AN INVITATION TO HIGHER ALGEBRA

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ABSTRACT. These are lecture notes from my course on homological algebra at Caltech (Math 128) during the winter 2021 quarter. They are **under construction**, and will be updated at the course website at the end of each lecture.

CONTENTS

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0. Miscellanea

0.1. Exercises.

0.1.1. In order to obtain a grade of 100% on the homework by the end of the quarter, you will need to earn 120 points.

0.1.2. Partial solutions may be submitted for partial credit.

0.1.3. Solutions to exercises should always be justified (even if e.g. the exercise is stated as merely a "yes or no" question).

0.1.4. In preparing your homework, please copy down the problem statement, since it is possible that the numbering may change.

0.2. Conventions.

0.2.1. As this document is a work in progress, I will frequently want to make changes to existing material (in addition to adding material as we cover it during lecture). Of course, it would be quite difficult for a reader to spot such changes, especially as the document grows. Therefore, any substantial such changes will be temporarily flagged by their change date (for easy searching) and additionally will be colored as follows: changes that have occurred since the most recent lecture are red, changes made between the past two lectures are blue, and changes made between two and three lectures ago are green. Older changes are no longer colored or dated.

0.2.2.   We use standard notation without comment, e.g. $\mathbb{Z}$ denotes the integers and Set denotes the category of sets. However, notation will very often only be "local": the meanings of various symbols will be fluid, and notation may change slightly through the document as needed.

0.2.3.   The term "natural number" (and the notation $\mathbb{N}$) sometimes will include 0 and sometimes will not. It will often be a good exercise to think through this boundary case, to see whether the given assertion holds (or even makes mathematical sense).

0.2.4.   We use the basic language of category theory freely. The canonical reference is [Mac71]. Many more efficient introductions are available, e.g. [Saf] or [Wei94, §A]. We consider posets as categories without comment. We write e.g. $\mathbb{N}^{\leqslant}$ and $\mathbb{Z}^{\leqslant}$ for the usual poset structures on the natural numbers and the integers. Given some datum in a category (e.g. an object or morphism), we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite category, although we may also omit this superscript when our meaning is sufficiently clear. We mostly ignore set-theoretic issues.[1] We refer the reader to [Shua] for a thorough discussion of the role of set theory in category theory.

0.2.5.   Especially in later sections, we will frequently give references to Lurie's books [Lur09] and [Lur]. Our exposition is nevertheless intended to be self-contained, with these references merely providing the reader with entry points for exploring those books further. For brevity, we will use the abbreviations "T" and "A" to refer to these works, and moreover we will omit environment names (except for the section symbol §). So for instance, we will refer to [Lur09, Theorem 4.1.3.1] simply as [T.4.1.3.1] and to [Lur, §1.3.3] simply as [§A.1.3.3].

0.2.6.   The term "(commutative) ring" means "associative unital (resp. commutative) ring". Likewise, modules are always unital (meaning that the unit element acts as the identity).

0.2.7.   In the interest of brevity, universal quantifiers will often be dropped. For instance, an assertion involving an integer n should generally be understood to refer to *all* integers n unless otherwise specified, and formulas involving arbitrary elements (e.g. of abelian groups) should generally be understood to refer to *all* elements unless otherwise specified.

0.2.8.   For brevity, we will often use a slash to make multiple statements at once. This idiom has two possible meanings; the specific meaning should always be clear from context. On the one hand, we will write e.g. "homotopy co/kernel sequence" as a stand-in for "sequences which are simultaneously homotopy cokernel sequences and homotopy kernel sequences – and let us not forget that it suffices to check either condition in order to deduce both". On the other hand, we will write e.g. "co/limits" as a stand in for "both colimits and limits".

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[1]Or, said differently, we implicitly work with respect to a fixed Grothendieck universe.

Part I. Homological algebra

1. Some motivation for homological algebra

1.1. **Intersection theory.** A basic endeavor in geometry is to understand *intersections*. For example, given a (smooth) manifold M and two submanifolds $N_0, N_1 \subseteq M$ of complementary dimensions, a fundamental question is to compute the algebraic intersection number

$$[N_0] \cdot [N_1] \in \mathbb{Z} \ .$$

If N_0 and N_1 intersect *transversely* (i.e. $T_p N_0 + T_p N_1 = T_p M$ for all $p \in (N_0 \cap N_1)$), then this is simply the (signed) sum of their intersection points. Moreover, this is invariant under small perturbations, as long as the intersection remains transverse.

However, if the intersection of N_0 and N_1 is not transverse, the situation is somewhat complicated. On the one hand, there will always exist arbitrarily small perturbations of either N_0 or N_1 that make the intersection transverse, and it is a fact that the resulting intersection number will not depend on the chosen perturbation.[2] However, this approach has a number of (related) drawbacks.

(1) Perturbations are noncanonical.
(2) Perturbations will generally destroy the inherent symmetries of the situation.[3]
(3) Even if one begins with algebraic varieties, the perturbations guaranteed by the genericity of transversality are generally only transcendental.[4]

A first application of homological algebra is to compute non-transverse intersections without perturbations. We will illustrate the failure of ordinary (i.e. non-homological) algebra in §1.4, after some preliminaries.

1.2. **Tensor products.** We first recall the notion of tensor product.

Let R be a commutative ring, and let M and N be R-modules. The (***relative***) ***tensor product*** of M and N over R,[5] denoted $M \otimes_R N$, is the universal abelian group equipped with an R-balanced bilinear function

$$M \times N \xrightarrow{\varphi} M \otimes_R N \ ,$$

i.e. a function satisfying the following axioms:

(1) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$ and $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$;

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[2] A good introduction to these ideas is [GP74].

[3] For instance, perturbations to transverse intersections need not exist in the equivariant context.

[4] It turns out that it is in some sense always possible to perturb of algebraic varieties that achieve transversality, however, at least when the ambient variety is sufficiently nice. This is *Chow's moving lemma*, where "perturb" means "change to a new but rationally equivalent algebraic cycle". It is fundamental in the classical approach to intersection theory in algebraic geometry [EH16].

[5] The word "relative" here is meant to emphasize that R is an arbitrary commutative ring. By contrast, the term "absolute tensor product" would emphasize that $R = \mathbb{Z}$.

(2) $\varphi(m \cdot r, n) = \varphi(m, r \cdot n).$[6]

In other words, for any abelian group A, precomposition with φ determines a canonical isomorphism

$$\{R\text{-bilinear functions } M \times N \to A\} \xleftarrow{\ \cong\ } \{\text{abelian group homomorphisms } M \otimes_R N \to A\} \ .$$

In the case that R is understood (and particularly when $R = \mathbb{Z}$ or when R is a field), we may simply write $\otimes := \otimes_R$.

The relative tensor product $M \otimes_R N$ is defined by a universal property, which does not a priori guarantee that it exists. However, it is also easy to construct explicitly. Namely, one begins with the abelian group $M \times N$ and quotients by the following relations:

(1) $(m + m', n) \sim (m, n) + (m', n)$ and $(m, n + n') \sim (m, n) + (m, n')$;
(2) $(m \cdot r, n) \sim (m, r \cdot n)$.

Exercise 1.1 (2 points). For any natural numbers $m, n \in \mathbb{N}$, prove that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/\gcd(m, n)$.

1.3. **Basic principles of algebraic geometry.** In order to illustrate intersection theory via tensor products, we recall a few basic principles of algebraic geometry. We work over $\mathbb{R}$ to adhere to geometric intuition, but the same ideas apply over any field. For further background, see [Har77, §I.1].

1.3.1.   The polynomial functions on $\mathbb{R}^n$ are the n-variate polynomials: $\mathcal{O}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$. We simply write $R = \mathcal{O}(\mathbb{R}^n)$ (leaving n implicit).

1.3.2.   By definition, an ***algebraic subset*** of $\mathbb{R}^n$ is a closed subset $Z \subseteq \mathbb{R}^n$ that is cut out by (i.e. equal to) the vanishing of some subset $S \subseteq R$ of polynomial functions on $\mathbb{R}^n$.[7] In this case we write $Z = V(S)$, and we say that Z is the ***vanishing locus*** of the elements of S. If $J \subseteq R$ is the ideal generated by a subset $S \subseteq R$, then $V(J) = V(S)$.[8]

1.3.3.   We write $I(Z) \subseteq R$ for the ideal of those functions that vanish along Z. Then, the ring of polynomial functions on Z is

$$\mathcal{O}(Z) = R/I(Z) \ .$$

1.3.4.   Conversely, any ideal $J \subseteq R$ has a corresponding vanishing locus

$$V(J) := \{p \in \mathbb{R}^n : f(p) = 0 \text{ for all } f \in I\} \subseteq \mathbb{R}^n \ .$$

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[6]The notation here stems from the fact that more generally, we can define the relative tensor product when R is merely an associative ring, M is a right R-module, and N is a left R-module.

[7]By definition of the Zariski topology, these are precisely the Zariski-closed subsets of $\mathbb{R}^n$.

[8]Since R is noetherian, any ideal is finitely generated. In other words, we may always take S to be a *finite* set of polynomial functions on $\mathbb{R}^n$.

1.3.5. These constructions determine functions

$$\{\text{subsets of } \mathbb{R}^n\} \xrightarrow[V]{I} \{\text{ideals in } R\} \quad .[9]$$

These are inclusion-reversing, and associate intersections of subsets with unions of ideals. In particular, given algebraic subsets $Z_0, Z_1 \subseteq \mathbb{R}^n$ and writing $I_i = I(Z_i)$, we have

$$\mathcal{O}(Z_0 \cap Z_1) \cong R/(I_0, I_1) \cong R/I_0 \otimes_R R/I_1 \ .$$

1.4. Intersections via tensor products.
We now proceed to study a few basic examples of intersections via tensor products.

1.4.1. Our first example merely illustrates the above principles.

Consider the curves $y = x^2$ and $y = x$ in the plane $\mathbb{R}^2$. Their intersection is the locus where $x = x^2$, or $x \cdot (x - 1) = 0$. Now, $\mathbb{R}$ is an integral domain (in fact, it is a field), and so the equation $r \cdot s = 0$ in $\mathbb{R}$ implies that $r = 0$ or $s = 0$. In this case, we find that the solutions are $x = 0$ and $x = 1$.

We now compute the same intersection, but using the above principles. The algebraic subsets

$$Z_0 = \{(x, y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2 \qquad \text{and} \qquad Z_1 = \{(x, y) \in \mathbb{R}^2 : y = x\} \subseteq \mathbb{R}^2$$

respectively correspond to the ideals

$$I_0 = I(Z_0) = (y - x^2) \subseteq R \qquad \text{and} \qquad I_1 = I(Z_1) = (y - x) \subseteq R \ .$$

So, the polynomial functions on $Z_0 \cap Z_1$ are

$$\begin{aligned}
\mathcal{O}(Z_0 \cap Z_1) \quad &\cong \quad R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x, y]/(y - x^2, y - x) \cong \mathbb{R}[x]/(x - x^2) \\
&= \quad \mathbb{R}[x]/(x \cdot (1 - x)) \cong \mathbb{R}[x]/x \times \mathbb{R}[x]/(1 - x) \cong \mathbb{R} \times \mathbb{R} \ ,
\end{aligned}$$

where the second-to-last isomorphism is via the Chinese remainder theorem (note that $\mathbb{R}[x]$ is a PID, in fact it is a Euclidean domain).[10] The fact that this is a 2-dimensional $\mathbb{R}$-algebra corresponds to the fact that $Z_0 \cap Z_1$ consists of two points.

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[9]Over an algebraically closed field $\Bbbk$, this construction restricts to a bijection between *closed* subsets of $\Bbbk^n$ (with respect to the Zariski topology) and *radical* ideals of $\Bbbk[x_1, \ldots, x_n]$. The composite $V \circ I$ carries a subset $Y \subseteq \Bbbk^n$ to its closure $\overline{Y} \subseteq \Bbbk^n$, while the composite $I \circ V$ carries an ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ to its radical $\sqrt{I} = \{f \in \Bbbk[x_1, \ldots, x_n] : \exists\, n > 0 \text{ s.t. } f^n \in I\} \subseteq R$. By contrast, over $\mathbb{R}$ the function V fails to be injective, e.g. $V(\mathbb{R}[x]) = V(x^2 + 1) = \varnothing$.

[10]An explicit inverse is given by carrying the pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ to the function $x \mapsto f_{a,b}(x) := a + (b - a) \cdot x$ (which has $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$), considered as an element of $\mathbb{R}[x]/(x \cdot (1 - x))$. One can check directly that this is a ring homomorphism. It is clearly injective. To see that it is surjective, for any $g \in \mathbb{R}[x]$ we claim that $g - f_{g(0),g(1)}$ lies in the ideal generated by $x \cdot (x - 1)$. Observe that $g - f_{g(0),g(1)}$ vanishes at $x = 0$ and $x = 1$. So this is simply the assertion that if a polynomial vanishes at $r \in \mathbb{R}$, then we can factor out $(x - r)$. (And this can be accomplished via the Euclidean algorithm.)

1.4.2. Our second example illustrates the power of *scheme theory*, i.e. the presence of nilpotent elements, which can in good situations detect the correct multiplicity of a non-transverse intersection point.

Consider the ideals $I_0 = (y - x^2)$ and $I_1 = (y)$ in R. These correspond to the curves $y = x^2$ and $y = 0$. These intersect "twice" at the origin. This can be seen in differential topology by taking derivatives (in fact, it can be seen in algebraic geometry that way too). Correspondingly, we compute that

$$R/I_0 \otimes_R R/I_1 \cong R/(I_0, I_1) \cong \mathbb{R}[x, y]/(y - x^2, y) \cong \mathbb{R}[x]/(x^2) \ .$$

The 2-dimensionality of this $\mathbb{R}$-algebra again reflects the fact that the two curves $V(I_0)$ and $V(I_1)$ intersect "with multiplicity two". Namely, this $\mathbb{R}$-algebra corresponds to "the origin along with infinitesimal fuzz in the direction of the x-axis". This is in contrast with the previous example, where the tensor product split as a cartesian product.

These techniques are quite robust.

Exercise 1.2 (4 points). Consider the curves $y = x^2$ and $y = -1$ in $\mathbb{R}^2$. Compute and interpret their scheme-theoretic intersection.

1.4.3. Here is the simplest example of a non-transverse intersection for which ordinary (as opposed to homological) algebra fails to give the correct answer.

Consider points $a, b \in \mathbb{R}^1$ as algebraic subsets. These correspond to the ideals $I_0 = (x - a) \subseteq R$ and $I_1 = (x - b) \subseteq R$. We compute the functions on their intersection to be

$$\mathcal{O}(\{a\} \cap \{b\}) \cong R/I_0 \otimes_R R/I_1 \cong \mathbb{R}[x]/(x - a, x - b) \cong \mathbb{R}/(a - b) \cong \begin{cases} \mathbb{R}, & a = b \\ 0, & a \neq b \end{cases} \ .$$

Generically, two points in the line do not intersect, and in this situation (i.e. when $a \neq b$) we obtain the expected intersection number of 0. However, in the non-generic situation where $a = b$, we obtain a 1-dimensional $\mathbb{R}$-algebra.

Using homological algebra, namely the notion of **derived tensor products**, we will be able to obtain the expected intersection number of 0 even when $a = b$.

1.4.4. The following exercise illustrates another source of failure of the expected dimension, introducing projective space along the way.

Exercise 1.3 (6 points). Generically, two lines in $\mathbb{R}^2$ intersect in a point. Of course, not all pairs of lines are in general position. For instance, consider the curves $y = x$ and $y = x + 1$ in $\mathbb{R}^2$.

  (a) Compute (the functions on) their intersection using tensor products.

The issue here is that these lines "just barely avoid intersecting": morally they should intersect "at infinity".[11] This issue is repaired by passing to the projective plane, i.e. the quotient

$$\mathbb{RP}^2 := (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^\times$$

by the scaling action. So, its points are specified by nonzero triples $[x : y : z]$, called *homogeneous coordinates*, which are governed by the relation that for any $\lambda \in \mathbb{R}^\times$ we have $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Moreover, there is an inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$ given by the formula $(x, y) \mapsto [x : y : 1]$.[12]

  (b) Show that a *homogenous* polynomial $g \in \mathbb{R}[x, y, z]$ (i.e. one for which $g(\lambda p) = \lambda^d \cdot g(p)$ for some $d \in \mathbb{N}$) has a well-defined vanishing locus $\tilde{V}(g) \subseteq \mathbb{RP}^2$.
  (c) Find *homogenizations* of $f_1 = y - x$ and $f_2 = y - x - 1$, i.e. homogenous polynomials $g_1, g_2 \in \mathbb{R}[x, y, z]$ such that $g_i([x : y : 1]) = f_i(x, y)$.
  (d) Compute and interpret the intersection of the vanishing loci $\tilde{V}(g_i) \subseteq \mathbb{RP}^2$.

1.4.5. As we have seen in §1.4.4, given two lines in $\mathbb{R}^2$, we are more likely to get the expected number if we intersect them (or rather their closures) in $\mathbb{RP}^2$: namely, this gives the correct answer even when the lines are parallel. However, this fails to give the correct answer when the two lines are *equal*. Derived tensor products repair this failure. Namely, the derived tensor product of a (projective) line with itself in $\mathbb{RP}^2$ is "a line, but with cardinality equal to that of a single point".

1.4.6. Of course, there are also examples that are not self-intersections where derived tensor products give the correct answer where ordinary tensor products do not. For this it is necessary to work in higher dimensions, see e.g. [EH16, Example 2.6].

2. Chain complexes, homology, and tensor products

We now proceed to introduce the basic objects of study in homological algebra.

2.1. **Algebra conventions.** For concreteness, we work in the context of ordinary algebra. Namely, we fix a commutative ring $\Bbbk$ and a $\Bbbk$-algebra R. For the most part, we will work in Mod_R, the category of (right) R-modules, and one may take $\Bbbk$ to be $\mathbb{Z}$. However, at times we will want to specialize to a commutative ring, and for this it is convenient for $\Bbbk$

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[11]A better way to say this would be to consider the equations $y = x$ and $y = tx + 1$: these are surfaces in $\mathbb{R}^3$, which may be considered as families of lines indexed by the parameter $t \in \mathbb{R}$. As $t \to 1^+$ their intersection point has $x \to -\infty$, while as $t \to 1^-$ their intersection point has $x \to +\infty$. This suggests that there should be a *single* point "at infinity" where they intersect in the case that $t = 1$.

[12]So, the "points at infinity" are those of the form $[x : y : 0]$. Since we disallow the possibility that $x = y = 0$, these form a copy of $\mathbb{RP}^1 := (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}^\times$. Note that each such point $[x : y : 0]$ may be uniquely identified with a slope $\frac{y}{x}$, where we declare that $\infty := \frac{y}{0}$ for $y \neq 0$ (this is the unique point in $\mathbb{RP}^1 \backslash \mathbb{R}^1$).

to be arbitrary.[13]   Moreover, we will study some interactions between $\Bbbk$-modules and R-modules. At the level of ordinary (i.e. non-homological) algebra, these are encapsulated by the following facts.

(1) The category $\mathsf{Mod}_\Bbbk$ is symmetric monoidal via the tensor product, which we denote by $\otimes := \otimes_\Bbbk$; its unit object is $\Bbbk$.

(2) The category Mod_R is naturally enriched in $\mathsf{Mod}_\Bbbk$. In other words, for any R-modules $M, N \in \mathsf{Mod}_R$, the set $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$ of R-linear homomorphisms carries the natural structure of a $\Bbbk$-module, and moreover composition in Mod_R is $\Bbbk$-multilinear.

(3) Moreover, $\Bbbk$-modules naturally act on R-modules in two different ways: for any $\Bbbk$-module $T \in \mathsf{Mod}_\Bbbk$ and any R-modules $M, N \in \mathsf{Mod}_R$ we have R-modules

$$T \otimes_\Bbbk M \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N) \ ,$$

where the (right) R-actions are induced from those on M and N, and these constructions participate in natural isomorphisms

$$\mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, \mathsf{hom}_{\mathsf{Mod}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Mod}_R}(T \otimes_\Bbbk M, N) \cong \mathsf{hom}_{\mathsf{Mod}_R}(M, \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T, N)) \ .$$

Of course, one may take $R = \Bbbk$ as a special case.[14]   As a result, the notions that we will develop relating to the interactions between $\Bbbk$-modules and R-modules will all be *generalizations* of the notions that we develop relating to $\Bbbk$-modules alone.

As we will see later, most of the theory works equally well for a general abelian category, although there will be some additional hiccups that do not arise when studying modules.[15]

2.2. **Chain complexes.** A ***chain complex*** of R-modules is a diagram

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of R-modules such that for all $n \in \mathbb{Z}$, the composite $d_n \circ d_{n+1} = 0$. One may simply write $M_\bullet$ for a chain complex; the bullet indicates that "all indices are being referred to at once". To emphasize the differentials, one may write $(M_\bullet, d_\bullet)$. Also, one may simply refer to a chain complex as a "complex".[16]   On the other hand, we also may omit the bullet and simply write $M := M_\bullet$ for simplicity. The integer n is called the ***degree*** or the ***dimension***. For an element $m \in M_n$, we may write $\deg(m) := n$.

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[13]Of course, we will apply results developed for R-modules to $\Bbbk$-modules without comment.

[14]The above facts then reduce to the assertion that $\mathsf{Mod}_\Bbbk$ is a *closed* symmetric monoidal category, i.e. that it carries a self-enrichment that is compatible with its symmetric monoidal structure.

[15]On the other hand, the *Freyd–Mitchell embedding theorem* states that any abelian category embeds fully faithfully into Mod_R for some ring R (although the choice of such a ring R is noncanonical). So in a sense, working at the level of abelian categories offers no additional generality.

[16]The word "chain" here is historical: the first example of a chain complex has in degree n the "n^{th} chain group" of a simplicial complex X, i.e. the group of chains (i.e. formal linear combinations) of n-simplices of X. (It was only later realized that chain complexes are worth studying in their own right.)

The morphisms d_n are called the **differentials** of the chain complex. We fix the convention that they are always indexed by their *source* (i.e. the source of d_n is M_n). However, one frequently omits the indices, in which case the equation $d_n \circ d_{n+1} = 0$ may be more simply written as $d^2 = 0$. On the other hand, when we wish to emphasize that these are the differentials of $M_\bullet$, we superscript them as d_n^M.

In this notation, a morphism of chain complexes $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ is a sequence of morphisms $M_n \xrightarrow{f_n} N_n$ of R-modules such that the diagram

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{n+2}^M} & M_{n+1} & \xrightarrow{d_{n+1}^M} & M_n & \xrightarrow{d_n^M} & M_{n-1} & \xrightarrow{d_{n-1}^M} & \cdots \\
& & \downarrow{f_{n+1}} & & \downarrow{f_n} & & \downarrow{f_{n-1}} & & \\
\cdots & \xrightarrow{d_{n+2}^N} & N_{n+1} & \xrightarrow{d_{n+1}^N} & N_n & \xrightarrow{d_n^N} & N_{n-1} & \xrightarrow{d_{n-1}^N} & \cdots
\end{array}
$$

commutes.[17] These morphisms are often referred to as **chain maps**. We write Ch_R for the category of chain complexes of R-modules.

In depicting a complex, it is customary to decorate the term in degree 0 with a squiggled underline when appropriate.

Any R-module $M \in \mathsf{Mod}_R$ determines a chain complex concentrated in degree 0:

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \cdots .$$

This construction determines a fully faithful embedding

$$\mathsf{Mod}_R \lhook\joinrel\longrightarrow \mathsf{Ch}_R .$$

As a result, we may not notationally distinguish between data in Mod_R and its image in Ch_R.

When a complex only has a few nonzero terms, for brevity one may omit the zero terms. For instance, the above complex may also be written as $\underset{\sim}{M}$.

2.3. **Homology.** Fix a chain complex $M_\bullet$. Its n-**cycles** and n-**boundaries** are the submodules

$$\mathsf{Z}_n(M_\bullet) := \mathsf{ker}(d_n) \subseteq M_n \qquad \text{and} \qquad \mathsf{B}_n(M_\bullet) := \mathsf{im}(d_{n+1}) \subseteq M_n .^{18}$$

Note that $\mathsf{B}_n(M_\bullet) \subseteq \mathsf{Z}_n(M_\bullet)$ because $d^2 = 0$. Then, the nth **homology** of $M_\bullet$ is the quotient R-module

$$\mathsf{H}_n(M_\bullet) := \frac{\mathsf{Z}_n(M_\bullet)}{\mathsf{B}_n(M_\bullet)} := \frac{\mathsf{ker}(d_n)}{\mathsf{im}(d_{n+1})} .$$

Although $\mathsf{H}_n(M_\bullet)$ is an R-module, it is common to refer to it merely as a *homology group*.

Exercise 2.1 (2 points). Verify that the constructions Z_n, B_n, and H_n define functors

$$\mathsf{Ch}_R \longrightarrow \mathsf{Mod}_R .$$

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[17]For typographical reasons, we will generally draw morphisms of chain complexes vertically in this way.

[18]The German words for "cycle" and "boundary" respectively begin with the letters "Z" and "B".

A morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R is called a ***quasi-isomorphism*** if the induced morphisms $H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We may indicate that a morphism is a quasi-isomorphism by decorating the arrow as $\xrightarrow{\approx}$.

We say that $M_\bullet$ is ***acyclic*** if $\mathsf{H}_n(M_\bullet) = 0$ for all n. So, $M_\bullet$ is acyclic if and only if the unique map $0 \to M_\bullet$ from the zero complex is a quasi-isomorphism, if and only if the unique map $M_\bullet \to 0$ to the zero complex is a quasi-isomorphism.

2.4. The derived category of R-modules.

2.4.1.   By and large, we would like to think of quasi-isomorphic chain complexes as "essentially interchangeable", with some representatives of a given quasi-isomorphism class (namely the *projective* and *injective* complexes introduced below) being "well-adapted" for certain purposes.[19] In other words, one should think of quasi-isomorphisms as if they are actual isomorphisms.

This can be made literally true by ***localizing*** the category Ch_R at the quasi-isomorphisms, i.e. by adjoining formal inverses for them. This yields a category that (for reasons that will become clear later) we will denote by $\mathsf{H}_0(\mathbf{D}_R)$ and refer to as ***the derived category of R-modules***; its objects are called ***derived R-modules***.[20] So by definition, there is a canonical functor

$$\mathsf{Ch}_R \longrightarrow \mathsf{H}_0(\mathbf{D}_R)$$

that carries all quasi-isomorphisms to isomorphisms, and moreover it is universal with respect to this requirement. Indeed, for any category $\mathcal{C}$, the restriction functor

$$\mathsf{Fun}(\mathsf{Ch}_R, \mathcal{C}) \longleftarrow \mathsf{Fun}(\mathsf{H}_0(\mathbf{D}_R), \mathcal{C})$$

is a fully faithful inclusion, whose image consists of those functors $\mathsf{Ch}_R \to \mathcal{C}$ that carry quasi-isomorphisms to isomorphisms.

Note that a derived R-module is a "purely homotopical" object: while it can by definition be presented by a chain complex of R-modules, one cannot speak e.g. of its underlying R-module in dimension 0, as this notion is not preserved under quasi-isomorphisms.[21] On the other hand, one *can* speak e.g. of its nth homology, as this notion is by definition preserved under quasi-isomorphisms.

---

[19]This is very closely akin to how one should think of equivalent categories as "essentially interchangeable", even when they are not isomorphic. However, in a precise sense, *all* categories are "equally well-adapted" for all purposes (in contrast with chain complexes).

[20]The placement of the word "derived" is admittedly slightly unfortunate, but this terminology is quite common.

[21]Likewise, one cannot speak of the underlying set of a weak homotopy equivalence class of topological spaces, nor can one speak of the set of objects of an equivalence class of categories.

2.4.2.   Essentially by construction, given two complexes $M, N \in \mathsf{Ch}_R$, morphisms from M to N in the derived category are given by equivalence classes of zigzags

$$M \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} N$$

(in which all backwards maps are quasi-isomorphisms). Thankfully, it will turn out that every equivalence class contains representatives of the forms

$$M \xleftarrow{\approx} \bullet \longrightarrow N \qquad \text{and} \qquad M \longrightarrow \bullet \xleftarrow{\approx} N \ ,$$

which makes the situation substantially more manageable.[22]

2.4.3.   Although we introduce the derived category now, we will not have much use for it: it contains too little information. The richer and more primitive object is $\mathbf{D}_R$, the **derived ∞-category** of R. This is a mathematical entity whose objects are still the derived R-modules, but whose hom-objects are more elaborate: namely, they are *derived $\Bbbk$-modules*. Of course, passing from $\mathbf{D}_R$ to $\mathsf{H}_0(\mathbf{D}_R)$ amounts to entracting only the 0th homology groups of these hom-objects.

2.4.4.   While quasi-isomorphic complexes have isomorphic homology groups, we will see that the converse is generally false: the obstruction will be encoded by **k-invariants**, as explained in §7.3.3.[23] That is, a quasi-isomorphism class of complexes is equivalent data to its homology groups along with all of its k-invariants. For this reason, we will generally consider (quasi-isomorphism classes of) complexes themselves as the "true" mathematical objects of lasting interest, while their homology groups are merely algebraic invariants that can be extracted therefrom.

2.5. **Tensor products.** Given complexes $M, N \in \mathsf{Ch}_\Bbbk$ of $\Bbbk$-modules, we define their **tensor product** complex

$$(M \otimes N)_\bullet := (M \otimes_\Bbbk N)_\bullet$$

as follows. First of all, we define its kth term to be

$$(M \otimes N)_k := \bigoplus_{i+j=k} (M_i \otimes N_j) := \bigoplus_{i+j=k} (M_i \otimes_\Bbbk N_j) \ .$$

Then, the differential is characterized by the fact that it carries a pure tensor

$$m \otimes n \in (M_i \otimes N_j) \subseteq (M \otimes N)_k$$

---

[22]These reductions are guaranteed by the existence of two different **model structures** on Ch_R, which respectively have the features that all objects are fibrant and that all objects are cofibrant.

[23]This same name is given to the (closely analogous) obstructions to spaces with the same homotopy groups being weak homotopy equivalent.

to the sum of pure tensors

$$d(m \otimes n) := d(m) \otimes n + (-1)^i \cdot m \otimes d(n) \ ,^{24}$$

(an element of $((M_{i-1} \otimes N_j) \oplus (M_i \otimes N_{j-1})) \subseteq (M \otimes N)_{k-1})$. More elaborately, this may be written as

$$d_k^{M \otimes N}(m \otimes n) := d_i^M(m) \otimes n + (-1)^i \cdot m \otimes d_j^N(n) \ .$$

Exercise 2.2 (2 points). Verify that this formula defines a complex.

In particular, in solving Exercise 2.2 you will see why the signs are necessary in the definition of the tensor product of complexes. In fact, many sign conventions are possible (and all give equivalent symmetric monoidal categories), but it is impossible to remove all signs from the theory (unless one works over $\mathbb{F}_2$).

From here, it is straightforward to see that the above construction defines a monoidal structure

$$\mathsf{Ch}_\mathbb{k} \times \mathsf{Ch}_\mathbb{k} \xrightarrow{\otimes} \mathsf{Ch}_\mathbb{k} \ ,$$

with unit object $\mathbb{k} := \underset{\approx}{\mathbb{k}} \in \mathsf{Ch}_\mathbb{k}$. In fact, this is a *symmetric* monoidal structure, with symmetry isomorphisms

$$M \otimes N \xrightarrow{\cong} N \otimes M$$

determined by the formula

$$m \otimes n \longmapsto (-1)^{\deg(m) \cdot \deg(n)} \cdot n \otimes m \ .^{25}$$

More generally, this same construction defines an action

$$\mathsf{Ch}_\mathbb{k} \times \mathsf{Ch}_R \xrightarrow{\otimes} \mathsf{Ch}_R$$

of the symmetric monoidal category $(\mathsf{Ch}_\mathbb{k}, \otimes, \mathbb{k})$ on the category Ch_R.

Exercise 2.3 (8 points). Fix two complexes $M, N \in \mathsf{Ch}_\mathbb{k}$.

(a) Verify that the formula $[m] \otimes [n] \mapsto [m \otimes n]$ determines a morphism

$$H_i(M) \otimes H_j(N) \longrightarrow H_{i+j}(M \otimes N)$$

of $\mathbb{k}$-modules.

It follows that we obtain a morphism

$$\bigoplus_{i+j=k} (H_i(M) \otimes H_j(N)) \longrightarrow H_k(M \otimes N)$$

of $\mathbb{k}$-modules.

---

[24]The factor $(-1)^i$ is determined by the *Koszul sign rule*, which is a general principle asserting that in commuting graded quantities past one another of degrees $\alpha, \beta \in \mathbb{Z}$ one should pick up a factor of $(-1)^{\alpha \cdot \beta}$. Namely, we consider the symbol "d" as an expression of degree -1 (which makes sense since it changes dimensions by 1).

[25]This formula is another instance of the Koszul sign rule.

(b) Prove that this is an isomorphism under the assumption that $\Bbbk$ is a field.

(c) Find an example where this is not an isomorphism.

3. Homotopies, homotopy co/kernels, and exact sequences

3.1. Homotopies.

3.1.1.   Let $M_1 \xrightarrow{f} M_0$ be a morphism of R-modules. This gives us a complex $M_\bullet := (M_1 \xrightarrow{f} M_0)$. Observe that this has a canonical morphism

$$
\begin{array}{ccc}
M_\bullet & & M_1 \xrightarrow{\ f\ } M_0 \\
\downarrow & := & \downarrow \qquad\qquad \downarrow \\
\mathsf{coker}(f) & & 0 \longrightarrow \mathsf{coker}(f)
\end{array}
$$

to the cokernel of f (considered as a complex in degree 0). Observe further that

$$
\mathsf{H}_n(M_\bullet) \cong \begin{cases} \mathsf{coker}(f)\,, & n = 0 \\ \mathsf{ker}(f)\,, & n = 1 \\ 0\,, & \text{otherwise} \end{cases} .
$$

Hence, the above map is a quasi-isomorphism iff f is an injection. One might think of $M_\bullet$ as a "presentation" of the underlying R-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$: the generators are M_0, the relations are M_1 (i.e. each $m \in M_1$ gives a relation $d(m) \sim 0$), but then $\mathsf{H}_1(M_\bullet)$ furthermore measures the "redundancy" of the relations. Said differently, $M_\bullet$ is a "homotopically correct" version of the cokernel of f, which remembers not only the literal cokernel but also the extent to which the relations are overdetermined. Indeed, it will be the *homotopy cokernel* of the morphism f.

3.1.2.   Let $M_\bullet, N_\bullet \in \mathsf{Ch}_R$ be complexes and let $f_\bullet, g_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ be morphisms. A (***chain***) ***homotopy*** from $f_\bullet$ to $g_\bullet$ is a set of morphisms

$$
M_n \xrightarrow{h_n} N_{n+1}
$$

satisfying the condition that

$$
g_n - f_n = d^N_{n+1} \circ h_n + h_{n-1} \circ d^M_n \ .
$$

We may write this as $f_\bullet \overset{h_\bullet}{\Rightarrow} g_\bullet$. A ***nullhomotopy*** of $g_\bullet$ is a homotopy $0 \overset{h_\bullet}{\Rightarrow} g_\bullet$ from the zero map. A ***contraction*** of a complex is a nullhomotopy of its identity map. If a complex admits a contraction, we say that it is ***contractible***.

Exercise 3.1 (3 points). Show that the relation of homotopy on $\mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ is an equivalence relation.

In fact, the argument of Exercise 3.1 easily upgrades to imply that we can enhance Ch_R from an ordinary category to a category enriched in groupoids (a.k.a. a $(2,1)$-category):[26] its objects are chain complexes, its 1-morphisms are chain maps, and its 2-morphisms are chain homotopies.[27] Indeed, the arguments for transitivity, reflexivity, and symmetry of the relation of homotopy respectively endow these hom-categories with their composition laws, identity morphisms, and inverses (so that they are indeed hom-groupoids). It is moreover clear that homotopies may be composed appropriately, either by definition or using Exercise 3.2 below.

3.1.3. For present and future use, we introduce the complex $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ and the morphisms $i_0, i_1 \in \mathsf{hom}_{\mathsf{Ch}_{\Bbbk}}(\Bbbk, \mathbb{I})$ according to the diagram

This object $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ (along with the two maps i_0 and i_1) is an ***interval object*** for the homotopy theory of chain complexes (which explains the notation).[28] The general definition of an interval object is suggested by the discussion of §8.1.2. In the present setting, this assertion amounts to the following result.

Exercise 3.2 (4 points). Given morphisms $f, g \in \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$, prove that a homotopy $f \Rightarrow g$ is equivalent data to a morphism $\mathbb{I} \otimes M \to N$ that makes the diagram

commute.

---

[26]As we will see, this is the homotopy $(2,1)$-category of a more fundamental object, namely an ∞-category (meaning an $(\infty, 1)$-category).

[27]This explains the notation $f_\bullet \Rightarrow g_\bullet$ just introduced.

[28]This is a particularly natural choice of an interval object: as we will see, it is the simplicial chains (with coefficients in $\Bbbk$) on the 1-simplex Δ^1 (whose underlying topological space is a closed interval), and the two maps i_0 and i_1 are the simplicial chains on the inclusions of its two 0-simplices (i.e. the endpoints of the closed interval).

commute.

3.1.4. A morphism $M \xrightarrow{f} N$ in Ch_R is a **homotopy equivalence** if there exists a morphism $N \xrightarrow{g} M$ and homotopies $\mathsf{id}_M \Rightarrow g \circ f$ and $f \circ g \Rightarrow \mathsf{id}_N$.[29] We may indicate that a morphism is a homotopy equivalence by decorating it as $\xrightarrow{\sim}$.

As a special case, a complex $M_\bullet$ is contractible if and only if either unique map $0 \to M_\bullet$ or $M_\bullet \to 0$ is a homotopy equivalence.

Exercise 3.3 (3 points). Show that homotopic maps on complexes give equal maps on homology.

It follows immediately from Exercise 3.3 that homotopy equivalences are quasi-isomorphisms, and in particular that contractible complexes are acyclic. However, the converse is false.

Exercise 3.4 (10 points).
   (a) Show that the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$$

   of $\mathbb{Z}/4$-modules is acyclic but not contractible.
   (b) Show that the complex

$$\cdots \xrightarrow{x} \mathbb{k}[x]/x^2 \xrightarrow{x} \mathbb{k}[x]/x^2 \xrightarrow{x} \mathbb{k}[x]/x^2 \xrightarrow{x} \cdots$$

   of $\mathbb{k}[x]/x^2$-modules is acyclic. Show that it is contractible as a complex of $\mathbb{k}$-modules, but not as a complex of $\mathbb{k}[x]/x^2$-modules (assuming that $\mathbb{k} \neq 0$).

3.1.5. We will generally consider homotopic maps as "essentially interchangeable". On the other hand, rather than merely positing the *existence* of a homotopy between two maps, we will always want to *keep track* of the homotopy that witnesses them as being homotopic.

3.2. Homotopy cokernels.

3.2.1. Recall that the cokernel of a morphism $M \xrightarrow{f} N$ in Mod_R is by definition an R-module $\mathsf{coker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$N \xrightarrow{u} \mathsf{coker}(f)$$

---

[29]The directions of these homotopies are intended to be suggestive of adjunctions (with f functioning as the left adjoint), but by Exercise 3.1 they are irrelevant. (Likewise, equivalences of categories are both left adjoints and right adjoints.)

satisfying the universal property that precomposition with u determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(\mathsf{coker}(f), T) \xrightarrow{\;\cong\;} \left\{ \begin{array}{c} \text{morphisms } N \to T \text{ such that the} \\ \text{composite } M \xrightarrow{f} N \to T \text{ is zero} \end{array} \right\}_{.}{}^{30}$$

From here, the principles indicated in §3.1.5 lead directly to the definition of a **homotopy cokernel** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R:[31] this is an object $\mathsf{hcoker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$N_\bullet \xrightarrow{\;u_\bullet\;} \mathsf{hcoker}(f_\bullet)$$

satisfying the universal property that precomposition with $u_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), T_\bullet) \xrightarrow{\;\cong\;} \left\{ \begin{array}{c} \text{morphisms } N_\bullet \to T_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } M_\bullet \xrightarrow{f_\bullet} N_\bullet \to T_\bullet \end{array} \right\}_{.}{}^{32}$$

In fact, we claim that we have already seen an example of a homotopy cokernel: namely, for any morphism $M \xrightarrow{f} N$ in Mod_R, the complex $(M \xrightarrow{f} \underset{\sim}{N}) \in \mathsf{Ch}_R$ equipped with the map

$$\begin{array}{ccc}
0 & \longrightarrow & \underset{\sim}{N} \\
\downarrow & & \downarrow{\scriptstyle \mathsf{id}_N} \\
M & \xrightarrow{\;f\;} & \underset{\sim}{N}
\end{array}$$

---

[30]Said differently, the cokernel of f is the pushout

$$\begin{array}{ccc}
M & \xrightarrow{\;f\;} & N \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathsf{coker}(f)
\end{array}_{.}$$

[31]This is more often referred to as the *cone*, but we prefer the term "homotopy cokernel" as it is more directly suggestive of the object's "role in life" (a.k.a. its *raison d'être*). It is also common to refer to this as the *homotopy cofiber*, but this deviates from the more standard terminology of "cokernel" (as opposed to "cofiber") in the context of an abelian category.

[32]Similarly, the homotopy cokernel of f may be characterized as the *homotopy* pushout

$$\begin{array}{ccc}
M_\bullet & \xrightarrow{\;f_\bullet\;} & N_\bullet \\
\downarrow & \nearrow & \downarrow \\
0 & \longrightarrow & \mathsf{hcoker}(f_\bullet)
\end{array}_{,}$$

i.e. the *initial homotopy-coherent cocone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$. Here, the symbol $\Rightarrow$ indicates that the square only commutes up to a (specified) homotopy.

is a homotopy cokernel of f (when considered in Ch_R). More generally, given a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R, consider the diagram

$$
\cdots \xrightarrow{\begin{pmatrix} -d_1^M & 0 \\ f_1 & d_2^N \end{pmatrix}} \begin{matrix} M_0 \\ \oplus \\ N_1 \end{matrix} \xrightarrow{\begin{pmatrix} -d_0^M & 0 \\ f_0 & d_1^N \end{pmatrix}} \begin{matrix} M_{-1} \\ \oplus \\ N_0 \end{matrix} \xrightarrow{\begin{pmatrix} -d_{-1}^M & 0 \\ f_{-1} & d_0^N \end{pmatrix}} \begin{matrix} M_{-2} \\ \oplus \\ N_{-1} \end{matrix} \xrightarrow{\begin{pmatrix} -d_{-2}^M & 0 \\ f_{-2} & d_{-1}^N \end{pmatrix}} \cdots
$$

of R-modules.

Exercise 3.5 (6 points)**.** Verify that the above diagram indeed defines a complex and moreover is a homotopy cokernel of $f_\bullet$.

In particular, we find that for a morphism $M \xrightarrow{f} N$ in Mod_R, we have a canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f)$, and this is a quasi-isomorphism iff f is injective. This is an instance of the general principle that homotopically sensitive constructions are equivalent (in this case quasi-isomorphic) to their ordinary variants in "simple" situations (in this case, when there is no redundacy in the relations). More generally, we have the following.

Exercise 3.6 (6 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R that is injective in each dimension.

(a) Show that the canonical morphism

$$\mathsf{hcoker}(f) \longrightarrow \mathsf{coker}(f)$$

to the levelwise cokernel is a quasi-isomorphism.
(b) Give an example showing that this map need not be a homotopy equivalence.

As the following exercise illustrates, homotopy cokernels give us a way of translating conditions on morphisms between chain complexes to conditions on chain complexes themselves.[33]

Exercise 3.7 (4 points)**.** Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

(a) Prove that f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic.[34]
(b) Prove that f is a homotopy equivalence iff $\mathsf{hcoker}(f)$ is contractible.

---

[33]In the paradigm of *Goodwillie calculus*, a category may be thought of as a "categorified manifold": objects and morphisms in the category respectively correspond to points and paths in the manifold. In this analogy, vector spaces correspond to *stable* categories, of which chain complexes give a fundamental example, and passage to homotopy cokernels corresponds to translation to the origin.

[34]The corresponding statement fails e.g. for spaces: there exist spaces X such that the morphism $X \to \mathsf{pt}$ is not a weak homotopy equivalence and yet $\Sigma X := \mathsf{hcoker}(X \to \mathsf{pt})$ is weakly contractible. Such spaces are called *acyclic*, as they are characterized by having the integral homology of a point.

3.2.2.   As a special case of a homotopy cokernel, we simply write

$$\Sigma M_\bullet := \mathsf{hcoker}(M_\bullet \longrightarrow 0) \, , {}^{35}$$

and refer to this as the ***suspension*** of $M_\bullet$.[36] So by definition, giving a chain map $\Sigma M_\bullet \to T_\bullet$ is equivalent to giving a nullhomotopy of the composite map $M_\bullet \to 0 \to T_\bullet$.

It is evident from the construction that suspension defines an autoequivalence

$$\mathsf{Ch}_R \xrightarrow{\Sigma} \mathsf{Ch}_R \, .$$

Namely,

$$\Sigma \left(\cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} \underset{\sim}{M_0} \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots \right) \cong \left(\cdots \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} \underset{\sim}{M_{-1}} \xrightarrow{-d_{-1}} M_{-2} \xrightarrow{-d_{-2}} \cdots \right) :$$

the operation of suspension simply shifts all terms up by one and negates all differentials. We write Σ^{-1} for its inverse, which we refer to as ***desuspension***. More generally, for any $k \in \mathbb{N}$ we write $\Sigma^k := \Sigma^{\circ k}$ and $\Sigma^{-k} := (\Sigma^{-1})^{\circ k}$. We also note for future reference the evident natural isomorphisms

$$\mathsf{H}_n \circ \Sigma^k \cong \mathsf{H}_{n-k}$$

for all $n, k \in \mathbb{Z}$.

Exercise 3.8 (6 points). Fix a morphism $M \xrightarrow{f} N$ in Ch_R.

(a) Using the formula for the homotopy cokernel above, establish a canonical homotopy equivalence

$$\Sigma M \simeq \mathsf{hcoker}(N \xrightarrow{u} \mathsf{hcoker}(f)) \, . {}^{37}$$

(b) Establish this same homotopy equivalence using the universal characterization of homotopy cokernels (as well as previously established properties of homotopies).

3.3. **Homotopy kernels.**

---

[35]Not coincidentally, when R is commutative this admits a canonical identification

$$\Sigma M_\bullet \cong (R \to \underset{\sim}{0}) \otimes M_\bullet$$

with the tensor product of $M_\bullet$ with the reduced simplicial chains on the simplicial circle $\Delta^1/\partial\Delta^1$. (Note that this is consistent with our sign convention for tensor products; the complex $M_\bullet \otimes (R \to \underset{\sim}{0})$ is different (although naturally isomorphic)).

[36]This is more often denoted $M_\bullet[1]$ and referred to as the *shift* of M, but we prefer the more blatantly topological notation and terminology.

[37]That these are quasi-isomorphic follows from Exercise 3.6(a).

3.3.1.   Dually, recall that the kernel of a morphism $M \xrightarrow{f} N$ is Mod_R is by definition an R-module $\mathsf{ker}(f) \in \mathsf{Mod}_R$ equipped with a morphism

$$\mathsf{ker}(f) \xrightarrow{v} M$$

satisfying the universal property that postcomposition with v determines a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(T, \mathsf{ker}(f)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T \to M \text{ such that the} \\ \text{composite } T \to M \xrightarrow{f} N \text{ is zero} \end{array} \right\} .^{38}$$

This leads to the dual notion of a ***homotopy kernel*** of a morphism $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ in Ch_R: this is an object $\mathsf{hker}(f_\bullet) \in \mathsf{Ch}_R$ equipped with a morphism

$$\mathsf{hker}(f_\bullet) \xrightarrow{v_\bullet} M_\bullet$$

satisfying the universal property that postcomposition with $v_\bullet$ determines a bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f_\bullet)) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{morphisms } T_\bullet \to M_\bullet \text{ equipped with a} \\ \text{nullhomotopy of the composite } T_\bullet \to M_\bullet \xrightarrow{f_\bullet} N_\bullet \end{array} \right\} .^{39}$$

Exercise 3.9 (4 points)**.** Given a morphism $M \xrightarrow{f} N$ in Ch_R, prove that the evident levelwise projection map

$$\Sigma^{-1}\mathsf{hcoker}(f) \longrightarrow M$$

is a homotopy kernel of f.[40]

By combining Exercises 3.6(a) and 3.9, it follows that if a morphism $M \xrightarrow{f} N$ in Ch_R is surjective in each dimension, then the canonical morphism

$$\mathsf{ker}(f) \longrightarrow \mathsf{hker}(f)$$

from the levelwise kernel is a quasi-isomorphism (though again it is not necessarily a homotopy equivalence).

---

[38]Said differently, the kernel of f is the pullback

$$\begin{array}{ccc} \mathsf{ker}(f) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array} .$$

[39]Similarly, the homotopy kernel of f may be characterized as the *homotopy* pullback

$$\begin{array}{ccc} \mathsf{hker}(f_\bullet) & \longrightarrow & 0 \\ \downarrow & \swarrow & \downarrow \\ M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \end{array} ,$$

i.e. the *terminal homotopy-coherent cone* over the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$.

[40]There are unfortunately some signs that should arise here. They could be removed by tweaking the construction of hker (giving a different but isomorphic formula), but they would then arise elsewhere.

3.3.2.   Although it will take some time to see why, the following feature is in some sense the *fundamental advantage* of working in Ch_R instead of in Mod_R.[41]

Exercise 3.10 (6 points)**.** Prove that for any morphism $M \xrightarrow{f} N$ in Ch_R the dashed canonical morphisms

are homotopy equivalences.[42]

Namely, this implies that up to homotopy equivalence, every homotopy cokernel sequence is a homotopy kernel sequence, and conversely. Of course, this fails drastically in Mod_R: a cokernel sequence is a kernel sequence iff the original map is injective, and a kernel sequence is a cokernel sequence iff the original map is surjective.

3.4. **Exact sequences.**

3.4.1.   A complex $M_\bullet \in \mathsf{Ch}_R$ is said to be **exact** at M_n if $\mathsf{H}_n(M_\bullet) = 0$. In particular, an acyclic complex is also called an **exact sequence**, or sometimes a **long exact sequence** to emphasize that it is (potentially) infinite in one or both directions.

As a special case, a **short exact sequence** is a three-term acyclic complex

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \ .$$

So, a short exact sequence is such a diagram satisfying the conditions that f is injective, g is surjective, and $\mathsf{ker}(g) = \mathsf{im}(f)$. In this case, one may also say that M is an **extension** of N by L. For instance,

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules, which expresses $\mathbb{Z}/4$ as an extension of $\mathbb{Z}/2$ by itself.

More generally, we may refer to any (possibly finite) sequence of morphisms in Mod_R as an **exact sequence** if it is exact at all interior terms. So for example, one may refer to the diagram

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

---

[41]Namely, this is the key property that makes chain complexes into a *stable* category.

[42]Note that these diagrams are only *homotopy-coherently* commutative (indeed, the canonical morphisms are induced by homotopy-coherent universal properties).

as an exact sequence.[43]

3.4.2.   The following is the source of almost every single long exact sequence in mathematics.[44]

Exercise 3.11 (2 points)**.** For any morphism $M \xrightarrow{f} N$ in Ch_R, show that the sequence

$$\mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f))$$

is exact.

Namely, from Exercises 3.8, 3.9, and 3.10 we obtain an infinite sequence

$$\cdots \xrightarrow{\Sigma^{-1}v} \Sigma^{-1}M \xrightarrow{\Sigma^{-1}f} \Sigma^{-1}N \xrightarrow{\Sigma^{-1}u} \Sigma^{-1}\mathsf{hcoker}(f)$$

$$\wr\|$$

$$\mathsf{hker}(f) \xrightarrow{v} M \xrightarrow{f} N \xrightarrow{u} \mathsf{hcoker}(f)$$

$$\wr\|$$

$$\Sigma\mathsf{hker}(f) \xrightarrow{\Sigma v} \Sigma M \xrightarrow{\Sigma f} \Sigma N \xrightarrow{\Sigma u} \cdots$$

of morphisms in Ch_R in which every composable pair of morphisms is a homotopy cokernel sequence up to homotopy equivalence. Thereafter, by Exercise 3.11, applying H_0 yields a long exact sequence

$$\cdots \longrightarrow \mathsf{H}_1(M) \longrightarrow \mathsf{H}_1(N) \longrightarrow \mathsf{H}_1(\mathsf{coker}(f))$$

$$\wr\|$$

$$\mathsf{H}_0(\mathsf{hker}(f)) \longrightarrow \mathsf{H}_0(M) \longrightarrow \mathsf{H}_0(N) \longrightarrow \mathsf{H}_0(\mathsf{hcoker}(f))$$

$$\wr\|$$

$$\mathsf{H}_{-1}(\mathsf{hker}(f)) \longrightarrow \mathsf{H}_{-1}(M) \longrightarrow \mathsf{H}_{-1}(N) \longrightarrow \cdots$$

in Mod_R.

Just as (quasi-isomorphism classes of) complexes encode more information than their homology groups (recall §2.4.4), so does a (quasi-isomorphism class of) morphism in Ch_R encode more information than the corresponding long exact sequence. Therefore, we view the homotopy co/kernel sequence as the more fundamental notion.

The following exercise illustrates the more classical approach to constructing long exact sequences.

---

[43]Note in particular that we are *not* implicitly extending the sequence by zero here.

[44]A notable exception is the long exact sequence on homotopy groups (of which this is actually a special case).

Exercise 3.12 (6 points). Suppose that $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a sequence of morphisms in Ch_R that is exact in each dimension.[45] Construct a long exact sequence

$$
\cdots \xrightarrow{\mathsf{H}_{n+1}(g)} \mathsf{H}_{n+1}(N)
$$

$$
\hookrightarrow \mathsf{H}_n(L) \xrightarrow{\mathsf{H}_n(f)} \mathsf{H}_n(M) \xrightarrow{\mathsf{H}_n(g)} \mathsf{H}_n(N)
$$

$$
\hookrightarrow \mathsf{H}_{n-1}(L) \xrightarrow{\mathsf{H}_{n-1}(f)} \cdots
$$

in Mod_R.

4. The dg-category of complexes

4.1. **Hom-complexes.** Given complexes $M, N \in \mathsf{Ch}_R$, we have seen that morphisms $M \to N$ may be related by *homotopies*. In fact, these homotopies can be related by *higher homotopies*, ad infinitum. Altogether, these data organize neatly into the **hom-complex** from M to N, denoted

$$
\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \in \mathsf{Ch}_{\Bbbk} \ .\ [46]
$$

Namely, in dimension n the hom-complex is given by

$$
\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_n := \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \ ,
$$

and its differential

$$
\prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \xrightarrow{d_n} \prod_{i \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})
$$

carries a tuple $\{f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n})\}_{i \in \mathbb{Z}}$ to the tuple

$$
\left\{ (d_{i+n}^N \circ f_i + (-1)^{n-1} \cdot f_{i-1} \circ d_i^M) \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1}) \right\}_{i \in \mathbb{Z}} \ .
$$

Exercise 4.1 (2 points). Verify that this formula defines a complex.

Directly from the definition, we have that $\mathsf{Z}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$. Moreover, given any morphism $M \xrightarrow{f} N$ in Ch_R, a nullhomotopy $0 \xRightarrow{h} f$ is equivalent data to an element

---

[45]Recall from Exercise 3.6(a) that this implies that the canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f) \cong N$ is a quasi-isomorphism.

[46]In fact, the hom-complex also contains "lower homotopies", in the sense that complexes are "like spaces, but with the possibility of cells in negative dimensions".

$h \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$ such that $d_1(h) = f$. So, $\mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N))$ is canonically isomorphic to the abelian group of *homotopy classes* of morphisms $M \to N$. Note too that

$$\Sigma^i \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(\Sigma^{-i}M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \Sigma^i N) \ ;$$

this gives an analogous description of all homology groups of the hom-complex, as $\mathsf{H}_n \cong \mathsf{H}_0 \circ \Sigma^{-n}$. On the other hand, these homology groups can also be understood in a more homotopical (although closely related) manner.

Exercise 4.2 (4 points). Given a chain map $M \overset{f}{\to} N$ and two nullhomotopies $0 \overset{h_0}{\Rightarrow} f$ and $0 \overset{h_1}{\Rightarrow} f$, define a notion of a homotopy $h_0 \Rightarrow h_1$ between homotopies, and prove that such a higher homotopy always exists precisely when $\mathsf{H}_1(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) = 0$.

If one views the complexes M and N as 0-cells, the maps $M \overset{0}{\to} N$ and $M \overset{f}{\to} N$ as 1-cells, and the homotopies h_i as 2-cells, then such a higher homotopy should be viewed as a 3-cell. Of course, there are analogs of this same interpretation for all the homology groups of $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$.

For brevity, we may simply write

$$\underline{\mathsf{hom}}(M, N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \ .$$

4.2. The dg-category of complexes. These hom-complexes, in turn, can be naturally assembled into a single object.

Exercise 4.3 (8 points). Fix any complexes $L, M, N, O \in \mathsf{Ch}_R$.

(a) Construct a map

$$\underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, N) \xrightarrow{\chi_{L,M,N}} \underline{\mathsf{hom}}(L, N)$$

in $\mathsf{Ch}_{\Bbbk}$ that encodes the composition of morphisms of complexes.

(b) Verify that these composition morphisms are associative, in the sense that the diagram

$$
\begin{array}{ccc}
\underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, N) \otimes \underline{\mathsf{hom}}(N, O) & \xrightarrow{\chi_{L,M,N} \otimes \mathsf{id}} & \underline{\mathsf{hom}}(L, N) \otimes \underline{\mathsf{hom}}(N, O) \\
\Big\downarrow{\scriptstyle \mathsf{id} \otimes \chi_{M,N,O}} & & \Big\downarrow{\scriptstyle \chi_{L,N,O}} \\
\underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, O) & \xrightarrow{\chi_{L,M,O}} & \underline{\mathsf{hom}}(L, O)
\end{array}
$$

commutes.[47]

---

[47]This implicitly uses the associativity of the operation $\otimes$ in $\mathsf{Ch}_{\Bbbk}$.

(c) Construct a map

$$\Bbbk \xrightarrow{\iota_M} \underline{\mathsf{hom}}(M, M)$$

using the identity morphism of $M \in \mathsf{Ch}_R$, and verify that it defines a two-sided identity for the above composition in the sense that the diagrams

$$\underline{\mathsf{hom}}(L, M) \otimes \Bbbk \xrightarrow{\mathsf{id} \otimes \iota_M} \underline{\mathsf{hom}}(L, M) \otimes \underline{\mathsf{hom}}(M, M)$$
$$\downarrow^{\chi_{L,M,M}}$$
$$\underline{\mathsf{hom}}(L, M)$$

and

$$\Bbbk \otimes \underline{\mathsf{hom}}(M, N) \xrightarrow{\iota_M \otimes \mathsf{id}} \underline{\mathsf{hom}}(M, M) \otimes \underline{\mathsf{hom}}(M, N)$$
$$\downarrow^{\chi_{M,M,N}}$$
$$\underline{\mathsf{hom}}(M, N)$$

commute (where the isomorphisms are the canonical ones coming from the fact that $\Bbbk \in \mathsf{Ch}_\Bbbk$ is the unit object).

Altogether, Exercise 4.3 yields a ($\Bbbk$-*linear*) *dg-category*,[48] i.e. a category enriched in the symmetric monoidal category $(\mathsf{Ch}_\Bbbk, \otimes, \Bbbk)$:[49] its objects are the chain complexes of R-modules and its hom-objects are the hom-complexes between them. We denote this dg-category by $\mathbf{K}_R$ and refer to it as *the dg-category of complexes of R-modules*.[50] In particular, we may also write $\mathsf{hom}_{\mathbf{K}_R}(M, N) := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)$; our chosen notation will depend on our desired emphasis.

The fundamental purpose of the dg-category $\mathbf{K}_R$ is to assemble the data of chain maps, homotopies, and higher (and lower) homotopies into a single object. In particular, it is the homology groups of the hom-complexes in $\mathbf{K}_R$ that are relevant. Therefore, it will be natural to view the hom-complexes in $\mathbf{K}_R$ as "only being important up to quasi-isomorphism".

---

[48]Here, "dg" is short for "differential graded". (In general, it is common to refer to a chain complex of R-modules as a *dg-R-module*.)

[49]Of course, a dg-category is a particular instance of a more general notion. Namely, given a monoidal category $\mathcal{V} := (\mathcal{V}, \otimes_{\mathcal{V}}, \mathbb{1}_{\mathcal{V}})$, there is a natural notion of a *category enriched in $\mathcal{V}$*, or simply a *$\mathcal{V}$-enriched category*: a $\mathcal{V}$-enriched category $\mathcal{C}$ consists of a set of objects, the data of hom-objects $\underline{\mathsf{hom}}_{\mathcal{C}}(X, Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{C}$, and the data of composition and identity morphisms

$$\underline{\mathsf{hom}}_{\mathcal{C}}(X, Y) \otimes_{\mathcal{V}} \underline{\mathsf{hom}}_{\mathcal{C}}(Y, Z) \xrightarrow{\chi_{X,Y,Z}} \underline{\mathsf{hom}}_{\mathcal{C}}(X, Z) \qquad \text{and} \qquad \mathbb{1}_{\mathcal{V}} \xrightarrow{\iota_Y} \underline{\mathsf{hom}}_{\mathcal{C}}(Y, Y)$$

in $\mathcal{V}$ for all $X, Y, Z \in \mathcal{C}$, subject to the evident associativity and unitality conditions. (As indicated, one sometimes uses an underline to emphasize that these are *enriched* hom-objects (as opposed to mere hom-sets).) On the other hand, note that the notation $\mathsf{hom}_{\mathcal{C}}(X, Y)$ is already unambiguous, as $\mathcal{C}$ is a $\mathcal{V}$-enriched category.

[50]The German word for "complex" begins with the letter "K".

As a special case, note that for any complex $M \in \mathsf{Ch}_R$, composition in $\mathbf{K}_R$ makes the complex $\underline{\hom}_{\mathsf{Ch}_R}(M, M) \in \mathsf{Ch}_\Bbbk$ into an *associative algebra object*, i.e. it is a **dg-algebra** (or **dga** for short) over $\Bbbk$.

Exercise 4.4 (4 points)**.** Show that the following conditions are equivalent.

(i) The complex $M \in \mathsf{Ch}_R$ is contractible.
(ii) The complex $\underline{\hom}_{\mathsf{Ch}_R}(M, M) \in \mathsf{Ch}_\Bbbk$ is acyclic.
(iii) The complex $\underline{\hom}_{\mathsf{Ch}_R}(M, M) \in \mathsf{Ch}_\Bbbk$ has that $\mathsf{H}_0(\underline{\hom}_{\mathsf{Ch}_R}(M, M)) = 0$.
(iv) The element $[\mathsf{id}_M] \in \mathsf{H}_0(\underline{\hom}_{\mathsf{Ch}_R}(M, M))$ has that $[\mathsf{id}_M] = 0$.

Of course, this construct can be reversed. Given a dg-algebra $A \in \mathsf{Alg}(\mathsf{Ch}_\Bbbk)$, we can form a dg-category $\mathfrak{B}A \in \mathsf{cat}^{\mathsf{dg}}$ as follows: it has a single object $*$, and we declare that $\hom_{\mathfrak{B}A}(*, *) := A$ (with composition and identity respectively defined as the multiplication and unit in A).

On the other hand, among the morphisms in the dg-category $\mathbf{K}_R$, the homotopy equivalences play a distinguished role.

Exercise 4.5 (2 points)**.** Given a homotopy equivalence $M \to N$ in $\mathbf{K}_R$, show that the natural transformations

$$\hom_{\mathbf{K}_R}(-, M) \longrightarrow \hom_{\mathbf{K}_R}(-, N) \qquad \text{and} \qquad \hom_{\mathbf{K}_R}(M, -) \longleftarrow \hom_{\mathbf{K}_R}(N, -)$$

are natural homotopy equivalences.

The quasi-isomorphisms cannot play any such distinguished role in $\mathbf{K}_R$. For instance, given any acyclic but noncontractible complex $A \in \mathbf{K}_R$ (recall Exercise 3.4), by Exercise 4.4 the morphism

$$0 \simeq \hom_{\mathbf{K}_R}(A, 0) \longrightarrow \hom_{\mathbf{K}_R}(A, A)$$

cannot be a quasi-isomorphism (let alone a homotopy equivalence).

4.3. **Basic features of the dg-category of complexes.** In order to to understand the hom-complexes in the dg-category $\mathbf{K}_R$, it is helpful to understand how they interact with other complexes of $\Bbbk$-modules.

We take our motivation from the expected tensor-hom adjunction for complexes of $\Bbbk$-modules: for any $T, M, N \in \mathsf{Ch}_\Bbbk$ we have a natural isomorphism

$$\hom_{\mathsf{Ch}_\Bbbk}(T \otimes_\Bbbk M, N) \cong \hom_{\mathsf{Ch}_\Bbbk}(T, \underline{\hom}_{\mathsf{Ch}_\Bbbk}(M, N)) \ .$$

The general situation is as follows. For any complex $T \in \mathsf{Ch}_\Bbbk$ of $\Bbbk$-modules and any complexes $M, N \in \mathsf{Ch}_R$ of R-modules, we may form the complexes

$$T \otimes_\Bbbk M \qquad \text{and} \qquad \underline{\hom}_{\mathsf{Ch}_\Bbbk}(T, N)$$

of R-modules, where the (right) R-actions are induced from those on M and N. These satisfy the universal properties that

$$\mathsf{hom}_{\mathsf{Ch}_\Bbbk}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) \cong \mathsf{hom}_{\mathsf{Ch}_R}(T \otimes_\Bbbk M, N) \cong \mathsf{hom}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T, N)) \,.^{51}$$

In fact, these satisfy enriched universal properties.

Exercise 4.6 (4 points). Prove that for any $T \in \mathsf{Ch}_\Bbbk$ and any $M, N \in \mathsf{Ch}_R$ we have natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_\Bbbk M, N) \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T, N))$$

in $\mathsf{Ch}_\mathbb{Z}$.

We can now show that the hom-complexes in R preserve homotopy co/kernel sequences separately in each variable.

Exercise 4.7 (8 points). Choose any complexes $M, N, T \in \mathsf{Ch}_R$ and any morphism $M \xrightarrow{f} N$. Using Exercise 4.6 and the universal property of the homotopy co/kernel, construct natural isomorphisms

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(T, \mathsf{hker}\left(M \xrightarrow{f} N\right)\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, M) \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(T, f)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, N)\right)$$

and

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(\mathsf{hcoker}\left(M \xrightarrow{f} N\right), T\right) \cong \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(N, T) \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(f, T)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, T)\right)$$

in $\mathsf{Ch}_\Bbbk$.

Of course, essentially identical reasoning shows that the hom-complex bifunctor

$$\mathsf{Ch}_\Bbbk^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_\Bbbk}(-,-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.[52]

Exercise 4.8 (4 points). Show that the tensor product bifunctor

$$\mathsf{Ch}_\Bbbk \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_\Bbbk(-)} \mathsf{Ch}_R$$

preserves homotopy co/kernel sequences separately in each variable.

---

[51]In the general context of enriched category theory, one says that $T \otimes M$ is the *tensoring* of M by T and that $\underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T, N)$ is the *cotensoring* of N by T.

[52]Alternatively, this follows from Exercise 4.7, the commutative square

$$\begin{array}{ccc} \mathsf{Ch}_\Bbbk^{\mathsf{op}} \times \mathsf{Ch}_R & \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_\Bbbk}(-,-)} & \mathsf{Ch}_R \\ {\scriptstyle \mathsf{id} \times \mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle \mathsf{fgt}} \\ \mathsf{Ch}_\Bbbk^{\mathsf{op}} \times \mathsf{Ch}_\Bbbk & \xrightarrow[\mathsf{hom}_{\mathsf{Ch}_\Bbbk}(-,-)]{} & \mathsf{Ch}_\Bbbk \end{array} \,,$$

and the fact that the forgetful functor both preserves and detects homotopy kernel sequences.

5. PROJECTIVE AND INJECTIVE RESOLUTIONS

5.1. **Motivation for resolutions.** The original motivation for homological algebra is the fact that many natural functors on ordinary modules do not preserve exact sequences. Equivalently but more fundamentally, they do not respect short exact sequences, i.e. they do not respect both kernels and cokernels.[53]

Exercise 5.1 (6 points).
  (a) Show that the functor

  $$\mathsf{Mod}_{\Bbbk} \times \mathsf{Mod}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Mod}_R$$

  does not generally preserve exact sequences in either variable.
  (b) Show that the functor

  $$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_R$$

  does not generally preserve exact sequences in either variable.

Namely, applying either bifunctor appearing in Exercise 5.1 to an exact sequence in one of its slots, one obtains a "half-exact" sequence: $(-)\otimes_{\Bbbk}(-)$ preserves cokernels separately in each variable, $\mathsf{hom}_{\mathsf{Mod}_R}(M,-)$ preserves kernels, and $\mathsf{hom}_{\mathsf{Mod}_R}(-,M)$ carries cokernels to kernels. It was originally desired for these half-exact sequences to extend to long exact sequences, whose additional terms would quantify these various failures of exactness.

   As illustrated by Exercises 4.7 and 4.8, towards resolving these issues it is fruitful to pass from ordinary modules and ordinary co/kernels to complexes of modules and homotopy co/kernels; the desired long exact sequences would then be those on homology discussed in §3.4, although of course we will take the perspective that the homotopy co/kernel sequences themselves are the more fundamental objects. However, given that we would like to consider quasi-isomorphisms as isomorphisms, the following results show that this maneuver does not suffice on its own.

Exercise 5.2 (6 points).
  (a) Show that the functor

  $$\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(-)\otimes_{\Bbbk}(-)} \mathsf{Ch}_R$$

  does not generally preserve acyclic objects in either variable.
  (b) Show that the functor

  $$\mathsf{Ch}_R^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(-,-)} \mathsf{Ch}_{\Bbbk}$$

  does not generally preserve acyclic objects in either variable.

---

[53]Here we use the word "respect" instead of "preserve" due to the contravariance of $\mathsf{hom}_{\mathsf{Mod}_R}(-,M)$: recognizing that $\mathsf{Mod}_R^{\mathsf{op}}$ is also an abelian category, one might hope that this would carry kernels to cokernels and cokernels to kernels.

Clearly, preservation of acyclics is necessary for the preservation of quasi-isomorphisms. But in fact, the converse is guaranteed by Exercise 3.7(a) (combined with Exercises 4.7 and 4.8). This motivates the notions that we introduce now.

5.2. Projective and injective complexes.

5.2.1.   We write $\mathbf{A}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the acyclic complexes.

We say that a complex $P \in \mathbf{K}_R$ is **projective** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(P, A) \in \mathsf{Ch}_{\mathbb{Z}}$ is acyclic. Because acyclic complexes are preserved under de/suspensions, we may equivalently demand simply that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(P, A)) = 0$ for every acyclic complex $A \in \mathbf{A}_R$. In other words, P is projective iff every morphism $P \to A$ to an acyclic complex admits a nullhomotopy. In turn, this is equivalent to the condition that for every solid diagram

$$
\begin{array}{ccc}
 & & \mathsf{hker}(\mathsf{id}_A) \\
 & \nearrow & \downarrow {\scriptstyle v} \\
P & \longrightarrow & A
\end{array}
$$

where $A \in \mathbf{A}_R$ is acyclic there exists a lift making the diagram commute. We write $\mathbf{P}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the projective complexes.

We observe for future reference that v is a levelwise surjective quasi-isomorphism: indeed, $\mathsf{hker}(\mathsf{id}_A)$ is acyclic by the long exact sequence in homology, and v is surjective by construction (or by its defining universal property). So, in order for a complex to be projective it suffices for it to have the analogous lifting property with respect to *all* levelwise surjective quasi-isomorphisms.

Dually, we say that a complex $I \in \mathbf{K}_R$ is **injective** if for every acyclic complex $A \in \mathbf{A}_R$ the complex $\mathsf{hom}_{\mathbf{K}_R}(A, I) \in \mathsf{Ch}_{\mathbb{Z}}$ is acyclic. Likewise, we may equivalently demand for all acyclic complexes $A \in \mathbf{A}_R$ that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(A, I)) = 0$, or that every morphism $A \to I$ admits a nullhomotopy, or that for every solid diagram

$$
\begin{array}{ccc}
A & \longrightarrow & I \\
{\scriptstyle u} \downarrow & \nearrow & \\
\mathsf{hcoker}(\mathsf{id}_A) & &
\end{array}
$$

there exists an extension making the diagram commute. We write $\mathbf{I}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory on the injective complexes.

It is easy to deduce the following facts directly from the definitions.

Exercise 5.3 (6 points).
  (a) Show that all quasi-isomorphisms in $\mathbf{P}_R$ are homotopy equivalences.[54]

---

[54]This is formally analogous to *Whitehead's theorem*, which states that a weak homotopy equivalence between cell complexes (or retracts thereof) is necessarily a homotopy equivalence.

(b) Show that all quasi-isomorphisms in $\mathbf{I}_R$ are homotopy equivalences.

In particular, clearly the zero complex is projective (resp. injective), so that an acyclic projective (resp. injective) complex must be contractible.

(c) Show that projective complexes are preserved under tensor product: if $P \in \mathbf{P}_\Bbbk$ and $Q \in \mathbf{P}_R$ then $P \otimes Q \in \mathbf{P}_R$.

(d) Show that for any projective complexes $P \in \mathbf{P}_\Bbbk$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_\Bbbk}(P,-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(Q,-)} \mathbf{K}_\Bbbk$$

preserve quasi-isomorphisms.

(e) Show that for any injective complex $I \in \mathbf{I}_R$, the functors

$$\mathbf{K}_\Bbbk^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_\Bbbk}(-,I)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(-,I)} \mathbf{K}_\Bbbk$$

preserve quasi-isomorphisms.

Moreover, it follows from Exercise 4.5 that the property of projectivity (resp. injectivity) is stable under homotopy equivalence: if a complex is homotopy equivalent to a projective (resp. injective) complex, then it itself is projective (resp. injective).

5.2.2.   Of course, the definitions of projective and injective complexes on their own are not so useful. What gives them their power is that every complex $M \in \mathbf{K}_R$ admits both a ***projective resolution*** $P \xrightarrow{\simeq} M$ and an ***injective resolution*** $M \xrightarrow{\simeq} I$ (i.e. quasi-isomorphisms as indicated). These are the promised representatives of the quasi-isomorphism class of M that are "well-adapted" for certain purposes. Specifically, we will respectively view the functors

$$(-) \otimes_\Bbbk P \,, \qquad \mathsf{hom}_{\mathbf{K}_R}(P,-) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-,I)$$

as "corrected" (a.k.a. "derived") versions of the functors

$$(-) \otimes_\Bbbk M \,, \qquad \mathsf{hom}_{\mathbf{K}_R}(M,-) \qquad \text{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(-,I)$$

(and similarly for projective resolutions of complexes of $\Bbbk$-modules).

As we will see, it is relatively straightforward to construct projective resolutions of *bounded-below* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \ll 0$) and to construct injective resolutions of *bounded-above* complexes (i.e. $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n \gg 0$). Note in particular that this will apply to ordinary R-modules via the inclusion $\mathsf{Mod}_R \subseteq \mathsf{Ch}_R$.

As for the general (i.e. unbounded) situation, it turns out to be much easier (and more conceptually satisfying) to construct projective resolutions than injective resolutions. On the other hand, we will not have any specific need for injective resolutions in general, beyond their existence. Therefore, we will discuss only projective resolutions in general, and refer the reader to [Spa88] (or to [Hov99, §2.3]) for the construction of injective resolutions in general.

It is easy to deduce the following facts directly from the existence of projective and resolutions.

Exercise 5.4 (4 points).

    (a) Show that if the complex $M \in \mathbf{K}_R$ has that $\underline{\hom}_{\mathbf{K}_R}(P, M)$ is acyclic for every projective complex $P \in \mathbf{P}_R$, then M is acyclic.[55]

    (b) Show that if the complex $M \in \mathbf{K}_R$ has that $\underline{\hom}_{\mathbf{K}_R}(M, I)$ is acyclic for every injective complex $I \in \mathbf{I}_R$, then M is acyclic.

    (c) Show that for any projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors

$$\mathbf{K}_R \xrightarrow{P \otimes_{\Bbbk} (-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_{\Bbbk} \xrightarrow{(-) \otimes_{\Bbbk} Q} \mathbf{K}_R$$

    preserve quasi-isomorphisms.

In particular, the tensor product of a projective complex and an acyclic complex is acyclic.

    We will eventually organize many of the facts enumerated in Exercises 5.3 and 5.4 in a more systematic way.

5.3. **Projective resolutions in the bounded-below case.** Recall that an R-module $P \in \mathsf{Mod}_R$ is called **_projective_** if mapping out of it preserves surjections, i.e. if for any surjection $M \twoheadrightarrow N$ in Mod_R the induced map $\hom_{\mathsf{Mod}_R}(P, M) \to \hom_{\mathsf{Mod}_R}(P, N)$ is a surjection. Said differently, given any solid diagram

$$
\begin{array}{ccc}
 & & M \\
 & \nearrow & \downarrow \\
P & \longrightarrow & N
\end{array}
$$

there exists a dashed lift making the diagram commute.

Exercise 5.5 (2 points). Show that an R-module $P \in \mathsf{Mod}_R$ is projective iff it is a summand of a free module (i.e. there exists an R-module $Q \in \mathsf{Mod}_R$ and an isomorphism $P \oplus Q \cong R^{\oplus S}$ with the free R-module on a set S).

Exercise 5.6 (2 points). Show that every projective $\mathbb{Z}$-module is free.

Exercise 5.7 (4 points). Give necessary and sufficient conditions on $n \in \mathbb{N}$ such that every projective $\mathbb{Z}/n$-module is free.

Exercise 5.8 (4 points). Show that an R-module $M \in \mathsf{Mod}_R$ is projective iff the corresponding complex $\underset{\sim}{M} \in \mathsf{Ch}_R$ is projective.

    We now turn from projective modules back to projective complexes.

Exercise 5.9 (4 points). Show that if $P \in \mathbf{P}_R$ is a projective complex, then $P_n \in \mathsf{Mod}_R$ is a projective R-module for all $n \in \mathbb{Z}$.

We have the following partial converse.

---

[55]Evidently, it is already sufficient just to take $P = R$ (which is projective by Exercises 5.5 and 5.10).

Exercise 5.10 (6 points). Show that if $P \in \mathbf{K}_R$ is a bounded-below complex such that each $P_n \in \mathsf{Mod}_R$ is projective, then $P \in \mathbf{P}_R$ is projective.

However, the bounded-below hypothesis in Exercise 5.10 is necessary: both parts of Exercise 3.4 give examples of unbounded complexes which are levelwise free (hence levelwise projective by Exercise 5.5) that cannot be projective. Indeed, an acyclic projective complex must be contractible, as its identity map must admit a nullhomotopy.

Exercise 5.11 (4 points). Given a bounded-below complex $M \in \mathbf{K}_R$, construct a projective resolution $P \xrightarrow{\approx} M$.

5.4. **Injective resolutions in the bounded-above case.** Recall that an R-module $I \in \mathsf{Mod}_R$ is called **injective** if mapping into it carries injections to surjections, i.e. if for any injection $M \hookrightarrow N$ in Mod_R the induced map $\mathsf{hom}_{\mathsf{Mod}_R}(N, I) \to \mathsf{hom}_{\mathsf{Mod}_R}(M, I)$ is surjective. Said differently, given any solid diagram

$$
\begin{array}{ccc}
M & \longrightarrow & I \\
\downarrow & \nearrow & \\
N & &
\end{array}
$$

there exists a dashed extension making the diagram commute.

Injective modules are much more bizarre than projective modules.

Exercise 5.12 (2 points). Assuming that R is a PID, show that an R-module $M \in \mathsf{Mod}_R$ is injective iff it is *divisible*, i.e. for every nonzero element $r \in R$ the map $M \xrightarrow{r} M$ is surjective.

So for instance, $\mathbb{Q} \in \mathsf{Mod}_{\mathbb{Z}}$ is injective while $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}$ is not.

It is easy to dualize the arguments of Exercises 5.9 and 5.10: injective complexes are levelwise injective, and bounded-above levelwise injective complexes are injective.

The argument of Exercise 5.11 uses the fact that every R-module admits a surjection from a projective R-module; one says in this situation that Mod_R **has enough projectives**. It can be easily dualized to show that any bounded-above complex $M \in \mathbf{K}_R$ admits an injective resolution $M \xrightarrow{\approx} I$, using the following result showing that Mod_R also **has enough injectives**.

Exercise 5.13 (12 points).
   (a) For any ring homomorphism $S \to R$, verify that the functors

$$
\begin{array}{c}
\overset{(-)\otimes_S R}{\overbrace{\qquad\qquad}} \\
\mathsf{Mod}_S \xleftarrow{\;\;\perp\;\;} \mathsf{fgt} \xrightarrow{\qquad} \mathsf{Mod}_R \\
\underset{\mathsf{hom}_{\mathsf{Mod}_S}(R,-)}{\underbrace{\qquad\qquad}}_{\perp}
\end{array}
$$

participate in adjunctions as indicated.[56]

(b) Given modules $M \in \mathsf{Mod}_R$ and $N \in \mathsf{Mod}_S$ and an injection

$$\mathsf{fgt}(M) \longhookrightarrow N$$

in Mod_S, show that the corresponding morphism

$$M \longrightarrow \mathsf{hom}_{\mathsf{Mod}_S}(R, N)$$

in Mod_R is also an injection.

(c) Deduce from the fact that fgt preserves injections that the functor $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$ preserves injective objects.[57]

(d) Show that injective objects are preserved under products.[58]

(e) Fix an abelian group $A \in \mathsf{Ab}$. Show that for every $a \in A$, there exists a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ carrying a to a nonzero element.[59] Deduce that there exists an injection

$$A \longhookrightarrow \prod_A \mathbb{Q}/\mathbb{Z} \,.^{[60]}$$

Namely, suppose we are given an R-module $M \in \mathsf{Mod}_R$. By (d) we have an injection

$$\mathsf{fgt}(M) \longhookrightarrow \prod_M \mathbb{Q}/\mathbb{Z} =: I$$

in Ab, and by (d) and Exercise 5.12 we see that $I \in \mathsf{Ab}$ is injective. Thereafter, by (b) we obtain an injection

$$M \longhookrightarrow \mathsf{hom}_{\mathsf{Ab}}(R, I) =: J$$

in Mod_R, and by (c) it follows that $J \in \mathsf{Mod}_R$ is injective.

5.5. **Cell complexes and lifting criteria.** For the purpose of constructing projective resolutions, we first introduce some auxiliary ideas.

For any $n \in \mathbb{Z}$, we define the objects and morphism

$$
\begin{array}{ccc}
S^n & & 0 \longrightarrow R \\
{\scriptstyle i_n}\big\downarrow & := & \big\downarrow \quad\quad \big\downarrow {\scriptstyle \mathsf{id}_R} \\
D^{n+1} & & R \xrightarrow[\mathsf{id}_R]{} R
\end{array}
$$

---

[56]In defining the functor $(-) \otimes_S R$ (resp. $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$), we use that R is an (S, R)-bimodule (resp. an (R, S)-bimodule).

[57]Dually, the functor $(-) \otimes_S R$ preserves projective objects because the functor fgt also preserves surjections.

[58]Dually, projective objects are preserved under coproducts.

[59]This uses Exercise 5.12.

[60]One says that $\mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$ is an ***injective cogenerator***. Dually, $R \in \mathsf{Mod}_R$ is a ***projective generator***: it is projective, and moreover for any $M \in \mathsf{Mod}_R$ there exists a surjection $R^{\oplus S} \twoheadrightarrow M$ for some set S.

in Ch_R, where the columns are in dimensions $n + 1$ and n.[61] For any complex $M \in \mathsf{Ch}_R$, we have natural isomorphisms

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \cong \mathsf{Z}_n(M) \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \cong M_{n+1} \ ;$$

precomposition with i_n induces the morphism

$$\mathsf{Z}_n(M) \xleftarrow{d_{n+1}} M_{n+1} \ .[62]$$

We say that a morphism $M \xrightarrow{f} N$ has the ***right lifting property*** with respect to i_n (or equivalently that i_n has the ***left lifting property*** with respect to f) if for every solid commutative diagram

$$\begin{array}{ccc}
S^n & \longrightarrow & M \\
{\scriptstyle i_n}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
D^{n+1} & \longrightarrow & N
\end{array}$$

there exists a dashed lift that makes the diagram commute. This is equivalent to saying that the canonical dashed morphism in the pullback

is surjective. We may abbreviate this by writing that $f \in \mathsf{rlp}(i_n)$. We also introduce the notation $I := \{i_n\}_{n \in \mathbb{Z}}$ and $\mathsf{rlp}(I) := \bigcap_{n \in \mathbb{Z}} \mathsf{rlp}(i_n)$.

Exercise 5.14 (4 points)**.** Prove that if $f \in \mathsf{rlp}(i_n)$ then $\mathsf{H}_n(f)$ is injective and $\mathsf{Z}_{n+1}(f)$ is surjective.

---

[61]Of course, this morphism will function as "the inclusion of the n-sphere into the $(n + 1)$-disk": indeed, for any $n \geqslant 0$ it is the reduced simplicial chains on the inclusion $\Delta^{\{0,\dots,n\}}/\partial\Delta^{\{0,\dots,n\}} \hookrightarrow \Delta^n/\Lambda_0^n$ (which is a particularly small simplicial model for this morphism). Note too that $S^n \cong \Sigma^n S^0$ and $D^n \cong \Sigma^n D^0$. While the former isomorphism is always an equality (so that we may take $\Sigma^n S^0$ to be the definition of S^n), the latter introduces an inconvenient sign when n is odd (which is why we do not take $\Sigma^n D^0$ to be the definition of D^n).

[62]Categorically speaking, the reason that the morphism

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -) \xleftarrow{(-)\circ i_n} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, -)$$

in $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_{\mathbb{Z}})$ lifts to $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Mod}_R)$ is that the morphism i_n is in fact a morphism of complexes of (R, R)-bimodules.

Now, if $f \in \mathsf{rlp}(I)$, then Exercise 5.14 immediately implies that f is a quasi-isomorphism. On the other hand, suppose that $f \in \mathsf{rlp}(i_{n-1}) \cap \mathsf{rlp}(i_n)$. Then, $\mathsf{Z}_n(f)$ is surjective by Exercise 5.14. It follows that f_{n+1} is surjective by the above reformulation of $\mathsf{rlp}(i_n)$. So, $f \in \mathsf{rlp}(I)$ implies as well that f is levelwise surjective.

In fact, the converse is also true.

Exercise 5.15 (4 points)**.** Prove that if $M \xrightarrow{f} N$ is a quasi-isomorphism that is levelwise surjective, then $f \in \mathsf{rlp}(I)$.

Given a morphism $M \xrightarrow{f} N$ in Ch_R, we write

$$I(f)_n := \mathsf{hom}_{\mathsf{Ar}(\mathsf{Ch}_R)}(i_n, f) \cong \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \underset{\mathsf{hom}_{\mathsf{Ch}_R}(S^n, N)}{\times} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, N)$$

for the set of morphisms

$$
\begin{array}{ccc}
S^n & \dashrightarrow & M \\
{\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f} \\
D^{n+1} & \dashrightarrow & N
\end{array}
$$

in the arrow category $\mathsf{Ar}(\mathsf{Ch}_R) := \mathsf{Fun}([1], \mathsf{Ch}_R)$ of Ch_R (i.e. pairs of morphisms $S^n \to M$ and $D^{n+1} \to N$ making the square commute), and we write

$$I(f) := \bigsqcup_{n \in \mathbb{Z}} I(f)_n$$

for their disjoint union over all $n \in \mathbb{Z}$. Given an element $\alpha \in I(f)$, we write $n_\alpha := n$ if $\alpha \in I(f)_n$. (That is, we write $I(f) \xrightarrow{n_{(-)}} \mathbb{Z}$ for the evident function from the disjoint union.)

5.6. Projective resolutions as cellular approximations.

5.6.1.   We now construct projective resolutions via a general procedure known as the **small object argument**.[63] In fact, given a morphism $M \xrightarrow{f} N$ in Ch_R, we will construct a factorization

$$
\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & N \\
 & {\scriptstyle c^{(\infty)}}\searrow \quad \nearrow{\scriptstyle f^{(\infty)}} & \\
 & M^{(\infty)} &
\end{array}
$$

satisfying the following conditions:

(1) the morphism $c^{(\infty)}$ has the left lifting property with respect to levelwise surjective quasi-isomorphisms, and

(2) the morphism $f^{(\infty)}$ is a quasi-isomorphism.

---

[63]This is formally analogous to the construction of a cell complex that is weak homotopy equivalent to a given topological space. The name arises from its crucial use of a certain "smallness" property of the sources of the elements of the set I. In the present situation, the relevant fact is given as Exercise 5.18.

Taking $M \xrightarrow{f} N$ to be the map $0 \to N$ then yields a projective resolution of N.[64]

5.6.2.   We make the following general construction. First of all, observe that we have a canonical commutative square

$$
\begin{array}{ccc}
\bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \longrightarrow & M \\
{\scriptstyle \oplus_{\alpha \in I(f)} i_{n_\alpha}} \downarrow & & \downarrow {\scriptstyle f} \\
\bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} & \longrightarrow & N
\end{array}
\quad .
$$

We write

$$
M^{(1)} := \mathsf{colim} \left(
\begin{array}{ccc}
\bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \longrightarrow & M \\
{\scriptstyle \oplus_{\alpha \in I(f)} i_{n_\alpha}} \downarrow & & \\
\bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} & &
\end{array}
\right)
$$

for the pushout,[65] and we write

$$
\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & N \\
 & {\scriptstyle c^{(1)}} \searrow \quad \nearrow {\scriptstyle f^{(1)}} & \\
 & M^{(1)} &
\end{array}
$$

for the morphisms in the induced factorization.

Exercise 5.16 (2 points). Show that the morphism $c^{(1)}$ has the left lifting property with respect to any levelwise surjective quasi-isomorphism.

---

[64]One may interpret the general case as providing a "relatively projective" resolution of N as an object under M. (Whereas a projective complex is like a cell complex (or a retract thereof), a relatively projective complex is like a relative cell complex (or a retract thereof).)

[65]Note that co/limits in chain complexes are levelwise. Indeed, it is a full subcategory of the functor category $\mathsf{Fun}((\mathbb{Z}^{\leqslant})^{\mathsf{op}}, \mathsf{Mod}_R)$ (where $\mathbb{Z}^{\leqslant} := \{\cdots \to -1 \to 0 \to 1 \to \cdots\}$), and co/limits in any functor category are computed pointwise. So, it suffices to see that the subcategory $\mathsf{Ch}_R \subseteq \mathsf{Fun}((\mathbb{Z}^{\leqslant})^{\mathsf{op}}, \mathsf{Mod}_R)$ is stable under co/limits. This follows from the more general fact that the full subcategory of the arrow category $\mathsf{Ar}(\mathsf{Mod}_R) := \mathsf{Fun}([1], \mathsf{Mod}_R)$ on those morphisms that factor through $0 \in \mathsf{Mod}_R$ is stable under co/limits; as co/limits in $\mathsf{Ar}(\mathsf{Mod}_R)$ are of course also computed pointwise, this follows from the fact that the zero object $0 \in \mathsf{Mod}_R$ is stable under co/limits (which follows from the fact that it is both initial and terminal).

Exercise 5.17 (2 points). Show that for every solid commutative diagram

$$
\begin{array}{ccc}
S^n & \longrightarrow & M \\
\downarrow & \searrow^{c^{(1)}} & \downarrow f \\
 & M^{(1)} & \\
D^{n+1} & \xrightarrow[\quad]{\quad f^{(1)}} & N
\end{array}
$$

there exists a dashed factorization making the diagram commute.

We now apply the above construction again, but to $f^{(1)}$ instead of to f: namely, we observe the canonical commutative square

$$
\begin{array}{ccc}
\bigoplus\limits_{\alpha \in I(f^{(1)})} S^{n_\alpha} & \longrightarrow & M^{(1)} \\
{\scriptstyle \bigoplus_{\alpha \in I(f^{(1)})} in_\alpha} \downarrow & & \downarrow {\scriptstyle f^{(1)}} \\
\bigoplus\limits_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} & \longrightarrow & N
\end{array} \quad ,
$$

we write

$$
M^{(2)} := \mathsf{colim} \left(\begin{array}{ccc}
\bigoplus\limits_{\alpha \in I(f^{(1)})} S^{n_\alpha} & \longrightarrow & M^{(1)} \\
{\scriptstyle \bigoplus_{\alpha \in I(f^{(1)})} in_\alpha} \downarrow & & \\
\bigoplus\limits_{\alpha \in I(f^{(1)})} D^{n_\alpha+1} & &
\end{array} \right)
$$

for the pushout, and we write

$$
\begin{array}{ccc}
M^{(1)} & \xrightarrow{\quad f \quad} & N \\
{\scriptstyle c^{(2)}} \searrow & & \nearrow {\scriptstyle f^{(2)}} \\
 & M^{(2)} &
\end{array}
$$

for the morphisms in the induced factorization. This gives us a commutative diagram

$$
\begin{array}{ccccc}
M & \xrightarrow{\qquad\qquad f \qquad\qquad} & & & N \\
{\scriptstyle c^{(1)}} \searrow & & & \nearrow & \\
 & M^{(1)} & \xrightarrow{\ f^{(1)}\ } & & \\
 & {\scriptstyle c^{(2)}} \searrow & & \nearrow {\scriptstyle f^{(2)}} & \\
 & & M^{(2)} & &
\end{array} \quad .
$$

Of course, we continue to iterate this construction in the obvious way. To conclude, we define

$$M^{(\infty)} := \mathsf{colim}\left(M \xrightarrow{c^{(1)}} M^{(1)} \xrightarrow{c^{(2)}} M^{(2)} \xrightarrow{c^{(3)}} \cdots \right) ;$$

we define the map $M \xrightarrow{c^{(\infty)}} M^{(\infty)}$ to be the canonical map to the colimit, and we define the map $M^{(\infty)} \xrightarrow{f^{(\infty)}} N$ to be induced by the universal property of the colimit.

5.6.3.   We now verify the two conditions given in §5.6.1.

(1) Let $B \xrightarrow{g} C$ be a levelwise surjective quasi-isomorphism. Given any commutative square

$$\begin{array}{ccc} M & \longrightarrow & B \\ {\scriptstyle c^{(\infty)}}\downarrow & & \downarrow{\scriptstyle g} \\ M^{(\infty)} & \longrightarrow & C \end{array} \ ,$$

we enlarge it to the solid commutative diagram

Here, applying Exercise 5.16 repeatedly we may inductively construct dashed lifts as indicated that make the diagram commute. By the universal property of the colimit, together these yield the desired lift in the original diagram.[66]

(2) In fact, we show that $f^{(\infty)} \in \mathsf{rlp}(I)$. For this, suppose we are given any commutative diagram

$$\begin{array}{ccc} S^n & \longrightarrow & M^{(\infty)} \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f^{(\infty)}} \\ D^{n+1} & \longrightarrow & N \end{array} \ .$$

---

[66]This may be phrased as choosing an element of a codirected limit of *surjective* functions between sets.

By Exercise 5.18 below, there exists some $k \in \mathbb{N}$ and a lift

$$
\begin{array}{ccc}
 & & M^{(k)} \\
 & \nearrow & \downarrow \\
S^n & \longrightarrow & M^{(\infty)}
\end{array}
$$

of the given map. Now, just as in Exercise 5.17, we obtain a lift

$$
\begin{array}{ccc}
S^n & \longrightarrow & M^{(k)} \\
\downarrow{\scriptstyle i_n} & & \\
 & M^{(k+1)} & \\
 & & M^{(\infty)} \\
D^{n+1} & \longrightarrow & N
\end{array}
$$

,

with arrows $c^{(k+1)}$, $f^{(k+1)}$, $f^{(\infty)}$

which proves the claim.

Exercise 5.18 (2 points)**.** Show that the object $S^n \in \mathsf{Ch}_R$ is ***compact***, i.e. that the functor

$$
\mathsf{Ch}_R \xrightarrow{\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -)} \mathsf{Set}
$$

commutes with filtered colimits.[67]

6. THE DERIVED ∞-CATEGORY

6.1. **The homotopy theory of the dg-category of complexes.**

6.1.1. We have by now obtained a diagram

$$
\begin{array}{ccc}
\mathbf{P}_R & & \\
 & \searrow & \\
(1) & & \mathbf{K}_R \longleftrightarrow \mathbf{A}_R \\
 & \nearrow & \\
\mathbf{I}_R & &
\end{array}
$$

---

[67]Note that a functor commutes with filtered colimits if and only if it commutes with directed colimits. This is one instance of the useful general principle of "coordinatization" of a class of colimits. As another example, given a functor between categories that admit finite colimits, it preserves them if and only if it preserves the initial object and pushouts. Another related fact is that given a functor between cocomplete categories (i.e. categories admitting all (small) colimits), it preserves all (small) colimits if and only if it preserves finite colimits and filtered colimits.

of fully faithful inclusions among dg-categories: we began with $\mathbf{A}_R \subseteq \mathbf{K}_R$, and then we defined $\mathbf{P}_R \subseteq \mathbf{K}_R$ (resp. $\mathbf{I}_R \subseteq \mathbf{K}_R$) to be what might be called its *homotopical left* (resp. *right*) *orthogonal*.

Recall that for any complex $M \in \mathbf{K}_R$ we have constructed a projective resolution $M' \xrightarrow{\approx} M$, i.e. a quasi-isomorphism from a projective object. Consider the resulting homotopy kernel sequence

$$A \longrightarrow M' \longrightarrow M$$

in Ch_R, where we write $A := \mathsf{hker}(M' \to M)$ because this complex is acyclic (by the long exact sequence in homology). Now, for any projective complex $P \in \mathbf{P}_R \subseteq \mathbf{K}_R$, by Exercise 4.7 we obtain a homotopy kernel sequence

$$\underline{\mathsf{hom}}(P, A) \longrightarrow \underline{\mathsf{hom}}(P, M') \longrightarrow \underline{\mathsf{hom}}(P, M)$$

in $\mathsf{Ch}_{\Bbbk}$. Because P is projective, the complex $\underline{\mathsf{hom}}(P, A)$ is acyclic, and hence we find that the morphism

$$\underline{\mathsf{hom}}(P, M') \xrightarrow{\approx} \underline{\mathsf{hom}}(P, M)$$

is a quasi-isomorphism (again by the long exact sequence). Heuristically, we have found that "the morphism $M' \to M$ appears to be an isomorphism when mapping from projective complexes (and considering hom-complexes up to quasi-isomorphism)".

6.1.2.   This sort of phenomenon is in fact quite ubiquitous. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between ordinary categories. Given an object $D \in \mathcal{D}$, a ***pointwise right adjoint*** to F at D consists of an object $C \in \mathcal{C}$ and a morphism $F(C) \to D$ such that the resulting composite morphism

$$\mathsf{hom}_{\mathcal{C}}(-, C) \xrightarrow{F} \mathsf{hom}_{\mathcal{D}}(F(-), F(C)) \longrightarrow \mathsf{hom}_{\mathcal{D}}(F(-), D)$$

in $\mathsf{Fun}(\mathcal{C}, \mathsf{Set})$ is a natural isomorphism. In this situation, we may refer to the morphism $F(C) \to D$ as the ***pointwise counit*** and denote it by ε_D (or simply by ε).

Exercise 6.1 (6 points)**.**
  (a) Show that if it exists, a pointwise right adjoint to F at D is unique up to unique isomorphism.[68]
  (b) Show that the datum of a right adjoint to F is equivalent data to a choice of pointwise right adjoint to F at every object $D \in \mathcal{D}$.[69]

So, the above discussion may be summarized as saying that in a homotopical sense – that is, if we only consider hom-complexes up to quasi-isomorphism – the fully faithful inclusion $\mathbf{P}_R \hookrightarrow \mathbf{K}_R$ "should" admit a right adjoint whose pointwise counits are projective resolutions. On the

---

[68]Said differently, the category of pointwise right adjoints to F at D is always either an empty or contractible groupoid.

[69]It follows that the category of right adjoints to F is always either an empty or contractible groupoid (because these are preserved under products).

other hand, because these morphisms on hom-complexes are only quasi-isomorphisms and not isomorphisms, we should not expect this structure to exist at the level of dg-categories.

6.1.3.   Of course, this discussion immediately dualizes: any complex $M \in \mathbf{K}_R$ has an injective resolution $M \xrightarrow{\approx} M''$ (i.e. a quasi-isomorphism to an injective object), and this has the property that for any injective complex $I \in \mathbf{I}_R \subseteq \mathbf{K}_R$ the morphism

$$\underline{\mathsf{hom}}(M'', I) \xrightarrow{\approx} \underline{\mathsf{hom}}(M, I)$$

is a quasi-isomorphism. Heuristically, "the morphism $M \to M''$ appears to be an isomorphism when mapping to an injective complex (and considering hom-complexes up to quasi-isomorphism)". So, in a homotopical sense the fully faithful inclusion $\mathbf{I}_R \hookrightarrow \mathbf{K}_R$ "should" admit a left adjoint whose pointwise units are injective resolutions.

6.2. A brief introduction to $\Bbbk$-linear ∞-categories.

6.2.1.   These adjoints that "should" exist in a homotopical sense actually *do* exist at the level of underlying $\Bbbk$-linear ∞-categories. So, we work within an ∞-categorical context.
    However, our present usage of ∞-category theory will be extremely "soft": it will not require any real familiarity with the foundations of the theory. So in the interest of maintaining the narrative thread, here we give an extremely brief summary of the immediately relevant features of ∞-category theory, and pursue a more systematic discussion later.

6.2.2.   Here is the most expedient definition of a $\Bbbk$-linear ∞-category.[70] We write cat for the category of ordinary categories, and we write $\mathsf{cat}^{\mathsf{dg}} \xrightarrow{\mathsf{H}_0} \mathsf{cat}$ for the functor given by applying H_0 to each hom-complex. Then, a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between dg-categories is called a **weak equivalence** if the following two conditions hold:

   (1) it is **homotopically fully faithful**, i.e. for all $C, C' \in \mathcal{C}$ the induced morphism $\mathsf{hom}_{\mathcal{C}}(C, C') \to \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathsf{Ch}_{\Bbbk}$ is a quasi-isomorphism; and
   (2) it is **homotopically essentially surjective**, i.e. the induced functor $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is essentially surjective.[71]

(Of course, if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence of dg-categories then $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is an equivalence of categories.)
    We localize the category $\mathsf{cat}^{\mathsf{dg}}_{\Bbbk}$ of $\Bbbk$-linear dg-categories at the weak equivalences. This yields a category that (for reasons that will become clear later) we will denote by $\mathsf{ho}(\mathsf{Cat}_{\Bbbk})$ and refer to as **the homotopy category of $\Bbbk$-linear ∞-categories**; its objects are called **$\Bbbk$-linear ∞-categories**. So by definition, there is a canonical functor

$$\mathsf{cat}^{\mathsf{dg}} \longrightarrow \mathsf{ho}(\mathsf{Cat}_{\Bbbk})$$

---

[70]This should be compared with the discussion of §2.4.

[71]It is possible to phrase homotopical essential surjectivity at the level of dg-categories, but this requires a bit of enriched category theory.

that carries all weak equivalences to isomorphisms. In particular, this functor is essentially surjective: every $\Bbbk$-linear ∞-category can be represented by a $\Bbbk$-linear dg-category.[72]

Given a dg-category $\mathcal{C}$, for the remainder of this section we will write $\mathcal{C}^\infty$ for its underlying $\Bbbk$-linear ∞-category. However, we will thereafter omit this superscript, as we will have finished drawing distinctions between dg-categories and their underlying $\Bbbk$-linear ∞-categories. We will generally refer to data in $\mathcal{C}$ as "point-set" and data in $\mathcal{C}^\infty$ as "∞-categorical".

Essentially by construction, given two dg-categories $\mathcal{C}, \mathcal{D} \in \mathsf{cat}^{\mathsf{dg}}$, morphisms $\mathcal{C}^\infty \to \mathcal{D}^\infty$ in $\mathsf{ho}(\mathsf{Cat}_\Bbbk)$ are given by equivalence classes of zigzags

$$\mathcal{C} \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \cdots \longrightarrow \bullet \xleftarrow{\approx} \mathcal{D}$$

in $\mathsf{cat}^{\mathsf{dg}}$ (in which all backwards maps are weak equivalences). However, it turns out that every equivalence class contains a representative of the form

$$\mathcal{C} \xleftarrow{\approx} \mathcal{C}' \longrightarrow \mathcal{D} \ .$$

In fact, dg-categories admit projective resolutions: every equivalence class contains a representative of this form for *any* fixed projective resolution $\mathcal{C}' \xrightarrow{\approx} \mathcal{C}$.

Once one is only considering hom-complexes up to quasi-isomorphism, homotopy equivalences become indistinguishable from isomorphisms. In general, one simply uses the term *equivalence* to refer to the ∞-categorical notion of isomorphism. We will denote equivalences in an ∞-category by $\xrightarrow{\sim}$.

For brevity, for the time being we will often simply refer to $\Bbbk$-linear ∞-categories as "∞-categories".

6.2.3.   To a first approximation, a $\Bbbk$-linear ∞-category is simply a category enriched in $\mathsf{H}_0(\mathbf{D}_\Bbbk)$, the derived category of $\Bbbk$ introduced in §2.4. In particular, passing from dg-categories to their underlying $\Bbbk$-linear ∞-categories does indeed "only remember their hom-complexes up to quasi-isomorphism". The distinction lies in the higher homotopies (e.g. those recording the homotopy-coherent associativity of composition) which are present in the underlying ∞-category of a dg-category but are lost in passing to its underlying $\mathsf{H}_0(\mathbf{D}_\Bbbk)$-enriched category. Namely, as we will see when we study ∞-categories more generally, given a dg-category $\mathcal{C}$, commutative diagrams in its underlying $\Bbbk$-linear ∞-category are represented by *homotopy-coherently* commutative diagrams in $\mathcal{C}$ itself. For instance, we will see the projective resolution

(2)

---

of the free strictly-commutative triangle $[2] = \{0 < 1 < 2\}$ by the free homotopy-coherently commutative triangle, and this persists upon passing to free dg-categories.[73]

6.2.4.   As a particular case, it turns out that homotopy co/kernel squares in dg-categories define co/limit diagrams (namely pushout/pullback squares) in their underlying ∞-categories.[74] For the moment, we will write coker^∞ and ker^∞ to denote ∞-categorical co/kernels.

It will also be convenient to discuss more general ∞-categorical pushout/pullback squares in a $\Bbbk$-linear ∞-category. As an expedient definition, we can declare that a commutative square

$$
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
Y & \xrightarrow{i} & Z
\end{array}
$$

in a $\Bbbk$-linear ∞-category is a **pushout** (resp. **pullback**) if the resulting commutative square

$$
\begin{array}{ccc}
W & \xrightarrow{(f,g)} & X \oplus Y \\
\big\downarrow & & \big\downarrow{\scriptstyle (h,-i)} \\
0 & \longrightarrow & Z
\end{array}
$$

is a cokernel (resp. kernel) square.

It turns out that pushout and pullback squares (and in particular, cokernel and kernel squares) coincide in *any* $\Bbbk$-linear ∞-category.[75] In order to refer to them in an unbiased way, we may refer to them as **exact squares**.

6.3. **The derived ∞-category.**

6.3.1.   As we will explain, the diagram (1) of dg-categories extends to a diagram

(3)

on underlying ∞-categories, in which the triangle involving $\rho_{\mathbf{P}}$, $\rho_{\mathbf{I}}$, and either vertical equivalence commutes. In particular, this yields a canonical equivalence $\mathbf{P}_R^\infty \simeq \mathbf{I}_R^\infty$ of ∞-categories.

---

[73]To be precise, this is a sort of projective resolution (actually a "cofibrant" resolution) among categories enriched in simplicial sets, and by "the free $\Bbbk$-linear dg-category" on a category enriched in simplicial sets we mean the dg-category obtained by taking $\Bbbk$-linear simplicial chains on its hom-objects.

[74]This should not be surprising, based on our discussion of the former.

[75]This makes crucial use of the $\Bbbk$-linearity: it is certainly not the case in a general ∞-category that pushout and pullback squares coincide.

We write $\mathbf{D}_R$ for their common value, which we refer to as ***the derived ∞-category of R-modules***.[76] (The fact that this is indeed the ∞-categorical localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms will be justified in §6.3.4.) Hence, collapsing the equivalences and using the unbiased notation $\mathbf{D}_R$, the above diagram may be expressed as a diagram

$$
(4) \qquad \mathbf{D}_R \underset{i_R}{\overset{i_L}{\underset{\perp}{\overset{\perp}{\longleftarrow \pi \longrightarrow}}}} \mathbf{K}_R^\infty \underset{a_R}{\overset{a_L}{\underset{\perp}{\overset{\perp}{\longleftarrow i_\mathbf{A} \longrightarrow}}}} \mathbf{A}_R^\infty \quad ,
$$

in which i_L (resp. i_R) denotes "the inclusion of $\mathbf{D}_R$ into $\mathbf{K}_R^\infty$ as the full subcategory of projective (resp. injective) complexes", which is the left (resp. right) adjoint to the projection functor π. This latter diagram forms what is called a ***recollement***, a sort of categorified extension sequence, which notion we will discuss more later.

6.3.2.   At this point, we can sharpen the perspective on the passage from a dg-category to its underlying ∞-category given in §6.2.3, in a way that will be helpful shortly. Namely, one can also *define* a $\Bbbk$-linear ∞-category to be a category enriched in the derived ∞-category $\mathbf{D}_\Bbbk$ of $\Bbbk$ (using the formalism of enriched ∞-categories of [GH15]). Then, the passage from a $\Bbbk$-linear dg-category to its underlying $\Bbbk$-linear ∞-category amounts to applying the functor

$$
\mathbf{K}_\Bbbk^\infty \overset{\pi}{\longrightarrow} \mathbf{D}_\Bbbk
$$

to each of its hom-complexes.

It is important to note that there are many commutative squares in $\mathbf{K}_R^\infty$ that may not be co/kernel squares themselves but nevertheless become so in $\mathbf{D}_R$, e.g. any commutative square of the form

$$
\begin{array}{ccc}
X & \overset{\approx}{\longrightarrow} & Y \\
\downarrow & & \downarrow \\
0 & \underset{\approx}{\longrightarrow} & Z
\end{array} \quad .
$$

In general, it is only after projective resolution or injective resolution that such a square becomes a homotopy co/kernel square.

6.3.3.   We now obtain the adjunctions and equivalences appearing in diagram (3).

First of all, the existence of the right adjoint $\rho_\mathbf{P}$ follows directly from (the ∞-categorical analog of) Exercise 6.1, and the existence of the left adjoint $\rho_\mathbf{I}$ follows from its dual. We refer to $\rho_\mathbf{P}$ (resp. $\rho_\mathbf{I}$) as the ***projective*** (resp. ***injective***) ***resolution*** functor.

---

[76]The superscript $(-)^\infty$ need not be applied here, as we have not defined $\mathbf{D}_R$ to be the underlying ∞-category of a specific dg-category: it is only well-defined up to canonical equivalence of ∞-categories in the first place.

Exercise 6.2 (6 points). Prove that the composite adjunction

$$\mathbf{P}_R^\infty \underset{\rho_{\mathbf{P}}}{\overset{i_{\mathbf{P}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{K}_R^\infty \underset{i_{\mathbf{I}}}{\overset{\rho_{\mathbf{I}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{I}_R^\infty$$

is an adjoint equivalence.

We now construct the adjoint a_L; the adjoint a_R will arises from dual considerations. We refer to a_L (resp. a_R) as the **left** (resp. **right**) **acyclification** functor.

For this, let us observe that for any complex $M \in \mathbf{K}_R$, the complex

$$a_L(M) := \mathsf{coker}^\infty \left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M)) \xrightarrow{\varepsilon_M} M \right)$$

is acyclic by Exercise 3.7(a).[77] In other words, the formula

$$a_L := \mathsf{coker}^\infty \left(i_{\mathbf{P}} \circ \rho_{\mathbf{P}} \xrightarrow{\varepsilon} \mathsf{id}_{\mathbf{K}_R} \right)$$

defines a functor $\mathbf{K}_R^\infty \xrightarrow{a_L} \mathbf{A}_R^\infty$.

We now verify that a_L defines a left adjoint to the inclusion $\mathbf{K}_R^\infty \xleftarrow{i_{\mathbf{A}}} \mathbf{A}_R^\infty$. At the ∞-categorical level, this is a straightforward computation. However, it is also a good opportunity to highlight the distinction between working in a dg-category and working in its underlying ∞-category, which makes it somewhat more subtle. Namely, the functor a_L is only well-defined at the ∞-categorical level, which implies that it is not literally meaningful to compute with it at the point-set level. On the other hand, by the dual of Exercise 6.1, it suffices to verify that *any* point-set representative of the morphism $M \to i_{\mathbf{A}} a_L(M)$ in $\mathbf{K}_R^\infty$ is a pointwise unit on underlying ∞-categories. Therefore, let us write

$$i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon_M'} M$$

for a specific but arbitrary point-set representative of the morphism ε_M (i.e. an arbitrary projective resolution of M), and let us write $a_L(M)' := \mathsf{hcoker}(\varepsilon_M')$; recall that homotopy cokernels in $\mathbf{K}_R$ compute ∞-categorical cokernels in $\mathbf{K}_R^\infty$, so the morphism $M \to a_L(M)'$ in $\mathbf{K}_R$ is indeed a point-set model for the morphism $M \to a_L(M)$ in $\mathbf{K}_R^\infty$. Now, for any acyclic complex $A \in \mathbf{A}_R^\infty$ we compute that

$$
\begin{aligned}
\mathsf{hom}_{\mathbf{A}_R^\infty}(a_L(M), A) &:= \pi \left(\mathsf{hom}_{\mathbf{A}_R}(a_L(M)', A) \right) \\
&:= \pi \left(\mathsf{hom}_{\mathbf{K}_R}(a_L(M)', A) \right) \\
&:= \pi \left(\mathsf{hom}_{\mathbf{K}_R} \left(\mathsf{hcoker} \left(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \xrightarrow{\varepsilon_M'} M \right), A \right) \right) \\
(5) \qquad &\simeq \pi \left(\mathsf{hker} \left(\mathsf{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(\varepsilon_M', A)} \mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A) \right) \right)
\end{aligned}
$$

---

[77]In view of the adjunction $i_{\mathbf{P}} \dashv \rho_{\mathbf{P}}$, one may think of $a_L(M)$ as being obtained by killing off the part of M that can be seen by projective complexes.

$$
\tag{6} \simeq \pi\left(\ker^\infty\left(\mathsf{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A)} \mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right)
$$

$$
\tag{7} \simeq \ker^\infty\left(\pi\left(\mathsf{hom}_{\mathbf{K}_R}(M, A)\right) \xrightarrow{\pi(\mathsf{hom}_{\mathbf{K}_R}(\varepsilon'_M, A))} \pi\left(\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right)\right)
$$

$$
\tag{8} \simeq \ker^\infty\left(\pi\left(\mathsf{hom}_{\mathbf{K}_R}(M, A)\right) \longrightarrow 0\right)
$$

$$
\simeq \pi(\mathsf{hom}_{\mathbf{K}_R}(M, A))
$$

$$
=: \mathsf{hom}_{\mathbf{K}_R^\infty}(M, A)
$$

as desired, by the following reasoning.

- Equivalence (5) follows from the fact that hom-complexes respect homotopy co/kernels (Exercise 4.7).
- Equivalence (6) follows from the fact that homotopy kernels in $\mathbf{K}_R$ compute ∞-categorical kernels in $\mathbf{K}_R^\infty$.
- As we have just seen, $\mathbf{K}_{\Bbbk}^\infty \xrightarrow{\pi} \mathbf{D}_{\Bbbk}$ is a right adjoint. It therefore preserves ∞-categorical limits, and in particular ∞-categorical kernels. This gives equivalence (7).
- By construction, the complex $i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))' \in \mathbf{K}_R$ is projective.[78] Hence, the complex $\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A) \in \mathbf{K}_{\Bbbk}$ is acyclic. It follows that we have an equivalence

$$
\pi\left(\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(\rho_{\mathbf{P}}(M))', A)\right) \simeq 0
$$

in $\mathbf{D}_{\Bbbk}$ (recall Exercise 5.3(a)), which explains equivalence (8).

Because cokernel sequences and kernel sequences in $\mathbf{K}_R^\infty$ agree, note that we can also view the projective (resp. injective) resolution functor as the kernel (resp. cokernel) of the left (resp. right) acyclification functor. Indeed, we have ∞-categorical co/kernel sequences

$$
\begin{array}{ccc}
i_{\mathbf{P}}\rho_{\mathbf{P}} \longrightarrow \mathsf{id}_{\mathbf{K}_R^\infty} & & i_{\mathbf{A}}a_R \longrightarrow \mathsf{id}_{\mathbf{K}_R^\infty} \\
\downarrow \qquad\qquad \downarrow & \text{and} & \downarrow \qquad\qquad \downarrow \\
0 \longrightarrow i_{\mathbf{A}}a_L & & 0 \longrightarrow i_{\mathbf{I}}\rho_{\mathbf{I}}
\end{array}
$$

in $\mathsf{Fun}(\mathbf{K}_R^\infty, \mathbf{K}_R^\infty)$.

6.3.4. We now simultaneously address the commutativity of the left triangle in diagram (3) and explain why $\mathbf{D}_R$ is the (∞-categorical) localization of $\mathbf{K}_R^\infty$ at the quasi-isomorphisms.

We say that an adjunction

$$
\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}
$$

is a **reflective localization adjunction** if the right adjoint G is fully faithful. In this situation, we refer to the functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ as a **reflective localization**.

Exercise 6.3 (6 points)**.**

---

[78]Recall that the property of projectivity is invariant under homotopy equivalence: any complex that is homotopy equivalent to a projective complex is itself projective.

(a) Show that G is fully faithful if and only if the counit $FG \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{D}}$ is a natural equivalence.

(b) Show that for a reflective localization adjunction $F \dashv G$, the morphism $F \xrightarrow{F\eta} FGF$ is a natural equivalence.

Given such a reflective localization adjunction, let us write $\mathbf{W} \subseteq \mathcal{C}$ for the subcategory consisting of those morphisms that are carried by F to equivalences. Then, we claim that the reflective localization $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is precisely the localization of $\mathcal{C}$ at $\mathbf{W}$.[79] To see this, fix any functor $\mathcal{C} \xrightarrow{T} \mathcal{E}$, and consider the diagram

$$
\begin{array}{ccc}
 & \mathcal{C} \xrightarrow{\quad T \quad} \mathcal{E} & \\
 G \nearrow \quad F \downarrow \quad {}^{T\eta}\!\!\Downarrow \quad {}_{TG}\!\!\nearrow & & \\
 \mathcal{D} \xrightarrow[\mathrm{id}_{\mathcal{D}}]{} \mathcal{D} & &
\end{array}
\qquad ,
$$

in which the left triangle commutes via the natural equivalence ε of Exercise 6.3(a). Considering just the solid diagram, we see that there is at most one choice of factorization of T through F, namely TG. On the other hand, we *always* have the indicated natural transformation $T \overset{T\eta}{\Longrightarrow} TGF$. So, it remains to check that $T\eta$ is a natural equivalence if and only if T carries the morphisms in $\mathbf{W}$ to equivalences. By Exercise 6.3(b), the components of η lie in $\mathbf{W} \subseteq \mathcal{C}$, which implies the "if" direction.

Exercise 6.4 (4 points). Prove the "only if" direction: that if $T\eta$ is a natural equivalence then T carries all morphisms in $\mathbf{W}$ to equivalences.

Now, by definition the adjunction $\rho_{\mathbf{I}} \dashv i_{\mathbf{I}}$ is a reflective localization adjunction.

Exercise 6.5 (2 points). Prove that a morphism in $\mathbf{K}_R^{\infty}$ is carried by $\rho_{\mathbf{I}}$ to an equivalence if and only if it is a quasi-isomorphism.

By what we have just seen, this implies that the functor $\mathbf{K}_R^{\infty} \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I}_R^{\infty}$ is indeed the localization of $\mathbf{K}_R^{\infty}$ at the quasi-isomorphisms.

Of course, dually the functor $\mathbf{K}_R^{\infty} \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_R^{\infty}$ is a *coreflective* localization, and so it is *also* the localization of $\mathbf{K}_R^{\infty}$ at the quasi-isomorphisms.

So, *both* functors $\rho_{\mathbf{P}}$ and $\rho_{\mathbf{I}}$ are the localization at the quasi-isomorphisms. In other words, they share the same universal property. Therefore, they must be canonically equivalent. Moreover, tracing through our proof that a reflective localization is a localization (and dualizing), we find that the canonical morphisms between $\mathbf{P}_R^{\infty}$ and $\mathbf{I}_R^{\infty}$ that are induced by their shared universal property (as $\Bbbk$-linear ∞-categories under $\mathbf{K}_R^{\infty}$) are precisely the

---

[79]To explain the terminology further, considering $\mathcal{D} \subseteq \mathcal{C}$ as a full subcategory via G, the functor F is also called the *reflector* of $\mathcal{C}$ into $\mathcal{D}$.

composites

(which are equivalences by Exercise 6.2). In particular, these triangles commute, as asserted.

It is also not hard to see that these triangles commute using the Yoneda lemma. For instance, given any $P \in \mathbf{P}_R^\infty$ and any $M \in \mathbf{K}_R^\infty$, we have the composite equivalence

$$\mathsf{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}} i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \simeq \mathsf{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), i_{\mathbf{I}} \rho_{\mathbf{I}}(M)) \xrightarrow{\sim} \mathsf{hom}_{\mathbf{K}_R^\infty}(i_{\mathbf{P}}(P), M) \simeq \mathsf{hom}_{\mathbf{P}_R^\infty}(P, \rho_{\mathbf{P}}(M))$$

in $\mathbf{D}_{\Bbbk}$, using the fact that $\mathsf{hom}_{\mathbf{K}_R}(i_{\mathbf{P}}(P), -)$ carries quasi-isomorphisms to quasi-isomorphisms.

6.3.5. The ∞-category $\mathbf{K}_{\Bbbk}^\infty$ of complexes over $\Bbbk$ is symmetric monoidal via tensor product, and the ∞-category $\mathbf{K}_R^\infty$ of complexes over R is a left module over it. It follows from Exercise 5.3(c) that the fully faithful inclusions

$$\mathbf{D}_{\Bbbk} \xhookrightarrow{i_L} \mathbf{K}_{\Bbbk}^\infty \qquad \text{and} \qquad \mathbf{D}_R \xhookrightarrow{i_L} \mathbf{K}_R^\infty$$

endow the derived ∞-categories with the same structures: $\mathbf{D}_{\Bbbk}$ is symmetric monoidal and $\mathbf{D}_R$ is a left module over it. It is customary to denote these tensor products as

$$(-) \overset{\mathbb{L}}{\otimes} (-) \, ,\text{[80]}$$

in order to emphasize that they are taking place in the context of *derived* ∞-categories.

Note that here we must use the inclusions i_L (not i_R). This is due to the handedness of the tensor-hom adjunction: tensor products satisfying a universal *mapping out* property (instead of a universal *mapping in* property). Said differently, it is the tensor product of *projective* complexes (and not injective complexes) that computes the tensor product of derived modules.

Relatedly, the localization $\mathbf{K}_{\Bbbk}^\infty \xrightarrow{\pi} \mathbf{D}_{\Bbbk}$ is *not* a symmetric monoidal localization. For instance, given ordinary $\Bbbk$-modules $M, N \in \mathsf{Mod}_{\Bbbk}$, as explained further in §7.2, we have a comparison morphism

$$(9) \qquad\qquad M \overset{\mathbb{L}}{\otimes} N := \rho_{\mathbf{P}}(M) \otimes \rho_{\mathbf{P}}(N) \longrightarrow M \otimes N$$

(note that the inclusion $\mathsf{Mod}_{\Bbbk} \hookrightarrow \mathbf{K}_{\Bbbk}$ is symmetric monoidal), but this morphism is not generally an equivalence. Rather, the functor π is *right-laxly* symmetric monoidal: it comes equipped with natural comparison morphisms (such as the morphism (9)) from the (iterated) tensor product in $\mathbf{D}_{\Bbbk}$ of its values to its value on the corresponding tensor product in $\mathbf{K}_{\Bbbk}^\infty$.

---

[80]The notation will be explained in §7.

Inasmuch as the derived tensor product behaves better than the ordinary tensor product, we view the fact that π is only right-laxly symmetric monoidal as a feature rather than a bug. Moreover, right-laxly symmetric monoidal functors still interact nicely with algebraic structures.[81] For instance, in the present situation, we have a canonical lift

$$
\begin{array}{ccc}
\mathsf{CAlg}(\mathbf{K}_\Bbbk^\infty) & \dashrightarrow^{\ \pi\ } & \mathsf{CAlg}(\mathbf{D}_\Bbbk) \\
{\scriptstyle\mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle\mathsf{fgt}} \\
\mathbf{K}_\Bbbk^\infty & \xrightarrow{\ \pi\ } & \mathbf{D}_\Bbbk
\end{array} \quad ,
$$

so that commutative algebra objects in $\mathbf{K}_\Bbbk^\infty$ define commutative algebra objects in $\mathbf{D}_\Bbbk$, and likewise for associative algebra objects.[82]

6.3.6. Fix an arbitrary abelian category $\mathcal{A}$. Given this, we may form the category $\mathsf{Ch}(\mathcal{A})$ of complexes in $\mathcal{A}$. This admits an enrichment in $\mathsf{Ch}_\mathbb{Z}$, i.e. it defines a $\mathbb{Z}$-linear dg-category, whose underlying $\mathbb{Z}$-linear ∞-category we denote by $\mathbf{K}(\mathcal{A}) := \mathbf{K}^\infty(\mathcal{A})$. From here, we may form the derived ∞-category $\mathbf{D}(\mathcal{A})$ by freely inverting the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$. There are also the three variants $\mathbf{D}^-(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$, and $\mathbf{D}^b(\mathcal{A})$: the bounded-above, bounded-below, and bounded derived ∞-categories.[83] All of these $\mathbb{Z}$-linear ∞-categories are **stable**: they are nonempty and admit all co/kernels.[84] Moreover, at least the latter three may admit various universal properties as such. The simplest is that of $\mathbf{D}^b(\mathcal{A})$, which is the universal ($\mathbb{Z}$-linear) stable ∞-category receiving a functor from $\mathcal{A}$ that carries all short exact sequences to co/kernel sequences [BCKW, Corollary 7.59]. The universal properties of $\mathbf{D}^\pm(\mathcal{A})$ are somewhat more subtle: they require hypotheses on $\mathcal{A}$ (namely the existence of enough projectives/injectives and/or the property of being a *Grothendieck* abelian category), and make reference to "analytic" notions (in the sense of convergence) involving t-structures [§A.1.3].

7. Derived functors

A substantial vein of current mathematical research takes place entirely within the derived realm, e.g. working only with the derived ∞-category $\mathbf{D}_R$ instead of with the abelian category Mod_R. We will take this point of view when we study sheaf theory. However, it is nevertheless worthwhile to connect back with the original approach to homological algebra. We explain the general principles in §7.1, and then illustrate how passing to classical derived functors amounts to working in the derived ∞-category in §§7.2-7.3. We then briefly discuss the

---

[81]Dually, left-laxly symmetric monoidal functors interact nicely with *coalgebraic* structures.

[82]It turns out that associative algebra objects in $\mathbf{D}_\Bbbk$ always lift to associative algebra objects in $\mathbf{K}_\Bbbk$ (i.e. dgas). When $\Bbbk$ has characteristic zero, commutative algebra objects in $\mathbf{D}_\Bbbk$ also always lift to commutative algebra objects in $\mathbf{K}_\Bbbk$ (i.e. cdgas). However, in general there are commutative algebra objects in $\mathbf{D}_\Bbbk$ that do not lift to cdgas; the obstructions effectively arise from the cohomology of symmetric groups (which vanish in characteristic zero).

[83]In the notations $\mathbf{D}^\pm(\mathcal{A})$, the superscript indicates the direction in which infinitude is permitted.

[84]We discuss stable ∞-categories (without $\mathbb{Z}$-linearity) in **??**.

example of group co/homology in §7.4. We refer the interested reader to [Wei94] for a more in-depth treatment of group co/homology, as well as for treatments of two other fundamental examples: Lie algebra co/homology and Hochschild co/homology. We conclude with a brief discussion of decategorification in §7.5, which applies to the derived approach to the counting problems described in §1.4.

7.1. **Derived functors.**

7.1.1.   As explained in §5.1, the original motivation for homological algebra was to repair certain failures of exactness. Namely, an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories is called

- *left exact* if it commutes with finite limits,
- *right exact* if it commutes with finite colimits, and
- *exact* if it is both left exact and right exact.

Indeed, if an object $A \in \mathcal{A}$ is given to us e.g. as a finite colimit, we can only deduce an analogous presentation of its image $F(A) \in \mathcal{B}$ if F is right exact.

The names "left exact" and "right exact" arise from the following facts (which also serve as mnemonics). Throughout, we refer to an exact sequence

$$(10) \qquad\qquad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathcal{A}$.

Exercise 7.1 (8 points)**.**
  (a) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is left exact if and only if it commutes with kernels, i.e. the sequence

$$(11) \qquad\qquad 0 \cong F(0) \longrightarrow F(L) \longrightarrow F(M) \longrightarrow F(N)$$

in $\mathcal{B}$ is also exact.
  (b) Show that $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is right exact if and only if it commutes with cokernels, i.e. the sequence

$$(12) \qquad\qquad F(L) \longrightarrow F(M) \longrightarrow F(N) \longrightarrow F(0) \cong 0$$

in $\mathcal{B}$ is also exact.

So, if F is only left exact, one is inclined to extend the sequence (11) to a long exact sequence

$$
\begin{array}{ccccccc}
0 & \longrightarrow & (\mathbb{R}^0 F)(L) & \longrightarrow & (\mathbb{R}^0 F)(M) & \longrightarrow & (\mathbb{R}^0 F)(N) \\
\\
& & (\mathbb{R}^1 F)(L) & \longrightarrow & (\mathbb{R}^1 F)(M) & \longrightarrow & (\mathbb{R}^1 F)(N) \\
\\
& & (\mathbb{R}^2 F)(L) & \longrightarrow & \cdots
\end{array}
$$

in $\mathcal{B}$. Here, we refer to $\mathbb{R}^n F$ as the nth ***right derived functor*** of F, and by definition $\mathbb{R}^0 F := F$. Dually, if F is only right exact, one is inclined to the extend the sequence (12) to a long exact sequence

$$
\begin{array}{ccccccc}
& & & & \cdots & \longrightarrow & (\mathbb{L}_2 F)(N) \\
\\
& & (\mathbb{L}_1 F)(L) & \longrightarrow & (\mathbb{L}_1 F)(M) & \longrightarrow & (\mathbb{L}_1 F)(N) \\
\\
& & (\mathbb{L}_0 F)(L) & \longrightarrow & (\mathbb{L}_0 F)(M) & \longrightarrow & (\mathbb{L}_0 F)(N) & \longrightarrow & 0
\end{array}
$$

in $\mathcal{B}$. Here, we refer to $\mathbb{L}_n F$ as the nth ***left derived functor*** of F, and by definition $\mathbb{L}_0 F := F$.[85]

7.1.2. As we saw in §3.4, long exact sequences in algebra arise from homotopy co/kernel sequences in homological algebra. Thus, we should expect to obtain the above long exact sequences from homotopy co/kernel sequences. More specifically, the exact sequence (10) in $\mathcal{A}$ may be thought of as a homotopy co/kernel sequence among complexes in $\mathcal{A}$ (recall Exercise 3.6(a)), to which we wish to apply some functor derived from F to obtain another homotopy co/kernel sequence among complexes in $\mathcal{B}$. We implement this in a slightly more general situation, and then specialize to the case of interest.

Let us write $\mathbf{K}(\mathcal{A})$ for the ($\mathbb{Z}$-linear) ∞-category of complexes in $\mathcal{A}$, and let us write

$$
\mathbf{K}(\mathcal{A}) \xrightarrow{\ \pi\ } \mathbf{D}(\mathcal{A})
$$

---

[85]A priori, it is not clear that these derived functors are well-defined or uniquely characterized by these conditions, although they do turn out to be so.

for its localization at the quasi-isomorphisms. Then, for any functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{T} \mathcal{C} \ ,$$

we define its **total right derived functor** and its **total left derived functor** to respectively be the terminal and initial extensions

(13)
$$
\begin{array}{ccc}
\mathbf{K}(\mathcal{A}) \xrightarrow{T} \mathcal{C} & & \mathbf{K}(\mathcal{A}) \xrightarrow{T} \mathcal{C} \\
{\scriptstyle\pi}\downarrow \quad {\scriptstyle\mathbb{R}T} & \text{and} & {\scriptstyle\pi}\downarrow \quad {\scriptstyle\mathbb{L}T} \\
\mathbf{D}(\mathcal{A}) & & \mathbf{D}(\mathcal{A})
\end{array}
$$

along π (which may not exist in general). Said differently, these total derived functors participate in comparison maps

$$\pi \circ \mathbb{L}T \longrightarrow T \longrightarrow \pi \circ \mathbb{R}T$$

in $\mathsf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C})$, and by definition they are the universal objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C})$ equipped with such comparison maps. Thereafter, the composites $\pi \circ \mathbb{L}T$ and $\pi \circ \mathbb{R}T$ should be thought of two dual ways of universally forcing the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to be sent to equivalences in $\mathcal{C}$.

We now return to our additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ between abelian categories. To apply the foregoing discussion, we take T to be the composite functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(F)} \mathbf{K}(\mathcal{B}) \xrightarrow{\pi} \mathcal{D}(\mathcal{B}) \ ,$$

and we denote the resulting total derived functors (which are objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathbf{D}(\mathcal{B}))$) simply by

$$\mathbb{R}F := \mathbb{R}(\pi \circ \mathbf{K}(F)) \qquad \text{and} \qquad \mathbb{L}F := \mathbb{L}(\pi \circ \mathbf{K}(F)) \ .$$

As we will see in Exercise 7.3, we thereafter recover the derived functors of §7.1.1 (which are objects of $\mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{B})$) as the composites

$$\mathbb{R}^n F \cong \mathsf{H}_{-n} \circ \mathbb{R}F \qquad \text{and} \qquad \mathbb{L}_n F \cong \mathsf{H}_n \circ \mathbb{L}F \ .$$

Examining the universal extensions (13), we see that these assignments $T \mapsto \mathbb{R}T$ and $T \mapsto \mathbb{L}T$ are nothing other than adjoints

(14)
$$
\mathsf{Fun}(\mathbf{K}(\mathcal{A}), \mathcal{C}) \xleftarrow{\ \mathsf{Fun}(\pi, \mathcal{C})\ } \mathsf{Fun}(\mathbf{D}(\mathcal{A}), \mathcal{C})
$$
with $\mathbb{R}(-)$ above and $\mathbb{L}(-)$ below, $\perp$.

to the restriction functor.[86] (Note the unfortunate conventional clash that taking the *right* derived functor is a *left* adjoint, and reversely.) There are formulas for computing such adjoints in general, but they are extremely easy to compute in certain special cases.

---

[86]In this generality, these adjoints may not exist. But when a particular total derived functor exists, it is nothing other than a pointwise adjoint as indicated.

Exercise 7.2 (4 points). Given any categories $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$, show that applying the functors $\mathsf{Fun}(\mathcal{F},-)$ and $\mathsf{Fun}(-,\mathcal{F})$ to an adjunction

$$\mathcal{D} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{E}$$

yields adjunctions

$$\mathsf{Fun}(\mathcal{F},\mathcal{D}) \underset{\mathsf{Fun}(\mathcal{F},R)}{\overset{\mathsf{Fun}(\mathcal{F},L)}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}(\mathcal{F},\mathcal{E}) \qquad \text{and} \qquad \mathsf{Fun}(\mathcal{D},\mathcal{F}) \underset{\mathsf{Fun}(L,\mathcal{F})}{\overset{\mathsf{Fun}(R,\mathcal{F})}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}(\mathcal{E},\mathcal{F}) \quad .$$

In the particular case that $\mathcal{A} = \mathsf{Mod}_R$ (so that $\mathbf{K}(\mathcal{A}) = \mathbf{K}(\mathsf{Mod}_R) =: \mathbf{K}_R$), the ∞-categorical analog of Exercise 7.2 identifies the adjoints (14) simply as

$$\mathsf{Fun}(\mathbf{K}_R, \mathcal{C}) \xleftarrow{\mathsf{Fun}(\pi,\mathcal{C})} \overset{\overset{\mathsf{Fun}(i_R,\mathcal{C})}{\underset{\perp}{\dashrightarrow}}}{\underset{\underset{\mathsf{Fun}(i_L,\mathcal{C})}{\underset{\perp}{\dashrightarrow}}}{}} \mathsf{Fun}(\mathbf{D}_R, \mathcal{C}) \quad .$$

In other words, passing to total right derived functors amounts to injective resolution

while passing to total left derived functors amounts to projective resolution

Exercise 7.3 (12 points). Verify that these formulas produce the long exact sequences sought in §7.1.1. In particular,

- in the case that F is left exact, prove that

$$\mathsf{H}_n \circ \mathbb{R}F \cong \begin{cases} F, & n = 0 \\ 0, & n > 0 \end{cases},$$

  and
- in the case that F is right exact, prove that

$$\mathsf{H}_n \circ \mathbb{L}F \cong \begin{cases} F, & n = 0 \\ 0, & n < 0 \end{cases}.$$

We have passed from an arbitrary abelian category $\mathcal{A}$ to the abelian category Mod_R in order to guarantee the existence of the adjoints i_L and i_R to the projection π. In particular, in this situation the adjoints (14) exist without any hypotheses on $\mathcal{C}$.

7.2. Tor.

7.2.1. We now study Tor, one of two classical examples of a derived functor. By definition, Tor is the left derived functor of relative tensor product. It will be illuminating to work relative to our base commutative ring $\Bbbk$, so we begin in that special case, and then proceed to discuss Tor over the associative $\Bbbk$-algebra R.

7.2.2. The relative tensor product bifunctor

$$\mathsf{Mod}_\Bbbk \times \mathsf{Mod}_\Bbbk \xrightarrow{(-)\otimes(-)} \mathsf{Mod}_\Bbbk$$

is right exact separately in each variable, but it is not left exact in either variable. Let us fix a $\Bbbk$-module $M \in \mathsf{Mod}_\Bbbk$ and consider the resulting right exact functor

$$\mathsf{Mod}_\Bbbk \xrightarrow{M\otimes(-)} \mathsf{Mod}_\Bbbk .$$

According to the prescription of §7.1.2, we define its (total) left derived functor $\mathbb{L}(M \otimes (-))$ as the composite

$$\mathbf{D}_\Bbbk \simeq \mathbf{P}_\Bbbk \xhookrightarrow{i_{\mathbf{P}}} \mathbf{K}_\Bbbk \xrightarrow{\mathbf{K}(M\otimes(-))} \mathbf{K}_\Bbbk \xrightarrow{\pi} \mathbf{D}_\Bbbk .$$

In particular, its value on an ordinary $\Bbbk$-module is obtained as the upper composite in the diagram

$$\mathsf{Mod}_\Bbbk \hookrightarrow \mathbf{K}_\Bbbk \xrightarrow{\rho_{\mathbf{P}}} \mathbf{P}_\Bbbk \xhookrightarrow{i_{\mathbf{P}}} \mathbf{K}_\Bbbk \xrightarrow{\mathbf{K}((-)\otimes N)} \mathbf{K}_\Bbbk \xrightarrow{\pi} \mathbf{D}_\Bbbk .$$

$$\varepsilon \Downarrow \qquad \mathsf{id}_{\mathbf{K}_\Bbbk}$$

Of course, we can equally well perform the same operation in the second slot. We are therefore led to consider *two* possible notions of "the left derived tensor product of M and N". Luckily, they agree: by Exercise 5.4(c), we have natural quasi-isomorphisms fitting into a commutative square

$$\mathbb{L}(M \otimes (-))(N)$$
$$\vdots\|$$
$$\begin{array}{ccc} \rho_{\mathbf{P}}(M) \otimes \rho_{\mathbf{P}}(N) & \xrightarrow{\approx} & M \otimes \rho_{\mathbf{P}}(N) \\ \wr\downarrow & & \downarrow \\ \rho_{\mathbf{P}}(M) \otimes N & \xrightarrow{\quad\quad} & M \otimes N \end{array}$$
$$\|\vdots$$
$$\mathbb{L}((-) \otimes N)(M)$$

in $\mathbf{K}_{\Bbbk}$ (omitting the functor $i_{\mathbf{P}}$ for simplicity). It is customary to simply write

$$M \overset{\mathbb{L}}{\otimes} N \in \mathbf{D}_{\Bbbk}$$

for the common value of these derived functors and refer to it as the **_derived tensor prod-_**
uct. Note that this is nothing more than the tensor product in $\mathbf{D}_{\Bbbk}$, as defined in §6.3.5.

For any $n \geqslant 0$, we define

$$\mathsf{Tor}_n(M, N) := \mathsf{Tor}_n^{\Bbbk}(M, N) := \mathsf{H}_n(M \otimes^{\mathbb{L}} N) \in \mathsf{Mod}_{\Bbbk} .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{L}_n(M \otimes (-))(N) \cong \mathsf{Tor}_n(M, N) \cong \mathbb{L}_n((-) \otimes N)(M) .$$

7.2.3.   The source of the name "Tor" is the following fundamental example.

Exercise 7.4 (6 points). Taking $\Bbbk = \mathbb{Z}$ and all possible combinations where $M \in \{\mathbb{Z}, \mathbb{Z}/m\}$
and $N \in \{\mathbb{Z}, \mathbb{Z}/n\}$, compute $\mathsf{Tor}_i(M, N)$ for all $i \geqslant 0$.

As each functor $\mathsf{Tor}_i(-, -)$ preserves finite sums separately in each variable, Exercise 7.4
effectively gives a computation of Tor for all finitely generated abelian groups.

7.2.4.   We now discuss Tor over the associative $\Bbbk$-algebra R. Of course, it is the left derived
functor of the bifunctor

$$\mathsf{Mod}_R \times {}_R\mathsf{Mod} \xrightarrow{(-)\otimes_R(-)} \mathsf{Mod}_{\Bbbk} ;$$

the subtlety is simply that this is not a monoidal structure on a single category. Once again,
it is sufficient to take a projective resolution in one variable or the other.

7.2.5.   It is worth mentioning a particular small (and therefore computable) model for the
derived relative tensor product of ordinary R-modules.[87] Namely, given a right R-module
$M \in \mathsf{Mod}_R$ and a left R-module $N \in {}_R\mathsf{Mod}$, we define the (**_two-sided_**) **_bar complex_**
$\mathsf{Bar}(M, R, N) \in \mathbf{K}_{\Bbbk}$ to be

$$\cdots \xrightarrow{d_3} M \otimes R^{\otimes 2} \otimes N \xrightarrow{d_2} M \otimes R \otimes N \xrightarrow{d_1} \underset{\sim\sim\sim}{M \otimes N} \longrightarrow 0 ,$$

where we define $d_i := \sum_{j=0}^{i}(-1)^j d_i^j$ and we define d_i^j by multiplying the jth and $(j+1)^{\text{st}}$
tensor factors (where we count starting from 0), i.e.

$$d_i^j(m \otimes r_1 \otimes \cdots \otimes r_i \otimes n) := \begin{cases} m \cdot r_1 \otimes r_2 \otimes \cdots \otimes r_i \otimes n , & j = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_{j-1} \otimes r_j \cdot r_{j+1} \otimes r_{j+2} \otimes \cdots \otimes r_i \otimes n , & 0 < j < i \\ m \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i \cdot n , & j = i \end{cases} .$$

Then, we have a canonical equivalence

$$(15) \qquad\qquad\qquad\qquad \mathsf{Bar}(M, R, N) \simeq M \overset{\mathbb{L}}{\underset{R}{\otimes}} N$$

---

[87]This admits a straightforward generalization to complexes of R-modules, which we do not address here.

in $\mathbf{D}_\Bbbk$. This is particularly easy to see if $\mathsf{fgt}(M) \in \mathsf{Mod}_\Bbbk$ is projective (or similarly if $\mathsf{fgt}(N) \in \mathsf{Mod}_\Bbbk$ is projective), e.g. if $\Bbbk$ is a field.[88] In general, if N carries a right action S-action that commutes with its left R-action, then we may consider $\mathsf{Bar}(M, R, N) \in \mathbf{K}_S$ as a complex of right S-modules. Taking $N = R = S$, we obtain a complex $\mathsf{Bar}(M, R, R) \in \mathbf{K}_R$ of right R-modules. By Exercise 5.10, this is projective: it is levelwise projective by the dual of Exercise 5.13(c), and moreover it is bounded below.

Exercise 7.5 (2 points)**.** Construct a natural quasi-isomorphism $\mathsf{Bar}(M, R, R) \xrightarrow{\approx} M$ in $\mathbf{K}_R$.

In other words, Exercise 7.5 implies that $\mathsf{Bar}(M, R, R) \xrightarrow{\approx} M$ is a projective resolution. Now, equivalence (15) follows from the evident isomorphism $\mathsf{Bar}(M, R, R) \otimes_R N \cong \mathsf{Bar}(M, R, N)$ in $\mathbf{K}_\Bbbk$.

7.3. Ext.

7.3.1. We now study Ext, the other classical example of a derived functor. By definition, Ext is the right derived functor of hom. The hom bifunctor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_\Bbbk$$

is left exact separately in each variable, but it is not right exact in either variable.[89] Now, following §7.1.2 we define its two possible right derived functors as in the diagrams

and

.[90]

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[88]The proof in the general case requires techniques that we do not discuss here; see [Wei94, §8.6] for an explanation.

[89]When considering contravariant functors, it is customary to put the $(-)^{\mathsf{op}}$ on the source category (e.g. for determining whether the functor $\mathsf{hom}_{\mathsf{Mod}_R}(-, N)$ should be considered as left exact or right exact).

[90]Note that passing to opposites exchanges projectives and injectives.

Once again, these agree: by Exercise 5.3(d)(e), we have natural quasi-isomorphisms fitting into a commutative square

$$\mathbb{R}(\mathsf{hom}_{\mathsf{Mod}_R}(M,-))(N)$$
$$\overset{..}{\|}$$

$$\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(M,\rho_{\mathbf{I}}(N))$$

$$\mathsf{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M),N) \overset{\approx}{\longrightarrow} \mathsf{hom}_{\mathbf{K}_R}(\rho_{\mathbf{P}}(M),\rho_{\mathbf{I}}(N))$$

$$\overset{\|}{.}$$
$$\mathbb{R}(\mathsf{hom}_{\mathsf{Mod}_R}(-,N))(M)$$

in $\mathbf{K}_{\Bbbk}$. It is customary to simply write

$$\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \in \mathbf{D}_{\Bbbk}$$

for the common value of these derived functors and refer to it as the ***derived hom***. Note that this is nothing more than the hom in $\mathbf{D}_R$:

$$\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N) \simeq \mathsf{hom}_{\mathbf{D}_R}(M,N) := \mathsf{hom}_{\mathbf{D}_R}(\pi(M),\pi(N))\ .$$

For any $n \geqslant 0$, we define

$$\mathsf{Ext}^n(M,N) := \mathsf{Ext}^n_R(M,N) := \mathsf{H}_{-n}(\mathbb{R}\mathsf{hom}_{\mathsf{Mod}_R}(M,N)) \in \mathsf{Mod}_{\Bbbk}\ .$$

By Exercise 7.3, we have natural isomorphisms

$$\mathbb{R}^n(\mathsf{hom}_{\mathsf{Mod}_R}(M,-))(N) \cong \mathsf{Ext}^n(M,N) \cong \mathbb{R}^n(\mathsf{hom}_{\mathsf{Mod}_R}(-,N))(M)\ .$$

7.3.2.   The name "Ext" arises from the fact that for any $n \geqslant 1$, $\mathsf{Ext}^n_R(M,N)$ classifies equivalence classes of exact sequences

$$(16) \qquad\qquad 0 \longrightarrow N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0\ ,$$

called n-***extensions*** of M by N, under the relation that there exists a commutative diagram

$$(17) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_{n-1} & \longrightarrow \cdots \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \mathsf{id}_N \downarrow & & \downarrow & & \downarrow & & \downarrow \mathsf{id}_M & & \\ 0 & \longrightarrow & N & \longrightarrow & X'_{n-1} & \longrightarrow \cdots \longrightarrow & X'_0 & \longrightarrow & M & \longrightarrow & 0 \end{array} \qquad .$$

Observe that we may view an n-extension (16) as a sequence of composable morphisms

$$
\begin{array}{ccc}
\Sigma^{n-1}N & & N \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \\
\downarrow f & & \downarrow \quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow \quad\quad \downarrow \\
X_\bullet & := & X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \underset{\sim}{X_0} \\
\downarrow g & & \downarrow \quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow \quad\quad \downarrow \\
M & & 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{\sim}{M}
\end{array}
$$

in Ch_R, such that the composition equals zero. Hence, we obtain a commutative square

$$
\begin{array}{ccc}
\Sigma^{n-1}N & \overset{f}{\longrightarrow} & X_\bullet \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & M
\end{array}
$$

in the ∞-category $\mathbf{K}_R$. This may not be a co/kernel square in $\mathbf{K}_R$, but by the long exact sequence in homology it does become a co/kernel square in $\mathbf{D}_R$. As such, it is completely determined by the morphism $X_\bullet \overset{g}{\to} M$, which in turn is completely specified by the morphism $M \to \mathsf{coker}(g)$. On the other hand, we have a canonical identification

$$
\mathsf{coker}(g) \simeq \Sigma^n N \quad :
$$

both squares in the diagram

(18)
$$
\begin{array}{ccccc}
\Sigma^{n-1}N & \overset{f}{\longrightarrow} & X_\bullet & \longrightarrow & 0 \\
\downarrow & & \downarrow g & & \downarrow \\
0 & \longrightarrow & M & \underset{k}{\longrightarrow} & \mathsf{hcoker}(g)
\end{array}
$$

in $\mathbf{D}_R$ are pushouts, and so the composite rectangle is also a pushout. Of course, this morphism k is precisely the element of $\mathsf{Ext}_R^n(M, N)$ that is classified by the n-extension (16). Note too that a morphism (17) of n-extensions determines a quasi-isomorphism

$$
X_\bullet \overset{\approx}{\longrightarrow} X_\bullet'
$$

in $(\mathsf{Ch}_R)_{\Sigma^{n-1}N//M}$, which gives a homotopy between the corresponding morphisms $M \to \Sigma^n N$ in $\mathbf{D}_R$.

Exercise 7.6 (10 points). Construct an inverse function from $\mathsf{Ext}_R^n(M, N)$ to the set of equivalence classes of n-extensions of M by N.

7.3.3. To simplify our discussion, let us assume that $n \geqslant 2$. Then, by the long exact sequence in homology, the complex $X \in \mathsf{Ch}_R$ determined by an n-extension (16) has homology groups

$$\mathsf{H}_i(X) \cong \begin{cases} M , & i = 0 \\ N , & i = n-1 \\ 0 , & \text{otherwise} \end{cases} .$$

Said differently, the derived R-module $X \in (\mathbf{D}_R)_{\Sigma^{n-1}N//M}$ has the property that its structure maps $\Sigma^{n-1}N \xrightarrow{f} X \xrightarrow{g} M$ respectively induce isomorphisms on H_{n-1} and H_0. On the other hand, as we have seen, this does *not* characterize X as an object of $(\mathbf{D}_R)_{\Sigma^{n-1}N//M}$: equivalences $X \to X'$ therein correspond to homotopies $k \Rightarrow k'$ in $\mathsf{hom}_{\mathbf{D}_R}(M, \Sigma^n N)$, which are obstructed by $\mathsf{H}_0(\mathsf{hom}_{\mathbf{D}_R}(M, \Sigma^n N)) \cong \mathsf{Ext}_R^n(M, N)$. Indeed, the equivalence class of the object $X \in (\mathbf{D}_R)_{\Sigma^{n-1}N//M}$ is *equivalent data* to the morphism $M \xrightarrow{k} \Sigma^n N$, which is called the $(n-1)^{st}$ *k-invariant* of X.

More generally, we can now make precise the assertion of §2.4.4 that a derived R-module is specified by the data of its homology groups and its k-invariants. To explain this, let us write $\mathbf{D}_R^{\leqslant n} \subseteq \mathbf{D}_R$ for the full subcategory of *n-truncated* derived R-modules, i.e. those with vanishing homology above dimension n. Its inclusion admits a left adjoint

$$\mathbf{D}_R \xrightleftharpoons[{}]{\tau_{\leqslant n}} \mathbf{D}_R^{\leqslant n} \quad ,^{[91]}$$

and from here it should be plausible that we have an equivalence

$$\mathsf{id}_{\mathbf{D}_R} \xrightarrow{\sim} \lim \left(\cdots \longrightarrow \tau_{\leqslant (n+1)} \longrightarrow \tau_{\leqslant n} \longrightarrow \tau_{\leqslant (n-1)} \longrightarrow \cdots \right)$$

in $\mathsf{Fun}(\mathbf{D}_R, \mathbf{D}_R)$:[92] every derived R-module $Y \in \mathbf{D}_R$ is canonically equivalent to the limit of its ***Postnikov tower***

$$\cdots \longrightarrow \tau_{\leqslant (n+1)}Y \longrightarrow \tau_{\leqslant n}Y \longrightarrow \tau_{\leqslant (n-1)}Y \longrightarrow \cdots .$$

Moreover, we see from the iterated co/kernel diagram

$$\begin{array}{ccccc} \Sigma^n \mathsf{H}_n(Y) & \longrightarrow & \tau_{\leqslant n}Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tau_{\leqslant (n-1)}Y & \xrightarrow{k_n(Y)} & \Sigma^{n+1}\mathsf{H}_n(Y) \end{array}$$

in $\mathbf{D}_R$ (akin to diagram (18)) that given $\tau_{\leqslant (n-1)}Y$, the data of $\tau_{\leqslant n}Y$ (equipped with its canonical n-truncation map to $\tau_{\leqslant (n-1)}Y$) is equivalent to the data of the nth ***k-invariant***

---

[91]A dual adjunction (from which this one may be immediately deduced) is essentially established in Exercise 8.7.

[92]This assertion is slightly subtle, because homology does *not* commute with (∞-categorical) inverse limits in $\mathbf{D}_R$ in general. However, it *does* commute with the limits of inverse systems whose structure maps are surjective on homology.

of Y, namely the morphism

$$k_n(Y) \in \hom_{\mathbf{D}_R}(\tau_{\leqslant(n-1)}Y, \Sigma^{n+1}\mathsf{H}_n(Y)) \ .$$

7.4. Group co/homology.

7.4.1. Let G be a discrete group. Then, a ($\Bbbk$-linear) G-**module** is a $\Bbbk$-module equipped with a (right) G-action. It is easy to see that these are equivalent to modules over the group algebra $\Bbbk[G]$.

Given any G-module $M \in \mathsf{Mod}_{\Bbbk[G]}$, its G-**invariants** and G-**coinvariants** are the $\Bbbk$-modules

$$M^G := \{m \in M : m \cdot g = m \text{ for all } g \in G\} \qquad \text{and} \qquad M_G := M/\{m - m \cdot g\}_{m \in M, g \in G} \ .$$

These are respectively the maximal $\Bbbk$-submodule and minimal quotient $\Bbbk$-module on which G act trivially.

Let us equip $\Bbbk$ itself with the trivial G-bimodule structure: $g \cdot \lambda := \lambda =: \lambda \cdot g$ for all $g \in G$ and for all $\lambda \in \Bbbk$. Then, it is clear that we have isomorphisms

$$M^G \cong \hom_{\mathsf{Mod}_{\Bbbk[G]}}(\Bbbk, M) \qquad \text{and} \qquad M_G \cong M \underset{\Bbbk[G]}{\otimes} \Bbbk \ .^{93}$$

This makes it clear how to derive these functors, and we define the **homotopy G-invariants** and **homotopy G-coinvariants** of M to be the derived $\Bbbk$-modules

$$M^{\mathsf{h}G} := \mathbb{R}\hom_{\mathsf{Mod}_{\Bbbk[G]}}(\Bbbk, M) \qquad \text{and} \qquad M_{\mathsf{h}G} := M \overset{\mathbb{L}}{\underset{\Bbbk[G]}{\otimes}} \Bbbk \ .$$

Said differently, we have an evident *augmentation* homomorphism $\Bbbk[G] \to \Bbbk$ given by the formula $g \mapsto 1$ for all $g \in G$, and these derived functors can then be interpreted as defining adjoints

(19)
$$\mathbf{D}_{\Bbbk[G]} \underset{(-)^{\mathsf{h}G}}{\overset{(-)_{\mathsf{h}G}}{\underset{\perp}{\overset{\perp}{\xleftarrow{\quad \mathsf{fgt} \quad}}}}} \mathbf{D}_{\Bbbk}$$

(as in Exercise 5.13(a)). Thereafter, we recover the **cohomology** and **homology** of G with coefficients in M as

$$\mathsf{H}^n(G; M) := \mathsf{H}_{-n}(M^{\mathsf{h}G}) =: \mathsf{Ext}^n_{\Bbbk[G]}(\Bbbk, M) \qquad \text{and} \qquad \mathsf{H}_n(G; M) := \mathsf{H}_n(M_{\mathsf{h}G}) =: \mathsf{Tor}^{\Bbbk[G]}_n(M, \Bbbk) \ .^{94}$$

---

[93]The first uses the right G-action on $\Bbbk$, while the second uses the left G-action on $\Bbbk$. So these both carry residual right G-actions, but of course these actions are trivial.

[94]As we will see, the G-module M defines a local system on the space $\mathsf{B}G$, and these are precisely its co/homology groups.

7.4.2. In view of Exercise 5.3(c)(d), to compute the co/homology of G it suffices to choose a projective resolution of $\Bbbk$ as a left or right $\Bbbk[G]$-module; this is certainly easier than injectively resolving M in order to compute cohomology, and if we do it once and for all then we need not projectively resolve M in order to compute homology. For this, it is standard to use the one-sidedly projective (in fact free) resolutions

$$\mathsf{Bar}(\Bbbk[G], \Bbbk[G], \Bbbk) \in \left({}_{\Bbbk[G]}\mathbf{P} \underset{{}_{\Bbbk[G]}\mathbf{K}}{\times} {}_{\Bbbk[G]}\mathbf{K}_{\Bbbk[G]} \right) \qquad \text{and} \qquad \mathsf{Bar}(\Bbbk, \Bbbk[G], \Bbbk[G]) \in \left(\mathbf{P}_{\Bbbk[G]} \underset{\mathbf{K}_{\Bbbk[G]}}{\times} {}_{\Bbbk[G]}\mathbf{K}_{\Bbbk[G]} \right) .$$

Exercise 7.7 (8 points). Use bar resolutions to compute the homology and cohomology of the cyclic group $\mathsf{C}_n := \mathbb{Z}/n$ with coefficients in the trivial module $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}[\mathsf{C}_n]}$.

In solving Exercise 7.7, one finds that group homology and group cohomology exhibit "mostly periodic" behavior. The corresponding complexes can be combined into a single periodic complex that computes *Tate cohomology*, a fascinating construction originating in the study of class field theory.

7.4.3. Homotopy co/invariants compose nicely. For instance, for any subgroup $H \subseteq G$, taking (homotopy) H-coinvariants is implemented by taking the (resp. derived) tensor product with $\Bbbk[G/H] \in {}_{\Bbbk[G]}\mathsf{Mod}$. When H is normal this carries a commuting right action of G/H,[95] and we find that

$$(-)_{\mathsf{h}(G/H)} \circ (-)_{\mathsf{h}H} := (-) \otimes^{\mathbb{L}}_{\Bbbk[G]} \Bbbk[G/H] \otimes^{\mathbb{L}}_{\Bbbk[G/H]} \Bbbk \simeq (-) \otimes^{\mathbb{L}}_{\Bbbk[G]} \Bbbk =: (-)_{\mathsf{h}G} .^{[96]}$$

Of course, an analogous relation holds for homotopy invariants. From the perspective of the diagram (19) describing homotopy co/invariants as adjoint functors, we can also view these relations as arising from a diagram

in which the upper and lower (hyperbolic) triangles commute by the the uniqueness of adjoints.

Homotopy co/invariants admit other subtler relations.

---

[95]More generally, this carries an action of the *Weyl group* of H in G, the quotient $\mathsf{W}_G(H) := \mathsf{N}_G(H)/H$ by it of its normalizer. (In fact, $\mathsf{W}_G(H)$ is precisely the monoid of automorphisms of G/H as a left G-set.)

[96]Here we use the facts that the derived relative tensor product is associative and the equivalence $\Bbbk[G/H] \otimes^{\mathbb{L}}_{\Bbbk[G/H]} \Bbbk \xrightarrow{\sim} \Bbbk$ of derived left $\Bbbk[G]$-modules.

Exercise 7.8 (12 points)**.** Let $G * H$ denote the coproduct (a.k.a. free product) of groups G and H. For any derived $(G * H)$-module $M \in \mathbf{D}_{\Bbbk[G*H]}$, construct exact squares

$$
\begin{array}{ccc}
M & \longrightarrow & M_{\mathsf{h}G} \\
\downarrow & & \downarrow \\
M_{\mathsf{h}H} & \longrightarrow & M_{\mathsf{h}(G*H)}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
M & \longleftarrow & M^{\mathsf{h}G} \\
\uparrow & & \uparrow \\
M^{\mathsf{h}H} & \longleftarrow & M^{\mathsf{h}(G*H)}
\end{array}
$$

in $\mathbf{D}_{\Bbbk}$.[97]

7.5. Decategorification and algebraic K-theory.

7.5.1. The term ***decategorification*** refers informally to the process of taking a mathematical object and extracting an invariant of it that lies at a lower categorical level – e.g. extracting a vector space from a (linear) category, or extracting a number from a vector space. Indeed, a first example – which appeared in §1 – is given by taking the dimension of a (finite-dimensional) vector space. This is generalized by taking the ***Euler characteristic*** of a (perfect) derived module. We explain this in the two relevant examples introduced in §1.4, both of which take place over the field $\mathbb{R}$, and then discuss Euler characteristics over more general rings.

7.5.2. Let us briefly recall the setup of §1. We wished to compute algebraic intersection numbers of subvarieties of complementary codimensions. That is, we wished to compute the "expected" number of intersection points that would be literally obtained by perturbing the intersection to be transverse, without actually making such a perturbation. For a transverse intersection (and in somewhat more generality (recall §1.4.2)), we saw that computing the dimension of a corresponding tensor product gave the correct answer. It was asserted that derived tensor products would give the correct answer unconditionally.

7.5.3. As we saw in §1.4.3, the first instance where ordinary tensor products fail to give the correct answer is for the self-intersection of a single point $a \in \mathbb{R}^1$. In this case, the corresponding ***derived intersection*** is computed by the derived tensor product

$$
(20) \qquad \mathbb{R}[x]/(x-a) \overset{\mathbb{L}}{\underset{\mathbb{R}[x]}{\otimes}} \mathbb{R}[x]/(x-a) \ .
$$

That is, we can enhance the ordinary intersection by passing to the world of ***derived algebraic geometry***: the derived intersection has an underlying variety given by the ordinary intersection (which is just the point $a \in \mathbb{R}^1$), but then it has a *derived* commutative $\mathbb{R}$-algebra of functions, namely this derived relative tensor product.[98]

---

[97]This may be seen as a Mayer–Vietoris decomposition for (twisted) co/homology via the wedge sum decomposition $\mathsf{B}(G * H) \simeq \mathsf{B}G \vee \mathsf{B}H$, as we will see below.

[98]Recall from §6.3.5 that projective resolution carries commutative algebras to commutative algebras.

In order to compute this derived relative tensor product, we form the projective resolution

$$P := \left(\mathbb{R}[x] \xrightarrow{(x-a)} \mathbb{R}[x] \right) \xrightarrow{\approx} \mathbb{R}[x]/(x-a)$$

(as a complex of $\mathbb{R}[x]$-modules). Then, the ordinary relative tensor product

$$P \underset{\mathbb{R}[x]}{\otimes} \mathbb{R}[x]/(x-a) \cong \left(\mathbb{R} \xrightarrow{0} \mathbb{R} \right)$$

in $\mathbf{K}_{\mathbb{R}}$ represents the derived relative tensor product (20) in $\mathbf{D}_{\mathbb{R}}$.[99]

The relevant numerical invariant of this derived $\mathbb{R}$-module is defined on the full subcategory $\mathbf{D}_{\mathbb{R}}^{\mathsf{perf}} \subseteq \mathbf{D}_{\mathbb{R}}$ of **perfect** (derived) $\mathbb{R}$-modules, i.e. those with finite-dimensional total homology (equivalently, those with finitely many nonzero homology groups that are each finite-dimensional $\mathbb{R}$-modules).[100] Namely, given a perfect $\mathbb{R}$-module $M \in \mathbf{D}_{\mathbb{R}}^{\mathsf{perf}}$, its **Euler characteristic** is the alternating sum

$$\chi(M) := \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{R}}(\mathsf{H}_n(M)) .$$

Here we find that

$$\chi \left(\mathbb{R}[x]/(x-a) \underset{\mathbb{R}[x]}{\overset{\mathbb{L}}{\otimes}} \mathbb{R}[x]/(x-a) \right) = \dim_{\mathbb{R}}(\mathbb{R}) - \dim_{\mathbb{R}}(\mathbb{R}) = 0 ,$$

as desired: generically, the intersection of two points in the line is empty.

7.5.4. We now explain how to compute the derived self-intersection of a projective line in $\mathbb{RP}^2$ (as asserted in §1.4.5), using some basic facts about the cohomology of line bundles on projective space (see e.g. [Har77, §III.5]).[101]

Given a curve $Z \subseteq X := \mathbb{RP}^2$, let us write $\mathcal{O}_Z$ for the corresponding ideal sheaf on X. Then, the **derived self-intersection** of Z with itself in X is the derived scheme

$$Y := Z \underset{X}{\overset{\mathbb{R}}{\times}} Z := \left(|Z|, \mathcal{O}_Z \underset{\mathcal{O}_X}{\overset{\mathbb{L}}{\otimes}} \mathcal{O}_Z \right)$$

(expressed as a topological space equipped with a sheaf of derived commutative $\mathbb{R}$-algebras, and omitting the pullback functor from the notation). Now, because the underlying variety of Y is no longer just a point, in order to extract a numerical invariant we must take the Euler characteristic of Y: that is, the Euler characteristic of the derived global sections (a.k.a. (hyper)cohomology) of its structure sheaf.[102]

---

[99]It is not hard to see that the commutative algebra structure is the evident one that exists at the point-set level on this particular chain complex representative.

[100]In general, the subcategory $\mathbf{D}_R^{\mathsf{perf}} \subseteq \mathbf{D}_R$ of perfect (derived) R-modules can be defined as the smallest full subcategory containing the object $R \in \mathbf{D}_R$ that is closed under co/kernels, de/suspensions, and retracts.

[101]In fact, essentially the same computation applies to the derived intersection of any pair of (possibly equal) algebraic curves in $\mathbb{RP}^2$, yielding the product of their degrees as the algebraic intersection number.

[102]The fact that Y arose as the intersection of two subvarieties of complementary codimension is reflected in its **virtual dimension**, which can be read off from its cotangent complex (as described below).

Exercise 7.9 (8 points)**.** For definiteness, let us take $Z \subseteq X := \mathbb{RP}^2$ to be the closure of the x-axis, i.e. the locus $\{[x : y : z] \in \mathbb{RP}^2 : y = 0\}$.

(a) Find a line bundle L over X and a section s of L such that $Z = s^{-1}(0)$.

This gives a short exact sequence $0 \to L^{-1} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_Z \to 0$, which we may view as defining a quasi-isomorphism $M := (L^{-1} \xrightarrow{s} \underset{\sim}{\mathcal{O}_X}) \xrightarrow{\approx} \mathcal{O}_Z$. Thereafter, we may compute the derived tensor product as $\mathcal{O}_Z \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_Z \simeq M \otimes_{\mathcal{O}_X} \mathcal{O}_Z$.[103]

(b) Compute the Euler characteristic of Y, i.e. the Euler characteristic of the derived global sections (a.k.a. hypercohomology) of its structure sheaf.[104]

7.5.5.   The Euler characteristic of derived modules over more general rings is not a priori well-defined. This is rectified by a universal construction known as ***algebraic K-theory***, as we now explain.

We begin with the following motivating observation: given a co/kernel sequence of derived $\mathbb{R}$-modules $L \to M \to N$, by the long exact sequence in homology we have an equality $\chi(M) = \chi(L) + \chi(N)$. This is referred to as the ***additivity*** of Euler characteristic.[105]

We define the full subcategory $\mathbf{D}^{\mathsf{perf}}_R \subseteq \mathbf{D}_R$ of ***perfect*** (derived) R-modules to be the smallest subcategory containing R and closed under retracts, co/kernels, and de/suspensions.[106] (Alternatively, these are the derived R-modules that can be presented by bounded complexes of finite-rank projective R-modules.) Then, the 0th ***algebraic K-group*** of $\mathbf{D}^{\mathsf{perf}}_R$ is the abelian group

$$\mathsf{K}_0(\mathbf{D}^{\mathsf{perf}}_R)$$

defined as follows: it has a generator $[M]$ for every perfect R-module $M \in \mathbf{D}^{\mathsf{perf}}_R$, and for every co/kernel sequence $L \to M \to N$ it has a relation $[M] = [L] + [N]$.[107] (In particular, if $M \simeq M'$ then $[M] = [M']$, because the co/kernel of an equivalence is zero.) By construction,

---

[103]That this indeed computes the derived tensor product follows from the fact that this is a *flat* resolution of $\mathcal{O}_X$-modules. (An $\mathcal{O}_X$-module is flat iff it is locally flat.) Note that there are no nonzero projective objects in the abelian category of $\mathcal{O}_X$-modules, so our previous considerations do not immediately apply here.

[104]Equivalently, one can compute the alternating sum of the Euler characteristics of a specific chain complex representative of the structure sheaf. This is an enhancement of the fact that given a bounded complex $M \in \mathsf{Ch}_{\Bbbk}$ of modules over a field $\Bbbk$, we have the alternative formula

$$\chi(M) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\Bbbk}(M_n)$$

for the Euler characteristic of its underlying derived $\Bbbk$-module.

[105]The Euler characteristic for (suitably finite) spaces is likewise additive for cofiber sequences.

[106]This finiteness restriction is to avoid the so-called "Eilenberg swindle": for instance, the short exact sequence

$$0 \longrightarrow R \xrightarrow{r \mapsto (r,0,0,\dots)} R^{\oplus \mathbb{N}} \xrightarrow{(r_1, r_2, \dots) \mapsto (r_2, r_3, \dots)} R^{\oplus \mathbb{N}} \longrightarrow 0$$

of R-modules would yield the relation that $[R] = 0$.

[107]The German word for "class" (as in "equivalence class") begins with the letter "K".

we have a "generalized Euler characteristic" function

$$\{\text{objects of } \mathbf{D}_R^{\mathsf{perf}}\} \xrightarrow{M \longmapsto [M]} \mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}}) \ ,$$

which is additive for co/kernel sequences.

In fact, it is not so hard to compute $\mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}})$. Let us define an abelian group $\mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}})$ as follows: it has a generator $[M]$ for each finitely generated projective R-module $M \in \mathsf{Mod}_R^{\mathsf{f.g.,proj}}$, and for every short exact sequence $0 \to L \to M \to N \to 0$ among such it has a relation $[M] = [L] + [N]$. Observe that there is an the evident homomorphism

(21) $$\mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}}) \longrightarrow \mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}}) \ .$$

Exercise 7.10 (10 points)**.** Prove that there is a well-defined homomorphism

$$\mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}}) \longrightarrow \mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}})$$
$$\cup \hspace{4.5cm} \cup$$
$$[M] \longmapsto \sum_{n \in \mathbb{Z}} (-1)^n \cdot [\mathsf{H}_n(M)]$$

that defines an inverse to the homomorphism (21).

In particular, we see that K-theory classes are insensitive to k-invariants.

As the notation indicates, it is possible to define higher (and in fact also lower) K-groups. As with the discussion of §7.1.1, these constructions are motivated by a desire to repair certain failures of exactness of the functor K_0, although now the notion of "exactness" becomes substantially more subtle. To a first approximation, the functor K takes values in derived $\mathbb{Z}$-modules, and we recover the nth algebraic K-group as $\mathsf{K}_n := \mathsf{H}_n \circ \mathsf{K}$. We refer the reader to [Wei13] for a comprehensive introduction to algebraic K-theory.

Part II. Higher category theory

In this part, we give a rapid introduction to ∞-category theory. We take the point of view that it is easiest to learn this theory through examples; those that we choose to highlight are provided by sheaf theory, which is the subject of **??**. So, the present part will contain relatively few examples, in the interest of reaching sheaf theory as quickly as possible. In particular, the present account of ∞-category theory is very far from exhaustive. Rather, it is intended to simultaneously explain

(1) how ∞-categories are actually used in practice, and
(2) how this usage connects with the rigorous definitions.

We address these goals in reverse order: the foundations of ∞-category theory are based in the theory of model categories, and so we begin with a brief summary of the latter topic.[108]

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[108]Also, some indication of how ∞-categories are used in practice already appeared in Part I.

The character of our exposition will change fairly dramatically here: we will introduce many results here without any semblance of proof, especially after §8 (although we will give references where relevant). Based on these black boxes, we will return to proving most of our assertions in **??**.

8. Model categories

8.1. Model categories, Quillen adjunctions, and Quillen equivalences.

8.1.1. A ***relative category*** is a pair $(\mathcal{C}, \mathbf{W})$ consisting of a category $\mathcal{C}$ equipped with a subcategory $\mathbf{W} \subseteq \mathcal{C}$, which is called the subcategory of ***weak equivalences***. We generally denote weak equivalences by $\xrightarrow{\approx}$. Given a relative category $(\mathcal{C}, \mathbf{W})$, its ***localization*** (also often called its ***homotopy category***) is the category $\mathcal{C}[\mathbf{W}^{-1}]$ obtained from $\mathcal{C}$ by freely inverting the morphisms in $\mathbf{W}$.[109]

A ***model structure*** on a relative category $(\mathcal{C}, \mathbf{W})$ consists of additional structure which makes it feasible to perform computations in the localization $\mathcal{C}[\mathbf{W}^{-1}]$. In particular, this gives a means of obtaining adjunctions and equivalences among localizations of relative categories.

We first give the relevant definitions, and then illustrate them through some examples. We note here that the examples will be much more important for us than the specific details of the definitions.

8.1.2. A ***model structure*** on a relative category $(\mathcal{C}, \mathbf{W})$ consists of subcategories $\mathbf{C}, \mathbf{F} \subseteq \mathcal{C}$, whose morphisms are respectively called ***cofibrations*** and ***fibrations***, which are respectively denoted by $\rightarrowtail$ and $\twoheadrightarrow$; morphisms in $\mathbf{W} \cap \mathbf{C}$ and $\mathbf{W} \cap \mathbf{F}$ are respectively called ***acyclic cofibrations*** and ***acyclic fibrations***. These data are required to satisfy the following axioms.

(1) The category $\mathcal{C}$ has all finite limits and colimits.
(2) The subcategory $\mathbf{W} \subseteq \mathcal{C}$ satisfies the two-out-of-three property: given any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, if any two of the morphisms f, g, and gf lie in $\mathbf{W}$ then so does the third.
(3) Every morphism $X \to Y$ admits factorizations

and .

(4) We have equalities $(\mathbf{W} \cap \mathbf{C}) = \mathsf{llp}(\mathbf{F})$ (or equivalently $\mathbf{F} = \mathsf{rlp}(\mathbf{W} \cap \mathbf{C})$) and $\mathbf{C} = \mathsf{llp}(\mathbf{W} \cap \mathbf{F})$ (or equivalently $(\mathbf{W} \cap \mathbf{F}) = \mathsf{rlp}(\mathbf{C})$).[110]

---

[109]Beware that we previously wrote $\mathcal{C}[\mathbf{W}^{-1}]$ to denote ∞-categorical localizations.

[110]Conditions 3 and 4 are sometimes expressed by saying that the pairs $(\mathbf{W} \cap \mathbf{C}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{W} \cap \mathbf{F})$ form *weak factorization systems* on $\mathcal{C}$.

In particular, axiom 4 implies that given solid commutative squares

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
\wr\downarrow & \nearrow & \downarrow \\
Y & \longrightarrow & Z
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow\wr \\
Y & \longrightarrow & Z
\end{array}
$$

there always exist dashed lifts making the diagrams commute. It additionally requires that these various classes of morphisms are *characterized* by these lifting conditions.[111]

In general, we will notate a model category and its various attendant data using a subscript that indicates the name of the model structure. However, we will notate the weak equivalences with a subscript that indicates the more standard term for them, when one exists; for instance, we will always write $\mathbf{W}_{\mathsf{q.i.}} \subseteq \mathsf{Ch}_R$ for the subcategory of quasi-isomorphisms.

Fix a model category $\mathcal{C}$. We say that an object $X \in \mathcal{C}$ is **cofibrant** if the unique map $\varnothing_{\mathcal{C}} \to X$ from the initial object is a cofibration, **fibrant** if the unique map $X \to \mathsf{pt}_{\mathcal{C}}$ to the terminal object is a fibration, and **bifibrant** if it is both cofibrant and fibrant. We respectively write $\mathcal{C}^c, \mathcal{C}^f, \mathcal{C}^{cf} \subseteq \mathcal{C}$ for the full subcategories on the cofibrant, fibrant, and bifibrant objects.

Note that every object is weakly equivalent to both a cofibrant object and a fibrant object. Indeed, we define a **cofibrant resolution** and a **fibrant resolution** of an object $X \in \mathcal{C}$ to respectively be factorizations

$$
\begin{array}{ccc}
\varnothing_{\mathcal{C}} & \longrightarrow & X \\
\searrow & & \nearrow_{\wr} \\
& X^c &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
X & \longrightarrow & \mathsf{pt}_{\mathcal{C}} \\
{}_{\wr}\searrow & & \nearrow \\
& X^f &
\end{array}
\quad ;
$$

these are not unique, but they are guaranteed to exist by axiom 3. Even further, axiom 3 can be used to construct a bifibrant object that is weakly equivalent to X (albeit generally only through a zigzag of weak equivalences).

8.1.3. The first purpose of co/fibrant objects is that they are "good for mapping from/to", as we now explain.[112]

---

[111]There are a number of slight variations on the axioms: notably, some authors strengthen axiom 1 to require that $\mathcal{C}$ have all (not necessarily finite) limits and colimits, and some authors strengthen axiom 3 to require that the factorizations can be made to be functorial. These stronger axioms hold in essentially all examples of interest.

[112]More generally, a cofibration $W \rightarrowtail X$ makes X "good for mapping from in $\mathcal{C}_{W/}$", while a fibration $Y \twoheadrightarrow Z$ makes Y "good for mapping to in $\mathcal{C}_{/Z}$". Indeed, a model structure on $\mathcal{C}$ determines a model structure on $\mathcal{C}_{W//Z}$ in an evident way.

Fix a model category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$. We define a **cylinder object** for X and a **path object** for Y to respectively be factorizations

these are not unique, but they are guaranteed to exist by axiom 3. Then, for any pair of morphisms $f, g \in \mathsf{hom}_{\mathcal{C}}(X, Y)$, we define a **left homotopy** and a **right homotopy** from f to g (with respect to these choices) to respectively be dashed morphisms

making the diagrams commute. We respectively denote the existence of such homotopies by $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$. Straightforward considerations then lead to the **fundamental theorem of model categories**: if X is cofibrant and Y is fibrant, then the map

$$\mathsf{hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathsf{hom}_{\mathcal{C}[\mathbf{W}^{-1}]}(X, Y)$$

is surjective, with the equivalence relation implementing it given by either left homotopy or right homotopy (i.e. these relations are both equivalence relations and moreover they coincide). We refer the reader to [Hov99, §1.2] for a proof; we include the following to give a representative sample of the arguments involved.

Exercise 8.1 (8 points)**.** Prove the following statements.

(a) Given a diagram

$$W \overset{e}{\longrightarrow} X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{h}{\longrightarrow} Z$$

in $\mathcal{C}$, $f \overset{l}{\sim} g$ implies $hf \overset{l}{\sim} hg$, and if Y is fibrant then $f \overset{l}{\sim} g$ implies $fe \overset{l}{\sim} ge$.

(b) If X is cofibrant then $\overset{l}{\sim}$ defines an equivalence relation on $\mathsf{hom}_{\mathcal{C}}(X, Y)$.

(c) If X is cofibrant and moreover $Y \xrightarrow{h} Z$ is either an acyclic fibration or a weak equivalence between fibrant objects, then postcomposition with h induces an isomorphism

$$
\begin{array}{ccc}
\mathsf{hom}_{\mathcal{C}}(X,Y) & \xrightarrow{\ h\circ(-)\ } & \mathsf{hom}_{\mathcal{C}}(X,Z) \\
\downarrow & & \downarrow \\
\mathsf{hom}_{\mathcal{C}}(X,Y)/\overset{l}{\sim} & \dashrightarrow[\cong] & \mathsf{hom}_{\mathcal{C}}(X,Z)/\overset{l}{\sim}
\end{array}
\qquad .
$$

(d) Suppose X is cofibrant. Then, for any $f,g \in \mathsf{hom}_{\mathcal{C}}(X,Y)$, $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Moreover, the relation of right homotopy is realized by any fixed path object for Y.

We emphasize that the fundamental theorem of model categories should be seen as quite striking: a priori, morphisms in $\mathcal{C}[\mathbf{W}^{-1}]$ are given by zigzags (or arbitrary length) in $\mathcal{C}$ in which the backwards maps are weak equivalences, whereas it implies that for any objects $X, Y \in \mathcal{C}$, every morphism in $\mathsf{hom}_{\mathcal{C}[\mathbf{W}^{-1}]}(X,Y)$ is represented by a zigzag

$$
X \xleftarrow{\ \approx\ } X^c \longrightarrow Y^f \xleftarrow{\ \approx\ } Y
$$

involving arbitrary but fixed co/fibrant resolutions of X and Y.

8.1.4.   Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A ***Quillen adjunction*** is an adjunction

$$
(22) \qquad\qquad\qquad \mathcal{C} \; \underset{G}{\overset{F}{\rightleftarrows}} \; \mathcal{D}
$$

such that F preserves cofibrations and acyclic cofibrations; by axiom 4, these conditions are respectively equivalent to requiring that G preserves acyclic fibrations and fibrations.

As we now explain, a Quillen adjunction (22) determines a diagram

where we simply write $\mathbf{W}$ for all relevant subcategories of weak equivalences. First of all, the curved arrows arise from the easy consequence that F (resp. G) preserves weak equivalences between cofibrant (resp. fibrant) objects [Hov99, Lemma 1.1.12]. Moreover, the horizontal

equivalences follow easily from the fundamental theorem of model categories. Hence, we may define the dashed arrows so that the curved regions commute; we respectively call $\mathbb{L}F$ and $\mathbb{R}G$ the **left derived functor** of F and the **right derived functor** of G.[113] So by definition, for any objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we respectively have isomorphisms

$$(\mathbb{L}F)(X) \cong F(X^c) \qquad \text{and} \qquad (\mathbb{R}G)(Y) \cong G(Y^f)$$

in $\mathcal{D}[\mathbf{W}^{-1}]$ and $\mathcal{C}[\mathbf{W}^{-1}]$ for any co/fibrant resolutions $X^c \xrightarrow{\approx} X$ and $Y \xrightarrow{\approx} Y^f$. The natural transformations arise from the resolution maps: their components are

$$(\mathbb{L}F)(X) \cong F(X^c) \longrightarrow F(X) \qquad \text{and} \qquad G(Y) \longrightarrow G(Y^f) \cong (\mathbb{R}G)(Y) \ .^{114}$$

Of course, implicit is the assertion that these are independent of the choices of co/fibrant resolutions.

The main point of a Quillen adjunction (22) is the following result.

Exercise 8.2 (8 points)**.** Prove that the derived functors participate in a canonical **derived adjunction**

$$\mathcal{C}[\mathbf{W}^{-1}] \underset{\mathbb{R}G}{\overset{\mathbb{L}F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}[\mathbf{W}^{-1}] \ .$$

The Quillen adjunction is called a **Quillen equivalence** if for every $X \in \mathcal{C}^c$ and every $Y \in \mathcal{D}^f$, a morphism $X \to G(Y)$ in $\mathcal{C}$ is a weak equivalence if and only if the adjunct morphism $F(X) \to Y$ in $\mathcal{D}$ is a weak equivalence. It is clear from (the solution of) Exercise 8.2 that the derived adjunction is an equivalence if and only if the Quillen adjunction is a Quillen equivalence.

In practice, most functors between model categories do *not* preserve weak equivalences. Indeed, if a Quillen functor preserves all weak equivalences, then its point-set values compute its derived values, obviating the need for co/fibrant resolutions. Given a functor between model categories, we will say that it is **automatically derived** if it preserves weak equivalences. For instance, a left (resp. right) Quillen functor is automatically derived if in its source all objects are cofibrant (resp. fibrant). We simply write $F := \mathbb{L}F$ for the left derived functor of a left Quillen functor F that is automatically derived, and similarly we simply write $G := \mathbb{R}G$ for the right derived functor of a right Quillen functor G that is automatically derived.

8.2. Model categories of derived modules.

8.2.1. The category Ch_R is a relative category via the subcategory $\mathbf{W}_{\mathsf{q.i.}} \subseteq \mathsf{Ch}_R$ of quasi-isomorphisms. Its localization is the derived category of R: $\mathsf{Ch}_R[\mathbf{W}^{-1}] \simeq \mathsf{H}_0(\mathbf{D}_R)$.

---

[113]As the notation and terminology suggest, these are closely related to the derived functors introduced in §7.1.2.

[114]Note that F and G do *not* generally preserve weak equivalences between arbitrary objects.

8.2.2.   The relative category $(\mathsf{Ch}_R, \mathbf{W}_{\mathsf{q.i.}})$ admits a ***projective model structure***, which is characterized by the fact that the fibrations are the levelwise surjective chain maps. So, all objects are fibrant, and the cofibrant objects are precisely the projective complexes. Hence, all weak equivalences are fibrant resolutions, and cofibrant resolutions are simply projective resolutions.

We now discuss axiom 3. For this, consider the sets

$$I := \{S^n \longrightarrow D^{n+1}\}_{n \in \mathbb{Z}} \qquad \text{and} \qquad J := \{0 \xrightarrow{\approx} D^n\}_{n \in \mathbb{Z}}$$

of morphisms in Ch_R, where I is the set introduced in §5.5. Now, the factorizations factorizations $\xrightarrow{\approx} \twoheadrightarrow$ were constructed in §5.6 using the small object argument applied to the set I. It is easy to see that the small object argument applied to the set J yields the factorizations $\rightarrowtail \xrightarrow{\approx} \twoheadrightarrow$.[115] We summarize this situation by saying that the projective model structure is ***cofibrantly generated*** by the sets I and J, which are respectively called the set of *generating cofibrations* and the set of *generating acyclic cofibrations*.

Let us say that a morphism is a ***relative I-cell complex*** if it can be expressed as a transfinite composition of pushouts of elements of I (such as the morphism $c^{(\infty)}$ constructed in §5.6).[116] Then, the relative I-cell complexes are cofibrations, and in fact every cofibration is a retract of a relative I-cell complex. Similarly, every acyclic cofibration is a retract of a relative J-cell complex. We also have $\mathbf{F} = \mathsf{rlp}(J)$ and $(\mathbf{W} \cap \mathbf{F}) = \mathsf{rlp}(I)$: to detect (resp. acyclic) fibrations, it suffices to check the right lifting property merely against the set J (resp. the set I). Of course, these statements hold in any cofibrantly generated model category.

8.2.3.   The relative category $(\mathsf{Ch}_R, \mathbf{W}_{\mathsf{q.i.}})$ also admits an ***injective model structure***, which is characterized by the fact that the cofibrations are the levelwise injective chain maps. So, all objects are cofibrant, and the fibrant objects are precisely the injective complexes. Hence, all weak equivalences are cofibrant resolutions, and fibrant resolutions are simply injective resolutions.[117]

8.2.4.   The identity adjunction defines a Quillen equivalence

$$(\mathsf{Ch}_R)_{\mathsf{proj}} \; \underset{\mathsf{id}_{\mathsf{Ch}_R}}{\overset{\mathsf{id}_{\mathsf{Ch}_R}}{\rightleftarrows}} \; (\mathsf{Ch}_R)_{\mathsf{inj}}$$

in which both adjoints are automatically derived, whose derived equivalence is the identity functor on the derived category $\mathsf{H}_0(\mathbf{D}_R) := \mathsf{Ch}_R[\mathbf{W}_{\mathsf{q.i.}}^{-1}]$.

---

[115]To apply the small object argument to the set I, we needed to know that the source objects $S^n \in \mathsf{Ch}_R$ were compact (Exercise 5.18). It is trivial to verify that the object $0 \in \mathsf{Ch}_R$ is compact, so that we can indeed apply the small object argument to the set J.

[116]To be precise, in §5.6 we constructed transfinite compositions of pushouts of *coproducts* of elements of I, but it is not hard to see that these are also I-cell complexes.

[117]This model structure is also cofibrantly generated, but the generating sets are quite inexplicit; see [Hov99, Theorem 2.3.13].

8.2.5. The model category $(\mathsf{Ch}_\Bbbk)_{\mathsf{proj}}$ is a ***symmetric monoidal model category***. In particular, this means that the bifunctor

$$(\mathsf{Ch}_\Bbbk)_{\mathsf{proj}} \times (\mathsf{Ch}_\Bbbk)_{\mathsf{proj}} \xrightarrow{(-)\otimes_\Bbbk(-)} (\mathsf{Ch}_\Bbbk)_{\mathsf{proj}}$$

is a *left Quillen bifunctor*, which in particular means that for any cofibrant object $P \in (\mathsf{Ch}_\Bbbk)^c_{\mathsf{proj}} = \mathbf{P}_\Bbbk$, the functors $P\otimes_\Bbbk(-)$ and $(-)\otimes_\Bbbk P$ are left Quillen functors (i.e. they preserve cofibrations and acyclic cofibrations). This also means that the internal hom bifunctor

$$(\mathsf{Ch}_\Bbbk)^{\mathsf{op}}_{\mathsf{proj}} \times (\mathsf{Ch}_\Bbbk)_{\mathsf{proj}} \xrightarrow{\hom_{\mathsf{Ch}_\Bbbk}(-,-)} (\mathsf{Ch}_\Bbbk)_{\mathsf{proj}}$$

is also suitably compatible with the projective model structure.

8.2.6. Similarly, the bifunctor

$$(\mathsf{Ch}_\Bbbk)_{\mathsf{proj}} \times (\mathsf{Ch}_R)_{\mathsf{proj}} \xrightarrow{(-)\otimes_\Bbbk(-)} (\mathsf{Ch}_R)_{\mathsf{proj}}$$

is a left Quillen bifunctor. This also means that its two-variable adjoints

$$(\mathsf{Ch}_R)^{\mathsf{op}}_{\mathsf{proj}} \times (\mathsf{Ch}_R)_{\mathsf{proj}} \xrightarrow{\hom_{\mathsf{Ch}_R}(-,-)} (\mathsf{Ch}_\Bbbk)_{\mathsf{proj}} \qquad \text{and} (\mathsf{Ch}_\Bbbk)^{\mathsf{op}}_{\mathsf{proj}} \times (\mathsf{Ch}_R)_{\mathsf{proj}} \xrightarrow{\hom_{\mathsf{Ch}_\Bbbk}(-,-)} (\mathsf{Ch}_R)_{\mathsf{proj}}$$

are also suitably compatible with the projective model structures.

8.3. Model categories of spaces.

8.3.1. The category Top is a relative category via the subcategory $\mathbf{W}_{\mathsf{w.h.e.}} \subset \mathsf{Top}$ of weak homotopy equivalences. We refer to objects of the localization $\mathsf{Top}[\mathbf{W}^{-1}_{\mathsf{w.h.e.}}]$ as ***spaces***.[118] So by definition, a space is a weak homotopy equivalence class of topological spaces. For reasons that will become clear later, we write $\mathsf{ho}(\mathcal{S}) := \mathsf{Top}[\mathbf{W}^{-1}_{\mathsf{w.h.e.}}]$ and refer to this as ***the homotopy category of spaces***.

8.3.2. The relative category $(\mathsf{Top}, \mathbf{W}_{\mathsf{w.h.e.}})$ admits a model structure known as the ***Quillen–Serre model structure***, which we denote by $\mathsf{Top}_{\mathsf{QS}}$. This model structure is cofibrantly generated by the sets

$$I_{\mathsf{QS}} := \{S^{n-1} \hookrightarrow D^n\}_{n\geqslant 0} \qquad \text{and} \qquad J_{\mathsf{QS}} := \{D^n \cong D^n \times \{0\} \xhookrightarrow{\approx} D^n \times [0,1]\}_{n\geqslant 0} \ .$$

So, the cofibrations are the retracts of relative cell complexes (in the standard sense), and the fibrations are precisely the Serre fibrations. In particular, the cofibrant objects are the cell complexes and their retracts, and all objects are fibrant.

In any model category $\mathcal{C}$, if X is fibrant and Y is cofibrant, then any weak equivalence $X \xrightarrow{\approx} Y$ admits a "weak inverse", i.e. a weak equivalence $Y \xrightarrow{\approx} X$ that becomes an inverse in $\mathcal{C}[\mathbf{W}^{-1}]$. Applied to $\mathsf{Top}_{\mathsf{QS}}$, this recovers (a slight strengthening of) ***Whitehead's theorem***.

---

[118]We will later also refer to spaces as ***∞-groupoids***.

8.3.3.   Recall the category $\boldsymbol{\Delta}$ of finite nonempty totally ordered sets and order-preserving functions, and recall the object $[n] := \{0 < 1 < \cdots < n\} \in \boldsymbol{\Delta}$ for every $n \geqslant 0$. Evidently, every object of $\boldsymbol{\Delta}$ is isomorphic to $[n]$ for some $n \geqslant 0$.

The category of ***cosimplicial objects*** in a category $\mathcal{C}$ is the category $\mathsf{c}\mathcal{C} := \mathsf{Fun}(\boldsymbol{\Delta}, \mathcal{C})$. In general, given a cosimplicial object $X \in \mathsf{c}\mathcal{C}$, we generally write $X^n := X([n]) \in \mathcal{C}$ for its value at the object $[n] \in \boldsymbol{\Delta}$; correspondingly, we may write $X^\bullet := X$ for emphasis.

Dually, the category of ***simplicial objects*** in a category $\mathcal{C}$ is the category $\mathsf{s}\mathcal{C} := \mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, \mathcal{C})$. Given a simplicial object $X \in \mathsf{s}\mathcal{C}$, we generally write $X_n := X([n]^\circ) \in \mathcal{C}$ for its value at the object $[n]^\circ \in \boldsymbol{\Delta}^{\mathsf{op}}$; correspondingly, we may write $X_\bullet := X$ for emphasis.

Simplicial sets can be used as models for spaces. That is, the category sSet admits a model structure known as the ***Kan–Quillen model structure***, which we denote by $\mathsf{sSet}_{\mathsf{KQ}}$, which participates in a Quillen equivalence

$$(23) \qquad\qquad \mathsf{sSet}_{\mathsf{KQ}} \; \xrightleftharpoons[\mathsf{Sing}]{\overset{|-|}{\perp}} \; \mathsf{Top}_{\mathsf{QS}} \quad ,$$

as we explain shortly. Of course, it will follow that we have an equivalence $\mathsf{sSet}[\mathbf{W}_{\mathsf{KQ}}^{-1}] \simeq \mathsf{ho}(\mathcal{S})$, and in particular that a space can *also* be defined as a weak equivalence class of simplicial set.

8.3.4.   We begin with some background on simplicial sets.

We write $\Delta^n := \mathsf{hom}_{\boldsymbol{\Delta}}(-, [n])$ for the (***combinatorial***) n-***simplex***. These assemble into a functor $\boldsymbol{\Delta} \xrightarrow{\Delta^\bullet} \mathsf{sSet}$, i.e. a cosimplicial object in sSet, which is simply the Yoneda embedding. Hence, given a finite nonempty totally ordered set $S \in \boldsymbol{\Delta}$ (e.g. a nonempty subset of $[n]$ with the induced ordering), we may write $\Delta^S := \mathsf{hom}_{\boldsymbol{\Delta}}(-, S)$ for the functor that it represents.

Given a simplicial set $X \in \mathsf{sSet}$, we refer to the set $X_n := X([n]) \cong \mathsf{hom}_{\mathsf{sSet}}(\Delta^n, X)$ as its set of n-***simplices***. We also refer to 0-simplices as ***vertices*** and to 1-simplices as ***edges***. An n-simplex of X is called ***degenerate*** if it arises as a composition $\Delta^n \to \Delta^i \to X$ for some $i < n$, and ***nondegenerate*** otherwise. The simplices of X assemble into a category

$$\boldsymbol{\Delta}_{/X} := \boldsymbol{\Delta} \underset{\mathsf{sSet}}{\times} \mathsf{sSet}_{/X} \; ,$$

which is called the ***category of simplices*** of X.

As in any presheaf category, limits and colimits in sSet are computed pointwise.[119]

Exercise 8.3 (4 points)**.** For every $n \geqslant 0$, determine the number of degenerate and nondegenerate n-simplices of $\Delta^1 \times \Delta^1$.

---

[119]More generally, in any functor category $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$, if $\mathcal{C}$ admits co/limits indexed over $\mathcal{I}$, then so does $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$ and these co/limits are computed pointwise. However, beware that there may exist co/limits in $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$ that are *not* computed pointwise (necessarily arising in the case that $\mathcal{C}$ does *not* admit such co/limits).

Exercise 8.4 (6 points)**.** Prove that any simplicial set $X \in \mathsf{sSet}$ is the colimit of its simplices, i.e. that the canonical morphism

$$\operatorname*{colim}_{(\Delta^n \downarrow X) \in \mathbf{\Delta}_{/X}} \Delta^n \longrightarrow X$$

is an isomorphism.

We can rephrase Exercise 8.4 as saying that the left Kan extension

$$\mathbf{\Delta} \xhookrightarrow{\Delta^{\bullet}} \mathsf{sSet}$$
$$\Delta^{\bullet} \Big\downarrow \qquad \nearrow$$
$$\mathsf{sSet}$$

is the identity functor.[120] Indeed, in general the left Kan extension

$$\mathcal{I} \xrightarrow{F} \mathcal{C}$$
$$\varphi \Big\downarrow \quad \overset{\mathcal{U}}{\underset{\varphi_! F}{\nearrow}}$$
$$\mathcal{J}$$

is given by the formula

$$j \longmapsto (\varphi_! F)(j) \cong \operatorname{colim} \left(\mathcal{I} \underset{\mathcal{J}}{\times} \mathcal{J}_{/j} \xrightarrow{\mathsf{fgt}} \mathcal{I} \xrightarrow{F} \mathcal{C} \right) .$$

(whenever these colimits in $\mathcal{C}$ exist).[121]

Exercise 8.5 (6 points)**.** Prove that any simplicial set $X \in \mathsf{sSet}$ is also the colimit of its *nondegenerate* simplices. That is, writing $\mathbf{\Delta}_{\mathsf{nondeg}/X} \subseteq \mathbf{\Delta}_{/X}$ for the full subcategory on the nondegenerate simplices, prove that the canonical morphism

$$\operatorname*{colim}_{(\Delta^n \downarrow X) \in \mathbf{\Delta}_{\mathsf{nondeg}/X}} \Delta^n \longrightarrow X$$

is an isomorphism.

In fact, Exercise 8.5 follows from the more general claim that *any* colimit over $\mathbf{\Delta}_{/X}$ is isomorphic to the colimit of the restriction to $\mathbf{\Delta}_{\mathsf{nondeg}/X}$, as we will see below.

---

[120]Note that Kan extensions along a fully faithful functor do not change the values on the full subcategory, so the natural transformation that should appear here is a natural isomorphism.

[121]Heuristically, this operation can be seen as replacing each object $j \in \mathcal{J}$ with the formal colimit of the diagram $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/j} \xrightarrow{\mathsf{fgt}} \mathcal{I}$, which is then carried to $(\varphi_! F)(j) \in \mathcal{C}$ by (an appropriate extension of) the functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$.

8.3.5. We now return to the Quillen equivalence (23).

For any $n \geq 0$, the **topological n-simplex** is the topological space

$$\Delta_{\text{top}}^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=0}^n x_i = 1 \text{ and } x_i \geq 0 \right\} \in \mathsf{Top} \ .$$

These assemble into a cosimplicial topological space $\boldsymbol{\Delta} \xrightarrow{\Delta_{\text{top}}^\bullet} \mathsf{Top}$.[122] We can now define the right adjoint Sing: it is simply the restricted Yoneda functor. That is, for any topological space $Y \in \mathsf{Top}$, its **singular simplicial set** is the functor

$$\mathsf{Sing}(Y)_\bullet : \boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\hom_{\mathsf{Top}}(\Delta_{\text{top}}^\bullet, Y)} \mathsf{Set} \ .$$

We refer to the set $\mathsf{Sing}(Y)_n := \hom_{\mathsf{Top}}(\Delta_{\text{top}}^n, Y)$ as the set of (**singular**) n-**simplices** of Y. Its left adjoint functor $|-|$ is called **geometric realization**. The most efficient definition is that it is the left Kan extension

$$
\begin{array}{ccc}
\boldsymbol{\Delta} & \xrightarrow{\Delta_{\text{top}}^\bullet} & \mathsf{Top} \\
{\scriptstyle \Delta^\bullet} \downarrow & \nearrow & \\
\mathsf{sSet} & \raisebox{1ex}{$\scriptstyle |-|$} &
\end{array}
$$

of $\Delta_{\text{top}}^\bullet$ along the Yoneda embedding $\Delta^\bullet$. So by definition, for any simplicial set $X \in \mathsf{sSet}$, we have

$$|X| := \mathsf{colim}\left(\boldsymbol{\Delta}_{/X} \xrightarrow{\mathsf{fgt}} \boldsymbol{\Delta} \xrightarrow{\Delta_{\text{top}}^\bullet} \mathsf{Top} \right) \ .$$

So, the topological space $|X| \in \mathsf{Top}$ is built by gluing together topological n-simplices, one for each combinatorial n-simplex of X.

Exercise 8.6 (4 points)**.** Verify the adjunction (23).

Now, we define the subcategory $\mathbf{W}_{\text{w.h.e.}} \subset \mathsf{sSet}$ of **weak homotopy equivalences** simply by pullback along $\mathsf{sSet} \xrightarrow{|-|} \mathsf{Top}$. These are the weak equivalences of the Kan–Quillen model structure. The cofibrations are the monomorphisms (i.e. the levelwise injections), so that every object is cofibrant. To describe the fibrations, we introduce some notation. First of all, for any $n \geq 0$, we write $\partial \Delta^n \subseteq \Delta^n$ for the largest simplicial subset not containing the unique

---

[122]The functoriality of $\Delta_{\text{top}}^\bullet$ is uniquely specified by the following requirements: all maps in its image are linear (in the evident sense), and the function $\mathsf{fgt}([n]) \to \mathsf{fgt}(\Delta_{\text{top}}^n)$ on underlying sets taking $i \in [n]$ to the ith unit basis vector assembles into a natural transformation

$$
\begin{array}{ccc}
\boldsymbol{\Delta} & \xrightarrow{\Delta_{\text{top}}^\bullet} & \mathsf{Top} \\
{\scriptstyle \mathsf{fgt}} \searrow & {\scriptstyle \nearrow} & \downarrow {\scriptstyle \mathsf{fgt}} \\
& \mathsf{Set} &
\end{array} \ .
$$

nondegenerate n-simplex. For any $0 \leqslant i \leqslant n$, the ith **face** of Δ^n is the (nondegenerate) $(n-1)$-simplex given by the morphism $[n-1] \xrightarrow{d_n^i} [n]$ in $\mathbf{\Delta}$ defined by

$$d_n^i(j) = \begin{cases} j, & 0 \leqslant j < i \\ j+1, & i \leqslant j \leqslant n \end{cases}.$$

Then, the ith **horn** of Δ^n is the largest simplicial subset $\Lambda_i^n \subseteq \Delta^n$ not containing the ith face. So for instance, we may depict Λ_0^2 as

$$
\begin{array}{c}
1 \\
\nearrow \\
0 \longrightarrow 2
\end{array}
.
$$

Finally, the Kan–Quillen model structure is cofibrantly generated by the sets

$$I_{\mathsf{KQ}} := \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \geqslant 0} \qquad \text{and} \qquad J_{\mathsf{KQ}} := \{\Lambda_i^n \xrightarrow{\approx} \Delta^n\}_{0 \leqslant i \leqslant n \geqslant 1}.$$

In particular, an object is fibrant iff it has the extension property with respect to all horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$. A fibrant object of $\mathsf{sSet}_{\mathsf{KQ}}$ is called a **Kan complex**.[123]

Now, it is immediate from the definitions that the left adjoint $\mathsf{sSet}_{\mathsf{KQ}} \xrightarrow{|-|} \mathsf{Top}_{\mathsf{QS}}$ is a left Quillen functor; indeed, it even carries the generating (resp. acyclic) cofibrations in $\mathsf{sSet}_{\mathsf{KQ}}$ to the generating (resp. acyclic) cofibrations in $\mathsf{Top}_{\mathsf{QS}}$ (up to certain straightforward homeomorphisms). To see that the Quillen adjunction (23) is in fact a Quillen equivalence, for any $X \in \mathsf{sSet}^c = \mathsf{sSet}$ and any $Y \in \mathsf{Top}^f = \mathsf{Top}$ we must have that a morphism $|X| \xrightarrow{f} Y$ lies in $\mathbf{W}_{\mathsf{QS}}$ iff its adjunct morphism $X \xrightarrow{f^\sharp} \mathsf{Sing}(Y)$ lies in $\mathbf{W}_{\mathsf{KQ}}$. By definition, the latter holds iff the morphism $|X| \xrightarrow{|f^\sharp|} |\mathsf{Sing}(Y)|$ lies in $\mathbf{W}_{\mathsf{QS}}$. Via the commutative triangle

$$
\begin{array}{ccc}
& & |\mathsf{Sing}(Y)| \\
& \nearrow^{|f^\sharp|} & \downarrow{\scriptstyle\varepsilon} \\
|X| & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

and the two-out-of-three property for $\mathbf{W}_{\mathsf{QS}} \subset \mathsf{Top}$, it is equivalent to verify that the counit morphism $|\mathsf{Sing}(Y)| \xrightarrow{\varepsilon} Y$ lies in $\mathbf{W}_{\mathsf{QS}}$. This is a nontrivial fact, but it should be plausible; indeed, this morphism is a cellular approximation (i.e. a weak equivalence from a cell complex).

We note that both adjoints of the Quillen equivalence (23) are automatically derived, simply because $\mathsf{sSet}_{\mathsf{KQ}}^c = \mathsf{sSet}$ and $\mathsf{Top}_{\mathsf{QS}}^f = \mathsf{Top}$. Also, $\mathsf{sSet}_{\mathsf{KQ}}$ is a symmetric monoidal

---

[123]It is worth noting that Kan complexes are necessarily very large. In fact, a Kan complex with any nondegenerate positive-dimensional simplices will necessarily have infinitely many simplices in all dimensions. In a strong sense, this largeness reflects the fundamental incalculability of unstable homotopy theory. This may be compared e.g. with the "small simplicial circle", namely $\Delta^1/\partial\Delta^1 \in \mathsf{sSet}$: it only has one nondegenerate edge, whereas a fibrant replacement must have at least as many edges as there are homotopy classes of maps $S^1 \to S^1$.

model categories with respect to cartesian product, and $\mathsf{Top}_{\mathsf{QS}}$ is as well once restricting to a "convenient" subcategory (so that cartesian product is a two-variable left adjoint).

8.4. Homology.

8.4.1.   As we now explain, homology of spaces essentially amounts to ***derived linearization***. Beyond its intrinsic interest, this will be relevant for us in understanding the relationship between ∞-categories and $\Bbbk$-linear ∞-categories.

8.4.2.   We begin with a brief digression.

We write $\mathsf{Ch}_R^{\geqslant 0} \subseteq \mathsf{Ch}_R$ for the full subcategory on the ***nonnegatively-graded complexes*** of R-modules, i.e. those $M \in \mathsf{Ch}_R$ such that $M_n = 0$ for all $n < 0$. We write

$$\mathsf{H}_0(\mathbf{D}_R^{\geqslant 0}) := (\mathsf{Ch}_R^{\geqslant 0})[\mathbf{W}_{\mathsf{q.i.}}^{-1}]$$

for its localization, and refer to it as the category of ***nonnegatively-graded derived R-modules***,[124] as justified by Exercise 8.7.

Exercise 8.7 (6 points)**.**
   (a) Prove that the projective model structure on Ch_R restricts to a ("projective") model structure on $\mathsf{Ch}_R^{\geqslant 0} \subseteq \mathsf{Ch}_R$.
   (b) Construct a Quillen coreflective localization adjunction

$$(\mathsf{Ch}_R^{\geqslant 0})_{\mathsf{proj}} \underset{\tau_{\geqslant 0}}{\overset{i_{\geqslant 0}}{\underset{\perp}{\rightleftarrows}}} (\mathsf{Ch}_R)_{\mathsf{proj}} \quad ,$$

   and verify that both adjoints are automatically derived.
   (c) Prove that the resulting derived adjunction

$$\mathsf{H}_0(\mathbf{D}_R^{\geqslant 0}) := \mathsf{Ch}_R^{\geqslant 0}[\mathbf{W}_{\mathsf{q.i.}}^{-1}] \underset{\tau_{\geqslant 0}}{\overset{i_{\geqslant 0}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ch}_R[\mathbf{W}_{\mathsf{q.i.}}^{-1}] =: \mathsf{H}_0(\mathbf{D}_R)$$

   is also a coreflective localization adjunction, and prove that the image of the derived left adjoint $i_{\geqslant 0} = \mathbb{L}(i_{\geqslant 0})$ consists of precisely those derived R-modules $M \in \mathsf{H}_0(\mathbf{D}_R)$ such that $\mathsf{H}_n(M) = 0$ for all $n < 0$.[125]

---

[124]It is also common to call these *connective* derived R-modules (which is meant to sound similar to "connected", but that requires the vanishing of H_0 as well). This terminology has the advantage that it carries over without change when one uses cohomological indexing conventions.

[125]In other words, a derived R-module with nonnegatively-graded homology can be presented by a nonnegatively-graded complex of R-modules.

8.4.3. We now study certain intermediate categories that are relevant in the definition of homology.

Given a category $\mathcal{C}$, we write $\mathsf{Mod}_R(\mathcal{C})$ for the category of right R-module objects in $\mathcal{C}$. Note that $\mathsf{Mod}_R(\mathsf{sSet}) \simeq \mathsf{sMod}_R(\mathsf{Set}) =: \mathsf{sMod}_R$.[126] As a special case, we have the category $\mathsf{Ab}(\mathcal{C}) := \mathsf{Mod}_{\mathbb{Z}}(\mathcal{C})$ of abelian group objects, and no intuition is lost by restricting to this special case.

It will be convenient to refer to certain *lifted* model structures. Namely, we have Quillen adjunctions

$$
\mathsf{sSet}_{\mathsf{KQ}} \underset{\mathsf{fgt}}{\overset{R\{-\}}{\underset{\perp}{\rightleftarrows}}} (\mathsf{sMod}_R)_{\mathsf{KQ}} \qquad \text{and} \qquad \mathsf{Top}_{\mathsf{QS}} \underset{\mathsf{fgt}}{\overset{R\{-\}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_R(\mathsf{Top})_{\mathsf{QS}} \ .^{127}
$$

The model structures on categories of R-module objects are "created by the right adjoints", in the sense that a morphism is a (resp. acyclic) fibration iff its image under fgt is a (resp. acyclic) fibration. Because every object of $\mathsf{Top}_{\mathsf{QS}}$ is fibrant, so is every object of $\mathsf{Mod}_R(\mathsf{Top})_{\mathsf{QS}}$.

Exercise 8.8 (2 points). Show that every object of $(\mathsf{sMod}_R)_{\mathsf{KQ}}$ is fibrant.

8.4.4. There is a close relationship between the categories sMod_R and $\mathsf{Ch}_R^{\geqslant 0}$, which is known as the **Dold–Kan correspondence**.

Namely, there are two functors $\overline{\mathsf{C}}_{\bullet}, \mathsf{C}_{\bullet} \in \mathsf{Fun}(\mathsf{sMod}_R, \mathsf{Ch}_R^{\geqslant 0})$, the *normalized* and *unnormalized chains* on a simplicial R-module. These are related by a commutative diagram

(24)
$$
\begin{array}{ccc}
\overline{\mathsf{C}}_{\bullet} & \overset{\approx}{\hookrightarrow} & \mathsf{C}_{\bullet} \\
& {\scriptstyle \mathrm{id}_{\overline{\mathsf{C}}_{\bullet}}} \searrow & \downarrow {\scriptstyle \approx} \\
& & \overline{\mathsf{C}}_{\bullet}
\end{array}
$$

of natural quasi-isomorphisms, and moreover $\overline{\mathsf{C}}_{\bullet}$ is an equivalence of categories [Wei94, §8.3].

We do not discuss the diagram (24), but we at least define the functors that it involves. First of all, for a simplicial R-module $X \in \mathsf{sMod}_R$, we define $\mathsf{C}_{\bullet}(X) \in \mathsf{Ch}_R^{\geqslant 0}$ by setting $\mathsf{C}_n(X) = X_n$ and defining the differentials $\mathsf{C}_n(X) \xrightarrow{d_n} \mathsf{C}_{n-1}(X)$ to be

$$
\mathsf{C}_n(X) := X_n \xrightarrow{\sum_{i=0}^{n} (-1)^i \cdot X(d_n^i)} X_{n-1} =: \mathsf{C}_{n-1}(X) \ .
$$

Then, $\overline{\mathsf{C}}_{\bullet}(X) \subseteq \mathsf{C}_{\bullet}(X)$ is the subcomplex defined levelwise by

$$
\overline{\mathsf{C}}_n(X) := \bigcap_{i=0}^{n-1} \mathsf{ker}\left(X_n \xrightarrow{X(d_n^i)} X_{n-1} \right) \ ,
$$

---

[126]This equivalence follows from the fact that products in sSet are computed pointwise.

[127]The free simplicial R-module on a simplicial set $X \in \mathsf{sSet}$ has $R\{X\}_n := R\{X_n\}$, i.e. it is given by applying the free R-module functor $\mathsf{Set} \xrightarrow{R\{-\}} \mathsf{Mod}_R$ levelwise. The free topological R-module on a topological space is the free R-module on its underlying set equipped with a suitable topology: its elements are given by finite unordered configurations of points in X labeled by elements of R, where nearby configurations with identical labels are nearby, and with addition and right R-action given pointwise.

so that its differential is simply $\overline{\mathsf{C}}_n(X) \xrightarrow{(-1)^n \cdot X(d_n^n)} \overline{\mathsf{C}}_{n-1}(X)$.

Exercise 8.9 (8 points)**.** For any simplicial R-module $M \in \mathsf{sMod}_R$ and any $n \geq 0$, establish a commutative diagram

$$
\begin{array}{ccc}
\mathsf{hom}_{\mathsf{sSet}_*}(\Delta^n/\partial\Delta^n, \mathsf{fgt}(M)) & \xrightarrow{\ \cong\ } & \mathsf{Z}_n(\overline{\mathsf{C}}_\bullet(M)) \\
\uparrow & & \uparrow{\scriptstyle d_{n+1}} \\
\mathsf{hom}_{\mathsf{sSet}_*}(\Delta^{n+1}/\Lambda_{n+1}^{n+1}, \mathsf{fgt}(M)) & \xrightarrow{\ \cong\ } & \overline{\mathsf{C}}_{n+1}(M)
\end{array}
$$

in Mod_R (where we consider $\mathsf{fgt}(M) := \mathsf{fgt}(0 \to M) \in \mathsf{sSet}_*$), and use this to deduce an isomorphism

$$\pi_n(\mathsf{fgt}(M)) \cong \mathsf{H}_n(\mathsf{C}_\bullet(M))$$

in Mod_R.

It follows from Exercise 8.9 and the above discussion that the functors $\overline{\mathsf{C}}_\bullet$ and $\mathsf{C}_\bullet$ induce canonically equivalent equivalences

$$\mathsf{sMod}_R[\mathbf{W}_{\mathsf{w.h.e.}}^{-1}] \xrightarrow{\ \sim\ } \mathsf{Ch}_R^{\geq 0}[\mathbf{W}_{\mathsf{q.i.}}^{-1}]$$

on localizations.[128] So, the model category $(\mathsf{sMod}_R)_{\mathsf{KQ}}$ is a presentation of $\mathsf{H}_0(\mathbf{D}_R^{\geq 0})$.

8.4.5.   Now, both singular and simplicial homology (with coefficients in R) may be located in the diagram

(25)

namely, **simplicial homology** is the composite functor

$$\mathsf{H}_*^\Delta(-; R) : \mathsf{sSet} \xrightarrow{R\{-\}} \mathsf{sMod}_R \xrightarrow{\mathsf{C}_\bullet} \mathsf{Ch}_R \xrightarrow{\mathsf{H}_*} \mathsf{Fun}(\mathbb{N}, \mathsf{Mod}_R) \ ,$$

---

[128]In fact, it is not hard to see that the equivalence of categories

$$(\mathsf{sMod}_R)_{\mathsf{KQ}} \xrightarrow{\overline{\mathsf{C}}_\bullet} (\mathsf{Ch}_R^{\geq 0})_{\mathsf{proj}}$$

is both a left and right Quillen equivalence (using the notation of Exercise 8.7).

and **singular homology** is the composite functor

$$\mathsf{H}_*^{\mathsf{sing}}(-;R) : \mathsf{Top} \xrightarrow{\mathsf{Sing}} \mathsf{sSet} \xrightarrow{\mathsf{H}_*^{\Delta}(-;R)} \mathsf{Fun}(\mathbb{N}, \mathsf{Mod}_R) \, . ^{129}$$

The adjunction $\mathsf{Mod}_R(|-| \dashv \mathsf{Sing})$ arises from the fact that both functors in the adjunction $|-| \dashv \mathsf{Sing}$ preserve finite products, and it is clear that the square in diagram (25) commutes after omitting all left adjoints or all right adjoints.

Exercise 8.10 (4 points)**.** Prove that the adjunction $\mathsf{Mod}_R(|-| \dashv \mathsf{Sing})$ is a Quillen equivalence.

Combining Exercise 8.10 with the discussion of §8.4.4, we see that the model category $\mathsf{Mod}_R(\mathsf{Top})_{\mathsf{QS}}$ is also a presentation of $\mathsf{H}_0(\mathbf{D}_R^{\geqslant 0})$ and that the triangle in diagram (25) commutes.

From diagram (25), we also see that homology with coefficients in R is indeed derived R-linearization; note that both composites

$$\mathsf{C}_\bullet^\Delta(-;R) : \mathsf{sSet} \xrightarrow{R\{-\}} \mathsf{sMod}_R \xrightarrow{\mathsf{C}_\bullet} \mathsf{Ch}_R$$

and

$$\mathsf{C}_\bullet^{\mathsf{sing}}(-;R) : \mathsf{Top} \xrightarrow{\mathsf{Sing}} \mathsf{sSet} \xrightarrow{\mathsf{C}_\bullet^\Delta(-;R)} \mathsf{Ch}_R$$

are automatically derived.[130] Indeed, diagram (25) yields the commutative diagram

---

[129]More generally, homology with coefficients in a (possibly derived) R-module is obtained by taking the (resp. derived) tensor product before taking homology of complexes.

[130]As always, we prefer to interpret these "homology" functors without actually passing to homology groups of complexes.

on localizations.[131] From this, we immediately deduce the **Dold–Thom theorem**: for any cofibrant object $X \in \mathsf{Top}_{\mathsf{QS}}$ (i.e. a cell complex or retract thereof), we have a canonical isomorphism

$$\pi_*(R\{X\}) \cong \mathsf{H}^{\mathsf{sing}}_*(X; R) \ .$$

Said differently, the singular homology of X with coefficients in R is computed by the homotopy groups of the free topological R-module on X. We also deduce a sort of free/forget adjunction

$$\mathsf{ho}(\mathcal{S}) := \mathsf{Top}[\mathbf{W}^{-1}_{\mathsf{w.h.e.}}] \underset{\mathsf{fgt} \circ \tau_{\geqslant 0}}{\overset{\mathsf{C}^{\mathsf{sing}}_\bullet(-;R)}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ch}_R[\mathbf{W}^{-1}_{\mathsf{q.i.}}] =: \mathsf{H}_0(\mathbf{D}_R)$$

between the homotopy category of spaces and the category of derived R-modules.[132]

8.5. Homotopy co/limits.

8.5.1.   Given a category $\mathcal{J}$ and a model category $\mathcal{C}$, we obtain a relative category $(\mathsf{Fun}(\mathcal{J}, \mathcal{C}), \mathbf{W})$ whose weak equivalences are the componentwise weak equivalences. Under various conditions (which are satisfied in all our cases of interest) there exist **projective** and **injective** model structures on this relative category, which are defined so that the adjunctions

$$\mathsf{Fun}(\mathcal{J}, \mathcal{C})_{\mathsf{proj}} \underset{\mathsf{const}}{\overset{\mathsf{colim}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C} \qquad \text{and} \qquad \mathcal{C} \underset{\mathsf{lim}}{\overset{\mathsf{const}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}(\mathcal{J}, \mathcal{C})_{\mathsf{inj}}$$

are Quillen adjunctions. The derived adjoints $\mathbb{L}\mathsf{colim}$ and $\mathbb{R}\mathsf{lim}$ are respectively called the **homotopy colimit** and the **homotopy limit**. In particular, homotopy co/limits preserve componentwise weak equivalences. We will discuss specific examples of homotopy co/limits below.

8.5.2.   Being a derived functor, the homotopy co/limit of a diagram is computed as the ordinary co/limit of an appropriate co/fibrant resolution of the diagram. However, there are various assumptions under which the ordinary co/limit of a diagram also computes its homotopy co/limit, which will be convenient for us.

   (1) (a) To compute a homotopy pushout (i.e. the homotopy colimit of a span), it suffices
             to assume that all objects are cofibrant and that at least one of the morphisms
             is a cofibration.

---

[131]That is, the two adjunctions are identified via the equivalences, so that in particular the diagram commutes upon omitting both left adjoints or both right adjoints.

[132]Note that this adjunction restricts to the free/forget adjunction

$$\mathsf{Set} \underset{\mathsf{fgt}}{\overset{R\{-\}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_R$$

between sets and R-modules. This analogy will be amplified below.

(b) Dually, to compute a homotopy pullback (i.e. the homotopy limit of a cospan), it suffices to assume that all objects are fibrant and that at least one of the morphisms is a fibration.

(2) (a) To compute a sequential homotopy colimit (i.e. a homotopy colimit over $\mathbb{N}^{\leqslant}$), it suffices to assume that all objects are cofibrant and that all morphisms are cofibrations.

(b) Dually, to compute a sequential homotopy limit (i.e. a homotopy limit over $(\mathbb{N}^{\leqslant})^{\mathsf{op}}$), it suffices to assume that all objects are fibrant and that all morphisms are fibrations.

8.6. A few other model categories.

8.6.1. We conclude §8 by briefly discussing three other model categories; the first will be relevant to us in §9, and the latter two are interesting (or at least curious) in their own rights.

8.6.2. The category cat of categories admits a **canonical model structure**, in which the weak equivalences are the equivalences of categories and the cofibrations are the functors that are injective on objects. The fibrations are the **isofibrations**, i.e. the functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ such that any isomorphism $F(C) \xrightarrow{\cong} D$ in $\mathcal{D}$ lifts to an isomorphism $C \xrightarrow{\cong} C'$ in $\mathcal{C}$. In particular, every object is bifibrant, and morphisms in $\mathsf{cat}[\mathbf{W}^{-1}]$ are equivalence classes of functors under the relation of natural isomorphism.

8.6.3. The category Top is a relative category via the subcategory $\mathbf{W}_{\mathsf{h.e.}} \subset \mathsf{Top}$ of homotopy equivalences. The relative category $(\mathsf{Top}, \mathbf{W}_{\mathsf{h.e.}})$ admits a **Strøm model structure**, whose fibrations are the Hurewicz fibrations. Because every homotopy equivalence is a weak homotopy equivalence and every Hurewicz fibration is a Serre fibration, the identity adjunction is a Quillen adjunction

$$\mathsf{Top}_{\mathsf{QS}} \underset{\mathsf{id}_{\mathsf{Top}}}{\overset{\mathsf{id}_{\mathsf{Top}}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Top}_{\mathsf{Strøm}} \quad .$$

The right adjoint is automatically derived, and from this it is easy to check that the derived adjunction

$$\mathsf{ho}(\mathcal{S}) := \mathsf{Top}[\mathbf{W}_{\mathsf{w.h.e.}}^{-1}] \underset{\mathbb{R}(\mathsf{id}_{\mathsf{Top}})}{\overset{\mathbb{L}(\mathsf{id}_{\mathsf{Top}})}{\underset{\perp}{\rightleftarrows}}} \mathsf{Top}[\mathbf{W}_{\mathsf{h.e.}}^{-1}]$$

is a right localization adjunction; its counit may be thought of as cellular approximation.

8.6.4. Let $\mathsf{CAlg}_{\Bbbk} := \mathsf{CAlg}(\mathsf{Mod}_{\Bbbk})$ denote the category of commutative $\Bbbk$-algebras. For any commutative $\Bbbk$-algebra $A \in \mathsf{CAlg}_{\Bbbk}$ and any A-module $M \in \mathsf{Mod}_A$, a $\Bbbk$-**linear derivation of** A **into** M **relative to** $\Bbbk$ is a $\Bbbk$-module homomorphism $A \xrightarrow{d} M$ such that the composite $\Bbbk \xrightarrow{\eta} A \xrightarrow{d} M$ is zero and satisfying the Leibniz rule that $d(f \cdot g) = f \cdot d(g) + g \cdot d(f)$. It

is not hard to construct a universal A-module equipped with a derivation of A, namely the A-module of **Kähler differentials**

$$\Omega^1_{A|\Bbbk} := A\{da\}_{a\in A} \left/ \left\{ \begin{array}{c} d(a+b) = da + db \text{ for all } a, b \in A \\ d(\eta(\lambda)) = 0 \text{ for all } \lambda \in \Bbbk \\ d(a \cdot b) = a \cdot db + b \cdot da \text{ for all } a, b \in A \end{array} \right\} \right. ,$$

with universal derivation $A \to \Omega^1_{A|\Bbbk}$ given by the formula $a \mapsto da$.

This construction plays the role of **differential 1-forms** in algebraic geometry, so that the A-module $\Omega^1_{A|\Bbbk}$ is the (relative) cotangent space. Therefore, it should not be surprising that it does not behave well when A is singular. It was for this reason that Quillen invented model categories: namely, in order to take the *derived functor of derivations*.

In order to explain this, we begin by reformulating the construction of Kähler differentials.

Exercise 8.11 (6 points)**.**

(a) Construct inverse equivalences

$$\mathsf{Ab}((\mathsf{CAlg}_\Bbbk)_{/A}) \underset{\text{square-zero extension}}{\overset{\text{kernel of map to } A}{\underset{\sim}{\longleftarrow}}} \mathsf{Mod}_A \quad .$$

(b) Prove that the left adjoint in the adjunction

(26) $$(\mathsf{CAlg}_\Bbbk)_{/A} \underset{\text{forget}}{\overset{\text{free}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ab}((\mathsf{CAlg}_\Bbbk)_{/A}) \underset{\text{square-zero extension}}{\overset{\text{kernel of map to } A}{\underset{\sim}{\longleftarrow}}} \mathsf{Mod}_A$$

carries $A \in (\mathsf{CAlg}_\Bbbk)_{/A}$ to $\Omega^1_{A|\Bbbk} \in \mathsf{Mod}_A$.

Now, Quillen endowed the category $\mathsf{sCAlg}_\Bbbk$ of *simplicial* commutative $\Bbbk$-algebras with a model structure whose weak equivalences and fibrations are created by the right adjoint in the adjunction

$$\mathsf{sSet}_{\mathsf{KQ}} \underset{\mathsf{fgt}}{\overset{\Bbbk[-]}{\underset{\perp}{\rightleftarrows}}} \mathsf{sCAlg}_\Bbbk$$

(as in §8.4.3). Because $\mathsf{sSet}_{\mathsf{KQ}}$ is cofibrantly generated, so is $(\mathsf{CAlg}_\Bbbk)_{\mathsf{Quillen}}$, and the cofibrant objects are the levelwise free commutative $\Bbbk$-algebras such that the degeracy maps carry generators to generators. Meanwhile, by Exercise 8.8, all objects of $(\mathsf{CAlg}_\Bbbk)_{\mathsf{Quillen}}$ are fibrant. As a result, applying $\mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, (-))$ to the adjunction (26) yields a Quillen adjunction

$$(\mathsf{s}(\mathsf{CAlg}_\Bbbk)_{/A})_{\mathsf{Quillen}} \overset{\perp}{\rightleftarrows} (\mathsf{sMod}_A)_{\mathsf{KQ}}$$

(because the right adjoint preserves fibrations and weak equivalences). The value of the derived left adjoint on the object A is called the **cotangent complex** of A, a derived A-module denoted

$$\mathbb{L}\Omega_{A|\Bbbk} \in (\mathsf{sMod}_A)[\mathbf{W}_{\mathsf{KQ}}^{-1}] \simeq \mathsf{H}_0(\mathbf{D}_A^{\geqslant 0}) ,$$

and its homology groups are called the **André–Quillen homology groups** of A.

Exercise 8.12 (6 points)**.** Take $A = \Bbbk[x]/x^2 \in \mathsf{CAlg}_\Bbbk \subseteq \mathsf{sCAlg}_\Bbbk$.

(a) Construct a cofibrant resolution of $A \in (\mathsf{sCAlg}_{\Bbbk})_{\mathsf{Quillen}}$ (which will also be a cofibrant resolution in $((\mathsf{sCAlg}_{\Bbbk})_{/A})_{\mathsf{Quillen}}$).

(b) Compute the André–Quillen homology groups of A.

9. Basic notions in ∞-category theory

9.1. Model categories of ∞-categories.

9.1.1. We now introduce ∞-categories. Our discussion is effectively a more in-depth treatment of the nonlinear version of the introduction to $\Bbbk$-linear ∞-categories given in §6.2. For more details, we refer the reader to [§T.1].

9.1.2. We write $\mathsf{cat}(\mathcal{V})$ for the category of categories enriched in a monoidal category $\mathcal{V} := (\mathcal{V}, \boxtimes)$. If $\mathcal{V}$ is *symmetric* monoidal, then $\mathsf{cat}(\mathcal{V})$ is symmetric monoidal as well: for any $\mathcal{C}, \mathcal{D} \in \mathsf{cat}(\mathcal{V})$, an object of $\mathcal{C} \boxtimes \mathcal{D}$ is a pair of objects of $\mathcal{C}$ and $\mathcal{D}$, and we define

$$\mathsf{hom}_{\mathcal{C} \boxtimes \mathcal{D}}((C, D), (C', D')) := \mathsf{hom}_{\mathcal{C}}(C, C') \boxtimes \mathsf{hom}_{\mathcal{D}}(D, D') \ .$$

As a particular case, we have $\mathsf{cat}(\mathsf{Ch}_{\Bbbk}) := \mathsf{cat}_{\Bbbk}^{\mathsf{dg}}$. Other than this example, $\mathcal{V}$ will always be symmetric monoidal via the cartesian product. We simply write $\mathsf{cat} := \mathsf{cat}(\mathsf{Set})$ for the category of ordinary unenriched categories. We refer to objects of $\mathsf{cat}(\mathsf{sSet})$ as **simplicial categories**. This is for brevity, but it is slightly abusive: these are *not* the same thing as simplicial objects in cat. Similarly, we refer to objects of $\mathsf{cat}(\mathsf{Top})$ as **topological categories**.[133]

9.1.3. Observe the adjunction

$$\mathsf{sSet} \ \underset{\mathsf{const}}{\overset{\pi_0}{\underset{\perp}{\rightleftarrows}}} \ \mathsf{Set} \ \ ,$$

where we write $\pi_0 := \pi_0 \circ |-|$ for simplicity.[134] Both functors preserve products, and so we obtain an adjunction

(27) $$\mathsf{cat}(\mathsf{sSet}) \ \overset{\mathsf{ho}}{\underset{\perp}{\rightleftarrows}} \ \mathsf{cat}(\mathsf{Set}) =: \mathsf{cat}$$

between simplicial categories and ordinary categories. We refer to the left adjoint as the **homotopy category** functor, and we do not give notation or terminology for the right adjoint.

A morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathsf{cat}(\mathsf{sSet})$ is called a **weak equivalence** if the following two conditions hold:

(1) it is **homotopically fully faithful**, i.e. for all $C, C' \in \mathcal{C}$ the induced morphism $\mathsf{hom}_{\mathcal{C}}(C, C') \to \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ in sSet is a weak homotopy equivalence;

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[133]As implied in §6.2.2, we reserve the notation Cat for the ∞-category of ∞-categories.

[134]This left adjoint can also be identified as the colimit functor.

(2) it is **homotopically essentially surjective**, i.e. the induced morphism $\mathsf{ho}(\mathcal{C}) \xrightarrow{\mathsf{ho}(F)} \mathsf{ho}(\mathcal{D})$ in cat is essentially surjective.[135]

(Of course, if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence of simplicial categories then $\mathsf{ho}(\mathcal{C}) \xrightarrow{\mathsf{ho}(F)} \mathsf{ho}(\mathcal{D})$ is an equivalence of categories.)

We localize the category $\mathsf{cat}(\mathsf{sSet})$ at the weak equivalences. This yields a category that (for reasons that will become clear presently) we denote by $\mathsf{ho}(\mathsf{Cat})$ and refer to as **the homotopy category of ∞-categories**; its objects are called **∞-categories**. So by definition, there is a canonical functor

$$\mathsf{cat}(\mathsf{sSet}) \longrightarrow \mathsf{ho}(\mathsf{Cat})$$

that carries all weak equivalences to isomorphisms. In particular, this functor is essentially surjective: every ∞-category can be represented by a simplicial category.

Equipped with these weak equivalences, $\mathsf{cat}(\mathsf{sSet})$ admits a **Bergner model structure**. This is characterized in terms of its fibrations. First of all, a morphism in $\mathcal{D} \in \mathsf{cat}(\mathsf{sSet})$ is called an **equivalence** (or a **homotopy equivalence**) if it becomes an isomorphism in $\mathsf{ho}(\mathcal{D})$; we denote these by $\xrightarrow{\sim}$. Then, a morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$ is a fibration iff the following conditions hold:

(1) for every $C, C' \in \mathcal{C}$, the induced morphism $\mathsf{hom}_{\mathcal{C}}(C, C') \to \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ lies in $\mathbf{F}_{\mathsf{KQ}} \subseteq \mathsf{sSet}$;

(2) any equivalence $F(C) \xrightarrow{\sim} D$ in $\mathcal{D}$ lifts to an equivalence $C \xrightarrow{\sim} C'$ in $\mathcal{C}$.

Note that condition 2 is a homotopical version of the notion of an isofibration (recall §8.6.2); indeed, it is not hard to see that the adjunction (27) becomes a Quillen adjunction

$$\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}} \underset{\longleftarrow}{\overset{\mathsf{ho}}{\underset{\perp}{\longrightarrow}}} \mathsf{cat}_{\mathsf{can}} \quad .$$

9.1.4. There is similarly a Bergner model structure on $\mathsf{cat}(\mathsf{Top})$, and applying the Quillen equivalence $|-| \dashv \mathsf{Sing}$ to hom-objects determines a Quillen equivalence

$$\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}} \underset{\mathsf{cat}(\mathsf{Sing})}{\overset{\mathsf{cat}(|-|)}{\underset{\perp}{\longrightarrow}}} \mathsf{cat}(\mathsf{Top})_{\mathsf{Bergner}} \quad .$$

It follows that we have an equivalence $\mathsf{cat}(\mathsf{Top})[\mathbf{W}^{-1}] \simeq \mathsf{ho}(\mathsf{Cat})$, and in particular an ∞-category can *also* be defined as a weak equivalence class of topological category.

9.1.5. The Bergner model structure on $\mathsf{cat}(\mathsf{sSet})$ is defined in terms of its fibrations. As a result, the cofibrations are relatively inexplicit. In particular, $\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$ has a fatal flaw: it is not a *monoidal* model category, because the cartesian product of cofibrant objects will not generally be cofibrant. This is a major issue, as it implies that the internal hom bifunctor among simplicial categories does not generally give an internal hom at the level

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[135]It is possible to phrase homotopical essential surjectivity at the level of simplicial categories, but this requires a bit of enriched category theory.

of underlying ∞-categories, even under co/fibrancy assumptions (recall §8.2.5).[136] It is of course absolutely imperative that any flavor of "category theory" have a well-behaved internal hom, i.e. a corresponding notion of "functor category". Of course, $\mathsf{cat}(\mathsf{Top})_{\mathsf{Bergner}}$ suffers from precisely the same fatal flaw.

In §9.2.3 we will introduce an alternative model category of ∞-categories that *is* a symmetric monoidal model category, and in particular has a homotopically well-behaved internal hom (as we will see in §9.5.1). In fact, this will be a model structure on the category sSet of simplicial sets, called the ***Joyal model structure***.

9.2. The Joyal model structure.

9.2.1. As background, we first discuss the relationship between simplicial sets and ordinary categories. By definition, the category $\mathbf{\Delta}$ is a full subcategory of cat. We write $\mathbf{\Delta} \xrightarrow{[\bullet]} \mathsf{cat}$ for the inclusion. Just as in §8.3, this determines an adjunction

$$\mathsf{sSet} \xrightarrow[\substack{\longleftarrow \\ \mathsf{N}}]{\substack{\mathsf{ho} \\ \perp}} \mathsf{cat} \quad,$$

as follows. First of all, the right adjoint N is the restricted Yoneda functor: for any category $\mathcal{C} \in \mathsf{cat}$, its ***nerve*** is the functor

$$\mathsf{N}(\mathcal{C})_\bullet : \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathsf{cat}}([\bullet],\mathcal{C})} \mathsf{Set} \ ,$$

so that $\mathsf{N}(\mathcal{C})_n$ is the set of sequences of n composable morphisms in $\mathcal{C}$. Then, the right adjoint ho is the left Kan extension

$$
\begin{array}{ccc}
\mathbf{\Delta} & \xrightarrow{\ [\bullet]\ } & \mathsf{cat} \\
{\scriptstyle \Delta^\bullet}\big\downarrow & \nearrow{\scriptstyle \mathsf{ho}} & \\
\mathsf{sSet} & &
\end{array}
$$

of $[\bullet]$ along the Yoneda embedding $\Delta^\bullet$; for reasons that will become clear shortly (see Exercise 9.5), we refer to this as the ***homotopy category*** functor.[137] So by definition, for any simplicial set $X \in \mathsf{sSet}$, its homotopy category is

$$\mathsf{ho}(X) := \mathsf{colim} \left(\mathbf{\Delta}_{/X} \xrightarrow{\mathsf{fgt}} \mathbf{\Delta} \xrightarrow{[\bullet]} \mathsf{cat} \right) \ .$$

So, the homotopy category $\mathsf{ho}(X) \in \mathsf{cat}$ is built by gluing together copies of the categories $[n] \in \mathsf{cat}$, one for each combinatorial n-simplex of X.

---

[136]Given $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{cat}(\mathsf{sSet})$, at the level of underlying ∞-categories we would like an internal hom object $\underline{\mathsf{hom}}(\mathcal{D}, \mathcal{E}) \in \mathsf{ho}(\mathsf{Cat})$ such that

$$\mathsf{hom}_{\mathsf{ho}(\mathsf{Cat})}(\mathcal{C}, \underline{\mathsf{hom}}(\mathcal{D}, \mathcal{E})) \cong \mathsf{hom}_{\mathsf{ho}(\mathsf{Cat})}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \ .$$

Because $\mathcal{C} \times \mathcal{D} \in \mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$ will not generally be cofibrant even if $\mathcal{C}$ and $\mathcal{D}$ are, the right side must in general be computed using a further cofibrant replacement of $\mathcal{C} \times \mathcal{D}$ (even if $\mathcal{E}$ is fibrant).

[137]Of course, this will be closely related to the functor $\mathsf{cat}(\mathsf{sSet}) \xrightarrow{\mathsf{ho}} \mathsf{cat}$ introduced in §9.1.3.

Exercise 9.1 (4 points).
  (a) Prove that the nerve functor N is fully faithful, and that its image consists of those simplicial sets $X \in \mathsf{sSet}$ that admit *unique* inner horn fillers.
  (b) Given a category $\mathcal{C} \in \mathsf{cat}$, prove that its nerve $\mathsf{N}(\mathcal{C}) \in \mathsf{sSet}$ admits outer horn fillers if and only if $\mathcal{C}$ is a groupoid.

9.2.2. Given any $i, j \in [n]$, we write P_{ij} for the poset of sequences $i = k_0 < k_1 < \cdots < k_l = j$ for $l \geqslant 0$, ordered by inclusion. Observe that for $i \leqslant j$, P_{ij} is simply the power set of the set $\{k \in [n] : i < k < j\}$, ordered by inclusion, while for $i > j$, P_{ij} is empty.

We define an object $\mathfrak{C}[n] \in \mathsf{cat}(\mathsf{sSet})$ as follows: its objects are those of $[n] \in \mathbf{\Delta} \subset \mathsf{cat}$, its hom-objects are given by the formula

$$\mathsf{hom}_{\mathfrak{C}[n]}(i, j) := \mathsf{N}(P_{ij}) \in \mathsf{sSet} ,$$

and its composition is given by the concatenation functors $P_{ij} \times P_{jk} \to P_{ik}$ (note that N preserves products, being a right adjoint). Now, power sets have weakly contractible nerves (they are isomorphic to cubes), and so the natural morphism

$$\mathfrak{C}[n] \longrightarrow [n]$$

is a weak equivalence in $\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$. This weak equivalence depicted for $n = 2$ in diagram (2); whereas $[2] \in \mathsf{cat}(\mathsf{sSet})$ corepresents *strictly* commutative triangles, $\mathfrak{C}[2] \in \mathsf{cat}(\mathsf{sSet})$ corepresents *homotopy-coherently* commutative triangles.

These simplicial categories assemble into a cosimplicial object $\mathbf{\Delta} \xrightarrow{\mathfrak{C}[\bullet]} \mathsf{cat}(\mathsf{sSet})$ in simplicial categories, which determines an adjunction

$$\mathsf{sSet} \overset{\mathfrak{C}}{\underset{\mathsf{N}^{\mathsf{h.c.}}}{\rightleftarrows}} \mathsf{cat}(\mathsf{sSet}) \quad ,$$

as follows. First of all, the right adjoint $\mathsf{N}^{\mathsf{h.c.}}$ is simply the restricted Yoneda functor, so that for any simplicial category $\mathcal{C} \in \mathsf{cat}(\mathsf{sSet})$, its **homotopy-coherent nerve** is the simplicial set

$$\mathsf{N}^{\mathsf{h.c.}}(\mathcal{C})_\bullet : \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathsf{cat}(\mathsf{sSet})}(\mathfrak{C}[\bullet], \mathcal{C})} \mathsf{Set} .$$

So in low dimensions, the simplicial set $\mathsf{N}^{\mathsf{h.c.}}(\mathcal{C}) \in \mathsf{sSet}$ may be described as follows: its 0-simplices are given by objects of $\mathcal{C}$, its 1-simplices are given by morphisms in $\mathcal{C}$, and its 2-simplices are given by homotopy-coherently commutative triangles in $\mathcal{C}$. And then, the left adjoint $\mathfrak{C}$ is the left Kan extension

$$
\begin{array}{ccc}
\mathbf{\Delta} & \xrightarrow{\mathfrak{C}[\bullet]} & \mathsf{cat}(\mathsf{sSet}) \\
{\scriptstyle \Delta^\bullet}\downarrow & \nearrow & \\
\mathsf{sSet} & {\scriptstyle \mathfrak{C}} &
\end{array}
$$

of $\mathfrak{C}[\bullet]$ along the Yoneda embedding $\Delta^\bullet$. So by definition, for any simplicial set $X \in \mathsf{sSet}$, we have

$$\mathfrak{C}X := \mathsf{colim}\left(\Delta_{/X} \xrightarrow{\mathsf{fgt}} \Delta \xrightarrow{\mathfrak{C}[\bullet]} \mathsf{cat}(\mathsf{sSet})\right) \ .$$

9.2.3.   Recall that a simplicial set $X \in \mathsf{sSet}$ is called a *Kan complex* if it has the extension property with respect to all horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$ for all $0 \leqslant i \leqslant n \geqslant 1$. For short, we will say that such a simplicial set **admits horn fillers**. Note that these horn fillers (i.e. extensions) are not generally unique (recall that the prototypical example of a Kan complex is $\mathsf{Sing}(Y) \in \mathsf{sSet}_{\mathsf{KQ}}^f$ for any $Y \in \mathsf{Top}$).

We refine this notion by declaring that a horn inclusion $\Lambda_i^n \hookrightarrow \Delta^n$ for $0 \leqslant i \leqslant n \geqslant 1$ is an **inner horn inclusion** if $0 < i < n \geqslant 2$ and is an **outer horn inclusion** otherwise. We then define a common weakening of the notion of a Kan complex and the nerve of a category (recall Exercise 9.1): a **quasicategory** is a simplicial set that admits (not necessarily unique) inner horn fillers. So, these four notions fit into the following diagram.

	fillers are unique	fillers are not necessarily unique
fillers for all horns	nerves of groupoids	Kan complexes
fillers for inner horns	nerves of categories	quasicategories

Now, the **Joyal model structure** on sSet is characterized by the facts that its weak equivalences are created by $\mathfrak{C}$ and that its cofibrations are the monomorphisms. Its fibrant objects are the quasicategories. It turns out that the adjunction

$$\mathsf{sSet}_{\mathsf{Joyal}} \underset{\mathsf{N}^{\mathsf{h.c.}}}{\overset{\mathfrak{C}}{\underset{\perp}{\rightleftarrows}}} \mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$$

is a Quillen equivalence [T.2.2.5.1]. Of course, it follows that we have an equivalence $\mathsf{sSet}[\mathbf{W}_{\mathsf{Joyal}}^{-1}] \simeq \mathsf{ho}(\mathsf{Cat})$, and in particular that an ∞-category can *also* be defined as a Joyal weak equivalence class of simplicial set.

A particular consequence of this result is the following.

Exercise 9.2 (8 points)**.** Given any fibrant simplicial category $\mathcal{C} \in \mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}^f$, prove that its homotopy-coherent nerve $\mathsf{N}^{\mathsf{h.c.}}(\mathcal{C}) \in \mathsf{sSet}$ is a quasicategory.

In general, we write e.g. $\mathsf{C} \in \mathsf{sSet}_{\mathsf{Joyal}}^f$, etc. to denote a quasicategory and $\mathcal{C} \in \mathsf{ho}(\mathsf{Cat})$ to denote its underlying ∞-category. Similarly, we will write e.g. $\mathsf{x} \in \mathsf{C}$ to denote an object of that represents an object $x \in \mathcal{C}$. This is intended to emphasize the difference between "underlying" objects and the objects that represent them. In fact, we will use this same font more generally to denote chosen presentations of underlying objects.

9.3. Basic notions in quasicategories.

9.3.1. Just as the prototypical example of a Kan complex is $\mathsf{Sing}(Y)$ for $Y \in \mathsf{Top} = \mathsf{Top}^f_{\mathsf{QS}}$, the prototypical example of a quasicategory is $\mathsf{N}^{\mathsf{h.c.}}(\mathcal{C})$ for $\mathcal{C} \in \mathsf{Cat}(\mathsf{sSet})^f_{\mathsf{Bergner}}$ (recall Exercise 9.2). This motivates much of the following basic discussion of how one "does category theory" in an arbitrary quasicategory $\mathsf{C} \in \mathsf{sSet}^f_{\mathsf{Joyal}}$.

9.3.2. An **object** of C is a vertex. Given an object $\mathsf{x} \in \mathsf{C}_0$, we often simply write $\mathsf{x} \in \mathsf{C}$.

A **morphism** in C is an edge. Given a morphism $\mathsf{f} \in \mathsf{C}_1 \cong \hom_{\mathsf{sSet}}(\Delta^1, \mathsf{C})$, its *source* is $\sigma(\Delta^{\{0\}}) \in \mathsf{C}_0$ and and its *target* is $\sigma(\Delta^{\{1\}}) \in \mathsf{C}_0$. Given an object $\mathsf{x} \in \mathsf{C}_0$, its *identity morphism* is the degenerate 1-simplex $\Delta^1 \to \Delta^0 \xrightarrow{\mathsf{x}} \mathsf{C}$, which we denote by $\mathsf{id}_\mathsf{x} \in \mathsf{C}_1$.

A 2-simplex $\sigma \in \mathsf{C}_2$ is called a (**homotopy-coherently**) **commutative triangle** in C. This may be depicted as

$$(28) \qquad \begin{array}{ccc} & \mathsf{y} & \\ {}^{\mathsf{f}}\nearrow & \simeq & \searrow^{\mathsf{g}} \\ \mathsf{x} & \xrightarrow[\mathsf{h}]{} & \mathsf{z} \end{array} \qquad .$$

We say that such a 2-simplex **witnesses h as a composite of f and g**. In this situation, we write $\mathsf{h} \simeq \mathsf{g} \circ \mathsf{f}$, even though the right side does *not* have prior meaning.

The data of the pair of composable morphisms

$$\begin{array}{ccc} & \mathsf{y} & \\ {}^{\mathsf{f}}\nearrow & & \searrow^{\mathsf{g}} \\ \mathsf{x} & & \mathsf{z} \end{array}$$

in C is specified by a morphism $\Lambda^2_1 \to \mathsf{C}$. Because C admits inner horn fillers, there always exists *some* composite of f and g (equipped with a witnessing 2-simplex). However, in general there is no preferred choice: any composite is "just as good as" any other one. This may sound bizarre, and in particular it may sound like a major deficiency of quasicategories compared to simplicial categories; but as we will see, it is in fact *the entire point of ∞-categories*. In particular, we will see that we gain power and flexibility by passing from $\mathcal{C} \in \mathsf{cat}(\mathsf{sSet})^f_{\mathsf{Bergner}}$ (in which strict compositions exist) to $\mathsf{N}^{\mathsf{h.c.}}(\mathcal{C}) \in \mathsf{sSet}^f_{\mathsf{Joyal}}$ (in which they do not).[138]

9.3.3. Suppose we are given a 2-simplex (28) in C. In the case that $\mathsf{y} = \mathsf{z}$ and $\mathsf{g} = \mathsf{id}_\mathsf{y} = \mathsf{id}_\mathsf{z}$, we say that σ is a **homotopy** from h to f.[139] Likewise, in the case that $\mathsf{x} = \mathsf{y}$ and $\mathsf{f} = \mathsf{id}_\mathsf{x} = \mathsf{id}_\mathsf{y}$, we say that σ is a **homotopy** from h to g.

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[138]We refer the reader to [MG] for a diatribe that attempts to make this point.

[139]The directionality is motivated by the definition of $\mathfrak{C}[2] \in \mathsf{cat}(\mathsf{sSet})$, but it is irrelevant by Exercise 9.3.

Exercise 9.3 (6 points). Show that for any objects $x, y \in C$, these two notions of homotopy give equivalent equivalence relations on the set

$$\lim \left(\begin{array}{c} \mathsf{hom}_{\mathsf{sSet}}(\Delta^1, C) \\ \downarrow {\scriptstyle (s,t)} \\ \mathsf{pt}_{\mathsf{Set}} \xrightarrow[{(x,y)}]{} C_0 \times C_0 \end{array} \right) \in \mathsf{Set}$$

of morphisms from x to y.

Exercise 9.4 (6 points). Given a pair of composable morphisms in C, show that any two composites are homotopic.

We will see a strengthening of Exercise 9.4 in §9.6.2.

The **homotopy category** of C is defined as follows: its objects are those of C and its morphisms are homotopy classes of morphisms in C. Its composition is well-defined by Exercise 9.4.

Exercise 9.5 (4 points). Show that the functor $\mathsf{sSet} \xrightarrow{\mathsf{ho}} \mathsf{cat}$ carries a quasicategory C to its homotopy category.

Exercise 9.6 (4 points). Show that the adjunction

$$(29) \qquad\qquad \mathsf{sSet}_{\mathsf{Joyal}} \overset{\mathsf{ho}}{\underset{\perp}{\rightleftarrows}} \mathsf{cat}_{\mathsf{can}}$$

is a Quillen adjunction.

Both functors in the Quillen adjunction (29) are automatically derived, and moreover the underlying adjunction is a reflective localization. Hence, the derived adjunction is a reflective localization adjunction

$$\mathsf{ho}(\mathsf{Cat}) := \mathsf{sSet}[\mathbf{W}^{-1}_{\mathsf{Joyal}}] \overset{\mathsf{ho}}{\underset{\perp}{\rightleftarrows}} \mathsf{cat}[\mathbf{W}^{-1}] \quad ,$$

whose left adjoint we also refer to as the **homotopy category** functor.

9.3.4. Given objects $x, y \in C$, there is not merely a set of morphisms from x to y, but in fact a *simplicial set* of morphisms from x to y. To define this, we use the internal hom of sSet: given simplicial sets $X, Y \in \mathsf{sSet}$, we may define a simplicial set

$$\underline{\mathsf{hom}}(X, Y) := \underline{\mathsf{hom}}_{\mathsf{sSet}}(X, Y) \in \mathsf{sSet}$$

by the formula

$$\underline{\mathsf{hom}}(X, Y)_n := \mathsf{hom}_{\mathsf{sSet}}(\Delta^n \times X, Y) .^{140}$$

So, a vertex of $\underline{\mathsf{hom}}(X, Y)$ is a morphism $X \to Y$, an edge is a morphism $\Delta^1 \times X \to Y$, and so on.

---

[140] More precisely, $\underline{\mathsf{hom}}(X, Y)_\bullet := \mathsf{hom}_{\mathsf{sSet}}(\mathbf{\Delta}^\bullet \times X, Y)$.

Exercise 9.7 (4 points)**.** Given any simplicial set $Z \in \mathsf{sSet}$, construct a natural isomorphism

$$\mathsf{hom}_{\mathsf{sSet}}(Z, \underline{\mathsf{hom}}(X, Y)) \cong \mathsf{hom}_{\mathsf{sSet}}(Z \times X, Y) \ .$$

Then, for any objects $\mathsf{x}, \mathsf{y} \in \mathsf{C}$, we define

$$\mathsf{hom}_{\mathsf{C}}(\mathsf{x}, \mathsf{y}) := \lim \left(\begin{array}{c} \underline{\mathsf{hom}}_{\mathsf{sSet}}(\Delta^1, \mathsf{C}) \\ \downarrow{\scriptstyle (s,t)} \\ \mathsf{pt}_{\mathsf{sSet}} \xrightarrow[(\mathsf{x},\mathsf{y})]{} \mathsf{C} \times \mathsf{C} \end{array} \right) \in \mathsf{sSet} \ .^{141}$$

We refer to the underlying space of $\mathsf{hom}_{\mathsf{C}}(\mathsf{x}, \mathsf{y}) \in \mathsf{sSet}$ as the ***hom-space*** from x to y in C. It turns out that there is a canonical weak homotopy equivalence

$$\mathsf{hom}_{\mathsf{C}}(x, y) \xrightarrow{\approx} \mathsf{hom}_{\mathfrak{C}(\mathfrak{C})}(x, y)$$

in $\mathsf{sSet}_{\mathsf{KQ}}$, which is an important consistency check.

By definition, a vertex of $\mathsf{hom}_{\mathsf{C}}(\mathsf{x}, \mathsf{y})$ is a morphism from x to y, and more generally an n-simplex of $\mathsf{hom}_{\mathsf{C}}(\mathsf{x}, \mathsf{y})$ is an extension

that makes the diagram commute. Although it is not immediate from the definition, morphisms from x to y are homotopic if and only if there exists a path between them (in either direction) in $\mathsf{hom}_{\mathsf{C}}(\mathsf{x}, \mathsf{y})$.

9.3.5.   We say that a 2-simplex

in C ***witnesses*** g ***as a left inverse of*** f, or equivalently that it ***witnesses*** f ***as a right inverse of*** g.[142] We say that a morphism in C is an ***equivalence*** if it admits both a left inverse and a right inverse.

Exercise 9.8 (6 points)**.** Show that for any equivalence $\mathsf{x} \xrightarrow{\mathsf{f}} \mathsf{y}$ in C, there exists a morphism $\mathsf{y} \xrightarrow{\mathsf{g}} \mathsf{x}$ that is both a left inverse and a right inverse of f.

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[141]Because $\mathsf{sSet}_{\mathsf{Joyal}}$ is compatibly self-enriched (see §9.5.4), this pullback is in fact a homotopy pullback in $\mathsf{sSet}_{\mathsf{Joyal}}$.

[142]These conventions align with the standard convention for function composition.

Exercise 9.9 (4 points)**.** Show that a morphism in $C \in \mathsf{sSet}^f_{\mathsf{Joyal}}$ becomes an isomorphism in $\mathsf{ho}(C) \in \mathsf{cat}$ if and only if it is an equivalence.

9.3.6.   Morphisms in $\mathbf{W}_{\mathsf{Joyal}} \subset \mathsf{sSet}$ are called ***categorical equivalences***. From Exercise 9.9, it is easy to see that equivalences in a quasicategory $C \in \mathsf{sSet}^f_{\mathsf{Joyal}} = \mathsf{sSet}^{cf}_{\mathsf{Joyal}}$ are carried to equivalences in its corresponding simplicial category $\mathfrak{C}(C) \in \mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$. Hence, recalling §9.1.3, we see that a morphism $C \xrightarrow{F} D$ between quasicategories is a categorical equivalence iff the following two conditions hold:

(1) it is ***homotopically fully faithful***, i.e. for all $c, c' \in C$ the induced morphism $\mathsf{hom}_C(c, c') \to \mathsf{hom}_D(F(c), F(c'))$ in sSet is a weak homotopy equivalence (i.e. it lies in $\mathbf{W}_{\mathsf{KQ}} := \mathbf{W}_{\mathsf{w.h.e.}}$);

(2) it is ***homotopically essentially surjective***, i.e. for every $d \in D$ there exists an object $c \in C$ and an equivalence $F(c) \xrightarrow{\sim} d$ in D.

9.3.7.   It is clear that if a quasicategory C is in fact a Kan complex, then $\mathsf{ho}(C)$ is a groupoid. It is an important and nontrivial fact that the converse holds: given a quasicategory C such that $\mathsf{ho}(C)$ is a groupoid, C is in fact a Kan complex. In view of Exercise 9.9, this implies that whereas quasicategories present ∞-categories, Kan complexes present ∞-***groupoids***.[143] Of course, this is just to say that "∞-groupoids" are synonymous with "spaces".[144]

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[143]An ∞-groupoid may also be called an $(\infty, 0)$-***category***: in general, for $0 \leqslant k \leqslant n \leqslant \infty$, an (n, k)-***category*** is an n-category in which every i-morphism is invertible for $k < i \leqslant n$, and an n-***groupoid*** is an $(n, 0)$-category. Another way of saying this is that an (n, k)-category should be a category enriched in $(n-1, k-1)$-categories (at least for $k > 0$ – the simplest fix is to declare here that $0-1 = 0$). Recalling §9.3.4, we see that what we have been calling "∞-categories" may be more properly termed "$(\infty, 1)$-categories".

The n-morphisms in an $(\infty, 1)$-category can be spelled out more concretely as follows. In general, an n-*morphism* should be a sort of morphism between $(n-1)$-morphisms; these $(n-1)$-morphisms are required to be *parallel*, which means that they share the same source and target. We begin by declaring that a *0-morphism* is simply an object, and that any two 0-morphisms are parallel. Then, for $n \geqslant 1$, an n-morphism in an $(\infty, 1)$-category is an $(n-1)$-disk in one of its hom-spaces. More precisely, the boundary $(n-2)$-sphere of this $(n-1)$-cell should be thought of as "hemispherically stratified", with the two hemispheres giving the source and target $(n-1)$-cells (and their agreement on their common boundary encoding the requirement that they be parallel). Note that these n-morphisms are indeed invertible for every $n \geqslant 2$, because $(n-1)$-morphisms in a space are invertible for every $n \geqslant 2$.

[144]This is a "definitional" assertion, which can be phrased more carefully as the assertion that "any good theory of $(\infty, 1)$-categories should have that the $(\infty, 0)$-groupoids are equivalent to spaces". This idea is motivated by the equivalence between the homotopy categories of n-groupoids and homotopy n-types (i.e. topological spaces with homotopy groups vanishing above dimension n) for small values of n; it is due to Grothendieck, and is referred to as the *homotopy hypothesis*.

Now, the identity adjunction defines a Quillen adjunction

$$\mathsf{sSet}_{\mathsf{Joyal}} \xrightarrow[\overset{\mathsf{id}_{\mathsf{sSet}}}{\underset{\mathsf{id}_{\mathsf{sSet}}}{\perp}}]{} \mathsf{sSet}_{\mathsf{KQ}} \quad .^{145}$$

As the left adjoint is automatically derived, the derived counit is a natural equivalence, and so the derived adjunction is a reflective localization adjunction

$$\mathsf{ho}(\mathsf{Cat}) \simeq \mathsf{sSet}[\mathbf{W}_{\mathsf{Joyal}}^{-1}] \xrightarrow[\underset{\perp}{}]{(-)^{\mathsf{gpd}}} \mathsf{sSet}[\mathbf{W}_{\mathsf{KQ}}^{-1}] \simeq \mathsf{ho}(\mathcal{S}) \quad .$$

We leave the derived right adjoint implicit, and we refer to the derived left adjoint as ∞-**groupoid completion**: given an ∞-category $\mathcal{C} \in \mathsf{ho}(\mathsf{Cat})$, the unit morphism $\mathcal{C} \to \mathcal{C}^{\mathsf{gpd}}$ is the initial morphism in $\mathsf{ho}(\mathsf{Cat})$ from $\mathcal{C}$ to an ∞-groupoid.

9.3.8.   There is another way of passing from an ∞-category to an ∞-groupoid: rather than freely adjoining inverses to all morphisms, we can also simply discard the morphisms that are not already invertible. This is called the **maximal subgroupoid** construction, which determines a right adjoint

$$(30) \qquad\qquad\qquad \mathsf{ho}(\mathcal{S}) \xrightarrow[\underset{\iota_0}{\longleftarrow}]{\hookrightarrow} \mathsf{ho}(\mathsf{Cat})$$

to the defining inclusion.

This construction can be modeled as follows. Given a quasicategory C, it turns out that the largest simplicial subset $\iota_0\mathsf{C} \subseteq \mathsf{C}$ containing only the equivalences is in fact a Kan complex. It follows that this construction defines a right adjoint

$$(31) \qquad\qquad\qquad \mathsf{sSet}_{\mathsf{KQ}}^f \xrightarrow[\underset{\iota_0}{\longleftarrow}]{\hookrightarrow} \mathsf{sSet}_{\mathsf{Joyal}}^f$$

to the inclusion from Kan complexes into quasicategories. We will see in Exercise 9.11 that this adjunction upgrades to an adjunction in $\mathsf{cat}(\mathsf{sSet})$, which on homotopy categories (in the sense of §9.1.3) presents the adjunction (30).

9.3.9.   A **subcategory** of an ∞-category $\mathcal{C} \in \mathsf{ho}(\mathsf{Cat})$ is equivalent data to a subcategory of its homotopy category $\mathsf{ho}(\mathcal{C}) \in \mathsf{cat}[\mathbf{W}^{-1}]$.[146] Namely, we can uniquely specify a subcategory $\mathcal{D} \subseteq \mathcal{C}$ by declaring which equivalence classes of objects and which homotopy classes of morphisms among them belong to $\mathcal{D}$, subject to the condition that those homotopy classes of morphisms are closed under composition.

By definition, the morphisms in a subcategory $\mathcal{D} \subseteq \mathcal{C}$ are stable under equivalence. Note that this implies that a subcategory inclusion is necessarily *fully faithful on equivalences*. So

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[145]Of course, the underlying adjunction of this Quillen adjunction is an equivalence. However, we use the notation $\perp$ instead of $\sim$, because this Quillen adjunction should not be thought of as an equivalence (because it is not a Quillen equivalence).

[146]We find the term "sub-∞-category" to be too awkward, so we omit the "∞" in this case. Likewise, we say "subgroupoid" instead of "sub-∞-groupoid".

for example, the forgetful functor $\mathbf{\Delta} \to \mathsf{Fin}$ is *not* the inclusion of a subcategory.[147] And in particular, a subcategory of an ∞-groupoid is an inclusion of path components (and is necessarily a *full* subcategory). We will generally refer to a subcategory of a space as a **subspace**.

Of course, given a quasicategory $\mathtt{C} \in \mathsf{sSet}^f_{\mathsf{Joyal}}$ representing $\mathcal{C} \in \mathsf{ho}(\mathsf{Cat})$, it is straightforward to construct a sub-quasicategory $\mathtt{D} \subseteq \mathtt{C}$ that represents a given subcategory $\mathcal{D} \subseteq \mathcal{C}$.

9.4. Underlying ∞-categories of model categories.

9.4.1. A **relative ∞-category** is a pair $(\mathcal{C}, \mathbf{W})$ consisting of an ∞-category $\mathcal{C} \in \mathsf{ho}(\mathsf{Cat})$ equipped with a subcategory $\mathbf{W} \subseteq \mathcal{C}$. Its ($\infty$-*categorical*) **localization** is the initial morphism

$$\mathcal{C} \longrightarrow \mathcal{C}[\![\mathbf{W}^{-1}]\!]$$

in $\mathsf{ho}(\mathsf{Cat})$ that sends all morphisms in the subcategory $\mathbf{W} \subseteq \mathcal{C}$ to equivalences.[148] Although we have not defined all the relevant terms yet, it should be plausible that this can be defined as an ∞-categorical pushout

$$\mathcal{C}[\![\mathbf{W}^{-1}]\!] \simeq \mathsf{colim} \left(\begin{array}{ccc} \mathbf{W} & \longhookrightarrow & \mathcal{C} \\ \downarrow & & \\ \mathbf{W}^{\mathsf{gpd}} & & \end{array} \right)$$

in Cat (the ∞-category of ∞-categories).

Following §9.3.7, we can construct the ∞-categorical localization explicitly as follows. We first choose any quasicategory $\mathtt{C}$ presenting $\mathcal{C}$, any sub-quasicategory $\mathtt{W} \subseteq \mathtt{C}$ presenting the subcategory $\mathbf{W} \subseteq \mathcal{C}$, and any fibrant resolution $\mathtt{W} \to \mathtt{W}^{f_{\mathsf{KQ}}}$ in $\mathsf{sSet}^f_{\mathsf{KQ}}$. Then, the localization $\mathcal{C}[\![\mathbf{W}^{-1}]\!] \in \mathsf{ho}(\mathsf{Cat})$ is presented in $\mathsf{sSet}_{\mathsf{Joyal}}$ by the pushout

$$\mathsf{colim} \left(\begin{array}{ccc} \mathtt{W} & \longhookrightarrow & \mathtt{C} \\ \downarrow & & \\ \mathtt{W}^{f_{\mathsf{KQ}}} & & \end{array} \right)$$

(which is also a homotopy pushout (recall §8.5.2)). Note that this will not generally be a quasicategory, however.

Exercise 9.10 (4 points). Given a relative ∞-category $(\mathcal{C}, \mathbf{W})$, prove that $\mathsf{ho}(\mathcal{C}[\![\mathbf{W}^{-1}]\!])$ and $\mathsf{ho}(\mathcal{C})[\mathbf{W}^{-1}]$ are canonically isomorphic in $\mathsf{cat}[\mathbf{W}^{-1}]$.

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[147]The utility of this definition is the fact that the inclusion of a subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ is a monomorphism: it is merely a *condition* for a functor to $\mathcal{C}$ to factor through $\mathcal{D}$.

[148]This notation is meant to be suggestive of the ordinary localization of relative categories, but equipped with "higher order" structure. In particular, even when $\mathcal{C}$ is an ordinary category, the ∞-categorical localization $\mathcal{C}[\![\mathbf{W}^{-1}]\!]$ will rarely be so (even up to equivalence).

9.4.2.   We now define our ∞-categories of lasting interest.[149]

First of all, given a model category, we may refer to its ∞-categorical localization as its **underlying ∞-category**: we have seen in §8 that the model structure gives a way of computing hom-sets in the localization, and in fact an elaboration of the same techniques gives a way of computing hom-*spaces* in the ∞-categorical localization (so that a model category may also be seen as a convenient presentation of its underlying ∞-category). In particular, it turns out that Quillen adjunctions give adjunctions between ∞-categories (to be defined more carefully below), and Quillen equivalences give equivalences between ∞-categories; we simply refer to these as the *derived adjunction* (resp. *derived equivalence*) of a Quillen adjunction (resp. Quillen equivalence).

We define **the ∞-category of ∞-categories** to be

$$\mathsf{Cat} := \mathsf{sSet}[\![\mathbf{W}_{\mathsf{Joyal}}^{-1}]\!] \simeq \mathsf{cat}(\mathsf{sSet})[\![\mathbf{W}_{\mathsf{Bergner}}^{-1}]\!] \simeq \mathsf{cat}(\mathsf{Top})[\![\mathbf{W}_{\mathsf{Bergner}}^{-1}]\!] \ .$$

Note that as we have defined it (and ignoring set-theoretic issues), this is an object

$$\mathsf{Cat} \in \mathsf{ho}(\mathsf{Cat})$$

of the homotopy category of ∞-categories. However, we will for the most part work inside of Cat itself.

We define **the ∞-category of spaces** (or **∞-groupoids**) to be

$$\mathcal{S} := \mathsf{Gpd} := \mathsf{sSet}[\![\mathbf{W}_{\mathsf{w.h.e.}}^{-1}]\!] \simeq \mathsf{Top}[\![\mathbf{W}_{\mathsf{w.h.e.}}^{-1}]\!] \ .$$

We write

$$\mathsf{sSet} \xrightarrow{|-|} \mathcal{S} \xleftarrow{\Pi_\infty} \mathsf{Top}$$

for the canonical functors. We refer to the former as the **geometric realization** functor, and we refer to the latter as the **fundamental ∞-groupoid** functor.[150]

We define **the ∞-category of derived R-modules** to be

$$\mathbf{D}_R := \mathsf{Ch}_R[\![\mathbf{W}_{\mathsf{q.i.}}^{-1}]\!] \ .$$

Below, we will upgrade this to the $\Bbbk$-linear ∞-category which we studied in §6 (using the same notation).

We write

$$\mathsf{Cat}_1 := \mathsf{cat}[\![\mathbf{W}^{-1}]\!] \qquad \text{and} \qquad \mathsf{Gpd}_1 := \mathsf{gpd}[\![\mathbf{W}^{-1}]\!] \ .$$

The hom-spaces in these two ∞-categories lie in $\mathsf{Gpd}_1 \subseteq \mathsf{Gpd} := \mathcal{S}$ (so that they are in fact $(2,1)$-categories). We henceforth use the terms **categories** and **groupoids** to refer to objects of these ∞-categories; we use the terms **strict category** and **strict groupoid**

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[149]Note that these are all of an essentially (higher-)algebraic flavor. However, there are also ∞-categories of interest that arise in differential topology, as we indicate in §10.5.

[150]Note that the notation $|-|$ and the terminology "geometric realization" were already given to the functor $\mathsf{sSet} \xrightarrow{|-|} \mathsf{Top}$. It should always be clear from context which notion is intended. The ∞-categorical notions will be explained further in §9.7 (and this abuse of notation is further extended in §9.7.5).

to respectively refer to objects of cat and gpd. So, a category is an equivalence class of strict category. For any categories $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_1$ and any representatives $\mathsf{C}, \mathsf{D} \in \mathsf{cat}$ by strict categories, the hom-groupoid $\mathsf{hom}_{\mathsf{Cat}_1}(\mathcal{C}, \mathcal{D}) \in \mathsf{Gpd}_1$ is the underlying groupoid of the strict groupoid $\underline{\mathsf{hom}}_{\mathsf{cat}}(\mathsf{C}, \mathsf{D}) \in \mathsf{gpd}$ (whose objects are functors and whose morphisms are natural equivalences).

9.4.3. Of course, all of the various derived adjunctions on 1-categorical localizations that we have seen previously upgrade to derived adjunctions on the ∞-categorical localizations introduced in §9.4.2; for instance, we have a diagram

in which the lower adjunction follows from Exercise 9.11 below. Also, we have a diagram

that commutes in the evident senses, in which we may refer to either functor Π_1 as the **fundamental groupoid** functor.[151]

We highlight one other derived adjunction for future use. Recall the composite Quillen adjunction

$$(32) \qquad \mathsf{sSet}_{\mathsf{KQ}} \xrightarrow[\substack{\perp \\ \mathsf{fgt}}]{R\{-\}} (\mathsf{sMod}_R)_{\mathsf{KQ}} \xrightarrow[\sim]{\overline{\mathsf{C}}_\bullet} (\mathsf{Ch}_R^{\geq 0})_{\mathsf{proj}} \xrightarrow[\substack{\perp \\ \tau_{\geq 0}}]{i_{\geq 0}} (\mathsf{Ch}_R)_{\mathsf{proj}}$$

from §8.4. This determines a composite derived adjunction

$$(33) \qquad \mathcal{S} \xrightarrow[\substack{\perp \\ \mathsf{fgt}}]{R\{-\}} \mathbf{D}_R^{\geq 0} \xrightarrow[\substack{\perp \\ \tau_{\geq 0}}]{i_{\geq 0}} \mathbf{D}_R$$

on underlying ∞-categories. We refer to the left adjoint $R\{-\}$ (or the composite left adjoint $i_{\geq 0} \circ R\{-\}$) as the **free derived R-module** functor.[152]

9.5. Enriched ∞-categories.

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[151]Of course, there is also a factorization $\mathsf{Top} \xrightarrow{\Pi_1} \mathsf{gpd} \to \mathsf{Gpd}_1$ through the category of strict groupoids.

[152]Recall from §8.4.5 that the R-homology of a space $X \in \mathcal{S}$ is by definition the homology of the derived R-module $R\{X\} \in \mathbf{D}_R$.

9.5.1. Given a monoidal ∞-category $\mathcal{V}$, there is an ∞-category $\mathsf{Cat}(\mathcal{V})$ of $\mathcal{V}$-**enriched** ∞-**categories**. We do not define these notions in full generality here, but here we explore a few basic instances of lasting interest. We begin by discussing a few general notions.

In general, we use an underline to denote an enrichment.

Given a monoidal ∞-category $\mathcal{V}$ with unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$, applying $\mathsf{Cat}(-)$ to the functor

$$\mathcal{V} \xrightarrow{\mathsf{hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -)} \mathcal{S}$$

determines a forgetful functor

$$\mathsf{Cat}(\mathcal{V}) \xrightarrow{\mathsf{fgt}} \mathsf{Cat}(\mathcal{S}) \simeq \mathsf{Cat} \ ,$$

which carries a $\mathcal{V}$-enriched ∞-category to its **underlying** ∞-**category**.

Suppose that V is a monoidal model category that presents the monoidal ∞-category $\mathcal{V}$. Then, there is a Bergner model structure on the category $\mathsf{cat}(\mathsf{V})$ of ordinary V-enriched categories that presents the ∞-category $\mathsf{Cat}(\mathcal{V})$.

In turn, suppose that $\mathcal{C}$ is a V-enriched model category, i.e. it is a model category and at the same time it is enriched in V in a way that is compatible with its model structure. Then, the V-enriched category

$$\underline{\mathcal{C}^{cf}} \in \mathsf{cat}(\mathsf{V})_{\mathsf{Bergner}}$$

of bifibrant objects is a presentation of a $\mathcal{V}$-enriched ∞-category

$$\underline{\mathcal{C}[\![\mathbf{W}^{-1}]\!]} \in \mathsf{Cat}(\mathcal{V})$$

that enhances the underlying (unenriched) ∞-category of $\mathcal{C}$:

$$\mathsf{fgt}(\underline{\mathcal{C}[\![\mathbf{W}^{-1}]\!]}) \simeq \mathcal{C}[\![\mathbf{W}^{-1}]\!] \ .$$

All of the examples that we discuss fall into this paradigm.

9.5.2. The model categories $\mathsf{sSet}_{\mathsf{KQ}}$ and $\mathsf{Top}_{\mathsf{QS}}$ are both compatibly self-enriched, relative to their cartesian product symmetric monoidal structures.[153] Hence, the ∞-category $\mathcal{S}$ of spaces is presented both by the simplicial category

$$\underline{\mathsf{sSet}^{cf}_{\mathsf{KQ}}} = \underline{\mathsf{sSet}^{f}_{\mathsf{KQ}}} \in \mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$$

of Kan complexes and by the topological category

$$\underline{\mathsf{Top}^{cf}_{\mathsf{QS}}} = \underline{\mathsf{Top}^{c}_{\mathsf{QS}}} \in \mathsf{cat}(\mathsf{Top})_{\mathsf{Bergner}}$$

of retracts of cell complexes.

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[153]In the latter case, one should restrict to a subcategory of "convenient" topological spaces. We elide this point here.

9.5.3. The model category $(\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}}$ is compatibly self-enriched, relative to its tensor product symmetric monoidal structure (as first discussed in §8.2.5). Therefore, $\mathsf{Ch}_{\Bbbk}$-enriched categories are presentations of $\mathbf{D}_{\Bbbk}$-enriched ∞-categories. We write

$$\mathsf{Cat}_{\Bbbk} := \mathsf{Cat}(\mathbf{D}_{\Bbbk}) \ ,$$

and refer to $\mathbf{D}_{\Bbbk}$-enriched ∞-categories as $\Bbbk$-*linear* ∞-*categories* (as first introduced in §6.2.2).

Because the model category $(\mathsf{Ch}_R)_{\mathsf{proj}}$ is compatibly enriched in $(\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}}$, we find that the ∞-category $\mathbf{D}_R := \mathsf{Ch}_R[\![\mathbf{W}_{\mathsf{q.i.}}^{-1}]\!] \in \mathsf{Cat}$ enhances to a $\Bbbk$-linear ∞-category, which is presented by the dg-category

$$\underline{(\mathsf{Ch}_R)_{\mathsf{proj}}^{cf}} = \underline{(\mathsf{Ch}_R)_{\mathsf{proj}}^{c}} = \mathbf{P}_R \in \mathsf{cat}(\mathsf{Ch}_{\Bbbk})_{\mathsf{Bergner}}$$

of projective complexes of R-modules. Of course, as we saw in §6, this same $\Bbbk$-linear ∞-category is also presented by the dg-category

$$\underline{(\mathsf{Ch}_R)_{\mathsf{inj}}^{cf}} = \underline{(\mathsf{Ch}_R)_{\mathsf{inj}}^{f}} = \mathbf{I}_R \in \mathsf{cat}(\mathsf{Ch}_{\Bbbk})_{\mathsf{Bergner}}$$

of injective complexes of R-modules.

We have a Quillen adjunction

$$\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}} \ \overset{\perp}{\underset{}{\rightleftarrows}} \ \mathsf{cat}(\mathsf{Ch}_{\Bbbk})_{\mathsf{Bergner}}$$

obtained by applying $\mathsf{cat}(-)$ to the Quillen adjunction (32). This yields a derived adjunction

(34)
$$\mathsf{Cat} \ \overset{\mathsf{Cat}(\Bbbk\{-\})}{\underset{\mathsf{Cat}(\mathsf{fgt})}{\rightleftarrows}} \ \mathsf{Cat}_{\Bbbk}$$

on underlying ∞-categories (obtained by applying $\mathsf{Cat}(-)$ to the adjunction (33) in the case that $R = \Bbbk$), whose right adjoint is the underlying ∞-category functor. Note that this coincides with the description given in §6.2.3.

9.5.4. The model category $\mathsf{sSet}_{\mathsf{Joyal}}$ is compatibly self-enriched, relative to its cartesian product symmetric monoidal structure. This gives a presentation of the self-enrichment of Cat, i.e. its structure as an $(\infty, 2)$-category:

$$\underline{\mathsf{Cat}} \in \mathsf{Cat}_2 := \mathsf{Cat}(\mathsf{Cat}) \ .$$

Since this structure is crucially important (indeed, it is the reason that we passed from $\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$ to $\mathsf{sSet}_{\mathsf{Joyal}}$), we describe it explicitly at the point-set level.

For a simplicial set $\mathsf{C} \in \mathsf{sSet}_{\mathsf{Joyal}}^{c} = \mathsf{sSet}$ and a quasicategory $\mathsf{D} \in \mathsf{sSet}_{\mathsf{Joyal}}^{f}$, we write

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}) := \underline{\mathsf{hom}}_{\mathsf{sSet}}(\mathsf{C}, \mathsf{D}) \in \mathsf{sSet}_{\mathsf{Joyal}}^{f}.$$

for the simplicial set of morphisms between them: this is a quasicategory, which gives a homotopically well-behaved internal hom.[154] We refer to it as the **quasicategory of functors** from C to D. We refer to its vertices as **functors** and its edges as **natural transformations**.

In particular, the fact that this construction is homotopically well-behaved implies that it is invariant up to categorical equivalence under categorical equivalence among co/fibrant objects in the source/target. In particular, we do not need to assume that C is a quasicategory: if $C \to C^{f_{\mathsf{Joyal}}}$ is a fibrant resolution in $\mathsf{sSet}_{\mathsf{Joyal}}$, then the restriction morphism

$$\mathsf{Fun}(C^{f_{\mathsf{Joyal}}}, D) \longrightarrow \mathsf{Fun}(C, D)$$

is a categorical equivalence – and it even lies in $(\mathbf{W} \cap \mathbf{F})_{\mathsf{Joyal}}$ if the morphism $C \to C^{f_{\mathsf{Joyal}}}$ lies in $(\mathbf{W} \cap \mathbf{C})_{\mathsf{Joyal}}$.[155]

Given a natural transformation $\Delta^1 \times C \to D$, its **component** at an object $x \in C$ is the morphism in D selected by the composite

$$\Delta^1 \xrightarrow{(\mathsf{id}_{\Delta^1}, \mathsf{const}_x)} \Delta^1 \times C \longrightarrow D .$$

Among the natural transformations are the **natural equivalences**; these are simply the equivalences in $\mathsf{Fun}(C, D)$. It is a nontrivial fact that a natural transformation is a natural equivalence if and only if all of its components are equivalences.

Of course, the self-enrichment of Cat restricts to a self-enrichment of Cat_1, which is presented by the self-enrichment of cat.

The above discussion allows us to enhance the category $\mathsf{sSet}^f_{\mathsf{Joyal}}$ of quasicategories to a $\mathsf{sSet}^f_{\mathsf{KQ}}$-enriched category as follows: for any $C, D \in \mathsf{sSet}^f_{\mathsf{Joyal}}$ we define the hom-simplicial set

$$\underline{\hom}_{\mathsf{sSet}^f_{\mathsf{Joyal}}}(C, D) := \iota_0 \mathsf{Fun}(C, D) \in \mathsf{sSet}^f_{\mathsf{KQ}} .^{[156]}$$

Exercise 9.11 (4 points). Show that the adjunction (31) in cat upgrades to an adjunction

$$\underline{\mathsf{sSet}}^f_{\mathsf{KQ}} \underset{\iota_0}{\overset{\perp}{\underset{\longleftarrow}{\longrightarrow}}} \underline{\mathsf{sSet}}^f_{\mathsf{Joyal}}$$

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[154]Namely, the bifunctor

$$\mathsf{sSet}_{\mathsf{Joyal}} \times \mathsf{sSet}_{\mathsf{Joyal}} \xrightarrow{(-) \times (-)} \mathsf{sSet}_{\mathsf{Joyal}}$$

is a left Quillen bifunctor (recall §8.2.5). In particular, the product of cofibrant objects in $\mathsf{sSet}_{\mathsf{Joyal}}$ is again cofibrant. This fails in $\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$ (recall §9.1.5).

[155]This strongly illustrates is the sense in which ∞-categories "want" to only have homotopy-coherently well-defined composition (as opposed to strictly well-defined composition), i.e. the sense in which $\mathsf{sSet}_{\mathsf{Joyal}}$ harmonizes with the true nature of ∞-categories while $\mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$ does not. Namely, if one presents ∞-categories $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}$ as strictly-enriched categories $C, D \in \mathsf{cat}(\mathsf{sSet})_{\mathsf{Bergner}}$, then even to construct the correct hom-set $\hom_{\mathsf{ho}(\mathsf{Cat})}(\mathcal{C}, \mathcal{D})$ one must in particular require that C be "equipped with coherent homotopies" (e.g. $\mathfrak{C}[n]$ is cofibrant whereas $[n]$ is not).

[156]Unfortunately, this enrichment does *not* make $\mathsf{sSet}_{\mathsf{Joyal}}$ into a $\mathsf{sSet}_{\mathsf{KQ}}$-enriched model category. This is one of the reasons that Lurie studies the category of *marked simplicial sets* in [§T.3]: it admits a $\mathsf{sSet}_{\mathsf{KQ}}$-enriched model structure that presents the ∞-category of ∞-categories.

in $\mathsf{cat}(\mathsf{sSet})$.

9.6. Uniqueness in quasicategory theory.

9.6.1. In ordinary category theory, co/representing objects are *unique up to unique iso-morphism* – assuming that they exist. For example, in any category, the subcategory of initial objects is an empty or contractible groupoid.[157] More generally, given a diagram in a category, the category of its colimits is an empty or contractible groupoid.[158]

In ∞-category theory, the notion of "uniqueness" must be slightly reenvisioned. Namely, co/representing objects form **contractible ∞-groupoids** – assuming that they exist. More generally, one often says that a given construction is well-defined "up to a contractible space of choices".

These notions are also captured by the technical usage of the word "essentially": for instance, one may say that an initial object in an ∞-category is *essentially unique*, just as one may say that a functor is *essentially surjective* – the word "essentially" essentially means "up to equivalences" (and equivalences between equivalences, etc., when appropriate). On the other hand, in situations where the meaning is clear, it is common to omit this modifier: one may simply say that an initial object in an ∞-category is *unique*, or that a functor is *surjective*. Similarly, one may often say "is" to mean "is canonically equivalent to".

In this subsection, we illustrate two basic examples of this phenomenon as it is implemented within the context of quasicategories.

9.6.2. Composition in an ∞-category is essentially unique. For example, if we work in $\mathsf{cat}(\mathsf{Top})_{\mathsf{Bergner}}$, then "$\infty$-categorical composition" is explicitly implemented by homotopy-coherent composition, i.e. extension along the cofibration $\mathsf{cat}(|-|)(\mathfrak{C}(\Lambda_1^2 \rightarrowtail \Delta^2))$ (whose source is simply $[2] \in \mathsf{cat} \subset \mathsf{cat}(\mathsf{Top})$). Given a pair of composable morphisms in a topological category, a homotopy-coherent composite is therefore a morphism equipped with a path from their strict composite. So in this context, the "essential uniqueness" claim follows from the fact that for any based topological space $X \in \mathsf{Top}_*$ the based path space $\{\gamma \in \underline{\mathsf{hom}}_{\mathsf{Top}}([0,1], X) : \gamma(0) = x\}$ is contractible.

Let us verify the essential uniqueness of composition within the context of quasicategories. Fix a quasicategory $\mathsf{C} \in \mathsf{sSet}^f_{\mathsf{Joyal}}$ and a map $\Lambda_1^2 \xrightarrow{\varphi} \mathsf{C}$ classifying a pair of composable morphisms. Recall from §9.3.3 that a *composition* is an extension

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{\ \varphi\ } & \mathsf{C} \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array} \quad .$$

---

[157]Recall that a *contractible groupoid* is a category in which all hom-sets are singletons (which is necessarily a groupoid).

[158]This is a special case of the previous example, as a colimit is an initial object in the category of cocones.

This is guaranteed to exist (because $\mathbf{C}$ is a quasicategory), and we saw in Exercise 9.4 that any two composites are homotopic. We can strengthen this as follows.

Exercise 9.12 (2 points). Show that the morphism $\Lambda_1^2 \to \Delta^2$ is an acyclic cofibration in $\mathsf{sSet}_{\mathsf{Joyal}}$.

Then, the compatible self-enrichment of $\mathsf{sSet}_{\mathsf{Joyal}}$ implies that the resulting restriction morphism

$$\mathsf{Fun}(\Delta^2, \mathbf{C}) \longrightarrow \mathsf{Fun}(\Lambda_1^2, \mathbf{C})$$

is an acyclic fibration in $\mathsf{sSet}_{\mathsf{Joyal}}$ between quasicategories (recall §9.5.4). In particular, we obtain a pullback diagram

$$
\begin{array}{ccc}
\mathsf{Fun}(\Delta^2, \mathbf{C})_\varphi & \longrightarrow & \mathsf{Fun}(\Delta^2, \mathbf{C}) \\
\wr\downarrow & & \downarrow\wr \\
\Delta^0 & \xrightarrow{\quad\varphi\quad} & \mathsf{Fun}(\Lambda_1^2, \mathbf{C})
\end{array}
$$

which is also a homotopy pullback diagram in $\mathsf{sSet}_{\mathsf{Joyal}}$ (recall §8.5.2). In particular, we find that the simplicial set of composites $\mathsf{Fun}(\Delta^2, \mathbf{C})_\varphi$ is a quasicategory (since its map to Δ^0 is a fibration) which is categorically equivalent to the Kan complex Δ^0. It follows that this quasicategory is in fact a contractible Kan complex.[159]

9.6.3.   Observe that the functor $[1] \to \mathsf{pt}$ is an ∞-groupoid completion.[160] It follows that for any ∞-category $\mathbf{C} \in \mathsf{Cat}$, the morphism therein classified by a functor $[1] \to \mathbf{C}$ is an equivalence if and only if there exists an extension

$$
\begin{array}{ccc}
[1] & \longrightarrow & \mathbf{C} \\
\downarrow & \nearrow & \\
\mathsf{pt} & &
\end{array}
\quad,
$$

and moreover that the space of such extensions is either empty or contractible. In this situation, it is almost tautological to extract its inverse: it is the lower composite in the diagram

(35)
$$
\begin{array}{ccccc}
S^0 & \longrightarrow & [1] & \longrightarrow & \mathbf{C} \\
\downarrow & & \downarrow & \nearrow & \\
{[1]}^{\mathsf{op}} & \longrightarrow & \mathsf{pt} & &
\end{array}
\quad,
$$

---

[159]This can also be seen as follows (without implicitly using the discussion of §9.3.6): we have $\mathbf{C}_{\mathsf{KQ}} = \mathbf{C}_{\mathsf{Joyal}}$, and so $(\mathbf{W} \cap \mathbf{F})_{\mathsf{KQ}} = (\mathbf{W} \cap \mathbf{F})_{\mathsf{Joyal}}$, and so the acyclic fibrant objects in $\mathsf{sSet}_{\mathsf{KQ}}$ and $\mathsf{sSet}_{\mathsf{Joyal}}$ coincide.

[160]Recall from §9.3.7 that this is just to say that the morphism $\Delta^1 \to \Delta^0$ lies in $\mathbf{W}_{\mathsf{w.h.e.}} =: \mathbf{W}_{\mathsf{KQ}} \subset \mathsf{sSet}_{\mathsf{KQ}}$.

in which the copy of $S^0 \in \mathcal{S} \subset \mathsf{Cat}$ witnesses the fact that the source and target have been exchanged.[161]

The passage to inverses is actually somewhat clearer in the context of quasicategories, because these are closer to the familiar context of strict categories. Then, we model the left square in diagram (35) in Cat as the diagram

$$
\begin{pmatrix} \Delta^0 \sqcup \Delta^0 \longrightarrow \Delta^1 \\ \downarrow \qquad\qquad \downarrow \\ (\Delta^1)^{\mathrm{op}} \longrightarrow J \end{pmatrix} \quad := \quad \mathsf{N} \begin{pmatrix} [0] \sqcup [0] \longrightarrow [1] \\ \downarrow \qquad\qquad \downarrow \\ [1]^{\mathrm{op}} \longrightarrow [1]^{\mathrm{gpd}} \end{pmatrix}
$$

in $\mathsf{sSet}^c_{\mathsf{Joyal}}$, where $[1]^{\mathrm{gpd}} \in \mathsf{gpd} \subset \mathsf{cat}$ denotes the strict groupoid completion of $[1] \in \mathsf{cat}$ (which has two objects and two nonidentity morphisms (which are inverse isomorphisms)).[162]

Now, we have the following elaboration of Exercise 9.8.

Exercise 9.13 (6 points)**.** Show that a morphism $x \xrightarrow{f} y$ in a quasicategory C is an equivalence if and only if there exists an extension

$$
\begin{array}{ccc}
\Delta^1 & \xrightarrow{\ f\ } & \mathsf{C} \\
\downarrow & \nearrow & \\
J & &
\end{array}
\quad .
$$

---

[161]A potential source of confusion here is the fact that every equivalence in $\mathcal{C}$ is equivalent in $\mathsf{Ar}(\mathcal{C})$ to an identity morphism. More generally, given two objects in an ∞-category, note that it is *additional data* to witness them as being equivalent: namely, this is the data of a factorization

$$
\begin{array}{ccc}
S^0 & \longrightarrow & \mathcal{C} \\
\downarrow & \nearrow & \\
\mathsf{pt} & &
\end{array}
\quad ,
$$

which is generally *not* unique even if it exists. (Note that it is *not* invariant under categorical equivalence of quasicategories to ask whether two vertices of a quasicategory are equal. Rather, it is only invariant to ask for a specified equivalence between them. (And it is *not* invariant under categorical equivalence to ask whether this equivalence is an identity morphism, i.e. a degenerate edge.) For instance, if $\mathcal{C} \in \mathsf{Gpd} \subset \mathsf{Cat}$ is an ∞-groupoid, then the space of such factorizations is precisely the *path space* (with fixed endpoints).) In particular, this applies to the source and target of the proposed inverse morphisms in $\mathcal{C}$. (Another illustrative example of this phenomenon is the fact that it is *additional data* to witness a morphism as an endomorphism.)

[162]In particular, note that J is a contractible Kan complex. (Indeed, the geometric realization $|J| \in \mathsf{Top}$ equipped with its resulting CW-complex structure is the standard "hemispherical" CW-structure on the infinite-dimensional sphere.)

In fact, there f is an equivalence, then there is a *contractible Kan complex* of such extensions. This yields a solid diagram

$$
\begin{array}{ccc}
& \mathsf{Fun}(\mathsf{J},\mathsf{C}) & \\
\mathsf{Fun}(\Delta^1,\mathsf{C}) \longleftarrow \mathsf{Fun}(\Delta^1,\mathsf{C})' \dashrightarrow & & \mathsf{Fun}((\Delta^1)^{\mathsf{op}},\mathsf{C})' \longleftrightarrow \mathsf{Fun}((\Delta^1)^{\mathsf{op}},\mathsf{C})
\end{array}
$$

in $\mathsf{sSet}_{\mathsf{Joyal}}$ (using its compatible self-enrichment), where $\mathsf{Fun}(\Delta^1,\mathsf{C})' \cong \mathsf{Fun}((\Delta^1)^{\mathsf{op}},\mathsf{C})'$ denotes the quasicategory of equivalences in C, and choosing a dashed section as indicated determines a functor $\mathsf{Fun}(\Delta^1,\mathsf{C})' \to \mathsf{Fun}((\Delta^1)^{\mathsf{op}},\mathsf{C})'$ carrying an equivalence in C to its inverse.[163] Of course, such a section is not unique; rather such sections assemble into a contractible Kan complex.

9.7. Co/limits in quasicategories.

9.7.1. We now illustrate the notion of a colimit in a quasicategory; this can be easily dualized to obtain the notion of a limit in a quasicategory. Of course, this will be an instance of the discussion of §9.6 regarding uniqueness in quasicategories.

9.7.2. We briefly summarize the situation in strict categories, which we will then mimic in the context of quasicategories. Let $\mathsf{J}, \mathsf{C} \in \mathsf{cat}$ be strict categories. The **right cone** on J is the strict category $\mathsf{J}^{\triangleright} \in \mathsf{cat}$ obtained by freely adjoining a terminal object. Then, given a functor $\mathsf{J} \xrightarrow{F} \mathsf{C}$, a **cocone under** F is an extension

$$
\begin{array}{ccc}
\mathsf{J} & \xrightarrow{\ F\ } & \mathsf{C} \\
\downarrow & \nearrow & \\
\mathsf{J}^{\triangleright} & &
\end{array} \quad .
$$

These assemble into a strict category

$$
\mathsf{C}_{F/} := \lim \left(\begin{array}{ccc} & & \mathsf{Fun}(\mathsf{J}^{\triangleright},\mathsf{C}) \\ & & \downarrow \\ \mathsf{pt} & \xrightarrow{\ F\ } & \mathsf{Fun}(\mathsf{J},\mathsf{C}) \end{array} \right) ,
$$

of which a **colimit** of F is an initial object. Observe that this is precisely the data of a pointwise left adjoint to the functor

$$
\mathsf{Fun}(\mathsf{J},\mathsf{C}) \xleftarrow{\ \mathsf{const}\ } \mathsf{C}
$$

at the object $F \in \mathsf{Fun}(\mathsf{J},\mathsf{C})$.

---

[163]This section can be constructed by working in the model category $(\mathsf{sSet}_{/\mathsf{Fun}(\Delta^1,\mathsf{C})'})_{\mathsf{Joyal}}$ (so that the map is literally a section and not merely a section up to homotopy).

9.7.3.   We fix quasicategories $I, C \in \mathsf{sSet}^f_{\mathsf{Joyal}}$.

We say that an object $x \in C$ is **initial** if for every object $y \in C$, the hom-space $|\mathsf{hom}_C(x, y)| \in$ S is contractible (i.e. the Kan complex $\mathsf{hom}_C(x, y) \in \mathsf{sSet}^f_{\mathsf{KQ}}$ is contractible).

Exercise 9.14 (4 points)**.** Show that the sub-quasicategory of initial objects in a quasicategory is an empty or contractible Kan complex.

The **right cone** on I is defined to be the pushout

$$I^{\triangleright} := \mathsf{colim} \left(\begin{array}{ccc} I \times \Delta^{\{1\}} & \longrightarrow & I \times \Delta^1 \\ \downarrow & & \\ \Delta^0 & & \end{array} \right) \in \mathsf{sSet}_{\mathsf{Joyal}} \,.^{164}$$

Given a functor $I \xrightarrow{F} C$, a **cocone under F** is an extension

$$\begin{array}{ccc} I & \xrightarrow{\ F\ } & C \\ \downarrow & \nearrow & \\ I^{\triangleright} & & \end{array} \,.$$

These assemble into a quasicategory

$$C_{F/} := \mathsf{lim} \left(\begin{array}{ccc} & & \mathsf{Fun}(I^{\triangleright}, C) \\ & & \downarrow \\ \Delta^0 & \xrightarrow{\ F\ } & \mathsf{Fun}(I, C) \end{array} \right) ,$$

of which a **colimit** of F is an initial object. Of course, (once we have defined this notion in quasicategories we will see that) this is likewise a pointwise left adjoint.

In fact (and correspondingly), colimits can be constructed in families, as follows. Let us write $\mathsf{Fun}(I, C)' \subseteq \mathsf{Fun}(I, C)$ for the full sub-quasicategory on those functors that admit colimits and $\mathsf{Fun}(I^{\triangleright}, C)' \subseteq \mathsf{Fun}(I^{\triangleright}, C)$ for the full sub-quasicategory on those functors $I^{\triangleright} \to C$ whose restrictions to I are colimits.

Exercise 9.15 (4 points)**.** Prove that we have a diagram

$$\begin{array}{ccc} \mathsf{Fun}(I^{\triangleright}, C)' & \longrightarrow & \mathsf{Fun}(I^{\triangleright}, C) \\ \wr \downarrow & & \downarrow \\ \mathsf{Fun}(I, C)' & \longrightarrow & \mathsf{Fun}(I, C) \end{array}$$

in $\mathsf{sSet}^f_{\mathsf{Joyal}}$.

---

[164]Note that this is a homotopy pushout in $\mathsf{sSet}_{\mathsf{Joyal}}$. Hence, (as $(-) \times \Delta^1$ preserves categorical equivalences) a categorical equivalence $I \xrightarrow{\approx} J$ in $\mathsf{sSet}_{\mathsf{Joyal}}$ determines a categorical equivalence $I^{\triangleright} \xrightarrow{\approx} J^{\triangleright}$. In particular, our constructions go through equally well even when I is not a quasicategory.

It follows that there exists a section

$$\begin{array}{c} \mathsf{Fun}(\mathtt{I}^\rhd, \mathtt{C})' \\ \Big\downarrow_{\approx} \; \Big\uparrow_{\substack{\cdot \\ \cdot \\ \cdot}} \\ \mathsf{Fun}(\mathtt{I}, \mathtt{C})' \end{array} \quad ,$$

which may be called a **_colimit functor_**. Of course, such a section is not unique; rather such sections assemble into a contractible Kan complex.

We have the following important consistency check for these constructions.

Exercise 9.16 (8 points)**.** Prove that $\mathtt{I}$-indexed colimits in $\mathtt{C}$ – and in particular, their existence – are invariant under categorical equivalence in both variables $\mathtt{I}$ and $\mathtt{C}$.

9.7.4. It turns out that homotopy co/limits in model categories present ∞-categorical co/limits in their underlying ∞-categories. This is nicely illustrated using the model category $\mathsf{Top}_{\mathsf{QS}}$.

Suppose we are given a span

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle g}\Big\downarrow & & \\ Z & & \end{array}$$

in $\mathsf{Top}_{\mathsf{QS}}$, and assume for simplicity that $X \in \mathsf{Top}_{\mathsf{QS}}^c$ is cofibrant. Then, to compute the homotopy pushout, it suffices to replace both maps by cofibrations. That is, for any factorizations

$$\begin{array}{ccc} X & \xrightarrow{\;\;\;f\;\;\;} & Y \\ & \searrow_{f'} \quad \nearrow_{\approx} & \\ & Y' & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{\;\;\;g\;\;\;} & Z \\ & \searrow_{g'} \quad \nearrow_{\approx} & \\ & Z' & \end{array} \quad ,$$

the homotopy pushout is computed as the pushout

$$(36) \qquad \mathsf{colim}\left(\begin{array}{ccc} X & \xrightarrowtail{\;f'\;} & Y' \\ {\scriptstyle g'}\Big\downarrow & & \\ Z' & & \end{array} \right) .$$

Now, a natural construction of the above factorizations is given by the _mapping cylinder_ construction: namely, we can define

$$Y' := \mathsf{colim}\left(\begin{array}{ccc} X \times \{1\} & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \approx}\Big\downarrow & & \\ X \times [0,1] & & \end{array} \right) \qquad \text{and} \qquad Z' := \mathsf{colim}\left(\begin{array}{ccc} X \times \{1\} & \xrightarrow{\;g\;} & Z \\ {\scriptstyle \approx}\Big\downarrow & & \\ X \times [0,1] & & \end{array} \right) ,$$

and define both maps f' and g' to be given by the formula $x \mapsto (x,0)$ (and both unlabeled weak homotopy equivalences to be the evident projection maps). In this case, we find that the colimit (36) in Top corepresents the data of a map from Y, a map from Z, and

a homotopy between their pullbacks to X. Note that this is indeed a model for the ∞-categorical colimit.[165]

9.7.5. Broadly speaking, simplicial objects in ordinary categories tend to be stand-ins for corresponding ∞-categorical colimits. For example, the functor $\mathsf{sSet} \xrightarrow{|-|} \mathcal{S}$ introduced in §9.4.2 is in fact the composite

$$\mathsf{sSet} := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \hookrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S}) \xrightarrow{\mathsf{colim}_{\mathbf{\Delta}^{\mathsf{op}}}} \mathcal{S} \ .$$

Similarly, the canonical functor

$$\mathsf{sMod}_R \longrightarrow \mathsf{sMod}_R[\![\mathbf{W}_{\mathsf{KQ}}^{-1}]\!] \simeq \mathbf{D}_R^{\geqslant 0}$$

is in fact the composite

$$\mathsf{sMod}_R := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Mod}_R) \hookrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathbf{D}_R^{\geqslant 0}) \xrightarrow{\mathsf{colim}_{\mathbf{\Delta}^{\mathsf{op}}}} \mathbf{D}_R^{\geqslant 0} \ .^{166}$$

In ∞-category theory, it is common to refer to the colimit of a simplicial object as its *geometric realization*, and to denote it by $|-| := \mathsf{colim}_{\mathbf{\Delta}^{\mathsf{op}}}(-)$.

9.8. Working model-independently.

9.8.1. Having discussed many point-set aspects of ∞-category theory, we conclude §9 by discussing the notion of *working model-independently* with ∞-categories. We will generally work model-independently for the remainder of this document.

9.8.2. One way of articulating the idea of working model-independently is to say that we work with quasicategories, but never make reference to *specific* quasicategories or their point-set features. Rather, we only perform operations on them that are invariant under categorical equivalence up to categorical equivalence. For example, the discussions of §9.6.2, §9.6.3, and §9.7.3 respectively imply that composition, inversion of equivalences, and formation of colimits can be discussed in a model-independent fashion, and in particular that these operations are *essentially well-defined* in this context (in the sense of §9.6.1). Indeed, every homotopically meaningful operation that one can perform on a quasicategory can be discussed in a model-independent fashion (which is exactly the point – or perhaps even a tautology). This manner of interacting with ∞-categories is formally analogous to the manner in which most mathematicians interact with the foundations of set theory.

---

[165]More precisely, (assuming Y and Z are cofibrant) this defines a homotopy-coherent commutative square in the topological category $\underline{\mathsf{Top}}_{\mathsf{QS}}^c \in \mathsf{cat}(\mathsf{Top})_{\mathsf{Bergner}} = \mathsf{cat}(\mathsf{Top})_{\mathsf{Bergner}}^f$ (and therefore also in the corresponding quasicategory $\mathsf{N}^{\mathsf{h.c.}}(\mathsf{cat}(\mathsf{Sing})(\underline{\mathsf{Top}}_{\mathsf{QS}}^c)) \in \mathsf{sSet}_{\mathsf{Joyal}}^f$), which is an ∞-categorical pushout diagram in the sense we have just defined.

[166]Note too that the inclusion $\mathbf{D}_R^{\geqslant 0} \hookrightarrow \mathbf{D}_R$ preserves colimits, being a left adjoint.

9.8.3.   If one wishes to be more philosophically aggressive, one could alternatively articulate the idea of working model-independently as saying that one is employing a Platonic theory of ∞-categories that exists independently of any particular model. One aim of the emerging field of homotopy type theory [Pro13] is to give mathematically rigorous foundations for such a philosophical posture.

9.8.4.   The model-independent notion of ∞-categories will be amplified in §10.2, where (for both technical and philosophical reasons) we will discuss a presentation of ∞-categories using the ∞-category of ∞-groupoids. Although it is of course logically circular, we find this to be the most "conceptually correct" definition of ∞-categories.[167]

9.8.5.   As a practical linguistic matter, when working model-independently we will essentially always omit technical uses of the word "essentially" (as discussed in § 9.6.1). For example, we will simply refer to *the* composite of a pair of composable morphisms, rather than saying "a composite, which is unique up to a contractible space of choices". Likewise, on a notational level we may implicitly invert equivalences, e.g. we may write $A \to B \xleftarrow{\sim} C \to D$ to denote a morphism $A \to D$ obtained by choosing an inverse (if it arises most naturally in this way).

9.8.6.   In particular, in this paradigm we generally implicitly consider strict categories as categories. So for example, "the ∞-category of simplicial objects in an ∞-category $\mathcal{C}$" may be interpreted to mean "the quasicategory $\mathsf{Fun}(\mathsf{N}(\mathbf{\Delta}^{\mathsf{op}}), \mathsf{C})$, for any particular quasicategory $\mathsf{C} \in \mathsf{sSet}^f_{\mathsf{Joyal}}$ representing the ∞-category $\mathcal{C} \in \mathsf{Cat}$" (and for any strict category $\mathbf{\Delta}^{\mathsf{op}} \in \mathsf{cat}$ representing the category $\mathbf{\Delta}^{\mathsf{op}} \in \mathsf{Cat}_1$ (e.g. a skeleton)).

9.8.7.   Let C be a model category with underlying ∞-category $\mathcal{C} := \mathsf{C}[\![\mathbf{W}^{-1}]\!]$. At least when C is sufficiently nice, for any strict category $\mathcal{I} \in \mathsf{cat}$ the functor ∞-category $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$ is the localization of $\mathsf{Fun}(\mathcal{I}, \mathsf{C})$ at the componentwise weak equivalences.[168]   For example, the ∞-category $\mathsf{s}\mathcal{S} := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$ is the localization of the category $\mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Top})$ at the componentwise weak homotopy equivalences. This fact may be helpful inasmuch as it gives a concrete way of thinking about certain functor ∞-categories (e.g. the ∞-category of simplicial spaces, which features heavily in §10), but of course actually choosing such rectifications is antithetical to the practice of model-independent mathematics.

10. Monoids, complete Segal spaces, and (symmetric) monoidal ∞-categories

10.1. **Monoids and groups.**

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[167]More generally (and less circularly), we find $\mathbf{\Theta}_n$-spaces to be the most "conceptually correct" definition of (∞, n)-categories [Rez10].

[168]This is a fairly nontrivial fact, and may be referred to as *rectification of diagrams.*

10.1.1. Let $\mathcal{X}$ be an ∞-category that admits finite products. A **monoid** (or **monoid object**, or even ∞-**monoid object**) in $\mathcal{X}$ is a simplicial object $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{M_\bullet} \mathcal{X}$ satisfying the following conditions:

(1) $M_\bullet$ is **reduced**, i.e. the canonical morphism $M_0 \to \mathsf{pt}_{\mathcal{X}}$ is an equivalence;
(2) $M_\bullet$ is **Segal**, i.e. for any $n \geqslant 2$ the nth **Segal map**

$$M_n := M_{\{0<1<\cdots<n\}} \longrightarrow M_{\{0<1\}} \times M_{\{1<2\}} \times \cdots \times M_{\{n-1<n\}} =: (M_1)^{\times n}$$

is an equivalence.

One thinks of $M := \mathsf{fgt}(M_\bullet) := M_1 \in \mathcal{X}$ as the underlying object, and of the maps

$$\mu : M_1 \times M_1 \xleftarrow{\sim} M_2 \xrightarrow{M_{[2] \leftarrow \{0<2\}}} M_1 \qquad \text{and} \qquad \eta : \mathsf{pt} \xleftarrow{\sim} M_0 \xrightarrow{M_{[0] \leftarrow [1]}} M_1$$

respectively as the multiplication and unit maps. The higher data record the *homotopy-coherent associativity* of the multiplication. We write $\mathsf{Mon}(\mathcal{X}) \subseteq \mathsf{s}\mathcal{X}$ for the full subcategory on the monoid objects. We simply write $\mathsf{Mon} := \mathsf{Mon}(\mathcal{S})$, and refer to its objects as **monoids** or ∞-**monoids**. A monoid object in Cat is called a **monoidal ∞-category**; we will discuss these further in §10.3.

It is common to abuse terminology by simply referring to an object $M \in \mathcal{X}$ as a monoid object. To be precise, this refers to a specific monoid object $M_\bullet \in \mathsf{Mon}(\mathcal{X})$ equipped with an equivalence $M \simeq M_1$ in $\mathcal{X}$, i.e. a lift of M through the forgetful functor $\mathsf{Mon}(\mathcal{X}) \xrightarrow{\mathsf{fgt}} \mathcal{X}$.

Exercise 10.1 (6 points).
(a) Prove that ∞-monoids in Set are equivalent to ordinary monoids.
(b) Prove that if $\mathcal{X}$ is a 1-category (e.g. Set), then the composite functor

$$\mathsf{Mon}(\mathcal{X}) \lhook\joinrel\longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{X}) \longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 3}, \mathcal{X})$$

is fully faithful.

In fact, Exercise 10.1(b) generalizes as follows: for any $n \geqslant 1$, if $\mathcal{X}$ is an $(n, 1)$-category, then the composite functor

$$\mathsf{Mon}(\mathcal{X}) \lhook\joinrel\longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{X}) \longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+2}, \mathcal{X})$$

is fully faithful. A particular case of interest is the $(2, 1)$-category Cat_1, in which situation Mac Lane's coherence theorem effectively asserts that it suffices to verify the *pentagon axiom*

for 4-fold associativity [ML63].[169] Of course, when $\mathfrak{X}$ is an arbitrary $(\infty, 1)$-category we must use the entire indexing category $\mathbf{\Delta}^{\mathsf{op}}$.

Exercise 10.2 (4 points)**.** Given monoid objects $M, N \in \mathsf{Mon}(\mathfrak{X})$, endow the product $M \times N \in \mathfrak{X}$ with the structure of a monoid object.

10.1.2.   Given any ∞-category $\mathcal{C} \in \mathsf{Cat}$ equipped with a distinguished object $x \in \mathcal{C}$, the endomorphism space

$$\mathsf{end}_{\mathcal{C}}(x) := \mathsf{hom}_{\mathcal{C}}(x, x) \in \mathcal{S}$$

admits a canonical ∞-monoid structure via composition. We simply write $\mathsf{Cat}_* \xrightarrow{\mathsf{end}} \mathsf{Mon}$ for this functor from pointed ∞-categories to ∞-monoids.

Exercise 10.3 (4 points)**.** Construct a cosimplicial object in pointed ∞-categories $E^{\bullet} \in \mathsf{cCat}_* := \mathsf{Fun}(\mathbf{\Delta}, \mathsf{Cat}_*)$ such that there is a natural equivalence

$$\mathsf{hom}_{\mathsf{Cat}_*}(E^{\bullet}, (\mathcal{C}, x)) \simeq \mathsf{end}_{\mathcal{C}}(x)_{\bullet}$$

in $\mathsf{Mon} \subset \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$ (i.e. such that $E^{\bullet}$ corepresents the composite functor $\mathsf{Cat}_* \xrightarrow{\mathsf{end}} \mathsf{Mon} \hookrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$).

As will be clear from the discussion of §10.2, there exists a fully faithful left adjoint

$$\mathsf{Mon} \underset{\mathsf{end}}{\overset{\mathfrak{B}}{\rightleftarrows}} \mathsf{Cat}_*$$

from ∞-monoids to pointed ∞-categories, which we refer to as the (***categorical***) ***deloop*** functor. This carries an ∞-monoid M to a pointed ∞-category $(* \in \mathfrak{B}M) \in \mathsf{Cat}_*$, which is characterized by the fact that it has a single equivalence class of object and that $\mathsf{end}_{\mathfrak{B}M}(*) \simeq M$ (as ∞-monoids).[170]

10.1.3.   Note that the functor $\mathcal{S} \xrightarrow{\pi_0} \mathsf{Set}$ preserves finite products. An ∞-monoid $M \in \mathsf{Mon}$ is called an ∞-***group*** if the ordinary monoid $\pi_0(M) \in \mathsf{Mon}(\mathsf{Set})$ is in fact a group. We write $\mathsf{Grp} \subset \mathsf{Mon}$ for the full subcategory on the ∞-groups. Given an ∞-group G, we

---

[169]There is a subtlety here: the classical definition of a monoidal category is a presentation of an object of $\mathsf{Mon}(\mathsf{Cat}_1)$ via data in cat. In particular, the associator and left and right unitors are natural isomorphisms in cat that are intended to present paths in hom-groupoids in Cat_1 that witness the *homotopy-coherent* associativity and unitality of the monoidal structure. Mac Lane's coherence theorem may be read as asserting that these data in cat do indeed suffice to define objects of $\mathsf{Mon}(\mathsf{Cat}_1)$. Objects of $\mathsf{Mon}(\mathsf{cat})$ may be called (strictly) monoidal strict categories; these are defined by requiring these natural isomorphisms to be natural *equalities*.

[170]Note that the maximal subgroupoid of $\mathfrak{B}M$ will be $\mathfrak{B}$ of the maximal sub-(∞-)group of M (as implicitly defined in §10.1.3).

write $\mathsf{B}G := \mathfrak{B}G \in \mathcal{S}_*$ for the corresponding pointed ∞-groupoid, which we refer to as its **classifying space** or its **deloop**.[171]

Given any ∞-groupoid $X \in \mathcal{S}$ equipped with a distinguished object $x \in X$, we have a canonical identification

$$\mathsf{end}_X(x) \simeq \Omega X := \Omega_x X := \mathsf{hom}_{\mathcal{S}_*}(S^1, (X, x))$$

with the **based loopspace**, under which composition of morphisms corresponds to concatenation of paths. Hence, this ∞-monoid is in fact an ∞-group, because the ordinary monoid $\pi_0(\Omega_x X) =: \pi_1(X, x)$ is a group. A fundamental theorem in homotopy theory [Sta63] asserts that *all* ∞-groups arise in this way. Indeed, the functors $\mathfrak{B}$ and B participate in a diagram

$$
\begin{array}{ccccc}
\mathsf{Mon} & \underset{\mathsf{end}}{\overset{\mathfrak{B}}{\underset{\sim}{\rightleftarrows}}} & \mathsf{Cat}_*^{\mathsf{one\ obj}} & \underset{}{\overset{\perp}{\rightleftarrows}} & \mathsf{Cat}_* \\[2mm]
(-)^{\mathsf{gp}} \Big\uparrow\!\!\Big\downarrow \mathsf{max'l\ subgp} & & (-)^{\mathsf{gpd}} \Big\uparrow\!\!\Big\downarrow \iota_0 & & (-)^{\mathsf{gpd}} \Big\uparrow\!\!\Big\downarrow \iota_0 \quad, \\[2mm]
\mathsf{Grp} & \underset{\Omega}{\overset{\mathsf{B}}{\underset{\sim}{\rightleftarrows}}} & \mathcal{S}_*^{\geqslant 1} & \underset{}{\overset{\perp}{\rightleftarrows}} & \mathcal{S}_*
\end{array}
$$

in which the full subcategory $\mathsf{Cat}_*^{\mathsf{one\ obj}} \subseteq \mathsf{Cat}_*$ (resp. $\mathcal{S}_*^{\geqslant 1} \subseteq \mathcal{S}_*$) is that on those pointed ∞-categories (resp. ∞-groupoids) that have exactly one equivalence class of object (and the right adjoints to their inclusions are given by restriction to the distinguished objects), and which commutes in the evident ways.[172] In particular, this explains the term "deloop" for the functor B: it carries an ∞-group to the unique pointed connected space of which it is the based loopspace.

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[171]The notation $\mathfrak{B}$ instead of B is meant to emphasize the categorical nature of the former. More generally, for an ∞-monoid M, the notation $\mathsf{B}M$ is sometimes used to denote the ∞-groupoid completion $(\mathfrak{B}M)^{\mathsf{gpd}} \in \mathcal{S}_*$. These letters "B" both stand for "bar construction", although this can be somewhat confusing: here we generally take "bar construction" to mean "corresponding simplicial object" (so that we have *defined* ∞-monoids in terms of their bar constructions), whereas historically the bar construction often referred to the geometric realization thereof. Namely, in terms of simplicial objects, for any ∞-monoid $M \in \mathsf{Mon} \subset \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathcal{S})$ we have an equivalence

$$(\mathfrak{B}M)^{\mathsf{gpd}} \simeq |M_\bullet| := \mathsf{colim}\left(\Delta^{\mathsf{op}} \xrightarrow{M_\bullet} \mathcal{S}\right)$$

of pointed spaces, with the basepoint on the right side coming from the canonical morphism $\mathsf{pt} \simeq G_0 \to |G_\bullet|$ to the colimit.

[172]A remarkable theorem [McD79] asserts that the long composite in the commutative diagram

$$
\begin{array}{ccc}
\mathsf{Mon}(\mathsf{Set}) \lhook\joinrel\longrightarrow \mathsf{Mon} & \xrightarrow[\sim]{\mathfrak{B}} & \mathsf{Cat}_*^{\mathsf{one\ obj}} \\[2mm]
(-)^{\mathsf{gp}} \Big\downarrow & & \Big\downarrow (-)^{\mathsf{gpd}} \\[2mm]
\mathsf{Grp} & \xrightarrow[\mathsf{B}]{\sim} & \mathcal{S}_*^{\geqslant 1}
\end{array}
$$

is surjective: every pointed connected space is equivalent to the classifying space of an ordinary monoid. (This result may also be read as an indication of the essential complexity of (ordinary) monoids: whereas groups present homotopy 1-types, monoids present all homotopy types.)

10.2. **Complete Segal spaces.**

10.2.1.   Building on the discussion of §10.1, we discuss a convenient and conceptually clar-ifying presentation of ∞-categories as ***complete Segal spaces***.[173] This discussion may be thought of as an ∞-categorical (in particular, model-independent) analog of §9.2.1.

10.2.2.   Recall the functor $\mathbf{\Delta} \xrightarrow{[\bullet]} \mathsf{cat} \to \mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}$. The ∞-***nerve*** functor is the restricted Yoneda embedding, i.e. the composite

$$\mathsf{N}_\infty : \mathsf{Cat} \xrightarrow{\hom_{\mathsf{Cat}}(=,-)} \mathsf{Fun}(\mathsf{Cat}^{\mathsf{op}}, \mathcal{S}) \xrightarrow{([\bullet]^{\mathsf{op}})^*} \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S}) =: \mathsf{s}\mathcal{S} \ .$$

In other words, given an ∞-category $\mathcal{C} \in \mathsf{Cat}$, its ∞-nerve is a simplicial space $\mathsf{N}_\infty(\mathcal{C})_\bullet \in \mathsf{s}\mathcal{S}$ whose n^{th} space is the space

$$\mathsf{N}_\infty(\mathcal{C})_n := \hom_{\mathsf{Cat}}([n], \mathcal{C}) \in \mathcal{S}$$

of n composable morphisms in $\mathcal{C}$.

10.2.3.   Let $C \in \mathsf{s}\mathcal{S}$ be a simplicial space. We say that C is a ***Segal space*** if for any $n \geqslant 2$ the $n^{\boldsymbol{th}}$ ***Segal map***

$$C_n := C_{\{0<1<\cdots<n\}} \longrightarrow C_{\{0<1\}} \underset{C_{\{1\}}}{\times} C_{\{1<2\}} \underset{C_{\{2\}}}{\times} \cdots \underset{C_{\{n-1\}}}{\times} C_{\{n-1<n\}} =: C_1 \underset{C_0}{\times} \cdots \underset{C_0}{\times} C_1$$

is an equivalence.[174] We write

$$\mathsf{S}\mathcal{S} \subset \mathsf{s}\mathcal{S}$$

for the full subcategory on the Segal spaces.

Exercise 10.4 (6 points)**.** Using the Joyal model structure, prove that we have an equiva-lence

$$\mathsf{colim}\left(\{0 < 1\} \coprod_{\{1\}} \{1 < 2\} \coprod_{\{2\}} \cdots \coprod_{\{n-1\}} \{n - 1 < n\} \right) \xrightarrow{\sim} [n]$$

in Cat.

It follows from Exercise 10.4 that the ∞-nerve of an ∞-category is a Segal space. Conversely, given any Segal space $C \in \mathsf{S}\mathcal{S}$, we can make the following heuristic definition of a corre-sponding ∞-category (which we will make rigorous momentarily): its objects are the points

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[173]Complete Segal spaces were originally introduced as the bifibrant objects in a model structure on the category $\mathsf{ssSet} := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})) \simeq \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}} \times \mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$ of bisimplicial sets [Rez01]. This model structure is a "left Bousfield localization" of a different model structure $\mathsf{s}(\mathsf{sSet}_{\mathsf{KQ}})_{\mathsf{Reedy}}$, which presents $\mathsf{s}\mathcal{S}$; this situation is a model-categorical counterpart of the present ∞-categorical discussion.

[174]Note that this reduces to the definition of an ∞-monoid in the case that C is reduced (i.e. that $C_0 \xrightarrow{\sim} \mathsf{pt}$).

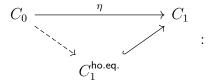of $C_0 \in \mathcal{S}$, for any $x, y \in C_0$ we define the hom-space

$$\mathsf{hom}_C(x, y) := \lim \left(\begin{array}{c} C_1 \\ \downarrow \\ \mathsf{pt} \xrightarrow[(x,y)]{} C_0 \times C_0 \end{array} \right) \in \mathcal{S} \ ,$$

and composition and identities are induced by the maps

$$\mu : C_1 \underset{C_0}{\times} C_1 \xleftarrow{\ \sim\ } C_2 \xrightarrow{C_{[2] \leftarrow \{0 < 2\}}} C_1 \qquad \text{and} \qquad \eta : C_0 \xrightarrow{C_{[0] \leftarrow [1]}} C_1 \ .$$

In particular, we consider C_n as "the space of n composable morphisms in C", and the Segal condition guarantees that this terminology is sound.

Given this definition, it is easy to define the notion of a **homotopy equivalence** in the Segal space C: it is a point of C_1 that admits left and right homotopy inverses. This notion is clearly invariant under equivalence (i.e. paths in C_1), so that we obtain a subspace $C_1^{\mathsf{ho.eq.}} \subseteq C_1$ of homotopy equivalences. Moreover, it is immediate that we have a factorization

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \ \eta\ \ } & C_1 \\ & \searrow \qquad \nearrow & \\ & C_1^{\mathsf{ho.eq.}} & \end{array} \quad :$$

every identity morphism is trivially a homotopy equivalence. We say that the Segal space C is **complete** if the morphism $C_0 \to C_1^{\mathsf{ho.eq.}}$ is an equivalence.[175] Heuristically, this says that the "externally-defined" space of objects C_0 is equivalent to the space of equivalences in the "internally-defined" ∞-category theory of C. We write

$$\mathsf{CSS} \subset \mathsf{SS}$$

for the full subcategory on the complete Segal spaces.

10.2.4. Because $[1]^{\mathsf{gpd}} \xrightarrow{\sim} [0]$, the ∞-nerve of an ∞-category is in fact a *complete* Segal space. In fact, the ∞-nerve functor defines an equivalence

$$\mathsf{Cat} \xrightarrow[\sim]{\ \mathsf{N}_\infty\ } \mathsf{CSS} \ .$$

Its inverse functor is the restriction to the subcategory $\mathsf{CSS} \subset \mathsf{sS}$ of the left Kan extension

$$\begin{array}{ccc} \boldsymbol{\Delta} & \xhookrightarrow{\ [\bullet]\ } & \mathsf{Cat} \\ {\scriptstyle \Delta^\bullet} \downarrow & \nearrow {\scriptstyle L} & \\ \mathsf{sS} & & \end{array} \quad ,$$

which carries a simplicial space $X \in \mathsf{sS}$ to the colimit

$$L(X) := \mathsf{colim} \left(\boldsymbol{\Delta}_{/X} \xrightarrow{\ \mathsf{fgt}\ } \boldsymbol{\Delta} \xrightarrow{\ [\bullet]\ } \mathsf{Cat} \right) \in \mathsf{Cat} \ .$$

---

[175]In the context of homotopy type theory [Pro13], this condition is also referred to as **univalence**.

So, the ∞-category $L(X) \in$ Cat is built by gluing together copies of the ∞-categories $[n] \in$ Cat, one for each point of $X_n \in \mathcal{S}$. Indeed, the functor L can also be thought of as defining a left adjoint

$$\mathsf{s}\mathcal{S} \underset{\longleftarrow}{\overset{L}{\underset{\perp}{\dashrightarrow}}} \mathsf{CSS} \underset{\mathsf{N}_\infty}{\overset{L}{\underset{\sim}{\rightleftarrows}}} \mathsf{Cat} \quad .$$

This gives an ∞-categorical perspective on the Joyal model structure: the localization functor $\mathsf{sSet} \to \mathsf{sSet}[\![\mathbf{W}_{\mathsf{Joyal}}^{-1}]\!] =:$ Cat coincides with the composite

$$\mathsf{sSet} \lhook\joinrel\longrightarrow \mathsf{s}\mathcal{S} \overset{L}{\longrightarrow} \mathsf{Cat} \ .^{176}$$

As it turns out, ∞-groupoids admit a particularly tautological description under the presentation of ∞-categories as complete Segal spaces.

Exercise 10.5 (12 points)**.**
   (a) Prove that if $\mathcal{G} \in$ Gpd is an ∞-groupoid, then $\mathsf{N}_\infty(\mathcal{G}) \in \mathsf{s}\mathcal{S}$ is the constant simplicial space $\mathbf{\Delta}^{\mathsf{op}} \overset{\mathsf{const}_\mathcal{G}}{\longrightarrow} \mathcal{S}$.
   (b) Prove directly that constant simplicial spaces are complete Segal spaces.
   (c) Prove that the functor $\mathcal{S} \overset{\mathsf{const}}{\longrightarrow} \mathsf{s}\mathcal{S}$ is fully faithful.
   (d) Prove that if a Segal space corresponds to an ∞-groupoid, then it is a complete Segal space if and only if it is constant.

Altogether, Exercise 10.5 yields a commutative diagram

$$\begin{array}{ccc} \mathsf{Cat} & \overset{\mathsf{N}_\infty}{\underset{\sim}{\longrightarrow}} & \mathsf{CSS} \\ \big\uparrow & & \big\uparrow{\scriptstyle\mathsf{const}} \\ \mathsf{Gpd} & \underset{\mathsf{N}_\infty}{\overset{\sim}{\longrightarrow}} & \mathcal{S} \ . \end{array}$$

Exercise 10.6 (6 points)**.** Describe the adjoints

$$\mathsf{Cat} \overset{(-)^{\mathsf{gpd}}}{\underset{\iota_0}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}} \mathcal{S}$$

in terms of complete Segal spaces.

Exercise 10.7 (4 points)**.** Given a space $X \in \mathcal{S}$, construct a Segal space with 0th space X whose corresponding ∞-category has all hom-spaces contractible.[177]

10.2.5.   The presentation of ∞-categories as complete Segal spaces is also compatible with the internal hom of simplicial spaces, which is defined by the formula

$$\underline{\mathsf{hom}}_{\mathsf{s}\mathcal{S}}(X, Y)_n := \mathsf{hom}_{\mathsf{s}\mathcal{S}}(\Delta^n \times X, Y) \ .$$

Namely, given ∞-categories $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}$, we have an equivalence

$$\mathsf{N}_\infty(\mathsf{Fun}(\mathcal{C}, \mathcal{D})) \simeq \underline{\mathsf{hom}}_{\mathsf{s}\mathcal{S}}(\mathsf{N}_\infty(\mathcal{C}), \mathsf{N}_\infty(\mathcal{D}))$$

of complete Segal spaces. In fact, more is true.

Exercise 10.8 (8 points)**.** Fix a simplicial space $X \in \mathsf{s}\mathcal{S}$ and a complete Segal space $C \in \mathsf{CS}\mathcal{S}$.

(a) Show that the internal hom-object $\underline{\mathsf{hom}}_{\mathsf{s}\mathcal{S}}(X, C) \in \mathsf{s}\mathcal{S}$ is again a complete Segal space.[178]

(b) Show that $\underline{\mathsf{hom}}_{\mathsf{s}\mathcal{S}}(X, C) \simeq \mathsf{N}_\infty(\mathsf{Fun}(L(X), L(C)))$.

Exercise 10.8(b) should be seen as an analog of the corresponding fact for quasicategories discussed in §9.5.4.

10.2.6.   In fact, Segal spaces (which are not necessarily complete) *also* carry intrinsic homotopical meaning: namely, a Segal space $C \in \mathsf{S}\mathcal{S}$ is equivalent data to its underlying ∞-category $L(C) \in \mathsf{Cat}$ along with the data of the resulting functor $C_0 \to L(C)$ from its 0th space. Conversely, any surjective functor from an ∞-groupoid to an ∞-category arises in this way. Indeed, writing $\mathsf{Ar}^{\mathsf{surj}}(\mathsf{Cat}) \subset \mathsf{Ar}(\mathsf{Cat})$ for the full subcategory on the surjective functors and writing

$$
\begin{array}{ccc}
\mathsf{Ar}^{\mathsf{surj}}(\mathsf{Cat})^{|\mathsf{Gpd}} & \longrightarrow & \mathsf{Gpd} \\
\downarrow & & \downarrow \\
\mathsf{Ar}^{\mathsf{surj}}(\mathsf{Cat}) & \longrightarrow \mathsf{Ar}(\mathsf{Cat}) \xrightarrow{\ s\ } & \mathsf{Cat}
\end{array}
$$

---

[177]Thus, this Segal space presents the (-1)-truncation

$$\tau_{\leqslant(-1)}X := \left\{ \begin{array}{ll} \mathsf{pt}\ , & X \not\simeq \varnothing \\ \varnothing\ , & X \simeq \varnothing \end{array} \right. \in \mathcal{S} \subset \mathsf{Cat}\ .$$

[178]Because of this, one may say that the subcategory $\mathsf{CS}\mathcal{S} \subset \mathsf{s}\mathcal{S}$ is an *exponential ideal*.

for the pullback, we have a diagram

that commutes in the evident senses.

Exercise 10.9 (6 points). Construct the equivalence $\mathsf{Ar}^{\mathsf{surj}}(\mathsf{Cat})^{|\mathsf{Gpd}} \xrightarrow{\sim} \mathsf{S\!S}$.

In fact, the inclusion $\mathsf{S\!S} \subset \mathsf{s\!S}$ admits a left adjoint, so that we obtain a diagram of adjunctions

$$\mathsf{s\!S} \underset{\longleftarrow}{\overset{\bot}{\dashrightarrow}} \mathsf{S\!S} \underset{\longleftarrow}{\overset{\bot}{\dashrightarrow}} \mathsf{C\!S\!S} \quad .$$

10.3. Monoidal ∞-categories.

10.3.1. Recall from §10.1 that by definition, a ***monoidal ∞-category*** is a reduced Segal object in simplicial ∞-categories.

Fix a monoidal ∞-category $\mathcal{V}_\bullet \in \mathsf{Mon}(\mathsf{Cat}) \subset \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat})$. We refer to the bifunctor

$$\mathcal{V} \times \mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_1 \xleftarrow{\sim} \mathcal{V}_2 \xrightarrow{\mu} \mathcal{V}_1 =: \mathcal{V}$$

as its ***monoidal structure***; we generally denote this by $(-) \boxtimes (-)$. Recall that this is homotopy-coherently associative, as encoded by the simplicial object $\mathcal{V}_\bullet$. We refer to the object selected by the functor

$$\mathsf{pt} \xleftarrow{\sim} \mathcal{V}_0 \xrightarrow{\eta} \mathcal{V}_1 =: \mathcal{V}$$

as the ***unit object***. We generally denote this by $\mathbb{1}_\mathcal{V}$, or possibly simply by $\mathbb{1} := \mathbb{1}_\mathcal{V}$ if $\mathcal{V}$ itself is clear.

Exercise 10.10 (2 points). Prove that the diagram

canonically commutes.

A morphism in $\mathsf{Mon}(\mathsf{Cat})$ is called a ***monoidal functor***. We will use the fact that $\mathsf{Mon}(\mathsf{Cat})$ is actually an $(\infty, 2)$-category: given monoidal ∞-categories $\mathcal{V}, \mathcal{W} \in \mathsf{Mon}(\mathsf{Cat})$, there is an ∞-category $\mathsf{Fun}^{\otimes}(\mathcal{V}, \mathcal{W})$ whose objects are monoidal functors and whose morphisms are *monoidal* natural transformations.[179]

---

[179]These notions will be implicitly defined in §11.

10.3.2. Given a monoidal category $\mathcal{V}$, there is a classical notion of an ***algebra object*** in $\mathcal{V}$: this is an object $A \in \mathcal{V}$ equipped with morphisms $A \boxtimes A \xrightarrow{\mu} A$ and $\mathbb{1}_\mathcal{V} \xrightarrow{\eta} A$, subject to the usual associativity and unitality conditions. For example, algebra objects in $\Bbbk$-modules are $\Bbbk$-algebras.

Here we introduce the ∞-category $\mathsf{Alg}(\mathcal{V}) := \mathsf{Alg}(\mathcal{V}^{\boxtimes})$ of algebra objects in a monoidal ∞-category $\mathcal{V} := (\mathcal{V}, \boxtimes)$. We will eventually see that this generalizes the notion of monoid objects introduced in §10.1 to the case of an arbitrary (not necessarily cartesian) monoidal structure, in the sense that when $\mathcal{X} := (\mathcal{X}, \times)$ is cartesian monoidal then we have an equivalence $\mathsf{Alg}(\mathcal{X}) := \mathsf{Alg}(\mathcal{X}^{\times}) \simeq \mathsf{Mon}(\mathcal{X})$.[180] (Note that we have not yet endowed $\mathcal{X}$ with a cartesian monoidal structure per se.)

We present the simplest definition; an equivalent but simultaneously more complex and more economical definition is discussed in §11.6.7. Let us write $\mathbf{\Delta}_+ \in \mathsf{Cat}_1$ for the ***augmented simplex category***, i.e. the category of finite totally ordered sets. This contains $\mathbf{\Delta} \subset \mathbf{\Delta}_+$ as a full subcategory; we may write $[-1] := \varnothing$ for the unique object not in its image. This is a monoidal category under the *join* functor $([m], [n]) \mapsto ([m] \star [n]) \cong [m + n + 1]$.

Exercise 10.11 (6 points).
  (a) Endow the object $[0] \in \mathbf{\Delta}_+$ with the structure of an algebra object.
  (b) Show that this is the *universal* algebra object in a monoidal category, i.e. that for any $\mathcal{V} \in \mathsf{Mon}(\mathsf{Cat}_1)$ the composite functor
  $$\mathsf{Fun}^{\otimes}(\mathbf{\Delta}_+, \mathcal{V}) \longrightarrow \mathsf{Fun}(\mathsf{Alg}(\mathbf{\Delta}_+), \mathsf{Alg}(\mathcal{V})) \xrightarrow{\mathsf{ev}_{[0]}} \mathsf{Alg}(\mathcal{V})$$
  is an equivalence of categories.

From Exercise 10.11(a), we obtain a monoidal ∞-category $\mathbf{\Delta}_+ := (\mathbf{\Delta}_+, \star) \in \mathsf{Mon}(\mathsf{Cat})$ (by applying the product-preserving inclusion $\mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}$). Inspired by Exercise 10.11(b), we then define
$$\mathsf{Alg}(\mathcal{V}) := \mathsf{Fun}^{\otimes}(\mathbf{\Delta}_+, \mathcal{V}) \ .$$

The fact that this is a robust definition is a philosophical assertion that is strongly justified by [Lur].[181] It follows immediately from the definition that a monoidal functor $\mathcal{V} \to \mathcal{W}$ determines a functor $\mathsf{Alg}(\mathcal{V}) \to \mathsf{Alg}(\mathcal{W})$ – indeed, that the construction Alg assembles as a functor $\mathsf{Mon}(\mathsf{Cat}) \xrightarrow{\mathsf{Alg}} \mathsf{Cat}$.

Exercise 10.12 (2 points). For any monoidal ∞-category $\mathcal{V} \in \mathsf{Mon}(\mathsf{Cat})$, endow the unit object $\mathbb{1}_\mathcal{V} \in \mathcal{V}$ with the structure of an algebra object.

---

[180]The linguistic distinction is primarily one of emphasis: traditionally the word "algebra object" is used in algebraic contexts (e.g. in $(\mathsf{Mod}_\Bbbk, \otimes)$), where the monoidal structure in *not* the cartesian product.

[181]This is an instance of the general paradigm that once ∞-category theory has been set up in a sufficiently homotopy-coherent way, homotopy-coherent objects can be defined as representations of their 1-categorical (and even strict) analogs. (The first example of this was the notion of a commutative triangle in an ∞-category $\mathcal{C}$, which we have ultimately simply defined as a functor $[2] \to \mathcal{C}$.)

There are alternative presentations of the augmented simplex category (see also Exercise 10.19).

Exercise 10.13 (4 points)**.** Construct a faithful embedding $\mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathsf{op}}$, and describe the resulting monoidal structure on its image.

10.3.3. One source of monoidal ∞-categories is monoidal model categories. Given a monoidal model category V, the subcategory $\mathsf{V}^c \subseteq \mathsf{V}$ of cofibrant objects is a monoidal subcategory, which is compatible with the weak equivalences (in the sense that it is a monoid object in relative categories). It follows that the localization $\mathsf{V}^c[\![\mathbf{W}^{-1}]\!]$ acquires the canonical structure of a monoidal ∞-category (because the localization functor from relative ∞-categories to ∞-categories preserves finite products). Moreover, the inclusion $\mathsf{V}^c \hookrightarrow \mathsf{V}$ in cat induces a canonical equivalence $\mathsf{V}^c[\![\mathbf{W}^{-1}]\!] \xrightarrow{\sim} \mathsf{V}[\![\mathbf{W}^{-1}]\!]$ in Cat on localizations, and so this is indeed a monoidal structure on the underlying ∞-category of V (up to equivalence). Of course, the monoidal structure in V only computes that of $\mathsf{V}[\![\mathbf{W}^{-1}]\!]$ when restricted to cofibrant objects.

10.4. Commutative monoids and symmetric monoidal ∞-categories.

10.4.1. We now discuss *commutative* monoid objects, and in particular *symmetric* monoidal ∞-categories. The basic facts largely parallel the associative analogs, and so we give fewer details and leave some of the corresponding assertions implicit.

10.4.2. The appropriate analog of $\mathbf{\Delta}^{\mathsf{op}}$ is the category Fin_* of finite pointed sets. We write $S_+ \in \mathsf{Fin}_*$ for a typical object, i.e. $S \in \mathsf{Fin}$ is a finite set and $S_+ := S \sqcup \{+\}$ is obtained by adjoining a basepoint. We write $\underline{n} := \{1, \dots, n\} \in \mathsf{Fin}$. For any $s \in S$, we write $S_+ \xrightarrow{\rho_s} \underline{1}_+$ for the morphism in Fin_* defined by the formula

$$\rho_s(t) := \begin{cases} 1, & t = s \\ +, & t \neq s \end{cases}.$$

To emphasize the dependence on $s \in S$, we may write $S_+ \xrightarrow{\rho_s} \{s\}_+$ for this morphism. We also write $S_+ \xrightarrow{\alpha} \underline{1}_*$ for the unique morphism in Fin_* such that $\alpha^{-1}(+) = \{+\}$.

10.4.3. Let $\mathfrak{X}$ be an ∞-category that admits finite products. Given a functor $\mathsf{Fin}_* \xrightarrow{M} \mathfrak{X}$, we often write $M_n := M(\underline{n}_+)$ for simplicity; correspondingly, we may write $M_\bullet := M$ to emphasize the fact that M is a functor. A ***commutative monoid object*** in $\mathfrak{X}$ is a functor $\mathsf{Fin}_* \xrightarrow{M_\bullet} \mathfrak{X}$ satisfying the following conditions:

(1) $M_\bullet$ is ***reduced***, i.e. the canonical morphism $M_0 \to \mathsf{pt}_{\mathfrak{X}}$ is an equivalence.
(2) $M_\bullet$ is ***Segal***, i.e. for any $n \geqslant 2$ the nth ***Segal map***

$$M_n := M(\underline{n}_+) \xrightarrow{(M(\rho_1), \dots, M(\rho_n))} M(\{1\}_+) \times \cdots \times M(\{n\}_+) =: (M_1)^{\times n}$$

is an equivalence.

Its underlying object is $M := \mathsf{fgt}(M_\bullet) := M_1 \in \mathcal{X}$, and the morphisms

$$\mu : M_1 \times M_1 \xleftarrow{\ \sim\ } M_2 \xrightarrow{M(\underline{2}_+ \xrightarrow{\alpha} \underline{1}_+)} M_1 \qquad \text{and} \qquad \eta : \mathsf{pt} \xleftarrow{\ \sim\ } M_0 \xrightarrow{M(\underline{0}_+ \xrightarrow{\alpha} \underline{1}_+)} M_1$$

are respectively its multiplication and unit maps. We write $\mathsf{CMon}(\mathcal{X}) \subseteq \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{X})$ for the full subcategory on the commutative monoid objects. We simply write $\mathsf{CMon} := \mathsf{CMon}(\mathcal{S})$ and refer to its objects as **commutative monoids** or **commutative ∞-monoids**. A commutative monoid object in Cat is called a **symmetric monoidal ∞-category**. Morphisms in $\mathsf{CMon}(\mathsf{Cat})$ are called **symmetric monoidal functors**. We will use the fact that $\mathsf{CMon}(\mathsf{Cat})$ is actually an $(\infty, 2)$-category, and simply write $\mathsf{Fun}^\otimes(-, -)$ for its hom-objects in Cat.

Exercise 10.14 (2 points)**.** Prove that the multiplication of a commutative monoid object $M \in \mathsf{CMon}(\mathcal{X})$ is indeed commutative, i.e. that the diagram

$$
\begin{array}{ccc}
M^{\times 2} & \xrightarrow{\ \ \tau\ \ } & M^{\times 2} \\
& {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\
& M &
\end{array}
$$

commutes (where τ denotes the transposition of factors).

Exercise 10.15 (4 points)**.** Prove that precomposition with the finite pointed simplicial set

$$\Delta^{\mathrm{op}} \xrightarrow{\ \Delta^1/\partial\Delta^1\ } \mathsf{Fin}_*$$

induces a functor

$$\mathsf{Mon}(\mathcal{X}) \longleftarrow \mathsf{CMon}(\mathcal{X})$$

carrying a commutative monoid object to its underlying monoid object.

10.4.4.   Let $\mathcal{V} \in \mathsf{CMon}(\mathsf{Cat})$ be a symmetric monoidal ∞-category. We define the ∞-category $\mathsf{CAlg}(\mathcal{V}) \in \mathsf{Cat}$ of **commutative algebra objects** in $\mathcal{V}$ via the following (which uses the corresponding notion in 1-categories). First of all, we endow the category Fin of finite sets with the coproduct symmetric monoidal structure.

Exercise 10.16 (6 points)**.**
  (a) Endow the object $\underline{1} \in \mathsf{Fin}$ with the structure of a commutative algebra object.
  (b) Show that this is the *universal* commutative algebra object in a symmetric monoidal category, i.e. that for any $\mathcal{V} \in \mathsf{CMon}(\mathsf{Cat}_1)$ the composite functor

$$\mathsf{Fun}^\otimes(\mathsf{Fin}, \mathcal{V}) \longrightarrow \mathsf{Fun}(\mathsf{CAlg}(\mathsf{Fin}), \mathsf{CAlg}(\mathcal{V})) \xrightarrow{\mathrm{ev}_{\underline{1}_+}} \mathsf{CAlg}(\mathcal{V})$$

  is an equivalence.

For an arbitrary symmetric monoidal ∞-category $\mathcal{V} \in \mathsf{CMon}(\mathsf{Cat})$, we then define

$$\mathsf{CAlg}(\mathcal{V}) := \mathsf{Fun}^\otimes(\mathsf{Fin}, \mathcal{V}) \ .$$

Exercise 10.17 (8 points)**.** Construct a functor

$$\mathsf{CAlg}(\mathcal{V}) \longrightarrow \mathsf{Alg}(\mathcal{V}) := \mathsf{Alg}(\mathsf{fgt}(\mathcal{V}))$$

carrying a commutative algebra object to its underlying algebra object (where $\mathsf{fgt}(\mathcal{V}) \in \mathsf{Mon}(\mathsf{Cat})$ arises from Exercise 10.15).

10.5. **Factorization homology.**

10.5.1.   As an application of the foregoing discussion, we briefly discuss the theory of ***factorization homology***, which combines higher algebra with differential topology. Namely, it gives a pairing between manifolds and higher-algebraic data; for instance, it gives a way of integrating an $\mathbb{E}_n$-*algebra* [AF15] or an (∞, n)-*category* [AFR18] over an n-manifold.[182] We outline the former variant of factorization homology.[183]

10.5.2.   We define the ∞-category $\mathcal{M}\mathsf{fld}_n$ to be that underlying the following topological category: its objects are (finitary smooth) manifolds, and its hom-objects are topological spaces of (smooth) embeddings (equipped with the compact-open topology).[184]   Because embeddings respect tangent spaces, there is a functor

$$\mathcal{M}\mathsf{fld}_n \longrightarrow \mathcal{S}_{/\mathsf{BGL}_n(\mathbb{R})}$$

carrying a manifold M to the morphism $\Pi_\infty(M) \xrightarrow{TM} \mathsf{BGL}_n(\mathbb{R})$ of spaces that classifies its tangent bundle. Thereafter, for any morphism $B \to \mathsf{BGL}_n(\mathbb{R})$, we obtain the ∞-category of B-*framed n-manifolds* as the pullback

$$
\begin{array}{ccc}
\mathcal{M}\mathsf{fld}_n^B & \longrightarrow & \mathcal{S}_{/B} \\
\downarrow & & \downarrow \\
\mathcal{M}\mathsf{fld}_n & \longrightarrow & \mathcal{S}_{/\mathsf{BGL}_n(\mathbb{R})}
\end{array} \quad .
$$

So, an object of $\mathcal{M}\mathsf{fld}_n^B$ is an n-manifold M equipped with a lift

$$
\begin{array}{ccc}
 & & B \\
 & \nearrow & \downarrow \\
\Pi_\infty(M) & \xrightarrow[TM]{} & \mathsf{BGL}_n(\mathbb{R})
\end{array}
$$

---

[182]Actually, there are many variants of factorization homology, and in these various variants the n-manifold may be required to satisfy certain conditions (e.g. compactness) and/or to be equipped with certain structure (e.g. a framing).

[183]The latter variant is substantially more powerful and at the same substantially more technical. Indeed, even the mechanism by which the relevant ∞-categories of manifolds are defined is itself nontrivial [AFR19].

[184]This definition can be interpreted as the philosophical assertion that topological spaces of embeddings are most mathematically meaningful when considered as spaces (i.e. after passing to their fundamental ∞-groupoids). Indeed, in practice one primarily studies their homotopical features (e.g. their homotopy groups (and in particular their nonemptiness)), not their point-set features.

of its tangent classifier, and a morphism in $\mathfrak{Mfld}_n^B$ is a morphism $M \xrightarrow{f} N$ in $\mathfrak{Mfld}_n$ along with the data of a commutative diagram

(37)

in $\mathcal{S}$ (involving the given lifts of TM and TN). For example, if $B = \mathsf{BGL}_n^+(\mathbb{R})$ then a B-framing is an orientation, and if $B = \mathsf{B}\{e\} \simeq \mathsf{pt}$ then a B-framing is a framing in the usual sense (namely a trivialization of the tangent bundle).[185] We use the suggestive notations $\mathfrak{Mfld}_n^{\mathsf{or}} := \mathfrak{Mfld}_n^{\mathsf{BGL}_n^+(\mathbb{R})}$ and $\mathfrak{Mfld}_n^{\mathsf{fr}} := \mathfrak{Mfld}_n^{\mathsf{pt}}$.

10.5.3. We endow $\mathfrak{Mfld}_n^B$ with a symmetric monoidal structure given by disjoint union.[186] We write $\mathfrak{Disk}_n^B \subseteq \mathfrak{Mfld}_n^B$ for the full (symmetric monoidal) subcategory on those objects of the form $(\mathbb{R}^n)^{\sqcup i}$ for $i \geqslant 0$. We likewise use the distinguished superscripts $B \in \{\varnothing, \mathsf{or}, \mathsf{fr}\}$ in the evident way.

Fix a symmetric monoidal ∞-category $\mathcal{V}$ (e.g. the derived ∞-category $\mathbf{D}_{\Bbbk}$). A B-**framed** $\mathbb{E}_n$-**algebra** (or simply an $\mathbb{E}_n^B$-**algebra**) in $\mathcal{V}$ is a symmetric monoidal functor

$$\mathfrak{Disk}_n^B \xrightarrow{A} \mathcal{V} \; ;$$

these assemble into an ∞-category

$$\mathsf{Alg}_{\mathbb{E}_n^B}(\mathcal{V}) := \mathsf{Fun}^{\otimes}(\mathfrak{Disk}_n^B, \mathcal{V}) \; .$$

So by definition, $\mathfrak{Disk}_n^B$ is the free symmetric monoidal ∞-category containing a B-framed $\mathbb{E}_n$-algebra. Given $A \in \mathsf{Alg}_{\mathbb{E}_n^B}(\mathcal{V})$, one thinks of $A := A(\mathbb{R}^n) \in \mathcal{V}$ as its underlying object; for each $i \geqslant 0$, this comes equipped with a parametrized family

$$\mathsf{hom}_{\mathfrak{Disk}_n^B}((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n) \xrightarrow{A} \mathsf{hom}_{\mathcal{V}}(A^{\boxtimes i}, A)$$

---

[185]It is of course possible to present these data in the context of topological spaces (modeling $B \to \mathsf{BGL}_n(\mathbb{R})$ by a fibration). Still, it is appropriate to require the diagram (37) to be homotopy-coherently commutative instead of strictly commutative, which is exactly the sort of maneuver that is much more easily handled in ∞-category theory.

[186]Note that the functor $\mathsf{cat}(\mathsf{Top}) \to \mathsf{cat}(\mathsf{Top})[\![\mathbf{W}^{-1}]\!] \simeq \mathsf{Cat}$ preserves products and so determines a functor $\mathsf{CMon}(\mathsf{cat}(\mathsf{Top})) \to \mathsf{CMon}(\mathsf{Cat})$.

of i-ary operations (using the symmetric monoidality of A), with compatibilities according to composition in $\mathcal{D}\mathsf{isk}_n^B$. We write

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{V}) := \mathsf{Alg}_{\mathbb{E}_n^{\mathsf{fr}}}(\mathcal{V}) := \mathsf{Fun}^\otimes(\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}, \mathcal{V}) \ ,$$

and refer to objects of this ∞-category simply as $\mathbb{E}_n$-*algebras* in $\mathcal{V}$.[187]

Exercise 10.18 (12 points).
    (a) Prove that the ∞-categories $\mathcal{D}\mathsf{isk}_1$ and $\mathcal{D}\mathsf{isk}_1^{\mathsf{fr}}$ are (equivalent to) 1-categories.
    (b) Give combinatorial descriptions of these categories (e.g. as strict categories), including descriptions of their symmetric monoidal structures.
    (c) Prove that $\mathcal{D}\mathsf{isk}_1^{\mathsf{fr}}$ is the free symmetric monoidal category containing an associative algebra object.
    (d) Formulate and prove an analogous universal characterization of $\mathcal{D}\mathsf{isk}_1$.

Exercise 10.19 (6 points).
    (a) Identify the category $\mathbf{\Delta}_+$ in terms of the category $\mathcal{D}\mathsf{isk}_1^{\mathsf{fr}}$.
    (b) Give a topological description of the embedding $\mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathsf{op}}$ of Exercise 10.13.

Exercise 10.20 (2 points). Give an explicit description of the ∞-category $\mathsf{Alg}_{\mathbb{E}_0}(\mathcal{V})$.

Exercise 10.21 (6 points). For any framed n-manifold $M \in \mathcal{M}\mathsf{fld}_n^{\mathsf{fr}}$, prove that evaluation at origins determines an equivalence

$$\mathsf{hom}_{\mathcal{M}\mathsf{fld}_n^{\mathsf{fr}}}((\mathbb{R}^n)^{\sqcup i}, M) \xrightarrow{\ \sim\ } \Pi_\infty(\mathsf{Conf}_n(M))$$

with the underlying space of the topological space $\mathsf{Conf}_n(M) \in \mathsf{Top}$ of configurations of n distinct points in M (a subspace of $M^{\times n} \in \mathsf{Top}$).

    For any $n \geqslant 0$, there is a canonical symmetric monoidal functor

$$\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \longrightarrow \mathcal{D}\mathsf{isk}_{n+1}^{\mathsf{fr}}$$

---

[187]Beware that this (extremely well-established) convention does *not* agree with the above convention that $\mathcal{D}\mathsf{isk}_n := \mathcal{D}\mathsf{isk}_n^{\mathsf{BGL}_n(\mathbb{R})}$. Even more confusingly, $\mathbb{E}_n^{\mathsf{BGL}_n(\mathbb{R})}$-algebras are sometimes referred to as "framed $\mathbb{E}_n$-algebras". This refers to the fact that morphisms in $\mathcal{D}\mathsf{isk}_n^{\mathsf{BGL}_n(\mathbb{R})} \simeq \mathcal{D}\mathsf{isk}_n$ are (∞-categorically) equivalent to the data of their images along with "framings"; for instance, differentiation (and in particular evaluation) at the origin determines an equivalence

$$\mathsf{hom}_{\mathcal{D}\mathsf{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\ \sim\ } \{(p \in \mathbb{R}^n, T_0\mathbb{R}^n \xrightarrow{\ \cong\ } T_p\mathbb{R}^n)\}$$

with the underlying space of the frame bundle of $\mathbb{R}^n$, ultimately yielding an equivalence $\mathsf{hom}_{\mathcal{D}\mathsf{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathsf{GL}_n(\mathbb{R})$ of ∞-monoids. We suggest the terminologies "unoriented $\mathbb{E}_n$-algebras" or "ribbon $\mathbb{E}_n$-algebras" for $\mathbb{E}_n^{\mathsf{BGL}_n(\mathbb{R})}$-algebras.

given by thickening embeddings of n-dimensional disks into embeddings of $(n+1)$-dimensional disks.[188] Clearly, this extends to a commutative triangle

$$
\begin{array}{ccc}
\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} & \longrightarrow & \mathcal{D}\mathsf{isk}_{n+1}^{\mathsf{fr}} \\
& \searrow{\scriptstyle \pi_0} & \downarrow{\scriptstyle \pi_0} \\
& & \mathsf{Fin}
\end{array}
$$

of symmetric monoidal ∞-categories. In fact, these symmetric monoidal functors determine an equivalence

$$
\mathsf{colim}^{\mathsf{CMon}(\mathsf{Cat})}\left(\mathcal{D}\mathsf{isk}_0^{\mathsf{fr}} \longrightarrow \mathcal{D}\mathsf{isk}_1^{\mathsf{fr}} \longrightarrow \mathcal{D}\mathsf{isk}_2^{\mathsf{fr}} \longrightarrow \cdots\right) \xrightarrow{\sim} \mathsf{Fin}
$$

of symmetric monoidal ∞-categories.[189] Hence, for any $\mathcal{V} \in \mathsf{CAlg}(\mathsf{Cat})$ we obtain an equivalence

$$
\mathsf{CAlg}(\mathcal{V}) \xrightarrow{\sim} \lim^{\mathsf{Cat}}\left(\cdots \longrightarrow \mathsf{Alg}_{\mathbb{E}_2}(\mathcal{V}) \longrightarrow \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{V}) \longrightarrow \mathsf{Alg}_{\mathbb{E}_0}(\mathcal{V})\right)
$$

of ∞-categories. For this reason, it is common to refer to commutative algebras as $\mathbb{E}_\infty$-**algebras**.[190] Note that an $\mathbb{E}_n$-algebra has an underlying $\mathbb{E}_i$-algebra for any $0 \leqslant i \leqslant n \leqslant \infty$.

Exercise 10.22 (6 points)**.** Prove that if $\mathcal{V}$ is a symmetric monoidal category and $n \geqslant 2$ then the forgetful functor

$$
\mathsf{CAlg}(\mathcal{V}) \longrightarrow \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{V})
$$

is an equivalence of categories.

10.5.4. We discuss $\mathbb{E}_n$-algebras in spaces (i.e. taking $(\mathcal{V}, \boxtimes) = (\mathcal{S}, \times)$). These are closely related to *n-fold loopspaces*. Namely, one-point compactification (and passage to underlying pointed spaces) determines a canonical functor

$$
\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \xrightarrow{(-)^+} \mathcal{S}_*^{\mathsf{op}} .[191]
$$

---

[188] More precisely, there is a contractible topological space of such functors at the level of topological categories, given by making continuous choices of radii for thickenings. (The necessity of such choices can be avoided at the expense of using a more rigid topological model for $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$ (as in the usual definition of the "little n-disks operad").)

[189] This is because the path components of the space $\mathsf{hom}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}}((\mathbb{R}^n)^{\sqcup i}, (\mathbb{R}^n)^{\sqcup j})$ become increasingly highly connected as n grows, yielding an equivalence

$$
\mathsf{colim}^{\mathcal{S}}\left(\mathsf{hom}_{\mathcal{D}\mathsf{isk}_0^{\mathsf{fr}}}((\mathbb{R}^0)^{\sqcup i}, (\mathbb{R}^0)^{\sqcup j}) \longrightarrow \mathsf{hom}_{\mathcal{D}\mathsf{isk}_1^{\mathsf{fr}}}((\mathbb{R}^1)^{\sqcup i}, (\mathbb{R}^1)^{\sqcup j}) \longrightarrow \mathsf{hom}_{\mathcal{D}\mathsf{isk}_2^{\mathsf{fr}}}((\mathbb{R}^2)^{\sqcup i}, (\mathbb{R}^2)^{\sqcup j}) \longrightarrow \cdots\right) \longrightarrow \mathsf{hom}_{\mathsf{Fin}}(\underline{i}, \underline{j})
$$

of spaces. For instance, taking $i = 2$ and $j = 1$, by Exercise 10.21 we have an equivalence

$$
\mathsf{hom}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}}((\mathbb{R}^n)^{\sqcup 2}, \mathbb{R}^n) \simeq \Pi_\infty(\mathsf{Conf}_2(\mathbb{R}^n)) \simeq S^{n-1}
$$

of spaces. (Note too that the forgetful functor $\mathsf{CMon}(\mathsf{Cat}) \to \mathsf{Cat}$ is conservative, and that the extraction of hom-spaces in ∞-categories commutes with filtered colimits (as can be seen via complete Segal spaces).)

[190] More precisely, there is a fairly evident symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}_\infty^{\mathsf{fr}}$ of infinite-dimensional framed disks, and this is also equivalent to Fin as such.

[191] Such notions are addressed systematically in [AF].

Moreover, this functor is canonically symmetric monoidal: it carries disjoint unions to wedge sums. Thereafter, a based space $X \in \mathcal{S}_*$ yields an $\mathbb{E}_n$-algebra

$$\Omega^n X := \left(\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \xrightarrow{(-)^+} \mathcal{S}_*^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathcal{S}_*}(-, X)} \mathcal{S} \right) \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S}) \ ,$$

whose underlying object is its n-fold loopspace

$$\mathsf{hom}_{\mathcal{S}_*}((\mathbb{R}^n)^+, X) := \mathsf{hom}_{\mathcal{S}_*}(\Pi_\infty((\mathbb{R}^n)^+), X) \simeq \mathsf{hom}_{\mathcal{S}_*}(S^n, X) =: \Omega^n X \in \mathcal{S} \ .$$

For $n \geqslant 1$, we say that an $\mathbb{E}_n$-algebra is **grouplike** if its image under the functor

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S}) \xrightarrow{\mathsf{Alg}_{\mathbb{E}_n}(\pi_0)} \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Set}) \longrightarrow \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Set}) \simeq \mathsf{Alg}(\mathsf{Set}) \simeq \mathsf{Mon}(\mathsf{Set})$$

is a group (using Exercise 10.18(c) and the evident equivalence $\mathsf{Alg}(\mathsf{Set}) := \mathsf{Alg}(\mathsf{Set}^\times) \simeq \mathsf{Mon}(\mathsf{Set})$ (which is as-yet unproved for ∞-categories) to identify the target). We write

$$\mathsf{Alg}_{\mathbb{E}_n}^{\mathsf{gp}}(\mathcal{S}) \subset \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S})$$

for the full subcategory on the grouplike $\mathbb{E}_n$-algebras.[192] Clearly, the $\mathbb{E}_n$-algebra $\Omega^n X$ is grouplike, as $\pi_0(\Omega^n X) =: \pi_n(X)$ is a group. A fundamental theorem in homotopy theory [BV73, May72] asserts that *all* grouplike $\mathbb{E}_n$-algebras arise in this way: indeed, we have a diagram

$$\mathsf{Alg}_{\mathbb{E}_n}^{\mathsf{gp}}(\mathcal{S}) \underset{\Omega^n}{\overset{\mathsf{B}^n}{\underset{\sim}{\rightleftarrows}}} \mathcal{S}_*^{\geqslant n} \overset{\hookrightarrow}{\underset{\perp}{\rightleftarrows}} \mathcal{S}_* \ . \quad {}^{193}$$

10.5.5.   The **factorization homology** functor of a B-framed $\mathbb{E}_n$-algebra $A \in \mathsf{Alg}_{\mathbb{E}_n^B}(\mathcal{V})$ is the left Kan extension

$$
\begin{array}{ccc}
\mathcal{D}\mathsf{isk}_n^B & \xrightarrow{\ A\ } & \mathcal{V} \\
\downarrow & \nearrow_{\int_{(-)} A} & \\
\mathcal{M}\mathsf{fld}_n^B & &
\end{array} \quad .
$$

In other words, its factorization homology over a B-framed n-manifold $M \in \mathcal{M}\mathsf{fld}_n^B$ is the colimit

$$\int_M A := \mathsf{colim}\left((\mathcal{D}\mathsf{isk}_n^B)_{/M} \xrightarrow{\mathsf{fgt}} \mathcal{D}\mathsf{isk}_n^B \xrightarrow{\ A\ } \mathcal{V} \right) \ .$$

This has a number of pleasant features. Most importantly for us, it admits a characterization akin to the Eilenberg–Steenrod axioms: every "homology theory for B-framed n-manifolds" H is given by factorization homology, with coefficients in the $\mathbb{E}_n$-algebra $H(\mathbb{R}^n)$ [AF15].[194]

---

[192]It will be clear that the equivalence $\mathsf{Alg}_{\mathbb{E}_1}(\mathcal{S}) \xrightarrow{\sim} \mathsf{Mon}(\mathcal{S}) =: \mathsf{Mon}$ factors as an equivalence $\mathsf{Alg}_{\mathbb{E}_1}^{\mathsf{gp}}(\mathcal{S}) \xrightarrow{\sim} \mathsf{Grp}(\mathcal{S}) =: \mathsf{Grp}$.

[193]This should be compared with the discussion of §10.1.3: more generally, $\mathbb{E}_n$-algebras are essentially equivalent to pointed (∞, n)-categories with a single equivalence class of object.

[194]In particular, this notion of "homology theory" involves a *multiplicative* form of excision (inspired by TQFT).

10.5.6.   A special case of classical interest is **Hochschild homology**, which is factorization homology over $S^1 \in \mathcal{M}\mathsf{fld}_1^{\mathsf{fr}}$. We briefly explain the original definition (as discussed e.g. in [Lod98]) and how factorization homology recovers it.

To simplify our notation, we write

$$\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} := (\mathcal{D}\mathsf{isk}_1^{\mathsf{fr}})_{/S^1} \ .$$

Exercise 10.23 (8 points).
  (a) Prove that the ∞-category $\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}$ is a 1-category.
  (b) Give a combinatorial description of this category.

The category $\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}$ is called the **paracyclic category**.[195]

Using Exercise 10.23 (which guarantees that no issues of homotopy-coherent arise), one can construct a canonical functor

$$(38) \qquad\qquad\qquad \boldsymbol{\Delta}^{\mathsf{op}} \longrightarrow \boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} \ ,$$

which carries $[n]^\circ \in \boldsymbol{\Delta}^{\mathsf{op}}$ to a configuration of $n+1$ framed 1-disks embedded in S^1, with face maps given by colliding adjacent disks and with degeneracy maps given by allowing new disks to appear. Then, for any $\mathbb{E}_1$-algebra $A \in \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{V})$, we define its **cyclic bar construction** to be the simplicial object

$$\mathsf{B}^{\mathsf{cyc}}(A) : \boldsymbol{\Delta}^{\mathsf{op}} \longrightarrow \boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} \xrightarrow{\ \mathsf{fgt}\ } \mathcal{D}\mathsf{isk}_1^{\mathsf{fr}} \xrightarrow{\ A\ } \mathcal{V} \ .$$

It turns out that the functor (38) is *final*: colimits over $\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}$ can be computed as colimits over $\boldsymbol{\Delta}^{\mathsf{op}}$.[196] Hence, the canonical morphism

$$|\mathsf{B}^{\mathsf{cyc}}(A)| := \mathsf{colim}_{\boldsymbol{\Delta}^{\mathsf{op}}}(\mathsf{B}^{\mathsf{cyc}}(A)) \longrightarrow \mathsf{colim}\left(\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} \xrightarrow{\ \mathsf{fgt}\ } \mathcal{D}\mathsf{isk}_1^{\mathsf{fr}} \xrightarrow{\ A\ } \mathcal{V} \right) =: \int_{S^1} A$$

is an equivalence.

Now, the classical definition is as follows: given an associative $\Bbbk$-algebra $A \in \mathsf{Alg}_\Bbbk := \mathsf{Alg}(\mathsf{Mod}_\Bbbk) \simeq \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Mod}_\Bbbk)$, its nth **Hochschild homology** is the $\Bbbk$-module

$$\mathsf{HH}_n(A) := \mathsf{H}_n(\mathsf{C}_\bullet(\mathsf{B}^{\mathsf{cyc}}(A))) \cong \pi_n(\mathsf{B}^{\mathsf{cyc}}(A)) \in \mathsf{Mod}_\Bbbk$$

(recall §8.4.4).

Exercise 10.24 (2 points). Give an explicit description of $\mathsf{HH}_0(A)$ in terms of A.

---

[195]Beware that the usual definition of the paracyclic category does not include the initial object ($\varnothing \hookrightarrow S^1$). We elide this distinction here to simplify the discussion (which is mathematically harmless because the inclusion of the usual paracyclic category into the one discussed here is final).

[196]More precisely, a functor $\mathcal{I} \xrightarrow{\varphi} \mathcal{J}$ is **final** if for any functor $\mathcal{J} \xrightarrow{F} \mathcal{C}$, the canonical morphism

$$\mathsf{colim}_{\mathcal{I}}(\varphi^* F) \longrightarrow \mathsf{colim}_{\mathcal{J}}(F)$$

in $\mathcal{C}$ is an equivalence (and in particular both colimits exist if either one does). We discuss finality more thoroughly below.

Exercise 10.25 (4 points)**.** Compute $\mathsf{HH}_*(\mathbb{k}[\varepsilon]/\varepsilon^2)$.

Exercise 10.26 (6 points)**.** Prove that for any commutative $\mathbb{k}$-algebra $A \in \mathsf{CAlg}_{\mathbb{k}}$ there is a canonical isomorphism $\mathsf{HH}_1(A) \cong \Omega^1_{A|\mathbb{k}}$ (recall §8.6.4).[197]

Of central importance is an action of the circle group on Hochschild homology.[198] In terms of factorization homology, this is immediate: it follows from the fact that factorization homology is a functor

$$\mathcal{M}\mathsf{fld}^{\mathsf{fr}}_1 \xrightarrow{\int_{(-)} A} \mathcal{V} \ ,$$

as the circle group acts on the object $S^1 \in \mathcal{M}\mathsf{fld}^{\mathsf{fr}}_1$. By contrast, in the classical definition, this circle action must be built in "by hand" via simplicial methods.

10.5.7.   Whereas ordinary co/homology of manifolds adheres to Poincaré duality, factorization homology participates in ***nonabelian Poincaré duality***. Fix a framed n-manifold $M \in \mathcal{M}\mathsf{fld}^{\mathsf{fr}}_n$. Given a based space $X \in \mathcal{S}_*$, there is a canonical map

$$(39) \qquad\qquad \int_M \Omega^n X \longrightarrow \mathsf{map}_c(M, X)$$

to the space of *compactly-supported maps* from M to X.[199] This can be described informally as follows: a point of $\int_M \Omega^n X$ is the data of a framed embedding $(\mathbb{R}^n)^{\sqcup i} \hookrightarrow M$ along with labels $\gamma_1, \ldots, \gamma_i \in \Omega^n X$, and the map (39) carries this to the composite

$$\Pi_\infty(M^+) \longrightarrow \Pi_\infty(((\mathbb{R}^n)^{\sqcup i})^+) \simeq (S^n)^{\vee i} \xrightarrow{(\gamma_1, \ldots, \gamma_i)} X$$

where the first map is the Pontrjagin–Thom collapse. Assuming that X is n-connected (i.e. that $\pi_j(X) = 0$ for $j < n$), the map (39) is an equivalence of pointed spaces (where the basepoints are the empty configuration and the constant map).[200] This may be thought of

---

[197]This fact is the basis of the celebrated *Hochschild–Kostant–Rosenberg* (or simply *HKR*) theorem [HKR62]: assuming that A is smooth over $\mathbb{k}$, there is a canonical isomorphism $\Omega^\bullet_{A|\mathbb{k}} \cong \mathsf{HH}_\bullet(A)$ of graded $\mathbb{k}$-algebras. A derived version of the HKR theorem (using the cotangent complex) is established in [TV11].

[198]In terms of the HKR theorem, this implements the de Rham differential.

[199]Because we have assumed that M is finitary, we have an equivalence

$$\mathsf{map}_c(M, X) \simeq \mathsf{hom}_{\mathcal{S}_*}(\Pi_\infty(M^+), X) \ .$$

More generally, given a topological space $T \in \mathsf{Top}$, writing $\mathsf{Cpct}(T)$ for its poset of compact subsets we have a composite functor

$$\mathsf{Cpct}(T)^{\mathsf{op}} \xrightarrow{K \longmapsto T/(T \setminus K)} \mathsf{Top}_* \xrightarrow{\Pi_\infty} \mathcal{S}_* \ ,$$

and using this we define the space of compactly-supported maps from T to the based space $X \in \mathcal{S}_*$ to be

$$\mathsf{map}_c(T, X) := \mathsf{colim}_{K \in \mathsf{Cpct}(T)} \mathsf{hom}_{\mathcal{S}_*}(\Pi_\infty(T/(T \setminus K)), X) \in \mathcal{S} \ .$$

[200]Of course, there is a generalization for B-framed n-manifolds. For this, one must replace compactly-supported maps with compactly-supported sections of a certain bundle (which is trivialized by a framing).

as a sort of "cellular approximation in families" result (as n-manifolds are n-dimensional cell complexes (e.g. via Morse theory)); its proof consists in showing that the functor

$$\mathcal{M}\mathsf{fld}_n^{\mathsf{fr}} \xrightarrow{\ \mathsf{map}_c(-,X)\ } \mathcal{S}$$

satisfies the same axioms characterizing $\int_{(-)} \Omega^n X$ that were alluded to in §10.5.5.

Given an abelian group A, this recovers ordinary Poincaré duality by taking $X = K(A,n)$ to be the corresponding Eilenberg–MacLane space, so that we have an equivalence $\Omega^n X \simeq A$ of $\mathbb{E}_n$-algebras. On the one hand, passing from embedded disks to their origins yields an equivalence

$$\int_M A \xrightarrow{\ \sim\ } \Pi_\infty(\mathbb{Z}\{M\} \otimes_{\mathbb{Z}} A)$$

with the space of configurations of points in M labeled by elements of A, and hence we obtain isomorphisms

$$\pi_j\left(\int_M A\right) \cong \mathsf{H}_j(M;A)$$

(as in §8.4.5). Meanwhile, we have isomorphisms

$$\pi_j\left(\mathsf{map}_c(M, K(A,n))\right) \cong \pi_0\left(\mathsf{map}_c(M, \Omega^j K(A,n))\right) \cong \pi_0\left(\mathsf{map}_c(M, K(A, n-j))\right) \cong \mathsf{H}_c^{n-j}(M;A)\,.$$

Hence, applying π_j to the equivalence (39) in $\mathcal{S}_*$ (in the case that $X = K(A,n)$) yields the Poincaré duality isomorphism

$$\mathsf{H}_j(M;A) \xrightarrow{\ \cong\ } \mathsf{H}_c^{n-j}(M;A)\,.$$

10.5.8. There is a largely analogous theory based on a corresponding ∞-category $\mathcal{M}\mathsf{fld}_n^{\mathsf{top}}$ of (finitary) *topological* n-manifolds. These only have tangent *microbundles*, which are classified by maps to $\mathsf{BTop}(n)$ (the classifying space of the topological group $\mathsf{Top}(n)$ of self-homeomorphisms of $\mathbb{R}^n$). These can likewise we equipped with B-framings for morphisms $B \to \mathsf{BTop}(n)$, yielding an ∞-category $\mathcal{M}\mathsf{fld}_n^{\mathsf{top},B}$ of B-framed topological n-manifolds. In fact, taking $B = \mathsf{BGL}_n(\mathbb{R})$, for $n \neq 4$ the forgetful functor

$$\mathcal{M}\mathsf{fld}_n \longrightarrow \mathcal{M}\mathsf{fld}_n^{\mathsf{top},\mathsf{BGL}_n(\mathbb{R})}$$

is an equivalence.

10.5.9. In a different direction, one can also incorporate manifolds with boundary (or corners). Let us write $\mathcal{M}\mathsf{fld}_n^\partial$ for the analogous symmetric monoidal ∞-category of smooth manifolds with boundary, and $\mathcal{D}\mathsf{isk}_n^\partial \subseteq \mathcal{M}\mathsf{fld}_n^\partial$ for the full (symmetric monoidal) subcategory on those objects of the form

$$(\mathbb{R}^n)^{\sqcup i} \sqcup (\mathbb{R}^{n-1} \times \mathbb{R}_{\geqslant 0})^{\sqcup j}$$

for $i,j \geqslant 0$.

Exercise 10.27 (6 points).
(a) Prove that the ∞-category $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr}}$ is a 1-category.

(b) Give a combinatorial description of it, including a description of its symmetric monoidal structure.

In fact, the paper [AFT17] develops the theory of factorization homology for *stratified spaces*. Of course, it continues to be the case that the local models determine the nature of the coefficients.

11. The Grothendieck construction

In its most general form, the ***Grothendieck construction*** over an ∞-category $\mathcal{B} \in$ Cat is an equivalence Gr between $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat})$ and a certain subcategory of $\mathsf{Cat}_{/\mathcal{B}}$. Given a functor $\mathcal{B} \xrightarrow{F} \mathsf{Cat}$, one says that the object $(\mathsf{Gr}(F) \to \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ – its Grothendieck construction – is *classified* by the functor F. In this sense, the Grothendieck construction generalizes more classical correspondences between parametrized objects and their classifying maps, such as those for covering spaces and G-bundles. We first describe the Grothendieck construction over ∞-groupoids in §11.1 (including those specific examples, see §11.1.2 and §11.1.7 respectively), and then discuss the general case in §§11.2.1-11.5. We conclude in §11.6 by discussing applications to the theory of (symmetric) monoidal ∞-categories.[201]

11.1. The Grothendieck construction over ∞-groupoids.

11.1.1.   Let $X \in \mathcal{S}$ be an ∞-groupoid. A ***local system*** on X with values in an ∞-category $\mathcal{C}$ is a functor $X \to \mathcal{C}$. These assemble into the ∞-category $\mathsf{Fun}(X, \mathcal{C})$. Here we study the examples where $\mathcal{C}$ is Set, $\mathcal{S}$, and Cat. In each of these cases, we will give an equivalent description of local systems on X as certain objects lying *over* the space X, via an equivalence Gr called the ***Grothendieck construction***.

11.1.2.   A ***covering space*** of X is a map $E \to X$ in $\mathcal{S}$ whose fibers are discrete. We generally consider a covering space as an object $(E \downarrow X) \in \mathcal{S}_{/X}$; covering spaces of X assemble into a full subcategory $\mathsf{Cov}_X \subseteq \mathcal{S}_{/X}$. Classical covering space theory can be phrased as an equivalence

$$(40) \qquad\qquad\qquad \mathsf{Fun}(X, \mathsf{Set}) \xrightarrow[\sim]{\mathsf{Gr}} \mathsf{Cov}_X$$

of ∞-categories. The Grothendieck construction carries a functor $X \xrightarrow{F} \mathsf{Set}$ to the covering space $\mathsf{Gr}(F) \downarrow X$ whose fiber over each point $x \in X$ is the discrete space $F(x) \in \mathsf{Set} \subset \mathcal{S}$.

Because $\mathsf{Set} \in \mathsf{Cat}_1 \subset \mathsf{Cat}$ is a 1-category, we have an equivalence

$$(41) \qquad\qquad \mathsf{Fun}(X, \mathsf{Set}) \xleftarrow{\sim} \mathsf{Fun}(\mathsf{ho}(X), \mathsf{Set}) =: \mathsf{Fun}(\Pi_1(X), \mathsf{Set}) \; ;$$

---

[201]We refer the reader to [MG19b] for a somewhat more thorough and leisurely discussion of the Grothendieck construction, including examples as well as a proof that the notions presented here coincide with those studied in [Lur09]. We refer the reader to [MG19a] for a number of more elaborate applications of the Grothendieck construction.

in particular, the equivalence (40) of ∞-categories is in fact an equivalence of 1-categories. Note too that if X is nonempty and connected, then choosing a basepoint we obtain an equivalence $X \simeq \mathsf{B}G$ where $G := \Omega X \in \mathsf{Grp}$, so that the equivalence (41) become an equivalence

$$\mathsf{Fun}(\mathsf{B}G, \mathsf{Set}) \xleftarrow{\sim} \mathsf{Fun}(\Pi_1(\mathsf{B}G), \mathsf{Set}) \simeq \mathsf{Fun}(\mathsf{B}(\pi_0(G)), \mathsf{Set}) =: \mathsf{Fun}(\mathsf{B}(\pi_1(X)), \mathsf{Set}) =: \mathsf{Mod}_{\pi_1(X)}(\mathsf{Set}) ,$$

whence the equivalence (40) reduces to the classical equivalence

$$\mathsf{Mod}_{\pi_1(X)}(\mathsf{Set}) \xrightarrow[\sim]{\mathsf{Gr}} \mathsf{Cov}_X .$$

Let us briefly pass to underlying groupoids in the equivalence (40) of categories, obtaining an equivalence

$$(42) \qquad \mathsf{hom}_{\mathsf{S}}(X, \iota_0\mathsf{Set}) \xrightarrow{\sim} \mathsf{hom}_{\mathsf{Cat}}(X, \mathsf{Set}) \xrightarrow[\sim]{\mathsf{Gr}} \iota_0(\mathsf{Cov}_X)$$

of groupoids. As this equivalence is natural in the space $X \in \mathsf{S}$, we find that the space $\iota_0\mathsf{Set} \in \mathsf{S}$ is *the classifying space for covering spaces*. In other words, it must carry a universal covering space, such that the composite equivalence (42) is implemented by pullback. And indeed, it is not hard to see that the universal covering space is the forgetful map

$$\iota_0 \left(\mathsf{Set}_* \xrightarrow{\mathsf{fgt}} \mathsf{Set} \right)$$

carrying a pointed set to its underlying set. Indeed, its fiber over an object $S \in \mathsf{Set}$ is the set $S \in \mathsf{Set} \subset \mathsf{S} \subset \mathsf{Cat}$ itself.

More generally, the equivalence (40) of categories is implemented by pullback of the forgetful map

$$\mathsf{Set}_* \xrightarrow{\mathsf{fgt}} \mathsf{Set} .$$

That is, morphisms in $\mathsf{Fun}(X, \mathsf{Set})$ are (by adjunction) equivalent data to functors $X \to \mathsf{hom}_{\mathsf{Cat}}([1], \mathsf{Set})$, which target carries the universal *morphism* of covering spaces.

Exercise 11.1 (4 points)**.** Give an explicit construction of the universal morphism of covering spaces.

More generally, functors $[n] \to \mathsf{Fun}(X, \mathsf{Set})$ are equivalent data to functors $X \to \mathsf{hom}_{\mathsf{Cat}}([n], \mathsf{Set})$. Of course, this is ultimately just to say that the functor category $\mathsf{Fun}(X, \mathsf{Set})$ has ∞-nerve given by the formula

$$\mathsf{N}_\infty(\mathsf{Fun}(X, \mathsf{Set})) \simeq \underline{\mathsf{hom}}_{\mathsf{sS}}(\mathsf{N}_\infty(X), \mathsf{N}_\infty(\mathsf{Set})) \simeq \underline{\mathsf{hom}}_{\mathsf{sS}}(\mathsf{const}_X, \mathsf{N}_\infty(\mathsf{Set})) \simeq \mathsf{hom}_{\mathsf{S}}(X, \mathsf{N}_\infty(\mathsf{Set})_\bullet)$$

(recall §10.2.5 and Exercise 10.5(a)).

11.1.3.   We note the functoriality of the equivalence (40) in the space X. Namely, for any morphism $Y \xrightarrow{\varphi} X$ of spaces, we have a commutative square

$$
\begin{array}{ccc}
\mathsf{Fun}(X, \mathsf{Set}) & \xrightarrow[\sim]{\mathsf{Gr}} & \mathsf{Cov}_X \\
{\scriptstyle (-) \circ f} \downarrow & & \downarrow {\scriptstyle f^*} \\
\mathsf{Fun}(Y, \mathsf{Set}) & \xrightarrow[\mathsf{Gr}]{\sim} & \mathsf{Cov}_Y
\end{array} \quad .
$$

In other words, via the Grothendieck construction, precomposition of functors to Set corresponds to pullback of covering spaces.

We mention once and for all that all of the various Grothendieck constructions that we discuss in this section enjoy analogous functoriality: precomposition of functors will correspond to pullback (of co/cartesian fibrations).

11.1.4.   We now remove the discreteness assumption of §11.1.2, so that the Grothendieck construction defines an equivalence

$$(43) \qquad\qquad\qquad \mathsf{Fun}(X, \mathcal{S}) \xrightarrow[\sim]{\mathsf{Gr}} \mathcal{S}_{/X} \ . $$

This carries a functor $X \xrightarrow{F} \mathcal{S}$ to the space $\mathsf{Gr}(F) \downarrow X$ over X whose fiber over each point $x \in X$ is the space $F(x) \in \mathcal{S}$. The universal bundle of spaces is the forgetful map

$$ \mathcal{S}_* \xrightarrow{\mathsf{fgt}} \mathcal{S} $$

(whose fiber over an object $S \in \mathcal{S}$ is the space $S \in \mathcal{S} \subset \mathsf{Cat}$ itself).

Let us suppose that X is nonempty and connected, so that choosing a basepoint we obtain an equivalence $X \simeq \mathsf{B}G$ where $G := \Omega X \in \mathsf{Grp}$. Then, the equivalence (43) becomes an equivalence

$$ \mathsf{Mod}_G(\mathcal{S}) := \mathsf{Fun}(\mathsf{B}G, \mathcal{S}) \xrightarrow[\sim]{\mathsf{Gr}} \mathcal{S}_{/\mathsf{B}G} \ . $$

In these terms, we can also identify the Grothendieck construction as the ***homotopy G-coinvariants*** functor, i.e. a lift of the left adjoint

$$ \mathsf{Mod}_G(\mathcal{S}) \xrightleftharpoons[\mathsf{const}]{\overset{(-)_{hG}}{\underset{\perp}{\longrightarrow}}} \mathcal{S} \quad : $$

any G-space $\mathsf{B}G \xrightarrow{M} \mathcal{S}$ has a canonical morphism to the terminal G-space $\mathsf{B}G \xrightarrow{\mathsf{pt}} \mathcal{S}$, and then we have that

$$ \mathsf{Gr}(M) \simeq \left(\begin{array}{c} M \\ \downarrow \\ \mathsf{pt} \end{array} \right)_{hG} \quad . $$

Intuitively, the homotopy G-coinvariants $M_{hG} \in \mathcal{S}$ is constructed by beginning with the space $M \in \mathcal{S}$ and adjoining for each point $m \in M$ and each element $g \in G$ an equivalence $m \xrightarrow{\sim} g \cdot m$. In particular, this implies that G-modules are completely specified by their

homotopy coinvariants (considered as spaces over $\mathsf{B}G$). We can also identify the ***homotopy G-invariants*** functor in these terms, i.e. the right adjoint

$$\mathcal{S} \underset{(-)^{hG}}{\overset{\mathsf{const}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_G(\mathcal{S}) \quad :$$

it is the composite

$$\mathsf{Mod}_G(\mathcal{S}) \underset{\sim}{\overset{\mathsf{Gr}}{\longrightarrow}} \mathcal{S}_{/\mathsf{B}G} \overset{\Gamma}{\longrightarrow} \mathcal{S} \ .$$

Intuitively, the homotopy G-invariants M^{hG} is constructed by selecting an element $m \in M$ (the image of the basepoint $* \in \mathsf{B}G$) equipped with a compatible system of equivalences $m \overset{\sim}{\to} g \cdot m$ for each element $g \in G$ (the image of the endomorphism $g \in \mathsf{end}_{\mathsf{B}G}(*) =: G$).

Of course, the functors $(-)_{hG}$ and $(-)^{hG}$ are simply the co/limit functors over $\mathsf{B}G$. More generally, we can view the Grothendieck construction (43) as a lift of the left adjoint

$$\mathsf{Fun}(X, \mathcal{S}) \underset{\mathsf{const}}{\overset{\mathsf{colim}_X(-)}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}$$

in an identical manner.[202] Thereafter, the functor $\mathsf{Fun}(X, \mathcal{S}) \xrightarrow{\mathsf{lim}_X(-)} \mathcal{S}$ is likewise given by the composite

$$\mathsf{Fun}(X, \mathcal{S}) \underset{\sim}{\overset{\mathsf{Gr}}{\longrightarrow}} \mathcal{S}_{/X} \overset{\Gamma}{\longrightarrow} \mathcal{S} \ .$$

11.1.5.   We now generalize §11.1.4 from local systems of ∞-groupoids to local systems of ∞-categories. Now, the Grothendieck construction defines an equivalence

$$\mathsf{Fun}(X, \mathsf{Cat}) \underset{\sim}{\overset{\mathsf{Gr}}{\longrightarrow}} \mathsf{Cat}_{/X} \ .^{203}$$

Just as in §11.1.4 (and in fact in §11.1.2 as well), this is nothing other than a canonical lift of the colimit functor $\mathsf{Fun}(X, \mathsf{Cat}) \xrightarrow{\mathsf{colim}_X(-)} \mathsf{Cat}$, and moreover the limit functor $\mathsf{Fun}(X, \mathsf{Cat}) \xrightarrow{\mathsf{lim}_X(-)} \mathsf{Cat}$ is equivalent to the composite

$$\mathsf{Fun}(X, \mathsf{Cat}) \underset{\sim}{\overset{\mathsf{Gr}}{\longrightarrow}} \mathsf{Cat}_{/X} \overset{\Gamma}{\longrightarrow} \mathsf{Cat}$$

(where now Γ denotes the ∞-*category* of sections).

11.1.6.   It is worth highlighting the case of a constant functor $\mathsf{B}G \xrightarrow{\mathsf{const}_{\mathcal{C}}} \mathsf{Cat}$, i.e. the trivial action of an ∞-group G on an ∞-category $\mathcal{C}$. Then, §11.1.5 specializes to give identifications

$$\mathcal{C}_{hG} \simeq \mathcal{C} \times \mathsf{B}G \qquad \text{and} \qquad \mathcal{C}^{hG} \simeq \Gamma \left(\begin{array}{c} \mathcal{C} \times \mathsf{B}G \\ \downarrow \\ \mathsf{B}G \end{array} \right) \simeq \mathsf{Fun}(\mathsf{B}G, \mathcal{C}) \ .$$

---

[202]In particular, the colimit of the constant functor $X \xrightarrow{\mathsf{pt}} \mathcal{S}$ is simply X itself.

[203]We will describe the universal bundle of ∞-categories (pullback of which implements this equivalence) in §11.4.6.

In particular, the homotopy G-invariants of the trivial G-action on $\mathcal{C}$ is the ∞-category of G-modules in $\mathcal{C}$.

11.1.7.   It is also worth unpacking the connection between the Grothendieck construction and principal bundles.

Let $G \in \mathsf{Grp}$ be an ∞-group. We define the ***regular representation*** of G to be the G-space

$$G/e := \left(\mathsf{B}G \xrightarrow{\mathsf{hom}_{\mathsf{B}G}(*,-)} \mathcal{S} \right) \in \mathsf{Fun}(\mathsf{B}G, \mathcal{S}) \ .$$

Exercise 11.2 (6 points).
  (a) Show that under the Grothendieck construction (i.e. the equivalence (43) in the case that $X = \mathsf{B}G$), the object $G/e \in \mathsf{Fun}(\mathsf{B}G, \mathcal{S})$ corresponds to the canonical map $(\mathsf{pt} \xrightarrow{*} \mathsf{B}G) \in \mathcal{S}_{/\mathsf{B}G}$ selecting the basepoint.
  (b) Use part (a) to deduce an equivalence $\mathsf{hom}_{\mathsf{Fun}(\mathsf{B}G,\mathcal{S})}(G/e, G/e) \simeq G$ of ∞-monoids.

It follows from Exercise 11.2(b) that the full subcategory of $\mathsf{Mod}_G(\mathcal{S}) := \mathsf{Fun}(\mathsf{B}G, \mathcal{S})$ on the regular representation is equivalent to $\mathsf{B}G$ itself.

Now, fix an arbitrary space $X \in \mathcal{S}$. A ***G-space over*** X is a functor $\mathsf{B}G \to \mathcal{S}_{/X}$. These assemble into the ∞-category $\mathsf{Fun}(\mathsf{B}G, \mathcal{S}_{/X})$. A G-space over X is called a ***principal G-bundle*** if its fiber over every point $x \in X$ is equivalent to the regular representation. We write

$$\mathsf{Bun}_G(X) \subseteq \mathsf{Fun}(\mathsf{B}G, \mathcal{S}_{/X})$$

for the full subcategory on the principal G-bundles.

Exercise 11.3 (6 points).
  (a) Establish equivalences

  $$\mathsf{Fun}(\mathsf{B}G, \mathcal{S}_{/X}) \simeq \mathsf{Fun}(\mathsf{B}G, \mathcal{S})_{/\mathsf{const}_X} \simeq \mathcal{S}_{/(\mathsf{B}G \times X)} \simeq \mathsf{Fun}(X, \mathcal{S}_{/\mathsf{B}G}) \simeq \mathsf{Fun}(X, \mathsf{Fun}(\mathsf{B}G, \mathcal{S})) \ .$$

  (b) Identify the principal G-bundles under the equivalent definitions of G-spaces over X given by part (a).

In particular, Exercise 11.3(b) yields the first equivalence in the composite equivalence

$$\mathsf{Bun}_G(X) \simeq \mathsf{Fun}(X, \mathsf{B}G) \simeq \mathsf{hom}_{\mathcal{S}}(X, \mathsf{B}G) \ ,$$

so that $\mathsf{B}G$ is indeed the classifying space for principal G-bundles (and the ∞-category $\mathsf{Bun}_G(X)$ is in fact an ∞-groupoid).

11.2. **Overview of the Grothendieck construction over ∞-categories.**

11.2.1. We now generalize the discussion of §11.1 by replacing the ∞-groupoid $X \in \mathcal{S}$ with an ∞-category $\mathcal{B} \in \mathsf{Cat}$. The situation is substantially more subtle now that not all morphisms are invertible. In fact, there will be both *covariant* and *contravariant* Grothendieck constructions, which are equivalences

$$\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) \xrightarrow[\sim]{\mathsf{Gr}} \mathsf{coCart}_{\mathcal{B}} \longleftrightarrow \mathsf{Cat}_{/\mathcal{B}} \longleftrightarrow \mathsf{Cart}_{\mathcal{B}} \xleftarrow[\sim]{\mathsf{Gr}^-} \mathsf{Fun}(\mathcal{B}^{\mathsf{op}}, \mathsf{Cat})$$

to certain (non-full) subcategories of the ∞-category $\mathsf{Cat}_{/\mathcal{B}}$ of ∞-categories over $\mathcal{B}$, whose objects are respectively called **cocartesian fibrations** and **cartesian fibrations** over $\mathcal{B}$.[204] The inverses to these equivalences are called the **straightening** functors, or respectively the **cocartesian straightening** and **cartesian straightening** functors in order to emphasize the variance.[205]

11.2.2. We give a heuristic description of the functors Gr and Gr^-, which we will make more precise in §11.5.

Given a functor $\mathcal{B} \xrightarrow{F} \mathsf{Cat}$, its covariant Grothendieck construction $\mathsf{Gr}(F)$ can be informally described as arising through the following procedure: begin with the disjoint union of the ∞-categories $F(b) \in \mathsf{Cat}$ for all $b \in \mathcal{B}$; adjoin for each morphism $b_0 \xrightarrow{\varphi} b_1$ and each object $e \in F(b_0)$ a morphism $e \to F(\varphi)(e)$; and impose higher relations according to composition in $\mathcal{B}$.

Dually, given a functor $\mathcal{B}^{\mathsf{op}} \xrightarrow{F} \mathsf{Cat}$, its contravariant Grothendieck construction $\mathsf{Gr}^-(F)$ can be informally described as arising through the following procedure: begin with the disjoint union of the ∞-categories $F(b^\circ) \in \mathsf{Cat}$ for all $b \in \mathcal{B}$; adjoin for each morphism $b_0 \xrightarrow{\varphi} b_1$ and each object $e \in F(b_1^\circ)$ a morphism $F(\varphi^\circ)(e) \to e$; and impose higher relations according to composition in $\mathcal{B}$.

In both cases, the projection functor to $\mathcal{B}$ is the evident one (which carries the newly adjoined morphisms described above to the morphism $b_0 \xrightarrow{\varphi} b_1$).

11.2.3. As implied in §11.2.2 (and explained further in §11.5), the Grothendieck constructions Gr and Gr^- can be respectively identified as the **left-lax colimit** and the **right-lax colimit**. In particular, for any functor $\mathcal{B} \xrightarrow{F} \mathsf{Cat}$, there is a canonical functor

$$\mathsf{Gr}(F) \longrightarrow \mathsf{colim}_{\mathcal{B}}(F)$$

---

[204]These respectively generalize the notions of *Grothendieck opfibrations* and *Grothendieck fibrations* in classical category theory.

[205]The explanation for this terminology stems from the point-set presentations of these equivalences, which are established in [Lur09] as Quillen equivalences between model categories: one of co/cartesian fibrations among quasicategories (which only encode diagrams of ∞-categories up to contractible choices) and another of morphisms in $\mathsf{cat}(\mathsf{sSet})$ (the category of simplicial categories) to a particular presentation of Cat. The left Quillen equivalences "straighten" fibrations into functors, while the right Quillen equivalences "unstraighten" functors into fibrations.

to the ("strict") colimit, which is a localization at the formally adjoined morphisms $e \to$ $F(\varphi)(e)$.[206] Dually, given any functor $\mathcal{B}^{\mathsf{op}} \xrightarrow{F} \mathsf{Cat}$, there is a canonical functor

$$\mathsf{Gr}^-(F) \longrightarrow \mathsf{colim}_{\mathcal{B}^{\mathsf{op}}}(F)$$

to the ("strict") colimit, which is a localization at the formally adjoined morphisms $F(\varphi^{\circ})(e) \to$ e. These facts generalize the assertion in §11.1.5 that the Grothendieck construction over an ∞-groupoid is simply the colimit functor (because for diagrams indexed by an ∞-groupoid, lax colimits and strict colimits coincide).

Similar considerations generalize the corresponding assertion in §11.1.5 regarding limits. Namely, given any functor $\mathcal{B} \xrightarrow{F} \mathsf{Cat}$, the ∞-category

$$\Gamma \left(\begin{array}{c} \mathsf{Gr}(F) \\ \downarrow \\ \mathcal{B} \end{array} \right)$$

of sections of the corresponding cocartesian fibration can be identified as the **left-lax limit** of F. Dually, given any functor $\mathcal{B}^{\mathsf{op}} \xrightarrow{F} \mathsf{Cat}$, the ∞-category

$$\Gamma \left(\begin{array}{c} \mathsf{Gr}^-(F) \\ \downarrow \\ \mathcal{B} \end{array} \right)$$

of sections of the corresponding cartesian fibration can be identified as the **right-lax limit** of F. In both cases, the strict limit can be identified as the full subcategory consisting of those sections that carry every morphism in $\mathcal{B}$ to one of the formally adjoined morphisms in the co/cartesian fibration.

We will not emphasize these points in the following discussion, but we encourage the reader to keep them in mind while continuing through §11. A more thorough treatment of lax limits (including those of lax functors, which can be a rather subtle matter) can be found in [AMGR, §A].

11.3. **The Grothendieck construction over** [1].

11.3.1.   We now study the Grothendieck construction in the special case that $\mathcal{B} = [1]$, the unique nonidentity morphism in which we denote by $0 \xrightarrow{\iota} 1$.

---

[206]Note that this colimit does not generally lie over $\mathcal{B}$, but it retains a canonical morphism to the ∞-category

$$\mathsf{colim}\left(\mathcal{B} \xrightarrow{\mathsf{const}_{\mathsf{pt}}} \mathsf{Cat} \right) \simeq \mathcal{B}^{\mathsf{gpd}}$$

(which equivalence follows from Exercise 12.3).

Given a functor $[1] \xrightarrow{F} \mathsf{Cat}$, its covariant Grothendieck construction is

$$
(44) \qquad \mathsf{Gr}(F) := \mathsf{colim} \left(\begin{array}{ccc} F(0) & \xrightarrow{F(\iota)} & F(1) \\ {\scriptstyle (\mathsf{id}_{F(0)}, \mathsf{const}_1)} \Big\downarrow & & \\ F(0) \times [1] & & \end{array} \right),
$$

with the canonical functor $\mathsf{Gr}(F) \to [1]$ being classified by the evident commutative square

$$
\begin{array}{ccc}
F(0) & \xrightarrow{F(\iota)} & F(1) \\
{\scriptstyle (\mathsf{const}_1, \mathsf{id}_{F(0)})} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathsf{const}_1} \\
[1] \times F(0) & \longrightarrow & [1]
\end{array} \quad .
$$

This may be thought of as a *directed mapping cylinder* of the functor $F(0) \xrightarrow{F(\iota)} F(1)$. [changed 7/8]
Moreover, it follows directly from the formula (44) that functors $\mathsf{Gr}(F) \to \mathcal{C}$ are equivalent data to lax-commutative triangles

which illustrates the fact that $\mathsf{Gr}(F)$ is the left-lax colimit of F.

Let us examine the ∞-category $\mathsf{Gr}(F)$ more closely. First of all, its fibers over $[1]$ are given by

$$
\mathsf{Gr}(F)_0 \simeq F(0) \qquad \text{and} \qquad \mathsf{Gr}(F)_1 \simeq F(1) \ .
$$

This determines the hom-spaces between objects that lie in the same fiber; the remaining hom-spaces are specified by the fact that for any $x \in \mathsf{Gr}(F)_0$ and any $y \in \mathsf{Gr}(F)_1$ we have an equivalence

$$
\mathsf{hom}_{\mathsf{Gr}(F)}(x, y) \xleftarrow{\sim} \mathsf{hom}_{\mathsf{Gr}(F)_1}(F(\iota)(x), y)
$$

given by precomposition with the distinguished morphism $x \to F(\iota)(x)$.

The existence of this distinguished morphism for each object $x \in \mathsf{Gr}(F)_0$ is the key point. Namely, given an arbitrary functor $\mathcal{E} \xrightarrow{p} [1]$ and objects $x \in \mathcal{E}_0$ and $x' \in \mathcal{E}_1$, we say that a morphism $x \xrightarrow{\varphi} x'$ in $\mathcal{E}$ is p-**cocartesian**, or that it is a p-**cocartesian lift** of the morphism $0 \xrightarrow{\iota} 1$ in $[1]$ at the object $x \in \mathcal{E}_0$, if for any object $y \in \mathcal{E}_1$ precomposition with φ determines an equivalence

$$
\mathsf{hom}_{\mathcal{E}}(x, y) \xleftarrow{\sim} \mathsf{hom}_{\mathcal{E}_1}(x', y) \ . \ ^{207}
$$

---

[207] This may be thought of as a directed version of the path-lifting condition in the definition of a Serre fibration.

In other words, the object $x' \in \mathcal{E}_1$ corepresents the functor

$$\mathcal{E}_1 \longhookrightarrow \mathcal{E} \xrightarrow{\mathsf{hom}_{\mathcal{E}}(x,-)} \mathcal{S} \ ,$$

with the universal datum being the element $\varphi \in \mathsf{hom}_{\mathcal{E}}(x, x')$. This makes it clear that a p-cocartesian lift of ι at x is unique if it exists. We may simply write $x \to \iota_* x$ for a p-cocartesian lift of ι at x, and refer to $\iota_* x$ as the **cocartesian pushforward** of x along ι. As the notation suggests, (essentially by Yoneda) this construction assembles into a partially-defined functor ι_* to $\mathcal{E}_1$ from the full subcategory of $\mathcal{E}_0$ on those objects that admit cocartesian pushforwards along ι. Indeed, this may be phrased alternatively as follows.

Exercise 11.4 (4 points)**.** Show that p-cocartesian lifts of ι are precisely pointwise left adjoints to the restriction functor $\mathcal{E}_0 \xleftarrow{\mathsf{ev}_0} \Gamma([1]; \mathcal{E})$.

In terms of Exercise 11.4, the partially-defined functor ι_* is simply the composite

$$\mathcal{E}_0 \xdashrightarrow{(\mathsf{ev}_0)^L} \Gamma([1]; \mathcal{E}) \xrightarrow{\mathsf{ev}_1} \mathcal{E}_1$$

of the partially-defined left adjoint $(\mathsf{ev}_0)^L$ of ev_0 followed by ev_1.

We say that a functor $\mathcal{E} \xrightarrow{p} [1]$ is a **cocartesian fibration** if for every $x \in \mathcal{E}_0$ there exists a p-cocartesian lift of ι at x. Given cocartesian fibrations $\mathcal{E} \xrightarrow{p} [1]$ and $\mathcal{F} \xrightarrow{q} [1]$, we say that a morphism

$$
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{F} \\
& {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\
& [1] &
\end{array}
$$

in $\mathsf{Cat}_{/[1]}$ is **strict** (or **cocartesian**) if it carries p-cocartesian morphisms in $\mathcal{E}$ to q-cocartesian morphisms in $\mathcal{F}$; otherwise, for emphasis we may say that it is **left-lax**.[208] We write

$$\mathsf{coCart}_{[1]} \subset \mathsf{coCart}_{[1]}^{\mathsf{l.lax}} \subset \mathsf{Cat}_{/[1]}$$

for the subcategories whose objects are the cocartesian fibrations over $[1]$ and whose morphisms are respectively the strict and left-lax morphisms between them. (So the latter inclusion is of a full subcategory, while the former inclusion is not.)

From this discussion, it should be plausible that the Grothendieck construction indeed determines an equivalence

$$\mathsf{Fun}([1], \mathsf{Cat}) \xrightarrow[\sim]{\mathsf{Gr}} \mathsf{coCart}_{[1]} \ ;$$

its inverse – the (**cocartesian**) **straightening** functor – carries a cocartesian fibration $\mathcal{E} \xrightarrow{p} [1]$ to the functor $[1] \to \mathsf{Cat}$ that selects the functor $\mathcal{E}_0 \xrightarrow{\iota_*} \mathcal{E}_1$.

---

[208]The term "cocartesian" is much more commonly used that "strict", but we find that the latter introduces substantially less awkward language.

More generally, the left-lax morphisms among cocartesian fibrations are so named because they are equivalent to *left-lax* natural transformations between functors $[1] \to \mathsf{Cat}$: namely, a left-lax morphism $\mathsf{Gr}(F) \xrightarrow{\alpha} \mathsf{Gr}(G)$ corresponds to a diagram

$$
(45) \qquad
\begin{array}{ccc}
F(0) & \xrightarrow{F(\iota)} & F(1) \\
{\scriptstyle\alpha_0}\downarrow & \nearrow & \downarrow{\scriptstyle\alpha_1} \\
G(0) & \xrightarrow[G(\iota)]{} & G(1)
\end{array}
$$

in Cat.[209] To see why this might be true, observe that α carries each cocartesian lift $x \to F(\iota)(x)$ in $\mathsf{Gr}(F)$ of ι to an arbitrary morphism $\alpha_0(x) \to \alpha_1(F(\iota)(x))$ in $\mathsf{Gr}(G)$, which admits a unique factorization

$$
(46) \qquad
\begin{array}{ccc}
\alpha_0(x) & \longrightarrow & \alpha_1(F(\iota)(x)) \\
 & \searrow & \uparrow \\
 & & G(\iota)(\alpha_0(x))
\end{array}
$$

through a cocartesian lift of ι at $\alpha_0(x)$. The component of the natural transformation in diagram (45) at the object $x \in F(0)$ is the vertical morphism in diagram (46), which lies in $G(1) \simeq \mathsf{Gr}(G)_1 \subseteq \mathsf{Gr}(G)$.

We now describe cartesian fibrations over $[1]$, which are similar but involve a change in handedness. Given a functor $[1]^{\mathsf{op}} \xrightarrow{F} \mathsf{Cat}$, its contravariant Grothendieck construction is

$$
(47) \qquad \mathsf{Gr}^-(F) := \mathsf{colim} \left(
\begin{array}{ccc}
F(1^\circ) & \xrightarrow{(\mathsf{const}_0,\mathsf{id}_{F(1^\circ)})} & [1] \times F(1^\circ) \\
{\scriptstyle F(\iota^\circ)}\downarrow & & \\
F(0^\circ) & &
\end{array}
\right),
$$

with the canonical functor $\mathsf{Gr}^-(F) \to [1]$ being classified by the evident commutative square

$$
\begin{array}{ccc}
F(1^\circ) & \xrightarrow{(\mathsf{const}_0,\mathsf{id}_{F(1^\circ)})} & [1] \times F(1^\circ) \\
{\scriptstyle F(\iota^\circ)}\downarrow & & \downarrow \\
F(0^\circ) & \xrightarrow[\mathsf{const}_0]{} & [1]
\end{array} \qquad .
$$

This may *also* be thought of as a directed mapping cylinder of the functor $F(0^\circ) \xleftarrow{F(\iota^\circ)} F(1^\circ)$, albeit one with the opposite handedness,[changed 7/8] and it follows directly from the formula (47)

---

[209]To be more precise, at the time of writing this correspondence and various other related statements are strongly expected to be true but have not been established in full generality; see the appendix of [GR17] for a systematic (but incomplete) treatment of $(\infty, 2)$-category theory.

that functors $\mathsf{Gr}^-(F) \to \mathcal{C}$ are equivalent data to lax-commutative triangles

$$
\begin{array}{ccc}
& F(1^\circ) & \\
F(\iota^\circ) \Big\downarrow & \nearrow & \\
& \Uparrow & \mathcal{C} \quad . \\
& \nearrow & \\
& F(0^\circ) &
\end{array}
$$

Now, the key point is that for any objects $x \in \mathsf{Gr}^-(F)_0 \simeq F(0^\circ)$ and $y \in \mathsf{Gr}^-(F)_1 \simeq F(1^\circ)$, we have an equivalence

$$
\mathsf{hom}_{\mathsf{Gr}^-(F)}(x,y) \xleftarrow{\ \sim\ } \mathsf{hom}_{\mathsf{Gr}^-(F)_0}(x, F(\iota^\circ)(y))
$$

given by postcomposition with the distinguished morphism $F(\iota^\circ)(y) \to y$, which is a **carte-sian lift** of the morphism $0 \xrightarrow{\iota} 1$ at the object $y \in \mathsf{Gr}^-(F)_1$.[210] A functor $\mathcal{E} \xrightarrow{p} [1]$ is a **cartesian fibration** if for every $y \in \mathcal{E}_1$ there exists a p-cartesian lift of ι at y, and we similarly define the subcategories

$$
\mathsf{Cart}_{[1]} \subset \mathsf{Cart}_{[1]}^{\mathsf{r.lax}} \subset \mathsf{Cat}_{/[1]} \ ,
$$

whose morphisms are respectively called **strict** (or **cartesian**) and **right-lax**. A right-lax morphism $\mathsf{Gr}^-(F) \xrightarrow{\alpha} \mathsf{Gr}^-(G)$ corresponds to a diagram

(48)
$$
\begin{array}{ccc}
F(0^\circ) & \xleftarrow{F(\iota^\circ)} & F(1^\circ) \\
\alpha_0 \Big\downarrow & \searrow & \Big\downarrow \alpha_1 \\
G(0^\circ) & \xleftarrow[G(\iota^\circ)]{} & G(1^\circ)
\end{array}
$$

in Cat.[211]

---

[210]Note that we have an equivalence

$$
\mathsf{Gr}^-(F)^{\mathsf{op}} \simeq \mathsf{Gr}\left(F(1^\circ)^{\mathsf{op}} \xrightarrow{F(\iota^\circ)^{\mathsf{op}}} F(1^\circ)^{\mathsf{op}} \right) \ :
$$

passing to opposites switches not just the directions of the co/cartesian morphisms, but also the directions of the morphisms in the fibers. Passing directly between cocartesian and cartesian fibrations *without* taking fiberwise opposites (i.e. passing through the composite equivalence $\mathsf{coCart}_\mathcal{B} \simeq \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) \simeq \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}}$) is nontrivial to implement directly [BGN18]. (In fact, it makes use of the *twisted arrow ∞-category* of $\mathcal{B}$, introduced in §11.5.2 below.)

[211]Our choice to discuss both cocartesian and cartesian fibrations over $\mathcal{B}$ (instead of cocartesian fibrations over $\mathcal{B}$ and cartesian fibrations over $\mathcal{B}^{\mathsf{op}}$) means that one must pass through the equivalence $[1] \simeq [1]^{\mathsf{op}}$ to compare the notions of left-lax and right-lax natural transformations between functors $[1] \to \mathsf{Cat}$ that we have just introduced. Specifically, horizontally reflecting diagram (48) yields a diagram which is comparable to diagram (45), but in which the 2-morphism runs in the opposite direction. These illustrate the general definitions of left-lax and right-lax natural transformations.

11.3.2.   After the discussion of §11.3.1, we can now rigorously define an ***adjunction*** between ∞-categories $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}$: it is a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{C} & \longrightarrow & \mathcal{A} & \longleftarrow & \mathcal{D} \\
\downarrow & & {\scriptstyle p}\downarrow & & \downarrow \\
\{0\} & \longrightarrow & [1] & \longleftarrow & \{1\}
\end{array}
$$

in Cat in which both squares are pullbacks (i.e. we have specified equivalences $\mathcal{C} \simeq \mathcal{A}_0$ and $\mathcal{D} \simeq \mathcal{A}_1$) and moreover p is both a cocartesian fibration and a cartesian fibration. Indeed, for any objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$ we obtain equivalences

$$
\mathsf{hom}_{\mathcal{C}}(x, \iota^* y) \xrightarrow{\sim} \mathsf{hom}_{\mathcal{A}}(x, y) \xleftarrow{\sim} \mathsf{hom}_{\mathcal{D}}(\iota_* x, y) \ ,
$$

i.e. the functor $\mathcal{A} \xrightarrow{p} [1]$ records the data of an adjunction

$$
\mathcal{C} \underset{\iota^*}{\overset{\iota_*}{\underset{\perp}{\rightleftarrows}}} \mathcal{D} \quad .
$$

Specifically, the left adjoint is the cocartesian straightening of p, while the right adjoint is the cartesian straightening of p.

Exercise 11.5 (4 points). Describe the unit and counit morphisms of an adjunction, and verify that they satisfy the desired universal properties (namely, they implement the equivalences between hom-spaces that constitute the adjunction).

   More generally, this same framework immediately generalizes to give a notion of *partial adjoints*. For instance, given a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, we may form the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{C} & \longrightarrow & \mathsf{Gr}(F) & \longleftarrow & \mathcal{D} \\
\downarrow & & {\scriptstyle p}\downarrow & & \downarrow \\
\{0\} & \longrightarrow & [1] & \longleftarrow & \{1\}
\end{array}
$$

in Cat in which both squares are pullbacks and p is a cocartesian fibration, and the partial right adjoint of F is the partially-defined functor ι^* to $\mathsf{Gr}(F)_0 \simeq \mathcal{C}$ from the full subcategory of $\mathsf{Gr}(F)_1 \simeq \mathcal{D}$ on those objects that admit cartesian pullbacks along ι.

11.4. **Co/cartesian fibrations.**

11.4.1.   We now rigorously define cocartesian fibrations; we leave it to the reader to dualize the discussion to apply to cartesian fibrations. Throughout, we fix a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$. To ease notation, we simply underline data in $\mathcal{E}$ to denote its image in $\mathcal{B}$.

We say that a morphism $e_0 \xrightarrow{\psi} e_1$ in $\mathcal{E}$ is p-***cocartesian***, or that it is a p-***cocartesian lift*** of the morphism $\underline{e_0} \xrightarrow{\underline{\psi}} \underline{e_1}$ in $\mathcal{B}$ at the object $e_0 \in \mathcal{E}_{\underline{e_0}}$, if the induced commutative square

(49)
$$
\begin{array}{ccc}
\mathcal{E}_{e_1/} & \xrightarrow{\psi\circ(-)} & \mathcal{E}_{e_0/} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\mathcal{B}_{\underline{e_1}/} & \xrightarrow[\underline{\psi}\circ(-)]{} & \mathcal{B}_{\underline{e_0}/}
\end{array}
$$

is a pullback square.[212] We can schematically depict this situation in the diagram

in which the commutative squares indicate compatibility of morphisms with repect to the functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ and the object $f \in \mathcal{E}$ is arbitrary.[213] Namely, in the commutative square (49), the canonical morphism to the pullback implements the passage from the datum of a dotted morphism to the data of the dashed morphisms. If the commutative square (49) is in fact a pullback square, then this passage is an equivalence, i.e. given the dashed morphisms we obtain a canonical dotted morphism. Said differently, the square

(50)
$$
\begin{array}{ccc}
\mathsf{hom}_{\mathcal{E}}(e_1, f) & \xrightarrow{(-)\circ\psi} & \mathsf{hom}_{\mathcal{E}}(e_0, f) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\mathsf{hom}_{\mathcal{B}}(\underline{e_1}, \underline{f}) & \xrightarrow[(-)\circ\underline{\psi}]{} & \mathsf{hom}_{\mathcal{B}}(\underline{e_0}, \underline{f})
\end{array}
$$

in $\mathcal{S}$ is a pullback. In fact, it turns out that if for all objects $f \in \mathcal{E}$ the square (50) is a pullback then the morphism $e_0 \xrightarrow{\psi} e_1$ is p-cocartesian.

Note that (in contrast with the special case where $\mathcal{B} = [1]$ discussed in §11.3.1) this definition makes reference to *all* morphisms out of e_1, not merely those within the fiber $\mathcal{E}_{\underline{e_1}}$ in which it lies. We discuss a useful weaker notion in §11.4.2.

---

[212]To be precise, given a morphism $b_0 \xrightarrow{\varphi} b_1$ in $\mathcal{B}$ and an object $e \in \mathcal{E}$, one can only speak of "a p-cocartesian lift of φ at e" after specifying the data necessary to witness e as lying in the fiber $\mathcal{E}_{b_0}$, namely an equivalence $p(e) \simeq b_0$ in $\mathcal{B}$.

[213]To be less schematic, one could rewrite this as an actual commutative diagram in the ∞-category $\mathsf{Gr}([1]) \xrightarrow{\mathcal{E} \xrightarrow{p} \mathcal{B}} \mathsf{Cat}$: the vertical assignments would then become cocartesian lifts of the morphism $0 \xrightarrow{\iota} 1$ in $[1]$.

Exercise 11.6 (2 points). Suppose that the morphism $\underline{e_0} \xrightarrow{\psi} \underline{e_1}$ in $\mathcal{B}$ is an equivalence. Show that the morphism ψ is p-cocartesian if and only if it is an equivalence.

We say that the functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a ***cocartesian fibration*** if for every object $e \in \mathcal{E}$ and every morphism $\underline{e} \xrightarrow{\varphi} b$ in $\mathcal{B}$ there exists a p-cocartesian lift of φ at e.

Alternatively, we can characterize cocartesian fibrations as follows. For each $b \in \mathcal{B}$, observe that we have a canonical fully functor

$$
\begin{array}{ccccc}
\mathcal{E} & \longleftarrow & \mathcal{E}_{/b} & \dashleftarrow & \mathcal{E}_b \\
{\scriptstyle p}\downarrow & & \downarrow & & \downarrow \\
\mathcal{B} & \underset{\mathsf{fgt}}{\longleftarrow} & \mathcal{B}_{/b} & \underset{\mathsf{id}_b}{\longleftarrow} & \mathsf{pt}
\end{array}
$$

between pullbacks (in which diagram both squares are pullbacks).

Exercise 11.7 (4 points). Prove that if $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a cocartesian fibration then for every object $b \in \mathcal{B}$ the functor

$$\mathcal{E}_{/b} \longleftarrow \mathcal{E}_b$$

admits a left adjoint.

Of course, such a left adjoint implements cocartesian pushforward, i.e. it acts according to the formula

$$
\left(
\begin{array}{c}
e \\
\vrule height 12pt depth 0pt width 0pt \downarrow \\
\underline{e} \xrightarrow{\ \varphi\ } b
\end{array}
\right)
\longmapsto
\left(
\begin{array}{c}
\varphi_* e \\
\downarrow \\
b
\end{array}
\right).
$$

Exercise 11.8 (2 points). Prove that for any ∞-category $\mathcal{B}$, the functors

$$\mathsf{Ar}(\mathcal{B}) \xrightarrow{\ s\ } \mathcal{B} \qquad \text{and} \qquad \mathsf{Ar}(\mathcal{B}) \xrightarrow{\ t\ } \mathcal{B}$$

are respectively cartesian and cocartesian fibrations.

Exercise 11.9 (6 points). Let VBdl denote the category whose objects are pairs of a topological space $X \in \mathsf{Top}$ and a vector bundle $E \downarrow X$, and whose morphisms $(X_0, E_0) \to (X_1, E_1)$ are given by commutative squares

$$
\begin{array}{ccc}
E_0 & \xrightarrow{\ \tilde{f}\ } & E_1 \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{\ f\ } & X_1
\end{array}
$$

in Top in which $\tilde{f}$ is fiberwise linear. Prove that the forgetful functor

$$\mathsf{VBdl} \xrightarrow{\ \mathsf{fgt}\ } \mathsf{Top}$$

is a cartesian fibration.

Exercise 11.10 (4 points)**.** Prove that the composition of two cartesian fibrations is a cartesian fibration.

Of course, we extend the notions of morphisms between cocartesian fibrations from that introduced in §11.3.1: we define subcategories

$$\text{(51)} \qquad\qquad \mathsf{coCart}_{\mathcal{B}} \subseteq \mathsf{coCart}_{\mathcal{B}}^{\text{l.lax}} \subseteq \mathsf{Cat}_{/\mathcal{B}} \ ,$$

whose morphisms are respectively called **strict** (or **cocartesian**) and **left-lax**.

The following exercise explains why the discussion of §11.1 was so much simpler than that of §11.2.1 (and illustrates the consistency of §11.2.1 with §11.1.5).

Exercise 11.11 (2 points)**.** Prove that if $\mathcal{B}$ is an ∞-groupoid, then the inclusions (51) are equivalences.

11.4.2.   We briefly discuss a useful weakening of the notion of a cocartesian fibration (again leaving the cartesian variant implicit).

We continue to use the notation of §11.4.1. Given a morphism $e_0 \xrightarrow{\psi} e_1$ in $\mathcal{E}$, we simply write $[1] \xrightarrow{\underline{\psi}} \mathcal{B}$ for the composite $[1] \xrightarrow{\psi} \mathcal{E} \xrightarrow{p} \mathcal{B}$, and we write

$$
\begin{array}{ccc}
\underline{\psi}^{*}\mathcal{E} & \longrightarrow & \mathcal{E} \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\
[1] & \xrightarrow{\ \underline{\psi}\ } & \mathcal{B}
\end{array}
$$

for the pullback. Then, we say that the morphism ψ is **locally p-cocartesian** (or that it is a **locally p-cocartesian lift** of $\underline{\psi}$ at e_0) if it defines a p'-cocartesian lift of ι.[214] Of course, we say that $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a **locally cocartesian fibration** if for every object $e \in \mathcal{E}$ and every morphism $\underline{e} \xrightarrow{\varphi} b$ in $\mathcal{B}$ there exists a locally p-cocartesian lift of φ at e.

We have the following refinement of Exercise 11.7.

Exercise 11.12 (4 points)**.** Prove that $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a locally cocartesian fibration if and only if for every object $b \in \mathcal{B}$ the functor

$$\mathcal{E}_{/b} \longleftrightarrow \mathcal{E}_b$$

admits a left adjoint.

---

[214]By definition we have a commutative triangle

$$
\begin{array}{ccc}
 & & \mathcal{E} \\
 & \nearrow^{\psi} & \downarrow{\scriptstyle p} \ , \\
[1] & \xrightarrow[\ \underline{\psi}\ ]{} & \mathcal{B}
\end{array}
$$

in which the diagonal morphism is equivalent data to a section of p'.

Note that the definition of a morphism in $\mathcal{E}$ being locally p-cocartesian does not make reference to the "global" structure of $\mathcal{E}$, in contrast with the definition of it being p-cocartesian.[215] Therefore, it is generally much easier to check that a functor is a locally cocartesian fibration than it is to check that it is a cocartesian fibration.

This makes the following fact extremely useful: a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a cocartesian fibration iff it is a locally cocartesian fibration and the locally p-cocartesian morphisms are stable under composition (in which case they are in fact p-cocartesian). Note that by induction it suffices merely to check two-fold compositions. Said differently, the functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a cocartesian fibration if and only for every functor $[n] \xrightarrow{\sigma} \mathcal{B}$ with $1 \leqslant n \leqslant 2$, the base change $\sigma^*\mathcal{E} \to [n]$ is a cocartesian fibration.

As a result, it should not be surprising that the minimal example of a locally cocartesian fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ which is not a cocartesian fibration has $\mathcal{B} = [2]$. Namely, it is the functor $\mathsf{sd}([2]) \xrightarrow{\mathsf{max}} [2]$ from the **subdivision** of $[2]$ (the poset of strings of nonidentity morphisms in $[2]$ (possibly of length zero), ordered by factorization) given by taking maxima, as depicted in the diagram

(where e.g. 02 and 012 respectively denote the strings $0 \to 2$ and $0 \to 1 \to 2$).

Exercise 11.13 (4 points). Verify that the functor $\mathsf{sd}([2]) \xrightarrow{\mathsf{max}} [2]$ is a locally cocartesian fibration but not a cocartesian fibration.

Indeed, whereas cocartesian fibrations over $\mathcal{B}$ are classified by functors from $\mathcal{B}$ to Cat, locally cocartesian fibrations over $\mathcal{B}$ are classified by **left-lax functors** from $\mathcal{B}$ to Cat: these take objects to objects and morphisms to morphisms, but they only respect composition laxly. For instance, given a locally cocartesian fibration $\mathcal{E} \xrightarrow{p} [2]$, for each object $e \in \mathcal{E}_0$

---

[215]Indeed, a p-cocartesian morphism is automatically locally p-cocartesian. Conversely, if there exist both a p-cocartesian lift and a locally p-cocartesian lift at $e \in \mathcal{E}$ of a morphism $\underline{e} \to b$ in $\mathcal{B}$, then these must coincide.

we obtain a cocartesian pushforward $(01)_* e \in \mathcal{E}_1$ and thereafter a cocartesian pushforward $(12)_* (01)_* e \in \mathcal{E}_2$, and hence a canonical and unique factorization

$$
\begin{array}{ccc}
 & (01)_* e & \\
\nearrow & & \searrow \\
e & & (12)_* (01)_* e \\
\dashrightarrow & & \uparrow \\
 & (02)_* e &
\end{array}
$$

of the composite.[216] This factorization is the component at e of a natural transformation

(52)

$$
\begin{array}{ccc}
 & \mathcal{E}_1 & \\
{}^{(01)_*} \nearrow & \Uparrow & \searrow {}^{(12)_*} \\
\mathcal{E}_0 & \xrightarrow{\;\;(02)_*\;\;} & \mathcal{E}_2
\end{array}
$$

.

Exercise 11.14 (4 points)**.** Determine the lax-commutative triangle of categories (i.e. the left-lax functor from $[2]$ to Cat_1) that classifies the locally cocartesian fibration $\mathsf{sd}([2]) \xrightarrow{\mathsf{max}} [2]$.

Exercise 11.15 (6 points)**.** Given a locally cocartesian fibration $\mathcal{E} \to [2]$, describe the data of a functor $\mathcal{E} \to \mathcal{C}$ in terms of the data of the lax-commutative triangle (52) that classifies it.

Dually, locally cartesian fibrations are classified by **right-lax functors** to Cat: for instance, a locally cartesian fibration over $[2]$ is classified by a lax-commutative triangle

$$
\begin{array}{ccc}
 & \mathcal{E}_1 & \\
{}^{(01)^*} \swarrow & \Downarrow & \nwarrow {}^{(12)_*} \\
\mathcal{E}_0 & \xleftarrow{\;\;(02)^*\;\;} & \mathcal{E}_2
\end{array}
$$

in Cat.

---

[216]Note that any morphism $(02)_* e \to (12)_* (01)_* e$ necessarily lies in the fiber $\mathcal{E}_2$ (because the object $2 \in [2]$ has no nontrivial endomorphisms).

11.4.3. We very briefly mention another useful notion, primarily to draw the reader's attention to its existence. Namely, a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is called an **exponentiable fibration** if there exists a right adjoint

$$(53) \qquad \mathsf{Cat}_{/\mathcal{B}} \underset{\mathsf{Fun}^{\mathsf{rel}}_{\mathcal{B}}(\mathcal{E},-)}{\overset{p^*}{\underset{\perp}{\rightleftarrows}}} \mathsf{Cat}_{/\mathcal{E}}$$

to the pullback functor. When it exists, this right adjoint is called the **relative functor ∞-category** construction; it may be seen as analogous to the internal hom among presheaves. We will see an example in §11.6.9.

Exercise 11.16 (2 points)**.** Show that the functor $[1] \xrightarrow{0<2} [2]$ is not an exponentiable fibration.

Cocartesian and cartesian fibrations are exponentiable. Conversely, a locally co/cartesian fibration is a co/cartesian fibration if and only if it is exponentiable. We refer the reader to [AF20] for more discussion of this notion and its uses.

11.4.4. A cocartesian fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ whose fibers are all ∞-groupoids is called a **left fibration**. Dually, a cartesian fibration whose fibers are all ∞-groupoids is called a **right fibration**. With the evident notation, the Grothendieck constructions of §11.2.1 factor as equivalences

$$\begin{array}{ccccccccc}
\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) & \xrightarrow[\sim]{\mathsf{Gr}} & \mathsf{coCart}_{\mathcal{B}} & \hookrightarrow & \mathsf{Cat}_{/\mathcal{B}} & \hookleftarrow & \mathsf{Cart}_{\mathcal{B}} & \xleftarrow[\sim]{\mathsf{Gr}^-} & \mathsf{Fun}(\mathcal{B}^{\mathsf{op}}, \mathsf{Cat}) \\
\big\uparrow & & \big\uparrow & & \big\downarrow & & \big\uparrow & & \big\uparrow \\
\mathsf{Fun}(\mathcal{B}, \mathcal{S}) & \dashrightarrow[\sim] & \mathsf{LFib}_{\mathcal{B}} & & & & \mathsf{RFib}_{\mathcal{B}} & \dashleftarrow[\sim] & \mathsf{Fun}(\mathcal{B}^{\mathsf{op}}, \mathcal{S})
\end{array}$$

between full subcategories.

Exercise 11.17 (4 points)**.** Show that a cocartesian fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a left fibration if and only if every morphism in $\mathcal{E}$ is p-cocartesian.

Exercise 11.18 (2 points)**.** Show that for any ∞-category $\mathcal{B}$ and any object $b \in \mathcal{B}$, the functors

$$\mathcal{B}_{b/} \xrightarrow{\mathsf{fgt}} \mathcal{B} \qquad \text{and} \qquad \mathcal{B}_{/b} \xrightarrow{\mathsf{fgt}} \mathcal{B}$$

are respectively a left fibration and a right fibration, and describe their unstraightenings.

Exercise 11.19 (4 points)**.** Show that a functor $\mathcal{E} \to \mathcal{B}$ is a left fibration if and only if the commutative square

$$\begin{array}{ccc}
\mathsf{Ar}(\mathcal{E}) & \xrightarrow{s} & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathsf{Ar}(\mathcal{B}) & \xrightarrow{s} & \mathcal{B}
\end{array}$$

in Cat is a pullback.

Exercise 11.20 (4 points). Given a cocartesian fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$, write $\mathcal{E}^{\mathsf{LFib}} \subseteq \mathcal{E}$ for the subcategory with the same objects but only the p-cocartesian morphisms.

(a) Prove that the composite

$$\mathcal{E}^{\mathsf{LFib}} \longhookrightarrow \mathcal{E} \xrightarrow{p} \mathcal{B}$$

is a left fibration, and describe its unstraightening.

(b) Prove that this construction defines a right adjoint

$$\mathsf{LFib}_{\mathcal{B}} \underset{(-)^{\mathsf{LFib}}}{\overset{\longhookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathsf{coCart}_{\mathcal{B}}$$

to the inclusion inclusion.

Of course, in the notation of Exercise 11.20, we may refer to $(\mathcal{E}^{\mathsf{LFib}} \downarrow \mathcal{B})$ as the **maximal sub-left fibration** of the cocartesian fibration $(\mathcal{E} \downarrow \mathcal{B})$.

As a cute repurposing of terminology, a functor $\mathcal{E} \to \mathcal{B}$ that is both a left fibration and a right fibration is often called a **Kan fibration**.[217] These are classified by functors to $\iota_0 \mathcal{S}$, which explains the lack of handedness: functors $\mathcal{B} \to \iota_0 \mathcal{S}$ are equivalent to functors $\mathcal{B}^{\mathsf{op}} \to \iota_0 \mathcal{S}$.

11.4.5. We illustrate the foregoing discussion with an example, namely that of quasicoherent sheaves.

To begin, observe the functor

$$\mathsf{CAlg}_{\Bbbk}^{\mathsf{op}} \xrightarrow{(\mathsf{Mod},\mathsf{res})} \mathsf{Cat}_1 \ ,$$

which carries a commutative $\Bbbk$-algebra to its category of modules and carries a morphism opposite to a morphism $R \xrightarrow{\varphi} S$ in $\mathsf{CAlg}_{\Bbbk}$ to the restriction functor

$$\mathsf{Mod}_R \xleftarrow{\mathsf{res}_\varphi} \mathsf{Mod}_S \ .[218]$$

We write

$$\mathcal{M}\mathrm{od} := \mathcal{M}\mathrm{od}_{\Bbbk} := \mathsf{Gr}^-(\mathsf{Mod}, \mathsf{res})$$
$$\downarrow$$
$$\mathsf{CAlg}_{\Bbbk}$$

for the cartesian fibration that it classifies.

---

[217]Note that here we are discussing a model-independent notion (unlike the original notion of a Kan fibration between simplicial sets, although of course they are closely related).

[218]This can be constructed as the composite

$$\mathsf{CAlg}_{\Bbbk}^{\mathsf{op}} \xrightarrow{\mathsf{fgt}} \mathsf{Alg}_{\Bbbk}^{\mathsf{op}} \xrightarrow{\mathfrak{B}} \mathsf{Cat}(\mathsf{Mod}_{\Bbbk}) \xrightarrow{\mathsf{Fun}_{\Bbbk}(-,\mathsf{Mod}_{\Bbbk})} \mathsf{Cat}(\mathsf{Mod}_{\Bbbk}) \xrightarrow{\mathsf{fgt}} \mathsf{Cat}_1 \ ,$$

where $\mathsf{Fun}_{\Bbbk}(-,-)$ denotes the internal hom in the category $\mathsf{Cat}(\mathsf{Mod}_{\Bbbk})$ of $\Bbbk$-linear categories.

Now, we claim that the functor $\mathcal{M}\mathrm{od} \to \mathsf{CAlg}_{\Bbbk}$ is a *cocartesian* fibration. To see this, we first verify that it is a locally cocartesian fibration.[219] Let us write $[1] \xrightarrow{\tilde{\varphi}} \mathsf{CAlg}_{\Bbbk}$ for the functor classifying a morphism $R \xrightarrow{\varphi} S$, and consider the pullback

(54)
$$\begin{array}{c} \tilde{\varphi}^*\mathcal{M}\mathrm{od} \\ \downarrow \\ [1] \end{array} \quad .$$

This is a cartesian fibration, which is classified by the composite functor

$$[1]^{\mathrm{op}} \xrightarrow{\tilde{\varphi}^{\mathrm{op}}} \mathsf{CAlg}_{\Bbbk}^{\mathrm{op}} \xrightarrow{(\mathsf{Mod},\mathsf{res})} \mathsf{Cat}_1$$

(recall §11.1.3). However, note the existence of the left adjoint induction functor

$$\mathsf{Mod}_R \underset{\mathsf{res}_\varphi}{\overset{\mathsf{ind}_\varphi := (-)\otimes_R S}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_S \quad .$$

It follows that the functor (54) is also a cocartesian fibration (recall §11.3.2). Hence, the functor $\mathcal{M}\mathrm{od} \to \mathsf{CAlg}_{\Bbbk}$ is a locally cocartesian fibration, with a locally cocartesian lift of the morphism $R \xrightarrow{\varphi} S$ at an object $M \in \mathsf{Mod}_R$ given by the morphism $M \to \mathsf{ind}_\varphi(M)$ (recall Exercise 11.5). Now, because adjunctions compose and adjoints are unique, it follows that the locally cocartesian morphisms compose, so that the functor $\mathcal{M}\mathrm{od} \to \mathsf{CAlg}_{\Bbbk}$ is indeed a cocartesian fibration. We may denote its straightening as

$$\mathsf{CAlg}_{\Bbbk} \xrightarrow{(\mathsf{Mod},\mathsf{ind})} \mathsf{Cat}_1 \ ,$$

so that we have

$$\mathsf{Gr}^-(\mathsf{Mod}, \mathsf{res}) =: \mathcal{M}\mathrm{od} =: \mathsf{Gr}(\mathsf{Mod}, \mathsf{ind}) \ .$$

Now, suppose that we are given a functor $\mathsf{CAlg}_{\Bbbk} \xrightarrow{X} \mathsf{Set}$. We define the category of **quasicoherent sheaves** on X to be

$$\mathsf{QCoh}(X) := \underline{\hom}_{\mathsf{coCart}_{\mathsf{CAlg}_{\Bbbk}}}(\mathsf{Gr}(X), \mathcal{M}\mathrm{od}) \ .^{220}$$

In other words, a quasicoherent sheaf $\mathcal{F} \in \mathsf{QCoh}(X)$ is the data of a strict morphism

$$\begin{array}{ccc} \mathsf{Gr}(X) & \xdashrightarrow{\ \ \mathcal{F}\ \ } & \mathcal{M}\mathrm{od} \\ & \searrow \quad \swarrow & \\ & \mathsf{CAlg}_{\Bbbk} & \end{array}$$

---

[219] It is of course possible, but somewhat more cumbersome, to verify directly that it is a cocartesian fibration.

[220] The category $\mathsf{Gr}(X)$ is often referred to as the *category of elements* of X.

between cocartesian fibrations over $\mathsf{CAlg}_{\Bbbk}$. Let us unpack this further. To begin, an object of $\mathsf{Gr}(X)$ is a pair $(R \in \mathsf{CAlg}_{\Bbbk}, \alpha \in X(R))$, and to this our quasicoherent sheaf $\mathcal{F}$ assigns an object $\mathcal{F}(\alpha) \in \mathsf{Mod}_R$. Next, a morphism

$$(R \in \mathsf{CAlg}_{\Bbbk}, \alpha \in X(R)) \longrightarrow (S \in \mathsf{CAlg}_{\Bbbk}, \beta \in X(S))$$

in $\mathsf{Gr}(X)$ is simply the data of a morphism $R \xrightarrow{\varphi} S$ in $\mathsf{CAlg}_{\Bbbk}$ such that $X(\varphi)(\alpha) = \beta$. This is a cocartesian morphism (recall Exercise 11.17), and so our quasicoherent sheaf $\mathcal{F}$ must assign it to a *cocartesian* morphism

(55) $$\mathcal{F}(\alpha) \longrightarrow \mathcal{F}(\beta) \ .$$

In other words, the morphism (55) admits a canonical factorization

in which the morphism $g_{\varphi,\alpha}$ lies in the fiber Mod_S, and to say that the morphism (55) is cocartesian is equivalently to say that the morphism $g_{\varphi,\alpha}$ is an isomorphism.

Exercise 11.21 (4 points). Determine criteria guaranteeing that the above assignments on objects and morphisms define a functor (i.e. respect identities and composition).

From a different perspective, one can equivalently define

$$\mathsf{QCoh}(X) := \underline{\hom}_{\mathsf{Fun}(\mathsf{CAlg}_{\Bbbk}, \mathsf{Cat}_1)}(X, (\mathsf{Mod}, \mathsf{ind})) \ .$$

In other words, quasicoherent sheaves are natural transformations (morphisms between functors), while morphisms between them are known as "modifications" (morphisms between natural transformations). That is, morphisms between quasicoherent sheaves are equivalently 3-morphisms in the 3-category of 2-categories. This illustrates the great extent to which the Grothendieck construction can reduce categorical dimension, which is one of the reasons it is so ubiquitous; and these advantages become even more pronounced when one passes from n-categories to (∞, n)-categories.

Exercise 11.22 (4 points). Fix a commutative $\Bbbk$-algebra $R \in \mathsf{CAlg}_{\Bbbk}$, and write

$$\mathsf{Spec}(R) := \hom_{\mathsf{CAlg}_{\Bbbk}}(R, -) \in \mathsf{Fun}(\mathsf{CAlg}_{\Bbbk}, \mathsf{Set}) \ .$$

Construct an equivalence of categories

(56) $$\mathsf{QCoh}(\mathsf{Spec}(R)) \xrightarrow{\sim} \mathsf{Mod}_R \ .$$

One advantage of our given definition of $\mathsf{QCoh}(X)$ is the fact that the equivalence (56) of Exercise 11.22 becomes relatively straightforward (rather than requiring the establishment of Zariski descent). On the other hand, from this perspective it is not so clear that $\mathsf{QCoh}(X)$

is an *abelian* category: it is a limit of a diagram of abelian categories, but the transition functors (namely the functors $\mathsf{Mod}_R \xrightarrow{\mathsf{ind}_\varphi} \mathsf{Mod}_S$ for morphisms $R \xrightarrow{\varphi} S$ in $\mathsf{CAlg}_\Bbbk$) are not exact.

11.4.6. Recall from §11.1.2 that the universal bundle of sets (i.e. the universal covering space) is the functor

$$\mathsf{Set}_* \xrightarrow{\mathsf{fgt}} \mathsf{Set} ,$$

and from §11.1.4 that the universal bundle of spaces is the functor

$$\mathcal{S}_* \xrightarrow{\mathsf{fgt}} \mathcal{S} .$$

Here we describe the universal cocartesian fibration and the universal cartesian fibration.

The universal cocartesian fibration must be a cocartesian fibration over Cat, the classifying object for cocartesian fibrations, and its fiber over each object $\mathcal{C} \in \mathsf{Cat}$ must be the ∞-category $\mathcal{C} \in \mathsf{Cat}$ itself. Precisely, it is the ***right-lax undercategory*** of the terminal object $\mathsf{pt} \in \mathsf{Cat}$, denoted $\mathsf{Cat}^{\mathsf{r.lax}}_{\mathsf{pt}/}$. This can be heuristically described as follows. First of all, an object of $\mathsf{Cat}^{\mathsf{r.lax}}_{\mathsf{pt}/}$ is the datum of a morphism $(\mathsf{pt} \xrightarrow{c} \mathcal{C})$ in Cat, i.e. an ∞-category $\mathcal{C} \in \mathsf{Cat}$ equipped with a chosen object $c \in \mathcal{C}$. Then, a morphism in $\mathsf{Cat}^{\mathsf{r.lax}}_{\mathsf{pt}/}$ from $(\mathsf{pt} \xrightarrow{c} \mathcal{C})$ to $(\mathsf{pt} \xrightarrow{d} \mathcal{D})$ is the datum of a lax-commutative triangle

i.e. a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ along with a morphism $Fc \to d$ in $\mathcal{D}$.

Exercise 11.23 (8 points).
   (a) Give a rigorous construction of the ∞-category $\mathsf{Cat}^{\mathsf{r.lax}}_{\mathsf{pt}/}$ as a complete Segal space (taking that of Cat as given).
   (b) Verify that the forgetful functor $\mathsf{Cat}^{\mathsf{r.lax}}_{\mathsf{pt}/} \xrightarrow{\mathsf{fgt}} \mathsf{Cat}$ is a cocartesian fibration.
   (c) Establish a pullback square

$$
\begin{array}{ccc}
\mathcal{S}_* & \lhook\joinrel\longrightarrow & \mathsf{Cat}^{\mathsf{r.lax}}_{\mathsf{pt}/} \\
{\scriptstyle \mathsf{fgt}}\downarrow & & \downarrow{\scriptstyle \mathsf{fgt}} \\
\mathcal{S} & \lhook\joinrel\longrightarrow & \mathsf{Cat}
\end{array}
$$

in which the horizontal functors are fully faithful.

Dually, the universal cartesian fibration must be a cartesian fibration over $\mathsf{Cat}^{\mathsf{op}}$ (it is classified by the functor $(\mathsf{Cat}^{\mathsf{op}})^{\mathsf{op}} \xrightarrow{\mathsf{id}_{\mathsf{Cat}}} \mathsf{Cat}$).

Exercise 11.24 (4 points)**.** Give a heuristic description of the universal cartesian fibration, and explain its relationship with the universal cocartesian fibration.

11.5. The Grothendieck construction over ∞-categories.

11.5.1.   For completeness, we explain the constructions of the Grothendieck construction functors

$$\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) \xrightarrow{\;\mathsf{Gr}\;} \mathsf{Cat}_{/\mathcal{B}} \xleftarrow{\;\mathsf{Gr}^-\;} \mathsf{Fun}(\mathcal{B}^{\mathsf{op}}, \mathsf{Cat})$$

(following [GHN17]). It will not be clear that these constructions land in the subcategories $\mathsf{coCart}_{\mathcal{B}} \subseteq \mathsf{Cat}_{/\mathcal{B}} \supseteq \mathsf{Cart}_{\mathcal{B}}$, let alone that they define equivalences thereto; to verify these assertions requires a substantially subtler analysis of colimits among ∞-categories than we undertake here.[221] Nevertheless, we find these constructions clarifying, and the techniques are broadly applicable. However, this material can be safely skipped.

11.5.2.   We first define an ∞-category $\mathsf{TwAr}(\mathcal{B}) \in \mathsf{Cat}$, the ***twisted arrow ∞-category*** of $\mathcal{B}$: an object is a morphism $b \xrightarrow{\varphi} b'$ in $\mathcal{B}$, and a morphism $\varphi_0 \to \varphi_1$ is a factorization

$$
\begin{array}{ccc}
b_0 & \xrightarrow{\varphi_0} & b_0' \\
\vdots\downarrow & & \uparrow\vdots \\
b_1 & \xrightarrow{\varphi_1} & b_1'
\end{array}
$$

of the source through the target. This comes equipped with a functor

$$
\begin{array}{c}
\mathsf{TwAr}(\mathcal{B}) \\
{\scriptstyle (s,t)}\Big\downarrow \qquad {\scriptstyle 222} \\
\mathcal{B} \times \mathcal{B}^{\mathsf{op}}
\end{array}.
$$

For instance, writing $ij := (i \to j) \in \mathsf{TwAr}([n])$ for any $0 \leqslant i \leqslant j \leqslant n$, we have

$$
\mathsf{TwAr}([2]) = \left\{
\begin{array}{ccccc}
 & & 02 & & \\
 & \swarrow & & \searrow & \\
 & 01 & & 12 & \\
 \swarrow & & \searrow \swarrow & & \searrow \\
00 & & 11 & & 22
\end{array}
\right\} ,
$$

with the source and target functors

$$
\begin{array}{c}
\mathsf{TwAr}([2]) \\
{\scriptstyle s}\swarrow \qquad \searrow{\scriptstyle t} \\
[2] \qquad\qquad [2]^{\mathsf{op}}
\end{array}
$$

---

[221]Indeed, this remains so even in the case that $\mathcal{B} = [1]$. (This issue was swept under the rug in §11.3.1.)

[222]Note that conventions regarding twisted arrows vary: what we call $\mathsf{TwAr}(\mathcal{B})$ is also sometimes called $\mathsf{TwAr}(\mathcal{B})^{\mathsf{op}}$ (which comes equipped with a functor $\mathsf{TwAr}(\mathcal{B})^{\mathsf{op}} \xrightarrow{(s,t)} \mathcal{B}^{\mathsf{op}} \times \mathcal{B}$).

given by projecting in the corresponding directions.

Exercise 11.25 (4 points)**.** Give a rigorous definition of the endofunctor $\mathsf{Cat} \xrightarrow{\mathsf{TwAr}(-)} \mathsf{Cat}$ and its natural morphism $\mathsf{TwAr}(-) \to (-) \times (-)^{\mathsf{op}}$ in terms of complete Segal spaces.

Exercise 11.26 (4 points)**.** Show that the functor $\mathsf{TwAr}(\mathcal{B}) \xrightarrow{(s,t)} \mathcal{B} \times \mathcal{B}^{\mathsf{op}}$ is a right fibration, and describe its unstraightening.

11.5.3.   We need one other ingredient in order to define the Grothendieck construction.

Exercise 11.27 (4 points)**.** Use the Grothendieck construction equivalences to respectively assemble the undercategories $\mathcal{B}_{b/}$ and overcategories $\mathcal{B}_{/b}$ as functors

$$\mathcal{B}^{\mathsf{op}} \xrightarrow{\mathcal{B}_{\bullet/}} \mathsf{Cat} \qquad \text{and} \qquad \mathcal{B} \xrightarrow{\mathcal{B}_{/\bullet}} \mathsf{Cat} \ .$$

Now, using Exercise 11.27, for any functor $\mathcal{B} \xrightarrow{F} \mathsf{Cat}$ we define its covariant Grothendieck construction to be

$$\mathsf{Gr}(F) := \mathsf{colim} \left(\mathsf{TwAr}(\mathcal{B}) \xrightarrow{(s,t)} \mathcal{B} \times \mathcal{B}^{\mathsf{op}} \xrightarrow{F \times \mathcal{B}_{\bullet/}} \mathsf{Cat} \right) \ ,$$

and for any functor $\mathcal{B}^{\mathsf{op}} \xrightarrow{F} \mathsf{Cat}$ we define its contravariant Grothendieck construction to be

$$\mathsf{Gr}^{-}(F) := \mathsf{colim} \left(\mathsf{TwAr}(\mathcal{B}) \xrightarrow{(s,t)} \mathcal{B} \times \mathcal{B}^{\mathsf{op}} \xrightarrow{\mathcal{B}_{/\bullet} \times F} \mathsf{Cat} \right) \ .^{223}$$

Here we simply write $(-) \times (-)$ for the external product of two functors to Cat, i.e. the product of the functors themselves postcomposed with the product functor $\mathsf{Cat} \times \mathsf{Cat} \xrightarrow{(-) \times (-)} \mathsf{Cat}$.

11.5.4.   Observe that the formulas of §11.5.3 reduce to the ones given in §11.3.1 in the case that $\mathcal{B} = [1]$.[224] Moreover, they make precise the heuristic descriptions given in §11.2.2, as we now explain in the case of the contravariant Grothendieck construction.

First of all, for each object $b \in \mathcal{B}$ we have the component

$$\mathcal{B}_{/b} \times F(b^{\circ}) \in \mathsf{Cat} \ ,$$

the value of the composite functor at the object $\mathsf{id}_b \in \mathsf{TwAr}(\mathcal{B})$; therein, we have the distinguished copy of $F(b^{\circ})$ given by the full subcategory $\{\mathsf{id}_b\} \times F(b^{\circ}) \subseteq \mathcal{B}_{/b} \times F(b^{\circ})$ (note that

---

[223]These constructions can be placed into a slightly broader categorical context, as explained in [GHN17, §2]. Namely, there is a general definition of a co/end, which specializes to the notion of a weighted co/limit, which specializes to the notion of a (right- or left-)lax co/limit; and the covariant (resp. contravariant) Grothendieck construction is precisely the left-lax (resp. right-lax) colimit.

[224]The pushouts (44) and (47) are both arranged so as to be indexed over

$$\mathsf{TwAr}([1]) = \left\{ \begin{array}{ccc} 01 & \longrightarrow & 11 \\ \downarrow & & \\ 00 & & \end{array} \right\} \ .$$

the object $\mathsf{id}_b \in \mathcal{B}_{/b}$ is terminal). Next, we can consider each morphism $b_0 \xrightarrow{\varphi} b_1$ in $\mathcal{B}$ as an object of $\mathsf{TwAr}(\mathcal{B})$, the component at which is

$$\mathcal{B}_{/b_0} \times F(b_1^\circ) \in \mathsf{Cat} \ .$$

The span

in $\mathsf{TwAr}(\mathcal{B})$ (in which all unlabeled morphisms on the right are all identity morphisms) is carried to a span

(57)
$$\mathcal{B}_{/b_0} \times F(b_1^\circ) \xrightarrow{\mathcal{B}_{/\varphi} \times \mathsf{id}_{F(b_1^\circ)}} \mathcal{B}_{/b_1} \times F(b_1^\circ)$$
$$\downarrow{\scriptstyle \mathsf{id}_{\mathcal{B}_{/b_0}} \times F(\varphi^\circ)}$$
$$\mathcal{B}_{/b_0} \times F(b_0^\circ)$$

in Cat. Let us consider an arbitrary object $y \in F(b_1^\circ)$, and let us write

$$\varphi^* y := F(\varphi^\circ)(y) \in F(b_0^\circ)$$

for simplicity. Then, the span (57) contains the assignments

$$
\begin{array}{ccc}
(\mathsf{id}_{b_0}, y) & \longmapsto & (\varphi, y) \\
\downarrow & & \\
(\mathsf{id}_{b_0}, \varphi^* y) & &
\end{array}
$$

.

Hence, in the colimit defining $\mathsf{Gr}^-(F)$, we obtain a distinguished equivalence between the objects

$$(\mathsf{id}_{b_0}, \varphi^* y) \in \mathcal{B}_{/b_0} \times F(b_0^\circ) \qquad \text{and} \qquad (\varphi, y) \in \mathcal{B}_{/b_1} \times F(b_1^\circ) \ .$$

The former lies in the distinguished copy of $F(b_0^\circ)$, while the latter comes equipped with a canonical morphism

$$(\varphi, y) \longrightarrow (\mathsf{id}_{b_1}, y)$$

in $\mathcal{B}_{/b_1} \times F(b_1^\circ)$ to an object lying in the distinguished copy of $F(b_1^\circ)$. So indeed, we have formally adjoined a morphism

$$F(b_0^\circ) \ni \varphi^* y \longrightarrow y \in F(b_1^\circ)$$

in the colimit. Of course, this will be the cartesian lift of the morphism $b_0 \xrightarrow{\varphi} b_1$ in $\mathcal{B}$ at the object $y \in F(b_1^\circ) \simeq \mathsf{Gr}^-(F)_{b_1}$.

Exercise 11.28 (6 points)**.** Show that these asserted cartesian lifts compose: given a composite $b_0 \xrightarrow{\varphi} b_1 \xrightarrow{\psi} b_2$ in $\mathcal{B}$ and any object $z \in F(b_2^\circ)$, obtain a commutative square

$$
\begin{array}{ccc}
\varphi^*(\psi^* z) & \longrightarrow & \psi^* z \\
\wr \downarrow & & \searrow \\
(\psi\varphi)^* z & \longrightarrow & z
\end{array}
$$

in $\mathsf{Gr}^-(F)$ in which all non-vertical morphisms are cartesian lifts.

Of course, Exercise 11.28 indicates just one of an infinite hierarchy of relations that are imposed in $\mathsf{Gr}^-(F)$ according to composition in $\mathcal{B}$.

11.6. Monoidal and symmetric monoidal ∞-categories, revisited.

11.6.1.   As we have seen, the Grothendieck construction cleanly accommodates various notions of laxness. We explain here how this applies to the theory of monoidal and symmetric monoidal ∞-categories. Because the stories are largely parallel, we first discuss the case of symmetric monoidal ∞-categories in detail, and then indicate the case of monoidal ∞-categories in §11.6.10.

11.6.2.   Recall that in §10 we defined the ∞-category of symmetric monoidal ∞-categories to be the full subcategory

$$\mathsf{CMon}(\mathsf{Cat}) \subset \mathsf{Fun}(\mathsf{Fin}_*, \mathsf{Cat})$$

on the reduced Segal objects (i.e. as the ∞-category of commutative monoid objects in Cat). In §11.6.4, we will connect this definition with the one given in [Lur] in terms of fibrations.

Recall too that given a symmetric monoidal ∞-category $\mathcal{V}$, in §10 we defined the ∞-category of commutative algebra objects in $\mathcal{V}$ to be the ∞-category

$$\mathsf{CAlg}(\mathcal{V}) := \mathsf{Fun}^\otimes(\mathsf{Fin}, \mathcal{V})$$

of symmetric monoidal functors to $\mathcal{V}$ from $\mathsf{Fin} := (\mathsf{Fin}, \sqcup)$, the free symmetric monoidal category on a commutative algebra object. In §11.6.7, we will connect this definition with the one given in [Lur] in terms of fibrations. The latter has the advantage that it is more economical, but in trade it effectively requires the usage of lax natural transformations, via the notion of a ***right-laxly symmetric monoidal functor*** that we introduce in §11.6.5. We study the relationship between ordinary ("strictly") symmetric monoidal functors and right-laxly symmetric monoidal functors in §11.6.6.

11.6.3.   The material in this subsection is rather technical. However, we have included it in order to make contact with the theory of symmetric monoidal ∞-categories developed in [Lur] (which is also rather technical). We briefly summarize the relationship before proceeding.

In fact, [Lur] develops the theory of ∞-***operads***, which are a generalization of symmetric monoidal ∞-categories. In essence, an ∞-operad is a symmetric monoidal ∞-category in which not all tensor products are required to exist. Thus, a symmetric monoidal ∞-category is a special case of an ∞-operad. This more general notion leads to a further reduction in combinatorial complexity (beyond the notion of right-laxly symmetric monoidal functor, which applies equally well to ∞-operads): heuristically, it allows us to avoid carrying around any tensor products that are not relevant for the problem at hand.[225]

11.6.4.   Choose any morphism

$$S_+ \xrightarrow{\varphi} T_+$$

in Fin_*. We say that the morphism φ is ***active*** if $\varphi^{-1}(+) = \{+\}$, i.e. if only the basepoint $+ \in S_+$ is sent to the basepoint $+ \in T_+$. We say that the morphism φ is ***inert*** if the factorization

$$
\begin{array}{ccc}
\varphi^{-1}(T) & \dashrightarrow & T \\
\downarrow & & \downarrow \\
S_+ & \xrightarrow{\varphi} & T_+
\end{array}
$$

is a bijection, i.e. if the preimage of each non-basepoint element $t \in T \subset T_+$ is a singleton. We write

$$\mathsf{Fin}_*^{\mathsf{act}} \subset \mathsf{Fin}_* \supset \mathsf{Fin}_*^{\mathsf{inrt}}$$

for the subcategories on the active and inert morphisms, respectively.

Exercise 11.29 (2 points)**.** Show that $(\mathsf{Fin}_*^{\mathsf{inrt}}; \mathsf{Fin}_*^{\mathsf{act}})$ defines a factorization system on the category Fin_*, i.e. that every morphism in Fin_* admits a unique factorization as the composite of an inert morphism followed by an active morphism.

Observe that the reduced Segal conditions on a functor $\mathsf{Fin}_* \to \mathsf{Cat}$ can be checked upon restriction to the subcategory $\mathsf{Fin}_*^{\mathsf{inrt}} \subset \mathsf{Fin}_*$: reducedness is simply contractibility of the value at $\underline{0}_+$, while the nth Segal map is induced by the inert morphisms $\underline{n}_+ \xrightarrow{\rho_i} \underline{1}_+$ for $i \in \underline{n}$.

---

[225]In particular, the notion of *symmetric monoidal envelope* introduced in §11.6.6 actually applies to ∞-operads, carrying them to symmetric monoidal ∞-categories (which explains the terminology).

Let us write $\mathsf{coCart}'_{\mathsf{Fin}^{\mathsf{inrt}}_*} \subset \mathsf{coCart}_{\mathsf{Fin}^{\mathsf{inrt}}_*}$ for the full subcategory on the cocartesian fibrations that correspond to reduced Segal functors $\mathsf{Fin}^{\mathsf{inrt}}_* \to \mathsf{Cat}$. Then, the Grothendieck construction induces a canonical pullback square

(58)
$$
\begin{array}{ccc}
\mathsf{CMon}(\mathsf{Cat}) & \longrightarrow & \mathsf{coCart}'_{\mathsf{Fin}^{\mathsf{inrt}}_*} \\
{\scriptstyle \mathrm{ff}}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{ff}} \\
\mathsf{coCart}_{\mathsf{Fin}_*} & \longrightarrow & \mathsf{coCart}_{\mathsf{Fin}^{\mathsf{inrt}}_*}
\end{array}
$$

among ∞-categories, where the lower horizontal functor is pullback along the inclusion $\mathsf{Fin}^{\mathsf{inrt}}_* \hookrightarrow \mathsf{Fin}_*$. In particular, a symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes) \in \mathsf{Cat}$ can be equivalently recorded as a cocartesian fibration over Fin_*, denoted

$$
\begin{array}{c}
\mathcal{V}^{\boxtimes} \\
\big\downarrow \quad . \\
\mathsf{Fin}_*
\end{array}
$$

Its fiber over each object $S_+ \in \mathsf{Fin}_*$ is the ∞-category $\mathcal{V}^{\times S}$, and e.g. the operation

$$
\mathcal{V}^{\times 5} \xrightarrow{(V_1,\ldots,V_5) \longmapsto (V_1 \boxtimes V_3, V_2)} \mathcal{V}^{\times 2}
$$

is recorded as the cocartesian monodromy functor over the morphism $\underline{5}_+ \xrightarrow{\varphi} \underline{2}_+$ in Fin_* characterized by the fact that $\varphi^{-1}(1) = \{1,3\}$ and $\varphi^{-1}(2) = \{2\}$. And conversely, any cocartesian fibration over Fin_* that satisfies the reduced Segal conditions (which can be checked over $\mathsf{Fin}^{\mathsf{inrt}}_*$) records a symmetric monoidal ∞-category, whose underlying ∞-category is its fiber over $\underline{1}_+ \in \mathsf{Fin}_*$. Moreover, symmetric monoidal functors $(\mathcal{V}, \boxtimes) \to (\mathcal{W}, \otimes)$ are equivalently recorded by morphisms

(59)
$$
\begin{array}{ccc}
\mathcal{V}^{\boxtimes} & \longrightarrow & \mathcal{W}^{\otimes} \\
& \searrow \quad \swarrow & \\
& \mathsf{Fin}_* &
\end{array}
$$

in $\mathsf{coCart}_{\mathsf{Fin}_*}$, i.e. by strict morphisms between their corresponding cocartesian fibrations over Fin_*.

In what follows, we will pass freely between these two perspectives.

11.6.5. We now define the notion of a **right-laxly symmetric monoidal functor** between symmetric monoidal ∞-categories.[226] These are the morphisms in the ∞-category

---

[226]It is also common to call a right-laxly monoidal functor "lax monoidal" and a left-laxly monoidal functor "oplax monoidal". We find the systematic use of handedness much more memorable (due to various accompanying mnemonics). Additionally, the adverb "laxly" emphasizes that it is the *monoidality* of the functor that is lax (rather than the functor itself). (We find this emphasis helpful, despite the fact that the notion of a "lax functor" is only meaningful when the target is an $(\infty, 2)$-category, as implied in §11.4.2.)

$\mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}}$ defined as the pullback of the back face in the diagram

(in which the front face is the pullback square (58) and the back lower horizontal functor is likewise pullback). In other words, a right-laxly symmetric monoidal functor $(\mathcal{V}, \boxtimes) \to (\mathcal{W}, \otimes)$ is the data of a morphism (59) in $\mathsf{coCart}_{\mathsf{Fin}_*}^{\mathsf{l.lax}}$ (a full subcategory of $\mathsf{Cat}_{/\mathsf{Fin}_*}$) such that its restriction to $\mathsf{Fin}_*^{\mathsf{inrt}}$ lies in $\mathsf{coCart}_{\mathsf{Fin}_*^{\mathsf{inrt}}} \subset \mathsf{Cat}_{/\mathsf{Fin}_*^{\mathsf{inrt}}}.$[227]

Let us describe the data of a right-laxly symmetric monoidal functor $(\mathcal{V}, \boxtimes) \xrightarrow{F} (\mathcal{W}, \otimes)$. For simplicity, we will also write $\mathcal{V} \xrightarrow{F} \mathcal{W}$ for its underlying functor (i.e. we write $F := F_{\underline{1}_+}$ for its restriction to fibers over the object $\underline{1}_+ \in \mathsf{Fin}_*$).

First of all, such a right-laxly symmetric monoidal functor consists of functors on fibers: over each $\underline{n}_+ \in \mathsf{Fin}_*$, it determines a functor

$$\mathcal{V}^{\times n} \simeq (\mathcal{V}^{\boxtimes})_{\underline{n}_+} \xrightarrow{F_{\underline{n}_+}} (\mathcal{W}^{\otimes})_{\underline{n}_+} \simeq \mathcal{W}^{\times n} \ .$$

---

[227]The fact that a *right*-laxly symmetric monoidal functor is defined as a *left*-lax natural transformation between reduced Segal functors $\mathsf{Fin}_* \to \mathsf{Cat}$ is an unfortunate clash, but the terminologies are both strongly motivated by other considerations. On the one hand, the handedness of left/right fibrations is already well-established, which explains the pairings "left $\leftrightarrow$ cocartesian" and "right $\leftrightarrow$ cartesian" for lax natural transformations. On the other hand, a fundamental fact is that given an adjunction

$$\mathcal{V} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{W}$$

between (symmetric) monoidal ∞-categories, the data enhancing the left adjoint F to be *left*-laxly (resp. symmetric) monoidal is equivalent to the data enhancing the right adjoint G to be *right*-laxly (resp. symmetric) monoidal. (In practice, the most frequent usage of this fact is when F is strictly (symmetric) monoidal, so that G is canonically right-laxly (resp. symmetric) monoidal. In this situation, both adjoints lift to functors on ∞-categories of (resp. commutative) algebra objects (see §11.6.7), and in fact these participate in an adjunction

$$\mathsf{Alg}(\mathcal{V}) \underset{\mathsf{Alg}(G)}{\overset{\mathsf{Alg}(F)}{\underset{\perp}{\rightleftarrows}}} \mathsf{Alg}(\mathcal{W})$$

(resp. an analogous adjunction obtained by replacing Alg by CAlg throughout).)

However, inert-cocartesianness implies that these are all specified by the underlying functor $\mathcal{V} \xrightarrow{F} \mathcal{W}$: for each $i \in \underline{n}$, the inert morphism $\underline{n}_+ \xrightarrow{\rho_i} \underline{1}_+$ determines a commutative square

$$
\begin{array}{ccc}
\mathcal{V}^{\times n} & \xrightarrow{F_{\underline{n}_+}} & \mathcal{W}^{\times n} \\
{\scriptstyle\mathrm{pr}_i}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}_i} \\
\mathcal{V} & \xrightarrow{\quad F \quad} & \mathcal{W}
\end{array} \quad ,
$$

so that altogether we obtain an identification

$$
\begin{array}{ccc}
\mathcal{V}^{\times n} & \xrightarrow{F_{\underline{n}_+}} & \mathcal{W}^{\times n} \\
{\scriptstyle(\mathrm{pr}_i)_{i\in\underline{n}}}\downarrow \wr & & \wr\downarrow{\scriptstyle(\mathrm{pr}_i)_{i\in\underline{n}}} \\
\mathcal{V}^{\times n} & \xrightarrow{\quad F^{\times n} \quad} & \mathcal{W}^{\times n}
\end{array} \quad .
$$

More generally, cocartesianness over $\mathsf{Fin}_*^{\mathrm{inrt}}$ implies that these identifications $F_{\underline{n}_+} \simeq F^{\times n}$ are all coherently compatible. For instance, over the inert morphism $\underline{5}_+ \xrightarrow{\varphi} \underline{2}_+$ characterized by the fact that $\varphi^{-1}(1) = \{4\}$ and $\varphi^{-1}(2) = \{2\}$, we obtain a commutative square

$$
\begin{array}{ccc}
\mathcal{V}^{\times 5} & \xrightarrow{F_{\underline{5}_+}} & \mathcal{W}^{\times 5} \\
{\scriptstyle(\mathrm{pr}_4,\mathrm{pr}_2)}\downarrow & & \downarrow{\scriptstyle(\mathrm{pr}_4,\mathrm{pr}_2)} \\
\mathcal{V}^{\times 2} & \xrightarrow{\quad F_{\underline{2}_+} \quad} & \mathcal{W}^{\times 2}
\end{array} \quad .
$$

In order to proceed, we introduce some convenient notation: for every $k \geqslant 0$ we write $\underline{k}_+ \xrightarrow{\alpha_n} \underline{1}_+$ for the unique active morphism, and for any morphism $\underline{m}_+ \xrightarrow{\varphi} \underline{n}_+$ in Fin_* we write θ_φ for the natural transformation

$$
\begin{array}{ccc}
\mathcal{V}^{\times m} & \xrightarrow{F^{\times m}} & \mathcal{W}^{\times m} \\
{\scriptstyle\varphi_*}\downarrow & \overset{\theta_\varphi}{\Swarrow} & \downarrow{\scriptstyle\varphi_*} \\
\mathcal{V}^{\times n} & \xrightarrow{\quad F^{\times n} \quad} & \mathcal{W}^{\times n}
\end{array}
$$

between functors $\mathcal{V}^{\times m} \to \mathcal{W}^{\times n}$ recorded by our right-laxly symmetric monoidal functor F (which is a natural equivalence when φ is inert). For simplicity, we may write $\theta_k := \theta_{\alpha_k}$. Our remaining goal is to describe the roles of the natural transformations θ_φ. A key feature is their composability, as follows.

Exercise 11.30 (4 points). Given a pair of composable morphisms $\underline{m}_+ \xrightarrow{\varphi} \underline{n}_+ \xrightarrow{\psi} \underline{p}_+$ in Fin_*, show that the composite natural transformation

$$
\begin{array}{ccc}
\mathcal{V}^{\times m} & \xrightarrow{F^{\times m}} & \mathcal{W}^{\times m} \\
\varphi_* \downarrow & \quad \theta_\varphi \Swarrow & \downarrow \varphi_* \\
\mathcal{V}^{\times n} & \xrightarrow{F^{\times n}} & \mathcal{W}^{\times n} \\
\psi_* \downarrow & \quad \theta_\psi \Swarrow & \downarrow \psi_* \\
\mathcal{V}^{\times p} & \xrightarrow{F^{\times p}} & \mathcal{W}^{\times p}
\end{array}
$$

is precisely $\theta_{\psi\varphi}$.

Of course, Exercise 11.30 only expresses the two-fold composability of the natural transformations θ_φ; these identifications are themselves associative in a fairly evident way that we will not make explicit.

Now, the fundamental examples of the natural transformations θ_φ are those associated to the active morphisms α_2 and α_0, i.e. the natural transformations

(60)
$$
\begin{array}{ccc}
\mathcal{V}^{\times 2} & \xrightarrow{F_{\underline{2}_+}} & \mathcal{W}^{\times 2} \\
\boxtimes := (\alpha_2)_* \downarrow & \theta_2 \Swarrow & \downarrow (\alpha_2)_* =: \otimes \\
\mathcal{V} & \xrightarrow{F} & \mathcal{W}
\end{array}
\qquad\text{and}\qquad
\begin{array}{ccc}
\mathsf{pt} & \xrightarrow[\sim]{F_{\underline{0}_+}} & \mathsf{pt} \\
\mathbb{1}_{\mathcal{V}} := (\alpha_0)_* \downarrow & \theta_0 \Swarrow & \downarrow (\alpha_0)_* =: \mathbb{1}_{\mathcal{W}} \\
\mathcal{V} & \xrightarrow{F} & \mathcal{W}
\end{array}
\; .
$$

Said differently, these are natural morphisms

$$
F(V_1) \otimes F(V_2) \xrightarrow{\theta_2} F(V_1 \boxtimes V_2) \qquad\text{and}\qquad \mathbb{1}_{\mathcal{W}} \xrightarrow{\theta_0} F(\mathbb{1}_{\mathcal{V}})
$$

in $\mathcal{W}$ (the former for any $(V_1, V_2) \in \mathcal{V}^{\times 2}$). As we will explain, all of the natural transformations θ_φ can be seen as collectively recording compatibilities between the natural transformations θ_2 and θ_0.

Preliminarily, observe that an arbitrary active morphism $\underline{m}_+ \xrightarrow{\varphi} \underline{n}_+$ admits a canonical identification

$$
\left(\underline{m}_+ \xrightarrow{\varphi} \underline{n}_+ \right) \simeq \bigvee_{i \in \underline{n}} \left(\varphi^{-1}(i)_+ \xrightarrow{\alpha_{\varphi^{-1}(i)}} \{i\}_+ \right)
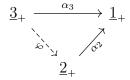$$

as a wedge sum of active morphisms to $\underline{1}_+$.

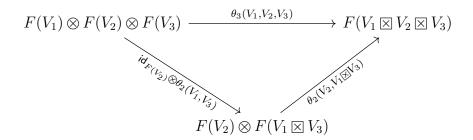**Exercise 11.31** (8 points)**.** Establish a canonical equivalence

$$
\left(
\begin{array}{ccc}
\mathcal{V}^{\times m} & \xrightarrow{F^{\times m}} & \mathcal{W}^{\times m} \\
\varphi_* \downarrow & \theta_\varphi \; \Swarrow & \downarrow \varphi_* \\
\mathcal{V}^{\times n} & \xrightarrow[F^{\times n}]{} & \mathcal{W}^{\times n}
\end{array}
\right)
\simeq
\prod_{i \in \underline{n}}
\left(
\begin{array}{ccc}
\mathcal{V}^{\times \varphi^{-1}(i)} & \xrightarrow{F^{\times \varphi^{-1}(i)}} & \mathcal{W}^{\times \varphi^{-1}(i)} \\
(\varphi_i)_* \downarrow & \theta_{\varphi^{-1}(i)} \; \Swarrow & \downarrow (\varphi_i)_* \\
\mathcal{V} & \xrightarrow[F]{} & \mathcal{W}
\end{array}
\right)
$$

between lax-commutative squares.

By Exercise 11.31, we can consider the natural transformations θ_φ for active morphisms φ as simply recording products of the natural transformations θ_k (such as $\theta_{\varphi^{-1}(i)} := \theta_{\alpha_{\varphi^{-1}(i)}}$), in a way that allows us to compose the latter. Thereafter, we can view the natural transformations θ_k as recording compatibilities between composites of the natural transformations θ_2 and θ_0, as follows. For definiteness, let us first consider the four factorizations

$$
\begin{array}{ccc}
\underline{3}_+ & \xrightarrow{\alpha_3} & \underline{1}_+ \\
 & {\scriptstyle \varphi} \searrow \quad \nearrow {\scriptstyle \alpha_2} & \\
 & \underline{2}_+ &
\end{array}
$$

via (necessarily active) surjections. Each of these corresponds to a distinct way of composing θ_2 with itself to give a ternary operation, and by Exercises 11.30 and 11.31, all such ternary operations must be equivalent to θ_3. For instance, in the case that $\varphi^{-1}(1) = \{2\}$, we obtain a natural identification

$$
\begin{array}{ccc}
F(V_1) \otimes F(V_2) \otimes F(V_3) & \xrightarrow{\theta_3(V_1, V_2, V_3)} & F(V_1 \boxtimes V_2 \boxtimes V_3) \\
{\scriptstyle \mathrm{id}_{F(V_2)} \otimes \theta_2(V_1, V_3)} \searrow & & \nearrow {\scriptstyle \theta_2(V_2, V_1 \boxtimes V_3)} \\
 & F(V_2) \otimes F(V_1 \boxtimes V_3) &
\end{array}
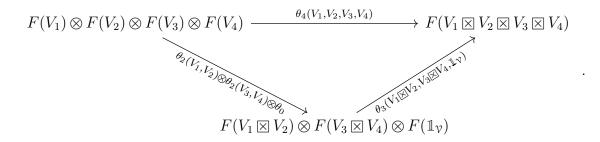$$

(for any $(V_1, V_2, V_3) \in \mathcal{V}^{\times 3}$, and where the symmetry (and associativity) of the symmetric monoidal structures is implicit). On the other hand, the commutative triangle
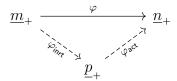
$$
\begin{array}{ccc}
\underline{4}_+ & \xrightarrow{\alpha_4} & \underline{1}_+ \\
 & {\scriptstyle \varphi} \searrow \quad \nearrow {\scriptstyle \alpha_3} & \\
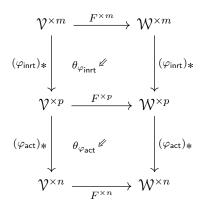 & \underline{3}_+ &
\end{array}
$$

in $\mathsf{Fin}_*^{\mathsf{act}}$ in which $\varphi^{-1}(1) = \{1,2\}$ and $\varphi^{-1}(2) = \{3,4\}$ leads to a natural identification

$$
\begin{array}{ccc}
F(V_1) \otimes F(V_2) \otimes F(V_3) \otimes F(V_4) & \xrightarrow{\;\theta_4(V_1,V_2,V_3,V_4)\;} & F(V_1 \boxtimes V_2 \boxtimes V_3 \boxtimes V_4) \\
& & \\
& F(V_1 \boxtimes V_2) \otimes F(V_3 \boxtimes V_4) \otimes F(\mathbb{1}_{\mathcal{V}}) &
\end{array}
$$

with $\theta_2(V_1,V_2) \otimes \theta_2(V_3,V_4) \otimes \theta_0$ and $\theta_3(V_1 \boxtimes V_2, V_3 \boxtimes V_4, \mathbb{1}_{\mathcal{V}})$.

For a general (not necessarily active) morphism $\underline{m}_+ \xrightarrow{\varphi} \underline{n}_+$ in Fin_*, recall from Exercise 11.29 we have a unique factorization

$$
\begin{array}{ccc}
\underline{m}_+ & \xrightarrow{\;\varphi\;} & \underline{n}_+ \\
& \searrow_{\varphi_{\mathsf{inrt}}} \qquad \nearrow_{\varphi_{\mathsf{act}}} & \\
& \underline{p}_+ &
\end{array}
$$

as the composite of an inert morphism followed by an active morphism. By Exercise 11.30, the natural transformation θ_φ must be the composite natural transformation in the diagram

$$
\begin{array}{ccc}
\mathcal{V}^{\times m} & \xrightarrow{\;F^{\times m}\;} & \mathcal{W}^{\times m} \\
{\scriptstyle (\varphi_{\mathsf{inrt}})_*} \downarrow \quad {\scriptstyle \theta_{\varphi_{\mathsf{inrt}}}}\!\!\Swarrow & & \downarrow {\scriptstyle (\varphi_{\mathsf{inrt}})_*} \\
\mathcal{V}^{\times p} & \xrightarrow{\;F^{\times p}\;} & \mathcal{W}^{\times p} \\
{\scriptstyle (\varphi_{\mathsf{act}})_*} \downarrow \quad {\scriptstyle \theta_{\varphi_{\mathsf{act}}}}\!\!\Swarrow & & \downarrow {\scriptstyle (\varphi_{\mathsf{act}})_*} \\
\mathcal{V}^{\times n} & \xrightarrow[\;F^{\times n}\;]{} & \mathcal{W}^{\times n}
\end{array}
$$

(in which $\theta_{\varphi_{\mathsf{inrt}}}$ is an equivalence). Hence, via the factorization system of Exercise 11.30, the natural transformations θ_φ for general morphisms φ may be seen as recording compatibilities between the natural transformations θ_φ for φ active and the natural equivalences θ_φ for φ inert.

11.6.6.   As we now explain, it turns out that there exists a left adjoint

$$
(61) \qquad\qquad\qquad \mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}} \underset{\longleftarrow}{\overset{\mathsf{Env}}{\dashrightarrow}} \mathsf{CMon}(\mathsf{Cat})
$$

to the inclusion, called the ***symmetric monoidal envelope*** functor.[228] For clarity, we write $\mathsf{Env}(\mathcal{V}, \boxtimes)$ for the underlying ∞-category of the symmetric monoidal envelope of a symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes) \in \mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}}$, and we write $\widetilde{\boxtimes}$ for its symmetric monoidal structure.

Let us write $\mathsf{Ar}^{\mathsf{act}}(\mathsf{Fin}_*) \subset \mathsf{Ar}(\mathsf{Fin}_*)$ for the full subcategory on the active morphisms. Then, $(\mathsf{Env}(\mathcal{V}, \boxtimes), \widetilde{\boxtimes}) \in \mathsf{CMon}(\mathsf{Cat})$ is defined as the pullback

$$
\begin{array}{ccc}
\mathsf{Env}(\mathcal{V}, \boxtimes)^{\widetilde{\boxtimes}} & \longrightarrow & \mathcal{V}^{\boxtimes} \\
\downarrow & & \downarrow \\
\mathsf{Ar}^{\mathsf{act}}(\mathsf{Fin}_*) & \xrightarrow{\ s\ } & \mathsf{Fin}_* \\
{\scriptstyle t}\downarrow & & \\
\mathsf{Fin}_* & &
\end{array} \quad ,
$$

with structure morphism given by the vertical composite.

Exercise 11.32 (6 points). Show that the composite functor $\mathsf{Env}(\mathcal{V}, \boxtimes)^{\widetilde{\boxtimes}} \to \mathsf{Ar}^{\mathsf{act}}(\mathsf{Fin}_*) \xrightarrow{t} \mathsf{Fin}_*$ is indeed a symmetric monoidal ∞-category (i.e. a cocartesian fibration that satisfies the reduced Segal conditions).

The symmetric monoidal envelope can be described informally as follows. First of all, an object of $\mathsf{Env}(\mathcal{V}, \boxtimes) := (\mathsf{Env}(\mathcal{V}, \boxtimes)^{\widetilde{\boxtimes}})_{\underline{1}_+}$ is given by a pair

$$
\left(S_+, S \xrightarrow{V_\bullet} \mathcal{V} \right)
$$

of a pointed finite set $S_+ \in \mathsf{Fin}_*$ (equipped with its unique active morphism $S_+ \to \underline{1}_+$) along with an object of the fiber $(\mathcal{V}^{\boxtimes})_{S_+} \simeq \mathcal{V}^{\times S} \simeq \mathsf{Fun}(S, \mathcal{V})$. Then, a morphism

$$
\left(S_+, S \xrightarrow{V_\bullet} \mathcal{V} \right) \longrightarrow \left(T_+, T \xrightarrow{W_\bullet} \mathcal{V} \right)
$$

is given by an active morphism $S_+ \xrightarrow{\varphi} T_+$ in Fin_* along with a collection of morphisms

$$
\left\{ \underset{s \in \varphi^{-1}(t)}{\boxtimes} V_s \longrightarrow W_t \right\}_{t \in T}
$$

in $\mathcal{V}$. Finally, the symmetric monoidal structure of $\mathsf{Env}(\mathcal{V}, \boxtimes)$ is described by the formula

$$
\left(S_+, S \xrightarrow{V_\bullet} \mathcal{V} \right) \widetilde{\boxtimes} \left(T_+, T \xrightarrow{W_\bullet} \mathcal{V} \right) := \left((S \sqcup T)_+, S \sqcup T \xrightarrow{(V_\bullet, W_\bullet)} \mathcal{V} \right) .
$$

---

[228]For disambiguation, this left adjoint may be denoted $\mathsf{Env}_{\mathsf{Comm}}$: more generally, any ∞-operad $\mathcal{O}$ has a corresponding $\mathcal{O}$-***monoidal envelope functor***, a left adjoint

$$
\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat})^{\mathsf{r.lax}} \underset{\overleftarrow{\hspace{1.2cm}}}{\overset{\mathsf{Env}_{\mathcal{O}}}{\dashrightarrow}} \mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \quad .
$$

By the adjunction (61), the symmetric monoidal envelope $(\mathsf{Env}(\mathcal{V}, \boxtimes), \widetilde{\boxtimes}) \in \mathsf{CMon}(\mathsf{Cat})$ is the initial symmetric monoidal ∞-category equipped with a right-laxly symmetric monoidal functor from $(\mathcal{V}, \boxtimes)$. Namely, the universal right-laxly symmetric monoidal functor

$$(\mathcal{V}, \boxtimes) \longrightarrow (\mathsf{Env}(\mathcal{V}, \boxtimes), \widetilde{\boxtimes})$$

has underlying functor $\mathcal{V} \to \mathsf{Env}(\mathcal{V}, \boxtimes)$ given by the formula

$$V \longmapsto \left(\underline{1}_+, \underline{1} \xrightarrow{V} \mathcal{V}\right) ,$$

and it is right-laxly symmetric monoidal via the evident morphisms

$$\left(S_+, S \xrightarrow{V_\bullet} \mathcal{V}\right) \longrightarrow \left(\underline{1}_+, \underline{1} \xrightarrow{\boxtimes_{s \in S} V_s} \mathcal{V}\right) .$$

On the other hand, considering the identity morphism $(\mathcal{V}, \boxtimes) \xrightarrow{\mathsf{id}} (\mathcal{V}, \boxtimes)$ as a right-laxly symmetric monoidal functor (i.e. as a morphism in $\mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}}$), we obtain a canonical extension

$$
\begin{array}{ccc}
(\mathcal{V}, \boxtimes) & \xrightarrow{\quad \mathsf{id} \quad} & (\mathcal{V}, \boxtimes) \\
\downarrow & \nearrow & \\
(\mathsf{Env}(\mathcal{V}, \boxtimes), \widetilde{\boxtimes}) & &
\end{array}
$$

to a strictly symmetric monoidal functor, which is given by the formula

$$\left(S_+, S \xrightarrow{V_\bullet} \mathcal{V}\right) \longmapsto \boxtimes_{s \in S} V_s .$$

11.6.7.   Let us consider the symmetric monoidal envelope of the terminal symmetric monoidal ∞-category $(\mathsf{pt}, \boxdot) \in \mathsf{CMon}(\mathsf{Cat})$.

Exercise 11.33 (4 points)**.**
   (a) Prove that the functor $\mathsf{pt}^{\boxdot} \to \mathsf{Fin}_*$ is an equivalence.
   (b) Establish a canonical equivalence

$$(\mathsf{Env}(\mathsf{pt}, \boxdot), \widetilde{\boxdot}) \simeq (\mathsf{Fin}, \sqcup)$$

   of symmetric monoidal $(\infty$-$)$categories.

Combining Exercise 11.33(b) with the $((\infty, 2)$-categorical) adjunction (61), for any symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes) \in \mathsf{CMon}(\mathsf{Cat})$ we obtain an identification

$$
\begin{aligned}
\mathsf{CAlg}(\mathcal{V}) &:= \mathsf{Fun}^{\otimes}((\mathsf{Fin}, \sqcup), (\mathcal{V}, \boxtimes)) \\
&:= \underline{\mathsf{hom}}_{\mathsf{CMon}(\mathsf{Cat})}((\mathsf{Fin}, \sqcup), (\mathcal{V}, \boxtimes)) \\
&\simeq \underline{\mathsf{hom}}_{\mathsf{CMon}(\mathsf{Cat})}((\mathsf{Env}(\mathsf{pt}, \boxdot), \widetilde{\boxdot}), (\mathcal{V}, \boxtimes)) \\
&\simeq \underline{\mathsf{hom}}_{\mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}}}((\mathsf{pt}, \boxdot), (\mathcal{V}, \boxtimes)) .
\end{aligned}
$$

In other words, commutative algebra objects in $(\mathcal{V}, \boxtimes)$ are equivalent to right-laxly symmetric monoidal functors to it from $(\mathsf{pt}, \boxdot)$. Thereafter, by Exercise 11.33(a), we see that these are nothing other than sections

$$
\begin{array}{c}
\mathcal{V}^{\boxtimes} \\
\nearrow \Big\uparrow \Big\downarrow \\
\mathsf{Fin}_*
\end{array}
$$

that carry inert morphisms to cocartesian morphisms. The relative simplicity of the object $\mathsf{Fin}_* \in \mathsf{Cat}$ in comparison with the object $(\mathsf{Fin}, \sqcup) \in \mathsf{CMon}(\mathsf{Cat})$ (recorded e.g. as a cocartesian fibration $\mathsf{Fin}^{\sqcup} \downarrow \mathsf{Fin}_*$) is the reduction in combinatorial complexity alluded to in §11.6.2.

In particular, it follows that the assignment $\mathcal{V} \mapsto \mathsf{CAlg}(\mathcal{V})$ defines not just a functor $\mathsf{CMon}(\mathsf{Cat}) \to \mathsf{Cat}$, but a functor $\mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}} \to \mathsf{Cat}$. That is, commutative algebra objects are preserved by *right-laxly* symmetric monoidal functors. This should be plausible: given a morphism $(\mathcal{V}, \boxtimes) \xrightarrow{F} (\mathcal{W}, \otimes)$, in the notation of §11.6.5, the functor

$$
\mathsf{CAlg}(\mathcal{V}, \boxtimes) \xrightarrow{\mathsf{CAlg}(F)} \mathsf{CAlg}(\mathcal{W}, \otimes)
$$

carries a commutative algebra object $A \in \mathsf{CAlg}(\mathcal{V}, \boxtimes)$ with multiplication and unit morphisms

$$
A \boxtimes A \xrightarrow{\mu} A \qquad \text{and} \qquad \mathbb{1}_{\mathcal{V}} \xrightarrow{\iota} A
$$

to a commutative algebra object $F(A) \in \mathsf{CAlg}(\mathcal{W}, \otimes)$ with multiplication and unit morphisms

$$
F(A) \otimes F(A) \xrightarrow{\theta_2(A,A)} F(A \boxtimes A) \xrightarrow{F(\mu)} F(A) \qquad \text{and} \qquad \mathbb{1}_{\mathcal{W}} \xrightarrow{\theta_0} F(\mathbb{1}_{\mathcal{V}}) \xrightarrow{F(\iota)} F(A) .
$$

11.6.8. Suppose that $\mathcal{C} \in \mathsf{Cat}$ is an ∞-category that admits finite products. This yields a symmetric monoidal ∞-category $\mathcal{C}^{\times} \to \mathsf{Fin}_*$.[229] It turns out that there exists a functor

$$
\tag{62} \mathcal{C}^{\times} \xrightarrow{\Pi} \mathcal{C} ,
$$

given informally by the formula

$$
\left(S_+, S \xrightarrow{C_\bullet} \mathcal{C} \right) \longmapsto \prod_{s \in S} C_s .
$$

Specifically, this assignment is functorial for *all* morphisms in $\mathcal{C}^{\times}$, not just those covering active morphisms in Fin_*; for example, it carries an inert-cocartesian morphism

$$
\left(S_+, S \xrightarrow{C_\bullet} \mathcal{C} \right) \longrightarrow \left(\{s_0\}_+, \{s_0\} \xrightarrow{C_{s_0}} \mathcal{C} \right)
$$

in $\mathcal{C}^{\times}$ to the projection morphism

$$
\prod_{s \in S} C_s \longrightarrow C_{s_0}
$$

in $\mathcal{C}$, which exists because the unit object of $(\mathcal{C}, \times)$ is terminal.

---

[229] The actual construction of this cocartesian fibration is slightly involved, and so we do not give it here.

Postcomposition with the functor (62) implements the equivalence between commutative algebra objects in $(\mathcal{C}, \times)$ and commutative monoids in $\mathcal{C}$. Namely, it turns out that the composite functor

$$\mathsf{CAlg}(\mathcal{C}, \times) \xrightarrow{\sim} \Gamma^{\mathsf{inrt.coCart}}(\mathcal{C}^{\times} \downarrow \mathsf{Fin}_*) \xrightarrow{\mathsf{fgt}} \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{C}^{\times}) \xrightarrow{\prod \circ (-)} \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{C})$$

(using §11.6.7) defines an equivalence onto the subcategory $\mathsf{CMon}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{C})$ on the reduced Segal objects.

Exercise 11.34 (4 points)**.** Verify that the composite functor $\mathsf{CAlg}(\mathcal{C}, \times) \to \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{C})$ takes values in the subcategory $\mathsf{CMon}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathsf{Fin}_*, \mathcal{C})$ of reduced Segal objects.

11.6.9.   Let $\mathcal{C} \in \mathsf{Cat}$ be an ∞-category that admits finite coproducts. We briefly indicate the construction of the corresponding symmetric monoidal ∞-category $\mathcal{C}^{\amalg} \to \mathsf{Fin}_*$.[230]

We write Fin_{**} for the category of **doubly-pointed finite sets**, i.e. the category of pairs $(S_+ \in \mathsf{Fin}_*, s \in S)$ of a finite pointed set equipped with a distinguished non-basepoint element. It turns out that the forgetful functor

$$\mathsf{Fin}_{**} \xrightarrow{\mathsf{fgt}} \mathsf{Fin}_*$$

is an *exponentiable fibration* (recall §11.4.3). Hence, we may form the relative functor ∞-category

$$\mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{Fin}_*}(\mathsf{Fin}_{**}, \underline{\mathcal{C}}) \in \mathsf{Cat}_{/\mathsf{Fin}_*} \ ,$$

where we write $\underline{\mathcal{C}} := \mathcal{C} \times \mathsf{Fin}_{**} \in \mathsf{Cat}_{/\mathsf{Fin}_{**}}$.[231] By definition (i.e. by the $((\infty, 2)$-categorical) adjunction (53)), this is characterized by the fact that for any $(\mathcal{K} \downarrow \mathsf{Fin}_*) \in \mathsf{Cat}_{/\mathsf{Fin}_*}$, we have an equivalence

$$\underline{\mathsf{hom}}_{\mathsf{Cat}_{/\mathsf{Fin}_*}}\left(\mathcal{K}, \mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{Fin}_*}(\mathsf{Fin}_{**}, \underline{\mathcal{C}})\right) \simeq \underline{\mathsf{hom}}_{\mathsf{Cat}_{/\mathsf{Fin}_{**}}}\left(\mathcal{K} \times_{\mathsf{Fin}_*} \mathsf{Fin}_{**}, \underline{\mathcal{C}}\right) \simeq \mathsf{Fun}\left(\mathcal{K} \times_{\mathsf{Fin}_*} \mathsf{Fin}_{**}, \mathcal{C}\right) \ .$$

In particular, the fiber over each object $S_+ \in \mathsf{Fin}_*$ is

$$\mathsf{Fun}\left(\{S_+\} \times_{\mathsf{Fin}_*} \mathsf{Fin}_{**}, \mathcal{C}\right) \simeq \mathsf{Fun}(S, \mathcal{C}) \ .$$

---

[230]Cocartesian symmetric monoidal structures are somewhat simpler to construct than cartesian symmetric monoidal structures. This asymmetry may be slightly surprising: after all, passing to opposites should define an involution on symmetric monoidal ∞-categories. Indeed, this involution is easy to construct using the definition given in §10: it is simply postcomposition of a reduced Segal functor $\mathsf{Fin}_* \to \mathsf{Cat}$ with the involution $\mathsf{Cat} \xrightarrow[\sim]{(-)^{\mathsf{op}}} \mathsf{Cat}$. However, this involution is more combinatorially delicate when phrased in terms of fibrations (namely, one must use [BGN18]). (Relatedly, it is an involution on $\mathsf{CMon}(\mathsf{Cat})$, but merely extends to inverse equivalences between $\mathsf{CMon}(\mathsf{Cat})^{\mathsf{r.lax}}$ and $\mathsf{CMon}(\mathsf{Cat})^{\mathsf{l.lax}}$.) This at least explains why the difference in combinatorial complexity is not unreasonable.

[231]This notation is suggested by that for constant (pre)sheaves. Indeed, the functor $\underline{\mathcal{C}} \to \mathsf{Fin}_{**}$ is both a cocartesian fibration and a cartesian fibration, both of which are classified by constant functors at the object $\mathcal{C} \in \mathsf{Cat}$.

Exercise 11.35 (10 points)**.** Prove that the functor

$$\mathcal{C}^{\amalg} := \mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{Fin}_*}(\mathsf{Fin}_{**}, \underline{\mathcal{C}}) \longrightarrow \mathsf{Fin}_*$$

encodes the cocartesian symmetric monoidal structure on the ∞-category $\mathcal{C}$. That is, prove that it is a reduced Segal cocartesian fibration, and that its cocartesian monodromy over an active morphism $S_+ \xrightarrow{\varphi} T_+$ in Fin_* is the functor

$$\mathsf{Fun}(S, \mathcal{C}) \longrightarrow \mathsf{Fun}(T, \mathcal{C})$$

given by the formula

$$\left(S \xrightarrow{C_\bullet} \mathcal{C} \right) \longmapsto \left(T \xrightarrow{t \longmapsto \amalg_{s \in \varphi^{-1}(t)} C_s} \mathcal{C} \right) .$$

11.6.10. We now briefly discuss the case of monoidal ∞-categories. In fact, the situation is sufficiently parallel to the symmetric monoidal case that we merely indicate the corresponding starting point – the factorization system on $\boldsymbol{\Delta}^{\mathsf{op}}$ – and leave it as an exercise for the reader to go back through the material earlier in this subsection and make the relevant substitutions.

Namely, we say that a morphism $[m] \xrightarrow{\psi} [n]$ in $\boldsymbol{\Delta}$ is **active** if it is surjective on endpoints, and **inert** if it is an interval inclusion (i.e. it is of the form $i \mapsto i + k$ for some fixed k), and we use the same terminology for the corresponding morphisms in $\boldsymbol{\Delta}^{\mathsf{op}}$.

Exercise 11.36 (2 points)**.** Show that $(\boldsymbol{\Delta}^{\mathsf{op},\mathsf{inrt}}; \boldsymbol{\Delta}^{\mathsf{op},\mathsf{act}})$ defines a factorization system on the category $\boldsymbol{\Delta}^{\mathsf{op}}$.

Recall from Exercise 10.15 that the forgetful functor $\mathsf{Mon}(\mathsf{Cat}) \leftarrow \mathsf{CMon}(\mathsf{Cat})$ is induced by precomposition with the finite pointed simplicial set $\boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\Delta^1/\partial\Delta^1} \mathsf{Fin}_*$. This suggests the following consistency check.

Exercise 11.37 (2 points)**.** Show that the functor $\boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\Delta^1/\partial\Delta^1} \mathsf{Fin}_*$ respects the respective inert-active factorization systems on the categories $\boldsymbol{\Delta}^{\mathsf{op}}$ and Fin_*.

It is also worth mentioning that the theory of enriched ∞-categories of [GH15] is essentially a "many-object" version of the theory of associative algebra objects in a monoidal ∞-category that is hinted at here.

12. Co/limits and presentable ∞-categories

12.1. **Co/limits, Kan extensions, and final/initial functors.**

12.1.1. Of course, we have been using co/limits throughout the foregoing text, and we have even used Kan extensions in various places (these were first recalled in §8.3.4). In this subsection, we discuss some less basic features of these operations. In order to avoid repetition, we will generally only discuss colimits and left Kan extensions, although we will occasionally discuss dual ideas where appropriate. We may also omit the recurring phrase "assuming this co/limit exists".

12.1.2. We briefly review two (closely related) perspectives on colimits, and discuss the relationship between them.

We first review the notion of colimit discussed in §9.7.3 (in the language of quasicategories). The **right cone** on an ∞-category $\mathfrak{I}$ is the ∞-category

$$\mathfrak{I}^{\triangleright} := \mathsf{Gr}\left([1] \xrightarrow{\mathfrak{I} \rightarrow \mathsf{pt}} \mathsf{Cat}\right) \simeq \mathsf{colim}\left(\begin{array}{ccc} & \mathfrak{I} & \longrightarrow \mathsf{pt} \\ {\scriptstyle(\mathsf{id}_{\mathfrak{I}}, \mathsf{const}_1)}\Big\downarrow & & \\ & \mathfrak{I} \times [1] & \end{array}\right).$$

By the discussion of §11.3.1, this construction formally adjoins a terminal object to $\mathfrak{I}$ (which is often denoted $+\infty$). Given a functor $\mathfrak{I} \xrightarrow{F} \mathfrak{C}$, a **cocone** on F is an extension

$$\begin{array}{ccc} \mathfrak{I} & \xrightarrow{F} & \mathfrak{C} \\ \Big\downarrow & \nearrow & \\ \mathfrak{I}^{\triangleright} & & \end{array}$$

over the right cone on $\mathfrak{I}$; these assemble into an ∞-category

$$\mathfrak{C}_{F/} := \mathsf{lim}\left(\begin{array}{c} \mathsf{Fun}(\mathfrak{I}^{\triangleright}, \mathfrak{C}) \\ \Big\downarrow \\ \mathsf{pt} \xrightarrow{F} \mathsf{Fun}(\mathfrak{I}, \mathfrak{C}) \end{array}\right).^{232}$$

Then, a **colimit** of F is an initial object of $\mathfrak{C}_{F/}$ (which is therefore unique if it exists), which is generally denoted

$$\mathsf{colim}_{\mathfrak{I}}(F) \in \mathfrak{C}_{F/}\ .$$

Of course, we often simply consider the colimit as an object of $\mathfrak{C}$ via the forgetful functor

$$\mathfrak{C}_{F/} \xrightarrow{\mathsf{ev}_{+\infty}} \mathfrak{C}\ .$$

Alternatively, a **colimit** of F is a pointwise left adjoint at F to the functor

$$\mathsf{Fun}(\mathfrak{I}, \mathfrak{C}) \xleftarrow{\mathsf{const}_{(-)}} \mathfrak{C}\ .$$

---

[232]The dual notion, an extension

$$\begin{array}{ccc} \mathfrak{I} & \xrightarrow{F} & \mathfrak{C} \\ \Big\downarrow & \nearrow & \\ \mathfrak{I}^{\triangleleft} & & \end{array}$$

over the **left cone**

$$\mathfrak{I}^{\triangleleft} := \mathsf{Gr}^{-}\left([1]^{\mathsf{op}} \xrightarrow{\mathsf{pt} \leftarrow \mathfrak{I}} \mathsf{Cat}\right)$$

on $\mathfrak{I}$ (whose cone point is often denoted $-\infty$), is called a **cone** on F.

By definition, this is an initial object of the ∞-category

$$\mathcal{C} \underset{\mathsf{Fun}(\mathcal{I},\mathcal{C})}{\times} \mathsf{Fun}(\mathcal{I},\mathcal{C})_{F/} := \lim \left(\begin{array}{c} \mathsf{Fun}(\mathcal{I},\mathcal{C})_{F/} \\ \downarrow{\scriptstyle\mathsf{fgt}} \\ \mathcal{C} \xrightarrow[\mathsf{const}_{(-)}]{} \mathsf{Fun}(\mathcal{I},\mathcal{C}) \end{array} \right)$$

of pairs of an object $C \in \mathcal{C}$ and a morphism $F \to \mathsf{const}_C$.

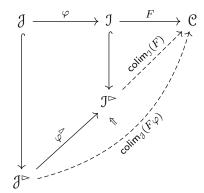**Exercise 12.1** (2 points). Establish an equivalence

$$\mathcal{C}_{F/} \simeq \mathcal{C} \underset{\mathsf{Fun}(\mathcal{I},\mathcal{C})}{\times} \mathsf{Fun}(\mathcal{I},\mathcal{C})_{F/} \ .$$

Exercise 12.1 guarantees that these two definitions of colimits are indeed equivalent.

These two definitions each have their advantages. On the one hand, the latter shows that if $\mathcal{C}$ admits all $\mathcal{I}$-indexed colimits, then the formation of colimits over $\mathcal{I}$ defines a left adjoint functor

$$\mathsf{Fun}(\mathcal{I},\mathcal{C}) \underset{\underset{\mathsf{const}_{(-)}}{\xleftarrow{\hspace{1cm}}}}{\overset{\mathsf{colim}_{\mathcal{I}}(-)}{\dashrightarrow}} \mathcal{C} \quad .$$

On the other hand, suppose we are given any functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ such that both functors $\mathcal{I} \xrightarrow{F} \mathcal{C}$ and $\mathcal{J} \xrightarrow{\varphi} \mathcal{I} \xrightarrow{F} \mathcal{C}$ admit colimits. Then, unwinding the definitions we obtain a canonical morphism

in $\mathcal{C}_{F\varphi/}$.[233]

In fact, these two functorialities can be combined. Namely, let us write

$$\mathsf{Cat}_{\mathsf{l.lax}/\mathcal{C}} := \mathsf{Gr}^- \left(\mathsf{Cat}^{\mathsf{op}} \xrightarrow{\mathsf{Fun}(-,\mathcal{C})} \mathsf{Cat} \right)$$

---

[233]Dually, given functors $\mathcal{J} \xrightarrow{\varphi} \mathcal{I} \xrightarrow{F} \mathcal{C}$ such that F and $F\varphi$ admit limits, we obtain a canonical morphism

$$\lim_{\mathcal{J}}(F\varphi) \longleftarrow \lim_{\mathcal{I}}(F) \ .$$

for the **left-lax overcategory** of $\mathcal{C}$: its objects are functors to $\mathcal{C}$, and a morphism from $(\mathcal{J} \xrightarrow{G} \mathcal{C})$ to $(\mathcal{I} \xrightarrow{F} \mathcal{C})$ is a lax-commutative triangle

(63)

$$
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{\varphi} & \mathcal{I} \\
& \nearrow & \\
G \searrow & & \swarrow \hat{F} \\
& \mathcal{C} &
\end{array}
$$ [234] .

Then, if $\mathcal{C}$ admits all colimits, the formation of colimits in $\mathcal{C}$ assembles as a functor

(64) $$ \mathsf{Cat}_{\mathsf{l.lax}/\mathcal{C}} \xrightarrow{\mathsf{colim}_{(-)}(-)} \mathcal{C} \ , $$

with functorialities for fiber morphisms and cartesian morphisms as respectively indicated above.[235]

[changed 7/8] We conclude this subsubsection with the following useful fact. Suppose that $\mathcal{C}$ admits $\mathcal{I}$-indexed colimits. Then, for any ∞-category $\mathcal{J}$, the ∞-category $\mathsf{Fun}(\mathcal{I}, \mathcal{C})$ also admits $\mathcal{I}$-indexed colimits, and moreover these are computed pointwise. More precisely, a cocone

$$
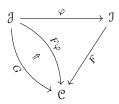\begin{array}{ccc}
\mathcal{J} & \xrightarrow{F} & \mathsf{Fun}(\mathcal{I}, \mathcal{C}) \\
\downarrow & \nearrow & \\
\mathcal{J}^{\triangleright} & \hat{F} &
\end{array}
$$

is a colimit if and only if for every object $I \in \mathcal{I}$ the composite cocone

$$
\begin{array}{ccccc}
\mathcal{J} & \xrightarrow{F} & \mathsf{Fun}(\mathcal{I}, \mathcal{C}) & \xrightarrow{\mathsf{ev}_I} & \mathcal{C} \\
\downarrow & & & \nearrow & \\
\mathcal{J}^{\triangleright} & & \mathsf{ev}_I \circ \hat{F} &
\end{array}
$$

is a colimit. This fact may be summarized as saying that "colimits in functor ∞-categories are pointwise".[236]

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[234]By definition, the forgetful functor $\mathsf{Cat}_{\mathsf{l.lax}/\mathcal{C}} \to \mathsf{Cat}$ is a cartesian fibration. The cartesian pullback of an object $(\mathcal{I} \xrightarrow{F} \mathsf{Cat})$ along a morphism $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ in Cat is the object $(\mathcal{J} \xrightarrow{\varphi} \mathcal{I} \xrightarrow{F} \mathsf{Cat})$, and indeed the lax-commutative triangle (63) is equivalently specified by the diagram

$$
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{\varphi} & \mathcal{I} \\
& \nearrow & \\
G \searrow & F\varphi \downarrow & \swarrow \hat{F} \\
& \mathcal{C} &
\end{array}
$$

(in which the triangle commutes).

[235]A slight generalization applies in the case that $\mathcal{C}$ does not admit all colimits [MG19a, Proposition 3.12].

[236]Beware that it is possible for a functor $\mathcal{J} \xrightarrow{F} \mathsf{Fun}(\mathcal{I}, \mathcal{C})$ to admit a colimit even if the pointwise colimits do not all exist.

12.1.3.   We now introduce an essential notion in the study of colimits. Namely, a functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is called **final** if for any functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$ that admits a colimit, the composite functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I} \xrightarrow{F} \mathcal{C}$ also admits a colimit and moreover the canonical morphism

$$\mathsf{colim}_{\mathcal{J}}(F\varphi) \longrightarrow \mathsf{colim}_{\mathcal{I}}(F)$$

in $\mathcal{C}$ is an equivalence.[237] Obviously, the inclusion of a final object is a final functor, which explains the terminology.

Finality can be characterized more strongly, as follows: a functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is final iff for any functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$, the induced functor

$$\mathcal{C}_{F\varphi/} \longleftarrow \mathcal{C}_{F/}$$

is an equivalence. This condition clearly implies finality, because a colimit of F (resp. of $F\varphi$) is an initial object of $\mathcal{C}_{F/}$ (resp. of $\mathcal{C}_{F\varphi/}$).

Finality can also be characterized more weakly, as follows: a functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is final iff for any functor $\mathcal{I} \xrightarrow{F} \mathcal{S}$, the induced morphism

$$\mathsf{colim}_{\mathcal{J}}(F\varphi) \longrightarrow \mathsf{colim}_{\mathcal{I}}(F)$$

in $\mathcal{S}$ is an equivalence.[238] In view of §11.2.3, it is equivalent to demand that for any left fibration $\mathcal{F} \to \mathcal{I}$, the upper horizontal functor in the pullback square

$$
\begin{array}{ccc}
\varphi^*\mathcal{F} & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{J} & \xrightarrow{\ \varphi\ } & \mathcal{I}
\end{array}
$$

in Cat becomes an equivalence upon passing to ∞-groupoid completions.[239] In particular, taking $\mathcal{F} = \mathcal{I} \xrightarrow{\mathsf{id}} \mathcal{I}$, we find that a final functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ must itself induce an equivalence on ∞-groupoid completions.

Since they will arise frequently in what follows, for brevity we refer to functors that induce equivalences on ∞-groupoid completions as **weak homotopy equivalences**.[240] Similarly,

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[237]Dually, the functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is called **initial** if for any functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$ that admits a limit, the composite functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{J} \xrightarrow{F} \mathcal{C}$ also admits a limit and moreover the canonical morphism

$$\lim_{\mathcal{J}}(F\varphi) \longleftarrow \lim_{\mathcal{I}}(F)$$

in $\mathcal{C}$ is an equivalence. Evidently, $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is initial iff $\mathcal{J}^{\mathsf{op}} \xrightarrow{\varphi^{\mathsf{op}}} \mathcal{J}^{\mathsf{op}}$ is final.

[238]Note that the ∞-category $\mathcal{S}$ admits all colimits.

[239]Dually, the functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is initial iff for every left fibration $\mathcal{F} \to \mathcal{I}$, the induced morphism

$$\Gamma(\varphi^*\mathcal{F} \downarrow \mathcal{J}) \longleftarrow \Gamma(\mathcal{F} \downarrow \mathcal{I})$$

in $\mathcal{S}$ is an equivalence. (Note furthermore that passing from left fibrations to right fibrations interchanges these criteria for finality and initiality.)

[240]Of course, this terminology is motivated by the Kan–Quillen and Joyal model structures on sSet.

to indicate that an ∞-category has contractible ∞-groupoid completion we may simply say that it is **weakly contractible**.

The following exercise shows that final functors satisfy two out of the three possible two-out-of-three properties.

Exercise 12.2 (4 points)**.**
   (a) Given a pair of composable functors $\mathcal{K} \xrightarrow{\psi} \mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ such that ψ is final, show that φ is final iff $\varphi\psi$ is final.
   (b) Give an example of a pair of composable functors $\mathcal{K} \xrightarrow{\psi} \mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ such that φ and $\varphi\psi$ are both final but ψ is not final.

The following exercise should be seen as rather remarkable, given that localizations are in general quite difficult to compute (and in particular, to verify Theorem A for (see below)).

Exercise 12.3 (4 points)**.** Prove that localizations are both final and initial.

Another remarkable fact is that every ∞-category admits a final functor (in particular a weak homotopy equivalence) from a poset.

The primary means of verifying finality is referred to as (**Quillen's**) **Theorem A**: a functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is final iff for every object $I \in \mathcal{I}$ the ∞-category

$$\mathcal{J}_{I/} := \mathcal{J} \underset{\mathcal{J}}{\times} \mathcal{I}_{I/}$$

is weakly contractible.[241]

Exercise 12.4 (2 points)**.** Prove that right adjoints are final.

In fact, Theorem A is *also* the primary means of verifying that a functor is a weak homotopy equivalence (although the conclusion of finality is strictly stronger). Note that it can be combined nontrivially with its dual to detect weak homotopy equivalences. For instance, if in the composite $\mathcal{K} \xrightarrow{\psi} \mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ the functor ψ is initial and the composite $\varphi\psi$ is final, then the functor φ is a weak homotopy equivalence.

12.1.4.   We now review the theory of left Kan extensions.

We begin by recalling the definition. We fix a functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ and an ∞-category $\mathcal{C}$. Precomposition with φ determines a functor

$$\mathsf{Fun}(\mathcal{J}, \mathcal{C}) \xleftarrow{\varphi^*} \mathsf{Fun}(\mathcal{I}, \mathcal{C}) \ .$$

Then, a **left Kan extension** of an object $F \in \mathsf{Fun}(\mathcal{J}, \mathcal{C})$ along φ is a pointwise left adjoint to φ^* at F, which we typically denote by $\varphi_! F \in \mathsf{Fun}(\mathcal{I}, \mathcal{C})$.[242] So by definition, a left Kan extension is an initial object in the ∞-category of pairs of a functor $G \in \mathsf{Fun}(\mathcal{I}, \mathcal{C})$ and a

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[241]A mnemonic for remembering that $\mathcal{J}_{I/}$ is relevant for finality is the fact that the inclusion of a final object is a final functor. (Dually, $\mathcal{J}_{I/}$ is relevant for initiality.)

[242]Dually, right Kan extension along φ is typically denoted by φ_*.

morphism $F \to \varphi^*G$ in $\mathsf{Fun}(\mathcal{J}, \mathcal{C})$. Such data – often referred to as a *left extension* of F along φ – is generally depicted as in the diagram

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\ F\ } & \mathcal{C} \\ {\scriptstyle\varphi}\downarrow & {\scriptstyle\mathcal{U}} \nearrow & \\ \mathcal{J} & \overset{G}{\dashrightarrow} & \end{array} \quad .$$

Left Kan extensions generalize colimits, as follows.

Exercise 12.5 (2 points)**.** Fix a functor $\mathcal{J} \xrightarrow{F} \mathcal{C}$ that admits a colimit.

(a) Writing $\mathcal{J} \xrightarrow{\varphi} \mathsf{pt}$, show that $\mathsf{pt} \xrightarrow{\varphi_! F} \mathcal{C}$ exists and computes the colimit of F.

(b) Writing $\mathcal{J} \xrightarrow{\varphi} \mathcal{J}^\triangleright$ for the inclusion into the right cone, show that $\mathcal{J}^\triangleright \xrightarrow{\varphi_! F} \mathcal{C}$ exists and carries the cone point to the colimit of F.

In fact, left Kan extensions can often be computed in terms of colimits, as we now explain. Consider an arbitrary left extension

$$\left(G, F \xrightarrow{\ \alpha\ } \varphi^*G\right) \in \mathsf{Fun}(\mathcal{J}, \mathcal{C}) \underset{\mathsf{Fun}(\mathcal{J},\mathcal{C})}{\times} \mathsf{Fun}(\mathcal{J}, \mathcal{C})_{F/} =: \mathsf{Fun}(\mathcal{J}, \mathcal{C})_{F/}$$

of F along φ. For any fixed object $I \in \mathcal{J}$, our left extension determines a functor

$$(65) \qquad\qquad \mathcal{J} \underset{\mathcal{J}}{\times} \mathcal{J}_{/I} =: \mathcal{J}_{/I} \longrightarrow \mathcal{C}_{/G(I)}$$

given by the formula

$$\left(J, \varphi(J) \xrightarrow{\ \gamma\ } I\right) \longmapsto \left(F(J) \xrightarrow{\alpha_J} G(\varphi(J)) \xrightarrow{G(\gamma)} G(I)\right) \ .$$

The functor (65) is equivalent data to a cocone

$$(66) \qquad\qquad \begin{array}{ccccc} \mathcal{J}_{/I} & \xrightarrow{\ \mathsf{fgt}\ } & \mathcal{J} & \xrightarrow{\ F\ } & \mathcal{C} \\ \downarrow & & & \nearrow & \\ (\mathcal{J}_{/I})^\triangleright & \dashrightarrow & & & \end{array}$$

that takes the cone point to the object $G(I) \in \mathcal{C}$.[243] Hence, if the colimit

$$(67) \qquad\qquad \mathsf{colim}\left(\mathcal{J}_{/I} \xrightarrow{\ \mathsf{fgt}\ } \mathcal{J} \xrightarrow{\ F\ } \mathcal{C}\right) \in \mathcal{C}$$

exists, then it comes equipped with a canonical morphism to $G(I) \in \mathcal{C}$. This colimit (67) is therefore the initial possible *value* at $I \in \mathcal{J}$ of a left extension of F along φ, and it is therefore

---

[243]_[changed 7/8] In case the natural transformation $F \xrightarrow{\alpha} \varphi^*G$ is an equivalence, the cocone (66) admits a simpler description: it arises from the commutative diagram

$$\begin{array}{ccccc} \mathcal{J}_{/I} & \xrightarrow{\ \mathsf{fgt}\ } & \mathcal{J} & \xrightarrow{\ F\ } & \mathcal{C} \\ \downarrow & & & & \uparrow{\scriptstyle G} \\ (\mathcal{J}_{/I})^\triangleright & \xrightarrow[(\varphi, +\infty \mapsto I)]{} & & & \mathcal{J} \end{array} \quad .$$

(More generally, α determines a downwards natural transformation.)

called the ***pointwise left Kan extension*** at $I \in \mathcal{I}$ of F along φ.[244] Conversely, if for every object $I \in \mathcal{I}$ the pointwise left Kan extension (67) of F along φ exists, then the left Kan extension $\varphi_! F$ exists and its value at each object $I \in \mathcal{I}$ is the pointwise left Kan extension (67).[245]

It is also straightforward to compute left Kan extensions along a left adjoint. Namely, given an adjunction

$$\mathcal{J} \xrightarrow[\substack{\perp \\ \varphi^R}]{\varphi} \mathcal{I} \quad ,$$

applying $\mathsf{Fun}(-, \mathcal{C})$ we obtain an adjunction

$$\mathsf{Fun}(\mathcal{J}, \mathcal{C}) \xrightarrow[\substack{\perp \\ \varphi^*}]{(\varphi^R)^*} \mathsf{Fun}(\mathcal{I}, \mathcal{C})$$

(recall Exercise 7.2). By the uniqueness of left adjoints, it follows that $\varphi_! \simeq (\varphi^R)^*$. This same conclusion also follows from the above considerations: each ∞-category $\mathcal{J}_{/I}$ has a terminal object $(\varphi^R(I), \varphi(\varphi^R(I)) \xrightarrow{\varepsilon_I} I)$, and so the pointwise left Kan extension at $I \in \mathcal{I}$ of any functor $\mathcal{J} \xrightarrow{F} \mathcal{C}$ along $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ (automatically exists and) is simply the value $F(\varphi^R(I)) \in \mathcal{C}$.

12.1.5.   We now discuss some further features of left Kan extensions, as well as their interactions.

First, we discuss the composability of left Kan extensions. Given functors $\mathcal{K} \xrightarrow{\psi} \mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ and an ∞-category $\mathcal{C}$, we clearly have a commutative diagram

$$\mathsf{Fun}(\mathcal{K}, \mathcal{C}) \xleftarrow{\psi^*} \mathsf{Fun}(\mathcal{J}, \mathcal{C}) \xleftarrow{\varphi^*} \mathsf{Fun}(\mathcal{I}, \mathcal{C}) \quad .$$
$$\underset{(\varphi\psi)^*}{\underleftarrow{\hspace{3cm}}}$$

By the uniqueness of (pointwise) left adjoints, it follows that left Kan extensions compose: given a functor $F \in \mathsf{Fun}(\mathcal{K}, \mathcal{C})$, if $\psi_! F$ and $\varphi_!(\psi_! F)$ both exist, then $(\varphi\psi)_! F$ exists and we

---

[244]Note that if the functor $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is the inclusion of a full subcategory, then for each $J \in \mathcal{J}$ there exists a terminal object $(J \xrightarrow{\mathsf{id}_J} J) \in \mathcal{I}_{/J}$. So in this case, the pointwise Kan extension always exists at each object $J \in \mathcal{J} \subseteq \mathcal{I}$, and is simply $F(J) \in \mathcal{C}$ itself.

[245]Beware that it is possible for a left Kan extension to exist even if the pointwise left Kan extensions do not all exist. [changed 7/8] A minimal example of this phenomenon is given by the commutative triangle

$$\begin{array}{ccc} \mathsf{pt} & \longrightarrow & S^0 \\ {\scriptstyle 1}\big\downarrow & \nearrow & \\ [1] & & \end{array} \quad ,$$

which is easily checked to be a left Kan extension diagram even though the pointwise left Kan extension of $\mathsf{pt} \to S^0$ along $\mathsf{pt} \xrightarrow{1} [1]$ does not exist. (Beware too that [Lur09] only *defines* left Kan extensions as (those that are) pointwise left Kan extensions, cf. [Lur09, Definitions 4.3.2.2 and 4.3.3.2]. This is presumably due to the fact that these notions agree in essentially all examples of interest.)

have a canonical equivalence

$$(\varphi\psi)_! F \simeq \varphi_!(\psi_! F) \ .$$

We now discuss left Kan extensions along cocartesian fibrations. Let us fix a cocartesian fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ and a functor $\mathcal{E} \xrightarrow{F} \mathcal{C}$. Recall from Exercise 11.7 that for every object $b \in \mathcal{B}$ the functor
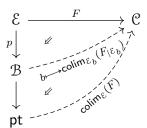
$$\mathcal{E}_{/b} \longleftrightarrow \mathcal{E}_b$$

is a right adjoint, and is therefore final by Exercise 12.4. Hence, we can compute the pointwise left Kan extension of F along p at an object $b \in \mathcal{B}$ (if it exists) as

$$\mathrm{colim}\left(\mathcal{E}_{/b} \xrightarrow{\mathsf{fgt}} \mathcal{E} \xrightarrow{F} \mathcal{C}\right) \xleftarrow{\sim} \mathrm{colim}\left(\mathcal{E}_b \longleftrightarrow \mathcal{E}_{/b} \xrightarrow{\mathsf{fgt}} \mathcal{E} \xrightarrow{F} \mathcal{C}\right) \ .$$

In particular, if these pointwise left Kan extensions all exist, then one may say that the left Kan extension

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ F\ } & \mathcal{C} \\
{\scriptstyle p}\downarrow & \overset{\nearrow}{\underset{\diagup\diagup p_!F}{\nearrow}} & \\
\mathcal{B} & &
\end{array}
$$

is computed by fiberwise colimit.[246,247] Combining this with Exercise 12.5(a) and the fact that left Kan extensions compose, we obtain a diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\quad F \quad} & \mathcal{C} \\
{\scriptstyle p}\downarrow & \quad & \\
\mathcal{B} & \dashrightarrow_{b \mapsto \mathrm{colim}_{\mathcal{E}_b}(F_{|\mathcal{E}_b})} & \\
\downarrow & \quad & \\
\mathsf{pt} & \dashrightarrow_{\mathrm{colim}_{\mathcal{E}}(F)} &
\end{array}
$$

in which both triangles are left Kan extension diagrams. Appealing again to Exercise 12.5(a), we obtain a formula

$$(68) \qquad\qquad \mathrm{colim}_{\mathcal{E}}(F) \simeq \mathrm{colim}_{b \in \mathcal{B}}\left(\mathrm{colim}_{\mathcal{E}_b}(F_{|\mathcal{E}_b})\right)$$

for the colimit over $\mathcal{E}$.

---

[246]Dually, right Kan extension along a cartesian fibration is computed by fiberwise limit.

[247]However, beware that "fiberwise colimit" does not generally define a functor. (For instance, given arbitrary functors $\mathcal{I} \to [1]$ and $\mathcal{I} \xrightarrow{F} \mathcal{C}$, there does not generally exist a canonical morphism $\mathrm{colim}_{\mathcal{I}_0}(F_{|\mathcal{I}_0}) \to \mathrm{colim}_{\mathcal{I}_1}(F_{|\mathcal{I}_1})$ in $\mathcal{C}$ (even if these colimits exist).) Nevertheless, in what follows (e.g. in formula (68)) we take as implicit the assertion that the indicated formula does indeed assemble as a functor. Indeed, this is essentially the proof that the formation of colimits in $\mathcal{C}$ assembles as a functor (64) from $\mathsf{Cat}_{\mathsf{l.lax}/\mathcal{C}}$. (On the other hand, left Kan extensions along *locally* cocartesian fibrations are also computed by fiberwise colimits by Exercise 11.12, and in fact more generally (and essentially by definition) left Kan extensions along "proper" functors are also computed by fiberwise colimits [Cis19, Proposition 6.4.3] (see [Cis19, Definition 4.4.1 and Proposition 4.4.4]). By contrast, these more general facts do not seem to have straightforward "straightened" interpretations (in terms of the $(\infty, 2)$-category Cat).)

A special case is when $\mathcal{E} = \mathcal{B} \times \mathcal{B}' \xrightarrow{\mathsf{pr}} \mathcal{B}$ is the projection from a product (which is a cocartesian fibration classified by the functor $\mathcal{B} \xrightarrow{\mathsf{const}_{\mathcal{B}'}} \mathsf{Cat}$). Then, the formula (68) for the colimit of a functor $\mathcal{B} \times \mathcal{B}' \xrightarrow{F} \mathcal{C}$ reduces to an equivalence

$$\mathsf{colim}_{\mathcal{B} \times \mathcal{B}'}(F) \simeq \mathsf{colim}_{b \in \mathcal{B}} \left(\mathsf{colim}_{b' \in \mathcal{B}'}(F(b, b')) \right) \ .$$

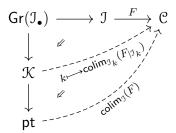Reversing the roles of $\mathcal{B}$ and $\mathcal{B}'$, we obtain a composite equivalence

$$\mathsf{colim}_{b \in \mathcal{B}} \left(\mathsf{colim}_{b' \in \mathcal{B}'}(F(b, b')) \right) \simeq \mathsf{colim}_{\mathcal{B} \times \mathcal{B}'}(F) \simeq \mathsf{colim}_{b' \in \mathcal{B}'} \left(\mathsf{colim}_{b \in \mathcal{B}}(F(b, b')) \right) \ ,$$

which is often referred to as **Fubini's theorem for colimits**.

Exercise 12.6 (2 points)**.** Show that if $\mathcal{J} \xrightarrow{\varphi} \mathcal{I}$ is a final (resp. initial) functor then so is $\mathcal{J} \times \mathcal{K} \xrightarrow{\varphi \times \mathsf{id}_{\mathcal{K}}} \mathcal{I} \times \mathcal{K}$.

Lastly, we explain a decomposition theorem for co/limits indexed over a colimit of ∞-categories. For this, let us fix a diagram $\mathcal{K} \xrightarrow{\mathcal{I}_\bullet} \mathsf{Cat}$ of ∞-categories, and let us write $\mathcal{I} := \mathsf{colim}_{k \in \mathcal{K}}(\mathcal{I}_k)$. Recall from §11.2.3 that there is a canonical localization functor $\mathsf{Gr}(\mathcal{I}_\bullet) \to \mathcal{I}$, which by Exercise 12.3 is final. Then, suppose we are given a functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$. Assuming that $\mathcal{C}$ admits the relevant pointwise left Kan extensions, we obtain a diagram

in which both triangles are left Kan extension diagrams. In other words, we obtain a formula

$$\mathsf{colim}_{\mathcal{I}}(F) \simeq \mathsf{colim}_{k \in \mathcal{K}} \left(\mathsf{colim}_{\mathcal{I}_k}(F_{|\mathcal{I}_k}) \right)$$

for the colimit over $\mathcal{I}$. Dually, there is a canonical localization functor $\mathsf{Gr}^-(\mathcal{I}_\bullet) \to \mathcal{I}$, which by Exercise 12.3 is initial, and assuming that $\mathcal{C}$ admits the relevant pointwise right Kan extensions we obtain a diagram

in which both triangles are right Kan extension diagrams, yielding a formula

$$\mathsf{lim}_{\mathcal{I}}(F) \simeq \mathsf{lim}_{k^\circ \in \mathcal{K}^{\mathsf{op}}} \left(\mathsf{lim}_{\mathcal{I}_k}(F_{|\mathcal{I}_k}) \right)$$

for the limit over $\mathcal{I}$. For instance, in the special case that $\mathcal{K}$ is merely a set (so that $\mathcal{I} \simeq \coprod_{k \in \mathcal{K}} \mathcal{I}_k$), we obtain formulas

$$\mathsf{colim}_{\mathcal{I}}(F) \simeq \coprod_{k \in \mathcal{K}} \mathsf{colim}_{\mathcal{I}_k}(F_{|\mathcal{I}_k}) \qquad \text{and} \qquad \mathsf{lim}_{\mathcal{I}}(F) \simeq \prod_{k \in \mathcal{K}} \mathsf{lim}_{\mathcal{I}_k}(F_{|\mathcal{I}_k})$$

(assuming these co/limits exist).

12.1.6. We now discuss co/limits over ∞-groupoids.

By the discussion of §12.1.5, in order to understand co/limits over ∞-groupoids, it suffices to understand co/limits over *connected* ∞-groupoids: more general ones are simply co/products of such. Now, a connected ∞-groupoid is necessarily of the form $\mathsf{B}G$ for some ∞-group $G \in \mathsf{Grp}(\mathcal{S})$, co/limits over which are by definition the G-co/invariants. Of course, these can be quite nontrivial (e.g. recall §7.4).

In a different direction, it is also interesting to study co/limits over ∞-groupoids of *constant* functors.[248] Given an ∞-groupoid $X \in \mathcal{S}$ and an object $C \in \mathcal{C}$, the **tensor** and **cotensor** of C over X are by definition

$$X \odot C := \mathsf{colim}\left(X \xrightarrow{\mathsf{const}_C} \mathcal{C}\right) \qquad \text{and} \qquad X \pitchfork C := \mathsf{lim}\left(X \xrightarrow{\mathsf{const}_C} \mathcal{C}\right) .$$

Exercise 12.7 (2 points). In the case that $\mathcal{C} = \mathcal{S}$, show that $X \pitchfork C \simeq \mathsf{hom}_{\mathcal{S}}(X, C)$.

It follows from Exercise 12.7 that the co/tensor of C over X satisfies the universal property that for any object $D \in \mathcal{C}$ we have

$$\mathsf{hom}_{\mathcal{C}}(X \odot C, D) \simeq \mathsf{hom}_{\mathcal{S}}(X, \mathsf{hom}_{\mathcal{C}}(C, D)) \qquad \text{and} \qquad \mathsf{hom}_{\mathcal{C}}(D, X \pitchfork C) \simeq \mathsf{hom}_{\mathcal{S}}(X, \mathsf{hom}_{\mathcal{C}}(D, C)) .$$

Hence, the co/tensor generalizes the indexed co/product: in the special case that X is merely a set, we have

$$X \odot C \simeq \coprod_X C \qquad \text{and} \qquad X \pitchfork C \simeq \prod_X C .$$

It is worth noting that more general co/limits sometimes reduce to those over ∞-groupoids. Namely, suppose that a functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$ factors through the maximal subgroupoid $\iota_0 \mathcal{C}$. This is equivalent to the existence of a factorization

through the ∞-groupoid completion of $\mathcal{I}$. The functor $\mathcal{I} \to \mathcal{I}^{\mathsf{gpd}}$ is a localization, hence both final and initial by Exercise 12.3. It follows that we obtain canonical equivalences

$$\mathsf{colim}_{\mathcal{I}}(F) \xrightarrow{\sim} \mathsf{colim}_{\mathcal{I}^{\mathsf{gpd}}}(\tilde{F}) \qquad \text{and} \qquad \mathsf{lim}_{\mathcal{I}}(F) \xleftarrow{\sim} \mathsf{lim}_{\mathcal{I}^{\mathsf{gpd}}}(\tilde{F})$$

(assuming these co/limits exist).

---

[248]Note that in general it is data (and not merely a condition) for a functor to be constant.

Exercise 12.8 (4 points)**.** Suppose that the object $C \in \mathcal{C}$ admits co/tensors over all spaces, so that there exist functors

$$\mathcal{S} \xrightarrow{(-) \odot C} \mathcal{C} \qquad \text{and} \qquad \mathcal{S} \xrightarrow{(-) \pitchfork C} \mathcal{C}^{\mathrm{op}} \, .^{249}$$

Prove that these functors preserve colimits.

Co/tensors also play a fundamental role in the theory of presentably (symmetric) monoidal ∞-categories, as discussed in **??**.[250]

12.1.7. We conclude this subsection by discussing a few common classes of co/limits. A recurring theme is the *generation* of a general class of co/limits by some sub-class thereof, as we now explain. Here we make some light use of a few set-theoretic notions (e.g. finiteness and κ-smallness), which will be discussed in §12.2.2.

A first example is the class of ***finite colimits***, i.e. colimits over finite ∞-categories. An ∞-category admits finite colimits iff it admits an initial object (i.e. a colimit of the empty diagram) and pushouts. Similarly, a functor between ∞-categories admitting finite colimits preserves them iff it preserves initial objects and pushouts. Hence, we say that initial objects and pushouts *generate* all finite colimits.[251]

This example generalizes to the class of κ-***small colimits*** for any regular cardinal κ (the previous example being recovered when $\kappa = \omega$). Namely, an ∞-category admits κ-small colimits iff it admits pushouts and κ-small coproducts. Similarly, a functor between ∞-categories admitting κ-small colimits preserves them iff it preserves pushouts and κ-small coproducts. In other words, κ-small colimits are generated by pushouts and κ-small coproducts. In fact, κ-small colimits are also generated by coequalizers and κ-small coproducts.

Another example of a similar flavor is that for any regular cardinal κ, small colimits are generated by κ-small colimits and κ-filtered colimits (the latter are introduced in **??**). In particular (taking $\kappa = \omega$), small colimits are generated by finite colimits and filtered colimits.

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[249]Precisely, these functors can be deduced from the Yoneda lemma, e.g. the latter is (opposite to) a factorization

$$\mathcal{S}^{\mathrm{op}} \xrightarrow{\hom_{\mathcal{S}}((-), \hom_{\mathcal{C}}(=, C))} \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

$$\xdashrightarrow{(-) \pitchfork C} \mathcal{C} \quad \uparrow \text{ff}$$

(whose existence can be checked objectwise).

[250]For instance, given a commutative ring $\Bbbk$, its derived ∞-category $\mathbf{D}_\Bbbk$ is presentably symmetric monoidal, and in this context co/tensors are also known as co/chains: for any derived R-module $M \in \mathbf{D}_R$ and any space $X \in \mathcal{S}$ we have canonical equivalences

$$X \odot M \simeq \Bbbk\{X\} \otimes_\Bbbk M =: \mathsf{C}_\bullet(X; M) \qquad \text{and} \qquad X \pitchfork M \simeq \underline{\hom}_{\mathbf{D}_\Bbbk}(\Bbbk\{X\}, M) =: \mathsf{C}^\bullet(X; M)$$

in $\mathbf{D}_\Bbbk$ (recall §8.4).

[251]Dually, terminal objects and pullbacks generate all finite limits.

An ∞-category $\mathfrak{I}$ is called **sifted** if it is nonempty and moreover the diagonal functor $\mathfrak{I} \to \mathfrak{I} \times \mathfrak{I}$ is final. The relevance of this notion arises in the study of functors of several variables (e.g. the functor $\mathfrak{C}^{\times n} \xrightarrow{\otimes} \mathfrak{C}$ for a (symmetric) monoidal ∞-category $(\mathfrak{C}, \otimes)$). For instance, suppose we are given a bifunctor

$$\mathfrak{C} \times \mathfrak{D} \xrightarrow{F} \mathcal{E}$$

such that all three ∞-categories $\mathfrak{C}, \mathfrak{D}, \mathcal{E} \in \mathsf{Cat}$ admit $\mathfrak{I}$-indexed colimits and the bifunctor F preserves $\mathfrak{I}$-indexed colimits separately in each variable.[252] If $\mathfrak{I}$ is sifted, then the functor F preserves $\mathfrak{I}$-indexed colimits (considered merely as a functor).

Exercise 12.9 (2 points)**.** Prove that a finite product of sifted ∞-categories is sifted.

The key example of a sifted ∞-category is $\mathbf{\Delta}^{\mathsf{op}}$, and indeed this alone generates the class of sifted colimits.

It is also worth mentioning the class of **weakly contractible colimits**. Fix an ∞-category $\mathfrak{C}$ and an object $C \in \mathfrak{C}$. We obtain a forgetful functor $\mathfrak{C}_{C/} \to \mathfrak{C}$. Clearly this commutes with limits.[253] In fact, it *also* commutes with weakly contractible colimits (e.g. pushouts). More generally, any left fibration commutes with weakly contractible colimits.[254]

We conclude by discussing *retracts*, which are somewhat subtler than the preceding notions.

We begin by defining the ∞-category

$$\mathsf{Ret} := \mathrm{colim} \left(\begin{array}{c} [1] \xrightarrow{0 < 2} [2] \\ \downarrow \\ \mathsf{pt} \end{array} \right) \in \mathsf{Cat}$$

(which is in fact merely a 1-category), whose objects and morphisms we denote as in the diagram

$$\mathsf{Ret} := \left(\begin{array}{c} X \\ {}^{s}\nearrow \quad \searrow^{r} \\ R \xrightarrow{\quad \sim \quad} R \end{array} \right).$$

By definition, Ret is the free ∞-category on a **section-retraction pair**; more specifically, we say that s is a **section** of r, or that r is a **retraction** of s. We write $\mathsf{Idem} \subset \mathsf{Ret}$ for the full subcategory on the object $X \in \mathsf{Ret}$. By definition, this corepresents the data of an

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[252]Explicitly, this means that for any objects $C \in \mathfrak{C}$ and $D \in \mathfrak{D}$ the functors

$$\mathfrak{C} \xrightarrow{F(-,D)} \mathcal{E} \qquad \text{and} \qquad \mathfrak{D} \xrightarrow{F(C,-)} \mathcal{E}$$

preserve $\mathfrak{I}$-indexed colimits.

[253]That is, given any functor to $\mathfrak{C}_{C/}$, it admits a limit iff its postcomposition to $\mathfrak{C}$ admits one, and in this case the forgetful functor preserves this limit.

[254]Dually, any right fibration commutes with weakly contractible limits.

idempotent endomorphism, i.e. the data of an object $C \in \mathcal{C}$, an endomorphism $C \xrightarrow{i} C$ (the image of the endomorphism $X \xrightarrow{sr} X$), an equivalence $i \simeq i \circ i$, and an infinite tower of coherence data.[255]

Now, it turns out that the inclusion $\mathsf{Idem} \hookrightarrow \mathsf{Ret}$ is an *epimorphism* in Cat: for any functor $\mathsf{Idem} \to \mathcal{C}$, the space of extensions

(69)

$$
\begin{array}{ccc}
\mathsf{Idem} & \longrightarrow & \mathcal{C} \\
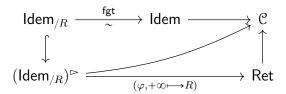\downarrow & \nearrow & \\
\mathsf{Ret} & &
\end{array}
$$

is always either empty or contractible. An extension (69) is generally referred to as a ***splitting*** of the given idempotent endomorphism, and the idempotent is called ***effective*** if it admits a splitting; the value of the splitting at the object $R \in \mathsf{Ret}$ is called the ***retract*** of the idempotent. Evidently, retracts are ***absolute***: they are preserved by any functor.

In fact, an extension (69) is automatically both a pointwise left Kan extension diagram and a pointwise right Kan extension diagram. Moreover, a retract of an idempotent $\mathsf{Idem} \to \mathcal{C}$ is both its colimit and its limit (and if any one of these exists then all three do). In particular, both limits and colimits indexed over Idem are absolute. These facts result from the following exercise.

Exercise 12.10 (8 points)**.**
    (a) Prove that there exists a retraction of the inclusion $\mathsf{Idem} \hookrightarrow \mathsf{Idem}^{\triangleright}$.
    (b) Use part (a) to prove that the inclusion $\mathsf{Idem} \xrightarrow{\varphi} \mathsf{Ret}$ is final.
Note that a commutative triangle (69) is a pointwise left Kan extension diagram iff the upper triangle in the commutative diagram

$$
\begin{array}{ccccc}
\mathsf{Idem}_{/R} & \xrightarrow[\sim]{\mathsf{fgt}} & \mathsf{Idem} & \longrightarrow & \mathcal{C} \\
\downarrow & & & \nearrow & \uparrow \\
(\mathsf{Idem}_{/R})^{\triangleright} & \xrightarrow[(\varphi,+\infty \longmapsto R)]{} & & & \mathsf{Ret}
\end{array}
$$

is a colimit (recall Footnote 243).
    (c) Obtain a factorization

$$
\begin{array}{ccc}
(\mathsf{Idem}_{/R})^{\triangleright} & \xrightarrow{(\varphi,+\infty \longmapsto R)} & \mathsf{Ret} \\
\mathsf{fgt} \downarrow \wr & & \uparrow \\
\mathsf{Idem}^{\triangleright} & \xrightarrow{\varphi^{\triangleright}} & \mathsf{Ret}^{\triangleright}
\end{array} \quad ,
$$

---

[255]More precisely, this last equivalence must be homotopy-coherently associative. For instance, it must be equipped with an equivalence between the equivalences $i \circ i \circ i = (i \circ i) \circ i \simeq i \circ i \simeq i$ and $i \circ i \circ i = i \circ (i \circ i) \simeq i \circ i \simeq i$. (This may be thought of as a nullhomotopy of a certain map $S^1 \to \mathsf{end}_{\mathcal{C}}(c)$.) Indeed, we have an equivalence $\mathsf{Idem} \simeq \mathfrak{B}M$, where $M \in \mathsf{Mon}(\mathsf{Set})$ is the two-element monoid $\{e, i\}$ with identity element e and with $i^2 = i$.

and use this and part (b) to prove that any commutative triangle (69) is a pointwise left Kan extension diagram.

(d) Prove that the inclusion $\mathsf{Idem} \hookrightarrow \mathsf{Ret}$ is self-opposite, i.e. establish an equivalence

$$\left(\mathsf{Idem} \xhookrightarrow{\;\varphi\;} \mathsf{Ret}\right)^{\mathsf{op}} \simeq \left(\mathsf{Idem} \xhookrightarrow{\;\varphi\;} \mathsf{Ret}\right) \ .$$

We have the following concrete construction of retracts.

Exercise 12.11 (4 points). Prove that the unique conservative functor $\mathbb{N}^{\leqslant} \to \mathsf{Idem}$ is final.

So if it exists, the retract of an idempotent $C \xrightarrow{i} C$ can be identified with

$$\mathsf{colim}\left(C \xrightarrow{\;i\;} C \xrightarrow{\;i\;} C \xrightarrow{\;i\;} \cdots\right) \in \mathcal{C} \ .$$

An ∞-category is called ***idempotent-complete*** if all of its idempotents are effective. Let us write $\mathsf{Cat}^{\mathsf{idem}} \subset \mathsf{Cat}$ for the full subcategory on the idempotent-complete ∞-categories. We will see in **[ref]** that there exists a left adjoint

$$\mathsf{Cat} \ \underset{\longleftarrow}{\overset{(-)^{\vee}}{\dashrightarrow}} \ \mathsf{Cat}^{\mathsf{idem}}$$

to the inclusion, called the ***idempotent completion*** functor. Moreover, it turns out that the components $\mathcal{C} \to \mathcal{C}^{\vee}$ of the unit of the adjunction are fully faithful inclusions.

Note that Idem is *not* a finite ∞-category (even though Ret is), so that an ∞-category $\mathcal{C}$ may admit finite colimits and/or finite limits without being idempotent-complete.[256] A concrete example is given by the ∞-category $\mathcal{S}^{\mathsf{fin}} \subseteq \mathcal{S}$ of finite spaces (i.e. those generated under finite colimits by the terminal object $\mathsf{pt} \in \mathcal{S}$); its failure to admit retracts is codified by *Wall's finiteness obstruction* [Wal65].[257]

12.2. Accessible ∞-categories.

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[256]On the other hand, if $\mathcal{C}$ is a 1-category, then the retract of an idempotent endomorphism $C \xrightarrow{i} C$ in $\mathcal{C}$ *can* be computed as a finite co/limit: namely, it is both the equalizer and coequalizer of the parallel pair $C \underset{\mathsf{id}_C}{\overset{i}{\rightrightarrows}} C$.

[257]This is formally analogous to the fact that in the derived ∞-category $\mathbf{D}_R$ of a ring R, the finitely presentable objects (i.e. those generated by $R \in \mathsf{Mod}_R \subseteq \mathbf{D}_R$ under finite colimits and desuspensions) are not generally closed under retracts. Indeed, the latter are closed under retracts if and only if every finitely generated projective R-module $M \in \mathsf{Mod}_R^{\mathsf{f.g.,proj}}$ is stably free. The stable freeness of M is equivalent to the vanishing of its corresponding class $[M]$ in the 0th reduced K-group

$$\tilde{\mathsf{K}}_0(R) := \mathsf{coker}\left(\mathbb{Z} \xrightarrow{n \longmapsto n \cdot [R]} \mathsf{K}_0(R)\right)$$

(writing $\mathsf{K}_0(R) := \mathsf{K}_0(\mathsf{Mod}_R^{\mathsf{f.g.,proj}}) \cong \mathsf{K}_0(\mathbf{D}_R^{\mathsf{perf}})$ for simplicity (recall §7.5.5)). Wall's finiteness obstruction is likewise an element of a 0th reduced K-group.

12.2.1.   Most (∞-)categories of lasting interest – for instance, those of sets, abelian groups, or spaces – are "large": their objects are too numerous to form a "set" of such, but rather form a "large set" (also sometimes called a *proper class*).[258] Nevertheless, it is generally the case that they are "controlled by" small ∞-categories, as are the functors among them of lasting interest. More precisely, most large ∞-categories (and functors thereamong) of lasting interest are *accessible*, as we explain in this subsection.

In fact, among the accessible ∞-categories are the *presentable* ∞-categories, and indeed most large ∞-categories of lasting interest are presentable. However, for expository purposes we discuss these notions separately, treating the latter in **??**.

Our discussion of accessible and presentable ∞-categories is tailored towards a typical end-user of the theory: we avoid many details, and in most cases we do not even begin to indicate proofs (which may be found in [Lur09, §5.4-5]). We also recommend [AR94] for a much more systematic (albeit 1-categorical) treatment of accessibility and presentability.

12.2.2.   We begin with a brief summary of the set-theoretic background that we will need. The interested reader may consult [Shub] for a more in-depth discussion of the interactions between set theory and category theory.

Evidently, the existence of a bijection between sets defines an equivalence relation. A **cardinal** is by definition an equivalence class. We typically write κ or τ for a cardinal. On the other hand, we may use the same notation to denote an arbitrary representative of a cardinal (as we will only be invoking isomorphism-invariant notions).

Given a cardinal κ, a set S is called κ-**small** if it has cardinality less than that of κ, i.e. there exists an injection $S \hookrightarrow \kappa$ and there does not exist an injection $\kappa \hookrightarrow S$. We write $\mathsf{Set}_{<\kappa} \subset \mathsf{Set}$ for the full subcategory on the κ-small sets.

We write ω for the cardinality of the natural numbers. So a set is ω-small iff it's finite, and more generally the terms "ω-small" and "finite" tend to be used interchangeably. Of course, ω is the unique countable ordinal, and in practice it can be somewhat exceptional: we'll often make statements that refer to uncountable cardinals, and treat the countable case separately. For this reason (and because it's more evocative), we'll typically use the term "finite" instead of the term "ω-small".

Here is a first example of this dichotomy. Fix an uncountable cardinal κ.

(1) A space $X \in \mathcal{S}$ is called
   (a) **finite** if it is generated by the terminal object $\mathsf{pt} \in \mathcal{S}$ under finite colimits, and
   (b) κ-**small** if the set $\pi_i(X, x)$ is κ-small for all points $x \in X$ and all $i \geqslant 0$.
(2) An ∞-category $\mathcal{C} \in \mathsf{Cat}$ is called

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[258]Indeed, in order to circumvent the possibility of the existence of "a set of all sets" (and the paradoxes that would result therefrom), we appeal to the device of *Grothendieck universes*, as indicated briefly in §12.2.2.

(a) **finite** if it is generated by the objects of the subcategory $\mathbf{\Delta} \subset \mathsf{Cat}$ under finite colimits,[259] and

(b) κ-**small** if the set $\pi_0(\mathcal{C}^\simeq)$ of equivalence classes of objects is κ-small and moreover the space $\mathsf{hom}_{\mathcal{C}}(C, C')$ is κ-small for all $C, C' \in \mathcal{C}$.

So for example, the circle $S^1 \in \mathcal{S}$ is a finite space (even though the set $\pi_1(S^1)$ is infinite), and similarly the free category on an endomorphism $\mathfrak{B}\mathbb{N} \in \mathsf{Cat}$ is a finite ∞-category (even though the hom-space $\mathsf{end}_{\mathfrak{B}\mathbb{N}}(*)$ is infinite). Of course, all objects of $\mathbf{\Delta}$ are finite colimits (in Cat) of the two objects $[0]$ and $[1]$, and so a finite ∞-category is equivalently one generated by those under finite colimits.

These notions have model-categorical characterizations. Let us call a simplicial set **finite** if it has finitely many nondegenerate simplices, and κ-**small** if it is levelwise so (for κ uncountable).[260] Then, for *any* infinite cardinal κ, a space is κ-small iff it can be represented by a κ-small simplicial set (in $\mathsf{sSet}_{\mathsf{KQ}}$); for $\kappa > \omega$, this holds iff it can be represented by a κ-small Kan complex. Similarly, for *any* infinite cardinal κ, an ∞-category is κ-small iff it can be represented by a κ-small simplicial set (in $\mathsf{sSet}_{\mathsf{Joyal}}$); for $\kappa > \omega$, this holds iff it can be represented by a κ-small quasicategory.

An infinite cardinal κ is called **regular** if the subcategory $\mathsf{Set}_{<\kappa} \subset \mathsf{Set}$ of κ-small sets is closed under κ-small colimits. It suffices to consider coproducts: κ is regular iff whenever $\kappa = \coprod_{a \in A} \lambda_a$ with $\lambda_a < \kappa$ for all $a \in A$, then $|A| \geqslant \kappa$. So for example, ω is regular, because $\mathsf{Fin} = \mathsf{Set}_{<\omega} \subset \mathsf{Set}$ is closed under finite colimits. Heuristically, the idea is that κ is regular if it cannot be broken down into a smaller number of smaller parts.

Exercise 12.12 (2 points)**.** Suppose that κ is a **successor** cardinal, i.e. that there exists some cardinal κ_0 such that κ is the least cardinal that is greater than κ_0. Prove that κ is regular.

In what follows, we will only ever refer to *regular* cardinals: the notations κ and τ should always be interpreted as such (even if we do not specify regularity).

Given regular cardinals κ and τ, we say that κ is **sharply smaller** than τ, and write $\kappa \ll \tau$, if for every $\tau_0 < \tau$ and $\kappa_0 < \kappa$ we have $(\tau_0)^{\kappa_0} := \mathsf{hom}_{\mathsf{Set}}(\kappa_0, \tau_0) < \tau$. Heuristically, the idea is that κ is sharply smaller than τ if performing fewer than τ operations to fewer than κ objects gives a result that is smaller than τ.

Exercise 12.13 (4 points)**.**

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[259]In order for this description not to be circular, one can replace the term "finite colimits" by "the initial object and pushouts" (recall §12.1.7).

[260]Note that having finitely many nondegenerate simplices is strictly stronger than being levelwise finite; for instance, the simplicial set

$$\left(\bigvee_{n \geqslant 0} \Delta^n / \partial \Delta^n \right) \in \mathsf{sSet}$$

is levelwise finite but has infinitely many nondegenerate simplices.

(a) Prove that the relation $\ll$ is transitive.

(b) Prove that for any set $\{\kappa_a\}_{a \in A}$ of regular cardinals, there exists a regular cardinal τ such that $\kappa_a \ll \tau$ for all $a \in A$.

An *uncountable* regular cardinal κ is called ***strongly inaccessible*** if $\kappa \ll \kappa$.

We can now indicate the set-theoretic device of ***Grothendieck universes***, under which we implicitly work.[261] Namely, we assume that for every cardinal κ there exists a strongly inaccessible cardinal τ such that $\tau \geqslant \kappa$. This allows us to fix strongly inaccessible cardinals $\zeta_0 < \zeta_1 < \zeta_2$; we then refer to objects (e.g. sets, spaces, or ∞-categories) as ***small*** if they're ζ_0-small, ***large*** if they're ζ_1-small, and ***huge*** if they're ζ_2-small.[262] Then, for instance we write Cat for the large ∞-category of small ∞-categories and $\widehat{\mathsf{Cat}}$ for the huge ∞-category of large ∞-categories.[263] The modifier "small" is essentially always implicit – for instance, the term "set" is a stand-in for the term "small set" – although it may be added for emphasis. Notably, when we say e.g. that an ∞-category admits all co/limits, we always really mean that it admits all *small* colimits.

There is a convenient (and commonly occurring) intermediate notion between those of small and large ∞-categories. Namely, we say that a large ∞-category is ***locally small*** if its hom-spaces are small. So for instance, Cat is locally small, despite not being small, and indeed most large ∞-categories of lasting interest are locally small. An ∞-category is locally small iff for every small set of objects, the full subcategory that they generate is small. Moreover, if $\mathcal{D} \in \mathsf{Cat}$ is small and $\mathcal{C} \in \widehat{\mathsf{Cat}}$ is locally small, then $\mathsf{Fun}(\mathcal{D}, \mathcal{C}) \in \widehat{\mathsf{Cat}}$ is again locally small.

12.2.3.   We now discuss some preliminary facts about ∞-categories of presheaves. Let us fix a (small) ∞-category $\mathcal{A} \in \mathsf{Cat}$. A ***presheaf*** on $\mathcal{A}$ is a functor $\mathcal{A}^{\mathsf{op}} \to \mathcal{S}$. These assemble into the (large but locally small) ∞-category

$$\mathcal{P}(\mathcal{A}) := \mathsf{Fun}(\mathcal{A}^{\mathsf{op}}, \mathcal{S}) \in \widehat{\mathsf{Cat}} \ .$$

We write

$$\mathcal{A} \xhookrightarrow{\ \mathsf{Yo}\ } \mathcal{P}(\mathcal{A}) := \mathsf{Fun}(\mathcal{A}^{\mathsf{op}}, \mathcal{S})$$
$$\rotatebox{90}{$\in$} \qquad\qquad\quad \rotatebox{90}{$\in$}$$
$$A \longmapsto \mathsf{Yo}(A) := \mathsf{hom}_{\mathcal{A}}(-, A)$$

---

[261]This choice, while standard in category theory, is not essential: it is only for convenience, in order to avoid more frequent consideration of cardinalities.

[262]It is shown in [Low] that standard categorical constructions (e.g. Kan extensions) do not depend on these choices.

[263]Note that under our terminological conventions, small objects are automatically large. We find this more convenient than e.g. calling $\widehat{\mathsf{Cat}}$ the huge ∞-category of *possibly-large* ∞-categories. Of course, the term "large" is nevertheless often used to mean "large, and not also small".

for the Yoneda embedding, which is of course fully faithful. Via the Grothendieck construction, the Yoneda embedding can be identified as the composite

$$
\begin{array}{ccc}
 & \mathsf{Fun}(\mathcal{A}^{\mathsf{op}}, \mathcal{S}) & \\
\overset{\mathsf{Yo}}{\nearrow} & \downarrow \wr \mathsf{Gr}^- & \\
\mathcal{A} \overset{\hookrightarrow}{\underset{A \mapsto (\mathcal{A}_{/A} \downarrow \mathcal{A})}{\longrightarrow}} & \mathsf{RFib}_{\mathcal{A}} &
\end{array} \quad .
$$

An essential fact is that the Yoneda embedding witnesses $\mathcal{P}(\mathcal{A})$ as the ***free cocompletion*** of $\mathcal{A}$. To explain this, let us fix a large ∞-category $\mathcal{C} \in \widehat{\mathsf{Cat}}$, which we assume to be cocomplete (i.e. to admit all (small) colimits). Then, writing $\mathsf{Fun}^{\mathsf{colim}}(\mathcal{P}(\mathcal{A}), \mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{P}(\mathcal{A}), \mathcal{C})$ for the full subcategory on the colimit-preserving functors, we have inverse equivalences

$$(70) \qquad \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \underset{\mathsf{Yo}^*}{\overset{\mathsf{Yo}_!}{\underset{\sim}{\rightleftarrows}}} \mathsf{Fun}^{\mathsf{colim}}(\mathcal{P}(\mathcal{A}), \mathcal{C}) \quad .$$

Let us unpack the left adjoint equivalence $\mathsf{Yo}_!$. Choose any functor $\mathcal{A} \overset{F}{\to} \mathcal{C}$. As $\mathcal{C}$ is cocomplete, the left Kan extension

$$
\begin{array}{ccc}
\mathcal{A} & \overset{F}{\longrightarrow} & \mathcal{C} \\
{\scriptstyle \mathsf{Yo}}\downarrow & \nearrow_{\mathsf{Yo}_!(F)} & \\
\mathcal{P}(\mathcal{A}) & &
\end{array}
$$

is computed by pointwise left Kan extension (recall §12.1.4); that is, it carries each presheaf $X \in \mathcal{P}(\mathcal{A})$ to the object

$$\mathsf{colim}\left(\mathcal{A}_{/X} \overset{\mathsf{fgt}}{\longrightarrow} \mathcal{A} \overset{F}{\longrightarrow} \mathcal{C} \right) \in \mathcal{C} \quad .$$

Note that the source ∞-category $\mathcal{A}_{/X}$ is precisely the cartesian unstraightening of the functor $\mathcal{A}^{\mathsf{op}} \overset{X}{\to} \mathcal{S}$ (and is therefore small), which is often referred to as the ∞-category of (***generalized***) ***points of*** X (a (*generalized*) *point* being a pair of an object $A \in \mathcal{A}$ and a point of $X(A) \simeq \mathsf{hom}_{\mathcal{P}(\mathcal{A})}(\mathsf{Yo}(A), X)$). Note too that because $\mathcal{A} \overset{\mathsf{Yo}}{\hookrightarrow} \mathcal{P}(\mathcal{A})$ is fully faithful, we have a canonical equivalence $F \overset{\sim}{\to} \mathsf{Yo}^*(\mathsf{Yo}_!(F))$, which observation is of course compatible with half of the assertion that the functors (70) are inverse equivalences.

Let us also examine the particular case that $\mathcal{C} = \mathcal{P}(\mathcal{A})$ (which is cocomplete because $\mathcal{S}$ is so and hence colimits in $\mathcal{P}(\mathcal{A}) := \mathsf{Fun}(\mathcal{A}^{\mathsf{op}}, \mathcal{S})$ exist and are computed pointwise). In this case, the object $\mathsf{id}_{\mathcal{P}(\mathcal{A})} \in \mathsf{Fun}^{\mathsf{colim}}(\mathcal{P}(\mathcal{A}), \mathcal{P}(\mathcal{A}))$ corresponds to the object $\mathsf{Yo} \simeq \mathsf{Yo}^*(\mathsf{id}_{\mathcal{P}(\mathcal{A})}) \in \mathsf{Fun}(\mathcal{A}, \mathcal{P}(\mathcal{A}))$, so that we have a left Kan extension diagram

$$
\begin{array}{ccc}
\mathcal{A} & \overset{\mathsf{Yo}}{\hookrightarrow} & \mathcal{P}(\mathcal{A}) \\
{\scriptstyle \mathsf{Yo}}\downarrow & \nearrow_{\mathsf{id}_{\mathcal{P}(\mathcal{A})}} & \\
\mathcal{P}(\mathcal{A}) & &
\end{array} \quad ;
$$

in other words, we have a canonical equivalence $\mathsf{Yo}_!(\mathsf{Yo}) \overset{\sim}{\to} \mathsf{id}_{\mathcal{P}(\mathcal{A})}$. As this left Kan extension is computed pointwise, we find that for every presheaf $X \in \mathcal{P}(\mathcal{A})$ we have a canonical

equivalence

$$\mathsf{colim}\left(\mathcal{A}_{/X} \xrightarrow{\mathsf{fgt}} \mathcal{A} \xhookrightarrow{\mathsf{Yo}} \mathcal{P}(\mathcal{A})\right) \xrightarrow{\sim} X \ ,$$

which is often articulated as the assertion that every presheaf is the colimit of its points (or equivalently, of representable presheaves). This is of course compatible with the other half of the assertion that the functors (70) are inverse equivalences: a colimit-preserving functor $\mathcal{P}(\mathcal{A}) \to \mathcal{C}$ is completely specified by its values on representable presheaves.

Beware that even though the Yoneda embedding $\mathcal{A} \xhookrightarrow{\mathsf{Yo}} \mathcal{P}(\mathcal{A})$ is the free cocompletion, it does *not* follow that it does not preserve any colimits. Namely, it (like any functor) preserves all absolute colimits. For instance, it clearly preserves colimits over $\mathsf{pt} \in \mathsf{Cat}$ (or over any ∞-category with a final object), and recalling §12.1.7 we see that it also preserves retracts.

Given a functor $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ between small ∞-categories, we obtain adjunctions

$$\mathcal{P}(\mathcal{A}) \overset{\displaystyle\overset{\varphi_!}{\overset{\perp}{\longleftarrow}}}{\underset{\displaystyle\underset{\varphi_*}{\underset{\perp}{\longrightarrow}}}{\xleftarrow{\ \varphi^* \ }}} \mathcal{P}(\mathcal{B})$$

(where both left and right Kan extensions exist and are computed pointwise because $\mathcal{S}$ is bicomplete (i.e. both cocomplete and complete (i.e. it admits all (small) limits)). As we will discuss in **??**, one generally takes $\varphi_!$ to be the functoriality of the construction taking a small ∞-category to its ∞-category of presheaves, i.e. of a functor

$$\mathsf{Cat} \xrightarrow{\mathcal{P}(-)} \widehat{\mathsf{Cat}} \ .$$

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