Harmonic analysis for relative trace formula

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This is an expository article on some local harmonic analysis related to relative trace formula.

1 An overview of the relative trace formula

Let G be a reductive group and H a subgroup both defined over a number field F. Let $\mathbb{A}$ denote the ring of adeles of F. Let π be a cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$. Then the automorphic period integral is a linear functional on π:

$$\mathscr{P}_{\mathrm{H}}(\phi) := \int_{\mathrm{H}(F)\backslash \mathrm{H}(\mathbb{A})} \phi(h)dh, \quad \phi \in \pi.$$

Many questions on special values of L-functions are tied to the study of period integrals of automorphic forms. A notable example is the formula of Waldspurger ([22]) that relates the toric period on GL_2 or its inner form to some central critical L-value. The conjecture of Gan, Gross and Prasad ([4]) started with the Gross-Prasad conjecture ([5]), as well as the refinement of Ichino and Ikeda ([8]) of the Gross-Prasad conjecture, vastly generalizes the Waldspurger formula to higher rank groups. We briefly state their conjecture. Let E be either F or a quadratic extension of F. If $E = F$ (resp., if E is a quadratic extension of F), let W_{n+1} be a quadratic space over $E = F$ (resp., a Hermitian space associated to E/F) of E-dimension $n + 1$. Let $W_n \subset W_{n+1}$ be a non-degenerate subspace of codimension one. Let G_i be $\mathrm{SO}(W_i)$ or $\mathrm{U}(W_i)$ for $i = n, n+1$. The Gan-Gross-Prasad period is attached to the pair (H, G) where $\mathrm{G} = \mathrm{G}_n \times \mathrm{G}_{n+1}$ and $\mathrm{H} \subset \mathrm{G}$ is the diagonal embedding of G_n. Assume further that π is tempered. The conjecture of Gan-Gross-Prasad ([4, Conjecture 24.1]) asserts that the following two statements are equivalent:

(i) For some automorphic representation in the Vogan L-packet (cf. [4]) of π, the linear functional $\mathscr{P}_{\mathrm{H}}$ does not vanish.

(ii) The central value $L(1/2, \pi, \mathrm{R})$ does not vanish, where R is a certain representation of the L-group ${}^LG$.

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In the Hermitian case, the author in [29] proves the conjecture for π satisfying a certain local condition. The tool is the relative trace formula first introduced by Jacquet to study period integrals. We give an outline of this strategy (cf. the survey articles [11], [14], [15]). We start with a triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$ consisting of a reductive group G and two suitable subgroups $\mathrm{H}_1, \mathrm{H}_2$. Here the two subgroups H_1 and H_2 are possibly the same. We associate to a test function $f \in \mathscr{C}_c^\infty(\mathrm{G}(\mathbb{A}))$ a kernel function:

$$K_f(x, y) := \sum_{\gamma \in \mathrm{G}(F)} f(x^{-1}\gamma y), \quad x, y \in \mathrm{G}(\mathbb{A}).$$

Then we consider the linear functional on $\mathscr{C}_c^\infty(\mathrm{G}(\mathbb{A}))$ defined by the double integral:

$$I(f) = \int_{\mathrm{H}_1(F)\backslash \mathrm{H}_1(\mathbb{A})} \int_{\mathrm{H}_2(F)\backslash H_2(\mathbb{A})} K_f(h_1, h_2) dh_1 dh_2. \tag{1}$$

We may also insert some "small" representation as a weight factor: for example, a character of $\mathrm{H}_i(F)\backslash \mathrm{H}_i(\mathbb{A})$, or the Weil representation. The relative trace formula attached to the triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$ is an identity between two different expansions of $I(f)$, known as the "*spectral expansion*" and the "*geometric expansion*" of $I(f)$, are equal. A cuspidal automorphic representation π contributes a term to the spectral expansion, which we will call a (global) *"spherical character"*, or *"relative character"*. This term is a distribution I_π on $\mathrm{G}(\mathbb{A})$ defined by

$$I_\pi(f) := \sum_{\phi \in \mathscr{B}(\pi)} \mathscr{P}_{\mathrm{H}_1}(\pi(f)\phi) \overline{\mathscr{P}_{\mathrm{H}_2}(\phi)},$$

where $\mathscr{B}(\pi)$ denotes an orthonormal basis of π. The terms in the geometric expansion are parameterized by double cosets $\gamma \in \mathrm{H}_1(F)\backslash \mathrm{G}(F)/\mathrm{H}_2(F)$. Given such a double coset γ, its contribution is the distribution $\mathrm{Orb}(\gamma, \cdot)$ defined by the orbital integral:

$$\mathrm{Orb}(\gamma, f) := \tau((\mathrm{H}_1 \times \mathrm{H}_2)_\gamma) \int_{(\mathrm{H}_1 \times \mathrm{H}_2)_\gamma(\mathbb{A})\backslash(\mathrm{H}_1 \times \mathrm{H}_2)(\mathbb{A})} f(h_1^{-1}\gamma h_2) dh_1 dh_2,$$

where $(\mathrm{H}_1 \times \mathrm{H}_2)_\gamma$ denotes the stabilizer of γ, and $\tau((\mathrm{H}_1 \times \mathrm{H}_2)_\gamma)$ is the volume of $(\mathrm{H}_1 \times \mathrm{H}_2)_\gamma(F)\backslash(\mathrm{H}_1 \times \mathrm{H}_2)_\gamma(\mathbb{A})$. Note that typically there is problem with convergence; in this expository article we will ignore such convergence issues. When we take the triple

$$(\mathrm{H} \times \mathrm{H}, \Delta_\mathrm{H}, \Delta_\mathrm{H}),$$

where $\Delta_\mathrm{H} \subset \mathrm{H} \times \mathrm{H}$ is the diagonal embedding of H, the associated relative trace formula is equivalent to the Arthur-Selberg trace formula associated to H. In general, Sakellaridis and Venkatesh ([19]) have initiated a conjectural framework which includes the case where each of the homogeneous spaces G/H_1 and G/H_2 is a *spherical variety* under G.

We will use RTF to stand for "relative trace formula". In application we usually need to compare two RTFs that are close to each other. In general the problem of identifying 'comparable' RTFs is a subtle one. To attack the Gan-Gross-Prasad conjecture in the Hermitian case, Jacquet and Rallis ([12]) constructed a pair of RTFs. The first RTF deals with the period integral and is associated to the triple $(\mathrm{G}, \mathrm{H}, \mathrm{H})$ where

$$\mathrm{H} = \Delta_{\mathrm{U}(W_n)}, \quad \mathrm{G} = \mathrm{U}(W_n) \times \mathrm{U}(W_{n+1}).$$

The second one deals with the L-values and is associated to the triple $(\mathrm{G}', \mathrm{H}_1, \mathrm{H}_2)$ where

$$\mathrm{G}' = \mathrm{Res}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_{n+1}),$$

and

$$\mathrm{H}_1 = \mathrm{Res}_{E/F}\mathrm{GL}_n, \quad \mathrm{H}_2 = \mathrm{GL}_n \times \mathrm{GL}_{n+1}.$$

Moreover it is necessary to insert a quadratic character $\eta_{n,n+1}$ of $\mathrm{H}_2(\mathbb{A})$:

$$\eta_{n,n+1} : \mathrm{H}_2(\mathbb{A}) \ni (h_n, h_{n+1}) \mapsto \eta^{n-1}(\det(h_n))\eta^n(\det(h_{n+1})),$$

where η is the quadratic character associated to E/F by class field theory.

We should take this discussion as an opportunity to mention a parallel version of the Jacquet–Rallis construction. As one way to generalize the Gross-Zagier formula ([6], [25]), there is also an *arithmetic* [1] version of the Gan-Gross-Prasad conjecture and the Ichino–Ikeda refinement ([27]). Inspired by the Jacquet-Rallis construction above, the author was led to a comparison of two '*arithmetic*' RTFs ([28]) to attack the problem. Now we resume with the same notation as in the Jacquet-Rallis construction above. For simplicity, we assume that $F = \mathbb{Q}$ and E is an imaginary quadratic field. We further assume that the Hermitian spaces W_n, W_{n+1} are of signatures $(n-1, 1)$ and $(n, 1)$ respectively. From the embedding $\mathrm{H} \subset \mathrm{G}$ one can construct an embedding of the associated Shimura varieties $\mathrm{Sh}_\mathrm{H} \subset \mathrm{Sh}_\mathrm{G}$. Then to a test function f in a *suitable* subspace of $\mathscr{C}_c^\infty(\mathrm{G}(\mathbb{A}))$, we may associate a Hecke correspondence $\mathrm{R}(f)$ on Sh_G. We then define an analogue of the distribution $I(f)$ in (1):

$$J(f) = \langle \mathrm{Sh}_\mathrm{H}, \mathrm{R}(f)\mathrm{Sh}_\mathrm{H} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Beilinson-Bloch height pairing between two algebraic cycles. When Sh_G is a curve and Sh_H is a divisor, the paring is the Neron-Tate height pairing. This serves as the first arithmetic RTF that deals with the height of algebraic cycles. For the second one, we modify the RTF associated to the triple $(\mathrm{G}', \mathrm{H}_1, \mathrm{H}_2)$ in the Jacquet–Rallis construction as follows. For $f' \in \mathscr{C}_c^\infty(\mathrm{G}'(\mathbb{A}))$, we may define a family of distributions parameterized by $s \in \mathbb{C}$:

$$I(f', s) = \int_{\mathrm{H}_1(F)\backslash \mathrm{H}_1(\mathbb{A})} \int_{\mathrm{H}_2(F)\backslash H_2(\mathbb{A})} K_{f'}(h_1, h_2)|\det(h_1)|^s \eta_{n,n+1}(h_2) dh_1 dh_2.$$

Its first derivative at $s = 0$ is intimately tied to the central derivative of the L-function $L(s, \pi, \mathrm{R})$. Then we expect that, for *suitable* 'matching' test functions f and f', we have an equality

$$\frac{d}{ds} I(f', s)|_{s=0} = J(f).$$

When $\dim \mathrm{Sh}_\mathrm{G} > 1$, substantial work needs to be done before we can prove this identity in any non-trivial case. Some partial progress is presented in [28].

Back to general RTFs, as in the case of Arthur-Selberg trace formula, the comparison of two RTFs leads us to at least two local questions: the *fundamental lemma* and the *existence of smooth transfer*. Due to the author's limited knowledge, we will not discuss the first one.

[1] Though it may be misleading, we will use the word "arithmetic" to indicate the case involving algebraic cycles.

Instead, we would like to focus on the second one and present some questions in related topics of harmonic analysis. Examples are provided by Waldspurger's work in the endoscopic case ([24]), Jacquet's work on the Jacquet-Ye conjecture for quadratic base change ([9], [10]), and the author's work on the Jacquet-Rallis RTFs ([29]) which leads to a proof of some cases of the Gan-Gross-Prasad conjecture for Hermitian spaces. The emphasis on the Jacquet–Rallis construction is due simply to the author's ignorance of the other cases.

2 A little bit of invariant theory

We will restrict ourselves to the case where $\mathrm{H}_1, \mathrm{H}_2$ and G are all reductive.[2] We first introduce some notation in a more abstract setting. Let F be a field of characteristic zero. Let H be a reductive group acting on an algebraic variety X over F. The *categorical quotient* of X by H (cf. [1], [16], [17]) consists of a pair (Y, π) where Y is an algebraic variety with the trivial action by H and $\pi : X \to Y$ is an H-morphism such that for any pair (Y', π') with $\pi' : X \to Y'$ an H-morphism, there exists a unique morphism $\phi : Y \to Y'$ such that $\pi' = \phi \circ \pi$. If such a pair exists, then it is unique up to a unique isomorphism. When X is affine, which we assume from now on, the categorical quotient always exists. Indeed we may construct this categorical quotient as follows. Consider the affine variety

$$X_{/\mathrm{H}} := \operatorname{Spec} \mathcal{O}(X)^{\mathrm{H}}$$

together with the obvious morphism

$$\pi = \pi_{X,\mathrm{H}} : X \to X_{/\mathrm{H}}.$$

Then $(X_{/\mathrm{H}}, \pi)$ defines a categorical quotient of X by H. By abuse of notation, we will also let π denote the induced map $X(F) \to X_{/\mathrm{H}}(F)$. We say that a point $x \in X(F)$ is

- H-semisimple if $\mathrm{H}x$ is Zariski closed in X (when F is a local field of characteristic zero, this is equivalent to requiring that $\mathrm{H}(F)x$ be closed in $X(F)$ for the analytic topology, cf. [18, p.109]).
- H-regular if the stabilizer H_x of x achieves the minimal dimension $\dim \mathrm{H}_y$ among all $y \in X$.

If no confusion can arise, we will simply use the words "semisimple" and "regular".We say that x is *regular semisimple* if it is regular and semisimple. We are interested in the following two cases

- For a triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$ above, we consider $X = \mathrm{G}$, and the product $\mathrm{H} = \mathrm{H}_1 \times \mathrm{H}_2$ where H_1 (H_2, resp.) acts by left (right, resp.) multiplication.
- An algebraic representation: $X = \mathrm{V}$ is a vector space (considered as an affine variety) with an action by a reductive group H. We will use (H, V) to denote the representation to emphasize the dependence on the group and the vector space on which the group acts. The action is then self-evident in our examples below.

[2]But there are indeed many interesting cases which involve non-reductive subgroups, for example, the Jacquet-Ye construction ([9], [10]).

Given an action of a reductive group H on an affine variety X, there is a procedure to obtain an algebraic representation from each semisimple point $x \in X(F)$. Indeed if $x \in X(F)$ is H-semisimple, the stabilizer H_x is a reductive subgroup of H defined over F. It preserves the normal space $\mathrm{N}^X_{\mathrm{H}x,x}$ [3] at x of the orbit $\mathrm{H}x$ in the ambient space X. The induced action of H_x on the vector space $\mathrm{N}^X_{\mathrm{H}x,x}$ is called the *sliced representation* at x. The sliced representation is a generalization of the Harish-Chandra's 'semisimple descent' ([13, §16]) in which case $X = \mathfrak{h}$ is the Lie algebra of a reductive group H endowed with the adjoint action of H, and the H_x-representation $\mathrm{N}^X_{\mathrm{H}x,x}$ is equivalent to the Lie algebra centralizer $\mathfrak{h}_x$ of $x \in \mathfrak{h}$ on which the stabilizer H_x operates by the adjoint action.

Roughly speaking, the sliced representation $(\mathrm{H}_x, \mathrm{N}^X_{\mathrm{H}x,x})$ models the action of H on X around the semisimple point x, at least when X is smooth. More precisely, if X is a smooth affine variety and $x \in X(F)$ is a semisimple point, there exists an *étale Luna slice* (cf. [1, A.2], [17, §6]), namely there exists a locally closed smooth H_x-invariant subvariety Z of X together with a strongly étale [4] H_x-morphism $Z \to \mathrm{N}^X_{\mathrm{H}x,x}$ such that the induced H-morphism $\mathrm{H} \times_{\mathrm{H}_x} Z \to X$ [5] by $(h, z) \mapsto hz$ is strongly étale. Because of this, we usually try to transfer questions on X to questions on the normal space $\mathrm{N}^X_{\mathrm{H}x,x}$. It is therefore important to study the harmonic analysis attached to various algebraic representations (H, V).

We list some examples of spliced representations. In the case of the RTF associated to a triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$, we will be interested in the sliced representations at $1 \in \mathrm{G}$ for the action of $\mathrm{H}_1 \times \mathrm{H}_2$ on G. We will call this sliced representation the '*Lie algebra*' of the RTF associated to the triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$, or simply the '*Lie algebra*' of the triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$.

1. Let $\mathfrak{h}$ be the Lie algebra of H. Then the adjoint representation $(\mathrm{H}, \mathfrak{h})$ is the sliced representation at $1 \in \mathrm{H}$ for the adjoint action of H on itself. We may also obtain some representations of similar flavor if we consider symmetric spaces and Vinberg's θ-group ([18, §6]).

2. The Gan–Gross–Prasad case: for the triple $(\mathrm{G}, \mathrm{H}, \mathrm{H})$, we obtain an $\mathrm{H} \times \mathrm{H}$-action on $\mathrm{G} = \mathrm{G}_n \times \mathrm{G}_{n+1}$ where $\mathrm{H} \subset \mathrm{G}$ is the diagonal embedding of G_n. The sliced representation at $1 \in \mathrm{G}$ can be identified with the pair $(\mathrm{H}, \mathfrak{g}_{n+1})$ where $\mathfrak{g}_{n+1}$ is the Lie algebra of G_{n+1} and the action of $\mathrm{H} \simeq \mathrm{G}_n$ is via the restriction of the adjoint action of G_{n+1} on $\mathfrak{g}_{n+1}$ to G_n.

3. The Jacquet–Rallis case: for the triple $(\mathrm{G}', \mathrm{H}_1, \mathrm{H}_2)$, the sliced representation at $1 \in \mathrm{G}'$ is given by the pair $(\mathrm{GL}_n, \mathfrak{s}_{n+1})$ where $\mathfrak{s}_{n+1}$ is the F-vector space consisting of $X \in$

[3]The vector space $\mathrm{N}^X_{\mathrm{H}x,x}$ is the quotient of the tangent space T^X_x of X at x by the tangent space $\mathrm{T}^{\mathrm{H}x}_x$ of the orbit $\mathrm{H}x$ at x.

[4]Let X and Y be two affine varieties with H-action. We say an H-morphism $\phi : X \to Y$ strongly étale if the induced morphism $\phi_{/\mathrm{H}} : X_{/\mathrm{H}} \to Y_{/\mathrm{H}}$ is étale and the following induced diagram is Cartesian:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ X_{/\mathrm{H}} \arrow[r, "\phi_{/\mathrm{H}}"] & Y_{/\mathrm{H}}. \end{tikzcd}$$

[5]The notation $\mathrm{H} \times_{\mathrm{H}_x} Z$ means the quotient of $\mathrm{H} \times Z$ by H_x via the free action $h \cdot (g, z) = (gh^{-1}, hz)$.

$M_{n+1}(E)$ ($(n+1)\times(n+1)$-matrices with coefficients in E) satisfying

$$X+\overline{X}=0,$$

where $\overline{X}$ is the Galois conjugate of X (entry-wise). The action of GL_n on $\mathfrak{s}_{n+1}$ is by conjugation. If we write the quadratic extension E as $F[\sqrt{\delta}]$ with $\delta\in F$, we have a non-canonical isomorphism $\mathfrak{s}_{n+1}\simeq\mathfrak{gl}_{n+1,F}$ (the Lie algebra of $\mathrm{GL}_{n+1,F}$) by $X\mapsto X/\sqrt{\delta}$.

In the case of an algebraic representation (H,V), there is a distinguished point in V as well as in $\mathrm{V}_{/\mathrm{H}}$, namely the zero vector in V and its image in $\mathrm{V}_{/\mathrm{H}}$ denoted still by 0. We define the H-nilpotent cone to be the fiber $\pi^{-1}(0)$, and denote it by $\mathcal{N}$. Elements in $\mathcal{N}$ are called H-nilpotent. An element is H-nilpotent if and only if the Zariski closure of the orbit $\mathrm{H}x$ contains $0\in\mathrm{V}$. The notions of H-semisimplicity, regularity and nilpotency coincide with the usual ones when $(\mathrm{H},\mathrm{V})=(\mathrm{H},\mathfrak{h})$ is the adjoint representation of H on its Lie algebra $\mathfrak{h}$. The geometry of the H-nilpotent cone in some sense dictates many aspects of the quotient morphism π. For example, the number of H-orbits in each fiber is bounded above by the number of H-orbits in the nilpotent cone.

It is an interesting question how to extend the above discussion to the case where H_1 and/or H_2 is non-reductive.

3 Local relative trace formula for (H,V)

Let (H,V) be an algebraic representation. Imitating the convention of Harish-Chandra in [7] for the adjoint action, we will write the action in an exponential way: $h\cdot X=X^h$ for $h\in\mathrm{H},X\in\mathrm{V}$. From now on we assume that F is a non-archimedean local field of characteristic zero (i.e., finite extension of $\mathbb{Q}_p$). We abuse notation and use V,H to denote the F-points of V,H. Let $\mathscr{C}_c^\infty(\mathrm{V})$ be the space of locally constant and compactly supported functions on V. For $h\in\mathrm{H}$, we denote by ${}^hf$ the function ${}^hf(X):=f(X^h)$. Let $\mathscr{D}(\mathrm{V})$ be the space of distributions on V, i.e.:

$$\mathscr{D}(\mathrm{V})=\mathrm{Hom}(\mathscr{C}_c^\infty(\mathrm{V}),\mathbb{C}).$$

By an H-*polarization*, we mean a non-degenerate symmetric F-bilinear pairing

$$\langle\cdot,\cdot\rangle:\mathrm{V}\times\mathrm{V}\to F,$$

that is H-invariant:

$$\langle X^h,Y^h\rangle=\langle X,Y\rangle,\quad X,Y\in\mathrm{V},h\in\mathrm{H}.$$

Then for a fixed nontrivial character $\psi:F\to\mathbb{C}^\times$ we may define an automorphism (of order 4) of $\mathscr{C}_c^\infty(\mathrm{V})$, i.e., the Fourier transform:

$$\widehat{f}(X)=\int_{\mathrm{V}}f(Y)\psi(\langle Y,X\rangle)dX.$$

We use the self-dual measure on V. The H-invariance of the pairing implies that the Fourier transform commutes with the H-action on $\mathscr{C}_c^\infty(\mathrm{V})$:

$$\widehat{{}^hf}={}^h\widehat{f},\quad h\in\mathrm{H}.$$

By duality each h also defines an automorphism of $\mathscr{D}(\mathrm{V})$.

To illustrate the idea of the local trace formula, we first assume that $\mathrm{H}(F)$ is compact (cf.[2] for the case of the usual local trace formula on groups). In particular, we may simply define the orbital integral as

$$\mathrm{Orb}(X, f) := \int_{\mathrm{H}} f(X^h)dh, \quad X \in \mathrm{V}.$$

The Parseval-Plancherel theorem asserts that the Fourier transform preserves the L^2-norm:

$$\int_{\mathrm{V}} f_1(X)\widehat{f_2}(X)dX = \int_{\mathrm{V}} \widehat{f_1}(X)f_2(X)dX.$$

We may replace f_1 by ${}^h f_1$:

$$\int_{\mathrm{V}} f_1(X^h)\widehat{f_2}(X)dX = \int_{\mathrm{V}} \widehat{f_1}(X^h)f_2(X)dX.$$

Since H is compact we may integrate the above identity over H and interchange the order of integration to obtain

$$\int_{\mathrm{V}} \mathrm{Orb}(X, f_1)\widehat{f_2}(X)dX = \int_{\mathrm{V}} \mathrm{Orb}(X, \widehat{f_1})f_2(X)dX. \tag{2}$$

We will call this *the local relative trace formula*, or simply *the local trace formula*, associated to (H, V). Equivalently, under a suitable choice of a measure on the quotient $\mathrm{V}_{/\mathrm{H}}$ [6]:

$$\int_{\mathrm{V}_{/\mathrm{H}}} \mathrm{Orb}(X, f_1)\mathrm{Orb}(X, \widehat{f_2})dX = \int_{\mathrm{V}_{/\mathrm{H}}} \mathrm{Orb}(X, \widehat{f_1})\mathrm{Orb}(X, f_2)dX.$$

Clearly if $\mathrm{H} = \{1\}$, the formula reduces to the Parseval-Plancherel theorem on Fourier transform. In general, one may view the local trace formula as an equivariant version of the Parseval-Plancherel theorem.

When the group H is non-compact, the formula needs to be regularized and some weighted orbital integrals may appear. It is much more difficult to establish such a formula. For example, in the case of the adjoint action $(\mathrm{H}, \mathfrak{h})$, Waldspurger ([23]) proves a local trace formula for Lie algebras which is closely related to Arthur's local trace formula for groups ([3]). In the case of symmetric spaces, Sparling has made some progress towards deriving a local trace formula for Lie algebras ([20], [21]).

It is plausible that the local trace formula for (H, V) may retain the simple form as in the compact case above when the stabilizer of every regular semisimple element of V is compact. Then we may expect the situation to be similar to the *elliptic part* of the local trace formula of Waldspurger for Lie algebras. Two examples where this happens are the local trace formulae associated to the representations in the second (the Gan-Gross-Prasad case) and the third example (the Jacquet-Rallis case) in §2. In these cases, the stabilizer of a regular semisimple element is always trivial. The local trace formula turns out to have the same shape as (2) above and is proved in [29].

[6]Strictly speaking, there may be an issue of "stability": there may be regular semisimple elements that are not $\mathrm{H}(F)$-equivalent but have the same image under $\mathrm{V} \to \mathrm{V}_{/\mathrm{H}}$.

There are some examples of sliced representations in the Vinberg theory of θ-group where the stabilizer of every regular semisimple element is always finite (hence compact). It may be interesting to investigate the local trace formula for these sliced representations.

In the context of the RTF for a triple $(\mathrm{G}, \mathrm{H}_1, \mathrm{H}_2)$, we will consider the sliced representation at the identity. In general it may be also necessary to consider the sliced representations at *all* semisimple elements. But in our examples above the sliced representations at other semisimple elements of G are the same as some sliced ones at the identity of some other RTFs. Therefore it suffices to study the local trace formula for sliced representations at the identity element for many RTFs.

One may also wonder about the Arthur's local trace formula for groups in the context of RTFs. The problem has not receive much attention and the author does not know any such example. However, for the study of existence of smooth transfer, it suffices to use the local trace formula for sliced representations.

4 Fourier transform of orbital integrals

Representability question. We retain the notations from the previous section. We say that an H-invariant function κ on V is *nice* if it is locally constant on the regular semisimple locus V_{rs} and locally integrable on V. It is important to consider the Fourier transform of orbital integrals for regular semisimple $X \in \mathrm{V}$:

$$\widehat{\mathrm{Orb}}(X, f) := \mathrm{Orb}(X, \widehat{f}), \quad f \in \mathscr{C}_c^\infty(\mathrm{V}).$$

We denote this distribution also by $\widehat{\mathrm{Orb}}_X$. When $\mathrm{V} = \mathfrak{h}$ is the Lie algebra of H, Harish-Chandra discovered that the Fourier transform of an orbital integral on Lie algebra $\mathfrak{h}$ plays the role of the character of an irreducible representation on the group H.

Question 4.1 Let $X \in \mathrm{V}$ be H-regular semisimple. Is the distribution $\widehat{\mathrm{Orb}}_X$ representable by a nice function?

In other words, the questions asks whether there is a nice function $\kappa(X, \cdot)$ such that for all $f \in \mathscr{C}_c^\infty(\mathrm{V})$:

$$\widehat{\mathrm{Orb}}(X, f) = \int_{\mathrm{V}} f(Y)\kappa(X, Y)dY.$$

The two-variable function κ is then a sort of kernel function which contains a great deal of the harmonic analysis of (H, V). In the case of the adjoint representation $(\mathrm{H}, \mathfrak{h})$, a theorem of Harish-Chandra ([7, Theorem 1.1]) answers the question affirmatively. The key ingredients are the local trace formula for $(\mathrm{H}, \mathfrak{h})$ [7] and *Howe's finiteness hypothesis* ([13, §26]). We recall the statement of Howe's finiteness hypothesis. Let ω be a compact subset of V and let ω^{H} be the set of elements $X^h, X \in \omega, h \in \mathrm{H}$. We denote by $J(\omega)$ the set of H-invariant distributions with support in ω^{H}. Let Λ be a lattice in V and let $\mathscr{C}_c^\infty(\mathrm{V}/\Lambda)$ be the subspace of $\mathscr{C}_c^\infty(\mathrm{V})$ consisting of Λ-translation invariant functions. We denote by $j_\Lambda^* J(\omega)$ the image of $J(\omega)$ under the homomorphism $j_\Lambda^* : \mathrm{Hom}(\mathscr{C}_c^\infty(\mathrm{V}), \mathbb{C}) \to \mathrm{Hom}(\mathscr{C}_c^\infty(\mathrm{V}/\Lambda), \mathbb{C})$. Then Howe's

[7]Harish-Chandra's original proof did not use the full strength of the local trace formula. He essentially used the elliptic part of the local trace formula, cf. [7] and [13, §26].

finiteness hypothesis for (H, V) states that

For any compact subset ω of V and any lattice Λ of V, the space $j_\Lambda^* J(\omega)$ is finite dimensional.

This was shown to be true for the adjoint representation $(\mathrm{H}, \mathfrak{h})$ by Howe in the case where $\mathrm{H} = \mathrm{GL}_n$ and in general by Harish-Chandra ([7]). Rader and Rallis ([18]) also proved the finiteness in the case of (H, V) arising from Vinberg's θ-group (hence, including the case of symmetric spaces).

However, when Howe's finiteness hypothesis fails, it seems to be a challenging question how to prove (or disprove) the representability of $\widehat{\mathrm{Orb}}_X$. Note that a counterexample to Howe's finiteness hypothesis already shows up in the Jacquet–Rallis case for $n = 2$: $\mathrm{H} = \mathrm{GL}_2$, $\mathrm{V} = \mathfrak{gl}_3$. Consider the lattice $\Lambda = \mathrm{V}(\mathcal{O}_F)$ and the characteristic functions

$$f_t = 1_{t^{-1}\Lambda}, \quad t \in F^\times, |t| \leq 1.$$

Then $f_t \in \mathscr{C}_c^\infty(\mathrm{V}/\Lambda)$. Let ω be any compact open neighborhood of 0 in V. Then ω^{H} contains the H-nilpotent cone $\mathcal{N}$. [8] In $\mathcal{N}$, there are infinitely many nilpotent orbits. A continuous family can be given as

$$n(u) := \begin{bmatrix} 0 & u & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad u \in F.$$

The stabilizer of $n(u)$ is the nilpotent group N consisting of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. The naive definition of orbital integral turns out to be absolutely convergent and defines an H-invariant distribution on V:

$$\mathrm{Orb}(n(u), f) = \int_{N\backslash \mathrm{H}} f(n(u)^h) dh.$$

Indeed, by the Iwaswa decomposition $\mathrm{H} = NAK$ (A being the diagonal subgroup, $K = \mathrm{GL}_2(\mathcal{O}_F)$), we have (for a suitable choice of measures)

$$\mathrm{Orb}(n(u), f) = \int_{F^2} f_K \begin{bmatrix} 0 & uxy & x \\ 0 & 0 & 0 \\ 0 & y & 0 \end{bmatrix} dxdy, \quad f_K(X) := \int_K f(X^k) dk.$$

This is clearly absolutely convergent. We see immediately

$$\mathrm{Orb}(n(u), f_t) = |t|^{-2} \mathrm{Orb}(n(t^{-1}u), f_1),$$

and, if we normalize the measures such that $\mathrm{vol}(\mathcal{O}_F) = \mathrm{vol}(\mathrm{GL}_2(\mathcal{O}_F)) = 1$:

$$\mathrm{Orb}(n(u), f_1) = \begin{cases} 1, & |u| \leq 1; \\ (-v_F(u)\zeta_F(1)^{-1} + 1)|u|^{-1}, & |u| > 1. \end{cases}$$

Here v_F is the valuation on F and $\zeta_F(s)$ is the local zeta function of F. From these facts we may deduce that the subspace of $j_\Lambda^* J(\omega)$ generated by the image of the various $\mathrm{Orb}_{n(u)}$ is not finite dimensional.

[8] To see this, note that if $X \in \mathcal{N}$, the closure of $\{X\}^{\mathrm{H}}$ contains zero. Hence $\{X\}^{\mathrm{H}} \cap \omega$ is non-empty.

Compatibility between smooth transfer and Fourier transform. In the comparison of two RTFs, we may first consider an infinitesimal version, namely the comparison between sliced representations at semisimple points, especially at 1. This simplifies the question and usually one can deduce the original questions from the infinitesimal comparison when we vary x in all semisimple points.

We again consider the Jacquet-Rallis construction for the Gan-Gross-Prasad conjecture in the Hermitian case. There is a canonical isomorphism between the categorical quotients

$$\mathfrak{u}(W_{n+1})_{/\mathrm{U}(W_n)} \simeq \mathfrak{s}_{n+1/\mathrm{GL}_n}. \tag{3}$$

For each F-point in the quotient, the fiber (possibly empty) under the map

$$\pi : \mathfrak{u}(W_{n+1})(F) \to \mathfrak{u}(W_{n+1})_{/\mathrm{U}(W_n)}(F)$$

contains at most *one* regular semisimple $\mathrm{U}(W_n)(F)$-orbit (note that there is no 'stability' issue here). The same holds for $\mathfrak{s}_{n+1/\mathrm{GL}_n}$. The isomorphism allows us to match orbits: given a regular semisimple $\mathrm{U}(W_n)(F)$-orbit in $\mathfrak{u}(W_{n+1})(F)$, say of $X \in \mathfrak{u}(W_{n+1})(F)$, it turns out that there exists a unique regular semisimple $\mathrm{GL}_n(F)$-orbit of Y in $\mathfrak{s}_{n+1}(F)$ whose image in the quotient is the same as that of X, under the isomorphism (3). Conversely, given a regular semisimple $\mathrm{GL}_n(F)$-orbit of Y in $\mathfrak{s}_{n+1}(F)$, there exists a pair of Hermitian spaces W_n, W_{n+1} and a unique $\mathrm{U}(W_n)(F)$-orbit in $\mathfrak{u}(W_{n+1})(F)$ with the same image in the quotient. In each of the two cases above, the stabilizer of a regular semisimple element is trivial. We thus simply take the orbital integrals as

$$\mathrm{Orb}(X, f) = \int_{\mathrm{U}(W_n)(F)} f(X^h)dh, \quad f \in \mathscr{C}_c^\infty(\mathfrak{u}(W_{n+1}(F)), X \in \mathfrak{u}(W_{n+1})(F),$$

and

$$\mathrm{Orb}(Y, f') = \int_{\mathrm{GL}_n(F)} f'(Y^h)\eta(h)dh, \quad f' \in \mathscr{C}_c^\infty(\mathfrak{s}_{n+1}(F)), Y \in \mathfrak{s}_{n+1}(F),$$

where $\eta(h)$ stands for $\eta(\det(h))$ and η is the quadratic character of $F^\times$ attached to the quadratic extension E/F. We say that f' and f are *smooth transfers* of each other if for all matching regular semisimple X and Y, we have

$$\mathrm{Orb}(X, f) = \eta'(Y)\mathrm{Orb}(Y, f'),$$

where $\eta'(Y) \in \{\pm 1\}$ is a certain transfer factor satisfying $\eta'(Y^h) = \eta(h)\eta'(Y)$ for all regular semisimple Y and $h \in \mathrm{GL}_n(F)$ ([29]). We may define Fourier transforms that are compatible in a suitable sense. The following compatibility result is proved in [29], basing partially on the local trace formula.

Theorem 4.2 *If f, f' are smooth transfers of each other, so are $\widehat{f}$ and $\lambda \cdot \widehat{f'}$, where λ is an explicit constant independent of f, f'.*

This is a relative variant of Waldspurger's result on the compatibility between the endoscopic transfer and Fourier transform ([24]). But in our case the proof is purely local and does not require the fundamental lemma, which has been proved by Yun ([26]). Together with other ingredients, this compatibility result implies the existence of smooth transfer for all f and f':

Theorem 4.3 *Given $f \in \mathscr{C}_c^\infty(\mathfrak{u}(W_{n+1}(F))$, there exists $f' \in \mathscr{C}_c^\infty(\mathfrak{s}_{n+1}(F))$ that is a smooth transfer of f. Conversely, given $f' \in \mathscr{C}_c^\infty(\mathfrak{s}_{n+1}(F))$, there exists $f \in \mathscr{C}_c^\infty(\mathfrak{u}(W_{n+1}(F))$ that is a smooth transfer of f'.*

Roughly speaking, Fourier transforms provide a way to generate more pairs of functions that are smooth transfers of each other, starting from certain basic pairs (for example, pairs of (f, f') with each of f, f' supported in the regular semisimple locus).

We would like to give a heuristic as to why we should expect the compatibility result of Theorem 4.2 to hold. Let F be a number field. For (H, V) being one of the two representations above, we may consider a θ-function attached to $f \in \mathscr{C}_c^\infty(\mathrm{V}(\mathbb{A}))$:

$$K_f(h) = \sum_{X \in \mathrm{V}(F)} f(X^h), \quad h \in \mathrm{H}(\mathbb{A}).$$

It is clearly absolutely convergent and $\mathrm{H}(F)$-invariant on the left. We consider the integral

$$I(f) := \int_{\mathrm{H}(F)\backslash \mathrm{H}(\mathbb{A})} K_f(h)dh.$$

We proceed formally to ignore the convergence problem. We may expand it as

$$I(f) = \sum_X \mathrm{Orb}(X, f) + \cdots .$$

where the sum is over a set of representatives for the regular semisimple orbits X, and where the non-regular-semisimple contribution needs to be regularized. On the other hand, by the Poisson summation formula we have

$$K_f(h) = K_{\widehat{f}}(h).$$

Therefore we have an alternative expansion of $I(f) = I(\widehat{f})$ and we conclude

$$\sum_X \mathrm{Orb}(X, f) + \cdots = \sum_X \mathrm{Orb}(X, \widehat{f}) + \cdots . \tag{4}$$

This can be viewed as an infinitesimal version, or a 'Lie algebraic version', of the global RTF. In the case of an adjoint representation $(\mathrm{H}, \mathfrak{h})$, if we compare the Lie algebraic version with the Arthur-Selberg trace formula for the group H, we may view the Fourier transform of an orbital integral on the Lie algebra $\mathfrak{h}$ as an analogue of an irreducible character of the group H. Now if we compare the equality (4) for two pairs of (H, V), we may naturally expect that the Fourier transform should respect the smooth transfer (possibly up to some local constant whose product over all places is equal to one).

In the endoscopic case too, such a compatibility result holds and was reduced to the fundamental lemma for Lie algebras by Waldspurger ([24]). In this case, the kernel function κ representing the Fourier transform of orbital integrals is known to exist but hard to evaluate. The fundamental lemma is used to apply a global argument to deduce an identity between the kernel functions. There is another example due to Jacquet ([9]), where the kernel function κ is computed explicitly and the fundamental lemma can then be deduced from the kernel function. In our case, it is not known whether the kernel function κ exists but a certain special feature of $(\mathrm{GL}_n, \mathfrak{s}_{n+1})$ allows us to circumvent the difficulty and prove the compatibility result of Theorem 4.2.

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