METAPLECTIC COVERS OF p-ADIC GROUPS AND QUANTUM GROUPS AT ROOTS OF UNITY

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ABSTRACT. We describe the structure of the Whittaker or Gelfand–Graev module on a n-fold metaplectic cover of a p-adic group G at both the Iwahori and spherical level. Following a conjecture of Gaitsgory–Lurie, we express our answer in terms of the representation theory of a quantum group at a root of unity attached to the Langlands dual group of G. To do so, we introduce an algebro-combinatorial model for our p-adic Whittaker modules and develop for them a Kazhdan–Lusztig theory involving new generic parameters. These parameters can either be specialized to Gauss sums to recover the p-adic theory or to the grading parameter in the representations theory of quantum groups. As applications, we deduce 'geometric' Casselman–Shalika type results for metaplectic covers as well as a variant of G. Savin's local Shimura type correspondences at the Whittaker level.

CONTENTS

1. Introduction

Let $\mathcal{F}$ be a non-archimedean local field and $G = \mathbf{G}(\mathcal{F})$ the points of a (split) simple and simply connected Chevalley group. To a pair (Q, ℓ) where ℓ is a positive integer and Q is an integer valued, Weyl group invariant, quadratic form on the coweight lattice of G, we may associate a metaplectic ℓ-fold covering group $\widetilde{G}$ of G. Let us denote by K a maximal compact subgroup of G. Fix a Borel subgroup and write $I^- \subset K$ for the Iwahori subgroup defined with respect to the opposed Borel. We denote lifts of these subgroups to $\widetilde{G}$ by the same symbols. Also fix ψ an additive character of $\mathcal{F}$ of conductor 0 and extend it to one for the unipotent radical of the chosen Borel (see §2.2.3 and §6.1.1). In the linear case (*i.e.* $\ell = 1$), the space of spherical Whittaker functions on G, otherwise called the spherical Gelfand–Graev representation and denoted $\mathcal{W}_\psi(G, K)$, is a free rank one module over the spherical Hecke algebra $\mathcal{H}(G, K)$ (see [42, §5.3]). One way to see this is by identifying its Iwahori–Whittaker analogue $\mathcal{W}_\psi(G, I^-)$ with the anti-spherical module of the Iwahori–Hecke algebra $\mathcal{H}(G, I^-)$ (see [29, Cor. 4.4] or [11, Equation (2)]). Moreover, from these descriptions of Whittaker spaces, one may deduce both the classical and 'geometric' Casselman–Shalika formulas that have played an important in the classical and geometric Langlands programs.

The aim of this paper is to extend such results to the non-linear or metaplectic case (*i.e.* $\ell > 1$). Since the pioneering work of Kazhdan–Patterson [67], it has been known that the metaplectic Gelfand–Graev representation $\mathcal{W}_\psi(\widetilde{G}, K)$ is not a rank one module over $\mathcal{H}(\widetilde{G}, K)$. Moreover, it was expected that the structure of $\mathcal{W}_\psi(\widetilde{G}, K)$ must be somewhat intricate to reflect both the algebraic and arithmetic complexities of known metaplectic Casselman–Shalika formulas [24, 32, 91]. A fundamentally new framework to understand these issues was provided by the program of D. Gaitsgory and J. Lurie (see especially [46, Conjecture 0.4]).

To explain this, note that for a specific choice of Q (essentially the 'minimal' or 'primitive' choice, see §3.2.8) and any ℓ, we can also construct another object, namely the (big) quantum group $\dot{\mathbf{U}}_\zeta\left(\check{\mathbf{G}}\right)$ of the dual group $\check{\mathbf{G}}$ at a 2ℓ-th root of unity $\zeta \in \mathbb{C}$. As G. Lusztig observed (see [83, §35]), the category of finite-dimensional representations $\mathrm{Rep}\left(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\right)$ is equipped with the action (via quantum Frobenius) of the representation category $\mathrm{Rep}\left(\check{\mathbf{G}}_\ell\right)$ of an auxiliary linear algebraic group $\check{\mathbf{G}}_\ell$ constructed from $\mathbf{G}$ and the chosen (Q, ℓ). The Gaitsgory–Lurie conjecture posits an equivalence between a certain category of twisted, Whittaker sheaves on the affine Grassmannian attached to $\mathbf{G}$ with $\mathrm{Rep}\left(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\right)$ in a manner compatible with the actions of Hecke algebra and quantum Frobenius, respectively. Whereas progress towards the full Gaitsgory–Lurie conjecture at the level of factorization categories seems ongoing (see [47, 48]), a version of this conjecture (at the level of DG-categories of *D*-modules) has already been established by Campbell–Dhillon–Raskin in [27].

The main results obtained in this paper can be summarized as follows:

(a) We obtain an explicit decomposition of $\mathcal{W}_\psi(\widetilde{G}, I^-)$ into $\mathcal{H}(\widetilde{G}, I^-)$-submodules, each of which can be identified precisely with a certain anti-spherical (or closely related) module for the corresponding affine Hecke algebra. This leads us to find and study new Kazhdan–Lusztig type bases for $\mathcal{W}_\psi(\widetilde{G}, K)$, in terms of which we can formulate various Casselman–Shalika type formulas. One such formula (see (1.12)) can be expressed in terms of q-Littlewood–Richardson coefficients, expressions of considerable combinatorial interest.

(b) We prove an isomorphism between $\mathcal{W}_\psi(\widetilde{G}, K)$ and a certain (enriched) graded Grothedieck ring of $\mathrm{Rep}\left(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\right)$ that intertwines the Hecke action with the $\mathrm{Rep}\left(\check{\mathbf{G}}_\ell\right)$ action.

As $\mathcal{W}_\psi(\widetilde{G}, K)$ is an analogue of the category of Whittaker sheaves, (b) can perhaps be construed as a p-adic version of the Gaitsgory–Lurie conjecture. This work is independent of and, as we explain in more detail in §1.2.8, does not appear to be just a 'decategorified' version of [27]. A decomposition of $\mathcal{W}_\psi(\widetilde{G}, I^-)$ similar to the one in (a) recently appeared in two places: in the geometric work of Campbell–Dhillon–Raskin [27] and in the algebraic work of Gao–Gurevich–Karasiewicz [49]; in both papers results at the pro-p level appear naturally. Our approach is different, seems to handle cases not covered by [49], and also led us, following ideas of (Lascoux)–Leclerc–Thibon [77], to the introduction of certain Gauss-sum twisted Kazhdan–Lusztig bases for $\mathcal{W}_\psi(\widetilde{G},K)$. These new bases, perhaps unexpected from the usual representation theory of (covers of) p-adic groups, are however natural in the geometric context of the Gaitsgory–Lurie program, and they play an important role in our approach to connecting $\mathcal{W}_\psi(\widetilde{G},K)$ with quantum groups. So, let us now say a few, non-technical words about the different types of bases we encounter in this work.

Recall first that $\mathcal{H}(\widetilde{G},K)$ has not one, but two standard choices for bases: the usual p-adic basis of K-double cosets and the 'geometric' basis which has played an important role in the geometric Satake correspodence, though it was first introduced algebraically by Lusztig [82]. As for $\mathcal{W}_\psi(\widetilde{G},K)$, one again has a natural p-adic basis. By introducing certain involutions into $\mathcal{W}_\psi(\widetilde{G},K)$, we can produce two new bases of Kazhdan–Lusztig type starting from this standard p-adic basis, or from any *rescaling* of it for that matter.[1] Indeed, a specific rescaling of the standard p-adic basis turns out to be important in this work. To explain this, we note that the explicit formulas for the Hecke action on Whittaker modules informing this work (see §4.1) exhibit two layers of complexity: an algebro-combinatorial one stemming from the failure of a multiplicity one result; and an arithmetic 'enhancement' which introduces certain Gauss sums into the formulas[2]. The algebraic complexities, as predicted in the Gaitsgory–Lurie program (and especially in a form made precise by S. Lysenko [85]) are matched by those from the affine Kazhdan-Lusztig theory governing $\mathrm{Rep}\left(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\right)$. As for the arithmetic complexities, we are able to manage them by rescaling the natural p-adic bases by certain explicit product of Gauss sums. This involves reinterpreting some of the previously mentioned (Lascoux–)Leclerc–Thibon combinatorics in terms of the metapletic Demazure–Lustzig operators of Chinta–Gunnells–Puskás (see [31]). This rescaling appears *a posteriori* in our approach and at the moment, we have no real p-adic intuition for it, though the rescaled bases do seem to be necessary for the match with the quantum group world. Now, whereas the D-module context of [27] and [85] naturally treats the algebro-combinatorial complexity, we do not understand if the 'arithmetic' complexities (*i.e.* the Gauss sums) may also be a shadow of some other aspect of the Gaitsgory–Lurie program and have a representation theoretic meaning in a (probably ℓ-adic) version of [27].

As noted above in (a), the coefficients we find in one of our geometric metaplectic Casselman–Shalika formula, *i.e.* in the left hand side of (1.12), are equal, up to an explicitly computable product of Gauss sums, to the so-called quantum Littlewood–Richardson (q-LR) polynomials. These polynomials appear in the Schur expansion of LLT polynomials, have connections to Macdonald polynomials, and enjoy certain positivity properties [53, 54, 77]. As far as we know, largely

[1] It might be worth remarking that in the linear case, all of these bases turn out to be the same and the rescaling is trivial.

[2] These Gauss sums are crucial for applications to number theory, for example to the theory of multiple Dirichlet series and to computing moments of L-functions [23, 39].

gone unnoticed by the p-adic community (unlike the usual Littlewood–Richardson coefficients). We refer to §1.2.4 for further remarks.

Although not explicitly used in the main body of this work, an *a priori* unrelated connection between $\mathcal{W}_\psi(\widetilde{G}, I^-)$ and another class of quantum obects, namely quantum affine groups at generic values of the deformation parameter, played an important role in our thinking[3]. Stemming from a study of the multiple Dirichlet series of Siegel [108] and their modern generalization called Weyl group multiple Dirichlet series of [25], this connection was discovered in type A by the first named author and his collaborators [17, 18] and has sinceled to a flurry of activity connecting p-adic groups and discrete integrable systems (see [19–22]). The presence of this second quantum group, distinct from the quantum group predicted by Gaitsgory-Lurie, yet still related to the same Whittaker space, has remained something of a mystery. One explanation for it can be given using the techniques from this paper. Indeed, as we will describe below, the central player in our work is a 'generic' version of the Whittaker space $\mathcal{W}_\psi(\widetilde{G}, K)$ which can be specialized to a graded version of the Grothendieck ring of a quantum group at a root of unity. The latter object is known to carry a natural action (at least in type A) of a quantum affine group at a generic value providing a representation theoretic interpretation of the quantum affine group appearing in [17, 18]. We refer the interested reader to §1.2.5 for further comments.

As we just signaled, our approach to connect the p-adic and quantum pictures involves a third 'generic' space $\mathbb{V}_{\mathrm{sph}}$ depending on certain formal parameters that can be specialized in two ways: via a map $\mathfrak{p}$ to the Gauss sums in the p-adic world; or alternatively, via a map $\mathfrak{q}$ to the grading within the representation category of the quantum group. The space $\mathbb{V}_{\mathrm{sph}}$ also carries an action of a certain spherical Hecke algebra $\widetilde{H}_{\mathrm{sph}}$, and the connections we describe in this work can be summarized pictorially as follows

$$\begin{tikzcd}
 & \mathbb{V}_{\mathrm{sph}} \arrow[loop above, "\overset{\widetilde{H}_{\mathrm{sph}}}{\star}"] \arrow[dl, "\mathfrak{p}"'] \arrow[dr, "\mathfrak{q}"] & \\
\mathcal{W}_\psi(\widetilde{G}, K) \arrow[loop above, "\overset{\mathcal{H}(\widetilde{G},K)}{\star}"] & & K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}}))) \arrow[loop above, "\overset{\mathrm{Rep}(\check{\mathbf{G}}_\ell)}{-\otimes \mathrm{Fr}}"]
\end{tikzcd}$$

Here $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})))$ is a graded version of the Grothendieck group (more precisely, the enriched right Grothendieck ring of [36]). In other words, as predicted by Gaitsgory–Lurie and as has been verified in the D-module context by Campbell-Dhillon-Raskin [27], the structure of $\mathcal{W}_\psi(\widetilde{G}, K)$ as a $\mathcal{H}(\widetilde{G}, K)$-module is essentially the same as that of $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}}))$ regarded as the $K_0(\mathrm{Rep}(\check{\mathbf{G}}_\ell))$-module.

[3]In fact, the technique of treating the Gauss sums as formal parameters and creating 'generic' spaces which can be specialized in different manners can be traced to the works describing this connection.

Though pre-dating their appearance by a few years, the representation theory of metaplectic p-adic groups is perhaps less well-studied than that of quantum groups at a root of unity. So our main result allows us to pair a number of curious phenomenon in the representation theory of p-adic covering groups to better studied counterparts in the theory of quantum groups. For example, throughout this introduction we focus on the case of 'geometric' Casselman–Shalika formulas for metaplectic groups which record (following conjectures of Lysenko [85]) the complexity of decomposing certain tensor products of quantum group representations, and perhaps for this reason remained somewhat obscure from a purely p-adic/number theoretic perspective. As another example, our work naturally suggests a local Shimura correspondence (see [90, 103]) at the level of the Gelfand–Graev representation which both encapsulates certain 'exceptional'[4] phenomenon in $\mathcal{W}_\psi(\widetilde{G}, K)$ observed by Gao–Shahidi–Szpruch [50] and also provides for these phenomena an interpretation in terms of tensor product theorems for certain irreducible (or indecomposable tilting) modules of quantum groups.

Finally, note that many of the techniques from this work were designed in such a way that they could be extended to (metaplectic covers of) p-adic Kac-Moody groups [15, 16][5]. Whittaker functions in this context have already been investigated in [96, 97]. Unlike the finite-dimensional situation, the representation theory of corresponding quantum object in this case, $U_\zeta(\widehat{\mathfrak{g}})$– a quantum group at a root of unity attached to an affine Kac–Moody algebra– is still in its infancy. Down the line, we hope that ideas from the p-adic world may spur its further development. For example the (double affine) Kazhdan–Lusztig type polynomials which presumably govern decomposition rules for $U_\zeta(\widehat{\mathfrak{g}})$ have not been defined, but the p-adic picture suggests certain (finite!) quantities to which these should be equal.

Before describing our main results in more detail, let us note that there are other areas in which similar types of affine Weyl group combinatorics arise, *e.g.* affine Lie algebras at negative level, representation theory of algebraic groups in positive characteristic, and so our results might equally link the metaplectic world to these areas.

1.1. **Geometric Casselman–Shalika Formulas for Metaplectic Covers.** Let us now describe our main results from the vantage point of 'geometric' Casselman–Shalika formulas.

1.1.1. *Formulas of Casselman–Shalika type for linear groups.* Before turning to covers, let us recall a few results of Casselman–Shalika type for linear groups. These results are well-known, but our presentation here is influenced by our work in the non-linear setting. As above, write $G := \mathbf{G}(\mathcal{F})$ and consider $\mathcal{H}(G, K)$, the spherical Hecke algebra of G with respect to a maximal compact subgroup K. It is an algebra under convolution and is equipped with a natural p-adic basis $h_{\check{\lambda}}$

[4]Here 'exceptional' in the metaplectic world means that it behaves 'normally,' or as in the linear group case.

[5]There are however, certain important and interesting obstacles related to the lack of Coxeter structure in the 'double affine' Weyl groups governing the Hecke algebras of [16] which need to be treated.

given by characteristic functions of double cosets $K\pi^{\check{\lambda}}K$ with $\check{\lambda}$ in Y_+, the set of dominant coweights (we shall also write Y for the full lattice of coweights). The Satake isomorphism [102], as reinterpreted by Langlands [73, Ch. 2], gives an identification

$$S : \mathcal{H}(G,K) \xrightarrow{\sim} K_0(\mathrm{Rep}(\check{\mathbf{G}}(\mathbb{C}))) \tag{1.1}$$

between the spherical Hecke algebra and the representation ring of the dual group $\check{\mathbf{G}}(\mathbb{C})$ of G. To each irreducible highest weight representation $V_{\check{\lambda}} \in \mathrm{Rep}(\check{\mathbf{G}}(\mathbb{C}))$ we can assign its preimage under S, namely the element $c_{\check{\lambda}} \in \mathcal{H}(G,K)$. The $c_{\check{\lambda}}$ are the 'geometric' basis mentioned above, and a formula expressing them in terms of the basis $\{h_{\check{\mu}}\}$ of $\mathcal{H}(G,K)$ was given by Kato and Lusztig [61, 82]; it involves certain affine, parabolic Kazhdan-Lusztig polynomials.

As above, let ψ be a non-trivial additive character of $\mathcal{F}$ of conductor 0 (see §2.2.3) extended to $U = \mathbf{U}(\mathcal{F})$, the unipotent radical of some fixed Borel subgroup, and consider $\mathcal{W}_\psi(G,K)$ the space of compactly supported functions on G which are (U,ψ)-left invariant and right K-invariant. To each coweight $\check{\mu} \in Y$, we can consider the unique function $\mathcal{J}_{\check{\mu}} \in \mathcal{W}_\psi(G,K)$ which takes the value 1 on $\pi^{\check{\mu}}$ and is supported on $U\pi^{\check{\mu}}K$. One finds that $\mathcal{J}_{\check{\mu}}$ is only well-defined when $\check{\mu} \in Y_+$ and that such elements form a $\mathbb{C}$-basis of $\mathcal{W}_\psi(G,K)$. Denoting by $\star$ the natural right convolution of $\mathcal{H}(G,K)$ on $\mathcal{W}_\psi(G,K)$, for $\check{\mu}, \check{\lambda} \in Y_+$, we may ask to rewrite the product $\mathcal{J}_{\check{\mu}} \star h_{\check{\lambda}}$ in terms of the basis $\{\mathcal{J}_{\check{\zeta}}\}_{\check{\zeta} \in Y_+}$. For $\check{\mu}$ fixed, and $\check{\lambda}$ large and dominant (compared to $\check{\mu}$), the answer to this question is given by the usual Casselman–Shalika formula [28]. To explain this, consider

$$\mathrm{CS}(\check{\mu}) := \prod_{a>0}(1 - q^{-1}Y_{-\check{a}})\chi_{\check{\mu}}(Y), \tag{1.2}$$

the formula for the unramified spherical Whittaker function found in *op. cit.*, where the notation used is as follows: $Y_{-\check{a}}$ represents an element in the group algebra of coweights $\mathbb{C}[Y]$ corresponding to the coroot $\check{a}$; $\chi_{\check{\mu}}(Y)$ is the Weyl character for the irreducible representation $V_{\check{\mu}}$ of $\check{\mathbf{G}}(\mathbb{C})$ of highest-weight $\check{\mu}$; q is the cardinality of the residue field of $\mathcal{F}$; and the product is over all positive roots of G. In analogy with [82, Thm 6.6], which works in the algebraic context of affine Hecke algebras (as opposed to p-adic groups), one shows

$$\mathcal{J}_{\check{\mu}} \star h_{\check{\lambda}} = \mathrm{CS}(\check{\mu}) \circ \mathcal{J}_{\check{\lambda}} \quad \text{where we set} \quad Y_{\check{\mu}} \circ \mathcal{J}_{\check{\zeta}} = \mathcal{J}_{\check{\mu}+\check{\zeta}}. \tag{1.3}$$

Note in passing that formulas for $\mathrm{CS}(0)$ play an important role in the theory of automorphic forms, [98, 106, 107]. One complication is understanding (1.3) is that upon applying $\circ$, elements of the form $\mathcal{J}_{\check{\lambda}+\check{\zeta}}$ where $\check{\lambda} + \check{\zeta}$ is no longer dominant may appear on the right hand side of (1.3). So to correctly interpret (1.3) as a formula in $\mathcal{W}_\psi(G,K)$, one needs to *formally* introduce elements $\mathcal{J}_{\check{\mu}}$ for $\check{\mu} \in Y \setminus Y_+$, subject to 'straightening' rules using the 'dot' action $\bullet$ of the affine Weyl group:

$$\mathcal{J}_{\check{\mu}} = \begin{cases} -\mathcal{J}_{\check{\mu}\bullet s_a} & \text{if } \langle \check{\mu} + \check{\rho}, a \rangle \neq 0 \\ 0 & \text{if } \langle \check{\mu} + \check{\rho}, a \rangle = 0, \end{cases} \tag{1.4}$$

where in the above formula $\check{\rho}$ is the half-sum of the positive coroots and a is any simple root such that $\langle \check{\mu} + \check{\rho}, a \rangle \leq 0$. Notice that if $\check{\mu}$ is fixed and $\check{\lambda}$ is chosen very large, we can arrange for all the terms in the right hand side of (1.3) to be dominant, *i.e.* no straightening rules are involved, and the $\circ$ in (1.3) amounts to just a $\check{\lambda}$-translation of $\mathrm{CS}(\check{\mu})$. For this reason, we might refer to the formula for $\mathrm{CS}(\check{\mu})$ in [28] as an *asymptotic* Casselman–Shalika formula.

From (1.3), (see [82, Corollary 6.8]), one deduces[6] the existence of $\mathcal{C}_{\check{\lambda}} \in \mathcal{H}(G,K)$ for $\check{\lambda} \in Y_+$ satisfying

$$\mathcal{J}_0 \star \mathcal{C}_{\check{\lambda}} = \mathcal{J}_{\check{\lambda}} \quad \text{and} \quad S(\mathcal{C}_{\check{\lambda}}) = [V_{\check{\lambda}}], \tag{1.5}$$

the latter of which implies that $\mathcal{C}_{\check{\lambda}} = c_{\check{\lambda}}$, the element introduced above. As the above expression has a natural description within the category of perverse sheaves on the affine Grassmannian (see [42, 44, 93, 94]), we shall refer to (1.5) as a *geometric* Casselman–Shalika formula, even though we are just in the p-adic setting. Using (1.5), we compute

$$\mathcal{J}_{\check{\mu}} \star c_{\check{\lambda}} = \mathcal{J}_0 \star c_{\check{\mu}} \star c_{\check{\lambda}} = \mathcal{J}_0 \star \sum_{\check{\eta}} c^{\check{\eta}}_{\check{\mu},\check{\lambda}} c_{\check{\eta}} = \sum_{\check{\eta}} c^{\check{\eta}}_{\check{\mu},\check{\lambda}} \mathcal{J}_{\check{\eta}}, \tag{1.6}$$

where $c^{\check{\eta}}_{\check{\mu},\check{\lambda}} := \dim_{\mathbb{C}} \mathrm{Hom}(V_{\check{\eta}}, V_{\check{\mu}} \otimes V_{\check{\lambda}})$ are the Littlewood–Richardson coefficients for $\check{\mathbf{G}}(\mathbb{C})$.

Remark. *Whereas* (1.6) *has a natural representation theoretic interpretation, such an interpretation of* (1.3) *seems more opaque. In* [82], (1.3) *was used to deduce* (1.6)*, but, as far as we know, deducing* (1.3) *from* (1.6) *has not been explicitly treated in the literature. One approach to this question would use an inverse to the so-called Kato–Lusztig formula, cf.* [5, 62, 81].

To summarize, the above results show that the map $h \mapsto \mathcal{J}_0 \star h$ from $\mathcal{H}(G,K) \to \mathcal{W}_\psi(G,K)$ is an isomorphism of $\mathcal{H}(G,K)$ modules sending $c_{\check{\lambda}}$ to $\mathcal{J}_{\check{\lambda}}$. Combined with (1.1), we obtain an isomorphism

$$\mathcal{W}_\psi(G,K) \simeq K_0\left(\mathrm{Rep}(\check{\mathbf{G}}(\mathbb{C}))\right) \tag{1.7}$$

intertwining the $\mathcal{H}(G,K)$ action on $\mathcal{W}_\psi(G,K)$ with the $K_0\left(\mathrm{Rep}(\check{\mathbf{G}}(\mathbb{C}))\right)$-action (by tensor product) on itself.

1.1.2. *Metaplectic Casselman–Shalika type problems.* Let us now turn to the case of an n-fold metaplectic cover $\widetilde{G}$ of G. One can again construct a space of (genuine) spherical Whittaker functions $\mathcal{W}_\psi(\widetilde{G},K)$ equipped with a right convolution by the spherical Hecke algebra $\mathcal{H}(\widetilde{G},K)$. The space $\mathcal{W}_\psi(\widetilde{G},K)$ has a basis denoted $\widetilde{\mathcal{J}}_{\check{\lambda}}, \check{\lambda} \in Y_+$ and constructed in a similar fashion to $\mathcal{J}_{\check{\lambda}}$. On the other hand, although one can introduce as above the elements $\widetilde{h}_{\check{\lambda}}$ in $\mathcal{H}(\widetilde{G},K)$ for any $\check{\lambda} \in Y_+$, unless $\check{\lambda}$ lies in a certain sublattice of finite index $\widetilde{Y}_+ \subset Y_+$, the element $\widetilde{h}_{\check{\lambda}}$ is not

[6] This manner of introducing $\mathcal{C}_{\check{\lambda}}$, due originally to G. Lusztig, is a bit different from the 'modern' point of view which introduces $\mathcal{C}_{\check{\lambda}}$ directly from a geometric point of view, see [42, 43]. We note that Lusztig's approach has the advantage that it also applies in the Kac-Moody setting.

well-defined. This 'reduction in support' is a hallmark of the metaplectic world. Nonetheless, one may work with $\mathcal{H}(\widetilde{G},K)$ in a similar manner to $\mathcal{H}(G,K)$, and in fact McNamara [90], building on the work of Savin [103], showed $\mathcal{H}(\widetilde{G},K) \cong K_0(\mathrm{Rep}(\check{\mathbf{G}}_{(\mathsf{Q},n)}(\mathbb{C})))$ for a complex group $\check{\mathbf{G}}_{(\mathsf{Q},n)}$ whose root data is determined from that of $\mathbf{G}$ and the metaplectic structure (Q,n) defining $\widetilde{G}$ (see also [41]).

One can define $\widetilde{c}_{\check{\lambda}} \in \mathcal{H}(\widetilde{G},K)$ (uniquely) for any $\check{\lambda} \in \widetilde{Y}_+$ to satisfy the second condition in (1.5). Although the first condition in (1.5) breaks down, *i.e.* we may have $\widetilde{\mathfrak{J}}_0 \star \widetilde{c}_{\check{\lambda}} \neq \widetilde{\mathfrak{J}}_{\check{\lambda}}$ for some $\check{\lambda} \in \widetilde{Y}_+$, we show that (1.3) persists if we replace $\mathrm{CS}(\check{\mu})$ with its metaplectic version $\widetilde{\mathrm{CS}}(\check{\mu})$ that was studied in [24, 32, 91, 95], *i.e.*

$$\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{h}_{\check{\lambda}} = \widetilde{\mathrm{CS}}(\check{\mu}) \circ \widetilde{\mathfrak{J}}_{\check{\lambda}}, \text{ where } \check{\mu} \in Y_+, \check{\lambda} \in \widetilde{Y}_+. \tag{1.8}$$

Making precise sense of the above expression was actually the starting point of this work.[7] Indeed, one runs into the same issue as in the linear case (*i.e.* what to do with non-dominant $\widetilde{\mathfrak{J}}_{\check{\lambda}+\check{\mu}}$) in the right hand side of (1.8). The fix now is more involved and uses what we call 'metaplectic' straightening rules, see Proposition 4.3.2. These recover (and were motivated by) the straightening rules found some time ago by Lascoux–Leclerc–Thibon [77] in type A (see [53] and [75] for the extension to general type). In our setup, these metaplectic straightening originate at the Iwahori level from explicit formulas for the action of $\mathcal{H}(\widetilde{G},I^-)$ on $\mathcal{W}_\psi(\widetilde{G},I^-)$.

Inspired by [77], we use our straightening rules to define an involutions of Kazhdan–Lusztig type. As mentioned above, this allows us to construct two new bases, called *canonical bases* in analogy with their quantum counterpart, $\{\widetilde{\mathcal{L}}_{\check{\lambda}}\}_{\check{\lambda} \in Y_+}$ and $\{\widetilde{\mathscr{T}}_{\check{\lambda}}\}_{\check{\lambda} \in Y_+}$ of $\mathcal{W}_\psi(\widetilde{G},K)$ starting from our original basis $\widetilde{\mathfrak{J}}_{\check{\lambda}}$. One should understand the bases $\widetilde{\mathcal{L}}_{\check{\lambda}}$, $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ and $\widetilde{\mathscr{T}}_{\check{\lambda}}$ as p-adic versions of the irreducible, (co)standard and indecomposable tilting modules for the corresponding quantum group at roots of unity. Let us note that the existence of these new bases suggests *three* natural metaplectic[8] analogues of the linear geometric Casselman–Shalika problem (1.5):

- compute $\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}}$ for $\check{\lambda} \in \widetilde{Y}_+$ and $\check{\mu} \in Y_+$; and
- compute $\widetilde{\mathcal{L}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}}$ and $\widetilde{\mathscr{T}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}}$ for $\check{\lambda} \in \widetilde{Y}_+$ and $\check{\mu} \in Y_+$.

The answers to each of these questions are naturally expressed using the representation theory of quantum groups at roots of unity, as was predicted by Gaitsgory–Lurie.

[7] Note that this is *not* the p-adic analogue of the geometric Casselman–Shalika problem formulated by Lysenko in [85, §11], which in the p-adic setting amounts to the computation of $\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}}$ (see (1.12)). It is unclear to us what the natural geometric significance of (1.8) might be (though its importance to the theory of automorphic forms and multiple Dirichlet series is clear). Again, note that going between formulas for $\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}}$ and $\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{h}_{\check{\lambda}}$ we run into the same issues as were mentioned in Remark 1.1.1. See also §1.2.4.

[8] If one performs the same procedure in the linear case, both canonical bases agree with the standard basis.

1.1.3. *Quantum groups at roots of unity.* The quantum groups of relevance in this paper are Lusztig's dotted version with divided powers. To define them, one constructs an 'integral' form (or rather a $\mathbb{Z}[u,u^{-1}]$ form, where u is the deformation parameter in the quantum group) and then specializes the variable u to a root of unity $\zeta \in \mathbb{C}$. The corresponding object will be called $\dot{\mathbf{U}}_\zeta(\mathbf{G})$ where $\mathbf{G}(\mathbb{C})$ is the complex group attached to some root datum. In the main body of this paper, we adopt a slighly different notation, but still emphasize that the quantum group we construct depends on a root datum, and not just a Cartan datum, *i.e.* not just the data needed to specify a semi-simple Lie algebra. What is of particular importance for us is not the quantum group itself, but the structure of its (graded) category of finite-dimensional modules $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G}))$[9]. When u is specialized to a root of unity, the representation theory of the quantum group diverges from the complex representation theory of semi-simple Lie algebras. For example, although irreducible highest weight modules from the complex semi-simple theory can be deformed to objects $\Delta_\lambda \in \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G}))$ for $\lambda \in X_+$ a dominant weight, such modules may be reducible. One can still compute their character by the Weyl character formula, and so these modules are often called Weyl or standard modules. Understanding their irreducible quotients L_λ was the subject of conjectures put forth by Lusztig and answered by combining the work of several groups, see [3, 59, 60, 64–66]. In addition to the irreducible and (co)standard modules, there is another important class of modules in $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G}))$ called the indecomposable tilting modules T_λ, again indexed by $\lambda \in X_+$. The relation between the T_λ and the (co)standard modules were the subject of conjectures and then theorems of Soergel [110, 111].

In contrast to the representation theory of complex Lie algebras, when u is specialized to a root of unity, the category $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G}))$ is no longer semi-simple as one has non-trivial extensions between irreducible objects. As such, the usual Grothendieck group construction loses important information about the category, and one may work with various enhancements to try to recapture this lost data. It seems to be well-understood that $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G}))$ is also graded in the sense of [109] and hence its Grothendieck group has a natural $\mathbb{Z}[\tau,\tau^{-1}]$-structure. We could not find an explicit reference in the literature outside of rank 1 (see [4]), so as a substitute we work here with the so-called left and right enriched Grothendieck rings $K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G})))$ and $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G})))$ of [36]. They are $\mathbb{Z}[\tau,\tau^{-1}]$-modules with basis indexed by the classes of standard objects $[\Delta_\lambda]$ and costandard objects $[\nabla_\lambda]$, respectively, where $\lambda \in X_+$. In these spaces one has expansions[10]

$$[L_\lambda] = \textstyle\sum_{\mu\in X_+\cap\lambda\bullet W_{\mathrm{aff}}} o^-_{\lambda,\mu}(\tau)[\nabla_\mu], \text{ and conjecturally } [T_\lambda] = \textstyle\sum_{\mu\in X_+\cap\lambda\bullet W_{\mathrm{aff}}} o^+_{\lambda,\mu}(\tau^{-1})[\Delta_\mu], \tag{1.9}$$

where $o^\pm_{\lambda,\mu}(\tau)$ are certain parabolic Kazhdan–Lusztig polynomials and $\bullet$ is the dilated dot action on the weight lattice. In contrast to this, working with the usual

[9]The dotted and non-dotted quantum groups at a root of unity have equivalent representation theory (see [84, §31] or [9, §3.7]).

[10]We could not find an explicit reference for the tilting expansion, but note that Z. Yun [118, Theorem 5.3.1] proves a mixed geometric analogue of this result.

Grothendieck ring, one finds similar relations, but with a specialization of the polynomials $o^{\pm}_{\lambda,\mu}(-1)$ that produces character formulas for irreducibles (Lusztig's conjecture) and for indecomposable tiltings (Soergel's conjecture [110, §7]).

For a certain *algebraic* group $\mathbf{G}_\ell$ constructed from the root data for $\mathbf{G}$ and the integer ℓ[11], there exists a functor called *quantum Frobenius* and denoted $\mathrm{Fr}: \mathrm{Rep}(\mathbf{G}_\ell(\mathbb{C})) \to \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G}))$ which equips the latter category with an action by the former [84, §35]. We write this as $W, V \mapsto W \otimes \mathrm{Fr}(V)$ where $W \in \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G})), V \in \mathrm{Rep}(\mathbf{G}_\ell(\mathbb{C}))$. One may now pose the following questions in $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G})))$:

- write $\nabla_\mu \otimes \mathrm{Fr}(V_\lambda)$ in terms of the ∇_η (or more precisely, understand a ∇-filtration of $\nabla_\mu \otimes \mathrm{Fr}(V_\lambda)$);
- write $L_\mu \otimes \mathrm{Fr}(V_\lambda)$ and $T_\mu \otimes \mathrm{Fr}(V_\lambda)$ in terms of the ∇_η.

The second question can be deduced from the Steinberg–Lusztig theorem, which states that $L_\mu \otimes \mathrm{Fr}(V_\lambda) \simeq L_{\mu+\lambda}$ whenever μ is in some 'restricted' set of weights and λ is in the dominant l-weight lattice $X_{l,+}$ of $\mathbf{G}_\ell$. The last question is the subject of the tilting tensor product theorem, which states that for μ 'restricted' one can define a new weight $\mu^\dagger$ such that $T_{\mu^\dagger} \otimes \mathrm{Fr}(V_\lambda) = T_{\mu^\dagger+\lambda}$ (see §7.3.6 for more details). As for the first question, the answer is given by the q-Littlewood–Richardson coefficients of Lascoux–Leclerc–Thibon [76,77] (type A) and Grojnowski–Haiman [53] (general type).

1.1.4. *Main result and some consequences.* As mentioned above, our approach[12] to connecting the metaplectic and quantum worlds goes through a combinatorial model consisting of a representation $\mathbb{V}_{\mathrm{sph}}$ of the spherical Hecke algebra $\widetilde{H}_{\mathrm{sph}}$. We define a ring $\mathbb{Z}_{\tau,\mathfrak{g}}$ (see §2.3.1) which depends on both a parameter τ as well as a family of other parameters $\mathfrak{g}_k, k \in \mathbb{Z}$ modelling the behavior of certain Gauss sums from the p-adic world. The space $\mathbb{V}_{\mathrm{sph}}$ is a $\mathbb{Z}_{\tau,\mathfrak{g}}$-module and can be obtained from a representation $\mathbb{V}$ of the affine Hecke algebra $\widetilde{H}_{\mathrm{aff}}$, where the action is via the metaplectic Demazure–Lusztig operators of [31,95]. Moreover, $\mathbb{V}_{\mathrm{sph}}$ has a natural basis $[\mathbb{v}_{\check{\mu}}], \check{\mu} \in Y_+$ and an involution of Kazhdan–Lusztig type which produces new bases denoted $[\mathbb{G}_{\check{\mu}}]$ and $[\mathbb{G}^-_{\check{\mu}}]$. Working with the basis $[\mathbb{v}_{\check{\mu}}]$, the space $\mathbb{V}_{\mathrm{sph}}$ behaves identically to the Fock spaces considered in Leclerc–Thibon [77] . To make a connection to the p-adic world however, we work with renormalizations $[Y_{\check{\mu}}]$, $[G_{\check{\mu}}]$, and $[G^-_{\check{\mu}}]$ of $[\mathbb{v}_{\check{\mu}}]$, $[\mathbb{G}_{\check{\mu}}]$, and $[\mathbb{G}^-_{\check{\mu}}]$, respectively involving Gauss sum parameters.

The space $\mathbb{V}_{\mathrm{sph}}$ recovers $\mathcal{W}_\psi(\widetilde{G}, K)$ (along with its action of the spherical Hecke algebra) under what we call a *p-adic specialization* $\mathfrak{p}$ of the parameters $\tau, \mathfrak{g}_k$ (see §2.3.3) and it recovers $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}}))$ under what we call a *quantum specialization* $\mathfrak{q}$ of these parameters (see §2.3.2). Each of the spaces $\mathcal{W}_\psi(\widetilde{G}, K)$, $\mathbb{V}_{\mathrm{sph}}$, and $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}}))$ carry natural actions by essentially isomorphic algebras $\mathcal{H}(\widetilde{G}, K)$,

[11]That the construction of this group $\mathbf{G}_\ell$ parallels the construction of the algebraic group controlling the spherical Hecke algebras of ℓ-fold covers was noted by M. Weissmann [117, Remark 1.4] several years after, and seemingly independently of, the Gaitsgory–Lurie conjecture.

[12]Note that there are other approaches that work in a geometric context, [27], [85], [48] which seem to follow a different path.

$\widetilde{H}_{\mathrm{sph}}$, and $K_0(\mathrm{Rep}(\check{\mathbf{G}}_l))$, respectively. Our main result asserts that $\mathfrak{p}$ and $\mathfrak{q}$ intertwine these actions.

Theorem (see Thm. 8.1.4)**.** *Let ℓ be a positive integer. Let (Q,ℓ) be a metaplectic twist on the group $\mathbf{G}$ and $\widetilde{G}$ the corresponding ℓ-fold metaplectic cover. Assume that $\mathbf{G}$ and $\mathbf{G}_{(\mathsf{Q},\ell)}$ are of simply-connected type.*

(1) There exists an isomorphism

$$\mathfrak{p} : \mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \xrightarrow{\simeq} \mathcal{W}_\psi(\widetilde{G},K) \quad \textit{sending} \quad [Y_{\check{\mu}}] \mapsto \widetilde{\mathfrak{J}}_{\check{\mu}} \textit{ for } \check{\mu} \in Y_+ \ , \tag{1.10}$$

intertwining the $\widetilde{H}_{\mathrm{sph}}$ and $\mathcal{H}(\widetilde{G},K)$ actions, and sending $[G^-_{\check{\mu}}]$ and $[G_{\check{\mu}}]$ to $\widetilde{\mathcal{L}}_{\check{\mu}}$ and $\widetilde{\mathcal{T}_{\check{\mu}}}$, respectively.

(2) If ℓ is both larger than the Coxeter number of $\check{\mathfrak{D}}$ and KL-good (see §7.5), there exists an isomorphism

$$\mathfrak{q} : \mathbb{Z}[\tau,\tau^{-1}] \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \xrightarrow{\simeq} K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}}))) \quad \textit{sending} \quad [Y_{\check{\mu}}] \mapsto [\nabla_{\check{\mu}}] \textit{ for } \check{\mu} \in Y_+ \tag{1.11}$$

intertwining the $\widetilde{H}_{\mathrm{sph}}$ and $K_0(\mathrm{Rep}(\check{\mathbf{G}}_\ell))$ actions, and sending $[G^-_{\check{\mu}}]$ to $[L_{\check{\mu}}]$.

We refer to §1.2.1 and §1.2.2 for some comments on the proof of this result.

Remark. *(1) A (conjectural) relation between $[G_{\check{\mu}}]$ and the tilting modules $T_{\check{\lambda}}$ is explained in §8.1.6. This depends on certain facts about graded decomposition numbers which we could not find in the literature on quantum groups (though we believe they may be known).*

(2) The hypothesis that $\mathbf{G}_{(\mathsf{Q},\ell)}$ is simply connected can be easily discarded if one works with extended affine Hecke algebras; we only impose it to simplify some of the exposition and computations. On the quantum side, the hypothesis that $\check{\mathbf{G}}$ is of adjoint type and the conditions on ℓ are a bit more serious: they allow us to use a generalization of Lusztig's conjecture whose proof depends on the Kazhdan–Lusztig equivalence [69]. We do not believe the KL-good condition is necessary for the proof of (1.11) (and therefore Lysenko's conjecture), see Remark 7.5.4. Note that if we worked in the setting of affine Lie algebras at negative level (instead of quantum groups at roots of unity), such conditions could be avoided.

As for the geometric Casselman–Shalika problems described above, one has

Corollary (See Thm. 6.3.2 and Prop. 7.5.4)**.** *Let $\check{\zeta} \in \widetilde{Y}_+$.*

(1) If $\check{\lambda} \in Y_+$ is 'restricted' (see §3.4.9) then $\widetilde{\mathcal{L}}_{\check{\lambda}} \star \widetilde{c}_{\check{\zeta}} = \widetilde{\mathcal{L}}_{\check{\lambda}+\check{\zeta}}$.

(2) If $\check{\lambda} \in Y_+$ is 'restricted' and we define $\check{\lambda}^\dagger := \check{\lambda}_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check{\rho})$, where $\widetilde{\rho}^\vee$ is the analogue of $\check{\rho}$ for $\mathbf{G}_{(\mathsf{Q},n)}$, then $\widetilde{\mathcal{T}_{\check{\lambda}^\dagger}} \star \widetilde{c}_{\check{\zeta}} = \widetilde{\mathcal{T}_{\check{\lambda}^\dagger+\check{\zeta}}}$.

(3) There exist ${}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_{\tau,\mathfrak{g}}$ such that with respect to $\mathfrak{p} : \mathbb{Z}_{\tau,\mathfrak{g}} \to \mathbb{C}$ and $\mathfrak{q} : \mathbb{Z}_{\tau,\mathfrak{g}} \to \mathbb{Z}[\tau,\tau^{-1}]$

$$\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}} = \textstyle\sum_{\check{\eta}} \mathfrak{p}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}) \widetilde{\mathfrak{J}}_{\check{\eta}} \quad \textit{and} \quad [\nabla_{\check{\mu}} \otimes \mathrm{Fr}(V_{\check{\lambda}})] = \textstyle\sum_{\check{\eta} \in Y_+} \mathfrak{q}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}) [\nabla_{\check{\eta}}]. \tag{1.12}$$

Remark. *We find that* $\mathfrak{q}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}})$ *is just the quantum Littlewood–Richardson or (untwisted) q-LR coefficient of* [53, 54, 77]. *We shall refer to both* ${}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ *and* $\mathfrak{p}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}})$ *as* twisted *q-LR coefficients. The twisted and untwisted q-LR coefficients are related by an explicit product of Gauss sums, see* (4.65).

Part (1) of the above is a p-adic analogue of a result of Lysenko [85, Thm. 7.1.1], which itself is an analogue of the Steinberg–Lusztig tensor product theorem from quantum groups.

Part (2) is an analogue of the tensor product theorem of Anderson and Donkin [6] for tilting modules, which, as far as we know, has not explicitly appeared in the literature on Whittaker sheaves/D-modules.

Part (3) answers a version of Lysenko [85, Conjecture 11.2.4] at the level of enriched Grothendieck groups; at the D-module level in which it was originally formulated, it follows from the main results in [27]. Our result seems simultaneously both 'less and more' than the original conjecture– less as [85] and [27] work in the derived category whereas we work at the level of the enriched Grothendieck group; and perhaps more since our arguments naturally produce the elements ${}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ containing the arithmetic Gauss sums which we do not (yet) appear in the geometric context.

1.2. **Summary of techniques and relations to existing literature.** We now summarize the rough structure of our arguments and describe some applications and connections to the existing literature.

1.2.1. *Iwahori and spherical level analysis.* The results described above are at the spherical level, but nearly everything we have said requires an Iwahori (or affine Hecke algebra) level analysis. Recalling that I^- is the Iwahori subgroup attached to the Borel with unipotent radical U^- (the one opposed to U). Consider $\mathcal{W}_\psi(\widetilde{G}, I^-)$ the space of (genuine) left (U,ψ) and right I^- invariant compactly supported functions on $\widetilde{G}$ which carries an action of the Iwahori–Hecke algebra $\mathcal{H}(\widetilde{G}, I^-)$. The structure of $\mathcal{H}(\widetilde{G}, I^-)$ was described by G. Savin and P. J. McNamara in [90, 103, 104], and as in the linear case, $\mathcal{H}(\widetilde{G}, I^-)$ has two different presentations, of Iwahori–Matsumoto description or Bernstein type, the latter of which identifies it with $\widetilde{H}_{\mathrm{aff}} := H_W \otimes \mathbb{C}[\widetilde{Y}]$. Now, as was observed in [95], the larger space $\mathbb{C}[Y]$ carries the action of certain metaplectic Demazure–Lusztig operators $\widetilde{\mathbf{T}}_w$ for $w \in W$, as well as the translation action of $\mathbb{C}[\widetilde{Y}]$ Hence it can be equipped with a $\mathcal{H}(\widetilde{G}, I^-)$-action, and we have the following result.

Proposition (see Prop. 6.1.6 and 6.2.1)**.** *Restricted to the big cell, the averaging map,*

$$\mathrm{Av}^{\mathrm{gen}}_{U^-} : \mathcal{W}_\psi(\widetilde{G}, I^-) \xrightarrow{\simeq} \mathbb{C}[Y] \tag{1.13}$$

is an isomorphism of vector spaces intertwining the actions of $\mathcal{H}(\widetilde{G}, I^-)$ *on* $\mathcal{W}_\psi(\widetilde{G}, I^-)$.

Our proof uses p-adic techniques from [95]. It gives a natural p-adic basis $\{\widetilde{\mathcal{Y}}_{\check{\lambda}}\}_{\check{\lambda}\in Y}$ of $\mathcal{W}_\psi(\widetilde{G},I^-)$ defined by $\mathrm{Av}_{U^-}^{\mathrm{gen}}(\widetilde{\mathcal{Y}}_{\check{\lambda}}) = Y_{\check{\lambda}}$ for $\check{\lambda} \in Y$ as well as an explicit description of the action of $\mathcal{H}(\widetilde{G},I^-)$ in this basis.

As mentioned before, in the linear case, $\mathcal{W}_\psi(G,I^-)$ reduces to the anti-spherical module for the corresponding affine Hecke algebra. A suitable replacement of this result in the metaplectic world is obtained by taking a cue from the work of (Lascoux)–Leclerc–Thibon [77] who find

$$\mathbb{Z}_\tau[Y] = \oplus_{\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{-,n}} \varepsilon^-_{J_\eta}\widetilde{H}_{\mathrm{aff}} \tag{1.14}$$

where $\overline{\mathcal{A}}^{\bullet}_{-,n}$ is the upper-closure of a certain alcove defined by an affine root system connected to $\widetilde{Y}$ and the $\varepsilon^-_{J_\eta}$ are certain (parabolic) anti-symmetrizers in $\widetilde{H}_{\mathrm{aff}}$, see (3.39) and (3.113). Observing that the action on $\mathbb{Z}_\tau[Y]$ by $\widetilde{H}_{\mathrm{aff}}$ looks 'essentially' like that of $\mathcal{H}(\widetilde{G},I^-)$ on $\mathcal{W}_\psi(\widetilde{G},I^-)$ [13], (1.14) leads us to a decomposition

$$\mathcal{W}_\psi(\widetilde{G},I^-) = \oplus_{\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{-,n}} \mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta}), \tag{1.15}$$

where each $\mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta})$ is a $\mathcal{H}(\widetilde{G},I^-)$-submodule generated by the basis vector $\widetilde{\mathcal{Y}}_{\check{\eta}}$ described above (this follows combining the above with Proposition 4.2.2). The above picture suggests other natural bases of $\mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta})$ mentioned above [14]:

- Starting from $\widetilde{\mathcal{Y}}_{\check{\eta}}$ and using a procedure as in [77, §5.1], we obtain the recaled basis, each term differing by a certain product of Gauss sums, see (4.40)
- Importing the natural Kazhdan–Lusztig involutions coming from the anti-spherical module $\varepsilon^-_{J_\eta}\widetilde{H}_{\mathrm{aff}}$, Leclerc–Thibon consructed their canonical bases. We proceed analogously.

The results above descend to the spherical level, and on the quantum group side the decomposition obtained in related to the weak linkage principle for blocks.

1.2.2. *Quantum connection.* To identify $K_0^R\left(\mathrm{Rep}\left(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\right)\right)$ and $\mathcal{W}_\psi(\widetilde{G},K)$ as modules over $K_0\left(\mathrm{Rep}\left(\check{\mathbf{G}}_\ell\right)\right)$ and $\mathcal{H}(\widetilde{G},K)$, respectively (recall that $K_0\left(\mathrm{Rep}\left(\check{\mathbf{G}}_\ell\right)\right) \cong \mathcal{H}(\widetilde{G},K)$), we first observe that both modules are free of finite type and of the same rank: for $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G})))$ this follows from the Steinberg–Lusztig theorem [84, §35.4], and for $\mathcal{W}_\psi(\widetilde{G},K)$ this follows from Corollary 1.1.4 (1). These results allow us to identify the canonical basis $\widetilde{\mathcal{L}}_{\check{\lambda}}$ with the basis given by irreducibles in $K_0^R\left(\mathrm{Rep}\left(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\right)\right)$.

[13] More precisely, one works in a space with formal parameters corresponding to Gauss sums, namely $\mathbb{Z}_{\mathfrak{g},\tau}[Y]$, and equips it with an action of $\widetilde{H}_{\mathrm{aff}}$ now using the 'generic' Demazure–Lusztig operators of §4.1 constructed using the Chinta–Gunnells action. This space has a decomposition similar to (1.14), see Proposition (4.2.2), and its Hecke action specializes in two ways, in one way recovering the $\widetilde{H}_{\mathrm{aff}}$ action on $\mathbb{Z}_\tau[Y]$ and in another recovering the $\mathcal{H}(\widetilde{G},I^-)$ action on $\mathcal{W}_\psi(\widetilde{G},I^-)$.

[14] We warn the reader that in the main body of this paper, we mostly consider such bases at the spherical level, though the same methods also apply to at the Iwahori level.

To identify the p-adic basis $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ with some natural class of modules in $\operatorname{Rep}\big(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\big)$, we deduce from our combinatorial analysis that the transition coefficients from the $\widetilde{\mathcal{L}}_{\check{\lambda}}$ basis to the $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ basis are certain (parabolic, singular) affine Kazhdan-Lusztig polynomials. Since the costandard modules and the irreducibles in $K_0^R(\operatorname{Rep}(\dot{\mathbf{U}}_\zeta(\mathbf{G})))$ are known to be in analogous relation[15], we may identify the $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ basis with the costandard basis in $\operatorname{Rep}\big(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})\big)$. Finally, to identify the $\widetilde{\mathscr{T}_{\check{\lambda}}}$ basis with the tilting basis on the quantum side, we compute the relation between $\widetilde{\mathscr{T}_{\check{\lambda}}}$ and $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ and use a (conjectural) connection[16] between indecomposable tiltings and costandards for the quantum group at a root of unity.

1.2.3. *Local Shimura correspondence and other applications.* The Iwahori analysis presented above allows us to formulate a Gelfand–Graev version of Savin's local Shimura correspondence; namely that the $\mathcal{H}(\widetilde{G},I^-)$-submodule $\mathcal{W}_\psi(\widetilde{G},I^-)(-\check{\rho})$ and the $\mathcal{H}(\widetilde{G},K)$-submodule $\mathcal{W}_\psi(\widetilde{G},K)(-\check{\rho})$ behave essentially like their counterparts in the linear (or non-metaplectic) setting for the group $\mathbf{G}_{(\mathsf{Q},n)}$ (see Propositions 8.2.3 and 8.2.4). We use this to prove in Proposition 8.2.4 a 'linear' *geometric* Casselman–Shalika formula in the metaplectic world (see [50] for a related *asymptotic* Casselman–Shalika formula).

Proposition. *Let $\widetilde{\rho}^\vee$ be the analogue of $\check{\rho}$ for $\mathbf{G}_{(\mathsf{Q},n)}$. For $\check{\mu}\in\widetilde{Y}_+$, $\widetilde{\mathfrak{J}}_{\widetilde{\rho}^\vee-\check{\rho}}\star\widetilde{c}_{\check{\mu}}=\widetilde{\mathfrak{J}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}$.*

A corollary of this result is the fact that for coweights of the form $\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}$, the p-adic elements $\widetilde{\mathfrak{J}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}$ and the canonical elements $\widetilde{\mathcal{L}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}$ and $\widetilde{\mathscr{T}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}$ are equal. This result has a nice interpretation on the quantum side: the irreducible, indecomposable tilting, and (co)standard modules with (Steinberg) weights $\widetilde{\rho}-\rho+\mu$ for μ in the dominant l-dilated weight lattice $X_{l,+}$ are equal (see [88, Corollary 6.8], [6]).

Other applications in §8.3 include relations between the transition coefficients from the basis $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ to the basis $\widetilde{\mathcal{L}}_{\check{\lambda}}$ and the strong linkage principle from the theory of quantum groups ([8, 9]), a $\check{\mu}$-large asymptotic version of the geometric Casselman-Shalika formula (not to be confused with the formula (1.3) discussed earlier where $\check{\mu}$ small compared to $\check{\lambda}$). We also add some speculations about the relation of our work to that of Frenkel–Hernandez [45] and McGerty [88].

1.2.4. *The combinatorics of structure coefficients.* As we hope we have made clear, the works of Lascoux–Leclerc–Thibon, especially [77], played a central role in this paper. We found the extensions of this work in the papers of Grojnowski–Haiman [53] and Lanini–Ram(–Sobaje) [74, 75] to be especially helpful, and our proof of the tensor product theorems follows the elegant argument in [74] using the Littelmann path model. We note here that the second tensor product theorem

[15] To see this, we use results of [71] which uses the Kazhdan–Lusztig equivalence between quantum groups and affine Lie algebras.

[16] See the right hand side of (1.9).

we prove in Proposition 3.9.1 (and which is inspired by Andersen's tensor product theorem for tilting modules [6]) seems to be new, though we emphasize that the method of proof is a simple modification of the argument in [74]. As a small repayment towards our debt to these works, let us perhaps mention that our Proposition 3.6.2 relating the Chinta–Gunnells action to the actions considered in [77].

As a consequence of our main result, combinatorial quantities of interest arise as structure coefficients appearing in the convolution action of $\mathcal{H}(\widetilde{G},K)$ on $\mathcal{W}_{\psi}(\widetilde{G},K)$. Let us focus on two examples.

(1) G. Chinta, independently and without knowledge of the quantum side, noticed that the twisted qLR coefficients $\mathfrak{p}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}})$ of Corollary 1.1.4 (3) exhibited some positivity properties[17]. From our work (see Theorem 6.3.2 part 1 and (3.133)), it follows that up to an explicit product of Gauss sums, these coefficients are equal to qLR coefficeints $\mathfrak{q}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}})$. Using the positivity of the q-LR coefficients by the (unpublished) work of Grojnowski–Haiman [53], one now has a path to understand the observation of Chinta. A. Aggarwal has also been able to relate Chinta's positivity observations to Grojnowski–Haiman's work, but now based on ideas from [17] and [2]. We also note here that there are no known combinatorial descriptions for the q-LR coefficients which make manifest their known positivity.

(2) The structure coefficients ${}^{\mathfrak{g}}A^{\check{\eta}}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_{\tau,\mathfrak{g}}$, defined so that we have $\widetilde{\mathcal{J}}_{\check{\mu}} \star \widetilde{h}_{\check{\lambda}} = \sum_{\check{\eta}} \mathfrak{p}({}^{\mathfrak{g}}A^{\check{\eta}}_{\check{\mu},\check{\lambda}})\widetilde{\mathcal{J}}_{\check{\eta}}$, do not seem to exhibit positivity properties. Precise combinatorial formulas for ${}^{\mathfrak{g}}A^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ seem hard to compute from the point of view of this paper since one needs to apply various *straightening* rules, a complicated recursive procedure. However, using recent techniques developed in the area of integrable models, by the work of [2, 17] and ideas of A. Aggarwal, in type A one can now compute them non-recursively (the first author will return to this in a future paper).

We might also add here that the noticeable disparity in combinatorial complexity between these two types of structure coefficients ${}^{\mathfrak{g}}A^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ and ${}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ reinforces our opinion that going between the 'geometric' and classical metaplectic Casselman–Shalika formula is quite non-trivial at the level of formulas, and so perhaps also subtle at the level of representation theory.

1.2.5. *Relation to works on integrable systems and quantum affine algebras.* As mentioned above, the program started in [17] which was aimed at studying metaplectic Whittaker functions using solvable lattice models [19, 20] has now led to connections to (super-symmetric) LLT polynomials in type A [21], metaplectic representations of the affine Hecke algebra [18], and a theory of vector-valued metaplectic Demazure–Lusztig operators [20]. Underpinning the connection to solvable lattice models is the fact, first observed in [17, Theorem 1.1], that certain

[17] We thank G. Chinta for sharing his observation with us; it seems to have arisen first for the double cover case where the Gauss sums are just $\pm\sqrt{q}$ and positivity is natural to formulate.

intertwining operators acting on Whittaker coinvariants of the principal series representations of a metaplectic cover of $\mathrm{GL}_r(\mathcal{F})$ can be identified with R-matrices for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}_n})$.

In future work of the first author, it will be shown that for $G = \mathrm{GL}_r(\mathcal{F})$, the space $\mathcal{W}_\psi(\widetilde{G}, I^-)$ and $\mathcal{W}_\psi(\widetilde{G}, K)$ can be equipped with an action of $U_q(\widehat{\mathfrak{gl}_n})$; and in fact $\mathcal{W}_\psi(\widetilde{G}, K)$ is isomorphic to the finite q-Fock space representation of $U_q(\widehat{\mathfrak{gl}_n})$. On the quantum side, one may (conjecturally) endow the graded Grothendieck group $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}})))$ with an $U_q(\widehat{\mathfrak{gl}_n})$ action coming from the theory of categorical actions developed by Chuang–Rouquier [33] (see also [99, §6] on how to explicitly construct this action in the $q = 1$ case). We can then extend the maps (1.10) and (1.11) to isomorphisms of $U_q(\widehat{\mathfrak{gl}_n})$ modules, which then recovers the identification of intertwiners with R-matrices from [17].

1.2.6. *Relation to the work of Sahi–Stokman–Venkateswaran.* The interesting recent work of Sahi–Stokman–Venkateswaran [100, 101] studies the metaplectic polynomial representation introduced in §4.2.1 and uses it to construct metaplectic generalizations of Macdonald polynomials. A highlight of [100, 101] is the extension of this space to a representation of the double affine Hecke algebra, which produces a natural basis depending on an additional parameter. This is used in the construction of 'metaplectic' Macdonald polynomials that are shown to specialize to values of metaplectic Iwahori–Whittaker functions. It would be interesting to better understand the relations between our work and theirs, in particular to understand the p-adic meaning of the DAHA structure. Moreover, our polynomials ${}^{\mathbb{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ may be used to construct $\mathbb{g}$-twisted LLT polynomials by using [53, eq. after (13)]; the expansion of SSV polynomials into $\mathbb{g}$-twisted LLT polynomials should lead to interesting combinatorial applications (*cf.* [53, 54] where the expansion of regular Macdonald polynomials into LLT is used to great effect).

1.2.7. *Relation to the work of Gurevich–Gao–Karasiewicz.* The recent work [49] approaches $\mathcal{W}_\psi(\widetilde{G}, I^-)$ by working at the level of pro-p Iwahori subgroups. Using this work, one can extract an alternate proof of Proposition 6.2.1. In the description of $\mathcal{W}_\psi(\widetilde{G}, I^-)$ obtained in [49], the $\widetilde{H}_{\mathrm{aff}}$-submodules corresponding to certain orbits $\widetilde{W}_{\mathrm{aff}}$ (termed 'splitting' in *op. cit.*) admit an explicit descriptions as anti-spherical modules. In many instances (*e.g.* under the *oasitic* condition, see [49, Cor. 6.4]), all the orbits are splitting and the description of $\mathcal{W}_\psi(\widetilde{G}, I^-)$ from [49] agrees with the present one. In our notation, it seems that the 'splitting' orbits correspond to $\mathrm{Stab}_{(\widetilde{W}_{\mathrm{aff}}, \bullet)}(\check{\eta})$ lying within the 'standard' finite Weyl group of $\widetilde{W}_{\mathrm{aff}}$.

We might also note the p-adic input to our proofs are based on [95], so we do not actually use any pro-p techniques here. Nonetheless, we hope that the pro-p methods and results are more than just a technical tool that can be circumvented, and may in fact be an essential feature of the theory.

1.2.8. *Relation to the work of Gaitsgory–Lurie and Campbell–Dhillon–Raskin.* There is by now a large and evolving body of work both inspired by and concerning the Gaitsgory–Lurie conjecture, now sometimes referred to as the quantum Langlands program in the literature. Within this fascinating web interweaving infinite-dimensional algebraic geometry and the representation theory of affine Lie algebras, we find the idea of relating quantum groups at roots of unity to metapletic covers, a cornerstone of this paper. Though we do not presently have the expertise to pinpoint all the points of contact between our work and the innovations of this program, let us try to comment briefly on the work of Campbell–Dhillon–Raskin [27] which provides a description of the (derived) category of certain twisted D-modules $\mathrm{Whit}_{\kappa}^{\mathrm{aff}}$ on an affine flag variety for $\mathbf{G}$ – this being a geometric analogue[18] of our $\mathcal{W}_{\psi}(\widetilde{G}, I^{-})$– in terms of the representations of the Langlands dual $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}$ of the affine Lie algebra $\widehat{\mathfrak{g}}_{\kappa}$ with level κ. As with us, a key step in [27] is to obtain a decomposition of $\mathrm{Whit}_{\kappa}^{\mathrm{aff}}$ into blocks stable under the action of certain Hecke categories $\mathcal{H}_{\lambda}$. These blocks are identified with corresponding blocks of a certain category of representations $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}$, but not directly. Indeed, first the blocks of $\mathrm{Whit}_{\kappa}^{\mathrm{aff}}$ are identified with blocks of a certain category of representations of $\widehat{\mathfrak{g}}_{\kappa}$; and second, these are identified with blocks for the dual $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}$ using Feibig's extension of Soergel's bimodule theory to Kac-Moody setting (see [40, Thm. 4.1]). This last step involves a combinatorial match of certain parameters (see [27, Theorem 3.5.6]) which required the authors of [27] to impose a certain condition on the levels allowed– the so-called 'κ-good' condition (see [27, §3.4.4]). [19]

The identification of blocks of $\mathrm{Whit}_{\kappa}^{\mathrm{aff}}$ with a corresponding block of representations of an affine Lie algebra (see [27, Thm. 3.3.3 and Cor. 3.4.8]) proceeds via the so-called affine Skyrabian equivalence (see [27, (3.17)] and references therein). As mentioned before, to make the connection with present work, it seems that one needs to upgrade this equivalence in some l-adic context in such a way that the Gauss-sums present in the Hecke action on the Whittaker side survive a categorical/geometric lift. As we understand, this (or perhaps something related) seems to be the subject of work in progress by Dhillon–Li–Yun–Zhu.

In our work, one obtains a description of the blocks on the Whittaker side directly in terms of anti-spherical modules (and their close relatives for orbits with non-trivial stabilizers) for an affine Hecke algebra. This requires what seems to be a different combinatorial analysis from what is contained in [27]. It is for this reason that we say that our constructions are not related by a simple 'de-categorifcation' process (again, ignoring the Gauss sum issues). We might also mention that our decomposition of $\mathcal{W}_{\psi}(\widetilde{G}, I^{-})$ into anti-spherical modules allows us to connect directly to the 'dual' quantum group, and, perhaps for this reason, the κ-good condition does not seem to be needed in our approach.

[18] In an l-adic context, one could make this analogy precise and connect to the case when our local field is of positive characteristic.

[19] We might note here something of a resemblance to the oasitic cover condition of [49, Def. 6.1] mentioned in the previous paragraph, soley based on the tabulations included in each of these papers.

Distancing ourselves somewhat from these differences, it does seem the approach from [27] and ours should admit a common refinement *à la* 'Riche–Williamson' [99], *i.e.*, $\mathrm{Whit}_\kappa^{\mathrm{aff}}$ should be acted upon by a certain version of a diagrammatic Hecke category/category of Soergel bi-modules which could rigidify $\mathrm{Whit}_\kappa^{\mathrm{aff}}$ into an appropriate tilting category of a quantum group at a root of unity. It remains to be understood what the action of reflection functors may correspond to on a p-adic level, but perhaps we can already mention that in type A, the categorical action of $\widehat{\mathfrak{gl}}_\ell$ that plays a central role in [99, Part 2] seems to correspond, on the p-adic level, to the identification of $\mathcal{W}_\psi(\widetilde{G}, K)$ as a finite quantum Fock space for $U_q(\widehat{\mathfrak{gl}_n})$ which was also mentioned in §1.2.5.

1.2.9. *Organization of the paper.* We refer the reader to the table of contents which hopefully makes clear our organizational scheme. The longest section of this work, Section 3, introduces both preliminaries, but also revisits the works of Leclerc–Thibon [77] in such a way that the applications to p-adic groups and the metaplectic polynomial representation follow quite easily. The key new results in Section 4 are the 'straightening' rules for the Chinta–Gunnells–Puskás operators which allow us to produce $\mathbb{g}$-twisted versions of the results in Section 3 and connect them to our p-adic results in §6.2. The main p-adic results in §6.3 make use of this connection. The proof of our main result in §8.1 follows immediately from what we established before. To conclude our paper, we present in §8.2 applications of our work to the 'local' Shimura correspondence and investigate natural combinatorial questions in §8.3.

For a table of frequently used notation, we refer to §2.4.

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2. Notations and Conventions

2.1. General Conventions.

2.1.1. Bold faced objects denote functors (of groups usually) and the corresponding roman letters will denote the field-valued points. For example, $\mathbf{G}$ will be a group-valued functor and G will denote $\mathbf{G}(F)$, where F is a field, assumed to be specified implicitly in our discussion.

2.1.2. For an abelian group H and (commutative, unital) ring R, we write $R[H]$ for the group algebra of H with coefficients in R and typically denote the elements in $R[H]$ as H_a, H_b etc. with $a, b \in H$ and with multiplication defined as $H_a H_b = H_{a+b}$.

2.1.3. *The Grothendieck group.* Let $\mathscr{C}$ be an abelian $\mathbb{C}$-linear monoidal category (not necessarily semi-simple) whose objects have finite length. The Grothendieck group of $\mathscr{C}$ shall be denoted by $K_0(\mathscr{C})$ and its complexification $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathscr{C})$ by $K_0^{\mathbb{C}}(\mathscr{C})$. To every object in $X \in \mathscr{C}$ we write its class as $[X] \in K_0(\mathscr{C})$.

2.1.4. We usually write F for a general field and $\mathcal{F}$ for a non-archimedean local field.

2.2. Notation for p-adic fields and Gauss sums.

2.2.1. *Non-archimedean local fields.* Let $\mathcal{F}$ be a non-archimedean local field with ring of integers $\mathscr{O}$ and valuation map $\mathsf{val} : \mathcal{F}^* \to \mathbb{Z}$. Let $\pi \in \mathscr{O}$ be a uniformizing element, $\kappa := \mathscr{O}/\pi\mathscr{O}$ be the residue field, $\omega : \mathscr{O} \to \kappa$ the natural surjection, and let us write q for the cardinality of κ. For $k \geq 0$, set $\mathscr{O}^*[k] = \{x \in \mathcal{F}^* \mid \mathsf{val}(x) = k\}$ and $\mathscr{O}(k) = \{x \in \mathcal{F}^* \mid \mathsf{val}(x) \geq k\}$ so that the units in $\mathscr{O}$ are $\mathscr{O}^* = \mathscr{O}^*[0]$, where $\mathscr{O}^*[k] = \mathscr{O}(k) \setminus \mathscr{O}(k+1)$.

2.2.2. *Hilbert symbols, assumptions on q and n.* For A an abelian group, a *bilinear Steinberg symbol* is a map $(\cdot,\cdot) : F^* \times F^* \to A$ such that $(\cdot,\cdot)$ is bimultiplicative, i.e., $(x, yz) = (x,y)(x,z)$ and $(xy, z) = (x,z)(y,z)$ and also $(x, 1-x) = 1$ if $x \neq 1$. Fix a positive integer n. In this paper, we focus exclusively on the case of (tame) Hilbert symbols. To define these, assume $q \equiv 1 \mod 2n$ and let $\mu_n \subset \mathcal{F}$ be the set of n-th roots of unity with $|\mu_n| = n$. The n-th order Hilbert symbol (see e.g. [105, §9.2, 9.3]) is a bilinear map $(\cdot,\cdot)_n : \mathcal{F}^* \times \mathcal{F}^* \to \mu_n$. As n is fixed throughout our paper, we often drop it from our notation. Note that $(\cdot,\cdot)$ is a bilinear Steinberg symbol (see [92, Chapter V, Proposition 3.2]) and is also unramified, *i.e.* $(x,y) = 1$ if $x, y \in \mathscr{O}^*$. To avoid certain sign issues, we assumed $q \equiv 1 \mod 2n$, since we now have $(-1,-1) = (-1,x) = 1$ for $x \in \mathcal{F}^*$, and also $(\pi,\pi) = 1$ and $(\pi, u) = \varpi(u)^{\frac{q-1}{n}}$ for $u \in \mathscr{O}^*$.

2.2.3. *Additive characters.* Let $\psi : \mathcal{F} \to \mathbb{C}^*$ be an additive character. For $a \in \mathbb{Z}$, if

$$\psi|_{\mathscr{O}(a)} = \mathrm{Id}|_{\mathscr{O}(a)} \text{ and } \psi\,|_{\mathscr{O}[a-1]} \text{ is non-trivial,}$$

we say that ψ has *conductor* a. We are chiefly interested in characters ψ of conductor 0[20], i.e. ψ is the identity on $\mathscr{O}$ and non-trivial on $\pi^{-1}\mathscr{O}$.

2.2.4. *Gauss sums.* For $\tau : \mathcal{F} \to \mathbb{C}^*$ an additive character and $\sigma : \mathcal{F}^* \to \mathbb{C}^*$ a multiplicative one, set $\mathbf{g}(\sigma, \tau) = \int_{\mathscr{O}^*} \sigma(u')\tau(u')du'$ where du' is the Haar measure on $\mathcal{F}$ giving $\mathscr{O}^*$ volume $q-1$. This is a *Gauss sum*. Fix ψ an additive character of conductor 0 and define for each integer k a multiplicative and additive character by using the Hilbert symbol $(\cdot,\cdot)_n$

$$\sigma(u) = (u,\pi)_n^{-k} \quad \text{and} \quad \tau(u) = \psi(-\pi^{-1}u), \ \text{ for } u \in \mathcal{F}^*, \tag{2.1}$$

respectively. Setting $\mathbf{g}_k := \mathbf{g}(\sigma, \psi)$, we note (see [92]) that

$$\mathbf{g}_k = \mathbf{g}_l \text{ if } n|k-l,\ \mathbf{g}_0 = -1, \text{ and if } k \neq 0 \mod n, \text{ then } \mathbf{g}_k\mathbf{g}_{-k} = q, \tag{2.2}$$

where for the last equality we must again assume that $q \equiv 1 \mod 2n$.

2.3. Generic ring $\mathbb{Z}_{\tau,\mathfrak{g}}$ and its quantum, and p-adic specializations.

2.3.1. *The generic ring* $\mathbb{Z}_{\tau,\mathfrak{g}}$. Fix a positive integer n, and let τ and t be a formal parameters related by $\tau^2 = t^{-1}$. Introduce a system of formal parameters $\mathfrak{g}_k$ for $k \in \mathbb{Z}$ satisfying the conditions

$$\mathfrak{g}_k = \mathfrak{g}_\ell \text{ if } k \equiv \ell \mod n,\ \mathfrak{g}_k\mathfrak{g}_{-k} = \tau^2 = t^{-1} \text{ for } k \neq 0, \text{ and } \mathfrak{g}_0 = -1. \tag{2.3}$$

A central role in this work will be played by the rings

$$\mathbb{Z}_{\tau,\mathfrak{g}} := \mathbb{Z}[\tau^{\pm 1}, \{\mathfrak{g}_k\}_{k\in\mathbb{Z}}]/\sim \quad \text{and} \quad \mathbb{C}_{\tau,\mathfrak{g}} := \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}_{\tau,\mathfrak{g}} \tag{2.4}$$

obtained by formally adjoining τ, τ^{-1} and the $\mathfrak{g}_k$ subject to the above relations. Note that

$$\mathbb{Z}_{\tau,\mathfrak{g}} \simeq \mathbb{Z}[\tau^{\pm 1}, \mathfrak{g}_1^{\pm 1}, \dots, \mathfrak{g}_{n-1}^{\pm 1}] \quad \text{and} \quad \mathbb{C}_{\tau,\mathfrak{g}} \simeq \mathbb{C}[\tau^{\pm 1}, \mathfrak{g}_1^{\pm 1}, \dots, \mathfrak{g}_{n-1}^{\pm 1}]. \tag{2.5}$$

2.3.2. *The quantum specialization* $\mathfrak{q}$. Let τ, t be indeterminates satisfying the condition $\tau^2 = t^{-1}$ and introduce the ring $\mathbb{Z}_\tau := \mathbb{Z}[\tau, \tau^{-1}]$ of Laurent polynomials in τ. It carries an involution sending $\tau \mapsto \tau^{-1}$. The map $\mathfrak{q} : \mathbb{Z}_{\tau,\mathfrak{g}} \to \mathbb{Z}_\tau$ which sends

$$\tau \mapsto \tau, \quad \mathfrak{g}_0 \mapsto -1, \quad \mathfrak{g}_i \mapsto \tau, \quad \text{for } i \in \{1, \dots, n-1\} \tag{2.6}$$

will be called the *quantum specialization*.

[20]Note that in [29, 49], the conductor of the additive character is taken to be -1; this introduces a "ρ"-shift when comparing formulas with these sources. On the other hand, our choice of conductor is in alignment with the classical work of Casselman–Shalika [28] and the literature on its geometric version [42, 44].

2.3.3. *The p-adic specialization* $\mathfrak{p}$. Suppose we are working over a p-adic field (i.e. non-archimedean local field) $\mathcal{F}$ of residue cardinality q (a power of a prime) which is also equipped with a non-degenerate additive character ψ (see §2.2.3 for more details). For each positive integer n, one can then define Gauss sums $\mathbf{g}_k \in \mathbb{C}$ for each $k \in \mathbb{Z}$ as in §2.2.4. These elements satisfy the same properties as (2.3) and so we can consider what we call the *p-adic specialization* $\mathfrak{p} : \mathbb{C}_{\tau,\mathbb{g}} \to \mathbb{C}$ defined by sending

$$t = \tau^{-2} \mapsto q^{-1}, \mathbb{g}_i \mapsto \mathbf{g}_i \text{ for } i \in \mathbb{Z}. \tag{2.7}$$

2.4. **Frequently used Notation.** We collect here some frequently used notation for easy reference.

Notation	Meaning	Location
$\mathcal{F}$, $\mathscr{O}$, q, π	Non-archimedean local field, ring of integers, residue char., uniformizer	§2.2.1
n, μ_n	degree of metaplectic cover, n-th roots of unity	§2.2.2
$(\cdot,\cdot) = (\cdot,\cdot)_n$	Hilbert n-symbol	§2.2.2
ψ	Additive character of $\mathcal{F}$ or unipotent group	§2.2.3, §6.1.1
τ, t	parameters for Hecke algebra	§3.3.1 and §3.3.2
$\mathbb{Z}_\tau$, $\mathbb{Z}_\tau^+$, $\mathbb{Z}_\tau^-$	$\mathbb{Z}[\tau,\tau^{-1}]$, $\tau\mathbb{Z}[\tau]$, $\tau^{-1}\mathbb{Z}[\tau^{-1}]$	(3.28)
$\mathbb{g}_k$ and $\mathbf{g}_i$	formal and p-adic Gauss sum parameters	§2.3.1 and §2.2.4
$\mathbb{Z}_{\tau,\mathbb{g}}$	generic parameter ring	(2.4)
$\mathfrak{p}$, $\mathfrak{q}$	p-adic and quantum specialization maps	§2.3.3 and 2.3.2
W, W_{aff}	finite and affine Weyl groups	§3.4
$\bullet$	dot action of (affine) Weyl group	§3.4.6
$\overline{\mathcal{A}}^{\bullet}_{-,n}$	upper closure of twisted negative alcove	(3.74)
$\square_{(\mathsf{Q},n)}, \square_l$	restricted weights	§3.4.9, (7.17)
H_{aff}, $\widetilde{H}_{\mathrm{aff}}$	affine Hecke algebra, twisted affine Hecke algebra	§3.5 and §3.6
$Y_{\check{\beta}}$	Bernstein elements of affine Hecke algebra (or basis element of $\mathbb{V}$, see below)	§3.5.1
$o_{t,u} = o^+_{t,u}$ and $o^-_{t,u}$	singular, parabolic affine Kazhdan-Lusztig polynomials	§3.3.7
ε	spherical symmetrizer	(3.86)
H_{sph}, $\widetilde{H}_{\mathrm{sph}}$	spherical and twisted spherical Hecke algebra	§3.5.3, start of §3.6
$\widetilde{V}$ and $\widetilde{V}_{\mathrm{sph}}$	quantum polynomial and spherical representations	§3.6.5 and §3.6.7
$\mathsf{v}_{\check{\lambda}}$ and $[\mathsf{v}_{\check{\mu}}]$	basis of $\widetilde{V}$ and $\widetilde{V}_{\mathrm{sph}}$	§3.6.5 and §3.6.7
$[\mathsf{G}_{\check{\lambda}}], [\mathsf{G}^-_{\check{\lambda}}]$	canonical bases of $\widetilde{V}_{\mathrm{sph}}$	§3.7.3
$\mathbb{V}$ and $\mathbb{V}_{\mathrm{sph}}$	generic polynomial and spherical representations	§4.2.2 and §4.3.3
$Y_{\check{\mu}}$ and $\mathbb{v}_{\check{\mu}}$	bases of $\mathbb{V}$	§4.2.2 and §4.2.3
$\kappa(\check{\mu})$	product of Gauss sums relating $Y_{\check{\mu}}$ and $\mathbb{v}_{\check{\mu}}$	(4.40)
$[\mathbb{G}_{\check{\lambda}}], [\mathbb{G}^-_{\check{\lambda}}]$	canonical bases of $\mathbb{V}_{\mathrm{sph}}$	§4.3.3
$[G_{\check{\lambda}}], [G^-_{\check{\lambda}}]$	($\mathbb{g}$-twisted) canonical bases of $\mathbb{V}_{\mathrm{sph}}$	§4.3.3
I^- and K	Iwahori and maximal compact of $\widetilde{G}$	§5.2.2, §5.2.6
$\mathcal{H}(\widetilde{G},I^-)$, $\mathcal{H}(\widetilde{G},K)$	Iwahori and spherical Hecke algebras	§5.3.2 and §5.3.6
$\mathcal{W}_\psi(\widetilde{G},I^-), \mathcal{W}_\psi(\widetilde{G},K)$	Iwahori and spherical Whittaker spaces for $\widetilde{G}$	§6.1.4
$\widetilde{\mathcal{Y}}_{\check{\lambda}}$	basis of the Whittaker space $\mathcal{W}_\psi(\widetilde{G},I^-)$	§6.2.1

Part 1. **Combinatorial models**

3. AFFINE HECKE ALGEBRAS AND CANONICAL BASES

In this section, we collect various algebro-combinatorial preliminaries needed in this paper. In §3.6.2 we interpret certain straightening rules in terms of a quantum specialization of metaplectic Demazure–Lusztig operators and use it in §3.8.2 to express Littlewood–Richardson polynomials in terms of the metaplectic Casselman–Shalika formula. These results and Proposition 3.9.1 (2) seem to be new.

3.1. **Cartan datum and associated structures.**

3.1.1. *Cartan datum.* Following [84, §1.1], we define a *Cartan datum* to be a pair $(I,\cdot)$, where I is a finite set and $\cdot$ is a $\mathbb{Z}$-valued symmetric bilinear form on the free module with basis I, namely $\mathbb{Z}[I]$, satisfying

$$i\cdot i \in \{2,4,6,\dots\} \text{ for } i\in I; \text{ and } \quad 2\frac{i\cdot j}{i\cdot i} \in \{0,-1,-2,\dots\} \text{ for } i,j\in I, i\neq j. \tag{3.1}$$

Given a Cartan datum $(I,\cdot)$, define its *generalized Cartan matrix* (gcm) as in [58, p.1]:

$$\mathsf{A} = \mathsf{A}(I,\cdot) = (a_{ij})_{i,j\in I} \text{ where } a_{ij} = 2\frac{i\cdot j}{i\cdot i}. \tag{3.2}$$

We assume throughout that the Cartan datum $(I,\cdot)$ is irreducible (see [84, §2.1.3]) which ensures that A is irreducible in the sense of [58, p.2]. We deal in this paper exclusively with such Cartan datum $(I,\cdot)$ of *finite type*, i.e. the symmetric matrix $(i\cdot j)_{i,j\in I}$ is positive definite, or of *affine type*, when the corresponding symmetric matrix has nullity one.

3.1.2. *Dual Cartan datum.* For $(I,\cdot)$ a Cartan datum, let m_I to be the smallest positive integer such that $\frac{m_I}{2(i\cdot i)} \in \mathbb{Z}$. Following [117, p.92], define the dual Cartan datum $(I,\check{\cdot})$ via:

$$i\,\check{\cdot}\, j := m_I \frac{i\cdot j}{(i\cdot i)(j\cdot j)} \text{ for } i,j\in I. \tag{3.3}$$

Writing $\check{\mathsf{A}} := \mathsf{A}(I,\check{\cdot})$ for the corresponding Cartan matrix, one finds that $\check{\mathsf{A}} = {}^t\mathsf{A}$.

3.1.3. *Minimal Cartan datum attached to a Cartan matrix.* Let $\mathsf{A} = (a_{ij})$ be an irreducible Cartan matrix of finite type and of size $k\times k$. Write $I := \{1,\dots,k\}$ and let D be a diagonal matrix such that $D\mathsf{A} = S$ is a symmetric matrix. Note that such a D always exists; it is unique up to rescaling the matrix by a constant, and so we can choose D to have positive, integral values. Denote by $D_{\min} = \mathrm{diag}(d_1,\dots,d_k)$, a symmetrizing matrix with the property that every other choice of D is a positive, integral multiple of $D_{\min}$. Using $D_{\min}$, we construct the *minimal Cartan datum* $(I,\cdot)$ *attached to a given Cartan matrix* A by setting $i\cdot j := d_i a_{ij}$.

3.1.4. *Untwisted affinizations and computing the symmetrization of* A. Denote the (untwisted) affinization of A by A_{aff}; it is defined as in [58, p.100, (7.4.3)]. The matrix A_{aff} is of size $(k+1)\times(k+1)$ and our convention is to identify the square matrix formed from the last k-rows and columns of A_{aff} with A. As such, we also write in this case

$$I_{\text{aff}} := I \sqcup \{0\}, \text{ where } I = \{1,\ldots,k\}. \tag{3.4}$$

The matrix A_{aff} has a kernel containing a unique vector with strictly positive, relatively prime entries $\delta(\mathsf{A}_{\text{aff}}) = (m_0(\mathsf{A}),\ldots,m_k(\mathsf{A}))$. For these untwisted affinizations, it turns out that $m_0(\mathsf{A}) = 1$ for all irreducible A. The transpose ${}^t\mathsf{A}$ is again a Cartan matrix of finite type and we denote by $\check{m}_i(\mathsf{A}) := m_i({}^t\mathsf{A})$. With these definitions, the matrix

$$D := \operatorname{diag}(\check{m}_0(\mathsf{A})/m_1(\mathsf{A}),\ldots,\check{m}_k(\mathsf{A})/m_k(\mathsf{A})) \tag{3.5}$$

symmetrizes A, i.e. DA is symmetric. However, D only has positive and rational entries in general, and multiplies this matrix by an appropriate integral multiple to obtain $D_{\min}$.

3.1.5. *Braid and Weyl groups.* Given a Cartan datum $(I,\cdot)$ with associated Cartan matrix A we define integers h_{ij} for $i,j \in I$ according the rules as in [84, §2.1.1], i.e. $\cos^2\frac{\pi}{h_{ij}} = \frac{a_{ij}a_{ji}}{4}$. The *braid group* $\mathcal{B}(I,\cdot)$, is the free group on symbols $s_i\,(i\in I)$ with relations

$$\underbrace{s_is_js_i\cdots}_{h_{ij}} = \underbrace{s_js_is_j\cdots}_{h_{ij}} \text{ for } i\neq j, \tag{3.6}$$

where both sides have $h_{ij} < \infty$ terms. If we further impose the relation $s_i^2 = 1$ for all $i\in I$ we obtain the *Weyl group* $\mathcal{W}(I,\cdot)$. As both $\mathcal{B}(I,\cdot)$ and $\mathcal{W}(I,\cdot)$ only depend on the associated gcm A, we often just write these as $\mathcal{B}(\mathsf{A})$ or $\mathcal{W}(\mathsf{A})$. In the case that A_{aff} is the untwisted affinitization of A, we often write

$$W_{\text{aff}} := \mathcal{W}(\mathsf{A}_{\text{aff}}) \text{ and } W := \mathcal{W}(\mathsf{A}). \tag{3.7}$$

3.1.6. *Coxeter groups and Bruhat order.* The pair $(\mathcal{W}(\mathsf{A}),S)$ where $S=\{s_i\}_{i\in I}$ described in the previous paragraph forms a Coxeter system and $\mathcal{W} := \mathcal{W}(\mathsf{A})$ is called a Coxeter group (see [14, Ch. IV, §1.3, Def. 3, p. 4]). Note that every element $s\in S$ satisfies $s^2=1$ (i.e. $S^{-1}=S$) so words in S are just products of elements from S. We refer to [14, Ch. IV, §1] for the definitions of reduced expressions, the length function $\ell : \mathcal{W}\to\mathbb{Z}$, etc. Let us write $\leq_S$, or just $\leq$ if the set S is not in question, for the Bruhat order on $\mathcal{W}(\mathsf{A})$ induced from the Coxeter structure (see [13, Ch. 2]). As usual, we write $x<y$ to mean that $x\leq y$ and $x\neq y$. If A is of finite type, there exists a unique element in $\mathcal{W}(\mathsf{A})$ which is maximal for the Bruhat order, which we write as w_0.

3.1.7. *Parabolic subgroups.* For $J \subset I$, if we define $\mathcal{W}_J \subset \mathcal{W}$ to be the subgroup generated by $s_j, j \in J$, then from [14, Ch. IV, §8, Thm. 2], one knows that $(\mathcal{W}_J, \{s_j\}_{j\in J})$ is itself Coxeter group. These are called *parabolic* subgroups. For example, if $\mathsf{A}_{\mathrm{aff}}$ is the affinization of A and $\mathcal{W}(\mathsf{A}_{\mathrm{aff}}) := \mathcal{W}(\mathsf{A}_{\mathrm{aff}})$, then we have $(\mathcal{W}_{\mathrm{aff}})_I = \mathcal{W}(\mathsf{A}) = W$ (where $I \subset I_{\mathrm{aff}}$ is an in (3.4)). We write w_0^J for the unique maximal length element of $\mathcal{W}_J$, whenever it exists. For example, w_0^J exists for any $J \subsetneq I_{\mathrm{aff}}$.

3.1.8. *Poincaré polynomials.* For q a formal variable, $J \subset I$ such that the parabolic subgroup $\mathcal{W}_J$ defined above is finite, we define the Poincare polynomial

$$\mathscr{P}_{\mathcal{W}_J}(\mathsf{q}) = \mathscr{P}_J(\mathsf{q}) = \sum_{w\in\mathcal{W}_J} \mathsf{q}^{\ell(w)}. \tag{3.8}$$

When $J = I$, we sometimes write $\mathscr{P}_{\mathcal{W}}(\mathsf{q})$ or even $\mathscr{P}(\mathsf{q})$.

3.1.9. *Cosets.* For any subset $J \subset I$, define the set $\mathcal{W}^J := \{w \in \mathcal{W} \mid \ell(ws_j) > \ell(w) \text{ for all } j \in J\}$, and note (see [13, Prop 2.4.4]) that every $w \in W$ has a unique factorization $w = w_1 w_2$ where $w_1 \in \mathcal{W}^J, w_2 \in \mathcal{W}_J$, and in this factorization $\ell(w) = \ell(w_1) + \ell(w_2)$. It follows from [37, Lemmas 3.1-3.2] that

$$\text{if } s \in S, \sigma \in \mathcal{W}^J, \text{ but } sw \notin \mathcal{W}^J, \text{ then } s\sigma = \sigma s_j \text{ for some } j \in J. \tag{3.9}$$

Similarly, consider the set ${}^J\mathcal{W} := \{w \in \mathcal{W} \mid \ell(s_j w) > \ell(w) \text{ for } j \in J\}$, which consists of minimal length representatives of the cosets $\mathcal{W}_J \setminus \mathcal{W}$. Finally, suppose now $J, K \subset I$ are given. Then it is known that the set

$${}^K\mathcal{W}^J := \{w \in \mathcal{W} \mid s_k w > w, ws_j > w \text{ for all } k \in K, j \in K\} \tag{3.10}$$

forms a set of minimal length representatives for the double cosets $\mathcal{W}_K \setminus \mathcal{W}/\mathcal{W}_J$. We say that such a double coset is *regular* if its stabilizer (or equivalently the stabilizer of any element in the double coset) under the action of $\mathcal{W}_K \times \mathcal{W}_J$ (acting on left and right, respectively) is trivial. Equivalently, an element $t \in {}^K\mathcal{W}^J$ is regular if $\mathcal{W}_K t \cap t\mathcal{W}_J = \emptyset$. The set of such regular double cosets will be denoted by $(\mathcal{W}_K \setminus \mathcal{W}/\mathcal{W}_J)_{\mathrm{reg}}$ and the set of minimal length regular representatives is denoted $\left({}^K\mathcal{W}^J\right)_{\mathrm{reg}}$.

3.2. **Root datum and their twists.** In this section $(I, \cdot)$ will be of finite type.

3.2.1. *Root datum.* By a *root datum* of type $(I, \cdot)$, we shall mean a quadruple $\mathfrak{D} = (Y, \{y_i\}_{i\in I}, X, \{x_i\}_{i\in I})$ where Y, X are free abelian groups of finite rank that are in duality under a pairing that we denote $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$, and where $\{y_i\}_{i\in I} \subset Y$ and $\{x_i\}_{i\in I} \subset X$ satisfy

$$\langle y_i, x_j \rangle = a_{ij} \text{ for } i, j \in I. \tag{3.11}$$

Here, as in §3.1.1, $\mathsf{A} = (a_{ij})$ is the associated gcm to $(I, \cdot)$. The root datum is said to be of *simply connected* type if Y is the free $\mathbb{Z}$-module with basis $\{y_j\}_{j\in I}$, and it is said to be of *adjoint type* if X is the free $\mathbb{Z}$-module with basis $\{x_i\}$.

If $\mathfrak{D} = (Y, \{y_i\}_{i\in I}, X, \{x_i\}_{i\in I})$ and $\mathfrak{D}' = (Y', \{y_i'\}_{i\in I}, X', \{x_i'\}_{i\in I})$ are two root datum for $(I, \cdot)$, we say that $\mathfrak{D}$ and $\mathfrak{D}'$ are *isomorphic root datum* and write $\mathfrak{D} \cong \mathfrak{D}'$

if there exist isomorphisms of abelian groups $Y \to Y'$ sending $y_i \mapsto y_i'$ and $X \to X'$ sending $x_i \mapsto x_i'$ compatible with the pairings $Y \times X \to \mathbb{Z}$ and $Y' \times X' \to \mathbb{Z}$.

3.2.2. *Dual root datum.* Recall the dual Cartan datum $(I, \check{\cdot})$ attached to $(I, \cdot)$ from §3.1.2. If $\mathfrak{D} = (Y, \{y_i\}_{i\in I}, X, \{x_i\}_{i\in I})$ is a root datum of Cartan type $(I, \cdot)$, we define its (Langlands) dual $(I, \check{\cdot}, \check{\mathfrak{D}})$ by setting $\check{\mathfrak{D}} := (X, \{x_i\}_{i\in I}, Y, \{y_i\}_{i\in I})$.

3.2.3. *Weyl group action.* Given a root datum $\mathfrak{D}$, let $\sigma_i : Y \to Y$ be the map defined by $\sigma_i(y) = y - \langle y, x_i\rangle y_i$ for $\check{\mu} \in Y$. We can similarly define an operation $\sigma_i : X \to X$. We note that the σ_i so defined satisfy the Coxeter presentation for the Weyl group. Hence $W(I, \cdot)$ acts on Y by sending $s_i \in W(I, \cdot)$ to σ_i for $i \in I$.

3.2.4. *Roots systems attached to* $\mathfrak{D}$. Fix $(I, \cdot, \mathfrak{D})$ as above. We refer to the set $\check{\Pi} := \{y_i\}_{i\in I}$ as the set of simple coroots, and $\check{\mathcal{R}} := W\check{\Pi}$ as the set of coroots of the root datum. Every $\check{a} \in \check{\mathcal{R}}$ can be written as a unique linear combination of the elements from $\check{\Pi}$ with all positive or all negative coefficients. In this way, we can define $\check{\mathcal{R}}_\pm$ as the set of positive and negative coroots. We also define the *coroot*, *root*, and *weight lattices* as

$$\check{\mathcal{Q}} := \bigoplus_{i\in I} \mathbb{Z} y_i, \quad \mathcal{Q} := \bigoplus_{i\in I} \mathbb{Z} x_i, \text{ and } \Lambda := \{x \in X \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle y_i, x\rangle \in \mathbb{Z} \text{ for } i \in I\}, \tag{3.12}$$

respectively. We define the positive root and coroot lattices by replacing $\mathbb{Z}$ with $\mathbb{Z}_{\geq 0}$ in the above, and also write $\Lambda_+ \subset \Lambda$ for the *dominant weights*, defined by the additional condition that $\langle y_i, x\rangle \geq 0$ for all $i \in I$. A *regular dominant weight* is one such that $\langle y_i, x\rangle > 0$ for all $i \in I$. These are denoted as $\Lambda_{+,\mathrm{reg}}$. We define the dominance order $\leq$ on Λ :

$$\mu \leq \lambda \text{ if } \lambda - \mu \in \mathcal{Q}_+ \text{ for } \lambda, \mu \in \Lambda. \tag{3.13}$$

In a dual manner, replacing the $\{x_i\}$ with $\{y_i\}$ in the above we can also define the simple roots Π, all roots $\mathcal{R}$, positive/negative roots $\mathcal{R}_\pm$, coweights $\Lambda^\vee$, dominant coweights $\Lambda^\vee_+$, and dominance order (again denoted $\leq$) on $\Lambda^\vee$. For each $a \in \mathcal{R}$, if $x_i \in \Pi$ is such that $wx_i = a$ then the element $\check{a} := wy_i \in \check{\mathcal{R}}$ is well-defined and independent of the choice of such y. The element $\check{a}$ is called the *coroot* attached to a. Let us also note here that since $(I, \cdot)$ is of finite-type, there is a unique root $\theta \in \mathcal{R}$ such that when we write

$$\theta := \sum_{i\in I} m_i x_i \text{ with } m_i \in \mathbb{Z}_{\geq 0}, \tag{3.14}$$

the m_i are maximized. It is known that all $m_i > 0$ and in fact $m_i = m_i(\mathsf{A})$ are the numbers introduced in §3.1.4. In a similar way we can define a highest coroot $\check{\theta}$ written as in (3.14) with m_i replaced by $\check{m}_i(\mathsf{A})$ and x_i replaced by y_i. We shall also introduce the elements $\rho, \check{\rho}$ in Λ and $\check{\Lambda}$ by the following definitions:

$$2\rho = \sum_{a\in\mathcal{R}} a \quad \text{and} \quad 2\check{\rho} = \sum_{\check{a}\in\check{\mathcal{R}}} \check{a}. \tag{3.15}$$

Note that this implies that $\langle y_i, \rho\rangle = 1$. A similar formula hold for $\check{\rho}$.

3.2.5. (Q,n)-*twists.* Fix $(I,\cdot,\mathfrak{D})$ be a root datum with associated Cartan matrix $\mathsf{A}=(a_{ij})$ as in §3.1.1. Writing $\mathfrak{D}:=(Y,\{y_i\},X,\{x_i\})$, a (Q,n)-*twist*[21] on $\mathfrak{D}$ is the data of an integer $n\geq 1$ and a $W=W(I,\cdot)$-invariant (for the action defined in §3.2.3) quadratic form Q on Y. Attached to Q, we define the symmetric, bilinear form $\mathsf{B}:Y\times Y\to\mathbb{Z}$, $\mathsf{B}(y_1,y_2):=\mathsf{Q}(y_1+y_2)-\mathsf{Q}(y_1)-\mathsf{Q}(y_2)$ for $y_1,y_2\in Y$. One verifies

$$\mathsf{B}(y_i,y)=\mathsf{Q}(y_i)\langle y,x_i\rangle \text{ for } y\in Y, i\in I \quad \text{ and hence } \quad \mathsf{Q}(y_j)/\mathsf{Q}(y_i)=a_{ji}/a_{ij}. \tag{3.16}$$

From [116, Proposition 3.10], there exists a unique W-invariant, quadratic form Q on $\check{\mathfrak{Q}}$ (thought of as a subset of Y) which takes the value 1 on all short coroots. We shall call such a structure *primitive* and note that every $\mathbb{Z}$-valued W-invariant form on $\check{\mathfrak{Q}}$ is an integer multiple of Q. Hence, simply-connected root datum admit primitive twists.

3.2.6. (Q,n)-*twisted root datum.* Let (Q,n) be a twist on $(I,\cdot,\mathfrak{D})$. Following see [117, Construction 1.3] and [84, §2.2.4], we may twist the Cartan datum $(I,\cdot)$ to obtain a new Cartan datum denoted $(I,\circ_{(\mathsf{Q},n)})=(I,\circ)$ and constructed as follows

$$i\circ j:=\frac{n^2}{n(y_i)\,n(y_j)}i\cdot j, \tag{3.17}$$

where for $i\in I$ we define $n(y_i)$ as the smallest positive integer satisfying

$$n(y_i)\,\mathsf{Q}(y_i)\equiv 0 \mod n. \tag{3.18}$$

The associated Cartan matrix is now $\widetilde{\mathsf{A}}:=\mathsf{A}_{(\mathsf{Q},n)}=\left(\frac{n(y_i)}{n(y_j)}a_{ij}\right)_{i,j\in I}$. Hence

$$B(I,\circ)\simeq B(I,\cdot) \quad \text{ and } \quad W(I,\circ)\simeq W(I,\cdot)\ . \tag{3.19}$$

If we now set

- $Y_{(\mathsf{Q},n)}:=\widetilde{Y}:=\{y\in Y \mid \mathsf{B}(y,y')\in n\mathbb{Z} \text{ for all } y'\in Y\}$,
- $y_{(\mathsf{Q},n),i}:=\widetilde{y_i}:=n(y_i)y_i$ for $i\in I$,
- $X_{(\mathsf{Q},n)}:=\widetilde{X}:=\{x\in X\otimes\mathbb{Q} \mid \langle y,x\rangle\in\mathbb{Z} \text{ for all } y\in\widetilde{Y}\}$,
- $x_{(\mathsf{Q},n),i}:=\widetilde{x_i}:=n(y_i)^{-1}x_i$ for $i\in I$,

one can verify, as in [84, §2.2.5], [90, §11], or [117, Construction 1.3], that

$$\mathfrak{D}_{(\mathsf{Q},n)}:=\widetilde{\mathfrak{D}}=(\widetilde{Y},\{\widetilde{y_i}\}_{i\in I},\widetilde{X},\{\widetilde{x_i}\}_{i\in I}) \tag{3.20}$$

is a root datum for $(I,\circ)=\left(I,\circ_{(\mathsf{Q},n)}\right)$. We adopt the convention that tildes will designate the corresponding notion for twisted root systems, but occasionally when we need to keep track of the precise twist under consideration, we revert to the more precise notations $\mathfrak{D}_{(\mathsf{Q},n)},Y_{(\mathsf{Q},n)}$, etc. So, for example $\widetilde{\mathcal{R}}$ (resp. $\widetilde{\mathcal{R}}^\vee$) will denote the set of roots (resp. coroots), etc. The quantities $\widetilde{\rho},\widetilde{\rho}^\vee$ are defined as in (3.15) using the twisted root systems.

[21] These are called metaplectic structures in [117], where they were first introduced

3.2.7. *A rank one example.* Consider the Cartan datum $I = \{a\}$ with $a \cdot a = 2$. We may associate to it the root datum $\mathfrak{D} = (Y, \check{a}, X, a)$, with $Y := \mathbb{Z}\check{a}$, $X := \frac{1}{2}\mathbb{Z}a$ and $\langle \check{a}, a\rangle = 2$. This is the root datum which specifies, over $\mathbb{C}$, the simply connected group $\mathbf{PGL}_2(\mathbb{C})$. The Langlands dual $\check{\mathfrak{D}} = \left(\frac{1}{2}\mathbb{Z}a, a, \mathbb{Z}\check{a}, \check{a}\right)$, which corresponds to the group $\mathbf{SL}_2(\mathbb{C})$, is not of simply-connected type. The primitive (Q, n) twist of $\mathfrak{D}$ is determined by specifying $\mathsf{Q}(\check{a}) = 1$. Then $n(\check{a}) = n$ and the associated bilinear form B on Y is specified by $\mathsf{B}(k\check{a}, k'\check{a}) = 2kk'$ for $k, k' \in \mathbb{Z}$. One may then check that

$$\mathfrak{D}_{(\mathsf{Q},n)} := \begin{cases} \left(n\mathbb{Z}\check{a}, n\check{a}, \frac{1}{2n}\mathbb{Z}a, \frac{1}{n}a\right) \simeq \mathfrak{D} \text{ if } n \text{ is odd,} \\ \left(\frac{n}{2}\mathbb{Z}\check{a}, n\check{a}, \frac{1}{n}\mathbb{Z}a, \frac{1}{n}a\right) \simeq \check{\mathfrak{D}} \text{ if } n \text{ is even.} \end{cases} \tag{3.21}$$

Remark. *In general, the primitive twist of a finite Cartan datum will be isomorphic to itself (if n is odd), or to its Langlands dual (if n is even). This is not always true for affine type (cf* [97, Table 2.3.2]*).*

3.2.8. *Example: l-twisted Cartan and root datum.* Fix a positive integer l and a root datum $(I, \cdot, \mathfrak{D})$ with $\mathfrak{D} = (Y, \{y_i\}, X, \{x_i\})$. In [84, 2.2.4-2.2.5], Lusztig has defined the notion of an l-twisted root datum. To introduce it, let l_i be the smallest positive integers such that $l_i\frac{i \cdot i}{2} \in l\mathbb{Z}$. Then one defines $(I, \circ_l)$ as the Cartan datum with $i \circ_l j = l_i l_j (i \cdot j)$ and attaches to it a root datum $\mathfrak{D}_l = (Y_l, \{y_{l,i}\}, X_l, \{x_{l,i}\})$ defined as follows:

$$X_l := \{\zeta \in X \mid \langle y_i, \zeta\rangle \in l_i\mathbb{Z}\}, Y_l := \mathrm{Hom}(X_l, \mathbb{Z}), x_{l,i} := l_i x_i, \quad \text{and } y_{l,i} := l_i^{-1} y_i. \tag{3.22}$$

This root datum $(I, \circ_l, \mathfrak{D}_l)$ can be subsumed into the theory of twists from §3.2.6.

We start with $\mathsf{A} = (a_{ij})$ an irreducible Cartan matrix and fix the minimal Cartan datum $(I, \cdot)$ as in §3.1.3 so that $i \cdot j = d_i a_{ij}$ with positive integers d_i as in §3.1.4. Hence

$$\tfrac{i \cdot i}{2} = \tfrac{d_i a_{ii}}{2} = d_i, \quad \text{and so defining} \quad l_i := l/(l, d_i) \text{ for } i \in I, \tag{3.23}$$

we find that l_i are the smallest positive integers such that $l_i\frac{i \cdot i}{2} = l_i d_i \in l\mathbb{Z}$. Fix $\mathfrak{D} = (Y, \{y_i\}, X, \{x_i\})$ a root datum of adjoint type, so that $\check{\mathfrak{D}}$ is of simply-connected type and hence admits a primitive twist $\check{\mathsf{Q}} : X \times X \to \mathbb{Z}$ in the sense of §3.2.5, i.e. $\check{\mathsf{Q}}(x_j) = 1$ for all short roots. Note that from (3.16) applied to the Cartan datum $(I, \check{\cdot})$ with Cartan matrix $\check{\mathsf{A}} = {}^t\mathsf{A}$, we have

$$\frac{\check{\mathsf{Q}}(x_j)}{\check{\mathsf{Q}}(x_i)} = \frac{a_{ij}}{a_{ji}} = \frac{i \cdot j}{d_i}\frac{d_j}{j \cdot i} = \frac{d_j}{d_i}. \tag{3.24}$$

If x_i is any long root attached to a short root x_j, then $\check{\mathsf{Q}}(x_j) = d_j/d_i = d_j$, since $d_i = 1$ for x_i short. In other words, if $\check{\mathsf{Q}}$ is the primitive twist on $\check{\mathfrak{D}}$, we have

$$\check{\mathsf{Q}}(x_i) = d_i \text{ for all } i \in I \quad \text{and} \quad n(x_i) = l_i \tag{3.25}$$

by using (3.18) which states that $n(x_i)$ is the smallest positive integer such that $\check{\mathsf{Q}}(x_i) \equiv 0 \mod n$.

Claim. *Let $(I,\cdot)$ be the minimal Cartan datum attached to a Cartan matrix A and $(I,\cdot,\mathfrak{D})$ a root datum of adjoint type. Writing $\check{\mathsf{Q}}$ is the primitive twist on $\check{\mathfrak{D}}$, we have an isomorphism of root datum*

$$(I,\circ_l,\mathfrak{D}_l) \simeq \left(I,((\check{\cdot})_{(\check{\mathsf{Q}},l)})^\vee, \left((\check{\mathfrak{D}})_{(\check{\mathsf{Q}},l)}\right)^\vee\right). \tag{3.26}$$

We apply this claim to $\check{\mathfrak{D}}$, which is assumed to be of adjoint type, in Part III. In this case, writing $\check{\mathfrak{D}}_l$ for the corresponding l-twist and Q for the primitive twist on the simply-connected $\mathfrak{D}$, we have

$$(I,\circ_l,\check{\mathfrak{D}}_l) \simeq (I,((\check{\cdot})_{(\mathsf{Q},l)})^\vee, (\mathfrak{D}_{(\mathsf{Q},l)})^\vee). \tag{3.27}$$

Remark. *For k a positive integer, let $(k\check{\mathsf{Q}},l)$ be a multiple of the primitive twist $(\check{\mathsf{Q}},l)$ as in the Claim. Then $\left((\check{\mathfrak{D}})_{(k\check{\mathsf{Q}},l)}\right)^\vee$ is isomorphic to $\mathfrak{D}_{l'}$, where $l' = \frac{l}{\gcd(l,k)}$.*

3.2.9. *A $\mathbf{GL}_r$-example.* Let us now give an example of a root system that does not have a unique primitive twist. Define the Cartan datum $(I,\cdot,\mathfrak{D})$, where $I := \{1,\dots,r-1\}$,

$$i\cdot j = \begin{cases} 2 & \text{if } i=j, \\ -1 & \text{if } i\neq j, \end{cases}$$

and $\mathfrak{D} = (Y,\{\check{a}_i\},X,\{\check{a}_i\})$ is the root datum constructed as follows: Y is the free $\mathbb{Z}$-module with basis $\{\check{e}_i, 1\le i\le r\}$, $\check{a}_i := \check{e}_i - \check{e}_{i+1}$, X is the free $\mathbb{Z}$-module with basis $\{e_i, 1\le i\le r\}$, $a_i := e_i - e_{i+1}$, and the pairing between Y and X is $\langle \check{e}_i, e_j\rangle := \delta_{ij}$. This is the root datum for the group $\mathbf{GL}_r$.

A (Q,n) twist on $\mathfrak{D}$ consists of a positive integer n and a quadratic form Q on Y. Such a twist is actually determined by two integers $\mathbf{p} := \frac{1}{2}\mathsf{B}(\check{e}_1,\check{e}_1) = \mathsf{Q}(\check{e}_1)$ and $\mathbf{q} := \mathsf{B}(\check{e}_1,\check{e}_2)$ (see [51, §4.1]). With this notation, we have that $\mathsf{Q}(\check{a}_i) = 2\mathbf{p}-\mathbf{q}$. As such, there is no *primitive* twist, and actually several classes of such covers (which are not multiples of one another) appear in the literature. For example, if $2\mathbf{p}-\mathbf{q} = -1$, one obtains the *Kazhdan-Patterson* (see [67]) covers used in the literature on automorphic forms. Note that this cover is the pull back of the opposite (i.e. $\mathsf{Q}(\check{a}_i) = -1$) of the primitive cover of $\mathbf{SL}_r$. The covers when $2\mathbf{p}-\mathbf{q} = -2$ features in the work of Savin (see [51, §4.1]). Finally, the coverings studied in the works [17, 18, 20] linking Whittaker functions with quantum *affine* groups and lattice models satisfy $\mathbf{p} = k$ and $\mathbf{q} = 0$.

It would be interesting to understand if Lusztig's definition of a quantum group at a root of unity based on l-twists (introduced in §3.2.8) can be generalized to all (Q,n) twists. In particular, if one can build a quantum group at a root of unity corresponding to the Kazhdan-Patterson twist and the Savin twist introduced above. We note here that for the $\mathbf{p} = k$ and $\mathbf{q} = 0$ twist, one may naturally build the l-twisted root datum $\widetilde{Y} := \frac{1}{l}Y, \widetilde{X} := lX$ and $\widetilde{a}_i^\vee = \frac{1}{l}\check{a}_i, \widetilde{a}_i = la_i$ and its associated quantum $\mathfrak{gl}_r$ at a root of unity.

3.3. Hecke algebras and (parabolic) Kazhdan–Lusztig theory of Coxeter groups.

Throughout this section, we work over the ring $\mathbb{Z}_\tau = \mathbb{Z}[\tau,\tau^{-1}]$ from §2.3.2. Recall that $\tau^2 = t^{-1}$ and define

$$\mathbb{Z}_\tau^+ := \tau\mathbb{Z}[\tau] \quad \text{and} \quad \mathbb{Z}_\tau^- := \tau^{-1}\mathbb{Z}[\tau^{-1}]. \tag{3.28}$$

3.3.1. *Hecke algebra of Coxeter group.* Let $(\mathcal{W}, S)$ a Coxeter group (so, we allow both finite and affine examples). Then we define its Hecke algebra $H_{\mathcal{W}} := H(\mathcal{W}, t)$ as the algebra over $\mathbb{Z}_\tau$ with linear basis $\{T_w\}_{w\in\mathcal{W}}$ and multiplication determined by the rules:

- **Quadratic:** for $s \in S$, $T_s^2 = t^{-1} + (t^{-1} - 1)T_s$, i.e. $(T_s - t^{-1})(T_s + 1) = 0$; and
- **Braid:** there exists a unique group homomorphism $\mathcal{B}(\mathsf{A}) \to H(\mathcal{W}, t)$ sending $s_i \mapsto T_{s_i}$.

The braid relations ensure that we may unambiguously define, for any $w \in W$, the element

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_r}} \in H_{\mathcal{W}} \text{ for any reduced decomposition } w = s_{i_1} \cdots s_{i_r}. \tag{3.29}$$

The quadratic relation ensures that T_{s_i} is invertible in $H_{\mathcal{W}}$, and we denote its inverse as $T_{s_i}^{-1}$,which is given explicitly as $T_{s_i}^{-1} = t^{-1}T_{s_i} + (t^{-1} - 1)$. Using the braid relations, we can define T_w^{-1} for any $w \in \mathcal{W}$.

3.3.2. *Renormalized basis.* We often work with the following renormalization of the generators. Let

$$H_w := \tau^{-\ell(w)}T_w \text{ for } w \in W, \tag{3.30}$$

which again satisfy the braid relations together with the new quadratic relation

$$(H_{s_i} - \tau)(H_{s_i} + \tau^{-1}) = 0, \text{ i.e. } H_{s_i}^2 + (\tau - \tau^{-1})H_{s_i} - 1 = 0. \tag{3.31}$$

Again all H_w are units, and one checks that

$$H_{s_i}^{-1} = H_{s_i} + (\tau^{-1} - \tau) \text{ for } i \in I. \tag{3.32}$$

3.3.3. *Kazhdan–Lusztig theory.* There exists exactly one ring homomorphism

$$d : H(\mathcal{W}, t) \to H(\mathcal{W}, t) \text{ such that } d(\tau) = \tau^{-1}, \text{ and } d(H_w) = (H_{w^{-1}})^{-1}. \tag{3.33}$$

The map d is an involution, called the *Kazhdan–Lusztig involution*, and its application to $f \in H(\mathcal{W}, t)$ is often just written as $f \mapsto \overline{f}$. An element $f \in H(\mathcal{W}, t)$ is called *self-dual* if $d(f) = f$. For future use, we record the following facts:

- For $w \in W$, there exist $r_{w',w} \in \mathbb{Z}_\tau$ so that (with respect to the Bruhat order)
$$\overline{H}_w = H_w + \sum_{w'<w} r_{w',w}H_{w'}. \tag{3.34}$$
- Suppose $\mathcal{W}$ is finite with longest element w_0. Then for any $\sigma \in \mathcal{W}$, we have (see [13, Prop. 3.2.2(2)]) $\ell(\sigma w_0) + \ell(\sigma^{-1}) = \ell(w_0)$, so that $H_{\sigma^{-1}}H_{\sigma w_0} = H_{w_0}$ and hence
$$\overline{H}_\sigma = H_{\sigma w_0}H_{w_0}^{-1}. \tag{3.35}$$

Using [84, Lemma 24.2.1] and the relation (3.34), we may conclude (see [68] or [110, Theorem 2.1 and Claim 2.3]) that for each $w \in \mathcal{W}$, there exists a unique self-dual element $\underline{H}_w \in H(\mathcal{W},t)$ such that

$$\underline{H}_w = H_w + \sum_{y<w} h_{y,w} H_y \text{ where } h_{y,w} \in \mathbb{Z}_\tau^+. \tag{3.36}$$

We could have worked in $\mathbb{Z}_\tau^-$ and showed there exists a unique self-dual element $\underline{H}_w^- \in H(\mathcal{W},t)$ such that

$$\underline{H}_w^- = H_w + \sum_{y<w} h_{y,w}^- H_y \text{ where } h_{y,w}^- \in \mathbb{Z}_\tau^-. \tag{3.37}$$

The elements $\{\underline{H}_w\}_{w\in W}$ (resp. $\{\underline{H}_w^-\}_{w\in W}$) form the *Kazhdan-Lusztig* $\mathbb{Z}_\tau$-basis of $H(\mathcal{W},t)$.

Remark. *Typically, the polynomials*

$$P_{y,x} = \tau^{\ell(x)-\ell(y)} h_{y,x}^- \in \mathbb{Z}[\tau,\tau^{-1}] \tag{3.38}$$

are called Kazhdan–Lusztig polynomials *and satisfy (*[110, §2]*)* $P_{y,x} \in \mathbb{Z}[\tau^{-2}]$ *with constant coefficient* 1.

3.3.4. *Symmetrizers and anti-symmetrizers.* Fix the same notation as in the previous paragraph and let $J \subset S$ be such that $\mathcal{W}_J$, the parabolic subgroup defined in §3.1.9, is finite so that w_0^J, the longest element in $\mathcal{W}_J$, is well-defined. Define the elements

$$\varepsilon_J^+ := \tau^{-\ell(w_0^J)} \textstyle\sum_{w\in\mathcal{W}_J} \tau^{\ell(w)} H_w \quad \text{and} \quad \varepsilon_J^- := (-\tau)^{\ell(w_0^J)} \textstyle\sum_{w\in\mathcal{W}_J} (-\tau^{-1})^{\ell(w)} H_w. \tag{3.39}$$

We may compute, see [86, (5.5.17)(i)], for $w \in \mathcal{W}_J$,

$$H_w \varepsilon_J^+ = \varepsilon_J^+ H_w = \tau^{\ell(w)} \varepsilon_J^+ \quad \text{and} \quad H_w \varepsilon_J^- = \varepsilon_J^- H_w = (-\tau)^{-\ell(w)} \varepsilon_J^+, \tag{3.40}$$

and also verify that (see [86, (5.5.17)(ii))]) in terms of the Poincaré polynomial defined in (3.8) we have

$$(\varepsilon_J^+)^2 = \tau^{-\ell(w_0^J)} \mathscr{P}_{\mathcal{W}_J}(t^{-1}) \varepsilon_J^+ \text{ and } (\varepsilon_J^-)^2 = (-\tau)^{\ell(w_0^J)} \mathscr{P}_{\mathcal{W}_J}(-t) \varepsilon_J^-. \tag{3.41}$$

Finally, it is known (see [110, Proposition 2.9]) that $\varepsilon_J^+ = \underline{H}_{w_0^J}$ and that ε_J^- is the 'signed Kazhdan–Lusztig' basis element denoted as $C_{w_0^J}$ in [68], so that in particular both $\varepsilon_J^\pm$ are self-dual, i.e. $d\left(\varepsilon_J^\pm\right) = \varepsilon_J^\pm$.

3.3.5. *The parabolic modules M_J and ${}_JN$.* Write $H_\mathcal{W} := H(\mathcal{W},t)$. Fix the same notation as in the previous paragraphs, so that J is chosen with $\mathcal{W}_J$ finite. As in §3.1.9, we write $\mathcal{W}^J$ for a set of minimal length representatives of W/W_J. Consider the A-modules

$$M_J := H_\mathcal{W} \varepsilon_J^+ := \{h\varepsilon_J^+ \mid h \in H_\mathcal{W}\} \quad \text{and} \quad {}_JN := \varepsilon_J^- H_\mathcal{W} := \{\varepsilon_J^- \cdot h \mid h \in H_\mathcal{W}\}. \tag{3.42}$$

From (3.40), we see that M_J and ${}_JN$ have A-bases $M_w := H_w \varepsilon_J^+$ and $N_w := \varepsilon_J^- H_w$ where w ranges over all elements in $\mathcal{W}^J$ and ${}^J\mathcal{W}$ respectively. As we shall need it later, we record here the following elementary computation: for $\sigma \in {}^J\mathcal{W}$ and $s \in S$

$$N_\sigma H_s = \begin{cases} N_{\sigma s} & \text{if } \sigma s \in {}^J\mathcal{W}, \sigma s > \sigma, \\ N_{\sigma s} + (\tau - \tau^{-1})N_\sigma & \text{if } \sigma s \in {}^J\mathcal{W}, \sigma s < \sigma, \\ -\tau^{-1} N_\sigma & \text{if } \sigma s \notin {}^J\mathcal{W}. \end{cases} \tag{3.43}$$

The first and second cases follow from the braid relations, and the third by also using (3.9) and (3.40).

3.3.6. *Parabolic Kazhdan–Lusztig polynomials.* The involution d induces one on M_J and ${}_JN$, respectively, so that one can again speak of self-dual elements in these A-modules. From Deodhar [38] (see [110, Theorem 3.1]) for $w \in \mathcal{W}^J, z \in {}^J\mathcal{W}$, there exist unique self-dual elements $\underline{M}_w, \underline{M}_w^-$ as well as $\underline{N}_z, \underline{N}_z^-$ such that

$$\underline{M}_w := M_w + \textstyle\sum_{y \le w} m_{y,w} M_y, \quad \underline{N}_z := N_z + \sum_{y \le z} n_{y,z} N_y, \quad \text{where} \quad m_{y,w}, n_{y,z} \in \mathbb{Z}_\tau^+, \tag{3.44}$$

$$\underline{M}_w^- := M_w + \textstyle\sum_{y \le w} m_{y,w}^- M_y, \quad \underline{N}_z^- := N_z + \sum_{y \le z} n_{y,z}^- N_y, \quad \text{where} \quad m_{y,w}^-, n_{y,z}^- \in \mathbb{Z}_\tau^-, \tag{3.45}$$

These elements satisfy the following properties (cf. [38], [110, Prop. 3.4]): for $y, w \in \mathcal{W}^J$,

$$m_{y,w}^- = h_{yw_0^J, ww_0^J}^- \quad \text{and} \quad n_{y,w}^- = \textstyle\sum_{z \in \mathcal{W}_J} (-\tau^{-1})^{\ell(z)} h_{zy,w}^-, \tag{3.46}$$

$$m_{y,w} = (-1)^{\ell(y)+\ell(w)} d(n_{y,x}^-) \quad \text{and} \quad n_{y,w} = (-1)^{\ell(y)+\ell(w)} d(m_{y,x}^-), \tag{3.47}$$

where the involution d on $\mathbb{Z}_\tau$ interchanges $\tau \leftrightarrow \tau^{-1}$. For the last line, see [110, Thm. 3.5].

Remark. *The elements $\tau^{\ell(w)-\ell(y)} m_{y,w}$ and $\tau^{\ell(w)-\ell(y)} n_{y,w}$ are examples of parabolic Kazhdan–Lusztig polynomials* [38]. *Our definitions of $m_{y,w}$ and $n_{y,w}$ match similarly named quantities in* [110] *after setting $v = \tau^{-1}$ in* op. cit.

3.3.7. *The module ${}_JO_K$.* Suppose $K \subset I$ is such that W_K is finite. Then we consider the vector space ${}_JO_K := \varepsilon_J^- H_\mathcal{W} \varepsilon_K$ whose basis consists of $O_t := \varepsilon_J^- H_t \varepsilon_K$ for $t \in ({}_J\mathcal{W}_K)_{\text{reg}}$. One can view it as a submodule ${}_JO_K \hookrightarrow {}_JN$ from which we may define $d : {}_JO_K \to {}_JO_K$ by restricting the involution d on ${}_JN$.

Proposition. *For all $u \in ({}_J\mathcal{W}_K)_{\text{reg}}$, there exists a unique self-dual element $\underline{O}_u^-$ such that*

$$\underline{O}_u^- = O_u + \sum_{t<u} o_{t,u}^- O_t, \text{ with } o_{t,u}^- \in \mathbb{Z}_\tau^-. \tag{3.48}$$

For this element, we have

$$o_{t,u}^- = n_{tw_0^K, uw_0^K}^- = \sum_{z \in \mathcal{W}_J} (-\tau^{-1})^{\ell(z)} m_{zt,u}^- = \sum_{z \in \mathcal{W}_J} (-\tau^{-1})^{\ell(z)} h_{ztw_0^K, uw_0^K}^-. \tag{3.49}$$

We also have unique self-dual elements $\underline{O}_u \in {}_JO_K$ satisfying

$$\underline{O}_u = O_u + \sum_{t<u} o_{t,u} O_t, \text{ with } o_{t,u} \in \mathbb{Z}_\tau^+. \tag{3.50}$$

For such elements, we have

$$o_{t,u} = m_{w_0^J t, w_0^J u} = \sum_{z \in \mathcal{W}_K} \tau^{\ell(z)} n_{tz,u} = \sum_{z \in \mathcal{W}_K} \tau^{\ell(z)} h_{w_0^J tz, w_0^J u}, \tag{3.51}$$

Proof. The existence of the unique self-dual elements $\underline{O}_u^{\pm}$ satisfying (3.48) follows from [84, Lemma 24.2.1] using the fact that the involution d on ${}_JO_K$ inherits the required triangularity property from that of the involution on ${}_JN$ (or $H_{\mathcal{W}}$.) Let us just show (3.49). From [110, Proposition 2.9],

$$\varepsilon_K^+ = \underline{H}_{w_0^K}^- = \sum_{w\in\mathcal{W}_K} \tau^{\ell(w)-\ell(w_0^K)} H_w. \tag{3.52}$$

Hence, for $u \in ({}_J\mathcal{W}_K)_{\mathrm{reg}}$, under the map $\zeta : {}_JO_K = {}_JN\varepsilon_K^+ \hookrightarrow {}_JN$ we have

$$\zeta(O_u^-) = \varepsilon_J^- H_u \underline{H}_{w_0^K}^- = \varepsilon_J^- \sum_{w\in\mathcal{W}_K} \tau^{\ell(w)-\ell(w_0^K)} H_{tw}. \tag{3.53}$$

Applying the involution to both sides of the above, using the fact that ζ is compatible with taking such involutions, and finally using the characterization of $\underline{N}_z^-$ for $z \in {}^JW$ reviewed in §3.3.6 we can verify that

$$\zeta(\underline{O}_u^-) = \varepsilon_J^- \underline{H}_{uw_0^K}^-. \tag{3.54}$$

Expanding the left hand side and using the fact that $H_z\varepsilon_K = \tau^{\ell(z)}\varepsilon_K$ for all $z \in \mathcal{W}_K$, we may write

$$\underline{O}_u^- = \sum_{x\in{}^J\mathcal{W}^K} o_{x,u}^- O_x = \sum_t o_{x,u}^- \varepsilon_J^- H_x \varepsilon_K = \sum_{\substack{x\in{}^J\mathcal{W}^K\\ z\in\mathcal{W}_K}} o_{x,u}^- \varepsilon_J^- H_{xz}\tau^{\ell(z)-\ell(w_0^K)} = \sum_{\substack{x\in{}^J\mathcal{W}^K\\ z\in\mathcal{W}_K}} o_{x,u}^- N_{xz}\tau^{\ell(z)-\ell(w_0^K)}.$$

As for the right hand side of (3.54), we expand using $\varepsilon_J^- H_w = (-\tau^{-1})^{\ell(w)}$ for $w \in \mathcal{W}_J$

$$\varepsilon_J^- \underline{H}_{uw^K} = \varepsilon_J^- \sum_{x\in\mathcal{W}} h_{x,uw_0^K}^- H_x = \varepsilon_J^- \sum_{\substack{y\in\mathcal{W}_J\\ z\in{}^J\mathcal{W}}} h_{yz,uw_0^K}^- H_{yz} = \varepsilon_J^- \sum_{\substack{y\in\mathcal{W}_J\\ z\in{}^J\mathcal{W}}} h_{yz,uw_0^K}^- (-\tau)^{-\ell(y)} H_z \tag{3.55}$$

$$= \sum_{\substack{y\in\mathcal{W}_J\\ z\in{}^J\mathcal{W}}} h_{yz,uw_0^K}^- (-\tau)^{-\ell(y)} N_z = \sum_{\substack{y\in\mathcal{W}_J\\ xz\in{}^J\mathcal{W}}} h_{yxz,uw_0^K}^- (-\tau)^{-\ell(y)} N_{xz} \tag{3.56}$$

Hence we obtain that

$$\textstyle\sum_{y\in\mathcal{W}_J} h_{yxz,uw^K}^- (-\tau)^{-\ell(y)} = o_{x,u}^- \tau^{\ell(z)-\ell(w_0^K)} \text{ and } \sum_{y\in\mathcal{W}_J} h_{yxw_0^K,uw_0^K}^- (-\tau)^{-\ell(y)} = o_{x,u}^-, \tag{3.57}$$

where we set $z = w_0^K$ in the first equation to get the second. The other equalities in (3.49) follow from (3.46).

One may verify (3.51) in a similar way. We leave the proof to the reader. □

3.4. **Affine Weyl groups.** In this section, $(I,\cdot)$ will be a Cartan datum of finite type with associated Cartan matrix A and untwisted affinization $\mathsf{A}_{\mathrm{aff}}$. Write $W := \mathcal{W}(\mathsf{A})$ for the finite Weyl group and $W_{\mathrm{aff}} := \mathcal{W}(\mathsf{A}_{\mathrm{aff}})$ for the affine Weyl group. Fix $(I,\cdot,\mathfrak{D})$which is assumed to be of simply-connected type and write $\mathfrak{D} := (Y,\{\check{a}_i\},X,\{a_i\})$. Using this root datum, we can define the Euclidean space $V := Y\otimes_{\mathbb{Z}}\mathbb{R}$. Extend the pairing $Y\times X\to\mathbb{Z}$ to one between V and $\check{V} := X\otimes_{\mathbb{Z}}\mathbb{R}$ by linearity and denote it by the same symbol. We write $\mathcal{R}\subset V^*$ to be the roots as in §3.2.4 , $\theta\in\mathcal{R}$ the highest positive root as in (3.14), $\mathcal{Q}$ for the root lattice, and $\check{\mathcal{Q}}\cong Y$ for the coroot lattice.

3.4.1. *Groups generated by affine reflections.* Define for each $i \in I$ and integer $k \in \mathbb{Z}$, the affine reflection $\sigma_{a_i,k} : V \to V$ by $y \cdot \sigma_{a_i,k} = y - (\langle y, a_i \rangle - k)\check{a}_i$ for $v \in V$. Note that the reflection $\sigma_{a_i,k}$ fixes the hyperplane $\mathfrak{H}_{a_i,k} := \{v \in V \mid \langle v, a_i \rangle = k\}$. Replacing a_i with a for any root $a \in \mathcal{Q}$ (the root lattice of $(I, \cdot, \mathfrak{D})$ in the formulas above, we define $\sigma_{a,k}$. We define $\mathfrak{H}_{a,k}$ as the fixed hyperplane of this reflection, and say that $y \in V$ is *positive (resp. negative)* for the hyperplane $\mathfrak{H}_{a,k}$ if $\langle y, a_i \rangle > k$ (resp. $\langle y, a_i \rangle < k$). Define now

$$W_{\mathrm{aff}}(I, \cdot, \mathfrak{D}) = \langle \sigma_{a_i,k} \mid i \in I, k \in \mathbb{Z} \rangle. \tag{3.58}$$

It contains as a subgroup $W = \mathcal{W}(I, \cdot) \cong \langle \sigma_{a_i,0} \mid i \in I \rangle$, as well as the subgroup $\mathtt{T}$ generated by the elements

$$\mathtt{t}(k\check{a}_i) := \sigma_{a_i,k} \circ \sigma_{a_i,0} \text{ for } i \in I, k \in \mathbb{Z}. \tag{3.59}$$

One may check that $y \cdot \mathtt{t}(k\check{a}_i) = y - k\check{a}_i$ for $y \in V$ and hence $\mathtt{t}(k\check{a}_i) \circ \mathtt{t}(m\check{a}_j) = \mathtt{t}(m\check{a}_j) \circ \mathtt{t}(k\check{a}_i)$. We denote this last element as $\mathtt{t}(k\check{a}_i + m\check{a}_j) \in \mathtt{T}$. In a similar way we may define for any $\check{a} \in \check{\mathcal{R}}$ a corresponding element $\mathtt{t}(\check{a}) \in \mathtt{T}$. One verifies $\mathtt{T} \cong \check{\mathcal{Q}} \cong Y$, where the last isomorphism follows from the simply connected hypothesis. Moreover, the subgroup $W := W(I, \cdot)$ normalizes $\mathtt{T}$ and

$$W_{\mathrm{aff}}(I, \cdot, \mathfrak{D}) \cong W \ltimes Y. \tag{3.60}$$

3.4.2. *Coxeter structure.* As we are in the simply-connected case, the group $W_{\mathrm{aff}}(I, \cdot, \mathfrak{D})$ is a Coxeter group with Coxeter structure defined by the set of simple reflections S_{aff} consisting of $s_i := \sigma_{a_i,0}$ for $i \in I$, and the new reflection $s_0 := \sigma_{\theta,1}$. In this simply-connected case, we then have $W_{\mathrm{aff}}(I, \cdot, \mathfrak{D}) \simeq \mathcal{W}(\mathsf{A}_{\mathrm{aff}})$ as Coxeter groups; we shall from now on just write

$$W_{\mathrm{aff}} := W_{\mathrm{aff}}(I, \cdot, \mathfrak{D}) \cong \mathcal{W}(\mathsf{A}_{\mathrm{aff}}). \tag{3.61}$$

One can then define the Bruhat order on $W_{\mathrm{aff}}(I, \cdot, \mathfrak{D})$ as well as the length function ℓ with respect to this Coxeter structure, and from ([56, p.20, Prop. 1.23]), one has the following useful formula for the length of an element in $W_{\mathrm{aff}}(I, \cdot, \mathfrak{D})$ with respect to the previous decomposition (3.60): for $w\mathtt{t}(\check{\beta})$ with $w \in W, \check{\beta} \in \check{\mathcal{Q}}$,

$$\ell\left(w\mathtt{t}(\check{\beta})\right) = \sum_{a \in \mathcal{R}_+ \cap w^{-1}\mathcal{R}_+} |\langle \check{\beta}, a \rangle| + \sum_{a \in \mathcal{R}_+ \cap w^{-1}(-\mathcal{R}_+)} |1 + \langle \check{\beta}, a \rangle|. \tag{3.62}$$

From this formula, (3.15), and the fact that $\ell(w)$ is equal to the cardinality of the set $\mathcal{R}_+ \cap w^{-1}(-\mathcal{R}_+)$, (see [14, Cor. 2, p.170]), we see that if $\langle \check{\beta}, a \rangle \geq 0$ for all $a \in \mathcal{R}$, then

$$\ell(w\mathtt{t}(\check{\beta})) = |\mathcal{R}_+ \cap w^{-1}(-\mathcal{R}_+)| + \sum_{a \in \mathcal{R}} \langle \check{\beta}, a \rangle = \ell(w) + \langle 2\rho, \check{\beta} \rangle. \tag{3.63}$$

3.4.3. *Chambers.* The Euclidean space $V := Y \otimes_{\mathbb{Z}} \mathbb{R}$, regarded as an affine space has two chamber structures, in the sense of [14, Ch. V, §1] corresponding to the action of W and $W_{\mathrm{aff}}(I, \cdot, \mathfrak{D})$ on V. The *chambers* are the connected components of $V \setminus \{L = \mathfrak{H}_{a,0}, a \in R\}$. The Weyl group W acts transitively on the collection of chambers. We define the dominant chamber and antidominant chambers as

$$\mathcal{C}_+ := \{v \in V \mid \langle v, a_i \rangle > 0 \text{ for } i \in I\} \text{ and } \mathcal{C}_- := \{v \in V \mid \langle v, a_i \rangle < 0 \text{ for } i \in I\}, \text{resp.} \tag{3.64}$$

3.4.4. *Alcoves.* Define the collection of *affine hyperplanes* as $\widehat{M} := \{\mathfrak{H}_{a,k}, a \in \mathcal{R}_+, k \in \mathbb{Z}\}$, and denote the connected components of $V \setminus \widehat{M}$ as *alcoves*. Write $\mathscr{A}$ for the collection of alcoves. For any $v \in V \setminus \widehat{M}$ and $a \in \mathcal{R}_+$, there exists an unique integer k_a so that $k_a < \langle v, a\rangle < k_a + 1$, *i.e.* v lies in the connected region between the walls $\mathfrak{H}_{a,k}$ and $\mathfrak{H}_{a,k+1}$. For any alcove $A \in \mathscr{A}$, there exist integers $k_a, a \in \mathcal{R}$ such that for $v \in A$, one has $k_a < \langle v, a\rangle < k_a + 1$ for all $a \in \mathcal{R}_+$. The anti-dominant fundamental alcoves will be defined as

$$\mathcal{A}_- = \{v \in V \mid -1 < \langle v, a\rangle < 0 \text{ for all } a \in \mathcal{R}_+\} \subset \mathcal{C}_-. \tag{3.65}$$

The *upper closure* of an alcove A, which will be denoted as $\overline{A}$, and it is obtained by replacing the upper bounds in the definition of an alcove with inequalities. Every $v \in V$ can be uniquely written as $v = x \cdot w$ for $w \in W_{\text{aff}}$ and $v \in \underline{\mathcal{A}}_-$, where we record here that

$$\overline{\mathcal{A}}_- := \{v \in V \mid -1 < \langle v, a\rangle \leq 0 \text{ for all } a \in \mathcal{R}_+\} \subset \mathcal{C}_-. \tag{3.66}$$

3.4.5. *A result on separating walls.* For $w \in W_{\text{aff}}$ define $\widehat{M}(w)$ to be the set of walls which separate $\mathcal{A}$ and $w\mathcal{A}$ (a wall is called *separating* if the alcoves lie on different half-spaces defined by the wall). In terms of a reduced decomposition of $w \in \widetilde{W}_{\text{aff}}$, say $w = w_{b_1} \cdots w_{b_d}$ where $b_j \in I_{\text{aff}} = \{0\} \sqcup I$, then we have :

$$\widehat{M}(w) = \{H_{b_1}, H_{b_2} w_{b_1}, \ldots, H_d w_{b_{d-1}} \cdots w_{b_1}\}, \tag{3.67}$$

where $H_j := H_{(a_j,0)}$ if $j \in I$ and $H_0 := H_{(\theta,1)}$ the walls defining simple reflections (see [55, Theorem 4.5]).

Remark. *In terms of affine root systems (which we have not formally introduced), these are the walls corresponding to positive affine roots which are flipped to negative by w^{-1}.*

3.4.6. *Dot action.* We are often interested in the following 'dot' action of W_{aff} on V

$$x \bullet w := (x + \check{\rho}) \cdot w - \check{\rho} \text{ for } w \in W_{\text{aff}}, x \in V, \tag{3.68}$$

where $\check{\rho}$ was defined in (3.15). Note that it is indeed an *action* of W. Under the dot action, $\sigma_{a,k}$ is a reflection through the hyperplane $\mathfrak{H}^\bullet_{a,k} := \{x \in V \mid \langle x + \check{\rho}, a\rangle = k\}$ since

$$x \bullet \sigma_{a,k} = x - (\langle x + \check{\rho}, a\rangle - k)\check{a} \text{ for } x \in V, a \in \mathcal{R}, k \in \mathbb{Z}. \tag{3.69}$$

One can again define a set of alcoves with respect to the hyperplanes $\{\mathfrak{H}^\bullet_{a,k} \mid a \in R, k \in \mathbb{Z}\}$ and we designate the (anti)-fundamental alcove and its upper closure as

$$\begin{aligned} \mathcal{A}^\bullet_- &:= \{x \in V \mid -1 < \langle v + \check{\rho}, a\rangle < 0 \text{ for all } a \in \mathcal{R}_+\}, && (3.70)\\ \overline{\mathcal{A}}^\bullet_- &:= \{x \in V \mid -1 < \langle x + \check{\rho}, a\rangle \leq 0 \text{ for all } a \in \mathcal{R}_+\}. && (3.71) \end{aligned}$$

3.4.7. *Example: (Q,n)-twisted affine Weyl group.* Let $(I,\cdot,\mathfrak{D})$ be an arbitrary root datum, (Q,n) a given twist with associated root datum $(I,\circ_{(\mathsf{Q},n)},\widetilde{\mathfrak{D}})$ which we assume is of simply connected type. We note that $\mathcal{W}(I,\cdot)\simeq\mathcal{W}(I,\circ_{(\mathsf{Q},n)})$ and both groups are denoted as W. Recalling the Cartan matrix of $(I,\circ_{(\mathsf{Q},n)})$ was denoted as $\widetilde{\mathsf{A}}$, we shall now write

$$\widetilde{W}_{\mathrm{aff}}:=W_{\mathrm{aff}}(I,\circ_{(\mathsf{Q},n)},\widetilde{\mathfrak{D}})\simeq\mathcal{W}(\widetilde{\mathsf{A}}_{\mathrm{aff}})\simeq W\ltimes\widetilde{Y}. \tag{3.72}$$

The dot action of $\widetilde{W}_{\mathrm{aff}}$ on $V:=Y\otimes_{\mathbb{Z}}\mathbb{R}$ is defined via

$$x\bullet\sigma_{\widetilde{a}_i,k}=v-(\langle v+\check{\rho},\widetilde{a}_i\rangle-k)\,\widetilde{a}_i^{\vee}\text{ for }v\in V,i\in I,k\in\mathbb{Z}. \tag{3.73}$$

Note that the $\check{\rho}$ which appears in the above formula is 'untwisted'. A fundamental domain for this action of $\widetilde{W}_{\mathrm{aff}}$ on V is given by the set

$$\begin{aligned}\overline{\mathcal{A}}^{\bullet}_{-,(\mathsf{Q},n)} &:= \{v\in V\mid -1<\langle v+\check{\rho},\widetilde{a}\rangle\leq 0\text{ for all }\widetilde{a}\in\widetilde{\mathcal{R}}_+\} &(3.74)\\ &= \{v\in V\mid -n(\check{a})<\langle v+\check{\rho},a\rangle\leq 0\text{ for all }a\in\mathcal{R}_+\}. &(3.75)\end{aligned}$$

When the choice of Q is implicitly understood, we abbreviate our notation and write

$$\overline{\mathcal{A}}^{\bullet}_{-,n}:=\overline{\mathcal{A}}^{\bullet}_{-,(\mathsf{Q},n)}. \tag{3.76}$$

3.4.8. *Stabilizers, orbits, and cosets.* For $\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{-,n}$, let $J\subsetneq I_{\mathrm{aff}}$ be such that

$$(\widetilde{W}_{\mathrm{aff}})_J:=\mathrm{Stab}_{(\widetilde{W}_{\mathrm{aff}},\bullet)}(\check{\eta}). \tag{3.77}$$

Then we may identity $(\widetilde{W}_{\mathrm{aff}})_J\setminus\widetilde{W}_{\mathrm{aff}}$ with elements in the orbit $\widetilde{W}_{\mathrm{aff}}\bullet\check{\eta}$. Writing $\check{\mu}\in\widetilde{W}_{\mathrm{aff}}\bullet\check{\eta}$ as $\check{\mu}=\check{\eta}\bullet w$ with $w\in\widetilde{W}_{\mathrm{aff}}$, we may fix w uniquely by requiring it to lie in ${}^J(\widetilde{W}_{\mathrm{aff}})$. In this way we get a bijective correspondence

$$\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}\leftrightarrow{}^J\widetilde{W}_{\mathrm{aff}}\text{ for }J\subsetneq I_{\mathrm{aff}}\text{ as in (3.77).} \tag{3.78}$$

We extend this result to conclude that for $\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{-,n}$ and J as in (3.77), there is a bijection

$$(\widetilde{W}_{\mathrm{aff}})_J\setminus\widetilde{W}_{\mathrm{aff}}/W\xleftrightarrow{1:1}(W,\bullet)\text{-orbits in }\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}, \tag{3.79}$$

where by $(W,\bullet)$-orbits, we mean orbits under the $\bullet$-action of W. The above correspondence can be further strengthened to one between regular double cosets (see the end of §3.1.9) and regular $(W,\bullet)$-orbits in $\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}$ (*i.e.* those orbits under the $\bullet$-action of W whose stabilizer is trivial). One may verify that each regular $(W,\bullet)$-orbit in $\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}$ has a unique dominant representative and so we obtain a bijection

$$Y_+\cap\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}\xleftrightarrow{1:1}\left({}^J\widetilde{W}^I_{\mathrm{aff}}\right)_{\mathrm{reg}}. \tag{3.80}$$

Denoting the Bruhat order on the left hand side by $\leq_B$ (the one induced from the Bruhat order on the right hand side). Then one knows that if $x,y\in\left({}^J\widetilde{W}^I_{\mathrm{aff}}\right)_{\mathrm{reg}}$ correspond to dominant coweights $\check{\lambda}_x,\check{\lambda}_y$, then

$$x\leq_B y\implies\check{\lambda}_y\leq\check{\lambda}_x \tag{3.81}$$

where the order $\leq$ in the right hand side is the dominance order.

3.4.9. *Boxes and restricted weights.* Keep the notation of the previous section and define the lower-closure of the box (in the terminology of [63]), or set of *restricted weights*, as

$$\square_{(\mathsf{Q},n)} := \{\check{\lambda} \in Y_+ \mid 0 \leq \langle \check{\lambda}, a_i \rangle < n(\check{a}_i) \text{ for } i \in I\}. \tag{3.82}$$

One can easily check that each $\check{\lambda} \in Y_+$ may be uniquely written as

$$\check{\lambda} = \check{\lambda}_0 + \check{\zeta}, \check{\lambda}_0 \in \square_{(\mathsf{Q},n)}, \check{\zeta} \in \widetilde{Y}_+. \tag{3.83}$$

Indeed, from $\check{\lambda} \in Y_+$, we may just subtract positive multiples of the fundamental coweights attached to $\widetilde{\mathfrak{D}}$ until we arrive at an element in the box (our simply-connected hypothesis ensures that these fundamental coweights lie in $\widetilde{Y}_+$).

3.5. **Affine Hecke algebras and their combinatorics.** In this subsection, we assume $(I, \cdot, \mathfrak{D})$ is of simply connected type and write $\mathfrak{D} = (Y, \{\check{a}_i\}, X, \{a_i\})$. Write W for the Weyl group of $(I, \cdot)$ and $W_{\mathrm{aff}} := W_{\mathrm{aff}}(I, \cdot, \mathfrak{D}) = W \ltimes Y$ for the corresponding affine Weyl group, which we know is a Coxeter group under the simply connected hypothesis. As such we can define two Hecke algebras: $H_W := H(W, t)$ will be called the *finite Hecke algebra* and $H_{\mathrm{aff}} := H(W_{\mathrm{aff}}, t)$ will be called the *affine Hecke algebra*.

3.5.1. *Bernstein presentation.* If $\check{\beta} \in Y_+$ we set $Y_{\check{\beta}} := T_{\mathfrak{t}(\check{\beta})}$ and we note by the length formula (3.62) that $Y_{\check{\beta}+\check{\gamma}} = Y_{\check{\beta}} Y_{\check{\gamma}} = Y_{\check{\gamma}} Y_{\check{\beta}}$ when $\check{\beta}, \check{\gamma} \in Y_+$. For a general element $\check{\beta} \in Y$, we may write $\check{\beta} = \check{\lambda} - \check{\mu}$ with $\check{\lambda}, \check{\mu} \in Y_+$ and then set $Y_{\check{\beta}} = Y_{\check{\lambda}} Y_{\check{\mu}}^{-1}$ which can be shown to be well-defined (i.e. independent of the choice of $\check{\lambda}, \check{\mu} \in Y_+$, cf. [86, p.40]). This definition extends to give a well-defined injective morphism of algebras $\iota : \mathbb{Z}_\tau[Y] \to H_{\mathrm{aff}}$. Let us also remark that one has the formula (see [70, Lemma 3.5] for a conceptual derivation)

$$\overline{Y}_{\check{\beta}} = H_{w_0} Y_{w_0 \check{\beta}} H_{w_0}^{-1}. \tag{3.84}$$

One can verify (see [86, p. 59, (4.2.7)]) that the elements $\{H_w Y_{\check{\beta}}\}$ for $w \in W, \check{\beta} \in Y$ (and similarly $\{Y_{\check{\beta}} H_w\}$) form an $\mathbb{Z}_\tau$-basis of H_{aff}, where one has the following relation

$$H_{s_i} Y_{\check{\beta}} - Y_{s_i \check{\beta}} H_{s_i} = (\tau^{-1} - \tau) \frac{Y_{s_i \check{\beta}} - Y_{\check{\beta}}}{1 - Y_{-\check{a}_i}}. \tag{3.85}$$

3.5.2. *The module $H_{\mathrm{aff}}\varepsilon$.* Regard $I \subset I_{\mathrm{aff}}$ in the natural way so that $(\widetilde{W}_{\mathrm{aff}})_I = W$ and set

$$\varepsilon := \frac{\tau^{\ell(w_0)}}{\mathscr{P}_W(t^{-1})} \varepsilon_I^+. \tag{3.86}$$

From (3.40) and (3.41) we find that

$$H_{s_i} \varepsilon = \tau \varepsilon = \varepsilon H_{s_i} \text{ for } i \in I \text{ and } \varepsilon^2 = \varepsilon. \tag{3.87}$$

The parabolic module $H_{\mathrm{aff}}\varepsilon = H_{\mathrm{aff}}\varepsilon_I^+$ defined as in §3.3.5 is sometimes also referred to as the *polynomial representation* in the literature, as the relation (3.85) and the first of the relations above show that the map ι above induces an isomorphism of $\mathbb{Z}_\tau$-modules, $\iota : \mathbb{Z}_\tau[Y] \xrightarrow{\cong} H_{\mathrm{aff}}\,\varepsilon$, see [86, §4.3]. The induced action of H_{aff} on $\mathbb{Z}_\tau[Y]$, denoted $h \mapsto h.p(Y), h \in H_{\mathrm{aff}}, p(Y) \in \mathbb{Z}_\tau[Y]$, takes the following form:

$$Y_{\check{\lambda}}.Y_{\check{\mu}} = Y_{\check{\lambda}+\check{\mu}} \text{ for } \check{\lambda},\check{\mu} \in Y, \quad \text{and} \quad Y_{\check{\mu}}.H_{s_i} = \tfrac{\tau^{-1}-\tau Y_{-\check{\alpha}_i}}{1-Y_{-\check{\alpha}_i}}\,Y_{s_i\check{\mu}} + \tfrac{\tau-\tau^{-1}}{1-Y_{-\check{\alpha}_i}}\,Y_{\check{\mu}}. \quad (3.88)$$

3.5.3. *The spherical Hecke algebra* H_{sph}. The *spherical subalgebra* is the $\mathbb{Z}_\tau$-module

$$H_{\mathrm{sph}} := \varepsilon\, H_{\mathrm{aff}}\, \varepsilon, \tag{3.89}$$

with algebra structure induced from H_{aff} using the fact that ε is an idempotent. The inverse ι from the previous paragraph induces an isomorphism of H_{aff}-subalgebras

$$S : H_{\mathrm{sph}} \xrightarrow{\cong} \mathbb{Z}_\tau[Y]^W, \tag{3.90}$$

where $\mathbb{Z}_\tau[Y]^W$ is the space of W-invariant elements in the group algebra of Y over $\mathbb{Z}_\tau$ (the action of W on Y induces the action of s_i on $\mathbb{Z}_\tau[Y]$ by the relation $s_iY_{\check{\beta}} = Y_{s_i\check{\beta}}$, *etc.*). One can show that the elements $h_{\check{\mu}} := \varepsilon\, Y_{\check{\mu}}\, \varepsilon$ for $\check{\mu} \in Y_+$ form a basis of H_{sph} and that

$$S(h_{\check{\mu}}) = \frac{\tau^{\langle \check{\mu}, 2\rho\rangle}}{\mathscr{P}_{W_{\check{\mu}}}(t^{-1})} \sum_{w\in W} \left(\prod_{a\in\mathcal{R}_+} \frac{1-\tau^{-2}Y_{-w\check{\alpha}}}{1-Y_{-w\check{\alpha}}} \cdot Y_{w\check{\mu}} \right), \tag{3.91}$$

where $\mathscr{P}_{W_{\check{\mu}}}(t^{-1})$ is the Poincaré polynomial of the stabilizer of $\check{\mu}$ in the finite Weyl group W see (3.8). The space $\mathbb{Z}_\tau[Y]^W$ has another distinguished basis, namely the characters $\chi_{\check{\lambda}}, \check{\lambda} \in \check{\Lambda}_+$ defined as:

$$\chi_{\check{\lambda}} := \chi_{\check{\lambda}}(Y) := \frac{\sum_{w\in W}(-1)^{\ell(w)}\, Y_{w(\check{\lambda}+\check{\rho})-\check{\rho}}}{\prod_{\check{\alpha}\in\mathcal{R}_+}(1-Y_{-\check{\alpha}})}. \tag{3.92}$$

Denoting by $c_{\check{\lambda}} \in H_{\mathrm{sph}}$ the element such that $S(c_{\check{\lambda}}) = \chi_{\check{\lambda}}$, the elements $\{c_{\check{\lambda}}\}_{\check{\lambda}\in Y_+}$ form another basis of H_{sph}, which is related to the basis $\{h_{\check{\lambda}}\}$ through the so-called Kato–Lusztig formula

$$c_{\check{\lambda}} = \textstyle\sum_{\check{\mu}\le\check{\lambda}} p_{\check{\mu},\check{\lambda}}(\tau)\, h_{\check{\mu}}, \quad \text{or equivalently} \quad \chi_{\check{\lambda}} = \sum_{\check{\mu}\le\check{\lambda}} p_{\check{\mu},\check{\lambda}}(\tau)\, S(h_{\check{\mu}}), \tag{3.93}$$

where $\le$ in the above refers to the dominance order on Y as in (3.13). For $\check{\zeta} \in Y_+$, if we write $w_{\check{\zeta}}$ for the longest element in $W\mathtt{t}(\check{\zeta})W$, then it can be deduced from (3.62) that

$$w_{\check{\zeta}} := w_0\mathtt{t}(\check{\zeta}) \text{ and } \ell(w_{\check{\zeta}}) = \ell(w_0) + \langle \check{\zeta}, 2\rho\rangle. \tag{3.94}$$

In terms of the Kazhdan–Lusztig polynomials (3.38), $p_{\check{\mu},\check{\lambda}}(\tau) = \tau^{\langle \check{\lambda}-\check{\mu}, 2\rho\rangle} P_{w_{\check{\mu}},w_{\check{\lambda}}}(\tau^2)$. It is known that $y \le w_{\check{\lambda}}$ in the Bruhat order if and only if there exists $\check{\mu} \in Y_+$ such that $\check{\mu} \le \check{\lambda}$ and $y \in W\mathtt{t}(\check{\mu})W$, see [61, (4.6)]. So it follows that if $\check{\mu}, \check{\lambda} \in Y_+$, then $\check{\mu} \le \check{\lambda}$ if and only if $w_{\check{\mu}} \le w_{\check{\lambda}}$.

3.5.4. *The modules ${}_JV$ and ${}_JV_{\mathrm{sph}}$.* Let $J \subset I \subsetneq I_{\mathrm{aff}}$ and consider the $\mathbb{Z}_\tau$-modules

$$ {}_JV := \varepsilon_J^- H_{\mathrm{aff}} \quad \text{and} \quad {}_JV_{\mathrm{sph}} := \varepsilon_J^- H_{\mathrm{aff}}\,\varepsilon. \tag{3.95}$$

We have already described a basis for the space ${}_JV$, namely $N_w := \varepsilon_J^- H_w$ where w ranges over the elements in ${}^JW_{\mathrm{aff}}$. In a similar way, we see that ${}_JV_{\mathrm{sph}}$ has a basis $B_u := \varepsilon_J^- H_u\varepsilon$ as u ranges over a set of regular, minimal length coset representatives $\left({}^JW_{\mathrm{aff}}^I\right)_{\mathrm{reg}}$. One can equip ${}_JV$ and ${}_JV_{\mathrm{sph}}$ with involutions inherited from the involution d on H_{aff}, and we continue to call these $d : {}_JV \to {}_JV$ and $d : {}_JV_{\mathrm{sph}} \to {}_JV_{\mathrm{sph}}$. Finally, let us note that ${}_JV$ carries a right action by H_{aff}, denoted as $\cdot$ and ${}_JV_{\mathrm{sph}}$ is equipped with a right action by H_{sph}, denote as $\star$, and computed as follows: for $h \in H_{\mathrm{aff}}, u \in \left({}^JW_{\mathrm{aff}}^I\right)_{\mathrm{reg}}$, we have $\left(\varepsilon_J^- H_u\varepsilon\right) \star (\varepsilon h\varepsilon) = \varepsilon_J^- \left(H_u\,\varepsilon\, h\right)\varepsilon$.

3.6. **Twisted affine Hecke algebras.** Throughout this section, we let $(I,\cdot,\mathfrak{D})$ be a root datum, written $\mathfrak{D} = (Y,\{\check{a}_i\},X,\{a_i\})$ and equipped with twist (Q,n) such that the associated root datum $(I,\circ_n,\widetilde{\mathfrak{D}})$, $\widetilde{\mathfrak{D}} = (\widetilde{Y},\{\widetilde{a}_i^\vee\},\widetilde{X},\{\widetilde{a}_i\})$, is of simply-connected type. Set

$$ W := \mathcal{W}(I,\cdot) \cong \mathcal{W}(I,\circ_{(\mathsf{Q},n)}), \widetilde{W}_{\mathrm{aff}} := W_{\mathrm{aff}}(I,\circ_{(\mathsf{Q},n)}) \cong W \ltimes \widetilde{Y} \tag{3.96}$$

and $\widetilde{H}_{\mathrm{aff}} := H_{\mathrm{aff}}(\widetilde{W}_{\mathrm{aff}},t) \cong H_W \otimes \mathbb{Z}_\tau[\widetilde{Y}]$. Denote the analogues of the modules in (3.95) as

$$ {}_J\widetilde{V} := \varepsilon_J^- \widetilde{H}_{\mathrm{aff}} \quad \text{and} \quad {}_J\widetilde{V}_{\mathrm{sph}} := \varepsilon_J^- \widetilde{H}_{\mathrm{aff}}\,\varepsilon. \tag{3.97}$$

3.6.1. *The module $\widetilde{V}(\check{\eta})$.* Pick $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ and suppose $J \subset I_{\mathrm{aff}}$ is such that $\mathrm{Stab}_{(\widetilde{W}_{\mathrm{aff}},\bullet)}(\check{\eta}) = (\widetilde{W}_{\mathrm{aff}})_J$. Define the A-module $\widetilde{V}(\check{\eta})$ to be a copy of the $\widetilde{H}_{\mathrm{aff}}$-module ${}_J\widetilde{V}$. We now describe a basis $\{\mathsf{v}_{\check{\mu}}\}$ of $\widetilde{V}(\check{\eta})$ indexed by elements $\check{\mu}$ in $\check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ (recall that, as we explained in (3.78), this indexing set is in bijection with ${}^J\widetilde{W}_{\mathrm{aff}}$, the set indexing the natural basis of ${}_J\widetilde{V}$). The elements $\mathsf{v}_{\check{\mu}}$ are defined as follows:

- for $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$, write $\check{\mu} = \check{\eta} \bullet w$ for a unique $w \in {}^J\widetilde{W}_{\mathrm{aff}}$;
- decompose $w = \sigma\,\mathtt{t}(\check{\beta})$ with $\check{\beta} \in \widetilde{Y}$ and $\sigma \in W$;
- set $\mathsf{v}_{\check{\mu}} := \varepsilon_J^- H_\sigma Y_{\check{\beta}}$.

The elements $\mathsf{v}_{\check{\mu}}$ with $\check{\mu} \in \widetilde{W}_{\mathrm{aff}} \bullet \check{\eta}$ satisfy

$$ \mathsf{v}_{\check{\mu}} = \varepsilon_J^- H_\sigma \text{ if } \check{\mu} = \check{\eta} \bullet \sigma, \sigma \in {}^J\widetilde{W}_{\mathrm{aff}}, \tag{3.98}$$

$$ \mathsf{v}_{\check{\mu}} \cdot Y_{\check{\lambda}} = \mathsf{v}_{\check{\lambda}+\check{\mu}} \text{ if } \check{\lambda} \in \widetilde{Y}, \check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}. \tag{3.99}$$

Lemma. *Let $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$, J defined as in (3.77), and $\check{\lambda} \in Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$. Then we may write $\check{\lambda} = \check{\eta} \bullet \sigma\mathtt{t}(\check{\beta})$ with $\check{\beta} \in \widetilde{Y}_+$ and $\sigma\mathtt{t}(\check{\beta}) \in {}^J\widetilde{W}_{\mathrm{aff}}$, and hence $\mathsf{v}_{\check{\lambda}} = \varepsilon_J^- H_x$ for the unique $x \in {}^J\widetilde{W}_{\mathrm{aff}}$ such that $\check{\eta} \bullet x = \check{\lambda}$.*

Proof. Observe that for each $\sigma \in W$, we have $-n(\check{a}_i) < \langle \check{\eta} \bullet \sigma, a_i\rangle < n(\check{a}_i)$. Hence writing $\check{\lambda} = \check{\eta} \bullet \sigma\mathtt{t}(\check{\beta}) = (\check{\eta} \bullet \sigma) + \check{\beta}$ for *any* $\sigma \in W, \check{\beta} \in \widetilde{Y}$, we must have $\langle \check{\beta}, a_i\rangle \geq 0$. Indeed, if $\check{\omega}_{\widetilde{a}_i} = n(\check{a}_i)\check{\omega}_{a_i}$ denotes the fundamental coweight in the root system attached to $\widetilde{\mathfrak{D}}$, we have (by the simply-connected hypothesis) that $\check{\omega}_{\widetilde{a}_i} \in \widetilde{Y}$. The exist $f_i \in \mathbb{Z}$ such that

$$\check{\beta} = \sum_{i \in I} f_i \, \check{\omega}_{\widetilde{a}_i} = \sum_i f_i \, n(\check{a}_i) \check{\omega}_{a_i}, \tag{3.100}$$

from which it follows that $\langle \check{\beta}, a_i \rangle \in \{0, \pm n(\check{a}_i), \pm 2n(\check{a}_i), \dots, \}$. The dominance of $\check{\beta}$ follows, and the second assertion of the Lemma follows from the (3.63) and the fact that if $\check{\beta} \in \widetilde{Y}_+$, then $H_{\mathrm{t}_{\check{\beta}}} = Y_{\check{\beta}}$. $\square$

3.6.2. *Quantum Demazure–Lusztig operators.* The action of the polynomial part of the affine Hecke algebra $\widetilde{H}_{\mathrm{aff}}$, i.e. of the group algebra $\mathbb{Z}_\tau[\widetilde{Y}]$ is given in the basis just introduced of $\widetilde{V}(\check{\eta})$ by formula (3.99). The action of the finite Hecke algebra $H_W \subset \widetilde{H}_{\mathrm{aff}}$ on $\widetilde{V}(\check{\eta})$ can be described by the quantum specialization (2.6) of the *metaplectic Demazure–Lusztig operators* that will be introduced in §4.1. Since we explain the properties of these (unspecialized) Demazure–Lusztig operators in more detail in *loc. cit.*, we introduce them in a somewhat *ad hoc* manner now. To begin, we introduce the expression

$$\upsilon(m) = \begin{cases} -1 & \text{if } m \equiv 0 \mod n \\ \tau & \text{if } m \not\equiv 0 \mod n \end{cases}, \tag{3.101}$$

then define the operators $\widetilde{\mathbf{T}}^{\flat}_{s_i}$ on $\widetilde{V}(\check{\eta})$, acting on the right, via the formulas :

$$\mathsf{v}_{\check{\lambda}} \cdot \left(\tau^{-1} \widetilde{\mathbf{T}}^{\flat}_{s_i}\right) := \begin{cases} \tau^{-1} \, \upsilon(\langle \check{\lambda} + \check{\rho}, a_i \rangle \mathsf{Q}(\check{a}_i)) \, \mathsf{v}_{\check{\lambda} \bullet s_i} + (\tau - \tau^{-1}) \displaystyle\sum_{\substack{k \geq 0 \\ kn(\check{a}) \leq \langle \check{\lambda}, a \rangle}} \mathsf{v}_{\check{\lambda} - kn(\check{a}_i)\check{a}_i} & \text{if } \langle \check{\lambda}, a_i \rangle \geq 0, \\ \tau^{-1} \, \upsilon(\langle \check{\lambda} + \check{\rho}, a_i \rangle \mathsf{Q}(\check{a}_i)) \, \mathsf{v}_{\check{\lambda} \bullet s_i} + (\tau^{-1} - \tau) \displaystyle\sum_{\substack{k > 0 \\ kn(\check{a}) < -\langle \check{\lambda}, a \rangle}} \mathsf{v}_{\check{\lambda} + kn(\check{a}_i)\check{a}_i} & \text{if } \langle \check{\lambda}, a_i \rangle < 0. \end{cases} \tag{3.102}$$

Using these formulas, we find that for $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$,

$$\mathsf{v}_{\check{\mu}} \cdot \left(\tau^{-1} \widetilde{\mathbf{T}}^{\flat}_{s_i}\right) = \begin{cases} \mathsf{v}_{\check{\mu} \bullet s_i} & \text{if } -n(\check{a}_i) < \langle \check{\mu} + \check{\rho}, a_i \rangle < 0, \\ \mathsf{v}_{\check{\mu} \bullet s_i} + (\tau - \tau^{-1}) \mathsf{v}_{\check{\mu}} & \text{if } 0 < \langle \check{\mu} + \check{\rho}, a_i \rangle < n(\check{a}_i), \\ -\tau^{-1} \mathsf{v}_{\check{\mu}} = -\tau^{-1} \mathsf{v}_{\check{\mu} \bullet s_i} & \text{if } \langle \check{\mu} + \check{\rho}, a_i \rangle = 0, \end{cases} \tag{3.103}$$

where in the first two cases, we note that $\langle \check{\mu} + \check{\rho}, a_i \rangle \mathsf{Q}(\check{a}_i) \not\equiv 0 \mod n$ by the definition of $n(\check{a}_i)$ in (3.18).

Proposition. *Let $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$. Then in $\widetilde{V}(\check{\eta})$, we have the relation*

$$\mathsf{v}_{\check{\mu}} \cdot H_{s_i} = \mathsf{v}_{\check{\mu}} \cdot \left(\tau^{-1} \widetilde{\mathbf{T}}^{\flat}_{s_i}\right) \text{ for } \check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}} \text{ and any } i \in I. \tag{3.104}$$

3.6.3. *Proof of Proposition 3.6.2, part 1.* Let $J := \{i \in I_{\mathrm{aff}} \mid \langle \check{\eta} + \check{\rho}, a_i \rangle = 0\}$, so that by definition $\widetilde{V}(\check{\eta}) \cong \widetilde{V}_J$. Suppose that $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ actually lies in the orbit of W, i.e.

$$\check{\mu} = \check{\eta} \bullet \sigma \text{ for } \sigma \in W. \tag{3.105}$$

Then (3.43) together with the fact that $\bullet$ defines an *action* of W implies for each $i \in I$

$$\mathsf{v}_{\check{\mu}} \cdot H_{s_i} = \varepsilon_J^- \cdot H_\sigma \cdot H_{s_i} = \begin{cases} \mathsf{v}_{\check{\mu} \bullet s_i} & \text{if } \sigma s_i > \sigma, \sigma s_i \in {}^J\widetilde{W}_{\mathrm{aff}}, \\ \mathsf{v}_{\mu \bullet s_i} + (\tau - \tau^{-1}) \mathsf{v}_{\check{\mu}} & \text{if } \sigma s_i < \sigma, \sigma s_i \in {}^J\widetilde{W}_{\mathrm{aff}}, \\ -\tau^{-1} \mathsf{v}_{\check{\mu}} & \text{if } \sigma s_i \notin {}^J\widetilde{W}_{\mathrm{aff}}. \end{cases} \tag{3.106}$$

Hence, we need to match up the conditions in (3.106) and (3.103).

Lemma. *For $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$ and $J \subset I_{\mathrm{aff}}$ such that (3.77) holds, we have*

(1) If $\sigma s_i \in {}^J\widetilde{W}_{\mathrm{aff}}$ and $\sigma s_i > \sigma$ then $-n(\check{a}_i) < \langle \check{\mu} + \check{\rho}, a_i \rangle < 0$.
(2) If $\sigma s_i \in {}^J\widetilde{W}_{\mathrm{aff}}$ and $\sigma s_i < \sigma$ then $0 < \langle \check{\mu} + \check{\rho}, a_i \rangle < n(\check{a}_i)$.
(3) If $\sigma s_i \notin {}^J\widetilde{W}_{\mathrm{aff}}$ if and only if $\langle \check{\mu} + \check{\rho}, a_i \rangle = 0$.

Proof. Towards (3), if $\sigma s_i \notin {}^J\widetilde{W}_{\mathrm{aff}}$, using (3.9) we write $\sigma s_i = s_j \sigma$ for a $j \in J$. Hence

$$(\check{\eta} \bullet \sigma) \bullet s_i = \check{\eta} \bullet (\sigma s_i) = \check{\eta} \bullet (s_j \sigma) = \check{\eta} \bullet s_j \bullet \sigma = \check{\eta} \bullet \sigma \tag{3.107}$$

by the assumption on the stabilizer of $\check{\eta}$. But $(\check{\eta} \bullet \sigma) \bullet s_i = \check{\eta} \bullet \sigma$ implies that $\langle \check{\eta} \bullet \sigma + \rho, a_i \rangle = \langle \check{\mu} + \rho, a_i \rangle = 0$. Conversely, if $\langle \check{\mu} + \check{\rho}, a_i \rangle = 0$, then $\check{\mu} \bullet s_i = \check{\mu}$. By definition of $\check{\mu}$, $(\check{\eta} \bullet \sigma) \bullet s_i = \check{\eta} \bullet \sigma$ and hence

$$\check{\eta} \bullet (\sigma s_i) = \check{\eta} \bullet \sigma. \tag{3.108}$$

This last equation means that $\sigma s_i \sigma^{-1} \in (\widetilde{W}_{\mathrm{aff}})_J$ or $\sigma s_i = \tau \sigma$ with $1 \neq \tau \in (\widetilde{W}_{\mathrm{aff}})_J$. Therefore $s_i \sigma \notin {}^J\widetilde{W}_{\mathrm{aff}}$ as σ is the minimal length representative for right $(\widetilde{W}_{\mathrm{aff}})_J$ cosets.

As for (1), first observe that

$$\langle \check{\mu} + \check{\rho}, a_i \rangle = \langle \check{\eta} \bullet \sigma + \check{\rho}, a_i \rangle = \langle (\check{\eta} + \check{\rho}) \cdot \sigma, a_i \rangle = \langle \check{\eta} + \check{\rho}, \sigma^{-1}(a_i) \rangle. \tag{3.109}$$

Using the definition $\widetilde{a}_i$ from §3.2.6, what we need to show is

$$-1 < \langle (\check{\eta} + \check{\rho}) \cdot \sigma, \widetilde{a}_i \rangle < 0. \tag{3.110}$$

By assumption, $-1 < \langle \check{\eta} + \check{\rho}, \widetilde{a}_k \rangle \leq 0$ for all $k \in I$. As $\sigma s_i > \sigma$, $(\widetilde{a}_i)\sigma^{-1} > 0$ and we find that $\langle \check{\mu} + \check{\rho}, a_i \rangle \leq 0$. However, we cannot have $\langle \check{\mu} + \check{\rho}, a_i \rangle = 0$ since this would contradict part (1), so the desired upper bound follows. With our assumption that $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{+,n}$, the lower bound is just the statement that $\check{\eta} + \check{\rho}$ and $(\check{\eta} + \check{\rho})\sigma$ are on the same side of the hyperplane defining the reflection $\sigma_{\widetilde{a}_i,1}$. But this is clear from the fact that $\sigma \in W$ and the description of the set $\widehat{M}(\sigma)$ from (3.67). Part (2) follows from a similar analysis. □

3.6.4. *Proof of Proposition 3.6.2, part 2.* In the general case, suppose $\check{\mu} = \check{\eta} \bullet w$, $w \in {}^J\widetilde{W}_{\mathrm{aff}}$, where $w = \sigma \mathtt{t}(\check{\beta})$ for $\sigma \in W$ and $\check{\beta} \in \widetilde{Y}$. Defining $\check{\lambda} := \check{\eta} \bullet \sigma$ it follows that $\sigma \in {}^J\widetilde{W}_{\mathrm{aff}}$ as well, so that

$$\mathsf{v}_{\check{\lambda}} := \varepsilon_J H_\sigma \text{ and } \mathsf{v}_{\check{\mu}} = \mathsf{v}_{\check{\lambda}} \cdot Y_{\check{\beta}}. \tag{3.111}$$

The Bernstein relation (3.85) for the metaplectic Iwahori Hecke algebra shows that

$$\mathsf{v}_{\check{\mu}} \cdot H_{s_i} = \mathsf{v}_{\check{\lambda}} \cdot Y_{\check{\beta}} \cdot H_{s_i} = \mathsf{v}_{\check{\lambda}} H_{s_i} Y_{\check{\beta}s_i} - (\tau - \tau^{-1})\, \mathsf{v}_{\check{\lambda}} \cdot \frac{Y_{\check{\beta}s_i} - Y_{\check{\beta}}}{1 - Y_{-\widetilde{a}_i^\vee}}. \tag{3.112}$$

We leave it as an exercise to verify that $(\mathsf{v}_{\check{\lambda}} \cdot Y_{\check{\beta}})(\tau^{-1}\widetilde{\mathbf{T}}^\flat_{s_i}) = \mathsf{v}_{\check{\lambda}+\check{\beta}}(\tau^{-1}\widetilde{\mathbf{T}}^\flat_{s_i})$ satisfies the same recursion as $\mathsf{v}_{\check{\lambda}+\check{\beta}} H_{s_i}$ using the explicit formulas given in (3.102).

Remark. *The operators $\widetilde{\mathbf{T}}^\flat_{s_i}$ can also be built from the so-called Chinta–Gunnells action of the Weyl group W (see §4.1.1 - 4.1.2 for more details). Using the special property of the Chinta–Gunnells action* (4.6)*, the recursion asserted above for $\widetilde{\mathbf{T}}^\flat_{s_i}$ then follows immediately from the formula* (4.8)*.*

3.6.5. *The module $\widetilde{V}$.* Keep the same notation as above and consider the sum

$$\widetilde{V} := \bigoplus_{\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}} \widetilde{V}(\check{\eta}). \tag{3.113}$$

as an $\widetilde{H}_{\text{aff}}$-module. Note that as a $\widetilde{H}_{\text{aff}}$-module, many of the summands $\widetilde{V}(\check{\eta})$ will be isomorphic to the same $\widetilde{V}_J$. We may write each $\check{\lambda} \in Y$ as $\check{\lambda} = \check{\eta} \bullet w$ for a unique $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$; defining $\mathsf{v}_{\check{\lambda}} \in \widetilde{V}(\check{\eta})$ as above, we see that $\{\mathsf{v}_{\check{\lambda}}\}_{\check{\lambda} \in Y}$ form an A-basis of V, i.e. we have as A-modules $\widetilde{V} \cong \mathbb{Z}_\tau[\widetilde{Y}]$. The action of $\widetilde{H}_{\text{aff}}$ on V and an involution d are obtained by adding together those on individual $\widetilde{V}(\check{\eta})$.

3.6.6. *On the modules $\widetilde{V}_{\text{sph}}(\check{\eta})$.* For $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ and J as in (3.77), we define

$$\widetilde{V}_{\text{sph}}(\check{\eta}) := \varepsilon_J^- \, \widetilde{H}_{\text{aff}}\, \varepsilon \quad \text{and let} \quad [\mathsf{v}_{\check{\mu}}] := \mathsf{v}_{\check{\mu}}\varepsilon \text{ for } \check{\mu} \in Y \cap \check{\eta} \bullet \widetilde{W}_{\text{aff}}. \tag{3.114}$$

The elements $[\mathsf{v}_{\check{\mu}}]$ will span $\widetilde{V}_{\text{sph}}(\check{\eta})$ by a simple application of the Bernstein presentation, but they will not be linearly independent, as the following relations hold.

Proposition. *Let $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ and let a_i be a simple root with corresponding reflection s_i.*

(1) If $\langle \check{\lambda} + \check{\rho}, a_i \rangle \geq 0$ and $\langle \check{\lambda} + \check{\rho}, a_i \rangle \equiv 0 \mod n(\check{a}_i)$ then $[\mathsf{v}_{\check{\lambda}}] + [\mathsf{v}_{\check{\lambda} \bullet s_i}] = 0$.

(2) If $0 < \langle \check{\lambda} + \check{\rho}, a_i \rangle < n(\check{a}_i)$, then $[\mathsf{v}_{\check{\lambda}}] - \tau[\mathsf{v}_{\check{\lambda} \bullet s_i}] = 0$.

(3) If $\langle \check{\lambda} + \check{\rho}, a_i \rangle > n(\check{a}_i)$ but $j := \langle \check{\lambda} + \check{\rho}, a_i \rangle \not\equiv 0 \mod n(\check{a}_i)$, then setting

$$\check{\lambda}_{(1)} := \check{\lambda} - \mathsf{res}_{n(\check{a}_i)}(\langle \check{\lambda} + \check{\rho}, a_i \rangle)\, \check{a}_i, \text{ we have } [\mathsf{v}_{\check{\lambda}}] = \tau[\mathsf{v}_{\check{\lambda} \bullet s_i}] + \tau[\mathsf{v}_{\check{\lambda}^{(1)}}] - [\mathsf{v}_{\check{\lambda}^{(1)} \bullet s_i}]. \tag{3.115}$$

In type A, the above is due to [77, Prop. 5.9]. The generalization to arbitrary type is in [53, Prop. 6.3(ii)], see also [75, Prop. 4.4]. As it is a specialization of Proposition 4.3.2, we defer the proof to Section 4.

3.6.7. *The space $\widetilde{V}_{\text{sph}}$.* As $\widetilde{V}_{\text{sph}}(\check{\eta}) \cong \varepsilon_J^- \widetilde{H}_{\text{aff}}\varepsilon$, we know that a basis consists of regular, minimal length representatives of the double cosets. On the other hand, we have a bijection, see (3.79):

$$Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\text{aff}} \overset{1:1}{\longleftrightarrow} \left({}^J\widetilde{W}^I_{\text{aff}}\right)_{\text{reg}}, \;\; \check{\mu} = \check{\eta} \bullet w \mapsto (\widetilde{W}_{\text{aff}})_J\, w W. \tag{3.116}$$

Using the straightening rules above, we may show that

Corollary. *Fix $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$. Then $\{[\mathsf{v}_{\check{\mu}}]\}$ for $\check{\mu} \in Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ forms a basis of $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$.*

If we now define the $\mathbb{Z}_\tau$-module

$$\widetilde{V}_{\mathrm{sph}} := \bigoplus_{\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}} \widetilde{V}_{\mathrm{sph}}(\check{\eta}), \tag{3.117}$$

then we find that, as an $\mathbb{Z}_\tau$-module, $\widetilde{V}_{\mathrm{sph}}$ has a basis $[\mathsf{v}_{\check{\lambda}}]$ with $\check{\lambda} \in Y_+$.

3.7. **Involutions and canonical bases in $\widetilde{V}_{\mathrm{sph}}$.** Fix the notation as in §3.6.

3.7.1. *Involution on $\widetilde{V}(\check{\eta})$.* Recall that each $\widetilde{V}(\check{\eta}) \simeq \varepsilon_J^- \widetilde{H}_{\mathrm{aff}} \varepsilon$, and so is equipped with an involution d. It takes the following explicit form with respect to the basis $\mathsf{v}_{\check{\mu}}$.

Lemma. [77, Prop. 5.4] *Let $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ with J as in (3.77). Then for $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$, we have*

$$\overline{\mathsf{v}_{\check{\mu}}} := d(\mathsf{v}_{\check{\mu}}) = (-\tau)^{-\ell(w_0^J)} \mathsf{v}_{\check{\mu} \bullet w_0} \cdot H_{w_0}^{-1}. \tag{3.118}$$

Proof. The proof in *loc. cit.* carries over. To begin, suppose $\check{\eta} \bullet w = \check{\mu}$ with $w \in {}^J\widetilde{W}_{\mathrm{aff}}$ written as $w = \sigma \mathtt{t}(\check{\beta})$, so that $\mathsf{v}_{\check{\mu}} = \varepsilon_J^- \cdot H_\sigma Y_{\check{\beta}}$. By definition and using (3.35) and (3.84), we find

$$\overline{\mathsf{v}_{\check{\mu}}} = \varepsilon_J^- \overline{H_\sigma} \cdot \overline{Y_{\check{\beta}}} = \varepsilon_J^- H_{\sigma w_0} Y_{w_0 \check{\beta}} H_{w_0}^{-1}. \tag{3.119}$$

Now we have that

$$\check{\mu} \bullet w_0 = \check{\eta} \bullet \sigma \mathtt{t}(\check{\beta}) w_0 = \check{\eta} \bullet (\sigma w_0) \cdot \mathtt{t}(w_0 \check{\beta}). \tag{3.120}$$

Writing $\omega := (\sigma w_0)\mathtt{t}(w_0\check{\beta})$, we leave it as an exercise to verify that if $\sigma \mathtt{t}(\check{\beta}) \in {}^J\widetilde{W}_{\mathrm{aff}}$, then $w_0^J \omega \in {}^J\widetilde{W}_{\mathrm{aff}}$. The result follows since $H_{s_i}, i \in J$ acts on ε_J via the scalar $-\tau^{-1}$. $\square$

Note that by Proposition 3.6.2, we could have equivalently defined

$$\overline{\mathsf{v}_{\check{\mu}}} := \tau^{-\ell(w_0)} (-\tau)^{-\ell(w_0^J)} \mathsf{v}_{\check{\mu} \bullet w_0} \cdot (\widetilde{\mathbf{T}}^{\flat}_{w_0})^{-1}. \tag{3.121}$$

Remark. *By definition, if $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$, then $\mathsf{v}_{\check{\eta}} = \varepsilon_J$ and so $\overline{\mathsf{v}_{\check{\eta}}} = \mathsf{v}_{\check{\eta}}$. Hence, the above Lemma implies that*

$$\overline{\mathsf{v}_{\check{\eta}}} = (-\tau)^{-\ell(w_0^J)} \mathsf{v}_{\check{\eta} \bullet w_0} H_{w_0}^{-1} = \mathsf{v}_{\check{\eta}}. \tag{3.122}$$

But the action of H_{w_0} on $\mathsf{v}_{\check{\eta}}$ can be described in terms of the formulas (3.103): we have

$$\mathsf{v}_{\check{\eta} \bullet w} H_{s_i} = \mathsf{v}_{\check{\eta} \bullet w} \cdot (\tau^{-1} \widetilde{\mathbf{T}}^{\flat}_{s_i}) = c_i \mathsf{v}_{\check{\eta} \bullet w s_i} \quad \text{if} \quad \ell(w s_i) > \ell(w), \tag{3.123}$$

where $c_i = 1$ if $i \notin J$ and $c_i = -\tau^{-1}$ if $i \in J$.

3.7.2. *The involution on* $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$. In order to construct a Kazhdan–Lusztig basis on the space $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$ introduced in §3.6.7, we need the following result.

Proposition. *Fix* $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ *with* $J \subset I_{\mathrm{aff}}$ *defined as in §3.6.1. Suppose* $\check{\lambda} \in Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$. *Then we have*

(1) *The involution* $d : \widetilde{V}_{\mathrm{sph}}(\check{\eta}) \to \widetilde{V}_{\mathrm{sph}}(\check{\eta})$ *takes the following form:*

$$d([\mathsf{v}_{\check{\lambda}}]) = (-1)^{\ell(w_0^J)} \tau^{-\ell(w_0)-\ell(w_0^J)} [\mathsf{v}_{\check{\lambda} \bullet w_0}]. \tag{3.124}$$

(2) *With respect to the dominance order* $<$, *for some* $a_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau$ *one has*

$$d([\mathsf{v}_{\check{\lambda}}]) = [\mathsf{v}_{\check{\lambda}}] + \sum_{\substack{\check{\mu} \in \eta \bullet \widetilde{W}_{\mathrm{aff}} \cap Y_+ \\ \check{\mu} < \check{\lambda}}} a_{\check{\mu},\check{\lambda}} [\mathsf{v}_{\check{\mu}}]. \tag{3.125}$$

Proof. We have already seen that $d(\mathsf{v}_{\check{\lambda}}) = (-\tau^{-1})^{\ell(w_0^J)} \mathsf{v}_{\check{\lambda} \bullet w_0} H_{w_0}^{-1}$. On the other hand,

$$d[\mathsf{v}_{\check{\lambda}}] = d(\mathsf{v}_{\check{\lambda}} \varepsilon) = d(\mathsf{v}_{\check{\lambda}}) \varepsilon = (-\tau^{-1})^{\ell(w_0^J)} \mathsf{v}_{\check{\lambda} \bullet w_0} H_{w_0}^{-1} \varepsilon. \tag{3.126}$$

Since $H_{w_0} \varepsilon = \tau^{\ell(w_0)} \varepsilon$, we have $H_{w_0}^{-1} \varepsilon = \tau^{-\ell(w_0)}$ and the result follows. The triangularity can either be deduced by inspecting the form of the straightening rules, or by resorting back to the triangularity of the involution in $\varepsilon_J^- \widetilde{H}_{\mathrm{aff}} \varepsilon$ and then using the comparison of the Bruhat order with the dominance order from (3.81). We will explain the former approach in more detail in the $\mathbb{g}$-twisted setting in the next section. $\square$

3.7.3. *The module* $\widetilde{V}_{\mathrm{sph}}$ *and its canonical bases.* For any $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$, using [84, Lemma 24.2.1] and part (2) of the previous Proposition applied to the set $Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ with the dominance order, we may construct a new basis of $\widetilde{V}_{\mathrm{sph}}$ starting from $[\mathsf{v}_{\check{\mu}}], \check{\mu} \in Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$. In fact, we have two possibilities: we may construct $[\mathsf{G}_{\check{\lambda}}]$ and $[\mathsf{G}^-_{\check{\lambda}}]$ for $\check{\lambda} \in Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ characterized uniquely by the property that they are self-dual with respect to the involution described above and satisfying

$$[\mathsf{G}_{\check{\lambda}}] = [\mathsf{G}_{\check{\lambda}}]^+ := [\mathsf{v}_{\check{\lambda}}] + \sum_{\check{\mu} < \check{\lambda}} o_{\check{\mu},\check{\lambda}} [\mathsf{v}_{\check{\mu}}], \text{ where } o_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau^+ \text{ and} \tag{3.127}$$

$$[\mathsf{G}^-_{\check{\lambda}}] = [\mathsf{v}_{\check{\lambda}}] + \sum_{\check{\mu} < \check{\lambda}} o^-_{\check{\mu},\check{\lambda}} [\mathsf{v}_{\check{\mu}}], \text{ where } o^-_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau^-, \tag{3.128}$$

respectively. If $y, x \in ({}^J W^I)_{\mathrm{reg}}$ are such that $\check{\eta} \bullet y = \check{\lambda}, \check{\eta} \bullet x = \check{\mu}$, then Lemma 3.6.1 implies $[\mathsf{v}_{\check{\lambda}}] = O_y$ and also $[\mathsf{v}_{\check{\mu}}] = O_x$. Hence we may conclude that

$$o_{\check{\mu},\check{\lambda}} = o_{y,x} \quad \text{and} \quad o^-_{\check{\mu},\check{\lambda}} = o^-_{y,x}, \tag{3.129}$$

where the right hand side of each equality were defined in Proposition 3.3.7 .

The collections $\{[\mathsf{G}_{\check{\lambda}}]\}$ and $\{[\mathsf{G}^-_{\check{\lambda}}]\}$ will be referred to as *canonical bases* of $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$. Putting these bases together (for all $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$), we get bases for $\widetilde{V}_{\mathrm{sph}}$ indexed by Y_+ which are again called canonical bases.

3.8. **Littlewood–Richardson polynomials and the $\widetilde{H}_{\mathrm{sph}}$-module structure on $\widetilde{V}_{\mathrm{sph}}$.**

3.8.1. *$\widetilde{H}_{\mathrm{sph}}$-module structure on $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$.* For $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ and J defined as in (3.77), the space $\widetilde{V}_{\mathrm{sph}}(\check{\eta}) = \varepsilon_J^- \widetilde{H}_{\mathrm{aff}}\varepsilon$ is naturally a right $\widetilde{H}_{\mathrm{sph}} = \varepsilon\widetilde{H}_{\mathrm{aff}}\varepsilon$-module under a right action $\star$ introduced in §3.5.4. Using the Satake isomorphism (3.90), we obtain an action of $\mathbb{Z}_\tau[\widetilde{Y}]^W$ on $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$, written as $\diamondsuit$, and defined as

$$\mathsf{v}\diamondsuit f = \mathsf{v} \star S^{-1}(f) \text{ for } f \in \mathbb{Z}_\tau[\widetilde{Y}]^W, \mathsf{v} \in \widetilde{V}_{\mathrm{sph}}(\check{\eta}). \tag{3.130}$$

Proposition. *Let $f(Y) = \sum_{\check{\lambda}\in\widetilde{Y}} c_{\check{\lambda}} Y_{\check{\lambda}} \in \mathbb{Z}_\tau[\widetilde{Y}]^W$ with $c_{\check{\lambda}} \in \mathbb{Z}_\tau$. Then*

$$[\mathsf{v}_{\check{\mu}}] \diamondsuit f = \sum_{\check{\lambda}} c_{\check{\lambda}} [\mathsf{v}_{\check{\lambda}+\check{\mu}}] \text{ for } \check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}. \tag{3.131}$$

Proof. It suffices to verify this for $f = S(h_{\check{\lambda}}), \check{\lambda} \in \widetilde{Y}_+$, since these form a basis of $\mathbb{Z}_\tau[Y]^W$. Writing $S(h_{\check{\lambda}}) = \sum_{\check{\zeta}\in\widetilde{Y}} c_{\check{\zeta},\check{\lambda}} Y_{\check{\zeta}}$ with $\check{\zeta} \in \widetilde{Y}$, we compute from the definitions

$$[\mathsf{v}_{\check{\mu}}]\diamondsuit S(h_{\check{\lambda}}) = [\mathsf{v}_{\check{\mu}}] \star h_{\check{\lambda}} = \mathsf{v}_{\check{\mu}}\varepsilon\,\varepsilon Y_{\check{\lambda}}\,\varepsilon = \mathsf{v}_{\check{\mu}} S(h_{\check{\lambda}})\varepsilon = \sum_{\check{\zeta}} c_{\check{\zeta},\check{\lambda}} [\mathsf{v}_{\check{\zeta}+\check{\lambda}}]. \tag{3.132}$$

□

3.8.2. *The Littlewood–Richardson polynomials.* The Littlewood–Richardson coefficients $c^{\check{\zeta}}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}$ for $\check{\lambda},\check{\mu},\check{\zeta} \in \widetilde{Y}_+$ are defined as decomposition numbers for the multiplication of characters $\chi_{\check{\lambda}} \in \mathbb{Z}[\widetilde{Y}]$ defined in (3.92): $\chi_{\check{\lambda}} \cdot \chi_{\check{\mu}} = \sum_{\check{\zeta}} c^{\check{\zeta}}_{\check{\mu},\check{\lambda}} \chi_{\check{\zeta}}$. A deformation of these numbers to polynomials in type A was introduced in [77] under the name q-Littlewood–Richardson coefficients see [53,75] for the extension to general type. In our setup, the polynomials $Q^{\check{\lambda}}_{\check{\zeta},\check{\mu}}(\tau)$ can be defined via the relation

$$[\mathsf{v}_{\check{\mu}}] \star c_{\check{\lambda}} = [\mathsf{v}_{\check{\mu}}]\diamondsuit\chi_{\check{\lambda}} = \sum_{\check{\zeta}\in Y_+} Q^{\check{\lambda}}_{\check{\zeta},\check{\mu}}(\tau)[\mathsf{v}_{\check{\zeta}}]. \tag{3.133}$$

Remark. *When $n = 1$, one has isomorphism of H_{sph}-modules*

$$V_{\mathrm{sph}} \cong H_{\mathrm{sph}} \cong \mathbb{Z}_\tau[Y]^W \tag{3.134}$$

under which $[\mathsf{v}_{\check{\mu}}]$ maps to $c_{\check{\mu}} \in H_{\mathrm{sph}}$ and $\chi_{\check{\mu}} \in \mathbb{Z}[Y]^W$, resp., reducing $Q^{\check{\lambda}}_{\check{\zeta},\check{\mu}}$ to $c^{\check{\lambda}}_{\check{\zeta},\check{\mu}}$.

Using Proposition 3.6.2 and (3.87), the action of ε on $\widetilde{V}$ is given, in terms of the basis $\mathsf{v}_{\check{\mu}}, \check{\mu} \in Y$, as

$$\mathsf{v}_{\check{\mu}}\varepsilon = \sum_{w\in W} \tau^{-\ell(w)} \mathsf{v}_{\check{\mu}} \widetilde{\mathbf{T}}^{\flat}_w \tau^{-\ell(w)}. \tag{3.135}$$

Let us introduce the element $\mathrm{CS}^\flat(\check{\mu}) \in \widetilde{V}$ (see below for some remarks on this notation)

$$\mathrm{CS}^\flat(\check{\mu}) := \mathsf{v}_{\check{\mu}} \cdot \varepsilon = \sum_{w\in W} \mathsf{v}_{\check{\mu}} \cdot \widetilde{\mathbf{T}}^{\flat}_w \tau^{-\ell(w)} \tag{3.136}$$

and write its image in $\widetilde{V}_{\mathrm{sph}}$ as $[\mathrm{CS}^\flat(\check{\mu})]$. Recall the polynomials $p_{\check{\tau},\check{\lambda}}$ from (3.93).

Proposition. *Let $\check{\lambda} \in \widetilde{Y}_+$ and $\check{\mu} \in Y_+$. Then we have*

$$\sum_{\check{\tau} \leq \check{\lambda}} p_{\check{\tau},\check{\lambda}} [\mathrm{CS}^\flat(\check{\mu}) \cdot Y_{\check{\tau}}] = \sum_{\check{\zeta} \in Y_+} Q^{\check{\lambda}}_{\check{\zeta},\check{\mu}}(\tau)[\mathsf{v}_{\check{\zeta}}]. \tag{3.137}$$

Note that in order to write the left hand side of the above relation (3.137) in terms of the basis elements $[\mathsf{v}_{\check{\zeta}}]$, we may need to use the straightening relations of Proposition 3.6.6.

Proof. It suffices to show, in light of the relation (3.93), that $[\mathsf{v}_{\check{\mu}}] \diamondsuit S(h_{\check{\lambda}}) = [\mathrm{CS}^\flat(\check{\mu}) \cdot Y_{\check{\lambda}}]$. This follows, since by definition, $[\mathsf{v}_{\check{\mu}}] \diamondsuit S(h_{\check{\lambda}}) = \mathsf{v}_{\check{\mu}} \varepsilon Y_{\check{\lambda}} \varepsilon = \mathrm{CS}^\flat(\check{\mu}) Y_{\check{\lambda}} \varepsilon$. $\square$

Remark. *The notation $\mathrm{CS}^\flat$ comes from the fact that in the metaplectic setting, the analogous expression $\widetilde{\mathrm{CS}}(\check{\mu})$ gives the metaplectic Casselman–Shalika formula that is featured prominently in the literature on multiple Dirichlet series. Motivated by this, McNamara* [89] *has proved combinatorial formulas expressing $\widetilde{\mathrm{CS}}(\check{\mu})$ as a sum over the crystal $\mathscr{B}_{\check{\mu}+\check{\rho}}$ (or alternatively as a sum over certain Gelfand–Tsetlin patterns in type A) and in* [17]*, lattice models are studied whose partition function yields $\widetilde{\mathrm{CS}}(\check{\mu})$. Although both results are proved in the metaplectic setting, one may treat the parameters introduced there as formal variables and perform a 'quantum specialization' to obtain what appears to be new combinatorial formulas for $Q^{\check{\lambda}}_{\check{\zeta},\check{\mu}}(\tau)$.*

3.9. **Tensor product theorems.** We formulate below two results for the bases $[\mathsf{G}_{\check{\lambda}}]$ and $[\mathsf{G}^-_{\check{\lambda}}]$ introduced in §3.7.3. Modeled on the Steinberg-Lusztig theorem (see Thm. 7.3.6 (1)), the result for $[\mathsf{G}^-_{\check{\lambda}}]$ seems to originate in the work of [77] in type A. The proof sketched below follows the nice argument due to Lanini-Ram [74] for general type. In fact, the argument in *op. cit.* can be easily modified to also give a tensor product theorem for $[\mathsf{G}_{\check{\lambda}}]$ which matches a result due to Andersen for tilting modules (see Thm. 7.3.6(2)).

3.9.1. *Tensor product theorem.* Recall the set $\square_{(\mathsf{Q},n)}$ from §3.4.9.

Proposition. *Let $\check{\lambda} \in Y_+$ be decomposed uniquely as $\check{\lambda} = \check{\lambda}_0 + \check{\zeta}$, where $\check{\zeta} \in \widetilde{Y}_+$ and $\check{\lambda}_0 \in \square_{(\mathsf{Q},n)}$.*

(1) [74, Thm 1.4] *We have $[\mathsf{G}^-_{\check{\lambda}_0}] \star \widetilde{c}_{\check{\zeta}} = [\mathsf{G}^-_{\check{\lambda}_0+\check{\zeta}}]$.*

(2) Writing $\check{\lambda}_0^\dagger := \check{\lambda}_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check{\rho})$ we have

$$[\mathsf{G}_{\check{\lambda}_0^\dagger}] \star \widetilde{c}_{\check{\zeta}} = [\mathsf{G}_{\check{\lambda}_0^\dagger+\check{\zeta}}]. \tag{3.138}$$

In both cases, the idea is to verify that the left hand side of the desired equality satisfies the characterizing properties of the right hand side (see §3.7.3). We shall just focus on the $[\mathsf{G}_{\check{\mu}}]$ variant. Since

$$\langle \check{\lambda}^\dagger, a_i \rangle = \langle \check{\lambda}_0, a_i \cdot w_0 \rangle + 2(n(\check{a}_i) - 1), \tag{3.139}$$

$\check{\lambda}_0 \in \square_{(\mathsf{Q},n)}$, and $a_i \cdot w_0$ is the negative of some simple root, it follows that $\check{\lambda}^\dagger \in Y_+$.

3.9.2. *Proof of Proposition 3.9.1, part 1: self-duality.* Pick $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ with J defined as in (3.77). We assume $[\mathsf{G}_{\check{\lambda}_0^\dagger}] \in \widetilde{V}_{\mathrm{sph}}(\check{\eta})$ is written as $[\mathsf{G}_{\check{\lambda}_0^\dagger}] = \varepsilon_J^- h\varepsilon$ for $h \in \widetilde{H}_{\mathrm{sph}}$. Write also $\widetilde{c}_{\check{\zeta}} = \varepsilon h'\varepsilon$ for some $h', h' \in \widetilde{H}_{\mathrm{aff}}$. As both of these elements are self-dual,

$$d\left([\mathsf{G}_{\check{\lambda}_0^\dagger}]\right) = \overline{\mathsf{v}_{\check{\eta}} h\varepsilon} = \mathsf{v}_{\check{\eta}}\overline{h}\varepsilon = \mathsf{v}_{\check{\eta}} h\varepsilon = [\mathsf{G}_{\check{\lambda}_0^\dagger}] \text{ and } d(\widetilde{c}_{\check{\zeta}}) = \overline{\varepsilon h'\varepsilon} = \varepsilon\overline{h'}\varepsilon = \widetilde{c}_{\check{\zeta}}, \tag{3.140}$$

we may apply the fact that d is a homomorphism to conclude

$$d\left([\mathsf{G}_{\check{\lambda}_0^\dagger}] \star \widetilde{c}_{\check{\zeta}}\right) = d(\mathsf{v}_{\check{\eta}} h\varepsilon\varepsilon h'\varepsilon) = \overline{\mathsf{v}_{\check{\eta}}}\overline{h}\varepsilon\overline{h'}\varepsilon = [\mathsf{G}_{\check{\lambda}_0^\dagger}] \star \widetilde{c}_{\check{\zeta}}. \tag{3.141}$$

3.9.3. *Proof of Proposition 3.9.1, part 2: triangularity.* We may write

$$[\mathsf{G}_{\check{\lambda}_0^\dagger}] = [\mathsf{v}_{\check{\lambda}_0}] + \textstyle\sum_{\check{\mu}<\check{\lambda}_0} o_{\check{\mu},\check{\lambda}_0}[\mathsf{v}_{\check{\mu}}] \quad \text{ and } \quad c_{\check{\zeta}} = h_{\check{\zeta}} + \textstyle\sum_{\check{\eta}<\check{\zeta}} p_{\check{\eta},\check{\zeta}} h_{\check{\eta}} \tag{3.142}$$

for elements $o_{\check{\mu},\check{\lambda}_0} \in \mathbb{Z}_\tau^+$ and $p_{\check{\eta},\check{a}} \in \mathbb{Z}_\tau^+$ defined earlier. From the straightening rules and the proof of Proposition 3.8.1, we have for any $\check{\lambda} \in Y_+$ and $\check{\mu} \in \widetilde{Y}_+$

$$[\mathsf{v}_{\check{\lambda}}] \star h_{\check{\mu}} = [\mathsf{v}_{\check{\lambda}+\check{\mu}}] + \sum_{\check{\xi}<\check{\lambda}+\check{\mu}} d_{\check{\xi},\check{\lambda}+\check{\mu}}[\mathsf{v}_{\check{\xi}}] \text{ for some } d_{\check{\xi},\check{\lambda}+\check{\mu}} \in \mathbb{Z}_\tau. \tag{3.143}$$

From here it follows that

$$[\mathsf{G}_{\check{\lambda}_0^\dagger}] \star \widetilde{c}_{\check{\zeta}} = [\mathsf{v}_{\check{\lambda}}] + \sum_{\check{\mu}<\check{\lambda}} a_{\check{\mu},\check{\lambda}}[\mathsf{v}_{\check{\mu}}], \text{ for } a_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau; \tag{3.144}$$

it remains to show $a_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau^+$. To do so, recall a few facts about Littelmann's path model for representations.

3.9.4. *Recollections on the path model.* We apply the path model of Littelmann [78] to the root system defined by $\widetilde{\mathfrak{D}}$. From *op. cit.*, the character of a highest representation is written as a *positive* sum over *paths* in the highest weight crystal $B(\check{\zeta})$, *i.e.*

$$\chi_{\check{\lambda}} := \sum_{p\in B(\check{\zeta})} Y_{\mathrm{wt}(p)}, \tag{3.145}$$

where $\mathrm{wt}(p) \in \widetilde{Y}$ denotes the end-point of the path. Recall also that there exists an involution on the set $\iota : B(\check{\zeta}) \setminus \{p_{\check{\zeta}}\}$, where $p_{\check{\zeta}}$ is the unique path in $B(\check{\zeta})$ with endpoint $\check{\zeta}$, which has the property that

$$\mathrm{wt}(\iota(p)) = \mathrm{wt}(p) \bullet s_i \text{ for a unique } i \in I. \tag{3.146}$$

3.9.5. *Proof of Proof of Proposition 3.9.1, part 3: straightening.* Write $a \equiv b$ to mean $a - b \in \oplus_{\check{\lambda}\in Y}\mathbb{Z}_\tau^+[Y]$, *i.e.* it is a linear combination of $Y_{\check{\mu}}$ with $\check{\mu} \in Y$ and the coefficients are in $\mathbb{Z}_\tau^+$. In this notation, the straightening rules in Proposition 4.3.2 imply that for $\check{\mu} \in Y_+$,

$$[Y_{\check{\mu}}] \equiv \begin{cases} 0 & \text{if } 0 \le \langle \check{\mu} + \check{\rho}, a_i \rangle < n(\check{a}_i), \\ -[Y_{\check{\mu}^{(1)}\bullet s_a}] & \text{otherwise.} \end{cases} \tag{3.147}$$

Lemma. [74, Lemma 1.5] *In the notation of part (2) of Proposition 3.9.1*

$$[Y_{\check{\lambda}_0^\dagger+\check{\zeta}}] \equiv -[Y_{\check{\lambda}_0^\dagger+\check{\zeta}\bullet s_i}] \quad \textit{for any} \quad i \in I. \tag{3.148}$$

Proof. Recall $\check{\lambda}^\dagger = \check{\lambda}_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check{\rho})$ and write $\check{\mu} = \check{\lambda}^\dagger + \check{\zeta}$. We compute

$$\langle \check{\mu} + \rho, a_i \rangle = \langle \check{\lambda}_0 \cdot w_0, a_i \rangle + 2(n(\check{a}_i) - 1) + 1 + \langle \check{\zeta}, a_i \rangle \tag{3.149}$$

$$= \langle \check{\lambda}_0 \cdot w_0, a_i \rangle + 2n(\check{a}_i) - 1 + \langle \check{\zeta}, a_i \rangle. \tag{3.150}$$

Using that $\langle \check{\zeta}, a_i \rangle \geq 0$ and $\check{\zeta} \in \widetilde{Y}$ as well as the fact that $-n(\check{a}_i) < \langle \check{\lambda}_0 \cdot w_0, a_i \rangle \leq 0$, we conclude that

$$\mathsf{res}_{n(\check{a}_i)} \langle \check{\mu} + \rho, a_i \rangle = \langle \check{\lambda}_0 \cdot w_0, a_i \rangle + n(\check{a}_i) - 1. \tag{3.151}$$

Using the definition that $\check{\mu}^{(1)} = \check{\mu}^{(1)} - \langle \check{\mu} + \check{\rho}, a_i \rangle \check{a}_i$ we obtain

$$\check{\mu}^{(1)} \bullet s_i = \check{\mu}^{(1)} \cdot s_i - \check{a}_i = \check{\lambda}^\dagger + \check{\zeta} - \mathsf{res}_{n(\check{a}_i)} \langle \check{\mu} + \rho, a_i \rangle - \check{a}_i \tag{3.152}$$

$$= \check{\lambda}_0 \cdot w_0 \cdot s_i + 2(\widetilde{\rho}^\vee - \check{\rho} - \check{a}_i) + \check{\zeta} \cdot s_i + \Big(\langle \check{\lambda}_0 \cdot w_0, a_i \rangle + n(\check{a}_i) - 1 \Big) \check{a}_i - \check{a}_i \tag{3.153}$$

$$= \check{\lambda}_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check{\rho}) + \check{\zeta} \cdot s_i - \check{a}_i. \tag{3.154}$$

The rest of the argument follows as in Lemma [74, Lemma 1.5]. $\square$

3.9.6. *Proof of Proof of Proposition 3.9.1, part 4: conclusion.* Since the $o_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau^+$

$$[\mathsf{G}_{\check{\lambda}_0^\dagger}] \star \widetilde{c}_{\check{\zeta}} \equiv [\mathsf{v}_{\check{\lambda}_0^\dagger}] \star \widetilde{c}_{\check{\zeta}}. \tag{3.155}$$

From our description of the action by $\widetilde{c}_{\check{\zeta}}$ in (3.133) and formula (3.145), this amounts to checking that

$$[\mathsf{v}_{\check{\lambda}_0^\dagger}] \diamondsuit \chi_{\check{\zeta}} = \sum_{p \in B(\check{\zeta})} [\mathsf{v}_{\check{\lambda}_0^\dagger + \mathrm{wt}(p)}] \equiv [\mathsf{v}_{\check{\lambda}_0^\dagger + \check{\zeta}}]. \tag{3.156}$$

Using the previous Lemma and what we recalled about ι in §3.9.4, we are done.

4. $\mathfrak{g}$-TWISTED CANONICAL BASES AND KAZHDAN-LUSZTIG POLYNOMIALS

Recall the generic ring $\mathbb{Z}_{\tau,\mathfrak{g}}$ from §2.3.1 which was defined in terms of formal parameters τ (where $\tau^2 = t^{-1}$) and the $\mathfrak{g}_k, k \in \mathbb{Z}$ subject to the relations as described in *loc. cit.* In this section, we introduce a '$\mathfrak{g}$-twisted' version of the constructions from the previous one. Under the quantum specialization (see §2.3.2) $\mathfrak{g}_k \mapsto \tau$, $k \neq 0$ we recover the constructions from the previous section, and under the p-adic specialization of §2.3.3), as we explain in Part II of this paper, the constructions of this section provide information about the structure of the Whittaker module on covers of p-adic groups.

We keep the same conventions as at the start of §3.6. Namely, we fix a root datum $(I, \cdot, \mathfrak{D})$, which is written $\mathfrak{D} = (Y, \{\check{a}_i\}, X, \{a_i\})$, and a twist (Q, n) on $\mathfrak{D}$ such that the twisted root datum $(I, \circ_{(\mathsf{Q},n)}, \widetilde{\mathfrak{D}})$ is of simply-connected type. Set $\widetilde{\mathfrak{D}} = (\widetilde{Y}, \{\widetilde{a}_i^\vee\}, \widetilde{X}, \{\widetilde{a}_i\})$ and write $\widetilde{H}_{\mathrm{aff}} := H_{\mathrm{aff}}(I, \circ_{(\mathsf{Q},n)}, \widetilde{\mathfrak{D}})$ and $\widetilde{H}_{\mathrm{sph}}$ for the corresponding affine Hecke algebra and spherical Hecke algebra, respectively. Recall that the Bernstein presentation for $\widetilde{H}_{\mathrm{aff}}$ allows us to identify it as

$$\widetilde{H}_{\mathrm{aff}} \cong H_W \otimes \mathbb{Z}_\tau[\widetilde{Y}], \tag{4.1}$$

where H_W is the finite Hecke algebra of $W := W(I,\cdot) \cong W(I, \circ_{(\mathsf{Q},n)})$ with generators $T_{s_i}, i \in I$, or their renormalizations $H_{s_i} = \tau^{-1} T_{s_i}$, satisfying the braid relations and the quadratic relations

$$(T_{s_i} - \tau^2)(T_{s_i} + 1) = 0 \text{ or equivalently } (H_{s_i} - \tau)(H_{s_i} - \tau^{-1}) = 0. \tag{4.2}$$

The key to our constructions in this section is to use a representations of the affine Hecke algebra $\widetilde{H}_{\mathrm{aff}}$ on a space of polynomials $\mathbb{Z}_{\tau,\mathbb{g}}[Y]$. Such a construction was introduced in [31, 95], and what is new here is the use of ideas from the previous section to equip $\mathbb{Z}_{\tau,\mathbb{g}}[Y]$ with an explicit decomposition into $\widetilde{H}_{\mathrm{aff}}$-submodules and an involution that allows us to introduce a $\mathbb{g}$-twisted version of Kazhdan–Lusztig theory.

4.1. **Metaplectic Demazure–Lusztig operators.** Our goal in this section is to construct a representation of $\widetilde{H}_{\mathrm{aff}}$ on a certain space of polynomials. Our computations will take place in the ring $\mathbb{Z}_{\tau,\mathbb{g}}[Y]$ (or $\mathbb{Z}_{\tau,\mathbb{g}}[\widetilde{Y}]$). We also introduce localizations $\mathbb{Z}_{\tau,\mathbb{g},\mathfrak{m}}[Y]$ and $\mathbb{Z}_{\tau,\mathbb{g},\mathfrak{m}}[\widetilde{Y}]$ by the ideal

$$\mathfrak{m} = \Big(1 - Y_{-\widetilde{a}_i^\vee}, 1 - \tau^{-2} Y_{-\widetilde{a}_i^\vee} \mid i \in I\Big) \subset \mathbb{C}_{\tau,\mathbb{g}}[Y], \tag{4.3}$$

and for a positive integer $n > 0$, define the residue map $\mathrm{res}_n : \mathbb{Z} \to \{0, 1, \ldots, n-1\}$.

4.1.1. *The Chinta–Gunnells action.* Following Chinta and Gunnells [30, Def. 3.1], define

$$Y_{\check{\lambda}} \star s_a = \frac{Y_{\check{\lambda}\cdot s_a}}{1 - \tau^{-2} Y_{-\widetilde{a}^\vee}} \left[(1-\tau^{-2}) Y_{\mathrm{res}_{n(\check{a})}\left(\frac{\mathsf{B}(\check{\lambda},\check{a})}{\mathsf{Q}(\check{a})}\right)\check{a}} - \tau^{-2} \mathbb{g}_{\mathsf{Q}(\check{a}) + \mathsf{B}(\check{\lambda},\check{a})} Y_{\widetilde{a}^\vee - \check{a}} (1 - Y_{-\widetilde{a}^\vee}) \right] \text{ for } \check{\lambda} \in Y, a \in \Pi, \tag{4.4}$$

where $\check{\lambda} \cdot s_a$ denotes the right action of s_a on $\check{\lambda}$ and Π denotes the set of simple roots attached to $\mathfrak{D}$. Extend this by $\mathbb{Z}_{\tau,\mathbb{g}}$-linearity to define $f \star s_a$ for every $f \in \mathbb{Z}_{\tau,\mathbb{g}}[Y]$ and use

$$\frac{f}{h} \star s_a = \frac{f \star s_a}{h^{s_a}} \text{ for } f \in \mathbb{Z}_{\tau,\mathbb{g}}[Y], h \in \mathfrak{m}, \tag{4.5}$$

to further extend this to an action $\star$ of W: $- \star s_a : \mathbb{Z}_{\tau,\mathbb{g},\mathfrak{m}}[Y] \to \mathbb{Z}_{\tau,\mathbb{g},\mathfrak{m}}[Y]$. Note that for $a \in \Pi$ we have that

$$(f \cdot h) \star s_a = (f \star s_a) \cdot h^{s_a} \text{ for } h \in \mathbb{Z}_{\tau,\mathbb{g}}[\widetilde{Y}], f \in \mathbb{Z}_{\tau,\mathbb{g}}[Y], \tag{4.6}$$

where h^{s_a} is the usual action of s_a on $\mathbb{Z}_{\tau,\mathbb{g}}[Y]$.

4.1.2. *Metaplectic Demazure–Lusztig operators.* Introduce the rational functions

$$\mathbf{b}(X) = \tfrac{\tau^2-1}{1-Y_{-X}} \quad \text{and} \quad \mathbf{c}(X) = \tfrac{\tau^2-Y_{-X}}{1-Y_X}. \tag{4.7}$$

For each $a \in \Pi$ with corresponding simple reflection s_a, we define the following elements in $\mathbb{Z}_{\tau,\mathbb{g}}(Y)[W]^\vee$:

$$\widetilde{\mathbf{T}}_{s_a} := \widetilde{\mathbf{T}}_a := [s_a]\mathbf{c}(\widetilde{a}^\vee) + [1]\mathbf{b}(\widetilde{a}^\vee). \tag{4.8}$$

Consider now the action on $\mathbb{C}_{\tau,\mathbb{g},\mathfrak{m}}[Y]$ of $\widetilde{\mathbf{T}}_a$ by the formulas

$$Y_{\check{\lambda}} \cdot \widetilde{\mathbf{T}}_a = \mathbf{c}(\widetilde{a}^\vee) Y_{\check{\lambda}} \star s_a + \mathbf{b}(\widetilde{a}^\vee) Y_{\check{\lambda}}. \tag{4.9}$$

Remark. *Note that our operators $\widetilde{\mathbf{T}}_a$ are the inverse of the ones in* [95, eq. (4.10)] *(if one replaces v with τ^{-2}). They also act on the right here, as opposed to the left in* op. cit.

The operators $\widetilde{\mathbf{T}}_a$ will be used to define a representation of the affine Hecke algebra on $\mathbb{C}_{\tau,\mathbb{g}}[Y]$ (see §4.2.1). Recall the dot action of the affine Weyl groups W_{aff} and $\widetilde{W}_{\mathrm{aff}}$ on Y from 3.4.6 and (3.73), respectively, as well as the fundamental domain $\overline{\mathcal{A}}^{\bullet}_{-,n}$ for the dot action from (3.74). From the computation in [95, (4.18) and (4.12)][22], we deduce the following formulas.

Lemma. *(1) For $a \in \Pi$ and $\check{\lambda} \in Y$, we have*

$$Y_{\check{\lambda}} \cdot \widetilde{\mathbf{T}}_a = \begin{cases} \mathbb{g}_{\langle \check{\lambda}+\check{\rho}, a\rangle\, \mathsf{Q}(\check{a})} Y_{\check{\lambda}\bullet s_a} + (\tau^2-1) \displaystyle\sum_{\substack{k\geq 0\\ kn(\check{a})\leq \langle\check{\lambda},a\rangle}} Y_{\check{\lambda}-kn(\check{a})\check{a}} & \text{if } \langle\check{\lambda},a\rangle \geq 0,\\ \mathbb{g}_{\langle \check{\lambda}+\check{\rho}, a\rangle\, \mathsf{Q}(\check{a})} Y_{\check{\lambda}\bullet s_a} + (1-\tau^2) \displaystyle\sum_{\substack{k> 0\\ kn(\check{a})< -\langle\check{\lambda},a\rangle}} Y_{\check{\lambda}+kn(\check{a})\check{a}} & \text{if } \langle\check{\lambda},a\rangle < 0. \end{cases} \tag{4.10}$$

(2) For $\check{\mu} = \check{\eta} \bullet W$ and $i \in I$, we have

$$Y_{\check{\mu}} \cdot \left(\tau^{-1}\widetilde{\mathbf{T}}_{s_i}\right) = \begin{cases} \tau^{-1}\,\mathbb{g}_{\langle\check{\mu}+\check{\rho},a_i\rangle\,\mathsf{Q}(\check{a}_i)} Y_{\check{\mu}\bullet s_i} & \text{if } -n(\check{a}_i) < \langle\check{\mu}+\check{\rho},a_i\rangle < 0,\\ \tau^{-1}\,\mathbb{g}_{\langle\check{\mu}+\check{\rho},a_i\rangle\,\mathsf{Q}(\check{a}_i)} Y_{\check{\mu}\bullet s_i} + (\tau-\tau^{-1})Y_{\check{\mu}} & \text{if } 0 < \langle\check{\mu}+\check{\rho},a_i\rangle < n(\check{a}_i),\\ -\tau^{-1}Y_{\check{\mu}} = -\tau^{-1}Y_{\check{\mu}\bullet s_i} & \text{if } \langle\check{\mu}+\check{\rho},a_i\rangle = 0. \end{cases} \tag{4.11}$$

(3) If $\check{\mu} \in Y, \check{\zeta} \in \widetilde{Y}$ and $i \in I$, we have

$$(Y_{\check{\mu}} Y_{\check{\zeta}}) \cdot (\tau^{-1}\widetilde{\mathbf{T}}_{s_i}) = Y_{\check{\mu}} \cdot (\tau^{-1}\widetilde{\mathbf{T}}_{s_i}) + (\tau-\tau^{-1}) \frac{Y_{\check{\zeta}\cdot s_i} - Y_{\check{\zeta}}}{1-Y_{-\check{a}}}. \tag{4.12}$$

Remark. *Note that the formulas in* (4.10) *are equivalent to the ones in* (4.11) *and* (4.12).

[22]Note that in (4.12) there is a mistake; the right hand side of the equality should be replaced by $Y_{\check{\lambda}} + v\mathbb{g}_{\mathsf{Q}(\check{a})} Y_{\check{\lambda}-\check{a}}$

4.1.3. *Statement of result.* The following computation plays a crucial role in our work. In making it we were inspired by the straightening formulas of [75, (1.3)] and [53, Prop. 6.3], see [77].

Proposition. *Let $\check{\lambda} \in Y$, $a \in \Pi$, and $n(\check{a})$ as in (3.18).*

(1) If $\langle \check{\lambda} + \check{\rho}, a \rangle \geq 0$ and $\langle \check{\lambda} + \check{\rho}, a \rangle \equiv 0 \mod n(\check{a})$ then

$$\left(Y_{\check{\lambda}} + Y_{\check{\lambda} \bullet s_a}\right) \widetilde{\mathbf{T}}_a = -\left(Y_{\check{\lambda}} + Y_{\check{\lambda} \bullet s_a}\right). \tag{4.13}$$

(2) If $0 < \langle \check{\lambda} + \check{\rho}, a \rangle < n(\check{a})$, then

$$\left(Y_{\check{\lambda}} - \mathfrak{g}_{\langle \check{\lambda} + \check{\rho}, a \rangle \, \mathsf{Q}(\check{a})} Y_{\check{\lambda} \bullet s_a}\right) \widetilde{\mathbf{T}}_a = -\left(Y_{\check{\lambda}} - \mathfrak{g}_{\langle \check{\lambda} + \check{\rho}, a \rangle \, \mathsf{Q}(\check{a})} Y_{\check{\lambda} \bullet s_a}\right). \tag{4.14}$$

(3) If $\langle \check{\lambda} + \check{\rho}, a \rangle > n(\check{a})$ but $j := \langle \check{\lambda} + \check{\rho}, a \rangle \not\equiv 0 \mod n(\check{a})$, then setting $\check{\lambda}^{(1)} := \check{\lambda} - \mathsf{res}_{n(\check{a})}(\langle \check{\lambda} + \check{\rho}, a \rangle)\, \check{a}$,

$$\left(Y_{\check{\lambda}} - \mathfrak{g}_{j \mathsf{Q}(\check{a})} Y_{\check{\lambda} \bullet s_a} - \mathfrak{g}_{j \mathsf{Q}(\check{a})} Y_{\check{\lambda}^{(1)}} + Y_{\check{\lambda}^{(1)} \bullet s_a}\right) \widetilde{\mathbf{T}}_a = -\left(Y_{\check{\lambda}} - \mathfrak{g}_{j \mathsf{Q}(\check{a})} Y_{\check{\lambda} \bullet s_a} - \mathfrak{g}_{j \mathsf{Q}(\check{a})} Y_{\check{\lambda}^{(1)}} + Y_{\check{\lambda}^{(1)} \bullet s_a}\right) \tag{4.15}$$

The proof will occupy the remainder of this section. Set $m := n(\check{a}) \geq 1$, $\widetilde{a}^{\vee} := m\check{a}$, then

$$\langle \check{\lambda} + \check{\rho}, a \rangle = j + md, \text{ where } j \in \{0, 1, \ldots, m-1\} \text{ and } d \in \mathbb{Z}. \tag{4.16}$$

Let us also record here that $\langle \check{\lambda} \bullet s_a, a \rangle = -\langle \check{\lambda}, a \rangle - 2$.

4.1.4. *Proof of Proposition 4.1.3 (1).* In case (1), we must have $j = 0$ and $d \geq 0$. Assume further $d = 0$ so that $\langle \check{\lambda} + \check{\rho}, a \rangle = 0$ and hence

$$\check{\lambda} \bullet s_a = (\check{\lambda} + \check{\rho}) s_a - \check{\rho} = \check{\lambda} + \check{\rho} - \check{\rho} = \check{\lambda}. \tag{4.17}$$

Since $\langle \check{\lambda}, a \rangle = \langle \check{\lambda} \bullet s_a, a \rangle = -1$ we may apply the second case of (4.10) to conclude that

$$\left(Y_{\check{\lambda}}\right) \widetilde{\mathbf{T}}_a = \left(Y_{\check{\lambda} \bullet s_a}\right) \widetilde{\mathbf{T}}_a = -Y_{\check{\lambda}}, \tag{4.18}$$

where we used $\mathfrak{g}_{\langle \check{\lambda} + \check{\rho}, a \rangle \, \mathsf{Q}(\check{a})} = \mathfrak{g}_0 = -1$. Next, assume that $d \geq 1$ in (4.16) and $j = 0$, so

$$\langle \check{\lambda}, a \rangle = md - 1 \geq 0 \text{ and } \langle \check{\lambda} \bullet s_a, a \rangle = -md - 1 < 0. \tag{4.19}$$

Applying the first and second case of (4.10) to $Y_{\check{\lambda}}$ and $Y_{\check{\lambda} \bullet s_a} = Y_{\check{\lambda} - dm\check{a}}$, resp., we find

$$\begin{aligned} \left(Y_{\check{\lambda}}\right) \widetilde{\mathbf{T}}_a &= -Y_{\check{\lambda} \bullet s_a} + (\tau^2 - 1) \sum_{0 \leq k \leq d-1} Y_{\check{\lambda} - k\widetilde{a}^{\vee}}, \text{ and} && (4.20) \\ \left(Y_{\check{\lambda} \bullet s_a}\right) \widetilde{\mathbf{T}}_a &= -Y_{\check{\lambda}} + (1 - \tau^2) \sum_{0 < k \leq d} Y_{\check{\lambda} - d\widetilde{a}^{\vee} + k\widetilde{a}^{\vee}}. && (4.21) \end{aligned}$$

If we add the above expressions, the terms in the sums over k cancel, which leaves with

$$\left(Y_{\check{\lambda}} + Y_{\check{\lambda} \bullet s_a}\right) \widetilde{\mathbf{T}}_a = -\left(Y_{\check{\lambda}} + Y_{\check{\lambda} \bullet s_a}\right). \tag{4.22}$$

4.1.5. *Proof of Proposition 4.1.3 (2).* In case (2), we must have $d=0$ and so $j=\langle\check{\lambda}+\check{\rho},a\rangle>0$. Hence $\langle\check{\lambda},a\rangle\geq 0$ and we may apply the first case of (4.10) to obtain

$$\left(Y_{\check{\lambda}}\right)\widetilde{\mathbf{T}}_a=\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}\bullet s_a}+(\tau^2-1)Y_{\check{\lambda}}. \tag{4.23}$$

On the other hand, to compute $\left(Y_{\check{\lambda}\bullet s_a}\right)\widetilde{\mathbf{T}}_a$ we note (see after (4.16)) that $\langle\check{\lambda}\bullet s_a,a\rangle=-(j+1)<0$ so that the second case of (4.10) yields

$$\left(Y_{\check{\lambda}\bullet s_a}\right)\widetilde{\mathbf{T}}_a=\mathfrak{g}_{-j\mathsf{Q}(\check{a})}Y_{\check{\lambda}}, \tag{4.24}$$

since the sum over k in the second expression of (4.10) has no terms since $j+1\leq m$. One then has

$$\begin{aligned}\left(\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}\bullet s_a}-Y_{\check{\lambda}}\right)\widetilde{\mathbf{T}}_a &= \mathfrak{g}_{j\mathsf{Q}(\check{a})}\mathfrak{g}_{-j\mathsf{Q}(\check{a})}Y_{\check{\lambda}}-\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}\bullet s_a}-(\tau^2-1)Y_{\check{\lambda}} &(4.25)\\ &= -\left(\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}\bullet s_a}-Y_{\check{\lambda}}\right), &(4.26)\end{aligned}$$

where we used that $\mathfrak{g}_{j\mathsf{Q}(\check{a})}\mathfrak{g}_{-j\mathsf{Q}(\check{a})}=\tau^2$ whenever $j\neq 0$ and $\mathsf{Q}(\check{a})\neq 0 \mod n$ (see §2.3.1).

4.1.6. *Proof of Proposition 4.1.3 (3).* In case (3), we have $d\geq 1$ and also $j\neq 0$ in (4.16). Hence $\langle\check{\lambda},a\rangle=j+md-1>0$, so that we may apply the first case of (4.10) to write

$$\left(Y_{\check{\lambda}}\right)\widetilde{\mathbf{T}}_a = \mathfrak{g}_{j\mathsf{Q}(\widetilde{a}^\vee)}Y_{\check{\lambda}\bullet s_a}+(\tau^2-1)\sum_{0\leq k\leq d}Y_{\check{\lambda}-k\widetilde{a}^\vee}. \tag{4.27}$$

Consider next $\check{\lambda}^{(1)}\bullet s_a=(\check{\lambda}-j\check{a})s_a-\check{a}=\check{\lambda}-md\check{a}$ from which we conclude

$$\langle\check{\lambda}^{(1)}\bullet s_a,a\rangle=-(\check{\lambda}^{(1)},a)+2=j-nd-1<0 \tag{4.28}$$

since $md\geq m$ and $0<j<m$. Hence we can apply the second case of (4.10) to obtain

$$(Y_{\check{\lambda}^{(1)}\bullet s_a})\widetilde{\mathbf{T}}_a=\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}^{(1)}}+(1-\tau^2)\sum_{\substack{k>0\\ km<md-j+1}}Y_{\check{\lambda}^{(1)}\bullet s_a+k\widetilde{a}^\vee}= \tag{4.29}$$

$$\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}^{(1)}}+(1-\tau^2)\sum_{0<k<d}Y_{\check{\lambda}-d\widetilde{a}^\vee+k\widetilde{a}^\vee}=\mathfrak{g}_{j\mathsf{Q}(\check{a})}Y_{\check{\lambda}^{(1)}}+(1-\tau^2)\sum_{0<k<d}Y_{\check{\lambda}-k\widetilde{a}^\vee}. \tag{4.30}$$

Hence, we find that

$$\left(Y_{\check{\lambda}}+Y_{\check{\lambda}^{(1)}\bullet s_a}\right)\widetilde{\mathbf{T}}_a=\mathfrak{g}_{j\mathsf{Q}(\check{a})}\left(Y_{\check{\lambda}\bullet s_a}+Y_{\check{\lambda}^{(1)}}\right)+(\tau^2-1)\left(Y_{\check{\lambda}}+Y_{\check{\lambda}^{(1)}\bullet s_a}\right). \tag{4.31}$$

If we now define

$$A:=Y_{\check{\lambda}}+Y_{\check{\lambda}^{(1)}\bullet s_a} \text{ and } B:=Y_{\check{\lambda}\bullet s_a}+Y_{\check{\lambda}^{(1)}}, \tag{4.32}$$

we may restate what we have proven as follows:

$$(A)\widetilde{\mathbf{T}}_a=\mathfrak{g}_{j\mathsf{Q}(\check{a})}B+(\tau^2-1)A. \tag{4.33}$$

Recalling the quadratic relation (4.36), we may apply $\widetilde{\mathbf{T}}_a$ to the previous relation to get

$$(\mathbb{g}_{j\mathsf{Q}(\check{a})}B)\widetilde{\mathbf{T}}_a = (A\widetilde{\mathbf{T}}_a - (\tau^2-1)A)\widetilde{\mathbf{T}}_a = \tau^2 A. \tag{4.34}$$

Using $\mathbb{g}_{j\mathsf{Q}(\check{a})}\mathbb{g}_{-j\mathsf{Q}(\check{a})} = \tau^2$ we get $(B)\widetilde{\mathbf{T}}_a = \mathbb{g}_{-j\mathsf{Q}(\check{a})}A$, hence we conclude using (4.33) that $A - \mathbb{g}_{j\mathsf{Q}(\check{a})}B$ is an eigenvector of $\widetilde{\mathbf{T}}_a$ with eigenvalue -1 as desired.

Remark. *In Proposition 4.1.3 we essentially compute eigenvectors with eigenvalue -1 for the operators $\widetilde{\mathbf{T}}_a$. These are used to define a certain "exterior power" quotient in §4.3.2 which models the metaplectic spherical Gelfand–Graev representation. A similar exterior power is known to model the representations theory of quantum groups at a root of unity; we'll make this more precise in §7. One may similarly compute* $\mathbf{g}$*-twisted eigenvectors with eigenvalue τ^2 for $\widetilde{\mathbf{T}}_a$ and use them to define "symmetric power" quotients. This does not have any immediate applications to the p-adic setting, so we will not pursue it here.*

4.1.7. *The classical straightening rules.* Note that Proposition 4.1.3 holds for any Q as defined in 3.2.5, including when $\mathsf{Q}(\check{a}) \equiv 0 \mod n$. In this case (owning to $\mathbb{g}_{mk} = -1$), equations (4.13) and (4.14) are the same and equation (4.15) is a linear combination of (4.13) for $\check{\lambda}$ and $\check{\lambda}^{(1)}$. The relations become classical in nature, and the straightening rules we develop in Proposition 4.3.2 will be the same to the classical straightening rules in (1.4). This is consistent with the fact that when $\mathsf{Q}(\check{a}) \equiv 0 \mod n$ for all $\check{a} \in \check{\Pi}$, then the corresponding metaplectic n-cover of $\mathbf{G}(\mathcal{F})$ is just the direct product $\mathbf{G}(\mathcal{F}) \times \mu_n$.

4.2. The metaplectic polynomial representation $\mathbb{V}$.

4.2.1. *The representation* $\mathbb{V}$. Let $\mathbb{V} := \mathbb{Z}_{\tau,\mathbb{g}}[Y]$ be the space of polynomials in the variables $Y_{\check{\lambda}}, \check{\lambda} \in Y$. The elements in $\mathbb{Z}_\tau[\widetilde{Y}]$ act on $\mathbb{V}$ by translations, i.e.

$$Y_{\check{\lambda}} \cdot Y_{\check{\mu}} = Y_{\check{\mu}+\check{\lambda}} \text{ for } \check{\mu} \in \widetilde{Y}, \check{\lambda} \in Y. \tag{4.35}$$

From the formulas in Lemma 4.1.2, one sees that $\widetilde{\mathbf{T}}_a$ preserves the space $\mathbb{V}$. In fact, one has the following.

Proposition. *Keep the notation above and recall the space $\widetilde{V}$ from §3.6.5 equipped with its $\widetilde{H}_{\text{aff}}$-action.*

(1) *The formulas (4.35) together with $Y_{\check{\lambda}} \cdot T_{s_i} := Y_{\check{\lambda}} \cdot \widetilde{\mathbf{T}}_{a_i}$ for $i \in I, \check{\lambda} \in Y$ define an action of $\widetilde{H}_{\text{aff}}$ on $\mathbb{V}$.*
(2) *The quantum specialization $\mathfrak{q} : \mathbb{Z}_{\tau,\mathbb{g}} \to \mathbb{Z}_\tau$ induces an isomorphism, which we continue to denote by the same name, $\mathfrak{q} : \mathbb{Z}_\tau \otimes_{\mathbb{Z}_{\tau,\mathbb{g}}} \mathbb{V} \xrightarrow{\cong} \widetilde{V}$ that is equivariant with respect to the $\widetilde{H}_{\text{aff}}$-actions.*

Proof. From the arguments in [31, Prop. 3.1] we know the operators $\{\widetilde{\mathbf{T}}_{a_i} \mid i \in I\}$ satisfy the braid relations as well as the quadratic relation

$$\widetilde{\mathbf{T}}_a^2 = (\tau^2-1)\widetilde{\mathbf{T}}_a + \tau^2 \text{ for } a \in \Pi. \tag{4.36}$$

This shows that the given formulas define an action on $\mathbb{V}$. A more conceptual proof of this fact that does not use the Chinta-Gunnells action is given in [100, Theorem 3.7]. The second part follows by observing that the formulas in Lemma 4.1.2 reduce to the formulas for $\widetilde{\mathbf{T}}_a^\flat$ under the quantum specialization. $\square$

4.2.2. *Decomposition of* $\mathbb{V}$. In analogy with the constructions of §3.6.1, for each $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$, define the right $\widetilde{H}_{\mathrm{aff}}$-module

$$\mathbb{V}(\check{\eta}) := Y_{\check{\eta}} \cdot \widetilde{H}_{\mathrm{aff}} := \{w \in \mathbb{V} \mid w = Y_{\check{\eta}} \cdot h \text{ for some } h \in \widetilde{H}_{\mathrm{aff}}\}. \tag{4.37}$$

The following result follows immediately from the explicit formulas in Lemma 4.1.2.

Proposition. *For each* $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$*,* $\mathbb{V}(\check{\eta})$ *is a* $\mathbb{Z}_{\tau,\mathfrak{g}}[\widetilde{Y}]$*-module of rank equal to the cardinality of the orbit* $\check{\eta} \bullet W$*, and we have* $\mathbb{V}(\check{\eta}) = \bigoplus_{\check{\zeta} \in \check{\eta} \bullet W} Y_{\check{\zeta}} \cdot \mathbb{Z}_{\tau,\mathfrak{g}}[\widetilde{Y}]$ *and a decomposition of* $\mathbb{V}$ *into* $\widetilde{H}_{\mathrm{aff}}$*-submodules*

$$\mathbb{V} := \mathbb{Z}_{\tau,\mathfrak{g}}[Y] \cong \bigoplus_{\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}} \mathbb{V}(\check{\eta}). \tag{4.38}$$

It is then clear that a basis of $\mathbb{V}(\check{\eta})$ is given by $Y_{\check{\mu}}$ for $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$.

4.2.3. *The basis* $\mathbb{v}_{\check{\mu}}$. Let $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$ and J as in (3.77). In analogy with the construction of the elements $\mathsf{v}_{\check{\mu}}$ from §3.6.1, we introduce elements $\mathbb{v}_{\check{\mu}}$ for $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ by the formula

$$\mathbb{v}_{\check{\mu}} := Y_{\check{\eta}} H_\sigma Y_{\check{\beta}} \quad \text{when} \quad \check{\mu} = \check{\eta} \bullet \sigma \mathtt{t}(\check{\beta}),\ \sigma \mathtt{t}(\check{\beta}) \in {}^J\widetilde{W}_{\mathrm{aff}},\ \sigma \in W, \check{\beta} \in \widetilde{Y}. \tag{4.39}$$

where we recall the action of $H_{s_i} = \left(\tau^{-1}\widetilde{\mathbf{T}}_{s_i}\right)$ on $\mathbb{Z}_{\tau,\mathfrak{g}}[Y]$ is computed in Lemma 4.1.2.

Lemma. *Let* $\check{\mu} = \check{\eta} \bullet \sigma \mathtt{t}(\check{\beta})$ *with* $\sigma \in {}^J W$ *of minimal length and* $\check{\beta} \in \widetilde{Y}$*. Then the new basis satisfies*

$$\mathbb{v}_{\check{\mu}} = \kappa(\check{\mu}) Y_{\check{\mu}} \quad \textit{where} \quad \kappa(\check{\mu}) = \prod_{a \in \mathcal{R}_+, (a)\sigma^{-1} > 0} (\tau^{-1} \mathbb{g}_{\langle \check{\eta}+\check{\rho}, a\rangle \mathsf{Q}(\check{a})}) \in \mathbb{Z}_{\tau,\mathfrak{g}} \tag{4.40}$$

Proof. By definition we have that $\mathbb{v}_{\check{\eta}} = Y_{\check{\eta}}$. If $\check{\mu} = \check{\eta} \bullet \sigma$ with $\sigma \in {}^J W$ of minimal length, then we may use induction on the length of σ and the argument follows by using the first case of (4.11) repeatedly. If $\check{\mu} = \check{\eta} \bullet \sigma \mathtt{t}(\check{\beta})$, then $\kappa(\check{\mu})$ only depends on σ. $\square$

Remark. *By construction, if we replace the* $Y_{\check{\mu}}$ *in Lemma 4.1.2 with* $\mathbb{v}_{\check{\mu}}$*, we obtain the exact relations from the previous section, namely* (4.11) *becomes* (3.102) *with* $\mathsf{v}_{\check{\mu}}$ *replaced by* $\mathbb{v}_{\check{\mu}}$*. In other words, all the Gauss sum contributions are absorbed into the* $\mathbb{v}_{\check{\mu}}$*. Finally, let us comment that under the quantum specialization of* (2.6)*,* $\kappa(\check{\mu})$ *is mapped to* 1*.*

4.2.4. *Examples.* Consider the $\mathbf{PGL}_3$ root datum $\mathfrak{D}$ with simple coroots $\check{\alpha}_1, \check{\alpha}_2$, simple roots α_1, α_2 such that $\langle \check{\alpha}_i, \alpha_j \rangle = -1 + 3\delta_{ij}$ and $\check{\rho} = \check{\alpha}_1 + \check{\alpha}_2$. Let $\mathfrak{D}_{(\mathsf{Q},n)}$ be the twist of $\mathfrak{D}$ with $n = 6$ and $\mathsf{Q}(\check{\alpha}_1) = \mathsf{Q}(\check{\alpha}_2) = \mathsf{Q}$. Consider $\check{\eta} = -2\check{\alpha}_1 - 2\check{\alpha}_2 \in \overline{\mathcal{A}}^{\bullet}_{-,n}$. Then we have by using the first case of Lemma 4.1.2(2):

$$\mathbb{v}_{-2\check{\alpha}_1-2\check{\alpha}_2} = Y_{-2\check{\alpha}_1-2\check{\alpha}_2}, \quad \mathbb{v}_{-\check{\alpha}_1-2\check{\alpha}_2} = \tau^{-1}\mathbb{g}_{-\mathsf{Q}} Y_{-\check{\alpha}_1-2\check{\alpha}_2}, \quad \mathbb{v}_{-2\check{\alpha}_1-\check{\alpha}_2} = \tau^{-1}\mathbb{g}_{-\mathsf{Q}} Y_{-2\check{\alpha}_1-\check{\alpha}_2},$$

$$\mathbb{v}_{-\check{\alpha}_1} = \tau^{-2}\mathbb{g}_{-\mathsf{Q}}\mathbb{g}_{-2\mathsf{Q}} Y_{-\check{\alpha}_1}, \quad \mathbb{v}_{-\check{\alpha}_2} = \tau^{-2}\mathbb{g}_{-\mathsf{Q}}\mathbb{g}_{-2\mathsf{Q}} Y_{-\check{\alpha}_2}, \quad \mathbb{v}_0 = \tau^{-3}\mathbb{g}_{-\mathsf{Q}}\mathbb{g}_{-2\mathsf{Q}}\mathbb{g}_{-\mathsf{Q}} Y_0.$$

These are all the elements $\mathbb{v}_{\check{\mu}}$ with $\check{\mu} \in \check{\eta} \bullet W$. The rest of the elements $\mathbb{v}_{\check{\mu}}$ in $\mathbb{V}(\check{\eta})$ can be obtained by multiplying with elements $Y_{\check{\beta}}$ for $\check{\beta} \in \widetilde{Y}$. If instead we work with $\check{\eta}' = -3\check{\alpha}_1 - 2\check{\alpha}_2$, then we have

$$\mathbb{v}_{-3\check{\alpha}_1-2\check{\alpha}_2} = Y_{-3\check{\alpha}_1-2\check{\alpha}_2}, \quad \mathbb{v}_{-2\check{\alpha}_2} = \tau^{-1}\mathbb{g}_{-3\mathsf{Q}} Y_{-2\check{\alpha}_2}, \quad \mathbb{v}_{2\check{\alpha}_1+3\check{\alpha}_2} = \tau^{-2}\mathbb{g}_{-3\mathsf{Q}}\mathbb{g}_{-3\mathsf{Q}} Y_{2\check{\alpha}_1+3\check{\alpha}_2}.$$

4.3. **$\mathbb{g}$-twisted canonical bases.** We now construct an involution on the space $\mathbb{V}$ and use it to study a version of Kazhdan–Lusztig theory on a spherical quotient $\mathbb{V}_{\mathrm{sph}}$ of $\mathbb{V}$.

4.3.1. *The involution on* $\mathbb{V}(\check{\eta})$. Let $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ and J as in (3.77). We would like to define an involution on $\mathbb{V}(\check{\eta})$ which is compatible with the action of $\widetilde{H}_{\mathrm{aff}}$ and reduces to our previous involution on $\widetilde{V}(\check{\eta})$ under the quantum specialization. We begin with the following simple observation which follows from Lemma 4.1.2(2) and the same ideas as in the remark concluding §3.7.1. For $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$, we have

$$Y_{\check{\eta}} \cdot \widetilde{\mathbf{T}}_{w_0} = \tau^{\ell(w_0)}(-\tau)^{-\ell(w_0^J)}\kappa(\check{\eta} \bullet w_0) Y_{\check{\mu}\bullet w_0} \quad \text{where} \quad \kappa(\check{\eta} \bullet w_0) = \prod_{\substack{a \in \mathcal{R}_+, \\ (a)(w_0^J)^{-1} > 0}} (\tau^{-1}\mathbb{g}_{\langle \check{\eta}+\check{\rho}, a\rangle \mathsf{Q}(\check{a})}),$$

or equivalently in terms of the basis $\mathbb{v}_{\check{\mu}}$ introduced in §4.2.3, we may write

$$\mathbb{v}_{\check{\mu}} \cdot H_{w_0} = (-\tau)^{-\ell(w_0^J)} \mathbb{v}_{\check{\mu}\bullet w_0}. \tag{4.41}$$

Inspired by (3.121) and the computation in Remark 3.7.1, we now set for $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$ either

$$d(Y_{\check{\mu}}) \;:=\; \overline{Y_{\check{\mu}}} := \tau^{\ell(w_0)}(-\tau)^{-\ell(w_0^J)}\kappa(\check{\mu} \bullet w_0)\kappa(\check{\mu}) Y_{\check{\mu}\bullet w_0}(\widetilde{\mathbf{T}}_{w_0})^{-1} \text{ or equiv} \tag{4.42}$$

$$d(\mathbb{v}_{\check{\mu}}) \;=\; (-\tau)^{-\ell(w_0^J)} \mathbb{v}_{\check{\mu}\bullet w_0} \cdot H_{w_0}^{-1}. \tag{4.43}$$

The equivalence of the two expressions is immediate from (4.40) and the fact that $\overline{\kappa(\check{\mu})} = \kappa(\check{\mu})^{-1}$. If $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$, then $\overline{Y_{\check{\eta}}} = Y_{\check{\eta}}$ follows from (3.122).

Proposition. *The formula (4.42) defines an involution on $\mathbb{V}(\check{\eta})$ which satisfies*

$$d(v \cdot h) = \overline{v} \cdot \overline{h} \quad \textit{for} \quad v \in \mathbb{V}(\check{\eta}), h \in \widetilde{H}_{\mathrm{aff}} \;. \tag{4.44}$$

Proof. First suppose $h = Y_{\check{\zeta}}$ with $\check{\zeta} \in \widetilde{Y}$ and $w = \mathbb{v}_{\check{\mu}}$ for any $\check{\mu} \in \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$. From (3.84), we have $\overline{Y_{\check{\zeta}}} = H_{w_0} Y_{\check{\zeta}\cdot w_0} H_{w_0}^{-1}$, and we wish to show that

$$d(\mathbb{v}_{\check{\mu}} Y_{\check{\zeta}}) = d(\mathbb{v}_{\check{\mu}+\check{\zeta}}) = (-\tau)^{-\ell(w_0^J)} \mathbb{v}_{(\check{\zeta}+\check{\mu})\bullet w_0} H_{w_0}^{-1} \text{ is equal to}$$
$$d(\mathbb{v}_{\check{\mu}})\overline{Y_{\check{\zeta}}} = (-\tau)^{-\ell(w_0^J)} \mathbb{v}_{\check{\mu}\bullet w_0} H_{w_0}^{-1} H_{w_0} Y_{\check{\zeta}\cdot w_0} H_{w_0}^{-1}.$$

Since $(\check{\zeta}+\check{\mu})\bullet w_0 = \check{\mu}\bullet w_0 + \check{\zeta}\cdot w_0$, (4.44) follows for $h = Y_{\check{\zeta}}$. Suppose now that $h = H_{s_i}, i \in I$ and $\check{\mu} = \check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$. It suffices to show show

$$d(\mathbb{v}_{\check{\eta}}\cdot H_{s_i}) = d(\mathbb{v}_{\check{\eta}})\cdot\overline{H_{s_i}} = \mathbb{v}_{\check{\eta}}\cdot H_{s_i}^{-1}. \tag{4.45}$$

Now $d(\mathbb{v}_{\check{\eta}}\cdot H_{s_i})$ may be computed using Lemma 4.1.2(2) as follows

$$d(\mathbb{v}_{\check{\eta}}\cdot H_{s_i}) = \begin{cases} (-\tau)^{-\ell(w_0^I)}\mathbb{v}_{\check{\eta}\bullet s_i w_0}H_{w_0}^{-1} & \text{if } \langle\check{\eta}+\check{\rho},a_i\rangle \neq 0 \\ (-\tau)^{-\ell(w_0^I)}(-\tau^{-1})\mathbb{v}_{\check{\eta}\bullet w_0}H_{w_0}^{-1} & \text{if } \langle\check{\eta}+\check{\rho},a_i\rangle = 0 \end{cases}. \tag{4.46}$$

The rest of the proof is an exercise in Coxeter combinatorics with the use of Lemma 4.1.2 (2). Note that the involutive property of d follows from (4.44) since each $v\ in \mathbb{V}(\check{\eta})$ is of the form $Y_{\check{\eta}}\cdot h$ for some $h\in\widetilde{H}_{\mathrm{aff}}$. $\square$

4.3.2. *On the space* $\mathbb{V}_{\mathrm{sph}}(\check{\eta})$. Recall the element ε from §3.5.2. For $\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{+,n}$, we define the $\mathbb{Z}_{\tau,\mathfrak{g}}$-module

$$\mathbb{V}_{\mathrm{sph}}(\check{\eta}) := \mathbb{V}(\check{\eta})\cdot\varepsilon := \mathrm{Span}_{\mathbb{Z}_{\tau,\mathfrak{g}}}\{[Y_{\check{\mu}}],\check{\mu}\in\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}\} = \mathrm{Span}_{\mathbb{Z}_{\tau,\mathfrak{g}}}\{[\mathbb{v}_{\check{\mu}}],\check{\mu}\in\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}\}, \tag{4.47}$$

where $[v] := v\cdot\varepsilon\in\mathbb{V}$ for any $v\in\mathbb{V}(\check{\eta})$. Here, the action of ε on v is obtained by writing $\varepsilon = \varepsilon_I^+$ as in (3.39) and identifying the action of $H_w, w\in W$ with that of the operator $\tau^{-\ell(w)}\widetilde{\mathbf{T}}_w$ from Lemma 4.1.2. As with $[\mathbb{v}_{\check{\mu}}]$, the elements $[Y_{\check{\mu}}]$ are not independent and satisfy the straightening rules below. Since $\varepsilon H_{s_i} = \tau\varepsilon$,

$$[Y_{\check{\mu}}]\cdot\widetilde{\mathbf{T}}_{s_i} = \tau^2[Y_{\check{\mu}}] \text{ for } i\in I. \tag{4.48}$$

As a consequence of Proposition 4.1.3 and (4.48), we obtain the following.

Proposition. *If $a = a_i, i\in I$, one has the following relations for elements $[Y_{\check{\lambda}}]$ (resp. $[\mathbb{v}_{\check{\lambda}}]$), $\check{\lambda}\in\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}$:*

(1) If $\langle\check{\lambda}+\check{\rho},a\rangle\geq 0$ and $\langle\check{\lambda}+\check{\rho},a\rangle\equiv 0 \mod n(\check{a})$ then

$$[Y_{\check{\lambda}}]+[Y_{\check{\lambda}\bullet s_a}] = 0 \quad or \quad [\mathbb{v}_{\check{\lambda}}]+[\mathbb{v}_{\check{\lambda}\bullet s_a}] = 0. \tag{4.49}$$

(2) If $0<\langle\check{\lambda}+\check{\rho},a\rangle<n(\check{a})$, then

$$[Y_{\check{\lambda}}] = \mathfrak{g}_{\langle\check{\lambda}+\check{\rho},\check{a}\rangle\mathsf{Q}(\check{a})}[Y_{\check{\lambda}\bullet s_a}] \quad or \quad [\mathbb{v}_{\check{\lambda}}] = \tau[\mathbb{v}_{\check{\lambda}\bullet s_a}]. \tag{4.50}$$

(3) If $\langle\check{\lambda}+\check{\rho},a\rangle>n(\check{a})$, $j:=\langle\check{\lambda}+\check{\rho},a\rangle\not\equiv 0 \mod n(\check{a})$, then setting $\check{\lambda}_{(1)} := \check{\lambda}-\mathsf{res}_{n(\check{a})}(\langle\check{\lambda}+\check{\rho},a\rangle)\check{a}$

$$\begin{aligned}&[Y_{\check{\lambda}}]-\mathfrak{g}_{j\mathsf{Q}(\check{a})}[Y_{\check{\lambda}\bullet s_a}]-\mathfrak{g}_{j\mathsf{Q}(\check{a})}[Y_{\check{\lambda}^{(1)}}]+[Y_{\check{\lambda}^{(1)}\bullet s_a}] = 0 \text{ or}\\ &[\mathbb{v}_{\check{\lambda}}]-\tau[\mathbb{v}_{\check{\lambda}\bullet s_a}]-\tau[\mathbb{v}_{\check{\lambda}^{(1)}}]+[\mathbb{v}_{\check{\lambda}^{(1)}\bullet s_a}] = 0.\end{aligned} \tag{4.51}$$

4.3.3. *On the space $\mathbb{V}_{\mathrm{sph}}$ and its canonical bases.* Fix $\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{-,n}$. Using the straightening rules above, we may show that

Corollary. *The collection $\{[Y_{\check{\mu}}]\}$ for $\check{\mu}\in Y_+\cap\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}$ forms a basis of $\widetilde{V}_{\mathrm{sph}}(\check{\eta})$, and similarly for $\{[\mathbb{v}_{\check{\mu}}]\}$.*

If we now define the $\mathbb{Z}_{\tau,\mathfrak{g}}$-module

$$\mathbb{V}_{\mathrm{sph}} := \bigoplus_{\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{-,n}} \mathbb{V}_{\mathrm{sph}}(\check{\eta}), \tag{4.52}$$

then we find that, as an $\mathbb{Z}_{\tau,\mathfrak{g}}$-module, $\mathbb{V}_{\mathrm{sph}}$ has a basis $\{[Y_{\check{\lambda}}]\}$ or $\{[\mathbb{v}_{\check{\lambda}}]\}$ where $\check{\lambda}$ ranges over Y_+. The involution $d:\mathbb{V}(\check{\eta})\to\mathbb{V}(\check{\eta})$ studied in §4.3.1 induces an involution on the subspace $\mathbb{V}_{\mathrm{sph}}(\check{\eta})$ by using Proposition 4.3.1 and the fact that $\overline{\varepsilon}=\varepsilon$. We have

$$d([\mathbb{v}_{\check{\mu}}]) := d(\mathbb{v}_{\check{\mu}}\varepsilon) = d(\mathbb{v}_{\check{\mu}})\varepsilon \text{ for } \check{\mu}\in\check{\eta}\bullet\widetilde{W}_{\mathrm{aff}}. \tag{4.53}$$

Combining the arguments in Proposition 3.7.2 and Proposition 4.3.1, we obtain the following formula for the involution $d:\mathbb{V}(\check{\eta})\to\mathbb{V}(\check{\eta})$ restricted to the subspace $\mathbb{V}_{\mathrm{sph}}(\check{\eta})$,

$$d([\mathbb{v}_{\check{\lambda}}]) = (-1)^{\ell(w_0^J)}\tau^{-\ell(w_0)-\ell(w_0^J)}[\mathbb{v}_{\check{\lambda}\bullet w_0}] \text{ or equivalently,} \tag{4.54}$$

$$d([Y_{\check{\lambda}}]) = (-1)^{\ell(w_0^J)}\tau^{-\ell(w_0)-\ell(w_0^J)}\kappa(\check{\lambda})\kappa(\check{\lambda}\bullet w_0)[Y_{\check{\lambda}\bullet w_0}]. \tag{4.55}$$

The involution d defined above naturally extends to one on $\mathbb{V}_{\mathrm{sph}}$.

Proposition. *For each $\check{\lambda}\in Y_+$, there exist unique self-dual elements $[\mathbb{G}_{\check{\lambda}}],[\mathbb{G}^-_{\check{\lambda}}]\in\mathbb{V}_{\mathrm{sph}}(\check{\eta})$ ($\check{\eta}\in\overline{\mathcal{A}}^{\bullet}_{+,n}$ is such that $\check{\lambda}=\check{\eta}\bullet w$ for $w\in\widetilde{W}_{\mathrm{aff}}$) which satisfy the triangularity condition*

$$\begin{aligned}&[\mathbb{G}_{\check{\lambda}}] = [\mathbb{v}_{\check{\lambda}}]+\textstyle\sum_{\check{\mu}<\check{\lambda}} a_{\check{\mu},\check{\lambda}}[\mathbb{v}_{\check{\mu}}], \textit{ for } a^+_{\check{\mu},\check{\lambda}}\in\mathbb{Z}^+_\tau \textit{ and}\\ &[\mathbb{G}^-_{\check{\lambda}}] = [\mathbb{v}_{\check{\lambda}}]+\textstyle\sum_{\check{\mu}<\check{\lambda}} a_{\check{\mu},\check{\lambda}}[\mathbb{v}_{\check{\mu}}], \textit{ for } a^-_{\check{\mu},\check{\lambda}}\in\mathbb{Z}^-_\tau.\end{aligned} \tag{4.56}$$

By the way they were constructed, one can verify that $a^{\pm}_{\check{\mu},\check{\lambda}}=o^{\pm}_{\check{\mu},\check{\lambda}}$ are the parabolic, singular Kazhdan–Lusztig polynomials from (3.127). Writing (4.56) in terms of the basis $[\mathbb{v}_{\check{\mu}}]=\kappa(\check{\mu})[Y_{\check{\mu}}]$ we find relations

$$\begin{aligned}&[\mathbb{G}_{\check{\lambda}}] = \kappa(\check{\lambda})[Y_{\check{\lambda}}]+\textstyle\sum_{\check{\mu}<\check{\lambda}}\kappa(\check{\mu})o^+_{\check{\mu},\check{\lambda}}(\tau^{-1})[Y_{\check{\mu}}] \text{ and}\\ &[\mathbb{G}^-_{\check{\lambda}}] = \kappa(\check{\lambda})[Y_{\check{\lambda}}]+\textstyle\sum_{\check{\mu}<\check{\lambda}}\kappa(\check{\mu})o^-_{\check{\mu},\check{\lambda}}(\tau^{-1})[Y_{\check{\mu}}]\end{aligned} \tag{4.57}$$

Defining $[G_{\check{\lambda}}],[G^-_{\check{\lambda}}]$ to satisfy $[\mathbb{G}_{\check{\lambda}}]=\kappa(\check{\lambda})[G_{\check{\lambda}}]$ and $[\mathbb{G}^-_{\check{\lambda}}]=\kappa(\check{\lambda})[G^-_{\check{\lambda}}]$, the previous equations become

$$\begin{aligned}&[G_{\check{\lambda}}] = [Y_{\check{\lambda}}]+\textstyle\sum_{\check{\mu}<\check{\lambda}}\kappa(\check{\mu})\kappa(\check{\lambda})^{-1}o^+_{\check{\mu},\check{\lambda}}(\tau^{-1})[Y_{\check{\mu}}] \text{ and}\\ &[G^-_{\check{\lambda}}] = [Y_{\check{\lambda}}]+\textstyle\sum_{\check{\mu}<\check{\lambda}}\kappa(\check{\mu})\kappa(\check{\lambda})^{-1}o^-_{\check{\mu},\check{\lambda}}(\tau^{-1})[Y_{\check{\mu}}].\end{aligned} \tag{4.58}$$

We shall refer to $[\mathbb{G}_{\check{\lambda}}]$ and $[\mathbb{G}^-_{\check{\lambda}}]$ as the canonical basis in $\mathbb{V}_{\mathrm{sph}}(\check{\eta})$ and $[G_{\check{\lambda}}],[G^-_{\check{\lambda}}]$ as the $\mathfrak{g}$-twisted canonical bases. We also set

$$^{\mathfrak{g}}o^{\pm}_{\check{\lambda},\check{\mu}} := \frac{\kappa(\check{\mu})}{\kappa(\check{\lambda})}o^{\pm}_{\check{\mu},\check{\lambda}}(\tau^{\pm1}) \tag{4.59}$$

and refer to these as $\mathfrak{g}$-twisted parabolic, singular Kazhdan–Lusztig polynomials.

4.4. $\mathbb{g}$**-twisted Littlewood–Richardson theory.** Recall the spherical subalgebra $\widetilde{H}_{\mathrm{sph}} := \varepsilon\widetilde{H}_{\mathrm{aff}}\varepsilon$ introduced in §3.5.3. It is equipped with two bases, $\widetilde{h}_{\check{\lambda}} := \varepsilon Y_{\check{\lambda}}\varepsilon$ and $\widetilde{c}_{\check{\lambda}}$, both indexed by $\check{\lambda} \in \widetilde{Y}_+$. Under the Satake isomorphism $S : \widetilde{H}_{\mathrm{sph}} \to \mathbb{Z}_\tau[\widetilde{Y}]^W$ we have $S(h_{\check{\lambda}})$ is given by the formula (3.91) and $S(\widetilde{c}_{\check{\lambda}}) = \chi_{\check{\lambda}}$ the Weyl character, see (3.92). The $\mathbb{Z}_{\tau,\mathbb{g}}$-modules $\mathbb{V}_{\mathrm{sph}}(\check{\eta})$ and $\mathbb{V}_{\mathrm{sph}}$ can be equipped with a right $\widetilde{H}_{\mathrm{sph}}$ action, denoted by $\star$, and defined as in §3.5.4. We shall also write $\diamondsuit$ for the action of $\mathbb{Z}_\tau[\widetilde{Y}]^W$ as in (3.130), *i.e.*

$$w\diamondsuit f = v\star S^{-1}(f) \text{ for } w\in\mathbb{V}_{\mathrm{sph}}, f\in\widetilde{H}_{\mathrm{sph}}. \tag{4.60}$$

4.4.1. *Action of* $\widetilde{h}_{\check{\lambda}}$. Let $[Y_{\check{\mu}}] := Y_{\check{\mu}}\varepsilon \in \mathbb{V}_{\mathrm{sph}}(\check{\eta})$ for $\check{\mu} \in Y_+ \cap \check{\eta} \bullet \widetilde{W}_{\mathrm{aff}}$. From §4.3.2

$$[Y_{\check{\mu}}]\star h_{\check{\lambda}} = Y_{\check{\mu}}\cdot\varepsilon Y_{\check{\lambda}}\varepsilon = [\widetilde{\mathrm{CS}}(\check{\mu})\cdot Y_{\check{\lambda}}] \quad \text{for} \quad \check{\lambda}\in\widetilde{Y}_+, \check{\mu}\in Y_+, \tag{4.61}$$

where we have written $\widetilde{\mathrm{CS}}(\check{\mu}) := Y_{\check{\mu}}\varepsilon$. The elements $\widetilde{\mathrm{CS}}(\check{\mu}) \in \mathbb{V}_{\mathrm{sph}}$ are precisely the values of the unramified spherical Whittaker function on metapletic covers (see [95] and the references therein). Alternatively, one may write the above expression in terms of the (Hall-Littlewood) polynomials $S(h_{\check{\lambda}})$ introduced in (3.91),

$$[Y_{\check{\mu}}]\star h_{\check{\lambda}} = [Y_{\check{\mu}}\cdot S(h_{\check{\lambda}})], \tag{4.62}$$

where again one needs to use the straightening rules to rewrite the above in terms of the basis $\{[Y_{\check{\mu}}]\}_{\check{\mu}\in Y_+}$. One may prove a version of Proposition 3.8.1 and

$$[Y_{\check{\lambda}}]\star\widetilde{c}_{\check{\zeta}} = [Y_{\check{\lambda}}]\diamondsuit\chi_{\check{\zeta}} = [Y_{\check{\lambda}}\cdot\chi_{\check{\zeta}}]. \tag{4.63}$$

4.4.2. $\mathbb{g}$*-twisted Littlewood–Richardson polynomials.* If we alternatively expand the product $[Y_{\check{\mu}}]\star c_{\check{\lambda}}$ for $\check{\mu},\check{\lambda}$ as in the previous paragraph, we obtain an expression of the form

$$[Y_{\check{\lambda}}]\star c_{\check{\zeta}} = \sum_{\check{\mu}\in Y_+} {}^{\mathbb{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}}[Y_{\check{\mu}}] \text{ for some } {}^{\mathbb{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_{\tau,\mathbb{g}}. \tag{4.64}$$

We call the elements ${}^{\mathbb{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}}$ $\mathbb{g}$-twisted Littlewood–Richardson polynomials and under the quantum specialization, one can easily show that $\mathfrak{q}({}^{\mathbb{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}}) = Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}}$, where the latter are the polynomials defined in (3.133). In fact, we can be even more precise. From Remark 4.2.3 it follows that the elements $[\mathbb{v}_{\check{\lambda}}]$ will satisfy equation (3.133) on the nose, and by comparing that with (4.64) and the use of (4.40) we get

$${}^{\mathbb{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}} = \kappa(\check{\mu})\kappa(\check{\lambda})^{-1}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}}. \tag{4.65}$$

4.4.3. $\mathbb{g}$*-twisted Tensor Product Theorems.* We now consider the action of $\widetilde{H}_{\mathrm{sph}}$ on the new bases $[\mathbb{G}_{\check{\mu}}], [\mathbb{G}^-_{\check{\mu}}]$, $\check{\mu}\in Y_+$ of the space $\mathbb{V}_{\mathrm{sph}}$ that were introduced in §4.3.3. The action is entirely determined from the following $\mathbb{g}$-twisted analogue of the tensor-product theorem [77], [74].

Proposition. *Let* $\check{\lambda}\in Y_+$ *be written uniquely as* $\check{\lambda} = \check{\lambda}_0 + \check{\zeta}$ *with* $\check{\lambda}_0 \in \square_{(\mathsf{Q},n)}$ *and* $\check{\zeta}\in\widetilde{Y}_+$.

(1) *We have* $[\mathbb{G}_{\check\lambda_0}] \star \widetilde{c}_{\check\zeta} = [\mathbb{G}_{\check\lambda_0+\check\zeta}]$

(2) *Writing* $\check\lambda_0^\dagger := \check\lambda_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check\rho)$ *we have*

$$[\mathbb{G}^-_{\check\lambda_0^\dagger}] \star \widetilde{c}_{\check\zeta} = [\mathbb{G}_{\check\lambda_0^\dagger+\check\zeta}]. \tag{4.66}$$

The proof given in Section §3.9 carries over here, where one uses the properties of the involution d from Proposition 4.3.1 as well as the fact that the straightening relations (see Proposition 4.3.2) when viewed with respect to the $[\mathbb{v}_{\check\lambda}]$ basis look the same as their 'quantum' counterparts from the previous section.

Remark. *The Proposition holds if we replace the bases* $[\mathbb{G}_{\check\lambda}]$ *and* $[\mathbb{G}^-_{\check\lambda}]$ *with the bases* $[G_{\check\lambda}]$ *and* $[G^-_{\check\lambda}]$*, respectively. To see this, note that* $[G_{\check\lambda}] = \kappa^{-1}(\check\lambda)[\mathbb{G}_{\check\lambda}]$ *and* $[G_{\check\lambda_0}] = \kappa^{-1}(\check\lambda_0)[\mathbb{G}_{\check\lambda_0}]$ *by definition, and use the fact that* $\kappa(\check\lambda) = \kappa(\check\lambda_0)$ *because* $\check\lambda - \check\lambda_0 = \check\eta \in \widetilde{Y}_+$*. The same argument works for the* $[G^-_{\check\lambda}]$ *basis.*

Part 2. p-adic models

Throughout this part, we let $(I,\cdot,\mathfrak{D})$be a root datum with $\mathfrak{D} = (Y,\{\check a_i\},X,\{a_i\})$ assumed to be of simply-connected type. Let (Q,n) be a twist as in §3.2.5 and $(I,\circ_{(\mathsf{Q},n)},\widetilde{\mathfrak{D}})$ the corresponding twisted root datum as in §3.2.6 again assumed to be of simply-connected type. Write $\widetilde{\mathfrak{D}} = (\widetilde{Y},\{\widetilde{a}_i^\vee\},\widetilde{X},\{\widetilde{a}_i\})$. Both simply-connected assumptions can be removed (for the purposes of this part) with known techniques, though the constructions become a bit more notationally involved. Denote also by

$$W_{\mathrm{aff}} := W_{\mathrm{aff}}(I,\circ,\mathfrak{D}) \quad \text{and} \quad \widetilde{W}_{\mathrm{aff}} := W_{\mathrm{aff}}(I,\circ_{(\mathsf{Q},n)},\widetilde{\mathfrak{D}}) \tag{4.67}$$

the affine Weyl groups attached to $\mathfrak{D}$ and $\widetilde{\mathfrak{D}}$. Again, by our assumptions both are Coxeter groups and we adopt the same terminology here as in §3.4. On occasion, we write A (resp. $\widetilde{\mathsf{A}}$) for the Cartan matrix attached to $(I,\cdot)$ (resp. $(I,\circ_{(\mathsf{Q},n)})$) and $\mathsf{A}_{\mathrm{aff}}$ and $\widetilde{\mathsf{A}}_{\mathrm{aff}}$ for their untwisted affinizations (see §3.1.4).

5. Recollections on p-adic groups, their covers, and associated Hecke algebras

In this section, we recall some facts about (covers of) p-adic groups and their associated Hecke algebras.

5.1. **Groups attached to root datum.** Throughout this section F will denote an arbitrary field. Denote by $\mathbf{G} := \mathbf{G}_{\mathfrak{D}}$ the corresponding split, simple algebraic group of simply-connected type whose points over any field F will be denoted as $\mathbf{G}(F) := \mathbf{G}_{\mathfrak{D}}(F)$. We review a few details of this construction following [115], and adopt the notation in §3.2.4 to describe the roots, weights, etc. associated to $(I,\cdot,\mathfrak{D})$.

5.1.1. *Torus* $\mathbf{H}$. The group $\mathbf{G}$ comes equipped with a (split) torus $\mathbf{H}$ such that $\mathbf{H}(F) \cong \mathrm{Hom}(X, F^*)$. For any $s \in F$ and $\check{\lambda} \in Y$ we write $s^{\check{\lambda}} \in \mathbf{H}(F)$ for the element which sends $\mu \in X$ to $s^{\langle \check{\lambda}, \mu \rangle}$. As $\mathfrak{D}$ was assumed simply-connected, Y has basis $\check{\Pi}$ and it follows that every element in $\mathbf{H}(F)$ may be written uniquely as $\prod_{i \in I} s_i^{\check{a}_i}$ with $s_i \in F^*$. There is a natural action of $W := W(I, \cdot)$ on $\mathbf{H}(F)$ induced from the corresponding action of W on X, and for $s \in W$ and $h \in \mathbf{H}(F)$ we denote this action by $s(h) \in \mathbf{H}(F)$.

5.1.2. *Unipotent subgroups.* For each $a \in \mathcal{R}$ there exists a subgroup $\mathbf{U}_a \subset \mathbf{G}$ and an isomorphism[23] $x_a : F \to \mathbf{U}_a(F)$. Recall that a *nilpotent* set of roots $\Psi \subset \mathcal{R}$ is a subset such that if $a, b \in \Psi$ and $a + b \in \mathcal{R}$, then $a + b \in \Psi$ as well. For a nilpotent set of the form $\{a, b\}$, we write $[a, b] = (\mathbb{N}a + \mathbb{N}b) \cap \mathcal{R}$ and also $]a, b[:= [a, b] \setminus \{a, b\}$. For each nilpotent pair $\{a, b\}$, there exist constants $k(a, b; c)$ that can be computed explicitly from $\mathfrak{D}$ with $c \in]a, b[$ such that $[x_a(s), x_b(t)] = \prod_c x_c(k(a, b; c) s^m t^n)$ for $a, b, c \in \mathcal{R}_+$ and where the products is taken over roots $c = ma + nb$ with $m, n \geq 1$. For any nilpotent set $\Psi \subset \mathcal{R}$, there exists a subgroup $\mathbf{U}_\Psi \subset \mathbf{G}$ equipped with inclusions $\mathbf{U}_a \to \mathbf{U}_\Psi$ and such that: i) $\mathbf{U}_\Psi(\mathbb{C})$ is the unipotent group corresponding to the nilpotent Lie algebra defined by Ψ; and ii) for any order on Ψ, we have an isomorphism of schemes, $\prod_{a \in \Psi} \mathbf{U}_a \to \mathbf{U}_\Psi$. The sets $\mathcal{R}_\pm$ are clearly nilpotent, and we write $\mathbf{U} := \mathbf{U}_{\mathcal{R}_+}$ and $\mathbf{U}^- := \mathbf{U}_{\mathcal{R}_-}$ for the corresponding 'positive' and 'negative' unipotent subgroups of $\mathbf{G}$. Often we drop the $+$ from the positive case.

5.1.3. *Some relations.* For each $a \in R$, consider elements in $\mathbf{G}(F)$,

$$w_a(s) := x_a(s) x_{-a}(-s^{-1}) x_a(s) \text{ for } x \in F^* \text{ and } h_a(s) := w_a(s) w_a(1)^{-1} \text{ for } s \in F^*. \quad (5.1)$$

Note that $w_a(s)^{-1} = w_a(-s)$.Write for $\dot{w}_a := w_a(-1)$. The following holds in $\mathbf{G}(F)$:

- $t x_a(s) t^{-1} = x_a(t(a)s)$ for $t \in \mathbf{H}(F) = \mathrm{Hom}(X, F^*)$ and $s \in F$ and $a \in \Pi \subset X$.
- For $a \in \Pi$ and $t \in \mathbf{H}(F)$, $\dot{w}_a t \dot{w}_a^{-1} = s_a(t)$ for $s_a \in W$ the simple reflection in W corresponding to a.
- For $a \in \Pi$ and $z \in F^*$, we have $h_a(z) = z^{\check{a}}$, where the latter elements were described in §5.1.1.
- For $a \in \Pi, b \in R$ and $x \in F$, we have $\dot{w}_a x_b(s) \dot{w}_a^{-1} = x_{w_a(b)}(\eta(a, b)s)$ for some $\eta(a, b) = \pm 1$ that can be determined explicitly from $\mathfrak{D}$ (cf. [87, Lemma 5.1(c)]).

5.1.4. *N and the Weyl group.* Denote by by $N := \langle w_a(s) \mid a \in \Pi, s \in F^* \rangle$ and note the above relations imply $\mathbf{H}(F) \subset N$. Then the map $W(\mathsf{A}) \to N$ which sends $s_i \mapsto \dot{w}_{a_i}$ induces an isomorphism of groups $W \to N / \mathbf{H}(F)$. Note that the lifts $\dot{w}_a$ satisfy the braid relations (see [114, Proposition 3]) so that for $w \in W$ with reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ we may unambiguously define $\dot{w} := \dot{w}_{a_{i_1}} \cdots \dot{w}_{a_{i_k}} \in \mathbf{G}(F)$. An explicit set of relations which the generators satisfy is also known (see [113]).

[23] To define x_a, a further choice is needed to fix some signs, and we refer to [115] for more details.

5.1.5. *BN-pairs.* Let $\mathbf{B}$ be the subgroup generated by $\mathbf{U}$ and $\mathbf{H}$. We know that $\mathbf{B}(F) = \mathbf{U}(F) \rtimes \mathbf{H}(F)$. In an analogous way, we may define $\mathbf{B}^-$ using $\mathbf{U}^-$. The tuple $(\mathbf{G}(F), \mathbf{B}(F),$
$N, S)$ where $S := \{\dot{w}_a, a \in \Pi\}$ satisfies the condition of a Tits system or BN-pair ([14, Def.1, Chap IV, §2.1]) and hence one has Bruhat decompositions for the group $\mathbf{G}(F)$. We can replace $\mathbf{B}$ with $\mathbf{B}^-$ and obtain a (conjugate) BN-pair.

5.1.6. Attached to the chosen twist Q as in §3.2.5 and an abelian group A with bilinear Steinberg symbol $(\cdot,\cdot) : F^* \times F^* \to A$ as in §2.2.2, one can construct a universal covering group E fitting into an exact sequence $0 \to A \to E \xrightarrow{p} \mathbf{G}(F) \to 1$. We follow the approach of Matsumoto [87], but adapt the terminology of [26] where the construction of covers for general reductive groups is given, and review a few aspects of this construction following [97] which treats the Kac-Moody (but simply-connected) case in this same notation.

5.1.7. *The torus $\widetilde{H}$.* There exists a subgroup $\widetilde{H} \subset E$ fitting into a sequence $0 \to A \to \widetilde{H} \to \mathbf{H}(F) \to 1$. More concretely, the group $\widetilde{H}$ is generated by A and symbols $\widetilde{h}_a(s)$ with $a \in \Pi$ and $s \in F^*$ and subject to the relations that $A \subset \widetilde{H}$ is an abelian subgroup, and that for $\check{a}, b^\vee \in \check{\Pi}$ and $s, t \in F^*$:

$$\widetilde{h}_a(s)\widetilde{h}_a(t)\widetilde{h}_a(st)^{-1} = (s,t)^{\mathsf{Q}(\check{a})} \quad \text{and} \quad [\widetilde{h}_a(s), \widetilde{h}_b(t)] = (s,t)^{\mathsf{B}(\check{a}, b^\vee)}. \tag{5.2}$$

5.1.8. *Some relations in E.* Next, we note that there exists a homomorphism $\mathbf{U}(F)^\pm \to E$ which splits the map p from (5.1.6). Abusing notation slightly, we shall henceforth write $x_a(s)$ for $a \in R$ and $s \in F$ to mean the corresponding element in E. Define elements $\widetilde{w}_a(s) := x_a(s)x_{-a}(-s^{-1})x_a(s) \in E$, and then set $\lambda_a := \widetilde{w}_a(-1)$. The elements λ_a satisfy the braid relations and $p(\lambda_a) = \dot{w}_a$ for each $a \in \Pi$. Moreover, one has the following relations

- $\widetilde{h}_a(s)x_b(t)\widetilde{h}_a(s)^{-1} = x_b(s^{\langle \check{a}, b\rangle}t)$,
- $\widetilde{w}_a(s)\widetilde{w}_a(-1) = \widetilde{h}_a(s)$,
- $\lambda_a^{-1}\widetilde{h}_b(s)\lambda_a^{-1} = \widetilde{h}_b(s)\widetilde{h}_a(s^{-\langle a, b^\vee\rangle})$,
- $\lambda_a^{-1}x_b(s)\lambda_a = \dot{w}_a^{-1}x_b(s)\dot{w}_a$.

5.1.9. *Weyl groups.* Define $\widetilde{N}$ to be the subgroup generated by $\widetilde{H}$ and the elements λ_a. Then the pair $(\widetilde{N}, \widetilde{B})$ where $\widetilde{B} := \widetilde{H} \rtimes \mathbf{U}(F)$ satisfies the axioms of a *BN*-pair. Replacing $\mathbf{U}(F)$ with $\mathbf{U}^-(F)$, we also obtain the BN-pair $(\widetilde{B}^-, \widetilde{N})$. The Weyl group of $\widetilde{N}/\widetilde{H}$ is isomorphic to $W := W(I, \cdot) \cong W(I, \circ_{(\mathsf{Q},n)})$.

5.2. **Structure of p-adic groups and their covers.** Let $\mathcal{F}$ be a non-archimedean local field as in §2.2.1 and write $G := \mathbf{G}(\mathcal{F})$ (recall the convention in §2.1.1). Let n be a positive integer such that $(q, 2n) = 1$, μ_n the group of n-th roots of unity in $\mathcal{F}$ (which, by our assumption, is of cardinality n), and write $(\cdot,\cdot)$ for the Hilbert n-symbol as in §2.2.2. We call $\widetilde{G}$ the central extension of G as in §5.1.6 specialized to these particular choices, see [97, §6] for more details in the same notation as this paper.

5.2.1. *Affine Weyl group.* Recall that we have an isomorphism $W \to N/H$ sending $s_i \mapsto w_{a_i}(-1)H$ for $i \in \Pi$. We can extend this to a map from the affine Weyl group $\zeta : W_{\text{aff}} \to N/H_{\mathcal{O}}$ as follows: let $x \in W_{\text{aff}}$ be written as $x = w\mathtt{t}(\check{\lambda})$, where $w \in W$ and $\check{\lambda} \in \check{\mathfrak{Q}}$ is written as $\check{\lambda} = \sum_{i \in I} b_i \check{a}_i$ with $b_i \in \mathbb{Z}$; define $\pi^{\check{\lambda}} := \prod_{i \in I} h_{a_i}(\pi^{m_i})$ and set $\zeta(x) = \zeta(w\mathtt{t}(\check{\lambda})) = \dot{w}\pi^{\check{\lambda}} H_{\mathcal{O}}$. The map ζ is an isomorphism, and we often write, for $x \in W_{\text{aff}}$, $\dot{x} := \dot{w}\pi^{\check{\lambda}}$.

5.2.2. *Compact and Iwahori Subgroups.* Let $K = \mathbf{G}(\mathcal{O})$ and let us continue to write ω for the reduction map $\mathbf{G}(\mathcal{O}) \to \mathbf{G}(\kappa)$. Let us define the Iwahori subgroup $I^- = \omega^{-1}(\mathbf{B}^-(\kappa))$. The integral torus and unipotent groups are defined as

$$H_{\mathcal{O}} := \mathbf{H}(F) \cap K = \mathbf{H}(\mathcal{O}) \quad \text{and } U_{\mathcal{O}}^{\pm} := K \cap \mathbf{U}^{\pm}(\mathcal{F}) = \mathbf{U}(\mathcal{O}), \tag{5.3}$$

where the second equality in each of the above expressions is verified using some simple representation theory. As $U^{\pm}(\mathcal{O}) \subset K$, we also define, using the map $\omega : U_{\mathcal{O}} \to \mathbf{U}(\kappa)$, the congruence subgroup $U_{\pi}^{\pm} := \omega^{-1}(1)$. One can verify that U_{π} is generated by elements $x_a(s)$ with $a \in R_+$ and $s \in \pi\mathcal{O}$; in fact one has a unique factorization of any $n \in U_{\pi}$ as $n = \prod_{a \in \mathcal{R}\_+} x_a(s_a)$ for some fixed order of $\mathcal{R}_+$ and where each $s_a \in \pi\mathcal{O}$. From ([56, Thm 2.5]), we have the Iwahori-Matsumoto decompositions $I^- = U_{\mathcal{O}}^- H_{\mathcal{O}} U_{\pi}$. We can also obtain different (though equivalent) decompositions by permuting the order of the factors in the above. From [56, Prop. 2.4], we have $K = \sqcup_{w \in W} I^- \dot{w} I^-$. From [56, Prop 3.2], for $x \in W_{\text{aff}}$, we have $|I^- \setminus I^- \dot{x} I^-| = q^{\ell(x)}$, and so, with respect to the natural Haar measure on K which is normalized to give I^- volume 1, we have

$$\mathsf{vol}(K) = \sum_{w \in W} q^{\ell(w)}. \tag{5.4}$$

5.2.3. *Some p-adic decompositions.* The *Iwasawa decomposition* for G asserts that $G = B^- K$. We can refine this further to assert the decomposition into disjoint pieces

$$G = \sqcup_{\check{\lambda} \in Y} U^- \pi^{\check{\lambda}} K. \tag{5.5}$$

In particular, to each $g \in G$ we may write $g = k\pi^{\check{\lambda}} u$ with $\check{\lambda} \in Y$ uniquely determined from g. Note that k and u are not unique, since we could replace u with any element from $\pi^{-\check{\lambda}} \mathbf{U}(\mathcal{O}) \pi^{\check{\lambda}}$. The *Cartan decomposition* asserts $G = \sqcup_{\check{\lambda} \in Y_+} K\pi^{\check{\lambda}} K$. Combining the Cartan and Iwasawa decomposition with the ones from the previous paragraph, we obtain what we refer to as the *Iwahori-Matsumoto decompositions*

$$G = \sqcup_{x \in W_{\text{aff}}} I^- \dot{x} I^- = \sqcup_{x \in W_{\text{aff}}} I^- \dot{x} I^- = \sqcup_{x \in W_{\text{aff}}} U^- \dot{x} I^-. \tag{5.6}$$

5.2.4. *The group $\widetilde{G}$.* As mentioned above, the group $\widetilde{G}$ splits over U and U^- and we fix a given splitting as above and continue to adopt the same notation (i.e. $x_a(s)$, etc.) for elements in these unipotent groups. We continue to denote the Weyl group of $\widetilde{G}$ by W and write, for $a \in \Pi$, $\widetilde{w}_a := \lambda_a$ for the elements as constructed in §5.1.8. As the λ_a satisfy the braid relations, we may also define $\widetilde{w}$ for any $w \in W$ in the natural way.

5.2.5. *On $\widetilde{H}$ and its abelian subgroups.* Fix an ordering on I. As we are in the simply-connected case, every $t \in \widetilde{H}$ may be written uniquely (with respect to the fixed order) as $t = \zeta \prod_{i \in I} \widetilde{h}_{a_i}(s_i)$ with $s_i \in \mathcal{F}^*$ and $\zeta \in \mu_n$. For $\check{\lambda} \in Y$, written in terms of the $\mathbb{Z}$-basis $\{\check{a}_i (i \in I)\}$ as $\check{\lambda} = \sum_{i \in I} c_i \check{a}_i$, we define the elements in $\widetilde{H}$, $\pi^{\check{\lambda}} := \prod_{i \in I} \widetilde{h}_{a_i}(\pi^{c_i})$. As $(\pi, \pi) = 1$ (see end of §2.2.2) we may use (5.2) to see that the above element does not depend on the choice of ordering on I. Define $\widetilde{H}_{\mathcal{O}}$ to be the subgroup generated by the elements $h_{a_i}(s)$ with $s \in \mathcal{O}^*$, $i \in I$. Under the assumption that $q \equiv 1 \mod 2n$, $\widetilde{H}_{\mathcal{O}}$ is in fact an abelian group and $\widetilde{H}_{\mathcal{O}} \backslash \widetilde{H} \simeq \mu_n \times Y$.

5.2.6. *Iwahori and compact subgroups of $\widetilde{G}$.* Recall the subgroups $K := \mathbf{G}(\mathcal{O})$ and $I^- \subset K$ constructed in §5.2.2. Let $\widetilde{I}^- \subset \widetilde{G}$ be the subgroup generated by the following elements:

$$\xi_a(s), s \in \mathcal{O}, a \in \mathcal{R}_-, \quad \xi_{-a}(t), a \in \mathcal{R}_+, t \in \pi\mathcal{O} \quad \text{and} \quad \widetilde{h} \in \widetilde{H}_{\mathcal{O}}. \tag{5.7}$$

Writing $\widetilde{I}_+^- := \widetilde{I} \cap U$ and $\widetilde{I}_-^- := \widetilde{I} \cap U^-$, one can show as in [56, Theorem 2.5] that $\widetilde{I}^- = \widetilde{I}_+^- \widetilde{H}_{\mathcal{O}} \widetilde{I}_-^-$. Moreover one knows [97, Lemma 6.1.3] that the covering map p satisfies $p|_{\widetilde{I}} : \widetilde{I}^- \to I^-$ is an isomorphism. For this reason, we continue to write I^- in place of $\widetilde{I}^-$ and will let context dictate what we mean.

Let $\widetilde{K} \subset \widetilde{G}$ be the subgroup generated by elements $\xi_a(s)$ with $s \in \mathcal{O}$, $a \in \mathcal{R}$. For $w \in W$, the elements $\widetilde{w}$ constructed earlier lie in K, and one has $\widetilde{K} = \bigsqcup_{w \in W} \widetilde{I} \widetilde{w} \widetilde{I}$. Moreover, $p(\widetilde{I} \widetilde{w} \widetilde{I}) = I \dot{w} I$. Thus if $x \in \widetilde{K}$ is such that $p(x) = 1$, then $x \in \widetilde{I}$ and, from what we said earlier about $p|_{\widetilde{I}}$ we have $x = 1$, i.e. $p : \widetilde{K} \to K$ is an isomorphism. From now on, we just identify K with $\widetilde{K}$ using p, and shall drop the notation $\widetilde{K}$.

5.2.7. *Decompositions of $\widetilde{G}$.* Recall the decompositions from §5.2.3 and let us explain the corresponding versions for $\widetilde{G}$ now. The Cartan decomposition for $\widetilde{G}$ asserts that $\widetilde{G} = \sqcup_{\check{\lambda} \in Y_+} \mu_n K \pi^{\check{\lambda}} K$ and easily follows from the corresponding result for $\widetilde{G}$. The analogue of the Iwasawa decomposition for $\widetilde{G}$ is as follows. Every $g \in \widetilde{G}$ can be written as $g = u^- a k$ with $k \in \widetilde{K}$, $u^- \in \widetilde{U}$, and $a \in \widetilde{H}$. The class of $a \in \widetilde{H}/\widetilde{H}_{\mathcal{O}}$ is unique in any such decomposition. For $g \in \widetilde{G}$ written (non-uniquely) as $g = u^- a k$, we denote the class of a in $\widetilde{H}/\widetilde{H}_{\mathcal{O}}$ by $\mathrm{Iw}_{\widetilde{H}}(g)$. Use what was said at the end of §5.2.5 to set

$$\ln(g) := \ln(\mathrm{Iw}_{\widetilde{H}}(g)) \in Y \quad \text{and} \quad \mathbf{z}(g) := \mathbf{z}(\mathrm{Iw}_{\widetilde{A}}(g)) \in \mu_n, \tag{5.8}$$

so that image of $\mathrm{Iw}_{\widetilde{A}}(g)$ in $Y \times \mu_n$ is $(\ln(g), \mathbf{z}(g))$.

5.3. **Hecke algebras.** We now review an algebraic (as opposed to measure theoretic) framework to associate convolution Hecke algebras on p-adic groups. We refer to [16, §4] for more details on this approach, which is equivalent to the usual one.

5.3.1. *Iwahori–Hecke algebras.* For $x \in W_{\mathrm{aff}}$, let $\mathcal{T}_x$ denote the characteristic function of $I^- \dot{x} I^-$. Let $\mathcal{H}(G, I^-)$ denote the space of $\mathbb{Z}$- linear combinations of the form $\sum_{x \in W_{\mathrm{aff}}} n_x \mathcal{T}_x$ with almost all $n_x = 0$. Equivalently, we are looking at $\mathbb{Z}$-valued, compactly supported I^--biinvariant functions on G. This $\mathbb{Z}$-module is equipped with the structure of an associative, unital algebra under convolution defined as follows. For $x, y \in W_{\mathrm{aff}}$, let $m_{x,y} : I^- \dot{x} I^- \times_{I^-} I^- \dot{y} I^- \to G$ and define (see [56, p. 44])

$$\mathcal{T}_x \star \mathcal{T}_y := \sum_{z \in W_{\mathrm{aff}}} n_z \mathcal{T}_z, \text{ where } n_z := |m^{-1}(\dot{z})| = |I^- \setminus I^- \dot{x}^{-1} I^- z \cap I^- \dot{y} I^-|. \tag{5.9}$$

Theorem. [56, Theorem 3.5] *Writing H_{aff} for the affine Hecke algebra attached to $(I, \cdot, \mathfrak{D})$ (see §3.5), the map $H_{\mathrm{aff}} \to \mathbb{Z}[q^{\pm 1/2}] \otimes \mathcal{H}(G, I^-)$ sending $T_{s_i} \mapsto \mathcal{T}_{s_i}$ for $i \in I_{\mathrm{aff}}$ is an isomorphism of algebras when $\tau \mapsto q^{1/2}$.*

Using this result, we construct elements in $\mathcal{H}(G, I^-)$ that we again denote (by abuse of notation) as H_w and $Y_{\check{\lambda}}$ for $w \in W, \check{\lambda} \in Y$ which correspond to similarly named elements in H_{aff}.

5.3.2. *Spherical Hecke algebra.* The *spherical Hecke algebra* $\mathcal{H}(G, K)$ consists of finite linear combinations of the characteristic functions $h_{\check{\lambda}}, \check{\lambda} \in Y_+$ of the double cosets $K \pi^{\check{\lambda}} K$ and multiplication defined as follows: for $\check{\lambda}, \check{\mu} \in Y_+$ and $m : K \pi^{\check{\lambda}} K \times_K K \pi^{\check{\mu}} K \to G$, set

$$h_{\check{\lambda}} \star h_{\check{\mu}} = \sum_{\check{\eta} \in Y_+} |m^{-1}(\pi^{\check{\eta}})| h_{\check{\eta}} = \sum_{\check{\eta} \in Y_+} |K \setminus K \pi^{-\check{\lambda}} K \pi^{\check{\eta}} \cap K \pi^{\check{\mu}} K| h_{\check{\eta}}. \tag{5.10}$$

Justifying the abuse of notation (since $h_{\check{\mu}}$ were already defined in §3.5.3), we have the following result.

Theorem. *The map from the spherical subalgebra $H_{\mathrm{sph}} := \varepsilon H_{\mathrm{aff}} \varepsilon \subset H_{\mathrm{aff}}$ to $\mathbb{Z}[q^{\pm 1/2}] \otimes \mathcal{H}(G, K)$ sending $\varepsilon Y_{\check{\mu}} \varepsilon$ to $h_{\check{\mu}}$ for $\check{\mu} \in Y_+$ is an isomorphism of algebras when $\tau \mapsto q^{1/2}$.*

5.3.3. *Relating $\mathcal{H}(G, I^-)$ and $\mathcal{H}(G, K)$.* To relate $\mathcal{H}(G, I^-)$ with $\mathcal{H}(G, K)$ we need to take into account the fact that convolution in the spherical Hecke algebra is with respect to a measure which assigns K volume 1, whereas in $\mathcal{H}(G, I^-)$, the convolution is with respect to a measure that assigns I^- volume 1. Recalling equation (5.4), let $\mathbf{e}_K := \frac{\mathbf{1}_K}{\mathsf{vol}(K)}$ where $\mathbf{1}_K$ is the characteristic function on G of the subgroup K. One may define a left and right convolution of $\mathcal{H}(G, I^-)$ with the function $\mathbf{e}_K$ which then produces an element in $\mathcal{H}(G, K)$, and we find that the map $\mathcal{H}(G, I^-) \to \mathcal{H}(G, K), f \mapsto \mathbf{e}_K \star f \star \mathbf{e}_K$ fits into the commutative diagram

$$\begin{tikzcd} \mathcal{H}(G, I^-) \arrow[r, "\mathbf{e}_K \star \cdot \star \mathbf{e}_K"] \arrow[d, "\cong"'] & \mathcal{H}(G, K) \arrow[d, "\cong"] \\ H_{\mathrm{aff}} \arrow[r, "\varepsilon \cdot \varepsilon"] & H_{\mathrm{sph}} \end{tikzcd} \tag{5.11}$$

where the bottom row is the map $h \mapsto \varepsilon h \varepsilon, h \in H_{\mathrm{aff}}$ and the vertical maps are from Theorems 5.3.1 and 5.3.2.

5.3.4. *ε-genuine functions.* Recall that $\mu_n \subset \mathcal{F}$ is assumed to have cardinality n. Fix a faithful embedding $\varepsilon : \mu_n \hookrightarrow \mathbb{C}^*$ which we use to define the Gauss sums as in §2.2.4. A function $f : \widetilde{G} \to \mathbb{C}$ is ε-genuine if

$$f(\zeta g) = \varepsilon(\zeta) f(g) \text{ for } g \in \widetilde{G}, \zeta \in \mu_n. \tag{5.12}$$

Let $\mathbb{C}_\varepsilon(\widetilde{G})$ denote the space of compactly supported ε-genuine functions on $f : \widetilde{G} \to \mathbb{C}$.

5.3.5. *Metaplectic spherical Hecke algebra.* Define the spherical Hecke algebra

$$\mathcal{H}(\widetilde{G}, K) := \{ f \in \mathbb{C}_\varepsilon(\widetilde{G}) \mid f(k_1 g k_2) = f(g) \text{ for } k_1, k_2 \in K \}. \tag{5.13}$$

The Cartan decomposition of $\widetilde{G}$ (see §5.2.7) ensures that the functions, for $\check{\lambda} \in Y_+$,

$$\widetilde{h}_{\check{\lambda}}(x) = \begin{cases} \varepsilon(\mathbf{z}(x)) & \text{if } x \in \mu_n K \pi^{\check{\lambda}} K, \\ 0 & \text{otherwise,} \end{cases} \tag{5.14}$$

span the $\mathbb{C}$-vector space $H_\varepsilon(\widetilde{G}, K)$. In fact, one can also verify that $\widetilde{h}_{\check{\lambda}}$ if well-defined only if $\check{\lambda} \in \widetilde{Y}$ since if $\check{\lambda} \notin \widetilde{Y}$ there exists $h_{\mathcal{O}} \in \widetilde{H}_{\mathcal{O}}$ such that $[h_{\mathcal{O}}, \pi^{\check{\lambda}}] \neq 1$. The collection $\{\widetilde{h}_{\check{\lambda}}\}_{\check{\lambda} \in \widetilde{Y}_+}$ forms a basis of $\mathcal{H}(\widetilde{G}, K)$ as a vector space. The vector space $\mathcal{H}(\widetilde{G}, K)$ is also equipped with a convolution structure, which is defined by considering the multiplication map $m_{\check{\lambda}, \check{\nu}} : \mu_n K \pi^{\check{\lambda}} K \times_K K \pi^{\check{\nu}} K \to \widetilde{G}$. For $\check{\mu} \in \widetilde{Y}_+$ and $x \in m^{-1}_{\check{\lambda}, \check{\nu}}(\pi^{\check{\mu}})$, written as $x = (a, b)$ with $a \in \mu_n$, $K \pi^{\check{\lambda}} K$ and $b \in K \pi^{\check{\nu}} K$, we set, in the terminology of §5.2.7, $\mathbf{z}(x) := \mathbf{z}(a) \in \mu_n$ as it only depends on x. Then we define the convolution product as

$$\widetilde{h}_{\check{\lambda}} \star \widetilde{h}_{\check{\nu}} = \sum_{\check{\mu} \in Y} \widetilde{h}_{\check{\mu}} \left(\sum_{x \in m^{-1}_{\check{\lambda}, \check{\nu}}(\pi^{\check{\mu}})} \varepsilon(\mathbf{z}(x)) \right). \tag{5.15}$$

The metaplectic Satake isomorphism ([90, §11]) gives an isomorphism of algebras

$$\widetilde{S} : \mathcal{H}(\widetilde{G}, K) \xrightarrow{\sim} \mathbb{C}[\widetilde{Y}]^W \cong K_0\left(\mathrm{Rep}(\check{\mathbf{G}}_{(\mathsf{Q},n)}(\mathbb{C}))\right), \tag{5.16}$$

where $\check{\mathbf{G}}_{(\mathsf{Q},n)}(\mathbb{C})$ is the complex group attached to $\widetilde{\mathfrak{D}}^\vee := \left(\mathfrak{D}_{(\mathsf{Q},n)}\right)^\vee$. So, for each $\check{\lambda} \in \widetilde{Y}_+$, we can define elements $\widetilde{c}_{\check{\lambda}} \in \mathcal{H}(\widetilde{G}, K)$ such that $\widetilde{S}(\widetilde{c}_{\check{\lambda}}) = \chi_{\check{\lambda}}$, i.e. $\widetilde{c}_{\check{\lambda}}$ corresponds to the class of the representation $V_{\check{\lambda}}$ of highest weight $\check{\lambda}$ of the group $\check{\mathbf{G}}_{(\mathsf{Q},n)}(\mathbb{C})$. For $\check{\lambda}, \check{\mu}, \check{\zeta} \in \widetilde{Y}_+$, we have the multiplication rule

$$\widetilde{c}_{\check{\lambda}} \star \widetilde{c}_{\check{\mu}} = \sum_{\check{\zeta} \in \widetilde{Y}_+} \dim \mathrm{Hom}_{\check{\mathbf{G}}_{(\mathsf{Q},n)}(\mathbb{C})}(V_{\check{\lambda}} \otimes V_{\check{\mu}}, V_{\check{\zeta}}) \widetilde{c}_{\check{\zeta}}. \tag{5.17}$$

5.3.6. *Metaplectic Iwahori–Hecke algebras.* We define $\mathcal{H}(\widetilde{G}, I^-)$ to be the space of functions $f \in \mathbb{C}_\varepsilon(\widetilde{G})$ which are also I^- bi-invariant. It has a convolution structure that can be defined starting from the multiplication map $m_{x,y} : I^-\dot{x}I^- \times_{I^-} I^-\dot{y}I^-\widetilde{G}$ and using a procedure as in §5.3.1, but taking into account the ε-genuine condition as in the previous paragraph. As an algebra, the structure of this Hecke algebra was first described by Savin in [103], [104, §6] (see [90, §12] for a proof that corrects an error in a rank 1 computation of Savin's original proof). In our notation, his results state

$$\mathcal{H}(\widetilde{G}, I^-) \cong \mathbb{C} \otimes_{\mathbb{Z}_\tau} \widetilde{H}_{\mathrm{aff}} := \mathbb{C} \otimes_{\mathbb{Z}_\tau} H_{\mathrm{aff}}(I, \circ_{(\mathsf{Q},n)}, \widetilde{\mathfrak{D}}) \cong H_W \otimes \mathbb{C}[\widetilde{Y}], \tag{5.18}$$

where the map $\mathbb{Z}_\tau \to \mathbb{C}$ sends $\tau \mapsto q^{-1/2}$. We shall write $Y_{\check{\lambda}}, \check{\lambda} \in \widetilde{Y}$ and $H_w, T_w, w \in W$ for the elements in $\mathcal{H}(\widetilde{G}, I^-)$ corresponding to similarly named elements in $\widetilde{H}_{\mathrm{aff}}$. To relate $\mathcal{H}(\widetilde{G}, I^-)$ and $\mathcal{H}(\widetilde{G}, K)$ one has a diagram analogous to (5.11), where we note that $\mathbf{e}_K$ is again defined as $\mathbf{1}_K/\mathsf{vol}(K)$ where $\mathsf{vol}(K)$ stays the same in the metaplectic context.

6. Spherical and Iwahori–Whittaker functions on covers of p-adic groups

Fix the same conventions as at the start of Part II and notations as in §5.2-5.3.

6.1. **Whittaker spaces for covering groups.**

6.1.1. *Character of unipotents.* Fix an additive character $\psi : \mathcal{F} \to \mathbb{C}^*$ as in §2.2.3. For $a \in \Pi$, using the chosen isomorphism $x_a : \mathcal{F} \to \mathbf{U}_a(\mathcal{F}), s \mapsto x_a(s)$, we regard ψ first as a character of $U_a := \mathbf{U}_a(\mathcal{F})$, and then, via the isomorphism $U \to U/[U,U] \cong \oplus_{a\in\Pi} U_a$, as a character, to be denoted by the same name, $\psi : U \to \mathbb{C}^*$. One can easily verify that $\psi|_{U(\mathcal{O})}$ is trivial, and $\psi|_{U(\pi^{-1}\mathcal{O})}$ is non-trivial. As we identify the unipotent group U with its preimage in the cover $\widetilde{G}$, we can regard ψ as a character of $U \subset \widetilde{G}$ as well.

6.1.2. *Spherical Whittaker space $\mathcal{W}_\psi(\widetilde{G}, K)$.* An ε-genuine function $f : \widetilde{G} \to \mathbb{C}$ is said to be (U, ψ) left invariant if $f(ug) = \psi(u) f(g)$ for any $u \in U$ and $g \in G$. Denote by $\mathcal{W}_\psi(\widetilde{G}, K)$ the space of ε-genuine functions on $\widetilde{G}$ which are right K-invariant, left (U, ψ)-invariant and supported on a finite union of sets of the form $\mu_n U \pi^{\check{\mu}} K$ with $\check{\mu} \in Y$. For $f \in \mathcal{W}_\psi(\widetilde{G}, K)$, if we define $\mathrm{Supp}(f) \subset Y$ as the set of $\check{\mu} \in Y$ such that $f(\pi^{\check{\mu}}) \neq 0$, then one can show that $\mathrm{Supp}(f) \subset Y_+$ for any $f \in \mathcal{W}_\psi(\widetilde{G}, K)$. For $\check{\mu} \in Y$, we define $\widetilde{\mathcal{J}}_{\check{\mu}} \in \mathcal{W}_\psi(\widetilde{G}, K)$ to be the unique ε-genuine function that takes value 1 on $\pi^{\check{\mu}}$ and is 0 outside of $\mu_n U \pi^{\check{\mu}} \widetilde{K}$. Using the Iwasawa decomposition from §5.2.7, one shows that the set $\{\widetilde{\mathcal{J}}_{\check{\mu}}\}_{\check{\mu} \in Y_+}$ is a basis of $\mathcal{W}_\psi(\widetilde{G}, K)$.

6.1.3. *Convolution action of $\mathcal{H}(\widetilde{G},K)$ on $\mathcal{W}_\psi(\widetilde{G},K)$.* One has a right convolution action, denoted again by $\star$, of $\mathcal{H}(\widetilde{G},K)$ on the space $\mathcal{W}_\psi(\widetilde{G},K)$. Let us recall here the definition of $\widetilde{\mathfrak{J}}_{\check{\mu}} \star h_{\check{\lambda}}$ with $\check{\mu} \in Y_+, \check{\lambda} \in \widetilde{Y}_+$. As we observed earlier, the support of such a function is a linear combination of $\widetilde{\mathfrak{J}}_{\check{\zeta}}$ with $\check{\zeta} \in Y_+$. For such a $\check{\zeta} \in Y_+$, consider the multiplication map $m_{\check{\mu},\check{\lambda}} : \mu_n U\pi^{\check{\mu}}K \times_K K\pi^{\check{\lambda}}K \to \widetilde{G}$ and let $(a,b) \in m^{-1}_{\check{\mu},\check{\lambda}}(\pi^{\check{\zeta}})$ with $a \in \mu_n U\pi^{\check{\mu}}K$ and $b \in K\pi^{\check{\lambda}}K$. Suppose we write $a = \omega u\pi^{\check{\mu}}k$ for $\omega \in \mu_n, u \in U, k \in K$. Write $\omega = \mathbf{z}(x)$ as one sees easily that it depends only on x. On the other hand, u is not well-defined as we can replace it by uu_1 with $\pi^{-\check{\mu}}u_1\pi^{\check{\mu}} \in K$, i.e. $n_1 \in \pi^{\check{\mu}}U_{\mathscr{O}}\pi^{-\check{\mu}}$. However, since $\check{\mu} \in Y_+$ and ψ is trivial on $U_{\mathscr{O}}$, we have $\psi(uu_1) = \psi(u)\psi(u_1) = \psi(n)$. In sum, if for any $x = (a,b) \in m^{-1}_{\check{\mu},\check{\lambda}}(\pi^{\check{\zeta}})$ where $a = \mathbf{z}(x)\,u\pi^{\check{\mu}}k$ we set $\mathbf{n}(x) = u$, then $\psi(\mathbf{n}(x))$ only depends on x. Using this notation, we define

$$\widetilde{\mathfrak{J}}_{\check{\mu}} \star_K \widetilde{h}_{\check{\lambda}} := \sum_{\check{\zeta} \in Y_+} q^{\langle \rho, \check{\zeta} \rangle} \left(\sum_{x \in m^{-1}_{\check{\mu},\check{\lambda}}(\pi^{\check{\zeta}})} \varepsilon(\mathbf{z}(x))\psi(\mathbf{n}(x)) \right) \mathfrak{J}_{\check{\zeta}}, \tag{6.1}$$

where $\check{\zeta} \leq \check{\lambda} + \check{\mu}$ in order for $b^{\check{\lambda}}_{\check{\mu}}(\check{\zeta}) := q^{\langle \rho, \check{\zeta} \rangle} \left(\sum_{x \in m^{-1}_{\check{\mu},\check{\lambda}}(\pi^{\check{\zeta}})} \psi(x)\varepsilon(\mathbf{z}(x)) \right) \neq 0$.

Remark. *Let du be the usual Haar measure on U which assigns $U(\mathscr{O})$ volume* 1. *Then we can also write*

$$b^{\check{\lambda}}_{\check{\mu}}(\check{\zeta}) \quad = \quad q^{\langle \rho, \check{\zeta} \rangle} \int_U \psi(u) h_{\check{\lambda}}(\pi^{-\check{\mu}+\check{\zeta}}u)du. \tag{6.2}$$

6.1.4. *Iwahori–Whittaker spaces.* We define $\mathcal{W}_\psi(\widetilde{G},I^-)$ by replacing K with I^- in the definitions given in §6.1.2. In light of the Iwahori-Matsumoto decomposition (5.6), we may regard functions in $\mathcal{W}_\psi(\widetilde{G},I^-)$ as (U,ψ)-left invariant, right I^--invariant, ε-genuine functions which are supported on a finite union of sets of the form $\mu_n U\dot{x}I^-$ with $x \in W_{\text{aff}} := W_{\text{aff}}(I,\cdot,\mathfrak{D})$. For each $x \in W_{\text{aff}}$, let $\mathbf{v}^{I^-}_{\psi,x}$ denote the unique function (if it exists) in $\mathcal{W}_\psi(\widetilde{G},I^-)$ which is supported on the subset $\mu_n U\dot{x}I^-$, and taking the value 1 on $\dot{x}$. If $x = \mathtt{t}(\check{\mu})$ with $\check{\mu} \in Y$, we shall just write $\mathbf{v}^{I^-}_{\psi,\check{\mu}}$ in place $\mathbf{v}^{I^-}_{\psi,\mathtt{t}(\check{\mu})}$ from now on. Note that $\mathbf{v}^{I^-}_{\psi,x}$ with $x \in W_{\text{aff}}$ is only well-defined when $\psi\,|_{xI^-x^{-1}\cap U} = 1$. The vector space $\mathcal{W}_\psi(\widetilde{G},I^-)$ carries a right action, denoted again by $\star$, under convolution by $\mathcal{H}(\widetilde{G},I^-)$. One can define its action in a similar manner to the one in which we defined the action of $\mathcal{H}(\widetilde{G},K)$ on $\mathcal{W}_\psi(\widetilde{G},K)$ (note that the modular character does not appear though).

6.1.5. *Averaging operator* $\mathrm{Av}^{\text{gen}}_{U^-}$. Define the operator $\mathrm{Av}^{\text{gen}}_{U^-} : \mathcal{W}_\psi(\widetilde{G},I^-) \to \mathbb{C}[Y]$ by linearly extending the following procedure. For $x \in W_{\text{aff}}$, let $f = \mathbf{v}^{I^-}_{\psi,x}$. Note that f is only well-defined if $\psi_{xI^-x^{-1}\cap U} = 1$.

Using the map $m = m_x : \mu_n U\dot{x}I^- \times I^-U^- \to \widetilde{G}$, for $z \in \widetilde{G}$ and $y = (a,b) \in m^{-1}(z)$ with $a = \mathbf{z}(y)\,uxi$ for $\mathbf{z}(y) \in \mu_n, u \in U$, and $i \in I^-$, we may set $\psi(y) = \psi(u)$. By our assumption on x, $\psi(y)$ is well-defined. Set

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(\mathbf{v}_{\psi,x}^{I^-}) = \sum_{\check{\mu}\in Y}\left(\sum_{y\in m_x^{-1}(\pi^{\check{\mu}})} \varepsilon(\mathbf{z}(y))\,\psi(y)\right) Y_{\check{\mu}}\, q^{\langle\rho,\check{\mu}\rangle}. \tag{6.3}$$

For example, since $U\pi^{\check{\lambda}}I^- \cap U^-\pi^{\check{\lambda}}I^- = \pi^{\check{\lambda}}I^-$ we have

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(q^{-\langle\rho,\check{\lambda}\rangle}\mathbf{v}_{\psi,\check{\lambda}}^{I^-}) = Y_{\check{\lambda}} \text{ for } \check{\lambda}\in Y_+. \tag{6.4}$$

Remark. *In the usual integral notation, one writes*

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(f) = \sum_{\check{\zeta}\in Y} q^{\langle\rho,\check{\zeta}\rangle}\left(\int_{U^-} f(\pi^{\check{\zeta}}u^-)du^-\right) Y_{\check{\zeta}}, \tag{6.5}$$

with du^- the Haar measure normalized so that $\mathbf{U}^-(\mathscr{O}) = K\cap U^-$ has total volume 1.

6.1.6. The following result can be deduced using arguments as in [29].

Proposition. *The map $\mathrm{Av}_{U^-}^{\mathrm{gen}}$ induces an isomorphism of vector spaces*

$$\mathrm{Av}_{U^-}^{\mathrm{gen}} : \mathcal{W}_\psi(\widetilde{G},I^-) \xrightarrow{\cong} \mathbb{C}[Y], \tag{6.6}$$

which satisfies the following equivariance condition:

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(w\star Y_{\check{\lambda}}) = \mathrm{Av}_{U^-}^{\mathrm{gen}}(w)Y_{\check{\lambda}} \text{ for } w\in\mathcal{W}_\psi(\widetilde{G},I^-), \check{\lambda}\in\widetilde{Y}, \tag{6.7}$$

where the action on the right hand side is just by multiplication in the group algebra.

One can prove the Proposition in a different manner than *op. cit.* as follows:

- We may write $\mathrm{Av}_{U^-}^{\mathrm{gen}}(\mathbf{v}_{\psi,x}^{I^-}) = \sum_{\check{\mu}\in S_x} c_{\check{\mu}} Y_{\check{\mu}}$ with $c_{\check{\mu}}\in\mathbb{C}$ and $S_x := \{\check{\mu}\in Y \mid \mu_n U\dot{x}I^- \cap \mu_n U^-\pi^{\check{\mu}}I^- \neq \emptyset\}$. One may introduce a natural order on Y such that $\mathrm{Av}_{U^-}^{\mathrm{gen}}$ is upper triangular with non-zero diagonal coefficients. This suffices to prove that the map is an isomorphism.
- To prove the equivariance, one uses that $Y_{\check{\lambda}} \star \mathbf{1}_{\mu_n I U^-} = \mathbf{1}_{\mu_n I^-\pi^{\check{\lambda}}U^-}$ for $\check{\lambda}\in\widetilde{Y}$, where for any $\check{\nu}\in\widetilde{Y}$, we write $\mathbf{1}_{\mu_n I^-\pi^{\check{\nu}}U^-}\in\mathbb{C}_\varepsilon(G)$ to be the unique function supported on $\mu_n I^-\pi^{\check{\nu}}U^-$ and taking value 1 on $\pi^{\check{\nu}}$.

6.2. **Structure of the Iwahori–Whittaker space $\mathcal{W}_\psi(\widetilde{G},I^-)$.** Recall that we constructed an action of $\widetilde{H}_{\mathrm{aff}}$ on $\mathbb{V} = \mathbb{Z}_{\tau,\mathfrak{g}}[Y]$ in §4.2 by means of the operators $\widetilde{\mathbf{T}}_{s_i}$ for $i\in I$, see §4.1. Using the p-adic specialization map $\mathfrak{p}: \mathbb{Z}_{\tau,\mathfrak{g}}\to\mathbb{C}$ defined as in 2.3.3 and the isomorphism $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{H}(\widetilde{G},I^-)\cong\mathbb{C}\otimes_{\mathbb{Z}_\tau}\widetilde{H}_{\mathrm{aff}}$, we obtain an action of $\mathcal{H}(\widetilde{G},I^-)$ on $\mathbb{C}[Y] = \mathbb{C}\otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}}\mathbb{Z}_{\tau,\mathfrak{g}}[Y]$.

6.2.1. Let $\widetilde{\mathcal{Y}}_{\check{\lambda}} \in \mathcal{W}_{\psi}(\widetilde{G}, I^-)$ be defined uniquely through the relation

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(\widetilde{\mathcal{Y}}_{\check{\lambda}}) = Y_{\check{\lambda}} \text{ for any } \check{\lambda} \in Y. \tag{6.8}$$

Proposition. *For any simple root $a \in \Pi$ and $\check{\lambda} \in Y$, we have*

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(\widetilde{\mathcal{Y}}_{\check{\lambda}} \star \widetilde{\mathcal{T}}_a) = Y_{\check{\lambda}} \cdot \widetilde{\mathbf{T}}_a. \tag{6.9}$$

Hence we have an isomorphism of $\mathbb{C}$-vector spaces that we denote as

$$\mathfrak{p} : \mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V} \xrightarrow{\simeq} \mathcal{W}_{\psi}(\widetilde{G}, I^-) \tag{6.10}$$

sending $Y_{\check{\lambda}} \mapsto \widetilde{\mathcal{Y}}_{\check{\lambda}}$ and intertwining the actions of $\widetilde{H}_{\mathrm{aff}}$ and $\mathcal{H}(\widetilde{G}, I^-)$.

6.2.2. *Proof of Proposition 6.2.1, part 1.* Suppose that $\check{\lambda} \in Y_+$ and consider the element $\mathbf{v}^{I^-}_{\psi,\check{\lambda}}$ constructed above which we showed satisfies $\mathrm{Av}_{U^-}^{\mathrm{gen}}(q^{-\langle\rho,\check{\lambda}\rangle}\mathbf{v}^{I^-}_{\check{\lambda}}) = Y_{\check{\lambda}}$. Hence $\widetilde{\mathcal{Y}}_{\check{\lambda}} = q^{-\langle\rho,\check{\lambda}\rangle}\mathbf{v}^{I^-}_{\psi,\check{\lambda}}$ for $\check{\lambda} \in Y_+$. We claim

$$\mathrm{Av}_{U^-}^{\mathrm{gen}}(q^{-\langle\rho,\check{\lambda}\rangle}\mathbf{v}^{I^-}_{\check{\lambda}} \star \widetilde{\mathcal{T}}_{s_a}) = Y_{\check{\lambda}} \cdot \widetilde{\mathbf{T}}_a \text{ when } \check{\lambda} \in Y_+ \tag{6.11}$$

follows from the arguments in [95, §5.7]. Since the setup in *op. cit.* is a bit different than the present context (in particular what we write as $\widetilde{\mathbf{T}}_{s_i}$ here is $\widetilde{\mathbf{T}}_{s_i}^{-1}$ in *op. cit.*), let us sketch the key ideas in the computation here. First, to determine the support of $\mathrm{Av}_{U^-}^{\mathrm{gen}}(\mathbf{v}^{I^-}_{\check{\lambda}} \star \widetilde{\mathcal{T}}_{s_a})$, we need to determine for which $\check{\mu} \in Y$

$$\emptyset \neq \mu_n U \pi^{\check{\lambda}} I^- \widetilde{w}_a I^- U^- \cap \mu_n U \pi^{\check{\mu}} U^- = \mu_n U \pi^{\check{\lambda}} U_{\pi} U_{-a,\mathscr{O}} \widetilde{w}_a U^- \cap \mu_n U \pi^{\check{\mu}} U^-, \tag{6.12}$$

where we have used the dominance of $\check{\lambda}$ and the Iwahori–Matsumoto factorization to obtain the equality in the last line. Apply [95, (5.24), (5.25)] or using the relations described in §5.1.8, for $x_{-a}(s) \in U_{-a}(\mathscr{O})$ with $s^{-1} = \pi^{-k}r, r \in \mathscr{O}^*$ and $k \geq 0$, we write

$$x_{-a}(s) = x_a(s^{-1})h_a(s^{-1})\widetilde{w}_a x_a(s^{-1}), \text{ where } h_a(s^{-1}) = h_a(r)\pi^{-k\check{a}}(r,\pi)^{k\mathsf{Q}(\check{a})}. \tag{6.13}$$

Defining $\Psi(x_{-a}(s)) = \varepsilon\left((r,\pi)^{k\mathsf{Q}(\check{a})}\right)$, one finds then that $\mathrm{Av}_{U^-}^{\mathrm{gen}}(q^{-\langle\rho,\check{\lambda}\rangle}\mathbf{v}^{I^-}_{\check{\lambda}} \star \widetilde{\mathcal{T}}_{s_a})$ is computed as the sum

$$\sum_{k\geq 0} q^{\langle\rho,\check{\lambda}-\check{\mu}\rangle} Y_{\check{\lambda}-k\check{a}} \int_{U_{-a}[k]} \psi(\pi^{\check{\lambda}} u_{-a} \pi^{\check{\lambda}}) \Psi(x_{-a}(s)) du_{-a}[k], \tag{6.14}$$

where $du_{-a}[k]$ is the restriction of the measure on U_{-a} giving $U_{-a}(\mathscr{O})$ total volume 1. Note that non-zero summands in the above expression may arise only for $Y_{\check{\zeta}}$ when $\check{\zeta} \in \{\check{\lambda}, \check{\lambda} - \widetilde{a}^{\vee}, \ldots, \check{\lambda} - k_0\widetilde{a}^{\vee}, \check{\lambda} \bullet w_a\}$, where k_0 is the largest positive integer such that $k_0\widetilde{a}^{\vee} \leq \langle\check{\lambda}, a\rangle$. We leave the rest of the computation to the reader as it follows from the same ideas at the end of [95, §5.7].

6.2.3. *Proof of Proposition 6.2.1, part 2.* Recalling the definition of $\overline{\mathcal{A}}^{\bullet}_{-,n}$ from (3.66), we choose $\check{\lambda} = \check{\eta} \bullet W$ for some $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$. As we observed earlier, this forces

$$-n(\check{a}_i) < \langle \check{\lambda} + \check{\rho}, a_i \rangle < n(\check{a}_i) \text{ for all } i \in I, \tag{6.15}$$

and one notes that verifying the Proposition for these $\check{\lambda}$ is equivalent (see Lemma 4.1.2(2)) to verifying :

$$\widetilde{\mathcal{Y}}_{\check{\lambda}} \cdot \widetilde{\mathcal{T}}_{s_i} = \begin{cases} \mathbf{g}_{\langle \check{\lambda}+\check{\rho}, a_i \rangle \mathsf{Q}(\check{a}_i)} \widetilde{\mathcal{Y}}_{\check{\lambda} \bullet s_i} & \text{if } -n(\check{a}_i) < \langle \check{\lambda} + \check{\rho}, a_i \rangle < 0, \\ \mathbf{g}_{\langle \check{\lambda}+\check{\rho}, a_i \rangle \mathsf{Q}(\check{a}_i)} \widetilde{\mathcal{Y}}_{\check{\lambda} \bullet s_i} + (q-1) \widetilde{\mathcal{Y}}_{\check{\lambda}} & \text{if } 0 < \langle \check{\lambda} + \check{\rho}, a_i \rangle < n(\check{a}_i), \\ -\widetilde{\mathcal{Y}}_{\check{\lambda}} & \text{if } \langle \check{\lambda} + \check{\rho}, a_i \rangle = 0. \end{cases} \tag{6.16}$$

Indeed, we may just apply the isomorphism $\mathrm{Av}^{\mathrm{gen}}_{U^-}$ to such relations and bear in mind that $\mathrm{Av}^{\mathrm{gen}}_{U^-} : \widetilde{\mathcal{Y}}_{\check{\lambda}} \mapsto Y_{\check{\lambda}}$. Now the second case above follows since $\check{\lambda} \in Y_+$, so we may apply part (1) and the general formulas for $\widetilde{\mathbf{T}}_{s_i}$ action. Assume then that $\check{\lambda}$ is as in the first case, and let us show the equivalent statement that $\mathbf{g}^{-1}_{\langle \check{\lambda}+\check{\rho}, a_i \rangle \mathsf{Q}(\check{a}_i)} \widetilde{\mathcal{Y}}_{\check{\lambda}} = \widetilde{\mathcal{Y}}_{\check{\lambda} \bullet s_i} \cdot \widetilde{\mathcal{T}}_{s_i}^{-1}$. Since

$$\mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda} \bullet s_i} \cdot \widetilde{\mathcal{T}}_{s_i}^{-1}) = q\, \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda} \bullet s_i} \cdot \widetilde{\mathcal{T}}_{s_i}) + (q-1)\, \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda} \bullet s_i}), \tag{6.17}$$

and $\langle \check{\lambda} \bullet s_i, a_i \rangle \geq 0$ by our assumption on $\check{\lambda}$, the right hand side of the above can be computed explicitly using part (1) and the desired result follows. The third case is proven similarly. The Proposition is thus proven for all $\check{\lambda} \in \overline{\mathcal{A}}^{\bullet}_{-,n} \bullet W$.

6.2.4. *Proof of Proposition 6.2.1, part 3.* Observe that for $\check{\zeta} \in \widetilde{Y}$, Proposition 6.1.6 gives

$$\widetilde{\mathcal{Y}}_{\check{\lambda}} Y_{\check{\zeta}} = \widetilde{\mathcal{Y}}_{\check{\lambda}+\check{\zeta}} \text{ for } \check{\lambda} \in \overline{\mathcal{A}}^{\bullet}_{-,n} \bullet W, \check{\zeta} \in \widetilde{Y}. \tag{6.18}$$

As every $\check{\mu} \in Y$ is of the form $\check{\mu} = \check{\lambda} + \check{\zeta}$ with $\check{\zeta} \in \widetilde{Y}$ and $\check{\lambda} \in \overline{\mathcal{A}}^{\bullet}_{-,n} \bullet W$, it suffices to show

$$\mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}+\check{\zeta}} \star \widetilde{\mathcal{T}}_a) = Y_{\check{\lambda}+\check{\zeta}} \cdot \widetilde{\mathbf{T}}_a \text{ for } a \in \Pi, \check{\lambda} \in \overline{\mathcal{A}}^{\bullet}_{-,n} \bullet W, \check{\zeta} \in \widetilde{Y}. \tag{6.19}$$

Using Proposition 6.1.6 we may compute

$$\begin{aligned} \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}+\check{\zeta}} \star \widetilde{\mathcal{T}}_a) &= \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}} \star Y_{\check{\zeta}} \cdot \widetilde{\mathcal{T}}_a) = \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}} \star \widetilde{\mathcal{T}}_a Y_{w_a \check{\zeta}}) + (q-1)\, \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}}) \star \frac{Y_{w_a \check{\zeta}} - Y_{\check{\zeta}}}{1 - Y_{-\check{a}}} &(6.20) \\ &= \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}} \star \widetilde{\mathcal{T}}_a) Y_{w_a \check{\zeta}} + (q-1)\, \mathrm{Av}^{\mathrm{gen}}_{U^-}(\widetilde{\mathcal{Y}}_{\check{\lambda}}) \star \frac{Y_{w_a \check{\zeta}} - Y_{\check{\zeta}}}{1 - Y_{-\check{a}}}. &(6.21) \end{aligned}$$

The terms on the right hand side are all computed from part 2, so (6.19) can be verified.

6.2.5. *A decomposition of $\mathcal{W}_\psi(\widetilde{G},I^-)$.* For $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$, define the $\mathcal{H}(\widetilde{G},I^-)$-module

$$\mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta}) := \mathrm{Span}_{\mathbb{C}}\{\, \widetilde{\mathcal{Y}}_{\check{\eta}} \star h \mid h \in \mathcal{H}(\widetilde{G},I^-)\}. \tag{6.22}$$

From the previous Proposition and the results of Section 4, we obtain the following.

Proposition. *For $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$, the map $\mathfrak{p}$ from Proposition 6.2.1 restricts to an isomorphism*

$$\mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}(\check{\eta}) \xrightarrow{\simeq} \mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta}) \tag{6.23}$$

and hence by Proposition 4.2.2 we have a decomposition into $\mathcal{H}(\widetilde{G},I^-)$-submodules

$$\mathcal{W}_\psi(\widetilde{G},I^-) := \oplus_{\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}} \mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta}). \tag{6.24}$$

6.3. Structure of the spherical Whittaker space $\mathcal{W}_\psi(\widetilde{G},K)$.

6.3.1. *Action of $\mathcal{H}(\widetilde{G},K)$ on $\mathcal{W}_\psi(\widetilde{G},K)$.* Let $\mathbb{V}_{\mathrm{sph}}$ be $\widetilde{H}_{\mathrm{sph}}$-module introduced in §4.3.3. The map $\mathfrak{p} : \mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V} \longrightarrow \mathcal{W}_\psi(\widetilde{G},I^-)$ from the Proposition 6.2.1 descends to a map that we denote as

$$\mathfrak{p}_K : \mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \longrightarrow \mathcal{W}_\psi(\widetilde{G},K). \tag{6.25}$$

Recall the diagram in (5.11) and the remarks at the end of §5.3.6 which explain the relation between the element $\mathbf{e}_K \in \mathcal{H}(\widetilde{G},I^-)$ and $\varepsilon \in \widetilde{H}_{\mathrm{aff}}$. The natural right convolution maps $\cdot \star \mathbf{e}_K : \mathcal{W}_\psi(\widetilde{G},I^-) \to \mathcal{W}_\psi(\widetilde{G},K)$.

Proposition. *The following diagram*

$$\begin{array}{ccc} \mathcal{W}_\psi(\widetilde{G},I^-) & \xrightarrow{\cdot\star\mathbf{e}_K} & \mathcal{W}_\psi(\widetilde{G},K) \\ {\scriptstyle\mathfrak{p}}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\mathfrak{p}_K} \\ \mathbb{C}\otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}}\mathbb{V} & \xrightarrow{\cdot\varepsilon} & \mathbb{C}\otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}}\mathbb{V}_{\mathrm{sph}} \end{array} \quad \textit{which sends} \quad \begin{array}{ccc} \widetilde{\mathcal{Y}}_{\check{\lambda}} & \longrightarrow & \widetilde{\mathcal{J}}_{\check{\lambda}} \\ \uparrow & & \uparrow \\ Y_{\check{\lambda}} & \longrightarrow & [Y_{\check{\lambda}}] \end{array} \quad \textit{for } \check{\lambda} \in Y$$

is commutative. The vertical maps in the left diagram are isomorphisms that intertwine the actions of the corresponding Hecke algebras: $\mathcal{H}(\widetilde{G},I^-) \simeq \widetilde{H}_{\mathrm{aff}}$ on the left and $\mathcal{H}(\widetilde{G},K) \simeq \widetilde{H}_{\mathrm{sph}}$ on the right.

We may define the $\mathcal{H}(\widetilde{G},K)$-submodules

$$\mathcal{W}_\psi(\widetilde{G},K)(\check{\eta}) := \mathcal{W}_\psi(\widetilde{G},I^-)(\check{\eta}) \star \mathbf{e}_K, \tag{6.26}$$

for $\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}$ and by the Proposition above we have that $\mathfrak{p}_K$ restricts to an isomorphism

$$\mathfrak{p}_K : \mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}}(\check{\eta}) \longrightarrow \mathcal{W}_\psi(\widetilde{G},K)(\check{\eta}) \tag{6.27}$$

which intertwines the $\mathcal{H}(\widetilde{G},K) \simeq \widetilde{H}_{\mathrm{sph}}$-actions. We can reformulate this using (4.52).

Corollary. *One has the following decomposition of $\mathcal{W}_\psi(\widetilde{G},K)$ into $\mathcal{H}(\widetilde{G},K)$-modules:*

$$\mathcal{W}_\psi(\widetilde{G},K) \simeq \oplus_{\check{\eta} \in \overline{\mathcal{A}}^\bullet_{-,n}} \mathcal{W}_\psi(\widetilde{G},K)(\check{\eta}). \tag{6.28}$$

For $\check{\lambda} \in Y_+$, define the natural bases $\widetilde{\mathcal{L}}_{\check{\lambda}}$ and $\widetilde{\mathscr{T}}_{\check{\lambda}}$ in $\mathcal{W}_\psi(\widetilde{G}, K)$ which correspond under the isomorphism $\mathfrak{p}_K$ to the bases $[G^-_{\check{\lambda}}]$ and $[G_{\check{\lambda}}]$ of $\mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}}(\check{\eta})$, respectively. We call these the *canonical* bases of $\mathcal{W}_\psi(\widetilde{G}, K)$.

6.3.2. *Metaplectic geometric Casselman–Shalika formulas.* Using Proposition 6.3.1 and the remarks in the previous paragraph, we can import the structures studied in Part I and especially in §4 to the study of $\mathcal{W}_\psi(\widetilde{G}, K)$. The results in §4.4.2 and §4.4.3 produce the following geometric Casselman–Shalika formulas for the 'p-adic basis' $\widetilde{\mathcal{J}}_{\check{\lambda}}$ and the canonical bases $\widetilde{\mathcal{L}}_{\check{\lambda}}$ and $\widetilde{\mathscr{T}}_{\check{\lambda}}$ of $\mathcal{W}_\psi(\widetilde{G}, K)$, respectively.

Theorem. *(1) Suppose $\check{\mu} \in Y_+$ and $\check{\lambda} \in \widetilde{Y}_+$. Writing ${}^{\mathbf{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}} := \mathfrak{p}({}^{\mathfrak{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}})$ for the p-adic specializations of the $\mathfrak{g}$-twisted Littlewood-Richardson coefficients defined in (4.64), we have*

$$\widetilde{\mathcal{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}} = \sum_{\check{\zeta} \in Y_+} {}^{\mathbf{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}} [Y_{\check{\zeta}}]. \tag{6.29}$$

(2) If $\check{\lambda}_0 \in \square_{(\mathsf{Q},n)}$ and $\check{\zeta} \in \widetilde{Y}_+$, then we have

$$\widetilde{\mathcal{L}}_{\check{\lambda}_0} \star \widetilde{c}_{\check{\zeta}} = \widetilde{\mathcal{L}}_{\check{\lambda}_0 + \check{\zeta}}. \tag{6.30}$$

(3) If $\check{\lambda}_0 \in \square_{(\mathsf{Q},n)}$ and $\check{\zeta} \in \widetilde{Y}_+$, then writing $\check{\lambda}_0^\dagger := \check{\lambda}_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check{\rho})$ we have

$$\widetilde{\mathscr{T}}_{\check{\lambda}_0^\dagger} \star \widetilde{c}_{\check{\zeta}} = \widetilde{\mathscr{T}}_{\check{\lambda}_0^\dagger + \check{\zeta}}. \tag{6.31}$$

Remark. *Note that the second and third parts, together with the well-known structure coefficients for $\widetilde{c}_{\check{\mu}} \in \mathcal{H}(\widetilde{G}, K)$ (see (5.17)), determine the action of $\widetilde{c}_{\check{\mu}}$ on the basis $\widetilde{\mathcal{L}}_{\check{\lambda}}$ and $\widetilde{\mathcal{T}}_{\check{\lambda}}$ as explained in (1.6) of §1.1.1.*

Part 3. Interpretation via quantum groups at roots of unity and applications

7. Quantum groups at a root of unity

After introducing in §7.1-7.3 quantum groups and their specializations (following Lusztig [80, 84]), we collect several results from the representation theory of these objects. We are working with Lusztig's *dotted* version of the quantum group and its root of unity and quasi-classical specializations, though we could have as well worked with the undotted versions of these objects as their representations theories are the same. The main results needed to connect the quantum setting to the p-adic one are contained in §7.5.

Fix throughout $(I, \cdot, \mathfrak{D})$ a root datum written as $\mathfrak{D} = (Y, \{\check{a}_i\}, X, \{a_i\})$ with Cartan matrix $\mathsf{A} = (a_{ij})$ attached to $(I, \cdot)$. Let u be a formal variable and, for each $i \in I$, set $u_i := u^{\frac{i \cdot i}{2}}$. These elements lie in the ring $\mathbb{Z}_u := \mathbb{Z}[u, u^{-1}]$. Throughout this section we shall consider commutative, unital rings A equipped with a ring homomorphisms $\phi : \mathbb{Z}_u \to A$. We shall need to impose stricter conditions on $\mathfrak{D}$ in §7.5 as we will explain there.

7.1. Algebras attached to root datum.

7.1.1. *The algebra* $\mathbf{f}$. Attached to the Cartan datum $(I,\cdot)$ with Cartan matrix $\mathsf{A} = (a_{ij})$, we construct $\mathbf{f} := \mathbf{f}(I,\cdot)$ the associative $\mathbb{Q}(u)$-algebra with unit defined as in [84, §1.2.5]. It is a deformation of the enveloping algebra of the 'positive' half of $\mathfrak{g}$, and defined as the unital algebra generated by elements $\theta_i\,(i \in I)$ subject to the quantum Serre relations: for $i, j \in I$, with $i \neq j$

$$\sum_{p,p' \in \mathbb{N}; p+p'=1-a_{ij}} (-1)^{p'} \left(\theta_i^p/[p]_i^!\right) \theta_j \left(\theta_i^{p'}/[p']_i^!\right) = 0 \text{ where } [p]_i^! = \prod_{s=1}^{p} \frac{u_i^s - u_i^{-s}}{u_i - u_i^{-1}}. \tag{7.1}$$

The algebra $\mathbf{f}$ has a decomposition $\mathbf{f} = \oplus_{\mu \in \mathbb{Z}[I]} \mathbf{f}_\mu$ where $\mathbf{f}_\mu$ is the vector space spanned by words in $\{\theta_j\}$ such that the number of appearances of θ_i is given by the coefficient of i in μ.

7.1.2. $\mathbb{Z}_u$*-forms for* $\mathbf{f}$. For each $i \in I, n \in \mathbb{Z}$, we define the divided power $\theta_i^{(n)} := \theta_i^n/[n]_i^!$ if $n \geq 0$ and $\theta_i^{(n)} = 0$ otherwise. Defining $\mathbf{f}_{\mathbb{Z}_u}$ to be the unital subalgebra of $\mathbf{f}$ generated by $\theta_i^{(n)}$ for all $i \in I, n \in \mathbb{Z}$, we have

$$\mathbf{f}_{\mathbb{Z}_u} = \oplus_{\mu \in \mathbb{Z}[I]} \mathbf{f}_{\mu,\mathbb{Z}_u} \text{ where } \mathbf{f}_{\mu,\mathbb{Z}_u} = \mathbf{f}_\mu \cap \mathbf{f}_{\mathbb{Z}_u}. \tag{7.2}$$

For any $\mathbb{Z}_u \to A$ as above, we set $\mathbf{f}_A := A \otimes_{\mathbb{Z}_u} \mathbf{f}_{\mathbb{Z}_u}$ and note again that

$$\mathbf{f}_A = \oplus_{\mu \in \mathbb{Z}[I]} \mathbf{f}_{\mu,A} \text{ where } \mathbf{f}_{\mu,A} = A \otimes_{\mathbb{Z}_u} \mathbf{f}_{\mu,\mathbb{Z}_u}. \tag{7.3}$$

7.1.3. *The quantum group* $\dot{\mathbf{U}}_A(\mathfrak{D})$. Let $\dot{\mathbf{U}}_A(\mathfrak{D})$ be the A-algebra generated by symbols $x^+ 1_\zeta y^-, x^- 1_\zeta y$ where $x \in \mathbf{f}_{\mu,A}, y \in \mathbf{f}_{\mu',A}$ for $\mu, \mu' \in \mathbb{Z}[I]$ and $\zeta \in X$ and subject to the relations described in [84, §31.1.3]. Note that these relations depend on the root datum $\mathfrak{D}$, not just on the Cartan datum $(I,\cdot)$. We also record here that the A-linear maps

$$\mathbf{f}_A \otimes_A \mathbb{Z}_\tau[X] \otimes_A \mathbf{f}_A \to \dot{\mathbf{U}}_A(\mathfrak{D}) \quad , \quad x \otimes X_\lambda \otimes y \mapsto x^+ 1_\lambda y^- \tag{7.4}$$

$$\mathbf{f}_A \otimes_A \mathbb{Z}_\tau[X] \otimes_A \mathbf{f}_A \to \dot{\mathbf{U}}_A(\mathfrak{D}) \quad , \quad x \otimes X_\lambda \otimes y \mapsto x^- 1_\lambda y^+ \tag{7.5}$$

are isomorphisms of A-modules and hence we have $\dot{\mathbf{U}}_A(\mathfrak{D}) = A \otimes_{\mathbb{Z}_u} \dot{\mathbf{U}}_{\mathbb{Z}_u}$. Note that $\dot{\mathbf{U}}_A$ does not have a unit, but instead a family of elements $(1_\lambda)_{\lambda \in X}$ such that $1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'}$. These give a decomposition $\dot{\mathbf{U}}_A = \sum_{\lambda,\lambda'} 1_\lambda \dot{\mathbf{U}}_A 1_{\lambda'}$. If $A = \mathbb{Q}(u)$, we often drop it from our notation and just write $\dot{\mathbf{U}}$ in this case; the natural map $\mathbb{Z}_u \subset \mathbb{Q}(u)$ allows us to view $\dot{\mathbf{U}}_A \subset \dot{\mathbf{U}}$ as an $\mathbb{Z}_u$-subalgebra, or, what we might call a $\mathbb{Z}_u$-lattice.

7.1.4. *Module categories.* Let $\mathfrak{C}_A(\mathfrak{D})$ denote the category whose objects $\dot{\mathbf{U}}_A(\mathfrak{D})$-modules M which are

- *unital* in the sense that for any $z \in M$, we have $1_\lambda z = 0$ for almost all $\lambda \in X$ and $\sum_{\lambda \in X} 1_\lambda z = z$,
- finitely generated when regarded as an A-module.

Objects $M \in \mathfrak{C}_A(\mathfrak{D})$ can be decomposed as $M = \oplus_{\lambda \in X} M_\lambda$ where $M_\lambda = 1_\lambda M$. One can equip $\mathfrak{C}_A(\mathfrak{D})$ with the structure of a monoidal tensor category (see [80, §1.6]) with product denoted $\otimes_A$. If $A = \mathbb{Q}(u)$, we drop it from our notation and just write $\mathfrak{C}(\mathfrak{D})$ in this case. An object $M \in \mathfrak{C}_A(\mathfrak{D})$ is said to be a *highest weight module* with highest weight $\lambda \in X$, see [84, §31.3] if there exists a vector $m \in M^\lambda$ such that

- setting $E_i^{(n)} := \sum_{\zeta \in X} (\theta_i^{(n)})^+ 1_\zeta$, we have $E_i^{(n)} m = 0$ for all i and $n > 0$;

- $M = \{x^- m \mid x \in \mathbf{f}_A\}$;
- M^λ is a free A-module of rank one with generator m.

We define a module M to be *integrable* if for every $m \in M$ and $i \in I$, there exists an n_0 such that $E_i^{(n_0)} m = F_i^{(n_0)} m = 0$ (see [84, §31.2.4]). For a field F, let $\mathrm{Rep}(\dot{\mathbf{U}}_F(\mathfrak{D}))$ be the full subcategory of $\mathfrak{C}_A(\mathfrak{D})$ consisting of integrable highest weight (unital) modules M whose weight spaces $M_{\check{\lambda}}$ are finite-dimensional F-vector spaces. For such modules we can define their character

$$\chi_M := \sum_{\lambda \in X} (\dim_F M_\lambda)\, X_\lambda \in F[X]. \tag{7.6}$$

7.1.5. *Standard or Weyl modules* Λ_λ. For $\lambda \in X_+$, define the $\mathbf{f}$-modules $T_\lambda = \sum_i \mathbf{f}\theta_i^{\langle \check{a}_i, \lambda \rangle + 1}$ and define $\Lambda_\lambda = \mathbf{f}/T_\lambda$. Letting $\eta_\lambda = 1 + T_\lambda$, we may regard Λ_λ as an element in $\mathfrak{C}(\mathfrak{D})$ see [84, 6.3.4, 23.1.4] in which

$$(\theta_i^+ 1_{\lambda'})\eta_\lambda = 0, i \in I, \lambda' \in X \quad \text{and} \quad (x^- 1_{\lambda'})\eta_\lambda = \delta_{\lambda,\lambda'} x + s, \text{ where } s \in T_\lambda \text{ for } x \in \mathbf{f}, \lambda' \in X. \tag{7.7}$$

We also write, given any $\mathbb{Z}_u$-algebra $\mathbb{Z}_u \to A$,

$$\Lambda_{\lambda,\mathbb{Z}_u} := \dot{\mathbf{U}}_{\mathbb{Z}_u}(\mathfrak{D})\eta_\lambda \quad \text{and} \quad \Lambda_{\lambda,A} := A \otimes_{\mathbb{Z}_u} \Lambda_{\lambda,A}\ . \tag{7.8}$$

One knows that $\Lambda_{\lambda,\mathbb{Z}_u}$ is an $\mathbb{Z}_u$-lattice in Λ_λ and $\Lambda_{\lambda,A} \in \mathfrak{C}_A(\mathfrak{D})$. These are called the *standard* modules. Over a field F, $\Lambda_{\lambda,k} \in \mathrm{Rep}(\dot{\mathbf{U}}_F(\mathfrak{D}))$ and their characters can be computed using the Weyl character formula.

7.1.6. *Irreducible modules.* Again we work over a field F. The structure of irreducible highest weight modules in $\mathrm{Rep}(\dot{\mathbf{U}}_F(\mathfrak{D}))$ is similar to that for ordinary Lie algebras, see [84, Prop. 31.3.2]: for each $\lambda \in X_+$ there is a simple object in $\mathrm{Rep}(\dot{\mathbf{U}}_F(\mathfrak{D}))$ which is a highest weight module with highest weight λ. We denote this object as $L_{\lambda,F}$ (or just L_λ if F is implicitly understood). One know that $L_{\lambda,F}$ and $L_{\lambda',F}$ are not isomorphic if $\lambda, \lambda' \in X$ are distinct. Moreover, if M is any highest weight module with highest weight module with highest weight λ, then M has a unique maximal subobject whose corresponding quotient is isomorphic to $L_{\lambda,F}$.

7.2. **Classical and quasi-classical specializations.**

7.2.1. *Classical and quasi-classical specializations.* Let $\phi : \mathbb{Z}_u \to A$ be a $\mathbb{Z}_u$- algebra such that $\phi(u_i) = \pm 1$ for all $i \in I$. In this case $\dot{\mathbf{U}}_A$ will be called a *quasi-classical* specialization of $\dot{\mathbf{U}}_{\mathbb{Z}_u}$. If $\phi(u) = 1$, then we say that $\dot{\mathbf{U}}_A$ is a *classical* specialization of $\dot{\mathbf{U}}_{\mathbb{Z}_u}$. We would like to describe the categories $\mathfrak{C}_A(\mathfrak{D})$ when A is taken to be one of these two specializations. As it turns out if $\phi : \mathbb{Z}_u \to A$ is a quasi-classical specialization, and we write $A_0 = A$ for a copy of A and define $\phi_0 : \mathbb{Z}_u \to A_0$ to be the unique map such that $\phi(u) = 1$, then from [84, Prop. 33.2.3], there is a unique isomorphism of $\mathbb{Z}_u$-algebras

$$\dot{\mathbf{U}}_{A_0}(\mathfrak{D}) \xrightarrow{\cong} \dot{\mathbf{U}}_A(\mathfrak{D}), \text{ and hence } \mathfrak{C}_{A_0}(\mathfrak{D}) \cong \mathfrak{C}_A(\mathfrak{D}). \tag{7.9}$$

Not only do the representation categories under classical and quasi-classical specializations agree, but from the construction in *op. cit.* it follows that if $M_0 \in \mathfrak{C}_{A_0}(\mathfrak{D})$ corresponds to $M \in \mathfrak{C}_A(\mathfrak{D})$ under this equivalence, then it also follows that $M_{0,\lambda} \cong M_\lambda$ for each $\lambda \in X$.

7.2.2. *Classical enveloping algebras attached to root datum.* Following [80, §5.1], define $\mathbf{U}_{\mathbb{Q}}(\mathfrak{D})$ be the $\mathbb{Q}$-algebra with unit generated by symbols x^+, x^- with $x \in \mathbf{f}_{\mathbb{Q}}$ and $\underline{y}, y \in Y$ subject to the relations:

- the maps $\mathbf{f}_{\mathbb{Q}} \to \mathbf{U}_{\mathbb{Q}}(\mathfrak{D}), x \mapsto x^{\pm}$ are $\mathbb{Q}$-algebra homomorphisms preserving 1,
- the map $Y \to \mathbf{U}_{\mathbb{Q}}, y \mapsto \underline{y}$ is $\mathbb{Z}$-linear,
- $\underline{yy'} = \underline{yy'}$ for $y, y' \in Y$,
- $\underline{y}\theta_i^{\pm} = \theta_i^{\pm}(\underline{y \pm \langle y, a_i \rangle})$ for $y \in Y, i \in I$,
- $\theta_i^+\theta_j^- \theta_j^- \theta_i^+ = \delta_{i,j}\underline{\check{a}_i}$ for $i, j \in I$.

This is a generalization of the usual universal enveloping algebra (attached to a semi-simple algebra), and will be called the *universal enveloping algebra attached to a root datum*. One may equip it with a Hopf algebra structure as in *op. cit.*

7.2.3. *Kostant type forms.* Let $\mathbf{U}_{\mathbb{Z}}(\mathfrak{D})$ be the subring of $\mathbf{U}_{\mathbb{Q}}(\mathfrak{D})$ generated by

$$\binom{\underline{y}}{k} := \frac{1}{k!}\underline{y}\,\underline{y-1} \cdots \underline{y-k+1} \text{ for } y \in Y, k \in \mathbb{N}$$

as well as $\left(\theta_i^{(m)}\right)^{\pm}$ for $i \in I, m \in \mathbb{N}$. It is a $\mathbb{Z}$-lattice in $\mathbf{U}_{\mathbb{Q}}(\mathfrak{D})$ which is the analogue of Kostant's $\mathbb{Z}$-form see [72], which it agrees with when $\mathfrak{D}$ is of adjoint type (we eventually need to restrict to this case due to existing limitations in the literature).

7.2.4. Let $\mathfrak{C}'_{\mathbb{Q}}(\mathfrak{D})$ denote the category of unital $\mathbf{U}_{\mathbb{Q}}(\mathfrak{D})$-modules M such that

$$M = \oplus_{\lambda \in X} M^{\lambda} \text{ where we set } M^{\lambda} = \{z \in M \mid \underline{y}z = \langle y, \lambda \rangle z \text{ for any } y \in Y\}. \quad (7.10)$$

The algebra form §7.1.3 attached to the map $\phi : \mathbb{Z}_u \to \mathbb{Q}, u \mapsto 1$ will be denoted $\dot{\mathbf{U}}_{\mathbb{Q}}(\mathfrak{D})$ and its category of unital modules written $\mathfrak{C}_{\mathbb{Q}}(\mathfrak{D})$. From [80, §5.4], one has an equivalence of categories $\mathfrak{C}_{\mathbb{Q}}(\mathfrak{D}) \xrightarrow{\cong} \mathfrak{C}'_{\mathbb{Q}}(\mathfrak{D})$. These categories are semi-simple with the simple objects $V_\lambda := \Lambda_{\lambda,\mathbb{Q}}, \lambda \in X_+$ introduced above. Letting $\mathbf{G}_{\mathfrak{D}}$ denote the algebraic group attached to the root datum $\mathfrak{D}$, it then also follows from [80] that there is an equivalence to the the category of finite-dimensional representations of $\mathbf{G}_{\mathfrak{D}}(\mathbb{C})$, *i.e.*

$$\mathrm{Rep}(\dot{\mathbf{U}}_{\mathbb{C}}(\mathfrak{D})) \simeq \mathrm{Rep}(\mathbf{G}_{\mathfrak{D}}(\mathbb{C})). \quad (7.11)$$

7.3. **Quantum groups at roots of unity.**

7.3.1. *Roots of unity.* Let ℓ be any positive integer. If ℓ is even, set $l = 2\ell$ while if ℓ is odd, we define $l = \ell$ or 2ℓ. Let $\mathbb{Z}_l = \mathbb{Z}_u/(\Phi_l(u))$ where Φ_d is the d-th cyclotomic polynomial. In this section, we only consider $\phi : \mathbb{Z}_u \to A$ which factors through $\mathbb{Z}_l$, i.e. $\Phi_l(\phi(u)) = 0$. For $\zeta \in \mathbb{C}$ a primitive l-th root of unity, we define the corresponding map $\phi_\zeta : \mathbb{Z}_u \to \mathbb{C}$ sending $u \mapsto \zeta$. With respect to this map, we then write

$$\dot{\mathbf{U}}_\zeta(\mathfrak{D}) := \mathbb{C} \otimes_{\mathbb{Z}_u} \dot{\mathbf{U}}_{\mathbb{Z}_u}(\mathfrak{D}). \quad (7.12)$$

7.3.2. *On* $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathfrak{D}))$. The category $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathfrak{D}))$ is isomorphic to the category of finite dimensional highest weight complex modules of type **1** of the (non-dotted) quantum group at a root of unity (as defined as in [8, Definition 1.1], see also [71]). For $\lambda \in X_+$, we denote by Δ_λ the specialization of the modules Λ_λ defined in §7.1.5. We denote by ∇_λ the costandard object of highest weight λ (for its construction, which is essentially 'dual' to that of Δ_λ, we refer to either [57] where they are denoted by $H^0(\lambda)$ in or to [36] where they are denoted as $A(\lambda)$). The characters of both standard and costandard modules are given by the Weyl character formula. Let L_λ be the simple module with highest weight $\lambda \in X_+$ in $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathfrak{D}))$. Their characters are given by Lusztig's conjecture (see, for example [57, Appendix H.12]).

7.3.3. *Tilting modules.* Tilting modules are a class of representations that naturally appear when studying $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathfrak{D}))$. They are objects admitting both a filtration by standard modules and a filtration by costandard modules. Andersen [6, Theorem 2.5] showed (using ideas from modular representation theory) that there exists a unique indecomposable tilting module in $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathfrak{D}))$ with highest weight $\lambda \in X_+$. We shall denote such a tilting by T_λ. See [7, §2] for an explicit construction of T_λ.

The characters of T_λ are known due to the work of Soergel [110, 111] in many cases.

7.3.4. *On* $\mathrm{Rep}(\dot{\mathbf{U}}(\mathfrak{D}_l))$. If $(I,\cdot,\mathfrak{D})$ is our given root datum, where $\mathfrak{D} = (Y, \{\check{a}_i\}, X, \{a_i\})$ is assumed of adjoint type, we may construct the l-twisted root datum $(I, \circ_l, \mathfrak{D}_l)$ as in §3.2.8. Denoting $\mathfrak{D}_l := (Y_l, \{\widetilde{a}_i^\vee\}, X_l, \{\widetilde{a}_i\})$, we have that $\widetilde{a}_i^\vee = l_i^{-1}\check{a}_i$, $\widetilde{a}_i = l_i a_i$ where l_i were defined in §3.2.8. With respect to $(I, \circ_l)$ we then have $u_i := u^{l_i^2}$ (cf. [84, 35.2.1]) so that if we are working with a map $\phi : \mathbb{Z}_u \to A$ as in the previous section, we are necessarily in the quasi-classical case. So, for example $\dot{\mathbf{U}}_{\mathbb{C}}(\mathfrak{D}_l)$ for the map $\mathbb{Z}_u \to \mathbb{C}, v \mapsto \zeta$ is a quasi-classical specialization of $\dot{\mathbf{U}}_{\mathbb{Z}_u}(\mathfrak{D}_l)$, and hence from (7.11)

$$\mathrm{Rep}_{\mathbb{C}}(\dot{\mathbf{U}}(\mathfrak{D}_l)) \cong \mathrm{Rep}(\mathbf{G}_l(\mathbb{C})), \tag{7.13}$$

where $\mathbf{G}_\ell$ is the algebraic group with root datum $\mathfrak{D}_l$.

7.3.5. *Quantum Frobenius.* For any $\phi : \mathbb{Z}_l \to A$ as in §7.3.1, the quantum Frobenius morphism of Lusztig, see [84, Ch. 35], is the unique A-algebra homomorphism $\mathrm{Fr} : \dot{\mathbf{U}}_A(\mathfrak{D}) \to \dot{\mathbf{U}}_A(\mathfrak{D}_l)$ such that for all $i \in I, n \in \mathbb{Z}$ and $\zeta \in X$ we have

$$\mathrm{Fr} : E_i^{(k)} \mapsto \begin{cases} E_i^{\left(\frac{k}{l_i}\right)} 1_\zeta & \text{if } k \equiv 0 \bmod l_i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \mathrm{Fr} : F_i^{(k)} \mapsto \begin{cases} F_i^{\left(\frac{k}{l_i}\right)} 1_\zeta & \text{if } k \equiv 0 \bmod l_i, \\ 0 & \text{otherwise .} \end{cases} \tag{7.14}$$

Using this map of algebras, we can define a fully faithful and exact embedding

$$\mathrm{Fr} : \mathfrak{C}_A(\mathfrak{D}_l) \to \mathfrak{C}_A(\mathfrak{D}). \tag{7.15}$$

For $\mathbb{Z}_u \to \mathbb{C}, v \mapsto \zeta$ as above, we obtain a functor $\mathrm{Fr} : \mathrm{Rep}(\mathbf{G}_{\mathfrak{D}_l}(\mathbb{C})) \to \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\mathfrak{D}))$. Replacing $\mathfrak{D}$ by $\check{\mathfrak{D}}$, writing $\check{\mathfrak{D}}_l = \left(\mathfrak{D}_{(\mathsf{Q},l)}\right)^\vee$ for (Q,l) as in §3.2.8, and setting $\check{\mathbf{G}}_\ell := \mathbf{G}_{\check{\mathfrak{D}}_l}$,

$$\mathrm{Fr} : \mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})) \to \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})). \tag{7.16}$$

The simples in $\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C}))$, which we'll denote from now on as $V_{\check{\lambda}}$, are parameterized by $\check{\lambda} \in Y_{l,+}$ which we regard as a subset of Y_+ (the latter is the set parameterizing simples $L_{\check{\mu}}$ in $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$.) As Fr is exact, $\mathrm{Fr}(V_{\check{\lambda}}) \in \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ for $\check{\lambda} \in Y_l$ is again simple. Unlike the general simple in the latter, one can explicitly write down the character of these simples using a dilation of the Weyl character formula [84, Prop. 35.3.2].

7.3.6. *Tensor product theorems.* Recall the notion of restricted weights from §3.4.9 and consider the box in the affine root system determined by $\check{\mathfrak{D}}_l$. Concretely,

$$\Box_l = \{\check{\lambda} \in Y \mid 0 \leq \langle \check{\lambda}, a_i \rangle < l_i, \text{ for all } i \in I\}. \tag{7.17}$$

Note that each $\check{\lambda} \in Y_+$ can be written uniquely as $\check{\lambda} = \check{\lambda}_0 + \check{\eta}$ where $\check{\lambda}_0 \in \Box_l, \check{\eta} \in Y_l$.

Theorem. *Let $\check{\lambda}_0 \in \Box_l$ and $\check{\eta} \in \widetilde{Y}_{l,+}$.*

(1) (Steinberg-Lusztig) We have an isomorphism of modules

$$L_{\check{\lambda}_0+\check{\eta}} \simeq L_{\check{\lambda}_0} \otimes \mathrm{Fr}(V_{\check{\eta}}). \tag{7.18}$$

(2) (Tilting) Writing $\check{\lambda}_0^\dagger := \check{\lambda}_0 \cdot w_0 + 2(\widetilde{\rho}^\vee - \check{\rho})$, we have

$$T_{\check{\lambda}_0^\dagger+\check{\eta}} \simeq L_{\check{\lambda}_0^\dagger} \otimes \mathrm{Fr}(V_{\check{\eta}}). \tag{7.19}$$

The first is the analogue Steinberg tensor product theorem from modular representation theory which has been proved for l odd by Lusztig [79] (for general l, see [9]). The second is due to Andersen [6, Corollary 5.10] , and follows closely from related results by Donkin in the mod p literature.

7.4. **Background on enriched Grothendieck rings.**

7.4.1. *Enriched Grothendieck rings of highest weight categories.* Given an abelian category $\mathscr{C}$, it is natural to consider the derived bounded category $D^b(\mathscr{C})$ (see [52, Chapter III] for definitions and basic properties of derived categories). The Grothendieck ring does not distinguish between the $\mathscr{C}$ and $D^b(\mathscr{C})$, namely $K_0(\mathscr{C}) \simeq K_0(D^b(\mathscr{C}))$. For a natural class of categories arising in representation theory, a finer invariant called the enriched Grothendieck group was introduced in [36] to partially remedy this deficiency. To introduce it, let τ be an indeterminate and consider the ring $\mathbb{Z}[\tau, \tau^{-1}]$ of Laurent polynomials[24]. Let $\mathscr{C}$ be a highest weight categories as introduced in [34] with weight poset Λ indexing simple objects in $\mathscr{C}$. Define the full additive subcategories $\hat{\mathscr{E}}^L \subset D^b(\mathscr{C})$ and $\hat{\mathscr{E}}^R \subset D^b(\mathscr{C})$ as in [36, Section 2]. Then $K_0^L(\mathscr{C})$ is the free abelian group generated by symbols $[X], X \in \hat{\mathscr{E}}^L$ subject to the relation $[X] + [Z] = [Y]$ if there is a distinguished triangle $X \to Y \to Z$ in $\hat{\mathscr{E}}^L$. One may define $K_0^R(\mathscr{C})$ similarly.

[24] At the categorical level, τ^{-1} will correspond to the exact functor $t : D^b(\mathscr{C}) \to D^b(\mathscr{C})$ that maps $\tau^{-1} : X \mapsto X[-1]$ on objects and $t : f \mapsto f[-1]$ on morphisms.

7.4.2. *Inner products.* Let $\mathscr{C}$ be a highest weight category equipped with weight poset Λ_+. For $\lambda \in \Lambda_+$, we denote by Δ_λ the standard object of highest weight λ (also known as the Weyl module) and by ∇_λ the costandard object of highest weight λ (which is the induced module and denoted $H^0(\lambda)$ in [57]). Let L_λ be the simple module with highest weight λ; it appears as the unique irreducible quotient of Δ_λ.

Proposition. [36, Proposition 2.3] *The spaces $K_0^L(\mathscr{C})$ and $K_0^R(\mathscr{C})$ are free $\mathbb{Z}[\tau,\tau^{-1}]$-modules with basis given by $\{[\Delta_\lambda],\lambda\in\Lambda_+\}$ and $\{[\nabla_\lambda],\lambda\in\Lambda_+\}$, respectively. Moreover, there is a natural non-degenerate, sesquilinear pairing*

$$\langle\cdot,\cdot\rangle : K_0^L(\mathscr{C})\times K_0^R(\mathscr{C})\to\mathbb{Z}[\tau,\tau^{-1}];\quad \langle[V],[W]\rangle=\mathbf{R}\operatorname{Hom}(V,W):=\textstyle\sum_{i\geq 0}\tau^{-i}\dim(\operatorname{Ext}^i_{\mathscr{C}}(V,W)).\tag{7.20}$$

Under this pairing $[\Delta_\mu],[\nabla_\eta]$ form dual bases, i.e., $\langle[\Delta_\mu],[\nabla_\eta]\rangle=\delta_{\mu,\eta}$ for $\mu,\eta\in\Lambda_+$.

Using this pairing, for $V\in\mathscr{C}$ its image V in $K_0^R(\mathscr{C})$ may be written as

$$[V]=\sum_\mu\langle[\Delta_\mu],[V]\rangle\,[\nabla_\mu].\tag{7.21}$$

Remark. *Note that in* (7.20) *we are using τ^{-1} instead of the t of* [36]*. This is done to make the matching with the results in §3.3 easier.*

7.4.3. *A trivial example.* Let $(I,\cdot,\mathfrak{D})$ be a root datum, and $\mathbf{G}$ the corresponding algebraic group. Then $\operatorname{Rep}(\mathbf{G}(\mathbb{C}))$, the semi-simple category of finite-dimensional representations of $\mathbf{G}(\mathbb{C})$, is a highest weight category and there are natural $\mathbb{Z}[\tau,\tau^{-1}]$ isomorphisms

$$K_0^L(\operatorname{Rep}(\mathbf{G}(\mathbb{C})))\simeq K_0^R(\operatorname{Rep}(\mathbf{G}(\mathbb{C})))\simeq\mathbb{Z}[\tau,\tau^{-1}]\otimes_{\mathbb{Z}}K_0(\operatorname{Rep}(\mathbf{G}(\mathbb{C}))).\tag{7.22}$$

Let us denote by $K_0^{\odot}(\operatorname{Rep}(\mathbf{G}(\mathbb{C})))$ the space $\mathbb{Z}[\tau,\tau^{-1}]\otimes_{\mathbb{Z}}K_0(\operatorname{Rep}(\mathbf{G}(\mathbb{C})))$.

7.5. **Extension formulas in** $\operatorname{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$**.** In this section we need to make more stringent requirements on $\mathfrak{D}$ and ℓ. Fix $\mathfrak{D}=(Y,\{\check{a}_i\},X,\{a_i\})$, which implies $\check{\mathfrak{D}}=(X,\{a_i\},Y,\{\check{a}_i\})$. We emphasize that all representations from now on are over $\mathbb{C}$. Our restrictions are:

- We assume $(I,\cdot,\mathfrak{D})$ is simply-connected. This means $\check{\mathfrak{D}}$ is of adjoint type and so admits a primitive twist $(\check{\mathrm{Q}},l)$ for any l. The quantum group $\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})$ is also isomorphic to the quantum group associated to a semi-simple Lie algebra $\dot{\mathbf{U}}_\zeta(\mathfrak{g})$.
- We choose l larger than the Coxeter number of the semi-simple Lie algebra $\mathfrak{g}$ of Cartan type given by $(I,\cdot)$ and also that l is KL-good (as defined in [112, §7]). Under these assumptions, one can invoke the Kazhdan–Lusztig equivalence [69].

Under the first assumption, it is known that $\operatorname{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ is a highest weight category see [35], see also [71, §2]. It has standard objects $\Delta_\lambda:=\Lambda_{\lambda,\mathbb{C}}$, costandard objects which we denote as ∇_λ and irreducibles L_λ for $\lambda\in Y_+$. The second assumption is needed because we use a result of Ko (Proposition 7.5.3).

7.5.1. Under the assumptions above, the spaces $K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})))$ and $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})))$ are free $\mathbb{Z}[\tau,\tau^{-1}]$-modules with (dual) bases given by the classes $[\Delta_{\check{\lambda}}]$ and $[\nabla_{\check{\lambda}}]$, respectively, for $\lambda \in Y_+$. We will mostly be working with $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})))$. Recall that $K_0^{\odot}(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})))$ defined in §7.4.3 has basis $[V_{\check{\eta}}], \check{\eta} \in Y_{l,+}$. The quantum Frobenius map (7.16) equips $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})))$ with the structure of a $K_0^{\odot}(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})))$-module defined as follows: for $V \in \mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C}))$ and $W \in \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$, then

$$[W] \odot [V] = [W \otimes \mathrm{Fr}(V)] \in K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})). \tag{7.23}$$

The $\mathbb{Z}[\tau,\tau^{-1}]$-module $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ also has another basis coming from simple modules $[L_{\check{\lambda}}], \check{\lambda} \in Y$, and Theorem 7.3.6(1) implies that for $\check{\lambda}_0 \in \square_l$ and $\check{\eta} \in Y_{l,+}$, one has

$$[L_{\check{\lambda}_0}] \odot [V_{\check{\eta}}] = [L_{\check{\lambda}_0+\check{\eta}}] \in K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})). \tag{7.24}$$

This relation determines the structure of $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ as a $K_0^{\odot}(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})))$-module in the following sense: if $\check{\lambda} \in Y_+$ and $\check{\eta} \in Y_{l,+}$, we write $\check{\lambda} = \check{\lambda}_0 + \check{\eta}'$ with $\check{\lambda}_0 \in \square_l$ and $\check{\eta}' \in Y_{l,+}$, and then we can compute

$$[L_{\check{\lambda}}] \odot [V_{\check{\eta}}] = \Big([L_{\check{\lambda}_0}] \odot [V_{\check{\eta}'}]\Big) \odot [V_{\check{\eta}}] = [L_{\check{\lambda}_0}] \odot ([V_{\check{\eta}'}] \cdot [V_{\check{\eta}}]), \tag{7.25}$$

where $[V_{\check{\eta}'}] \cdot [V_{\check{\eta}}]$ is computed in $K_0^{\odot}(\mathrm{Rep}(\check{\mathbf{G}}_l(\mathbb{C}))$ using the usual Littlewood–Richardson coefficients.

Alternatively, the $\mathbb{Z}[\tau,\tau^{-1}]$-module $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ also has another basis consisting of indecomposable tilting modules $[T_{\check{\lambda}}], \check{\lambda} \in Y$ and if $\check{\lambda} = \check{\lambda}_0 + \check{\eta}$ with $\check{\lambda}_0 \in \square_l$ and $\check{\eta} \in Y_{l,+}$, Theorem 7.3.6 implies that

$$[T_{\check{\lambda}_0^\dagger}] \odot [V_{\check{\eta}}] = [T_{\check{\lambda}^\dagger}] \in K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})). \tag{7.26}$$

7.5.2. Recall $\widetilde{H}_{\mathrm{sph}} = \widetilde{H}_{\mathrm{sph}}(\mathfrak{D}) \simeq H_{\mathrm{sph}}(\mathfrak{D}_{(\mathsf{Q},l)})$ from §3.8.1. It is a standard fact that

$$K_0^{\odot}(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C}))) \simeq \mathbb{Z}[Y_l]^W \simeq \widetilde{H}_{\mathrm{sph}}, \tag{7.27}$$

where in §3.8.1 we used $\widetilde{Y}$ for Y_l. The space $\widetilde{V}_{\mathrm{sph}}$ from §3.7.3 is a $\widetilde{H}_{\mathrm{sph}}$-module, where we take the twist (Q,l) to be a multiple of a primitive twist $(\mathsf{Q}_{\mathrm{prim}},l)$. Using Proposition 3.9.1 and Theorem 7.3.6(1), we deduce:

Proposition. *The map $\Phi : K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})) \to \widetilde{V}_{\mathrm{sph}}$ sending $[L_{\check{\lambda}}] \mapsto [\mathsf{G}^-_{\check{\lambda}}]$ for $\check{\lambda} \in Y_+$ is an isomorphism of $\mathbb{Z}[\tau,\tau^{-1}]$-modules which intertwines the $K_0^{\odot}(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C}))) \simeq \widetilde{H}_{\mathrm{sph}}$ actions. In particular,*

$$\Phi(v) \star c_{\check{\mu}} = v \odot [V_{\check{\mu}}] \quad \textit{for} \quad v \in K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})), \check{\mu} \in Y_{l,+}. \tag{7.28}$$

7.5.3. Let $\widetilde{W}_{\mathrm{aff}} := W_{\mathrm{aff}}(I, \circ_l, \mathfrak{D}_l^\vee)$. Let us also recall the definition of $\overline{\mathcal{A}}^{\bullet}_{-,n}$ from §3.4.7, which is again defined for root datum $(I, \circ_l, \mathfrak{D}_l^\vee)$. The following result of Ko will be used in our matching.

Proposition. [71, Thm. 4.10] *Fix $\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}$ and $J \subset I_{\mathrm{aff}}$ as in (3.77) and let $x, y \in W_{\mathrm{aff}}$ such that $\check{\eta} \bullet x, \check{\eta} \bullet y \in Y_+$. Then*

$$\langle [\Delta_{\check{\eta}\bullet y}], [L_{\check{\eta}\bullet x}] \rangle = \sum_{z \in {}_J(\widetilde{W}_l)} (-\tau^{-1})^{\ell(z)} m^-_{zy,x}. \tag{7.29}$$

Proof. This is essentially the result of [71, Thm. 4.10]; we will now explain how to translate it into our setting. Unlike *op. cit*, we use a right action of $\widetilde{W}_l$ on the weight lattice as opposed to a left action. As such, $x \bullet \lambda$ in *op. cit* corresponds to our $\lambda \bullet x^{-1}$. Now the t in *op. cit* corresponds to our τ^{-1} and $t^{\ell(x)-\ell(y)} P_{y,x}(t^{-1})$ from (2.3.1) in *op. cit* (which, we note, matches with the $m_{y,x}(t)$ in [57, II.C.2] needs to be replaced with $m^-_{y^{-1},x^{-1}}$ in our conventions, since M_J in (3.42) is a left module for $H_{\mathcal{W}}$ while [57] works with right modules and Jantzen's m is our m^-). Finally, we note that the expression

$$t^{\ell(x)-\ell(y)} P^J_{y,x}(t^{-1}) := t^{\ell(x)-\ell(y)} \sum_{z \in (\widetilde{W}_{\mathrm{aff}})_J} (-1)^{l(z)} P_{yz,x}(t^{-1}) = \sum_{z \in (\widetilde{W}_{\mathrm{aff}})_J} (-t)^{l(z)} t^{\ell(x)-\ell(yz)} P_{yz,x}(t^{-1})$$

in [71, §4.2] matches to Jantzen's [57] $\sum_{z \in (\widetilde{W}_l)_J} (-t^{-1})^{\ell(z)} m_{yz,x}(t)$ which in turn matches to our $\sum_{z \in {}_J(\widetilde{W}_l)} (-\tau^{-1})^{\ell(z)} m^-_{zy,x}$. □

7.5.4. The following result follows immediately by combining Propositions 7.5.2 and 7.5.3, (7.27) and (3.127) and the definition of $Q^{\check{\lambda}}_{\check{\eta},\check{\mu}}$ in (3.133).

Proposition. *For any $\check{\mu} \in Y_+$ and $\check{\lambda} \in Y_{l,+}$ we have*

$$[\nabla_{\check{\mu}}] \odot [V_{\check{\lambda}}] = \sum_{\check{\eta} \in Y} Q^{\check{\lambda}}_{\check{\eta},\check{\mu}} [\nabla_{\check{\eta}}]. \tag{7.30}$$

Proof. It suffices to show that for any $\check{\mu} \in Y_+$, we may write

$$[L_{\check{\mu}}] = [\nabla_{\check{\mu}}] + \sum_{\check{\xi} < \check{\mu}} o^-_{\check{\mu},\check{\xi}} [\nabla_{\check{\xi}}]. \tag{7.31}$$

We know that $o^-_{\check{\mu},\check{\xi}} = o^-_{y,x}$, so the result above follows by (7.21), (3.128) and (7.29). □

Remark. *The proof presented for this result is not optimal as it assumes (via the dependence on [71]) the graded version of Lusztig's conjecture. The restrictions on l which we place (especially l being KL-good) stem from the application of Kazhdan–Lusztig equivalence that is used to prove this conjecture. We believe*[25] *there should be a proof that bypasses this work and just uses the fact $\nabla_{\check{\mu}} \otimes \mathrm{Fr}(V_{\check{\zeta}})$ has a ∇ filtration where the successive quotients can be expressed combinatorially.*

[25]We thank Hankyung Ko and Catharina Stroppel for discussions and advice on this point.

7.5.5. *On tiltings and dualities.* Using the $\mathbb{Z}_\tau$-module $K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ allows us to naturally connect (at least conjecturally) the tilting modules with the polynomials $o_{\check{\lambda},\check{\mu}}$. Recall that $o_{\check{\lambda},\check{\mu}} \in \mathbb{Z}_\tau^+$, so we do not expect it to arise from an inner-product of the type introduced in Proposition 7.4.2, which would take values in τ^{-1}. On the other hand, one may just take $\overline{o}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau^-$ and posit that

$$\langle T_{\check{\lambda}}, \nabla_{\check{\mu}} \rangle = \overline{o}_{\check{\mu},\check{\lambda}}. \tag{7.32}$$

If this were true, then in $K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$, we would have

$$[T_{\check{\lambda}}] \quad = \quad [\Delta_{\check{\lambda}}] + \sum_{\check{\mu}<\check{\lambda}} \langle T_{\check{\lambda}}, \nabla_{\check{\mu}} \rangle [\Delta_{\check{\mu}}] = [\Delta_{\check{\lambda}}] + \sum_{\check{\mu}<\check{\lambda}} \overline{o}_{\check{\mu},\check{\lambda}} [\Delta_{\check{\mu}}] \tag{7.33}$$

We could not find a reference for (7.32) in the literature, though it does seem to hold at the level of specialized characters from the work of Soergel [111]. Actually, (7.32) seems to be the content of certain Koszul type dualities at the categorical level (see [1, §1.3]); it is also consistent with the known multiplicity of $\nabla_{\check{\mu}}$ in a ∇-flag of $T_{\check{\lambda}}$ (the later of which produces character formulas for many $T_{\check{\lambda}}$, see [110, §7] and [111]).

8. Main result and applications

In this section we formulate the main connection between the results in Part I, II, and §7. In §8.1, the main result is stated and the consequences for Lysenko's conjecture in the 'quantum' Geometric Langlands program are drawn. In §8.2, an extension of Savin's local Shimura correspondence to the Whittaker level is formulated. We use this to highlight certain classical behaviors within metaplectic Whittaker spaces connecting to recent work of Gao–Shahidi–Szpruch [50]. In §8.3, we describe some combinatorial properties and a formulate some questions concerning the different basis of $\mathcal{W}_\psi(\widetilde{G}, K)$.

8.1. **The main result.** Let us review the different settings which our main result will bring together.

8.1.1. *The generic case.* Fix $(I, \cdot, \mathfrak{D})$ a root datum, written $\widetilde{\mathfrak{D}} = (Y, \{\check{a}_i\}, X, \{a_i\})$, a (Q, n) a twist on $\mathfrak{D}$, and associated twisted root datum $(I, \circ_{(\mathsf{Q},n)}, \widetilde{\mathfrak{D}})$ where $\mathfrak{D} = (\widetilde{Y}, \{\widetilde{a}_i^\vee\}, X, \{\widetilde{a}_i\})$. Assume that $\widetilde{\mathfrak{D}}$ is of simply-connected type and construct the Hecke algebra $\widetilde{H}_{\mathrm{aff}} := H_{\mathrm{aff}}(\widetilde{\mathfrak{D}})$ and its module $\mathbb{V}$ as in §4.2. $\mathbb{V}$ has a decomposition $\mathbb{V} = \oplus_{\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}} \mathbb{V}(\check{\eta})$ into $\widetilde{H}_{\mathrm{aff}}$-submodules and bases $\{\mathbb{v}_{\check{\lambda}}\}$ and $\{Y_{\check{\lambda}}\}$ for $\check{\lambda} \in Y$.

At the spherical level, we have $\widetilde{H}_{\mathrm{sph}} := \varepsilon \widetilde{H}_{\mathrm{aff}} \varepsilon$ and its module $\mathbb{V}_{\mathrm{sph}} = \{[w] \mid w \in \mathbb{V}\}$, where $[w] := w\varepsilon$ for any $w \in \mathbb{V}$. The algebra $\widetilde{H}_{\mathrm{sph}}$ is equipped a basis $\widetilde{c}_{\check{\lambda}}$ while the module $\mathbb{V}_{\mathrm{sph}}$ is equipped with bases $[\mathbb{v}_{\check{\lambda}}]$, $[\mathbb{G}_{\check{\lambda}}]$, and $[\mathbb{G}^-_{\check{\lambda}}]$ (all indexed by $\check{\lambda} \in Y_+$), as well as their rescaled versions $[Y_{\check{\lambda}}]$, $[G_{\check{\lambda}}]$, $[G^-_{\check{\lambda}}]$ for $\check{\lambda} \in Y_+$, where

$$[\mathbb{v}_{\check{\lambda}}] = \kappa(\check{\lambda})[Y_{\check{\lambda}}], \quad [\mathbb{G}^-_{\check{\lambda}}] = \kappa(\check{\lambda})[G^-_{\check{\lambda}}], \quad \text{and } [\mathbb{G}_{\check{\lambda}}] = \kappa(\check{\lambda})[G_{\check{\lambda}}], \tag{8.1}$$

and $\kappa(\check{\lambda})$ is a product involving the Gauss sum parameters as in (4.40).

8.1.2. *The quantum case.* Assume that $\mathfrak{D}$ is of simply-connected type (i.e., $\check{\mathfrak{D}}$ is of adjoint type.) For any positive integer ℓ, define the integer l as in §7.3.1 and construct the corresponding quantum group at a primitive l-th root of unity $\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})$. Write (Q,l) for the primitive twist on $\mathfrak{D}$ (see §3.2.8) so that $(\mathfrak{D}_{(\mathsf{Q},l)})^\vee$ is equal to $\check{\mathfrak{D}}_l$, the l-twist from the theory of quantum groups at roots of unity. In §7.5.1 we constructed the module $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})))$ for the algebra $K_0^\odot(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})))$. The module is equipped with bases of simple modules $\{[L_{\check{\lambda}}]\}$, indecomposable tilting modules $[T_{\check{\lambda}}]$, and costandard modules $\{[\nabla_{\check{\lambda}}]\}$ for $\check{\lambda} \in Y_+$. On the other hand, the algebra $K_0^\odot(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})))$ comes equipped with bases $[V_{\check{\eta}}], \check{\eta} \in Y_{l,+}$, and its (right) action on $K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})))$ is denoted by $\odot$, *i.e.* $[W]\odot[V] = [W \otimes \mathrm{Fr}(V)]$ for $W \in \mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ and $V \in \mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C}))$.

8.1.3. *The p-adic case.* Fix ψ an additive character of conductor 0 of the local field $\mathcal{F}$ of residue characteristic q, an integer n such that $q \equiv 1 \mod 2n$, and $\varepsilon : \mu_n \to \mathbb{C}^*$ as in §5.3.4. We assume now that both $(I,\cdot,\mathfrak{D})$ and $(I,\circ_{(\mathsf{Q},n)},\widetilde{\mathfrak{D}})$ are of simply-connected type. In Part II, we have constructed a metaplectic group $\widetilde{G}$ with associated spherical and Iwahori–Hecke algebras of genuine functions $\mathcal{H}(\widetilde{G},I^-)$ and $\mathcal{H}(\widetilde{G},K)$ as well as their Whittaker modules $\mathcal{W}_\psi(\widetilde{G},I^-)$ and $\mathcal{W}_\psi(\widetilde{G},K)$, respectively. The right actions of the Hecke algebras were denoted by $\star$. We recall that $\mathcal{W}_\psi(\widetilde{G},K)$ has a natural p-adic basis which we write as $\mathcal{J}_{\check{\mu}}, \check{\mu} \in Y_+$ as well as corresponding canonical bases that were denoted as $\widetilde{\mathcal{L}}_{\check{\mu}}$ and $\widetilde{\mathcal{T}_{\check{\mu}}}$. The algebra $\mathcal{H}(\widetilde{G},K)$ has a natural basis $\widetilde{c}_{\check{\lambda}}$ for $\check{\lambda} \in \widetilde{Y}_+$ which, under the Satake isomorphism, is mapped to the character of the associated irreducible, highest weight representation $V_{\check{\lambda}}$ of the group $\mathbf{G}_{(\widetilde{\mathfrak{D}})^\vee}(\mathbb{C})$.

8.1.4. Connecting the above constructions, we have the following.

Theorem. *Let ℓ be a positive integer satisfying the assumptions in §7.5 and l defined as in §7.3.1. Fix $(I,\cdot,\mathfrak{D})$ of simply connected type with primitive twist (Q,l) such that $(\mathfrak{D}_{(\mathsf{Q},l)})^\vee = \check{\mathfrak{D}}_l$ is also of simply connected type. The maps $\mathfrak{p} : \mathbb{Z}_{\tau,\mathfrak{g}} \to \mathbb{C}$ (see §2.3.3) and $\mathfrak{q} : \mathbb{Z}_{\tau,\mathfrak{g}} \to \mathbb{Z}_\tau$ (see §2.3.2) extend to isomorphisms of $\mathbb{C}$ and $\mathbb{Z}_\tau$-modules respectively that we continue to denote by the same name,*

$$\mathfrak{p} : \mathbb{C} \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \xrightarrow{\simeq} \mathcal{W}_\psi(\widetilde{G},K) \quad \textit{sending } [Y_{\check{\mu}}] \mapsto \mathcal{J}_{\check{\mu}} \textit{ for } \check{\mu} \in Y_+ \tag{8.2}$$

$$\mathfrak{q} : \mathbb{Z}[\tau,\tau^{-1}] \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \xrightarrow{\simeq} K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathbf{G}}))) \quad \textit{sending } [Y_{\check{\mu}}] \mapsto [\nabla_{\check{\mu}}] \textit{ for } \check{\mu} \in Y_+ \tag{8.3}$$

The map $\mathfrak{p}$ intertwines the $\widetilde{H}_{\mathrm{sph}}$ and $\mathcal{H}(\widetilde{G},K)$ actions, while $\mathfrak{q}$ intertwines the $\widetilde{H}_{\mathrm{sph}}$ and $K_0^\odot(\mathrm{Rep}(\check{\mathbf{G}}_\ell(\mathbb{C})))$ actions. Moreover, $\mathfrak{p}$ maps $[G^-_{\check{\mu}}]$ to $\widetilde{\mathcal{L}}_{\check{\mu}}$, $[G_{\check{\mu}}]$ to $\widetilde{\mathcal{T}_{\check{\mu}}}$. On the other hand, $\mathfrak{q}$ sends $[G^-_{\check{\mu}}]$ (or $[\mathbb{G}^-_{\check{\mu}}]$) to $[L_{\check{\mu}}]$.

Remark. *In §8.1.6, we comment on the image of $[G_{\check{\mu}}]$ under the 'dual' of $\mathfrak{q}$. Let us also mention here that we may work on the p-adic side with a twist that is a multiple of a primitive twist, then the main theorem will still hold after choosing l accordingly, see Remark 3.2.8.*

As for the proof of this theorem: first on the quantum side, the theorem follows from the main results of §7: Theorem 7.3.6 and Proposition 7.5.4; on the p-adic side, the theorem follows from Proposition 6.3.1.

8.1.5. *Lysenko's Conjecture.* Within the quantum Langlands program (see [46]), Lysenko [85, Conjecture 11.2.4] formulated the Casselman–Shalika problem as giving an interpretation of a certain quantity (see [85, equation (62)])– essentially, the function-sheaf equivalent of our $\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}}$ – in terms of quantum groups at a root of unity. He conjectured a precise relation in Conjecture 11.2.4 of *loc. cit.* and the following result seems to answer the version of his conjecture that can be seen at the level of functions.

Corollary. *For $\check{\lambda} \in Y_{+,l}$, $\check{\mu}, \check{\eta} \in Y_+$ there exist ${}^{\mathfrak{g}}Q^{\check{\xi}}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_{\tau,\mathfrak{g}}$ defined via (4.64), such that*

$$\widetilde{\mathfrak{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}} = \textstyle\sum_{\check{\eta}} \mathfrak{p}({}^{\mathfrak{g}}Q^{\check{\lambda}}_{\check{\eta},\check{\mu}})\widetilde{\mathfrak{J}}_{\check{\eta}} \quad \textit{and} \quad [\nabla_{\check{\mu}}] \odot [V_{\check{\lambda}}] = \textstyle\sum_{\check{\eta}} \mathfrak{q}({}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}})\,[\nabla_{\check{\eta}}]\,. \tag{8.4}$$

The p-adic statement is Theorem 6.3.2, the quantum statement is Proposition 7.5.4.

8.1.6. *Tiltings, dualities, and inner-products.* In this paragraph only, denote the map $\mathfrak{q} : \mathbb{Z}_\tau \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \xrightarrow{\simeq} K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ which sends $[\mathrm{v}_{\check{\lambda}}] \mapsto \nabla_{\check{\lambda}}$ by $\mathfrak{q}^L$ now, and let us define a corresponding 'left'-variant

$$\mathfrak{q}^L : \mathbb{Z}_\tau \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} \xrightarrow{\simeq} K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})) \quad \text{by sending } \overline{[\mathrm{v}_{\check{\lambda}}]} \mapsto \Delta_{\check{\lambda}}, \tag{8.5}$$

where $\overline{[\mathrm{v}_{\check{\lambda}}]}$ is the image of $[\mathrm{v}_{\check{\lambda}}]$ under the involution d. Dualizing (3.127) one obtains

$$[\mathsf{G}_{\check{\lambda}}] \quad = \quad \overline{[\mathrm{v}_{\check{\lambda}}]} + \sum_{\check{\mu}<\check{\lambda}} \overline{o}_{\check{\mu},\check{\lambda}}\,\overline{[\mathrm{v}_{\check{\mu}}]}, \text{ where } \overline{o}_{\check{\mu},\check{\lambda}} \in \mathbb{Z}_\tau^-. \tag{8.6}$$

Applying $\mathfrak{q}^L$ to the right hand side, we obtain $[\Delta_{\check{\lambda}}] + \sum_{\check{\mu}<\check{\lambda}} \overline{o}_{\check{\mu},\check{\lambda}}\,[\Delta_{\check{\mu}}]$, which under assumption (7.32) is just equal to $[T_{\check{\lambda}}]$, *i.e.* $\mathfrak{q}^L([G_{\check{\lambda}}]) = [T_{\check{\lambda}}]$ under this assumption. The map $\mathfrak{q}^L$ may be understood as follows. The pairing of Proposition 7.4.2

$$K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})) \times K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})) \to \mathbb{Z}_\tau \tag{8.7}$$

is uniquely defined such that $\Delta_{\check{\lambda}}$ and $\nabla_{\check{\lambda}}$ are dual bases. Using the map $\mathfrak{q}^L$ and the diagram below, one obtains a unique pairing on $\mathbb{Z}_\tau \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}}$

$$\begin{array}{ccccc}
\mathbb{Z}_\tau \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} & \times & \mathbb{Z}_\tau \otimes_{\mathbb{Z}_{\tau,\mathfrak{g}}} \mathbb{V}_{\mathrm{sph}} & \longrightarrow & \mathbb{Z}_\tau \\
\Big\downarrow{\scriptstyle \mathfrak{q}^L} & & \Big\downarrow{\scriptstyle \mathfrak{q}^R} & & \\
K_0^L(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})) & \times & K_0^R(\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}})) & \longrightarrow & \mathbb{Z}_\tau
\end{array} \tag{8.8}$$

such that $[\mathrm{v}_{\check{\lambda}}]$ and $\overline{[\mathrm{v}_{\check{\lambda}}]}$ are dual bases.

Remark. *Presumably there is an inner product on $\mathcal{W}_\psi(\widetilde{G},K)$ given in terms of p-adic integrals, defined independently of our main result, but consistent with it. Such an inner product would allows us to give a consistent p-adic interpretation of all quantum objects $L_{\check{\lambda}}, \Delta_{\check{\lambda}}, \nabla_{\check{\lambda}}$ and $T_{\check{\lambda}}$ simultaneously.*

8.2. **On the local Shimura correspondence.**

8.2.1. Let $(I,\cdot,\mathfrak{D})$ be any root datum equipped with a twist (Q,n) and corresponding twisted root datum written now as $\mathfrak{D}_{(\mathsf{Q},n)} = (Y_{(\mathsf{Q},n)}, \{\check{a}_{(\mathsf{Q},n),i}\}, X_{(\mathsf{Q},n)}, \{a_{(\mathsf{Q},n),i}\})$. Denote the algebraic group attached to this twisted root datum as $\mathbf{G}_{(\mathsf{Q},n)}$. Write $G_{(\mathsf{Q},n)} := \mathbf{G}_{(\mathsf{Q},n)}(\mathcal{F})$, and let $I^{-}_{(\mathsf{Q},n)}$ and $K_{(\mathsf{Q},n)}$ be Iwahori subgroups and compact subgroups defined in analogy with I^{-} and K, respectively, where we note that a choice of dominant chamber for the root system defined by $\mathfrak{D}$ also picks one out in $\mathfrak{D}_{(\mathsf{Q},n)}$. In this notation, the local Shimura correspondence of Savin [103] and McNamara [90], see (5.18) and (5.16), states

$$\mathcal{H}(\widetilde{G},I^{-}) \simeq \mathcal{H}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)}) \quad \text{and} \quad \mathcal{H}(\widetilde{G},K) \simeq \mathcal{H}(G_{(\mathsf{Q},n)},K_{(\mathsf{Q},n)}). \tag{8.9}$$

8.2.2. Let ψ be an additive character on $\mathcal{F}$ of conductor 0 extended to the unipotent subgroup $U_{(\mathsf{Q},n)}$ of $\mathbf{G}_{(\mathsf{Q},n)}$ as in §6.1.1. We may then define $\mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)})$ and $\mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},K_{(\mathsf{Q},n)})$ as the Iwahori and spherical Whittaker spaces for the *linear* group $G_{(\mathsf{Q},n)}$. The structure of these modules over $\mathcal{H}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)})$ and $\mathcal{H}(G_{(\mathsf{Q},n)},K_{(\mathsf{Q},n)})$ was partially reviewed in §1.1.1. Let us also mention here some additional facts.

- It is known (see [29]) that $\mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)})$ is the anti-spherical module for $\mathcal{H}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)})$ with anti-spherical vector $\mathbf{v}^{I^{-}_{(\mathsf{Q},n)}}_{\psi,-\widetilde{\rho}^{\vee}} \in \mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)})$, *i.e.* $\mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)}) \cong \mathbb{C}[Y_{(\mathsf{Q},n)}]$ as vector spaces and as a module over $\mathcal{H}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)}) \cong H_W \otimes \mathbb{C}[Y_{(\mathsf{Q},n)}]$, the structure is specified as follows: $\mathbb{C}[Y_{(\mathsf{Q},n)}]$ acts by translation and for each $i \in I$, H_{s_i} acts on $\mathbf{v}^{I^{-}_{(\mathsf{Q},n)}}_{\psi,-\widetilde{\rho}^{\vee}}$ via the scalar $-\tau^{-1}$.
- The space $\mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},K_{(\mathsf{Q},n)})$ has a basis $\mathcal{J}_{(\mathsf{Q},n),\check{\mu}}$ for $\check{\mu} \in Y_{(\mathsf{Q},n),+}$ and $\mathcal{H}(G_{(\mathsf{Q},n)},K_{(\mathsf{Q},n)})$ has basis $c_{\check{\lambda}}$ for $\check{\lambda} \in Y_{(\mathsf{Q},n),+}$, see (1.5). One has $\mathcal{J}_{(\mathsf{Q},n),0} \star c_{\check{\lambda}} = \mathcal{J}_{(\mathsf{Q},n),\check{\lambda}}$.

8.2.3. Decompose $\mathcal{W}_{\psi}(\widetilde{G},I^{-}) \cong \mathbb{C}[Y] \cong \oplus_{\check{\eta} \in \overline{\mathcal{A}}^{\bullet}_{-,n}} \mathcal{W}_{\psi}(\widetilde{G},I^{-})(\check{\eta})$ into $\mathcal{H}(\widetilde{G},I^{-})$ submodules as in Proposition 6.2.5. Using Theorem 6.2.1 together with the observation that $-\check{\rho} \bullet w = -\check{\rho}$ for all $w \in W$, we conclude that $\mathcal{W}_{\psi}(\widetilde{G},I^{-})(-\check{\rho})$ has rank one as a module over $\mathbb{C}[\widetilde{Y}] = \mathbb{C}[Y_{(\mathsf{Q},n)}] \subset \mathcal{H}(\widetilde{G},I^{-})$ and the $H_{s_i}, i \in I$ act on $\widetilde{\mathcal{Y}}_{-\check{\rho}}$ via the scalar $-\tau^{-1}$. The following is an Iwahori–Whittaker variant of Savin's local Shimura correspondence. It follows by using the description of $\mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},I_{(\mathsf{Q},n)})$ in the previous paragraph.

Proposition. *There maps $\widetilde{\mathcal{Y}}_{-\check{\rho}} \mapsto \mathbf{v}^{I^{-}_{(\mathsf{Q},n)}}_{\psi,-\widetilde{\rho}^{\vee}}$ extends to an isomorphism of $\mathcal{H}(\widetilde{G},I^{-})$-modules*

$$\mathcal{W}_{\psi}(\widetilde{G},K)(-\check{\rho}) \xrightarrow{\simeq} \mathcal{W}_{\psi}(G_{(\mathsf{Q},n)},I^{-}_{(\mathsf{Q},n)}). \tag{8.10}$$

Remark. *This isomorphism must send* $\widetilde{\mathcal{Y}}_{-\check{\rho}+\widetilde{\rho}^\vee} = \widetilde{\mathcal{Y}}_{-\check{\rho}} Y_{\widetilde{\rho}^\vee}$ *to* $\mathbf{v}^{I^-_{(\mathsf{Q},n)}}_{\psi,-\widetilde{\rho}^\vee} \mathcal{Y}_{\widetilde{\rho}^\vee} = \mathbf{v}^{I^-_{(\mathsf{Q},n)}}_{\psi,0}$. *This computation explains the appearance of the coweight* $-\check{\rho} + \widetilde{\rho}^\vee$ *in the next paragraph.*

8.2.4. From Corollary 6.3.1, we have a decomposition $\mathcal{W}_\psi(\widetilde{G},K) = \oplus_{\check{\eta}\in\overline{\mathcal{A}}^\bullet_{-,n}} \mathcal{W}_\psi(\widetilde{G},K)(\check{\eta})$ into $\mathcal{H}(\widetilde{G},K)$-modules. One observes that $\mathcal{W}_\psi(\widetilde{G},K)(-\check{\rho}) := \mathcal{W}_\psi(\widetilde{G},I^-)(-\check{\rho})\star\mathbf{e}_K$ is a free rank one $\mathcal{H}(\widetilde{G},K)$-module. Within it, consider the elements $\widetilde{\mathcal{J}}_{-\check{\rho}+\widetilde{\rho}^\vee+\check{\mu}} = \widetilde{\mathcal{Y}}_{-\check{\rho}+\widetilde{\rho}^\vee+\check{\mu}} \star \mathbf{e}_K$ for $\check{\mu} \in \widetilde{Y}_+ = Y_{(\mathsf{Q},n),+}$, where we note that $\check{\mu} \in \widetilde{Y}_+$ implies $-\check{\rho} + \widetilde{\rho}^\vee + \check{\mu} \in Y_+$. The spherical extension of the previous result can now be stated:

Proposition. *There is an isomorphism of* $\mathcal{H}(\widetilde{G},K)$*-modules*

$$\mathcal{W}_\psi(\widetilde{G},K)(-\check{\rho}) \xrightarrow{\simeq} \mathcal{W}_\psi(G_{(\mathsf{Q},n)},K_{(\mathsf{Q},n)}) \tag{8.11}$$

which is defined by sending $\widetilde{\mathcal{J}}_{\widetilde{\rho}^\vee-\check{\rho}} \mapsto \mathcal{J}_{(\mathsf{Q},n),0}$. *For* $\check{\mu} \in \widetilde{Y}_+$, *one has (cf. (1.5))*

$$\widetilde{\mathcal{J}}_{\widetilde{\rho}^\vee-\check{\rho}} \star \widetilde{c}_{\check{\mu}} = \widetilde{\mathcal{J}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}, \tag{8.12}$$

Equation (8.12) together with Theorem 6.3.2 imply

Corollary. *Let* $\check{\mu} \in \widetilde{Y}_+$. *Then* $\widetilde{\mathcal{J}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}} = \widetilde{\mathcal{L}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}} = \widetilde{\mathscr{T}}_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}$.

8.2.5. *Quantum group interpretation.* On the quantum group side, Corollary 8.2.4 and equation (8.12) correspond to the fact that in $\mathrm{Rep}(\dot{\mathbf{U}}_\zeta(\check{\mathfrak{D}}))$ the representation $L_{\widetilde{\rho}^\vee-\check{\rho}+\check{\mu}}$ for $\check{\mu} \in \widetilde{Y}_+$ is equal to the costandard and indecomposable tilting corresponding to the same weights (see [88, Corollary 6.8] and [6]). In particular, equation (8.12) 'corresponds' on the quantum side to [88, Lemma 5.2]. The space $L_{\widetilde{\rho}^\vee-\check{\rho}}$ is called the Steinberg representation and plays an important role on the quantum side (see [6]).

8.2.6. *Relation to the p-adic literature.* An asymptotic or usual Casselman–Shalika formula (in the terminology of our introduction) was found in [50] for the same coweight $\widetilde{\rho}^\vee - \check{\rho}$ at the level of unramified Whittaker functions on the metaplectic group. In [22, Section 5] a similar result is shown using integrable systems techniques for certain metaplectic covers of GL_r. Actually in *op. cit.* multiple coweights where similar phenomenon occur are shown. This stems from the fact that $\mathfrak{gl}_r$ is not semi-simple. In the semi-simple case, one expects that $\widetilde{\rho}^\vee - \check{\rho}$ will be the only point at which such phenomenon can occur.

We also note here a proof of the classical (non-metaplectic) Casselman–Shalika formula in terms of the Steinberg-Lusztig theorem was given in [74, Theorem 2.3].

8.3. **Whittaker functions and** $\mathbb{g}$**-twisted combinatorics.** In this section we discuss in more detail the $\mathbb{g}$-coefficients which arise in our work and mention a few questions that can be solved using the connections to the theory of quantum groups. We fix the same notation and hypotheses as in §8.2.

8.3.1. *The $\check{\mu}$-large asymptotic Casselman–Shalika formula.* Let $\check{\mu} \in Y_+$ and $\check{\lambda} \in \widetilde{Y}_+$. As in (6.29), write

$$\begin{aligned} &\widetilde{\mathcal{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}} = \textstyle\sum_{\check{\zeta} \in Y_+} {}^{\mathbf{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}} \widetilde{\mathcal{J}}_{\check{\zeta}} \text{ or equivalently} \\ &[Y_{\check{\mu}}] \star \widetilde{c}_{\check{\lambda}} = [Y_{\check{\mu}}] \diamondsuit \chi_{\check{\lambda}} = [Y_{\check{\mu}} \cdot \chi_{\check{\lambda}}(Y)] = \textstyle\sum_{\check{\zeta} \in Y_+} {}^{\mathfrak{g}}Q^{\check{\zeta}}_{\check{\mu},\check{\lambda}} [Y_{\check{\zeta}}]. \end{aligned} \tag{8.13}$$

In general, straightening rules may be needed to rewrite the product $[Y_{\check{\mu}} \cdot \chi_{\check{\lambda}}(Y)]$ in terms of the basis $[Y_{\check{\lambda}}]$ with $\check{\lambda} \in Y_+$. However, if $\check{\mu}$ is large compared to $\check{\lambda}$ in the sense that $\check{\mu} + \check{\zeta} \in Y_+$ for any $\check{\zeta}$ appearing as a weight of the representation $V_{\check{\lambda}}$, then no such straightening relations are necessary. We shall say that $\check{\mu}$ is $\check{\lambda}$*-stable* if this condition is satisfied. Equations (4.63) and (4.65) can be used to show for $a_{\check{\lambda},\check{\mu}} \in \mathbb{Z}$:

Proposition. *Let $\check{\lambda} \in \widetilde{Y}_+$ and let $\check{\mu} \in Y_+$ be $\check{\lambda}$-stable. Then writing $\chi_{\check{\lambda}}(Y) = \sum_{\check{\zeta} \in \widetilde{Y}} a_{\check{\lambda},\check{\mu}} Y_{\check{\mu}}$,*

$$[Y_{\check{\mu}}] \star c_{\check{\lambda}} = \textstyle\sum_{\check{\zeta} \in \widetilde{Y}} {}^{\mathfrak{g}}a_{\check{\lambda},\check{\zeta}} [Y_{\check{\mu}+\check{\zeta}}] \quad \textit{where} \quad {}^{\mathfrak{g}}a_{\check{\zeta},\check{\mu}} := \kappa(\check{\mu})\kappa(\check{\lambda})^{-1} a_{\check{\zeta},\check{\mu}}. \tag{8.14}$$

There are well-known combinatorial interpretations of the coefficients $a_{\check{\lambda},\check{\zeta}}$. For example, in type A the Weyl character is written as a sum over Gelfand–Tsetlin patterns and the coefficients are the number of certain Young Tableaux (there are generalizations of this result to orthogonal and symplectic characters). This result and Proposition 6.3.1 produce simple combinatorial formulas for $\widetilde{\mathcal{J}}_{\check{\lambda}} \star \widetilde{c}_{\check{\zeta}}$ in the stable range:

$$\widetilde{\mathcal{J}}_{\check{\mu}} \star \widetilde{c}_{\check{\lambda}} = \sum_{\check{\zeta} \in \widetilde{Y}} {}^{\mathbf{g}}a_{\check{\lambda},\check{\zeta}} \widetilde{\mathcal{J}}_{\check{\mu}+\check{\zeta}}. \tag{8.15}$$

8.3.2. *On $\mathfrak{g}$-twisted LLT polynomials.* As already mentioned, $Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ are known to have non-negative coefficients by the work of Grojnowski and Haiman [53, Theorem 5.9]. We may use ${}^{\mathfrak{g}}Q^{\check{\eta}}_{\check{\mu},\check{\lambda}}$ to define $\mathfrak{g}$-twisted LLT polynomials by using the formula after [53, Eq. (13)]. Among the connections between LLT polynomials and Macdonald polynomials, let us note that it is proven in *op. cit.* that the Schur polynomial expansion of transformed Macdonald polynomials are positive by using an expression of transformed Macdonald polynomials in terms of LLT polynomials. This suggests there might exist connections between $\mathfrak{g}$-twisted LLT polynomials and the metaplectic Macdonald polynomials of [100, 101].

8.3.3. *On the inverse $\mathfrak{g}$-twisted Kazhdan–Lusztig polynomials.* Consider the basis $\widetilde{\mathcal{J}}_{\check{\lambda}}$ and $\widetilde{\mathcal{L}}_{\check{\lambda}}$ of the Whittaker space $\mathcal{W}_{\psi}(\widetilde{G}, K)$ introduced in §6.3.1. If we define ${}^{\mathbf{g}}d_{\check{\lambda},\check{\mu}} \in \mathbb{C}$ as the expansion coefficients in

$$\widetilde{\mathcal{J}}_{\check{\lambda}} = \sum_{\check{\mu} \in Y_+} {}^{\mathbf{g}}d_{\check{\lambda},\check{\mu}} \widetilde{\mathcal{L}}_{\check{\mu}}, \tag{8.16}$$

then it is a natural question to understand when these coefficients are non-zero.

As in the representation theory of quantum groups, we say that $\check{\lambda} \in Y_+$ is *strongly linked* to $\check{\mu} \in Y_+$ (see [8, §3]) if there exists a chain $\check{\mu} = \check{\mu}_1, \check{\mu}_2, \cdots, \check{\mu}_r = \check{\lambda}$ of elements in Y_+ such that

$$\check{\mu}_{j-1} = \check{\mu}_i \bullet s_{\check{\beta}_j} + m_j l_{\check{\beta}_j} \check{\beta}_j \leq \check{\mu}_i \quad \text{for} \quad \check{\beta}_j \in \check{\mathfrak{R}}_+, m_i \in \mathbb{Z}, i = 2, \cdots, r \ . \tag{8.17}$$

In the equation above l_{α_j} is the integer l_j defined in §3.2.8 when α_j is a simple positive coroot. Otherwise, for any coroot $\check{\beta}$, there exists a $w \in W$ such that $\check{\beta} = w(\check{\alpha}_j)$, $j \in I$ and we set $l_{\check{\beta}} = l_{\check{\alpha}_j}$. The following is a p-adic analogue of the strong linkage principle from the theory of quantum groups see [8, 10].

Proposition. *If the coefficient* ${}^{\mathbf{g}}d_{\check{\lambda},\check{\mu}} \neq 0$*, then* $\check{\mu}$ *is strongly linked to* $\check{\lambda}$*.*

Proof. By the discussion in §4.3.3, the coefficients ${}^{\mathbf{g}}d_{\check{\lambda},\check{\mu}} \neq 0$ will be non-zero if and only if their quantum version $d_{\check{\lambda},\check{\mu}}$ is non-zero (they will differ by a quotient of κ factors which is product of Gauss sums). The quantum coefficients are non-zero if a filtration of $\nabla_{\check{\lambda}}$ by irreducibles contains $L_{\check{\mu}}$. The result follows by the strong linkage principle see [10, Theorem 8.1][8, Theorem 3.13]. □

Remark. *It might be interesting to see if there is a purely p-adic proof of the above result. We also remark that a similar result may be stated for the relation between* $\widetilde{\mathfrak{J}}_{\check{\lambda}}$ *and* $\widetilde{\mathfrak{T}}_{\check{\lambda}}$*, whose proof would again rest of the validity of* (7.32).

8.3.4. *On the principal submodule* $\mathcal{W}_\psi(\widetilde{G}, K)(0)$. In §8.2.4 we related the subspace $\mathcal{W}_\psi(\widetilde{G}, K)(-\check{\rho})$ with the non-metaplectic version of the spherical Gelfand–Graev representation. Let us now look the *principal* subspace $\mathcal{W}_\psi(\widetilde{G}, K)(0)$. Let $\check{\lambda} \in \widetilde{Y}_+ \subset 0 \bullet \widetilde{W}_{\text{aff}} \cap Y_+$.

Recall that $[Y_{\check{\lambda}}] := Y_{\check{\lambda}} \varepsilon$ can be thought of as an element in $\mathbb{Z}_{\tau,\mathfrak{g}}[Y]$ obtained by acting with ε on $Y_{\check{\lambda}}$, where ε is defined in (3.86) and the action of the H_{s_i} on $\mathbb{Z}_{\tau,\mathfrak{g}}[Y]$ is the one from §4.2.1. On the other hand, in (3.88), we introduced an action of H_W on $\mathbb{Z}_\tau[\widetilde{Y}]$ (this is *not* the quantum action considered above). Using it, to each $\check{\mu} \in \widetilde{Y}$, we define an element $[\widetilde{Y}_{\check{\mu}}] := \widetilde{Y}_{\check{\mu}} \varepsilon$ where $\widetilde{Y}_{\check{\mu}}$ is a typical element in the group algebra $\mathbb{Z}_\tau[\widetilde{Y}]$.

Denote by $\Pi_{(\mathsf{Q},n)} : \mathbb{Z}_{\tau,\mathfrak{g}}[Y] \to \mathbb{Z}_{\tau,\mathfrak{g}}[\widetilde{Y}]$ the natural projection sending $Y_{\check{\mu}} \mapsto 0$ if $\check{\mu} \notin \widetilde{Y}$ and $Y_{\check{\mu}} \mapsto Y_{\check{\mu}}$ if $\check{\mu} \in \widetilde{Y}$. Now, one expects (as in [45, 88]) that $\Pi_{(\mathsf{Q},n)}([Y_{\check{\lambda}}]) \in \mathbb{C}[\widetilde{Y}]$ can be expanded to an identity in $\mathbb{Z}_{\tau,\mathfrak{g}}[\widetilde{Y}]$.

$$\Pi_{(\mathsf{Q},n)}([Y_{\check{\lambda}}]) = \sum_{\check{\mu}} {}^{\mathfrak{g}}m_{\check{\lambda}}^{\check{\mu}} \, [\widetilde{Y}_{\check{\mu}}]. \tag{8.18}$$

Specializing $\tau \mapsto -1$ in the identity above, which on the quantum group side means working with the regular Grothendieck group instead of the enriched one, the corresponding specialized coefficients, denoted ${}^{\mathbf{g}}m_{\check{\lambda}}^{\check{\mu}}$, will be equal to (up to signs) the Littlewood–Richardson coefficients $c_{\widetilde{\rho}^\vee - \check{\rho}, \check{\lambda}}^{\check{\mu} + \widetilde{\rho}^\vee - \check{\rho}}$ for the group $\check{G}_{(\mathsf{Q},n)}$. This essentially follows by comparing the work of Frenkel–Hernandez [45] and McGerty [88] (especially Proposition 4.4 and Theorem 5.5 in *loc. cit.*) to our setting. We pose the problem to study the coefficients ${}^{\mathfrak{g}}m^{\check{\mu}}_{\check{\lambda}}$ for generic τ and give them a p-adic interpretation.

REFERENCES

[1] Pramod N. Achar, Shotaro Makisumi, Simon Riche, and Geordie Williamson, *Koszul duality for Kac-Moody groups and characters of tilting modules*, J. Amer. Math. Soc. **32** (2019), no. 1, 261–310.

[2] Amol Aggarwal, Alexei Borodin, and Michael Wheeler, *Colored fermionic vertex models and symmetric functions*, 2021. Preprint, arXiv:2101.01605.

[3] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque **220** (1994), 321.

[4] H. H. Andersen and D. Tubbenhauer, *Diagram categories for $\mathbf{U}_q$-tilting modules at roots of unity*, Transform. Groups **22** (2017), no. 1, 29–89.

[5] Henning Haahr Andersen, *An inversion formula for the Kazhdan-Lusztig polynomials for affine Weyl groups*, Adv. in Math. **60** (1986), no. 2, 125–153. MR840301

[6] ———, *Tensor products of quantized tilting modules*, Comm. Math. Phys. **149** (1992), no. 1, 149–159.

[7] ———, *Tilting modules for algebraic and quantum groups*, Algebra—representation theory (Constanta, 2000), 2001, pp. 1–21. MR1858029

[8] ———, *The strong linkage principle for quantum groups at roots of 1*, 2003, pp. 2–15. Special issue celebrating the 80th birthday of Robert Steinberg.

[9] Henning Haahr Andersen and Jan Paradowski, *Fusion categories arising from semisimple Lie algebras*, Comm. Math. Phys. **169** (1995), no. 3, 563–588.

[10] Henning Haahr Andersen, Patrick Polo, and Ke Xin Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), no. 1, 1–59.

[11] Sergey Arkhipov and Roman Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group*, Israel J. Math. **170** (2009), 135–183. With an appendix by Bezrukavnikov and Ivan Mirković. MR2506322

[12] Roman Bezrukavnikov and Simon Riche, *Hecke action on the principal block*, Compos. Math. **158** (2022), no. 5, 953–1019. MR4454848

[13] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.

[14] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[15] Alexander Braverman and David Kazhdan, *The spherical Hecke algebra for affine Kac-Moody groups I*, Ann. of Math. (2) **174** (2011), no. 3, 1603–1642. MR2846488

[16] Alexander Braverman, David Kazhdan, and Manish M. Patnaik, *Iwahori-Hecke algebras for p-adic loop groups*, Invent. Math. **204** (2016), no. 2, 347–442.

[17] Ben Brubaker, Valentin Buciumas, and Daniel Bump, *A Yang-Baxter equation for metaplectic ice*, Commun. Number Theory Phys. **13** (2019), no. 1, 101–148.

[18] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Solomon Friedberg, *Hecke modules from metaplectic ice*, Selecta Math. (N.S.) **24** (2018), no. 3, 2523–2570.

[19] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Henrik P. A. Gustafsson, *Colored vertex models and Iwahori Whittaker functions*, 2019. Preprint, arXiv:1906.04140.

[20] ———, *Metaplectic Iwahori Whittaker functions and supersymmetric lattice models*, 2020. Preprint, arXiv:2012.15778.

[21] ———, *Vertex operators, solvable lattice models and metaplectic Whittaker functions*, Comm. Math. Phys. **380** (2020), no. 2, 535–579.

[22] ———, *Iwahori-metaplectic duality*, 2021. Preprint, arXiv:2112.14670.

[23] Ben Brubaker, Daniel Bump, and Solomon Friedberg, *Twisted Weyl group multiple Dirichlet series: the stable case*, Eisenstein series and applications, 2008, pp. 1–26.

[24] ———, *Weyl group multiple Dirichlet series: type A combinatorial theory*, Annals of Mathematics Studies, vol. 175, Princeton University Press, Princeton, NJ, 2011. MR2791904

[25] Benjamin Brubaker, Daniel Bump, Gautam Chinta, Solomon Friedberg, and Jeffrey Hoffstein, *Weyl group multiple Dirichlet series. I*, Multiple Dirichlet series, automorphic forms, and analytic number theory, 2006, pp. 91–114. MR2279932

[26] Jean-Luc Brylinski and Pierre Deligne, *Central extensions of reductive groups by* $\mathbf{K}_2$, Publ. Math. Inst. Hautes Études Sci. **94** (2001), 5–85.

[27] Justin Campbell, Gurbir Dhillon, and Sam Raskin, *Fundamental local equivalences in quantum geometric Langlands*, Compos. Math. **157** (2021), no. 12, 2699–2732.

[28] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function*, Compositio Math. **41** (1980), no. 2, 207–231.

[29] Kei Yuen Chan and Gordan Savin, *Iwahori component of the Gelfand-Graev representation*, Math. Z. **288** (2018), no. 1-2, 125–133.

[30] Gautam Chinta and Paul E. Gunnells, *Constructing Weyl group multiple Dirichlet series*, J. Amer. Math. Soc. **23** (2010), no. 1, 189–215. MR2552251

[31] Gautam Chinta, Paul E. Gunnells, and Anna Puskás, *Metaplectic Demazure operators and Whittaker functions*, Indiana Univ. Math. J. **66** (2017), no. 3, 1045–1064.

[32] Gautam Chinta and Omer Offen, *A metaplectic Casselman-Shalika formula for* GL_r, Amer. J. Math. **135** (2013), no. 2, 403–441. MR3038716

[33] Joseph Chuang and Raphaël Rouquier, *Derived equivalences for symmetric groups and* $\mathfrak{sl}_2$*-categorification*, Ann. of Math. (2) **167** (2008), no. 1, 245–298. MR2373155

[34] E. Cline, B. Parshall, and L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[35] ———, *The homological dual of a highest weight category*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 294–316.

[36] Edward Cline, Brian Parshall, and Leonard Scott, *Abstract Kazhdan-Lusztig theories*, Tohoku Math. J. (2) **45** (1993), no. 4, 511–534.

[37] Vinay V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math. **39** (1977), no. 2, 187–198.

[38] ———, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra **111** (1987), no. 2, 483–506.

[39] Adrian Diaconu, Dorian Goldfeld, and Jeffrey Hoffstein, *Multiple Dirichlet series and moments of zeta and L-functions*, Compositio Math. **139** (2003), no. 3, 297–360. MR2041614

[40] Peter Fiebig, *The combinatorics of category* $\mathscr{O}$ *over symmetrizable Kac-Moody algebras*, Transform. Groups **11** (2006), no. 1, 29–49. MR2205072

[41] Michael Finkelberg and Sergey Lysenko, *Twisted geometric Satake equivalence*, J. Inst. Math. Jussieu **9** (2010), no. 4, 719–739.

[42] E. Frenkel, D. Gaitsgory, D. Kazhdan, and K. Vilonen, *Geometric realization of Whittaker functions and the Langlands conjecture*, J. Amer. Math. Soc. **11** (1998), no. 2, 451–484.

[43] E. Frenkel, D. Gaitsgory, and K. Vilonen, *Whittaker patterns in the geometry of moduli spaces of bundles on curves*, Ann. of Math. (2) **153** (2001), no. 3, 699–748.

[44] ———, *On the geometric Langlands conjecture*, J. Amer. Math. Soc. **15** (2002), no. 2, 367–417.

[45] Edward Frenkel and David Hernandez, *Langlands duality for representations of quantum groups*, Math. Ann. **349** (2011), no. 3, 705–746.

[46] D. Gaitsgory, *Twisted Whittaker model and factorizable sheaves*, Selecta Math. (N.S.) **13** (2008), no. 4, 617–659.

[47] D. Gaitsgory and S. Lysenko, *Parameters and duality for the metaplectic geometric Langlands theory*, Selecta Math. (N.S.) **24** (2018), no. 1, 227–301.

[48] Dennis Gaitsgory and Sergey Lysenko, *Metaplectic Whittaker category and quantum groups : the "small" FLE*, 2019. Preprint, arXiv:1903.02279.

[49] Fan Gao, Nadya Gurevich, and Edmund Karasiewicz, *Genuine pro-p Iwahori–Hecke algebras, Gelfand–Graev representations, and some applications*, 2022. Preprint, arXiv:2204.13053.

[50] Fan Gao, Freydoon Shahidi, and Dani Szpruch, *Local coefficients and gamma factors for principal series of covering groups*, 2019. to appear in Memoirs of the AMS, arXiv:1902.02686.

[51] Fan Gao and Martin H. Weissman, *Whittaker models for depth zero representations of covering groups*, Int. Math. Res. Not. IMRN **11** (2019), 3580–3620.

[52] Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, Second, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[53] Ian Grojnowski and Mark Haiman, *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, 2007. Preprint.

[54] J. Haglund, M. Haiman, and N. Loehr, *A combinatorial formula for Macdonald polynomials*, J. Amer. Math. Soc. **18** (2005), no. 3, 735–761.

[55] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[56] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48.

[57] Jens Carsten Jantzen, *Representations of algebraic groups*, Second, Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.

[58] Victor G. Kac, *Infinite-dimensional Lie algebras*, Third, Cambridge University Press, Cambridge, 1990.

[59] Masaki Kashiwara and Toshiyuki Tanisaki, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. J. **77** (1995), no. 1, 21–62.

[60] ———, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. II. Nonintegral case*, Duke Math. J. **84** (1996), no. 3, 771–813.

[61] Shin-ichi Kato, *Spherical functions and a q-analogue of Kostant's weight multiplicity formula*, Invent. Math. **66** (1982), no. 3, 461–468.

[62] ———, *On the Kazhdan-Lusztig polynomials for affine Weyl groups*, Adv. in Math. **55** (1985), no. 2, 103–130. MR772611

[63] ———, *On the Kazhdan-Lusztig polynomials for affine Weyl groups*, Adv. in Math. **55** (1985), no. 2, 103–130.

[64] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras. I, II*, J. Amer. Math. Soc. **6** (1993), no. 4, 905–947, 949–1011.

[65] ———, *Tensor structures arising from affine Lie algebras. III*, J. Amer. Math. Soc. **7** (1994), no. 2, 335–381.

[66] ———, *Tensor structures arising from affine Lie algebras. IV*, J. Amer. Math. Soc. **7** (1994), no. 2, 383–453.

[67] D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*, Inst. Hautes Études Sci. Publ. Math. **59** (1984), 35–142.

[68] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[69] ———, *Affine Lie algebras and quantum groups*, Internat. Math. Res. Notices **2** (1991), 21–29.

[70] Friedrich Knop, *On the Kazhdan-Lusztig basis of a spherical Hecke algebra*, Represent. Theory **9** (2005), 417–425.

[71] Hankyung Ko, *Cohomology in singular blocks for a quantum group at a root of unity*, Algebr. Represent. Theory **22** (2019), no. 5, 1109–1132.

[72] Bertram Kostant, *Groups over Z*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), 1966, pp. 90–98.

[73] Robert P. Langlands, *Euler products*, Yale Mathematical Monographs, vol. 1, Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967.

[74] Martina Lanini and Arun Ram, *The Steinberg-Lusztig tensor product theorem, Casselman-Shalika, and LLT polynomials*, Represent. Theory **23** (2019), 188–204.

[75] Martina Lanini, Arun Ram, and Paul Sobaje, *A Fock space model for decomposition numbers for quantum groups at roots of unity*, Kyoto J. Math. **59** (2019), no. 4, 955–991.

[76] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties*, J. Math. Phys. **38** (1997), no. 2, 1041–1068.

[77] Bernard Leclerc and Jean-Yves Thibon, *Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials*, Combinatorial methods in representation theory (Kyoto, 1998), 2000, pp. 155–220.

[78] Peter Littelmann, *Paths and root operators in representation theory*, Ann. of Math. (2) **142** (1995), no. 3, 499–525.

[79] G. Lusztig, *Modular representations and quantum groups*, Classical groups and related topics (Beijing, 1987), 1989, pp. 59–77.

[80] ———, *Study of a* $\mathbf{Z}$*-form of the coordinate ring of a reductive group*, J. Amer. Math. Soc. **22** (2009), no. 3, 739–769.

[81] George Lusztig, *Hecke algebras and Jantzen's generic decomposition patterns*, Adv. in Math. **37** (1980), no. 2, 121–164.

[82] ———, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), 1983, pp. 208–229.

[83] ———, *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), no. 1-3, 89–113.

[84] ———, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[85] Sergey Lysenko, *Twisted Whittaker models for metaplectic groups*, Geom. Funct. Anal. **27** (2017), no. 2, 289–372.

[86] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, Cambridge, 2003.

[87] Hideya Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62.

[88] Kevin McGerty, *Langlands duality for representations and quantum groups at a root of unity*, Comm. Math. Phys. **296** (2010), no. 1, 89–109.

[89] Peter J. McNamara, *Metaplectic Whittaker functions and crystal bases*, Duke Math. J. **156** (2011), no. 1, 1–31.

[90] ———, *Principal series representations of metaplectic groups over local fields*, Multiple Dirichlet series, L-functions and automorphic forms, 2012, pp. 299–327.

[91] ———, *The metaplectic Casselman-Shalika formula*, Trans. Amer. Math. Soc. **368** (2016), no. 4, 2913–2937. MR3449262

[92] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[93] B. C. Ngô and P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, J. Algebraic Geom. **10** (2001), no. 3, 515–547.

[94] Bao Châu Ngô, *Preuve d'une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen pour les groupes linéaires généraux*, Israel J. Math. **120** (2000), no. part A, 259–270.

[95] Manish Patnaik and Anna Puskás, *On Iwahori-Whittaker functions for metaplectic groups*, Adv. Math. **313** (2017), 875–914.

[96] Manish M. Patnaik, *Unramified Whittaker functions on p-adic loop groups*, Amer. J. Math. **139** (2017), no. 1, 175–213.

[97] Manish M. Patnaik and Anna Puskás, *Metaplectic covers of Kac-Moody groups and Whittaker functions*, Duke Math. J. **168** (2019), no. 4, 553–653.

[98] I. I. Piatetski-Shapiro, *Euler subgroups.*, Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), 1975, pp. 597–620. MR406935

[99] Simon Riche and Geordie Williamson, *Tilting modules and the p-canonical basis*, Astérisque **397** (2018), ix+184.
[100] Siddhartha Sahi, Jasper V. Stokman, and Vidya Venkateswaran, *Metaplectic representations of Hecke algebras, Weyl group actions, and associated polynomials*, Selecta Math. (N.S.) **27** (2021), no. 3, Paper No. 47, 42.
[101] ———, *Quasi-polynomial representations of double affine Hecke algebras*, 2022. Preprint, arXiv:2204.13729.
[102] Ichirô Satake, *Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields*, Inst. Hautes Études Sci. Publ. Math. **18** (1963), 5–69.
[103] Gordan Savin, *Local Shimura correspondence*, Math. Ann. **280** (1988), no. 2, 185–190.
[104] ———, *On unramified representations of covering groups*, J. Reine Angew. Math. **566** (2004), 111–134.
[105] Jean-Pierre Serre, *Corps locaux*, Publications de l'Université de Nancago, No. VIII, Hermann, Paris, 1968. Deuxième édition.
[106] Freydoon Shahidi, *Eisenstein series and automorphic L-functions*, American Mathematical Society Colloquium Publications, vol. 58, American Mathematical Society, Providence, RI, 2010. MR2683009
[107] J. A. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. (2) **100** (1974), 171–193. MR348047
[108] Carl Ludwig Siegel, *Die Funktionalgleichungen einiger Dirichletscher Reihen*, Math. Z. **63** (1956), 363–373. MR74533
[109] Wolfgang Soergel, *Gradings on representation categories*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1995, pp. 800–806.
[110] ———, *Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114.
[111] ———, *Character formulas for tilting modules over Kac-Moody algebras*, Represent. Theory **2** (1998), 432–448.
[112] Toshiyuki Tanisaki, *Character formulas of Kazhdan-Lusztig type*, Representations of finite dimensional algebras and related topics in Lie theory and geometry, 2004, pp. 261–276.
[113] J. Tits, *Normalisateurs de tores. I. Groupes de Coxeter étendus*, J. Algebra **4** (1966), 96–116.
[114] ———, *Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples*, Inst. Hautes Études Sci. Publ. Math. **31** (1966), 21–58.
[115] Jacques Tits, *Uniqueness and presentation of Kac-Moody groups over fields*, J. Algebra **105** (1987), no. 2, 542–573.
[116] Martin H. Weissman, *Managing metaplectiphobia: covering p-adic groups*, Harmonic analysis on reductive, p-adic groups, 2011, pp. 237–277.
[117] ———, *Split metaplectic groups and their L-groups*, J. Reine Angew. Math. **696** (2014), 89–141.
[118] Zhiwei Yun, *Weights of mixed tilting sheaves and geometric Ringel duality*, Selecta Math. (N.S.) **14** (2009), no. 2, 299–320. MR2480718

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