection $R : W \to W_1 := \mathbf{G}_a$ is called the restriction map (or the 0-th ghost map). It is a homomorphism of ring schemes.
- $F : W \to W$ is the Witt vector Frobenius. It is a homomorphism of ring schemes. Moreover, the induced map on the special fibre $W|_{\mathrm{Spec}(\mathbf{F}_p)}$ coincides with the Frobenius endomorphism of the special fibre, i.e., F is a lift of the Frobenius.
- $V : F_*W \to W$ is the Verschiebung map. It is W-module map, is injective on points, and identifies F_*W with $\ker(W \xrightarrow{R} \mathbf{G}_a)$.

These maps satisfy some standard relations, e.g., $FV = p$ always, and $VF = p$ if we restrict the input to characteristic p rings. The following lemma realizes the group scheme $\mathbf{G}_a^\sharp$ via the Witt vectors; it is analogous to (and in fact relies on) the description of divided powers via δ-structures.

Lemma 2.6.1 ([Dri20, Lemma 3.2.6], [BL22a, Variant 3.4.12])**.** *The Frobenius $F : W \to W$ is faithfully flat. Moreover, the composition $W[F] \subset W \xrightarrow{R} \mathbf{G}_a$ lifts uniquely to an isomorphism $W[F] \simeq \mathbf{G}_a^\sharp$.*

Sketch of proof. We shall use the relationship of Witt vectors to δ-rings due to Joyal, so we recall the dictionary next; see [Joy85, Bor16]. One identifies $W = \mathrm{Spec}(\mathbf{Z}_{(p)}\{x\})$ with the spectrum of the free $\mathbf{Z}_{(p)}$-δ-algebra $\mathbf{Z}_{(p)}\{x\}$ on 1 generator. As a ring, we have

$$\mathbf{Z}_{(p)}\{x\} = \mathbf{Z}_{(p)}[x, \delta(x), \delta^2(x), ...].$$

Writing $x = \delta^0(x)$, the δ-structure is the obvious one from the notation. The corresponding Frobenius lift $\phi : \mathbf{Z}_{(p)}\{x\} \to \mathbf{Z}_{(p)}\{x\}$ (determined as the unique δ-endomoprhism satisfying $\phi(x) = x^p + p\delta(x)$) gives rise to the Witt vector Frobenius $F : W \to W$. The addition and multiplication maps on W are determined by the δ-ring maps $\mathbf{Z}_{(p)}\{x\} \to \mathbf{Z}_{(p)}\{a, b\}$ given by $x \mapsto a+b$ and $x \mapsto ab$ respectively. The n-th ghost map $\gamma_n : W \xrightarrow{R\circ F^n} \mathbf{G}_a$ is determined by the function $\phi^n(x) \in \mathbf{Z}_{(p)}\{x\}$. It is easy to see that the resulting map

$$\gamma : W \to \prod_{i\geq 0} \mathbf{G}_a$$

of ring schemes intertwines F with the shift map, and is an isomorphism after inverting p.

We now begin the proof. By the last point above, the map $F : W \to W$ is certainly faithfully flat after inverting p: it corresponds to the shift map on $\prod_{i\geq 0} \mathbf{G}_a$ under γ. As a map of p-torsionfree rings is faithfully flat iff it is faithfully flat after inverting p and after reducing modulo p, we are reduced to checking that $\phi : \mathbf{Z}_{(p)}\{x\} \to \mathbf{Z}_{(p)}\{x\}$ gives a faithfully flat map modulo p. But ϕ is a lift of the Frobenius modulo p, so the claim follows as Frobenius is faithfully flat on $\prod_{i\geq 0} \mathbf{A}^1$ over $\mathbf{F}_p$.

For the final assertion, the uniqueness of the lift is clear: by the previous paragraph, $W[F]$ is $\mathbf{Z}_{(p)}$-flat, while $\mathbf{G}_a^\sharp \to \mathbf{G}_a$ becomes an isomorphism on inverting p. Moreover, the map $W[F] \to \mathbf{G}_a$

is also an isomorphism after inverting p by the description via ghost maps. To analyze the integral picture, let us rename $x_i = \delta^i(x)$. By the above description of W, the group scheme $W[F]$ is identified with the spectrum of

$$\mathcal{O}(W[F]) := \mathbf{Z}_{(p)}[x_0, x_1, x_2, ...]/(x_0^p + px_1, x_1^p + px_2, ...).$$

and the map $W[F] \to \mathbf{G}_a$ determined by x_0. To lift this map (necessarily uniquely) to $\mathbf{G}_a^\sharp$, we must show that the divided powers $\gamma_n(x) := \frac{x^n}{n!} \in \mathcal{O}(W[F])[\frac{1}{p}]$ actually lie in $\mathcal{O}(W[F])$. The equations above show that $\gamma_p(x_0) \in \mathcal{O}(W[F])$. As $\mathcal{O}(W[F])$ is a p-torsionfree δ-ring, an elementary argument with δ-rings (see [BS19, Lemma 2.35]) shows that $\gamma_n(x_0) \in \mathcal{O}(W[F])$ for all $n \geq 1$, which gives the desired lift $W[F] \to \mathbf{G}_a^\sharp$ of $W[F] \to \mathbf{G}_a$. As the latter is a map of group schemes, so is the former by $\mathbf{Z}_{(p)}$-flatness. The resulting map $\mathcal{O}(\mathbf{G}_a^\sharp) \to \mathcal{O}(W[F])$ is injective (as it is an isomorphism after inverting p and both sides are p-torsionfree) and surjective: the equations show that

$$x_n = \pm \frac{x_0^{p^n}}{p^{1+p+...+p^{n-1}}},$$

which equals the p^n-th divided power $\gamma_{p^n}(x_0)$ up to a unit in $\mathbf{Z}_{(p)}^*$ since $p^{1+p+...+p^{n-1}}$ equals $(p^n)!$ up to a unit in $\mathbf{Z}_{(p)}^*$. The lemma follows. □

Remark 2.6.2 (Reformulation via W-module schemes)**.** The formula $xVy = V(Fxy)$ for Witt vector multiplication shows that $VW = \ker(W \twoheadrightarrow \mathbf{G}_a)$ annihilates $W[F]$, so the W-module structure on $W[F]$ factors uniquely through a $\mathbf{G}_a$-module structure. The isomorphism in Lemma 2.6.1 is a W-module isomorphism, and hence $\mathbf{G}_a$-module isomorphism as well. Consequently, Lemma 2.6.1 can be rewritten as a short exact sequence

$$0 \to \mathbf{G}_a^\sharp \to W \xrightarrow{F} F_*W \to 0 \tag{2.6.1}$$

of W-module schemes for the flat topology.

Variant 2.6.3 (The group scheme $\mathbf{G}_m^\sharp$ via the Witt vectors)**.** Work over $\mathbf{Z}_{(p)}$. Let $\mathbf{G}_m^\sharp$ be the PD-hull of $1 \in \mathbf{G}_m$, and let $W^*[F] = \ker(F : W^* \to W^*)$. Then an argument similar to Lemma 2.6.1 shows that the map $W^*[F] \subset W^* \to \mathbf{G}_m$ induced by restriction lifts uniquely to an isomorphism $W^*[F] \simeq \mathbf{G}_m^\sharp$; see [BL22a, Lemma 3.4.11] or [Dri20, §3.3.3]. For future reference, observe that the Teichmuller map $[\cdot] : \mathbf{G}_m \to W^*$ is a multiplicative section to the restriction map $W^* \to \mathbf{G}_m$, and that the F-action on W^* restricts to the p-power map $(-)^p$ on $\mathbf{G}_m \overset{[\cdot]}{\hookrightarrow} W^*$. In particular, passing to the kernel, we obtain an inclusion $\mu_p \subset W^*[F] \simeq \mathbf{G}_m^\sharp$ lifting the inclusion $\mu_p \subset \mathbf{G}_m$. The natural W^* action on W then restricts to an action on $\mathbf{G}_m^\sharp = W^*[F]$ on $\mathbf{G}_a^\sharp = W[F]$; unwinding identifications, one learns that the resulting action of $\mu_p \subset \mathbf{G}_m^\sharp$ on $\mathbf{G}_a^\sharp$ is the natural scalar action (coming from the $\mathbf{G}_m$-action on $\mathbf{G}_a^\sharp$).

For future reference, we note that the existence of divided powers gives a logarithm homomorphism $\log : \mathbf{G}_m^\sharp \to \mathbf{G}_a^\sharp$ of group schemes $\mathrm{Spf}(\mathbf{Z}_p)$, and it can be shown that this map gives an isomorphism $\mathbf{G}_m^\sharp/\mu_p \simeq \mathbf{G}_a^\sharp$ ([BL22a, Lemma 3.5.18]).

The sequence (2.6.1) plays a fundamental role in Drinfeld's approach [Dri20] to syntomification. In the rest of this section, we collect some more results and remarks on related themes that will be useful in the sequel. First, we obtain explicit conditions ensuring $\mathbf{G}_a^\sharp$-acyclicity:

Corollary 2.6.4 (Vanishing of higher $\mathbf{G}_a^\sharp$ cohomology on semiperfect rings)**.** *If R is a semiperfect $\mathbf{F}_p$-algebra, then $H^{>0}_{fl}(\mathrm{Spec}(R), \mathbf{G}_a^\sharp) = 0$.*

This corollary gives a convenient basis for the flat topology on $\mathbf{F}_p$-algebras where $R\Gamma_{fl}(-, \mathbf{G}_a^\sharp)$ is concentrated in degree 0. Restricted to this basis, functors like $\mathbf{G}_a^{dR}(-)$ as well as various variants encountered in the sequel have quite explicit descriptions, e.g., $\mathbf{G}_a^{dR}(R) = \mathrm{Cone}(\mathbf{G}_a^\sharp(R) \to R)$ for R semiperfect.

Proof. The assumption on R ensures that $F : W(R) \to W(R)$ is surjective (as it is given by the Frobenius on the Witt components), so the sequence in Remark 2.6.1 implies the claim on applying $R\Gamma_{fl}(\mathrm{Spec}(R), -)$. $\square$

Next, we obtain a presentation of $\mathbf{G}_a^\sharp$ as a profinite group scheme:

Corollary 2.6.5 ($\mathbf{G}_a^\sharp$ as an inverse limit of finite flat group schemes)**.** *Work over $\mathbf{F}_p$. Then the $\mathbf{G}_a$-module $\mathbf{G}_a^\sharp$ is the inverse limit of the following tower of finite flat $\mathbf{G}_a$-module schemes with finite flat and surjective transition maps:*

$$....W_{n+1}[F] \to W_n[F] \to .... \to W_1[F] = \alpha_p.$$

Moreover $\ker(W_{n+1}[F] \to W_n(F)) \simeq F_*^n \alpha_p$ *as $\mathbf{G}_a$-modules for all n.*

Proof. Throughout this proof, we repeatedly use the fact that Frobenius is an endomorphism of the identity functor on schemes of characteristic p, i.e., any map of $\mathbf{F}_p$-schemes commutes with the Frobenius.

The description as the inverse limit is immediate from (2.6.1) as the V-adic filtration on W is Frobenius stable over $\mathbf{F}_p$. As the map $F : W_n \to W_n$ is simply the Frobenius endomorphism of the smooth $\mathbf{F}_p$-scheme W_n, it is finite flat. Consequently, each $W_n[F]$ is a finite flat group scheme. Using the faithful flatness of Frobenius on $F_*^n \mathbf{G}_a = \ker(W_{n+1} \xrightarrow{R} W_n)$, one checks that $W_{n+1}[F] \to W_n[F]$ is surjective as flat sheaves with kernel $F_*^n \mathbf{G}_a[F] = F_*^n \alpha_p$. This implies that $W_{n+1}[F] \to W_n[F]$ is finite flat and surjective with kernel $F_*^n \alpha_p$, as wanted. $\square$

Remark 2.6.6 (The V-adic filtration on $\mathbf{G}_a^\sharp$)**.** Working over $\mathbf{F}_p$, we refer to the filtration on $\mathbf{G}_a^\sharp$ induced by the inverse limit description in Corollary 2.6.5 as the V-adic filtration. Note that the composition $\mathbf{G}_a^\sharp \xrightarrow{gr_V^0} \alpha_p \xrightarrow{\mathrm{can}} \mathbf{G}_a$ is the canonical map. Moreover, the subgroup $\mathrm{Fil}^1 \mathbf{G}_a^\sharp = \ker(\mathbf{G}_a^\sharp \to \alpha_p)$ identifies with $F_* \mathbf{G}_a^\sharp$ as a W-module: indeed, since $FV = VF = p$ as endomorphisms of W, the map $V : F_* W \to W$ induces the desired identification $F_* \mathbf{G}_a^\sharp \simeq \mathrm{Fil}^1 \mathbf{G}_a^\sharp$. Inductively, one can also check that for any $n \geq 1$, we have $F_*^n \mathbf{G}_a^\sharp \simeq \mathrm{Fil}^n \mathbf{G}_a^\sharp$ via V^n.

Remark 2.6.7 (The Cartier dual of $\mathbf{G}_a^\sharp$)**.** An alternate perspective on Corollary 2.6.5 is obtained by identifying the Cartier dual of $\mathbf{G}_a^\sharp$ with $\widehat{\mathbf{G}_a}$: the above inverse limit presentation is then dual to the standard colimit preservation $\mathrm{colim}_n\, \alpha_{p^n} = \widehat{\mathbf{G}}_a$. To prove this identification of Cartier duals, one can argue as in the proof of Proposition 2.2.13.

An important consequence of Lemma 2.6.1 for the sequel is the following, giving a rather different, and much more "finitistic", model for the ring stack $\mathbf{G}_a^{dR}$.

Corollary 2.6.8 (The Witt vector model for $\mathbf{G}_a^{dR}$)**.** *There is a natural quasi-isomorphism between the animated W-algebras $F_* W/p := \mathrm{Cone}(F_* W \xrightarrow{p} F_* W)$ and $\mathbf{G}_a^{dR}$ (as sheaves in the flat topology), where the latter is viewed as a W-algebra via the restriction $W \to \mathbf{G}_a$.*

Proof. In this proof, we freely identify the map $\mathbf{G}_a^\sharp \to \mathbf{G}_a$ with $W[F] \subset W \xrightarrow{R} \mathbf{G}_a$ via Lemma 2.6.1. Consider the following diagram:

$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[d, "\mathrm{can}"] & \mathbf{G}_a^\sharp \oplus F_*W \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d, "(\mathrm{can}, V)"] & F_*W \arrow[d, "p"] \\
\mathbf{G}_a & W \arrow[l, "R"'] \arrow[r, "F"] & F_*W.
\end{tikzcd}$$

This is a commutative diagram of W-module schemes. Moreover, each column is a quasi-ideal: for the middle column, this amounts to the Witt vector identity $xVy = V((Fx)y) = 0$ if $x \in \mathbf{G}_a^\sharp = W[F]$ and $y \in VW = F_*W$. The left and right squares each give maps of quasi-ideals. As $V : F_*W \to W$ is injective with image the kernel of the surjection $R : W \to \mathbf{G}_a$, the left square gives an isomorphism

$$\mathrm{Cone}(\mathbf{G}_a^* \oplus F_*W \to W) \simeq \mathbf{G}_a^{dR}$$

animated W-algebras. Similarly, as $\mathrm{can}\colon \mathbf{G}_a^\sharp \to W$ injective with image the kernel of the surjection $F : W \to W$ (Lemma 2.6.1), the right square gives an isomorphism

$$\mathrm{Cone}(\mathbf{G}_a^* \oplus F_*W \to W) \simeq F_*W/p$$

of animated W-algebras. Combining these gives the desired quasi-isomorphism $\mathbf{G}_a^{dR} \simeq F_*W/p$. $\square$

Remark 2.6.9. We shall later see that the W-algebra stack $W/p = \mathrm{Cone}(W \xrightarrow{p} W)$ captures prismatic cohomology relative to a crystalline prism via transmutation, so Corollary 2.6.8 is essentially a form of the crystalline comparison theorem for prismatic cohomology.

Warning 2.6.10. Work over a base ring k of characteristic p. The animated W-algebra structure on F_*W/p is given by $W \xrightarrow{F} F_*W \to F_*W/p$, so the animated W-algebra F_*W/p is naturally the restriction of scalars of the W-algebra W/p along the Frobenius on W. However, the k-algebra stack $\mathbf{G}_a^{dR}$ is not naturally the restriction of scalars of another k-algebra under the Frobenius on $\mathbf{G}_a$; this will only be true for certain k and will in any case require auxiliary choices[25]. Thus, one cannot use Corollary 2.6.8 to conclude that the theory of algebraic de Rham cohomology relative to k is naturally a Frobenius pullback.

Finally, let us use Lemma 2.6.1 to explicitly compute the homology of $\mathbf{G}_a^{dR}$. Note that, unlike the case of characteristic 0, the homology sheaves are representable and flat over the base.

Corollary 2.6.11 (The homology of $\mathbf{G}_a^{dR}$)**.** *Work over $\mathbf{F}_p$. As sheaves of W-modules for the flat topology, we can naturally identify*

$$\pi_0(\mathbf{G}_a^{dR}) = F_*\mathbf{G}_a \quad \textit{and} \quad \pi_1(\mathbf{G}_a^{dR}) = F_*\mathbf{G}_a^\sharp.$$

In particular, the animated W-algebra $\mathbf{G}_a^{dR}$ is a square-zero extension of $F_*\mathbf{G}_a$ by $BF_*\mathbf{G}_a^\sharp$.

This result will be useful in the sequel (e.g., Proposition 2.7.1) as it lets us understand $\mathbf{G}_a^{dR}(R)$-valued points of a scheme X in terms of deformation theory.

[25]For example, if $k = A/p$ for a p-torsionfree δ-ring A, then for any k-algebra R, the map $A \to k \to R$ refines uniquely to a δ-map $A \to W(R)$, which then induces a map $k = A/p \to W/p(R)$ refining the structure map $k \to R$. This construction realizes $W/p(-)$ as a k-algebra stack, and its restriction of scalars along the Frobenius on k is the k-algebra stack $F_*W/p \simeq \mathbf{G}_a^{dR}$.

Proof. The last part is a general statement: any 1-truncated animated ring R is a square-zero extension of $\pi_0(R)$ by its $\pi_1(R)[1]$. For the first two, we use the model F_*W/p provided by Corollary 2.6.8. Since we are working in characteristic p, we have $p = V(1)$ whence $xp = xV(1) = VFx$. As F is surjective and V is injective, it follows that

$$\pi_0(F_*W/p) = F_*W/F_*VW \simeq F_*\mathbf{G}_a$$

and

$$\pi_1(F_*W/p) = F_* \ker(F) = F_*\mathbf{G}_a^\sharp,$$

as wanted. $\square$

Corollary 2.6.11 implies that the cohomology sheaves of the complex $\left(\mathbf{G}_a^\sharp \xrightarrow{\text{can}} \mathbf{G}_a\right)$ are isomorphic (up to Frobenius twists) to the terms of the same complex. This is similar to what the Cartier isomorphism provides over a perfect field k of characteristic p: the de Rham complex $\Omega^\bullet_{X/k}$ of a smooth k-scheme X has cohomology sheaves $\Omega^i_{X^{(1)}/k}$, which are themselves (Frobenius twists of the) terms of the complex $\Omega^\bullet_{X/k}$. We shall soon see that these phenomena are related: the former implies the latter via transmutation.

2.7 The conjugate filtration and the Deligne–Illusie theorem

In this section, let k be a commutative ring of characteristic p. For any smooth k-scheme X, recall that in §2.5, we have defined the stack $(X/k)^{dR}$ on k-algebras given by

$$(X/k)^{dR}(R) = \mathrm{Map}_k(\mathrm{Spec}(\mathbf{G}_a^{dR}(R)), X),$$

where maps are computed in derived algebraic geometry over k. In this section, we use this perspective on de Rham cohomology to reconstruct the conjugate filtration via stacks. In fact, we give two constructions of the latter: one via a global geometric property of the stack $(X/k)^{dR}$, and other via transmutation from a filtration on the animated k-algebra stack $\mathbf{G}_a^{dR}$.

2.7.1 The stack $(X/k)^{dR}$ as a gerbe

Recall that we have already seen in Theorem 2.5.6

$$R\Gamma((X/k)^{dR}, \mathcal{O}) \simeq R\Gamma(X, \Omega^\bullet_{X/k}),$$

like the corresponding result (Theorem 2.3.6) in characteristic 0. As in the latter case, the evident map $X \to (X/k)^{dR}$, induced via transmutation from the map $\mathbf{G}_a \to \mathbf{G}_a^{dR}$ of k-algebra stacks, is a surjection of flat sheaves geometrizing the degree 0 part of the Hodge filtration. However, unlike characteristic 0, the space $(X/k)^{dR}$ itself has a nice algebro-geometric structure. In fact, we already in Remark 2.5.7 that $(X/k)^{dR}$ is very close to being an algebraic stack. In the present characteristic p context, the situation is even better and there is a global statement: the stack $(X/k)^{dR}$ is naturally a gerbe over a smooth k-scheme.

Proposition 2.7.1 (The de Rham space as a gerbe)**.** *The k-stack $(X/k)^{dR}$ is naturally identified with a $B\mathbf{V}(T_{X^{(1)}/k})^\sharp$-torsor over the Frobenius twist $X^{(1)}$ of X relative to k.*

Proof. For a k-algebra R, Corollary 2.6.11 gives a map $\mathbf{G}_a^{dR}(R) \to F_*\mathbf{G}_a(R) = F_*R$ of animated k-algebras and moreover realizes $\mathbf{G}_a^{dR}(R)$ as a square-zero extension of F_*R by $B\mathbf{G}_a^\sharp(R)$. Via transmutation, this gives a map $(X/k)^{dR} \to X^{(1)}$ of k-stacks and moreover realizes the source as a $B\mathbf{V}(T_{X^{(1)}/k})^\sharp$-torsor (by deformation theory, as X/k is smooth), as wanted. $\square$

We shall write $\nu : (X/k)^{dR} \to X^{(1)}$ for the structure map used above. This geometrizes the idea that the complex $F_{X/k,*}\Omega^\bullet_{X/k}$ is $\mathcal{O}_{X^{(1)}}$-linear. We can then go further and obtain the conjugate filtration as the Leray filtration for this map:

Corollary 2.7.2 (The conjugate filtration)**.** *Let $\nu : (X/k)^{dR} \to X^{(1)}$ be the structure map from Proposition 2.7.1.*

1. *There is a natural isomorphism*

$$\bigoplus_i R^i \nu_* \mathcal{O}_{(X/k)^{dR}} \simeq \wedge^* \Omega^*_{X^{(1)}/k}$$

of graded algebras on $X^{(1)}$.

2. *The Leray filtration on $R\Gamma((X/k)^{dR}, \mathcal{O})$ induced via map ν coincides with the conjugate filtration on $R\Gamma(X, \Omega^\bullet_{X/k})$ via the isomorphism of Theorem 2.5.6.*

3. *The pushforward $R\nu_*\mathcal{O}_{(X/k)^{dR}}$ is naturally identified with $F_{X/k,*}\Omega^\bullet_{X/k}$ in $D_{qc}(X^{(1)})$.*

Proof. 1. The claim would be clear from the calculation in Remark 2.4.6 if the gerbe in Proposition 2.7.1 was trivial (or, rather, trivialized). Even though the gerbe is typically not trivial, the calculation still suffices by the following general statement about gerbes:

($*$) Fix a scheme Y and a flat affine commutative group scheme G/Y. If $f : X \to Y$ and $g : Z \to Y$ are BG-torsors for the flat topology, then there is a natural identification

$$R^i f_* \mathcal{O}_X \simeq R^i g_* \mathcal{O}_Z$$

of quasi-coherent sheaves on Y for all i[26].

Consider the sheaf $T := \underline{\mathrm{Isom}}_{BG}(X, Z)$ of BG-equivariant isomorphisms $X \simeq Z$ on Y-schemes. This sheaf is itself a torsor for BG, whence $\mathcal{D}_{qc}(T)$ has a well-behaved "standard" t-structure with the pullback along $T \to Y$ being t-exact (corrsponding to flatness of this map). Moreover, the statement ($*$) in question is clearly true after pullback along $T \to Y$. It therefore suffices to prove that the pullback functor $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(T)$ is fully faithful. By descent, this can be checked flat locally on Y, so we may assume $T = BG$. But then we have faithfully flat maps $Y \xrightarrow{b} T \xrightarrow{a} Y$ with the composition being the identity. Consider the pullback functors $\mathrm{QCoh}(Y) \xrightarrow{a^*} \mathrm{QCoh}(T) \xrightarrow{b^*} \mathrm{QCoh}(Y)$. By faithful flatness, both a^* and b^* are faithful. But $b^* \circ a^* = \mathrm{id}^*$ is the identity, so it immediately follows that a^* is fully faithful.

2. When X is affine, the claim is automatic as both filtrations are simply the canonical filtrations of $R\Gamma(X, \Omega^\bullet_{X/k})$. The claim then follows in general as both filtrations form étale sheaves by the description of their associated graded pieces (coming from part (1) and the classical Cartier isomorphism respectively).

3. It suffices to construct a natural such identification for affine X, so assume $X = \mathrm{Spec}(R)$ is affine. Unwinding definitions, our task is to identify $R\Gamma((X/k)^{dR}, \mathcal{O}_{(X/k)^{dR}})$ with $R\Gamma(X, \Omega^\bullet_{X/k})$ in an $R^{(1)}$-linear fashion. In fact, Theorem 2.5.6 already supplies an isomorphism of the underlying k-complexes, so it suffices to show this identification is $R^{(1)}$-linear. This can be seen by running the degeneration argument proving Theorem 2.5.6 over $F_*\mathbf{G}_a$, i.e., using the

[26] We do not claim that this comes from an isomorphism between the complexes $Rf_*\mathcal{O}_X$ and $Rg_*\mathcal{O}_Y$; in fact, these will typically not be isomorphic.

map $\mathbf{G}_a^{dR,+} \to F_*\mathbf{G}_a$ of $\mathbf{G}_a$-algebra stacks over $\mathbf{A}^1/\mathbf{G}_m$ induced by the Frobenius $\mathbf{G}_a \to F_*\mathbf{G}_a$. We omit the details.

□

Remark 2.7.3 ($\mathbf{G}_a^{dR}$ as a sheaf of groupoids)**.** Corollary 2.6.11 gives a fibre sequence

$$BF_*\mathbf{G}_a^\sharp \to \mathbf{G}_a^{dR} \to F_*\mathbf{G}_a$$

of sheaves of $\mathbf{G}_a$-modules in the flat topology. In particular, regarded merely as sheaves of groupoids (which allows us to suppress F_*), this sequence realizes the stack $\mathbf{G}_a^{dR}$ as a $B\mathbf{G}_a^\sharp$-torsor over $\mathbf{G}_a$ or equivalently a $\mathbf{G}_a^\sharp$-gerbe over the affine line $\mathbf{G}_a$. Such gerbes are classified by $H^2(\mathbf{G}_a, \mathbf{G}_a^\sharp)$, and thus vanish by Lemma 2.4.7. Thus, we learn that there exists an isomorphism

$$\mathbf{G}_a^{dR} \simeq \mathbf{G}_a \times B\mathbf{G}_a^\sharp$$

of sheaves of groupoids[27]. Such an isomorphism cannot be upgraded to an isomorphism of ring stacks, i.e., $\mathbf{G}_a^{dR}$ is *not* isomorphic (as an $\mathbf{F}_p$-algebra stack) to the split square-zero extension $\mathbf{G}_a \oplus B\mathbf{G}_a^\sharp$. Indeed, if it were, the gerbe from Proposition 2.7.1 would be trivial for all smooth X, whence Corollary 2.7.2 would imply that the conjugate spectral sequence must always degenerate for any smooth X; this is well-known to be false. In fact, Petrov recently found examples of smooth projective k-varieties X that admit a flat lift to $W(k)$ with the property that the conjugate spectral sequence does not degenerate (see [Ill22]), answering an old problem from [DI87]. The existence of such examples also implies that $\mathbf{G}_a^{dR}$ is not isomorphic to the split square-zero extension $\mathbf{G}_a \oplus B\mathbf{G}_a^\sharp$ as a $\mathbf{Z}$-algebra stack. It is not clear to the author if this reasoning can be reversed: how far is the previous statement from the existence of examples mentioned in the statement before?

Remark 2.7.4 (Splitting the gerbe and the Azumaya property)**.** For a smooth k-variety X, the preceding discussion gives a factorization

$$X \xrightarrow{\pi} (X/k)^{dR} \xrightarrow{\nu} X^{(1)} \tag{2.7.1}$$

over k of the relative Frobenius $F_{X/k}$, obtained via transmutation from the composition

$$\mathbf{G}_a \to \mathbf{G}_a^{dR} \to \pi_0(\mathbf{G}_a^{dR}) = F_*\mathbf{G}_a$$

of maps of k-algebra stacks. The map π is a flat surjection (proof of Theorem 2.5.6), while the map ν is a $B\mathbf{V}(T_{X^{(1)}/k})^\sharp$-torsor (Proposition 2.7.1) with the property that $R\nu_*\mathcal{O}_{(X/k)^{dR}} \simeq F_{X/k,*}\Omega^\bullet_{X/k}$ (Corollary 2.7.2). Using quasi-syntomic descent, one may upgrade the last statement to a categorical one: the ∞-category $\mathcal{D}_{qc}((X/k)^{dR})$ identifies with the ∞-category $\mathrm{Crys}(X/k)$ of crystals of quasi-coherent complexes on the crystalline site $(X/k)_{\mathrm{crys}}$ or equivalently with derived ∞-category $\mathcal{D}_{qc,nilp}(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules whose homology sheaves are quasi-coherent and have locally nilpotent p-curvature; see [BL22b, Theorem 6.5] for a variant of this assertion for prismatic crystals. Under this equivalence, pullback along π corresponds to passage to the underlying $\mathcal{O}_X$-module, while the pullback ν^* corresponds to observing that the pullback $F_{X/k}^*M$ of $M \in \mathcal{D}_{qc}(X^{(1)})$ has a canonical $\mathcal{D}_X$-module structure with p-curvature 0. The above factorization $F_{X/k} = \nu \circ \pi$ gives a natural splitting of the torsor ν after pullback along the relative Frobenius $F_{X/k} : X \to X^{(1)}$, so $F_{X/k}^*(X/k)^{dR} \to X$ identifies with $B\mathbf{V}(F_{X/k}^*T_{X^{(1)}/k})^\sharp \simeq B\mathbf{V}(F_X^*T_{X/k})^\sharp$, whence

$$\mathcal{D}_{qc}(F_{X/k}^*(X/k)^{dR}) \simeq \mathcal{D}_{qc}(\mathbf{V}(\widehat{F_X^*\Omega^1_{X/k}})) \tag{2.7.2}$$

[27] When k is a perfect field, we will see later that the choice of a lift of $\mathbf{G}_a$ to $W(k)$ together with a lift of the Frobenius gives an explicit such splitting.

by Remark 2.4.6; this description can be regarded as an analog at the level of de Rham stacks of the Azumaya property of $\mathcal{D}_{X/k}$[28] [BMRR08, §2]. (The ideas in this remark were independently observed by Petrov–Vologodksy.)

Remark 2.7.5 (F-split varieties and Hodge-to-de Rham degeneration, following Petrov)**.** For X/k smooth, we observed in Remark 2.7.4 that the $B\mathbf{V}(T_{X^{(1)}/k})^\sharp$-torsor $\nu : (X/k)^{dR} \to X^{(1)}$ is canonically split after pullback along the relative Frobenius $F_{X/k} : X \to X^{(1)}$. Thus, we have a preferred identification $F^*_{X/k}(X/k)^{dR} \simeq B\mathbf{V}(T_{X/k})^\sharp$ over X. Passing to cohomology and using Proposition 2.4.5, we obtain an isomorphism

$$F^*_{X/k} F_{X/k,*} \Omega^\bullet_{X/k} \simeq F^*_{X/k} R\nu_* \mathcal{O}_{(X/k)^{dR}} \simeq F^*_{X/k}\left(\bigoplus_i \Omega^i_{X^{(1)}/k}[-i]\right),$$

i.e. the pullback $F^*_{X/k} F_{X/k,*}\Omega^\bullet_{X/k}$ is canonically a direct sum of its cohomology sheaves. This has the following consequence, first observed by Petrov:

Corollary 2.7.6 (Petrov)**.** *Assume X/k is relatively F-split, i.e., the map $\mathcal{O}_{X^{(1)}} \to F_{X/k,*}\mathcal{O}_X$ is $\mathcal{O}_{X^{(1)}}$-linearly split. Then $F_{X/k,*}\Omega^\bullet_{X/k}$ is a direct sum of its cohomology sheaves. In particular, the conjugate spectral sequence degenerates. (Thus, if k is a field, the Hodge-to-de Rham spectral sequence also degenerates.)*

If $\dim(X/k) < p$ then this corollary can be proven via [DI87] as F-split varieties are liftable (see [Zda18, §3.1] for a quick proof without smoothness constraints). However, in general, despite its relatively elementary formulation, we are not aware of a proof of Corollary 2.7.6 that does not pass through the de Rham stack. For completeness, we also remark that Petrov has also extended the preceding to quasi-F-splittings in the sense of [KTT+22] using the Witt vector models for $\mathbf{G}_a^{dR}$.

2.7.2 The conjugate filtration via transmutation

We shall now realize the conjugate filtration via transmutation by interpreting as a filtration on the stack $\mathbf{G}_a^{dR}$ itself, analogously to what we did for the Hodge filtration in §2.5. In fact, we actually give two quasi-ideal models for this filtration: a $\mathbf{G}_a$-algebra model $\mathbf{G}_a^{dR,c}$ with underlying non-filtered object $\left(\mathbf{G}_a^\sharp \to \mathbf{G}_a\right)$ (Construction 2.7.8), and a W-algebra model $\mathbf{G}_a^{dR,c,W}$ with underlying non-filtered object $\left(F_*W \xrightarrow{p} F_*W\right)$ (Construction 2.7.11); one advantage of the latter model is that it allows us to see certain hidden symmetries in de Rham cohomology of W_2-liftable varieties (first noticed by Drinfeld), leading to a refinement of the Deligne–Illusie theorem.

To construct $\mathbf{G}_a^{dR,c}$, the basic idea is to degenerate the cdga $\mathbf{G}_a^{dR}$ to its cohomology algebra (as computed in Corollary 2.6.11) rather than the underlying graded algebra (as we did in Definition 2.5.1). Since the same idea also recurs in constructing $\mathbf{G}_a^{dR,c,W}$, let us first recall the following general construction for degenerating 1-truncated cdgas to their cohomology.

[28] This property asserts that the associative $\mathcal{O}_{X^{(1)}}$-algebra $E := F_{X/k,*}\mathcal{D}_X$ is Azumaya over its center Z, and is naturally split after pullback along the faithfully flat map $F_{X/k}$. More precisely, one first identifies $Z \simeq \mathrm{Sym}_{\mathcal{O}_{X^{(1)}}}(T_{X^{(1)}/k})$ using p-curvatures, and then shows that $F^*_{X/k}E$ is naturally a split Azumaya algebra over $F^*_{X/k}Z \simeq \mathrm{Sym}_{\mathcal{O}_X}(F^*_X T_{X/k})$, whence Morita theory gives an equivalence

$$\mathcal{D}_{qc}(X, F^*_{X/k}E) \simeq \mathcal{D}_{qc}(X, \mathrm{Sym}_{\mathcal{O}_X}(F^*_X T_{X/k})) \simeq \mathcal{D}_{qc}(\mathbf{V}(F^*_X \Omega^1_{X/k})),$$

which can be compared in (2.7.2) above, with the tautological action of $B\mathbf{V}(T^\sharp_{X^{(1)}/k})$ on the torsor $(X/k)^{dR} \to X^{(1)}$ providing a geometric analog of the p-curvature map $\mathrm{Sym}_{\mathcal{O}_{X^{(1)}}}(T_{X^{(1)}/k}) \to E$.

Construction 2.7.7 (Explicitly degenerating a 1-truncated cdga to its cohomology algebra)**.** Work over some fixed base ring A. Let $d : I \to B$ be a quasi-ideal over A, representing an animated A-algebra B/I. We may then endow B/I with the (decreasing) Postnikov filtration, normalized so that $\mathrm{gr}^1 = K[1] := \ker(d)$, $\mathrm{gr}^0 = B/d(I)$, and $\mathrm{gr}^i = 0$ for $i \neq 0, 1$. Under the Rees equivalence, this filtered animated A-algebra can be seen as a 1-truncated animated algebra $\widetilde{B/I}$ in quasi-coherent sheaves on $\mathbf{A}^1/\mathbf{G}_m$ that recovers B/I over $\mathbf{G}_m/\mathbf{G}_m$ and gives $K(1)[1] \oplus B/d(I)$ over $B\mathbf{G}_m$, i.e., $\widetilde{B/I}$ degenerates the animated A-algebra B/I to its cohomology algebra. In fact, since we have an explicit quasi-ideal model for B/I, we also obtain an explicit quasi-ideal $\widetilde{d} : \widetilde{I} \to \widetilde{B}$ representing $\widetilde{B/I}$ in $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m) \simeq \mathrm{Mod}_{\mathrm{gr}}(A[t])$ that can be described as follows: $\widetilde{B}$ is the graded $A[t]$-algebra $B[t]$, while the graded $B[t]$-module $\widetilde{I}$ is defined via the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & K[t] \arrow[r] \arrow[d] & I[t] \arrow[r] \arrow[d] & I/K[t] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \frac{1}{t}K[t] \arrow[r] & \widetilde{I} \arrow[r] & I/K[t] \arrow[r] & 0
\end{tikzcd}$$

of exact sequences where the map on the left is the obvious one. The differential $\widetilde{d} : \widetilde{I} \to \widetilde{B} = B[t]$ is determined by the map $d[t] : I[t] \to B[t]$ and the 0 map $\frac{1}{t}K[t] \to B[t]$ via the pushout description. This an evident map $\frac{1}{t}K[t][1] \to \widetilde{B}/\widetilde{I}$ of graded $B[t]$-complexes identifying the source with $\pi_1(\widetilde{B}/\widetilde{I})[1]$; similarly, there is an obvious map $\widetilde{B}/\widetilde{I} \to B/d(I)[t]$ identifying the target with $\pi_0(\widetilde{B}/\widetilde{I})$. Thus, the homology of $\widetilde{B/I}$ is "constant" over $\mathbf{A}^1/\mathbf{G}_m$. For future reference, we note that the construction carrying the quasi-ideal $(d : I \to B)$ over A to the quasi-ideal $\left(\widetilde{d} : \widetilde{I} \to \widetilde{B}\right)$ over $\mathbf{A}^1/\mathbf{G}_m$ is functorial, and preserves quasi-isomorphisms.

Specializing this construction to $\mathbf{G}_a^{dR} = \mathrm{Cone}(\mathbf{G}_a^\sharp \to \mathbf{G}_a)$ leads to the following object:

Construction 2.7.8 (The conjugate filtered de Rham stack $\mathbf{G}_a^{dR,c}$)**.** Over the stack $\mathbf{A}^1/\mathbf{G}_m$ over $\mathrm{Spec}(\mathbf{F}_p)$ with tautological section $u : \mathcal{O} \to \mathcal{O}(1)$, consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_*\mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] & \mathbf{G}_a^\sharp \arrow[r, "\mathrm{pr}"] \arrow[d] \arrow[dd, bend right, "\mathrm{can}"'] & \alpha_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_*\mathbf{V}(\mathcal{O}(1))^\sharp \arrow[r] \arrow[d, "0"] & G_u \arrow[r] \arrow[d, "d_u"] & \alpha_p \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbf{G}_a \arrow[r, equal] & \mathbf{G}_a \arrow[r] & 0.
\end{tikzcd} \tag{2.7.3}$$

Here all rows are exact; $u^\sharp$ is the map induced by the tautological section u; the first row is the exact sequence coming from gr^0 of the V-adic filtration on $\mathbf{G}_a^\sharp = W[F]$ (Remark 2.6.6), while the second row is induced via pushout from the first; the map d_u is the unique map making the rest of the diagram commute. One checks that the diagram commutes and moreover that d_u is a quasi-ideal. We then define the animated $\mathbf{G}_a$-algebra stack $\mathbf{G}_a^{dR,c}$ over $\mathbf{A}^1/\mathbf{G}_m$ as

$$\mathbf{G}_a^{dR,c} := \mathrm{Cone}(G_u \xrightarrow{d_u} \mathbf{G}_a).$$

This stack has the following features:

- Homotopy: One has identifications

$$\pi_0(\mathbf{G}_a^{dR,c}) \simeq F_*\mathbf{G}_a \quad \text{and} \quad \pi_1(\mathbf{G}_a^{dR,c}) \simeq F_*\mathbf{V}(\mathcal{O}(1))^\sharp,$$

  induced by the Frobenius $\mathbf{G}_a \to \mathbf{G}_a/\alpha_p \simeq F_*\mathbf{G}_a$ and the inclusion $F_*\mathbf{V}(\mathcal{O}(1))^\sharp \to G_u$. In particular, $\mathbf{G}_a^{dR,c} \to \pi_0(\mathbf{G}_a^{dR,c}) = F_*\mathbf{G}_a$ is naturally a square-zero extension of the target by $BF_*\mathbf{V}(\mathcal{O}(1))^\sharp$ in animated $\mathbf{G}_a$-algebra stacks over $\mathbf{A}^1/\mathbf{G}_m$.

- The underlying non-filtered animated $\mathbf{G}_a$-algebra stack: $\mathbf{G}_a^{dR,c}|_{\mathbf{G}_m/\mathbf{G}_m} \simeq F_*W/p \simeq \mathbf{G}_a^{dR}$. In fact, by the middle column in the above diagram, there is a map $\mathbf{G}_a^{dR} \to \mathbf{G}_a^{dR,c}$ over $\mathbf{A}^1/\mathbf{G}_m$ that becomes an isomorphism over the open substack $\{u \neq 0\} = \mathbf{G}_m/\mathbf{G}_m$.

- The associated graded[29] animated $\mathbf{G}_a$-algebra stack: $\mathbf{G}_a^{dR,c}|_{B\mathbf{G}_m} \simeq F_*\mathbf{G}_a \oplus BF_*\mathbf{V}(\mathcal{O}(1))^\sharp$.

As before, given a smooth k-scheme X, transmutation gives a filtered stack $\pi_X : (X/k)^{dR,c} \to \mathbf{A}^1/\mathbf{G}_m$:

$$(X/k)^{dR,c}(\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m) = \mathrm{Map}_k(\mathrm{Spec}(\mathbf{G}_a^{dR,c}(R)), X).$$

We can now prove the promised geometrization of the conjugate filtration via $X^{dR,c}$:

Theorem 2.7.9 (The conjugate filtration via ring stacks)**.** *For X/k smooth and finitely presented and $\pi_X : (X/k)^{dR,c} \to \mathbf{A}^1/\mathbf{G}_m$ as above, the pushforward $\mathcal{H}_{dR,c}(X) := R\pi_{X,*}\mathcal{O}_{(X/k)^{dR,c}}$ is quasi-coherent and complete. The corresponding object of $\mathcal{DF}(k)$ is identified with $\mathrm{Fil}_*^{conj} R\Gamma(X, \Omega^\bullet_{X/k})$.*

Proof. Write $G = \mathbf{V}(\mathcal{O}(1))^\sharp = \pi_1(\mathbf{G}_a^{dR,c})$. The map $\mathbf{G}_a^{dR,c} \to \pi_0(\mathbf{G}_a^{dR,c}) = F_*\mathbf{G}_a$ of k-algebra stacks over $\mathbf{A}^1/\mathbf{G}_m$ is a square-zero extension of the target by BG (Construction 2.7.8). Via transmutation and derived deformation theory, it gives us a map

$$\nu : (X/k)^{dR,c} \to X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m$$

of stacks over $\mathbf{A}^1/\mathbf{G}_m$ which is a $B\mathbf{V}(T_{X^{(1)}/k}(1))^\sharp$-torsor. To prove the theorem, it suffices to identify the pushforward $R\nu_*\mathcal{O}_{(X/k)^{dR,c}} \in \mathcal{D}_{qc}(X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m)$ under the Rees equivalence with the filtered object $\mathrm{Fil}_*^{\mathrm{conj}} F_{X/k,*}\Omega^\bullet_{X/k} \in \mathcal{DF}_{qc}(X^{(1)})$. We already know that the underlying non-filtered objects are identified by Corollary 2.7.2. As the conjugate filtration on $F_{X/k,*}\Omega^\bullet_{X/k}$ is simply the canonical filtratiom, we may use Remark 2.2.9 to reduce to checking that $R\nu_*\mathcal{O}_{(X/k)^{dR,c}}$ is complete and that each $\mathcal{H}^i(R\nu_*\mathcal{O}_{(X/k)^{dR,c}})(i) \in \mathrm{QCoh}(X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m)$ is pulled back from $X^{(1)}$. But the relative cohomology sheaves of a BG-torsor are insensitive to the torsor structure by the proof of Corollary 2.7.2 (1). We may thus replace the BG-torsor ν with the trivial torsor $\nu' : BG \to X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m$ in the statement, i.e., it suffices to show that $R\nu'_*\mathcal{O}_{BG}$ is complete and that each $\mathcal{H}^i(R\nu'_*\mathcal{O}_{BG})(i) \in \mathrm{QCoh}(X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m)$ is pulled back from $X^{(1)}$. Both these properties follow from the Koszul description $R\nu'_*\mathcal{O}_{BG}$ in Remark 2.4.6. $\square$

Remark 2.7.10 (Vector bundles on $X^{dR,c}$)**.** Parallel to Remark 2.5.8, it can be shown via quasisyntomic descent (but we do not do so here) that vector bundles on $(X/k)^{dR,c}$ provide the natural notion of coefficients for conjugate filtered de Rham cohomology: they correspond to triples $(E, \mathrm{Fil}_\bullet, \nabla)$, where E is a vector bundle on X, $\nabla : E \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} E$ is a flat connection, and $\mathrm{Fil}_\bullet$ is a filtration of E subbundles such that ∇ preserves $\mathrm{Fil}_\bullet$ with the induced connection on $\mathrm{gr}_\bullet(E)$ having p-curvature 0. (Note that these structures force ∇ to have nilpotent p-curvature; to extend such assertions to describe more general quasi-coherent sheaves, one must enforce that ∇ has nilpotent p-curvature.)

[29] Note that the twist appearing here is opposite to the one appearing in the Hodge filtered version $\mathbf{G}_a^{dR,+}$ in Definition 2.5.1. This change of signs is expected: the conjugate filtration is an *increasing* $\mathbf{N}$-indexed filtration, while the Hodge filtration is a *decreasing* $\mathbf{N}$-indexed filtration.

In the rest of this section, we shall give a Witt vector model for the conjugate filtered de Rham stack $\mathbf{G}_a^{dR,c}$ from Construction 2.7.8 and discuss an application. To motivate the construction, recall that the Witt vector model for $\mathbf{G}_a^{dR}$ is given by $F_*W/p = F_*\mathrm{Cone}(W \xrightarrow{p} W)$. The idea, like Construction 2.7.8, is to degenerate this cdga to its cohomology algebra by pushing out $\pi_1(F_*W/p) = F_*\mathbf{G}_a^\sharp$ (Corollary 2.6.11) along the rescaling map $F_*\mathbf{G}_a^\sharp \xrightarrow{u} F_*\mathbf{G}_a^\sharp$ (for a scalar parameter u).

Construction 2.7.11 (The Witt vector model for $\mathbf{G}_a^{dR,c}$)**.** Consider the stack $\mathbf{A}^1/\mathbf{G}_m$ over $\mathrm{Spec}(\mathbf{F}_p)$ with tautological section labelled $u : \mathcal{O} \to \mathcal{O}(1)$. We shall build a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dd, bend right, red, "p=0"'] & W \arrow[r, "F"] \arrow[d] \arrow[dd, bend right, red, "p"'] & F_*W \arrow[r] \arrow[d, equal] \arrow[dd, bend right, red, "p"'] & 0 \\ 0 \arrow[r] & \mathbf{V}(\mathcal{O}(1))^\sharp \arrow[r] \arrow[d, "0"] & M_u \arrow[r] \arrow[d, "d_u"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0. \end{tikzcd} \tag{2.7.4}$$

Here all rows are exact sequences; the top and bottom row are the sequence (2.6.1) with the map between them (indicated in red) being multiplication by p; the second row is defined via pushout from the first row along the map $u^\sharp$; the fact that the left vertical composition (which is clearly 0) equals multiplication by p is simply because $\mathbf{G}_a^\sharp$ is annihilated by p as we work in characteristic p; the map $d_u : M_u \to W$ necessarily factors over the quotient $M_u \twoheadrightarrow F_*W$, and is defined as the composition $M_u \twoheadrightarrow F_*W \xrightarrow{V} W$, where V is Verschiebung. One checks that the diagram commutes and moreover that d_u is a quasi-ideal. We then define the animated W-algebra stack $\mathbf{G}_a^{dR,c,W}$ over $\mathbf{A}^1/\mathbf{G}_m$ as

$$\mathbf{G}_a^{dR,c,W} := F_*\mathrm{Cone}(M_u \xrightarrow{d_u} W).$$

This construction has the following features:

- As the differential d_u factors as $M_u \twoheadrightarrow F_*W \xrightarrow{V} W$, it is easy to see that one has identifications

$$\pi_0(\mathbf{G}_a^{dR,c,W}) \simeq F_*\mathbf{G}_a \quad \text{and} \quad \pi_1(\mathbf{G}_a^{dR,c,W}) \simeq F_*\mathbf{V}(\mathcal{O}(1))^\sharp,$$

  as in the case of $\mathbf{G}_a^{dR,c}$ (Construction 2.7.8).

- We have the following commutative square of W-module maps:

$$\begin{tikzcd} F_*W \arrow[r] \arrow[d, "V"] & F_*M_u \arrow[d, "d_t"] \\ W \arrow[r, "F"] & F_*W. \end{tikzcd}$$

  Here the top horizontal arrow comes by applying F_* to the map $W \to M_u$ appearing in the diagram (2.7.4), and the commutativity amounts to the observation $FV = p$. Each vertical arrow is a quasi-ideal, and the square can be regarded as a map of quasi-ideals. As the left quasi-ideal is a model for $\mathbf{G}_a$, this gives an animated $\mathbf{G}_a$-algebra structure on $\mathbf{G}_a^{dR,c,W}$, and hence an animated k-algebra structure.

The following is now obligatory:

Proposition 2.7.12. *The animated $\mathbf{G}_a$-algebra stacks $\mathbf{G}_a^{dR,c,W}$ and $\mathbf{G}_a^{dR,c}$ are quasi-isomorphic.*

Proof. This is obtained by applying the last sentence from Construction 2.7.7 to Corollary 2.6.8. Let us spell it out for simplicity. Consider the following commutative diagram (explained after the diagram).

$$\begin{tikzcd}
F_*\mathbf{G}_a^\sharp \arrow[r, equal] \arrow[d, "\mathrm{can}"] & F_*\mathbf{G}_a^\sharp \arrow[r, equal] \arrow[d, "(\mathrm{can},-\mathrm{can})"] & F_*\mathbf{G}_a^\sharp \arrow[d, "-\mathrm{can}"] \\
\mathbf{G}_a^\sharp \arrow[d] \arrow[dd, bend right=60, "\mathrm{can}"'] & \mathbf{G}_a^\sharp \oplus F_*W \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d] \arrow[dd, bend right=60, "(\mathrm{can},V)"'] & F_*W \arrow[d] \arrow[dd, bend right=60, "p"'] \\
G_u \arrow[d] & \widetilde{G}_u \arrow[l] \arrow[r] \arrow[d] & F_*M_u \arrow[d] \\
\mathbf{G}_a & W \arrow[l] \arrow[r] & F_*W.
\end{tikzcd} \tag{2.7.5}$$

The subdiagram spanned by the second and fourth rows is the one in the proof of Corollary 2.6.8. The first row is self-explanatory, and the map from the first row to the second comes from the canonical maps $F_*\mathbf{G}_a^\sharp \to \mathbf{G}_a^\sharp$ (Remark 2.6.6) and $F_*\mathbf{G}_a^\sharp \simeq F_*W[F] \subset F_*W$ (from (2.6.1)). Note the first row mapping to the second row is injective realizes the kernel of the second row mapping to the fourth row. The third row and the map to the fourth row are then obtained via pushing out the second row along the tautological map $F_*(u^\sharp) : F_*\mathbf{G}_a^\sharp \to F_*\mathbf{V}(\mathcal{O}(1))^\sharp$.

One then checks that the maps relating the third row to the fourth row are quasi-ideals, and that the squares relating the third and fourth rows give quasi-isomorphisms

$$\mathbf{G}_a^{dR,c} := \mathrm{Cone}(G_u \to \mathbf{G}_a) \xleftarrow{\simeq} \mathrm{Cone}(\widetilde{G}_u \to W) \xrightarrow{\simeq} \mathrm{Cone}(F_*M_u \to F_*W) =: \mathbf{G}_a^{dR,c,W}$$

of animated ring stacks. Moreover, one checks that each of the quasi-isomorphisms appearing above is a map over $\mathrm{Cone}(F_*W \xrightarrow{V} W) \simeq \mathbf{G}_a$, and hence the composite quasi-isomorphism is naturally a $\mathbf{G}_a$-algebra isomorphism. $\square$

Remark 2.7.13 (A $\mathbf{G}_m^\sharp$-action on W/M_u and $\mathbf{G}_a^{dR,c}$)**.** Recall that we introduced the group scheme $\mathbf{G}_m^\sharp = W^*[F]$ in Variant 2.6.3. We now observe that there is a natural action of $W^*[F]$ on the animated ring stack W/M_u over $\mathbf{A}^1/\mathbf{G}_m$ (over k) from Construction 2.7.11. In fact, it simply given by the action on the quasi-ideal $d_u : M_u \to W$ determined by the natural $W^*[F]$-action on M_u and the trivial $W^*[F]$-action on W; to verify the compatibility with d_u, observe that for a local section $\epsilon \in W^*[F]$, one has the following commutative diagram:

$$\begin{tikzcd}
\mathbf{V}(\mathcal{O}(1))^\sharp \oplus W \arrow[r, "\epsilon"] \arrow[d, "\mathrm{can}"] \arrow[dd, bend right=60, "(0{,}p)"'] & \mathbf{V}(\mathcal{O}(1))^\sharp \oplus W \arrow[d, "\mathrm{can}"] \arrow[dd, bend right=60, "(0{,}p)"'] \\
M_u \arrow[r, "\epsilon"] \arrow[d, "d_u"] & M_u \arrow[d, "d_u"] \\
W \arrow[r, equal] & W.
\end{tikzcd}$$

Here the two maps labelled can come from the pushout description of M_u in (2.7.4), the top square commutes by W-linearity of everything in sight, and the outer square commutes as $p\epsilon = V(1)\epsilon = V(F\epsilon) = V(1) = p$ (since $\epsilon \in W^*[F]$ and $p = V(1)$ as we work in characteristic p). This determines

$W^*[F]$-action on the animated W-algebra W/M_u over $\mathbf{A}^1/\mathbf{G}_m$. In fact, this is even an action of $W^*[F]$ on W/M_u regarded as an animated W/p^2-algebra stack: for a local section $\epsilon \in W^*[F]$, the diagram

$$\begin{tikzcd} & \left(p^2W \xrightarrow{\mathrm{can}} W\right) \arrow[dl, "\mathrm{can}"'] \arrow[dr, "\mathrm{can}"] & \\ \left(M_u \xrightarrow{d_u} W\right) \arrow[rr, "\epsilon"] & & \left(M_u \xrightarrow{d_u} W\right) \end{tikzcd}$$

of maps of quasi-ideals over $\mathbf{A}^1/\mathbf{G}_m$ commutes by a similar calculation to the one appearing above to justify the action; here p^2W is simply a copy of W labelled with a generator p^2 to make the maps clear. The resulting action of the subgroup $\mu_p \subset W^*[F]$ on the W/p^2-algebra stack W/M_u has the property that it acts trivially on $\pi_0(W/M_u) = \mathbf{G}_a$ and acts via the natural scalar action on $\pi_1(W/M_u) = \mathbf{V}(\mathcal{O}(1))^\sharp$. By transport of structure, we also obtain a $\mathbf{G}_m^\sharp$-action (and hence μ_p-action) on the animated W/p^2-algebra stack $\mathbf{G}_a^{dR,c}$ over $\mathbf{A}^1/\mathbf{G}_m$; we do not know how to construct this action without passage to the Witt vector model.

Using the action in Remark 2.7.13, we obtain Drinfeld's refinement of the seminal Deligne–Illusie theorem [DI87] on degeneration of the conjugate spectral sequence:

Corollary 2.7.14 (Drinfeld's refinement of Deligne–Illusie)**.** *Assume k is perfect. Let X be a smooth k-scheme equipped with a flat lift to $W_2(k)$. Then there is a natural (in the lift) μ_p-action on $F_{X/k,*}\Omega^\bullet_{X/k}$ giving $\Omega^i_{X^{(1)}/k} \simeq \mathcal{H}^i(F_{X/k,*}\Omega^\bullet_{X/k})$ weight $-i$. In particular, for any integers $a \leq b \leq a+p-1$, the truncation $\tau^{[a,b]}F_{X/k,*}\Omega^\bullet_{X/k}$ is naturally split.*

The proof sketched below follows the perspective advocated in [LM21a]; see [BL22a, Remark 4.7.18] or [Dri20, §5.12.1] for a version of this argument phrased in terms of the prismatization of $W_2(k)$. A slightly weaker variant of the last sentence of Corollary 2.7.14 was first proven via the method of [DI87] and homological algebra in [AS21].

Proof sketch. Let $G = \mathbf{V}(\mathcal{O}(1))^\sharp$, so we have a BG-torsor

$$\nu : (X/k)^{dR,c} \to X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m$$

over $\mathbf{A}^1/\mathbf{G}_m$ in the proof of Theorem 2.7.9, constructed via transmutation from the square-zero extension $\mathbf{G}_a^{dR,c} \to \pi_0(\mathbf{G}_a^{dR,c}) = F_*\mathbf{G}_a$ of animated k-algebra stacks over $\mathbf{A}^1/\mathbf{G}_m$. Theorem 2.7.9 shows that $R\nu_*\mathcal{O}_{(X/k)^{dR,c}} \in \mathcal{D}_{qc}(X^{(1)} \times \mathbf{A}^1/\mathbf{G}_m)$ identifies with $\mathrm{Fil}^{\mathrm{conj}}_* F_{X/k,*}\Omega^\bullet_{X/k} \in \mathcal{DF}_{qc}(X^{(1)})$ under the Rees equivalence. In particular, via the description in Remark 2.2.9, it suffices to construct a μ_p-action on $(X/k)^{dR,c}$ over ν with the feature that the induced action on $R\nu_*\mathcal{O}_{(X/k)^{dR}}$ gives $\mathcal{H}^i$ weight $-i$. Given a flat lift $\widetilde{X}/W_2(k)$ of X/k, we can rewrite the functor of points of $X^{dR,c} \to \mathbf{A}^1/\mathbf{G}_m$ as follows:

$$(X/k)^{dR,c}(\mathrm{Spec}(R) \to \mathbf{A}^1/\mathbf{G}_m) = \mathrm{Map}_k(\mathrm{Spec}(\mathbf{G}_a^{dR,c}(R)), X) \simeq \mathrm{Map}_{W_2(k)}(\mathrm{Spec}(\mathbf{G}_a^{dR,c}(R)), \widetilde{X}).$$

It is now clear that the μ_p-action on the $W_2(k)$-algebra stack $\mathbf{G}_a^{dR,c}$ constructed in Remark 2.7.13 gives a μ_p-action on $(X/k)^{dR,c}$. Moreover, as the μ_p-action on $\mathbf{G}_a^{dR,c}$ acts trivially on $\pi_0(\mathbf{G}_a^{dR,c})$, the resulting μ_p-action on $(X/k)^{dR,c}$ is linear over ν, and hence induces a μ_p-action on $R\nu_*\mathcal{O}_{(X/k)^{dR,c}}$. Our task is to show that this action gives $\mathcal{H}^i(R\nu_*\mathcal{O}_{(X/k)^{dR,c}})$ weight $-i$. But we saw in the proof of Theorem 2.7.9 that $\mathcal{H}^i(R\nu_*\mathcal{O}_{(X/k)^{dR,c}})(i)$ was pulled back from $X^{(1)}$ along the projection $X^{(1)} \times$

$\mathbf{A}^1/\mathbf{G}_m \to X^{(1)}$. It is therefore enough to check that the μ_p-action on $\mathcal{H}^i(R\nu_*\mathcal{O}_{(X/k)^{dR,c}})|_{B\mathbf{G}_m}$ has weight $-i$. Since we know that the formation of cohomology commutes with base change along ν, we are then reduced to checking the analogous assertion for pushforward along $\overline{\nu} : X^{dR,c}|_{B\mathbf{G}_m} \to X^{(1)} \times B\mathbf{G}_m$. But now the situation is very explicit: the stack $\overline{\nu}$ is obtained via transmutation from the split-square extension $\pi_0(\mathbf{G}_a^{dR,c}|_{B\mathbf{G}_m}) \oplus \pi_1(\mathbf{G}_a^{dR,c}|_{B\mathbf{G}_m})[1] = F_*\mathbf{G}_a \oplus F_*\mathbf{V}(\mathcal{O}(1))^\sharp[1]$ of $F_*\mathbf{G}_a$ over $B\mathbf{G}_m$, the map $\overline{\nu}$ corresponds to the projection $F_*\mathbf{G}_a \oplus F_*\mathbf{V}(\mathcal{O}(1))^\sharp[1] \to F_*\mathbf{G}_a$, and the μ_p-action is induced by the trivial action on $F_*\mathbf{G}_a$ and the action on $F_*\mathbf{V}(\mathcal{O}(1))^\sharp$ that is the standard scalar action on the underlying group scheme $\mathbf{V}(\mathcal{O}(1)^\sharp)$. The stack $X^{dR,c}|_{B\mathbf{G}_m}$ then identifies with the classifying stack $B\mathbf{V}(T_{X^{(1)}/k} \boxtimes \mathcal{O}(1))^\sharp$ over $X^{(1)} \times B\mathbf{G}_m$ with μ_p-action induced by the second factor. In this case, one can compute explicitly that $R^i\overline{\nu}_*\mathcal{O}_{X^{dR,c}|_{B\mathbf{G}_m}} \simeq \Omega^i_{X^{(1)}/k} \boxtimes \mathcal{O}(-i) \in \mathrm{QCoh}(X^{(1)} \times B\mathbf{G}_m)$, which has the desired μ_p-weight. $\square$

For completeness, we remark that [DI87] gave the first purely algebraic proof of the degeneration of the Hodge-to-de Rham spectral sequence in characteristic 0: one reduces to the analogous statement mod p by spreading out, then applies Corollary 2.7.14 in the case $p > \dim(X)$ to conclude that conjugate spectral sequence degenerates, which then formally implies that the Hodge-to-de Rham spectral sequence also degenerates.

Remark 2.7.15. The core of the proof of Corollary 2.7.14 shows the following statement: given a flat lift $\widetilde{X}/W_2(k)$ of X/k, there is a natural (in the lift) $\mathbf{G}_m^\sharp$-action on the conjugate filtered de Rham stack $(X/k)^{dR,c} \to \mathbf{A}^1/\mathbf{G}_m$ with certain properties. In particular, given any map $f : Y \to X$ of smooth k-schemes that lifts to $W_2(k)$, the pushforward $Rf_*\mathcal{O}_{(Y/k)^{dR,c}} \in \mathcal{D}_{qc}((X/k)^{dR,c})$ is naturally (in the lift) a $\mathbf{G}_m^\sharp$-equivariant object; this leads to more results analogous to Corollary 2.7.14.

2.8 Glueing the Hodge and conjugate filtrations

Fix a perfect field k of characteristic p. Given a smooth variety X/k, consider the Frobenius twisted[30] de Rham complex

$$E := \phi_* R\Gamma(X, \Omega^\bullet_{X/k}) \simeq (\phi^{-1})^* R\Gamma(X, \Omega^\bullet_{X/k}) \in \mathcal{D}(k),$$

where ϕ is the Frobenius on k. The object E has the following features[31]:

- $\phi^* E$ admits a decreasing Hodge filtration.
- E admits an increasing conjugate filtration.
- The associated graded pieces of both the above filtrations are identified with the Hodge cohomology complex $R\Gamma(X, \Omega^H_{X/k}) := \bigoplus_i R\Gamma(X, \Omega^i_{X/k})[-i]$.

In this section, we construct a ring stack that encodes these structures via transmutation. We have already explained how to see the Hodge and conjugate filtrations individually via transmutation (Theorems 2.5.6 and 2.7.9), so the current task is to glue these descriptions together. This relies on glueing together (up to Frobenius twists) the Witt vector model of $\mathbf{G}_a^{dR,c}$ (Construction 2.7.11) with a Witt vector model for $\mathbf{G}_a^{dR,+}$, described next; these models will eventually naturally lead us to the Nygaard filtration as well. The Witt vector model for $\mathbf{G}_a^{dR,+}$ is the following, directly inspired by the proof Corollary 2.6.8.

[30] The reason for the Frobenius twist is compatibility with the prismatic theory: the object E identifies k-linearly with the Hodge-Tate complex $R\Gamma_{\overline{\Delta}}(X/W(k)) \simeq R\Gamma_{\Delta}(X/W(k))/p$.

[31] Abstract vector spaces with such structures have been studied under the name of F-zips, see [Wed08, §1.8].

Construction 2.8.1 (The Witt vector model for $\mathbf{G}_a^{dR,+}$)**.** Consider the stack $\mathbf{A}^1/\mathbf{G}_m$ over $\mathbf{F}_p$, with tautological section labelled $t : \mathcal{O} \to \mathcal{O}(1)$. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{V}(\mathcal{O}(-1))^\sharp \arrow[r, "i_1"] \arrow[d, "t^\sharp"] & \mathbf{V}(\mathcal{O}(-1))^\sharp \oplus F_*W \arrow[r, "\mathrm{pr}"] \arrow[d, "(t^\sharp, V)"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0, \end{tikzcd} \tag{2.8.1}$$

where both rows are exact sequences. It is relatively straightforward to check that the middle vertical map is a quasi-ideal. Moreover, one has a commutative square

$$\begin{tikzcd} \mathbf{V}(\mathcal{O}(-1))^\sharp \oplus F_*W \arrow[r, "\mathrm{pr}_1"] \arrow[d, "(t^\sharp, V)"] & \mathbf{V}(\mathcal{O}(-1))^\sharp \arrow[d, "t^\sharp"] \\ W \arrow[r, "R"] & \mathbf{G}_a \end{tikzcd}$$

giving a map of quasi-ideals from the left column to the right column. As $V : F_*W \to W$ is exactly the kernel of $R : W \to \mathbf{G}_a$, the above diagram yields a quasi-isomorphism

$$\mathrm{Cone}(\mathbf{V}(\mathcal{O}(-1))^\sharp \oplus F_*W \xrightarrow{(t^\sharp, V)} W) \simeq \mathbf{G}_a^{dR,+},$$

of animated ring stacks, so we can regard the LHS as a Witt vector model for $\mathbf{G}_a^{dR,+}$ (Definition 2.5.1). In fact, the above map is compatible with the natural map from $\mathrm{Cone}(F_*W \xrightarrow{V} W) \simeq \mathbf{G}_a$ to either side, so this quasi-isomorphism is naturally an animated $\mathbf{G}_a$-algebra quasi-isomorphism.

Next, we will build a stack where we can, essentially, glue together the diagrams (2.8.1) and (2.7.4) to capture both the Hodge and conjugate filtrations as well as an isomorphism of their associated graded pieces (up to some Frobenius twists). To handle both an increasing and a decreasing filtration simultaneously, it is quite natural to contemplate the following base, replacing $\mathbf{A}^1/\mathbf{G}_m$ in our previous discussions that involved a single filtration:

Construction 2.8.2 (The stack C[32])**.** Let $R = k[u,t]/(ut)$, regarded as a graded ring where u has degree -1 and t has degree 1. Consider the quotient stack $C = \mathrm{Spec}(k[u,t]/(ut))/\mathbf{G}_m$; it comes equipped with a structure map $C \to B\mathbf{G}_m$. Two further descriptions of this stack are:

- Moduli-theoretic: Given a k-algebra S, the groupoid of k-maps $\eta : \mathrm{Spec}(S) \to C$ identifies with the groupoid of triples (L, u', t'), where $L \in \mathrm{Pic}(S)$ and $u' : S \to L$ and $t' : L \to S$ are maps such that $u't' = 0$ (whence $t'u' = 0$ as well).
- Geometric: The stack C is a union of two components $C_{u=0}$ and $C_{t=0}$, each of which is isomorphic to $\mathbf{A}^1/\mathbf{G}_m$ via the choice of co-ordinates t and u respectively (so the isomorphism $C_{u=0} \simeq \mathbf{A}^1/\mathbf{G}_m$ is compatible with the structure map to $B\mathbf{G}_m$, while the isomorphism $C_{t=0} \simeq \mathbf{A}^1/\mathbf{G}_m$ lives of $-1 : B\mathbf{G}_m \to B\mathbf{G}_m$ on account of u having degree -1); the intersection of these components is a copy of $B\mathbf{G}_m$. In particular, the stack C has 3 points: two open copies of $\mathrm{Spec}(k)$, and one closed k-point with stabilizer $\mathbf{G}_m$.

[32]We shall later learn that $C = k^{\mathcal{N}} \otimes \mathbf{F}_p$.

The moduli interpretation for C gives a universal $L \in \mathrm{Pic}(C)$ together with maps

$$\mathcal{O} \xrightarrow{u} L \quad \text{and} \quad L \xrightarrow{t} \mathcal{O} \quad \text{such that} \quad ut = 0.$$

These correspond to the graded R-module maps

$$R \xrightarrow{u} R(-1) \quad \text{and} \quad R(-1) \xrightarrow{t} R.$$

Bounded above quasi-coherent complexes on C have a relatively simple description: the restriction maps give an equivalence

$$\mathcal{D}^-_{qc}(C) \simeq \mathcal{D}^-_{qc}(C_{t=0}) \times_{\mathcal{D}^-_{qc}(B\mathbf{G}_m)} \mathcal{D}^-_{qc}(C_{u=0}),$$

as one checks by reducing to the analogous statement for the ring R itself (see [Lur18, Theorem 16.2.0.2] for a modern treatment). Translating through the Rees equivalence, one can arrives at the following informal description of quasi-coherent sheaves on C: a quasi-coherent complex on C consists of an increasingly filtered object $G_* \in \mathrm{Fun}((\mathbf{Z}, \leq), \mathcal{D}^-(k)) \simeq \mathcal{D}^-_{qc}(C_{t=0})$, a decreasingly filtered object $F^* \in \mathrm{Fun}((\mathbf{Z}, \leq)^{op}, \mathcal{D}^-(k)) \simeq \mathcal{D}^-_{qc}(C_{u=0})$, and an identification $\mathrm{gr}^*_F \simeq \mathrm{gr}^G_*$ of their associated gradeds in $\mathrm{Fun}(\mathbf{Z}^{\mathrm{disc}}, \mathcal{D}^-(k)) \simeq \mathcal{D}^-_{qc}(B\mathbf{G}_m)$.

In particular, the complex $E = \phi_* R\Gamma(X, \Omega^\bullet_{X/k})$ mentioned at the start of this section, equipped with the conjugate and Hodge filtrations, gives rise to a quasi-coherent sheaf on C via the description mentioned in Construction 2.8.2. Our goal is to obtain E via transmutation, so we need suitable ring stacks. More precisely, over C, we need to glue the ring stack $\mathbf{G}_a^{dR,c}$ over $C_{t=0}$ (up to a Frobenius twist) with the ring stack $\mathbf{G}_a^{dR,+}$ over $C_{u=0}$ to obtain the desired ring stack capturing both conjugate and Hodge filtrations at once. This can be accomplished gracefully via the Witt vector models (Constructions 2.8.1 and 2.7.11) as follows:

Construction 2.8.3 (The ring stack $\mathbf{G}_a^C$ over C[33])**.** Over the stack C, one can build the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dd, bend right, "p=0"'] & W \arrow[r, "F"] \arrow[d] \arrow[dd, bend right, "p"'] & F_*W \arrow[r] \arrow[d, equal] \arrow[dd, bend right, "p"'] & 0 \\
0 \arrow[r] & \mathbf{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp"] & M_u \arrow[r] \arrow[d, "d_{u,t}"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0.
\end{tikzcd} \tag{2.8.2}$$

This diagram is constructed similarly to (2.7.4). A slightly tedious check shows that the map $d_{u,t}$ is a quasi-ideal, so we can consider the ring stack

$$\mathbf{G}_a^C := \mathrm{Cone}(M_u \xrightarrow{d_{u,t}} W).$$

This construction has the following specializations :

- Over the component $C_{t=0}$: the identification $C_{t=0} \simeq \mathbf{A}^1/\mathbf{G}_m$ carries L to $\mathcal{O}(1)$. Under this isomorphism, diagram (2.8.2) identifies with the diagram (2.7.4). In particular, we learn that

$$F_* \mathbf{G}_a^C|_{C_{t=0}} \simeq \mathbf{G}_a^{dR,c}$$

under the identification $C_{t=0} \simeq \mathbf{A}^1/\mathbf{G}_m$.

[33] We shall later see that $\mathbf{G}_a^C \to C$ agrees with the restriction of $\mathbf{G}_a^{\mathcal{N}} \to k^{\mathcal{N}}$ to the divisor $k^{\mathcal{N}} \otimes \mathbf{F}_p \subset k^{\mathcal{N}}$.

- Over the component $C_{u=0}$: the dentification $C_{u=0} \simeq \mathbf{A}^1/\mathbf{G}_m$ carries L to $\mathcal{O}(-1)$. In particular, we learn that the subdiagram spanned by the second and third row in diagram (2.8.2) identifies with diagram (2.8.1), whence

$$\mathbf{G}_a^{\mathcal{N}}|_{C_{u=0}} \simeq \mathbf{G}_a^{dR,+}$$

under the isomorphism $C_{u=0} \simeq \mathbf{A}^1/\mathbf{G}_m$.

- Over the two open copies $j_{HT} : \mathrm{Spec}(k) = C_{u\neq 0} \subset C$ and $j_{dR} : \mathrm{Spec}(k) = C_{t\neq 0} \subset C$: we have isomorphisms

$$j_{HT}^* \mathbf{G}_a^C \simeq W/p \quad \text{and} \quad j_{dR}^* \mathbf{G}_a^C \simeq F_* W/p$$

of W-algebra stacks, giving an isomorphism

$$F_* j_{HT}^* \mathbf{G}_a^C \simeq j_{dR}^* \mathbf{G}_a^C. \tag{2.8.3}$$

of W-algebra stacks over $\mathrm{Spec}(k)$.

- Over the closed point $i_{Hodge} : B\mathbf{G}_m \to C$ (normalized to be inverse to the tautological map $C \to B\mathbf{G}_m$): we have an isomorphism

$$i_{Hodge}^* \mathbf{G}_a^C \simeq \mathbf{G}_a^{Hodge} := \mathbf{G}_a \oplus B\mathbf{V}(L)^\sharp$$

of W-algebra stacks on $B\mathbf{G}_m$.

Applying transmutation to the k-algebra stack $\mathbf{G}_a^C \to C$ then gives:

Definition 2.8.4 (The mod p filtered prismatization)**.** Given a smooth k-variety X, define[34] a C-stack $\pi_X : X^C \to C$ via $X^C(R) = X(\mathbf{G}_a^C(R))$.

Remark 2.8.5 (Local models for X^C)**.** Given a smooth k-scheme X, let us a describe a local model for X^C, analogous to what Remark 2.5.7 provided for $(X/k)^{dR}$. Over the stack C, the middle vertical column in diagram (2.8.2) gives a natural map $W/p \to \mathbf{G}_a^C$ of k-algebra stacks which is surjective fpqc locally, and whose fibre identifies with $\mathrm{Cone}(\mathbf{G}_a^\sharp \xrightarrow{u^\sharp} L^\sharp)$. As the map $L^\sharp \to W$ factors over $\mathbf{G}_a^\sharp \subset W$, we learn the following: for a $\mathbf{G}_a^\sharp$-acyclic test algebra R over $\mathbf{A}^1/\mathbf{G}_m$, the map $W/p(R) \to \mathbf{G}_a^C(R)$ is a map of animated rings which is surjective on π_0 with locally nilpotent kernel. Via transmutation, for a smooth k-scheme X, this gives a surjection[35] $(X^{(-1)}/k)^{dR} \times C \to X^C$ of fpqc sheaves over C with the property that if $X \to Y$ is étale, then the square

$$\begin{CD}
((X^{(-1)}/k)^{dR} \times C @>>> X^C \\
@VVV @VVV \\
(Y^{(-1)}/k)^{dR} \times C @>>> Y^C
\end{CD}$$

is Cartesian with horizontal maps being fpqc surjections. Arguing as in Remark 2.5.7, one then learns that the construction $X \mapsto X^C$ from smooth k-schemes to C-stacks preserves Tor independent finite limits, and carries étale maps (resp. covers) to representable étale maps (resp. covers).

[34] We shall later see that X^C agrees with the divisor $X^{\mathcal{N}} \otimes \mathbf{F}_p \subset X^{\mathcal{N}}$, so we call it the mod p filtered prismatization of X.

[35] Applying transmutation to the k-algebra stack $\mathbf{G}_a^{dR} = F_* W/p$ gives rise to $(X/k)^{dR}$, so applying transmutation to the k-algebra stack W/p gives rise to $(X^{(-1)}/k)^{dR}$, explaining the appearance of the latter. More conceptually, this is the mod p prismatization of X (to be defined later).

In particular, we obtain the following local models: if $f : X \to \mathbf{A}^n$ is an affine étale map of schemes, then $X^C \to (\mathbf{A}^n)^C = (W/M_u)^n$ is an affine étale map of C-stacks, so X^C is affine étale over the quotient of a flat affine C-scheme by a flat affine group C-scheme, and is thus itself[36] the quotient of a flat affine C-scheme by a flat affine group C-scheme.

We can now almost obtain the desired glueing:

Theorem 2.8.6 (Glueing the Hodge and conjugate filtered stacks at their origins)**.** *Given a smooth k-variety X, consider the map $\pi_X : X^C \to C$ from Definition 2.8.4. Then the pushforward*

$$\mathcal{H}_C(X) := R\pi_* \mathcal{O}_{X^C} \in \mathcal{D}_{qc}(C)$$

has the following features:

1. *The restriction $\mathcal{H}_C(X)|_{C_{t=0}}$ to $C_{t=0} \simeq \mathbf{A}^1/\mathbf{G}_m$ identifies (under the Rees equivalence) with $\phi_* R\Gamma(X, \Omega^\bullet_{X/k})$ equipped with the conjugate filtration. In particular, $\mathcal{H}_C(X)|_{C_{u\neq 0}} \simeq \phi_* R\Gamma(X, \Omega^\bullet_{X/k})$ in $\mathcal{D}(k)$.*
2. *The restriction $\mathcal{H}_C(X)|_{C_{u=0}}$ to $C_{u=0} \simeq \mathbf{A}^1/\mathbf{G}_m$ identifies (under the Rees equivalence) with $R\Gamma(X, \Omega^\bullet_{X/k})$ equipped with the Hodge filtration. In particular, $\mathcal{H}_C(X)|_{C_{t\neq 0}} \simeq R\Gamma(X, \Omega^\bullet_{X/k})$ in $\mathcal{D}(k)$.*
3. *The restriction $\mathcal{H}_C(X)|_{B\mathbf{G}_m}$ to the intersection $B\mathbf{G}_m = C_{u=0} \cap C_{t=0} \subset C$ identifies (under the Rees equivalence) with the graded Hodge complex $R\Gamma(X, \Omega^H_{X/k})$.*

Comments on the proof. If the formation of the pushforward $\mathcal{H}_C(X) = R\pi_* \mathcal{O}_{X^C}$ commuted with restriction to the loci $C_{u=0}$, $C_{t=0}$ and $B\mathbf{G}_m$ inside C then the claim would follow from Theorems 2.5.6 and 2.7.9. This base change compatibility is not automatic. Indeed, while the flatness of the local models from Remark 2.8.5 shows that $X^C \to C$ recovers $X^{dR,+} \to C_{u=0}$ and $X^{dR,c} \to C_{t=0}$ by pullback even in the derived sense, we do not formally get a similar assertion at the level of cohomology due to a problem of commuting an infinite limit (computing $R\pi_*$) with an infinite colimit (computing the base change): the cohomological dimension of $X^C \to C$ is *a priori* infinite and the closed immersions $C_{u=0} \to C$, $C_{t=0} \to C$ and $B\mathbf{G}_m \to C$ are not relatively perfect (as C is not a regular stack). This problem can be gracefully resolved by deforming the entire picture to the Witt vectors by replacing the graded k-algebra R with the graded $W(k)$-algebra $\widetilde{R} = W(k)[u,v]/(uv - p)$ as the latter is regular. As this idea naturally leads to the Nygaard filtration, we shall elaborate on it later. ☐

The stack C does not yet encode the fact that the Hodge and conjugate filtration live on the same object (up to a Frobenius twist). For this, we must glue the open points $C_{u\neq 0}$ and $C_{t\neq 0}$ themselves together (with a Frobenius twist) as follows.

Construction 2.8.7 (The stack $\overline{C}$[37])**.** We construct a stack $\overline{C}$ by glueing together the two open points j_{HT} and j_{dR} inside C via the Frobenius, i.e., via a colimit diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \sqcup \mathrm{Spec}(k) \arrow[r, "(j_{HT}, j_{dR})"] \arrow[d, "(\phi^{-1}, \mathrm{id})"] & C \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \overline{C} \end{tikzcd}.$$

[36] This follows from a generality: in any topos, given a group G acting on an object V and a map $Z \to V/G$ of stacks, we have $Z = W/G$, where $W \to Z$ is the G-torsor obtained as the pullback of the G-torsor $V \to V/G$ to Z.

[37] We shall later learn that $\overline{C} = k^{\mathrm{Syn}} \otimes \mathbf{F}_p$.

The k-algebra stack $\mathbf{G}_a^C \to C$ then descends to a k-algebra stack[38] $\mathbf{G}_a^{\overline{C}} \to \overline{C}$ via the isomorphism (2.8.3): to get the Frobenius twists right, observe that the $W(k)$-algebra stack $(\phi^{-1})^* W$ identifies with the $W(k)$-algebra stack $W(\phi_*^{-1}(-)) = F_* W(-)$. Via transmutation, for a smooth k-scheme X, we obtain $X^{\overline{C}} \to \overline{C}$. Pushing forward the structure sheaf along this map then gives a stacky explanation of the last sentence in Theorem 2.8.6 (1) and (2): the object E from Theorem 2.8.6 comes equipped with a canonical identification $\phi^* j_{HT}^* E \simeq j_{dR}^* E$.

[38] We shall later see that $\mathbf{G}_a^{\overline{C}} \to \overline{C}$ agrees with the restriction of $\mathbf{G}_a^{\mathrm{Syn}} \to k^{\mathrm{Syn}}$ along the divisor $k^{\mathrm{Syn}} \otimes \mathbf{F}_p \subset k^{\mathrm{Syn}}$.

Chapter 3

Crystalline cohomology, the Nygaard filtration and Mazur's theorem

Notation 3.0.1. In this section, we fix a perfect field k of characteristic $p > 0$, write $W = W(k)$. For $M \in \mathcal{D}(W)$, we shall write $M/p\{i\} := M \otimes^L_W p^i W/p^{i+1}W$; this is of course isomorphic to M/p, but it is useful to keep track of the twist with an eye towards mixed characteristic generalizations.

Given a smooth k-scheme X, one has its crystalline cohomology algebra $R\Gamma_{\mathrm{crys}}(X/W) \in \mathrm{CAlg}(\mathcal{D}(W))$, giving a functorial lift to W of the de Rham cohomology algebra $R\Gamma(X, \Omega^*_{X/k})$; one definition was given in Remark 2.5.12 in terms of the crystallization functor, i.e., via transmutation from the k-algebra stack $F_*W/p \simeq \mathbf{G}_a^{dR}$ on p-nilpotent W-algebras. The crystalline cohomology complex $R\Gamma_{\mathrm{crys}}(X/W)$ carries a natural filtration called the *Nygaard filtration*; this was first constructed in [Nyg81], though our homological perspective via the Beilinson t-structure will be closer to that of [BO78, BLM21]. To a first approximation, one might regard this filtration as an analog of the Hodge filtration in the setting of crystalline cohomology. Our goal in this section is to understand this filtration via transmutation. As a consequence of its basic properties, we shall explain a proof of Mazur's theorem [Maz73] determining the Hodge filtration from crystalline cohomology in certain situations; besides intrinsic interest, the purity argument we use to establish this theorem will also foreshadow some analogous arguments in mixed characteristic.

3.1 The prismatization over k

In this section, we explain how to compute a Frobenius twist of crystalline cohomology via transmutation; in fact, since we already know how to do this without a twist (Remark 2.5.12), the exposition will be quite brief as k is assumed to be perfect.

Construction 3.1.1 (The prismatization)**.** The ring stack $\mathbf{G}_a^{\Delta} := W/p$ can be regarded as a k-algebra stack on p-nilpotent W-algebras: the canonical map $W(k)/p \to k$ is an isomorphism as k is perfect. Given a smooth k-scheme X, we thus obtain a p-adic formal stack X^{Δ} over W via transmutation:

$$X^{\Delta}(R) = \mathrm{Map}_k(\mathrm{Spec}(W(R)/p), X)$$

for a p-nilpotent W-algebra R. We refer to X^{Δ} as *the prismatization*[39] of X, and write $R\Gamma_{\Delta}(X/W) := R\Gamma(X^{\Delta}, \mathcal{O})$ for its prismatic cohomology. It has the following features:

[39] Strictly speaking, we should write $(X/W)^{\Delta}$ as we are working with relative prismatizations over W. However, since W is perfect, this object identifies canonically with the absolute prismatizaton X^{Δ}. For compactness of notation and with an eye towards mixed characteristic generalizations, we shall thus simply write X^{Δ} (and later $X^{\mathcal{N}}$ and X^{Syn}).

1. Relation to crystallization: since we have an isomorphism $F_* W/p = \mathbf{G}_a^{dR}$ of k-algebra stacks (Corollary 2.6.8), we obtain
$$\phi^* X^{\mathbb{\Delta}} \simeq (X/W)^{\mathrm{crys}}$$
where ϕ denotes the Frobenius on the Witt vectors, and the RHS is defined in Remark 2.5.12. In particular, we have identifications
$$\phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \simeq R\Gamma_{\mathrm{crys}}(X/W) \quad \text{and} \quad \phi^* R\Gamma_{\mathbb{\Delta}}(X/W)/p \simeq R\Gamma(X, \Omega^\bullet_{X/k}),$$
so $R\Gamma_{\mathbb{\Delta}}(X/W)/p$ admits an increasing k-linear conjugate filtration with associated graded $R\Gamma(X, \Omega^{Hodge}_{X/k})$ (without Frobenius twists!).

2. Frobenius lifts: There is a natural map $\phi_{X^{\mathbb{\Delta}}} : X^{\mathbb{\Delta}} \to X^{\mathbb{\Delta}}$ living over $\phi : W \to W$. This can either be defined via functoriality of the Frobenius endomorphism of X, or can also be described as the map induced by the ϕ-semilinear endomorphism $F : W/p(-) \to W/p(-)$ of k-algebra stacks over $\mathrm{Spf}(W)$ coming from the Witt vector Frobenius F; the latter perspective is relevant for mixed characteristic analogs.

3. The Hodge-Tate gerbe: We define the Hodge–Tate stack $X^{HT} = X^{\mathbb{\Delta}} \times_{\mathrm{Spf}(W)} \mathrm{Spec}(k)$ to be the mod p reduction of $X^{\mathbb{\Delta}}$. Then $X^{HT} = (X^{(-1)}/k)^{dR}$ by unwinding definitions, so we can regard X^{HT} as a $B\mathbf{V}(T_{X/k})^\sharp$-torsor over X by Proposition 2.7.1. (Note that the Frobenius twists appearing in the latter have now disappeared!)

4. Local structure of $X^{\mathbb{\Delta}}$: The functor $X \mapsto X^{\mathbb{\Delta}}$ from k-schemes to p-adic formal W-stacks commutes with Tor independent finite limits and carries representable étale maps (resp. covers) to representable étale maps (resp. covers); moreover, each $X^{\mathbb{\Delta}}$ admits a Zariski open cover by quotient of flat affine $\mathrm{Spf}(W)$-schemes by flat affine group $\mathrm{Spf}(W)$-schemes, e.g., $\mathbf{G}_a^{\mathbb{\Delta}}$ is the quotient stack $\mathrm{Cone}(W \xrightarrow{p} W)$. (All of these assertions follow from (1) and Remark 2.5.12.)

5. Quasi-coherent complexes on $X^{\mathbb{\Delta}}$: Tautologically, quasi-coherent sheaves on $X^{\mathbb{\Delta}}$ provide a notion of "coefficients" for prismatic or crystalline cohomology. This notion turns out to be equivalent to the classical notion of crystals. More precisely, using quasi-syntomic descent, one can show the following:

   - $\mathcal{D}_{qc}(X^{\mathbb{\Delta}})$ identified with the ∞-category $\mathrm{Crys}_{\mathbb{\Delta}}(X/W)$ prismatic crystals on $(X/W)_{\mathbb{\Delta}}$, i.e., with the full subcategory of $\mathcal{D}((X/W)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ spanned by objects K such that for every map
$$((B, J), \mathrm{Spf}(B/J) \to X) \to ((A, I), \mathrm{Spf}(A/I) \to X)$$
in $(X/W)_{\mathbb{\Delta}}$, we have
$$K((A, I), \mathrm{Spf}(A/I) \to X) \widehat{\otimes}^L_A B \simeq K((B, J), \mathrm{Spf}(B/J) \to X)$$
via the natural map; this is essentially shown in [BL22a, Theorem 6.5, Lemma 6.3 and Lemma 6.1]. For the purposes of these notes, one can take $\mathcal{D}_{qc}(X^{\mathbb{\Delta}})$ as a definition of $\mathrm{Crys}_{\mathbb{\Delta}}(X/W)$.
   - $\phi^* \mathcal{D}_{qc}(X^{\mathbb{\Delta}})$ identifies with the ∞-category of crystals on $(X/W)_{\mathrm{crys}}$, i.e., the full subcategory of $\mathcal{D}((X/W)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}})$ spanned by objects K such that for every map
$$(U, T, \delta) \to (U', T', \delta')$$
of affine objects in $(X/W)_{\mathrm{crys}}$, we have
$$K(U', T', \delta') \otimes^L_{\mathcal{O}(T')} \mathcal{O}(T) \simeq K(U, T, \delta)$$

via the natural map; see [GR22, Proposition 6.4] for the statement involving perfect complexes.

6. Relationship to δ-lifts: In this remark, we assume the reader is familiar with δ-structures and their relationship to the Witt vectors. If X/k is a smooth k-scheme equipped with a lift $\widetilde{X}/\mathrm{Spf}(W)$ together with a Frobenius $\phi_{\widetilde{X}} : \widetilde{X} \to \widetilde{X}$ over $\phi : W \to W$, then there is a natural map $\widetilde{X} \to X^{\triangle}$ which is a flat cover: the map is defined by noting that any map $\mathrm{Spec}(R) \to \widetilde{X}$ over $\mathrm{Spf}(W)$ extends uniquely to a δ-map $\mathrm{Spf}(W(R)) \to \widetilde{X}$, which then defines a k-scheme map $\mathrm{Spec}(W(R)/p) \to X$ by reduction mod p. (The fact that this map is a flat cover can be seen using derived deformation theory; we do not give the argument here.)

   Example 3.1.2. Say $X = \mathbf{A}^1 = \mathrm{Spec}(k[x])$. We then have the standard lift $\widetilde{X} = \mathbf{A}^1_{\mathrm{Spf}(W)} = \mathrm{Spf}(V[x]^\wedge)$ with the Frobenius lift determined by $x \mapsto x^p$. The above construction gives a map $\widetilde{X} \to X^{\triangle}$ that can be described explicitly as the map

   $$\mathbf{A}^1_{\mathrm{Spf}(W)} \xrightarrow{[\cdot]} W(-)/p$$

   induced by the Teichmuller representative.

3.2 Review of the Nygaard filtration

In this section, we recall the basic properties that characterize the Nygaard filtration on crystalline cohomology. We will *not* recall the classical construction via the de Rham–Witt complex; rather, in §3.3, we construct this filtration via transmutation.

Theorem 3.2.1 (Characterizing of the Nygaard filtration)**.** *For any smooth affine k-scheme X, there is a natural filtered refinement*

$$\phi^*_{X/W} : \mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W) \to p^\bullet R\Gamma_{\mathrm{crys}}(X/W)$$

of the crystalline Frobenius $\phi^* R\Gamma_{\mathrm{crys}}(X/W) \to R\Gamma_{\mathrm{crys}}(X/W)$, uniquely characterized by the following properties:

1. $\mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W)$ *is complete as a filtered object.*
2. *The map $\phi^*_{X/W}$ induces an isomorphism*

$$\mathrm{gr}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W) \simeq \tau^{\leq i} \mathrm{gr}^i_{p^\bullet} R\Gamma_{\mathrm{crys}}(X/W) \simeq \mathrm{Fil}^{conj}_i R\Gamma(X, \Omega^\bullet_{X/k})\{i\}.$$

The filtration $\mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W)$ *is called* the Nygaard filtration*.*

Idea. We shall construct such a filtration using transmutation in §3.3, so let us simply explain why a filtration as in the theorem is unique. We shall use the Beilinson t-structure formalism; essentially, we are spelling out [BL22a, Remark D.11] in our situation.

Property (2) ensures that $\mathrm{gr}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W) \in \mathcal{D}^{\leq i}$ for all i, so $\mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W) \in {}^B\mathcal{DF}^{\leq 0}(W)$ belongs to the connective part of the Beilinson t-structure (Construction 2.2.2 (5)). Hence, the filtered map $\phi^*_{X/W}$ factors uniquely as

$$\mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W) \xrightarrow{\widetilde{\phi^*_{X/W}}} {}^B\tau^{\leq 0} p^\bullet R\Gamma_{\mathrm{crys}}(X/W) \xrightarrow{can} p^\bullet R\Gamma_{\mathrm{crys}}(X/W).$$

We shall prove this map is an isomorphism, which then gives the desired uniqueness. By description of connective covers in the Beilinson t-structure (see [BMS19, Theorem 5.4 (2)]), the isomorphism in property (2) also ensures that the refined Frobenius $\widetilde{\phi^*_{X/W}}$ induces an isomorphism on gr^*. As the source is complete for the filtration by property (1), it suffices to show the same for the target. By generalities on connective covers in the Beilinson t-structure (see [BL22a, Remark D.10]), it suffices to show that $p^\bullet R\Gamma_{\mathrm{crys}}(X/W)$ is complete for the filtration, which is clear. ☐

Remark 3.2.2 (Extension to non-affine smooth k-schemes)**.** For a smooth affine k-scheme X, Theorem 3.2.1 implies that $\mathrm{Fil}^\bullet_{\mathcal{N}}\phi^*R\Gamma_{\mathrm{crys}}(X/W)$ is complete and that $\mathrm{gr}^i_{\mathcal{N}}\phi^*R\Gamma_{\mathrm{crys}}(X/W)$ is described via differential forms. It follows that the assignment $X \mapsto \mathrm{Fil}^\bullet_{\mathcal{N}}\phi^*R\Gamma_{\mathrm{crys}}(X/W)$ is a $\mathcal{DF}(W)$-valued sheaf on smooth k-affines. By descent, for any smooth k-scheme X, one obtains a natural filtered refinement

$$\phi^*_{X/W} : \mathrm{Fil}^\bullet_{\mathcal{N}}\phi^*R\Gamma_{\mathrm{crys}}(X/W) \to p^\bullet R\Gamma_{\mathrm{crys}}(X/W)$$

of the relative Frobenius satisfying (1) and (2) in Theorem 3.2.1; we refer to the left hand side as the Nygaard filtration on $\phi^*R\Gamma_{\mathrm{crys}}(X/W)$.

Remark 3.2.3 (Prismatic realization of the Nygaard filtration)**.** Via Frobenius twisting, one obtains the following variant of Theorem 3.2.1: for a smooth affine k-scheme X, there is a natural filtered refinement $\phi^*_{X/W}\mathrm{Fil}^\bullet_{\mathcal{N}}\phi^*R\Gamma_{\Delta}(X/W) \to p^\bullet R\Gamma_{\Delta}(X/W)$ of the prismatic Frobenius $\phi^*R\Gamma_{\Delta}(X/W) \to R\Gamma_{\Delta}(X/W)$ characterized by the following properties:

1. $\mathrm{Fil}^\bullet_{\mathcal{N}}\phi^*R\Gamma_{\Delta}(X/W)$ is complete.
2. For X affine, the map $\phi^*_{X/W}$ induces an isomorphism

$$\mathrm{gr}^i_{\mathcal{N}}\phi^*R\Gamma_{\Delta}(X/W) \simeq \tau^{\leq i}\mathrm{gr}^i_{p^\bullet}R\Gamma_{\Delta}(X/W)$$

for all i.

Moreover, as in Remark 3.2.2, this globalizes.

Example 3.2.4 (The Nygaard filtration on W itself)**.** For $X = \mathrm{Spec}(k)$ itself, the Nygaard filtration is the p-adic filtration on $\phi^*W \simeq W$.

Example 3.2.5 (The Nygaard filtration in the F-liftable case)**.** Let $X = \mathrm{Spec}(R_0)$ be a smooth affine k-scheme. Assume we are given a lift $\widetilde{X} = \mathrm{Spf}(R)$ to flat p-adic formal W-scheme together an endomorphism $\phi_R : R \to R$ that lifts the Frobenius on $R_0 = R/p$ and is compatible with the Frobenius on W. Then the (continuous) de Rham complex $\Omega^\bullet_{R/W}$ computes $R\Gamma_{\mathrm{crys}}(X/W)$. The Nygaard filtration on $\phi^*\Omega^*_{X/W}$ can be described explicitly as a filtration at the level of complexes. For this, consider the subcomplexes

$$F^i = \left(p^i\phi^*R \xrightarrow{d} p^{i-1}\phi^*\Omega^1_{R/W} \xrightarrow{d} p^{i-2}\phi^*\Omega^2_{R/W} \to \cdots \xrightarrow{d} p^0\phi^*\Omega^i_{R/W} \xrightarrow{d} \phi^*\Omega^{i+1}_{R/W} \xrightarrow{d} \cdots\right)$$

of the complex

$$\phi^*\Omega^*_{R/W} = \left(\phi^*R \xrightarrow{d} \phi^*\Omega^1_{R/W} \xrightarrow{d} \phi^*\Omega^2_{R/W} \to \cdots\right),$$

We claim that the filtration $F^\bullet$ of the complex $\phi^*\Omega^*_{R/W}$ gives the Nygaard filtration $\mathrm{Fil}^\bullet_{\mathcal{N}}\phi^*\Omega^\bullet_{R/W}$ on passage to the derived category. Let us briefly explain why. It is clear that the filtration $F^\bullet$ is complete. Moreover, the relative Frobenius $\phi^*_{R/W} : \phi^*\Omega^*_{R/W} \to \Omega^*_{R/W}$ is divisible by p^i on $\phi^*\Omega^i_{R/W}$, and hence carries $F^\bullet$ to $p^\bullet\Omega^*_{R/W}$, thus giving a filtered map

$$\widetilde{\phi} : F^\bullet \to p^\bullet\Omega^*_{R/W},$$

refining the relative Frobenius $\phi^*_{R/W}$. It is enough to show that this map satisfies property (2) in Theorem 3.2.1. For this, first observe that the graded pieces of both sides are very easy to compute: we have

$$\mathrm{gr}^i_F \phi^*\Omega^*_{R/W} = \left(\phi^* R_0\{i\} \xrightarrow{0} \phi^*\Omega^1_{R_0/k}\{i-1\} \xrightarrow{0} \ldots \xrightarrow{0} \phi^*\Omega^i_{R_0/k} \xrightarrow{0} 0 \xrightarrow{0} 0 \cdots\right) \tag{3.2.1}$$

and

$$\mathrm{gr}^i_{p^\bullet}\Omega^*_{R/W} = \left(R_0\{i\} \xrightarrow{d} \Omega^1_{R_0/k}\{i\} \xrightarrow{d} \Omega^2_{R_0/k}\{i\} \to \ldots\right).$$

Undoing the twist, the resulting map $\mathrm{gr}^i(\widetilde{\phi}) : \mathrm{gr}^i_F\phi^*\Omega^*_{R/W} \to \mathrm{gr}^i_{p^\bullet}\Omega^*_{R/W}$ is induced by the map of complexes given in degree j by the map

$$\phi^*\Omega^j_{R_0/k} \xrightarrow{\frac{\phi^*_{R/W}}{p^j}} \Omega^j_{R_0/k}$$

for $j \leq i$, and the 0 map for $j > i$ (as the j-th term of $\mathrm{gr}^j_F\phi^*\Omega^*_{R/W}$ is 0 for such j). But it is classical that this map induces the Cartier isomorphism, i.e., its image is contained in the cycles, and the resulting map to cohomology gives an isomorphism

$$\phi^*\Omega^j_{R_0/k} \simeq H^j(\Omega^\bullet_{R_0/k}).$$

Thus, the map $\mathrm{gr}^i(\widetilde{\phi})$ induces an isomorphism

$$\mathrm{gr}^i_F\phi^*\Omega^*_{R/W} \simeq \tau^{\leq i}\mathrm{gr}^i_{p^\bullet}\Omega^*_{R/W},$$

as wanted.

Warning 3.2.6 (The naive Nygaard filtration is sometimes too naive)**.** Example 3.2.5 has an evident globalization to a smooth p-adic formal scheme Y/W lifting Y_0/k and equipped with a lift ϕ_Y of the Frobenius compatible with the Frobenius on W: the W-linear filtration $F^\bullet$ on the de Rham complex $\phi^*\Omega^*_{Y/W}$ defined by the subcomplexes

$$F^i := \left(p^i\phi^*\mathcal{O}_Y \xrightarrow{d} p^{i-1}\phi^*\Omega^1_{Y/W} \xrightarrow{d} p^{i-2}\phi^*\Omega^2_{Y/W} \to \cdots \xrightarrow{d} p^0\phi^*\Omega^i_{Y/W} \xrightarrow{d} \phi^*\Omega^{i+1}_{Y/W} \xrightarrow{d} \cdots\right)$$

induces a filtration on $R\Gamma(Y, \phi^*\Omega^*_{Y/W}) \simeq R\Gamma_{\mathrm{crys}}(Y_0/W)$ that is naturally (in the pair $(Y, \phi_{Y/W})$) identified with the Nygaard filtration; this assertion follows immediately from Example 3.2.5 by sheafification. However, we warn the reader that this identificaton critically needs the Frobenius lift ϕ_Y (even though the definition of $F^\bullet$ makes no reference to it), and can fail even for smooth projective Y/W without Frobenius lifts[40] . More precisely, we claim that if the filtered object $R\Gamma(Y, F^\bullet)$ is even abstractly isomorphic to $\mathrm{Fil}^\bullet_N\phi^*R\Gamma_{\mathrm{crys}}(Y_0/W)$, then the conjugate spectral sequence must degenerate for Y_0: indeed, for $i \gg 0$, we have

$$R\Gamma(Y, \mathrm{gr}^i_F) \simeq \bigoplus_j R\Gamma(Y_0, \phi^*\Omega^j_{Y_0/k})[-j]$$

by globalizing (3.2.1) and ignoring twists, so even if we have an abstract isomorphism

$$R\Gamma(Y, \mathrm{gr}^i_F) \simeq R\Gamma(Y, \mathrm{gr}^i_{p^\bullet}\Omega^*_{Y/W}) = R\Gamma(Y_0, \Omega^*_{Y_0/k}),$$

then dimension considerations force the conjugate spectral sequence to degenerate. The warning is then justified as Petrov recently constructed smooth projective Y/W such that the conjugate spectral sequence does not degenerate for Y_0/k (see [Ill22]).

[40]This warning was first brought to our attention by Arpon Raksit, and verifies the expectation formulated in [BMS19, opening paragraph of §8.1.2]; Raksit's argument predated Petrov's example, and relied instead on showing that the ring stacks corresponding to the two filtrations are non-isomorphic.

3.3 The (Nygaard) filtered prismatization

Given a smooth k-scheme X, the Nygaard filtration $\mathrm{Fil}^\bullet_{\mathcal{N}}\phi^* R\Gamma_{\mathrm{crys}}(X/W)$ from Theorem 3.2.1 and Remark 3.2.2 has many remarkable features that encode the known structures on de Rham cohomology of X/k (such as the Hodge and conjugate filtrations as well as the identification of their associated graded objects). Our goal in this section is to construct the Nygaard filtration via transmutation in a manner that makes these structures transparent. The basic geometric object over which all the action takes place is the following:

Construction 3.3.1 (The Rees stack of the p-adic filtration)**.** The Rees algebra of the p-adic filtration on W is the graded $W[t]$-algebra by

$$\mathrm{Rees}(p^\bullet W) := \bigoplus_{i\in\mathbf{Z}} p^{\max(i,0)} W t^{-i}.$$

Writing $u = pt^{-1} \in pWt^{-1}$ for the displayed degree -1 element of the Rees algebra, we obtain a more direct description of this graded ring:

$$\mathrm{Rees}(p^\bullet W) = W[u,t]/(ut-p). \tag{3.3.1}$$

Let $k^{\mathcal{N}}$ be the associated Rees stack on p-nilpotent W-algebras, i.e.,

$$k^{\mathcal{N}} := \mathrm{Spf}(W[u,t]/(ut-p))/\mathbf{G}_m.$$

We shall refer to $k^{\mathcal{N}}$ as *the filtered prismatization of* $\mathrm{Spec}(k)$. This stack will be a key player in what follows, so let us discuss some basic features:

- *Local properties and tautological classes:* The stack $k^{\mathcal{N}}$ is noetherian and even regular and W-flat: indeed, $W[u,t]/(ut-p)$ is a (ramified) p-torsionfree regular ring of dimension 2, and $k^{\mathcal{N}}$ is the quotient of $\mathrm{Spf}(W[u,t]/(ut-p))$ by $\mathbf{G}_m$. Moreover, $k^{\mathcal{N}}$ carries a tautological line bundle L (which is the pullback of $\mathcal{O}(-1)$ under the structure map $k^{\mathcal{N}} \to B\mathbf{G}_m$) with sections

  $$\mathcal{O} \xrightarrow{u} L \quad \text{and} \quad L \xrightarrow{t} \mathcal{O}$$

  whose composition is p. In fact, $k^{\mathcal{N}}$ is the universal p-adic formal W-stack equipped with such a triple (L,u,t).

- *The points of $k^{\mathcal{N}}$:* The special fibre $(k^{\mathcal{N}})_{p=0}$ is exactly the stack C studied in Construction 2.8.2, so the stack $k^{\mathcal{N}}$ has three physical points: the open points[41]

  $$\tilde{j}_{dR} : \mathrm{Spf}(W) = (k^{\mathcal{N}})_{t\neq 0} \hookrightarrow k^{\mathcal{N}} \quad \text{and} \quad j_{HT} : \mathrm{Spf}(W) = (k^{\mathcal{N}})_{u\neq 0} \hookrightarrow k^{\mathcal{N}},$$

  as well as the closed point

  $$i_H : [\mathrm{Spec}(k)/\mathbf{G}_m] = (k^{\mathcal{N}})_{u=t=0} \hookrightarrow k^{\mathcal{N}},$$

  giving a section to the projection $k^{\mathcal{N}} \to B\mathbf{G}_m$ modulo p. Moreover, the zero loci $(k^{\mathcal{N}})_{u=0}$ and $(k^{\mathcal{N}})_{t=0}$ of the sections $u \in H^0(L)$ and $t \in H^0(L^{-1})$ are Cartier divisors given by copies of $\mathbf{A}^1/\mathbf{G}_m$ (over k) that meet transversally along the locus $(k^{\mathcal{N}})_{u=t=0} \simeq B\mathbf{G}_m$.

[41] We use the notation $\tilde{j}_{dR}$ instead of the more suggestive j_{dR} for compatibility with what appears later: j_{dR} will be reserved for a Frobenius twist of $\tilde{j}_{dR}$.

- *Quasi-coherent sheaves on $k^{\mathcal{N}}$:* A coherent sheaf on $k^{\mathcal{N}}$ is the same thing as a finitely generated graded $W[u,t]/(ut-p)$-module. More generally, the ∞-category $\mathcal{D}_{qc}(k^{\mathcal{N}})$ can be seen as any of the following:

  1. $\mathcal{D}_{p-comp,\mathrm{gr}}(W[u,t]/(ut-p))$.
  2. $\mathrm{Mod}_{\mathrm{Rees}(p^{\bullet}W)}(\mathcal{D}_{p-comp,\mathrm{gr}}(W[t]))$.
  3. $\mathrm{Mod}_{p^{\bullet}W}(\mathcal{DF}_{p-comp}(W))$.

  The equivalence with (1) is general stack theory; the equivalence of (1) and (2) comes from the identification (3.3.1), while the equivalence of (2) and (3) is the Rees equivalence. It will later be convenient to switch between these descriptions: (3) is quite conceptual, while (1) is more easily amenable to algebraic arguments. We shall sometimes refer to objects of $\mathcal{D}_{qc}(k^{\mathcal{N}})$ as *gauges over k* (see Definition 3.4.1).

To proceed further via transmutation, we need ring stacks. In fact, as $(k^{\mathcal{N}})|_{p=0} = C$, it is reasonable to expect that the ring stack $\mathbf{G}_a^C$ from Construction 2.8.3 deforms to $k^{\mathcal{N}}$, which it does:

Construction 3.3.2 (The ring stack $\mathbf{G}_a^{\mathcal{N}}$ over $k^{\mathcal{N}}$)**.** Over the stack $k^{\mathcal{N}}$, consider the following diagram of group schemes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^{\sharp} \arrow[r] \arrow[d, "u^{\sharp}"] \arrow[dd, bend right, "p=ut"'] & W \arrow[r, "F"] \arrow[d] \arrow[dd, bend right, "p"'] & F_*W \arrow[r] \arrow[d, equal] \arrow[dd, bend right, "p"'] & 0 \\
0 \arrow[r] & \mathbf{V}(L)^{\sharp} \arrow[r] \arrow[d, "t^{\sharp}"] & M_u \arrow[r] \arrow[d, "d_{u,t}"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^{\sharp} \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0.
\end{tikzcd} \tag{3.3.2}$$

This diagram is constructed similarly to (2.8.2). One checks that the map $d_{u,t}$ is a quasi-ideal, so we can consider the W-algebra stack

$$\mathbf{G}_a^{\mathcal{N}} := \mathrm{Cone}(M_u \xrightarrow{d_{u,t}} W)$$

over $k^{\mathcal{N}}$. In fact, by the diagram, $\mathbf{G}_a^{\mathcal{N}}$ is naturally a W/p-algebra stack, and thus a k-algebra stack since $(W/p)(k) \simeq k$ via the restriction map $W \to \mathbf{G}_a$. Over the locus $C = (k^{\mathcal{N}})_{p=0}$, the diagram above restricts to (2.8.2), whence $\mathbf{G}_a^{\mathcal{N}}|_C = \mathbf{G}_a^C$.

Via transmutation, this gives:

Definition 3.3.3 (The filtered prismatization)**.** Let X/k be a smooth k-scheme. Its *filtered prismatization* is the stack $\pi : X^{\mathcal{N}} \to k^{\mathcal{N}}$ defined via transmutation from $\mathbf{G}_a^{\mathcal{N}}$:

$$X^{\mathcal{N}}(\mathrm{Spec}(R) \to k^{\mathcal{N}}) = \mathrm{Map}_k(\mathrm{Spec}(\mathbf{G}_a^{\mathcal{N}}(R)), X).$$

For X quasi-comapct, we write $\mathcal{H}_{\mathcal{N}}(X) := R\pi_* \mathcal{O}_{X^{\mathcal{N}}} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$.

Remark 3.3.4 (Relating $X^{\mathcal{N}}$ to X^{Δ})**.** There are two natural maps relating $X^{\mathcal{N}}$ to X^{Δ}.

- There is a map $W/p \to \mathbf{G}_a^{\mathcal{N}}$ of k-algebra stacks over $k^{\mathcal{N}}$ coming from the middle vertical column in (3.3.2). Via transmutation, for any smooth k-scheme X, this defines a map

  $$X^{\Delta} \times_W k^{\mathcal{N}} \to X^{\mathcal{N}}$$

  of stacks over $k^{\mathcal{N}}$. Using the local nilpotence of divided powers, one checks that this map is a flat cover. Moreover, by construction, it is an isomorphism over $(k^{\mathcal{N}})_{u\neq 0}$.

- Similarly, the bottom right square in (3.3.2) gives a map $\mathbf{G}_a^{\mathcal{N}} \to F_* W/p$ of k-algebra stacks over $k^{\mathcal{N}}$, which then induces via transmutation a map

$$X^{\mathcal{N}} \to \phi^* X^{\Delta} \times_W k^{\mathcal{N}}$$

over $k^{\mathcal{N}}$; this map is also a flat cover, and is an isomorphism over $(k^{\mathcal{N}})_{t\neq 0}$. Thus, from the perspective of the Rees dictionary with respect to the parameter t, we can regard $X^{\mathcal{N}}$ as a filtration on the stack $\phi^* X^{\Delta}$ with associated graded $(X^{\mathcal{N}})_{t=0}$ (which in turn is described in Remark 3.4.2 as a filtration on the Hodge–Tate stack X^{HT} using the parameter u).

The composition of the two above maps is induced via transmutation from the map $W/p \to F_* W/p$ of k-algebra stacks over $k^{\mathcal{N}}$, and hence can be identified with the map $(X^{\Delta} \xrightarrow{\phi_X} \phi^* X^{\Delta}) \times_W k^{\mathcal{N}}$ induced by the Frobenius on X^{Δ}.

The name given to $X^{\mathcal{N}}$ is justified by part (1) of the following:

Theorem 3.3.5 (The Nygaard filtration via transmutation)**.** *Let X/k be a smooth quasi-compact k-scheme. The object $\mathcal{H}_{\mathcal{N}}(X) \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ has the following comparisons:*

1. *The Nygaard filtration: under the identification of $\mathcal{D}_{qc}(k^{\mathcal{N}})$ with filtered modules over $p^\bullet W$ in $\mathcal{DF}_{p-comp}(W)$, the object $\mathcal{H}_{\mathcal{N}}(X)$ identifies with $\phi_* \mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\mathrm{crys}}(X/W) = \mathrm{Fil}^\bullet_{\mathcal{N}} \phi^* R\Gamma_{\Delta}(X/W)$.*
2. *The de Rham pullback: the object $\tilde{j}^*_{dR} \mathcal{H}_{\mathcal{N}}(X)$ identifies with $R\Gamma_{\mathrm{crys}}(X/W) = \phi^* R\Gamma_{\Delta}(X/W)$.*
3. *The Hodge-Tate pullback: the object $j^*_{HT} \mathcal{H}_{\mathcal{N}}(X)$ identifies with $\phi_* R\Gamma_{\mathrm{crys}}(X/W) = R\Gamma_{\Delta}(X/W)$.*
4. *The Hodge filtration: via the isomorphism $(k^{\mathcal{N}})_{u=0} \simeq \mathbf{A}^1_k/\mathbf{G}_m$ determined by the degree 1 element t, the object $\mathcal{H}_{\mathcal{N}}(X)|_{(k^{\mathcal{N}})|_{u=0}}$ identifies (under the Rees equivalence) with $\mathrm{Fil}^\bullet_H R\Gamma(X, \Omega^*_{X/k})$.*
5. *The conjugate filtration: via the isomorphism $(k^{\mathcal{N}})_{t=0} \simeq \mathbf{A}^1_k/\mathbf{G}_m$ determined by the degree -1 element u, the object $\mathcal{H}_{\mathcal{N}}(X)|_{(k^{\mathcal{N}})|_{t=0}}$ identifies (under the Rees equivalence) with $\phi_* \mathrm{Fil}^{conj}_\bullet R\Gamma(X, \Omega^*_{X/k})$.*
6. *The Hodge pullback: via the identification $\mathrm{Spec}(k)/\mathbf{G}_m \simeq (k^{\mathcal{N}})|_{u=t=0}$, the object $\mathcal{H}_{\mathcal{N}}(X)|_{(k^{\mathcal{N}})|_{u=0}}$ identifies with $\bigoplus_i R\Gamma(X, \Omega^i_{X/k})[-i](i)$.*

Proof. The strategy of the proof is to first prove the identifications (2) - (6), and then deduce the comparison in (1) by checking the properties in Theorem 3.2.1.

First, we observe a general base change property deduced from the regularity of $k^{\mathcal{N}}$. Say we are given a $k^{\mathcal{N}}$-stack Y such that Y can be written as the colimit of a simplicial flat affine $k^{\mathcal{N}}$-scheme $U^\bullet$; note that this applies to $Y = X^{\mathcal{N}}$ by the argument in Remark 2.8.5. Then we claim that the formation of $R\pi_* \mathcal{O}_Y$ commutes with pullback along any locally closed immersion $i : Z \to k^{\mathcal{N}}$ appearing in (2) - (6), i.e., $i^* R\pi_* \mathcal{O}_Y \simeq R\pi'_* \mathcal{O}_{Y_Z}$, where $\pi' : Y_Z \to Z$ is the projection. (In fact, this property holds for any locally closed substack Z of $k^{\mathcal{N}}$.) To see this, using the presentations $U^\bullet \to Y$ and $U^\bullet_Z \to Y_Z$, we are reduced to checking that the formation of cosimplicial totalizations in $\mathcal{D}^{\geq 0}_{qc}(k^{\mathcal{N}})$ commutes[42] with i^*. Now $i_* \mathcal{O}_Z$ a perfect complex over a localization of $\mathcal{O}_{k^{\mathcal{N}}}$ in all cases, so the claim follows as the formation of cosimplicial totalizations in $\mathcal{D}^{\geq 0}_{qc}(k^{\mathcal{N}})$ commutes with both filtered colimits and tensoring with perfect complexes.

Applying the observation in the previous paragraph to $X^{\mathcal{N}} \to k^{\mathcal{N}}$, we arrive at a simplification: the pullbacks appearing in (2) - (6) above can be computed by first restricting $\mathbf{G}_a^{\mathcal{N}}$ to the corresponding locus, and then applying transmutation. We next prove (2) - (6) and deduce (1).

[42] Here $\mathcal{D}^{\geq 0}_{qc}(k^{\mathcal{N}})$ denotes the full subcategory of $K \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ whose pullback to $D_{qc}(\mathrm{Spf}(W[u,t]/(ut-p))) \simeq D_{p-comp}(W[u,t]/(ut-p))$ lies in $\mathcal{D}^{\geq 0}$. Alternately, one could also use the coconnective part of the torsion t-structure.

Thanks to the observation in the previous paragraph, the pullback $\tilde{j}_{dR}^{*}\mathcal{H}_{\mathcal{N}}(X)$ identifies with $R\Gamma_{\mathbb{\Delta}}(X/W)$ by Corollary 2.6.8 and our definition of crystalline cohomology (Remark 2.5.12); this gives (2). The same reasoning shows that $j_{HT}^{*}\mathcal{H}_{\mathcal{N}}(X) \simeq \phi_* R\Gamma_{\mathbb{\Delta}}(X/W)$ as well, giving (3).

Assertion (5) is similarly a consequence of the recovery of the conjugate filtration from the animated k-algebra stack $\mathbf{G}_a^{dR,c}$ over $\mathbf{A}^1/\mathbf{G}_m$ with parameter u (Theorem 2.7.9) and the k-algebra isomorphism $\mathbf{G}_a^{dR,c} \simeq \mathbf{G}_a^{dR,c,W} = F_*\mathbf{G}_a^{\mathcal{N}}|_{(k^{\mathcal{N}})|_{t=0}}$ (Construction 2.7.11 and Proposition 2.7.12), while assertion (4) follows similarly from Theorem 2.5.6 and Construction 2.8.1. Either of these implies assertion (6) by passage to the associated graded.

It remains to prove (1), i.e., to check that the filtered object corresponding $\mathcal{H}_{\mathcal{N}}(X)$ carries the structures/properties required by the Nygaard filtration in Remark 3.2.3. The strategy is the following: we obtain the comparison map using maps of ring stacks coming from the diagram (3.3.2), then obtain the completeness of the Nygaard filtration using (4), and then finally check the shape of the associated graded using (5).

By sheafyness, we can assume X is a smooth affine k-scheme. As in Remark 3.3.4, we have faithfully flat maps

$$W/p \xrightarrow{\psi} \mathbf{G}_a^{\mathcal{N}} \xrightarrow{\eta} F_*W/p$$

of W-algebra stacks with η being an isomorphism over the locus $(k^{\mathcal{N}})|_{t\neq 0}$ and ψ being an isomorphism over the locus $(k^{\mathcal{N}})|_{u\neq 0}$. Via transmutation, this gives rise[43] to maps

$$\mathrm{Rees}(p^{\bullet}\phi^* R\Gamma_{\mathbb{\Delta}}(X/W)) \xrightarrow{\eta^*} \mathcal{H}_{\mathcal{N}}(X) \xrightarrow{\psi^*} \mathrm{Rees}(p^{\bullet} R\Gamma_{\mathbb{\Delta}}(X/W))$$

of commutative algebras in $\mathcal{D}_{qc}(k^{\mathcal{N}})$ with composition induced by the crystalline Frobenius; the map η^* (resp. ψ^*) is an isomorphism over the locus $(k^{\mathcal{N}})|_{t\neq 0}$ (resp. $(k^{\mathcal{N}})|_{u\neq 0}$). Translating back under the Rees equivalence, the above diagram then gives a composition

$$p^{\bullet}\phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \xrightarrow{\alpha} \mathrm{Fil}^{\bullet}\phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \xrightarrow{\beta} p^{\bullet} R\Gamma_{\mathbb{\Delta}}(X/W)$$

of maps of commutative algebras in $\mathcal{DF}_{p-comp}(W)$ with composition being the crystalline Frobenius; moreover, the first map being an isomorphism on underlying non-filtered objects as η^* was an isomorphism over $(k^{\mathcal{N}})_{t\neq 0}$, so the middle vertex is a filtration on $\phi^* R\Gamma_{\mathbb{\Delta}}(X/W)$, justifying the notation. By Theorem 3.2.1, it suffices to show the following two assertions:

(a) $\mathrm{Fil}^{\bullet}\phi^* R\Gamma_{\mathbb{\Delta}}(X/W)$ is complete as a filtered object.

(b) The map β induces an isomorphism

$$\mathrm{gr}^i_F \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \simeq \tau^{\leq i}\mathrm{gr}^i_{p^{\bullet}} R\Gamma_{\mathbb{\Delta}}(X/W) \left(\simeq \phi_* \mathrm{Fil}_i^{\mathrm{conj}} R\Gamma(X, \Omega^{\bullet}_{X/k})\{i\}\right)$$

for all $i \geq 0$, where the parenthetical isomorphism comes from the de Rham comparison $R\Gamma_{\mathbb{\Delta}}(X/W)/p \simeq \phi_* R\Gamma(X, \Omega^{\bullet}_{X/k})$.

For (a): via the Rees equivalence, it is enough to show that $\mathcal{H}_{\mathcal{N}}(X) \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ is t-complete. As everything is p-complete, it suffices to show the same after reduction modulo p. As $\{u, t\}$ give a regular sequence of length 2 in $W[u,t]/(ut-p)$, we have a short exact sequence

$$0 \to \mathcal{O}_{k^{\mathcal{N}}}/p \to \mathcal{O}_{k^{\mathcal{N}}}/u \oplus \mathcal{O}_{k^{\mathcal{N}}}/t \to \mathcal{O}_{k^{\mathcal{N}}}/(u,t) \to 0$$

[43] We are implicitly using the following assertion: for any $M \in \mathcal{D}_{p-comp}(W)$, the pullback of M along $k^{\mathcal{N}} \to \mathrm{Spf}(W)$ corresponds to $\mathrm{Rees}(p^{\bullet}W \otimes_W M) =: \mathrm{Rees}(p^{\bullet}M) \in \mathcal{D}_{qc}(k^{\mathcal{N}})$.

of coherent sheaves on $k^{\mathcal{N}}$. Tensoring $\mathcal{H}_{\mathcal{N}}(X)$ with this sequence, it is then enough to show that $\mathcal{H}_{\mathcal{N}}(X)/u$ is t-complete (as t is zero on the other two terms). This follows from the identification in (4) and the completeness of the Hodge filtration.

For (b), consider the following diagram in $\mathcal{D}_{\mathrm{gr}}(k[u])$:

$$\begin{tikzcd} \mathcal{H}_{\mathcal{N}}(X)/t \arrow[r, "\overline{\psi^*}"] \arrow[d, equal] & \mathrm{Rees}(p^\bullet R\Gamma_{\Delta}(X/W))/t \arrow[d, equal] \\ \bigoplus_i \mathrm{gr}^i_{\mathrm{Fil}} \phi^* R\Gamma_{\Delta}(X/W)(i) \arrow[r] & \bigoplus_i \mathrm{gr}^i_{p^\bullet} R\Gamma_{\Delta}(X/W)(i) \end{tikzcd}$$

The top horizontal arrow $\overline{\psi^*}$ is the mod t reduction of the map ψ^*, and the vertical identifications are general features of the Rees equivalence. Our task is to show that the bottom horizontal map of graded W-complexes realizes the degree i term on the LHS as $\tau^{\leq i}$ of the degree i term on the RHS. But the map $\overline{\psi^*}$ can also be studied geometrically: it is the map on graded $k[u]$-modules obtained via transmutation from map $W/p \to W/M_u$ of k-algebra stacks over $[\mathrm{Spec}(k[u])/\mathbf{G}_m]$ coming from middle vertical composition in (2.7.4). The map $\overline{\psi^*}$ then identifies, via the argument proving asssertion (5) above, with the graded $k[u]$-linear map

$$\bigoplus_i \phi_* \mathrm{Fil}_i^{\mathrm{conj}} R\Gamma(X, \Omega^\bullet_{X/k}) u^i \to \bigoplus_i \phi_* R\Gamma(X, \Omega^\bullet_{X/k}) u^i$$

coming from the Rees construction (with respect to u) applied to the conjugate filtration. But this map clearly realizes the degree i term on the LHS as $\tau^{\leq i}$ of the degree i term on the RHS (as X is a smooth affine), so we win. $\square$

Remark 3.3.6 (Perfectness of Nygaard filtered crystalline cohomology)**.** For a smooth proper k-scheme X, we claim that $\mathcal{H}_{\mathcal{N}}(X) \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ is actually a perfect complex with Tor amplitude in $[0, 2d]$. For perfectness, by p-completeness, it suffices to show that $\mathcal{H}_{\mathcal{N}}(X)/p$ is perfect on $(k^{\mathcal{N}})|_{p=0}$. But $(k^{\mathcal{N}})_{p=0}$ is the stack $C = \mathrm{Spec}(k[ut]/(ut))/\mathbf{G}_m$ from Construction 2.8.2. A quasi-coherent complex M on C is perfect if and only the corresponding complexes $M|_{u=0}$ on $C_{u=0}$ and $M|_{t=0}$ on $C_{t=0}$ are perfect[44]. In the case $M = \mathcal{H}_{\mathcal{N}}(X)/p$, using Theorem 3.3.5 (4) and (5), it suffices to show that the filtered objects $\mathrm{Fil}^\bullet_H R\Gamma(X, \Omega^*_{X/k})$ and $\phi_* \mathrm{Fil}^{\mathrm{conj}}_\bullet R\Gamma(X, \Omega^*_{X/k})$ give perfect complexes on $\mathbf{A}^1/\mathbf{G}_m$ via the Rees construction; this follows as each filtration is a finite filtration on a perfect k-complex whose associated graded complex is also k-perfect. The claim about Tor amplitudes is proven similarly once one observes that the Tor amplitude of a perfect complex on a noetherian scheme can be computed fibrewise.

3.4 Gauges over k

Using the filtered prismatization $X^{\mathcal{N}}$ from Theorem 3.3.5, we obtain our desired notion of gauges.

[44]Given a noetherian scheme X and two closed subschemes $Y, Z \subset X$ such that $|X| = |Y| \cup |Z|$, we claim that an object $M \in \mathcal{D}_{qc}(X)$ is perfect if and only if $M|_Y$ and $M|_Z$ are perfect on Y and Z respectively. The "only if" direction is clear. For the "if" direction, we may replace X, Y, and Z by their reduced closed subschemes as the condition of being perfect can be detected after pulling back along a nil-immersion. In this case, we have a triangle $\mathcal{O}_X \to \mathcal{O}_Y \times \mathcal{O}_Z \to Q$ of coherent sheaves on X with Q set-theoretically supported on $Y \cap Z$. As $M|_Y$ and $M|_Z$ are perfect are on Y and Z respectively, so is $M|_{Y\cap Z}$ on $Y \cap Z$, and thus $M \otimes_{\mathcal{O}_X} Q$ is bounded coherent on X. Tensoring the previous triangle with M then shows that M is bounded coherent on X. Now the perfectness of a bounded coherent complex on X can be detected after pulling back to points of X, so we win since pullback to a point of X factors over restricting to Y or Z.

Definition 3.4.1 (Gauges over X)**.** Given a smooth k-scheme X, write $\mathrm{Gauge}_{\Delta}(X) := \mathcal{D}_{qc}(X^{\mathcal{N}})$; we refer to such objects as as *(prismatic) gauges on* X.

Prismatic gauges provide a natural notion of "coefficients" for Nygaard filtered prismatic cohomology. We next make a series of remarks aimed towards demystifying this notion.

Remark 3.4.2 (What is $X^{\mathcal{N}}$ made of?)**.** Fix a smooth k-scheme X. The stack $X^{\mathcal{N}}$ contains two open substacks

$$(X^{\mathcal{N}})_{t\neq 0} \simeq \phi^* X^{\Delta} = (X/W)^{\mathrm{crys}} \quad \text{and} \quad (X^{\mathcal{N}})_{u\neq 0} \simeq X^{\Delta}$$

as well as the complementary closed substacks

$$(X^{\mathcal{N}})_{t=0} = X^{HT,c} := \phi_*(X/k)^{dR,c} \quad \text{and} \quad (X^{\mathcal{N}})_{u=0} = (X/k)^{dR,+}$$

with (transverse) intersection

$$(X^{\mathcal{N}})_{u=t=0} = (X/k)^{Hodge}.$$

Write $j_{HT}, j_{dR} : X^{\Delta} \to X^{\mathcal{N}}$ for the open immersions arising as

$$j_{HT} : X^{\Delta} \simeq (X^{\mathcal{N}})_{u\neq 0} \hookrightarrow X^{\mathcal{N}} \quad \text{and} \quad j_{dR} : X^{\Delta} \overset{\phi_W^{-1}}{\simeq} \phi^* X^{\Delta} \simeq (X^{\mathcal{N}})_{t\neq 0} \subset X^{\mathcal{N}}$$

Note that j_{HT} is W-linear, while j_{dR} lives over ϕ_W^{-1} on $\mathrm{Spf}(W)$.

Remark 3.4.3 (Gauges as filtered objects)**.** For a smooth k-scheme X, if we regard $X^{\mathcal{N}}$ as living over the t-line $\mathbf{A}^1/\mathbf{G}_m$, then a vector bundle E on $X^{\mathcal{N}}$ gives a vector bundle $j^{dR,*}E$ on $X^{\Delta} \simeq (X^{\mathcal{N}})_{t\neq 0}$ (aka a prismatic crystal on X, see Construction 3.1.1) together with additional filtration data specifying the extension across $(X^{\mathcal{N}})_{t\neq 0} \subset X^{\mathcal{N}}$; this additional data can roughly be regarded as providing a "Nygaard" filtration on the cohomology of the crystal $j^{dR,*}E$ compatible with that on the prismatic cohomology of X itself. In fact, we shall later prove a much more precise statement along these lines using Theorem 3.3.5: for a qrsp ring R, the stack $\mathrm{Spf}(R)^{\mathcal{N}}$ identifies with the Rees stack of the Nygaard filtration $\mathrm{Fil}^{\bullet}_{\mathcal{N}}\Delta_R$, so $\mathrm{Gauge}_{\Delta}(R)$ is exactly the ∞-category of filtered modules over $\mathrm{Fil}^{\bullet}_{\mathcal{N}}\Delta_R$. Thus, for general X, we may informally regard $\mathrm{Gauge}_{\Delta}(X)$ as the ∞-category of filtered modules over the filtered Nygaard complex, regarded as a sheaf on X[45].

Remark 3.4.4 (Gauges over X, concretely)**.** In this remark, we outline two expectations for descriptions of gauges in more explicit terms. The first of these is essentially accessible using quasi-syntomic descent, while the second is the subject of ongoing discussions with Dodd.

1. Reinterpreting the construction in [FJ13, §6] in modern language, one obtains a sheaf of graded $W[u,t]/(ut-p)$-algebras G on the quasi-syntomic site of k, determined by the following requirement: if R is a qrsp k-algebra, then $G(R) = \mathrm{Rees}(\oplus_{i\in\mathbf{Z}}\mathrm{Fil}^i\Delta_R t^{-i})$ (naturally in R) with $u = pt^{-1}$. Let G_X denote the restriction of G to the quasi-syntomic site X_{qsyn} of X (as in [BMS19, Variant 4.35]). Then $\mathrm{Gauge}_{\Delta}(X)$ identifies naturally with $\mathcal{D}_{qc,\mathrm{gr}}(X_{qsyn}, G)$.
2. Given a flat lift $\mathfrak{X}/\mathrm{Spf}(W)$ of X/k, Dodd has recently defined a notion also called gauges using filtered $\mathcal{D}$-module theory on $\mathfrak{X}$ (see [Dod22, §4]. We expect that $\mathrm{Gauge}_{\Delta}(X)$ embeds fully faithfully into Dodd's category, with essential image given by nilpotence constraints.

[45]Our original motivation for considering this category (and thus the stack $\mathrm{Spf}(R)^{\mathcal{N}}$ for R qrsp) as a natural coefficient category was by analogy with complex algebraic geometry: the analogous object there is equivalent, via a form of Koszul duality, to the category of coherent $\mathcal{D}$-modules equipped with a good filtration. In fact, one can use this perspective with quasi-syntomic descent to even construct $X^{\mathcal{N}}$ directly. However, the functor of points description provided by the perspective [Dri20] is quite mysterious from this approach.

Precise statements with proofs will appear elsewhere.

Remark 3.4.5 (Explicit local charts)**.** Fix a smooth k-scheme X/k with a lift $\widetilde{X}/\mathrm{Spf}(W)$ together with a Frobenius $\phi_{\widetilde{X}} : \widetilde{X} \to \widetilde{X}$ over $\phi : W \to W$. We then saw in Construction 3.1.1 that there is a (Frobenius equivariant) flat cover $\widetilde{X} \to X^{\Delta}$. Via the first map in Remark 3.3.4, we thus obtain a flat cover $\widetilde{X} \times k^{\mathcal{N}} \to X^{\mathcal{N}}$. For instance, with notation as in Example 3.1.2, this gives a flat cover

$$\mathrm{Spf}(W[x,u,t]/(ut-p)^{\wedge}) \to (\mathbf{A}^1)^{\mathcal{N}},$$

suggesting that $(\mathbf{A}^1)^{\mathcal{N}}$ can be regarded as a 3-dimensional object, consistent with the analogy between curves over finite fields and 3-manifolds in étale cohomology. This analysis also shows that $X^{\mathcal{N}}$ admits a flat cover by a regular noetherian p-adic formal scheme, so there is a reasonable theory of coherent sheaves on $X^{\mathcal{N}}$.

In the rest of this section, we shall study gauges over k and their cohomology. First, let us describe how one pictures a gauge, following the original perspective in [FJ13].

Remark 3.4.6 (How to picture a gauge?)**.** Fix a gauge $E \in \mathcal{D}_{qc}(k^{\mathcal{N}})$. This corresponds to an object[46] $M^\bullet \in \mathcal{D}_{p-comp,\mathrm{gr}}(W[u,t]/(ut-p))$. Let us describe this object explicitly. Consider the diagram

$$\cdots \underset{u}{\overset{t}{\rightleftarrows}} \mathcal{O}(-(i+1)) \underset{u}{\overset{t}{\rightleftarrows}} \mathcal{O}(-i) \underset{u}{\overset{t}{\rightleftarrows}} \mathcal{O}(-(i-1)) \underset{u}{\overset{t}{\rightleftarrows}} \cdots$$

of vector bundles on $k^{\mathcal{N}}$, where $ut = tu = p$. The above diagram describes a graded $W[u,t]/(ut-p)$-module object in $\mathrm{Vect}(k^{\mathcal{N}})$. Tensoring this object with E and taking global sections then gives the corresponding $M^\bullet \in \mathcal{D}_{p-comp,\mathrm{gr}}(W[u,t]/(ut-p))$. Informally, we can picture $M^\bullet$ (and thus E) as follows: set $M^i = R\Gamma(k^{\mathcal{N}}, E(-i)) \in \mathcal{D}_{p-comp}(W)$, so the maps $t : \mathcal{O}(-1) \to \mathcal{O}$ and $u : \mathcal{O} \to \mathcal{O}(-1)$ on $k^{\mathcal{N}}$ then give rise to a diagram

$$\cdots \underset{u}{\overset{t}{\rightleftarrows}} M^{i+1} \underset{u}{\overset{t}{\rightleftarrows}} M^i \underset{u}{\overset{t}{\rightleftarrows}} M^{i-1} \underset{u}{\overset{t}{\rightleftarrows}} \cdots \tag{3.4.1}$$

of p-complete W-complexes, where $p = ut = tu$. Write $M^{-\infty} = \mathrm{colim}\, M^{-i}$ for the (p-completed) colimit along the t maps, and $M^{\infty} = \mathrm{colim}_i\, M^i$ for the (p-completed) colimit along the u maps; more conceptually, $M^\infty = E|_{(k^{\mathcal{N}})|_{u\neq 0}}$ and $M^{-\infty} = E|_{(k^{\mathcal{N}})|_{t\neq 0}}$. Thus, we have a correspondence

$$M^\infty \xleftarrow{u^\infty} M^0 \xrightarrow{t^\infty} M^{-\infty}$$

in $\mathcal{D}_{p-comp}(W)$. Note that if $E \in \mathrm{Perf}(k^{\mathcal{N}})$, then the map t induces an isomorphism $M^i \to M^{i-1}$ for $i \ll 0$, and the map u induces an isomorphism $M^i \to M^{i+1}$ for $i \gg 0$. In this case, it follows that both maps appearing in the correspondence above are p-isogenies, and thus we have a natural identification $M^\infty[1/p] \simeq M^{-\infty}[1/p]$ in $\mathrm{Perf}(W[1/p])$.

Example 3.4.7 (The gauge $\mathcal{H}_{\mathcal{N}}(X)$)**.** Say $E = \mathcal{H}_{\mathcal{N}}(X)$, where X is a smooth quasi-compact k-scheme X. Let us describe the diagram (3.4.1) corresponding to the gauge E. Since we have

$$M^i = R\Gamma(k^{\mathcal{N}}, E(-i)) = \mathrm{Fil}^i_{\mathcal{N}} \phi^* R\Gamma_{\Delta}(X/W),$$

by Theorem 3.3.5, this diagram takes the form

$$\cdots \rightleftarrows \mathrm{Fil}^{i+1}_{\mathcal{N}} \phi^* R\Gamma_{\Delta}(X/W) \underset{u}{\overset{t}{\rightleftarrows}} \mathrm{Fil}^{i}_{\mathcal{N}} \phi^* R\Gamma_{\Delta}(X/W) \underset{u}{\overset{t}{\rightleftarrows}} \mathrm{Fil}^{i-1}_{\mathcal{N}} \phi^* R\Gamma_{\Delta}(X/W) \rightleftarrows \cdots,$$

[46] We have chosen cohomological indexing for the M^i's, to be consistent with the indexing for the Nygaard filtration. Thus, the behaviour of t and u is swapped: the map t decreases degree by 1, while the map u increases degree by 1.

where the map t is the obvious inclusion, while the map u is induced by multiplication by $p \in \mathrm{Fil}^1_{\mathcal{N}}\phi^*W = pW$. Note that the t map induces isomorphisms

$$\phi^* R\Gamma_{\Delta}(X/W) =: M^0 \xrightarrow{t,\simeq} M^{-1} \xrightarrow{t,\simeq} M^{-2} \xrightarrow{t,\simeq} ....,$$

so the map $t^\infty : M^0 \to M^{-\infty}$ is an isomorphism, and then $u^\infty : M^0 \to M^\infty$ gets identified with the prismatic Frobenius $\phi^*_{X/W} : \phi^* R\Gamma_{\Delta}(X/W) \to R\Gamma_{\Delta}(X/W)$ via the proof of Theorem 3.3.5. Let us highlight two special features of this example:

- *Frobenius is an isogeny:* As $\mathrm{Fil}^{i+1}_{\mathcal{N}}\phi^* R\Gamma_{\Delta}(X/W)/p\mathrm{Fil}^i_{\mathcal{N}}\phi^* R\Gamma_{\Delta}(X/W) \simeq \mathrm{Fil}^{i+1}_H R\Gamma(X, \Omega^\bullet_{X/k})$, we learn that the map $u : M^i \to M^{i+1}$ is also an isomorphism for $i + 1 > \dim(X)$, so $M^{-\infty}[1/p] \simeq M^\infty[1/p]$, just as for perfect complex gauges (even though $\mathcal{H}_{\mathcal{N}}(X)$ need not be perfect if X is not proper).
- *F-gauge structure:* Note that $M^\infty = R\Gamma_{\Delta}(X/W)$ and $M^{-\infty} = \phi^* R\Gamma_{\Delta}(X/W)$ are off by a Frobenius twist: this special feature of $\mathcal{H}_{\mathcal{N}}(X)$ corresponds to the fact that it is naturally an F-gauge in the sense of the upcoming §4.1.

The next notion is motivated from the above discussion, and will be quite useful later.

Definition 3.4.8 (Effective gauges)**.** We say that a gauge $E \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ is *effective* if the maps

$$t : R\Gamma(k^{\mathcal{N}}, E(-i)) \to R\Gamma(k^{\mathcal{N}}, E(-(i-1)))$$

are isomorphisms for $i \leq 0$. In other words, in terms of the diagram (3.4.1), the maps labelled t emanating at M^i are required to be isomorphisms for $i \leq 0$.

Similarly, we say that a gauge $E \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ is *anti-effective* if the maps

$$u : R\Gamma(k^{\mathcal{N}}, E(-i)) \to R\Gamma(k^{\mathcal{N}}, E(-(i+1)))$$

are isomorphisms for $i \leq 0$. In other words, in terms of the diagram (3.4.1), the maps labelled u emanating at M^i are required to be isomorphisms for $i \geq 0$.

Example 3.4.9. The gauge $\mathcal{H}_{\mathcal{N}}(X)$ for a smooth quasi-compact k-scheme is always effective (see Example 3.4.7), and the gauge $\mathcal{H}_{\mathcal{N}}(X)(-\dim(X))$ is anti-effective by Theorem 3.3.5 (4).

Remark 3.4.10. For any $E \in \mathrm{Perf}(k^{\mathcal{N}})$, the twist $E(i)$ is effective for $i \gg 0$: twisting by $\mathcal{O}(1)$ has the effect of shifting diagram (3.4.1) to the left one step.

With an eye towards Mazur's theorem, we are especially interested in vector bundles on $k^{\mathcal{N}}$. For this, the following result, essentially capturing the "properness" of $k^{\mathcal{N}}$, will be quite useful:

Lemma 3.4.11 (Formal GAGA for $k^{\mathcal{N}}$)**.** *Write $k^{\mathcal{N}}_{alg} = \mathrm{Spec}(W[u,t]/(ut-p))/\mathbf{G}_m$, so we have natural map $\pi : k^{\mathcal{N}} \to k^{\mathcal{N}}_{alg}$ realizing the source as the p-adic formal completion of the target.*

1. *Pullback along π indues equivalences on* $\mathrm{Vect}(-)$, $\mathrm{Coh}(-)$ *and* $\mathrm{Perf}(-)$.
2. *$R\Gamma(k^{\mathcal{N}}, -)$ carries* $\mathrm{Perf}(k^{\mathcal{N}})$ *into* $\mathrm{Perf}(W)$.

Proof. We first make some preliminary remarks. Write $X = \mathrm{Spec}(W[u,t]/(ut-p))$, and let $\widehat{X}$ be its p-adic formal completion. As X and thus $\widehat{X}$ are noetherian and regular, we have $\mathrm{Vect}(k^{\mathcal{N}}) \subset \mathrm{Coh}(k^{\mathcal{N}}) \subset \mathrm{Perf}(k^{\mathcal{N}})$, and similarly for $k^{\mathcal{N}}_{alg}$. Moreover, we have a Cech presentation

$$\left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \mathbf{G}_m \times \mathbf{G}_m \times \widehat{X} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathbf{G}_m \times \widehat{X} \substack{\longrightarrow\\\longrightarrow} \widehat{X} \right) \simeq k^{\mathcal{N}} \tag{3.4.2}$$

of p-adic formal stacks, as well as a Cech presentation

$$\left(\cdots \mathbf{G}_m \times \mathbf{G}_m \times X \rightrightarrows \mathbf{G}_m \times X \rightrightarrows X \right) \simeq k^{\mathcal{N}}_{alg} \tag{3.4.3}$$

of schemes. Both augmented simplicial diagrams have flat face maps, and we can use them to compute categories of quasi-coherent sheaves/complexes by descent. Arguing this way, one learns that there is a standard t-structure on $\mathrm{Perf}(k^{\mathcal{N}})$ with heart $\mathrm{Coh}(k^{\mathcal{N}})$, and similarly $k^{\mathcal{N}}_{alg}$; moreover, the pullback π^* is t-exact. We can now begin the proof.

1. First, we check full faithfulness. Let L be the tautological line bundle on $k^{\mathcal{N}}_{alg}$ (i.e., the pullback $\mathcal{O}(1) \in \mathrm{Pic}(B\mathbf{G}_m)$). As X is an affine scheme, the category $\mathcal{D}_{qc}(k^{\mathcal{N}}_{alg})$ is generated under colimits by $\{L^i\}_{i \in \mathbf{Z}}$. Applying $\mathrm{RHom}_{k^{\mathcal{N}}_{alg}}(L^i, -)$, it is then easy to see that the natural $\mathcal{O}_{k^{\mathcal{N}}_{alg}} \to R\pi_* \mathcal{O}_{k^{\mathcal{N}}}$ is an isomorphism: this amounts to observing that $(W[u,t]/(ut-p))_{\deg = i}$ is a finite W-module and hence already p-complete. By the projection formula, we deduce that π^* is fully faithful on $\mathrm{Perf}(-)$, and hence on all categories under consideration. For future use, let us also note that the functor $\pi^* : \mathrm{Perf}(k^{\mathcal{N}}_{alg}) \to \mathrm{Perf}(k^{\mathcal{N}})$ is t-exact as the pullback functor $\mathrm{Perf}(X) \to \mathrm{Perf}(\widehat{X})$ is t-exact by flatness of completion in the noetherian case.

   To prove essential surjectivity, we first explain the claim for $\mathrm{Coh}(-)$. By full faithfulness, it suffices to show that any $M \in \mathrm{Coh}(k^{\mathcal{N}})$ is a quotient of $\oplus_{i=1}^n L^{m_i}$ for suitable integers n, m_i. An object $M \in \mathrm{Coh}(k^{\mathcal{N}})$ is given by a compatible system $\{M_n \in \mathrm{Coh}_{\mathrm{gr}}(X, \mathcal{O}_X/p^n)\}$ of graded coherent sheaves. As M_1 is coherent and X is affine, we can then choose a surjection $\oplus_{i=1}^n L^{m_i} \to M_1$, corresponding to homogeneous elements $x_1, ..., x_n \in M_1$. Each such element can be lifted compatibly to homogeneous elements in each M_j; choosing such lifts for each j and each x_i separately, then gives a map $\oplus_{i=1}^n L^{m_i} \to M$ in $\mathrm{Coh}(k^{\mathcal{N}})$ which is surjective modulo p. By Nakayama, this map is surjective, so we win.

   It remains to prove essential surjectivity for $\mathrm{Perf}(-)$ and $\mathrm{Vect}(-)$. In fact, the claim for $\mathrm{Perf}(-)$ is immediate from the claim for $\mathrm{Coh}(-)$ using the full faithfulness and the standard t-structure. For $\mathrm{Vect}(-)$, using the claim for $\mathrm{Coh}(-)$, it suffices to show the following: given $M \in \mathrm{Coh}(k^{\mathcal{N}}_{alg})$, if $\pi^* M$ is a vector bundle, so is M. Measuring local freeness via Fitting ideals, it suffices to show the following: if a homogeneous ideal $I \subset W[u,t]/(ut-p)$ gives the unit ideal modulo p, then it must be the unit ideal. If I gives the unit ideal modulo p, then we have an equation of the form $1 + p\epsilon = \sum_i a_i x_i$ in $W[u,t]/(ut-p)$ with x_i being the homogeneous generators of I, and $a_i, \epsilon \in W[u,t]/(ut-p)$. Projecting to homogeneous components in degree 0, then shows that $1 + p\epsilon_0 \in I_{\deg=0}$ for some $\epsilon_0 \in W = (W[u,t]/(ut-p))_{\deg=0}$. But $1 + p\epsilon_0 \in W$ is a unit, so I contains a unit, as wanted.

2. Thanks (1) and its proof, any perfect complex on $k^{\mathcal{N}}$ is a retract of a finite complex whose terms are finite direct sums of powers of L. We are therefore reduced to checking that $R\Gamma(k^{\mathcal{N}}, L^i) \in \mathrm{Perf}(W)$, which we already saw in the proof of (1) above. □

Using the previous lemma, we next completely explicitly describe vector bundles on $k^{\mathcal{N}}$ via a standard purity result for schemes.

Proposition 3.4.12 (Vector bundles on $k^{\mathcal{N}}$ via purity)**.** *Restriction to the loci $(k^{\mathcal{N}})_{u \neq 0}$ and $(k^{\mathcal{N}})_{t \neq 0}$ identifies the category $\mathrm{Vect}(k^{\mathcal{N}})$ with the category of modifications of W-lattices, i.e., the category of triples (M_u, M_t, Ψ), where $M_u, M_t \in \mathrm{Vect}(W)$ and $\Psi : M_u[1/p] \simeq M_t[1/p]$ is a $W[1/p]$-linear isomorphism.*

Proof. Consider the diagram where all functors are induced by pullback:

$$\mathrm{Vect}(k^{\mathcal{N}}) \leftarrow \mathrm{Vect}(k^{\mathcal{N}}_{alg}) \to \mathrm{Vect}\left((k^{\mathcal{N}}_{alg} - V(u,t)\right) \to \mathrm{Vect}((k^{\mathcal{N}}_{alg})_{u\neq 0}) \times_{\mathrm{Vect}((k^{\mathcal{N}}_{alg})_{ut\neq 0})} \mathrm{Vect}((k^{\mathcal{N}}_{alg})_{u\neq 0}).$$

We claim all arrows are equivalences. This is Lemma 3.4.11 for the first arrow; for the second arrow, it follows by descent from purity of vector bundles[47], while the last arrow is simply an equivalence by Zariski descent. Now observe that

$$(k^{\mathcal{N}}_{alg})_{u\neq 0} = \mathrm{Spec}(W[u^{\pm 1}])/\mathbf{G}_m \simeq \mathrm{Spec}(W) \quad \text{and} \quad (k^{\mathcal{N}}_{alg})_{t\neq 0} = \mathrm{Spec}(W[t^{\pm 1}])/\mathbf{G}_m \simeq \mathrm{Spec}(W),$$

while

$$(k^{\mathcal{N}}_{alg})_{ut\neq 0} = \mathrm{Spec}(W[1/p]).$$

As all identifications are W-linear, the claim follows. □

Warning 3.4.13. The equivalence in Proposition 3.4.12 is not exact: the functor from $\mathrm{Vect}(k^{\mathcal{N}})$ to the category of modifications is exact, but the inverse functor is not. Correspondingly, Proposition 3.4.12 does not admit an obvious extension to quasi-coherent sheaves or higher cohomology: via the algebraization for $\mathrm{Perf}(-)$ in Lemma 3.4.11, the construction in Proposition 3.4.12 does derive to an exact functor

$$\Psi : \mathrm{Perf}(k^{\mathcal{N}}) \to \mathrm{Perf}(W) \times_{\mathrm{Perf}(W[1/p])} \mathrm{Perf}(W),$$

but this functor is *not* an equivalence; in fact, it fails to be fully faithful. For instance, $\mathrm{RHom}_{k^{\mathcal{N}}}(M,N)$ is W-perfect for $M,N \in \mathrm{Perf}(k^{\mathcal{N}})$ by Lemma 3.4.11, while this is essentially never true on the right (e.g., $\mathrm{RHom}(\Psi(\mathcal{O}),\Psi(\mathcal{O})) = \mathrm{fib}(W \times W \xrightarrow{(a,b)\mapsto a-b} W[1/p])$ is not W-perfect).

Example 3.4.14 (The Picard group of $k^{\mathcal{N}}$)**.** The equivalence in Proposition 3.4.12 is induced by pullback functors, and is thus symmetric monoidal. Consequently, we can identify the groupoid $\mathcal{P}\mathrm{ic}(k^{\mathcal{N}})$ of line bundles on $k^{\mathcal{N}}$ with the category of triples $(L, L', \Psi : L[1/p] \simeq L'[1/p])$, where L and L' are invertible W-modues. Up to isomorphism, any such triple has the form $(W, p^n W, \mathrm{can} : W[1/p] \simeq p^n W[1/p])$ for a unique integer n, and the map remembering n gives an isomorphism $\mathrm{Pic}(k^{\mathcal{N}}) \simeq \mathbf{Z}$, with $\mathcal{O}(1)$ corresponding to $+1 \in \mathbf{Z}$. Moreover, $H^0(k^{\mathcal{N}}, \mathcal{O}^*) = (W[u,t]/(ut-p))^*_{\deg 0} = W^*$, so it follows that pullback along $k^{\mathcal{N}} \to B\mathbf{G}_m$ induces an isomorphism

$$\mathbf{Z} \times BW^* \simeq \mathcal{P}\mathrm{ic}(k^{\mathcal{N}})$$

of Picard groupoids (aka objects in $\mathcal{D}^{[-1,0]}(\mathbf{Z})$).

Remark 3.4.15 (Explictly relating vector bundles to modifications of lattices)**.** For future reference, it will be convenient to know the explicit construction describing the equivalence in Proposition 3.4.12, its inverse, and its interaction with computing global sections.

1. *From vector bundles on $k^{\mathcal{N}}$ to modifications.* Let us explain how to explicitly construct the triple (M_u, M_t, Ψ) corresponding to a vector bundle $E \in \mathrm{Vect}(k^{\mathcal{N}})$. As in Remark 3.4.6, let $M^\bullet \in \mathrm{Vect}_{\mathrm{gr}}(W[u,t]/(ut-p))$ be the associated finite projective graded $W[u,t]/(ut-p)$-module. We then have a correspondence

$$M^{\infty} \xleftarrow{u^{\infty}} M^0 \xrightarrow{t^{\infty}} M^{-\infty}.$$

[47]For a regular scheme X and a closed subset $Z \subset X$ of everywhere codimension ≥ 2, the restriction functor $\mathrm{Vect}(X) \to \mathrm{Vect}(X - Z)$ is fully faithful in general, and an equivalence if $\dim(X) = 2$. Applying this to the terms of the simplicial presentation in (3.4.3) then gives the claim by smooth descent for vector bundles.

Note that both u^∞ and t^∞ are p-isogenies, so the above diagram gives an isomorphism

$$\Psi_E : M^\infty[1/p] \simeq M^{-\infty}[1/p].$$

One then checks that the triple (M_u, M_t, Ψ) corresponding to E under the equivalence in Proposition 3.4.12 is then explicitly given by $(M^\infty, M^{-\infty}, \Psi_E)$.

2. *Global sections of a vector bundle on $k^{\mathcal{N}}$ via the modifications.* Given a vector bundle $E \in \mathrm{Vect}(k^{\mathcal{N}})$ with corresponding triple (M_u, M_t, Ψ), one computes that

$$R\Gamma(k^{\mathcal{N}}, E(i)) = p^{-i} M_u \cap M_t,$$

where the intersection takes place inside $M_u[1/p] \overset{\Psi}{\simeq} M_t[1/p]$. The $\mathbf{Z}^{op}$-indexed diagram

$$... \xrightarrow{t} R\Gamma(k^{\mathcal{N}}, E(-i)) \xrightarrow{t} R\Gamma(k^{\mathcal{N}}, E(-(i-1)) \xrightarrow{t} ...$$

then identifies with the $\mathbf{Z}^{op}$-indexed diagram

$$... \subset p^i M_u \cap M_t \subset p^{i-1} M_u \cap M_t \subset ...$$

describing a filtration of M_t by W-submodules.

3. *From modifications to vector bundles.* Given a triple (M_u, M_t, Ψ), the corresponding vector bundle E, regarded as graded $W[u,t]/(ut-p)$-module $M = \oplus_i M^i$, is determined by setting $M^i = p^i M_u \cap M_t$, with $t : M^i \to M^{i-1}$ being the obvious inclusion, and $u : M^i \to M^{i+1}$ induced by multiplication by p; this follows by reversing (3).

In particular, it follows from the above descriptions that a vector bundle E corresponding to a triple (M_u, M_t, Ψ) is effective (Definition 3.4.8) if and only if Ψ carries the lattice $M_t \subset M_t[1/p]$ into $M_u \subset M_u[1/p]$.

3.5 Mazur's theorem

Theorem 3.3.5 explains how to attach gauges over k to a smooth proper k-scheme X. In this section, we use this formalism to present a proof of a theorem of Mazur relating the Hodge filtration to the crystalline Frobenius for certain X's; see [Maz72, Maz73] for more context surrounding this result, including its application to a conjecture of Katz relating the Newton and Hodge polygons[48].

Theorem 3.5.1 (Mazur)**.** *Let X/k be a smooth proper k-scheme. Assume that the crystalline cohomology of X is p-torsionfree, and that the Hodge-to-de Rham spectral sequence degenerates*[49]*. Then the F-crystal $\phi_{X/W} : H^n_{\mathrm{crys}} \to \phi_* H^n_{\mathrm{crys}}$ determines the Hodge filtration on $H^n_{dR} := H^n(X, \Omega^\bullet_{X/k})$ for all n. More precisely, for each k, we have*

$$\mathrm{Fil}^k_H H^n_{dR} = \mathrm{Im}\left(\phi^{-1}_{X/W}(p^k \phi_* H^n_{\mathrm{crys}}) \xrightarrow{can} H^n_{\mathrm{crys}}(X/W)/p \simeq H^n_{dR}\right) \subset H^n_{dR}.$$

Remark 3.5.2. In the situation of Theorem 3.5.1, if X/k admits a smooth proper lift $\mathfrak{X}/W$ with generic fibe $\mathfrak{X}_\eta$, then Theorem 3.5.1 implies that the Hodge numbers $h^{i,j}(\mathfrak{X}_\eta)$ of the generic fibre $\mathfrak{X}_\eta$ are determined by the F-crystal $(H^{i+j}_{\mathrm{crys}}(X/W), \phi^*_{X/W})$ attached to the special fibre.

[48] The idea of proving Theorem 3.5.1 using the present formalism of F-gauges goes back at least to Fontaine (see [Eke86, FJ13]); the author benefitted from a talk by Dodd on [Dod22].

[49] This is satisfied, for instance, if X admits a flat lift to W with p-torsionfree Hodge cohomology.

Remark 3.5.3. In the situation of Theorem 3.5.1, one can also recover the conjugate filtration from the F-crystals: one has

$$\mathrm{Fil}_k^{\mathrm{conj}} H^n_{dR} = \mathrm{Im}\left(\phi^{-1}_{X/W}(p^k \phi_* H^n_{\mathrm{crys}}) \xrightarrow{\frac{\phi^*_{X/W}}{p^k}} \phi_* H^n_{\mathrm{crys}} \to \phi_* H^n_{\mathrm{crys}}/p \simeq \phi_* H^n_{dR}\right).$$

This follows by a similar argument given below to prove Theorem 3.5.1.

The strategy for proving this theorem is the following: we reinterpret the assumption on X as stating that the cohomology sheaves of $\mathcal{H}_{\mathcal{N}}(X) \in \mathrm{Perf}(k^{\mathcal{N}})$ are vector bundles, and then deduce the theorem using purity of vector bundles on $k^{\mathcal{N}}$ (Proposition 3.4.12). To implement this strategy, we need the following criterion for recognizing when the cohomology groups are a perfect complex are vector bundles.

Lemma 3.5.4 (When do perfect complexes have finite projective cohomology?)**.** *Let R be a reduced ring*[50]*, and let $M \in \mathrm{Perf}(R)$ be a perfect complex. Assume that for all i, the function*

$$x \in \mathrm{Spec}(R) \mapsto \dim_{\kappa(x)} H^i(M \otimes^L_R \kappa(x)) \in \mathbf{N}$$

is locally constant. Then each cohomology group of M is a finite projective R-module.

Proof. By devissage, it suffices to show that the top cohomology group N of M is locally free. As N is the top cohomology group of a perfect complex, it must be finitely presented; moreover, the assumption on M ensures that $x \mapsto \dim_{\kappa(x)} N \otimes_R \kappa(x)$ is locally constant. We claim that this forces N to be finite projective. By a limit argument, it suffices to show that N is finite projective after localizing at points of $\mathrm{Spec}(R)$: the functor carrying a ring to its category of finitely presented modules (resp. finite projective modules) commutes with direct limits. We may thus assume R is a reduced local ring with maximal ideal $\mathfrak{m}$ and residue field k. In this case, we must show N is free. Choose a surjection $\pi : R^n \to N$ with n minimal, so $n = \dim_k(N \otimes_R k)$ and π gives an isomorphism after tensoring with k. By the assumption on N, it follows that π induces an isomorphism after tensoring with $\kappa(x)$ for all $x \in \mathrm{Spec}(R)$. If $\ker(\pi) \neq 0$, then we can find an element $0 \neq v \in \ker(\pi)$. But then v gives a nonzero map $R \xrightarrow{1 \mapsto v} R^n$ which vanishes after tensoring with $\kappa(x)$ for all $x \in \mathrm{Spec}(R)$, so $v = (v_1, ..., v_n) \in R^n$ must have the property that each $v_i \in R$ maps to 0 in every residue field of R. But we know that $v_i \neq 0$ for some i. As R is reduced, this means $R[1/v_i] \neq 0$, so tensoring $R \xrightarrow{1 \mapsto v} R^n$ with $\kappa(x)$ for any $x \in \mathrm{Spec}(R[1/v_i]) \subset \mathrm{Spec}(R)$ gives a contradiction. ☐

We are now in a position to implement our stated strategy for proving Mazur's theorem:

Proof of Mazur's theorem (Theorem 3.5.1). In this proof, we shall regard $\mathcal{H}_{\mathcal{N}}(X) \in \mathrm{Perf}(k^{\mathcal{N}})$ as an object of $\mathrm{Perf}(k^{\mathcal{N}}_{alg})$ by Lemma 3.4.11.

We first claim that, under the hypotheses in Theorem 3.5.1, the cohomology sheaves of $\mathcal{H}_{\mathcal{N}}(X)$ are vector bundles. Using the criterion in Lemma 3.5.4, it suffices to show that the fibrewise rank function is constant in all cohomological degrees. By the comparison in Theorem 3.3.5, the fibres of $\mathcal{H}_{\mathcal{N}}(X)$ are given by one of the following complexes of vector spaces (up to Frobenius twists over W, which we can ignore for rank considerations):

$$R\Gamma_{\mathrm{crys}}(X/W)[1/p],\ R\Gamma(X, \Omega^\bullet_{X/k}),\ R\Gamma(X, \Omega^{Hodge}_{X/k}).$$

[50] I thank Shiji Lyu and Longke Tang for suggesting removal of noetherian hypotheses from a previous version of this lemma.

The assumption that X has p-torsionfree crystalline cohomology and that its Hodge-to-de Rham spectral sequence degenerates guarantees that all these complexes of vector spaces have the same Betti numbers, so Lemma 3.5.4 ensures that the cohomology groups of $\mathcal{H}_{\mathcal{N}}(X)$ are indeed vector bundles. Write $\mathcal{H}^i_{\mathcal{N}}(X) := \mathcal{H}^i(\mathcal{H}_{\mathcal{N}}(X))$ for simplicity from now on.

Next, let us make the bundle $\mathcal{H}^i_{\mathcal{N}}(X)$ explicit as a graded $W[u,t]/(ut-p)$-module.. Applying Proposition 3.4.12, we learn that each $\mathcal{H}^i_{\mathcal{N}}(X)$ is corresponds to a triple (M^i_u, M^i_t, Ψ). Unwinding how this identification is constructed and using Theorem 3.3.5, we learn that $M^i_t = H^i_{\mathrm{crys}}(X/W)$ and $M^i_u = \phi_* H^i_{\mathrm{crys}}(X/W)$ with the identification $\Psi : M^i_u[1/p] \simeq M^i_t[1/p]$ induced by the crystalline Frobenius $\phi_{X/k} : H^i_{\mathrm{crys}}(X/W) \to \phi_* H^i_{\mathrm{crys}}(X/W)$ on inverting p. In other words, we learn that $\mathcal{H}^i_{\mathcal{N}}(X)$ is functorially determined by the F-crystal $(H^i_{\mathrm{crys}}(X/W), \phi_{X/k}[1/p])$. From the explicit description in Remark 3.4.15, we then learn that the graded $W[u,t]$-module $M = \oplus_j M^j$ corresponding to $\mathcal{H}^i_{\mathcal{N}}(X)$ is determined by

$$M^j = \phi^{-1}_{X/k}(p^j \phi_* H^i_{\mathrm{crys}}(X/W)) \cap H^i_{\mathrm{crys}}(X/W) \subset H^i_{\mathrm{crys}}(X/W)[1/p]$$

for all j. But $\phi_{X/k}[1/p]$ is effective, so this simplifies to give $M^j = \phi^{-1}_{X/k}(p^j H^i_{\mathrm{crys}}(X/W))$ for $j \geq 0$.

We now turn to the problem at hand. As each $\mathcal{H}^i_{\mathcal{N}}(X)$ is a vector bundle, the same is true for its specializations, so the comparison in Theorem 3.3.5 (4) holds true at the level of individual cohomology groups, and not just at the level of complexes. In particular, we have

$$\mathrm{Fil}^j_{\mathcal{N}} H^i_{\mathrm{crys}}(X/W) \simeq R\Gamma(k^{\mathcal{N}}, \mathcal{H}^i_{\mathcal{N}}(X)(-j)) \quad \text{and} \quad R\Gamma(k^{\mathcal{N}}, \mathcal{H}^i_{\mathcal{N}}(X)/u(-j)) \simeq \mathrm{Fil}^j_H H^i(X, \Omega^\bullet_{X/k})$$

for all $j \geq 0$, with the natural map from the LHS to the RHS being surjective. But the previous paragraph identifies the LHS with the subspace $\phi^{-1}_{X/k}(p^j H^i_{\mathrm{crys}}(X/W)) \subset H^i_{\mathrm{crys}}(X/W)$; plugging this into the above surjection (and identifying maps) then yields the theorem. $\square$

Chapter 4

Syntomic cohomology in characteristic p and duality

We work over a perfect field k of characteristic p. In this chapter, we introduce the primary object of study in characteristic p in these notes: the syntomification. To each smooth k-scheme X, we shall attach (via transmutation) a stack X^{Syn}, called the syntomification of X, functorially in X. For proper smooth maps $f : X \to Y$, the induced map f^{Syn} enjoys good finiteness properties, e.g., pushforward along f^{Syn} preserves perfect complexes. This device serves multiple purposes:

1. One can regard $\mathcal{D}_{qc}(k^{\mathrm{Syn}})$ as a category of "p-adic cohomology theories" on smooth k-schemes: for each $M \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$, we obtain a cohomology theory on smooth k-schemes via
$$(f : X \to \mathrm{Spec}(k)) \mapsto R\Gamma(X^{\mathrm{Syn}}, f^*M).$$
Examples of cohomology theories that arise this way include: (graded pieces of) Hodge cohomology, (Hodge/conjugate filtered layers of) de Rham cohomology, (Nygaard filtered layers of) crystalline cohomology, p-adic étale cohomology, the p-completion of the cohomology of $\mathbf{G}_m$, and more generally syntomic cohomology (which we shall define).
2. When k is a finite field and X/k is proper smooth, the stack X^{Syn} enjoys good absolute finiteness properties as a stack over $\mathbf{F}_p$ (and not merely relative to k^{Syn}). A concrete manifestation is a duality theorem (essentially due to Milne) expressing the idea that X has "p-cohomological dimension $2\dim(X) + 1$" — when $\dim(X) = 0$, this captures the idea that $\mathrm{Spec}(\mathbf{F}_q)$ behaves like a circle.

The stack k^{Syn} is constructed, roughly, a quotient of the filtered prismatization $k^{\mathcal{N}}$ by Frobenius. In general, for smooth k-schemes X, the stack X^{Syn} can then be constructed in two ways: either as a similar quotient of $X^{\mathcal{N}}$, or via transmutation from a suitable ring stack over $k^{\mathcal{N}}$.

4.1 Syntomification in characteristic p

For a smooth k-scheme X, Theorem 3.3.5 constructed the Nygaard filtered prismatization $X^{\mathcal{N}} \to k^{\mathcal{N}}$ via transmutation from the ring stack $\mathbf{G}_a^{\mathcal{N}} \to k^{\mathcal{N}}$. To proceed further, we construct a new stack X^{Syn} as a quotient of $X^{\mathcal{N}}$:

Definition 4.1.1 (The syntomification and F-gauges)**.** Fix a smooth k-scheme X. The stack $X^{\mathcal{N}}$ contains two open substacks isomorphic to $X^{\mathbb{\Delta}}$ as in Remark 3.4.2. Write X^{Syn} for the result of glueing these two open substacks together, i.e., we have a pushout square

$$\begin{tikzcd} X^{\Delta} \sqcup X^{\Delta} \arrow[r, "j_{HT}, j_{dR}"] \arrow[d, "\mathrm{can}"] & X^{\mathcal{N}} \arrow[d, "j_{\mathcal{N}}"] \\ X^{\Delta} \arrow[r, "j_{\Delta}"] & X^{\mathrm{Syn}} \end{tikzcd} \tag{4.1.1}$$

where all maps are étale (and even local isomorphisms). We call X^{Syn} the *syntomification* of X.

Remark 4.1.2 (Geometry of X^{Syn})**.** Let us describe certain loci in X^{Syn}:

- The stack k^{Syn} has only two physical points: an open point $\mathrm{Spf}(W)$ corresponding to either of $(k^{\mathcal{N}})_{t\neq 0}$ or $(k^{\mathcal{N}})_{u\neq 0}$ inside $k^{\mathcal{N}}$, and a closed point based at $\mathrm{Spec}(k)/\mathbf{G}_m \to k^{\mathrm{Syn}}$ coming from the closed point of $k^{\mathcal{N}}$. Via pullback, this gives an open-closed decomposition

$$X^{\Delta} \xrightarrow{j_{\Delta}} X^{\mathrm{Syn}} \xleftarrow{i_H} X^{Hodge}$$

of X^{Syn}, where $X^{Hodge} = (X/k)^{Hodge} = (BT^{\sharp}_{X/k})/\mathbf{G}_m$.

- The inclusion of the two divisors $(X/k)^{dR,+} = (X^{\mathcal{N}})_{u=0} \subset X^{\mathcal{N}}$ and $X^{HT,c} = (X^{\mathcal{N}})_{t=0} \subset X^{\mathcal{N}}$ gives rise to maps

$$h_{dR,+} : (X/k)^{dR,+} \to X^{\mathrm{Syn}} \quad \text{and} \quad h_{HT,+} : X^{HT,c} \to X^{\mathrm{Syn}}$$

of stacks. When $X = \mathrm{Spec}(k)$, these are simply given by maps

$$h_{dR,+} : \mathbf{A}^1/\mathbf{G}_m \to X^{\mathrm{Syn}} \quad \text{and} \quad h_{dR,c} : \mathbf{A}^1/\mathbf{G}_m \to X^{\mathrm{Syn}}$$

and have the feature that $h^*_{dR,+}\mathcal{H}_{\mathcal{N}}(X) = \mathcal{H}_{dR,+}(X)$ and $h^*_{HT,c}\mathcal{H}_{\mathcal{N}}(X) = \mathcal{H}_{HT,+}(X) := \phi_*\mathcal{H}_{dR,c}(X)$ for any smooth k-scheme X.

Warning 4.1.3. Fix a smooth k-scheme X.

- The stack X^{Syn} is not a W-stack but only a $\mathbf{Z}_p$-stack: the map $j_{HT} : X^{\Delta} \to X^{\mathcal{N}}$ is W-linear, while the map $j_{dR} : X^{\Delta} \to X^{\mathcal{N}}$ is linear over ϕ_W^{-1}, so X^{Syn} only has a $\mathbf{Z}_p$-structure and not a W-structure. (This is a feature and not a bug: it allows us to capture $\mathbf{Z}_p$-linear cohomology theories, such as $R\Gamma_{et}(X, \mathbf{Z}_p)$ or $R\Gamma_{et}(X, \mathbf{G}_m)/p^n$, as coherent cohomology on X^{Syn}.)

- The stack X^{Syn} is not separated: indeed, even k^{Syn} is not separated as there are two visible specializations from the open point to the closed point, one coming from each divisor $(k^{\mathcal{N}})_{u=0}$ and $(k^{\mathcal{N}})_{t=0}$ in $k^{\mathcal{N}}$. Nevertheless, the situation is somewhat tame: if X has affine diagonal, one can show the same for X^{Syn}.

- The maps $h_{dR,+} : (X/k)^{dR,+} \to X^{\mathrm{Syn}}$ and $h_{HT,+} : X^{HT,c} \to X^{\mathrm{Syn}}$ are locally closed immersions but not closed immersions. Indeed, even when $X = \mathrm{Spec}(k)$, the pullback of $h_{dR,+} : \mathbf{A}^1/\mathbf{G}_m \to k^{\mathrm{Syn}}$ along $j_{\mathcal{N}} : k^{\mathcal{N}} \to k^{\mathrm{Syn}}$ identifies with $(k^{\mathcal{N}})_{u=0} \sqcup (k^{\mathcal{N}})_{t\neq 0} \to k^{\mathcal{N}}$, which is a union of disjoint closed and open immersions.

Remark 4.1.4 (Stability properties)**.** The commutative square (4.1.1) can be regarded as a natural transformation of functors on smooth k-schemes. Moreover, for any map $f : X \to Y$ of smooth k-schemes, the induced map gives a base change identification

$$X^{\mathcal{N}} \simeq X^{\mathrm{Syn}} \times_{Y^{\mathrm{Syn}}} Y^{\mathcal{N}}.$$

As the map $j_{\mathcal{N}} : Y^{\mathcal{N}} \to Y^{\mathrm{Syn}}$ is an étale cover, we deduce that the functor $X \mapsto X^{\mathrm{Syn}}$ from smooth k-schemes to stacks over k^{Syn} enjoys the same nice properties as the functor $X \mapsto X^{\mathcal{N}}$ from smooth k-schemes to stacks over $k^{\mathcal{N}}$: it commutes with Tor independent finite limits, and carries (affine) étale maps (resp. covers) to (affine) étale maps (resp. covers).

Remark 4.1.5 (Noetherianness)**.** The stack k^{Syn} is a noetherian regular p-adic formal stack: indeed, the same is true for $k^{\mathcal{N}}$, and we have a surjective étale map $j_{\mathcal{N}} : k^{\mathcal{N}} \to k^{\mathrm{Syn}}$. Consequently, there is a sensible notion of coherent sheaves on k^{Syn}, and moreover we have a standard t-structure on $\mathrm{Perf}(k^{\mathrm{Syn}})$ with heart the abelian category $\mathrm{Coh}(k^{\mathrm{Syn}})$ of coherent sheaves. In fact, by Remark 3.4.5, the same remarks apply to X^{Syn} for any smooth k-scheme X.

The syntomification can also be defined via transmutation from the stack $\mathbf{G}_a^{\mathrm{Syn}}$.

Remark 4.1.6 (The k-algebra stack $\mathbf{G}_a^{\mathrm{Syn}}$)**.** Let us explain why $\mathbf{G}_a^{\mathrm{Syn}}$ is naturally a k-algebra stack. In fact, the compatibility with finite products shows that $\mathbf{G}_a^{\mathrm{Syn}}$ is naturally an animated algebra stack over k^{Syn}. To get the k-algebra structure, it is easiest to directly explain why the k-algebra structure on $\mathbf{G}_a^{\mathcal{N}}$ descends to one on $\mathbf{G}_a^{\mathrm{Syn}}$.

The diagram in Construction 3.3.2 gives an isomorphism

$$(\mathbf{G}_a^{\mathcal{N}})|_{u\neq 0} \simeq W/p$$

of k-algebra stacks on $(k^{\mathcal{N}})_{u\neq 0}$, as well as an isomorphism

$$(\mathbf{G}_a^{\mathcal{N}})|_{t\neq 0} \simeq \phi_* W/p$$

as k-algebra stacks on $(k^{\mathcal{N}})_{t\neq 0}$. The first isomorphism may be viewed as a natural identification $j_{HT}^* \mathbf{G}_a^{\mathcal{N}} \simeq W/p$ as k-algebra stacks on $k^{\mathbb{\Delta}}$. Moreover, the ϕ_* appearing in the second isomorphism coupled with the ϕ^{-1}-linearity of j_{dR} then also gives a natural identification $j_{dR}^* \mathbf{G}_a^{\mathcal{N}} \simeq (\phi^{-1})^* F_* W/p \simeq W/p$ as k-algebra stacks on $k^{\mathbb{\Delta}}$[51]. Via these identifications, the k-algebra structure on $\mathbf{G}_a^{\mathcal{N}}$ descends to a k-algebra structure on $\mathbf{G}_a^{\mathrm{Syn}}$ over k^{Syn} along $j_{\mathcal{N}} : k^{\mathcal{N}} \to k^{\mathrm{Syn}}$.

For a general smooth k-scheme X, we leave it to the reader to check that the stack $X^{\mathrm{Syn}} \to k^{\mathrm{Syn}}$ can also be constructed via transmutation from the k-algebra stack $\mathbf{G}_a^{\mathrm{Syn}} \to k^{\mathrm{Syn}}$.

4.2 Generalities on F-gauges

Definition 4.2.1 (F-gauges)**.** For a smooth k-scheme X, the ∞-category $\mathrm{F\text{-}Gauge}_{\mathbb{\Delta}}(X)$ is defined to be $\mathcal{D}_{qc}(X)$; objects here are called *F-gauges on X*. There is a natural map $\pi : X^{\mathrm{Syn}} \to k^{\mathrm{Syn}}$, and we write $\mathcal{H}_{\mathrm{Syn}}(X) := R\pi_* \mathcal{O}_{X^{\mathrm{Syn}}} \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$ when X is quasi-compact.

Let us first make some general remarks on F-gauges and their cohomology.

Remark 4.2.2 (Realization functors)**.** For a smooth k-scheme X, we have maps $j_{\mathbb{\Delta}} : X^{\mathbb{\Delta}} \to X^{\mathrm{Syn}}$, $h_{dR,+} : X^{dR,+} \to X^{\mathrm{Syn}}$, $h_{HT,c} : X^{HT,c} \to X^{\mathrm{Syn}}$ and $i_H : X^{Hodge} \to X^{\mathrm{Syn}}$ as described in Remark 4.1.2. Thus, given a vector bundle F-gauge $E \in \mathrm{Vect}(X^{\mathrm{Syn}})$, its pullback along any of these maps gives a more classical object, e.g., $j_{\mathbb{\Delta}}^* E \in \mathrm{Vect}(X^{\mathbb{\Delta}})$ is a prismatic crystal on X (Construction 3.1.1 (5)), $h_{dR,+}^* E \in \mathrm{Vect}(X^{dR,+})$ is a vector bundle with a flat connection

[51]Let us check this on the functor of points. Given a map $\alpha : k \to R$ of commutative rings, the k-algebra $\left((\phi^{-1})^* F_* W/p\right)(k \xrightarrow{\alpha} R)$ identifies with the k-algebra $F_* W/p(k \xrightarrow{\alpha\circ\phi^{-1}} R)$, which equals the k-algebra $\phi_* W/p(k \xrightarrow{\alpha\circ\phi^{-1}} R)$, which is simply the k-algebra $W(R)/p$, regarded as a k-algebra via the unique lift $k \to W(R)/p$ of $\alpha : k \to R$.

with nilpotent p-curvature equipped with a Griffiths transversal Hodge filtration (Remark 2.5.8), $i^*_{Hodge}E \in \mathrm{Vect}(X^{Hodge})$ is a nilpotent Higgs bundle on X/k (Remark 2.5.9), etc. Thanks to this picture, we often regard an F-gauge as a universal coefficient object for p-adic cohomology theories on X, and the aforementioned pullback functors can be regarded as "realization functors".

Remark 4.2.3 (Finiteness of cohomology)**.** For X/k smooth and proper of dimension d, we claim that $\mathcal{H}_{\mathrm{Syn}}(X) \in \mathrm{Perf}(k^{\mathrm{Syn}})$ with Tor amplitude in $[0, 2d]$. Indeed, this can be checked after pullback along the local isomorphism $k^{\mathcal{N}} \to k^{\mathrm{Syn}}$. But $X^{\mathcal{N}} \simeq X^{\mathrm{Syn}} \times_{k^{\mathrm{Syn}}} k^{\mathcal{N}}$, so we are reduced to showing that $\mathcal{H}_{\mathcal{N}}(X)$ is perfect with Tor amplitude in $[0, 2d]$, which was explained in Remark 3.3.6. The same reasoning combined with the proof of Theorem 3.5.1 also shows the following: if each $H^i_{\mathrm{crys}}(X/W)$ is p-torsionfree and the Hodge-to-de Rham spectral sequence fo X/k degenerates, then each cohomology sheaf $\mathcal{H}^i_{\mathrm{Syn}}(X) := \mathcal{H}^i(\mathcal{H}_{\mathrm{Syn}}(X))$ is a vector bundle.

Remark 4.2.4 (F-gauges on k in terms of gauges and glueing)**.** Recall that $k^{\mathcal{N}}$ contains two open copies of $\mathrm{Spf}(W)$ as a W-stack: one given by $(k^{\mathcal{N}})_{u\neq 0} \to k^{\mathcal{N}}$, and the other by $(k^{\mathcal{N}})_{t\neq 0} \to k^{\mathcal{N}}$. The stack k^{Syn} is obtained by identifying these together along the Frobenius. Keeping track of twists, one learns that specifying an F-gauge on k is equivalent to specifying a gauge $E \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ on k together with isomorphisms $\phi^*(E|_{(k^{\mathcal{N}})_{u\neq 0}}) \simeq E|_{(k^{\mathcal{N}})_{t\neq 0}}$.

Remark 4.2.5 (Cohomology of F-gauges, conceptually)**.** For smooth X/k, elaborating on Remark 4.2.4, the pushout square in Eq. (4.1.1) gives an equalizer diagram

$$\mathcal{D}_{qc}(X^{\mathrm{Syn}}) \simeq \mathrm{Eq}(\mathcal{D}_{qc}(X^{\mathcal{N}}) \overset{j^*_{HT}}{\underset{j^*_{dR}}{\rightrightarrows}} \mathcal{D}_{qc}(X^{\mathbb{A}}))$$

of stable ∞-categories. In particular, if $E \in \mathcal{D}_{qc}(X^{\mathrm{Syn}})$ be an F-gauge on X, then there is a natural triangle

$$R\Gamma(X^{\mathrm{Syn}}, E) \to R\Gamma(X^{\mathcal{N}}, E|_{X^{\mathcal{N}}}) \xrightarrow{j^*_{HT} - j^*_{dR}} R\Gamma(X^{\mathbb{A}}, E|_{X^{\mathbb{A}}}) \tag{4.2.1}$$

where we abusively write $E|_{X^{\mathcal{N}}} = j^*_{\mathcal{N}}E$ and $E|_{X^{\mathbb{A}}} = j^*_{\mathbb{A}}E$. The map j^*_{HT} is W-linear, while the map j^*_{dR} is not W-linear, so the second map appearing above is only $\mathbf{Z}_p$-linear, whence the fibre term on the left is only $\mathbf{Z}_p$-linear as well.

Using the above, we can produce some basic calculations of F-gauges and their cohomology.

Example 4.2.6 (Weight 0 syntomic cohomology)**.** Let us calculate $R\Gamma(X^{\mathrm{Syn}}, \mathcal{O})$ using the fibre sequence (4.2.1). Using Theorem 3.3.5 to compute the middle term in (4.2.1), we obtain a fibre sequence

$$R\Gamma(X^{\mathrm{Syn}}, \mathcal{O}) \to R\Gamma(X^{\mathbb{A}}, \mathcal{O}) \xrightarrow{\phi^*_X - 1} R\Gamma(X^{\mathbb{A}}, \mathcal{O}),$$

where we recall that $\phi^* R\Gamma(X^{\mathbb{A}}, \mathcal{O}) \simeq R\Gamma((X/W)^{\mathrm{crys}}, \mathcal{O})$ by §3.1. A standard argument[52] then shows that

$$R\Gamma(X^{\mathrm{Syn}}, \mathcal{O}) \simeq R\Gamma_{et}(X, \mathbf{Z}_p),$$

so syntomic cohomology of the structure sheaf is p-adic étale cohomology.

[52]By étale descent for prismatic/crystalline cohomology (which follows from the de Rham comparison), the assignment carrying U to the fibre $R\Gamma(U^{\mathbb{A}}, \mathcal{O})^{\phi^*_U = 1} := \mathrm{fib}\left(R\Gamma(U^{\mathbb{A}}, \mathcal{O}) \xrightarrow{\phi^*_U - 1} R\Gamma(U^{\mathbb{A}}, \mathcal{O})\right)$ is a p-complete étale sheaf of complexes on X_{et}, so there is a natural map $R\Gamma_{et}(X, \mathbf{Z}_p) \to R\Gamma(X^{\mathbb{A}}, \mathcal{O})^{\phi^*_X = 1}$. To check this is an isomorphism, one can reduce modulo p. Via the de Rham comparison for crystalline cohomology, we are reduced to checking that the natural map $R\Gamma_{et}(X, \mathbf{F}_p) \to R\Gamma(X, \Omega^\bullet_{X/k})^{\phi^*_X = 1}$ is an isomorphism. As ϕ^*_X kills differential forms of positive degree, this reduces to checking that $R\Gamma_{et}(X, \mathbf{F}_p) \to R\Gamma(X, \mathcal{O}_X)^{\phi^*_X = 1}$ is an isomorphism, which follows from the Artin–Schreier sequence.

Remark 4.2.7 (From local systems of F-gauges)**.** For an étale map $U \to X$ of smooth k-schemes, the map $U^{\mathrm{Syn}} \to X^{\mathrm{Syn}}$ is also étale by Remark 4.1.4. This construction is compatible with pullbacks and disjoint unions, and preserves surjections. Consequently, we have a natural morphism

$$((X^{\mathrm{Syn}})_{et}, \mathcal{O}/p^n) \to (X_{et}, \mathbf{Z}/p^n)$$

of ringed sites for all n. As the construction is local on X_{et}, the pullback of a local system along this map is a vector bundle. Passing to derived categories, we obtain pullback functor

$$\mathrm{RH} : \mathcal{D}^b_{lisse}(X_{et}, \mathbf{Z}/p^n) \to \mathrm{Perf}((X^{\mathrm{Syn}})_{p^n=0})$$

and hence, on passage to the limit, a pullback functor

$$\mathrm{RH} : \mathcal{D}^b_{lisse}(X_{et}, \mathbf{Z}_p) \to \mathrm{Perf}(X^{\mathrm{Syn}}).$$

Extending the analsysis in Example 4.2.6, one can show that both these functors are fully faithful. In particular, we have

$$R\Gamma(X_{et}, L) \simeq R\Gamma(X^{\mathrm{Syn}}, T(L)),$$

so we can regarded the theory of F-gauges as an enlargement of theory of lisse $\mathbf{Z}_p$-sheaves on X.

The F-gauges coming from étale local systems as above are very special; for instance, objects of geometric origin (e.g., $\mathcal{H}_{\mathcal{N}}(X)$ for $\dim(X) \geq 1$) essentially never have this form. To work with the cohomology of general F-gauges effectively, we need to explicitly understand the cohomology of an F-gauge on k itself; let us record an explicit description of this theory obtained from the explicit description of gauges in Remark 3.4.6.

Remark 4.2.8 (Cohomology of F-gauges on k, explicitly)**.** Let $E \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$ be an F-gauge on $\mathrm{Spec}(k)$. Let $M^\bullet \in \mathcal{D}_{p-comp,\mathrm{gr}}(W[u,t]/(ut-p))$ be the graded $W[u,t]/(ut-p)$-complex corresponds to the gauge $E|_{k^{\mathcal{N}}}$ as in Remark 3.4.6; thus, $M^i = R\Gamma(k^{\mathcal{N}}, E(-i))$, and we view the $W[u,t]/(ut-p)$-linear structure on $M^\bullet$ as a diagram

$$\cdots \underset{u}{\overset{t}{\rightleftarrows}} M^{i+1} \underset{u}{\overset{t}{\rightleftarrows}} M^i \underset{u}{\overset{t}{\rightleftarrows}} M^{i-1} \underset{u}{\overset{t}{\rightleftarrows}} \cdots$$

where $ut = tu = p$. Write $M^{-\infty} = \mathrm{colim}\, M^{-i}$ for the p-completed colimit along the t maps, and $M^{\infty} = \mathrm{colim}_i\, M^i$ for the p-completed colimit along the u maps, so we have a correspondence

$$M^{\infty} \xleftarrow{u^\infty} M^0 \xrightarrow{t^\infty} M^{-\infty}.$$

As E comes from k^{Syn}, we are given an isomorphism $\tau : \phi^* M^{\infty} \simeq M^{-\infty}$ (Remark 4.2.4), so we obtain an induced map $\tau\phi u^\infty : M^0 \xrightarrow{u^\infty} M^\infty \overset{\phi}{\simeq} \phi^* M_\infty \overset{\tau}{\simeq} M^{-\infty}$. Unwinding definitions, one finds that the complex $R\Gamma(k^{\mathrm{Syn}}, E)$ can then be computed as

$$R\Gamma(k^{\mathrm{Syn}}, E) = \mathrm{fib}\left(M^0 \xrightarrow{t^\infty - \tau\phi u^\infty} M^{-\infty}\right).$$

This formula will be quite useful in the sequel.

Let us end this section by observing that a small modification of mod p prismatic F-gauges on X naturally produces a category linear over the "twistor line"

Remark 4.2.9 (Relation to canonical global quantization)**.** In the recent paper [BKTV22], the authors introduce (amongst others) the following construction: given a smooth k-scheme X, there is a quasi-coherent sheaf of categories $\mathcal{C}_X$ over $X^{(1)} \times \mathbf{P}^1/\mathbf{G}_m$ (which the authors call the *twistor category*) with the following specializations:

1. The restriction of $\mathcal{C}_X$ to the open substack $X^{(1)} \times \{1\} \subset X^{(1)} \times \mathbf{P}^1/\mathbf{G}_m$ is the category $\mathcal{D}_X$-Mod of quasi-coherent $\mathcal{D}_X$-modules on X.
2. The restriction of $\mathcal{C}_X$ along $X^{(1)} \times \{0\} \to X^{(1)} \times \mathbf{P}^1/\mathbf{G}_m$ is the category $\mathcal{H}\mathrm{iggs}_X$ of quasi-coherent Higgs bundles on X.
3. The restriction of $\mathcal{C}_X$ along $X^{(1)} \times \{\infty\} \to X^{(1)} \times \mathbf{P}^1/\mathbf{G}_m$ is the category $\mathcal{H}\mathrm{iggs}_{X^{(1)}}$ of quasi-coherent Higgs bundles on $X^{(1)}$.

Let us explain how to recover a similar picture using mod p prismatic F-gauges; we are grateful to Dmitry Kubrak for discussions on this point.

Consider the normalization map $k^h \to (k^{\mathrm{Syn}})_{p=0}$. Thus, the $\mathbf{F}_p$-stack k^h is obtained by glueing two copies of $\mathbf{A}^1/\mathbf{G}_m$ over k to each other along their open points using the Frobenius automorphism. This stack is isomorphic (via an isomorphism that is the identity on one component and ϕ_k^{-1} on the other) to the k-stack $\Xi/\mathbf{G}_m$, where $\Xi = \mathbf{A}^1 \cup_{\mathbf{G}_m} \mathbf{A}^1$ is the affine line with the doubled origin. Moreover, we have a natural identification $\Xi/\mathbf{G}_m \simeq \mathbf{P}^1/\mathbf{G}_m$ of k-stacks[53], so we henceforth identify $k^h = \mathbf{P}^1/\mathbf{G}_m$.

Next, let $X^h = X^{\mathrm{Syn}} \times_{k^{\mathrm{Syn}}} k^h$, so there is a natural projection map $X^h \to k^h = \mathbf{P}^1/\mathbf{G}_m$. Starting from the pushout description of k^{Syn} and unwinding definitions, we learn that the stack X^h sits in a pushout square

$$\begin{tikzcd} (X^{\mathcal{N}})_{t=0,u\neq 0} \sqcup (X^{\mathcal{N}})_{u=0,t\neq 0} \arrow[r] \arrow[d] & (X^{\mathcal{N}})_{t=0} \sqcup (X^{\mathcal{N}})_{u=0} \arrow[d] \\ X^{HT} \arrow[r] & X^h. \end{tikzcd}$$

Via the modular description, one can also define a natural map $X^h \to X$, uniquely determined by the requirement that it extend the structure map $X^{HT} \to X$ on the bottom left term above.

Combining the above two paragraphs, we obtain a natural map $X^h \to X \times \mathbf{P}^1/\mathbf{G}_m$. The category $\mathrm{QCoh}(X^h)$ thus sheafifies to a quasi-coherent sheaf of categories $\mathcal{C}'_X$ over $X \times \mathbf{P}^1/\mathbf{G}_m$. One can check that $\mathcal{C}'_X$ has similar specializations to (1) - (3) above, with the following two changes: (a) one must add nilpotence constraints on the p-curvature in (1) and the Higgs fields in (2) and (3), and (b) there is one fewer Frobenius twist appearing in the specializations describing $\mathcal{C}'_X$. In fact, it seems quite likely that $\mathcal{C}'_X$ is exactly the full subcategory of $\mathcal{C}_X$ determined by the constraints in (a) after twisting by the Frobenius.

4.3 Vector bundles on k^{Syn}

To discuss syntomic cohomology in higher weight, we need to understand line bundles on k^{Syn} better. In fact, vector bundles on k^{Syn} are quite classical objects:

[53] This looks slightly funny, but is obtained by simply noticing that both stacks are obtained by glueing two copies of $\mathbf{A}^1/\mathbf{G}_m$ along their open point, and there is only one such glueing as the point has no automorphisms. However, the resulting isomorphism $\Xi/\mathbf{G}_m \simeq \mathbf{P}^1/\mathbf{G}_m$ is not linear over $B\mathbf{G}_m$: indeed, if it were, we would get the absurd conclusion that $\Xi \simeq \mathbf{P}^1$.

Proposition 4.3.1 (Vector bundles on k^{Syn} as F-crystals)**.** *The category* $\mathrm{Vect}(k^{\mathrm{Syn}})$ *is identified with the category of (not necessarily effective) F-crystals, i.e., pairs $(M, \tau : \phi^*M[1/p] \simeq M[1/p])$, where M is a finite projective W-module and τ is an isomorphism of $W[1/p]$-modules.*

Proof. We saw in Proposition 3.4.12 that the category $\mathrm{Vect}(k^{\mathcal{N}})$ identifies with the category of modifications $(M_u, M_t, \Psi : M_u[1/p] \simeq M_t[1/p])$ of W-lattices. Using the pushout square defining k^{Syn} and unwinding identifications, one then learns that $\mathrm{Vect}(k^{\mathrm{Syn}})$ identifies with the category of pairs $((M_u, M_t, \Psi : M_u[1/p] \simeq M_t[1/p]), \tau : \phi^*M_u \simeq M_t)$, where $(M_u, M_t, \Psi : M_u[1/p] \simeq M_t[1/p])$ is a modification of W-lattices and τ is a W-module isomorphism. This simplifies to the category in the lemma. $\square$

Remark 4.3.2 (Effective and anti-effective F-gauges)**.** We call an F-gauge $E \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$ *effective* (resp. *anti-effective*) if its pullback to $k^{\mathcal{N}}$ is effective (resp. anti-effective) in the sense of Definition 3.4.8. One then checks that a vector bundle $E \in \mathrm{Vect}(k^{\mathrm{Syn}})$ is effective (resp. anti-effective) if and only the corresponding F-crystal (M, τ) (from Proposition 4.3.1) is effective (resp. anti-effective) in the classical sense, i.e., the isomorphism $\tau : \phi^*M[1/p] \simeq M[1/p]$ satisfies $\tau(\phi^*M) \subset M$ (resp. $\tau(\phi^*M) \supset M$).

Example 4.3.3 (The Picard groupoid of k^{Syn})**.** We shall determine the groupoid $\mathcal{P}\mathrm{ic}(k^{\mathrm{Syn}})$ of line bundles on k^{Syn}, regarded as a Picard groupoid (i.e., an object of $\mathcal{D}^{[-1,0]}(\mathbf{Z})$). Given $\alpha \in W[1/p]^*$, we obtain an invertible F-gauge L_α via the description in Lemma 4.3.1: it corresponds to the F-crystal (W, τ_α), where $\tau_\alpha : \phi^*W[1/p] \simeq W[1/p]$ is the map determined by $1 \mapsto \alpha$. This construction is multiplicative, so it extends to a symmetric monoidal functor $W[1/p]^* \to \mathcal{P}\mathrm{ic}(k^{\mathrm{Syn}})$. It is easy to see this functor is essential surjective. To determine the "relations", observe that the composition

$$W^* \xrightarrow{x \mapsto \phi(x)x^{-1}} W[1/p]^* \to \mathcal{P}\mathrm{ic}(k^{\mathrm{Syn}})$$

is canonically trivialized: indeed, for $x \in W^*$, one has the isomorphism

$$(W, \tau_1) \overset{1 \mapsto x}{\simeq} (W, \tau_{\phi(x)x^{-1}})$$

of F-crystals. This construction determines a map

$$\mathrm{Cone}\left(W^* \xrightarrow{x \mapsto \phi(x)x^{-1}} W[1/p]^*\right) \to \mathcal{P}\mathrm{ic}(k^{\mathrm{Syn}})$$

that one checks is an equivalence of Picard groupoids. In this description, the effective invertible F-gauges are exactly those lying in the image of $W - \{0\} \subset W[1/p]^*$ in $\mathcal{P}\mathrm{ic}(k^{\mathrm{Syn}})$. In two important special cases, this can be made much more concrete:

- If k is separably closed, then $W^* \xrightarrow{x \mapsto \phi(x)x^{-1}} W^*$ is surjective with kernel $\mathbf{Z}_p^*$ by Artin-Schreier theory, giving an isomorphism
$$\mathcal{P}\mathrm{ic}(k^{\mathrm{Syn}}) \simeq \mathbf{Z} \times B\mathbf{Z}_p^*$$
of Picard groupoids.

- If $k = \mathbf{F}_p$, then $W = \mathbf{Z}_p$ and ϕ is the identity, so we learn that
$$\mathcal{P}\mathrm{ic}(\mathbf{F}_p^{\mathrm{Syn}}) \simeq \mathbf{Q}_p^* \times B\mathbf{Z}_p^* \simeq \mathbf{Z}_p^* \times \mathbf{Z} \times B\mathbf{Z}_p^* \tag{4.3.1}$$
of Picard groupoids.

The fundamental remaining input in defining syntomic cohomology is the following:

Definition 4.3.4 (Breuil–Kisin twists in characteristic p)**.** The Breuil–Kisin (BK) twist $\mathcal{O}\{-1\} \in \mathcal{P}\mathrm{ic}(\mathbf{F}_p^{\mathrm{Syn}})$ is defined to be the essential image $p \in \mathbf{Q}_p^*$ under the map $\mathbf{Q}_p^* \to \mathcal{P}\mathrm{ic}(\mathbf{F}_p^{\mathrm{Syn}})$ from (4.3.1). We use the same notation for its pullback to X^{Syn} for any smooth k-scheme X.

Thus, under the description in Proposition 4.3.1, the line bundle $\mathcal{O}\{-1\}$ corresponds to the effective F-crystal (M, τ), where $M = W$, and τ is determined by $\phi^* M \xrightarrow{1 \mapsto p} M$. The corresponding invertible gauge $\mathcal{O}\{-1\}|_{k^{\mathcal{N}}}$ is identified with $\mathcal{O}(1)$: this follows by plugging the effective F-crystal definition of $\mathcal{O}\{-1\}$ into the description of $\mathcal{P}\mathrm{ic}(k^{\mathcal{N}})$ given in Example 3.4.14.

Remark 4.3.5 (BK twists and cohomology)**.** Let $E \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$ be an F-gauge. Keep notation as in Remark 4.2.8, so $M^j = R\Gamma(k^{\mathcal{N}}, E|_{k^{\mathcal{N}}}(-j))$, etc. For any integer i, we claim that

$$R\Gamma(k^{\mathrm{Syn}}, E\{i\}) = \mathrm{fib}\left(M^i \xrightarrow{t^{\infty+i} - \tau\phi u^{\infty-i}} M^{-\infty}\right),$$

where $t^{\infty+i} : M^i \to M^{-\infty}$ is the map induced by the t maps, $u^{\infty-i} : M^i \to M^{\infty}$ is the map induced by the u maps, and $\tau\phi u^{\infty-i}$ is the map $M^i \xrightarrow{u^{\infty-i}} M^{\infty} \overset{\phi}{\simeq} \phi^* M^{\infty} \overset{\tau}{\simeq} M^{-\infty}$. To prove this, let $N^{\bullet} \in \mathcal{D}_{p-comp,\mathrm{gr}}(W[u,t]/(ut-p))$ be the graded $W[u,t]/(ut-p)$-complex corresponding to $E\{i\}|_{k^{\mathcal{N}}}$. It will suffice to show that $N^0 = M^i$, $N^{\pm\infty} = M^{\pm\infty}$, and then to identify the maps. To understand the terms, note that $\mathcal{O}\{-1\}|_{k^{\mathcal{N}}} \simeq \mathcal{O}(1)$ as explained above, whence $N^0 = M^i$ and $N^{\pm\infty} = M^{\pm\infty}$, as claimed. One can then explicitly track down the effect of BK twists on the Frobenius identification τ to obtain the desired.

Remark 4.3.6 (Realizing BK twists geometrically)**.** Consider the F-gauge $\mathcal{H}_{\mathcal{N}}(\mathbf{P}^1) \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$. By Remark 4.2.3, this is a perfect complex whose cohomology groups are vector bundles. Classical calculations of de Rham cohomology then show that these cohomology groups nonzero only in degrees 0 and 2, where they are line bundles. We claim that $\mathcal{H}^2_{\mathrm{Syn}}(\mathbf{P}^1)$ identifies with $\mathcal{O}\{-1\}$. Using the equivalence with F-crystals in Proposition 4.3.1 and its construction, we are reduced to checking that $H^2_{\mathrm{crys}}(\mathbf{P}^1/W) \to \phi_* H^2_{\mathrm{crys}}(X/W)$ identifies with $\phi^* W \xrightarrow{1 \mapsto p} W$, which is a classical calculation in crystalline cohomology. More generally, similar reasoning shows that $\mathcal{H}^{2n}_{\mathrm{Syn}}(\mathbf{P}^n) \simeq \mathcal{O}\{-n\}$.

4.4 Syntomic cohomology in characteristic p

In this section, we introduce syntomic cohomology for smooth k-schemes. Its definition is immediate from the syntomification functor and the notion of BK twists. The resulting cohomology theory can be reasonably regarded as "p-adic étale motivic cohomology" for smooth k-schemes.

Definition 4.4.1 (Syntomic cohomology)**.** For a smooth k-scheme X and an integer i, we define its *weight i syntomic cohomology* as

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i)) := R\Gamma(X^{\mathrm{Syn}}, \mathcal{O}\{i\}).$$

Syntomic cohomology in weight 0 is p-adic étale cohomology, as we saw in Example 4.2.6 via the Artin-Schreier sequence. More generally, in any weight, there is an Artin-Schreier style description of syntomic cohomology:

Proposition 4.4.2 (Syntomic cohomology via the Nygaard filtration)**.** *Fix a smooth k-scheme X. For any integer i, there is a natural fibre sequence*

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i)) \to \mathrm{Fil}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \xrightarrow{can - \phi_i} R\Gamma_{\mathbb{\Delta}}(X/W), \tag{4.4.1}$$

where can is the canonical map $\mathrm{Fil}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \to \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \overset{\phi^{-1}}{\simeq} R\Gamma_{\mathbb{\Delta}}(X/W)$*, while* ϕ_i *is the "divided Frobenius map"* $\mathrm{Fil}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \xrightarrow{\phi^*_{X/W}} p^i R\Gamma_{\mathbb{\Delta}}(X/W) \overset{1/p^i}{\simeq} R\Gamma_{\mathbb{\Delta}}(X/W)$.

Proof. Remark 4.3.5 and the description in Theorem 3.3.5 give a fibre sequence

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i)) \to \mathrm{Fil}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \xrightarrow{\alpha - \psi} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W),$$

where α is the canonical map forgetting the filtration, and ψ is the map

$$\mathrm{Fil}^i_{\mathcal{N}} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \xrightarrow{\frac{\phi^*_{X/W}}{p^i}} R\Gamma_{\mathbb{\Delta}}(X/W) \overset{\phi}{\simeq} \phi^* R\Gamma_{\mathbb{\Delta}}(X/W).$$

The map can $- \phi_i$ appearing in the proposition is obtained by postcoming the map $\alpha - \psi$ appearing above with the automorphism $\phi^* R\Gamma_{\mathbb{\Delta}}(X/W) \overset{\phi^{-1}}{\simeq} R\Gamma_{\mathbb{\Delta}}(X/W)$. As postcomposing with an automorphism does not change the fibre, the claim follows. $\square$

Proposition 4.4.3 (Very (anti-)effective F-gauges have vanishing cohomology)**.** *Let $E \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$. If E is effective, then $R\Gamma(k^{\mathrm{Syn}}, E\{-i\}) = 0$ for $i > 0$.*

Dually, if E is anti-effective, then $R\Gamma(k^{\mathrm{Syn}}, E\{i\}) = 0$ for any $i > 0$.

Proof. We only explain the first assertion as the second one follows by a similar argument. Moreover, it suffices to explain the claim when $i = 1$. We use the notation and recipe in Remark 4.3.5. By the formula there, we have

$$R\Gamma(k^{\mathrm{Syn}}, E\{-1\}) = \mathrm{fib}\left(M^{-1} \xrightarrow{t^{\infty-1} - \tau u^{\infty+1}} M^{-\infty}\right),$$

so we must show the map appearing on the right is an automorphism. Now as E is effective, the maps $M^0 \xrightarrow{t} M^{-1} \xrightarrow{t} M^{-2} \xrightarrow{t} \ldots$ induced by t are all isomorphisms, and thus the maps $M^0 \xleftarrow{u} M^{-1} \xleftarrow{u} M^{-2} \xleftarrow{u} \ldots$ are all identified with multiplication by p. Consequently, the map $t^{\infty-1}$ appearing above is an isomorphism, while the map $\tau u^{\infty+1}$ is divisible by p. But then the map $t^{\infty-1} - \tau u^{\infty+1}$ is an isomorphism modulo p, and thus an isomorphism by derived Nakayma. $\square$

Corollary 4.4.4 (Negative weight syntomic cohomology vanishes)**.** *For a smooth k-scheme and an integer $i > 0$, we have $R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(-i)) = 0$ and $R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(\dim(X) + i)) = 0$.*

Proof. It suffices to show this when X is quasi-compact. But then it is immediate from Proposition 4.4.3 as $\mathcal{H}_{\mathrm{Syn}}(X)$ is effective and $\mathcal{H}_{\mathrm{Syn}}(X)\{\dim(X)\}$ is anti-effective (Example 3.4.9). $\square$

Example 4.4.5 (The syntomic cohomology of k itself)**.** We have already seen that $R\Gamma_{\mathrm{Syn}}(k, \mathbf{Z}_p(0)) = R\Gamma_{et}(\mathrm{Spec}(k), \mathbf{Z}_p)$ (Example 4.2.6). Since $\dim(X) = 0$, Corollary 4.4.4 implies that $R\Gamma_{\mathrm{Syn}}(k, \mathbf{Z}_p(i))$ vanishes for $i \neq 0$.

Remark 4.4.6 (Generating the category of F-gauges)**.** Say k is an algebraically closed field of characteristic p. It follows from Example 4.4.5 that $\mathrm{RHom}_{k^{\mathrm{Syn}}}(\mathcal{O}\{i\}, \mathcal{O}\{j\})$ vanishes for $i \neq j$, and equals $\mathbf{Z}_p$ for $i = j$. Thus, pullback along the map $k^{\mathrm{Syn}} \to B\mathbf{G}_m$ classifying $\mathcal{O}\{1\}$ gives a fully faithful embedding $\mathcal{D}_{qc}(B\mathbf{G}_m) \to \mathcal{D}_{qc}(k^{\mathrm{Syn}})$. However, this map is not surjective, i.e., $\mathcal{D}_{qc}(k^{\mathrm{Syn}})$ is not generated under colimits by $\{\mathcal{O}\{i\}\}_{i \in \mathbf{Z}}$. Indeed, if this functor were essentially surjective, then it would be an equivalence. But this would imply any perfect complex on $(k^{\mathrm{Syn}})_{p=0}$ would have to be isomorphic to a finite direct sum of shifts of vector bundles (as the same holds true for $(B\mathbf{G}_m)_{p=0}$). In particular, this would imply that for any smooth proper X/k, the perfect complex $\mathcal{H}_{\mathrm{Syn}}(X)/p \in \mathrm{Perf}((k^{\mathrm{Syn}})_{p=0})$ has vector bundle cohomology sheaves. Pulling back to $k^{\mathcal{N}}$ and using Theorem 3.3.5, we would learn that the Hodge-to-de Rham and conjugate spectral sequences for X/k must always degenerate, which clearly need not be the case.

The preceding analysis raises the following question: is $\mathcal{D}_{qc}(k^{\mathrm{Syn}})$ generated (under colimits) by the compact objects $\{\mathcal{H}_{\mathcal{N}}(X)/p\}$ as X ranges over all smooth proper k-schemes?

We end this section by explaining why syntomic cohomology in weight 1 is the p-completion of the étale cohomology of $\mathbf{G}_m$ (in fact, we only prove it mod p); in any reasonable theory of motives, one expects $\mathbf{Z}(1)$ to be closely related to $\mathbf{G}_m$, so this calculation supports the view that $\mathcal{D}_{qc}(k^{\mathrm{Syn}})$ might be viewed as a category of p-adic étale motives, with $\mathcal{H}_{\mathcal{N}}(X)$ providing the "motive of X".

Proposition 4.4.7 (Syntomic cohomology in weight 1)**.** *For any smooth k-scheme X, there is a natural identification*

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(1)) \simeq R\Gamma_{et}(X, \mathbf{G}_m)^{\wedge}[-1],$$

where the completion is p-adic. Reducing mod p^n, we learn that

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(1))/p^n \simeq R\Gamma_{fl}(X, \mu_{p^n}).$$

Proof of the mod p analogue. We will only prove the mod p assertion: we shall prove that there is a natural identification

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(1))/p \simeq R\Gamma_{et}(X, \mathbf{G}_m)/p[-1] \left(\simeq R\Gamma_{fl}(X, \mu_p)\right),$$

where the parenthetical isomorphism comes from Kummer theory. Moreover, for simplicitly, we only give the argument when $X = \mathrm{Spec}(R)$ is a smooth affine $\mathbf{F}_p$-scheme; this allows us to suppress Frobenius twists. Let $\Delta_R = R\Gamma_{\Delta}(X/\mathbf{Z}_p)$ be the prismatic complex, so $\Delta_R/p \simeq \Omega^\bullet_{R/\mathbf{F}_p}$ is the de Rham complex. Let $Z^1\Omega_{R/\mathbf{F}_p} \subset \Omega^1_{R/\mathbf{F}_p}$ be the subgroup of 1-cycles, i.e., the kernel of d. The proof has two parts:

1. *Reinterpretation via differential forms:* We claim that there is a natural quasi-isomorphism

$$\left(\mathrm{Fil}^1_{\mathcal{N}}\Delta_R/p \xrightarrow{\mathrm{can}-\phi_i} \Delta_R/p\right) \simeq \left(Z^1\Omega_{R/\mathbf{F}_p} \xrightarrow{1-C} \Omega^1_{R/\mathbf{F}_p}\right)[-1] \tag{4.4.2}$$

   where the map 1 on the right denotes the canonical inclusion, and C denotes the projection $Z^1\Omega_{R/\mathbf{F}_p} \twoheadrightarrow H^1(\Omega^\bullet_{R/\mathbf{F}_p}) \simeq \Omega^1_{R/\mathbf{F}_p}$, with the last map being the Cartier isomorphism.

   For this, consider the sheaf $\mathcal{H}_{\mathcal{N}}(X) \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$. We have an exact triangle

$$\mathcal{H}_{\mathcal{N}}(X)/p \xrightarrow{can} \mathcal{H}_{\mathcal{N}}(X)/t \oplus \mathcal{H}_{\mathcal{N}}(X)/u \xrightarrow{(a,b)\mapsto a-b} \mathcal{H}_{\mathcal{N}}(X)/(u,t)$$

   on $\mathbf{F}_p^{\mathrm{Syn}}$, where all quotients are interpreted in the Koszul sense (this arises by tensoring the analogous triangle for $\mathcal{O}_{k^{\mathrm{Syn}}}$ with $\mathcal{H}_{\mathcal{N}}(X)$, as in the proof of Theorem 3.3.5). Identifying these objects with graded $\mathbf{Z}_p[u,t]/(ut-p)$-complexes and passing to the degree 1 summand, this gives an exact triangle

$$\mathrm{Fil}^1_{\mathcal{N}}\Delta_R/p \xrightarrow{\alpha:=(\phi_1,\mathrm{can})} \mathrm{Fil}_1^{\mathrm{conj}}\Omega^\bullet_{R/\mathbf{F}_p} \oplus \mathrm{Fil}^1_H\Omega^\bullet_{R/\mathbf{F}_p} \xrightarrow{\delta} \Omega^1_{R/\mathbf{F}_p}[-1] \tag{4.4.3}$$

in $\mathcal{D}(\mathbf{F}_p)$, where the map ϕ_1 is induced by the divided Frobenius, the map can is the canonical map, and δ is the map induced by the difference of the obvious maps from each term. On the other hand, recalling that

$$\mathrm{Fil}_1^{\mathrm{conj}}\Omega^\bullet_{R/\mathbf{F}_p} = \left(R \xrightarrow{d} Z^1\Omega_{R/\mathbf{F}_p}\right), \tag{4.4.4}$$

we obtain a natural fibre sequence

$$Z^1\Omega_{R/\mathbf{F}_p}[-1] \xrightarrow{\beta:=(\pi,-\mathrm{inc})} \mathrm{Fil}_1^{\mathrm{conj}}\Omega^\bullet_{R/\mathbf{F}_p} \oplus \mathrm{Fil}^1_H\Omega^\bullet_{R/\mathbf{F}_p} \xrightarrow{\sigma:=\mathrm{sum}} \Omega^\bullet_{R/\mathbf{F}_p} \simeq \Delta_R/p, \tag{4.4.5}$$

where π comes from the inclusion of $Z^1\Omega_{R/\mathbf{F}_p}$ as the last term of (4.4.4), while inc comes from the inclusion $Z^1\Omega_{R/\mathbf{F}_p} \subset \Omega^1_{R/\mathbf{F}_p}$: indeed, this is a general statement about any chain complex. Consider the map

$$\mathrm{Fil}^1_{\mathcal{N}}\Delta_R/ \xrightarrow{\sigma\circ\alpha} \Delta_R/p$$

obtained by composing α and σ from (4.4.3) with (4.4.5). Unwinding definitions, this map identifies with the map $\phi_1 - \mathrm{can}$ appearing on the LHS of (4.4.2), so it suffices to identify the fibre of this map with the RHS of (4.4.2). For this, consider the following diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & Z^1\Omega_{R/\mathbf{F}_p}[-1] \arrow[r, "\delta\circ\beta"] \arrow[d, "\beta"] & \Omega^1_{R/\mathbf{F}_p}[-1] \arrow[d, equal] \\
\mathrm{Fil}^1\Delta_R/p \arrow[r, "\alpha"] \arrow[d, "\sigma\circ\alpha"] & \mathrm{Fil}_1^{\mathrm{conj}}\Omega^\bullet_{R/\mathbf{F}_p} \oplus \mathrm{Fil}^1_H\Omega^\bullet_{R/\mathbf{F}_p} \arrow[r, "\delta"] \arrow[d, "\sigma"] & \Omega^1_{R/\mathbf{F}_p}[-1] \\
\Delta_R/p \arrow[r, equal] & \Delta_R/p &
\end{tikzcd}$$

where one first draws the middle column and row from (4.4.5) and (4.4.3) respectively, then defines F to make the square on the top left a fibre square, and then computes that the top row and left column are also exact. In particular, the term F is both the fibre of $\sigma\alpha$ as well as $\delta\beta$. To finish proving the claim, it suffices to identify $\delta\beta$ with $C-1$; this is immediate from the definitions and chasing diagrams, so we win.

Remark 4.4.8. The preceding homological argument is not specific to the F-gauge $\mathcal{H}_{\mathcal{N}}(X)$ and can be abstractly explained as follows (where we ignore Frobenius twists by assuming $k = \mathbf{F}_p$). The stack $(k^{\mathrm{Syn}})_{p=0}$ admits the following descriptions:

(a) Glue two copies of $\mathbf{A}^1/\mathbf{G}_m$ at their closed point (which is a copy of $B\mathbf{G}_m$) to get $(k^{\mathcal{N}})_{p=0}$, and then identify the two open copies of $\mathrm{Spec}(k)$ with each other to obtain $(k^{\mathrm{Syn}})_{p=0}$.

(b) Glue two copies of $\mathbf{A}^1/\mathbf{G}_m$ at their open point to get a stack Ξ, and then identify the two closed points (which are copies of $B\mathbf{G}_m$) together to obtain $(k^{\mathrm{Syn}})_{p=0}$.

Each of these glueing descriptions leads to a fibre seqeunce computing $R\Gamma(k^{\mathrm{Syn}}, E)$ for a mod p F-gauge $E \in \mathcal{D}_{qc}((k^{\mathrm{Syn}})_{p=0})$ on k. The first recipe gave the fibre sequence (4.2.1), which for $E = \mathcal{H}_{\mathcal{N}}(X)\{i\}$ leads to the left side of (4.4.2); the second recipe gives the right side of (4.4.2).

2. *Identifying* $\mathbf{G}_m/p$ *via differential forms, following Cartier*[54]: It suffices to show that there is a natural quasi-isomorphism

$$R\Gamma_{et}(\mathrm{Spec}(R), \mathbf{G}_m/p) \simeq \left(Z^1\Omega_{R/\mathbf{F}_p} \xrightarrow{1-C} \Omega^1_{R/\mathbf{F}_p}\right)[-1].$$

Moreover, the terms on the right are quasi-coherent sheaves on a Frobenius twisted copy of X. The claim will then follow by applying $R\Gamma_{et}(X, -)$ to the following local assertion:

($*$) There is a natural short exact sequence

$$0 \to \mathbf{G}_m/p \xrightarrow{d\log} Z^1\Omega_{X/\mathbf{F}_p} \xrightarrow{C-1} \Omega^1_{X/\mathbf{F}_p} \to 0$$

on X_{et}, where $d\log(f) = \frac{df}{f}$.

This exactness can be checked on stalks, so pick a geometric point $x \to X$, and let $\mathcal{O}_{X,x}$ be the corresponding strictly henselian local ring of X. Recall that the Cartier map C has the following properties:

- C is p^{-1}-linear: it is $\mathcal{O}_X$-linear when viewed as a map $F_*Z^1\Omega^1_{X/\mathbf{F}_p} \to \Omega^1_{X/\mathbf{F}_p}$. Concretely, we have $C(f^p\omega) = f\omega$ for $f \in \mathcal{O}_{X,x}$ and $\omega \in Z^1\Omega_{X/\mathbf{F}_p,x}$.
- We have $C(f^{p-1}df) = df$ for $f \in \mathcal{O}_{X,x}$.
- We have $C(\frac{df}{f}) = \frac{df}{f}$ for $f \in \mathcal{O}^*_{X,x}$: this follows from the previous one by dividing by f.

The third property ensures that the sequence of maps in ($*$) above gives a complex. Let us now prove exactness of the sequence.

Exactness on the left: given $f \in \mathcal{O}^*_{X,x}$, if $\frac{df}{f} = 0$, then $df = 0$, whence $f = g^p$ by the Cartier isomorphism, so f is 0 in $\mathcal{O}^*_{X,x}/(\mathcal{O}^*_{X,x})^p$, as wanted.

Exactness on the right: Pick invertible étale co-ordinates $t_1, ..., t_n \in \mathcal{O}_{X,x}$, so $\{\frac{dt_i}{t_i}\}$ forms a basis of $\Omega^1_{X/\mathbf{F}_p,x}$ over $\mathcal{O}_{X,x}$. It suffices to show that any element of the form $f\frac{dt_i}{t_i} \in \Omega^1_{X/\mathbf{F}_p,x}$ has the form $(C-1)(\omega)$ for a closed 1-form $\omega \in Z^1\Omega_{X/\mathbf{F}_p,x}$. As $\mathcal{O}_{X,x}$ is strictly henselian, we can find $g \in \mathcal{O}_{X,x}$ such that $g - g^p = f$. The form $\omega = g^p\frac{dt_i}{t_i}$ is closed and satisfies

$$(C-1)(g^p\frac{dt_i}{t_i}) = g\frac{dt_i}{t_i} - g^p\frac{dt_i}{t_i} = (g-g^p)\frac{dt_i}{t_i} = f\frac{dt_i}{t_i},$$

so $C-1$ is indeed surjective.

Exactness in the middle: This is the trickiest part of the proof, and we only give the argument using the following unproven property: if $\omega \in Z^1\Omega^1_{X/\mathbf{F}_p,x}$ is a closed 1-form, the associated flat connection $(M = \mathcal{O}_{X,x}, \nabla_\omega := d + \omega)$ has p-curvature 0 exactly when $(C-1)(\omega) = 0$. Granting this, we must show that if (M, d_ω) has p-curvature 0, then $\omega = \frac{df}{f}$ for some unit f at x. But if (M, ∇_ω) has p-curvature 0, then, by Cartier descent, it must have a ∇-horizontal basis, i.e., there is some generator (aka unit) $g \in M = \mathcal{O}_{Y,y}$ such that $\nabla_\omega(g) = 0$. The latter means $dg + g\omega = 0$, whence $\omega = -\frac{dg}{g} = \frac{df}{f}$ for $f = g^{-1}$, as wanted. □

Remark 4.4.9. For a smooth k-scheme X, one can give a description of $R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(n))$ in terms of logarithmic de Rham–Witt forms for all n, see [BMS19, Corollary 8.21].

[54]This result goes back to Cartier, see [Ses58, §2, Theorem 1]. The proof we sketch is adapted from [Mil76, Lemma 1.3] and [Ill79, Corollary 0. 2.1.18].

Remark 4.4.10 (An analog of the Lefschetz $(1,1)$ theorem)**.** The conclusions of Proposition 4.4.7 apply much more generally than just for smooth k-schemes: for any k-scheme X, there is a fibre sequence

$$R\Gamma_{et}(X, \mathbf{G}_m)/p \to R\Gamma(X, Z^1 L_{X/k}) \xrightarrow{1-C} R\Gamma(X, L_{X/k}),$$

which gives a tool for calculating the p-primary part of Picard/Brauer groups of X in terms of coherent cohomology, somewhat similar to the exponential sequence over $\mathbf{C}$. This result was essentially proven in [BMS19, §7] in the quasi-syntomic case, and [BL22a, Theorem 7.5.6] in general; see also [CZ22] for further exposition as well as some concrete geometric applications. Mixed characteristic analogs of this sequence were studied in [CS19] (for perfectoids, via arc descent) with applications to Gabber's conjectures on Picard and Brauer groups; see also [BL22a, Theorem 8.5.7] for an exposition via quasi-syntomic descent.

Remark 4.4.11 (From quasi-coherent sheaves to cohomology theories)**.** As noted earlier, we can regard $\mathcal{D}_{qc}(k^{\mathrm{Syn}})$ as a category of p-adic cohomology theories on smooth k-schemes: each $F \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$ gives rise to the cohomology theory carrying a smooth map $f : X \to \mathrm{Spec}(k)$ to the complex $R\Gamma(X^{\mathrm{Syn}}, f^*F) \simeq R\Gamma(k^{\mathrm{Syn}}, F \otimes \mathcal{H}_{\mathcal{N}}(X))$. We make a short list of some of the theories captured by this construction:

Quasi-coherent complex $F \in \mathcal{D}_{qc}(k^{\mathrm{Syn}})$	**Cohomology theory** $R\Gamma(k^{\mathrm{Syn}}, \mathcal{H}_{\mathrm{Syn}}(X) \otimes F)$
$\mathcal{O}_{k^{\mathrm{Syn}}}\{i\}$	$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i))$
$j_{\Delta,*}\mathcal{O}_{k^{\Delta}}$	$R\Gamma_{\Delta}(X/W)$
$j_{\mathcal{N},*}\mathcal{O}_{k^{\mathcal{N}}}\{i\}$	$\mathrm{Fil}^i_{\mathcal{N}}\phi^* R\Gamma_{\Delta}(X/W)$
$h_{dR,+,*}\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}\{i\}$	$\mathrm{Fil}^i_H R\Gamma(X, \Omega^\bullet_{X/k})$
$h_{HT,c,*}\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}\{i\}$	$\phi_*\mathrm{Fil}^{\mathrm{conj}}_i R\Gamma(X, \Omega^\bullet_{X/k})$
$i_{H,*}\mathcal{O}_{\mathrm{Spec}(k)}\{i\}$	$R\Gamma(X, \Omega^i_{X/k})[-i]$

Moreover, the natural maps between cohomology theories on the right are induced by maps of quasi-coherent sheaves on left (e.g., $\mathcal{O}_{k^{\mathrm{Syn}}}\{i\} \to j_{\mathcal{N},*}\mathcal{O}_{k^{\mathcal{N}}}\{i\}$ corresponds to the natural map $R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i)) \to \mathrm{Fil}^i_{\mathcal{N}}\phi^* R\Gamma_{\Delta}(X/W)$ appearing in Proposition 4.4.2). In particular, this table shows the flexibility of working with quasi-coherent complexes on k^{Syn} that are not perfect complexes (or equivalently not coherent complexes) is quite useful: the middle 4 entries in the table above come from quasi-coherent sheaves that are not perfect and yet give cohomology theories of interest with good finiteness properties.

4.5 Serre duality on $\mathbf{F}_p^{\mathrm{Syn}}$

In this section, we record a Serre duality theorem for the cohomology of perfect complexes on the stack $\mathbf{F}_p^{\mathrm{Syn}}$. This duality theorem (or, better, its analog for a curve over a finite field) is a "toy model" of the duality result mentioned in Theorem 1.3.1 (3). As a concrete application, we use it to deduce a duality theorem in the syntomic cohomology of smooth proper varieties over finite fields (due to Milne) as well as some concrete consequences in low dimensions.

In order to explain why $\mathbf{F}_p^{\mathrm{Syn}}$ satisfies a form of Serre duality, the following result, morally expressing the properness of $\mathbf{F}_p^{\mathrm{Syn}}$ over $\mathbf{F}_p$, is obligatory:

Proposition 4.5.1. *The functor* $R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, -)$ *carries* $\mathrm{Perf}(\mathbf{F}_p^{\mathrm{Syn}})$ *into* $\mathrm{Perf}(\mathbf{Z}_p)$. *Moreover, it has cohomological dimension* ≤ 1*, i.e.,* $H^{>1}(\mathbf{F}_p^{\mathrm{Syn}}, -)$ *vanishes on* $\mathrm{Coh}(\mathbf{F}_p^{\mathrm{Syn}})$.

Proof. For the first statement, using the sequence (4.2.1), it suffices to show a similar statement for the stacks $\mathbf{F}_p^{\mathcal{N}}$ and $\mathbf{F}_p^{\Delta}$. The former was shown in Lemma 3.4.11, while the latter is obvious since $\mathbf{F}_p^{\Delta} = \mathrm{Spf}(\mathbf{Z}_p)$. The second statement then follows immediately as $H^{>0}(\mathbf{F}_p^{\mathcal{N}}, -)$ and $H^{>0}(\mathbf{F}_p^{\Delta}, -)$ both vanish on coherent sheaves: the latter is clear as $\mathbf{F}_p^{\Delta}$ is affine, while the former follows as $\mathbf{F}_p^{\mathcal{N}}$ is the quotient of an affine p-adic formal scheme by the linearly reductive group $\mathbf{G}_m$. $\square$

The main theorem of this chapter is the following form of Serre duality for $\mathbf{F}_p^{\mathrm{Syn}}$; this can be regarded as a p-adic counterpart of the familiar assertion that, for prime-to-p finite coefficients, $\mathrm{Spec}(\mathbf{F}_p)$ has the étale homotopy type of a circle.

Theorem 4.5.2 (Serre duality on $\mathbf{F}_p^{\mathrm{Syn}}$)**.** *The following hold true.*

1. Cohomology of the structure sheaf: *There is a natural isomorphism*

$$t : R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, \mathcal{O}) \simeq \mathbf{Z}_p \oplus \mathbf{Z}_p[-1]$$

   in $\mathrm{Perf}(\mathbf{Z}_p)$.

2. Duality: *For any* $E \in \mathrm{Perf}(\mathbf{F}_p^{\mathrm{Syn}})$*, the pairing*

$$R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E) \otimes R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E^\vee) \to R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, \mathcal{O}) \xrightarrow{t} \mathbf{Z}_p[-1]$$

   coming from (1) induces an isomorphism

$$R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E) \simeq R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E^\vee)^\vee[-1]$$

   in $\mathrm{Perf}(\mathbf{Z}_p)$.

Note that it makes sense to formulate the statement in (2) without demanding that E is perfect. In fact, as the proof below shows, the claim holds true for some E's which are not perfect, and these are critical to the method of the proof.

Proof. For (1), the calculation of weight 0 syntomic cohomology (Example 4.2.6) gives

$$R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, \mathcal{O}) \simeq R\Gamma(\mathrm{Spec}(\mathbf{F}_p), \mathbf{Z}_p) \simeq R\Gamma_{cts}(\mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p), \mathbf{Z}_p),$$

where the last term denotes continuous group cohomology. We have the standard isomorphism $\widehat{\mathbf{Z}} \simeq \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$ coming from the Frobenius element. For any continuous $\widehat{\mathbf{Z}}$-module M, we have a natural identification

$$R\Gamma_{cts}(\widehat{\mathbf{Z}}, M) \simeq \left(M \xrightarrow{\gamma - 1} M\right),$$

where $\gamma \in \widehat{\mathbf{Z}}$ is a topological generator. When $M = \mathbf{Z}_p$ has the trivial action, the differential $\gamma - 1$ vanishes, which gives the claim in (1).

The proof of (2) is via devissage on E and will entail proving the assertion for objects $E \in \mathcal{D}_{qc}(\mathbf{F}_p^{\mathrm{Syn}})$ that are not necessarily perfect. Thus, let $\mathcal{C}$ be the full subcategory of $\mathcal{D}_{qc}(\mathbf{F}_p^{\mathrm{Syn}})$ spanned by those E for which the map

$$\eta_E : R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E) \to R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E^\vee[1])^\vee$$

coming from (1) is an isomorphism, where internal $(-)^\vee$ denotes $\underline{\mathrm{RHom}}_{\mathbf{F}_p^{\mathrm{Syn}}}(-, \mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}})$, computed in $\mathcal{D}_{qc}(\mathbf{F}_p^{\mathrm{Syn}})$[55]. Note that $E \in \mathcal{C}$ if and only if $E/p \in \mathcal{C}$. Our goal is to show that $\mathrm{Perf}(\mathbf{F}_p^{\mathrm{Syn}}) \subset \mathcal{C}$, which we shall do by a series of reductions.

(a) We have $\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}} \in \mathcal{C}$: this follows by the calculation in (1) and the construction of the pairing giving rise to the comparison map $\eta_{\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}}}$.

(b) We have $\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}}\{i\} \in \mathcal{C}$ for all $i \in \mathbf{Z}$: we just checked it for $i = 0$ in (a) and the claim is vacuous for $i \neq 0$ (as both sides vanish by Example 4.4.5).

(c) We claim that $E := j_{\Delta,*}\mathcal{O} \in \mathcal{C}$. For this, let us first compute $E^\vee/p$ using the following exact sequence

$$\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}}/p \to j_{\mathcal{N},*}\mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}}/u \oplus j_{\mathcal{N},*}\mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}}/t \to j_{\Delta,*}\mathcal{O}_{\mathbf{F}_p^{\Delta}}/p \oplus i_{H,*}\mathcal{O}_{\mathbf{F}_p^H},$$

which can be obtained by regarding $(\mathbf{F}_p^{\mathrm{Syn}})_{p=0}$ as obtained by glueing together two copies of $\mathbf{A}^1/\mathbf{G}_m$ along their open and closed points separately (and we have written $j_\Delta : \mathbf{F}_p^\Delta \to \mathbf{F}_p^{\mathrm{Syn}}$ and $i_H : \mathbf{F}_p^H \to \mathbf{F}_p^{\mathrm{Syn}}$ for the corresponding open and closed points in $\mathbf{F}_p^{\mathrm{Syn}}$). Consider the functor $F(-) := \underline{\mathrm{RHom}}_{\mathbf{F}_p^{\mathrm{Syn}}}(E, -)$. We claim that $F(-)$ kills $j_{\mathcal{N},*}\mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}}/u$, $j_{\mathcal{N},*}\mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}}/t$, and $i_{H,*}\mathcal{O}_{\mathbf{F}_p^H}$. As E is pushed forward along j_Δ which does not meet i_H, the claim for the last term is clear. For the first two, by adjunction, one reduces to the following assertion: if $j : \mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ is the inclusion of the open point, then $\underline{\mathrm{RHom}}_{\mathbf{A}^1/\mathbf{G}_m}(j_*\mathcal{O}, \mathcal{O}) = 0$, which can be proven using the completeness of perfect complexes on $\mathbf{A}^1/\mathbf{G}_m$ (Remark 2.2.7). Thus, applying $F(-)$ to the exact sequence above then gives a triangle

$$E^\vee/p := F(\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}}/p) \to 0 \to F(j_{\Delta,*}\mathcal{O}_{\mathbf{F}_p^\Delta}/p) \simeq E/p,$$

where the last isomorphism reflects full faithfulness of $j_{\Delta,*}$. In other words, we learn that

$$E^\vee/p[1] \simeq E/p,$$

so the comparison map η_E, reduced mod p, takes the form

$$R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E/p) \to R\Gamma(\mathbf{F}_p^{\mathrm{Syn}}, E/p)^\vee.$$

As $E/p = j_{\Delta,*}\mathcal{O}/p$, both sides above are naturally identified with $\mathbf{F}_p[0]$, and one checks by unwinding identifications that the map is an isomorphism.

(d) For any $E \in \mathrm{Perf}(\mathbf{F}_p^\Delta)$, we have $j_{\Delta,*}E \in \mathcal{C}$: indeed, since $\mathbf{F}_p^\Delta = \mathrm{Spf}(W)$, any such E can be finitely built from copies of $\mathcal{O}_{\mathbf{F}_p^\Delta}$, so the claim follows from (c).

(e) For $E \in \mathrm{Perf}(\mathbf{F}_p^{\mathcal{N}})$, we claim $j_{\mathcal{N},*}E \in \mathcal{C}$. For this, recall (as in the proof of Lemma 3.4.11) that $\mathrm{Perf}(\mathbf{F}_p^{\mathcal{N}})$ is generated under finite colimits, shifts and retracts by the twists $\mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}}(i)$ for $i \in \mathbf{Z}$. Now $j_{\mathcal{N}}^*\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}}\{i\} = \mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}}(-i)$, so it suffices (by the projection formula) to show that $(j_{\mathcal{N},*}\mathcal{O})\{i\} \in \mathcal{C}$ for all $i \in \mathbf{Z}$. For this, consider the exact triangle

$$\mathcal{O}_{\mathbf{F}_p^{\mathrm{Syn}}} \to j_{\mathcal{N},*}\mathcal{O}_{\mathbf{F}_p^{\mathcal{N}}} \to j_{\Delta,*}\mathcal{O}_{\mathbf{F}_p^\Delta}$$

resulting from the description of $\mathbf{F}_p^{\mathrm{Syn}}$ as the result of glueing two copies of $\mathbf{F}_p^\Delta$ inside $\mathbf{F}_p^{\mathcal{N}}$. After twisting by $\mathcal{O}\{i\}$, the outer two terms lie in $\mathcal{C}$ by (b) and (d), so we win.

[55]Note that $E \mapsto E^\vee$ defined as above is, a priori, a strange operation for non-perfect E, e.g., its formation does not commute with restriction to open subsets. We will only use this operation in cases where one has good control on the output. In particular, in all cases we consider, the complex E will be reflexive, i.e., the biduality map $E \to (E^\vee)^\vee$ is an isomorphism.

(f) Any $E \in \mathrm{Perf}(\mathbf{F}_p^{\mathrm{Syn}})$ lies in $\mathcal{C}$: this follows by tensoring the triangle in (e) with E, and using the conclusion of (d) and (e). $\square$

To extract tangible consequences of this form of Serre duality, we need the following version of Poincare duality for crystalline cohomology, formulated at the level of F-gauges.

Theorem 4.5.3 (Geometric Poincare duality via F-gauges, (Berthelot [Ber97], Tang [Tan22]))**.** *Fix a perfect field k and a smooth proper geometrically connected k-scheme X of dimension d.*

1. *There is a natural isomorphism $\mathcal{H}^{2d}_{\mathrm{Syn}}(X)\{d\} \simeq \mathcal{O}_{k^{\mathrm{Syn}}}$.*
2. *The pairing*

$$\mathcal{H}_{\mathrm{Syn}}(X) \otimes \mathcal{H}_{\mathrm{Syn}}(X) \to \mathcal{H}_{\mathrm{Syn}}(X) \xrightarrow{\mathrm{tr}_{X/k}} \mathcal{O}\{-d\}[-2d]$$

(where the last map comes from (1) and Remark 4.2.3) is perfect, giving an isomorphism

$$\mathcal{H}_{\mathrm{Syn}}(X)^\vee \simeq \mathcal{H}_{\mathrm{Syn}}(X)\{d\}[2d]$$

in $\mathrm{Perf}(k^{\mathrm{Syn}})$.

Remark 4.5.4 (Classical geometric Poincare duality from F-gauge version)**.** The Poincare duality in Theorem 4.5.3 gives rise to more classical forms of Poincare duality under various "realization" functors (Remark 4.2.2), e.g., after pulling back along $j_{\Delta} : \mathrm{Spf}(W) \to k^{\mathrm{Syn}}$ (resp., $i_H : B\mathbf{G}_m \to k^{\mathrm{Syn}}$, $h_{dR,+} : \mathbf{A}^1/\mathbf{G}_m \to k^{\mathrm{Syn}}$, and $h_{dR,c} : \mathbf{A}^1/\mathbf{G}_m$), one recovers, at least up to p-adic units, the known form of Poincare duality for crystalline cohomology (resp. Hodge cohomology, Hodge filtered de Rham cohomology, and conjugate filtered de Rham cohomology).

Remark 4.5.5 (Geometric Poincare duality with coefficients)**.** Fix X/k as in Theorem 4.5.3 and write $\pi : X^{\mathrm{Syn}} \to k^{\mathrm{Syn}}$ for the structure map. Using Theorem 4.5.3, a version of Remark 3.3.6 with coefficients as well as classical Serre duality for X/k, one can show the following (but we do not explain it here): for any $E \in \mathrm{Perf}(X^{\mathrm{Syn}})$, the pushforward $R\pi_* E$ is a perfect complex, and the resulting pairing

$$R\pi_* E \otimes R\pi_* E^\vee\{d\}[2d] \to R\pi_* \mathcal{O}_{X^{\mathrm{Syn}}}\{d\}[2d] = \mathcal{H}_{\mathrm{Syn}}\{d\}[2d] \to \mathcal{H}^{2d}_{\mathrm{Syn}}(X)\{d\} \simeq \mathcal{O}_{k^{\mathrm{Syn}}}$$

is perfect, i.e., we have

$$(R\pi_* E)^\vee \simeq R\pi_*(E^\vee)\{-d\}[-2d],$$

which can be regarded as a form of classical Poincare duality with coefficients in an F-gauge.

Combining Theorem 4.5.2 with Theorem 4.5.3, one obtains the following result of Milne [Mil76, Theorem 1.9], which was originally conjectured by Artin [Art74].

Corollary 4.5.6 (Milne)**.** *Let X/k be a smooth proper k-scheme of dimension d. For each integer i, there is a natural isomorphism*

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i)) \simeq R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(d-i))^\vee[-2d-1]$$

in $\mathrm{Perf}(\mathbf{Z}_p)$.

Proof. Since $R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i)) \simeq R\Gamma(k^{\mathrm{Syn}}, \mathcal{H}_{\mathcal{N}}(X)\{i\})$, the first claim follows by applying Theorem 4.5.2 to $E = \mathcal{H}_{\mathrm{Syn}}(X)\{i\}$, noting that $E^\vee \simeq \mathcal{H}_{\mathrm{Syn}}(X)\{d-i\}[2d]$ by Theorem 4.5.3. The last claim is then immediate from Corollary 4.4.4. $\square$

Let us explain the concrete information present in Corollary 4.5.6 in low dimensions.

Example 4.5.7 (From duality for curves to unramified p-primary class field theory)**.** Say X/k is a smooth proper curve over $\mathbf{F}_q$. Recall that unramified global class field theory for X gives an isomorphism

$$\mathrm{Pic}(X)^{\wedge} \simeq \pi_1(X)^{ab}$$

(with certain specified features) where the completion on the left denotes the profinite completion. Modulo p^n, this amounts to an identification

$$\mathrm{Pic}(X)/p^n \simeq \pi_1(X)^{ab}/p^n.$$

We explain how to recover this from Corollary 4.5.6; the analogous ℓ-adic deduction is classical.

Reducing the $i = 1$ case of the isomorphism in Corollary 4.5.6 modulo p^n and using Proposition 4.4.7, we obtain a natural identification

$$R\Gamma(X, \mu_{p^n}) \simeq R\Gamma(X, \mathbf{Z}/p^n)^{\vee}[-3], \tag{4.5.1}$$

where $(-)^{\vee}$ denotes $R\mathrm{Hom}_{\mathbf{Z}/p^n}(-, \mathbf{Z}/p^n)$. As this functor is t-exact (as $\mathbf{Z}/p^n$ is artinian and Gorenstein), we obtain

$$H^2(X, \mu_{p^n}) \simeq H^1(X, \mathbf{Z}/p^n)^{\vee} \simeq \mathrm{Hom}_{cts}(\pi_1(X), \mathbf{Z}/p^n)^{\vee} \simeq \pi_1(X)^{ab}/p^n.$$

The Kummer sequence gives a SES

$$0 \to \mathrm{Pic}(X)/p^n \to H^2(X, \mu_{p^n}) \to H^2(X, \mathbf{G}_m)[p^n] \to 0 \tag{4.5.2}$$

so it suffices to show that $H^2(X, \mathbf{G}_m)[p^n] = 0$, which then reduces to the corresponding statement for $n = 1$. This can be shown directly using Lang's theorem on the triviality of torsors for connected group schemes over finite fields; we give an argument using Corollary 4.5.6 for the sake of variety. Observe that taking Euler characteristics over the $n = 1$ case of (4.5.1) gives the formula

$$\dim H^0(X, \mu_p) + \dim H^2(X, \mu_p) - \dim H^1(X, \mu_p) - \dim H^3(X, \mu_p) = -\chi(X, \mathbf{Z}/p).$$

Noting that $\chi(X, \mathbf{Z}/p) = 0$ by Artin–Schreier theory, $\dim H^0(X, \mu_p) = 0$ as X is reduced, and $H^3(X, \mu_p) \simeq H^0(X, \mathbf{Z}/p)^{\vee} \simeq \mathbf{Z}/p$, this gives

$$\dim H^1(X, \mu_p) + 1 = \dim H^2(X, \mu_p).$$

The Kummer sequence shows that $H^1(X, \mu_p) = \mathrm{Pic}(X)[p]$. so the above can be rewritten using (4.5.2) as

$$\#\mathrm{Pic}(X)[p] \cdot p = \#\mathrm{Pic}(X)/p \cdot \#H^2(X, \mathbf{G}_m)[p].$$

But the degree map $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$ surjects onto $n\mathbf{Z}$ for some nonzero n[56], and can then be split to give an abstract isomorphism $\mathrm{Pic}(X) \simeq \mathrm{Pic}^0(X) \times \mathbf{Z}$ with $\mathrm{Pic}^0(X)$ finite. Contemplating multiplication by p on any such product shows that

$$\#\mathrm{Pic}(X)[p] \cdot p = \#\mathrm{Pic}(X)/p,$$

so we must have $\#H^2(X, \mathbf{G}_m)[p] = 1$, as wanted.

[56] In fact, one has $n = 1$: X carries a degree 1 line bundle, by either Lang's theorem or the Lang–Weil estimate.

Remark 4.5.8 (Duality for surfaces)**.** Corollary 4.5.6 for smooth proper geometrically connected surfaces $X/\mathbf{F}_q$ takes the form of a self-duality

$$R\Gamma_{fl}(X, \mu_{p^n}) \simeq R\Gamma_{fl}(X, \mu_{p^n})^{\vee}[-5] \tag{4.5.3}$$

for any $n \geq 0$. This duality formalism was used in (at least) two important results in positive characteristic algebraic geometry: Artin's work on supersingular K3 surfaces (including, e.g., Tate's conjecture for elliptic supersingular K3 surfaces) [Art74], and Milne's work on deducing the refined formula of B-SD type from Tate's conjecture for divisors on a surface [Mil75]. A concrete geometric observation in the latter is the construction of a well-behaved pairing on the Brauer group of X from (4.5.3), so let us quickly recall how this works.

Combining the above duality theorem with corresponding ℓ-adic statement, one can replace p^n with any integer m in (4.5.3). One then obtains a pairing

$$\mathrm{Br}(X) \times \mathrm{Br}(X) \xrightarrow{(a,b) \mapsto \langle a,b \rangle} \mathbf{Q}/\mathbf{Z}$$

by passage to the direct limit over n of the maps

$$H^2_{fl}(X, \mu_n) \times H^2_{fl}(X, \mu_n) \xrightarrow{(1,\beta_n)} H^2_{fl}(X, \mu_n) \times H^3_{fl}(X, \mu_n) \xrightarrow{\cup} H^5(X, \mu_n) \simeq \mathbf{Z}/n,$$

where β_n is the Bockstein map associated to the extension μ_{n^2} of μ_n by μ_n. Note that the pairing $\langle \cdot, \cdot \rangle$ is skew-symmetric: this amounts to using that β_n is a derivation together with the fact that $\beta_n : H^4_{fl}(X, \mu_n) \to H^5_{fl}(X, \mu_n) \simeq \mathbf{Z}/n$ is 0 as the canonical map $\mathbf{Z}/n \simeq H^5_{fl}(X, \mu_n) \to \mathbf{Z}/n^2 \simeq H^5_{fl}(X, \mu_{n^2})$ is injective. Moreover, this pairing kills divisible elements on either side, and induces (thanks to the duality with finite coefficients) a perfect skew-symmetric pairing on the quotient of $\mathrm{Br}(X)$ by its divisible elements.

Remark 4.5.9 (Serre duality for X^{Syn})**.** Combining Remark 4.5.5 with Theorem 4.5.2, we learn the following: for $X/\mathbf{F}_p$ proper smooth of dimension d and $E \in \mathrm{Perf}(X^{\mathrm{Syn}})$, there is a canonical isomorphism

$$R\Gamma(X^{\mathrm{Syn}}, E) \simeq R\Gamma(X^{\mathrm{Syn}}, E^{\vee})^{\vee}\{-d\}[-2d-1].$$

Thus, one might regard X as being "p-adic Poincare duality manifold of dimension $2d+1$", in analogy with what happens in étale cohomology. The $d = 1$ case of this result can be regarded as a characteristic p analog of Theorem 1.3.1 (3).

Chapter 5

Filtered prismatization in mixed characteristic

In this section, we turn to mixed characteristic. Given a bounded p-adic formal scheme X, the main object of study in this chapter is its prismatiziation X^{Δ} in §5.1 as well as its filtered enlargement $X^{\Delta} \subset X^{\mathcal{N}}$ in §5.3. We introduce these stacks through their moduli descriptions, and then explain why they specialize in characteristic p to the constructions from Chapter 3, justifying the notation. The primary theorem of this chapter is Corollary 5.5.11, describing the filtered prismatization explicitly (in terms of the Nygaard filtration on prismatic cohomology) for "quasiregular semiperfectoid rings", thereby enabling one to study quasi-coherent sheaves on the filtered prismatization stacks relatively explicitly via descent[57]; note that the connection with the Nygaard filtration was anticipated in [Dri20, Footnote 3].

Unlike previous chapters, it will be useful in this chapter if the reader has some prior exposure to the prismatic theory from [BS19] as well as parts of [BL22a], both for motivational purposes as well as understanding Corollary 5.5.11.

5.1 The prismatization

In this section, we introduce the prismatization X^{Δ} of a bounded p-adic formal scheme X. Roughly, the goal here is to geometrize the theory of prismatic crystals and their cohomology: the quasi-coherent sheaf theory of X^{Δ} should reflect the theory of prismatic crystals on X, and thus $R\Gamma(X^{\Delta}, \mathcal{O}_{X^{\Delta}})$ should equal the prismatic cohomology $R\Gamma_{\Delta}(X)$ in good cases. Note that saw variants in characteristic p for de Rham and crystalline cohomology in §2.5 and §3 respectively. To get started, we shall need the language of δ-rings and the concomittant theory of Witt vectors.

Construction 5.1.1 (δ-rings)**.** A δ-ring (A, δ) consists of a commutative ring A and a map $\delta : A \to A$ of sets such that $\delta(0) = \delta(1) = 0$,

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$

and

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

[57]Essentially all the material in this chapter comes from [Dri20, BL22a, BL22b, BL]. The definition of the filterered prismatization $X^{\mathcal{N}}$ is due to [Dri20]; the comparison theorem mentioned above will appear in [BL].

for all $x, y \in A$. Given a δ-ring (A, δ), the induced map

$$\phi : A \to A, \quad \phi(x) = x^p + p\delta(x)$$

is a ring homomorphism and a is a lift of the Frobenius on A/p. Conversely, if A is p-torsionfree, then any lift of the Frobenius comes from a δ-structure. In this section, we shall assume familiarity with the basic theory of δ-rings. In particular, we shall use the following: the forgetful functor from δ-rings to all rings has a right adjoint given by the p-typical Witt vector functor $W(-)$, with the counit $W(R) \to R$ of the adjunction called γ_0 (aka the 0-th ghost component), and given by projection to the 0-th component in the standard description of $W(-)$. Moreover, the induced δ-structure on $W(-)$ has the corresponding Frobenius lift given by the Witt vector Frobenius $F : W(R) \to W(R)$; see the references given in Lemma 2.6.1 for more on this relationship.

Recall that a generalized Cartier divisor[58] on a scheme X is given by an invertible $\mathcal{O}_X$-module $\mathcal{I}$ together with a map $d : \mathcal{I} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules; any such map is automatically a quasi-ideal. The following proposition studies a variant of this notion:

Proposition 5.1.2. *Fix a p-nilpotent ring R, an invertible $W(R)$-module I, and a map $\alpha : I \to W(R)$ of $W(R)$-modules. The following are equivalent:*

1. *For any map $R \to k$ with k a perfect field, the base change of α along the induced map $W(R) \to W(k)$ identifies (necessarily uniquely) with $pW(k) \xrightarrow{can} W(k)$.*
2. *The composition $I \xrightarrow{\alpha} W(R) \xrightarrow{\gamma_0 := restriction} R$ is nilpotent, and the image of the composition $I \xrightarrow{\alpha} W(R) \xrightarrow{\delta} W(R)$ generates the unit ideal.*
3. *For any Zariski open cover $\{\mathrm{Spec}(R_j)\}_{j \in J}$ of $\mathrm{Spec}(R)$ such that $I \otimes_{W(R)} W(R_i)$ is principal and any generator $d_j \in I \otimes_{W(R)} W(R_j)$, the Witt vector $\alpha(d_i) = (a_0, a_1, a_2, ...) \in W(R)$ is* distinguished, *i.e., a_0 is nilpotent and a_1 is a unit.*

Proof. (1) $\Rightarrow$ (2):

First, let us show that $I \xrightarrow{\alpha} W(R) \xrightarrow{\gamma_0} R$ has nilpotent image. Pick a point $x \in \mathrm{Spec}(R)$. We must show that $\gamma_0(\alpha(I))$ is 0 in $\kappa(x)$. Writing $k = \kappa(x)_{\mathrm{perf}}$, it suffices to show that $\gamma_0(\alpha(I))$ is 0 in k. We know from (1) that the composition $I \xrightarrow{\alpha} W(R) \xrightarrow{\tau} W(k)$ has image in $pW(k)$, where $\tau : W(R) \to W(k)$ is the induced map. But then $I \xrightarrow{\alpha} W(R) \xrightarrow{\tau} W(k) \xrightarrow{\gamma_0} k$ is 0. As τ commutes with restrictions, the desired claim follows.

Next, we we show that the image of $I \xrightarrow{\alpha} W(R) \xrightarrow{\delta} W(R)$ generates the unit ideal. Arguing as in the previous paragraph, it suffices to show that for any perfect field k with a map $R \to k$, if one writes $\tau : W(R) \to W(k)$ for the induced map, then the composition $I \xrightarrow{\alpha} W(R) \xrightarrow{\tau} W(k) \xrightarrow{\delta} W(k)$ generates the unit ideal; here we implicitly use that τ commutes with δ. By the assumption in (1), we know that $\tau(\alpha(I)) = pW(k)$, so we can find $f_1, ..., f_n \in I$ and $a_1, ..., a_n \in W(k)$ such that

$$\sum_i a_i \tau(\alpha(f_i)) = p.$$

[58]This notion has many other names in the literature, including virtual Cartier divisors, or “divisors” (with the quotes!). Note that specifying a generalized Cartier divisor $d : \mathcal{I} \to \mathcal{O}_X$ on X is equivalent to giving a map $\pi : X \to \mathbf{A}^1/\mathbf{G}_m$. The derived preimage of $B\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ along this map gives a closed derived subscheme $i : X_{d=0} \subset X$ with $i_*\mathcal{O}_{X_{d=0}} = \mathrm{Cone}(d)$; one regards the closed immersion i as the geometric incarnation of the generalized Cartier divisor. When d is injective, $X_{d=0}$ is a scheme, and the map i is a classical Cartier divisor on X.

As $\tau(\alpha(f_i)) \in pW(k)$ by the previous paragraph, applying $\delta(-)$ and reducing mod p then gives the equation

$$\sum_i \delta(a_i \tau(\alpha(f_i))) \equiv \delta(p) = \text{unit} \mod pW(k).$$

Expanding this out, again using that $\tau(\alpha(f_i)) \in pW(k)$, we conclude that at least one $\delta(\tau(\alpha(f_i)))$ must be a unit, as wanted.

(2) $\Rightarrow$ (1): Assume (2) and a fix a map $R \to k$ as in (1), with induced map $\tau : W(R) \to W(k)$. Note that τ commutes with both δ and γ_0. As $I \xrightarrow{\alpha} W(R) \xrightarrow{\gamma_0} R$ has nilpotent image, so does $I \xrightarrow{\alpha} W(R) \xrightarrow{\tau} W(k) \xrightarrow{\gamma_0} k$, whence $\tau(\alpha(I)) \subset pW(k)$. Moreover, as $\delta(\alpha(I)) \subset W(R)$ generates the unit ideal, so does $\delta(\tau(\alpha(I))) \subset W(k)$. Since $\delta(p^n W(k)) \subset p^{n-1}W(k)$ for any $n \geq 1$, it follows that $\tau(\alpha(I))$ must be either $pW(k)$ or $W(k)$; since we already ruled out the latter possibility, we must have $\tau(\alpha(I)) = pW(k)$. It remains to show that the induced map $I \otimes_{W(R)} W(k) \to pW(k)$ is an isomorphism. But this is a surjective $W(k)$-linear map of invertible $W(k)$-modules, and hence must be an isomorphism for general reasons: it admits a splitting by projectivity of invertible modules, so its kernel is projective of rank 0, whence 0.

(1) $\Rightarrow$ (3): Choose $\{\mathrm{Spec}(R_j)\}_{j\in J}$ and d_j as in (3). We want to show $\alpha(d_j) \in W(R_j)$ is distinguished. Note that for any element of any ring, the condition of being nilpotent or being a unit can be detected after mapping to perfected residue fields. Thanks (1), we are then reduced to checking the following: for any perfect field k, any generator of $pW(k)$ is distinguished. Such a generator has the form pu for $u \in W(k)^*$. If $u = \sum_{i\geq 0} V^i([u_i])$ is the Witt vector expansion, then we compute that

$$pu = V(1) \cdot (\sum_{i\geq 0} V^i([u_i])) = V(F(\sum_{i\geq 0} V^i([u_i])) \equiv V([u_0^p]) \mod V^2W(k).$$

Now u_0 is a unit, so the above formula shows that pu is distinguished, as wanted.

(3) $\Rightarrow$ (1): By the same logic used in the previous implication, it suffices to show the following: for any perfect field k, any distinguished element $d = (a_0, a_1, a_2, ...)$ of $W(k)$ is a generator of $W(k)$. As k is reduced and a_0 is nilpotent, we must have $a_0 = 0$. As a_1 is a unit, we then have $d = V(u)$ for $u \in W(k)^*$. As k is perfect, we can write $u = Fu'$ for a unique $u' \in W(k)^*$. But then $d = VFu' = u'V(1) = u'p$, which is indeed a generator of $W(k)$. $\square$

The following definition is key to the geometrization of prismatic cohomology:

Definition 5.1.3 (Cartier–Witt divisors)**.** Fix a p-nilpotent ring R.

1. A *Cartier–Witt divisor on* $\mathrm{Spf}(R)$ *(or just R)* is given by a map $\alpha : I \to W(R)$ satisfying the conditions in Proposition 5.1.2.
2. For a Cartier–Witt divisor $I \to W(R)$, write $\overline{W(R)} = \mathrm{Cone}(W(R)/I)$ if there is no ambiguity; we regard $\overline{W(R)}$ as a 1-truncated animated ring and write $\mathrm{Spec}(\overline{W(R)})$ for the corresponding derived scheme[59]
3. A *morphism* $(I \xrightarrow{\alpha} W(R)) \to (J \xrightarrow{\beta} W(R))$ is given by a map $I \to J$ of $W(R)$-modules intertwining α and β. The category of all *Cartier–Witt divisors* on R is denoted by $\mathrm{WCart}(R)$.

Example 5.1.4 (Two standard examples)**.** For any p-nilpotent ring R, one always has the following two Cartier–Witt divisors: $W(R) \xrightarrow{p} W(R)$ and $W(R) \xrightarrow{V(1)} W(R)$. We sometimes informally refer

[59]While it would be natural to endow $\overline{W(R)}$ with the p-adic topology and work with the corresponding derived formal scheme $\mathrm{Spf}(\overline{W(R)})$, this is actually not necessary. Indeed, $\overline{W(R)}$ is p-nilpotent, see [BL22b, Lemma 3.3].

to these as the de Rham and Hodge–Tate Cartier–Witt divisors respectively. If R is an $\mathbf{F}_p$-algebra, then $p = V(1)$, so these coincide. However, they are non-isomorphic in general.

The following lemma, which is a version of the irreducibility lemma for distinguished elements, is one of the most useful tools is understanding the behaviour of the prismaitzation and later its filtered counterpart.

Lemma 5.1.5 (Rigidity of maps)**.** *For a p-nilpotent ring R, the category* $\mathrm{WCart}(R)$ *is a groupoid, i.e., any map* $(I \xrightarrow{\alpha} W(R)) \to (J \xrightarrow{\beta} W(R))$ *in* $\mathrm{WCart}(R)$ *is an isomorphism.*

Proof. We must show that the induced map $I \to J$ is an isomorphism of $W(R)$-modules. As both sides are finite projective modules, it suffices to show the map is surjective, and this can be checked after tensoring with the residue field at a closed point of $\mathrm{Spec}(W(R))$ by Nakayama. But any closed point of $\mathrm{Spec}(W(R))$ lies in $\mathrm{Spec}(R) \xrightarrow{\gamma_0} \mathrm{Spec}(W(R))$, so it suffices to show the following: for any map $R \to k$ with k a perfect field, the induced map $I \otimes_{W(R)} k \to J \otimes_{W(R)} k$ is surjective. But Proposition 5.1.2 (1) implies that the map $I \otimes_{W(R)} W(k) \to J \otimes_{W(R)} W(k)$ is already an isomorphism, so it is certainly true after further tensoring with k. $\square$

Given any map $R \to S$ of p-nilpotent rings, there is an induced base change functor $\mathrm{WCart}(R) \to \mathrm{WCart}(S)$ given by sending a Cartier–Witt divisor $(I \xrightarrow{\alpha} W(R))$ to $(I \xrightarrow{\alpha} W(R)) \otimes_{W(R)} W(S) := (I \otimes_{W(R)} W(S) \xrightarrow{\alpha\otimes\mathrm{id}} W(S))$. Thus, we can regard $\mathrm{WCart}(-)$ can be regarded as a presheaf of groupoids over $\mathrm{Spf}(\mathbf{Z}_p)$. In fact, using fpqc descent for quasi-coherent sheaves, one can even show that $\mathrm{WCart}(-)$ is a sheaf for the fpqc topology.

Definition 5.1.6 (The prismatization)**.** For any bounded p-adic formal scheme X, we define a presheaf X^{Δ} over $\mathrm{Spf}(\mathbf{Z}_p)$ as follows: for a p-nilpotent ring R, the groupoid $X^{\Delta}(R)$ consists of Cartier–Witt divisors $(I \xrightarrow{\alpha} W(R))$ together with a map $\mathrm{Spec}(\overline{W(R)}) \to X$ of derived $\mathrm{Spf}(\mathbf{Z}_p)$-schemes.

When $X = \mathrm{Spf}(S)$ is affine, we shall often write S^{Δ} instead of $\mathrm{Spf}(S)^{\Delta}$. Note that a similar notation was used in Chapter 3 for an *a priori* distinct notion. However, we shall verify (Example 5.1.12) that the notions are equivalent, so we have not overloaded the notation.

Remark 5.1.7 (The prismatization via transmutation)**.** The construction carrying a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ to the animated quotient $\overline{W(R)}$ gives a natural (animated) ring stack over $\mathbf{Z}_p^{\Delta}$. Unwinding definitions, this ring stack is represented by $\mathbf{G}_a^{\Delta} \to \mathbf{Z}_p^{\Delta}$. Moreover, for general X, the stack $X^{\Delta} \to \mathbf{Z}_p^{\Delta}$ is obtained via transmutation from $\mathbf{G}_a^{\Delta}$. Thus, the prismatization construction $X \mapsto X^{\Delta}$ fits into the transmutation picture studied previously in these lectures.

Example 5.1.8 (The stack $\mathbf{Z}_p^{\Delta}$)**.** We have $\mathbf{Z}_p^{\Delta} = \mathrm{WCart}$ by definition. It is relatively straightforward from Proposition 5.1.2 (3) to see that this stack has the following presentation as a quotient stack (where quotients can be computed as sheaves in any topology between Zariski and fpqc):

$$\mathbf{Z}_p^{\Delta} = W_{dist}/W^*,$$

where $W_{dist} \subset W$ denotes the subfunctor of the affine pro-smooth p-adic formal scheme W parametrizing distinguished elements, i.e., W_{dist} is the formal completion of the affine open p-adic formal subscheme $W_{a_1 \neq 0} \subset W$ along the locus $\{a_0 = 0\}$ (where the a_i's are the Witt co-ordinate functions on the scheme W). In particular, $\mathbf{Z}_p^{\Delta}$ is flat over $\mathrm{Spf}(\mathbf{Z}_p)$ in the appropriate sense. In fact, using prisms, one can even write down flat surjective maps $\mathrm{Spf}(\mathbf{Z}_p[\![u]\!]) \to \mathbf{Z}_p^{\Delta}$, where $\mathbf{Z}_p[\![u]\!]$ is given the (p,u)-adic topology, which gives a good theory of coherent sheaves on $\mathbf{Z}_p^{\Delta}$.

Example 5.1.9 (The prismatization of a perfect field)**.** Let k be a perfect field of characteristic $p > 0$. We claim that $\mathrm{Spf}(W(k)) \simeq k^{\Delta}$. (The argument give below extends, *mutatis mutandis*, to show that $\mathrm{Spf}(\Delta_R) \simeq R^{\Delta}$ any perfectoid ring R.)

First, let us build a map $\rho : \mathrm{Spf}(W(k)) \to k^{\Delta}$ using the functor of points. Given a p-nilpotent $W(k)$-algebra R, the structure map $W(k) \to R$ lifts uniquely to a δ-map $W(k) \to W(R)$. Base change of the Cartier–Witt divisor $W(k) \xrightarrow{p} W(k)$ along this map gives a Cartier–Witt divisor $W(R) \xrightarrow{p} W(R)$ together with a map $k = \overline{W(k)} \to \overline{W(R)}$, thus giving a point of k^{Δ}, and thereby building the map ρ.

Conversely, we build a map $\eta : k^{\Delta} \to \mathrm{Spf}(W(k))$. Given a p-nilpotent ring R, an R-valued point of k^{Δ} is given by a Cartier–Witt divisor $(I \xrightarrow{\alpha} W(R))$ together with a map $k \to \overline{W(R)}$. Observe that both the maps $R \xleftarrow{\gamma_0} W(R) \to \overline{W(R)}$ are pro-infinitesimal thickenings. As $\mathbf{Z}_p \to W(k)$ is p-completely étale, we learn that the induced maps

$$\mathrm{Map}(W(k), R) \simeq \mathrm{Map}_{\delta}(W(k), W(R)) \to \mathrm{Map}(W(k), W(R)) \to \mathrm{Map}(W(k), \overline{W(R)})$$

are all equivalences. In particular, the given map $k \to \overline{W(R)}$ deforms uniquely to a δ-map $W(k) \to W(R)$, and the latter is adjoint to a map $W(k) \to R$, yielding an R-valued point of $\mathrm{Spf}(W(k))$. This construction yields the map η.

It is clear from the construction that $\eta \circ \rho = \mathrm{id}_{\mathrm{Spf}(W)}$. Conversely, we claim that $\rho \circ \eta$ is naturally isomorphic to $\mathrm{id}_{k^{\Delta}}$. Fix an R-valued point $(I \xrightarrow{\alpha} W(R), \tau : k \to \overline{W(R)})$ of k^{Δ}. As in the previous paragraph, the map τ deforms uniquely to a δ-map $W(k) \to W(R)$. But this deformation yields a map $(W(k) \xrightarrow{p} W(k)) \to (I \xrightarrow{\alpha} W(R))$ of Cartier–Witt divisors. By Lemma 5.1.5, this map induces an isomorphism

$$(W(k) \xrightarrow{p} W(k)) \otimes_{W(k)} W(R) = (W(R) \xrightarrow{p} W(R)) \simeq (I \xrightarrow{\alpha} W(R))$$

of Cartier–Witt divisors carrying the map $k \simeq W(k)/p \to W(R)/p$ to the map τ. Thus, we have naturally identified $(\rho \circ \eta)(\alpha, \tau)$ with (α, τ), as wanted.

Remark 5.1.10 (The Frobenius on X^{Δ})**.** Fix a p-nilpotent ring R and a Cartier–Witt divisor $(I \xrightarrow{\alpha} W(R)) \in \mathrm{WCart}(R)$. Then $(F^*I \xrightarrow{F^*\alpha} W(R))$ is also a Cartier–Witt divisor by the characterisation in Proposition 5.1.2 (1). Moreover, the Witt vector Frobenius F induces a map $W(R)/I \to W(R)/F^*I$ of animated rings. Passing to prismatizations, we learn that for any bounded p-adic formal scheme, there is a natural map $F_X : X^{\Delta} \to X^{\Delta}$ induced by the Witt vector Frobenius. When $X = \mathbf{Z}_p$, we simply write $F = F_X$ if there is no confusion. In general, if R is an $\mathbf{F}_p$-algebra, the Witt vector Frobenius $F : W(R) \to W(R)$ identifies with $W(\varphi)$; this observation implies that $F_X : X^{\Delta} \to X^{\Delta}$ is a lift of the Frobenius on $(X^{\Delta})_{p=0}$.

To make further computations, the following remark will be useful.

Remark 5.1.11 (The fibres of the prismatization map)**.** Let $f : X \to Y$ be a map of bounded p-adic formal schemes. For a p-nilpotent ring R, we obtain an induced map $f^{\Delta}(R) : X^{\Delta}(R) \to Y^{\Delta}(R)$ of groupoids. Given a point $(I \xrightarrow{\alpha} W(R), \tau : \mathrm{Spec}(\overline{W(R)}), Y) \in Y^{\Delta}(R)$, the fibre of $f^{\Delta}(R)$ over (α, τ) is given by the mapping space $\mathrm{Map}_Y(\mathrm{Spec}(\overline{W(R)}), X)$.

As promised, the prismatization in characteristic p coincides with the notion from §3.1.

Example 5.1.12 (The case of $\mathbf{F}_p$-schemes)**.** Fix a perfect field k of characteristic p and let X/k be any scheme. By Example 5.1.9, we can regard X^{Δ} as a stack over $\mathrm{Spf}(W(k)) = k^{\Delta}$. We claim that for any p-nilpotent ring $W(k)$-algebra R, we have

$$X^{\Delta}(R) = \mathrm{Map}_k(\mathrm{Spec}(W(R)/p), X),$$

where the points on the left are computed over $W(k)$, and the maps on the right are computed in derived k-schemes. Indeed, this follows from Example 5.1.9 and Remark 5.1.11. In particular, we learn that the prismatization $X^{\mathbb{\Delta}}$ as in Definition 5.1.6 agrees with the object considered in Construction 3.1.1.

In the rest of this section, we turn attention to the Hodge–Tate theory, which is the tautological (and perhaps most computable) specialization of the prismatic theory.

Construction 5.1.13 (The Hodge–Tate locus in $\mathbf{Z}_p^{\mathbb{\Delta}}$)**.** Given a p-nilpotent ring R and a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$, we can base change α along $W(R) \xrightarrow{\gamma_0} R$ to obtain a generalized Cartier divisor $I \otimes_{W(R)} R \xrightarrow{\gamma_0^*(\alpha)} R$ over R. As α is a Cartier–Witt divisor, the image of $\gamma_0^*(\alpha)$ is nilpotent. Thus, the assignment carrying α to $\gamma_0^*(\alpha)$ can be regarded as a map

$$u : \mathbf{Z}_p^{\mathbb{\Delta}} \to \widehat{\mathbf{A}^1}/\mathbf{G}_m,$$

over $\mathrm{Spf}(\mathbf{Z}_p)$, where $\widehat{\mathbf{A}^1}$ denotes the formal completion of $\mathbf{A}^1$ at 0. Write $\mathbf{Z}_p^{HT} \subset \mathbf{Z}_p^{\mathbb{\Delta}}$ for the preimage of $B\mathbf{G}_m \subset \widehat{\mathbf{A}^1}/\mathbf{G}_m$. Using the quotient description in Example 5.1.8, one checks that $\mathbf{Z}_p^{HT} \subset \mathbf{Z}_p^{\mathbb{\Delta}}$ is actually an effective Cartier divisor.

The following result describes $\mathbf{Z}_p^{HT}$ explicitly as the classifying stack of the group scheme $\mathbf{G}_m^{\sharp}$ from Variant 2.6.3.

Proposition 5.1.14 (Describing $\mathbf{Z}_p^{HT}$ explicitly)**.** *There is a natural isomorphism $\mathbf{Z}_p^{HT} \simeq B\mathbf{G}_m^{\sharp}$ of fpqc stacks over* $\mathrm{Spf}(\mathbf{Z}_p)$.

Proof. There is a natural map $\eta : \mathrm{Spf}(\mathbf{Z}_p) \to \mathbf{Z}_p^{HT}$ determined by the Cartier–Witt divisor $W(\mathbf{Z}_p) \xrightarrow{V(1)} W(\mathbf{Z}_p)$. To prove the proposition, we shall show that this map is an fpqc surjection, and that the automorphism group scheme of η identifies with $\mathbf{G}_m^{\sharp}$.

η is a flat surjection: fix a p-nilpotent ring R and a point $(I \xrightarrow{\alpha} W(R)) \in \mathbf{Z}_p^{HT}(R)$. Our goal is to lift this point along η after replacing R by a flat cover. After making such replacements, we may assume that $I = W(R)$ is principal, so α is determined by a distinguished element $d = (a_0, a_1, ...) \in W(R)$. By distinguishedness, a_1 is a unit. Moreover, the condition that $\alpha \in \mathbf{Z}_p^{HT}(R) \subset \mathbf{Z}_p^{\mathbb{\Delta}}(R)$ implies that $a_0 = 0$, so we have $d = V(u)$ for $u \in W(R)^*$. By replacing R with further flat covers, we may assume (thanks to Lemma 2.6.1) that $u = F(u')$ for some $u' \in W(R)^*$. But then $d = V(Fu') = u'V(1)$, so multiplication by u' defines an isomorphism of α with $W(R) \xrightarrow{V(1)} W(R)$, thus showing that α lifts along η.

The automorphism group $G = \mathrm{Aut}(\eta)$ of η: by definition, for a p-nilpotent ring R, the group $G(R)$ is the group of automorphisms of the Cartier–Witt divisor $W(R) \xrightarrow{V(1)} W(R)$. Such automorphisms are determined by $W(R)$-linear maps u making the following diagram commute:

$$\begin{tikzcd} W(R) \arrow[r, "V(1)"] \arrow[d, "u"] & W(R) \arrow[d, equal] \\ W(R) \arrow[r, "V(1)"] & W(R). \end{tikzcd}$$

In other words, $G(R) = \{u \in W(R)^* \mid uV(1) = V(1)\}$. But $uV(1) = V(Fu)$ and V is injective, so this simplifies to $G(R) = \{u \in W(R)^* \mid Fu = 1\}$, which we identified with $\mathbf{G}_m^{\sharp}$ earlier (Variant 2.6.3). $\square$

Remark 5.1.15. Proposition 5.1.14 is quite useful in understanding absolute prismatic cohomology. For instance, it formed the basis of Drinfeld's original proof of Corollary 2.7.14, see [BL22a, Remark 4.7.18]. Secondly, one can use Proposition 5.1.14 together with Cartier duality to give a concrete linear algebraic description of $\mathcal{D}_{qc}(\mathbf{Z}_p^{HT})$, which plays a foundational role throughout [BL22a]. Moreover, the analog of Proposition 5.1.14 describing $\mathcal{O}_K^{HT}$ (for $K/\mathbf{Q}_p$ finite) in [BL22b, Example 9.6] was used recently in [AHB22] to give a geometric perspective on Sen theory.

Remark 5.1.16 (The Hodge–Tate ideal sheaf and BK twists)**.** Consider the map $u : \mathbf{Z}_p^{\Delta} \to \widehat{\mathbf{A}^1}/\mathbf{G}_m$ from Construction 5.1.13. By definition, the pullback $u^*\mathcal{O}(-1) \in \mathcal{P}\mathrm{ic}(\mathbf{Z}_p^{\Delta})$ of $\mathcal{O}(-1) \in \mathcal{P}\mathrm{ic}(\widehat{\mathbf{A}^1}/\mathbf{G}_m)$ is identified with the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbf{Z}_p^{\Delta}}$ of the Hodge–Tate locus. Moreover, the special fibre of u is a map $u_0 : B\mathbf{G}_m^\sharp \to B\mathbf{G}_m$ to Proposition 5.1.14. It is straightforward to verify that this map is the map on classifying stacks induced by the tautological map $\mathbf{G}_m^\sharp \to \mathbf{G}_m$ of group schemes. It then follows from the second sentence of this remark that $\mathcal{I}/\mathcal{I}^2 = u_0^*\mathcal{O}(-1) \in \mathcal{P}\mathrm{ic}(B\mathbf{G}_m^\sharp)$; in particular, this line bundle is nonzero (as it is already nonzero after restriction along $B\mu_p \to B\mathbf{G}_m^\sharp$). We shall write $\mathcal{O}_{\mathbf{Z}_p^{HT}}\{1\} := \mathcal{I}/\mathcal{I}^2$ and refer to it as the BK twist; write $M \mapsto M\{n\}$ to denote the n-fold twist by this line bundle on $\mathcal{D}_{qc}(\mathbf{Z}_p^{HT})$. In Remark 5.1.19, we explain how to lift this line bundle on $\mathbf{Z}_p^{HT}$ to a line bundle on $\mathbf{Z}_p^{\Delta}$ in an interesting way, defining BK twists over $\mathbf{Z}_p^{\Delta}$.

Remark 5.1.17 ($\mathbf{Z}_p^{\Delta}$ as a formal stack)**.** The closed substack $(B\mathbf{G}_m^\sharp)_{p=0} = \left(\mathbf{Z}_p^{HT}\right)_{p=0} \subset \mathbf{Z}_p^{\Delta}$ is a closed substack of definition, i.e., every field valued point factors through this closed substack, and moreover this closed substack is fpqc-algebraic (in fact, it admits an affine faithfully flat cover by a scheme). Thus, we can regard $\mathbf{Z}_p^{\Delta}$ as a formal stack with the ideal of definition given by $(p, \mathcal{I}) \subset \mathcal{O}_{\mathbf{Z}_p^{\Delta}}$, i.e., it is roughly a 2-parameter formal thickening of an algebraic stack. In the future, we shall study various formal stacks over $\mathbf{Z}_p^{\Delta}$; these will essentially always be *adic*, i.e., the preimage of $(B\mathbf{G}_m^\sharp)_{p=0} \subset \mathbf{Z}_p^{\Delta}$ will be a closed substack of definition.

Construction 5.1.18 (The Hodge–Tate stack X^{HT})**.** More generally, for any p-adic formal scheme X, we define the Hodge–Tate locus $X^{HT} \subset X^{\Delta}$ as the pullback of $\mathbf{Z}_p^{HT} \subset \mathbf{Z}_p^{\Delta}$. This construction has the following features:

1. *The Hodge–Tate structure map:* The ring stack $\mathbf{G}_a^{HT} := \left(\mathbf{G}_a^{\Delta}\right)|_{\mathbf{Z}_p^{HT}}$ has the following properties:
$$\pi_0(\mathbf{G}_a^{HT}) = \mathbf{G}_a \quad \text{and} \quad \pi_1(\mathbf{G}_a^{HT}) = \mathbf{G}_a^\sharp\{1\},$$
where the twist is defined as in Remark 5.1.16. This follows from the following calculation which we leave to the reader: if $I \xrightarrow{\alpha} W(R)$ is a Cartier–Witt divisor lying in $\mathbf{Z}_p^{HT}(R) \subset \mathbf{Z}_p^{\Delta}(R)$ and R is $\mathbf{G}_a^\sharp$-acyclic, then $\mathrm{coker}(\alpha) = W(R)/VW(R) \simeq R$, while $\ker(\alpha) = I \otimes_{W(R)} \mathbf{G}_a^\sharp(R)$. In particular, the induced map $\mathbf{G}_a^{HT} \to \pi_0(\mathbf{G}_a^{HT}) = \mathbf{G}_a$ of ring stacks over $\mathbf{Z}_p^{HT}$ gives, via transmutation, a natural map
$$X^{HT} \to X$$
for any bounded p-adic formal scheme X; we call this the *Hodge–Tate structure map.*
2. *The case of characteristic p:* For smooth schemes X over a perfect field k of characteristic p, the stack X^{HT} coincides with the object defined in Construction 3.1.1 as in Example 5.1.12: this amounts to checking that the composition $\mathrm{Spf}(W(k)) \simeq \mathbf{F}_p^{\Delta} \to \mathbf{Z}_p^{\Delta} \to \widehat{\mathbf{A}^1}/\mathbf{G}_m$ classifies $p \in W(k)$, which is immediate from the definitions.
3. *The Hodge–Tate gerbe:* If $f : X \to Y$ is a smooth map of bounded p-adic formal schemes, then the induced map $X^{HT} \to Y^{HT} \times X$ is a $B\mathbf{V}(T_{X/Y}\{1\})^\sharp$-torsor by (1) above and the same deformation-theoretic argument used in Footnote 18.

Remark 5.1.19 (Breuil–Kisin twists)**.** For any bounded prism (A, I), one has a natural line bundle $A\{1\} \in \mathcal{P}\mathrm{ic}(A)$ equipped with an isomorphism $\varphi_{A\{1\}} : I \otimes_A \varphi^* A\{1\} \simeq A\{1\})$: heuristically, we have $A\{1\} = \otimes_{k \geq 0} (\varphi^k)^* I$, with $\varphi_{A\{1\}}$ being the evident isomorphism (see [BL22a, §2] or [Dri20, §4.9]). Varying over all (A, I) then gives rise to a line bundle $\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \in \mathcal{P}\mathrm{ic}(\mathbf{Z}_p^{\Delta})$ together with an isomorphism $\mathcal{I} \otimes F^* \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \simeq \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}$, where $\mathcal{I} \subset \mathcal{O}_{\mathbf{Z}_p^{\Delta}}$ is the Hodge–Tate ideal sheaf from Remark 5.1.16 and $F : \mathbf{Z}_p^{\Delta} \to \mathbf{Z}_p^{\Delta}$ is the Frobenius endomorphism (Remark 5.1.10).

For any prism (A, I), the line bundles $(\varphi^k)^* I$ are canonically trivialized by I for $k \geq 1$, and thus $A\{1\} \otimes_A A/I \simeq I/I^2$. Varying over all prisms then shows that $\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}|_{\mathbf{Z}_p^{HT}}$ agrees with the line bundle from Remark 5.1.16, justifying our notation.

This line bundle can also be realized geometrically: if $f : (\mathbf{P}^1)^{\Delta} \to \mathbf{Z}_p^{\Delta}$ is the structure morphism, then $\mathcal{H}^2_{\Delta}(X) := \mathcal{H}^2(Rf_* \mathcal{O}_{(\mathbf{P}^1)^{\Delta}})$ is naturally identified with $\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{-1\}$.

Using the above objects, one can geometrize the theory of prismatic crystals and their cohomology; this was done in [BL22a, BL22b]. In the rest of this chapter, we shall turn attention to Nygaard filtered refinements of this picture.

5.2 Invertible and admissible W-modules

In this section, W denotes the ring scheme of Witt vectors over $\mathrm{Spf}(\mathbf{Z}_p)$. The main goal of this section is to introduce (Definition 5.2.4) the key notion of an *admissible W-module* over a p-nilpotent ring from [Dri20] and study some of its basic examples and properties. Before that, however, we shall need some calculations of Hom's and Ext's in the category of W-module schemes; these are always computed in the category of sheaves for the fpqc topology unless otherwise specified. We shall also freely and frequently use that restriction of scalars along surjective maps of ring schemes, such as the Frobenius $F : W \to F_* W$ or the restriction $\gamma_0 : W \to \mathbf{G}_a$, is fully faithful.

Let us begin with a calculation that can be extracted from [Dri20, §3].

Proposition 5.2.1 (Some Hom and Ext calculations)**.** *Work over $\mathbf{Z}_{(p)}$.*

1. *The $\mathbf{G}_a$-action on $\mathbf{G}_a^\sharp$ gives an isomorphism $\mathbf{G}_a \simeq \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp)$.*
2. *There is a natural identification*
$$\underline{\mathrm{Ext}}_W(F_* W, \mathbf{G}_a^\sharp) := \tau^{\leq 0}(\underline{\mathrm{RHom}}_W(F_* W, \mathbf{G}_a^\sharp)[1]) \simeq \mathbf{G}_a^{dR} := \mathrm{Cone}(\mathbf{G}_a^\sharp \to \mathbf{G}_a)$$
of W-complexes, with $1 \in RHS$ corresponding to the extension (2.6.1).
3. *$\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, F_* W) = 0$.*
4. *The $\mathbf{G}_a^\sharp$-torsion in W identifies with $F_* W \xhookrightarrow{V} W$: applying $\underline{\mathrm{Hom}}_W(F_* W, -)$ to the map $V : F_* W \to W$ induces an isomorphism*
$$\underline{\mathrm{Hom}}_W(F_* W, F_* W) \simeq \underline{\mathrm{Hom}}_W(F_* W, W)$$
of W-modules.
5. *The $F_* W$-torsion in W (via $F_* W \xrightarrow{V} W$) identifies with $\mathbf{G}_a^\sharp \hookrightarrow W$: applying $\underline{\mathrm{Hom}}_W(\mathbf{G}_a, -)$ to $\mathbf{G}_a^\sharp \to W$ induces an isomorphism*
$$\underline{\mathrm{Hom}}_W(\mathbf{G}_a, \mathbf{G}_a^\sharp) \simeq \underline{\mathrm{Hom}}_W(\mathbf{G}_a, W)$$
of W-modules

6. *The natural map* $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a)$ *is injective.*
7. $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \underline{\mathrm{Hom}}_W(F_*W, \mathbf{G}_a^\sharp)) = 0$.

Proof. 1. Note that $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \simeq \underline{\mathrm{Hom}}_{\mathbf{G}_a}(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp)$ as the W-module structure on $\mathbf{G}_a^\sharp$ factors over $\gamma_0 : W \twoheadrightarrow \mathbf{G}_a$. Now any $\mathbf{G}_a$-module map is in particular a $\mathbf{G}_m$-equivariant map. The graded Cartier dual of $\mathbf{G}_a^\sharp$ is simply $\mathbf{G}_a$, so we learn that $\underline{\mathrm{Hom}}_{\mathbf{G}_a}(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \simeq \underline{\mathrm{Hom}}_{\mathbf{G}_a}(\mathbf{G}_a, \mathbf{G}_a) \simeq \mathbf{G}_a$.

2. Apply $\underline{\mathrm{Hom}}_W(-, \mathbf{G}_a^\sharp)$ to the sequence (2.6.1) to obtain

$$\mathrm{Cone}\left(\underline{\mathrm{Hom}}_W(W, \mathbf{G}_a^\sharp) \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp)\right) \simeq \underline{\mathrm{Ext}}_W(F_*W, \mathbf{G}_a^\sharp).$$

The first term appearing on the left is clearly $\mathbf{G}_a^\sharp$, while the second term is $\mathbf{G}_a$ by (1). One checks that the map is also the standard one, giving $\mathbf{G}_a^{dR} \simeq \underline{\mathrm{Ext}}_W(F_*W, \mathbf{G}_a^\sharp)$, as wanted.

3. By filtering F_*W for the V-adic filtration, it suffices to show that any W-module map $f : \mathbf{G}_a^\sharp \to F_*^n \mathbf{G}_a$ over any $\mathbf{Z}_{(p)}$-algebra R vanishes if $n > 0$. By the W-equivariance, any such map is also $\mathbf{G}_m$-equivariant (via the Teichmulller $\mathbf{G}_m \subset W$). Thus, f is determined by an element $\alpha(t) \in R[\{\frac{t^n}{n!}\}_{n \geq 1}]_{\deg = p^n}$ with the property that $\Delta(\alpha(t)) = \alpha(x) + \alpha(y) \in R[\{\frac{x^n}{n!}\}_{n\geq 1}, \{\frac{y^n}{n!}\}_{n \geq 1}]$. We can write $\alpha(t) = r \cdot \frac{t^{p^n}}{(p^n)!}$ for a unique $r \in R$, and we must show $r = 0$. We then compute

$$\Delta(\alpha(t)) = \Delta\left(r \cdot \frac{t^{p^n}}{(p^n)!}\right) = r\Delta\left(\frac{t^{p^n}}{(p^n)!}\right) = r\frac{(x+y)^{p^n}}{(p^n)!} = r \sum_{i+j=p^n, i,j\geq 0} \frac{x^i}{i!}\frac{y^j}{j!},$$

so $\Delta(\alpha(t)) = \alpha(x) + \alpha(y)$ forces

$$r \sum_{i+j=p^n} \frac{x^i}{i!}\frac{y^j}{j!} = r\frac{x^{p^n}}{(p^n)!} + r\frac{y^{p^n}}{(p^n)!}$$

or equivalently that

$$r \sum_{i+j=p^n, i,j\neq 0} \frac{x^i}{i!}\frac{y^j}{j!} = 0.$$

As the divided power monomials $\frac{x^i}{i!}\frac{y^j}{j!}$ form a basis of $R[\{\frac{x^m}{m!}\}_{m\geq 1}, \{\frac{y^m}{m!}\}_{m\geq 1}]$, the above is impossible for $n > 0$ unless $r = 0$, so we win.

4. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_*W \arrow[r, "V"] & W \arrow[r] \arrow[d, "\simeq"] & \mathbf{G}_a \simeq \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathrm{Hom}}_W(F_*W, W) \arrow[r] & \underline{\mathrm{Hom}}_W(W, W) \arrow[r] & \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W) &
\end{tikzcd}$$

where the top row is the standard exact sequence with isomorphism on the top right coming from (1), the bottom row is exact and comes by applying $\underline{\mathrm{Hom}}_W(-, W)$ to the standard exact sequence (2.6.1), the middle vertical arrow is the obvious isomorphism, and the right vertical arrow is induced by applying $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, -)$ to the inclusion $\mathbf{G}_a^\sharp \to W$. As the right vertical arrow is injective, a diagram chase shows that one can uniquely fill in the above diagram with an isomorphism $F_*W \to \underline{\mathrm{Hom}}_W(F_*W, W)$; explicitly, this is given by sending a local section $x \in F_*W$ to the map $F_*W \xrightarrow{x} F_*W \xrightarrow{V} W$, which proves the claim.

5. Applying $\underline{\mathrm{Hom}}_W(-, W)$ to the sequence $F_*W \xrightarrow{V} W \xrightarrow{\gamma_0} \mathbf{G}_a$, we learn that

$$\underline{\mathrm{Hom}}_W(\mathbf{G}_a, W) = \ker\left(\underline{\mathrm{Hom}}_W(W, W) \to \underline{\mathrm{Hom}}_W(F_*W, W)\right).$$

   By (4), we can identify the map appearing on the right as $W \xrightarrow{F} F_*W$, so the claim follows.

6. Using (5), it suffices to show that the map $\underline{\mathrm{Hom}}(\mathbf{G}_a^\sharp, W) \to \underline{\mathrm{Hom}}(\mathbf{G}_a^\sharp, \mathbf{G}_a)$ induced by $\gamma_0 : W \to \mathbf{G}_a$ is injective. The kernel of this map is $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, F_*W)$, which vanishes by (3).

7. Using (2), it suffices to show that any map $\mathbf{G}_a^\sharp \to \ker(\mathbf{G}_a^\sharp \to \mathbf{G}_a)$ vanishes, which follows from (6). □

Using the previous calculation, we can isolate and classify certain W-module schemes that will be the building blocks of all W-module schemes that we shall encounter in the sequel.

Construction 5.2.2 (Twisted forms of W, F_*W and $\mathbf{G}_a^\sharp$)**.** Given a p-nilpotent ring R and an affine W-module scheme M over R, we have:

1. Invertible W-modules: M is fpqc locally (on R) isomorphic to the W-module W if and only if $M \simeq L \otimes_{W(R)} W$, where $L \in \mathrm{Pic}(W(R))$: indeed, this follows as the groupoid of such M's is given by

$$\left(\tau^{\leq 1} R\Gamma(\mathrm{Spec}(R), W^*)\right)[1] \simeq \left(\tau^{\leq 1} R\Gamma(\mathrm{Spec}(W(R)), \mathbf{G}_m)\right)[1].$$

   We will call such W-module schemes *invertible*, so the construction $L \mapsto L \otimes_{W(R)} W$ gives an equivalence between $\mathcal{P}\mathrm{ic}(W(R))$ and the groupoid of invertible W-modules (and more generally between the corresponding categories, where we also allow non-invertible maps).

2. Invertible F_*W-modules: M is fpqc locally isomorphic to the W-module F_*W if and only if $M \simeq F_*L$ for an invertible W-module L. Indeed, this follows from (1) as well as the surjectivity of $F : W \to F_*W$.

3. $\sharp$-invertible W-modules: M is fpqc locally isomorphic to the W-module $\mathbf{G}_a^\sharp$ if and only if $M \simeq \mathbf{V}(L)^\sharp$ for a line bundle $L \in \mathrm{Pic}(R)$: the "if" direction is clear, and the "only if" direction essentially follows from Proposition 5.2.1 (1) as the groupoid of twisted forms of the W-module M is given by $\left(\tau^{\leq 1} R\Gamma(\mathrm{Spec}(R), \mathrm{Aut}_W(M))\right)[1] \simeq \left(\tau^{\leq 1} R\Gamma(\mathrm{Spec}(R), \mathbf{G}_m)\right)[1]$. We will call such W-module schemes *$\sharp$-invertible*, so the construction $L \mapsto \mathbf{V}(L)^\sharp$ gives an equivalence between $\mathcal{P}\mathrm{ic}(R)$ and the groupoid of $\sharp$-invertible W-modules. In fact, by the same reasoning, the same statement is also true for the corresponding categories where we allow possibly non-invertible maps.

One can reformulate the theory of Cartier–Witt divisors in the above terms:

Remark 5.2.3 (Reinterpreting Cartier–Witt divisors via W-module schemes)**.** Fix a p-nilpotent ring R. Thanks to the equivalence in Construction 5.2.2 (1), we can pass freely between the category of invertible $W(R)$-modules and the category of invertible W-module schemes over R. In particular, the groupoid $\mathbf{Z}_p^{\Delta}(R)$ of Cartier–Witt divisors over R admits a fully faithful functor to the category of pairs $(M, d : M \to W)$, where M is an invertible W-module scheme and d is a map of W-module schemes. The essential image of this embedding is characterized by the analog of the pointwise condition in Proposition 5.1.2 (1), formulated at the level of module schemes.

The following notion, along with its companion in Definition 5.3.1, is one of the fundamental definitions in [Dri20]:

Definition 5.2.4 (Admissible W-modules)**.** Fix a p-nilpotent ring R. An *admissible* W-module over R is an affine W-module scheme M which can be realized an extension of an invertible F_*W-module by a $\sharp$-invertible W-module. Write $\mathrm{Adm}(R)$ for the groupoid of admissible modules.

Our previous calculations imply that the property of being admissible is remarkably robust; in particular, the extension witnessing admissibility is essentially unique.

Remark 5.2.5 (The admissible sequence associated to an admissible module)**.** Fix a p-nilpotent ring R and an affine W-module scheme M. By Construction 5.2.2, M is admissible if and only if it sits in a short exact sequence

$$0 \to \mathbf{V}(L_M)^\sharp \to M \to F_*M' \to 0 \tag{$*_M$}$$

with $L_M \in \mathrm{Pic}(R)$ and M' invertible W-module scheme. We claim that the sequence $(*_M)$ is unique up to unique isomorphism; we refer to it as the *admissible sequence for* M. In fact, by Proposition 5.2.1 (3) and Construction 5.2.2, any W-module map $M \to N$ of admissible W-modules lifts uniquely to a map $(*_M) \to (*_N)$ of admissible sequences, which proves the uniqueness of $(*_M)$. By the uniqueness, the property of being admissible for a W-module scheme can be tested fpqc locally. We shall say that M is *split* if the sequence $(*_M)$ admits a splitting.

Invertible W-modules are admissible. In fact, given an invertible module, there are multiple modification procedures to build admissible modules that are not necessarily invertible; these are summarized in the next two examples, and will also appear in the sequel.

Example 5.2.6 (Admissible modules as pushouts of invertible modules)**.** For a p-nilpotent ring R, any invertible W-module scheme M has the form $I \otimes_{W(R)} W$ by Construction 5.2.2. We claim it is also an admissible module: indeed, it sits in a SES

$$0 \to I \otimes_{W(R)} \mathbf{G}_a^\sharp \to I \otimes_{W(R)} W \xrightarrow{\mathrm{id}\otimes F} I \otimes_{W(R)} F_*W \to 0. \tag{$*_{I\otimes_{W(R)}W}$}$$

More generally, given a line bundle $L \in \mathcal{P}\mathrm{ic}(R)$ and a map $u : I \otimes_{W(R)} R \to L$ of line bundles, we obtain an admissible module M_u via pushout of the above sequence $(*_{I\otimes_{W(R)}M})$ along the induced map $u^\sharp : I \otimes_{W(R)} \mathbf{G}_a^\sharp \to \mathbf{V}(L)^\sharp$, i.e., via the following map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_{W(R)} \mathbf{G}_a^\sharp \arrow[r] \arrow[d] & I \otimes_{W(R)} W \arrow[r] \arrow[d] & I \otimes_{W(R)} F_*W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{V}(L)^\sharp \arrow[r] & M_u \arrow[r] & I \otimes_{W(R)} F_*W \arrow[r] & 0.
\end{tikzcd}$$

Note that we have already seen such constructions earlier in the definition of the module M_u used to construct the ring stack $\mathbf{G}_a^{\mathcal{N}}$ in characteristic p (see Construction 3.3.2). In fact, we shall soon see that any admissible module over R arises by this construction fpqc locally on R (see Lemma 5.2.8).

Example 5.2.7 (Admissible modules via pullbacks of invertible modules)**.** Fix a p-nilpotent ring R and an invertible F_*W-module F_*N' (which necessarily has the form $F_*(J \otimes_{W(R)} W)$ for a uniquely determined invertible $W(R)$-module J). Given a map $d : F_*N' \to F_*W$ of W-modules, we can construct a W-module scheme M via the following pullback diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & F_*N' \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0.
\end{tikzcd}$$

Thanks to the top row, we learn that M is an admissible W-module; in fact, the top row is necessarily the admissible sequence $(*_M)$.

The next lemma implies that every admissible module arises, at least locally, via the modification procedure in Example 5.2.6.

Lemma 5.2.8 (Constructing all admissible modules via pushouts)**.** *Fix a p-nilpotent ring R and an admissible W-module scheme M over R. Then, fpqc locally on R, there a map of admissible sequences $(*_W) \to (*_M)$, displayed as*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d] & W \arrow[r, "F"] \arrow[d] & F_*W(R) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathbf{V}(L_M)^\sharp \arrow[r] & M \arrow[r] & F_*M' \arrow[r] & 0, \end{tikzcd}$$

with the right vertical arrow an isomorphism. In particular, the square on the left is a pushout square.

Proof. Consider the admissible sequence $(*_M)$ attached to M. Working fpqc locally on R, we may trivialize M' and L_M to assume this sequence has the form

$$0 \to \mathbf{G}_a^\sharp \to M \to F_*W \to 0.$$

Proposition 5.2.1 (1) and (2) show that any such sequence is, locally on $\mathrm{Spf}(R)$, the pushout of the standard sequence $(*_W)$ along some map $u \in \mathrm{Hom}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \simeq \mathbf{G}_a(R)$, determined uniquely up to elements of $\mathrm{Hom}_W(W, \mathbf{G}_a^\sharp) \simeq \mathbf{G}_a^\sharp(R)$. $\square$

The proof of Lemma 5.2.8 can be refined to give an explicit description of a certain gerbe over $\mathrm{Adm}(-)$, which we record next for completeness.

Remark 5.2.9 (An explicit description of $\mathrm{Adm}(-)$)**.** Fix a p-nilpotent ring R and consider the fibre product diagram

$$\begin{tikzcd} \mathcal{F}(R) \arrow[r] \arrow[d] & \mathcal{P}\mathrm{ic}(R) \times \mathcal{P}\mathrm{ic}(W(R)) \arrow[d, "(\mathrm{id}, F^*)"] \\ \mathrm{Adm}(R) \arrow[r, "M \mapsto (L_M, M')"] & \mathcal{P}\mathrm{ic}(R) \times \mathcal{P}\mathrm{ic}(W(R)) \end{tikzcd}$$

of groupoids defining $\mathcal{F}(R)$. Then the fibre of top horizontal map over $(L, I) \in \mathcal{P}\mathrm{ic}(R) \times \mathcal{P}\mathrm{ic}(W(R))$ is identified with the groupoid $\mathbf{G}_a^{dR}(R) \otimes_R \mathrm{Hom}_{W(R)}(I, L)$ via an examination of the proof of Lemma 5.2.8. As the Frobenius map $W^* \to W^*$ is fpqc locally surjective with kernel $\mathbf{G}_m^\sharp$ (Lemma 2.6.1, Variant 2.6.3), the vertical map on the right in the square above is a $B\mathbf{G}_m^\sharp(R)$-torsor, and hence the same holds true for the left vertical map. The fibre $\mathcal{F}(R)$ is identified with the groupoid $\widetilde{\mathrm{Adm}}(R)$ from [Dri20, §3.15][60].

[60]Let us formulate an expectation for an explicit quotient description of $\mathrm{Adm}(-)$. Consider the stack

$$\mathbf{G}_a^{dR} := \mathrm{Cone}\left(\mathbf{G}_a^\sharp = \mathrm{Hom}_W(W, \mathbf{G}_a^\sharp) \to \mathbf{G}_a = \mathrm{Hom}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp)\right).$$

The complex on the right, and hence also the quotient stack $\mathbf{G}_a^{dR}$, has a natural action of $W^* \times \mathbf{G}_m$ induced by the action of W^* on W and $\mathbf{G}_m$ on $\mathbf{G}_a^\sharp$ respectively. The quotient $\mathbf{G}_a^{dR}/(W^* \times \mathbf{G}_m)$ is identified with $\mathcal{F}(-)$. To proceed further, observe that the W^*-action on the stack $\mathbf{G}_a^{dR}$ actually factors over $W^* \to W^*/\mathbf{G}_m^\sharp \simeq F_*W^*$ via the following observation: for any local section $\lambda \in \mathbf{G}_m^\sharp$ and any W-module map $f : \mathbf{G}_a^\sharp \to \mathbf{G}_a^\sharp$, the map $\lambda f - f : \mathbf{G}_a^\sharp \to \mathbf{G}_a^\sharp$, which has the form ϵf for some $\epsilon \in \mathbf{G}_a$ with a specified lift to $\mathbf{G}_a^\sharp$, extends canonically along $\mathbf{G}_a^\sharp \hookrightarrow W$. Thus, we obtain an action of $F_*W^* \times \mathbf{G}_m$ on $\mathbf{G}_a^{dR}$, and we expect that the quotient is identified with $\mathrm{Adm}(-)$.

5.3 Filtered Cartier-Witt divisors

The goal of this section is to introduce and study the notion of a filtered Cartier–Witt divisor over p-nilpotent ring R. To motivate the definition, note that Cartier–Witt divisors on R can be regarded as (certain) maps $d: M \to W$ of W-module schemes over R with M an invertible W-module. Relaxing invertibility to admissibility leads to the following fundamental notion:

Definition 5.3.1 (Filtered Cartier–Witt divisors)**.** Given a p-nilpotent ring R, a *filtered Cartier-Witt divisor over* R consists of an admissible W-module scheme M over R and a map $d: M \to W$ of W-modules such that the induced map $F_*M' \to F_*W$ of associated invertible F_*W-modules comes from (after undoing F_*) a Cartier–Witt divisor over R via the construction in Remark 5.2.3.

A morphism $(d: M \to W) \to (e: N \to W)$ of filtered Cartier–Witt divisors is provided by a map $\tau: M \to N$ of admissible W-modules such that $e\tau = d$. (Note that the induced map $M' \to N'$ of invertible W-modules is automatically an isomorphism by the rigidity in Lemma 5.1.5.)

Write $\mathbf{Z}_p^{\mathcal{N}}(R)$ for the groupoid of all filtered Cartier–Witt divisors over R; as R varieties, this defines a stack $\mathbf{Z}_p^{\mathcal{N}}$ on p-nilpotent rings.

Given a p-nilpotent ring R and a filtered Cartier–Witt divisor $d: M \to W$, we obtain an induced map $(*_M) \to (*_W)$ of admissible sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{V}(L_M)^\sharp \arrow[r] \arrow[d, "\sharp(d)"] & M \arrow[r] \arrow[d, "d"] & F_*M' \arrow[r] \arrow[d, "F_*(d')"] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0 \end{tikzcd} \tag{5.3.1}$$

where the map d' is a Cartier–Witt divisor. We shall analyse the structure of $\mathbf{Z}_p^{\mathcal{N}}$ by using various components of this picture. But first, let us explain one procedure for turning a Cartier–Witt divisors into a filtered ones:

Construction 5.3.2 (From Cartier–Witt divisors to filtered Cartier–Witt divisors via j_{HT})**.** Given a p-nilpotent ring R and a Cartier–Witt divisor $\left(I \xrightarrow{\alpha} W(R)\right) \in \mathbf{Z}_p^{\Delta}(R)$, the induced map $\left(M := I \otimes_{W(R)} W \xrightarrow{d_\alpha} W\right)$ is a filtered Cartier–Witt divisor: indeed, M is invertible and thus admissible (Example 5.2.6), and the map $M' \to W$ is identified with the map $\left(F^*I \xrightarrow{F^*\alpha} W(R)\right) \otimes_{W(R)} W$, which satisfies the necessary condition by Remark 5.1.10.

Conversely, we claim that if $\left(M \xrightarrow{d} W\right) \in \mathbf{Z}_p^{\mathcal{N}}(R)$ is any filtered Cartier–Witt divisor with M invertible, then it arises by the preceding construction. Indeed, if M is invertible, then the map d' identifies with $F^*(d)$, so the claim follows from the observation that condition (1) in Proposition 5.1.2 can be tested after Frobenius pullback.

Thus, we have identified $\mathbf{Z}_p^{\Delta}$ with the substack $(\mathbf{Z}_p^{\mathcal{N}})_{\mathrm{inv}} \subset \mathbf{Z}_p^{\mathcal{N}}$ parametrizing filtered Cartier–Witt divisors $d: M \to W$ with M invertible. Write

$$j_{HT}: \mathbf{Z}_p^{\Delta} = (\mathbf{Z}_p^{\mathcal{N}})_{\mathrm{inv}} \to \mathbf{Z}_p^{\mathcal{N}}$$

for the resulting map; we leave it to the reader to check that this map is an open immersion.

Next, we try to justify the term "filtered Cartier–Witt divisor". First, observe that $\mathbf{Z}_p^{\mathcal{N}}$ is naturally a filtered stack.

Construction 5.3.3 (The structure and Rees maps for $\mathbf{Z}_p^{\mathcal{N}}$)**.** Fix a p-nilpotent ring R and a filtered Cartier–Witt divisor $d : M \to W$, and consider the diagram (5.3.1). We can then extract the following observables from the situation:

1. The map $d' : M' \to W$ is a Cartier–Witt divisor by definition. Sending d to d' thus defines a map
$$\pi : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{Z}_p^{\mathbb{\Delta}} = \mathrm{WCart}$$
that we refer to as the *structure map*.
2. The map $\sharp(d)$ is a map of $\sharp$-invertible modules. By Construction 5.2.2, it has the form $t(d)^\sharp$ for a unique map $t(d) : L_M \to R$ of invertible R-modules. Sending d to $t(d)$ then defines a morphism
$$t : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{A}^1/\mathbf{G}_m$$
that we refer to as the *Rees map* (for reasons to be explained later).

The following construction attaches filtered Cartier–Witt divisors to generalized Cartier divisors, and provides left inverse to the Rees map.

Construction 5.3.4 (Cartier divisors to filtered Cartier-Witt divisors)**.** Given a p-nilpotent ring R, a line bundle L and a map $L \to R$ corresponding to a point of $\mathbf{A}^1/\mathbf{G}_m(R)$, we can construct the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp"] & \mathbf{V}(L)^\sharp \oplus F_*W \arrow[r] \arrow[d, "d:=(t^\sharp, V)"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0.
\end{tikzcd}$$

This diagram is a map between admissible sequences of admissible modules, and one verifies that d is in fact a filtered Cartier–Witt divisor. The construction carrying t to d thus defines a map
$$i_{dR} : \mathbf{A}^1/\mathbf{G}_m \to \mathbf{Z}_p^{\mathcal{N}}$$
of stacks that we call the de Rham map.

Regarding $\mathbf{Z}_p^{\mathcal{N}}$ as a filtered stack via the Rees map, we observe next that the underlying non-filtered stack is a copy of the stack $\mathbf{Z}_p^{\mathbb{\Delta}}$ parametrizing Cartier–Witt divisors, explaining why we refer to points of $\mathbf{Z}_p^{\mathcal{N}}$ as filtered Cartier–Witt divisors.

Construction 5.3.5 (From Cartier–Witt divisors to filtered Cartier–Witt divisors via j_{dR})**.** The preimage of $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ under the Rees map $t : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{A}^1/\mathbf{G}_m$ from Construction 5.3.3 gives an open substack $(\mathbf{Z}_p^{\mathcal{N}})_{t\neq 0} \subset \mathbf{Z}_p^{\mathcal{N}}$. This locus identifies with $\mathbf{Z}_p^{\mathbb{\Delta}}$ via the structure map π from Construction 5.3.3, with inverse given by the construction in Example 5.2.7. Write
$$j_{dR} : \mathbf{Z}_p^{\mathbb{\Delta}} = (\mathbf{Z}_p^{\mathcal{N}})_{t\neq 0} \to \mathbf{Z}_p^{\mathcal{N}}$$
for the resulting open immersion. Thus, we can regard $\mathbf{Z}_p^{\mathcal{N}}$ as a filtration on the stack $\mathbf{Z}_p^{\mathbb{\Delta}}$.

The maps $j_{dR}, j_{HT} : \mathbf{Z}_p^{\mathbb{\Delta}} \to \mathbf{Z}_p^{\mathcal{N}}$ constructed above are disjoint.

Remark 5.3.6 (The two open copies of $\mathbf{Z}_p^{\Delta}$ in $\mathbf{Z}_p^{\mathcal{N}}$ are disjoint)**.** We have defined two open immersion $j_{HT}, j_{dR} : \mathbf{Z}_p^{\Delta} \to \mathbf{Z}_p^{\mathcal{N}}$ in Constructions 5.3.2 and 5.3.3 respectively. We claim these two copies are disjoint. To see this, fix a p-nilpotent ring R. Then the image of $j_{dR}(R)$ consists of those filtered Cartier–Witt divisors $d : M \to W$ for which the map $\sharp(d)$ in Eq. (5.3.1) is an isomorphism. On the other hand, for the filtered Cartier–Witt divisors $e : N \to W$ lying in $j_{HT}(R)$, the map $\sharp(e)$ has the form $t^\sharp$ where $t : I \otimes_{W(R)} R \to R$ is the map coming from a Cartier–Witt divisor $\alpha : I \to W(R)$; by definition of the latter, the map t is nilpotent, and hence $t^\sharp$ can never be an isomorphism by Proposition 5.2.1 (1).

The following result, which is a filtered refinement of Proposition 5.1.14, is important in understanding the geometry of $\mathbf{Z}_p^{\mathcal{N}}$.

Proposition 5.3.7. *The fibre $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}$ of the Rees map over $B\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ is naturally identified with $\mathbf{G}_a^{dR}/\mathbf{G}_m := \mathbf{A}^1/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$.*

Proof. Fix a p-nilpotent ring R. By definition, an R-valued point of $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}$ is given by a filtered Cartier–Witt divisor $d : M \to W$ over R with $\sharp(d) = 0$. The latter ensures that $F_*(d')$ lifts across $F : W \to F_*W$, i.e., we can add a unique arrow $\widetilde{d}$ in the diagram (5.3.1) to obtain a new commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{V}(L_M)^\sharp \arrow[r] \arrow[d, "\sharp(d)=0"] & M \arrow[r] \arrow[d, "d"] & F_*M' \arrow[r] \arrow[d, "F_*(d')"] \arrow[dl, "\widetilde{d}"'] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0.
\end{tikzcd}$$

In this situation, we first claim that $\widetilde{d}$ identifies its source F_*M' with $F_*W \overset{V}{\hookrightarrow} W$; this then identifies $F_*(d')$ with $p : F_*W \to F_*W$. For the first claim, note that the image is contained in $F_*W \overset{V}{\hookrightarrow} W$ by Proposition 5.2.1, so $\widetilde{d}$ factors as $F_*M' \xrightarrow{F_*(e)} F_*W \overset{V}{\hookrightarrow} W$. We must show e is an isomorphism. But then the commutativity of the triangle based at the bottom right vertex shows that $d' = pe$. We can then regard e as a map $(M' \xrightarrow{d'} W) \to (W \xrightarrow{p} W)$ of Cartier–Witt divisors, so it must be an isomorphism by the rigidity lemma, as wanted.

Thanks to the previous paragraph, we have identified $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}(R)$ with the groupoid consisting of line bundles $L \in \mathcal{P}ic(R)$ together with a W-module extension of F_*W by $\mathbf{V}(L)^\sharp$. This groupoid can be identified with $(\mathbf{G}_a^{dR}/\mathbf{G}_m)(R)$ by (passage to the $\mathbf{G}_m$-equivariant version of) Proposition 5.2.1, so we win. □

To construct the filtered prismatization of a general bounded p-adic formal scheme via transmutation, we need to produce a natural animated ring stack over $\mathbf{Z}_p^{\mathcal{N}}$, which ultimately comes from the following:

Proposition 5.3.8. *Fix a p-nilpotent ring R and a filtered Cartier–Witt divisor $\left(M \xrightarrow{d} W\right)$ over R. Then d is a quasi-ideal.*

One can prove this result "by hand", using the presentation of admissible modules provided by Lemma 5.2.8. The slick proof given below is essentially borrowed from [Dri20, Lemma 3.12.12].

Proof. We may work locally, so we can assume M is an extension of F_*W by $\mathbf{G}_a^\sharp$, so d sits in a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sharp(d)"] & M \arrow[r] \arrow[d, "d"] & F_*W \arrow[r] \arrow[d, "F_*(d')"] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0 \end{tikzcd}$$

with exact rows. For local sections $a, b \in M$, we must show that $\epsilon(a,b) := ad(b) - bd(a) \in M$ is 0. The map ϵ gives an alternating pairing $M \times M \to M$ or equivalently as a map $\epsilon : \wedge^2 M \to M$ (where $\wedge^2$ is computed in W-module sheaves). We make a series of reductions using Proposition 5.2.1 to conclude that ϵ must vanish.

First, as $F_*(d')$ is a quasi-ideal, the composition $\wedge^2 M \xrightarrow{\epsilon} M \twoheadrightarrow F_*W$ vanishes, so we can regard ϵ as a map $\wedge^2 M \to \mathbf{G}_a^\sharp$.

Next, observe that the restriction $d|_{\mathbf{G}_a^\sharp}$ is a quasi-ideal: this follows as the map $\sharp(d)$ is given by scalar multiplication by a section of W thanks to Proposition 5.2.1 (1). It follows that the composition $\wedge^2 \mathbf{G}_a^\sharp \to \wedge^2 M \xrightarrow{\epsilon} \mathbf{G}_a^\sharp$ vanishes, and thus we can regard ϵ as a map $\wedge^2 M / \wedge^2 \mathbf{G}_a^\sharp \xrightarrow{\epsilon} \mathbf{G}_a^\sharp$.

Regarding the admissible sequence $(*_M)$ as a filtration of M, we learn that $\wedge^2 M / \wedge^2 \mathbf{G}_a^\sharp$ admits a subobject H which is naturally a quotient of $\mathbf{G}_a^\sharp \otimes_W F_*W$, and a quotient object identified with $\wedge^2 F_*W$. But F_*W is a cyclic W-module (as $F : W \to F_*W$ is surjective), so $\wedge^2 F_*W = 0$, so we learn that $H \simeq \wedge^2 M / \wedge^2 \mathbf{G}_a^\sharp$ is a quotient of $\mathbf{G}_a^\sharp \otimes_W F_*W$. It is then enough to show that the composition

$$\mathbf{G}_a^\sharp \otimes_W F_*W \twoheadrightarrow H \simeq \wedge^2 M / \wedge^2 \mathbf{G}_a^\sharp \xrightarrow{\epsilon} \mathbf{G}_a^\sharp$$

is 0, which follows from Proposition 5.2.1 (6). ☐

Corollary 5.3.9 (From filtered Cartier–Witt divisors to animated rings)**.** *For a p-nilpotent ring R and a filtered Cartier–Witt divisor $\left(M \xrightarrow{d} W\right) \in \mathbf{Z}_p^{\mathcal{N}}(R)$, the $W(R)$-complex $R\Gamma(\mathrm{Spec}(R), W/M)$ is naturally a 1-truncated animated $W(R)$-algebra.*

Proof. Thanks to Proposition 5.3.8, it suffices to show that $R\Gamma_{fl}(\mathrm{Spec}(R), W/M) \in \mathcal{D}^{[-1,0]}$. We have $R\Gamma_{fl}(\mathrm{Spec}(R), W) = W(R)[0]$, so it is enough to check that $R\Gamma_{fl}(\mathrm{Spec}(R), M) \in \mathcal{D}^{[0,1]}$. By the admissible sequence of M, it suffices to show the following: for line bundle $L \in \mathcal{P}\mathrm{ic}(R)$ and $I \in \mathcal{P}\mathrm{ic}(W(R))$, we have $R\Gamma_{fl}(\mathrm{Spec}(R), \mathbf{V}(L)^\sharp), R\Gamma_{fl}(\mathrm{Spec}(R), I \otimes_{W(R)} W) \in \mathcal{D}^{[0,1]}$. We check this by computing both objects. For the second, one checks that $I \simeq R\Gamma_{fl}(\mathrm{Spec}(R), I \otimes_{W(R)} W)$ via the natural map, which gives the desired containment. For the former, we compute that $L \otimes_R R\Gamma(\mathrm{Spec}(R), \mathbf{G}_a^\sharp) \simeq R\Gamma_{fl}(\mathrm{Spec}(R), \mathbf{V}(L)^\sharp)$, which gives the claim by Lemma 2.4.7. ☐

By Corollary 5.3.9, the construction carrying a filtered Cartier–Witt divisor $\left(M \xrightarrow{d} W\right) \in \mathbf{Z}_p^{\mathcal{N}}(R)$ to $(W/M)(R) := R\Gamma(\mathrm{Spec}(R), W/M)$ yields an animated W-algebra stack $\mathbf{G}_a^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$. Moreover, the right square in the diagram (5.3.1) gives a map $\mathbf{G}_a^{\mathcal{N}} \to \pi^* \mathbf{G}_a^{\mathbb{\Delta}}$ of animated ring stacks[61], where $\pi : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{Z}_p^{\mathbb{\Delta}}$ is the structure map. Via transmutation, this yields:

Definition 5.3.10 (The filtered prismatization)**.** Fix a bounded p-adic formal scheme X.

1. The *filtered prismatization of* X is the stack $X^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$ obtained via transmutation with from the animated ring stack $\mathbf{G}_a^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$.
2. The category of *gauges over* X is defined as $\mathrm{Gauge}_{\mathbb{\Delta}}(X) := \mathcal{D}_{qc}(X^{\mathcal{N}})$.

[61] Both sides have natural W-algebra structures, and the map is linear over $F : W \to W$.

3. The *structure map* $\pi_X : X^{\mathcal{N}} \to X^{\Delta}$ is the map obtained via transmutation from the map $\mathbf{G}_a^{\mathcal{N}} \to \pi^*\mathbf{G}_a^{\Delta}$; it lives over $\pi : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{Z}_p^{\Delta}$.

4. The *Rees map* $t_X : X^{\mathcal{N}} \to \mathbf{A}^1/\mathbf{G}_m$ is simply the composition of the structure map $X^{\mathcal{N}} \to \mathbf{Z}_p^{\Delta}$ with the Rees map $t : \mathbf{Z}_p^{\Delta} \to \mathbf{A}^1/\mathbf{G}_m$.

5. Pullback of the open immersions $j_{HT}, j_{dR} : \mathbf{Z}_p^{\Delta} \to \mathbf{Z}_p^{\mathrm{Syn}}$ (from Constructions 5.3.2 and 5.3.5) defines, via transmutation, the open immersions $j_{HT}, j_{dR} : X^{\Delta} \to X^{\mathcal{N}}$ with disjoint image (Remark 5.3.6).

Remark 5.3.11. The compositions of various maps introduced above can be understood via the functor of points directly. For instance, we have $\pi_X \circ j_{dR} = \mathrm{id}_{X^{\Delta}}$, that $\pi_X \circ j_{HT} = F_X$ (where F_X is the Frobenius from Remark 5.1.10), that $t_X \circ j_{dR}$ factors over $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$, and that $t_X \circ j_{HT} : X^{\Delta} \to \mathbf{A}^1/\mathbf{G}_m$ is classified by the map $\mathcal{I} \to \mathcal{O}_{X^{\Delta}}$ given by the Hodge–Tate ideal sheaf (Remark 5.1.16).

Example 5.3.12 (The structure gauge)**.** Given a quasi-syntomic qcqs p-adic formal scheme X with structure map $\pi : X^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$, we obtain a gauge $\mathcal{H}_{\mathcal{N}}(X) := R\pi_*\mathcal{O}_{X^{\mathcal{N}}} \in \mathcal{D}_{qc}(\mathbf{Z}_p^{\mathcal{N}})$ over $\mathbf{Z}_p$ that we sometimes call the *structure gauge* of X.

Construction 5.3.13 (The de Rham map)**.** Fix a bounded p-adic formal scheme X. Its (absolute) Hodge-filtered de Rham stack $X^{dR,+}$ is the stack over $\mathbf{A}^1/\mathbf{G}_m$ defined[62] via transmutation from the $\mathbf{G}_a$-algebra stack $\mathbf{G}_a^{dR,+} = \mathrm{Cone}\left(\mathbf{V}(\mathcal{O}(-1))^{\sharp} \xrightarrow{t^{\sharp}} \mathbf{G}_a\right)$, where $t : \mathcal{O}(-1) \to \mathcal{O}$ is the tautological section over $\mathbf{A}^1/\mathbf{G}_m$. Thus, there are natural maps $X \times \mathbf{A}^1/\mathbf{G}_m \to X^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$ with the composition being the projection. Now observe there is a natural quasi-isomorphism

$$\left(\mathbf{V}(\mathcal{O}(-1))^{\sharp} \oplus F_*W \xrightarrow{(t^{\sharp},V)} W\right) \simeq \left(\mathbf{V}(\mathcal{O}(-1))^{\sharp} \xrightarrow{t^{\sharp}} \mathbf{G}_a\right)$$

of quasi-ideals induced by the restriction map $W \to \mathbf{G}_a$ (see Construction 2.8.1; the characteristic p assumption is not necessary there). As the quasi-ideal appearing on the left above is naturally a filtered Cartier–Witt divisor while the one on the right represents an animated $\mathbf{G}_a$-algebra, we deduce via transmutation that there is a natural map

$$i_{dR} : X^{dR,+} \to X^{\mathcal{N}}.$$

By construction, this is a map of stacks over $\mathbf{A}^1/\mathbf{G}_m$ (where we use the Rees map on the target), so it can be regarded as a filtered map of filtered stacks. We write $T_{dR}(-) = i_{dR}^*$ and call it the *de Rham realization* functor for gauges over X. The reason for the name is the following: when X is smooth over $\mathbf{Z}_p$, the functor $T_{dR}(-)$ carries vector bundle gauges over $X^{\mathcal{N}}$ to filtered vector bundles on X equipped with a Griffiths-transversal connection relative to $\mathbf{Z}_p$ (Remark 2.5.8). A similar remark applies in the relative context of Footnote 62 as well.

Remark 5.3.14 (Hodge–Tate weights)**.** Fix a bounded p-adic formal scheme X. Restricting the composition $X \times \mathbf{A}^1/\mathbf{G}_m \to X^{dR,+} \to X^{\mathcal{N}}$ from Construction 5.3.13 to the closed point of point of $\mathbf{A}^1/\mathbf{G}_m$ gives a map $\iota : X \times B\mathbf{G}_m \to X^{\mathcal{N}}$. In particular, given a gauge $E \in \mathcal{D}_{qc}(X^{\mathcal{N}})$, the pullback ι^*E can be regarded as a graded quasi-coherent complex $\oplus_i M_i$ on X. The set $\{i \in \mathbf{Z} \mid M_i \neq 0\}$

[62] The relative variant of this construction was studied in §2.5 to define $(X/V)^{dR,+}$ for X smooth over a bounded p-complete ring V. Note that there is a natural map $(X/V)^{dR,+} \to X^{dR,+}$: in fact, we have $(X/V)^{dR,+} \simeq X^{dR,+} \times_{\mathrm{Spf}(V)^{dR,+}} \mathrm{Spf}(V)$. Thus, one can readily relativize the constructions explained in this paragraph.

is called the set of *Hodge–Tate weights* of E. For example, if $f : Y \to X$ is a smooth qcqs map of relative dimension d, then using the comparisons in §5.5, one can show that $\iota^* Rf_*^{\mathcal{N}} \mathcal{O}_{Y^{\mathcal{N}}}$ identifies with $\oplus_i Rf_* \Omega^i_{Y/X}[-i]$ as a graded quasi-coherent complex on X, whence the set of Hodge–Tate weights of $Rf_*^{\mathcal{N}} \mathcal{O}_{Y^{\mathcal{N}}}$ is contained in $\{0, 1, ..., d\}$.

In the next sections, we shall describe these objects more concretely in certain cases.

5.4 Filtered Cartier-Witt divisors over k

In this section, fix a perfect field k of characteristic p as well as a smooth k-scheme X. Definition 5.3.10 introduced a stack $X^{\mathcal{N}}$ attached to X. Another object with the same notation was introduced in Definition 3.3.3; call the latter $X^{\mathcal{N}'}$ in this section. The goal of this section is to show that these stacks are naturally identified.

Theorem 5.4.1. *There is a natural isomorphism $X^{\mathcal{N}'} \simeq X^{\mathcal{N}}$, compatible with the Rees maps.*

As both stacks under consideration are defined via transmutation, it suffices to prove the following:

Proposition 5.4.2. *There is a natural isomorphism $k^{\mathcal{N}} \simeq k^{\mathcal{N}'}$ of stacks over $\mathbf{A}^1/\mathbf{G}_m$ that carries the animated k-algebra stack $\mathbf{G}_a^{\mathcal{N}}$ to the animated k-algebra stack $\mathbf{G}_a^{\mathcal{N}'}$.*

Proof. Recall from Construction 3.3.1 that $k^{\mathcal{N}'}$ is the Rees stack of the p-adic filtration on $W(k)$. Let us first construct a map $\eta : k^{\mathcal{N}'} \to k^{\mathcal{N}}$. Thus, we must build a filtered Cartier–Witt divisor $\left(M \xrightarrow{d} W\right)$ over $k^{\mathcal{N}'}$ together with a map $k \to R\Gamma(k^{\mathcal{N}'}, W/M)$ of animated rings. This is provided by Construction 3.3.2: the map $\left(M_u \xrightarrow{d_{u,t}} W\right)$ in diagram (3.3.2) is a filtered Cartier–Witt divisor, and the middle vertical column in the same diagram gives a map $\left(W \xrightarrow{p} W\right) \to \left(M_u \xrightarrow{d_{u,t}} W\right)$ of quasi-ideals, which then yields an animated $W(k)$-algebra map $k \simeq W(k)/p \to R\Gamma(k^{\mathcal{N}'}, W/M)$.

Conversely, let us construct a map $\tau : k^{\mathcal{N}} \to k^{\mathcal{N}'}$ by doing so on points. Given a p-nilpotent ring R, a point $x \in k^{\mathcal{N}}(R)$ is given a filtered Cartier–Witt divisor $\left(M \xrightarrow{d} W\right)$ over R together with a map $k \to (W/M)(R)$ of animated rings. By deformation theory, this map lifts uniquely to a map $W(k) \to W(R)$ of animated rings, giving a commutative diagram

$$\begin{tikzcd} W(k) \arrow[r] \arrow[d] & W(R) \arrow[d] \\ k \arrow[r] & (W/M)(R) \end{tikzcd}$$

of animated rings. Passing to fibres then gives a commutative diagram

$$\begin{tikzcd} W(k) \arrow[r, "\alpha"] \arrow[d, "p"] & M(R) \arrow[d] \\ W(k) \arrow[r] & W(R) \end{tikzcd}$$

of quasi-ideals, which is adjoint to a map $\alpha : \left(W \xrightarrow{p} W\right) \to \left(M \xrightarrow{d} W\right)$ of filtered Cartier–Witt divisors. Writing each admissible module appearing in this map in terms of its admissible sequence gives a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dd, bend right, red, "p=\sharp(p)"'] & W \arrow[r, "F"] \arrow[d, "\alpha"] \arrow[dd, bend right, red, "p"'] & F_*W \arrow[r] \arrow[d, "\alpha'"] \arrow[dd, bend right, red, "p"'] & 0 \\
0 \arrow[r] & \mathbf{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp=\sharp(d)"] & M \arrow[r] \arrow[d, "d"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0,
\end{tikzcd}$$

where we have used Proposition 5.2.1 to write the maps appearing in the left column as $u^\sharp$ and $t^\sharp$ for uniquely determined sections $u : R \to L$ and $t : L \to R$. Note that the map α' is an isomorphism by Lemma 5.1.5, and moreover that $ut = p$. Thus, the datum (L, u, t) determines an R-valued point $\tau(x)$ of $k^{\mathcal{N}'}(R)$. This construction is natural in x, giving a map $\tau : k^{\mathcal{N}} \to k^{\mathcal{N}'}$.

One can now check that τ and η are mutually inverse isomorphisms carrying $\mathbf{G}_a^{\mathcal{N}}$ to $\mathbf{G}_a^{\mathcal{N}'}$. In fact, these isomorphisms carry the diagram above for $k^{\mathcal{N}}$ to the diagram (3.3.2) over $k^{\mathcal{N}'}$. $\square$

5.5 The filtered prismatization of a qrsp ring

In this section, we assume the reader is familiar with the notion of perfectoid rings, the quasi-syntomic site and quasi-regular semiperfectoid (qrsp) rings, and most importantly the prismatic cohomology of qrsp rings[63]. Our main goal in this section is to explain why the filtered prismatization $R^{\mathcal{N}}$ of a qrsp ring R has a concrete description in terms the Nygaard filtration $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ on the prismatic cohomology of R (Corollary 5.5.11); this in turn gives a purely algebraic description of quasi-coherent sheaves on $R^{\mathcal{N}}$.

5.5.1 Review of the prismatic cohomology of qrsp rings

In this subsection, we recall some basic prismatic structures attached to a qrsp ring. Thus, let R be a qrsp ring. One then has the following objects attached to R.

1. *The prism of R:* There is an initial object $(\mathbb{\Delta}_R, I)$ of the prismatic site of R, i.e.,
   - $\mathbb{\Delta}_R$ is a naturally δ-ring with δ-structure classified by a map $\mathbb{\Delta}_R \to W(\mathbb{\Delta}_R)$ splitting the restriction map.

[63]Quasi-syntomic rings are bounded p-complete rings whose cotangent complex over $\mathbf{Z}_p$ has p-complete Tor amplitude in $[-1, 0]$; examples include noetherian lci rings, perfectoid rings, and smooth (or even syntomic) extensions of these. A particularly important subcategory is given by qrsp rings; these rings admit maps from perfectoid rings, are semiperfect modulo p, and have the feature that their cotangent complex has p-complete Tor amplitude concentrated in degree -1, with typical examples being regular quotients of a perfectoid ring. We do not discuss these notions here, referring the reader to [BMS19, §4] for more details on the quasi-syntomic site, and [BS19, §12] for the prismatic cohomology of qrsp rings. But, since qrsp rings are exotic from the perspective of classical algebraic geometry, let us at least explain why they play an important technical (in fact, fundamental) role in the prismatic theory. The explanation rests on the following features of this class of rings:

1. For a qrsp ring R, essentially all natural prismatic invariants attached to R (e.g., the prismatic cohomology $\mathbb{\Delta}_R$, the layers $\mathrm{Fil}^i_{\mathcal{N}}\mathbb{\Delta}_R$ or graded pieces $\mathrm{gr}^i_{\mathcal{N}}\mathbb{\Delta}_R$ of the Nygaard filtration, the Hodge–Tate complex $\overline{\mathbb{\Delta}}_R$,...) are concentrated in degree 0.
2. The collection of all qrsp rings forms a basis for the quasi-syntomic site.

The combination of these properties ensures the centrality of qrsp rings in the prismatic formalism: (2) allows us to reduce general theorems to the qrsp case, and then (1) ensures we can proceed quite explicitly.

- $I \subset \mathbb{\Delta}_R$ is an invertible ideal such that $\mathbb{\Delta}_R$ is (p, I)-adically complete with $\overline{\mathbb{\Delta}}_R := \mathbb{\Delta}_R/I$ having bounded p-power torsion; henceforth, give $\mathbb{\Delta}_R$ (and its modules) the (p, I)-adic topology.
- I is locally generated by a distinguished element (i.e., $I \otimes_{\mathbb{\Delta}_R} W(\mathbb{\Delta}_R) \to W(\mathbb{\Delta}_R)$ is a Cartier–Witt divisor on $\mathrm{Spf}(\mathbb{\Delta}_R)$).
- There is a map $R \to \overline{\mathbb{\Delta}}_R := \mathbb{\Delta}_R/I$ universal with respect to the preceding data.

For $M \in \mathcal{D}(\overline{\mathbb{\Delta}}_R)$, write $M\{i\} := M \otimes_{\overline{\mathbb{\Delta}}_R} I^i/I^{i+1}$; it is called the *Breuil–Kisin* twist.

2. *The Nygaard filtration:* The prism $(\mathbb{\Delta}_R, I)$ carries a natural Nygaard filtration $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ defined via $\mathrm{Fil}^i_{\mathcal{N}}\mathbb{\Delta}_R = \varphi^{-1}(I^i) \subset \mathbb{\Delta}_R$. In particular, we have a map $\varphi : \mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R \to I^\bullet$ of filtered rings that we sometimes call the filtered Frobenius.

   Example 5.5.1 (The case of a perfectoid ring)**.** Say R is perfectoid. Consider Fontaine's ring $A_{\mathrm{inf}}(R) = W(R^\flat) = W((R/p)^{\mathrm{perf}})$. The natural surjection $A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(R)/p = (R/p)^{\mathrm{perf}} \to R/p$ deforms uniquely to a surjection $\theta : A_{\mathrm{inf}}(R) \to R$ whose kernel is a principal ideal generated by a distinguished element. The pair $(A_{\mathrm{inf}}(R), \ker(\theta))$ with the isomorphism $R \simeq A_{\mathrm{inf}}(R)/\ker(\theta)$ then realizes $(A_{\mathrm{inf}}(R), \ker(\theta))$ as the initial prism $(\mathbb{\Delta}_R, I)$ over R. As the comparison map $R \to \overline{\mathbb{\Delta}}_R$ is an isomorphism in this case, we have a notion of BK twists on $\mathcal{D}(R)$, and hence also on $\mathcal{D}(S)$ for any R-algebra S, compatibly with the one coming from (1) in the case $S = \overline{\mathbb{\Delta}}_T$ for a qrsp ring T. Moreover, φ is an automorphism, so $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R = \varphi^{-1}(I^\bullet)$ is the filtration by powers of a nonzerodivisor, and the filtered Frobenius induces an identification
   $$\mathrm{gr}^*_{\mathcal{N}}\mathbb{\Delta}_R \simeq \mathrm{Sym}_R(R\{1\}) \tag{5.5.1}$$
   of graded rings. The choice of a generator $d \in I$ (or even just its image in $R_0\{1\} = I/I^2$) identifies the right hand side with $R_0[u]$.

3. *The picture relative to a chosen perfectoid:* Choose a perfectoid ring R_0 and a map $R_0 \to R$. Then there is a natural (in the map $R_0 \to R$) conjugate filtration $\mathrm{Fil}^{conj}_\bullet\overline{\mathbb{\Delta}}_R \subset \overline{\mathbb{\Delta}}_R$ with the following two features:

   (a) The associated graded $\mathrm{gr}^{conj}_*\overline{\mathbb{\Delta}}_R$ is naturally identified with the graded ring $\Omega^H_{R/R_0} := \wedge^* L_{R/R_0}\{-*\}[-*]$ (where the notion of BK twists comes from R_0).

   (b) The associated graded of the filtered Frobenius map $\varphi : \mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R \to I^\bullet$ factors[64] as
   $$\mathrm{gr}^i_{\mathcal{N}}\mathbb{\Delta}_R \simeq \mathrm{Fil}^{conj}_i\overline{\mathbb{\Delta}}_R\{i\} \subset \overline{\mathbb{\Delta}}_R\{i\} = \mathrm{gr}^i_{I^\bullet}\mathbb{\Delta}_R, \tag{5.5.2}$$
   where the BK twist on the second term is defined using the map $\overline{\mathbb{\Delta}}_{R_0} \simeq R_0 \to R$. Under this description, the inclusion $\mathrm{Fil}^{conj}_i\overline{\mathbb{\Delta}}_R\{i\} \otimes_{R_0} R_0\{1\} \subset \mathrm{Fil}^{conj}_{i+1}\overline{\mathbb{\Delta}}_R\{i+1\}$ of the stages of the filtration coming from (1) corresponds to the natural map
   $$\mathrm{gr}^i_{\mathcal{N}}\mathbb{\Delta}_R \otimes_{\mathrm{gr}^0_{\mathcal{N}}\mathbb{\Delta}_{R_0}} \mathrm{gr}^1_{\mathcal{N}}\mathbb{\Delta}_{R_0} \to \mathrm{gr}^{i+1}_{\mathcal{N}}\mathbb{\Delta}_R$$
   under the isomorphism (5.5.1).

[64] As the filtered Frobenius map is independent of the choice of R_0, (5.5.2) implies that the subgroup $\mathrm{Fil}^{conj}_i\overline{\mathbb{\Delta}}_R\{i\} \subset \overline{\mathbb{\Delta}}_R\{i\}$ is independent of the choice of R_0. However, as $\mathrm{Fil}^{conj}_i\overline{\mathbb{\Delta}}_R\{i\}$ is merely a module over $R = \mathrm{Fil}^{conj}_0\overline{\mathbb{\Delta}}_R$ and not over $\overline{\mathbb{\Delta}}_R$, we are not allowed to twist this inclusion by $\overline{\mathbb{\Delta}}_R\{-i\}$ to conclude that $\mathrm{Fil}^{conj}_i\overline{\mathbb{\Delta}}_R$ is independent of the choice of R_0. In fact, it is provably not, see [BS19, Example 12.3].

(c) Combining (a) and (b), we obtain a natural isomorphism

$$\mathrm{gr}^*_{\mathcal{N}}{\Delta}_R \otimes^L_{\mathrm{gr}^*_{\mathcal{N}}{\Delta}_{R_0}} R_0 \simeq \Omega^H_{R/R_0}$$

of graded rings.

(d) The filtered Frobenius yields a map $\bigoplus_{i\in\mathbf{Z}} \mathrm{Fil}^i_{\mathcal{N}}{\Delta}_R t^{-i} \to \bigoplus_{i\in\mathbf{Z}} I^i t^{-i}$ of $\mathbf{Z}$-graded rings that exhibits the target as a localization of the source (in the (p,I)-complete sense): indeed, one simply needs to invert a generator of $\mathrm{Fil}^1_{\mathcal{N}}{\Delta}_{R_0} t^{-1}$ and use (b) as well as the fact that conjugate filtration is exhaustive. Note that while the choice of element to invert depends on the choice of R_0, the assertion that the map is a localization is independent of the choice of R_0.

For our future use, it will be convenient to introduce a name and notation for the Rees stack associated to the Nygaard filtration.

Definition 5.5.2 (The Rees stack of $\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R$)**.** Let $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$ be the extended Rees algebra of the Nygaard filtration on ${\Delta}_R$, i.e.

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R) = \bigoplus_{i\in\mathbf{Z}} \mathrm{Fil}^i_{\mathcal{N}}{\Delta}_R t^{-i} \subset {\Delta}_R[t, t^{-1}].$$

This is naturally a graded ring (where $\deg(t) = 1$), and we define $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$ to be the corresponding Rees stack, i.e.:

$$\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R) := \mathrm{Spf}(\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R))/\mathbf{G}_m.$$

The parameter t gives a map $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R) \to \mathbf{A}^1/\mathbf{G}_m$ that we call *the Rees map*; the inclusion of the degree 0 subring yields a map $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R) \to \mathrm{Spf}({\Delta}_R)$ that we call *the structure map.*

Remark 5.5.3 (The functor of points of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$)**.** For a p-nilpotent test ring S, the groupoid $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)(S)$ is the groupoid consisting of:

1. A line bundle $L \in \mathcal{P}\mathrm{ic}(S)$ with a map $t : L \to S$; this pair yields a map $\mathrm{Spf}(S) \to \mathbf{A}^1/\mathbf{G}_m$
2. A map of graded rings $\oplus_{i\geq 0}\mathrm{Fil}^i_{\mathcal{N}}{\Delta}_R t^{-i} \to \oplus_{i\geq 0} L^i t^{-i}$ that kills some power of $(p, I) \subset {\Delta}_R$ and intertwines multiplication by t on the source (corresponding to filtration level inclusions) with multiplication by t on the target (corresponding to multiplying by the section t from (1)).

In particular, for test rings S where the map t in (1) is injective (e.g., if $\mathrm{Spf}(S) \to \mathbf{A}^1/\mathbf{G}_m$ is flat), the data in (2) is simply a map ${\Delta}_R \to S$ that kills some power of (p, I) and carries $\mathrm{Fil}^i_{\mathcal{N}}{\Delta}_R$ into $L^i \overset{t^i}{\subset} S$ for all $i \geq 0$.

Remark 5.5.4 (Quasi-coherent sheaves on $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$)**.** By the Rees equivalence, the quasi-coherent derived category $\mathcal{D}_{qc}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R))$ identifies with the filtered derived category $\mathcal{DF}_{(p,I)\text{-comp}}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$ of (p, I)-complete complexes over ${\Delta}_R$ endowed with the structure of a $\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R$-module, i.e., we can roughly regard a quasi-coherent complex on $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$ as a (p, I)-complete ${\Delta}_R$-complex equipped with a filtration compatible with the Nygaard filtration.

Remark 5.5.5 (Some open substacks of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$)**.** The stack $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$ contains two distinguished open copies of $\mathrm{Spf}({\Delta}_R)$:

1. The open immersion j_{dR}: The fibre over $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ of the Rees map $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R) \to \mathbf{A}^1/\mathbf{G}_m$ identifies with $\mathrm{Spf}({\Delta}_R)$ by generalities on the Rees construction, giving an open immersion $j_{dR} : \mathrm{Spf}({\Delta}_R) \subset \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R)$. The composition $\mathrm{Spf}({\Delta}_R) \xrightarrow{j_{dR}} \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\Delta}_R) \xrightarrow{\pi} \mathrm{Spf}({\Delta}_R)$ is the identity, where π is the structure map.

2. The open immersion j_{HT}: The filtered Frobenius yields a map $\bigoplus_{i\in\mathbf{Z}} \mathrm{Fil}^i_{\mathcal{N}}\mathbbl{\Delta}_R t^{-i} \to \bigoplus_{i\in\mathbf{Z}} I^i t^{-i}$ of $\mathbf{Z}$-graded rings which is a localization at a single element. Passing to the corresponding stacks then defines an open immersion

$$j_{HT} : \mathrm{Spf}(\mathbbl{\Delta}_R) \simeq \mathrm{Spf}(\bigoplus_{i\in\mathbf{Z}} I^i t^{-i})/\mathbf{G}_m \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R).$$

   The composition $\mathrm{Spf}(\mathbbl{\Delta}_R) \xrightarrow{j_{HT}} \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R) \xrightarrow{\pi} \mathrm{Spf}(\mathbbl{\Delta}_R)$ is the Frobenius, where π is the structure map.

It is easy to see that these copies are disjoint: the Rees parameter t is invertible after pullback along j_{dR}, and is topologically nilpotent after pullback along j_{HT}.

In the next example, we explain why the Rees stack introduced above takes on a familiar form for perfectoid R: we obtain the same structures that we saw earlier in §3.3 when R was a perfect field of characteristic p.

Example 5.5.6 (The Nygaard filtered Rees stack for perfectoids)**.** If R is a perfectoid ring, then $\mathrm{Fil}^\bullet_{\mathcal{N}} R = \varphi^{-1}(I)^\bullet$. Choose a generator $d \in I$. Then $u = \varphi^{-1}(d)t^{-1} \in \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R)_{\deg=-1}$ is a generator satisfies $ut = \varphi^{-1}(d)$. In fact, through this choice, we obtain a presentation

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R) = \mathbbl{\Delta}_R[u, t]/(ut - \varphi^{-1}(d)).$$

More intrinsically, $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R)$ has the following moduli description: for a p-nilpotent ring R, the groupoid $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R)$ is identified with the groupoid of maps $\mathbbl{\Delta}_R \to S$ that kill some power of (p, I) together with a line bundle $L \in \mathcal{P}\mathrm{ic}(S)$ and a factorization $\varphi^{-1}(I) \otimes_{\mathbbl{\Delta}_R} S \xrightarrow{u} L \xrightarrow{t} S$ of canonical map. In particular, we can regard u as a canonically defined section of a line bundle on $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R)$, so its (non-)vanishing locus is well-defined, independent of any choices. In fact, the non-vanishing loci of u and t correspond respectively to the open immersions j_{HT} and j_{dR} from Remark 5.5.5.

For future reference, let us explain why the ring stack from Construction 3.3.2, which concerns the case where R is a perfect field of characteristic p, has a natural variant over any perfectoid R as here. To explain this, observe that for any δ-ring A, there is a natural δ-map $A \to W$ of sheaves of δ-rings on the big site of all A-schemes: indeed, for any A-algebra R, the structure map $A \to R$ refines uniquely to a δ-map $A \to W(R)$. In particular, there is a natural δ-map $\mathbbl{\Delta}_R \to W$ of presheaves of δ-rings over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R)$. One can then imitate the reasoning used to construct (3.3.2) to obtain the following diagram of W-module schemes over

$$\begin{tikzcd}
0 \arrow[r] & \varphi^{-1}(I)\otimes_{\mathbbl{\Delta}_R}\mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dd, bend right, "\mathrm{can}"'] & \varphi^{-1}(I)\otimes_{\mathbbl{\Delta}_R} W \arrow[r, "\mathrm{id}\otimes F"] \arrow[d] \arrow[dd, bend right, "\mathrm{can}"'] & \varphi^{-1}(I)\otimes_{\mathbbl{\Delta}_R} F_*W \simeq F_*(I\otimes_{\mathbbl{\Delta}_R} W) \arrow[r] \arrow[d, equal] \arrow[dd, bend right, "\mathrm{can}"'] & 0 \\
0 \arrow[r] & \mathbf{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp"] & M_u \arrow[r] \arrow[d, "d_{u,t}"] & \varphi^{-1}(I)\otimes_{\mathbbl{\Delta}_R} F_*W \simeq F_*(I\otimes_{\mathbbl{\Delta}_R} W) \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0.
\end{tikzcd} \tag{5.5.3}$$

The map $d_{u,t}$ is a quasi-ideal (Proposition 5.3.8), leading to an animated W-algebra stack

$$\mathbf{G}_a^{\mathcal{R}} := \mathrm{Cone}(d_{u,t})$$

over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbbl{\Delta}_R)$. The middle vertical column shows that $\mathbf{G}_a^{\mathcal{R}}$ is naturally an animated ring over $W/\varphi^{-1}(I)$, and hence also an animated ring over $\mathbbl{\Delta}_R/\varphi^{-1}(I) \overset{\varphi}{\simeq} \varphi_*\mathbbl{\Delta}_R/I = \varphi_* R$, where the φ_*

decoration gives the compatibility between this R-algebra structure and the original Δ_R-algebra structure on $\mathbf{G}_a^{\mathcal{R}}$. Note that ring stack is actually valued in p-nilpotent animated $\varphi_* R$-algebras by [BL22b, Lemma 3.3] and the $W/\varphi^{-1}(I)$-algebra structure. In the sequel, we shall thus regard $\mathbf{G}_a^{\mathcal{R}}$ as a p-nilpotent animated $\varphi_* R$-algebra valued stack on (p, I)-nilpotent on Δ_R-algebras over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R)$.

5.5.2 Identifying the prismatization

In this subsection, let R be a qrsp ring. Our goal is to sketch a proof of the following theorem identifying the prismatization R^{Δ} in classical terms; as qrsp rings form a basis for the quasi-syntomic site, this justifies the name "prismatization" given the functor $X \mapsto X^{\Delta}$ on quasi-syntomic schemes.

Theorem 5.5.7 (The prismatization via Δ_R)**.** *There is a natural isomorphism* $\mathrm{Spf}(\Delta_R) \simeq R^{\Delta}$, *where we use the* (p, I)*-adic topology on* Δ_R *on the left.*

The proof we sketch is borrowed from [BL22b], and we refer to [BL22b, Lemma 6.1, Corollary 7.18] for the missing details details.

Proof sketch. The δ-structure on Δ_R is classified by a map $w : \Delta_R \to W(\Delta_R)$ splitting the restriction map γ_0. Reducing w modulo I and composing with the natural map $R \to \Delta_R/I$ then defines a map $R \to W(\Delta_R)/I$. This construction can be regarded as giving a map $\eta_R : \mathrm{Spf}(\Delta_R) \to R^{\Delta}$ by definition of points of the target. We claim this map is an isomorphism.

As η_R is naturally defined, we can make choices to prove it is an isomorphism. Thus, choose a perfectoid ring R_0 and a map $R_0 \to R$. By enlarging R_0, we may also assume $R_0 \to R$ is surjective. We only explain the argument in the key case R_0 is p-torsionfree and $R = R_0/(f_1, ..., f_r)$ for a sequence $f_1, ..., f_r$ in R_0 that is regular on R_0/p; the general case can be reduced to this case by an elaborate devissage. Write (Δ_{R_0}, I_0) for the prism of R_0, so we have an induced map $(\Delta_{R_0}, I_0) \to (\Delta_R, I)$ of prisms, and thus an identification $I_0 \otimes_{\Delta_{R_0}} \Delta_R \simeq I$.

First, the argument proving Example 5.1.9 adapts to prove the statement in the perfectoid case, i.e., η_{R_0} gives an isomorphism $\mathrm{Spf}(\Delta_{R_0}) \simeq R_0^{\Delta}$. It then suffices to prove that the fibres of η_R over η_{R_0} are identified, i.e., to show the following: for a p-nilpotent ring S and an S-valued point $x : \mathrm{Spec}(S) \to \mathrm{Spf}(\Delta_{R_0})$, the fibre F_x of $\mathrm{Spf}(\Delta_R)(S) \to \mathrm{Spf}(\Delta_{R_0})(S)$ over x identifies with the fibre G_x of $R^{\Delta}(S) \to R_0^{\Delta}(S)$ over $\eta(x)$ via the map η. We can identify the fibres explicitly as

$$F_x = \mathrm{Map}_{\Delta_{R_0}}(\Delta_R, S) \quad \text{and} \quad G_x = \mathrm{Map}_{R_0}(R, \overline{W(S)}),$$

where $\overline{W(S)} = \mathrm{Cone}(I \otimes_{\Delta_{R_0}} W(S) \to W(S))$, with the implicit map $\Delta_{R_0} \to W(S)$ being the unique δ-lift adjoint of the given map $\Delta_{R_0} \to S$. To prove this, let us describe the map η_R explicitly. First, by adjunction, we have

$$F_x = \mathrm{Map}_{\Delta_{R_0}}(\Delta_R, S) \simeq \mathrm{Map}_{\Delta_{R_0}, \delta}(\Delta_R, W(S)).$$

Given this description, the map $F_x \to G_x$ induced by η_R is the composition

$$\mathrm{Map}_{\Delta_{R_0}, \delta}(\Delta_R, W(S)) \to \mathrm{Map}_{\overline{\Delta}_{R_0}}(\overline{\Delta}_R, \overline{W(S)}) \to \mathrm{Map}_{R_0}(R, \overline{W(S)}),$$

where the first map is by forgetting the δ-structure and reducing modulo I_0, and the second map is induced by $R \to \overline{\Delta}_R$ (and the observation that $R_0 \simeq \overline{\Delta}_{R_0}$). To check this is an isomorphism, we can use presentations for the first factor: R is obtained from R_0 by freely setting $f_1 = 0, f_2 = 0, ..., f_r = 0$ in p-complete animated rings, while Δ_R is obtained from Δ_{R_0} by freely adjoining $\frac{f_1}{I_0}, ..., \frac{f_r}{I_0}$ in the world of (p, I)-complete δ-rings by [BS19, Example 7.9]. Using these presentations, one explicitly checks that the above map is an isomorphism. $\square$

5.5.3 Identifying the filtered prismatization

In this subsection, we shall describe the filtered prismatization of qrsp rings explicitly as the Rees stack of the Nygaard filtration on prismatic cohomology (Corollary 5.5.11), extending Theorem 5.5.7 describing the (non-filtered) prismatization. In fact, as qrsp rings admit maps from perfectoids, it is convenient to fix a base perfectoid ring R_0 for the rest of this subsection (even though the the final result is independent of this choice). Let us first begin by describing $R_0^{\mathcal{N}}$ itself.

Proposition 5.5.8 (The filtered prismatization of a perfectoid ring)**.** *There is a natural identification $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0}) \simeq R_0^{\mathcal{N}}$ of stacks that intertwines the Rees and structure maps, and carries the animated W-algebra stack $\mathbf{G}_a^{\mathcal{R}}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ (from Example 5.5.6) to the animated W-algebra stack $\mathbf{G}_a^{\mathcal{N}}$ over $R_0^{\mathcal{N}}$.*

Proof. Using the perfectoid case of Theorem 5.5.7, this follows by essentially the same argument used in Proposition 5.4.2 □

Via transmutation, one already obtains a description of $X^{\mathcal{N}}$ that avoids contemplating admissible modules (or, rather, their moduli) directly.

Corollary 5.5.9 (Concrete description of $X^{\mathcal{N}}$)**.** *For any bounded p-adic formal R_0-scheme X and p-nilpotent ring S, there is a natural identification of $X^{\mathcal{N}}(S)$ with the groupoid of pairs (a, b), where a is a point of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})(S)$, and b is an φ_*R_0-map $\mathrm{Spec}(\mathbf{G}_a^{\mathcal{R}}(S)) \to \varphi_*X$.*

The main theorem of this chapter in the following, giving a concrete description of the filtered prismatization of a qrsp ring and thus justifying the name "filtered prismatization":

Theorem 5.5.10 (The filtered prismatization over R_0 via the Nygaard filtration)**.** *For a qrsp R_0-algebra R, there is a natural identification*

$$R^{\mathcal{N}} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$$

of derived stacks over the identification $R_0^{\mathcal{N}} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ from Proposition 5.5.8.

The two sides of the isomorphism in Theorem 5.5.7 depend only on the qrsp ring R, and not on its structure as an algebra over a perfectoid ring. However, *a priori*, the isomorphism we will construct certainly relies heavily on this choice. Nevertheless, it is in fact independent of the choice for relatively soft reasons:

Corollary 5.5.11 (Independence of perfectoid base)**.** *For any qrsp ring R, there is a natural isomorphism*

$$c_R : R^{\mathcal{N}} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$$

of stacks over $\mathbf{A}^1/\mathbf{G}_m$ (via the Rees map), uniquely determined by the requirement that it restricts over $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ to the isomorphism $R^{\Delta} \simeq \mathrm{Spf}(\Delta_R)$ from Theorem 5.5.7.

Proof. First, note that Theorem 5.5.10 provides such an isomorphism, say $c_{R_0\to R}$, for each choice of perfectoid ring R_0 mapping to R. Moreover, the Rees map from both $R^{\mathcal{N}}$ and $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$ to $\mathbf{A}^1/\mathbf{G}_m$ is independent of the choice of R_0 by construction. As the fibre $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)_{t\neq 0} \simeq \mathrm{Spf}(\Delta_R)$ over $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$ is dense[65] in $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$, it follows that $(R^{\mathcal{N}})_{t\neq 0}$ is dense in $R^{\mathcal{N}}$ as well. In particular, any isomorphism $R^{\mathcal{N}} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$ of stacks over $\mathbf{A}^1/\mathbf{G}_m$ is uniquely determined by its behaviour over $(-)_{t\neq 0}$ locus. Now the fibre $c_{R_0\to R}|_{t\neq 0}$ agrees with the isomorphism $R^{\Delta} \simeq \mathrm{Spf}(\Delta_R)$ from Theorem 5.5.7, and is thus independent of choice of R_0. It thus follows that $c_{R_0\to R}$ is also independent of the choice of R_0, and provides the natural isomorphism c_R in the corollary. The characterization mentioned in the corollary follows from the proof. □

[65]This amounts to observing that $\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R$ is an honest filtration, i.e., $\mathrm{Fil}^i_{\mathcal{N}}\Delta_R \to \mathrm{Fil}^{i-1}_{\mathcal{N}}\Delta_R$ is injective for all i.

Remark 5.5.12 (Why should one have expected the Rees description of $R^{\mathcal{N}}$?)**.** We are aware of two motivations for predicting the statement of Theorem 5.5.10 or Corollary 5.5.11; they rest on the naturality of contemplating $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$ from two different perspectives.

First, for a (say) smooth variety $X/\mathbf{C}$, consider the ∞-category $\mathcal{DF}_{coh}(\mathrm{Fil}^\bullet_H\Omega^\bullet_{X/\mathbf{C}})$ of complete filtered modules over the Hodge filtered de Rham complex with the property that the associated graded is coherent over X. This is a good category of coefficients for (Hodge filtered) de Rham cohomology: it can be identified (via a version of Koszul duality) with the filtered derived ∞-category $\mathcal{DF}_{good}(\mathcal{D}_X)$ of coherent $\mathcal{D}_X$-modules equipped with a good filtration. Motivated by this as well as the analogy of the Nygaard filtration on prismatic cohomology with the Hodge filtration on de Rham cohomology, it is natural to guess that $\mathcal{DF}_{(p,I)\text{-comp}}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$ is a good category of coefficients for (Nygaard filtered) prismatic cohomology. This reasoning (and Remark 5.5.4) led us to the statement of Corollary 5.5.11 in the first place.

Secondly, as was first remarked to us by Scholze, the Rees stack $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$ can also be presented via a Breuil–Kisin twisted version of the Rees algebra that appears extremely naturally in the theory of topological Hochschild homology (up to a completion, see Remark 5.5.16).

We now turn to the proof of Theorem 5.5.10. For a slightly technical reason that comes up in the proof, it will be quite convenient to know that the moduli stacks under consideration have natural modular extensions to the derived world, so we record this next.

Remark 5.5.13 (Derived extensions)**.** Regard $\mathbf{G}_a^{\mathcal{R}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ as a map of derived stacks over $\mathrm{Spf}(\Delta_R)$ in the natural way, i.e., via left Kan extension of the inclusion of affine schemes into derived stacks. Then $\mathbf{G}_a^{\mathcal{R}}$ can be regarded as a p-nilpotent animated R_0-algebra stack over the derived $\mathrm{Spf}(\Delta_{R_0})$-stack $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$[66]. It follows by transmutation that $X^{\mathcal{N}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ also has a natural extension to derived stacks with the following feature: given a p-nilpotent animated ring S and a point $x \in \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})(S)$, the fibre of $X^{\mathcal{N}}(S) \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})(S)$ over x identifies with the space of maps $\mathrm{Map}_{R_0}(\mathrm{Spec}(\mathbf{G}_a^{\mathcal{R}}(S)), X)$ of derived $\mathrm{Spf}(R_0)$-schemes.

We now sketch the proof of Theorem 5.5.10; more details, together with an alternative proof that directly constructs a filtered Cartier–Witt divisor over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$ unlike the proof below, will appear in [BL].

Proof sketch. In the proof below, all geometric objects that appear are regarded as derived stacks over $\mathrm{Spf}(\Delta_{R_0})$ via the recipe given in Remark 5.5.13; the main reason to pass to the derived world is that we need to contemplate derived pullbacks to run the argument (e.g., see item (4) below), and it is much easier to understand moduli-theoretically the derived pullback of a map between derived moduli stacks than the derived pullback of a map between classical moduli stacks. It follows from the proofs that the relevant derived stacks are actually classical. Also, we write $\varphi : \Delta_{R_0} \to \Delta_{R_0}$ for the Frobenius automorphism; the appearance of many Frobenius twists in the discussion below is a consequence of our desire to work relative to $\mathrm{Spf}(\Delta_{R_0})$ and can be ignored at first pass.

Let us begin by introducing a constructible stratification of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ that plays an important role, both implicitly and explicitly, in the arguments that follow. Using the sections u, t from

[66] The content here is the following. For a functor F from rings to an ∞-category admitting all colimits, write LF for its animation. Then the natural maps give an equivalence $LW \simeq \lim_n LW_n$ as animated ring valued functors on animated rings, so LW is both left Kan extended from polynomial rings (by definition) as welll as an fpqc sheaf (as each $W_n(-)$ is so, and limits of sheaves are sheaves). The same remarks also apply to functors finitely built from W, such as $R\Gamma_{fl}(\mathrm{Spec}(-), \mathbf{G}_a^\sharp) := \mathrm{fib}(W \xrightarrow{F} W)$ (regarded as a functor from rings to $\mathcal{D}(\mathrm{Ab})$). Applying this reasoning to all terms of (5.5.3) pulled back to the Cech nerve of a flat cover of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ then yields the structure of a p-nilpotent animated R_0-algebra valued stack on the derived stack $\mathbf{G}_a^{\mathcal{R}}$ by descent.

Example 5.5.6, we obtain the constructible stratification

$$\{\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})_{t\neq 0} = \mathrm{Spf}(\Delta_{R_0}), \quad \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})_{t=0,u\neq 0} = \mathrm{Spf}(\varphi_* R_0), \quad \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})_{u=t=0} = B\mathbf{G}_{m,\varphi_* R_0}\} \tag{5.5.4}$$

of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$. The pullback of $\mathbf{G}_a^{\mathcal{R}}$ to this stratification is given (as a p-nilpotent animated $\varphi_* R_0$-algebra valued derived stack) by

$$\{\varphi^*\mathbf{G}_a^{\Delta} = F_*(W/I), \quad \varphi^*\mathbf{G}_a^{HT} = W/\varphi^{-1}(I), \quad \mathbf{G}_a^{Hodge} = \mathbf{G}_a \oplus B\mathbf{V}(L)^\sharp\}$$

by analysing the diagram (5.5.3). Consequently, via transmutation, the pullback of $X^{\mathcal{N}}$ to this stratification has pieces described by

$$\{X^{\Delta} \to \mathrm{Spf}(\Delta_{R_0}), \quad \varphi_* X^{HT} \to \mathrm{Spf}(\varphi_* R_0), \quad \varphi_* X^{Hodge} \to B\mathbf{G}_{m,\varphi_* R_0}\} \tag{5.5.5}$$

We now begin the proof.

1. *Geometry of $X^{\mathcal{N}}$ for X/R_0 smooth:* Fix a smooth quasi-compact scheme $X/\mathrm{Spf}(R_0)$ and consider the derived stack $\pi : X^{\mathcal{N}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$. We claim that π is flat (and thus $X^{\mathcal{N}}$ is classical) and that formation of pushforward of quasi-coherent complexes along π commutes with base change along maps with finite Tor amplitude (such as the inclusions of the strata of the constructible stratification (5.5.4)). To see this, one first proves (as in Remark 2.8.5) that the functor $X \mapsto X^{\mathcal{N}}$ from smooth quasi-compact $\mathrm{Spf}(R_0)$-schemes to derived stacks over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$ commutes with finite limits and preserves affine étale maps (resp. covers); this implies that if $X \to \mathbf{A}^n$ is an affine étale map over $\mathrm{Spf}(R_0)$, then $X^{\mathcal{N}} \to (\mathbf{A}^n)^{\mathcal{N}} = (\mathbf{G}_a^{\mathcal{R}})^n = (W/M_u)^n$ is an affine étale, so $X^{\mathcal{N}}$ is the quotient of a flat affine $\mathrm{Spf}(R_0)$-scheme by the flat affine $\mathrm{Spf}(R_0)$-group scheme $(M_u)^n$ (where M_u is the admissible W-module scheme constructed in diagram (5.5.3)). This description gives the desired flatness as well as base change compatibility by Remark 2.5.5.

2. *Cohomology of $X^{\mathcal{N}}$ for X/R_0 smooth:* With notation as in (1), we claim that $R\pi_*\mathcal{O}_{X^{\mathcal{N}}} \in \mathcal{D}_{qc}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0}))$ identifies with $\mathrm{Fil}^\bullet_{\mathcal{N}}R\Gamma_{\Delta}(X) \in \mathcal{DF}_{comp}(\Delta_{R_0})$ under the Rees equivalence. Thanks to the base change compatibilities in (1), this follows by the same argument used to establish Theorem 3.3.5.

3. *The natural transformation $\eta : R^{\mathcal{N}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$:* Recall that the Nygaard filtration on the prismatic cohomology of p-complete animated R_0-algebras is left Kan extended from the smooth ones. Thus, it follows from (2) that for any bounded p-adic formal scheme X with structure map $\pi : X^{\mathcal{N}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$, one obtains a natural map

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}R\Gamma_{\Delta}(X)) \to R\pi_*\mathcal{O}_{X^{\mathcal{N}}}$$

of commutative algberas in $\mathcal{D}_{qc}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0}))$. Specializing to $X = \mathrm{Spf}(R)$ for our qrsp R, the left hand side is concentrated in degree 0, so the above map defines, by adjunction, a map

$$\eta : R^{\mathcal{N}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R)$$

of stacks over $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_{R_0})$.

4. *The isomorphy of η:* It suffices[67] to prove that η is an isomorphism after (derived) pullback to the pieces of the stratification (5.5.4). The pullback of $R^{\mathcal{N}}$ to the pieces of this stratification is given by

$$\{R^{\Delta}, \quad \varphi_* R^{HT}, \quad \varphi_* R^{Hodge}\}$$

[67] We are using the following fact: given an animated ring A, an element $f \in \pi_0(A)$, and a map $\tau : M \to N$ in $\mathcal{D}(A)$, the map τ is an isomorphism if and only if $\tau[1/f]$ and $\tau \otimes_A A/f$ are so.

as in (5.5.5) above; the pullback of $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R)$ to the pieces is given by

$$\{\mathrm{Spf}({\mathbb{\Delta}}_R), \quad \mathrm{Spf}(\varphi_*\overline{{\mathbb{\Delta}}}_R), \quad \varphi_*\mathrm{Spf}(\wedge^* L_{R/R_0}\{-*\}[-*])/\mathbf{G}_m\}$$

by known features of prismatic cohomology, as summarized in §5.5.1. On the first two pieces, the map can then be checked to be an isomorphism via known properties of prismatization as in Theorem 5.5.7. The claim for the last piece follows by analogous arguments. □

Remark 5.5.14. We leave it to the reader to check that the open substacks from Remark 5.5.5 match, under the isomorphism in Corollary 5.5.11, with the correspondingly named open substacks of $R^{\mathcal{N}}$ coming via pullback from Constructions 5.3.2 and 5.3.3 respectively.

Remark 5.5.15 (The BK twist over $R^{\mathcal{N}}$)**.** In [Dri20, §5.9], Drinfeld extends the line bundle $\mathcal{O}_{\mathbf{Z}_p^{\mathbb{\Delta}}}\{1\} \in \mathcal{P}\mathrm{ic}(\mathbf{Z}_p^{\mathbb{\Delta}})$ from Remark 5.1.19 along either inclusion $j_{HT}, j_{dR} : \mathbf{Z}_p^{\mathbb{\Delta}} \subset \mathbf{Z}_p^{\mathcal{N}}$ by constructing $\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} \in \mathcal{P}\mathrm{ic}(\mathbf{Z}_p^{\mathcal{N}})$, defined as

$$\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} = \pi^*\mathcal{O}_{\mathbf{Z}_p^{\mathbb{\Delta}}}\{1\} \otimes t^*\mathcal{O}(-1),$$

where π and t are the structure and Rees maps (Construction 5.3.3). As the isomorphism in Corollary 5.5.11 is compatible with the structure and Rees maps, the pullback $\mathcal{O}_{R^{\mathcal{N}}}\{1\}$ of $\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\}$ to $R^{\mathcal{N}} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R)$ coincides with the line bundle defined by the graded $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R)$-module $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R)\{1\}(-1)$. In particular, we have $R\Gamma(R^{\mathcal{N}}, \mathcal{O}_{R^{\mathcal{N}}}\{1\}) = (\mathrm{Fil}^1_{\mathcal{N}}{\mathbb{\Delta}}_R)\{1\}$ and more generally $R\Gamma(R^{\mathcal{N}}, \mathcal{O}_{R^{\mathcal{N}}}\{i\}) = (\mathrm{Fil}^i_{\mathcal{N}}{\mathbb{\Delta}}_R)\{i\}$ for all $i \in \mathbf{Z}$, as expected.

Remark 5.5.16 (Relation to TC)**.** For any line bundle $L \in \mathcal{P}\mathrm{ic}({\mathbb{\Delta}}_R)$, the twisted version of the Rees construction gives a $\mathbf{Z}$-graded ring

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R \otimes_{{\mathbb{\Delta}}_R} L^\bullet) = \bigoplus_{i\in\mathbf{Z}} (\mathrm{Fil}^i_{\mathcal{N}}{\mathbb{\Delta}}_R \otimes_{{\mathbb{\Delta}}_R} L^i)t^{-i} \subset \bigoplus_{i\in\mathbf{Z}} L^i t^{-i},$$

where the last term is the algebra of functions on the $\mathbf{G}_m$-torsor attached to L. Using the functor of points interpretation from Remark 5.5.3 as well as a twisted version, one checks that there is a natural identification

$$\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R) \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R \otimes_{{\mathbb{\Delta}}_R} L^\bullet)$$

of the corresponding Rees stacks (not compatible with the map to $B\mathbf{G}_m$!). In particular, taking $L = \mathcal{O}_{R^{\mathcal{N}}}\{1\}$ from Remark 5.5.15 and using Corollary 5.5.11, we learn that

$$R^{\mathcal{N}} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R\{\bullet\}).$$

The latter stack is an extremely natural object to consider from the perspective of topological cyclic homology: indeed, by [BMS19, Theorem 1.12], we have a natural identification of graded rings

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}}\widehat{{\mathbb{\Delta}}}_R\{\bullet\}) = \pi_*\mathrm{TC}^-(R; \mathbf{Z}_p),$$

where the $\widehat{(-)}$ denotes the Nygaard completion. In this description, the two open points

$$j_{dR}, j_{HT} : \mathrm{Spf}({\mathbb{\Delta}}_R) \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}{\mathbb{\Delta}}_R\{\bullet\})$$

from Remark 5.5.5 match with the two maps

$$\mathrm{can}, \varphi : \pi_*\mathrm{TC}^-(R; \mathbf{Z}_p) \to \pi_*\mathrm{TP}(R; \mathbf{Z}_p)$$

coming from the canonical and (cyclotomic) Frobenius maps respectively.

Having understood the filtered prismatization of qrsp rings (and thus implicitly the theory of gauges of such rings), let us define gauges explicitly and make some remarks on how to understand them by descent.

Definition 5.5.17 (Gauges)**.** For a bounded p-adic formal scheme X, its category of *(prismatic) gauges* is defined as

$$\mathrm{Gauge}_{\Delta}(X) := \mathcal{D}_{qc}(X^{\mathcal{N}}).$$

The category of gauges can be studied via descent thanks to Corollary 5.5.11.

Remark 5.5.18 (Descent to arbitrary quasi-syntomic p-adic formal schemes)**.** Given a quasi-syntomic cover $f : X \to Y$ of quasi-compact quasi-syntomic p-adic formal schemes, one can show that the induced map $X^{\mathcal{N}} \to Y^{\mathcal{N}}$ is also a cover (see [BL22b, Proposition 2.17] for the analogous statement for $(-)^{\Delta}$). Using the behaviour of $(-)^{\mathcal{N}}$ under finite limits as well as the basis property of qrsp rings for the quasi-syntomic site, we obtain the following consequences for a quasi-syntomic scheme X:

1. The natural map
$$\operatorname*{colim}_{R \in X_{\mathrm{qrsp}}} \mathcal{R}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} \Delta_R) \to X^{\mathcal{N}}$$
is an equivalence, where the X_{qrsp} is the category of all qrsp rings R equipped with a map $\mathrm{Spf}(R) \to X$. In particular, this formula gives an alternative approach to the filtered prismatization construction.
2. Pullback along the maps in (1) induces an equivalence
$$\mathrm{Gauge}_{\Delta}(X) = \lim_{R \in X_{\mathrm{qrsp}}} \mathrm{Gauge}_{\Delta}(\mathrm{Spf}(R)) = \lim_{R \in X_{\mathrm{qrsp}}} \mathcal{DF}_{(p,I)-\mathrm{comp}}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} \Delta_R).$$
One may summarize this as follows: specifying a gauge on X is equivalent to specifying, in a base change compatible fashion, a filtered module over $\mathrm{Fil}^{\bullet}_{\mathcal{N}} \Delta_R$ for each qrsp ring R over X.

Remark 5.5.19 (A construction of charts for $X^{\mathcal{N}}$ with X regular)**.** Say (A, I) is a prism such that A is p-torsionfree and A/p is regular (whence A is noetherian with a flat p-complete cotangent complex over $\mathbf{Z}_p$). The ring $R = A/I$ is then a regular p-complete ring; moreover, any complete noetherian regular local ring has this form by the Cohen structure theorem. We shall describe a particularly convenient chart for the stack $R^{\mathcal{N}}$ resulting from the choice of the prism (A, I).

Consider the Rees stack $\mathcal{R}(I^{\bullet}) := \mathrm{Spf}(\mathrm{Rees}(I^{\bullet}))/\mathbf{G}_m$ attached to the I-adic filtration on A (endowed with the (p, I)-adic topology). For instance, if $I = (d)$ is principal, then $\mathrm{Rees}(I^{\bullet} A) = A[u, t]/(ut - d)$ with t being the degree 1 Rees parameter and u having degree -1; this description shows that $\mathcal{R}(I^{\bullet})$ is always a regular stack[68]. More canonically, given a p-nilpotent ring S, a point $x \in \mathcal{R}(I^{\bullet})(S)$ is given by a map $A \to S$ that kills some power of I, a line bundle $L \in \mathcal{P}\mathrm{ic}(S)$, and a factorization $I \otimes_A S \xrightarrow{u} L \xrightarrow{t} S$ of the canonical map. From this interpretation as well as the δ-structure on A to get a δ-A-algebra structure on the Witt ring scheme W over S, one can

[68] The ring $B := A[u, t]/(ut - d)$ contains a nonzerodivisor u such that $B/u = A[t]/(d) = R[t]$ is a regular ring, whence B must be a regular ring.

construct the following diagram of W-modules over S, as in Example 5.5.6:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_A \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dd, bend right, "\mathrm{can}"'] & I \otimes_A W \arrow[r, "\mathrm{id}\otimes F"] \arrow[d] \arrow[dd, bend right, "\mathrm{can}"'] & I \otimes_A F_*W \arrow[r] \arrow[d, equal] \arrow[dd, bend right, "\mathrm{can}"'] & 0 \\
0 \arrow[r] & \mathbf{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp"] & M_u \arrow[r] \arrow[d, "d_{u,t}"] & I \otimes_A F_*W \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"] & F_*W \arrow[r] & 0.
\end{tikzcd}$$

Here each row is an admissible sequence for an admissible module, the map $d_{u,t}$ is a quasi-ideal representing an animated W-algebra stack W/M_u over $\mathrm{Spf}(S)$, and the map labelled can in the bottom right square is a Cartier–Witt divisor. Consequently, the map $d_{u,t}$ gives a point $\overline{\pi}(x)$ of $\mathbf{Z}_p^{\mathcal{N}}(S)$. Moreover, the middle column yields a map $R = A/I \to R\Gamma(\mathrm{Spf}(S), W/M_u)$ of animated rings (and even A-algebras), so the point $\overline{\pi}(x) \in \mathbf{Z}_p^{\mathcal{N}}(S)$ is naturally lifted to a point $\pi(x) \in R^{\mathcal{N}}(S)$. The assignment $x \mapsto \pi(x)$ then defines a morphism $\pi : \mathcal{R}(I^\bullet) \to R^{\mathcal{N}}$ of stacks. One can check this map is faithfully flat by reducing to the perfectoid case treated in Proposition 5.5.8[69]. Moreover, the map is adic in the sense of Remark 5.1.17; more precisely, the preimage of the locus $(B\mathbf{G}_m^\sharp)_{p=0} \subset \mathbf{Z}_p^{\mathbb{\Delta}}$ along the composite map $\mathcal{R}(I^\bullet) \to R^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{Z}_p^{\mathbb{\Delta}}$ is a closed substack of definition. It follows[70] that $\mathrm{Perf}(R^{\mathcal{N}})$ admits a "standard" t-structure whose heart $\mathrm{Coh}(R^{\mathcal{N}})$ is the abelian category of coherent sheaves, and such that the pullback functor $\mathrm{Perf}(R^{\mathcal{N}}) \to \mathrm{Perf}(\mathcal{R}(I^\bullet))$ is t-exact for any prism (A, I) with $A/I = R$. In particular, $\mathrm{Coh}(R^{\mathcal{N}})$ is a noetherian category.

Example 5.5.20 (Charts for $\mathcal{O}_K^{\mathcal{N}}$)**.** Let $K/\mathbf{Q}_p$ be a completely discretely valued extension with perfect residue field k. Let $\pi \in \mathcal{O}_K$ be a uniformizer. Attached to this choice, we have a BK prism (A, I) where $A = W(k)[\![x]\!]$, and $I = (E(x))$ is the kernel of the $W(k)$-algebra map $A \to \mathcal{O}_K$ determined by $x \mapsto \pi$. The construction in Remark 5.5.19 then gives a faithfully flat cover $\mathrm{Spf}(A[u,t]/(ut - E(x))) \to \mathcal{O}_K^{\mathcal{N}}$. Note that $A[u,t]/(ut - E(x))$ has Krull dimension 3, so one might imagine $\mathcal{O}_K^{\mathcal{N}}$ as analogous to a 3-manifold.

[69]If A_∞ denotes the (p, I)-completed colimit perfection of A, then $A \to A_\infty$ is a (p, I)-complete quasi-syntomic cover, whence $R \to R_\infty := A_\infty/IA_\infty$ is a quasi-syntomic cover as well. Proposition 5.5.8 implies that the analogously constructed map $\mathcal{R}(I^\bullet A_\infty) \to R_\infty^{\mathcal{N}}$ is an isomorphism. As the maps $\mathcal{R}(I^\bullet A_\infty) \to \mathcal{R}(I^\bullet)$ and $R_\infty^{\mathcal{N}} \to R^{\mathcal{N}}$ are both faithfully flat covers, the claim follows.

[70]Let us first explain an abstract statement. Fix a cosimplicial commutative ring $A^\bullet$ with a Cartesian cosimplicial ideal $J^\bullet \subset A^\bullet$. Assume that A^0 is noetherian and regular, each A^j is derived J^j-complete, and that each face map $A^i \to A^j$ is J^i-completely flat. In this situation, note that $\mathrm{Perf}(\mathrm{Spf}(A^\bullet)) \simeq \mathrm{Perf}(A^\bullet)$ by $J^\bullet$-completeness. Moreover, by regularity and noetherianness of A^0, the standard t-structure on $\mathcal{D}(A^0)$ restricts to a t-structure on $\mathrm{Perf}(A^0)$. We claim that $\mathcal{C} := \lim \mathrm{Perf}(A^\bullet)$ also admits a "standard" t-structure for which the projection functors $\mathcal{C} \to \mathrm{Perf}(A^i) \subset \mathcal{D}(A^i)$ are t-exact for all i (and, in particular, truncations in $\mathcal{D}(A^i)$ of objects in $\mathrm{Perf}(A^i)$ that lift to $\mathcal{C}$ also lie in $\mathrm{Perf}(A^i)$). To see this, it suffices to show that for any face map $A^0 \to A^i$, the pullback functor $\mathrm{Perf}(A^0) \to \mathrm{Perf}(A^i) \subset \mathcal{D}(A^i)$ preserves coconnectivity. But this follows as the map $A^0 \to A^i$ is genuinely flat by J^0-complete flatness and noetherianness of A^0 (see [Bha20, Lemma 5.15]).

To obtain the desired t-structure on $R^{\mathcal{N}}$, one applies the result in the previous paragraph by taking $\mathrm{Spf}(A^\bullet)$ to be the Cech nerve of $\mathrm{Spf}(\mathrm{Rees}(I^\bullet)) \to X^{\mathcal{N}}$ (where $J^\bullet \subset A^\bullet$ is the pullback of $(p, \mathcal{I}) \subset \mathcal{O}_{\mathbf{Z}_p^{\mathbb{\Delta}}}$, and thus gives an ideal of definition). The independence of choice of charts can be proven similarly by comparing charts.

Chapter 6

Syntomification and duality

In this chapter, we study[71] syntomification and F-gauges in mixed characteristic, analogous to what Chapter 4 accomplished in positive characteristic.

To each bounded p-adic formal scheme X, we functorially attach a stack X^{Syn} called the *syntomification of* X; the category of F-gauges on X is defined to $\mathcal{D}_{qc}(X^{\mathrm{Syn}})$. The subcategory $\mathrm{Perf}(X^{\mathrm{Syn}})$ can be regarded as a universal home for (derived) p-adic local systems on X: besides being stable under proper smooth pushforward and arbitrary pullback, it admits (pushforward/pullback compatible) realization functors to more classical notions of p-adic local systems, e.g., one has $T_{dR}(-)$ towards Hodge-filtered de Rham local systems on X (Construction 5.3.13), $T_{crys}(-)$ towards crystalline local systems on $X_{p=0}$ (Remark 6.1.5), and $T_{et}(-)$ towards étale local systems on the rigid generic fibre X_η (Construction 6.3.2). It is the appearance of the generic fibre X_η that makes the mixed characteristic story both more subtle and more interesting than the characteristic p case from Chapter 4.

Beyond the basic definitions, our main focus in this chapter is on F-gauges over $\mathrm{Spf}(\mathbf{Z}_p)$ as well as their cohomology, especially in relation to Galois representations of the absolute Galois group $G_{\mathbf{Q}_p}$. We shall prove the following statements[72]:

1. *Relationship to crystalline Galois representations:* For any coherent sheaf $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$, the étale realization $T_{et}(M)$, regarded as a continuous representation of $G_{\mathbf{Q}_p}$ on a finitely generated $\mathbf{Z}_p$-module, is crystalline on inverting p (Corollary 6.7.2). Moreover, this construction establishes an equivalence between reflexive coherent sheaves on $\mathbf{Z}_p^{\mathrm{Syn}}$ and $\mathbf{Z}_p$-lattices in crystalline $G_{\mathbf{Q}_p}$-representations (Theorem 6.6.13).

   The proofs of these statements rely critically on the equivalence is [BS21] and quasi-syntomic descent; we also need an analysis of the kernel of the étale realization functor (Remark 6.3.5).

2. *Lagrangian refinement of Tate duality:* The functor $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, -)$ carries $\mathrm{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})$ to $\mathrm{Perf}(\mathbf{Z}_p)$ (Proposition 6.4.3). Moreover, for any $M \in \mathrm{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})$, there is a natural fibre sequence

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, M) \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(M)) \to \mathrm{RHom}_{\mathbf{Z}_p}(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, M^*\{1\}[2]), \mathbf{Z}_p)$$

[71] Essentially all the material in this chapter comes from [Dri20, BL]. In particular, the definition of the syntomification X^{Syn} as well as the geometry of the reduced locus $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}} \subset \mathbf{Z}_p^{\mathrm{Syn}}$ is due to Drinfeld [Dri20]. The results on coherent cohomology on these stacks will appear in [BL]. The relationship between reflexive coherent sheaves and crystalline Galois representations (Theorem 6.6.13) was discovered thanks to conversations with Toby Gee and Peter Scholze, who arrived at the statement in the context of their joint work with George Boxer and Matt Emerton.

[72] Implicit in all of these statements is that the stack $\mathbf{Z}_p^{\mathrm{Syn}}$ admits a reasonable theory of coherent sheaves and perfect complexes, analogous to a noetherian regular scheme, via Remark 5.5.19.

of perfect $\mathbf{Z}_p$-complexes, where M^* denotes the $\mathcal{O}$-linear dual, the first map is induced by the étale realization functor for M, while the second map is induced by the same functor for M^* as well classical local Tate duality for $T_{et}(M)$ (Corollary 6.5.22. We regard this theorem as an analog for $\mathbf{Z}_p^{\mathrm{Syn}}$ of the Serre duality for $\mathbf{F}_p^{\mathrm{Syn}}$ established in Theorem 4.5.2 (see Remark 6.5.24). Moreover, the fibre sequence above is deduced from a stronger statement for mod p F-gauges M': we formulate and prove purely geometrically a filtered refinement of the classical local Tate duality for $T_{et}(M')$ (Theorem 6.4.4).

The primary tool for proving these statements is a divisor $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}} \subset (\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$, whose geometry is described in §6.2 following [Dri20]. We use this this description in §6.5 to obtain a concrete linear algebraic description of quasi-coherent sheaves on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$, which is then used to prove our main theorems. The picture can be summarized as follows: quasi-coherent sheaf theory on the divisor $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ captures "mod p Sen theory", while quasi-coherent sheaf theory on the complement of this divisor captures the theory of mod p Galois representations (Remark 6.3.6), whence the total space $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$ provides a mechanism to understand certain Galois representations via (the typically much more explicit and linear algebraic) Sen theory.

3. *Relation to the Bloch–Kato Selmer group:* For any coherent sheaf $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$, the comparison map

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, M) \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(M))$$

is injective on $H^*(-)$ after inverting p (but not before), with the image given by the Bloch–Kato Selmer groups $H^*_f(G_{\mathbf{Q}_p}, -)$ from [BK07, §3] (Proposition 6.7.3, Remark 6.7.4).

These statements are deduced from those in (1) and (2).

6.1 Syntomification

We build the syntomification in mixed characteristic exactly as in characteristic p: by glueing two copies of the prismatization inside the filtered prismatization together.

Definition 6.1.1 (The syntomification and F-gauges)**.** Fix a bounded p-adic formal scheme X. The stack $X^{\mathcal{N}}$ contains two open substacks isomorphic to X^{Δ} as in Remark 3.4.2. Write X^{Syn} for the result of glueing these two open substacks together, i.e., we have a pushout square

$$\begin{tikzcd} X^{\Delta} \sqcup X^{\Delta} \arrow[r, "j_{HT}, j_{dR}"] \arrow[d, "\mathrm{can}"] & X^{\mathcal{N}} \arrow[d, "j_{\mathcal{N}}"] \\ X^{\Delta} \arrow[r, "j_{\Delta}"] & X^{\mathrm{Syn}} \end{tikzcd} \tag{6.1.1}$$

where all maps are étale (and even local isomorphisms). We then make the following definitions:

1. The stack X^{Syn} is called the *syntomification of* X.
2. The category $\text{F-Gauge}_{\Delta}(X) := \mathcal{D}_{qc}(X^{\mathrm{Syn}})$ is called *the category of prismatic F-gauges on X*.
3. Pullback along the étale cover $j_{\mathcal{N}}^*$ induces a functor $j_{\mathcal{N}}^* : \text{F-Gauge}_{\Delta}(X) \to \mathrm{Gauge}_{\Delta}(X)$ that we refer to as *passage to the underlying gauge.*

Example 6.1.2 (The structure F-gauge)**.** As in Example 5.3.12, given a quasi-syntomic qcqs p-adic formal scheme X with structure map $\pi : X^{\mathrm{Syn}} \to \mathbf{Z}_p^{\mathrm{Syn}}$, we obtain an F-gauge

$$\mathcal{H}_{\mathrm{Syn}}(X) := R\pi_* \mathcal{O}_{X^{\mathrm{Syn}}} \in \mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}})$$

on $\mathbf{Z}_p$ that we call the *structure F-gauge* of X; note that the underlying gauge of $\mathcal{H}_{\mathrm{Syn}}(X)$ is the structure gauge $\mathcal{H}_{\mathcal{N}}(X)$.

Remark 6.1.3 (F-gauges and their cohomology)**.** The discussion in Remark 4.2.5 transports to the present setting. In particular, we have an equalizer diagram

$$\mathcal{D}_{qc}(X^{\mathrm{Syn}}) \simeq \mathrm{Eq}(\mathcal{D}_{qc}(X^{\mathcal{N}}) \underset{j_{dR}^*}{\overset{j_{HT}^*}{\rightrightarrows}} \mathcal{D}_{qc}(X^{\Delta}))$$

of stable ∞-categories. Moreover, if $E \in \mathcal{D}_{qc}(X^{\mathrm{Syn}})$ be an F-gauge on X, then there is a natural triangle

$$R\Gamma(X^{\mathrm{Syn}}, E) \to R\Gamma(X^{\mathcal{N}}, E|_{X^{\mathcal{N}}}) \xrightarrow{j_{HT}^* - j_{dR}^*} R\Gamma(X^{\Delta}, E|_{X^{\Delta}}) \tag{6.1.2}$$

where we abusively write $E|_{X^{\mathcal{N}}} = j_{\mathcal{N}}^* E$ and $E|_{X^{\Delta}} = j_{\Delta}^* E$.

Remark 6.1.4 (F-gauges in characteristic p)**.** Say X is a smooth scheme over a perfect field k of characteristic p. Then the notions from Definition 6.1.1 coincide with those studied in Chapter 4 thanks to Proposition 5.4.2 and transmutation. In particular, we have not overloaded the notation.

Remark 6.1.5 (The crystalline realization)**.** Given any $\mathbf{Z}_p$-flat p-adic formal scheme X, functoriality applied to $X_{p=0} \hookrightarrow X$ yields a pullback functor $T_{crys}(-) : \mathrm{F\text{-}Gauge}_{\Delta}(X) \to \mathrm{F\text{-}Gauge}_{\Delta}(X_{p=0})$ that we refer to as the *crystalline realization functor.*

Example 6.1.6 (F-gauges over a perfectoid)**.** Say R is a perfectoid ring with corresponding perfect prism (Δ_R, I). Let $d \in I$ be a generator. Then $\mathrm{Gauge}_{\Delta}(R)$ identifies with $\mathcal{DF}_{(p,I)\text{-comp}}(\phi^{-1}(I)^{\bullet}\Delta_R) \simeq \mathcal{D}_{\mathrm{gr}}(\Delta_R[u,t]/(ut - \varphi^{-1}(d))$ thanks to Proposition 5.5.8 as well as the explicit description in Example 5.5.6. As in characteristic p (Remark 3.4.6), we can picture an object M in this category as a diagram

$$\cdots \underset{u}{\overset{t}{\rightleftarrows}} M^{i+1} \underset{u}{\overset{t}{\rightleftarrows}} M^{i} \underset{u}{\overset{t}{\rightleftarrows}} M^{i-1} \underset{u}{\overset{t}{\rightleftarrows}} \cdots \tag{6.1.3}$$

of (p,I)-complete Δ_R-complexes, where $\varphi^{-1}(d) = ut = tu$. Writing $M^{-\infty} = \mathrm{colim}\, M^{-i}$ for the $((p,I)$-completed) colimit along the t maps, and $M^{\infty} = \mathrm{colim}_i\, M^i$ for the ((p,I)-completed) colimit along the u maps, we obtain a correspondence

$$M^{\infty} \xleftarrow{u^{\infty}} M^0 \xrightarrow{t^{\infty}} M^{-\infty}$$

where both maps are $\varphi^{-1}(d)$-isogenies if M is in fact a perfect complex over $R^{\mathcal{N}}$. More conceptually, we have $M^{-\infty} = j_{dR}^* M$ and $\varphi^* M^{\infty} = j_{HT}^* M$. In particular, lifting $M \in \mathcal{D}_{qc}(R^{\mathcal{N}})$ to an object $\widetilde{M} \in \mathcal{D}_{qc}(R^{\mathrm{Syn}})$ amounts to specifiying an isomorphism $\tau : \varphi^*(M^{\infty}) \simeq M^{-\infty}$; the complex $R\Gamma(R^{\mathrm{Syn}}, \widetilde{M})$ is then computed by the fibre of $t^{\infty} - \tau\phi u^{\infty} : M^0 \to M^{-\infty}$, as in Remark 4.2.8.

Example 6.1.7 (F-gauges over a qrsp ring)**.** Fix a qrsp ring R and an F-gauge $E \in \mathcal{D}_{qc}(R^{\mathrm{Syn}})$; let us explain the concrete data extracted from E. First, we obtain $M(E) = E|_{R^{\Delta}} \in \mathcal{D}_{(p,I)\text{-comp}}(\Delta_R)$, the underlying prismatic crystal. The restriction $E|_{R^{\mathcal{N}}}$ can be viewed as a filtration $\mathrm{Fil}^{\bullet} M(E) \in \mathcal{DF}_{(p,I)\text{-comp}}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}\Delta_R)$ on $M(E)$, and the F-gauge structure on E gives a map $\widetilde{\varphi}_{M(E)} : \mathrm{Fil}^{\bullet} M(E) \to I^{\bullet} M := I^{\bullet}\Delta_R \otimes_{\Delta_R} M$ linear over the Frobenius map $\mathrm{Fil}^{\bullet}_{\mathcal{N}}\Delta_R \to I^{\bullet}\Delta_R$ (where $I^{\bullet}\Delta_R$ denotes the full $\mathbf{Z}$-indexed I-adic filtration on $\Delta_R[1/I]$) such that the linearized map

$$\mathrm{Fil}^{\bullet} M \widehat{\otimes}_{\mathrm{Fil}^{\bullet}_{\mathcal{N}}\Delta_R} I^{\bullet}\Delta_R \to I^{\bullet} M$$

is an isomorphism in $\mathcal{DF}_{(p,I)\text{-comp}}(I^{\bullet}\Delta_R) \simeq \mathcal{D}_{(p,I)\text{-comp}}(\Delta_R)$. Conversely, the filtered object $\mathrm{Fil}^{\bullet} M(E) \in \mathcal{DF}_{(p,I)\text{-comp}}(\mathrm{Fil}^{\bullet}_{\mathcal{N}}\Delta_R)$ recovers the gauge $E|_{R^{\mathcal{N}}}$ via the Rees construction, and then the entire triple $(M(E), \mathrm{Fil}^{\bullet} M(E), \widetilde{\varphi}_{M(E)})$ provides the relevant descent data needed to reconstruct the F-gauge E. In fact, the ∞-category of such triples identifies with $\mathcal{D}_{qc}(R^{\mathrm{Syn}})$. In the future, we shall typically represent an object of $\mathcal{D}_{qc}(R^{\mathrm{Syn}})$ by such a triple.

Example 6.1.8 (The BK twist as an F-gauge)**.** Recall from Remark 5.5.15 that we have

$$\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} = \pi^*\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \otimes t^*\mathcal{O}(-1) \in \mathcal{P}\mathrm{ic}(\mathbf{Z}_p^{\mathcal{N}}).$$

We shall construct a natural isomorphism $\tau : j_{dR}^*\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} \simeq j_{HT}^*\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\}$, so the pair $(\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\}, \tau)$ yields an F-gauge $\mathcal{O}_{\mathbf{Z}_p^{\mathrm{Syn}}}\{1\} \in \mathcal{P}\mathrm{ic}(\mathbf{Z}_p^{\mathrm{Syn}})$ via descent. To construct τ, we first claim that applying j_{dR}^* to the above formula defining $\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\}$ gives

$$j_{dR}^*\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} = \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}.$$

Indeed, this follows as $\pi \circ j_{dR} = \mathrm{id}_{\mathbf{Z}_p^{\Delta}}$ and because $t \circ j_{dR} : \mathbf{Z}_p^{\Delta} \to \mathbf{A}^1/\mathbf{G}_m$ factors over $\mathbf{G}_m/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$. On the other hand, applying j_{HT}^* shows that

$$j_{HT}^*\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} = F^*\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \otimes j_{HT}^*t^*\mathcal{O}(-1),$$

where $F = \pi \circ j_{HT} : \mathbf{Z}_p^{\Delta} \to \mathbf{Z}_p^{\Delta}$ is the Frobenius (Remark 5.1.10). The definition of BK twists over $\mathbf{Z}_p^{\Delta}$ (Remark 5.1.19) provides a natural identification $F^*\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \simeq \mathcal{I}^{-1} \otimes \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}$, where $\mathcal{I} \subset \mathcal{O}_{\mathbf{Z}_p^{\Delta}}$ is the Hodge–Tate ideal sheaf; moreover, we have $j_{HT}^*t^*\mathcal{O}(-1) = \mathcal{I}$ as $t \circ j_{HT} : \mathbf{Z}_p^{\Delta} \to \mathbf{A}^1/\mathbf{G}_m$ classifies the Cartier divisor $\mathbf{Z}_p^{HT} \subset \mathbf{Z}_p^{\Delta}$ by the functor of points description. Thus the above simplifies to give a natural identification

$$j_{HT}^*\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} \simeq \mathcal{I}^{-1} \otimes \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \otimes \mathcal{I} \simeq \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\},$$

as desired.

One can prove several stability properties of the construction $X \mapsto X^{\mathrm{Syn}}$ by reduction to statements about prismatic cohomology via the comparison in Remark 5.5.18, e.g., if $f : X \to Y$ is a proper smooth map of quasi-syntomic p-adic formal schemes, then $Rf_*^{\mathrm{Syn}} : \mathcal{D}_{qc}(X^{\mathrm{Syn}}) \to \mathcal{D}_{qc}(Y^{\mathrm{Syn}})$ preserves perfectness. As these follow relatively standard arguments, we do not elaborate on them here; instead, in the rest of the chapter, we focus on geometry of the stack $\mathbf{Z}_p^{\mathrm{Syn}}$ and the cohomology of F-gauges.

6.2 The reduced locus

In this section, we will always work with mod p coefficients.

The basic tool in understanding the geometry of $\mathbf{Z}_p^{\mathrm{Syn}}$ (and thus in understanding F-gauges and their cohomology) is the reduced locus $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$: this is a certain divisor inside $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$ that can be described very explicitly. The goal of this section is to construct this divisor, and recall the explicit geometric description coming from [Dri20]. First, let us explain where the divisor (or, rather, its defining equation) comes from:

Construction 6.2.1 (The section v_1 via quasi-syntomic descent)**.** We shall define a natural nonzero section

$$v_1 \in H^0(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{p-1\}/p).$$

Its pullback to X^{Syn} for any quasi-syntomic p-adic formal scheme X will be denoted $v_{1,X}$. In fact, to define v_1 by quasi-syntomic descent, it suffices to construct, for each p-torsionfree qrsp ring R with prism (Δ_R, I), a nonzero section $v_{1,R} \in H^0(R^{\mathrm{Syn}}, \mathcal{O}\{p-1\}/p)$ compatible with maps between such qrsp rings. To construct this section, we must define a nonzero section $\widetilde{v_{1,R}} \in$

$H^0(R^{\mathcal{N}}, \mathcal{O}\{p-1\}/p)$ satisfying a compatibility under the Frobenius. Thanks to Remark 5.5.15, we have $H^0(R^{\mathcal{N}}, \mathcal{O}\{p-1\}/p) = \mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R\{p-1\}/p$. We first claim that there is a natural identification

$$\Delta_R\{p-1\}/p = I^{-1}/p$$

as line bundles in $\mathcal{P}\mathrm{ic}(\Delta_R)$: heuristically, this follows from the "formula"

$$\Delta_R\{1\}/p = \bigotimes_{j\geq 0}(\varphi^j)^* I/p = \bigotimes_{i\geq 0}(I/p)^{\otimes p^j}$$

and the geometric series formula $(p-1)(1+p+p^2+...) = \frac{p-1}{1-p} = -1$. Thanks to this formula, to construct $\widetilde{v_{1,R}}$, it suffices to define a natural nonzero map $I/p \to \mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R/p$ of Δ_R-modules. But we claim that there is a natural inclusion

$$\theta : I/p \subset \mathrm{Fil}_{\mathcal{N}}^{p}\Delta_R/p$$

of submodules of Δ_R. Indeed, as R is p-torsionfree, we have $\mathrm{Fil}_{\mathcal{N}}^{i}\Delta_R/p = \varphi^{-1}(I^i/p)$ as submodules of Δ_R/p[73], which then clearly implies that $I/p \subset \mathrm{Fil}_{\mathcal{N}}^{p}\Delta_R$. Thus, we have constructed the promised nonzero map

$$I/p \overset{\theta}{\subset} \mathrm{Fil}_{\mathcal{N}}^{p}\Delta_R/p \subset \mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R/p,$$

yielding the nonzero section $\widetilde{v_{1,R}} \in H^0(R^{\mathcal{N}}, \mathcal{O}\{p-1\}/p)$. Using the formula $\phi(d)/d^{p-1} = d \mod p\Delta_R$, one can check that this section satisfies the desired compatibility, yielding the promised nonzero section $v_{1,R} \in H^0(R^{\mathrm{Syn}}, \mathcal{O}\{p-1\}/p)$.

Remark 6.2.2 (Other constructions of v_1)**.** An alternative and purely moduli-theoretic construction of a nonzero section in $H^0(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{p-1\}/p)$ is provided by [Dri20, §5.10.8 and Proposition 8.11.2]; one can show this section agrees with v_1 though we do not verify this here (but we shall use it). Another construction, this time via the prismatic logarithm, was provided in [BM22, Construction 2.7]; the element $\widetilde{\theta}$ in [BM22, Construction 2.8] corresponds to the map $\theta : I/p \to \mathrm{Fil}_{\mathcal{N}}^{p}\Delta_R/p$ in Construction 6.2.1 (see [BM22, Remark 2.10]). Yet another construction comes from homotopy theory and was in fact the original motivation for considering this element in [BM22] as well as its name (see [BM22, paragraph after Construction 2.7]).

Remark 6.2.3 (Explicitly identifying v_1 via distinguished elements)**.** Say R is a perfectoid ring with corresponding perfect prism (Δ_R, I); fix a generator d. We can then identify

$$R^{\mathcal{N}} = \mathrm{Spf}(\Delta_R[u,t]/(ut - \phi^{-1}(d)))/\mathbf{G}_m$$

as in Proposition 5.5.8. In this description, the section $\widetilde{v_{1,R}} \in H^0(R^{\mathcal{N}}, \mathcal{O}\{p-1\}/p)$ from Construction 6.2.1 corresponds to the degree $(p-1)$ element

$$u^p t = dt^{-p+1} \in \mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R/p \cdot t^{-p+1} \subset \Delta_R[u,t]/(ut - \phi^{-1}(d), p),$$

where we use the choice of d to identify $\mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R\{p-1\}/p = \mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R \otimes I^{-1}/p$ with $\mathrm{Fil}_{\mathcal{N}}^{p-1}\Delta_R/p$. The formulae $v_1 = u^p t$ and $d^{1/p} = ut$ in Δ_R/p show that the open subset $\mathrm{Spec}(\mathrm{Fil}_{\mathcal{N}}^{\bullet}\Delta_R/p)_{v_1\neq 0}$ coincides with the open subset $\mathrm{Spec}(\mathrm{Fil}_{\mathcal{N}}^{\bullet}\Delta_R/p)_{d\neq 0}$.

[73] More precisely, as R is p-torsionfree, the filtered Frobenius $\mathrm{Fil}_{\mathcal{N}}^{\bullet}\Delta_R \to I^{\bullet}$ induces, for each integer n, a map $\mathrm{gr}_{\mathcal{N}}^{n}\Delta_R \to I^n/I^{n+1}$ that is a map of p-torsionfree R-modules that is injective and remains injective modulo p; this implies by induction on i that $\mathrm{Fil}_{\mathcal{N}}^{i}\Delta_R/p \subset \Delta_R/p = \varphi^{-1}(I^i/p)$.

Definition 6.2.4 (The reduced locus of $\mathbf{Z}_p^{\mathrm{Syn}}$ and $\mathbf{Z}_p^{\mathcal{N}}$)**.** The *reduced locus*[74] $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}} \subset \mathbf{Z}_p^{\mathrm{Syn}}$ is defined to be the common zero locus $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0,v_1=0}$ of (p, v_1); the pullback of this closed substack to $\mathbf{Z}_p^{\mathcal{N}}$ is called the *reduced locus* of $\mathbf{Z}_p^{\mathcal{N}}$ and denoted $\mathbf{Z}_{p,\mathrm{red}}^{\mathcal{N}}$. For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, we write

$$E/v_1 = \mathrm{Cone}\left(E\{-(p-1)\} \xrightarrow{v_1} E\right)$$

for the restriction of E to $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$.

Remark 6.2.5 (The geometry of the reduced locus)**.** Drinfeld [Dri20, §7] has given a concrete algebro-geometric description of the stack $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$; we recall next the key elements of this description. First, to describe the stack $\mathbf{Z}_{p,\mathrm{red}}^{\mathcal{N}}$, we need the following two components:

1. The Hodge–Tate component $D_{HT} := (\mathbf{Z}_p^{\mathcal{N}})_{p=t=0}$: this is the closed substack of $(\mathbf{Z}_p^{\mathcal{N}})_{p=0}$ cut out by $t = 0$, as the notation suggests. As in Proposition 5.3.7, this component can be described as $\mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$ and thus has the following stratification:
   - An open substack $D_{und} := B\mathrm{Stab}_{\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m}(1 \in \mathbf{G}_a) \simeq B\mathbf{G}_m^\sharp = (\mathbf{Z}_p^{HT})_{p=0}$. This substack can be conceptually realized as $j_{HT}(\mathbf{Z}_p^{\Delta}) \cap (\mathbf{Z}_p^{\mathcal{N}})_{\mathrm{red}}$.
   - A closed substack $D_{Hod} := \alpha_p/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m) \simeq B(F_*\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$, which is the reduced complement of the open substack mentioned in the previous item.

   Remark 6.2.6. Note that D_{HT} is the mod p reduction of the $\mathbf{Z}_p$-flat stack $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}$, which can be described as $\mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$ over $\mathrm{Spf}(\mathbf{Z}_p)$ by Proposition 5.3.7. However, the closed substack $D_{Hod} \subset D_{HT}$ described above does not lift to a $\mathbf{Z}_p$-flat closed substack of $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}$.

2. The de Rham component $D_{dR} := (\mathbf{Z}_p^{\mathcal{N}})_{p=u^p=0}$: this is the closed substack of $(\mathbf{Z}_p^{\mathcal{N}})_{p=0}$ parametrizing filtered Cartier–Witt divisors $d : M \to W$ where M is locally split (i.e., the admissible sequence $(*_M)$ is locally split, see Remark 5.2.5). To describe this component concretely, note that we have the Rees map $t_{dR} : D_{dR} \subset (\mathbf{Z}_p^{\mathcal{N}})_{p=0} \to \mathbf{A}^1/\mathbf{G}_m$ as well its left inverse provided by the de Rham map $i_{dR} : \mathbf{A}^1/\mathbf{G}_m \to D_{dR}$ from Construction 5.3.13. It is relatively straightforward to see that i_{dR} is an fpqc cover; this implies that the map i_{dR} identifies the map $t_{dR} : D_{dR} \to \mathbf{A}^1/\mathbf{G}_m$ as the structure map for the classifying stack $B_{\mathbf{A}^1/\mathbf{G}_m}\mathcal{G}$ of the group scheme $\mathcal{G} \to \mathbf{A}^1/\mathbf{G}_m$ given by $\mathrm{Aut}(i_{dR})$. Drinfeld computes (see [Dri20, Corollary 7.5.5 (ii)]) that $\mathcal{G}$ is the "$[-t^p]$-rescaled" degeneration of $\mathbf{G}_m^\sharp$ to $\mathbf{G}_a^\sharp$; more canonically, we can realize $\mathcal{G}$ as an open subscheme of $\mathbf{V}(\mathcal{O}(-p))^\sharp$, endowed with a group law that degenerates multiplication to addition (see [Dri20, §7.4.3]). In particular, this stack admits the following stratification:
   - An open substack $D_{und} := B\mathbf{G}_m^\sharp$ given as the fibre of $B_{\mathbf{A}^1/\mathbf{G}_m}\mathcal{G} \to \mathbf{A}^1/\mathbf{G}_m$ over $\mathbf{G}_m/\mathbf{G}_m$. In fact, this identifies with $j_{dR}(\mathbf{Z}_p^{\Delta}) \cap (\mathbf{Z}_p^{\mathcal{N}})_{\mathrm{red}}$.
   - The closed substack $D_{Hod} := \left(B_{\mathbf{A}^1/\mathbf{G}_m}\mathcal{G}\right)|_{t=0} \simeq B(F_*\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$ given as the fibre over $B\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m$; this is also the reduced complement of the open substack mentioned in the previous item.

   Remark 6.2.7. Unlike the Hodge–Tate component, the substack $D_{dR} \subset (\mathbf{Z}_p^{\mathcal{N}})_{p=0}$ does not admit a flat lift to a closed substack of $\mathbf{Z}_p^{\mathcal{N}}$.

[74]This terminology can be justified by noticing that v_1 is a formal parameter on $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$, and that $\mathbf{Z}_{p,\mathrm{red}}$ is reduced.

To obtain $\mathbf{Z}_{p,\mathrm{red}}^{\mathcal{N}}$, we glue the two stacks D_{HT} and D_{dR} transversely along the closed substack D_{Hod} described above for each component; thus, $\mathbf{Z}_{p,\mathrm{red}}^{\mathcal{N}}$ has two irreducible components given by Hodge–Tate and de Rham components respectively, and the complement of the common intersection in each component is the copy of D_{und} described above.

The stack $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ is obtained $\mathbf{Z}_{p,\mathrm{red}}^{\mathcal{N}}$ by identifying the two copies of D_{und} mentioned above with each other. Thus, the stack $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ admits an open substack identified with D_{und} whose reduced closed complement is given by D_{Hod}

6.3 From prismatic F-gauges to Galois representations

In this section, using the explicit description of the filtered prismatization given in Chapter 5, we explain why prismatic F-gauges in perfect complexes over a quasi-syntomic p-adic formal scheme X naturally give rise to local systems on the rigid generic fibre X_η of X. Essentially, the idea is that such prismatic F-gauges give prismatic F-crystals, which then give local systems via a standard construction. To explain it more quantitatively, it is convenient to proceed via quasi-syntomic descent, and the key qrsp case is the following:

Construction 6.3.1 (From prismatic F-gauges to local systems for qrsp rings)**.** Let R be a p-torsionfree qrsp ring with corresponding prism $(\mathbbl{\Delta}_R, I)$. Write $\mathcal{D}^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbf{Z}_p)$ for the full subcategory of $\mathcal{D}(\mathrm{Spa}(R[1/p], R)_{proet}, \mathbf{Z}_p)$ spanned by bounded complexes with lisse cohomology; this can also be defined algebraically exactly the same way by replacing $\mathrm{Spa}(R[1/p], R)$ by $\mathrm{Spec}(R[1/p])$. We shall construct a $\mathbf{Z}_p$-linear symmetric monoidal *étale realization* functor

$$T_{et} : \mathrm{Perf}(R^{\mathrm{Syn}}) \to \mathcal{D}^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbf{Z}_p).$$

In fact, to construct this functor, it would suffice to assume R is perfectoid (via p-complete arc descent for the target when regarded as a functor of R); nonetheless, we prefer to work in the qrsp generality as some intermediate steps yield stronger results.

For any fixed integer $n \geq 1$, consider the diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbbl{\Delta}_R/p^n))/\mathbf{G}_m & \mathrm{Spec}(\mathbbl{\Delta}_R/p^n) \arrow[l, shift right, "j_{HT}"'] \arrow[l, shift left, "j_{dR}"] \\
\mathrm{Spec}(\mathbbl{\Delta}_R/p^n[1/I]) \arrow[u] & \mathrm{Spec}(\mathbbl{\Delta}_R/p^n[1/I]) \arrow[u] \arrow[l, shift right, "\varphi"'] \arrow[l, shift left, "1"]
\end{tikzcd}$$

Here each row can be regarded as a coequalizer diagram; the vertical arrows provide a map between coequalizer diagrams and are themselves open immersions obtained by pulling back the open immersion $\mathrm{Spec}(\mathbbl{\Delta}_R/p^n)_{I\neq 0} \subset \mathrm{Spec}(\mathbbl{\Delta}_R/p^n)$ along the structure map $\mathrm{Spec}(\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbbl{\Delta}_R/p^n))/\mathbf{G}_m \to \mathrm{Spec}(\mathbbl{\Delta}_R/p^n)$ (equivalently, by the last sentence of Remark 6.2.3, we simply invert (a lift of) v_1 to obtain the vertical arrows). These diagrams are compatible as n changes. Applying $\mathrm{Perf}(-)$, using the algebraization equivalence

$$\mathrm{Perf}(\mathrm{Spec}(\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbbl{\Delta}_R/p^n))/\mathbf{G}_m) \simeq \mathrm{Perf}((R^{\mathcal{N}})_{p^n=0}),$$

and taking a limit over n then gives a diagram

$$\begin{tikzcd}
\mathrm{Perf}(R^{\mathcal{N}}) \arrow[r, shift left, "j^*_{HT}"] \arrow[r, shift right, "j^*_{dR}"'] \arrow[d] & \mathrm{Perf}(R^{\mathbbl{\Delta}}) \arrow[d] \\
\mathrm{Perf}(\mathbbl{\Delta}_R[1/I]^\wedge) \arrow[r, shift left, "\varphi^*"] \arrow[r, shift right, "1"'] & \mathrm{Perf}(\mathbbl{\Delta}_R[1/I]^\wedge),
\end{tikzcd}$$

where the rows are equalizer diagrams, while the vertical maps provide a morphism of equalizer diagrams and are obtained by algebraizing and inverting (a lift of) v_1 modulo any power of p. Passing to equalizers, we obtain a map

$$T : \mathrm{Perf}(R^{\mathrm{Syn}}) \to \mathrm{Perf}(\Delta_R[1/I]^\wedge)^{\varphi=1}.$$

The target was called the ∞-category of *Laurent F-crystals* in [BS21]. Moreover, a variant of Artin-Schreier theory (see [BS21, Corollary 3.7] or [Wu21]) shows that the functor of taking Frobenius fixed points gives a symmetric monoidal $\mathbf{Z}_p$-linear identification

$$\mathrm{Perf}(\Delta_R[1/I]^\wedge)^{\varphi=1} \simeq \mathcal{D}^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbf{Z}_p),$$

so we can regard T as a symmetric monoidal $\mathbf{Z}_p$-linear functor

$$T_{et} : \mathrm{Perf}(R^{\mathrm{Syn}}) \to \mathcal{D}^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbf{Z}_p).$$

We refer to this functor as *the étale realization* functor for prismatic F-gauges.

Once we have the étale realization over qrsp rings (or just perfectoid rings), it is relatively formal to extend to the general case:

Construction 6.3.2 (The étale realization in general)**.** Let X be a bounded p-adic formal scheme. Let $X_{\Delta,\mathrm{perf}}$ be the perfect prismatic site of X, i.e., the (opposite of the) category of perfectoid rings R equipped with a map $\mathrm{Spf}(R) \to X$. By pullback and Construction 6.3.1, we have a natural map

$$\mathrm{Perf}(X^{\mathrm{Syn}}) \to \lim_{R \in X_{\Delta,\mathrm{perf}}} \mathrm{Perf}(R^{\mathrm{Syn}}) \to \lim_{R \in X_{\Delta,\mathrm{perf}}} \mathcal{D}^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbf{Z}_p).$$

Let X_η denote the generic fibre of X, regarded as a pro-étale sheaf on the category of all affinoid perfectoid spaces over $\mathbf{Q}_p$. One thus has a natural pullback map

$$\mathcal{D}^b_{lisse}(X_\eta, \mathbf{Z}_p) \to \lim_{R \in X_{\Delta,\mathrm{perf}}} \mathcal{D}^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbf{Z}_p)$$

which can be shown to be an equivalence via p-complete arc descent. Composing the inverse of this equivalence with the previous functor then gives a natural $\mathbf{Z}_p$-linear functor

$$T_{et} : \mathrm{Perf}(X^{\mathrm{Syn}}) \to \mathcal{D}^b_{lisse}(X_\eta, \mathbf{Z}_p)$$

that we call *the étale realiztion functor* for prismatic F-gauges on X.

Example 6.3.3 (BK twists and Tate twists)**.** In this example, we work over $\mathbf{Z}_p^{\mathrm{Syn}}$; fix a completed algebraic closure $\mathbf{C}_p$ of $\mathbf{Q}_p$, and write $G_{\mathbf{Q}_p}$ for the corresponding absolute Galois group of $\mathbf{Q}_p$, so there is a natural continuous action of $G_{\mathbf{Q}_p}$ on $\mathbf{C}_p$. Then the étale realization functor

$$T_{et} : \mathrm{Perf}(\mathbf{Z}_p^{\mathrm{Syn}}) \to \mathcal{D}^b_{lisse}(\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p), \mathbf{Z}_p) \simeq D^b_{fg}(G_{\mathbf{Q}_p}, \mathbf{Z}_p)$$

takes values in the derived category of continuous representations of the absolute Galois group $G_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ on finitely generated $\mathbf{Z}_p$-modules. One can show that this functor carries the BK twist $\mathcal{O}_{\mathbf{Z}_p^{\mathrm{Syn}}}\{1\}$ to the Tate twist $\mathbf{Z}_p(1)$ and the map $v_1 : \mathcal{O}_{\mathbf{Z}_p^{\mathrm{Syn}}}/p \to \mathcal{O}_{\mathbf{Z}_p^{\mathrm{Syn}}}\{p-1\}/p$ to the natural isomorphism $\mathbf{Z}/p \simeq (\mathbf{Z}/p(1))^{\otimes p-1} \simeq \mathbf{Z}/p(p-1)$. In particular, the étale realization of v_1 is an isomorphism, and thus the étale realization functor is not faithful.

In the rest of this section, we make some remarks on the constructions appearing above; these are not necessary to our main results.

Remark 6.3.4 (From prismatic F-gauges to prismatic F-crystals)**.** Construction 6.3.1 gave a rather explicit local procedure to attach Laurent F-crystals to prismatic F-gauges. One can in fact also attach prismatic F-crystals to prismatic F-gauges directly globally, as we briefly explain next; this will not be seriously used in the sequel but the author finds it psychologically useful.

Let X be a quasi-syntomic scheme. Write $\mathrm{Perf}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta})$ for the ∞-category of prismatic F-crystals on X, i.e., pairs (E, φ_E), where $E \in \mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta})$ is a prismatic crystal in perfect complexes on X, and $\varphi_E : \varphi^* E[\frac{1}{\mathcal{I}}] \simeq E[\frac{1}{\mathcal{I}}]$ is an isomorphism in $\mathcal{D}(X_{\Delta}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}])$; the vector bundle variant of this notion was studied in [BS21][75]. We claim that there is a natural symmetric monoidal functor

$$G : \mathrm{Perf}(X^{\mathrm{Syn}}) \to \mathrm{Perf}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}).$$

Indeed, given any $M \in \mathrm{Perf}(X^{\mathcal{N}})$, we obtain a natural correspondence

$$j_{HT}^* M \xleftarrow{a} \phi^* \pi_* M \xrightarrow{b} \phi^* j_{dR}^* M$$

where b is defined by applying ϕ^* to the canonical map $\pi_* M \to j_{dR}^* M$ (arising via functoriality of pullback along $j_{dR} : X^{\Delta} \to X^{\mathcal{N}}$ as a map of stacks over X^{Δ}), while a is defined by adjunction from the analogous map $\pi_* M \to j_{HT}^* M$ as well as the observation that $j_{HT} \circ \pi$ equals the Frobenius φ on X^{Δ}. Since M is perfect, a local calculation by descent to the qrsp case shows that both a and b are $\mathcal{I}$-isogenies. If $M = j_{\mathcal{N}}^* N$ comes from an F-gauge $N \in \mathrm{Perf}(X^{\mathrm{Syn}})$, then we have canonical identifications $j_{HT}^* M \simeq j_{dR}^* M$, so inverting I in the diagram above provides the desired object $G(N) \in \mathrm{Perf}^{\varphi}(X_{\Delta})$.

Using the functor G, we obtain the composition

$$\mathrm{Perf}(X^{\mathrm{Syn}}) \xrightarrow{G} \mathrm{Perf}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{(-)[\frac{1}{\mathcal{I}}]^{\wedge}} \mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge})^{\varphi^*=1} \simeq \mathcal{D}^b_{lisse}(X_\eta, \mathbf{Z}_p)$$

which can be identified with the étale realization $T_{et} : \mathrm{Perf}(X^{\mathrm{Syn}}) \to \mathcal{D}^b_{lisse}(X_\eta, \mathbf{Z}_p)$.

Remark 6.3.5 (The kernel of the étale realization)**.** Example 6.3.3 gives an example of an $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$ is annihilated by T_{et}; this sheaf M was annihilated by (p, v_1) and it is clear from the construction of T_{et} (which involves inverting v_1 when working mod p) that any sheaf annihilated by (p, v_1) must be killed by $T_{et}(-)$. The converse is also true: any $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$ with $T_{et}(M) = 0$ must be annihilated by $(p, v_1)^n$ for $n \gg 0$, i.e., it is annihilated by p^n for $n \gg 0$, and that M/pM is annihilated by v_1^n for $n \gg 0$. We sketch why next (using [BMS18] and [BS21]). Thus, for the rest of this remark, fix $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$ annihilated by T_{et}.

First, we observe that M/pM is annihilated by v_1^n for $n \gg 0$: this can be checked after pullback to $\mathcal{O}_{\mathbf{C}_p}^{\mathrm{Syn}}$, and then it follows as the étale realization functor for mod p F-gauges on $\mathcal{O}_{\mathbf{C}_p}$ is constructed by inverting v_1 on $\mathcal{O}_{\mathbf{C}_p}^{\mathcal{N}}$ (see Construction 6.3.1). It thus remains to show that M is p-power torsion.

Next, we show that $M|_{\mathbf{Z}_p^{\Delta}}$ is p-power torsion. As $T_{et}(M) = 0$, the prismatic F-crystal $M|_{\mathbf{Z}_p^{\Delta}} \in \mathrm{Coh}^{\varphi}(\mathbf{Z}_p^{\Delta})$ (coming from the functor G in Remark 6.3.4) has vanishing étale realization. By the structure theory of such crystals (see [BMS18, Proposition 4.3]), we must have $p^n M|_{\mathbf{Z}_p^{\Delta}} = 0$ for $n \gg 0$, as desired.

[75]Note that we have used the prismatic site X_{Δ} instead of the prismatization X^{Δ}. The reason is that even though we have $\mathrm{Perf}(X^{\Delta}) \simeq \mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta})$ by [BL22b], we need the larger category $\mathcal{D}(X_{\Delta}, \mathcal{O}_{\Delta})$ attached to the site X_{Δ} to make sense of inverting $\mathcal{I}$ meaningfully.

We now show that M itself is p-power torsion. If not, then by replacing M with a quotient, we may assume M is p-torsionfree. The previous paragraph then implies that $M|_{\mathbf{Z}_p^{\Delta}} = 0$. In particular, when applied at the de Rham point of $\mathbf{Z}_p^{\mathcal{N}}$, we learn that $N := M|_{\mathbf{Z}_p^{\mathcal{N}}}$ is annihilated by t^n, where t is the Rees parameter. We must show N is annihilated by some power of p. Filtering by powers of t, we may assume $tN = 0$, so N is a coherent sheaf on $(\mathbf{Z}_p^{\mathcal{N}})_{t=0} = \mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$ (Proposition 5.3.7). By the previous paragraph (applied now at the Hodge–Tate point of $\mathbf{Z}_p^{\mathcal{N}}$), the sheaf N vanishes after pullback to $B\mathbf{G}_m^\sharp = \mathbf{G}_m/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m) \subset \mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$. Writing u for the co-ordinate on $\mathbf{G}_a$, the sheaf N can be seen as a finitely generated graded $\mathbf{Z}_p[u]^\wedge$-module with a graded connection $D : N \to N\frac{du}{u}$ such that $N[\frac{1}{u}]^\wedge = 0$. Thanks to the grading, the p-completion plays essentially no role in the previous sentence: N is the p-completion a finitely generated graded $\mathbf{Z}_p[u]$-module N_0 with a graded connection $D_0 : N_0 \to N_0\frac{du}{u}$ such that $N_0[\frac{1}{u}] = 0$. But $N_0[1/p]$ must be locally free as coherent sheaves with a flat connection on a smooth variety in characteristic 0 are locally free, so $N_0[1/u] = 0$ implies that $N_0[1/p] = 0$, i.e., N_0 is p-power torsion, as wanted.

Remark 6.3.6 (Obtaining all local systems geometrically)**.** Fix a quasi-syntomic p-adic formal scheme X. We then have the section $v_{1,X} \in H^0(X^{\mathrm{Syn}}, \mathcal{O}\{p-1\}/p)$, coming from Construction 6.2.1. The vanishing locus of $v_{1,X}$ contains all the physical points of the formal stack X^{Syn} over $\mathrm{Spf}(\mathbf{Z}_p)$, so it does not make sense invert v_1 in an algebro-geometric sense. Nonetheless, it seems plausible, based on the descent results in [Mat22], one can define a p-adic formal analytic stack $(X^{\mathrm{Syn}})_{v_1\neq 0}$ for the "v_1-adic topology" with the feature that $\mathrm{Perf}((X^{\mathrm{Syn}})_{v_1\neq 0}) \simeq \mathcal{D}^b_{lisse}(X_\eta, \mathbf{Z}_p)$. The isomorphy of (6.4.1) in Construction 6.4.1 below as well as Remark 6.3.5 above can be regarded as partial justification for this picture.

6.4 Syntomic cohomology and Galois cohomology: statements

In this section, fix $\mathbf{C}_p$ and $G_{\mathbf{Q}_p}$ as in Example 6.3.3. We formulate the main theorems of this chapter in this section: we relate the cohomology of F-gauges on $\mathbf{Z}_p^{\mathrm{Syn}}$ and the Galois cohomology of the corresponding representation of $G_{\mathbf{Q}_p}$ with mod p coeffecients. The passage between the two will be mediated by the following filtraiton, which will be one of our primary tools for understanding cohomology on $\mathbf{Z}_p^{\mathrm{Syn}}$:

Construction 6.4.1 (The syntomic filtration)**.** Consider a mod p F-gauge $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$. We then have a $\mathbf{Z}$-filtered diagram

$$E\{\bullet(p-1)\} := \Big(... \xrightarrow{v_1} E\{-(p-1)\} \xrightarrow{v_1} E \xrightarrow{v_1} E\{p-1\} \xrightarrow{v_1} ...\Big)$$

in $\mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$ defined by multiplication by v_1. Note that the corresponding $\mathbf{Z}$-filtered object of $\mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}})$ is complete: the section v_1 is topologically nilpotent as it cuts out the reduced locus. Write

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] := \mathrm{colim}\Big(... \xrightarrow{v_1} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E\{-(p-1)\}) \xrightarrow{v_1} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \xrightarrow{v_1} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E\{p-1\} \xrightarrow{v_1} ...\Big)$$

for the displayed colimit[76]; we refer to the resulting $\mathbf{Z}$-filtered object $\mathrm{Fil}^{\mathrm{Syn}}_\bullet R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$ coming from the colimit description as *the syntomic filtration* on $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$. Note that the filtration

[76]Note that we cannot commute the colimit past the $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, -)$ as $\mathbf{Z}_p^{\mathrm{Syn}}$ is a formal stack. In fact, v_1 is a topologically nilpotent section, so $\mathrm{colim}\, E\{\bullet(p-1)\} \in \mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}})$ is just 0, while $\mathrm{colim}\, R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E\{\bullet(p-1)\}) \simeq R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$ is not always 0: it is given by Galois cohomology, as explained later in Construction 6.4.1.

is complete since $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, -)$ preserves limits. By construction, we have a canonical isomorphism

$$\mathrm{gr}_\bullet^{\mathrm{Syn}}\left(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]\right) = R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E/v_1\{\bullet(p-1)\})$$

of $\mathbf{Z}$-graded $\mathbf{F}_p$-complexes, where we recall that $E/v_1 := \mathrm{Cone}(E\{-(p-1)\} \xrightarrow{v_1} E)$.

By Example 6.3.3, the étale realization functor carries multiplication by v_1 to an isomorphism, so we obtain a natural map

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)). \tag{6.4.1}$$

An examination of the argument in Construction 6.3.1 shows that this map is an isomorphism. Consequently, we also obtain a natural $\mathbf{Z}$-indexed filtration $\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E))$ on the Galois cohomology complex that we also call the *syntomic filtration.* This is an complete increasing exhaustive filtration, and thus gives a useful tool in computing Galois cohomology.

Variant 6.4.2. Given a mod p F-gauge $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, we can truncate the syntomic filtration $\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$ in degrees ≤ 0 to obtain a complete $\mathbf{N}^{op}$-indexed filtration $\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)$ on $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)$; we also call this the *syntomic filtration* if no confusion arises. Again, by construction, we have a natural isomorphism

$$\mathrm{gr}_\bullet^{\mathrm{Syn}}\left(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)\right) = R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E/v_1\{\bullet(p-1)\})$$

of $\mathbf{N}$-graded $\mathbf{F}_p$-complexes. As this filtration is complete, it gives a useful tool for controlling the cohomology of F-gauges themselves.

The syntomic filtration has a number of pleasant finiteness properties

Proposition 6.4.3 (Finiteness properties for F-gauges on $\mathbf{Z}_p$)**.** *We have the following:*

1. *For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, the filtered $\mathbf{F}_p$-complex $\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$ is finite, i.e., it is complete as a filtered object, each gr_i is a perfect complex over $\mathbf{F}_p$ and vanishes for $|i| \gg 0$. In particular, $\mathrm{Fil}_0^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] \simeq R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)$ is a perfect $\mathbf{F}_p$-complex.*
2. *The functors $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, -)$ and $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, -)$ have cohomological dimension ≤ 2.*

These will be prove in §6.5. Using the syntomic filtration as well as its finiteness properties, we can formulate our main theorem on the cohomology of mod p F-gauges over $\mathbf{Z}_p$.

Theorem 6.4.4 (Filtered refinement of Tate duality)**.** *We have the following:*

1. The fundamental class: *There is a natural isomorphism*

$$\mathrm{Fil}_\bullet^{\mathrm{Syn}} H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{1\}/p)[\frac{1}{v_1}] \simeq \mathbf{F}_p\langle -1\rangle,$$

where the target is $\mathbf{F}_p$ regarded as a complete $\mathbf{Z}$-filtered vector space with $\mathrm{gr}_i = 0$ unless $i \neq 1$.

2. The duality theorem: *For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, the natural map*

$$\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] \otimes \mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^\vee\{1\})[\frac{1}{v_1}] \to \mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{1\}/p)[\frac{1}{v_1}] \to \mathbf{F}_p\langle -1\rangle[-2]$$

coming from (1) is a perfect pairing of $\mathbf{Z}$-filtered objects[77] *in $\mathcal{D}(\mathbf{F}_p)$.*

[77]The Rees construction of a finite filtered complex is a perfect complex on $\mathbf{A}^1/\mathbf{G}_m$. In this language, the assertion that a pairing of finite filtered complexes is perfect amounts to saying that the corresponding pairing on Rees constructions is perfect as a pairing of perfect complexes; equivalently, by perfectness, the corresponding pairing on the associated graded objects is a pairing pairing of graded objects.

The above results will be proven later in §6.5. In this section, we extract some consequences. First, by ignoring the filtration, we recover local Tate duality for the relevant representations:

Corollary 6.4.5 (Local Tate duality for crystalline Galois representations (Tate))**.** *For $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, one has a natural identification*

$$R\Gamma(G_{\mathbf{Q}_p}, T_{et}(V)) \simeq \mathrm{RHom}_{\mathbf{F}_p}(R\Gamma(G_{\mathbf{Q}_p}, T_{et}^*(1)[2]), \mathbf{F}_p)$$

in $\mathrm{Perf}(\mathbf{F}_p)$.

Proof. The étale realization functor $T_{et} : \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}) \to D^b_{fg}(G_{\mathbf{Q}_p}, \mathbf{F}_p)$ is symmetric monoidal and hence commutes with taking duals. Moreover, it carries BK twists to Tate twists. Since we have

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] \simeq R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E))$$

and similarly for $E^*\{1\}$, the claim follows by passing to the colimit in Theorem 6.4.4 (and using the finiteness from Proposition 6.4.3 (1) to see that the formation of duals commutes with forgetting the filtration). □

Next, by analysing the duality theorem in filtration degree 0, we obtain the following, which is the mod p version of the result promised in the introduction in Theorem 1.3.1 (3); the full integral version can be deduced from the (proof of the) mod p version, see Corollary 6.5.22.

Corollary 6.4.6 (Lagrangian refinement of Tate duality)**.** *For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, there is a natural fibre sequence*

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)) \to \mathrm{RHom}_{\mathbf{F}_p}(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^*\{1\}[2]), \mathbf{F}_p),$$

where E^* denotes the $\mathcal{O}$-linear dual on $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$, the first map is induced by the functor T_{et} for E, while the second map is induced by T_{et} for E^* as well as the Tate duality isomorphism

$$R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)) \simeq \mathrm{RHom}_{\mathbf{F}_p}(R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E^*\{1\}[2])), \mathbf{F}_p)$$

Proof. This follows from Theorem 6.4.4 (2) applied to filtration level 0. For simplicity of notation, write $M = R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, T_{et}(E))[\frac{1}{v}]$ and $N = R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, T_{et}(E^*\{1\}[2]))[\frac{1}{v}]$, each endowed with the syntomic filtration; thus, M and $N\langle 1\rangle$ are filtered Tate dual by Theorem 6.4.4(2). We have a tautological fibre sequence

$$\mathrm{Fil}_0^{\mathrm{Syn}} M \to M \to M/\mathrm{Fil}_0^{\mathrm{Syn}} M.$$

Our task is then to identify $M/\mathrm{Fil}_0^{\mathrm{Syn}} M$ with $(\mathrm{Fil}_0^{\mathrm{Syn}} N)^\vee$. Recall that for any finite filtered complex $\mathrm{Fil}_\bullet K$ over any field k, we have an identification $\mathrm{Fil}_i\left((\mathrm{Fil}_\bullet K)^\vee\right) = (K/\mathrm{Fil}_{-i-1} K)^\vee$ of k-complexes[78]. Applying this to $K = M$ with $i = -1$, we learn that

$$M/\mathrm{Fil}_0^{\mathrm{Syn}} M \simeq \left(\mathrm{Fil}_{-1}((\mathrm{Fil}_\bullet^{\mathrm{Syn}} M)^\vee)\right)^\vee = (\mathrm{Fil}_{-1}(N\langle 1\rangle))^\vee = (\mathrm{Fil}_0 N)^\vee,$$

as wanted. □

[78]One hint for getting the indices right is to recall that we must have $\mathrm{gr}_i((\mathrm{Fil}_\bullet K)^\vee) = (\mathrm{gr}_{-i} K)^\vee$.

Remark 6.4.7 (Analogy with 3-manifolds)**.** In topology, the following assertion is classical: Given a 3-manifold M with boundary a compact Riemann surface Σ, the natural map $\mathrm{Loc}(M) \to \mathrm{Loc}(\Sigma)$ on spaces of local systems induces a Lagrangian map on tangent spaces, where the symplectic structure on $\mathrm{Loc}(\Sigma)$ is induced by Poincare duality on Σ. Under the analogy between 3-manifolds and number rings, the statement in Corollary 6.4.6 can be roughly regarded as an arithmetic counterpart of this assertion: $\mathbf{Z}_p^{\mathrm{Syn}}$ replaces the 3-manifold M, the étale site $\mathrm{Spa}(\mathbf{Q}_p)_{et}$ replaces the boundary surface Σ, and local Tate duality replaces Poincare duality. In fact, the statement of Corollary 6.4.6 was inspired by this analogy, and formed the starting point of the work described in this chapter.

Finally, let us explain how to compute the Euler characteristic of the cohomology of an F-gauge as well as the Galois cohomology of the associated Galois representation in terms of its Hodge–Tate weights; this will fall out of the method of proving Theorem 6.4.4, so the proofs will appear in §6.5.

Corollary 6.4.8 (Euler characteristic formula)**.** *For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, write $E_{Hod} \in \mathcal{D}_{qc}(B\mathbf{G}_m)$ for the pullback of E along the Hodge map $B\mathbf{G}_m \to D_{Hod} \subset (\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$. We then have the following formulae:*

- $\chi(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E/v_1)) = -\Big(\dim(\mathrm{gr}^{-1}_{Hod}(E)) + \dim(\mathrm{gr}^{-2}_{Hod}(E)) + \ldots + \dim(\mathrm{gr}^{-(p-1)}_{Hod}(E))\Big)$
- $\chi(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)) = -\sum_{i<0} \dim(\mathrm{gr}^i_{Hod}(E))$
- $\chi(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]) = -\sum_{i\in\mathbf{Z}} \dim(\mathrm{gr}^i_{Hod}(E)) = -\mathrm{rank}(E)$

Remark 6.4.9 (Bloch–Kato Selmer groups)**.** Given a crystalline $\mathbf{Q}_p$-representation V of $G_{\mathbf{Q}_p}$, Bloch–Kato define in [BK07, §3] a subspace $H^1_f(G_{\mathbf{Q}_p}, V) \subset H^1(G_{\mathbf{Q}_p}, V)$ (corresponding to crystalline extensions), and prove in [BK07, Proposition 3.8] an analog the Lagrangianness property in Corollary 6.4.6 for this subspace, i.e., they show that $H^1_f(G_{\mathbf{Q}_p}, V) \subset H^1(G_{\mathbf{Q}_p}, V)$ and $H^1_f(G_{\mathbf{Q}_p}, V^*(1)) \subset H^1(G_{\mathbf{Q}_p}, V^*(1))$ are exact annihilators of each other under the Tate duality pairing. The integral version of Corollary 6.4.6 will be shown later to recover the Bloch–Kato result on inverting p (see §6.7). From this guise, Corollary 6.4.8 (2) is a mod p analog of the rational statement in [BK07, Corollary 3.8.4].

6.5 Cohomology on $\mathbf{Z}_p^{\mathrm{Syn}}$

The goal of this section is to sketch a proof of the assertions from §6.4, including the filtered Tate duality theorem (Theorem 6.4.4). Even though these statements concern cohomology on $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$, the bulk of the proof consists of understanding the coherent sheaves and cohomology on the stack $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$, described in §6.2. Recall the following objects introduced while explaining Drinfeld's description of $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ in Remark 6.2.5:

- $D_{HT} = (\mathbf{Z}_p^{\mathcal{N}})_{p=t=0} = \mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$.
- $D_{dR} = (\mathbf{Z}_p^{\mathcal{N}})_{p=u^p=0} = B_{\mathbf{A}^1/\mathbf{G}_m}\mathcal{G}$.
- $D_{Hod} = B(F_*\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$, regarded as a closed substack of both D_{HT} and D_{dR}.
- $D_{und} = B\mathbf{G}_m^\sharp$, regarded as an open substack of both D_{HT} and D_{dR}.

Thus, $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ is obtained by glueing D_{HT} and D_{dR} along both D_{Hod} and D_{und}. We adopt a similar notation for global sections: for a complex E on any substack of $\mathbf{Z}_p^{\mathrm{Syn}}$ containing D_{und}, write $F_{und}(E) = R\Gamma(D_{und}, E|_{D_{und}})$, and similarly for all the other substacks introduced above. (This notation is slightly ambiguous as the notion of restriction depends on the stack E is defined over, but will be clear from context.)

In the rest of this section, we shall explain how to compute quasi-coherent sheaves and their cohomology in concrete linear algebraic terms on each of the 4 stacks appearing above, as well as on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$. The desired theorems will then be deduced relatively easily from these calculations.

Remark 6.5.1 (Geometric context)**.** For any quasi-syntomic p-adic formal scheme X, one has a complex $\mathcal{H}_{\mathrm{Syn}}(X) \in \mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}})$ defined via pushforward. The explicit descriptions of quasi-coherent sheaves on the loci appearing above, when specialized to $\mathcal{H}_{\mathrm{Syn}}(X)$, translate to certain (sometimes new) structures on mod p cohomology theories attached to X that we shall elaborate below.

6.5.1 Cohomology on D_{und}

Via Cartier duality, the category $\mathcal{D}_{qc}(D_{und})$ identifies with $\mathcal{D}_{(\Theta^p-\Theta)\text{-nilp}}(\mathbf{F}_p[\Theta])$, with the standard tensor product being carried to convolution, the structure sheaf corresponding to $(\mathbf{F}_p, \Theta = 0)$, and the BK twist $\mathcal{O}\{i\}$ corresponds to $(\mathbf{F}_p, \Theta = i)$; see [BL22a, §3.5]. In particular, if we view an object $E \in \mathcal{D}_{qc}(D_{und})$ is a $\mathbf{F}_p$-complex V with an operator $\Theta : V \to V$, then we have

$$F_{und}(E) = \mathrm{fib}(V \xrightarrow{\Theta} V), \tag{6.5.1}$$

so $F_{und}(-)$ preserves perfectness. One then checks:

Proposition 6.5.2. *The functor $F_{und}(-) : \mathrm{Perf}(B\mathbf{G}_m^\sharp) \to \mathrm{Perf}(\mathbf{F}_p)$ satisfies Serre duality in dimension 1 with dualizing sheaf $\mathcal{O}\{p\} \simeq \mathcal{O}$, i.e., we have a natural identification*

$$F_{und}(E) \simeq F_{und}(E^*\{p\}[1])^\vee$$

for $E \in \mathrm{Perf}(B\mathbf{G}_m^\sharp)$.

Note that $\mathcal{O}\{p\}$ is trivial on $B\mathbf{G}_m^\sharp$, so it might seem slightly strange to use it on the RHS above. Nonetheless, for consistency with what happens on other substacks of $\mathbf{Z}_p^{\mathrm{Syn}}$ below, it is easier to use $\mathcal{O}\{p\}$.

Remark 6.5.3 (Geometric context)**.** For a quasi-syntomic p-adic formal scheme X, the restriction $\mathcal{H}_{\mathrm{Syn}}(X)|_{D_{und}}$, when regarded as an object of $\mathcal{D}_{(\Theta^p-\Theta)\text{-nilp}}(\mathbf{F}_p[\Theta])$, corresponds to the mod p diffracted Hodge complex $\Omega_X^{\not D}/p$ of X equipped with its Sen operator $\Theta_X : \Omega_X^{\not D}/p \to \Omega_X^{\not D}/p$, as studied in [BL22a, §4.7]. Note that we can identify $\Omega_X^{\not D}/p$ with the derived de Rham complex $\mathrm{dR}_{X/\mathbf{F}_p}/p$ by the de Rham comparison theorem for prismatic cohomology.

6.5.2 Cohomology on D_{Hod}

Via Cartier duality over $B\mathbf{G}_m$, the category $\mathcal{D}_{qc}(D_{Hod})$ is identified with $\mathcal{D}_{\mathrm{gr},\Psi\text{-nilp}}(\mathbf{F}_p[\Psi])$, where $\deg(\Psi) = -p$; this equivalence carries tensor product to convolution, and the BK twist $\mathcal{O}\{1\}$ to the 1-dimensional $\mathbf{F}_p$-vector space concentrated in grading degree -1 with (necessarily) trivial Ψ. In particular, quasi-coherent sheaves E on $B(D_{Hod})$ correspond to $\mathbf{Z}$-graded $\mathbf{F}_p$-vector spaces $V := \oplus_i V_i$ with an $\Psi_i : V_i \to V_{i-p}$ for all i such that $\Psi = \oplus_i \Psi_i$ is locally nilpotent on V. In particular, we have

$$F_{Hod}(E) = \mathrm{fib}(V_0 \xrightarrow{\Psi} V_{-p}) \tag{6.5.2}$$

so F_{Hod} preserves perfectness. One then shows:

Proposition 6.5.4. *The functor $F_{Hod}(-) : \mathrm{Perf}(D_{Hod}) \to \mathrm{Perf}(\mathbf{F}_p)$ satisfies Serre duality in dimension 1 with dualizing sheaf $\mathcal{O}\{p\}$, i.e., we have a natural identification*

$$F_{Hod}(E) \simeq F_{Hod}(E^*\{p\}[1])^\vee$$

for $E \in \mathrm{Perf}(B(F_*\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m))$.

Remark 6.5.5 (Geometric context)**.** For a quasi-syntomic p-adic formal scheme X, the restriction $\mathcal{H}_{\mathrm{Syn}}(X)|_{D_{Hod}}$, when regarded as an object of $\mathcal{D}_{\mathrm{gr},\Psi\text{-nilp}}(\mathbf{F}_p[\Psi])$, corresponds to the Hodge complex $\Omega^*_{X/\mathbf{Z}_p}/p$ of X, with Ψ induced by the Sen operator via the p-Griffiths transversality discussed in Remark 6.5.8.

6.5.3 Cohomology on D_{dR}

In Remark 6.2.5, we recalled Drinfeld's identification $D_{dR} \simeq B_{\mathbf{A}^1/\mathbf{G}_m}\mathcal{G}$, where the group scheme $\mathcal{G} \to \mathbf{A}^1/\mathbf{G}_m$ is given by the $[-t^p]$-rescaled degeneration of $\mathbf{G}_m^\sharp$ to $\mathbf{G}_a^\sharp$. One then shows:

Proposition 6.5.6. *Let H be the formal completion of $\mathbf{V}(\mathcal{O}(p)) := \mathrm{Spec}(\mathbf{F}_p[t,\Theta])/\mathbf{G}_m$ at the finite flat subgroup of $\mathbf{V}(\mathcal{O}(p))$ determined by the homogeneous element $\Theta^p - t^{p^2-p}\Theta$ (so H is a formal commutative subgroup scheme of $\mathbf{V}(\mathcal{O}(p))$ over $\mathbf{A}^1/\mathbf{G}_m$). One then has a natural equivalence*

$$\mathcal{D}_{qc}(D_{dR}) \simeq \mathcal{D}_{qc}(H)$$

via a symmetric monoidal equivalence carrying tensor products to convolution, and $\mathcal{O}\{p\}$ being the line bundle $\mathcal{O}(-p) \in \mathcal{P}\mathrm{ic}(\mathbf{A}^1/\mathbf{G}_m)$, regarded as a coherent sheaf on H sitting at the 0 section.

Proof idea. Recall that the group scheme $\mathcal{G}$ is defined as, as a scheme over $\mathrm{Spec}(\mathbf{F}_p[t])/\mathbf{G}_m = \mathbf{A}^1/\mathbf{G}_m$, as $\mathbf{G}_a^\sharp/\mathbf{G}_m = \mathrm{Spec}(\mathbf{F}_p[t]\langle x\rangle)/\mathbf{G}_m$, where $\deg(t) = 1$ and $\deg(x) = -p$. The comultiplication is determined by $\nabla(x) = x + y + [t^p]xy$. One then proceeds as in [BL22a, Theorem 3.5.8], using the differential operator $\widetilde{\Theta} = (1 + t^p x)\frac{d}{dx}$ (which has degree p) in lieu of the operator $t\frac{d}{dt}$ in *loc. cit.*. Alternately, one can directly identify H with the Cartier dual of $\mathcal{G}$ following [Dri21a, Appendix B]. ☐

Thus, $E \in \mathcal{D}_{qc}(D_{dR})$ corresponds to (decreasingly) $\mathbf{Z}$-filtered complex $F^*V \in \mathcal{DF}(\mathbf{F}_p)$ together with an operator $\Theta : F^*V \to F^{*-p}V$ such that $\Theta^p - t^{p^2-p}\Theta$ is locally nilpotent as an endomorphism of the Rees construction $\mathrm{Rees}(F^*V) = \bigoplus_i F^iV \cdot t^{-i}$ of F^*V[79]. The object $F_{dR}(E)$ is then given by

$$F_{dR}(E) = \mathrm{fib}(F^0V \xrightarrow{\Theta} F^{-p}V), \tag{6.5.3}$$

and hence preserves finiteness properties. We have a natural map

$$a_E : F_{dR}(E) \to F_{und}(E) \times F_{Hodge}(E).$$

given by restriction. This map is "Lagrangian" with respect to Serre duality on the target:

[79]Note that Θ induces an operator $\Theta^{-\infty} : V \to V$ on the underlying non-filtered complex $V = F^{-\infty}V$. The pair $(V, \Theta^{-\infty})$ is a Sen complex, i.e., its generalized eigenvalues lie in $\mathbf{F}_p$: this follows by the nilpotence assumption on $\Theta^p - t^{p^2-p}\Theta = t^{p^2} \cdot ((\frac{\Theta}{t^p})^p - \frac{\Theta}{t^p})$. Conversely, if F^*V is a perfect complex on $\mathbf{A}^1/\mathbf{G}_m$, then $F^i = 0$ for $i \gg 0$ and the transition map $t : F^i \to F^{i-1}$ is an isomorphism for $i \ll 0$. In this case, any $(-p)$-shifted filtered endomorphism $\Theta : F^*V \to F^{*-p}V$ whose underlying non-filtered endomorphism $\Theta^{-\infty} : V \to V$ defines a Sen complex satisfies the necessary nilpotence condition to yield a quasi-coherent complex on H.

Proposition 6.5.7. *For any $E \in \mathrm{Perf}(D_{dR})$, the restriction map a_E fits into a fibre sequence*

$$F_{dR}(E) \xrightarrow{a_E} F_{und}(E) \times F_{Hodge}(E) \to (F_{dR}(E^*\{p\}[1]))^\vee, \tag{Lag$_1$}$$

where the second map is induced by the Serre duality isomorphism

$$F_{und}(E) \times F_{Hodge}(E) \simeq (F_{und}(E^*\{p\}[1]) \times F_{Hodge}(E^*\{p\}[1]))^\vee$$

coming from Proposition 6.5.2 and 6.5.4 composed with the map $a^\vee_{E^*\{p\}[1]}$.

Proof idea. Given $E = (F^*V, \Theta) \in \mathrm{Perf}((\mathbf{Z}_p^{\mathcal{N}})_{p=u^p=0})$, we have

$$F_{dR}(E) = \mathrm{fib}\left(F^0V \xrightarrow{\Theta} F^{-p}V\right)$$

while

$$F_{und}(E) = \mathrm{fib}\left(V \xrightarrow{\Theta} V\right), \qquad F_{Hodge}(E) = \mathrm{fib}\left(\mathrm{gr}^0V \xrightarrow{\Theta} \mathrm{gr}^{-p}V\right),$$

with the map a_E being the obvious map. One then computes that

$$\mathrm{Cone}(a_E) = \mathrm{fib}\left(V/F^1V \xrightarrow{\Theta} V/F^{1-p}V\right).$$

Now for a finite decreasingly $\mathbf{Z}$-filtered complex Fil^*W, we have $\mathrm{Fil}^i((\mathrm{Fil}^*W)^\vee) = (W/\mathrm{Fil}^{-i+1}W)^\vee$. Applying this to E and noting that twisting by $\mathcal{O}\{p\}$ increases filtration by p (i.e., $\mathrm{Fil}^i(\mathrm{Fil}^*W\{p\}) = \mathrm{Fil}^{i+p}W$) without changing Θ, we have

$$(F_{dR}(E^*\{p\}[1]))^\vee = \left(\mathrm{fib}\left((V/F^{1-p}V)^\vee \xrightarrow{\Theta} (V/F^1V)^\vee\right)\right)^\vee[-1].$$

This simplifies to the expression for $\mathrm{Cone}(a_E)$ above, so the claim follows. ☐

Remark 6.5.8 (Geometric context)**.** For a quasi-syntomic p-adic formal scheme, the restriction $\mathcal{H}_{\mathrm{Syn}}(X)|_{D_{dR}}$, when regarded as an object of $\mathcal{D}_{qc}(H)$, corresponds to the Hodge filtered derived de Rham complex $\mathrm{Fil}^*_H\mathrm{dR}_{X/\mathbf{Z}_p}/p$, with the operator Θ corresponding to the Sen operator Θ_X under the identification $\mathrm{dR}_{X/\mathbf{Z}_p}/p \simeq \Omega^{\not{D}}_X/p$. In particular, we learn that the Sen operator on Θ_X satisfies p-Griffiths transversality, i.e., it fails to preserve the Hodge filtration by at most p. The associated graded of this statement was already used in Remark 6.5.5.

Via the comparison in Corollary 5.5.11, the complex $\mathcal{H}_{\mathrm{Syn}}(X)|_{D_{dR}}$ computes the Hodge–Tate complex of X modulo p equipped with the twisted Nygaard filtration[80], i.e., if one writes $t : D_{dR} \to \mathbf{A}^1/\mathbf{G}_m$ for the Rees, map, then we have

$$Rt_*\mathcal{H}_{\mathrm{Syn}}(X)|_{D_{dR}}\{i\} \simeq \mathrm{Rees}(\mathrm{Fil}^\bullet_{\tilde{\mathcal{N}}}R\Gamma(X, \overline{\Delta}_X/p\{i\})),$$

where $\mathrm{Fil}^\bullet_{\tilde{\mathcal{N}}}\Delta_X/p$ is defined by quasi-syntomic descent from the construction $R \mapsto \mathrm{Cone}(I \otimes_{\Delta_R} \mathrm{Fil}^{\bullet-p}_{\mathcal{N}}\Delta_R/p \xrightarrow{\theta} \mathrm{Fil}^\bullet_{\mathcal{N}}\Delta_R/p)$ on qrsp rings, where the map θ is the intermediate map appearing in Construction 6.2.1. The formula (6.5.3) then gives a fibre sequence calculating this filtered object in terms of the filtered Sen operator on $\mathrm{dR}_{X/\mathbf{Z}_p}/p$.

[80]This filtered object plays an essential role in [BM22].

6.5.4 Cohomology on D_{HT}

In Remark 6.2.5 and Proposition 5.3.7, we identified the Hodge–Tate component $D_{HT} \simeq \mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m) \simeq \mathbf{G}_a^{dR}/\mathbf{G}_m$, whence $\mathcal{D}_{qc}(D_{HT})$ identifies with the derived category of $\mathbf{G}_m$-equivariant crystals on $\mathbf{A}^1$ over $\mathbf{F}_p$. Concretely, if $\mathcal{A}_1 = \mathbf{F}_p\{x, D\}/(Dx - xD - 1)$ is the 1-dimensional Weyl algebra, then we have

$$\mathcal{D}_{qc}(D_{HT}) \simeq \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1).$$

Under this description,

- The global sections functor

$$F_{HT}(-) : \mathrm{Perf}(D_{HT}) \to \mathcal{D}(\mathbf{F}_p)$$

  corresponds to

$$M \in \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1) \quad \mapsto \quad \mathrm{fib}(M_0 \xrightarrow{D} M_{-1}), \tag{6.5.4}$$

  and hence preserves finiteness properties.

- The "restriction to open point" functor

$$\mathrm{Perf}(D_{HT}) \to \mathrm{Perf}(D_{und}) \simeq \mathcal{D}_{(\Theta^p-\Theta)\text{-nilp}}(\mathbf{F}_p[\Theta])$$

  is given concretely via

$$M \in \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1) \quad \mapsto \quad (M[\frac{1}{x^p}]_{\deg=0}, \Theta = xD).$$

  The resulting map on global sections is the map

$$\mathrm{fib}(M_0 \xrightarrow{D} M_{-1}) \to \mathrm{fib}(M[\frac{1}{x^p}]_{\deg=0} \xrightarrow{\Theta=xD} M[\frac{1}{x^p}]_{\deg=0})$$

  induced by multiplication by x on the second term.

- To understand the "restriction to closed point functor"

$$\mathrm{Perf}(D_{HT}) \to \mathrm{Perf}(D_{Hod})$$

  explicitly, recall that $D_{Hod} \subset D_{HT}$ is the closed substack $\alpha_p/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m) \subset \mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$. Consequently, $\mathcal{D}_{qc}(D_{Hod}) \simeq \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1/x^p)$, where we recall that $x^p \in \mathcal{A}_1$ is central. The above restriction functor is then simply given by

$$M \in \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1) \quad \mapsto \quad M/x^p.$$

  The resulting map on global sections is the obvious map

$$\mathrm{fib}(M_0 \xrightarrow{D} M_{-1}) \to \mathrm{fib}((M/x^p)_0 \xrightarrow{D} (M/x^p)_{-1})$$

  of complexes.

Remark 6.5.9. Let us reconcile the description of $\mathcal{D}_{Hod}$ used above with the one in §6.5.2. The key point is that the map $\mathbf{F}_p[\Psi] \xrightarrow{\Psi\mapsto D^p} \mathcal{A}_1/x^p$ realizes the target as a split Azumaya algebra over the source, with splitting module given by $K := \mathcal{A}_1/\mathcal{A}_1 x$, everything compatibly with the grading. In particular, we can identify $\mathcal{D}_{\mathrm{gr},\Psi\text{-nilp}}(\mathbf{F}_p[\Psi]) \simeq \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1)$ via $N \mapsto K \otimes_{\mathbf{F}_p[\Psi]} N$, with inverse given by $M \mapsto \mathrm{RHom}_{\mathcal{A}_1/x^p}(K, M)$. Moreover, there is a natural identification $\mathcal{A}_1/(\mathcal{A}_1 x^p, \mathcal{A}_1 D) \simeq \mathcal{A}_1/(\mathcal{A}_1 x, \mathcal{A}_1 D^p)\langle -(p-1)\rangle$ of graded left $\mathcal{A}_1$-modules determined by $1 \mapsto D^{p-1}$; using this, for any $M \in \mathcal{D}_{\mathrm{gr}}(\mathcal{A}_1)$, one identifies

$$\mathrm{fib}\left(M/x^p \xrightarrow{D(-)} M/x^p\langle -1\rangle\right) \simeq \mathrm{fib}\left(M/x \xrightarrow{D^p(-)} M/x\langle -p\rangle\right),$$

as graded $\mathbf{F}_p$-complexes. On global sections (i.e., in degree 0), the RHS above becomes

$$\mathrm{fib}\left(\mathrm{gr}_0 M \xrightarrow{D^p} \mathrm{gr}_{-p} M\right),$$

which matches the description of the global sections used above with that coming from §6.5.2.

In particular, for any $E \in \mathrm{Perf}(D_{HT})$, we have a natural map

$$b_E : F_{HT}(E) \to F_{und}(E) \times F_{Hodge}(E).$$

given by restriction. This map is "Lagrangian" with respect to Serre duality on the target:

Proposition 6.5.10. *For any $E \in \mathrm{Perf}(D_{HT})$, the restriction map b_E fits into a fibre sequence*

$$F_{HT}(E) \xrightarrow{b_E} F_{und}(E) \times F_{Hodge}(E) \to (F_{HT}(E^*\{p\}[1]))^\vee, \tag{Lag$_2$}$$

where the second map is induced by the Serre duality isomorphism

$$F_{und}(E) \times F_{Hodge}(E) \simeq \left(F_{und}(E^*\{p\}[1]) \times F_{Hodge}(E^*\{p\}[1])\right)^\vee$$

coming from Proposition 6.5.2 and 6.5.4 composed with the map $b^\vee_{E^*\{p\}[1]}$.

Proof idea. Write $M \in \mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1)$ for the left $\mathcal{A}_1$-complex corresponding to E. It is convenient to regard M as an (increasingly) $\mathbf{Z}$-filtered $\mathbf{F}_p$-complex F_*V via the inverse of the Rees construction where the Rees parameter is x, i.e., $F_iV = M_i$, multiplication by x corresponds to the transition map $F_iV \to F_{i+1}V$, and $V = M[\frac{1}{x^p}]_{\deg=0}$. Using the explicit descriptions used above, the map under consideration is

$$b_E : \mathrm{fib}\left(F_0V \xrightarrow{D} F_{-1}V\right) \to \mathrm{fib}\left(V \xrightarrow{D} V\right) \times \mathrm{fib}\left(F_0V/F_{-p}V \xrightarrow{D} F_{-1}V/F_{-1-p}V\right).$$

The cone of b_E then identifies with

$$\mathrm{fib}\left(V/F_{-p}V \xrightarrow{xD} V/F_{-1-p}V\right)$$

Using the description of the dual of a finite filtered $\mathbf{F}_p$-complex given in Corollary 6.4.6 as well as the fact that $\mathcal{O}\{p\} \in \mathrm{Perf}(D_{HT})$ corresponds to the grading shift $\mathbf{F}_p[x]\langle p\rangle$ as a left $\mathcal{A}_1$-module, one then identifies this cone with $(F_{HT}(E^*\{p\}[1]))^\vee$. $\square$

Remark 6.5.11 (Geometric context)**.** For a quasi-syntomic p-adic formal scheme X, the object $\mathcal{H}_{\mathrm{Syn}}(X)|_{D_{HT}}$ can be regarded as an object of $\mathcal{D}_{\mathrm{gr},D\text{-nilp}}(\mathcal{A}_1)$. This object is the Rees construction (with parameter x) for $\mathrm{Fil}_*^{\mathrm{conj}}\Omega_X^{\not{D}}/p$, with the Sen operator on $\Omega_X^{\not{D}}/p$ computed by the Euler operator xD, and the rest of the $\mathcal{A}_1$-module structure encoding the divisibility properties of Θ_X on the conjugate filtration (as in [BL22a, Notation 4.7.2]). Indeed, for any graded $\mathcal{A}_1$-complex M, one has a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "xD"] & \frac{1}{x}M \arrow[r] \arrow[d, "xD-1"] & \dots \arrow[r] & \frac{1}{x^i}M \arrow[r] \arrow[d, "xD-i"] & \dots \arrow[r] & {} \\ M \arrow[r] & \frac{1}{x}M \arrow[r] & \dots \arrow[r] & \frac{1}{x^i}M \arrow[r] & \dots. \arrow[r] & {} \end{tikzcd}$$

of graded $\mathbf{F}_p$-complexes with displayed maps. Taking the colimit gives a map $\Theta_\infty : M[\frac{1}{x}] \to M[\frac{1}{x}]$ of graded $\mathbf{F}_p$-complexes. Passing to degree 0, we obtain an $\mathbf{F}_p$-complex $V = M[\frac{1}{x}]_{\mathrm{deg}=0}$ with a filtration Fil_*V defined by the diagram, i.e., $\mathrm{Fil}_iV = (\frac{1}{x^i}M)_{\mathrm{deg}=0}$ and a filtered endomorphism $\Theta : \mathrm{Fil}_*V \to \mathrm{Fil}_*V$ (coming from Θ_∞) such that $\Theta + i : \mathrm{Fil}_iV \to \mathrm{Fil}_iV$ comes equipped with a factorization through $\mathrm{Fil}_{i-1}V$ (and thus $\mathrm{gr}_i(\Theta) = -i$) thanks to the x-divisibility of $(xD - i) + i$; this is exactly the structure on the diffracted Hodge complex studied in [BL22a, Notation 4.7.2].

Let us also note that via the comparison theorems in Corollary 5.5.11 and quasi-syntomic descent, it follows that $R\Gamma(D_{HT}, \mathcal{H}_{\mathrm{Syn}}(X)|_{D_{HT}}\{i\}) \simeq \mathrm{gr}^i_{\mathcal{N}} R\Gamma(X, \Delta_X/p)$. The description of these global sections coming from the explicit description of the underlying quasi-coherent sheaf in the previous paragraph together with the formula (6.5.4) for $F_{HT}(-)$ then recovers the Nygaard fibre sequence in [BL22a, Remark 5.5.8]

6.5.5 Cohomology on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$.

Having studied the components D_{HT} and D_{dR} of $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ as well as the overlaps D_{und} and D_{Hod}, we can access cohomology on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ itself. More precisely, combining the sequences (Lag_1) and (Lag_2), we prove that $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ has Serre duality in dimension 2 with dualizing sheaf $\mathcal{O}\{p\}$:

Proposition 6.5.12 (Serre duality on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$)**.** *For any $E \in \mathrm{Perf}(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}})$, we have a natural identification*

$$R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E)^\vee \simeq R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E^*\{p\}[2])^\vee.$$

In particular, $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ and $\mathbf{Z}_p^{\mathrm{Syn}}$ have cohomological dimension ≤ 2.

Proof. The claim for the cohomological dimension of $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ is immediate from the duality assertion, while that for $\mathbf{Z}_p^{\mathrm{Syn}}$ reduces to its analog for $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$ by p-completeness, and the latter follows from the statement for $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ by completeness of the syntomic (aka v_1-adic) filtration. So it remains to prove the duality theorem. For this, by the glueing description of $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$, we have

$$R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E) = \mathrm{fib}\left(F_{dR}(E) \times F_{HT}(E) \xrightarrow{a_E - b_E} F_{und}(E) \times F_{Hodge}(E)\right) \tag{6.5.5}$$

Using (Lag_1) and (Lag_2), a diagram chase then shows that this fibre is identified with the fibre of

$$F_{und}(E^*\{p\}[1])^\vee \times F_{Hodge}(E^*\{p\}[1])^\vee \xrightarrow{(a_{E^*\{p\}[1]} - b_{E^*\{p\}[1]})^\vee} F_{dR}(E^*\{p\}[1])^\vee \times F_{HT}(E^*\{p\}[1])^\vee.$$

The dual of this is then identified with $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E^*\{p\}[2])^\vee$, as wanted. $\square$

One can use the description of cohomology on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ given in (6.5.5) to make explicit calculations of cohomology of perfect complexes in terms of filtered vector spaces with operators. Let us record some examples:

Example 6.5.13 (Some explicit calculations)**.**

- $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, \mathcal{O}) = \mathrm{fib}(\mathbf{F}_p^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}} \mathbf{F}_p^2) \simeq \mathbf{F}_p \oplus \mathbf{F}_p[-1]$.

- $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, \mathcal{O}\{1\}) = \mathbf{F}_p[-1]$: to compute this, one calculates that $F_{dR}(\mathcal{O}\{1\}) = \mathbf{F}_p[-1]$ while $F_{HT}(\mathcal{O}\{1\}) = F_{und}(\mathcal{O}\{1\}) = F_{Hodge}(\mathcal{O}\{1\}) = 0$ (using that the Sen operator is $+1$ to kill $F_{und}(\mathcal{O}\{1\})$).

- $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, \mathcal{O}\{p\}) = \mathbf{F}_p[-1] \oplus \mathbf{F}_p[-2]$ via duality from the case of the structure sheaf.

- Say $E \in \mathrm{Perf}(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}})$ is *very effective*, i.e., the graded object $E|_{B\mathbf{G}_m}$ has vanishing gr_i for $i \leq 0$ (or equivalently that the Hodge–Tate weights in the sense of Remark 5.3.14 are all ≥ 0). Then $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E) = 0$. This applies to $\mathcal{O}\{i\}$ for $i < 0$.

- If $E \in \mathrm{Perf}(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}})$ is $(p+1)$-antieffective (i.e., its Hodge–Tate weights are all $\leq -(p+1)$), then $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E) = 0$ by duality from the previous item. This applies to $\mathcal{O}\{i\}$ for $i > p$.

Finally, let us also record the Euler characteristic statement for $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$:

Corollary 6.5.14 (Euler charateristics on $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$)**.** *For any $E \in \mathrm{Perf}(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}})$ with pullback $E_{Hod} \in \mathcal{D}_{qc}(B\mathbf{G}_m)$ as in Corollary 6.4.8,*

$$\chi(R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E)) = -\Big(\dim(\mathrm{gr}^{-1}_{Hod}(E) + \dim(\mathrm{gr}^{-2}_{Hod}(E)) + \ldots + \dim(\mathrm{gr}^{-(p-1)}_{Hod}(E))\Big) \tag{6.5.6}$$

Proof. We shall use the explicit description of quasi-coherent sheaves on D_{HT}, D_{dR}, D_{und} and D_{Hod} in terms of filtered/graded vector spaces with operators given in §6.5.1, §6.5.2, §6.5.3 and §6.5.4 to explicitly compute the Euler characteristic. The perfect complex E defines objects (V, F^*, F_*, Θ), where V is a perfect $\mathbf{F}_p$-complex, F^* and F_* are finite decreasing and increasing filtrations with an identification $\mathrm{gr}^i_F V \simeq \mathrm{gr}^F_i V$ for all i, and $\Theta : V \to V$ is an operator satisfying certain compatibilities (in the derived sense) with F^* and F_*. Thus, using the formula (6.5.5) to compute cohomology, it suffices to show that

$$\chi(F_{HT}(E)) + \chi(F_{dR}(E)) - \chi(F_{und}(E)) - \chi(F_{Hod}(E)) = -\sum_{i=1}^{p-1} \dim \mathrm{gr}_F^{-i} V \tag{χ_{red}}$$

Using (6.5.4), (6.5.3), (6.5.1), and (6.5.2), we can compute each term on the left explicitly:

$$\chi(F_{HT}(E)) = \dim F_0 - \dim F_{-1} = \dim \mathrm{gr}_0^F V \quad \text{and} \quad \chi(F_{dR}(E)) = \dim F^0 - \dim F^{-p} = -\sum_{i=1}^{p} \mathrm{gr}^i_F V,$$

while

$$\chi(F_{und}(E)) = \dim V - \dim V = 0 \quad \text{and} \quad \chi(F_{Hod}(E)) = \dim gr^0_F V - \dim \mathrm{gr}_F^{-p} V.$$

Using all 4 quantities, the LHS of (χ_{red}) equals the RHS, so we win. $\square$

Remark 6.5.15 (Syntomic cohomology in small Hodge–Tate weights)**.** Let $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$ be a mod p F-gauge such that $E\{-(p-1)\}$ is very effective, i.e., the Hodge–Tate weights of E are all $\geq -(p-2)$. Then $E\{-i(p-1)\}$ is very effective for all $i > 0$, so the calculations in Example 6.5.13 show that $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E/v_1\{-i(p-1)\}) = 0$ for $i > 0$, whence the complete syntomic filtration collapses to give

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \simeq R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E/v_1).$$

Moreover, one can also check that the map $F_{HT}(E) \to F_{Hodge}(E)$ is an isomorphism thanks to the condition on the weights. Writing $F_\bullet V$ and $F^\bullet V$ for the filtered $\mathbf{F}_p$-complexes underlying $E|_{D_{HT}}$ and $E|_{D_{dR}}$ respectively and using Eq. (6.5.5), one then calculates that

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \simeq R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E/v_1) \simeq \mathrm{fib}(F^0V \xrightarrow{1-\tau} V),$$

where τ is the map given by $F^0V \xrightarrow{can} \mathrm{gr}^0 V \xrightarrow{\iota} V$; here ι is the inclusion of the eigenspace (with eigenvalue 0) of the semisimple Sen operator on $F_0V/F_pV = F_0V$ (the isomorphism coming from the condition on the Hodge–Tate weights) followed by the canonical map $F_0V \to V$. In fact, the reader can check that the preceding discussion applies to any $E \in \mathcal{D}_{qc}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$ such that $E\{-(p-1)\}$ is very effective, i.e., we do not need finiteness conditions on E. In particular, taking $E = \mathcal{H}_{\mathrm{Syn}}(X)\{i\}$ for a smooth p-adic formal scheme X and $0 \leq i \leq p-2$, we learn that

$$R\Gamma_{\mathrm{Syn}}(X, \mathbf{Z}_p(i))/p \simeq \mathrm{fib}\left(\mathrm{Fil}^i_H R\Gamma_{dR}(X)/p \xrightarrow{1-\tau} R\Gamma_{dR}(X)/p\right).$$

Via Kan extensions, this compares with [AMMN22, Theorem F (2)]. On the other hand, [AMMN22, Theorem F (1)] contains a simiar statement for arbitrary i at the expense of working up to p-isogeny over $\mathbf{Z}_p$; we do not know how to obtain such a statement from the perspective of syntomification.

Remark 6.5.16 (Continuity of cohomology)**.** As $\mathbf{Z}_p^{\mathrm{Syn}}$ or even the substack $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$ is a formal stack, the formation of cohomology does not commute with filtered colimits of quasi-coherent complexes (see Footnote 76 for an explicit example that came up earlier). However, on the positive side, one has the following assertion:

Lemma 6.5.17. *Fix a filtered diagram $\{E_i\}$ in $\mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}})$. Assume that the Hodge–Tate weights of all the E_i's are uniformly bounded below. Then the natural map*

$$\operatorname{colim}_i R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E_i) \to R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, \operatorname{colim}_i E_i)$$

is an isomorphism after p-completion.

Proof. By reduction modulo p, We may assume $\{E_i\}$ is a diagram in $\mathcal{D}_{qc}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$. Write E_∞ for the colimit. Using (6.5.5) as well as the explicit description of the constituent functors, it is clear that the functor $R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, -)$ commutes with filtered colimits. In particular, the functor carrying $E \in \mathcal{D}_{qc}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$ to $\mathrm{gr}_i^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \simeq R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E/v_1\{-i(p-1)\})$ commutes with colimits for all $i \geq 0$. As the syntomic filtration is induced by the complete v_1-adic filtration on E and is itself complete, it then suffices to show that there exists some $c \geq 0$ such that $\mathrm{gr}_{\geq c}^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E_i)$ for all E_i in our diagram. But this follows from Example 6.5.13 as the Hodge–Tate weights of E_i are uniformly bounded below since twisting by $\mathcal{O}\{-1\}$ increases weights by 1. $\square$

The above lemma is a variant for F-gauges of known continuity properties in p-adic geometry, including that for syntomic cohomology ([BL22a, Proposition 7.4.8]), étale cohomology of p-adic formal schemes ([BS19, Lemma 9.2]), and topological cyclic homology ([CMM21, Theorem G]).

6.5.6 Filtered Tate duality on $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$.

We can now prove the promised filtered Tate duality theorem. For this, we first construct the fundamental class for $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$; this is relatively straightforward using the calculations of Example 6.5.13

Proposition 6.5.18 (The fundamental class for $(\mathbf{Z}_p^{\mathrm{Syn}})_{p=0}$)**.** *We have*

$$\mathrm{Fil}_\bullet^{\mathrm{Syn}} H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}/p\{1\})[\frac{1}{v_1}] = \mathbf{F}_p\langle -1\rangle.$$

Proof. Note that $\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}$ has cohomological dimension 2 by Proposition 6.5.12. It follows that the complete $\mathbf{Z}$-indexed diagram

$$\mathrm{Fil}_\bullet^{\mathrm{Syn}} H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}/p\{1\})[\frac{1}{v_1}]$$

has $\mathrm{gr}_i = H^2(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, \mathcal{O}\{1+i(p-1)\})$. It is then enough to show that $H^2(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, \mathcal{O}\{1+i(p-1)\})$ equals 0 if $i \neq 1$ and equals $\mathbf{F}_p$ if $i = 1$. This follows from the calculations in Example 6.5.13. ☐

Remark 6.5.19. Forgetting filtrations in Proposition 6.5.18 shows that

$$c_{\mathbb{A}} : H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}/p\{1\})[\frac{1}{v_1}] = \mathbf{F}_p.$$

Let us call this the *F-gauge fundamental class.* On the other hand, via the étale realization, the LHS is naturally identified with $H^2(G_{\mathbf{Q}_p}, \mathbf{F}_p(1))$, which in turn is identified with $\mathbf{F}_p$ via Tate duality; let us call the resulting isomorphism

$$c_{et} : H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}/p\{1\})[\frac{1}{v_1}] = \mathbf{F}_p.$$

We expect these are the same isomorphism; for our purposes, it will suffice to note that $c_{\mathbb{A}}$ and c_{et} differ by an element of $\mathbf{F}_p^*$, which is clear.

Having constructed the fundamental class, we obtain our promised theorem relatively easily from passage to the associated graded of the syntomic filtration and Proposition 6.5.12.

Theorem 6.5.20 (Filtered Tate duality)**.** *For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$, the map*

$$\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] \simeq \mathrm{RHom}_{\mathbf{F}_p}(\mathrm{Fil}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^\vee\{1\})[\frac{1}{v_1}], \mathbf{F}_p)\langle -1\rangle[-2]$$

of filtered objects induced by the fundamental class in Proposition 6.5.18 is an isomorphism.

Proof. As all filtrations in sight are finite, it suffices to prove the claim at the associated graded level, i.e., it suffices to show that the associated graded of the fundamental class induces an isomorphism

$$\mathrm{gr}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] \simeq \mathrm{RHom}_{\mathbf{F}_p}(\mathrm{gr}_\bullet^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^\vee\{1\})[\frac{1}{v_1}], \mathbf{F}_p)\langle -1\rangle[-2].$$

In degree i, the LHS is given by

$$\mathrm{gr}_i^{\mathrm{Syn}} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}] = R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E/v_1\{i(p-1)\}) = R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E/v_1\{ip-i\}), \qquad (6.5.7)$$

while the RHS is given by

$$\mathrm{gr}_{i-1}\left((\mathrm{gr}^{\mathrm{Syn}}_{\bullet} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^\vee\{1\})[\frac{1}{v_1}])^\vee\right)[-2] = \left(\mathrm{gr}_{1-i} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^\vee\{1\})[\frac{1}{v_1}]\right)^\vee [-2],$$

which simplifies to

$$R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E^\vee/v_1\{1+(1-i)(p-1)\})^\vee[-2] \simeq R\Gamma(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E^\vee/v_1\{i+p-ip\})^\vee[-2]. \tag{6.5.8}$$

The rightmost terms in (6.5.7) and (6.4.4) are identified under the Serre duality in Proposition 6.5.12 which is induced by the same class as the one used here, so we win. ☐

Finally, we obtain an Euler characteristic formula for $\mathbf{Z}_p^{\mathrm{Syn}}$ and also recover the Euler characteristic formula in local Galois cohomology

Corollary 6.5.21 (Euler charateristics on $\mathbf{Z}_p^{\mathrm{Syn}}$)**.** *For any $E \in \mathrm{Perf}((\mathbf{Z}_p^{\mathrm{Syn}})_{p=0})$ with pullback $E_{Hod} \in \mathcal{D}_{qc}(B\mathbf{G}_m)$ as in Corollary 6.4.8,*

$$\chi(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)) = -\sum_{i<0} \dim(\mathrm{gr}^i_{Hod}(E))$$

and

$$\chi(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]) = -\sum_{i\in\mathbf{Z}} \dim(\mathrm{gr}^i_{Hod}(E)) = -\mathrm{rank}(E).$$

Proof. Observe that we have a (finite) syntomic filtration $\mathrm{Fil}^{\mathrm{Syn}}_{\leq 0} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$ on $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)) = \mathrm{Fil}^{\mathrm{Syn}}_{\leq 0} R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]$. We thus learn that

$$\chi(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)) = \sum_{i\leq 0} \chi(\mathrm{gr}^{\mathrm{Syn}}_i R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E)[\frac{1}{v_1}]) = \sum_{i\leq 0} \chi(\mathbf{Z}_{p,\mathrm{red}}^{\mathrm{Syn}}, E/v_1\{i(p-1)\}).$$

The first formula then follows from Corollary 6.5.14. The second formula follows similarly using the full $\mathbf{Z}$-indexed syntomic filtration. ☐

So far, we have worked with mod p coefficients. But let us at least indicate how this can be used to deduce integral statements by explaining how to deduce Theorem 1.3.1 (3) from Corollary 6.4.6:

Corollary 6.5.22 (Lagrangianness with integral coefficients)**.** *For any $E \in \mathrm{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})$, there is a natural fibre sequence*

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)) \to \mathrm{RHom}_{\mathbf{Z}_p}(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^*\{1\}[2]), \mathbf{Z}_p),$$

where E^* denotes the $\mathcal{O}$-linear dual, the first map is induced by the functor T_{et} for E, while the second map is induced by T_{et} for E^* as well Tate duality.

Proof. First, we claim that $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{1\}) = 0$: as the cohomological dimension of $\mathbf{Z}_p^{\mathrm{Syn}}$ is 2, it suffices to prove $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{1\}/p) = 0$, which follows from Proposition 6.5.18 by passage to Fil_0. Thanks to this, it follows that the Tate duality pairing

$$R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)) \otimes R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E^*\{1\}[2])) \to H^2(G_{\mathbf{Q}_p}, \mathbf{Z}_p(1)) \simeq \mathbf{Z}_p$$

vanishes after precomposition with

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \otimes R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^*\{1\}[2]) \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)) \otimes R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E^*\{1\}[2])),$$

yielding a "complex"

$$R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E) \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(E)) \to \mathrm{RHom}_{\mathbf{Z}_p}(R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, E^*\{1\}[2]), \mathbf{Z}_p).$$

To show this complex is a fibre sequence, it suffices to do so after reduction modulo p. In this case, we are done by Corollary 6.4.6 (thanks to the compatibility of the étale and F-gauge fundamental classes in Remark 6.5.19). $\square$

Remark 6.5.23. Corollary 6.5.22 has some antecedents in the literature. Notably, specializing to $E = \mathcal{H}_{\mathrm{Syn}}(X)$ for a smooth proper $X/\mathrm{Spf}(\mathbf{Z}_p)$, we obtain from Corollary 6.5.22 a duality theorem for the syntomic cohomology of X itself; such a duality was first established in [Sat07] using very different techniques (and in fact [Sat07] allows more general X's). Using the comparison in [BM22, Theorem 5.8], we expect the two dualities to coincide.

Remark 6.5.24 (A higher dimensional analog)**.** It is natural to wonder if the Serre duality in Theorem 4.5.2 and the Lagrangianness property in Corollary 6.5.22 admit a common generalization. To this end, we expect (and have an outline of proof of) the following:

Conjecture 6.5.25 (Lagrangianness for proper regular $\mathbf{Z}_p$-schemes)**.** *Say X is a regular p-adic formal scheme that is proper over* $\mathrm{Spf}(\mathbf{Z}_p)$ *and with absolute dimension d. Then for any $E \in$* $\mathrm{Perf}(X^{\mathrm{Syn}})$*, there is a fibre sequence*

$$R\Gamma(X^{\mathrm{Syn}}, E) \to R\Gamma(X_\eta, T_{et}(E)) \to R\Gamma(X^{\mathrm{Syn}}, E^*\{d\}[2d])^\vee$$

of perfect $\mathbf{Z}_p$-complexes, where E^* denotes the $\mathcal{O}$-linear dual of a perfect complex on X^{Syn}, and $(-)^\vee$ denotes the $\mathbf{Z}_p$-linear dual of a perfect $\mathbf{Z}_p$-complex.

For $X = \mathrm{Spf}(\mathbf{Z}_p)$, this recovers Corollary 6.5.22, while for $X = \mathrm{Spec}(\mathbf{F}_p)$, this gives Theorem 4.5.2 (as the generic fibre is empty). Moreover, one can then also deduce the conjecture for X proper smooth over $\mathrm{Spec}(\mathbf{F}_p)$ or $\mathrm{Spf}(\mathbf{Z}_p)$ using geometric Poincare duality from [Tan22]. Granting Conjecture 6.5.25 when $\dim(X) = 1$, all the results in this chapter for F-gauges over $\mathbf{Z}_p$ extend to the ramified case, i.e., for F-gauges over $\mathcal{O}_K$ where $K/\mathbf{Q}_p$ is any finite extension.

6.6 From crystalline representations to F-gauges

In this section, fix a complete discretely valued extension $K/\mathbf{Q}_p$ with perfect residue field. Fix a completed algebraic closure $C/\mathbf{Q}_p$, and let G_K denote the corresponding absoluste Galois group of K. We shall explain the relationship between F-gauges over $\mathcal{O}_K$ and crystalline Galois representations of G_K: the latter correspond to "reflexive coherent sheaves" on $\mathcal{O}_K^{\mathrm{Syn}}$ (Theorem 6.6.13). The proof relies critically on the main theorem of [BS21], giving a relationship between crystalline G_K-representations and prismatic F-crystals over $\mathcal{O}_K$, as well as quasi-syntomic descent.

Remark 6.6.1. The main theorem in [BS21] was extended to smooth formal $\mathcal{O}_K$-schemes $\mathfrak{X}$ in [DLMS22, GR22] in the form an equivalence between crystalline $\mathbf{Z}_p$-local systems on $\mathfrak{X}_\eta$ and "analytic" prismatic F-crystals on $\mathfrak{X}$. It seems reasonable to expect that Theorem 6.6.13 also admits a generalization to such $\mathfrak{X}$'s (or, ideally, to any p-adic formal scheme $\mathfrak{X}$ that is regular and topologically finitely presented over $\mathcal{O}_K$ with K as above).

6.6.1 Reflexive F-gauges over a perfectoid field

In this subsection, we introduce reflexive coherent sheaves on $\mathcal{O}_C^{\mathrm{Syn}}$, and show that these correspond to prismatic F-crystals in vector bundles over $\mathcal{O}_C$. The functor in the forward direction is simply restriction along $j_{\Delta} : \mathcal{O}_C^{\Delta} \subset \mathcal{O}_C^{\mathrm{Syn}}$ (coupled with some observations concerning Frobenius structures). For the reverse construction, the idea is to interpret a prismatic F-crystal in vector bundles over $\mathcal{O}_C$ as a vector bundle on the complement of the closed point in $\mathcal{O}_C^{\mathrm{Syn}}$, and then construct an F-gauge by taking a $*$-extension. Due to the formal completions involved, the argument below is phrased more algebraically. For simplicity of notation, we work in slightly more generality.

Notation 6.6.2. Let (A, I) be a perfect prism with the property that $\overline{A}$ is a perfectoid valuation ring, and choose a generator $d \in I$ for notational simplicity. We then have the Rees ring $A[u,t]/(ut-d)$ of the I-adic filtration on A.

First, let us explain how to understand reflexive modules over the Rees algebra in terms of isogenies of vector bundles, mimicing what we saw in characteristic p in Proposition 3.4.12

Proposition 6.6.3. *The following two categories are equivalent:*

1. *The category* $\mathrm{Isog}(A, I)$ *of triples* (M, N, τ)*, where* $M, N \in \mathrm{Vect}(A)$ *and* $\tau : M[1/I] \simeq N[1/I]$ *is an isomorphism.*
2. *The category* $\mathrm{Coh}^{\mathrm{refl}}_{\mathrm{gr}}(A[u,t]/(ut-d)) \subset \mathrm{Perf}_{\mathrm{gr}}(A[u,t]/(ut-d))$ *spanned by those* E *that satisfy:*
   - *(a)* $E[1/u]$ *and* $E[1/t]$ *are locally free over* $A[u,u^{-1}]$ *and* $A[t,t^{-1}]$ *respectively.*
   - *(b)* E *is* (u,t)*-regular, i.e.,* $\mathrm{Kos}(E; u, t)$ *is discrete.*

The functor relating (1) to (2) is essentially $*$-extension along $\mathrm{Spec}(A[u,t]/(ut-d)) - V(u,t) \subset \mathrm{Spec}(A[u,t]/(ut-d))$. It would be nice to formulate a more general non-graded statement that would imply the proposition on passage to graded pieces.

Proof. Let us first define a functor $G : \mathrm{Coh}^{\mathrm{refl}}_{\mathrm{gr}}(A[u,t]/(ut-d)) \to \mathrm{Isog}(A, I)$. Given an object $E \in \mathrm{Coh}^{\mathrm{refl}}_{\mathrm{gr}}(A[u,t]/(ut-d))$, set $M(E) = E[1/u]_{\deg 0}$ and $N(E) = E[1/t]_{\deg 0}$. Then we have a natural correspondence $\widetilde{\tau}$:

$$M(E) \xleftarrow{u^\infty} E_{\deg 0} \xrightarrow{t^\infty} N(E).$$

By perfectness of E, both maps above become isomorphisms on inverting d, so we obtain an object $G(E) := (M(E), N(E), \tau = \widetilde{\tau}[1/d]) \in \mathrm{Isog}(A, I)$. This construction is clearly functorial, so we obtain the functor G.

Conversely, let us define a functor $F : \mathrm{Isog}(A, I) \to \mathrm{Coh}^{\mathrm{refl}}_{\mathrm{gr}}(A[u,t]/(ut-d))$. Given $(M, N, \tau) \in \mathrm{Isog}(A, I)$, consider the I-adically filtered A-module $\mathrm{Fil}^\bullet N$ given by $\mathrm{Fil}^i N = I^i M \cap N$ (where the intersection takes place inside $N[\frac{1}{I}] \simeq M[\frac{1}{I}]$ via τ). We then define

$$F(M, N, \tau) = \mathrm{Rees}(\mathrm{Fil}^\bullet N) \in \mathrm{Mod}_{\mathrm{gr}}\left(A[u,t]/(ut-d)\right).$$

The construction is evidently functorial, so, in order to define the functor F, we have to check that $F(M, N, \tau)$ is perfect and satisfies conditions (a) and (b) above.

For condition (a), we observe that $\mathrm{Fil}^i N = I^i M$ for $i \gg 0$ and $\mathrm{Fil}^i N = N$ for $i \ll 0$. Under the Rees dictionary, this implies that $\mathrm{Rees}(\mathrm{Fil}^\bullet N)[1/u] = M[u, u^{-1}]$ and $\mathrm{Rees}(\mathrm{Fil}^\bullet N)[1/t] = N[t, t^{-1}]$, so we have verified (a).

For condition (b), note that $\mathrm{Fil}^\bullet N$ is an honest filtration on an A-module, so t acts injectively on $\mathrm{Rees}(\mathrm{Fil}^\bullet N)$. Moreover, this filtration satisfies the property that an element $x \in \mathrm{Fil}^i N =$

$I^iM \cap N$ with $dx \in \mathrm{Fil}^{i+2}N = I^{i+2}M \cap N$ must satisfy $x \in I^{i+1}M \cap N = \mathrm{Fil}^i N$; this implies that multiplication by u acts injectively on $\mathrm{Rees}(\mathrm{Fil}^\bullet N)/t$, so $\mathrm{Rees}(\mathrm{Fil}^\bullet N)$ is indeed (u,t)-regular.

Next, let us check the $A[u,t]/(ut-d)$-perfectness of $\mathrm{Rees}(\mathrm{Fil}^\bullet N)$. Choose integers $a \leq b$ such that $I^bM \subset N \subset I^aM$. Consider the strict SES

$$0 \to \mathrm{Fil}^{\geq \max(\bullet, b)} N \to \mathrm{Fil}^\bullet N \to \mathrm{Fil}^\bullet Q \to 0$$

of I-adically filtered A-modules defining the last term. Applying $\mathrm{Rees}(-)$ to this sequence produces a short exact sequence. Moreover, by our choice of b, we can identify $\mathrm{Rees}(\mathrm{Fil}^{\bullet \geq b} N)$ with $I^bM \otimes_A A[u,t]/(ut-d)$ up to a twist, which is clearly perfect. So it suffices to show that $\mathrm{Rees}(\mathrm{Fil}^\bullet Q)$ is $A[u,t]/(ut-d)$-perfect. Now $\mathrm{Fil}^{\geq b} Q = 0$ by construction, and $\mathrm{Fil}^i Q = \mathrm{Fil}^{i-1} Q$ for $i \leq a$ as the same is true for $\mathrm{Fil}^\bullet N$. Thus, $\mathrm{Fil}^\bullet Q$ is a finite I-adic filtration on an I^{b-a}-torsion module. By devissage, it is enough to show that $\mathrm{Rees}(\mathrm{gr}^i Q)$ is $A[u,t]/(ut-d)$-perfect for each $a \leq i \leq b$, where $\mathrm{gr}^i(Q)$ is regarded as an I-adically filtered A-module in the trivial way: it is an $\overline{A}$-module with a trivial filtration, concentrated in grading degree i. In particular, we have $\mathrm{Rees}(\mathrm{gr}^i Q) = \mathrm{gr}^i Q[t]$ (up to a shift) with u acting by 0, so it suffices check that $\mathrm{gr}^i Q$ is perfect over $\overline{A}$. We have $\mathrm{gr}^i Q = \mathrm{gr}^i N = (I^iM \cap N)/(I^{i+1}M \cap N)$ for the relevant i's. As $\overline{A}$ is a valuation ring, it then suffices to show that $(I^iM \cap N)/(I^{i+1}M \cap N)$ is finitely presented over $\overline{A}$ or equivalently over A. But we have the formula

$$(I^iM \cap N)/(I^{i+1}M \cap N) = \Big((I^iM \cap N)/I^bM\Big) / \Big((I^{i+1}M \cap N)/I^bM\Big),$$

so $\mathrm{gr}^i Q$ is the cokernel of a map between intersections of finitely presented submodules of the finite projective A/I^{b-a}-module I^aM/I^bM for the relevant i's. As A/I^n is coherent for all $n \geq 1$ (by devissage down to the $n=1$ case of $\overline{A} = A/I$), any such intersection is finitely presented, so the claim follows.

Having defined functors in both directions, we must check they are mutually inverse equivalences. It is clear from the construction that $GF \simeq \mathrm{id}$. Let us show the same for FG. Given $E \in \mathrm{Coh}^{\mathrm{refl}}_{\mathrm{gr}}(A[u,t]/(ut-d))$, we first observe that E is (u,t) complete by perfectness[81], so the (u,t)-regularity of E implies E is discrete. In particular, both t and u act injectively on E. Let $(M,N,\tau) = G(E)$, and let $\mathrm{Fil}^\bullet N = E_{\deg = -\bullet}$ be the corresponding I-adically filtered A-complex. By the t-regularity of E, we know that $\mathrm{Fil}^\bullet N$ is an honest decreasing filtration on the locally free A-module N. Our task is to check that $\mathrm{Fil}^i N = I^iM \cap N$ inside $\tau : M[1/d] \simeq N[1/d]$. By shifting, it is enough to show the claim when $i = 0$. By construction of G, it is clear that $\mathrm{Fil}^0 N \subset M \cap N$. For the reverse inclusion, choose $x \in M \cap N$. Under the Rees dictionary, the element x gives a degree 0 section of $E|_{\mathrm{Spec}(A[u,t]/(ut-d)) - V(u,t)}$. But the (u,t)-regularity of E implies that E and $E|_{\mathrm{Spec}(A[u,t]/(ut-d)) - V(u,t)}$ have the same global sections, so we conclude that x is a global section of E, as wanted. $\square$

Motivated by the above, we isolate the following class of F-gauges on $\overline{A}$:

Definition 6.6.4. The category $\mathrm{Coh}^{\mathrm{refl}}(\overline{A}^{\mathrm{Syn}})$ of *reflexive F-gauges on $\overline{A}$* is the full subcategory of $\mathrm{Perf}(\overline{A}^{\mathrm{Syn}})$ spanned by E's satisfying:

1. $E|_{\overline{A}^{\Delta}}$ is locally free.
2. $E|_{\overline{A}^{\mathcal{N}}} \in \mathrm{Perf}(\overline{A}^{\mathcal{N}}) \simeq \mathrm{Perf}_{\mathrm{gr}}(A[u,t]/(ut - \varphi^{-1}(d)))$ is (u,t)-regular.

[81] Indeed, any object in $\mathrm{Perf}_{\mathrm{gr}}(A[u,t]/(ut-d))$ is derived (u,t)-complete in $\mathcal{D}_{\mathrm{gr}}(A[u,t]/(ut-d))$ as the same holds true for the ring itself.

Incorporating Frobenius structures in Proposition 6.6.3 then shows:

Corollary 6.6.5. *The following two categories are equivalent:*

1. *The category* $\mathrm{Vect}^{\varphi}(A) \simeq \mathrm{Vect}^{\varphi}((\overline{A})_{\Delta})$ *of prismatic F-crystals in vector bundles on* $\overline{A}$.
2. *The category* $\mathrm{Coh}^{\mathrm{refl}}(\overline{A}^{\mathrm{Syn}})$ *of reflexive F-gauges on* $\overline{A}$.

6.6.2 Nygaardian filtrations on prismatic F-crystals over qrsp rings

The goal of this subsection is record a lemma (Lemma 6.6.10) that characterizes F-gauges over qrsp rings obtained via suitable base change from the reflexive F-gauges considered in Section 6.6.1 intrinsically (i.e., in terms of the underlying F-crystal, without the base change data). For this purpose, the following slightly ad hoc definition will be useful:

Definition 6.6.6 (Nygaardian filtrations)**.** Let R be a qrsp ring and $(M, \tau : \varphi^* M[1/I] \simeq M[1/I]) \in \mathrm{Mod}^{\varphi}_{fp}(\Delta_R)$ be a prismatic F-crystal in finitely presented modules on $\mathrm{Spf}(R)$.

1. A *Nygaardian filtration* is a filtration $\mathrm{Fil}^{\bullet} M$ of M in (p, I)-complete Δ_R-modules such that the map
$$M \xrightarrow{\mathrm{can}} \varphi^* M \subset \varphi^* M[1/I] \overset{\tau}{\simeq} M[1/I] \tag{6.6.1}$$
carries $\mathrm{Fil}^i M$ into $I^i M$ for all $i \in \mathbf{Z}$.
2. A Nygaardian filtration $\mathrm{Fil}^{\bullet} M$ is called *saturated* if it is the maximal such filtration, i.e., $\mathrm{Fil}^i M$ is the preimage of $I^i M \subset M[1/I]$ under the above map[82]. In particular, there is a unique saturated Nygaardian filtration on any prismatic F-crystal.

Write $\mathrm{Mod}^{\varphi}_{fp}(\Delta_R)^+$ for the category of prismatic F-crystals $(M, \tau) \in \mathrm{Mod}^{\varphi}_{fp}(\Delta_R)$ equipped with a Nygaardian filtration $\mathrm{Fil}^{\bullet} M$; the full subcategory of $\mathrm{Mod}^{\varphi}_{fp}(\Delta_R)^+$ spanned by saturated Nygaardian filtrations identifies with $\mathrm{Mod}^{\varphi}_{fp}(\Delta_R)$ via the forgetful functor.

Example 6.6.7 (Nygaardian filtration from F-gauges)**.** Fix a qrsp ring R and an F-gauge $E \in \mathcal{D}_{qc}(R^{\mathrm{Syn}})$, corresponding to a triple $(M(E), \mathrm{Fil}^{\bullet} M(E), \widetilde{\varphi}_{M(E)} : \mathrm{Fil}^{\bullet} M(E) \to I^{\bullet} M(E))$ as in Example 6.1.7. If $M(E)$ is a finitely presented Δ_R-module and $\mathrm{Fil}^{\bullet} M(E)$ is an honest filtration, then $\mathrm{Fil}^{\bullet} M(E)$ is a Nygaardian filtration, as the map (6.6.1) is simply the map on underlying non-filtered objects associated to the filtered map $\widetilde{\varphi}_{M(E)}$.

Example 6.6.8 (Saturated Nygaardian filtrations in the perfectoid case)**.** Say $R = \mathcal{O}_C$ for $C/\mathbf{Q}_p$ a perfectoid field. Then any $E \in \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C)$ gives rise to a triple $(M(E), \mathrm{Fil}^{\bullet} M(E), \widetilde{\varphi}_{M(E)})$ as in Example 6.1.7. Condition (a) in Definition 6.6.4 ensures that $M(E)$ is naturally a prismatic F-crystal in vector bundles, while condition (b) in the same definition ensures that the filtration $\mathrm{Fil}^{\bullet} M(E)$ is saturated Nygaardian for the F-crystal $M(E)$ (see proof of Proposition 6.6.3).

Remark 6.6.9. Fix a qrsp ring R as well as $(M, \tau_M, \mathrm{Fil}^{\bullet} M), (N, \tau_N, \mathrm{Fil}^{\bullet} N) \in \mathrm{Mod}^{\varphi}_{fp}(\Delta_R)^+$. If $\mathrm{Fil}^{\bullet} N$ is a saturated Nygaardian filtration, then we have
$$\mathrm{Hom}_{\mathrm{Mod}^{\varphi}_{fp}(\Delta_R)^+}((M, \tau_M, \mathrm{Fil}^{\bullet} M), (N, \tau_N, \mathrm{Fil}^{\bullet} N)) \simeq \mathrm{Hom}_{\mathrm{Mod}^{\varphi}_{fp}(\Delta_R)}((M, \tau_M), (N, \tau_N))$$
via the forgetful functor $\mathrm{Mod}^{\varphi}_{fp}(\Delta_R)^+ \to \mathrm{Mod}^{\varphi}_{fp}(\Delta_R)$. In the other words, the formation of the saturated Nygaardian filtration provides a right adjoint to the forgetful functor.

[82]This preimage is automatically (p, I)-complete as it can be described as a fibre product of the diagram $M \to I^{-c} M \leftarrow I^i M$ for $c \gg 0$ in (p, I)-complete Δ_R-modules by finite presentation of M.

Our main lemma about these objects is the following, observing that reflexive F-gauges over a perfectoid valuation ring base change to yield saturated Nygaardian filtrations over flat qrsp algebras over the ring.

Lemma 6.6.10. *Fix a perfectoid valuation ring $\mathcal{O}_C$ as well as a p-torsionfree qrsp $\mathcal{O}_C$-algebra R. Given $E \in \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C^{\mathrm{Syn}})$, let $E_R \in \mathrm{Perf}(R^{\mathrm{Syn}})$ be its base change. Then, with notation as in Example 6.6.7, the filtration $\mathrm{Fil}^\bullet M(E_R)$ is the saturated Nygaardian filtration for the F-crystal $M(E_R)$ (and thus only depends on the latter F-crystal).*

Proof. Consider the triple $(M(E), \mathrm{Fil}^\bullet M(E), \widetilde{\varphi}_{M(E)})$ attached to E as in Example 6.1.7. As E is reflexive, $M(E)$ is a finite free $\Delta_{\mathcal{O}_C}$-module and $\mathrm{Fil}^\bullet M(E)$ is an honest filtration on $M(E)$. The triple $(M(E_R), \mathrm{Fil}^\bullet M(E_R), \widetilde{\varphi}_{M(E_R)})$ corresponding to E_R is obtained as the ((p, I)-completed, derived) base change of $(M(E), \mathrm{Fil}^\bullet M(E), \widetilde{\varphi}_{M(E)})$. Our task is to show the following:

1. $\mathrm{Fil}^\bullet M(E_R)$ is an honest filtration.
2. For each integer i, the subgroup $\mathrm{Fil}^i M(E_R) \subset M(E_R)$ is the preimage of $I^i M(E_R)$ under the non-filtered map underlying $\widetilde{\varphi}_{M(E_R)}$.

Before embarking on the proof, let us make some preliminary observations about $N := \mathrm{gr}^\bullet_{\mathrm{Fil}} M(E)$, regarded as a graded module over $\mathrm{gr}^\bullet_{\mathcal{N}} \Delta_{\mathcal{O}_C} = \mathcal{O}_C[u]$: we shall explain that both N and $N[1/u]/N$ have bounded p-power torsion. As E is perfect, N is a finitely presented graded $\mathcal{O}_C[u]$-module. Moreover, the assumption that E is (u,t)-regular implies that u acts injectively on N. Under the Rees dictionary (with respect to the parameter u), we then learn that N is the Rees module for a finite filtration $G_\bullet N_0$ on a finitely presented $\mathcal{O}_C$-module N_0 with the further property that $\mathrm{gr}^G_\bullet N_0$ is also finitely presented over $\mathcal{O}_C$; explicitly, we have $N = \oplus_i G_i N_0 u^i$. As finitely presented $\mathcal{O}_C$-modules have bounded p-power torsion, we learn that N has bounded p-power torsion. Moreover, the p-power torsion in N/u^n is bounded uniformly in n by the explicit formula for N, whence $N[1/u]/N$ also has bounded p-power torsion, as asserted.

We can now begin the proof. As $\mathrm{Fil}^\bullet M(E)$ is saturated, we have $\mathrm{Fil}^{-n} M(E) = M(E)$ for all $n \gg 0$, whence the same holds true after base change. Thus, to show (1), it suffices to show that $\mathrm{gr}^\bullet_{\mathrm{Fil}} M(E_R)$ is coconnective. But we have

$$\mathrm{gr}^\bullet_{\mathrm{Fil}} M(E_R) \simeq \mathrm{gr}^\bullet_{\mathrm{Fil}} M(E) \widehat{\otimes}^L{}_{\mathrm{gr}^\bullet_{\mathcal{N}} \Delta_{\mathcal{O}_C}} \mathrm{gr}^\bullet_{\mathcal{N}} \Delta_R. \tag{6.6.2}$$

Now $N = \mathrm{gr}^\bullet_{\mathrm{Fil}} M(E)$ is discrete by assumption and has bounded p-power torsion as explained above. The coconnectivity of the RHS above then follows as $\mathrm{gr}^\bullet_{\mathcal{N}} \Delta_R$ is p-completely flat over $\mathrm{gr}^\bullet_{\mathcal{N}} \Delta_{\mathcal{O}_C}$.

To show (2), since we already saw that $\mathrm{Fil}^{-n} M(E_R) = M(E_R)$ for $n \gg 0$, it suffices to show the map

$$\mathrm{gr}^\bullet(\widetilde{\varphi}_{M(E_R)}) : \mathrm{gr}^\bullet_{\mathrm{Fil}} M(E_R) \to \mathrm{gr}^\bullet_{I^\bullet M(E_R)} M(E_R)$$

is injective or equivalently has coconnective cone. As E is an F-gauge, this map identifies the RHS with the p-completed localization at $u \in \mathrm{gr}^1_{\mathcal{N}} \Delta_{\mathcal{O}_C}$ of the LHS, so we must show that

$$(\mathrm{gr}^\bullet_{\mathrm{Fil}} M(E_R)[1/u] / \mathrm{gr}^\bullet_{\mathrm{Fil}} M(E_R))^\wedge$$

is coconnective. By (6.6.2), the object above identifies with the p-completed base change of $N[1/u]/N$ along the p-completely flat map $\mathrm{gr}^\bullet_{\mathcal{N}} \Delta_{\mathcal{O}_C} \to \mathrm{gr}^\bullet_{\mathcal{N}} \Delta_R$, so the claim again follows as the $\mathcal{O}_C[u]$-module $N[1/u]/N$ has bounded p-power torsion as explained above. $\square$

6.6.3 Reflexive F-gauges over $\mathcal{O}_K$

Recall that we have fixed $K/\mathbf{Q}_p$, C/K and G_K at the start of the section. In Remark 5.5.19 and Example 5.5.20, we explained that $\mathcal{O}_K^{\mathrm{Syn}}$ admits faithfully flat adic covers by affine noetherian formal regular schemes of dimension 3, and in particular that there is a reasonable theory of coherent sheaves on $\mathcal{O}_K^{\mathrm{Syn}}$. In this section, we shall focus attention on those coherent sheaves which are reflexive, and relate them to crystalline Galois representations.

Definition 6.6.11. An F-gauge $E \in \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$ is called *reflexive* if its pullback to $\mathcal{O}_C^{\mathrm{Syn}}$ is reflexive (Definition 6.6.4); write $\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K^{\mathrm{Syn}})$ for the full subcategory spanned by such F-gauges.

Remark 6.6.12 (Relationship to reflexivity in commutative algebra)**.** Choose a prism (A, I) with $A/I = \mathcal{O}_K$. Let $B = \mathrm{Rees}(I^\bullet)$ be the Rees algebra for the I-adic filtration on A, and let $\widehat{B}$ be the (p, I)-completion of B. Thus, we obtain a faithfully flat map $\mathcal{R}(I^\bullet) = \mathrm{Spf}(B)/\mathbf{G}_m \to \mathcal{O}_K^{\mathcal{N}}$ as in Remark 5.5.19. Let us explain why reflexivity in the sense of Definition 6.6.11 is equivalent to reflexivity in the classical commutative algebra sense after pullback to $\mathrm{Spf}(\widehat{B})$.

First, the canonical map $\mathcal{O}_C^{\mathcal{N}} \to \mathcal{O}_K^{\mathcal{N}}$ can then be factored as $\mathcal{O}_C^{\mathcal{N}} \to \mathcal{R}(I^\bullet) \to \mathcal{O}_K^{\mathcal{N}}$, with both maps being faithfully flat (via the argument in Footnote 69 as well as the equivalence of perfectoid rings with perfect prisms). It follows that reflexive F-gauges as in Definition 6.6.11 are indeed coherent sheaves on $\mathcal{O}_K^{\mathrm{Syn}}$, justifying the notation for the category in Definition 6.6.11.

Secondly, given an F-gauge $E \in \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$ with pullback $E_{\widehat{B}} \in \mathrm{Perf}(\widehat{B})$, the following are equivalent:

1. E is reflexive in the sense of Definition 6.6.11.
2. $E_{\widehat{B}}$ is a reflexive[83] $\widehat{B}$-module (regarded as a complex by being placed in degree 0).

This equivalence, which follows[84] from standard commutative algebra arguments over the 3-dimensional regular ring $\widehat{B}$, provides justification for the name "reflexive" given to the concept in Definition 6.6.11.

Our main theorem about reflexive F-gauges is the following:

Theorem 6.6.13. *The following three categories are equivalent:*

1. *The category* $\mathrm{Rep}^{\mathrm{crys}}_{\mathbf{Z}_p}(G_K)$ *of crystalline* G_K*-representations on finite free* $\mathbf{Z}_p$*-modules.*
2. *The category* $\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta})$ *of prismatic* F*-crystals on* $\mathcal{O}_K$*.*
3. *The category* $\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K^{\mathrm{Syn}})$ *of* reflexive F*-gauges on* $\mathcal{O}_K$*,*

[83] Recall that $\widehat{B}$ is a 3-dimensional regular noetherian domain. A finitely generated $\widehat{B}$-module M is called reflexive if it is the (non-derived) $*$-extension of a vector bundle from the complement of some codimension ≥ 2 closed subset of $\mathrm{Spec}(\widehat{B})$. In fact, the stalks of such a module over codimension 2 points are locally free by Auslander-Buchsbaum and regularity of $\widehat{B}$, so it is equivalent to assume that M is the (non-derived) $*$-extension of a vector bundle from the complement of a codimension ≥ 3 closed subset.

[84] Say $(R, \mathfrak{m})$ is a 3-dimesional regular local ring. Fix a regular sequence $x, y \in \mathfrak{m}$ of length 2. Then a finitely generated R-module M is reflexive if and only if $M[1/x]$ and $M[1/y]$ are free, and $\mathrm{Kos}(M; x, y)$ is discrete. Indeed, the "if" direction follows as the hypotheses ensure that M is $*$-extended from a vector bundle on $\mathrm{Spec}(R) - V(x, y)$. For the "only if" direction, the local freeness after inverting x and y is clear from the reflexivity of M. Moreover, if $j : U = \mathrm{Spec}(R) - \{\mathfrak{m}\} \hookrightarrow \mathrm{Spec}(R)$ is the punctured spectrum, then by assumption we have $M \simeq R^0 j_* E$ with $E = M|_U$ being a vector bundle, so one can apply $\mathrm{Kos}(-; x, y)$ to the local cohomology exact triangle $R\Gamma_{\mathfrak{m}}(M) \to M \to Rj_*E$ to conclude that M satisfies the desired condition on the Koszul complex since $\mathrm{Kos}(E; x, y)$ is discrete by reduction to the case $E = \mathcal{O}_U$.

The functor from (3) to (2) is given by restriction to an open substack $\mathcal{O}_K^{\Delta} \subset \mathcal{O}_K^{\mathrm{Syn}}$ as in Remark 6.3.4, while the functor from (2) to (1) is étale realization.

We warn the reader that the equivalences in Theorem 6.6.13 are not exact. More precisely, the functors

$$\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K^{\mathrm{Syn}}) \to \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}) \to \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{crys}}(G_K)$$

described in theorem are exact, while the inverses are not.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is realized by the étale realization functor for prismatic F-crystals (Remark 6.3.4) by the main theorem of [BS21], so we focus on (2) $\Leftrightarrow$ (3). Restriction to the open point gives a functor

$$\Phi : \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K) \to \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}).$$

We shall show it is an equivalence by constructing an inverse. Write $R^\bullet$ for the p-completed Cech nerve of $\mathcal{O}_K \to \mathcal{O}_C$. We then have descent equivalences

$$\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}) \simeq \lim \mathrm{Vect}^{\varphi}(R^\bullet) \quad \text{and} \quad \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}}) \simeq \lim \mathrm{Perf}((R^\bullet)^{\mathrm{Syn}}).$$

For each i, let $\mathcal{C}^i \subset \mathrm{Perf}((R^i)^{\mathrm{Syn}})$ to be the full subcategory spanned by those prismatic F-gauges whose F-gauges E^i such that the corresponding triple $(M(E^i), \mathrm{Fil}^\bullet M(E^i), \widetilde{\varphi}_{M(E^i)})$ (as in Example 6.1.7) has the following properties:

1. Each $M(E^i)$ is a vector bundle on Δ_{R^i}.
2. Each $\mathrm{Fil}^\bullet M(E^i)$ gives a saturated Nygaardian filtration for the F-crystal $M(E^i)$.
3. The previous two conditions hold true after base change along any map $R^i \to R^j$ in the cosimplicial ring $R^\bullet$.

By construction, we obtain a cosimplicial category $\mathcal{C}^\bullet$ as well as natural functors

$$\mathrm{Perf}((R^\bullet)^{\mathrm{Syn}}) \xleftarrow{a^\bullet} \mathcal{C}^\bullet \xrightarrow{b^\bullet} \mathrm{Vect}^{\varphi}((R^\bullet)_{\Delta})$$

of cosimplicial categories, where $a^\bullet$ is the defining inclusion, while $b^\bullet$ is defined by restricting F-gauges to the open point. As a is termwise fully faithful, the limit functor

$$\lim a^\bullet : \lim \mathcal{C}^\bullet \to \lim \mathrm{Perf}((R^\bullet)^{\mathrm{Syn}}) \simeq \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$$

is also fully faithful. On the other hand, each b^i is fully faithful by construction (as the saturated Nygaardian filtration is determined by the F-crystal and preserved by maps between these); moreover, by Corollary 6.6.5 and Lemma 6.6.10, the functor b^0 is an equivalence. This implies that the limiting functor

$$\lim b^\bullet : \lim \mathcal{C}^\bullet \to \lim \mathrm{Vect}^{\varphi}(R^\bullet) \simeq \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta})$$

is an equivalence. Composing the inverse of this equivalence with $\lim a^\bullet$ then gives a fully faithful functor

$$\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}) \to \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$$

whose image is easily seen to lie inside $\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K)$ by construction. This gives a fully faithful functor

$$\Psi : \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}) \to \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K).$$

One can then check that Φ and Ψ are mutually inverse equivalences. $\square$

The following observation will be useful later in understanding maps to reflexive F-gauges.

Proposition 6.6.14. *Let $M \in \mathrm{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ and $N \in \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K)$. Then there is a natural identification*

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{Syn}}}(M, N) \simeq \mathrm{Hom}_{\mathrm{Coh}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}})}(M|_{\mathcal{O}_K^{\mathbb{\Delta}}}, N|_{\mathcal{O}_K^{\mathbb{\Delta}}}).$$

Proof. This follows by descent from Remark 6.6.9 as well as Lemma 6.6.10. □

6.7 Relation to the Bloch–Kato Selmer group

In this section, we use combine the results from §6.5 and §6.6 to give an exact relationship between the cohomology of a coherent sheaf on $\mathbf{Z}_p^{\mathrm{Syn}}$ and Galois cohomology of the corresponding Galois representation after inverting p.

Our starting point is the observation that any p-torsionfree F-gauge admits a reflexive hull with the difference supported in codimension ≥ 2.

Proposition 6.7.1 (Existence of the reflexive hull)**.** *For any p-torsionfree $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$, there is a universal reflexive F-gauge $\overline{M} \in \mathrm{Coh}^{\mathrm{refl}}(\mathbf{Z}_p^{\mathrm{Syn}})$ equipped with a map $\eta_M : M \to \overline{M}$. Moreover, $\mathrm{Cone}(\eta_M)$ is annihilated by $(p, v_1)^n$ for $n \gg 0$.*

By passage to the p-torsionfree quotient, the proposition also implies that any coherent F-gauge N on $\mathbf{Z}_p$ admits a universal map $N \to \overline{N}$ to a reflexive F-gauge, and moreover that the cone of this map is annihilated by p^n for $n \gg 0$.

Proof. The restriction $M^\circ := M|_{\mathbf{Z}_p^{\mathbb{\Delta}}}$ is naturally a prismatic F-crystal in finitely presented p-torsionfree modules. By the classification of such F-crystals, any such M° has a universal map $M^\circ \to \overline{M^\circ}$ to a prismatic F-crystal $\overline{M^\circ}$ in vector bundles such that the cone of $M^\circ \to \overline{M^\circ}$ is supported on the codimension 2 locus $(\mathbf{Z}_p^{\mathbb{\Delta}})_{p=\mathcal{I}=0} \subset \mathbf{Z}_p^{\mathbb{\Delta}}$ (and, in particular, the two F-crystals have the same étale realization). Let $\overline{M}$ be the unique reflexive F-gauge lifting $\overline{M^\circ}$ (Theorem 6.6.13). Then the map $M^\circ \to \overline{M^\circ}$ lifts uniquely to a map $M \to \overline{M}$ of F-gauges by Proposition 6.6.14. The same proposition also gives then the desired universal property for this map. Moreover, by construction, the cone of $M \to \overline{M}$ has vanishing étale realization, and must thus be killed by $(p, v_1)^n$ for $n \gg 0$ by Remark 6.3.5. □

Consequently, we learn that Galois representations coming from coherent sheaves on $\mathbf{Z}_p^{\mathrm{Syn}}$ are always crystalline[85].

Corollary 6.7.2. *For any $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$, the $G_{\mathbf{Q}_p}$-representation $T_{et}(M)[1/p]$ is crystalline.*

Proof. By replacing M with its p-torsionfree quotient and then by its reflexive hull as in Proposition 6.7.1, we may assume M is reflexive; the claim then follows from (the easy direction of) Theorem 6.6.13. □

The promised relationship between cohomology on $\mathbf{Z}_p^{\mathrm{Syn}}$ and Galois cohomology after inverting p, which follows next, roughly says that the former picks out the "crystalline part" of the latter.

[85] We warn the reader that this is false if we drop the finiteness condition on the sheaf, even in situations where the étale realization has good finiteness properties. For example, say $X/\mathbf{Z}_p$ is any proper lci scheme. Then the F-gauge $\mathcal{H}_{\mathrm{Syn}}(X) \in \mathcal{D}_{qc}(\mathbf{Z}_p^{\mathrm{Syn}})$ is not perfect or even pseudocoherent in general (unless X is smooth over $\mathbf{Z}_p$). However, its étale realization $T_{et}(\mathcal{H}_{\mathrm{Syn}}(X))$ is lisse: it identifies with the $G_{\mathbf{Q}_p}$-representation on the perfect $\mathbf{Z}_p$-complex $R\Gamma(X_{\overline{\mathbf{Q}_p}}, \mathbf{Z}_p)$. Moreover, the cohomology groups of the latter do not typically give crystalline $G_{\mathbf{Q}_p}$-representations on inverting p unless one assumes X is smooth over $\mathbf{Z}_p$.

Proposition 6.7.3. *For any $M \in \mathrm{Coh}(\mathbf{Z}_p^{\mathrm{Syn}})$, consider the map*

$$\eta : R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, M)[1/p] \to R\Gamma(G_{\mathbf{Q}_p}, T_{et}(M))[1/p]$$

of perfect $\mathbf{Q}_p$-complexes coming from functoriality. Writing $\eta^i = H^i(\eta)$, we have:

1. *η^0 is an isomorphism.*
2. *η^1 is injective with image the subspace of $H^1(G_{\mathbf{Q}_p}, T_{et}(M)[1/p])$ spanned by crystalline extensions of $\mathbf{Q}_p$ by $T_{et}(M)[1/p]$.*
3. *The group $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M)[1/p]$ vanishes.*

Proof. Since we are only interested in cohomology after inverting p, thanks to Proposition 6.7.1, we may assume M is a reflexive F-gauge. The exact triangle of Corollary 6.5.22 with p inverted gives a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbf{Z}_p^{\mathrm{Syn}}, M)[1/p] \arrow[r, "\eta^0"] & H^0(G_{\mathbf{Q}_p}, T_{et}(M))[1/p] \arrow[r] & H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M^*\{1\})[1/p]^\vee \arrow[dll, rounded corners] \\
& H^1(\mathbf{Z}_p^{\mathrm{Syn}}, M)[1/p] \arrow[r, "\eta^1"] & H^1(G_{\mathbf{Q}_p}, T_{et}(M))[1/p] \arrow[r] & H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M^*\{1\})[1/p]^\vee \arrow[dll, rounded corners] \\
& H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M)[1/p] \arrow[r] & H^2(G_{\mathbf{Q}_p}, T_{et}(M))[1/p] \arrow[r] & H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M^*\{1\})[1/p]^\vee \arrow[r] & 0
\end{tikzcd}$$

where the $(-)^\vee$ denotes duality in $\mathbf{Q}_p$-vector spaces, and we have used that the cohomological dimension of $\mathbf{Z}_p^{\mathrm{Syn}}$ is ≤ 2 to get the 0's on the first and last term. Let us explain how to deduce (3) assuming (1) and (2). Indeed, assuming the latter, the diagram would then show that $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M^*\{1\})[1/p]^\vee$ for all reflexive F-gauges M. By finiteness, the same holds true before dualizing, i.e., $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, M^*\{1\})[1/p] = 0$ for all reflexive F-gauges M. But then Proposition 6.7.1 and duality imply that $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, N)[1/p] = 0$ for all coherent sheaves N, as wanted in (3). It thus remains to prove (1) and (2).

For (1), as M is reflexive, the map η^0 is bijective by the full faithfulness in Theorem 6.6.13. Similarly, the injectivity in (2) follows from the full faithfulness of Theorem 6.6.13 again by interpreting H^1 via extensions. For the description of the image, again by interpreting H^1 via extensions, it suffices to show the following: the equivalence $\mathrm{Rep}^{\mathrm{crys}}_{\mathbf{Z}_p}(G_K) \to \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K^{\mathrm{Syn}})$ (inverse to étale realization) coming from Theorem 6.6.13 carries a SES to a sequence whose homology is killed by p^n for $n \gg 0$. This follows from Remark 6.3.5 as étale realization of the latter sequence is a SES by construction. $\square$

Remark 6.7.4 (Interpretation via Bloch–Kato Selmer groups)**.** Given a $G_{\mathbf{Q}_p}$-representation on a finite dimensional $\mathbf{Q}_p$-vector space V, Bloch–Kato introduce subspaces $H^i_f(G_{\mathbf{Q}_p}, V) \subset H^i(G_{\mathbf{Q}_p}, V)$ in [BK07, §3]: $H^{\geq 2}_f(-)$ vanishes by fiat, $H^0_f(G_{\mathbf{Q}_p}, V) = H^0(G_{\mathbf{G}_p}, V)$, and $H^1_f(G_{\mathbf{Q}_p}, V)$ is the subspace of $H^1(G_{\mathbf{Q}_p}, V)$ spanned by crystalline extensions of $\mathbf{Q}_p$ by V. A reformulation of Proposition 6.7.3 is thus that the étale realization functor induces isomorphisms

$$H^i(\mathbf{Z}_p^{\mathrm{Syn}}, M)[1/p] \simeq H^i_f(G_{\mathbf{Q}_p}, T_{et}(M))[1/p],$$

for all i. Note that unlike the RHS, the vanishing of the LHS for $i = 2$ is not formal: the proof uses the Lagrangianness result from Corollary 6.5.22, and the statement would be false if we did not invert p[86]. This suggests the following picture: given a motive X over $\mathrm{Spf}(\mathbf{Z}_p)$ with F-gauge realization $M(X)$, the cohomology $R\Gamma(\mathbf{Z}_p^{\mathrm{Syn}}, M(X))$ is reasonable candidate for the "motivic part" of Galois cohomology $R\Gamma(G_{\mathbf{Q}_p}, T)$ of the (integral) Galois representation $T = T_{et}(M(X))$ coming from the corresponding motive X_η over $\mathbf{Q}_p$.

Remark 6.7.5. Based on the results of [EK99] as well as the comparison in Proposition 6.7.3 and the equivalence in Theorem 6.6.13, it would be interesting to understand the difference (if any) between the isogeny category $\mathrm{Perf}(\mathbf{Z}_p^{\mathrm{Syn}})_{\mathbf{Q}_p}$ and the bounded derived category of the abelian category of finite dimensional crystalline $\mathbf{Q}_p$-representations of $G_{\mathbf{Q}_p}$.

[86] In fact, $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{p\}) \neq 0$: as the cohomological dimension of $\mathbf{Z}_p^{\mathrm{Syn}}$ is ≤ 2, the natural map $H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{p\}) \to H^2(\mathbf{Z}_p^{\mathrm{Syn}}, \mathcal{O}\{p\}/(p, v_1))$ is surjective, and the target is nonzero by Example 6.5.13.

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