

Some structure theorems for algebraic groups

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ABSTRACT. These are extended notes of a course given at Tulane University for the 2015 Clifford Lectures. Their aim is to present structure results for group schemes of finite type over a field, with applications to Picard varieties and automorphism groups.

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1. Introduction

The algebraic groups of the title are the group schemes of finite type over a field; they occur in many questions of algebraic geometry, number theory and representation theory. To analyze their structure, one seeks to build them up from algebraic groups of a specific geometric nature, such as smooth, connected, affine, proper... A first result in this direction asserts that *every algebraic group G has a largest connected normal subgroup scheme G^0 , the quotient G/G^0 is finite and étale, and the formation of G^0 commutes with field extensions*. The main goal of this expository text is to prove two more advanced structure results:

THEOREM 1. *Every algebraic group G over a field k has a smallest normal subgroup scheme H such that the quotient G/H is affine. Moreover, H is smooth, connected and contained in the center of G^0 ; in particular, H is commutative. Also, $\mathcal{O}(H) = k$ and H is the largest subgroup scheme of G satisfying this property. The formation of H commutes with field extensions.*

THEOREM 2. *Every algebraic group G over k has a smallest normal subgroup scheme N such that G/N is proper. Moreover, N is affine and connected. If k is perfect and G is smooth, then N is smooth as well, and its formation commutes with field extensions.*

In particular, every smooth connected algebraic group over a perfect field is an extension of an *abelian variety* (i.e., a smooth connected proper algebraic group) by a smooth connected algebraic group which is affine, or equivalently *linear*. Both building blocks, abelian varieties and linear algebraic groups, have been extensively studied; see e.g. the books [41] for the former, and [7, 56] for the latter.

Also, every algebraic group over a field is an extension of a linear algebraic group by an *anti-affine* algebraic group H , i.e., every global regular function on H is constant. Clearly, every abelian variety is anti-affine; but the converse turns out to be incorrect, unless k is a finite field or the algebraic closure of such a field (see §5.5). Still, the structure of anti-affine groups over an arbitrary field can be reduced to that of abelian varieties; see [10, 52] and also §5.5 again.

As a consequence, taking for G an anti-affine group which is not an abelian variety, one sees that the natural maps $H \rightarrow G/N$ and $N \rightarrow G/H$ are generally not isomorphisms with the notation of the above theorems. But when G is smooth and connected, one may combine these theorems to obtain more information on its structure, see §5.1.

The above theorems have a long history. Theorem 1 was first obtained by Rosenlicht in 1956 for smooth connected algebraic groups, see [48, Sec. 5]. The version presented here is due to Demazure and Gabriel, see [22, III.3.8]. In the setting of smooth connected algebraic groups again, Theorem 2 was announced by Chevalley in the early 1950's. But he published his proof in 1960 only (see [17]), as he had first to build up a theory of Picard and Albanese varieties. Meanwhile, proofs of Chevalley's theorem had been published by Barsotti and Rosenlicht (see [4], and [48, Sec. 5] again). The present version of Theorem 2 is a variant of a result of Raynaud (see [47, IX.2.7]).

The terminology and methods of algebraic geometry have much evolved since the 1950's; this makes the arguments of Barsotti, Chevalley and Rosenlicht rather hard to follow. For this reason, modern proofs of the above results have been made available recently: first, a scheme-theoretic version of Chevalley's proof of his structure theorem by Conrad (see [18]); then a version of Rosenlicht's proof for smooth connected algebraic groups over algebraically closed fields (see [14, Chap. 2] and also [40]).

In this text, we present scheme-theoretic proofs of Theorems 1 and 2, with (hopefully) modest prerequisites. More specifically, we assume familiarity with the contents of Chapters 2 to 5 of the book [35], which will be our standard reference for algebraic geometry over an arbitrary field. Also, we do not make an explicit use of sheaves for the fpqc or fppf topology, even if these notions are in the background of several arguments.

To make the exposition more self-contained, we have gathered basic notions and results on group schemes over a field in Section 2, referring to the books [22] and [SGA3] for most proofs. Section 3 is devoted to the proof of Theorem 1, and Section 4 to that of Theorem 2. Although the statements of both theorems are very similar, the first one is actually much easier. Its proof only needs a few preliminary results: some criteria for an algebraic group to be affine (§3.1), the notion of affinization of a scheme (§3.2) and a version of the rigidity lemma for “anti-affine” schemes (§3.3). In contrast, the proof of Theorem 2 is based on quite a few results on abelian varieties. Some of them are taken from [41], which will be our standard reference on that topic; less classical results are presented in §§4.1 and 4.2.

Section 5 contains applications and developments of the above structure theorems, in several directions. We begin with the Rosenlicht decomposition, which reduces somehow the structure of smooth connected algebraic groups to the linear and anti-affine cases (§5.1). We then show in §5.2 that every homogeneous space admits a projective equivariant compactification. §5.3 gathers some known results on the structure of commutative algebraic groups. In §5.4, we provide details on semi-abelian varieties, i.e., algebraic groups obtained as extensions of an abelian variety by a torus; these play an important rôle in various aspects of algebraic and arithmetic geometry. §5.5 is devoted to the classification of anti-affine algebraic groups, based on results from §§5.3 and 5.4. The final §5.6 contains developments on algebraic groups in positive characteristics, including a recent result of Totaro (see [57, §2]).

Further applications, of a geometric nature, are presented in Sections 6 and 7. We give a brief overview of the Picard schemes of proper schemes in §6.1, referring to [31] for a detailed exposition. §6.2 is devoted to structure results for the Picard

variety of a proper variety X , in terms of the geometry of X . Likewise, §7.1 surveys the automorphism group schemes of proper schemes. §7.2 presents a useful descent property for actions of connected algebraic groups. In the final §7.3, based on [11], we show that every smooth connected algebraic group over a perfect field is the connected automorphism group of some normal projective variety.

Each section ends with a paragraph of notes and references, which also contains brief presentations of recent work, and some open questions. A general problem, which falls out of the scope of these notes, asks for a version of Theorem 2 in the setting of group schemes over (say) discrete valuation rings. A remarkable analogue of Theorem 1 has been obtained by Raynaud in that setting (see [SGA3, VIB.12.10]). But Chevalley's structure theorem admits no direct generalization, as abelian varieties degenerate to tori. So finding a meaningful analogue of that theorem over a ring of formal power series is already an interesting challenge.

Notation and conventions. Throughout this text, we fix a ground field k with algebraic closure \bar{k} ; the characteristic of k is denoted by $\text{char}(k)$.

We denote by k_s the separable closure of k in \bar{k} and by Γ the Galois group of k_s over k . Also, we denote by k_i the perfect closure of k in \bar{k} , i.e., the largest subfield of \bar{k} that is purely inseparable over k . If $\text{char}(k) = 0$ then $k_s = \bar{k}$ and $k_i = k$; if $\text{char}(k) = p > 0$ then $k_i = \bigcup_{n \geq 0} k^{1/p^n}$.

We consider separated schemes over $\text{Spec}(k)$ unless otherwise stated; we will call them k -schemes, or just schemes if this creates no confusion. Morphisms and products of schemes are understood to be over $\text{Spec}(k)$. For any k -scheme X , we denote by $\mathcal{O}(X)$ the k -algebra of global sections of the structure sheaf \mathcal{O}_X . Given a field extension K of k , we denote the K -scheme $X \times \text{Spec}(K)$ by X_K .

We identify a scheme X with its *functor of points* that assigns to any scheme S the set $X(S)$ of morphisms $f : S \rightarrow X$. When S is affine, i.e., $S = \text{Spec}(R)$ for an algebra R , we also use the notation $X(R)$ for $X(S)$. In particular, we have the set $X(k)$ of k -rational points.

A *variety* is a geometrically integral scheme of finite type. The function field of a variety X will be denoted by $k(X)$.

2. Basic notions and results

2.1. Group schemes.

DEFINITION 2.1.1. A *group scheme* is a scheme G equipped with morphisms $m : G \times G \rightarrow G$, $i : G \rightarrow G$ and with a k -rational point e , which satisfy the following condition:

For any scheme S , the set $G(S)$ is a group with multiplication map $m(S)$, inverse map $i(S)$ and neutral element e .

This condition is equivalent to the commutativity of the following diagrams:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \text{id}} & G \times G \\ \text{id} \times m \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

(i.e., m is associative),

$$\begin{array}{ccccc} G & \xrightarrow{e \times \text{id}} & G \times G & \xleftarrow{\text{id} \times e} & G \\ & \searrow \text{id} & \downarrow m & \swarrow \text{id} & \\ & & G & & \end{array}$$

(i.e., e is the neutral element), and

$$\begin{array}{ccccc} G & \xrightarrow{\text{id} \times i} & G \times G & \xleftarrow{i \times \text{id}} & G \\ & \searrow e \circ f & \downarrow m & \swarrow e \circ f & \\ & & G & & \end{array}$$

(i.e., i is the inverse map). Here $f : G \rightarrow \text{Spec}(k)$ denotes the structure map.

We will write for simplicity $m(x, y) = xy$ and $i(x) = x^{-1}$ for any scheme S and points $x, y \in G(S)$.

REMARKS 2.1.2. (i) For any k -group scheme G , the base change under a field extension K of k yields a K -group scheme G_K .

(ii) The assignment $S \mapsto G(S)$ defines a *group functor*, i.e., a contravariant functor from the category of schemes to that of groups. In fact, the group schemes are exactly those group functors that are representable (by a scheme).

(iii) Some natural group functors are not representable. For example, consider the functor that assigns to any scheme S the group $\text{Pic}(S)$ of isomorphism classes of invertible sheaves on S , and to any morphism of schemes $f : S' \rightarrow S$, the pull-back map $f^* : \text{Pic}(S) \rightarrow \text{Pic}(S')$. This yields a commutative group functor that we still denote by Pic . For any local ring R , we have $\text{Pic}(\text{Spec}(R)) = 0$. If Pic is represented by a scheme X , then every morphism $\text{Spec}(R) \rightarrow X$ is constant for R local; hence every morphism $S \rightarrow X$ is locally constant. As a consequence, $\text{Pic}(\mathbb{P}^1) = \text{Hom}(\mathbb{P}^1, X) = 0$, a contradiction.

DEFINITION 2.1.3. Let G be a group scheme. A *subgroup scheme* of G is a (locally closed) subscheme H such that $H(S)$ is a subgroup of $G(S)$ for any scheme S . We say that H is *normal in* G , if $H(S)$ is a normal subgroup of $G(S)$ for any scheme S . We then write $H \trianglelefteq G$.

DEFINITION 2.1.4. Let G, H be group schemes. A morphism $f : G \rightarrow H$ is called a *homomorphism* if $f(S) : G(S) \rightarrow H(S)$ is a group homomorphism for any scheme S .

The *kernel* of the homomorphism f is the group functor $\text{Ker}(f)$ such that $\text{Ker}(f)(S) = \text{Ker}(f(S) : G(S) \rightarrow H(S))$. It is represented by a closed normal subgroup scheme of G , the fiber of f at the neutral element of H .

DEFINITION 2.1.5. An *algebraic group* over k is a k -group scheme of finite type.

This notion of algebraic group is somewhat more general than the classical one. More specifically, the “algebraic groups defined over k ” in the sense of [7, 56] are the *geometrically reduced* k -group schemes of finite type. Yet both notions coincide in characteristic 0, as a consequence of the following result of Cartier:

THEOREM 2.1.6. *When $\text{char}(k) = 0$, every algebraic group over k is reduced.*

PROOF. See [22, II.6.1.1] or [SGA3, VIB.1.6.1]. A self-contained proof is given in [41, p. 101]. \square

EXAMPLE 2.1.7. The *additive group* \mathbb{G}_a is the affine line \mathbb{A}^1 equipped with the addition. More specifically, we have $m(x, y) = x + y$ and $i(x) = -x$ identically, and $e = 0$.

Consider a subgroup scheme $H \subseteq \mathbb{G}_a$. If $H \neq \mathbb{G}_a$, then H is the zero scheme $V(P)$ for some non-constant polynomial $P \in \mathcal{O}(\mathbb{G}_a) = k[x]$; we may assume that P has leading coefficient 1. We claim that P is an *additive polynomial*, i.e.,

$$P(x + y) = P(x) + P(y)$$

in the polynomial ring $k[x, y]$.

To see this, note that $P(0) = 0$ as $0 \in H(k)$, and

$$P(x + y) \in (P(x), P(y))$$

(the ideal of $k[x, y]$ generated by $P(x)$ and $P(y)$), as the addition $\mathbb{G}_a \times \mathbb{G}_a \rightarrow \mathbb{G}_a$ sends $H \times H$ to H . Thus, there exist $A(x, y), B(x, y) \in k[x, y]$ such that

$$P(x + y) - P(x) - P(y) = A(x, y)P(x) + B(x, y)P(y).$$

Dividing $A(x, y)$ by $P(y)$, we may assume that $\deg_y A(x, y) < \deg(P)$ with an obvious notation. Since $\deg_y(P(x + y) - P(x) - P(y)) < \deg(P)$, it follows that $B = 0$. Likewise, we obtain $A = 0$; this yields the claim.

We now determine the additive polynomials. The derivative of any such polynomial P satisfies $P'(x + y) = P'(x)$, hence P' is constant. When $\text{char}(k) = 0$, we obtain $P(x) = ax$ for some $a \in k$, hence H is just the (reduced) point 0. Alternatively, this follows from Theorem 2.1.6, since $H(\bar{k})$ is a finite subgroup of $(\bar{k}, +)$, and hence is trivial.

When $\text{char}(k) = p > 0$, we obtain $P(x) = a_0x + P_1(x^p)$, where P_1 is again an additive polynomial. By induction on $\deg(P)$, it follows that

$$P(x) = a_0x + a_1x^p + \cdots + a_nx^{p^n}$$

for some positive integer n and $a_0, \dots, a_n \in k$. As a consequence, \mathbb{G}_a has many subgroup schemes in positive characteristics; for example,

$$\alpha_{p^n} := V(x^{p^n})$$

is a non-reduced subgroup scheme supported at 0.

Note finally that the additive polynomials are exactly the endomorphisms of \mathbb{G}_a , and their kernels yield all subgroup schemes of that group scheme (in arbitrary characteristics).

EXAMPLE 2.1.8. The *multiplicative group* \mathbb{G}_m is the punctured affine line $\mathbb{A}^1 \setminus 0$ equipped with the multiplication: we have $m(x, y) = xy$ and $i(x) = x^{-1}$ identically, and $e = 1$.

The subgroup schemes of \mathbb{G}_m turn out to be \mathbb{G}_m and the subschemes

$$\mu_n := V(x^n - 1)$$

of n th roots of unity, where n is a positive integer; these are the kernels of the endomorphisms $x \mapsto x^n$ of \mathbb{G}_m . Moreover, μ_n is reduced if and only if n is prime to $\text{char}(k)$.

EXAMPLE 2.1.9. Given a vector space V , the *general linear group* $\text{GL}(V)$ is the group functor that assigns to any scheme S , the automorphism group of the sheaf of \mathcal{O}_S -modules $\mathcal{O}_S \otimes_k V$. When V is of finite dimension n , the choice of a basis identifies V with k^n and $\text{GL}(V)(S)$ with $\text{GL}_n(\mathcal{O}(S))$, the group of invertible $n \times n$

matrices with coefficients in the algebra $\mathcal{O}(S)$. It follows that $\mathrm{GL}(V)$ is represented by an open affine subscheme of the affine scheme \mathbb{A}^{n^2} (associated with the linear space of $n \times n$ matrices), the complement of the zero scheme of the determinant. This defines a group scheme GL_n , which is smooth, connected, affine and algebraic.

DEFINITION 2.1.10. A group scheme is *linear* if it is isomorphic to a closed subgroup scheme of GL_n for some positive integer n .

Clearly, every linear group scheme is algebraic and affine. The converse also holds, see Proposition 3.1.1 below.

Some natural classes of group schemes arising from geometry, such as automorphism group schemes and Picard schemes of proper schemes, are generally not algebraic. Yet they turn out to be locally of finite type; this motivates the following:

DEFINITION 2.1.11. A *locally algebraic group* over k is a k -group scheme, locally of finite type.

PROPOSITION 2.1.12. *The following conditions are equivalent for a locally algebraic group G with neutral element e :*

- (1) G is smooth.
- (2) G is geometrically reduced.
- (3) $G_{\bar{k}}$ is reduced at e .

PROOF. Clearly, (1) \Rightarrow (2) \Rightarrow (3). We now show that (3) \Rightarrow (1). For this, we may replace G with $G_{\bar{k}}$ and hence assume that k is algebraically closed.

Observe that for any $g \in G(k)$, the local ring $\mathcal{O}_{G,g}$ is isomorphic to $\mathcal{O}_{G,e}$ as the left multiplication by g in G is an automorphism that sends e to g . It follows that $\mathcal{O}_{G,g}$ is reduced; hence every open subscheme of finite type of G is reduced as well. Since G is locally of finite type, it must be reduced, too. Thus, G contains a smooth k -rational point g . By arguing as above, we conclude that G is smooth. \square

2.2. Actions of group schemes.

DEFINITION 2.2.1. An *action* of a group scheme G on a scheme X is a morphism $a : G \times X \rightarrow X$ such that the map $a(S)$ yields an action of the group $G(S)$ on the set $X(S)$, for any scheme S .

This condition is equivalent to the commutativity of the following diagrams:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{id}} & G \times X \\ \mathrm{id} \times a \downarrow & & \downarrow a \\ G \times X & \xrightarrow{a} & X \end{array}$$

(i.e., a is “associative”), and

$$\begin{array}{ccc} X & \xrightarrow{e \times \mathrm{id}} & G \times X \\ & \searrow \mathrm{id} & \downarrow a \\ & & X \end{array}$$

(i.e., the neutral element acts via the identity).

We may view a G -action on X as a homomorphism of group functors

$$a : G \longrightarrow \mathrm{Aut}_X,$$

where Aut_X denotes the *automorphism group functor* that assigns to any scheme S , the group of automorphisms of the S -scheme $X \times S$. The S -points of Aut_X are those morphisms $f : X \times S \rightarrow X$ such that the map

$$f \times p_2 : X \times S \longrightarrow X \times S, \quad (x, s) \longmapsto (f(x, s), s)$$

is an automorphism of $X \times S$; they may be viewed as families of automorphisms of X parameterized by S .

DEFINITION 2.2.2. A scheme X equipped with an action a of G will be called a G -scheme; we then write for simplicity $a(g, x) = g \cdot x$ for any scheme S and $g \in G(S)$, $x \in X(S)$.

The action is *trivial* if a is the second projection $p_2 : G \times X \rightarrow X$; equivalently, $g \cdot x = x$ identically.

REMARK 2.2.3. For an arbitrary action a , we have a commutative triangle

$$\begin{array}{ccc} G \times X & \xrightarrow{u} & G \times X \\ & \searrow a & \downarrow p_2 \\ & & X, \end{array}$$

where $u(g, x) := (g, a(g, x))$. Since u is an automorphism (with inverse the map $(g, x) \mapsto (g, a(g^{-1}, x))$), it follows that the morphism a shares many properties of the scheme G . For example, a is always faithfully flat; it is smooth if and only if G is smooth.

In particular, the multiplication $m : G \times G \rightarrow G$ is faithfully flat.

DEFINITION 2.2.4. Let X, Y be G -schemes with actions a, b . A *morphism of G -schemes* $\varphi : X \rightarrow Y$ is a morphism of schemes such that the following square commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \text{id} \times \varphi \downarrow & & \downarrow \varphi \\ G \times Y & \xrightarrow{b} & Y. \end{array}$$

In other words, $\varphi(g \cdot x) = g \cdot \varphi(x)$ identically; we then say that φ is *G -equivariant*. When Y is equipped with the trivial action of G , we say that φ is *G -invariant*.

DEFINITION 2.2.5. Let X be a G -scheme with action a , and Y a closed subscheme of X .

The *normalizer* (resp. *centralizer*) of Y in G is the group functor $N_G(Y)$ (resp. $C_G(Y)$) that associates with any scheme S , the set of those $g \in G(S)$ which induce an automorphism of $Y \times S$ (resp. the identity of $Y \times S$).

The *kernel* of a is the centralizer of X in G , or equivalently, the kernel of the corresponding homomorphism of group functors.

The action a is *faithful* if its kernel is trivial; equivalently, for any scheme S , every non-trivial element of $G(S)$ acts non-trivially on $X \times S$.

The *fixed point functor* of X is the subfunctor X^G that associates with any scheme S , the set of all $x \in X(S)$ such that for any S -scheme S' and any $g \in G(S')$, we have $g \cdot x = x$.

THEOREM 2.2.6. *Let G be a group scheme acting on a scheme X .*

- (1) *The normalizer and centralizer of any closed subscheme $Y \subseteq X$ are represented by closed subgroup schemes of G .*
- (2) *The functor of fixed points is represented by a closed subscheme of X .*

PROOF. See [22, II.1.3.6] or [SGA3, VIB.6.2.4]. \square

In particular, $N_G(Y)$ is the largest subgroup scheme of G that acts on Y , and $C_G(Y)$ is the kernel of this action. Moreover, X^G is the largest subscheme of X on which G acts trivially. We also say that $N_G(Y)$ *stabilizes* Y , and $C_G(Y)$ *fixes* Y *pointwise*.

When Y is just a k -rational point x , we have $N_G(Y) = C_G(Y) =: C_G(x)$. This is the *stabilizer* of x in G , which is clearly represented by a closed subgroup scheme of G : the fiber at x of the *orbit map*

$$a_x : G \longrightarrow X, \quad g \longmapsto g \cdot x.$$

We postpone the definition of orbits to §2.7, where homogeneous spaces are introduced; we now record classical properties of the orbit map:

PROPOSITION 2.2.7. *Let G be an algebraic group acting on a scheme of finite type X via a .*

- (1) *The image of the orbit map a_x is locally closed for any closed point $x \in X$.*
- (2) *If k is algebraically closed and G is smooth, then there exists $x \in X(k)$ such that the image of a_x is closed.*

PROOF. (1) Consider the natural map $\pi : X_{\bar{k}} \rightarrow X$. Since π is faithfully flat and quasi-compact, it suffices to show that $\pi^{-1}(a_x(G))$ is locally closed (see e.g. [EGA, IV.2.3.12]). As $\pi^{-1}(a_x(G))$ is the image of the orbit map $(a_x)_{\bar{k}}$, we may assume k algebraically closed. Then $a_x(G)$ is constructible, and hence contains a dense open subset U of its closure. The pull-back $a_x^{-1}(U)$ is a non-empty open subset of the underlying topological space of G ; hence that space is covered by the translates $ga_x^{-1}(U)$, where $g \in G(k)$. It follows that $a_x(G)$ is covered by the translates gU , and hence is open in its closure.

(2) Choose a closed G -stable subscheme $Y \subseteq X$ of minimal dimension and let $x \in Y(k)$. If $a_x(G)$ is not closed, then $Z := \overline{a_x(G)} \setminus a_x(G)$ (equipped with its reduced subscheme structure) is a closed subscheme of Y , stable by $G(k)$. Since the normalizer of Z is representable and $G(k)$ is dense in G , it follows that Z is stable by G . But $\dim(Z) < \dim(a_x(G)) \leq \dim(Y)$, a contradiction. \square

EXAMPLE 2.2.8. Every group scheme G acts on itself by left multiplication, via

$$\lambda : G \times G \longrightarrow G, \quad (x, y) \longmapsto xy.$$

It also acts by right multiplication, via

$$\rho : G \times G \longrightarrow G, \quad (x, y) \longmapsto yx^{-1}$$

and by conjugation, via

$$\text{Int} : G \times G \longrightarrow G, \quad (x, y) \longmapsto xyx^{-1}.$$

The actions λ and ρ are both faithful. The kernel of Int is the *center* of G .

DEFINITION 2.2.9. Let G, H be two group schemes and $a : G \times H \rightarrow H$ an action by group automorphisms, i.e., we have $g \cdot (h_1 h_2) = (g \cdot h_1)(g \cdot h_2)$ identically.

The *semi-direct product* $G \ltimes H$ is the scheme $G \times H$ equipped with the multiplication such that

$$(g, h) \cdot (g', h') = (gg', (g'^{-1} \cdot h)h'),$$

the neutral element $e_G \times e_H$, and the inverse such that $(g, h)^{-1} = (g^{-1}, g \cdot h^{-1})$.

By using the Yoneda lemma, one may readily check that $G \ltimes H$ is a group scheme. Moreover, H (identified with its image in $G \ltimes H$ under the closed immersion $i : h \mapsto (e_G, h)$) is a closed normal subgroup scheme, and G (identified with its image under the closed immersion $g \mapsto (g, e_H)$) is a closed subgroup scheme having a retraction

$$r : G \ltimes H \longrightarrow G, \quad (g, h) \longmapsto g$$

with kernel H . The given action of G on H is identified with the action by conjugation in $G \ltimes H$.

REMARKS 2.2.10. (i) With the above notation, G is a normal subgroup scheme of $G \ltimes H$ if and only if G acts trivially on H .

(ii) Conversely, consider a group scheme G and two closed subgroup schemes N, H of G such that H normalizes N and the inclusion of H in G admits a retraction $r : G \rightarrow H$ which is a homomorphism with kernel N . Form the semi-direct product $H \ltimes N$, where H acts on N by conjugation. Then one may check that the multiplication map

$$H \ltimes N \longrightarrow G, \quad (x, y) \longmapsto xy$$

is an isomorphism of group schemes, with inverse being the morphism

$$G \longrightarrow H \ltimes N, \quad z \longmapsto (r(z), r(z)^{-1}z).$$

2.3. Linear representations.

DEFINITION 2.3.1. Let G be a group scheme and V a vector space. A *linear representation* ρ of G in V is a homomorphism of group functors $\rho : G \rightarrow \mathrm{GL}(V)$. We then say that V is a G -module.

More specifically, ρ assigns to any scheme S and any $g \in G(S)$, an automorphism $\rho(g)$ of the sheaf of \mathcal{O}_S -modules $\mathcal{O}_S \otimes_k V$, functorially on S . Note that $\rho(g)$ is uniquely determined by its restriction to V (identified with $1 \otimes_k V \subseteq \mathcal{O}(S) \otimes_k V$, where 1 denotes the unit element of the algebra $\mathcal{O}(S)$), which yields a linear map $V \rightarrow \mathcal{O}(S) \otimes_k V$.

A linear subspace $W \subseteq V$ is a G -submodule if each $\rho(g)$ normalizes $\mathcal{O}_S \otimes_k W$. More generally, the notions of quotients, exact sequences, tensor operations of linear representations of abstract groups extend readily to the setting of group schemes.

EXAMPLES 2.3.2. (i) When $V = k^n$ for some positive integer n , a linear representation of G in V is a homomorphism of group schemes $\rho : G \rightarrow \mathrm{GL}_n$ or equivalently, a linear action of G on the affine space \mathbb{A}^n .

(ii) Let X be an affine G -scheme with action a . For any scheme S and $g \in G(S)$, we define an automorphism $\rho(g)$ of the \mathcal{O}_S -algebra $\mathcal{O}_S \otimes_k \mathcal{O}(X)$ by setting

$$\rho(g)(f) := f \circ a(g^{-1})$$

for any $f \in \mathcal{O}(X)$. This yields a representation ρ of G in $\mathcal{O}(X)$, which uniquely determines the action in view of the equivalence of categories between affine schemes and algebras.

For instance, if G acts linearly on a finite-dimensional vector space V , then $\mathcal{O}(V) \cong \text{Sym}(V^*)$ (the symmetric algebra of the dual vector space) as G -modules.

(iii) More generally, given any G -scheme X , one may define a representation ρ of G in $\mathcal{O}(X)$ as above. But in general, the G -action on X is not uniquely determined by ρ . For instance, if X is a proper G -variety, then $\mathcal{O}(X) = k$ and hence ρ is trivial.

LEMMA 2.3.3. *Let X, Y be quasi-compact schemes. Then the map*

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \longrightarrow \mathcal{O}(X \times Y), \quad f \otimes g \longmapsto ((x, y) \mapsto f(x)g(y))$$

is an isomorphism of algebras. In particular, we have a canonical isomorphism

$$\mathcal{O}(X) \otimes_k R \xrightarrow{\cong} \mathcal{O}(X_R)$$

for any quasi-compact scheme X and any algebra R .

PROOF. The assertion is well-known when X and Y are affine.

When X is affine and Y is arbitrary, we may choose a finite open covering $(V_i)_{1 \leq i \leq n}$ of Y ; then the intersections $V_i \cap V_j$ are affine as well. Also, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(Y) \longrightarrow \prod_i \mathcal{O}(V_i) \xrightarrow{d_Y} \prod_{i,j} \mathcal{O}(V_i \cap V_j),$$

where $d_Y((f_i)_i) := (f_i|_{V_i \cap V_j} - f_j|_{V_i \cap V_j})_{i,j}$. Tensoring with $\mathcal{O}(X)$ yields an exact sequence

$$0 \longrightarrow \mathcal{O}(X) \otimes_k \mathcal{O}(Y) \longrightarrow \prod_i \mathcal{O}(X \times V_i) \xrightarrow{d_{X,Y}} \prod_{i,j} \mathcal{O}(X \times (V_i \cap V_j)),$$

where $d_{X,Y}$ is defined similarly. Since the $X \times V_i$ form an open covering of $X \times Y$, the kernel of $d_{X,Y}$ is $\mathcal{O}(X \times Y)$; this proves the assertion in this case.

In the general case, we choose a finite open affine covering $(U_i)_{1 \leq i \leq m}$ of X and obtain an exact sequence

$$0 \longrightarrow \mathcal{O}(X) \otimes_k \mathcal{O}(Y) \longrightarrow \prod_i \mathcal{O}(U_i \times Y) \longrightarrow \prod_{i,j} \mathcal{O}((U_i \cap U_j) \times Y),$$

by using the above step. The assertion follows similarly. \square

The quasi-compactness assumption in the above lemma is a mild finiteness condition, which is satisfied e.g. for affine or noetherian schemes.

PROPOSITION 2.3.4. *Let G be an algebraic group and X a G -scheme of finite type. Then the G -module $\mathcal{O}(X)$ is the union of its finite-dimensional submodules.*

PROOF. The action map $a : G \times X \rightarrow X$ yields a homomorphism of algebras $a^\# : \mathcal{O}(X) \rightarrow \mathcal{O}(G \times X)$. In view of Lemma 2.3.3, we may view $a^\#$ as a homomorphism $\mathcal{O}(X) \rightarrow \mathcal{O}(G) \otimes_k \mathcal{O}(X)$. Choose a basis (φ_i) of the vector space $\mathcal{O}(G)$. Then for any $f \in \mathcal{O}(X)$, there exists a family (f_i) of elements of $\mathcal{O}(X)$ such that $f_i \neq 0$ for only finitely many i 's, and

$$a^\#(f) = \sum_i \varphi_i \otimes f_i.$$

Thus, we have identically

$$\rho(g)(f) = \sum_i \varphi_i(g^{-1}) f_i.$$

Applying this to the action of G on itself by left multiplication, we obtain the existence of a family $(\psi_{ij})_j$ of elements of $\mathcal{O}(G)$ such that $\psi_{ij} \neq 0$ for only finitely many i 's, and

$$\varphi_i(h^{-1}g^{-1}) = \sum_j \varphi_i(g^{-1}) \psi_{ij}(h^{-1})$$

identically on $G \times G$. It follows that

$$\rho(g)\rho(h)(f) = \sum_{i,j} \varphi_i(g^{-1}) \psi_{ij}(h^{-1}) f_i.$$

As a consequence, the span of the f_i 's in $\mathcal{O}(G)$ is a finite-dimensional G -submodule, which contains $f = \sum_i \varphi_i(e) f_i$. \square

PROPOSITION 2.3.5. *Let G be an algebraic group and X an affine G -scheme of finite type. Then there exists a finite-dimensional G -module V and a closed G -equivariant immersion $\iota : X \rightarrow V$.*

PROOF. We may choose finitely many generators f_1, \dots, f_n of the algebra $\mathcal{O}(X)$. By Proposition 2.3.4, each f_i is contained in some finite-dimensional G -submodule $W_i \subseteq \mathcal{O}(X)$. Thus, $W := W_1 + \dots + W_n$ is a finite-dimensional G -submodule of $\mathcal{O}(X)$, which generates that algebra. This defines a surjective homomorphism of algebras $\text{Sym}(W) \rightarrow \mathcal{O}(X)$, equivariant for the natural action of G on $\text{Sym}(W)$. In turn, this yields the desired closed equivariant immersion. \square

Examples of linear representations arise from the action of the stabilizer of a k -rational point on its infinitesimal neighborhoods, which we now introduce.

EXAMPLE 2.3.6. Let G be an algebraic group acting on a scheme X via a and let $Y \subseteq X$ be a closed subscheme. For any non-negative integer n , consider the n th infinitesimal neighborhood $Y_{(n)}$ of Y in X ; this is the closed subscheme of X with ideal sheaf \mathcal{I}_Y^{n+1} , where $\mathcal{I}_Y \subseteq \mathcal{O}_X$ denotes the ideal sheaf of Y . The subschemes $Y_{(n)}$ form an increasing sequence, starting with $Y_{(0)} = Y$.

Next, assume that G stabilizes Y . Then $a^{-1}(Y) = p_2^{-1}(Y)$, and hence

$$a^{-1}(\mathcal{I}_Y)\mathcal{O}_{G \times X} = p_2^{-1}(\mathcal{I}_Y)\mathcal{O}_{G \times X}.$$

It follows that

$$a^{-1}(\mathcal{I}_Y^{n+1})\mathcal{O}_{G \times X} = p_2^{-1}(\mathcal{I}_Y^{n+1})\mathcal{O}_{G \times X}.$$

Thus, $a^{-1}(Y_{(n)}) = p_2^{-1}(Y_{(n)})$, i.e., G stabilizes $Y_{(n)}$ as well.

As a consequence, given a (say) locally noetherian G -scheme X equipped with a k -rational point $x = \text{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x)$, the algebraic group $C_G(x)$ acts on each infinitesimal neighborhood $x_{(n)} = \text{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1})$, which is a finite scheme supported at x . This yields a linear representation ρ_n of G on $\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ by algebra automorphisms. In particular, $C_G(x)$ acts linearly on $\mathfrak{m}_x/\mathfrak{m}_x^2$ and hence on the Zariski tangent space, $T_x(X) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Applying the above construction to the action of G on itself by conjugation, which fixes the point e , we obtain a linear representation of G in $\mathfrak{g} := T_e(G)$, called the *adjoint representation* and denoted by

$$\text{Ad} : G \longrightarrow \text{GL}(\mathfrak{g}).$$

This yields in turn a linear map

$$\text{ad} := d\text{Ad}_e : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g})$$

(where the right-hand side denotes the space of endomorphisms of the vector space \mathfrak{g}), and hence a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (x, y) \longmapsto \operatorname{ad}(x)(y).$$

One readily checks that $[x, x] = 0$ identically; also, $[\cdot, \cdot]$ satisfies the Jacobi identity (see e.g. [22, II.4.4.5]). Thus, $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra, called the *Lie algebra of G* ; we denote it by $\operatorname{Lie}(G)$.

Denote by T_G the tangent sheaf of G , i.e., the sheaf of derivations of \mathcal{O}_G . By [22, II.4.4.6], we may also view $\operatorname{Lie}(G)$ as the Lie algebra $H^0(G, T_G)^G = \operatorname{Der}^G(\mathcal{O}_G)$ consisting of those global derivations of \mathcal{O}_G that are invariant under the G -action via right multiplication; this induces an isomorphism

$$T_G \cong \mathcal{O}_G \otimes_k \operatorname{Lie}(G).$$

We have $\dim(G) \leq \dim \operatorname{Lie}(G)$ with equality if and only if G is smooth, as follows from Proposition 2.1.12. Also, every homomorphism of algebraic groups $f : G \rightarrow H$ differentiates to a homomorphism of Lie algebras

$$\operatorname{Lie}(f) := df_{e_G} : \operatorname{Lie}(G) \longrightarrow \operatorname{Lie}(H).$$

More generally, every action a of G on a scheme X yields a homomorphism of Lie algebras

$$\operatorname{Lie}(a) : \operatorname{Lie}(G) \longrightarrow H^0(X, T_X) = \operatorname{Der}(\mathcal{O}_X)$$

(see [22, II.4.4]).

When $\operatorname{char}(k) = p > 0$, the p th power of any derivation is a derivation; this equips $\operatorname{Lie}(G) = \operatorname{Der}^G(\mathcal{O}_G)$ with an additional structure of *p -Lie algebra*, also called *restricted Lie algebra* (see [22, II.7.3]). This structure is preserved by the above homomorphisms.

2.4. The neutral component. Recall that a scheme X is étale (over $\operatorname{Spec}(k)$) if and only if its underlying topological space is discrete and the local rings of X are finite separable extensions of k (see e.g. [22, I.4.6.1]). In particular, every étale scheme is locally of finite type. Also, X is étale if and only if the k_s -scheme X_{k_s} is constant; moreover, the assignment $X \mapsto X(k_s)$ yields an equivalence from the category of étale schemes (and morphisms of schemes) to that of discrete topological spaces equipped with a continuous action of the Galois group Γ (and Γ -equivariant maps); see [22, I.4.6.2, I.4.6.4].

Next, let X be a scheme, locally of finite type. By [22, I.4.6.5], there exists an étale scheme $\pi_0(X)$ and a morphism

$$\gamma = \gamma_X : X \longrightarrow \pi_0(X)$$

such that every morphism of schemes $f : X \rightarrow Y$, where Y is étale, factors uniquely through γ . Moreover, γ is faithfully flat and its fibers are exactly the connected components of X . The formation of the *scheme of connected components* $\pi_0(X)$ commutes with field extensions in view of [22, I.4.6.7].

As a consequence, given a morphism of schemes $f : X \rightarrow Y$, where X and Y are locally of finite type, we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \gamma_X \downarrow & & \downarrow \gamma_Y \\ \pi_0(X) & \xrightarrow{\pi_0(f)} & \pi_0(Y), \end{array}$$

where $\pi_0(f)$ is uniquely determined. Applying this construction to the two projections $p_1 : X \times Y \rightarrow X$, $p_2 : X \times Y \rightarrow Y$, we obtain a canonical morphism $\pi_0(X \times Y) \rightarrow \pi_0(X) \times \pi_0(Y)$, which is in fact an isomorphism (see [22, I.4.6.10]). In particular, the formation of the scheme of connected components commutes with finite products.

It follows easily that for any locally algebraic group scheme G , there is a unique group scheme structure on $\pi_0(G)$ such that γ is a homomorphism. Moreover, given an action a of G on a scheme X , locally of finite type, we have a compatible action $\pi_0(a)$ of $\pi_0(G)$ on $\pi_0(X)$.

THEOREM 2.4.1. *Let G be a locally algebraic group and denote by G^0 the connected component of e in G .*

- (1) G^0 is the kernel of $\gamma : G \rightarrow \pi_0(G)$.
- (2) The formation of G^0 commutes with field extensions.
- (3) G^0 is a geometrically irreducible algebraic group.
- (4) The connected components of G are irreducible, of finite type and of the same dimension.

PROOF. (1) This holds as the fibers of γ are the connected components of G .

(2) This follows from the fact that the formation of γ commutes with field extensions.

(3) Consider first the case of an algebraically closed field k . Then the reduced neutral component G_{red}^0 is smooth by Proposition 2.1.12, and hence locally irreducible. Since G_{red}^0 is connected, it is irreducible.

Returning to an arbitrary ground field, G^0 is geometrically irreducible by (2) and the above step. We now show that G^0 is of finite type. Choose a non-empty open subscheme of finite type $U \subseteq G^0$; then U is dense in G^0 . Consider the multiplication map of G^0 , and its pull-back

$$n : U \times U \longrightarrow G^0.$$

We claim that n is faithfully flat.

Indeed, n is flat by Remark 2.2.3. To show that n is surjective, let $g \in G^0(K)$ for some field extension K of k . Then $U_K \cap gi(U_K)$ is non-empty, since G_K^0 is irreducible. Thus, there exists a field extension L of k and $x, y \in G^0(L)$ such that $x = gy^{-1}$. This yields the claim.

By that claim and the quasi-compactness of $U \times U$, we see that G^0 is quasi-compact as well. But G^0 is also locally of finite type; hence it is of finite type.

(4) Let $X \subseteq G$ be a connected component. Since G is locally of finite type, we may choose a closed point $x \in X$; then the residue field $\kappa(x)$ is a finite extension of k . Thus, we may choose a field extension K of $\kappa(x)$, which is finite and stable under $\text{Aut}_k(\overline{\kappa(x)})$. The structure map $\pi : X_K \rightarrow X$ is finite and faithfully flat, hence open and closed; moreover, every point x' of $\pi^{-1}(x)$ is K -rational (as $\kappa(x')$ is a quotient field of $K \otimes_k \kappa(x)$). Thus, the fiber of γ_K at x' is the translate $x'G_K^0$ (since $x'^{-1}\gamma_K^{-1}\gamma_K(x')$ is a connected component of G_K and contains e). As a consequence, $\pi(x'G_K^0)$ is irreducible, open and closed in G , and contains $\pi(x') = x$; so $\pi(x'G_K^0) = X$. This shows that X is irreducible of dimension $\dim(G^0)$. To check that X is of finite type, observe that $X_K = \bigcup_{x' \in \pi^{-1}(x)} x'G_K^0$ is of finite type, and apply descent theory (see [EGA, IV.2.7.1]).

□

With the notation and assumptions of the above theorem, G^0 is called the *neutral component* of G . Note that G is equidimensional of dimension $\dim(G^0)$.

REMARKS 2.4.2. (i) Let G be a locally algebraic group acting on a scheme X , locally of finite type. If k is separably closed, then every connected component of X is stable by G^0 .

(ii) A locally algebraic group G is algebraic if and only if $\pi_0(G)$ is finite.

(iii) By [22, II.5.1.8], the category of étale group schemes is equivalent to that of discrete topological groups equipped with a continuous action of Γ by group automorphisms, via the assignement $G \mapsto G(k_s)$. Under this equivalence, the finite étale group schemes correspond to the (abstract) finite groups equipped with a Γ -action by group automorphisms.

These results reduce the structure of locally algebraic groups to that of algebraic groups; we will concentrate on the latter in the sequel.

2.5. Reduced subschemes. Recall that every scheme X has a largest reduced subscheme X_{red} ; moreover, X_{red} is closed in X and has the same underlying topological space. Every morphism of schemes $f : X \rightarrow Y$ sends X_{red} to Y_{red} .

PROPOSITION 2.5.1. *Let G be a smooth algebraic group acting on a scheme of finite type X .*

- (1) X_{red} is stable by G .
- (2) Let $\eta : \tilde{X} \rightarrow X_{\text{red}}$ denote the normalization. Then there is a unique action of G on \tilde{X} such that η is equivariant.
- (3) When k is separably closed, every irreducible component of X_{red} is stable by G .

PROOF. (1) As G is geometrically reduced, $G \times X_{\text{red}}$ is reduced by [EGA, IV.6.8.5]. Thus, $G \times X_{\text{red}} = (G \times X)_{\text{red}}$ is sent to X_{red} by a .

(2) Likewise, as G is geometrically normal, $G \times \tilde{X}$ is normal by [EGA, IV.6.8.5] again. So the map $\text{id} \times \eta : G \times \tilde{X} \rightarrow G \times X$ is the normalization. This yields a morphism $\tilde{a} : G \times \tilde{X} \rightarrow \tilde{X}$ such that the square

$$(2.5.1) \quad \begin{array}{ccc} G \times \tilde{X} & \xrightarrow{\tilde{a}} & \tilde{X} \\ \text{id} \times \eta \downarrow & & \downarrow \eta \\ G \times X_{\text{red}} & \xrightarrow{a} & X_{\text{red}}, \end{array}$$

commutes, where a denotes the G -action. Since η induces an isomorphism on a dense open subscheme of \tilde{X} , we have $\tilde{a}(e, \tilde{x}) = \tilde{x}$ identically on \tilde{X} . Likewise, $\tilde{a}(g, \tilde{a}(h, \tilde{x})) = \tilde{a}(gh, \tilde{x})$ identically on $G \times G \times \tilde{X}$, i.e., \tilde{a} is an action.

(3) Let Y be an irreducible component of X_{red} . Then the normalization \tilde{Y} is a connected component of \tilde{X} , and hence is stable by G^0 (Remark 2.4.2 (i)). Using the surjectivity of the normalization map $\tilde{Y} \rightarrow Y$ and the commutative square (2.5.1), it follows that Y is stable by G . \square

When the field k is perfect, the product of any two reduced schemes is reduced (see [22, I.2.4.13]). It follows that the natural map $(X \times Y)_{\text{red}} \rightarrow X_{\text{red}} \times Y_{\text{red}}$ is an isomorphism; in particular, the formation of X_{red} commutes with field extensions. This implies easily the following:

PROPOSITION 2.5.2. *Let G be a group scheme over a perfect field k .*

- (1) *Any action of G on a scheme X restricts to an action of G_{red} on X_{red} .*
- (2) *If G is locally algebraic, then G_{red} is the largest smooth subgroup scheme of G .*

Note that G_{red} is not necessarily normal in G , as shown by the following:

EXAMPLE 2.5.3. Consider the \mathbb{G}_m -action on \mathbb{G}_a by multiplication. If $\text{char}(k) = p$, then every subgroup scheme $\alpha_{p^n} = V(x^{p^n}) \subset \mathbb{G}_a$ is normalized by this action (since x^{p^n} is homogeneous), but not centralized (since \mathbb{G}_m acts non-trivially on $\mathcal{O}(\alpha_{p^n}) = k[x]/(x^{p^n})$). Thus, we may form the corresponding semi-direct product $G := \mathbb{G}_m \ltimes \alpha_{p^n}$. Then G is an algebraic group; moreover, $G_{\text{red}} = \mathbb{G}_m$ is not normal in G by Remark 2.2.10 (i).

To obtain a similar example with G finite, just replace \mathbb{G}_m with its subgroup scheme μ_ℓ of ℓ -th roots of unity, where ℓ is prime to p .

We now obtain a structure result for finite group schemes:

PROPOSITION 2.5.4. *Let G be a finite group scheme over a perfect field k . Then the multiplication map induces an isomorphism $G_{\text{red}} \ltimes G^0 \xrightarrow{\cong} G$.*

PROOF. Consider, more generally, a finite scheme X . We claim that the morphism $\gamma : X \rightarrow \pi_0(X)$ restricts to an isomorphism $X_{\text{red}} \cong \pi_0(X)$. To check this, we may assume that X is irreducible; then $X = \text{Spec}(R)$ for some local artinian k -algebra R with residue field K being a finite extension of k . Since k is perfect, K lifts uniquely to a subfield of R , which is clearly the largest subfield of that algebra. Then γ_X is the associated morphism $\text{Spec}(R) \rightarrow \text{Spec}(K)$; this yields our claim.

Returning to our finite group scheme G , we obtain an isomorphism of group schemes $G_{\text{red}} \xrightarrow{\cong} \pi_0(G)$ via γ . This yields in turn a retraction of G to G_{red} with kernel G^0 . So the desired statement follows from Remark 2.2.10 (ii). \square

With the notation and assumptions of the above proposition, G_{red} is a finite étale group scheme, which corresponds to the finite group $G(\bar{k})$ equipped with the action of the Galois group Γ . Also, G^0 is finite and its underlying topological space is just the point e ; such a group scheme is called *infinitesimal*. Examples of infinitesimal group schemes include α_{p^n} and μ_{p^n} in characteristic $p > 0$. When $\text{char}(k) = 0$, every infinitesimal group scheme is trivial by Theorem 2.1.6.

Proposition 2.5.4 can be extended to the setting of algebraic groups over perfect fields, see Corollary 2.8.7. But it fails over any imperfect field, as shown by the following example of a finite group scheme G such that G_{red} is not a subgroup scheme:

EXAMPLE 2.5.5. Let k be an imperfect field, i.e., $\text{char}(k) = p > 0$ and $k \neq k^p$. Choose $a \in k \setminus k^p$ and consider the finite subgroup scheme $G \subset \mathbb{G}_a$ defined as the kernel of the additive polynomial $x^{p^2} - ax^p$. Then $G_{\text{red}} = V(x(x^{p(p-1)} - a))$ is smooth at 0 but not everywhere, since $x^{p(p-1)} - a = (x^{p-1} - a^{1/p})^p$ over k_i . So G_{red} admits no group scheme structure in view of Proposition 2.1.12.

2.6. Torsors.

DEFINITION 2.6.1. Let X be a scheme equipped with an action a of a group scheme G , and $f : X \rightarrow Y$ a G -invariant morphism of schemes.

We say that f is a G -torsor over Y (or a *principal G -bundle over Y*) if it satisfies the following conditions:

- (1) f is faithfully flat and quasi-compact.
- (2) The square

$$(2.6.1) \quad \begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ p_2 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian.

REMARKS 2.6.2. (i) The condition (2) may be rephrased as follows: for any scheme S and any points $x, y \in X(S)$, we have $f(x) = f(y)$ if and only if there exists $g \in G(S)$ such that $y = g \cdot x$; moreover, such a g is unique. This is the scheme-theoretic version of the notion of principal bundle in topology.

(ii) Consider a group scheme G and a scheme Y . Let G act on $G \times Y$ via left multiplication on itself. Then the projection $p_2 : G \times Y \rightarrow Y$ is a G -torsor, called the *trivial G -torsor over Y* .

(iii) One easily checks that a G -torsor $f : X \rightarrow Y$ is trivial if and only if f has a section. In particular, a G -torsor X over $\text{Spec}(k)$ is trivial if and only if X has a k -rational point. When G is algebraic, this holds of course if k is algebraically closed, but generally not over an arbitrary field k . Assume for instance that k contains some element t which is not a square, and consider the scheme $X := V(x^2 - t) \subset \mathbb{A}^1$. Then X is normalized by the action of μ_2 on \mathbb{A}^1 via multiplication; this yields a non-trivial μ_2 -torsor over $\text{Spec}(k)$.

(iv) For any G -torsor $f : X \rightarrow Y$, the topology of Y is the quotient of the topology of X by the equivalence relation defined by f (see [EGA, IV.2.3.12]). As a consequence, the assignment $Z \mapsto f^{-1}(Z)$ yields a bijection from the open (resp. closed) subschemes of Y to the open (resp. closed) G -stable subschemes of X .

DEFINITION 2.6.3. Let G be a group scheme acting on a scheme X . A morphism of schemes $f : X \rightarrow Y$ is a *categorical quotient* of X by G , if f is G -invariant and every G -invariant morphism of schemes $\varphi : X \rightarrow Z$ factors uniquely through f .

In view of its universal property, a categorical quotient is unique up to unique isomorphism.

PROPOSITION 2.6.4. *Let G be an algebraic group, and $f : X \rightarrow Y$ be a G -torsor. Then f is a categorical quotient by G .*

PROOF. Consider a G -invariant morphism $\varphi : X \rightarrow Z$. Then $\varphi^{-1}(U)$ is an open G -stable subscheme for any open subscheme U of Z . Thus, f restricts to a G -torsor $f_U : \varphi^{-1}(U) \rightarrow V$ for some open subscheme $V = V(U)$ of Y . To show that φ factors uniquely through f , it suffices to show that $\varphi_U : \varphi^{-1}(U) \rightarrow U$ factors uniquely through f_U for any affine U . Thus, we may assume that Z is affine. Then φ corresponds to a G -invariant homomorphism $\mathcal{O}(Z) \rightarrow \mathcal{O}(X)$, i.e., to a homomorphism $\mathcal{O}(Z) \rightarrow \mathcal{O}(X)^G$ (the subalgebra of G -invariants in $\mathcal{O}(X)$). So it suffices to check that the map

$$f^\# : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)^G$$

is an isomorphism.

Since f is faithfully flat, it suffices in turn to show that the natural map

$$\mathcal{O}_X = f^*(\mathcal{O}_Y) \rightarrow f^*(f_*(\mathcal{O}_X)^G)$$

is an isomorphism. We have canonical isomorphisms

$$f^*(f_*(\mathcal{O}_X)) \cong p_{2*}(a^*(\mathcal{O}_X)) \cong p_{2*}(\mathcal{O}_{G \times X}) \cong \mathcal{O}(G) \otimes_k \mathcal{O}_X,$$

where the first isomorphism follows from the cartesian square (2.6.1) and the faithful flatness of f , and the third isomorphism follows from Lemma 2.3.3. Moreover, the composition of these isomorphisms identifies the G -action on $f^*(f_*(\mathcal{O}_X))$ with that on $\mathcal{O}(G) \otimes_k \mathcal{O}_X$ via left multiplication on $\mathcal{O}(G)$. Thus, taking G -invariants yields the desired isomorphism. \square

PROPOSITION 2.6.5. *Let $f : X \rightarrow Y$ be a G -torsor.*

- (1) *The morphism f is finite (resp. affine, proper, of finite presentation) if and only if so is the scheme G .*
- (2) *When Y is of finite type, f is smooth if and only if G is smooth.*

PROOF. (1) This follows from the cartesian diagram (2.6.1) together with descent theory (see [EGA, IV.2.7.1]).

Likewise, (2) follows from [EGA, IV.6.8.3]. \square

REMARKS 2.6.6. (i) As a consequence of the above proposition, every torsor $f : X \rightarrow Y$ under an algebraic group G is of finite presentation. In particular, f is surjective on \bar{k} -rational points, i.e., the induced map $X(\bar{k}) \rightarrow Y(\bar{k})$ is surjective. But f is generally not surjective on S -points for an arbitrary scheme S (already for $S = \text{Spec}(k)$). Still, f satisfies the following weaker version of surjectivity:

For any scheme S and any point $y \in Y(S)$, there exists a faithfully flat morphism of finite presentation $\varphi : S' \rightarrow S$ and a point $x \in X(S')$ such that $f(x) = y$.

Indeed, viewing y as a morphism $S \rightarrow Y$, we may take $S' := X \times_Y S$, $\varphi := p_2$ and $x := p_1$.

(ii) Consider a G -scheme X , a G -invariant morphism of schemes $f : X \rightarrow Y$ and a faithfully flat quasi-compact morphism of schemes $v : Y' \rightarrow Y$. Form the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ u \downarrow & & \downarrow v \\ X & \xrightarrow{f} & Y. \end{array}$$

Then there is a unique action of G on X' such that u is equivariant and f' is invariant. Moreover, f is a G -torsor if and only if f' is a G -torsor. Indeed, this follows again from descent theory, more specifically from [EGA, IV.2.6.4] for the condition (2), and [EGA, IV.2.7.1] for (3).

(iii) In the above setting, f is a G -torsor if and only if the base change f_K is a G_K -torsor for some field extension K of k .

(iv) Consider two G -torsors $f : X \rightarrow Y$, $f' : X' \rightarrow Y$ and a G -equivariant morphism $\varphi : X \rightarrow X'$ of schemes over Y . Then φ is an isomorphism: to check this, one may reduce by descent to the case where f and f' are trivial. Then φ is identified with an endomorphism of the trivial torsor. But every such endomorphism is of the form $(g, y) \mapsto (g\psi(y), y)$ for a unique morphism $\psi : Y \rightarrow G$, and hence is an automorphism with inverse $(g, y) \mapsto (g\psi(y)^{-1}, y)$.

EXAMPLE 2.6.7. Let G be a locally algebraic group. Then the homomorphism $\gamma : G \rightarrow \pi_0(G)$ is a G^0 -torsor.

Indeed, recall from §2.4 that the formation of γ commutes with field extensions. By Remark 2.6.6 (iii), we may thus assume k algebraically closed. Then the finite étale scheme $\pi_0(G)$ just consists of finitely many k -rational points, say x_1, \dots, x_n , and the fiber F_i of γ at x_i contains a k -rational point, say g_i . Recall that F_i is a connected component of G ; thus, the translate $g_i^{-1}F_i$ is a connected component of G through e , and hence equals G^0 . It follows that G is the disjoint union of the translates $g_i G^0$, which are the fibers of γ ; this yields our assertion.

2.7. Homogeneous spaces and quotients.

PROPOSITION 2.7.1. *Let $f : G \rightarrow H$ be a homomorphism of algebraic groups.*

- (1) *The image $f(G)$ is closed in H .*
- (2) *f is a closed immersion if and only if its kernel is trivial.*

PROOF. As in the proof of Proposition 2.2.7, we may assume that k is algebraically closed.

(1) Consider the action a of G on H given by $g \cdot h := f(g)h$. By Proposition 2.2.7 again, there exists $h \in H(k)$ such that the image of the orbit map a_h is closed. But $a_h(G) = a_e(G)h$ and hence $a_e(G) = f(G)$ is closed.

(2) Clearly, $\text{Ker}(f)$ is trivial if f is a closed immersion. Conversely, if $\text{Ker}(f)$ is trivial then the fiber of f at any point $x \in X$ consists of that point; in particular, f is quasi-finite. By Zariski's Main Theorem (see [EGA, IV.8.12.6]), f factors as an immersion followed by a finite morphism. As a consequence, there exists a dense open subscheme U of $f(G)$ such that the restriction $f^{-1}(U) \rightarrow U$ is finite. Since the translates of $U_{\bar{k}}$ by $G(\bar{k})$ cover $f(G_{\bar{k}})$, it follows that $f_{\bar{k}}$ is finite; hence f is finite as well. Choose an open affine subscheme V of $f(G)$; then so is $f^{-1}(V)$, and $\mathcal{O}(f^{-1}(V))$ is a finite module over $\mathcal{O}(V)$ via $f^\#$. Moreover, the natural map

$$\mathcal{O}(V)/\mathfrak{m} \longrightarrow \mathcal{O}(f^{-1}(V))/\mathfrak{m}\mathcal{O}(f^{-1}(V)) = \mathcal{O}(f^{-1}(\text{Spec } \mathcal{O}(V)/\mathfrak{m}))$$

is an isomorphism for any maximal ideal \mathfrak{m} of $\mathcal{O}(V)$. By Nakayama's lemma, it follows that $f^\#$ is surjective; this yields the assertion. \square

As a consequence of the above proposition, *every subgroup scheme of an algebraic group is closed.*

We now come to an important existence result:

THEOREM 2.7.2. *Let G be an algebraic group and $H \subseteq G$ a subgroup scheme.*

- (1) *There exists a G -scheme G/H equipped with a G -equivariant morphism*

$$q : G \longrightarrow G/H,$$

which is an H -torsor for the action of H on G by right multiplication.

- (2) *The scheme G/H is of finite type. It is smooth if G is smooth.*
- (3) *If H is normal in G , then G/H has a unique structure of algebraic group such that q is a homomorphism.*

PROOF. See [SGA3, VIA.3.2]. \square

REMARKS 2.7.3. (i) With the notation and assumptions of the above theorem, q is the categorical quotient of G by H , in view of Proposition 2.6.4. In particular, q is unique up to unique isomorphism; it is called the *quotient morphism*. The

homogeneous space G/H is equipped with a k -rational point $x := q(e)$, the *base point*. The stabilizer $C_G(x)$ equals H , since it is the fiber of q at x .

(ii) By the universal property of categorical quotients, the homomorphism of algebras $q^\# : \mathcal{O}(G/H) \rightarrow \mathcal{O}(G)^H$ is an isomorphism.

(iii) The morphism q is faithfully flat and lies in a cartesian diagram

$$\begin{array}{ccc} G \times H & \xrightarrow{n} & G \\ p_1 \downarrow & & \downarrow q \\ G & \xrightarrow{q} & G/H, \end{array}$$

where n denotes the restriction of the multiplication $m : G \times G \rightarrow G$. Also, q is of finite presentation in view of Proposition 2.6.5.

(iv) Since q is flat and G, H are equidimensional, we see that G/H is equidimensional of dimension $\dim(G) - \dim(H)$.

(v) We have $(G/H)(\bar{k}) = G(\bar{k})/H(\bar{k})$ as follows e.g. from Remark 2.6.6 (i). In particular, if k is perfect (so that G_{red} is a subgroup scheme of G), then the scheme G/G_{red} has a unique \bar{k} -rational point. Since that scheme is of finite type, it is finite and local; its base point is its unique k -rational point.

Next, we obtain two further factorization properties of quotient morphisms:

PROPOSITION 2.7.4. *Let $f : G \rightarrow H$ be a homomorphism of algebraic groups, $N := \text{Ker}(f)$ and $q : G \rightarrow G/N$ the quotient homomorphism. Then there is a unique homomorphism $\iota : G/N \rightarrow H$ such that the triangle*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ q \downarrow & \nearrow \iota & \\ G/N & & \end{array}$$

commutes. Moreover, ι is an isomorphism onto a subgroup scheme of H .

PROOF. Clearly, f is N -invariant; thus, it factors through a unique morphism $\iota : G/N \rightarrow H$ by Theorem 2.7.2. We check that ι is a homomorphism: let S be a scheme and $x, y \in (G/N)(S)$. By Remark 2.6.6, there exist morphisms of schemes $\varphi : T \rightarrow S$, $\psi : U \rightarrow S$ and points $x_T \in G(T)$, $y_U \in G(U)$ such that $q(x_T) = x$, $q(y_U) = y$. Using the fibered product $S' := T \times_S U$, we thus obtain a morphism $f : S' \rightarrow S$ and points $x', y' \in G(S')$ such that $q(x') = x$, $q(y') = y$; then $q(x'y') = xy$. Since $f(x'y') = f(x')f(y')$, we have $\iota(xy) = \iota(x)\iota(y)$. One may check likewise that $\text{Ker}(\iota)$ is trivial. Thus, ι is a closed immersion; hence its image is a subgroup scheme in view of Proposition 2.7.1. \square

PROPOSITION 2.7.5. *Let G be an algebraic group, X a G -scheme of finite type and $x \in X(k)$. Then the orbit map $a_x : G \rightarrow X$, $g \mapsto g \cdot x$ factors through a unique immersion $j_x : G/C_G(x) \rightarrow X$.*

PROOF. See [22, III.3.5.2] or [SGA3, V.10.1.2]. \square

With the above notation and assumptions, we may define the *orbit* of x as the locally closed subscheme of X corresponding to the immersion j_x .

2.8. Exact sequences, isomorphism theorems.

DEFINITION 2.8.1. Let $j : N \rightarrow G$ and $q : G \rightarrow Q$ be homomorphisms of group schemes. We have an *exact sequence*

$$(2.8.1) \quad 1 \longrightarrow N \xrightarrow{j} G \xrightarrow{q} Q \longrightarrow 1$$

if the following conditions hold:

- (1) j induces an isomorphism of N with $\text{Ker}(q)$.
- (2) For any scheme S and any $y \in Q(S)$, there exists a faithfully flat morphism $f : S' \rightarrow S$ of finite presentation and $x \in G(S')$ such that $q(x) = y$.

Then G is called an *extension* of Q by N .

We say that q is an *isogeny* if N is finite.

REMARKS 2.8.2. (i) The condition (1) holds if and only if the sequence of groups

$$1 \longrightarrow N(S) \xrightarrow{j(S)} G(S) \xrightarrow{q(S)} Q(S)$$

is exact for any scheme S .

(ii) The condition (2) holds whenever q is faithfully flat of finite presentation, as already noted in Remark 2.6.6(i).

(iii) As for exact sequences of abstract groups, one may define the push-forward of the exact sequence (2.8.1) under any homomorphism $N \rightarrow N'$, and the pull-back under any homomorphism $Q' \rightarrow Q$. Also, exactness is preserved under field extensions.

Next, consider an algebraic group G and a normal subgroup scheme N ; then we have an exact sequence

$$(2.8.2) \quad 1 \longrightarrow N \longrightarrow G \xrightarrow{q} G/N \longrightarrow 1$$

by Theorem 2.7.2 and the above remarks. Conversely, given an exact sequence (2.8.1) of algebraic groups, j is a closed immersion and q factors through a closed immersion $\iota : G/N \rightarrow Q$ by Proposition 2.7.4. Since q is surjective, ι is an isomorphism; this identifies the exact sequences (2.8.1) and (2.8.2).

As another consequence of Proposition 2.7.4, *the category of commutative algebraic groups is abelian*. Moreover, the above notion of exact sequence coincides with the categorical notion. In this setting, the set of isomorphism classes of extensions of Q by N has a natural structure of commutative group, that we denote by $\text{Ext}^1(Q, N)$.

We now extend some classical isomorphism theorems for abstract groups to the setting of group schemes, in a series of propositions:

PROPOSITION 2.8.3. *Let G be an algebraic group and $N \trianglelefteq G$ a normal subgroup scheme with quotient $q : G \rightarrow G/N$. Then the assignment $H \mapsto H/N$ yields a bijective correspondence between the subgroup schemes of G containing N and the subgroup schemes of G/N , with inverse the pull-back. Under this correspondence, the normal subgroup schemes of G containing N correspond to the normal subgroup schemes of G/N .*

PROOF. See [SGA3, VIA.5.3.1]. □

PROPOSITION 2.8.4. *Let G be an algebraic group and $N \subseteq H \subseteq G$ subgroup schemes with quotient maps $q_N : G \rightarrow G/N$, $q_H : G \rightarrow G/H$.*

- (1) *There exists a unique morphism $f : G/N \rightarrow G/H$ such that the triangle*

$$\begin{array}{ccc} G & \xrightarrow{q_H} & G/H \\ q_N \downarrow & \nearrow f & \\ G/N & & \end{array}$$

commutes. Moreover, f is G -equivariant and faithfully flat of finite presentation. The fiber of f at the base point of G/H is the homogeneous space H/N .

- (2) *If N is normal in H , then the action of H on G by right multiplication factors through an action of H/N on G/N that centralizes the action of G . Moreover, f is an H/N -torsor.*
- (3) *If H and N are normal in G , then we have an exact sequence*

$$1 \longrightarrow H/N \longrightarrow G/N \xrightarrow{f} G/H \longrightarrow 1.$$

PROOF. (1) The existence of f follows from the fact that q_N is a categorical quotient. To show that f is equivariant, let S be a scheme, $g \in G(S)$ and $y \in (G/N)(S)$. Then there exists a morphism $S' \rightarrow S$ and $y' \in G(S')$ such that $q_N(y') = y$. So

$$f(g \cdot y) = f(g \cdot q_N(y')) = (f \circ q_N)(gy') = q_H(gy') = g \cdot q_H(y') = g \cdot y.$$

One checks similarly that the fiber of f at the base point x equals H/N .

Next, note that the multiplication map $n : G \times H \rightarrow H$ yields a morphism $r : G \times H/N \rightarrow G/N$. We claim that the square

$$(2.8.3) \quad \begin{array}{ccc} G \times H/N & \xrightarrow{r} & G/N \\ p_1 \downarrow & & \downarrow f \\ G & \xrightarrow{q_H} & G/H \end{array}$$

is cartesian. The commutativity of this square follows readily from the equivariance of the involved morphisms. Let S be a scheme and $g \in G(S)$, $y \in (G/N)(S)$. Then $q_H(g) = f(y)$ if and only if $f(g^{-1} \cdot y) = q_H(e) = f(x)$, i.e., $g^{-1}y \in (H/N)(S)$. It follows that the map $G \times H/N \rightarrow G \times_{G/H} G/N$ is bijective on S -points; this yields the claim.

Since q_H and p_2 are faithfully flat of finite presentation, the same holds for f in view of the cartesian square (2.8.3).

(2) The existence of the action $G/N \times H/N \rightarrow G/N$ follows similarly from the universal property of the quotient $G \times H \rightarrow G/N \times H/N$. One may check by lifting points as in the proof of (1) that this action centralizes the G -action. Finally, f is a G -torsor in view of the cartesian square (2.8.3) again.

(3) This follows readily from (1) together with Proposition 2.7.4 (or argue by lifting points to check that f is a homomorphism). \square

PROPOSITION 2.8.5. *Let G be an algebraic group, $H \subseteq G$ a subgroup scheme and $N \trianglelefteq G$ a normal subgroup scheme. Consider the semi-direct product $H \ltimes N$, where H acts on N by conjugation.*

- (1) *The map*

$$f : H \ltimes N \longrightarrow G, \quad (x, y) \longmapsto xy$$

is a homomorphism with kernel $H \cap N$ identified with a subgroup scheme of $H \ltimes N$ via $x \mapsto (x^{-1}, x)$.

- (2) The image $H \cdot N$ of f is the smallest subgroup scheme of G containing H and N .
- (3) The natural maps $H/H \cap N \rightarrow H \cdot N/N$ and $N/H \cap N \rightarrow H \cdot N/H$ are isomorphisms.
- (4) If H is normal in G , then $H \cdot N$ is normal in G as well.

PROOF. The assertions (1) and (2) are easily checked.

(3) We have a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & H \ltimes N/N \\ \downarrow & & \downarrow \\ H/H \cap N & \longrightarrow & H \cdot N/N, \end{array}$$

where the top horizontal arrow is an isomorphism and the vertical arrows are $H \cap N$ -torsors. This yields the first isomorphism by using Proposition 2.6.4. The second isomorphism is obtained similarly.

(4) This may be checked as in the proof of Proposition 2.7.4. \square

We also record a useful observation:

LEMMA 2.8.6. *Keep the notation and assumptions of the above proposition. If $G = H \cdot N$, then $G(\bar{k}) = H(\bar{k})N(\bar{k})$. The converse holds when G/N is smooth.*

PROOF. The first assertion follows e.g. from Remark 2.6.6 (i).

For the converse, consider the quotient homomorphism $q : G \rightarrow G/N$: it restricts to a homomorphism $H \rightarrow G/N$ with kernel $H \cap N$, and hence factors through a closed immersion $i : H/H \cap N \rightarrow G/N$ by Proposition 2.7.4. Since $G(\bar{k}) = H(\bar{k})N(\bar{k})$, we see that i is surjective on \bar{k} -rational points. As G/N is smooth, i must be an isomorphism. Thus, $H \cdot N/N = G/N$. By Proposition 2.8.3, we conclude that $H \cdot N = G$. \square

We may now obtain the promised generalization of the structure of finite group schemes over a perfect field (Proposition 2.5.4):

COROLLARY 2.8.7. *Let G be an algebraic group over a perfect field k .*

- (1) $G = G_{\text{red}} \cdot G^0$.
- (2) $G_{\text{red}} \cap G^0 = G_{\text{red}}^0$ is the smallest subgroup scheme H of G such that G/H is finite.

PROOF. (1) This follows from Lemma 2.8.6, since $G/G^0 \cong \pi_0(G)$ is smooth and $G(\bar{k}) = G_{\text{red}}(\bar{k})$.

(2) Let $H \subseteq G$ be a subgroup scheme. Since G/H is of finite type, the finiteness of G/H is equivalent to the finiteness of $(G/H)(\bar{k}) = G(\bar{k})/H(\bar{k}) = G(\bar{k})/H_{\text{red}}(\bar{k})$. Thus, G/H_{red} is finite if and only if so is G/H . Likewise, using the finiteness of H/H^0 , one may check that G/H is finite if and only if so is G/H_{red}^0 . Under these conditions, the homogeneous space $G_{\text{red}}^0/H_{\text{red}}^0$ is finite as well; since it is also smooth and connected, it follows that $G_{\text{red}}^0 = H_{\text{red}}^0$, i.e., $G_{\text{red}}^0 \subseteq H$.

To complete the proof, it suffices to check that G/G_{red}^0 is finite, or equivalently that $G(\bar{k})/G^0(\bar{k})$ is finite. But this follows from the finiteness of G/G^0 . \square

DEFINITION 2.8.8. An exact sequence of group schemes (2.8.1) is called *split* if $q : G \rightarrow Q$ has a section which is a homomorphism.

Any such section s yields an endomorphism $r := s \circ q$ of the group scheme G with kernel N ; moreover, r may be viewed as a retraction of G to the image of s , isomorphic to H . By Remark 2.2.10 (ii), this identifies (2.8.1) with the exact sequence

$$1 \longrightarrow N \xrightarrow{i} H \ltimes N \xrightarrow{r} H \longrightarrow 1.$$

2.9. The relative Frobenius morphism. Throughout this subsection, we assume that the ground field k has characteristic $p > 0$.

Let X be a k -scheme and n a positive integer. The n th *absolute Frobenius morphism* of X is the endomorphism

$$F_X^n : X \longrightarrow X$$

which is the identity on the underlying topological space and such that the homomorphism of sheaves of algebras $(F_X^n)^\# : \mathcal{O}_X \rightarrow (F_X^n)_*(\mathcal{O}_X) = \mathcal{O}_X$ is the p^n th power map, $f \mapsto f^{p^n}$.

Clearly, every morphism of k -schemes $f : X \rightarrow Y$ lies in a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ F_X^n \downarrow & & \downarrow F_Y^n \\ X & \xrightarrow{f} & Y. \end{array}$$

Note that F_X^n is generally not a morphism of k -schemes, since the p^n th power map is generally not k -linear. To address this, define a k -scheme $X^{(n)}$ by the cartesian square

$$\begin{array}{ccc} X^{(n)} & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \xrightarrow{F_k^n} & \mathrm{Spec}(k), \end{array}$$

where π denotes the structure map and $F_k^n := F_{\mathrm{Spec}(k)}^n$ corresponds to the p^n th power map of k . Then F_X^n factors through a unique morphism of k -schemes

$$F_{X/k}^n : X \longrightarrow X^{(n)},$$

the n th *relative Frobenius morphism*. Equivalently, the above cartesian square extends to a commutative diagram

$$\begin{array}{ccccc} X & & & & \\ F_{X/k}^n \downarrow & \searrow F_X^n & & & \\ X^{(n)} & \longrightarrow & X & & \\ \downarrow & & \downarrow \pi & & \\ \mathrm{Spec}(k) & \xrightarrow{F_k^n} & \mathrm{Spec}(k). & & \end{array}$$

The underlying topological space of $X^{(n)}$ is X again, and the structure sheaf is given by

$$\mathcal{O}_{X^{(n)}}(U) = \mathcal{O}_X(U) \otimes_{F^n k} k$$

for any open subset $U \subseteq X$, where the right-hand side denotes the tensor product of $\mathcal{O}_X(U)$ and k over k acting on $\mathcal{O}_X(U)$ via scalar multiplication, and on k via the p^n th power map. Thus, we have in $\mathcal{O}_X(U) \otimes_{F^n} k$

$$tf \otimes u = f \otimes t^{p^n} u$$

for any $f \in \mathcal{O}_X(U)$ and $t, u \in k$. The k -algebra structure on $\mathcal{O}_X(U) \otimes_{F^n} k$ is defined by

$$t(f \otimes u) = f \otimes tu$$

for any such f, t and u . The morphism $F_{X/k}^n$ is again the identity on the underlying topological spaces; the associated homomorphism of sheaves of algebras is the map

$$(2.9.1) \quad (F_{X/k}^n)^\# : \mathcal{O}_X(U) \otimes_{F^n} k \longrightarrow \mathcal{O}_X(U), \quad f \otimes t \longmapsto tf^{p^n}.$$

Using this description, one readily checks that *the formation of the n th relative Frobenius morphism commutes with field extensions*. Moreover, for any positive integers m, n , we have an isomorphism of schemes

$$(X^{(m)})^{(n)} \cong X^{(m+n)}$$

that identifies the composition $F_{X^{(m)}/k}^n \circ F_{X/k}^m$ with $F_{X/k}^{m+n}$. In particular, $F_{X/k}^n$ may be seen as the n th iterate of the relative Frobenius morphism $F_{X/k}$.

Also, note that *the formation of $F_{X/k}^n$ is compatible with closed subschemes and commutes with finite products*. Specifically, any morphism of k -schemes $f : X \rightarrow Y$ induces a morphism of k -schemes $f^{(n)} : X^{(n)} \rightarrow Y^{(n)}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ F_{X/k}^n \downarrow & & \downarrow F_{Y/k}^n \\ X^{(n)} & \xrightarrow{f^{(n)}} & Y^{(n)} \end{array}$$

commutes. If f is a closed immersion, then so is $f^{(n)}$. Moreover, for any two schemes X, Y , the map

$$p_1^{(n)} \times p_2^{(n)} : (X \times Y)^{(n)} \longrightarrow X^{(n)} \times Y^{(n)}$$

is an isomorphism and the triangle

$$\begin{array}{ccc} X \times Y & \xrightarrow{F_{X \times Y/k}^n} & (X \times Y)^{(n)} \\ & \searrow F_{X/k}^n \times F_{Y/k}^n & \downarrow \cong \\ & & X^{(n)} \times Y^{(n)} \end{array}$$

commutes.

We now record some geometric properties of the relative Frobenius morphism:

LEMMA 2.9.1. *Let X be a scheme of finite type and n a positive integer.*

- (1) *The morphism $F_{X/k}^n$ is finite and purely inseparable.*
- (2) *The scheme-theoretic image of $F_{X/k}^n$ is geometrically reduced for $n \gg 0$.*

PROOF. (1) Since $F_{X/k}^n$ is the identity on the underlying topological spaces, we may assume that X is affine. Let $R := \mathcal{O}(X)$, then the image of the homomorphism $(F_{X/k}^n)^\# : R \otimes_{F^n} k \rightarrow R$ is the k -subalgebra kR^{p^n} generated by the p^n th powers.

Thus, $F_{X/k}^n$ is integral, and hence finite since R is of finite type. Also, $F_{X/k}^n$ is clearly purely inseparable.

(2) Let $I \subset R$ denote the ideal consisting of nilpotent elements. Since the algebra R is of finite type, there exists a positive integer n_0 such that $f^n = 0$ for all $f \in I$ and all $n \geq n_0$. Choose n_1 such that $p^{n_1} \geq n_0$, then $(F_{X/k}^n)^\#$ sends I to 0 for any $n \geq n_1$. Thus, the image of $F_{X/k}^n$ is reduced for $n \gg 0$. Since the formation of $F_{X/k}^n$ commutes with field extensions, this completes the proof. \square

PROPOSITION 2.9.2. *Let G be a k -group scheme.*

- (1) *There is a unique structure of k -group scheme on $G^{(n)}$ such that $F_{G/k}^n$ is a homomorphism.*
- (2) *If G is algebraic, then $\text{Ker}(F_{G/k}^n)$ is infinitesimal. Moreover, $G/\text{Ker}(F_{G/k}^n)$ is smooth for $n \gg 0$.*

PROOF. (1) This follows from the fact that the formation of the relative Frobenius morphism commutes with finite products.

(2) This is a consequence of the above lemma together with Proposition 2.1.12. \square

Notes and references.

Most of the notions and results presented in this section can be found in [22] and [SGA3] in a much greater generality. We provide some specific references:

Proposition 2.1.12 is taken from [SGA3, VIA.1.3.1]; Proposition 2.2.7 follows from results in [22, II.5.3]; Lemma 2.3.3 is a special case of [22, I.2.2.6]; Theorem 2.4.1 follows from [22, II.5.1.1, II.5.1.8]; Proposition 2.5.4 holds more generally for locally algebraic groups, see [22, II.2.2.4]; Example 2.5.5 is in [SGA3, VIA.1.3.2].

Our definition of torsors is somewhat ad hoc: what we call G -torsors over Y should be called G_Y -torsors, where G_Y denotes the group scheme $p_2 : G \times Y \rightarrow Y$ (see [22, III.4.1] for general notions and results on torsors).

Proposition 2.6.4 is a special case of a result of Mumford, see [42, Prop. 0.1]; Proposition 2.7.1 is a consequence of [22, II.5.5.1]; Proposition 2.7.4 is a special case of [SGA3, VIA.5.4.1].

Theorem 2.7.2 (on the existence of homogeneous spaces) is a deep result, since no direct construction of these spaces is known in this generality. In the setting of affine algebraic groups, homogeneous spaces may be constructed by a method of Chevalley; this is developed in [22, III.3.5].

Propositions 2.8.4 and 2.8.5 are closely related to results in [SGA3, VIA.5.3]. We have provided additional details to be used later.

Proposition 2.9.2 (2) holds more generally for locally algebraic groups, see [SGA3, VII.8.3].

Many interesting extensions of algebraic groups are not split, but quite a few of them turn out to be *quasi-split*, i.e., split after pull-back by some isogeny. For example, the extension

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$

is quasi-split for any algebraic group G (see [8, Lem. 5.11] when G is smooth and k is algebraically closed of characteristic 0; the general case follows from [13, Thm. 1.1]). Further instances of quasi-split extensions will be obtained in Theorems 4.2.5, 5.3.1

and 5.6.3 below. On the other hand, the group G of upper triangular unipotent 3×3 matrices lies in an extension

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow \mathbb{G}_a^2 \longrightarrow 1,$$

which is not quasi-split. It would be interesting to determine those classes of algebraic groups that yield quasi-split extensions.

3. Proof of Theorem 1

3.1. Affine algebraic groups. In this subsection, we obtain several criteria for an algebraic group to be affine, which will be used throughout the sequel. We begin with a classical result:

PROPOSITION 3.1.1. *Every affine algebraic group is linear.*

PROOF. Let G be an affine algebraic group. By Proposition 2.3.5, there exist a finite-dimensional G -module V and a closed G -equivariant immersion $\iota : G \rightarrow V$, where G acts on itself by left multiplication. Since the latter action is faithful, the G -action on V is faithful as well. In other words, the corresponding homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$ has a trivial kernel. By Proposition 2.7.1, it follows that ρ is a closed immersion. \square

Next, we relate the affineness of algebraic groups with that of subgroup schemes and quotients:

PROPOSITION 3.1.2. *Let H be a subgroup scheme of an algebraic group G .*

- (1) *If H and G/H are both affine, then G is affine as well.*
- (2) *If G is affine, then H is affine. If in addition $H \trianglelefteq G$, then G/H is affine.*

PROOF. (1) Since H is affine, the quotient morphism $q : G \rightarrow G/H$ is affine as well, in view of Proposition 2.6.5 and Theorem 2.7.2 (3). This yields the statement.

(2) The first assertion follows from the closedness of H in G (Proposition 2.7.1). The second assertion is proved in [22, III.3.7.3], see also [SGA3, VIB.11.7]. \square

REMARK 3.1.3. With the notation and assumptions of the above proposition, G is smooth (resp. proper, finite) if H and G/H are both smooth (resp. proper, finite), as follows from the same argument. Also, G is connected if H and G/H are both connected; since all these schemes have k -rational points, this is equivalent to geometric connectedness.

The above proposition yields that every algebraic group has an “affine radical”:

LEMMA 3.1.4. *Let G be an algebraic group.*

- (1) *G has a largest smooth connected normal affine subgroup scheme, $L(G)$.*
- (2) *$L(G/L(G))$ is trivial.*
- (3) *The formation of $L(G)$ commutes with separable algebraic field extensions.*

PROOF. (1) Let L_1, L_2 be two smooth connected normal affine subgroup schemes of G . Then the product $L_1 \cdot L_2 \subseteq G$ is a normal subgroup scheme by Proposition 2.8.5. Since $L_1 \cdot L_2$ is a quotient of $L_1 \times L_2$, it is smooth and connected. Also, by using the isomorphism $L_1 \cdot L_2/L_1 \cong L_2/L_1 \cap L_2$ together with Proposition 3.1.2, we see that $L_1 \cdot L_2$ is affine.

Next, take L_1 as above and of maximal dimension. Then $\dim(L_1 \cdot L_2/L_1) = 0$ by Proposition 2.7.4. Since $L_1 \cdot L_2/L_1$ is smooth and connected, it must be trivial. It follows that $L_2 \subseteq L_1$; this proves the assertion.

(2) Denote by $M \subseteq G$ the pull-back of $L(G/L(G))$ under the quotient map $G \rightarrow G/L(G)$. By Proposition 2.8.3, M is a normal subgroup scheme of G containing $L(G)$. Moreover, M is affine, smooth and connected, since so are $L(G)$ and $M/L(G)$. Thus, $M = L(G)$; this yields the assertion by Proposition 2.8.3 again.

(3) This follows from a classical argument of Galois descent, see [53, V.22]. More specifically, it suffices to check that the formation of $L(G)$ commutes with Galois extensions. Let K be such an extension of k , and \mathcal{G} the Galois group. Then \mathcal{G} acts on $G_K = G \times \text{Spec}(K)$ via its action on K . Let $L' := L(G_K)$; then for any $\gamma \in \mathcal{G}$, the image $\gamma(L')$ is also a smooth connected affine normal K -subgroup scheme of G_K . Thus, $\gamma(L') \subseteq L'$. Since this also holds for γ^{-1} , we obtain $\gamma(L') = L'$. As G_K is covered by \mathcal{G} -stable affine open subschemes, it follows (by arguing as in [53, V.20]) that there exists a unique subscheme $M \subseteq G$ such that $L' = M_K$. Then M is again a smooth connected affine normal subgroup scheme of G , and hence $M \subseteq L(G)$. On the other hand, we clearly have $L(G)_K \subseteq L'$; we conclude that $M = L(G)$. \square

REMARK 3.1.5. In fact, the formation of L commutes with separable field extensions that are not necessarily algebraic. This can be shown by adapting the proof of [20, 1.1.9], which asserts that the formation of the unipotent radical commutes with all separable field extensions. That proof involves methods of group schemes over rings, which go beyond the scope of this text.

Our final criterion for affineness is of geometric origin:

PROPOSITION 3.1.6. *Let $a : G \times X \rightarrow X$ be an action of an algebraic group on an irreducible locally noetherian scheme and let $x \in X(k)$. Then the quotient group scheme $C_G(x)/\text{Ker}(a)$ is affine.*

PROOF. We may replace G with $C_G(x)$, and hence assume that G fixes x . Consider the n th infinitesimal neighborhoods, $x_{(n)} := \text{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1})$, where n runs over the positive integers; these form an increasing sequence of finite subschemes of X supported at x . As seen in Example 2.3.6, each $x_{(n)}$ is stabilized by G ; this yields a linear representation ρ_n of G in $\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1} =: V_n$, a finite-dimensional vector space. Denote by N_n the kernel of ρ_n ; then N_n contains $\text{Ker}(a)$. As ρ_n is a quotient of ρ_{n+1} , we have $N_{n+1} \subseteq N_n$. Since G is of finite type, it follows that there exists n_0 such that $N_n = N_{n_0} =: N$ for all $n \geq n_0$. Then N acts trivially on each subscheme $x_{(n)}$. As X is locally noetherian and irreducible, the union of these subschemes is dense in X ; it follows that N acts trivially on X , by using the representability of the fixed point functor X^G (Theorem 2.2.6). Thus, $N = \text{Ker}(a)$. So $\rho_{n_0} : G \rightarrow \text{GL}(V_{n_0})$ factors through a closed immersion $j : G/\text{Ker}(a) \rightarrow \text{GL}(V_{n_0})$ by Proposition 2.7.1. \square

COROLLARY 3.1.7. *Let G be a connected algebraic group and Z its center. Then G/Z is affine.*

PROOF. Consider the action of G on itself by inner automorphisms. Then the kernel of this action is Z and the neutral element is fixed. So the assertion follows from Proposition 3.1.6. \square

The connectedness assumption in the above corollary cannot be removed in view of Example 4.2.2 below.

3.2. The affinization theorem. Every scheme X is equipped with a morphism to an affine scheme, namely, the canonical morphism

$$\varphi = \varphi_X : X \rightarrow \operatorname{Spec} \mathcal{O}(X).$$

The restriction of φ_X to any affine open subscheme $U \subseteq X$ is the morphism $U \rightarrow \operatorname{Spec} \mathcal{O}(X)$ associated with the restriction homomorphism $\mathcal{O}(X) \rightarrow \mathcal{O}(U)$. Moreover, φ satisfies the following universal property: every morphism $f : X \rightarrow Y$, where Y is an affine scheme, factors uniquely through φ . We say that φ is the *affinization morphism* of X , and denote $\operatorname{Spec} \mathcal{O}(X)$ by $\operatorname{Aff}(X)$. When X is of finite type, $\operatorname{Aff}(X)$ is not necessarily of finite type; equivalently, the algebra $\mathcal{O}(X)$ is not necessarily finitely generated (even when X is a quasi-projective variety, see Example 3.2.3 below).

Also, every morphism of schemes $f : X \rightarrow Y$ lies in a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi_X \downarrow & & \downarrow \varphi_Y \\ \operatorname{Aff}(X) & \xrightarrow{\operatorname{Aff}(f)} & \operatorname{Aff}(Y), \end{array}$$

where $\operatorname{Aff}(f)$ is the morphism of affine schemes associated with the ring homomorphism $f^\# : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$.

For quasi-compact schemes, the formation of the affinization morphism commutes with field extensions and finite products, as a consequence of Lemma 2.3.3. It follows that for any algebraic group G , there is a canonical group scheme structure on $\operatorname{Aff}(G)$ such that φ_G is a homomorphism. Moreover, given an action a of G on a quasi-compact scheme X , the map $\operatorname{Aff}(a)$ is an action of $\operatorname{Aff}(G)$ on $\operatorname{Aff}(X)$, compatibly with a .

With these observations at hand, we may make an important step in the proof of Theorem 1:

THEOREM 3.2.1. *Let G be an algebraic group, $\varphi : G \rightarrow \operatorname{Aff}(G)$ its affinization morphism and $H := \operatorname{Ker}(\varphi)$. Then H is the smallest normal subgroup scheme of G such that G/H is affine. Moreover, $\mathcal{O}(H) = k$ and $\operatorname{Aff}(G) = G/H$. In particular, $\mathcal{O}(G) = \mathcal{O}(G/H)$; thus, the algebra $\mathcal{O}(G)$ is finitely generated.*

PROOF. Consider a normal subgroup scheme N of G such that G/N is affine. Then we have a commutative diagram of homomorphisms

$$\begin{array}{ccc} G & \xrightarrow{q} & G/N \\ \varphi_G \downarrow & & \downarrow \varphi_{G/N} \\ \operatorname{Aff}(G) & \xrightarrow{\operatorname{Aff}(q)} & \operatorname{Aff}(G/N), \end{array}$$

where q is the quotient morphism and $\varphi_{G/N}$ is an isomorphism. Since H is the fiber of φ_G at the neutral element e_G , it follows that $H \subseteq N$.

We now claim that H is the kernel of the action of G on $\mathcal{O}(G)$ via left multiplication. Denote by N the latter kernel; we check that $H(R) = N(R)$ for any algebra R . Note that $H(R)$ consists of those $x \in G(R)$ such that $f(x) = f(e)$ for

all $f \in \mathcal{O}(G)$ (since $\mathcal{O}(G \times \text{Spec}(R)) = \mathcal{O}(G) \otimes_k R$). Also, $N(R)$ consists of those $x \in G(R)$ such that $f(xy) = f(y)$ for all $f \in \mathcal{O}(G \times \text{Spec}(R'))$ and $y \in G(R')$, where R' runs over all R -algebras. In particular, $f(x) = f(e)$ for all $f \in \mathcal{O}(G)$, and hence $N(R) \subseteq H(R)$.

To show the opposite inclusion, choose a basis $(\varphi_i)_{i \in I}$ of the k -vector space $\mathcal{O}(G)$; then the R' -module $\mathcal{O}(G \times \text{Spec}(R')) = \mathcal{O}(G) \otimes_k R'$ is free with basis $(\varphi_i)_{i \in I}$. Thus, for any $f \in \mathcal{O}(G) \otimes_k R'$, there exists a unique family $(\psi_i = \psi_i(f))_{i \in I}$ in $\mathcal{O}(G) \otimes_k R'$ such that $f(xy) = \sum_i \psi_i(x) \varphi_i(y)$ identically. So the equalities $f(xy) = f(y)$ for all $y \in G(R')$ are equivalent to the equalities

$$\sum_i (\psi_i(x) - \psi_i(e)) \varphi_i(y) = 0$$

for all such y . Since the latter equalities are satisfied for any $x \in H(R)$, this yields the inclusion $H(R) \subseteq N(R)$, and completes the proof of the claim.

By Proposition 2.3.4, there exists an increasing family of finite-dimensional G -submodules $(V_i)_{i \in I}$ of $\mathcal{O}(G)$ such that $\mathcal{O}(G) = \bigcup_i V_i$. Denoting by N_i the kernel of the corresponding homomorphism $G \rightarrow \text{GL}(V_i)$, we see that $H = N$ is the decreasing intersection of the N_i . Since the topological space underlying G is noetherian and each N_i is closed in G , there exists $i \in I$ such that $H = N_i$. It follows that G/H is affine. We have proved that H is the smallest normal subgroup scheme of G having an affine quotient.

The affinization morphism φ_G factors through a unique morphism of affine schemes $\iota : G/H \rightarrow \text{Aff}(G)$. The associated homomorphism

$$\iota^\# : \mathcal{O}(\text{Aff}(G)) = \mathcal{O}(G) \rightarrow \mathcal{O}(G/H) = \mathcal{O}(G)^H$$

is an isomorphism; thus, so is ι . This shows that $\text{Aff}(G) = G/H$.

Next, consider the kernel N of the affinization morphism φ_H . Then $N \trianglelefteq H$ and the quotient groups H/N is affine. Since G/H is affine as well, it follows by Proposition 3.1.2 that the homogeneous space G/N is affine. Thus, the quotient morphism $G \rightarrow G/N$ factors through a unique morphism $\text{Aff}(G) \rightarrow G/N$. Taking fibers at e yields that $H \subseteq N$; thus, $H = N$. Hence the action on H on itself via left multiplication yields a trivial action on $\mathcal{O}(H)$. As $\mathcal{O}(H)^H = k$, we conclude that $\mathcal{O}(H) = k$. \square

COROLLARY 3.2.2. *Let G be an algebraic group acting faithfully on an affine scheme X . Then G is affine.*

PROOF. The action of G on X factors through an action of $\text{Aff}(G)$ on $\text{Aff}(X) = X$. Thus, the subgroup scheme H of Theorem 3.2.1 acts trivially on X . Hence H is trivial; this yields the assertion. \square

EXAMPLE 3.2.3. Let E be an elliptic curve equipped with an invertible sheaf \mathcal{L} such that $\deg(\mathcal{L}) = 0$ and \mathcal{L} has infinite order in $\text{Pic}(E)$. (Such a pair (E, \mathcal{L}) exists unless k is algebraic over a finite field, as follows from [55]; see also [59]). Choose an invertible sheaf \mathcal{M} on E such that $\deg(\mathcal{M}) > 0$. Denote by L, M the line bundles on E associated with \mathcal{L}, \mathcal{M} and consider their direct sum,

$$\pi : X := L \oplus M \longrightarrow E.$$

Then X is a quasi-projective variety and

$$\pi_*(\mathcal{O}_X) \cong \bigoplus_{\ell, m} \mathcal{L}^{\otimes \ell} \otimes_{\mathcal{O}_E} \mathcal{M}^{\otimes m},$$

where the sum runs over all pairs of non-negative integers. Thus,

$$\mathcal{O}(X) \cong \bigoplus_{\ell, m} H^0(E, \mathcal{L}^{\otimes \ell} \otimes_{\mathcal{O}_E} \mathcal{M}^{\otimes m}).$$

In particular, the algebra $\mathcal{O}(X)$ is equipped with a bi-grading. If this algebra is finitely generated, then the pairs (ℓ, m) such that $\mathcal{O}(X)_{\ell, m} \neq 0$ form a finitely generated monoid under componentwise addition; as a consequence, the convex cone $C \subset \mathbb{R}^2$ generated by these pairs is closed. But we have $\mathcal{O}(X)_{\ell, 0} = 0$ for any $\ell \geq 1$, since $\mathcal{L}^{\otimes \ell}$ is non-trivial and has degree 0. Also, given any positive rational number t , we have $\mathcal{O}(X)_{n, tn} \neq 0$ for any positive integer n such that tn is integer, since $\deg(\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_E} \mathcal{M}^{\otimes tn}) > 0$. Thus, C is not closed, a contradiction. We conclude that the algebra $\mathcal{O}(X)$ is not finitely generated.

3.3. Anti-affine algebraic groups.

DEFINITION 3.3.1. An algebraic group G over k is *anti-affine* if $\mathcal{O}(G) = k$.

By Lemma 2.3.3, G is anti-affine if and only if G_K is anti-affine for some field extension K of k .

LEMMA 3.3.2. *Every anti-affine algebraic group is smooth and connected.*

PROOF. Let G be an algebraic group. Recall that the group of connected components $\pi_0(G) \cong G/G^0$ is finite and étale. Also, $\mathcal{O}(\pi_0(G)) \cong \mathcal{O}(G)^{G^0}$ by Remark 2.7.3 (ii). If G is anti-affine, then it follows that $\mathcal{O}(\pi_0(G)) = k$. Thus, $\pi_0(G)$ is trivial, i.e., G is connected.

To show that G is smooth, we may assume that k is algebraically closed. Then G_{red} is a smooth subgroup scheme of G ; moreover, the homogeneous space G/G_{red} is finite by Remark 2.7.3(v). As above, it follows that $G = G_{\text{red}}$. \square

We now obtain a generalization of a classical rigidity lemma (see [41, p. 43]):

LEMMA 3.3.3. *Let X, Y, Z be schemes such that X is quasi-compact, $\mathcal{O}(X) = k$ and Y is locally noetherian and irreducible. Let $f : X \times Y \rightarrow Z$ be a morphism. Assume that there exist k -rational points $x_0 \in X, y_0 \in Y$ such that $f(x, y_0) = f(x_0, y_0)$ identically. Then $f(x, y) = f(x_0, y)$ identically.*

PROOF. Let $z_0 := f(x_0, y_0)$; this is a k -rational point of Z . As in Example 2.3.6, consider the n th infinitesimal neighborhoods of this point,

$$z_{0, (n)} := \text{Spec}(\mathcal{O}_{Z, z_0} / \mathfrak{m}_{z_0}^{n+1}),$$

where n runs over the positive integers. These form an increasing sequence of finite subschemes of Z supported at z_0 , and one checks as in the above example that $X \times y_{0, (n)}$ is contained in the fiber of f at $z_{0, (n)}$, where $y_{0, (n)} := \text{Spec}(\mathcal{O}_{Y, y_0} / \mathfrak{m}_{y_0}^{n+1})$. In other words, f restricts to a morphism $f_n : X \times y_{0, (n)} \rightarrow z_{0, (n)}$. Consider the associated homomorphism of algebras $f_n^\# : \mathcal{O}(z_{0, (n)}) \rightarrow \mathcal{O}(X \times y_{0, (n)})$. By Lemma 2.3.3 and the assumptions on X , we have $\mathcal{O}(X \times y_{0, (n)}) = \mathcal{O}(X) \otimes_k \mathcal{O}(y_{0, (n)}) = \mathcal{O}(y_{0, (n)})$. Since $z_{0, (n)}$ is affine, it follows that f_n factors through a morphism $g_n : y_{0, (n)} \rightarrow Z$, i.e., $f_n(x, y) = g_n(y)$ identically. In particular, $f(x, y) = f(x_0, y)$ on $X \times y_{0, (n)}$.

Next, consider the largest closed subscheme $W \subseteq X \times Y$ on which $f(x, y) = f(x_0, y)$, i.e., W is the pull-back of the diagonal in $Z \times Z$ under the morphism $(x, y) \mapsto (f(x, y), f(x_0, y))$. Then W contains $X \times y_{0, (n)}$ for all n . Since Y is locally

noetherian and irreducible, the union of the $y_{0,(n)}$ is dense in Y . It follows that the union of the $X \times y_{0,(n)}$ is dense in $X \times Y$; we conclude that $W = X \times Y$. \square

PROPOSITION 3.3.4. *Let H be an anti-affine algebraic group, G an algebraic group and $f : H \rightarrow G$ a morphism of schemes such that $f(e_H) = e_G$. Then f is a homomorphism and factors through the center of G^0 .*

PROOF. Since H is connected by Lemma 3.3.2, we see that f factors through G^0 . Thus, we may assume that G is connected.

Consider the morphism

$$\varphi : H \times H \longrightarrow G, \quad (x, y) \longmapsto f(xy)f(y)^{-1}f(x)^{-1}.$$

Then $\varphi(x, e_H) = e_G = \varphi(e_H, e_H)$ identically; also, H is irreducible in view of Lemma 3.3.2. Thus, the rigidity lemma applies, and yields $\varphi(x, y) = \varphi(e_H, y) = e_G$ identically. This shows that f is a homomorphism.

The assertion that f factors through the center of G is proved similarly by considering the morphism

$$\psi : H \times G \longrightarrow G, \quad (x, y) \longmapsto f(x)yf(x)^{-1}y^{-1}.$$

\square

In particular, every anti-affine group G is commutative. Also, note that G/H is anti-affine for any subgroup scheme $H \subseteq G$ (since $\mathcal{O}(G/H) = \mathcal{O}(G)^H$).

We may now complete the proof of Theorem 1 with the following:

PROPOSITION 3.3.5. *Let G be an algebraic group and H the kernel of the affinization morphism of G .*

- (1) *H is contained in the center of G^0 .*
- (2) *H is the largest anti-affine subgroup of G .*

PROOF. (1) By Theorem 3.2.1, H is anti-affine. So the assertion follows from Lemma 3.3.2 and Proposition 3.3.4, or alternatively, from Corollary 3.1.7.

(2) Consider another anti-affine subgroup $N \subseteq G$. Then the quotient group $N/N \cap H$ is anti-affine, and also affine (since $N/N \cap H$ is isomorphic to a subgroup of G/H , and the latter is affine). As a consequence, $N/N \cap H$ is trivial, that is, N is contained in H . \square

We will denote the largest anti-affine subgroup of an algebraic group G by G_{ant} . For later use, we record the following observations:

LEMMA 3.3.6. *Let G be an algebraic group and $N \trianglelefteq G$ a normal subgroup scheme. Then the quotient map $G \rightarrow G/N$ yields an isomorphism*

$$G_{\text{ant}}/G_{\text{ant}} \cap N \xrightarrow{\cong} (G/N)_{\text{ant}}.$$

PROOF. By Proposition 2.7.4, we have a closed immersion of algebraic groups $G_{\text{ant}}/G_{\text{ant}} \cap N \rightarrow G/N$; moreover, $G_{\text{ant}}/G_{\text{ant}} \cap N$ is anti-affine. So we obtain a closed immersion of commutative algebraic groups $j : G_{\text{ant}}/G_{\text{ant}} \cap N \rightarrow (G/N)_{\text{ant}}$. Denote by C the cokernel of j ; then C is anti-affine as a quotient of $(G/N)_{\text{ant}}$. Also, C is a subgroup of $(G/N)/(G_{\text{ant}}/G_{\text{ant}} \cap N)$, which is a quotient group of G/G_{ant} . Since the latter group is affine, it follows that C is affine as well, by using Proposition 3.1.2. Thus, C is trivial, i.e., j is an isomorphism. \square

LEMMA 3.3.7. *The following conditions are equivalent for an algebraic group G :*

(1) G is proper.

(2) G_{ant} is an abelian variety and G/G_{ant} is finite.

Under these conditions, we have $G_{\text{ant}} = G_{\text{red}}^0$; in particular, G_{red}^0 is a smooth connected algebraic group and its formation commutes with field extensions.

PROOF. (1) \Rightarrow (2) As G_{ant} is smooth, connected and proper, it is an abelian variety. Also, G/G_{ant} is proper and affine, hence finite.

(2) \Rightarrow (1) This follows from Remark 3.1.3.

For the final assumption, note that the quotient group scheme G^0/G_{ant} is finite and connected, hence infinitesimal. So the algebra $\mathcal{O}(G^0/G_{\text{ant}})$ is local with residue field k (via evaluation at e). It follows that $(G^0/G_{\text{ant}})_{\text{red}} = e$ and hence that $G_{\text{red}}^0 \subseteq G_{\text{ant}}$; this yields the assertion. \square

Notes and references.

Some of the main results of this section originate in Rosenlicht's article [48]. More specifically, Corollary 3.1.7 is a scheme-theoretic version of [48, Thm. 13], and Theorem 3.2.1, of [48, Cor. 3, p. 431].

Also, Theorem 3.2.1, Lemma 3.3.2 and Proposition 3.3.4 are variants of results from [22, III.3.8].

The rigidity lemma 3.3.3 is a version of [52, Thm. 1.7].

4. Proof of Theorem 2

4.1. The Albanese morphism. Throughout this subsection, A denotes an abelian variety, i.e., a smooth connected proper algebraic group. Then A is commutative by Corollary 3.1.7. Thus, we will denote the group law additively; in particular, the neutral element will be denoted by 0. Also, the variety A is projective (see [41, p. 62]).

LEMMA 4.1.1. *Every morphism $f : \mathbb{P}^1 \rightarrow A$ is constant.*

PROOF. We may assume that k is algebraically closed. Suppose that f is non-constant and denote by $C \subseteq A$ its image, with normalization $\eta : \tilde{C} \rightarrow C$. Then f factors through a surjective morphism $\mathbb{P}^1 \rightarrow \tilde{C}$. By Lüroth's theorem, it follows that $\tilde{C} \cong \mathbb{P}^1$. Thus, f factors through the normalization $\eta : \mathbb{P}^1 \rightarrow C$ and hence it suffices to show that η is constant. In other words, we may assume that f is birational to its image. Then the differential

$$df : T_{\mathbb{P}^1} \longrightarrow f^*(T_A)$$

is non-zero at the generic point of \mathbb{P}^1 and hence is injective. Since the tangent sheaf T_A is trivial and $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, we obtain an injective map $\mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus n}$, where $n := \dim(A)$. This yields a contradiction, since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$ while $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. \square

THEOREM 4.1.2. *Let X be a smooth variety and $f : X \dashrightarrow A$ a rational map. Then f is a morphism.*

PROOF. Again, we may assume k algebraically closed. View f as a morphism $U \rightarrow A$, where $U \subseteq X$ is a non-empty open subvariety. Denote by $Y \subseteq X \times A$ the closure of the graph of f , with projections $p_1 : Y \rightarrow X$, $p_2 : Y \rightarrow A$. Then p_1 is proper (since so is A) and birational (since it restricts to an isomorphism over U). Assume that p_1 is not an isomorphism. Then p_1 contracts some rational curve

in Y , i.e., there exists a non-constant morphism $g : \mathbb{P}^1 \rightarrow Y$ such that $p_1 \circ g$ is constant (see [21, Prop. 1.43]). It follows that $p_2 \circ g : \mathbb{P}^1 \rightarrow A$ is non-constant; but this contradicts Lemma 4.1.1. \square

LEMMA 4.1.3. *Let X, Y be varieties equipped with k -rational points x_0, y_0 and let $f : X \times Y \rightarrow A$ be a morphism. Then we have identically*

$$f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0) = 0.$$

PROOF. By a result of Nagata (see [44, 45], and [36] for a modern proof), we may choose a compactification of X , i.e., an open immersion $X \rightarrow \bar{X}$, where \bar{X} is a proper variety. Replacing \bar{X} with its normalization, we may further assume that \bar{X} is normal. Also, we may assume k algebraically closed from the start.

Denote by U the smooth locus of \bar{X} and by V the smooth locus of Y . By Theorem 4.1.2, the rational map $f : \bar{X} \times Y \dashrightarrow A$ yields a morphism $g : U \times V \rightarrow A$. Also, note that the complement $\bar{X} \setminus U$ has codimension at least 2, since \bar{X} is normal and $k = \bar{k}$. It follows that $\mathcal{O}(U) = \mathcal{O}(\bar{X}) = k$. Using again the assumption that $k = \bar{k}$, we may choose points $x_1 \in U(k)$, $y_1 \in V(k)$. Consider the morphism

$$\varphi : U \times V \rightarrow A, \quad (x, y) \mapsto g(x, y) - g(x_1, y) - g(x, y_1) + g(x_1, y_1).$$

Then $\varphi(x, y_1) = 0$ identically, and hence $\varphi = 0$ by rigidity (Lemma 3.3.3). It follows that $f(x, y) - f(x_1, y) - f(x, y_1) + f(x_1, y_1) = 0$ identically on $X \times Y$. This readily yields the desired equation. \square

PROPOSITION 4.1.4. *Let G be a smooth connected algebraic group.*

- (1) *Let $f : G \rightarrow A$ be a morphism to an abelian variety sending e to 0. Then f is a homomorphism.*
- (2) *There exists a smallest normal subgroup scheme N of G such that the quotient G/N is an abelian variety. Moreover, N is connected.*
- (3) *For any abelian variety A , every morphism $f : G \rightarrow A$ sending e to 0 factors uniquely through the quotient homomorphism $\alpha : G \rightarrow G/N$.*
- (4) *The formation of N commutes with separable algebraic field extensions.*

PROOF. (1) This follows from Lemma 4.1.3 applied to the map $G \times G \rightarrow A$, $(x, y) \mapsto f(xy)$ and to $x_0 = y_0 = e$.

(2) Consider two normal subgroup schemes N_1, N_2 of G such that $G/N_1, G/N_2$ are abelian varieties. Then $N_1 \cap N_2$ is normal in G and $G/N_1 \cap N_2$ is proper, since the natural map

$$G/N_1 \cap N_2 \rightarrow G/N_1 \times G/N_2$$

is a closed immersion of algebraic groups (Proposition 2.7.1). Since $G/N_1 \cap N_2$ is smooth and connected, it is an abelian variety as well. It follows that there exists a smallest such subgroup scheme, say N .

We claim that the neutral component N^0 is normal in G . To check this, we may assume k algebraically closed. Then $G(k)$ is dense in G and normalizes N^0 ; this yields the claim.

The natural homomorphism $G/N^0 \rightarrow G/N$ is finite, since it is a torsor under the finite group N/N^0 (Propositions 2.6.5 and 2.8.4). As a consequence, G/N^0 is proper; hence $N = N^0$ by the minimality assumption. Thus, N is connected.

(3) By (1), f is a homomorphism; denote its kernel by H . Then G/H is an abelian variety in view of Proposition 2.7.4. Thus, H contains N , and the assertion follows from Proposition 2.8.4.

(4) This is checked by a standard argument of Galois descent, as in the proof of Lemma 3.1.4. \square

REMARK 4.1.5. In fact, the formation of N commutes with all separable field extensions. This may be proved as sketched in Remark 3.1.5.

With the notation and assumptions of Proposition 4.1.4, we say that the quotient morphism

$$\alpha : G \longrightarrow G/N$$

is the *Albanese homomorphism* of G , and $G/N =: \text{Alb}(G)$ the *Albanese variety*.

Actually, Proposition 4.1.4 (3) extends to any *pointed variety*, i.e., a variety equipped with a k -rational point:

THEOREM 4.1.6. *Let (X, x) be a pointed variety. Then there exists an abelian variety A and a morphism $\alpha = \alpha_X : X \rightarrow A$ sending x to 0, such that for any abelian variety B , every morphism $X \rightarrow B$ sending x to 0 factors uniquely through α .*

PROOF. See [54, Thm. 5] when k is algebraically closed; the general case is obtained in [62, Thm. A1]. \square

The morphism $\alpha : X \rightarrow A$ in the above theorem is uniquely determined by (X, x) ; it is called the *Albanese morphism*, and A is again the *Albanese variety*, denoted by $\text{Alb}(X)$. Combining that theorem with Theorem 4.1.2 and Lemma 4.1.3, we obtain readily:

COROLLARY 4.1.7. (1) *For any smooth pointed variety (X, x) and any open subvariety $U \subseteq X$ containing x , we have a commutative square*

$$\begin{array}{ccc} U & \xrightarrow{\alpha_U} & \text{Alb}(U) \\ j \downarrow & & \downarrow \text{Alb}(j) \\ X & \xrightarrow{\alpha_X} & \text{Alb}(X), \end{array}$$

where $j : U \rightarrow X$ denotes the inclusion and $\text{Alb}(j)$ is an isomorphism.

(2) *For any pointed varieties (X, x) , (Y, y) , we have $\alpha_{X \times Y} = \alpha_X \times \alpha_Y$. In particular, the natural map*

$$\text{Alb}(X \times Y) \longrightarrow \text{Alb}(X) \times \text{Alb}(Y)$$

is an isomorphism.

(3) *Any action a of a smooth connected algebraic group G on a pointed variety X yields a unique homomorphism $f = f_a : \text{Alb}(G) \rightarrow \text{Alb}(X)$ such that the square*

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \alpha_{G \times X} \downarrow & & \downarrow \alpha_X \\ \text{Alb}(G \times X) & \xrightarrow{f + \text{id}} & \text{Alb}(X) \end{array}$$

commutes.

The formation of the Albanese morphism commutes with separable algebraic field extensions, by Galois descent again. But it does not commute with arbitrary field extensions, as shown by Example 4.2.7 below.

4.2. Abelian torsors.

LEMMA 4.2.1. *Let A an abelian variety and n a non-zero integer. Then the multiplication map*

$$n_A : A \longrightarrow A, \quad x \longmapsto nx$$

is an isogeny.

PROOF. See [41, p. 62]. □

EXAMPLE 4.2.2. With the above notation, consider the semi-direct product $G := \mathbb{Z}/2 \ltimes A$, where $\mathbb{Z}/2$ (viewed as a constant group scheme) acts on A via $x \mapsto \pm x$. One may check that the center Z of G is the kernel of the multiplication map 2_A and hence is finite; thus, G/Z is not affine. So Corollary 3.1.7 does not extend to disconnected algebraic groups.

Also, note that G is not contained in any connected algebraic group, as follows from Proposition 3.3.4.

LEMMA 4.2.3. *Let G be a smooth connected commutative algebraic group, with group law denoted additively, and let $f : X \rightarrow \operatorname{Spec}(k)$ be a G -torsor. Then there exists a positive integer n and a morphism $\varphi : X \rightarrow G$ such that*

$$\varphi(g \cdot x) = \varphi(x) + ng$$

identically on $G \times X$.

PROOF. If X has a k -rational point x , then the orbit map $a_x : G \rightarrow X, g \mapsto g \cdot x$ is a G -equivariant isomorphism, where G acts on itself by translation. So we may just take $\varphi = a_x^{-1}$ and $n = 1$ (i.e., φ is G -equivariant).

In the general case, since X is a smooth variety, it has a K -rational point x_1 for some finite Galois extension of fields K/k . Denote by \mathcal{G} the corresponding Galois group and by x_1, \dots, x_n the distinct conjugates of x_1 under \mathcal{G} . Then the first step yields G_K -equivariant isomorphisms $g_i : X_K \rightarrow G_K$ for $i = 1, \dots, n$, such that $x = g_i(x) \cdot x_i$ identically. Consider the morphism

$$\phi : X_K \longrightarrow G_K, \quad x \longmapsto g_1(x) + \dots + g_n(x).$$

Then ϕ is equivariant under \mathcal{G} , since that group permutes the x_i 's and hence the g_i 's. Also, we have $\phi(g \cdot x) = \phi(x) + ng$ identically. So ϕ descends to the desired morphism $X \rightarrow G$. □

PROPOSITION 4.2.4. *Let A be an abelian variety and $f : X \rightarrow Y$ an A -torsor, where X, Y are smooth varieties. Then there exists a positive integer n and a morphism $\varphi : X \rightarrow A$ such that $\varphi(a \cdot x) = \varphi(x) + na$ identically on $A \times X$.*

PROOF. Let $\eta_Y : \operatorname{Spec} k(Y) \rightarrow Y$ be the generic point. Then the base change $X \times_Y \operatorname{Spec} k(Y) \rightarrow \operatorname{Spec} k(Y)$ is an $A_{k(Y)}$ -torsor. Using Lemma 4.2.3, we obtain a map $\psi : X \times_Y \operatorname{Spec} k(Y) \rightarrow A_{k(Y)}$ satisfying the required covariance property. Composing ψ with the natural maps $\eta_X \times f : \operatorname{Spec} k(X) \rightarrow X \times_Y \operatorname{Spec} k(Y)$ and $\pi : A_{k(Y)} \rightarrow A$ yields a map $\operatorname{Spec} k(X) \rightarrow A$, which may be viewed as a rational map $X \dashrightarrow A$ and hence (by Theorem 4.1.2) as a morphism $\varphi : X \rightarrow A$. Clearly, φ satisfies the same covariance property as ψ . □

THEOREM 4.2.5. *Let G be a smooth connected algebraic group and $A \subseteq G$ an abelian subvariety. Then $A \subseteq Z(G)$ and there exists a connected normal subgroup scheme $H \subseteq G$ such that $G = A \cdot H$ and $A \cap H$ is finite. If k is perfect, then we may take H smooth.*

PROOF. By Proposition 3.3.4, A is contained in the center of G . The quotient map $q : G \rightarrow G/A$ is an A -torsor; also, G/A is smooth, since so is G . Thus, Proposition 4.2.4 yields a map $\varphi : G \rightarrow A$ such that $\varphi(ag) = \varphi(g) + na$ identically, for some integer $n > 0$. Composing φ with a translation of A , we may assume that $\varphi(e_G) = 0$. Then φ is a homomorphism in view of Proposition 4.1.4; its restriction to A is the multiplication n_A .

We claim that $G = A \cdot \text{Ker}(\varphi)$. Since G is smooth, it suffices by Lemma 2.8.6 to show the equality $G(\bar{k}) = A(\bar{k}) \text{Ker}(\varphi)(\bar{k})$. Let $g \in G(\bar{k})$; by Lemma 4.2.1, there exists $a \in A(\bar{k})$ such that $\varphi(g) = na$. Thus, $\varphi(a^{-1}g) = 0$; this yields the claim.

Let $H := \text{Ker}(\varphi)^0$. Then $A \cap H$ is finite, since it is contained in $A \cap \text{Ker}(\varphi) = \text{Ker}(n_A)$. We now show that $G = A \cdot H$. The homogeneous space $G/A \cdot H$ is smooth and connected, since so is G . On the other hand, $G/A \cdot H = A \cdot \text{Ker}(\varphi)/A \cdot H$ is homogeneous under $\text{Ker}(\varphi)/H = \pi_0(\text{Ker}(\varphi))$, and hence is finite. Thus, $G/A \cdot H$ is trivial; this yields the assertion. Since H centralizes A , it follows that H is normal in G .

Finally, if k is perfect, then we may replace H with H_{red} in the above argument. \square

COROLLARY 4.2.6. *Let A be an abelian variety and $B \subseteq A$ an abelian subvariety. Then there exists an abelian subvariety $C \subseteq A$ such that $A = B + C$ and $B \cap C$ is finite.*

PROOF. By the above theorem, there exists a connected subgroup scheme H of A such that $A = B + H$ and $B \cap H$ is finite. Now replace H with H_{red} , which is an abelian subvariety by Lemma 3.3.7. \square

The following example displays several specific features of algebraic groups over imperfect fields. It is based on Weil restriction; we refer to Appendix A of [20] for the definition and main properties of this notion.

EXAMPLE 4.2.7. Let k be an imperfect field. Choose a non-trivial finite purely inseparable extension K of k . Let A' be a non-trivial abelian variety over K ; then the Weil restriction $R_{K/k}(A') =: G$ is a smooth connected commutative algebraic group over k , of dimension $[K : k] \dim(A')$. Moreover, there is an exact sequence of algebraic groups over K

$$(4.2.1) \quad 1 \longrightarrow U' \longrightarrow G_K \xrightarrow{q} A' \longrightarrow 1,$$

where U' is non-trivial, smooth, connected and unipotent (see [20, A.5.11]). It follows readily that q is the Albanese homomorphism of G_K .

Let H be a smooth connected affine algebraic group over k . We claim that every homomorphism $f : H \rightarrow G$ is constant. Indeed, the morphisms of k -schemes $f : H \rightarrow G$ correspond bijectively to the morphisms of K -schemes $f' : H_K \rightarrow A'$, via the assignement $f \mapsto f' := q \circ f_K$ (see [20, A.5.7]). Since q is a homomorphism, this bijection sends homomorphisms to homomorphisms. As every homomorphism $H_K \rightarrow A'$ is constant, this proves the claim.

Next, consider the Albanese homomorphism $\alpha : G \rightarrow \text{Alb}(G)$. If $\alpha_K = q$, then $\text{Ker}(\alpha)_K = \text{Ker}(\alpha_K) = U'$; as a consequence, $\text{Ker}(\alpha)$ is smooth, connected and affine. By the claim, it follows that $\text{Ker}(\alpha)$ is trivial, i.e., G is an abelian variety. Then so is G_K , but this contradicts the non-triviality of U . So *the formation of the Albanese morphism does not commute with arbitrary field extensions.*

Note that $U = L(G_K)$ (the largest smooth connected normal affine subgroup scheme of G_K , introduced in Lemma 3.1.4). Also, $L(G)$ is trivial by the claim. Thus, *the formation of $L(G)$ does not commute with arbitrary field extensions.*

Likewise, *G is not an extension of an abelian variety by a smooth connected affine algebraic group.* We will see in Theorem 4.3.2 that every smooth connected algebraic group over a *perfect* field lies in a unique such extension.

For later use, we analyze the structure of G in more detail. We claim that *there is a unique exact sequence*

$$1 \longrightarrow A \longrightarrow G \longrightarrow U \longrightarrow 1,$$

where A is an abelian variety and U is unipotent. This is equivalent to the assertion that G_{ant} is an abelian variety, and G/G_{ant} is unipotent. So it suffices to show the corresponding assertion for G_K . We may choose a positive integer n such that $p_{U'}^n = 0$. Thus, the extension (4.2.1) is trivialized by push-out via $p_{U'}^n$. It follows that (4.2.1) is also trivialized by pull-back via $p_{A'}^n$, and hence $G_K \cong (U' \times A')/F$ for some finite group scheme F (isomorphic to the kernel of $p_{A'}^n$). So the image of A' in G_K is an abelian variety with unipotent quotient; this proves the claim.

Next, we claim that *there exists a connected unipotent subgroup scheme $V \subseteq G$ such that $G = A \cdot V$ and $A \cap V$ is finite.* This follows from Theorem 4.2.5, or directly by taking for V the neutral component of $\text{Ker}(p_G^m)$, where m is chosen so that $p_U^m = 0$. Yet *there exists no smooth connected subgroup scheme $H \subseteq G$ such that $G = A \cdot H$ and $A \cap H$ is finite.* Otherwise, the quotient homomorphism $G \rightarrow U$ restricts to a homomorphism $H \rightarrow U$ with finite kernel. Thus, H is affine; also, G is an extension of the abelian variety $A/A \cap H$ by H . This yields a contradiction.

4.3. Completion of the proof of Theorem 2. The final ingredient in Rosenlicht's proof of the Chevalley structure theorem is the following:

LEMMA 4.3.1. *Every non-proper algebraic group over an algebraically closed field contains an affine subgroup scheme of positive dimension.*

We now give a brief outline of the proof of this lemma, which is presented in detail in [14, Sec. 2.3]; see also [40, Sec. 4]. Let G be a non-proper algebraic group over $k = \bar{k}$. Then G_{red}^0 is non-proper as well, and hence we may assume that G is smooth and connected. By [44, 45], there exists a compactification X of G , i.e., X is a proper variety containing G as an open subvariety; then the boundary $X \setminus G$ is non-empty. The action of G by left multiplication on itself induces a faithful rational action

$$a : G \times X \dashrightarrow X.$$

One shows (this is the main step of the proof) that there exists a proper variety X' and a birational morphism $\varphi : X' \rightarrow X$ such that the induced birational action $a' : G \times X' \dashrightarrow X'$ normalizes some irreducible divisor $D \subset X'$, i.e., a' induces a rational action $G \times D \dashrightarrow D$. Then one considers the “orbit map” a'_x for a general point $x \in D$, and the corresponding “stabilizer” $C_G(x)$ (these have to be defined appropriately). By adapting the argument of Proposition 3.1.6, one shows that $C_G(x)$ is affine; it has positive dimension, since $\dim(G/C_G(x)) \leq \dim(D) = \dim(G) - 1$. So $C_G(x)$ is the desired subgroup scheme.

We now show how to derive Theorem 2 from Lemma 4.3.1, under the assumptions that k is perfect and G is smooth and connected. We then have the following more precise result, which is a version of Chevalley's structure theorem:

THEOREM 4.3.2. *Let G be a smooth connected algebraic group over a perfect field k , and $L = L(G)$ the largest smooth connected affine normal subgroup scheme.*

- (1) *L is the kernel of the Albanese homomorphism of G .*
- (2) *The formation of L commutes with field extensions.*

PROOF. (1) Recall that the existence of $L(G)$ has been obtained in Lemma 3.1.4, as well as the triviality of $L(G/L(G))$. We may thus replace G with G/L and assume that L is trivial. Also, since the formations of $L(G)$ and of the Albanese homomorphism commute with algebraic field extensions (see Lemma 3.1.4 again, and Proposition 4.1.4), we may assume that k is algebraically closed.

Consider the center Z of G . If Z is proper, then its reduced neutral component Z_{red}^0 is an abelian variety, say A . Since G/Z is affine (Corollary 3.1.7) and Z/A is finite, G/A is affine as well by Proposition 3.1.2. Also, by Theorem 4.2.5, there exists a smooth connected normal subgroup scheme $H \trianglelefteq G$ such that $G = A \cdot H$ and $A \cap H$ is finite. Then $H/A \cap H \cong G/A$ is affine, and hence H is affine by Proposition 3.1.2 again. It follows that H is trivial, and $G = A$ is an abelian variety.

On the other hand, if Z is not proper, then it contains an affine subgroup scheme N of positive dimension by Lemma 4.3.1. Thus, N_{red}^0 is a non-trivial smooth connected central subgroup scheme of G . But this contradicts the assumption that $L(G)$ is trivial.

- (2) Let K be a field extension of k . Then the exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 1$$

yields an exact sequence of smooth connected algebraic groups over K

$$1 \longrightarrow L_K \longrightarrow G_K \longrightarrow A_K \longrightarrow 1,$$

where L_K is affine and A_K is an abelian variety. It follows readily that L_K equals $L(G_K)$ and is the kernel of the Albanese homomorphism of G_K . \square

REMARK 4.3.3. With the notation and assumptions of the above theorem, every smooth connected affine subgroup scheme of G (not necessarily normal) has a trivial image in the abelian variety A . Thus, L is the largest smooth connected affine subgroup scheme of G .

We now return to an arbitrary field and obtain the following:

THEOREM 4.3.4. *Let G be a smooth algebraic group and denote by N the kernel of the Albanese homomorphism of G^0 . Then N is the smallest normal subgroup scheme of G such that the quotient is proper. Moreover, N is affine and connected.*

PROOF. By Proposition 4.1.4, N is connected. Also, as G^0/N is proper and G/G^0 is finite, G/N is proper.

We now show that N is normal in G . For this, we may assume that k is separably closed, since the formation of N commutes with separable algebraic field extensions (Proposition 4.1.4 again). Then $G(k)$ is dense in G by smoothness, and normalizes N by the uniqueness of the Albanese homomorphism. Thus, G normalizes N .

Next, we show that N is contained in every normal subgroup scheme $H \trianglelefteq G$ such that G/H is proper. Indeed, one sees as above that $H^0 \trianglelefteq G$ and G/H^0 is proper as well; hence G^0/H^0 is an abelian variety. Thus, $H^0 \supseteq N$ as desired.

Finally, we show that N is affine. For this, we may assume that G is connected. By Theorem 4.3.2, there exists an exact sequence of algebraic groups over the perfect closure k_i ,

$$1 \longrightarrow L_i \longrightarrow G_{k_i} \longrightarrow A_i \longrightarrow 1,$$

where L_i is smooth, connected and affine, and A_i is an abelian variety. This exact sequence is defined over some subfield $K \subseteq k_i$, finite over k . In other words, there exists a finite purely inseparable field extension K of k and a smooth connected affine normal subgroup scheme $L' \trianglelefteq G_K$ such that G_K/L' is an abelian variety.

By Lemma 4.3.5 below, we may choose a subgroup scheme $L \subseteq G$ such that $L_K \supseteq L'$ and L_K/L' is finite. As a consequence, $L_K \trianglelefteq G_K$ and hence $L \trianglelefteq G$. Also, L_K is affine and hence L is affine. Finally, $(G/L)_K \cong G_K/L_K \cong (G_K/L')/(L_K/L')$ is an abelian variety, and hence G/L is an abelian variety. Thus, $N \subseteq L$ is affine. \square

LEMMA 4.3.5. *Let G be an algebraic group over k . Let K be a finite purely inseparable field extension of k , and $H' \subseteq G_K$ a K -subgroup scheme. Then there exists a k -subgroup scheme $H \subseteq G$ such that $H' \subseteq H_K$ and H_K/H' is finite.*

PROOF. We may choose a positive integer n such that $K^{p^n} \subseteq k$. Consider the n th relative Frobenius homomorphism

$$F^n := F_{G_K/K}^n : G_K \longrightarrow G_K^{(n)}$$

as in §2.9. Denote by H'_n the pull-back under F^n of the subgroup scheme $H'^{(n)}$ of $G_K^{(n)}$. Then $H' \subseteq H'_n$ and the quotient H'_n/H' is finite, since F^n is the identity on the underlying topological spaces and remains so over \bar{k} . Denote by $\mathcal{I}' \subset \mathcal{O}_{G_K}$ the sheaf of ideals of H' ; then the sheaf of ideals \mathcal{I}'_n of H'_n is generated by the p^n th powers of local sections of \mathcal{I}' , as follows from (2.9.1). Since $K^{p^n} \subseteq k$, every such power lies in \mathcal{O}_G . Thus, $\mathcal{I}'_n = \mathcal{I}_K$ for a unique sheaf of ideals $\mathcal{I} \subset \mathcal{O}_G$. The corresponding closed k -subscheme $H \subseteq G$ satisfies $H_K = H'_n$, and hence is the desired subgroup scheme. \square

REMARK 4.3.6. Let G be a smooth connected algebraic group over k with Albanese homomorphism $\alpha : G \rightarrow \text{Alb}(G)$ and consider a field extension K of k . Then the homomorphism $\alpha_K : G_K \rightarrow \text{Alb}(G)_K$ factors through a unique homomorphism

$$f : \text{Alb}(G_K) \longrightarrow \text{Alb}(G)_K.$$

Since α is surjective, f is surjective as well. Moreover, $\text{Ker}(f)$ is infinitesimal: indeed, denoting by $\alpha' : G_K \rightarrow \text{Alb}(G_K)$ the Albanese homomorphism, we have $\text{Ker}(\alpha') \subseteq \text{Ker}(\alpha_K)$ and $\text{Ker}(f) \cong \text{Ker}(\alpha_K)/\text{Ker}(\alpha')$. In particular, $\text{Ker}(f)$ is affine and connected. But $\text{Ker}(f)$ is also a subgroup scheme of the abelian variety $\text{Alb}(G_K)$; hence it must be finite and local.

In loose terms, the formation of the Albanese homomorphism commutes with field extensions up to purely inseparable isogenies.

Since every algebraic group is an extension of a smooth algebraic group by an infinitesimal one (Proposition 2.9.2), Theorem 4.3.4 implies readily:

COROLLARY 4.3.7. *Every connected algebraic group G is an extension of an abelian variety by a connected affine algebraic group.*

Yet G may contain no smallest connected affine subgroup scheme with quotient an abelian variety, as shown by the following:

EXAMPLE 4.3.8. Assume that $\text{char}(k) = p > 0$ and let A be a non-trivial abelian variety. Then A contains two non-trivial infinitesimal subgroup schemes I, J such that $J \subsetneq I$: we may take $I = \text{Ker}(F_{A/k})$ and $J = \text{Ker}(F_{A/k}^2)$, where $F_{A/k}^n$ denotes the n th relative Frobenius morphism as in §2.9. Thus,

$$G := (A \times I)/\text{diag}(J)$$

is a connected commutative algebraic group, where $\text{diag}(x) := (x, x)$. Consider the infinitesimal subgroup schemes $N_1 := (J \times I)/\text{diag}(J)$ and $N_2 := \text{diag}(I)/\text{diag}(J)$ of G . Then $G/N_1 \cong A/J$ and $G/N_2 \cong A$ are abelian varieties. Also, $N_1 \cap N_2$ is trivial, and G is not an abelian variety.

To complete the proof of Theorem 2, it remains to treat the general case, where G is an arbitrary algebraic group over an arbitrary field. We then have to prove:

LEMMA 4.3.9. *Any algebraic group G has a smallest normal subgroup scheme N such that G/N is proper. Moreover, N is affine and connected.*

PROOF. The existence and connectedness of N are obtained as in the proof of Proposition 4.1.4. To show that N is affine, it suffices to find an affine normal subgroup $H \trianglelefteq G$ such that G/H is proper. For this, we may reduce to the case where G is smooth by using Proposition 2.9.2 if $\text{char}(k) > 0$. Then we may take for H the kernel of the Albanese homomorphism of G^0 . \square

Notes and references.

Theorem 4.1.2 is called *Weil's extension theorem*. The proof presented here is taken from that of [21, Cor. 1.44].

Many results of this section are due to Rosenlicht. More specifically, Lemma 4.2.3 is a version of [48, Thm. 14], and Theorem 4.2.5, of [48, Cor. , p. 434]; Lemma 4.3.1 is [48, Lem. 1, p. 437].

Corollary 4.2.6 is called *Poincaré's complete reducibility theorem*; it is proved e.g. in [41, p. 173] over an algebraically closed field, and in [19, Cor. 3.20] over an arbitrary field. It implies that every abelian variety is isogenous to a product of simple abelian varieties, and these are uniquely determined up to isogeny and reordering.

Example 4.2.7 develops a construction of Raynaud, see [SGA3, XVII.C.5]. The proof of Lemma 4.3.5 is taken from [9, 9.2 Thm. 1]. Corollary 4.3.7 is due to Raynaud, see [47, IX.2.7].

5. Some further developments

5.1. The Rosenlicht decomposition. Throughout this section, G denotes a smooth connected algebraic group. By Theorem 2, G has a smallest connected normal affine subgroup scheme G_{aff} with quotient being an abelian variety; also, recall that G_{aff} is the kernel of the Albanese homomorphism $\alpha : G \rightarrow \text{Alb}(G)$. On the other hand, by Theorem 1, every algebraic group H has a largest anti-affine subgroup scheme that we will denote by H_{ant} ; moreover, H_{ant} is smooth, connected, and contained in the center of H^0 . Also, H_{ant} is the smallest normal subgroup scheme of H having an affine quotient.

We will analyze the structure of G in terms of those of G_{aff} and G_{ant} . Note that $(G_{\text{aff}})_{\text{ant}}$ is trivial (since G_{aff} is affine), but $(G_{\text{ant}})_{\text{aff}}$ may have positive dimension as we will see in §5.5.

THEOREM 5.1.1. *Keep the above notation and assumptions.*

- (1) $G = G_{\text{aff}} \cdot G_{\text{ant}}$.
- (2) $G_{\text{aff}} \cap G_{\text{ant}}$ contains $(G_{\text{ant}})_{\text{aff}}$.
- (3) The quotient $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$ is finite.

PROOF. (1) By Proposition 2.8.5, $G_{\text{aff}} \cdot G_{\text{ant}}$ is a normal subgroup scheme of G . Moreover, the quotient $G \rightarrow G/G_{\text{aff}} \cdot G_{\text{ant}}$ factors through a homomorphism $G/G_{\text{aff}} \rightarrow G/G_{\text{aff}} \cdot G_{\text{ant}}$ and also through a homomorphism $G/G_{\text{ant}} \rightarrow G/G_{\text{aff}} \cdot G_{\text{ant}}$, in view of Proposition 2.8.4. In particular, $G/G_{\text{aff}} \cdot G_{\text{ant}}$ is a quotient of an abelian variety, and hence is an abelian variety as well. Also, $G/G_{\text{aff}} \cdot G_{\text{ant}}$ is a quotient of an affine algebraic group, and hence is affine as well (Proposition 3.1.2). Thus, $G/G_{\text{aff}} \cdot G_{\text{ant}}$ is trivial; this proves the assertion.

(2) Proposition 2.8.5 yields an isomorphism $G_{\text{ant}}/G_{\text{ant}} \cap G_{\text{aff}} \cong G/G_{\text{aff}}$. In particular, $G_{\text{ant}}/G_{\text{ant}} \cap G_{\text{aff}}$ is an abelian variety. Since $G_{\text{ant}} \cap G_{\text{aff}}$ is affine, the assertion follows from Theorem 2.

(3) Consider the quotient $\bar{G} := G/(G_{\text{ant}})_{\text{aff}}$. Then $\bar{G} = \bar{G}_{\text{aff}} \cdot \bar{G}_{\text{ant}}$, where $\bar{G}_{\text{aff}} := G_{\text{aff}}/(G_{\text{ant}})_{\text{aff}}$ and $\bar{G}_{\text{ant}} := G_{\text{ant}}/(G_{\text{ant}})_{\text{aff}}$. Moreover, \bar{G}_{aff} is affine (as a quotient of G_{aff}) and \bar{G}_{ant} is an abelian variety. Thus, $\bar{G}_{\text{ant}} \cap \bar{G}_{\text{ant}}$ is finite. This yields the assertion in view of Proposition 2.8.3. \square

REMARK 5.1.2. The above theorem is equivalent to the assertion that *the multiplication map of G induces an isogeny*

$$(G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}} \longrightarrow G,$$

where $(G_{\text{ant}})_{\text{aff}}$ is viewed as a subgroup scheme of $G_{\text{aff}} \times G_{\text{ant}}$ via $x \mapsto (x, x^{-1})$.

Theorem 5.1.1 may also be reformulated in terms of the two Albanese varieties $\text{Alb}(G) = G/G_{\text{aff}}$ and $\text{Alb}(G_{\text{ant}}) = G_{\text{ant}}/(G_{\text{ant}})_{\text{aff}}$. Namely, the inclusion $\iota : G_{\text{ant}} \rightarrow G$ yields a homomorphism $\text{Alb}(\iota) : \text{Alb}(G_{\text{ant}}) \rightarrow \text{Alb}(G)$ with kernel isomorphic to $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$. The assertion (1) is equivalent to the surjectivity of $\text{Alb}(\iota)$, and (3) amounts to the finiteness of its kernel. So Theorem 5.1.1 just means that $\text{Alb}(\iota)$ is an isogeny.

PROPOSITION 5.1.3. *With the above notation and assumptions, G_{ant} is the smallest subgroup scheme $H \subseteq G$ such that $G = H \cdot G_{\text{aff}}$.*

PROOF. Let $H \subseteq G$ be a subgroup scheme. Then $H_{\text{ant}} \cdot G_{\text{aff}} \subseteq G$ is a normal subgroup scheme, since H_{ant} is central in G ; moreover, $G/H_{\text{ant}} \cdot G_{\text{aff}}$ is a quotient of G/G_{aff} , hence an abelian variety. If $G = H \cdot G_{\text{aff}}$, then

$$G/H_{\text{ant}} \cdot G_{\text{aff}} \cong H/H \cap (H_{\text{ant}} \cdot G_{\text{aff}}).$$

Moreover, $H \cap (H_{\text{ant}} \cdot G_{\text{aff}})$ is a normal subgroup scheme of H containing H_{ant} ; thus, the quotient $H/H \cap (H_{\text{ant}} \cdot G_{\text{aff}})$ is affine. So $G/H_{\text{ant}} \cdot G_{\text{aff}}$ is trivial, i.e., $G = H_{\text{ant}} \cdot G_{\text{aff}}$. By Proposition 2.8.4, it follows that $G/H_{\text{ant}} \cong G_{\text{aff}}/G_{\text{aff}} \cap H_{\text{ant}}$. The right-hand side is the quotient of an affine algebraic group by a normal subgroup scheme, and hence is affine. By Theorem 1, it follows that $H_{\text{ant}} \supseteq G_{\text{ant}}$. \square

REMARK 5.1.4. By the above proposition, the extension

$$1 \longrightarrow G_{\text{aff}} \longrightarrow G \longrightarrow \text{Alb}(G) \longrightarrow 1$$

is split if and only if $G_{\text{aff}} \cap G_{\text{ant}}$ is trivial. But this fails in general; in fact, $G_{\text{aff}} \cap G_{\text{ant}}$ is generally of positive dimension, and hence the above extension does not split after pull-back by any isogeny (see Remark 5.5.6 below for specific examples).

We now present two applications of Theorem 5.1.1; first, to the derived subgroup $\mathcal{D}(G)$. Recall from [22, II.5.4.8] (see also [SGA3, VIB.7.8]) that $\mathcal{D}(G)$ is the subgroup functor of G that assigns to any scheme S , the set of those $g \in G(S)$ such that g lies in the commutator subgroup of $G(S')$ for some scheme S' , faithfully flat and of finite presentation over S . Moreover, the group functor $\mathcal{D}(G)$ is represented by a smooth connected subgroup scheme of G , and $\mathcal{D}(G)(\bar{k})$ is the commutator subgroup of $G(\bar{k})$.

COROLLARY 5.1.5. *With the above notation and assumptions, we have $\mathcal{D}(G) = \mathcal{D}(G_{\text{aff}})$. In particular, $\mathcal{D}(G)$ is affine. Also, G is commutative if and only if G_{aff} is commutative.*

Our second application of Theorem 5.1.1 characterizes Lie algebras of algebraic groups in characteristic 0:

COROLLARY 5.1.6. *Assume that $\text{char}(k) = 0$ and consider a finite-dimensional Lie algebra \mathfrak{g} over k , with center \mathfrak{z} . Then the following conditions are equivalent:*

- (1) $\mathfrak{g} = \text{Lie}(G)$ for some algebraic group G .
- (2) $\mathfrak{g} = \text{Lie}(L)$ for some linear algebraic group L .
- (3) $\mathfrak{g}/\mathfrak{z}$ (viewed as a Lie subalgebra of $\text{Lie}(\text{GL}(\mathfrak{g}))$) via the adjoint representation) is the Lie algebra of some algebraic subgroup of $\text{GL}(\mathfrak{g})$.

PROOF. Since (2) \Leftrightarrow (3) follows from [16, V.5.3] and (2) \Rightarrow (1) is obvious, it suffices to show that (1) \Rightarrow (2).

Let G be an algebraic group such that $\mathfrak{g} = \text{Lie}(G)$. We may assume that G is connected. Then $G = G_{\text{aff}} \cdot G_{\text{ant}} = G_{\text{aff}} \cdot Z$ and hence $\mathfrak{g} = \mathfrak{g}_{\text{aff}} + \mathfrak{z}$ with an obvious notation. Thus, we have a decomposition of Lie algebras $\mathfrak{g} = \mathfrak{g}_{\text{aff}} \oplus \mathfrak{a}$ for some linear subspace $\mathfrak{a} \subseteq \mathfrak{z}$, viewed as an abelian Lie algebra. Let $n := \dim(\mathfrak{a})$, then $L := G_{\text{aff}} \times \mathbb{G}_a^n$ is a connected linear algebraic group with Lie algebra isomorphic to \mathfrak{g} . \square

The Lie algebras satisfying the condition (2) above are called *algebraic*.

5.2. Equivariant compactification of homogeneous spaces.

DEFINITION 5.2.1. Let G be an algebraic group and $H \subseteq G$ a subgroup scheme. An *equivariant compactification* of the homogeneous space G/H is a proper G -scheme X equipped with an open equivariant immersion $G/H \rightarrow X$ with schematically dense image.

Equivalently, X is a G -scheme equipped with a base point $x \in X(k)$ such that the G -orbit of x is schematically dense in X and the stabilizer $C_G(x)$ equals H .

THEOREM 5.2.2. *Let G be an algebraic group and $H \subseteq G$ a subgroup scheme. Then G/H has an equivariant compactification by a projective scheme.*

PROOF. Consider a homogeneous space G/H . If G is affine, then H is the stabilizer of a line ℓ in some finite-dimensional G -module V (see e.g. [22, II.2.3.5]). Then the closure of the G -orbit of ℓ in the projective space of lines of V yields the desired projective equivariant compactification X , in view of Proposition 2.7.5. Note that X is equipped with an ample G -linearized invertible sheaf in the sense of [42, I.3 Def. 1.6].

If G is proper, then the homogeneous space G/H is proper as well. So it suffices to check that G/H is projective. For this, we may assume k algebraically closed

by using [EGA, IV.9.1.5]. Note that $H \subseteq H \cdot G^0 \subseteq G$ and the quotient scheme $G/H \cdot G^0$ is finite and étale, hence constant. So the scheme G/H is a finite disjoint union of copies of $H \cdot G^0/G^0 \cong G^0/G^0 \cap H$. Thus, we may assume that G is connected. If $\text{char}(k) = 0$, then G , and hence G/H , are abelian varieties and thus projective. If $\text{char}(k) > 0$, then Proposition 2.9.2 yields that $G/\text{Ker}(F_{G/k}^n)$ is an abelian variety for $n \gg 0$; thus, $G/H \cdot \text{Ker}(F_{G/k}^n)$ is an abelian variety as well, and hence is projective. Moreover, the natural map $f : G/H \rightarrow G/H \cdot \text{Ker}(F_{G/k}^n)$ is the quotient by the action of the infinitesimal group scheme $\text{Ker}(F_{G/k}^n)$, and hence is finite. It follows that G/H is projective.

For an arbitrary algebraic group G , Theorem 2 yields an affine normal subgroup scheme $N \subseteq G$ such that G/N is proper. Then $H \cdot N$ is a subgroup scheme of G and $G/H \cdot N \cong (G/N)/(H \cdot N/N)$ is proper as well, hence projective.

We now claim that it suffices to show the existence of a projective $H \cdot N$ -equivariant compactification Y of $H \cdot N/H$ having an $H \cdot N$ -linearized ample line bundle. Indeed, by [42, Prop. 7.1] applied to the projection $p_1 : G \times Y \rightarrow G$ and to the $H \cdot N$ -torsor $q : G \rightarrow G/H \cdot N$, there exists a unique cartesian square

$$\begin{array}{ccc} G \times Y & \xrightarrow{p_1} & G \\ r \downarrow & & \downarrow q \\ X & \xrightarrow{f} & G/H \cdot N, \end{array}$$

where X is a G -scheme, r and f are G -equivariant, and r is an $H \cdot N$ torsor for the action defined by $x \cdot (g, y) := (gx^{-1}, xy)$, where $x \in H \cdot N$, $g \in G$ and $y \in Y$. Moreover, f is projective, and hence so is X . (We may view $f : X \rightarrow G/H \cdot N$ as the homogeneous fiber bundle with fiber Y associated with the principal bundle $q : G \rightarrow G/H \cdot N$ and the $H \cdot N$ -scheme Y). Also, as in the proof of Proposition 2.8.4, we have a cartesian square

$$\begin{array}{ccc} G \times H \cdot N/H & \xrightarrow{p_1} & G \\ n \downarrow & & \downarrow q \\ G/H & \xrightarrow{f} & G/H \cdot N, \end{array}$$

where n is obtained from the multiplication map $G \times H \cdot N \rightarrow G$ (indeed, this square is commutative and the horizontal arrows are $H \cdot N/H$ -torsors). Since Y is an equivariant compactification of $H \cdot N/H$, it follows that X is the desired equivariant compactification of G/H . This proves the claim.

In view of the first step of the proof (the case of an affine group G), it suffices in turn to check that $H \cdot N$ acts on $H \cdot N/H$ via an affine quotient group.

By Lemma 3.3.6, $(H \cdot N)_{\text{ant}}$ is a quotient of $(H \ltimes N)_{\text{ant}}$. The latter is the fiber at the neutral element of the affinization morphism $H \ltimes N \rightarrow \text{Spec } \mathcal{O}(H \ltimes N)$. Also, recall that $H \ltimes N \cong H \times N$ as schemes, N is affine, and the affinization morphism commutes with finite products; thus, $(H \ltimes N)_{\text{ant}} = H_{\text{ant}}$. As a consequence, we have $(H \cdot N)_{\text{ant}} = H_{\text{ant}}$. Since $H \cdot N/H \cong (H \cdot N/H_{\text{ant}})/(H/H_{\text{ant}})$, and $H \cdot N/H_{\text{ant}} = H \cdot N/(H \cdot N)_{\text{ant}}$ is affine, this completes the proof. \square

REMARKS 5.2.3. (i) With the notation and assumptions of the above theorem, the homogeneous space G/H is quasi-projective. In particular, every algebraic group is quasi-projective.

(ii) Assume in addition that G is smooth. Then G/H has an equivariant compactification by a *normal* projective scheme, as follows from Proposition 2.5.1.

(iii) If $\text{char}(k) = 0$ then every homogeneous space has a *smooth* projective equivariant compactification. This follows indeed from the existence of an equivariant resolution of singularities; see [34, Prop. 3.9.1, Thm. 3.36].

(iv) Over any imperfect field k , there exist smooth connected algebraic groups G having no smooth compactification (equivariant or not). For example, choose $a \in k \setminus k^p$, where $p := \text{char}(k)$, and consider the subgroup scheme $G \subseteq \mathbb{G}_a^2$ defined as the kernel of the homomorphism

$$\mathbb{G}_a^2 \longrightarrow \mathbb{G}_a, \quad (x, y) \longmapsto y^p - x - ax^p.$$

Then G_{k_i} is the kernel of the homomorphism $(x, y) \mapsto (y - a^{1/p}x)^p - x$; thus, $G_{k_i} \cong (\mathbb{G}_a)_{k_i}$ via the map $(x, y) \mapsto y - a^{1/p}x$. As a consequence, G is smooth, connected and unipotent. Moreover, G is equipped with a compactification X , the zero subscheme of $y^p - xz^{p-1} - ax^p$ in \mathbb{P}^2 ; the complement $X \setminus G$ consists of a unique k_i -rational point P with homogeneous coordinates $(1, a^{1/p}, 0)$. One may check that P is a regular, non-smooth point of X . Since G is a regular curve, it follows that X is its unique regular compactification.

5.3. Commutative algebraic groups. This section gives a brief survey of some structure results for the groups of the title. We will see how to build them from two classes: the (commutative) unipotent groups and the groups of multiplicative type, i.e., those algebraic groups M such that $M_{\bar{k}}$ is isomorphic to a subgroup scheme of some torus $\mathbb{G}_{m, \bar{k}}^n$. Both classes are stable under taking subgroup schemes, quotients, commutative group extensions, and extensions of the base field (see [22, IV.2.2.3, IV.2.2.6] for the unipotent groups, and [22, IV.1.2.4, IV.1.4.5] for those of multiplicative type). Also, every homomorphism between groups of different classes is constant (see [22, IV.2.2.4]).

The structure of commutative unipotent algebraic groups is very simple if $\text{char}(k) = 0$: every such group G is isomorphic to its Lie algebra via the exponential map (see [22, IV.2.4]). In particular, G is a *vector group*, i.e., the additive group of a finite-dimensional vector space, uniquely determined up to isomorphism of vector spaces. In characteristic $p > 0$, the commutative unipotent groups are much more involved (see [22, V.1] for structure results over a perfect field); we just recall that for any such group G , the multiplication map p_G^n is zero for $n \gg 0$.

Next, we consider algebraic groups of multiplicative type. Any such group M is uniquely determined by its *character group* $X^*(M)$, consisting of the homomorphisms of k_s -group schemes $G_{k_s} \rightarrow (\mathbb{G}_m)_{k_s}$. Also, $X^*(M)$ is a finitely generated abelian group equipped with an action of the Galois group Γ . Moreover, the assignment $M \mapsto X^*(M)$ yields an anti-equivalence from the category of algebraic groups of multiplicative type (and homomorphisms) to that of finitely generated abelian groups equipped with a Γ -action (and Γ -equivariant homomorphisms). Also, M is a torus (resp. smooth) if and only if the group $X^*(M)$ is torsion-free (resp. p -torsion free if $\text{char}(k) = p > 0$).

It follows readily that every algebraic group of multiplicative type M has a largest subtorus, namely, M_{red}^0 ; its character group is the quotient of $X^*(M)$ by

the torsion subgroup. Moreover, the formation of the largest subtorus commutes with arbitrary field extensions (see [22, IV.1.3] for these results).

THEOREM 5.3.1. *Let G be a commutative affine algebraic group.*

- (1) *G has a largest subgroup scheme of multiplicative type M , and the quotient G/M is unipotent.*
- (2) *When k is perfect, G also has a largest unipotent subgroup scheme U , and $G = M \times U$.*
- (3) *Returning to an arbitrary field k , there exists a subgroup scheme $H \subseteq G$ such that $G = M \cdot H$ and $M \cap H$ is finite.*
- (4) *Any exact sequence of algebraic groups*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

induces exact sequences

$$1 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 1, \quad 1 \rightarrow U_1 \rightarrow U \rightarrow U_2 \rightarrow 1$$

with an obvious notation.

PROOF. The assertions (1) and (2) are proved in [22, IV.3.1.1] and [SGA3, XVII.7.2.1].

(3) We argue as in the proof of Theorem 4.3.4. By (2), we have $G_{k_i} = M_{k_i} \times U_i$ for a unique unipotent subgroup scheme $U_i \subseteq G_{k_i}$. Thus, U_i is defined over some subfield $K \subseteq k_i$, finite over k , and $G_K = M_K \times U'$ with an obvious notation. By Lemma 4.3.5, there exists a subgroup scheme $H \subseteq G$ such that $H_K \supseteq U'$ and H_K/U' is finite. Then $G_K = M_K \cdot H_K$ and $M_K \cap H_K$ is finite. This yields the assertion.

(4) We have $M_1 = M \cap G_1$ by construction. It follows that the quotient map $q : G \rightarrow G_2$ induces a closed immersion of group schemes $\iota : M/M_1 \rightarrow M_2$. Let N be the scheme-theoretic image of ι ; then N is of multiplicative type and G_2/N is a quotient of G/M , hence is unipotent. Thus, $N = M_2$; this yields the first exact sequence, and in turn the second one. \square

The assertion (2) of the above proposition fails over any imperfect field, as shown by the following variant of Example 4.2.7:

EXAMPLE 5.3.2. Let k be an imperfect field and choose a non-trivial finite purely inseparable field extension K of k . Then $G := R_{K/k}(\mathbb{G}_{m,K})$ is a smooth connected commutative algebraic group of dimension $[K : k]$. Moreover, there is a canonical exact sequence

$$(5.3.1) \quad 1 \longrightarrow U' \longrightarrow G_K \xrightarrow{q} \mathbb{G}_{m,K} \longrightarrow 1,$$

where U' is a non-trivial smooth connected unipotent group (see [20, A.5.11]). Also, we have a closed immersion of algebraic groups

$$j : \mathbb{G}_{m,k} \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) = G$$

in view of [20, A.5.7]. Thus, $j_K : \mathbb{G}_{m,K} \rightarrow G_K$ a closed immersion of algebraic groups as well, which yields a splitting of the exact sequence (5.3.1). Moreover, we have an exact sequence

$$(5.3.2) \quad 1 \longrightarrow \mathbb{G}_{m,k} \xrightarrow{j} G \longrightarrow U \longrightarrow 1,$$

where U is smooth, connected and unipotent (since U_K is isomorphic to U').

We claim that (5.3.2) is not split. Indeed, it suffices to show that every homomorphism $f : H \rightarrow G$ is constant, where H is a smooth connected unipotent group. But as in Example 4.2.7, these homomorphisms correspond bijectively to the homomorphisms $f' : H_K \rightarrow \mathbb{G}_{m,K}$ via the assignment $f \mapsto f' := q \circ f$. Moreover, every such homomorphism is constant; this completes the proof of the claim.

Next, consider a smooth connected commutative algebraic group (not necessarily affine) over a perfect field k . By combining Theorems 4.3.2 and 5.3.1, we obtain an exact sequence

$$(5.3.3) \quad 1 \longrightarrow T \times U \longrightarrow G \xrightarrow{q} A \longrightarrow 1,$$

where $T \subseteq G$ is the largest subtorus, and $U \subseteq G$ the largest smooth connected unipotent subgroup scheme. This yields in turn two exact sequences

$$1 \longrightarrow T \longrightarrow G/U \longrightarrow A \longrightarrow 1, \quad 1 \longrightarrow U \longrightarrow G/T \longrightarrow A \longrightarrow 1$$

and a homomorphism

$$(5.3.4) \quad f : G \longrightarrow G/U \times_A G/T$$

which is readily seen to be an isomorphism.

Thus, the structure of G is reduced to those of G/U and G/T . The former will be described in the next section. We now describe the latter under the assumption that $\text{char}(k) = 0$; we then consider extensions

$$(5.3.5) \quad 1 \longrightarrow U \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where U is unipotent and A is a prescribed abelian variety. As U is a vector group, such an extension is called a *vector extension* of A .

When $U \cong \mathbb{G}_a$, every extension (5.3.5) yields a \mathbb{G}_a -torsor over A ; this defines a map

$$\text{Ext}^1(A, \mathbb{G}_a) \rightarrow H^1(A, \mathcal{O}_A)$$

which is in fact an isomorphism (see e.g. [46, III.17]). For an arbitrary vector group U , we obtain an isomorphism

$$\text{Ext}^1(A, U) \cong H^1(A, \mathcal{O}_A) \otimes_k U.$$

Viewing the right-hand side as $\text{Hom}_{\text{gp.sch.}}(H^1(A, \mathcal{O}_A)^*, U)$, it follows that there is a *universal vector extension*

$$(5.3.6) \quad 1 \longrightarrow H^1(A, \mathcal{O}_A)^* \longrightarrow E(A) \longrightarrow A \longrightarrow 1,$$

from which every extension (5.3.5) is obtained via push-out by a unique linear map $U \rightarrow H^1(A, \mathcal{O}_A)^*$.

REMARKS 5.3.3. (i) When $\text{char}(k) > 0$, every abelian variety A still has a universal vector extension $E(A)$, as shown by the above arguments. Yet note that $E(A)$ classifies extensions of A by vector groups, which form a very special class of smooth connected unipotent commutative groups in this setting.

(ii) Little is known on the structure of commutative algebraic groups over imperfect fields, due to the failure of Chevalley's structure theorem. We will present a partial remedy to that failure in Corollary 5.6.8, as a consequence of a structure result for commutative algebraic groups in positive characteristics (Theorem 5.6.3).

5.4. Semi-abelian varieties.

DEFINITION 5.4.1. A *semi-abelian variety* is an algebraic group obtained as an extension

$$(5.4.1) \quad 1 \longrightarrow T \longrightarrow G \xrightarrow{q} A \longrightarrow 1,$$

where T is a torus and A an abelian variety.

REMARKS 5.4.2. (i) With the above notation, q is the Albanese homomorphism. It follows that T and A are uniquely determined by G .

(ii) Every semi-abelian variety is smooth and connected; it is also commutative in view of Corollary 5.1.5. Moreover, the multiplication map n_G is an isogeny for any non-zero integer n , since this holds for abelian varieties (by Lemma 4.2.1) and for tori.

(iii) Given a semi-abelian variety G over k and an extension of fields K of k , the K -algebraic group G_K is a semi-abelian variety.

In the opposite direction, we will need:

LEMMA 5.4.3. *Let G be an algebraic group. If $G_{\bar{k}}$ is a semi-abelian variety, then G is a semi-abelian variety.*

PROOF. Clearly, G is smooth, connected and commutative. Also, arguing as at the end of the proof of Theorem 4.3.4, we obtain a finite extension K of k and an exact sequence of algebraic groups over K

$$1 \longrightarrow T' \longrightarrow G_K \xrightarrow{q'} A' \longrightarrow 1,$$

where T' is a torus and A' an abelian variety; here q' is the Albanese homomorphism. On the other hand, Theorem 4.3.4 yields an exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{q} A \longrightarrow 1,$$

where N is connected and affine, and A is an abelian variety; also, q is the Albanese homomorphism. By Remark 4.3.6, N_K contains T' and the quotient N_K/T' is infinitesimal; as a consequence, T' is the largest subtorus of N_K . Consider the largest subgroup scheme $M \subseteq N$ of multiplicative type; then N/M is unipotent and connected. Since the formation of M commutes with field extensions, we have $(N/M)_K = N_K/M_K$. The latter is a quotient of N_K/T' ; hence N_K/M_K is infinitesimal, and so is N/M . Let $T := M_{\text{red}}^0$; this is the largest subtorus of N and M/T is infinitesimal, hence so is N/T . It follows that $G/T =: A$ is proper; since G is smooth and connected, A is an abelian variety. Thus, G is a semi-abelian variety; moreover, $T_K = T'$ and $A_K = A'$ in view of Remark 5.4.2 (i). \square

REMARK 5.4.4. More generally, if G_K is a semi-abelian variety for some field extension K of k , then G is a semi-abelian variety as well. This may be checked as in Remarks 3.1.5 and 4.1.5, by adapting the proof of [20, 1.1.9].

Next, we obtain a geometric characterization of semi-abelian varieties:

PROPOSITION 5.4.5. *Let G be a smooth connected algebraic group over a perfect field k . Then G is a semi-abelian variety if and only if every morphism (of schemes) $f : \mathbb{A}^1 \rightarrow G$ is constant.*

PROOF. Assume that G is semi-abelian and consider the exact sequence (5.4.1). Then $q \circ f : \mathbb{A}^1 \rightarrow A$ is constant by Lemma 4.1.1. Thus, there exists $g \in G(\bar{k})$ such that $f_{\bar{k}}$ factors through the translate $gT_{\bar{k}} \subseteq G_{\bar{k}}$. So we may view $f_{\bar{k}}$ as a morphism $\mathbb{A}_{\bar{k}}^1 \rightarrow (\mathbb{A}_{\bar{k}}^1 \setminus 0)^n$ for some positive integer n . Since every morphism $\mathbb{A}_{\bar{k}}^1 \rightarrow \mathbb{A}_{\bar{k}}^1 \setminus 0$ is constant, we see that f is constant.

Conversely, assume that every morphism $\mathbb{A}^1 \rightarrow G$ is constant. By [56, 14.3.10], it follows that every smooth connected unipotent subgroup of G is trivial. In particular, the unipotent radical of G_{aff} is trivial, i.e., G_{aff} is reductive. Also, G_{aff} has no root subgroups and hence is a torus. So G is a semi-abelian variety. \square

As a consequence, semi-abelian varieties are stable under group extensions in a strong sense:

COROLLARY 5.4.6. *Let G be a smooth connected algebraic group and $N \subseteq G$ a subgroup scheme.*

- (1) *If G is a semi-abelian variety, then so is G/N . If in addition N is smooth and connected, then N is a semi-abelian variety as well.*
- (2) *If N and G/N are semi-abelian varieties, then so is G .*
- (3) *Let $f : G \rightarrow H$ be an isogeny, where H is a semi-abelian variety. Then G is a semi-abelian variety.*

PROOF. We may assume k algebraically closed by Lemma 5.4.3.

(1) We have a commutative diagram of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & T \cap N & \longrightarrow & N & \longrightarrow & P \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A \longrightarrow 1, \end{array}$$

where T is a torus, A an abelian variety, and P a subgroup scheme of A . This yields an exact sequence

$$1 \longrightarrow T/T \cap N \longrightarrow G/N \longrightarrow A/B \longrightarrow 1,$$

where $T/T \cap N$ is a torus and A/B is an abelian variety. Thus, G/N is a semi-abelian variety. The assertion on N follows from Proposition 5.4.5.

(2) Denote by $q : G \rightarrow G/N$ the quotient morphism. Let $f : \mathbb{A}^1 \rightarrow G$ be a morphism; then $q \circ f$ is constant by Proposition 5.4.5 again. Translating by some $g \in G(k)$, we may thus assume that f factors through N . Then f is constant; this yields the assertion.

(3) Consider the Albanese homomorphism $\alpha_H : H \rightarrow \text{Alb}(H)$ and the kernel N of the composition $\alpha_H \circ f$. Then N is an extension of $\text{Ker}(\alpha_H)$ (a torus) by $\text{Ker}(f)$ (a finite group scheme). As a consequence, N is affine and N/T is finite, where $T \subseteq N$ denotes the largest subtorus. Since $G/N \cong \text{Alb}(H)$ is an abelian variety, it follows that G/T is an abelian variety as well. Thus, G is a semi-abelian variety. \square

REMARKS 5.4.7. (i) Proposition 5.4.5 fails over any imperfect field k . Indeed, as in Remark 5.2.3 (iv), consider the subgroup scheme $G \subset \mathbb{G}_a^2$ defined by $y^p = x + ax^p$, where $p := \text{char}(k)$ and $a \in k \setminus k^p$. Then every morphism $f : \mathbb{A}^1 \rightarrow G$ is just given by $x(t), y(t) \in k[t]$ such that

$$y(t)^p = x(t) + ax(t)^p.$$

Thus, $x(t) = z(t)^p$ for a unique $z(t) \in k^{1/p}[t]$ such that

$$y(t) = z(t) + a^{1/p} z(t)^p.$$

Consider the monomial of highest degree in $z(t)$, say $a_n t^n$. If $n \geq 1$, then the monomial of highest degree in $y(t)$ is $a^{1/p} a_n^p t^{np}$. Since $a^{1/p} \notin k$ and $a_n^p \in k$, this contradicts the fact that $y(t) \in k[t]$. So $n = 0$, i.e., $z(t)$ is constant; then so are $x(t)$ and $y(t)$.

(ii) Consider a semi-abelian variety G and a semi-abelian subvariety $H \subseteq G$. Then the induced homomorphism between Albanese varieties, $\text{Alb}(H) \rightarrow \text{Alb}(G)$, has a finite kernel that may be arbitrary large. For example, let H be a non-trivial abelian variety over an algebraically closed field; then H contains a copy of the constant group scheme \mathbb{Z}/ℓ for any prime number $\ell \neq \text{char}(k)$ (see [41, p. 64]). Choosing a root of unity of order ℓ in k , we obtain a closed immersion $j : \mathbb{Z}/\ell \rightarrow \mathbb{G}_m$ and, in turn, a commutative diagram of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}/\ell & \longrightarrow & H & \longrightarrow & A \longrightarrow 1 \\ & & \downarrow j & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G & \longrightarrow & A \longrightarrow 1, \end{array}$$

where A is an abelian variety. So G is a semi-abelian variety containing H , and the kernel of the homomorphism $H = \text{Alb}(H) \rightarrow \text{Alb}(G) = A$ is \mathbb{Z}/ℓ .

(iii) Consider again a semi-abelian variety G and let $H \subseteq G$ be a subgroup scheme. Then H_{red}^0 is a semi-abelian variety. To see this, we may replace G , H with G/H_{ant} , H/H_{ant} , and hence assume that H is affine. Let T be the largest torus of G , then $H/T \cap H$ is affine and isomorphic to a subgroup scheme of the abelian variety G/T . Thus, $H/T \cap H$ is finite; it follows that $(H/T \cap H)_{\text{red}}^0$ is trivial, and hence $H_{\text{red}}^0 \subseteq T$. So $H_{\text{red}}^0 = (T \cap H)_{\text{red}}^0$ is a torus.

Next, we extend Lemma 4.1.3 to morphisms to semi-abelian varieties:

LEMMA 5.4.8. *Let X , Y be varieties equipped with k -rational points x_0 , y_0 and let $f : X \times Y \rightarrow G$ be a morphism to a semi-abelian variety. Then we have identically*

$$f(x, y) f(x, y_0)^{-1} f(x_0, y)^{-1} f(x_0, y_0) = e.$$

PROOF. We may assume that $f(x_0, y_0) = e$. Let $q : G \rightarrow A$ denote the Albanese homomorphism. By Lemma 4.1.3, we have identically

$$(q \circ f)(x, y) = (q \circ f)(x, y_0) (q \circ f)(x_0, y).$$

Thus, the morphism

$$\varphi : X \times Y \longrightarrow G, \quad (x, y) \longmapsto f(x, y) f(x, y_0)^{-1} f(x_0, y)^{-1}$$

factors through the torus $T := \text{Ker}(q)$. Also, we have identically $\varphi(x, y_0) = \varphi(x_0, y) = e$. We now show that φ is constant. For this, we may assume k algebraically closed; then $T \cong \mathbb{G}_m^n$ and accordingly, $\varphi = \varphi_1 \times \cdots \times \varphi_n$ for some $\varphi_i : X \times Y \rightarrow \mathbb{G}_m$. Equivalently, $\varphi \in \mathcal{O}(X \times Y)^*$ (the unit group of the algebra $\mathcal{O}(X \times Y)$). By [23, Lem. 2.1], there exists $u_i \in \mathcal{O}(X)^*$ and $v_i \in \mathcal{O}(Y)^*$ such that $\varphi_i(x, y) = u_i(x) v_i(y)$ identically. As $\varphi_i(x, y_0) = \varphi_i(x_0, y) = 1$, it follows that $u_i = 1 = v_i$; thus, φ factors through e . \square

REMARKS 5.4.9. (i) In view of Corollary 5.4.6 and Lemma 5.4.8, *every smooth connected algebraic group G admits a universal homomorphism to a semi-abelian variety, which satisfies the assertions of Proposition 4.1.4.* In particular, the kernel N of this homomorphism is connected. If k is perfect, then one may check that $N = \mathcal{R}_u(G_{\text{aff}}) \cdot \mathcal{D}(G_{\text{aff}})$, where \mathcal{R}_u denotes the unipotent radical (and \mathcal{D} the derived subgroup); as a consequence, N is smooth. This fails over an arbitrary field k , as shown by Example 4.2.7 again (then N is the kernel of the Albanese homomorphism).

(ii) More generally, *every pointed variety admits a universal morphism to a semi-abelian variety*, as follows from [53, Thm. 7] over an algebraically closed field, and from [62, Thm. A1] over an arbitrary field. This universal morphism, which gives back that of Theorem 4.1.6, is still called the Albanese morphism.

Finally, we discuss the structure of semi-abelian varieties by adapting the approach to the classification of vector extensions (§5.3). Consider first the extensions

$$(5.4.2) \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow G \xrightarrow{q_{k_s}} A \longrightarrow 1,$$

where A is a prescribed abelian variety. Any such extension yields a \mathbb{G}_m -torsor over A , or equivalently a line bundle over that variety. This defines a map

$$\text{Ext}^1(A, \mathbb{G}_m) \longrightarrow \text{Pic}(A).$$

By the Weil-Barsotti formula (see e.g. [46, III.17, III.18]), this map is injective and its image is the subgroup of translation-invariant line bundles over A ; this is the group of k -rational points of the dual abelian variety \hat{A} . We may thus identify $\text{Ext}^1(A, \mathbb{G}_m)$ with $\hat{A}(k)$. Also, using a Poincaré sheaf, we will view the points of $\hat{A}(k)$ as the algebraically trivial invertible sheaves \mathcal{L} on A equipped with a rigidification at 0, i.e., an isomorphism $k \xrightarrow{\cong} \mathcal{L}_0$.

Denoting by $\mathcal{L} \in \hat{A}(k)$ the invertible sheaf that corresponds to the extension (5.4.2), we obtain an isomorphism of sheaves of algebras over A

$$q_*(\mathcal{O}_G) \otimes k_s \cong \bigoplus_{n=-\infty}^{\infty} \mathcal{L}^{\otimes n},$$

since G is the \mathbb{G}_m -torsor over A associated with \mathcal{L} . The multiplication in the right-hand side is given by the natural isomorphisms

$$\mathcal{L}^{\otimes m} \otimes_{\mathcal{O}_A} \mathcal{L}^{\otimes n} \xrightarrow{\cong} \mathcal{L}^{\otimes (m+n)}.$$

For an arbitrary extension (5.4.1), consider the associated extension

$$(5.4.3) \quad 1 \longrightarrow T_{k_s} \longrightarrow G_{k_s} \xrightarrow{q} A_{k_s} \longrightarrow 1$$

and denote by Λ the character group of T . Then the split torus T_{k_s} is canonically isomorphic to $\text{Hom}_{\text{gp.sch.}}(\Lambda, (\mathbb{G}_m)_{k_s})$. Moreover, every $\lambda \in \Lambda$ yields an extension of the form (5.4.2) via pushout by $\lambda : T_{k_s} \rightarrow (\mathbb{G}_m)_{k_s}$. This defines a map

$$\gamma_{k_s} : \text{Ext}^1(A_{k_s}, T_{k_s}) \rightarrow \text{Hom}_{\text{gp.}}(\Lambda, \hat{A}(k_s))$$

which is readily seen to be an isomorphism by identifying T_{k_s} with $(\mathbb{G}_m)_{k_s}^n$ and accordingly Λ with \mathbb{Z}^n . Likewise, we obtain an isomorphism of sheaves of algebras over A_{k_s} :

$$(5.4.4) \quad q_*(\mathcal{O}_G)_{k_s} \cong \bigoplus_{\lambda \in \Lambda} c(\lambda),$$

where $c : \Lambda \rightarrow \widehat{A}(k_s)$ denotes the map classifying the extension. Here again, the multiplication of the right-hand side is given by the natural isomorphisms

$$c(\lambda) \otimes_{\mathcal{O}_A} c(\mu) \xrightarrow{\cong} c(\lambda + \mu).$$

By construction, γ_{k_s} is equivariant for the natural actions of Γ . Composing γ_{k_s} with the base change map $\text{Ext}^1(A, T) \rightarrow \text{Ext}^1(A_{k_s}, T_{k_s})$ (which is Γ -invariant), we thus obtain a map

$$(5.4.5) \quad \gamma : \text{Ext}^1(A, T) \rightarrow \text{Hom}_{\text{gp}}^{\Gamma}(\Lambda, \widehat{A}(k_s)).$$

By Galois descent, this yields:

PROPOSITION 5.4.10. *Let A be an abelian variety and T a torus. Then the map (5.4.5) is an isomorphism.*

5.5. Structure of anti-affine groups. Let G be an anti-affine group. Recall from Lemma 3.3.2 and Proposition 3.3.4 that G is smooth, connected and commutative; also, every quotient of G is anti-affine.

PROPOSITION 5.5.1. *When $\text{char}(k) = p > 0$, every anti-affine group over k is a semi-abelian variety.*

PROOF. By Lemma 5.4.3, we may assume k algebraically closed. Using the isomorphism (5.3.4), we may further assume that G lies in an exact sequence

$$1 \longrightarrow U \longrightarrow G \xrightarrow{q} A \longrightarrow 1,$$

where U is unipotent and A is an abelian variety. We may then choose a positive integer n such that the power map p_U^n is zero. Then p_G^n factors through a morphism $f : G/U = A \rightarrow G$. Since f is surjective and the square

$$\begin{array}{ccc} G & \xrightarrow{p_G^n} & G \\ q \downarrow & & \downarrow q \\ A & \xrightarrow{p_A^n} & A \end{array}$$

commutes, the composition $q \circ f : A \rightarrow A$ equals p_A^n and hence is an isogeny. Let B be the image of f ; then B is an abelian subvariety of G , isogenous to A via q . It follows that $G = U \cdot B$ and $U \cap B$ is finite. Thus, $U/U \cap B \cong G/B$ is affine (as a quotient of U) and anti-affine (as a quotient of G), hence trivial. Since U is smooth and connected, it must be trivial as well. \square

Returning to an arbitrary field, we now characterize those semi-abelian varieties that are anti-affine:

PROPOSITION 5.5.2. *Let G be a semi-abelian variety, extension of an abelian variety A by a torus T with character group Λ , and let $c : \Lambda \rightarrow \widehat{A}(k_s)$ be the map classifying this extension. Then G is anti-affine if and only if c is injective.*

PROOF. Since G is anti-affine if and only if $G_{\bar{k}}$ is anti-affine, we may assume k algebraically closed. Then the isomorphism (5.4.4) yields

$$\mathcal{O}(G) = H^0(A, q_*(\mathcal{O}_G)) \cong \bigoplus_{\lambda \in \Lambda} H^0(A, c(\lambda)).$$

Thus, G is anti-affine if and only if $H^0(A, c(\lambda)) = 0$ for all non-zero λ . So it suffices to check that $H^0(A, \mathcal{L}) = 0$ for any non-zero $\mathcal{L} \in \widehat{A}(k)$.

If $H^0(A, \mathcal{L}) \neq 0$, then $\mathcal{L} \cong \mathcal{O}_A(D)$ for some non-zero effective divisor D on A . Since \mathcal{L} is algebraically trivial, the intersection number $D \cdot C$ is zero for any irreducible curve C on A . Now choose a smooth point $x \in D_{\text{red}}(k)$; then there exists an irreducible curve C through x which intersects D_{red} transversally at that point. Then $D \cdot C > 0$, a contradiction. \square

COROLLARY 5.5.3. *The anti-affine semi-abelian varieties over a field k are classified by the pairs (A, Λ) , where A is an abelian variety over k and $\Lambda \subset \widehat{A}(k_s)$ is a Γ -stable free abelian subgroup of finite rank.*

If the ground field k is finite, then the group $\widehat{A}(k_s) = \widehat{A}(\bar{k})$ is the union of the subgroups $\widehat{A}(K)$, where K runs over all finite extensions of k . Since all these subgroups are finite, $\widehat{A}(k_s)$ is a torsion group. In view of Proposition 5.5.1 and Corollary 5.5.3, this readily yields:

COROLLARY 5.5.4. *Any anti-affine group over a finite field is an abelian variety.*

Using the Rosenlicht decomposition (Theorem 5.1.1), this yields in turn:

COROLLARY 5.5.5. *Let G be a smooth connected algebraic group over a finite field k . Then there is a unique decomposition $G = L \cdot A$, where $L \trianglelefteq G$ is smooth, connected and affine, and $A \subseteq G$ is an abelian variety. Moreover, $L \cap A$ is finite.*

REMARK 5.5.6. Returning to an abelian variety A over an arbitrary field k , assume that there exists an invertible sheaf $\mathcal{L} \in \widehat{A}(k)$ of infinite order (such a pair (A, \mathcal{L}) exists unless k is algebraic over a finite field, as seen in Example 3.2.3). Consider the associated extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow A \longrightarrow 1.$$

Then G is anti-affine by Proposition 5.5.2, and hence the above extension is not split. In fact, it does not split after pull-back by any isogeny $B \rightarrow A$: otherwise, it would split after pull-back by $n_A : A \rightarrow A$ for some positive integer n , or equivalently, after push-forward under the n th power map of \mathbb{G}_m . But this push-forward amounts to replacing \mathcal{L} with $\mathcal{L}^{\otimes n}$, which is still of infinite order.

Next, we turn to the classification of anti-affine groups in characteristic 0. As a first step, we obtain:

LEMMA 5.5.7. *Let G be a connected commutative algebraic group over a field of characteristic 0. Let $T \subseteq G$ be the largest torus and $U \subseteq G$ the largest unipotent subgroup scheme. Then G is anti-affine if and only if G/U and G/T are anti-affine.*

PROOF. If G is anti-affine, then so are its quotients G/U and G/T . Conversely, assume that G/U and G/T are anti-affine. Then $G/G_{\text{ant}} \cdot U$ is anti-affine, and also affine as a quotient of G/G_{ant} . Thus, $G = G_{\text{ant}} \cdot U$ and hence $G/G_{\text{ant}} \cong U/U \cap G_{\text{ant}}$ is unipotent. Likewise, one shows that G/G_{ant} is a torus. Thus, G/G_{ant} is trivial. \square

With the notation and assumptions of the above lemma, G/U is a semi-abelian variety and G/T is a vector extension of A . So to complete the classification, it remains to characterize those vector extensions that are anti-affine:

PROPOSITION 5.5.8. *Assume that $\text{char}(k) = 0$. Let G be an extension of an abelian variety A by a vector group U and denote by $\gamma : H^1(A, \mathcal{O}_A)^* \rightarrow U$ the linear map classifying the extension. Then G is anti-affine if and only if γ is surjective.*

PROOF. We have to show that the universal vector extension $E(A)$ is anti-affine, and every anti-affine vector extension (5.3.5) is a quotient of $E(A)$.

Let $V := H^1(A, \mathcal{O}_A)^*$; then $V = E(A)_{\text{aff}}$ with the notation of the Rosenlicht decomposition. By that decomposition, $(E(A)_{\text{ant}})_{\text{aff}} \subseteq V \cap E(A)_{\text{aff}}$ and the quotient is finite. Since $V \cap E(A)_{\text{aff}}$ is a vector group, we obtain the equality $(E(A)_{\text{ant}})_{\text{aff}} = V \cap E(A)_{\text{aff}} =: W$. In view of Remark 5.1.2, this yields a commutative diagram of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & W & \longrightarrow & E(A)_{\text{ant}} & \longrightarrow & A \longrightarrow 1 \\ & & \downarrow \iota & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & V & \longrightarrow & E(A) & \longrightarrow & A \longrightarrow 1, \end{array}$$

where ι is injective. As a consequence, every extension of A by \mathbb{G}_a is obtained by pushout from the top extension, i.e., the resulting map $W^* \rightarrow \text{Ext}^1(A, \mathbb{G}_a)$ is surjective. Since the bottom extension is universal, it follows that $W = V$ and $E(A)_{\text{ant}} = E(A)$.

Next, let G be a vector extension of A . If G is anti-affine, then the classifying map $\gamma : E(A) \rightarrow G$ is surjective by Lemma 3.3.6. The converse assertion follows from the fact that every quotient of an anti-affine group is anti-affine. \square

As a consequence, *the anti-affine vector extensions of A are classified by the linear subspaces $V \subseteq H^1(A, \mathcal{O}_A)$ by assigning to any such extension, the image of the transpose of its classifying map. Combining this result with the isomorphism (5.3.4), Lemma 5.5.7 and Proposition 5.5.2, we obtain the desired classification:*

THEOREM 5.5.9. *When $\text{char}(k) > 0$, the anti-affine groups are classified by the pairs (A, Λ) , where A is an abelian variety over k and $\Lambda \subseteq \hat{A}(k_s)$ is a Γ -stable free abelian subgroup of finite rank.*

When $\text{char}(k) = 0$, the anti-affine groups are classified by the triples (A, Λ, V) , where (A, Λ) is as above and $V \subseteq H^1(A, \mathcal{O}_A)$ is a linear subspace.

5.6. Commutative algebraic groups (continued). We first show that every group as in the title has a “semi-abelian radical”:

LEMMA 5.6.1. *Let G be a commutative algebraic group.*

- (1) *G has a largest semi-abelian subvariety that we will denote by G_{sab} . Moreover, $(G/G_{\text{sab}})_{\text{sab}}$ is trivial.*
- (2) *The formation of G_{sab} commutes with algebraic field extensions.*

PROOF. (1) This follows from the stability of semi-abelian varieties under taking products, quotients and extensions (Corollary 5.4.6) by arguing as in the proof of Lemma 3.1.4.

(2) When $\text{char}(k) = 0$, the statement is obtained by Galois descent as in the proof of Lemma 3.1.4 again. When $\text{char}(k) = p > 0$, we have $G_{\text{ant}} \subseteq G_{\text{sab}}$ by Proposition 5.5.1; also, $G_{\text{sab}}/G_{\text{ant}} = (G/G_{\text{ant}})_{\text{sab}}$ as follows from Corollary 5.4.6 again. Since G/G_{ant} is affine, $(G/G_{\text{ant}})_{\text{sab}}$ is just its largest subtorus. As the formations of G_{ant} and of the largest subtorus commute with field extensions, the assertion follows. \square

Next, we characterize those commutative algebraic groups that have a trivial semi-abelian radical:

LEMMA 5.6.2. *If $\text{char}(k) = p > 0$, then the following conditions are equivalent for a commutative algebraic group G :*

- (1) G_{sab} is trivial.
- (2) G is affine and its largest subgroup of multiplicative type is finite.
- (3) The multiplication map n_G is zero for some positive integer n .

If in addition G is smooth and connected, these conditions are equivalent to G being unipotent.

PROOF. (1) \Rightarrow (2) Note that G_{ant} is trivial in view of Proposition 5.5.1. Thus, G is affine. Also, G contains no non-trivial torus; this yields the assertion.

(2) \Rightarrow (3) By Theorem 5.3.1, we have an exact sequence

$$1 \longrightarrow M \longrightarrow G \longrightarrow U \longrightarrow 1,$$

where M is of multiplicative type and U is unipotent. Then M is finite, and hence killed by n_M for some positive integer n . Also, recall that U is killed by p_U^m for some m . It follows that $np_G^m = 0$.

(3) \Rightarrow (1) This follows from the fact that $n_H \neq 0$ for any non-trivial semi-abelian variety H and any $n \neq 0$.

When G is smooth and connected, the condition (2) implies that G is unipotent, in view of Theorem 5.3.1 again. Conversely, if G is unipotent, then it clearly satisfies the condition (2). \square

We say that a commutative algebraic group G has *finite exponent* if it satisfies the above condition (3). When $\text{char}(k) = 0$, this just means that G is finite; when $\text{char}(k) > 0$, this amounts to G being an extension of a unipotent algebraic group by a finite group scheme of multiplicative type.

THEOREM 5.6.3. *Assume that $\text{char}(k) = p > 0$ and consider a commutative algebraic group G and its largest semi-abelian subvariety G_{sab} .*

- (1) *The quotient G/G_{sab} has finite exponent.*
- (2) *There exists a subgroup scheme $H \subseteq G$ such that $G = G_{\text{sab}} \cdot H$ and $G_{\text{sab}} \cap H$ is finite.*
- (3) *If G is smooth and connected, then G/G_{sab} is unipotent. If in addition k is perfect, then we may take for H the largest smooth connected unipotent subgroup scheme of G .*

PROOF. (1) This follows from Lemmas 5.6.1 and 5.6.2.

(2) By (1), we may choose $n > 0$ such that $n_{G/G_{\text{sab}}} = 0$. Then n_G factors through G_{sab} ; also, recall that $n_{G_{\text{sab}}}$ is an isogeny. It follows that $G = G_{\text{sab}} \cdot \text{Ker}(n_G)$ and $G_{\text{sab}} \cap \text{Ker}(n_G)$ is finite.

(3) The first assertion is a consequence of Lemma 5.6.2 again.

Assume k perfect and consider the Rosenlicht decomposition $G = G_{\text{aff}} \cdot G_{\text{ant}}$. Then $G_{\text{aff}} = T \times U$, where T is a torus and U a smooth connected unipotent group. Thus, $G = T \cdot U \cdot G_{\text{ant}}$. Also, $T \cdot G_{\text{ant}}$ is a semi-abelian subvariety of G ; moreover, $G/T \cdot G_{\text{ant}}$ is isomorphic to a quotient of U , and hence contains no semi-abelian variety. Thus, $T \cdot G_{\text{ant}} = G_{\text{sab}}$ and hence $G = G_{\text{sab}} \cdot U$. Moreover, $G_{\text{sab}} \cap U$ is clearly finite. \square

In analogy with the structure of commutative affine algebraic groups (Theorem 5.3.1), note that the class of commutative algebraic groups of finite exponent is stable under taking subgroup schemes, quotients, commutative group extensions,

and extensions of the ground field. Moreover, when G is a semi-abelian variety and H a commutative algebraic group of finite exponent, every homomorphism $\varphi : G \rightarrow H$ is constant, and every homomorphism $\psi : H \rightarrow G$ factors through a finite subgroup scheme of G .

Also, note the following analogue of Lemma 3.3.6, which yields that the assignment $G \mapsto G_{\text{sab}}$ is close to being exact:

LEMMA 5.6.4. *Assume that $\text{char}(k) = p$. Let G be a commutative algebraic group and $H \subseteq G$ a subgroup scheme.*

- (1) $H_{\text{sab}} \subseteq G_{\text{sab}} \cap H$ and the quotient is finite.
- (2) The quotient map $q : G \rightarrow G/H$ yields an isomorphism

$$G_{\text{sab}}/G_{\text{sab}} \cap H \xrightarrow{\cong} (G/H)_{\text{sab}}.$$

PROOF. (1) This follows readily from Corollary 5.4.6 and Lemma 5.6.1.

(2) Observe that q restricts to a closed immersion of group schemes

$$\iota : G_{\text{sab}}/G_{\text{sab}} \cap H \longrightarrow G/H$$

that we will regard as an inclusion. Also, by Theorem 5.6.3, the quotient $G/G_{\text{sab}} \cdot H$ has finite exponent. Since $G/G_{\text{sab}} \cdot H \cong (G/H)/(G_{\text{sab}} \cdot H/H)$ is isomorphic to the cokernel of ι , it follows that $G_{\text{sab}}/G_{\text{sab}} \cap H \supseteq (G/H)_{\text{sab}}$. But the opposite inclusion holds by Corollary 5.4.6 again; this yields the assertion. \square

REMARK 5.6.5. When $\text{char}(k) = 0$, the quotient G/G_{sab} is not necessarily unipotent for a smooth connected commutative group G . This happens for example when G is the universal vector extension of a non-trivial abelian variety; then G_{sab} is trivial.

Still, the structure of such a group G reduces somehow to that of its anti-affine part. Specifically, one may check that there exists a subtorus $T \subseteq G$ and a vector subgroup $U \subseteq G$ such that the multiplication map $T \times U \times G_{\text{ant}} \rightarrow G$ is an isogeny; moreover, T is uniquely determined up to isogeny, and U is uniquely determined.

As an application of Theorem 5.6.3, we present a remedy to (or a measure of) the failure of Chevalley's structure theorem over imperfect fields. To state it, we need the following:

DEFINITION 5.6.6. A smooth connected algebraic group G is called a *pseudo-abelian variety* if it does not contain any non-trivial smooth connected affine normal subgroup scheme.

REMARKS 5.6.7. (i) By Lemma 3.1.4, every smooth connected algebraic group G lies in a unique exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where L is smooth, connected and linear, and Q is a pseudo-abelian variety.

(ii) If k is perfect, then every pseudo-abelian variety is just an abelian variety, as follows from Theorem 4.3.2. But there exist non-proper pseudo-abelian varieties over any imperfect field, as shown by Example 4.2.7.

COROLLARY 5.6.8. *Let G be a pseudo-abelian variety. Then G is commutative and lies in a unique exact sequence*

$$(5.6.1) \quad 1 \longrightarrow A \longrightarrow G \longrightarrow U \longrightarrow 1,$$

where A is an abelian variety and U is unipotent.

PROOF. The commutativity of G follows from Corollary 5.1.5.

For the remaining assertion, we may assume that $\text{char}(k) = p > 0$. By Theorem 5.6.3, G lies in a unique extension

$$1 \longrightarrow G_{\text{sab}} \longrightarrow G \longrightarrow U \longrightarrow 1,$$

where G_{sab} is a semi-abelian variety and U is unipotent. Since G contains no non-trivial torus, G_{sab} is an abelian variety. This shows the existence of the exact sequence (5.6.1); for its uniqueness, just observe that $A = G_{\text{ant}}$. \square

Notes and references.

Theorem 5.1.1 is due to Rosenlicht (see [48, Cor. 5, p. 140]). This result is very useful for reducing questions about general algebraic groups to the linear and anti-affine cases; see [39] for a recent application.

When $\text{char}(k) = p > 0$, characterizing Lie algebras of *smooth* algebraic groups among p -Lie algebras seems to be an open problem. It is well-known that every finite-dimensional p -Lie algebra is the Lie algebra of some *infinitesimal* group scheme; see [22, II.7.3, II.7.4] for this result and further developments.

The proof of Theorem 5.2.2 is adapted from [13, Thm. 4.13]. The quasi-projectivity of homogeneous spaces is a classical result, see e.g. [47, Cor. VI.2.6]. Also, the existence of equivariant compactifications of certain homogeneous spaces having no separable point at infinity has attracted recent interest, see [24, 25].

In positive characteristics, the existence of (equivariant or not) *regular* projective compactifications of homogeneous spaces is an open question.

Example 5.3.2 is due to Raynaud, see [SGA3, XVII.C.5].

The algebraic group considered in Remark 5.4.7 (i) is an example of a k -wound unipotent group G , i.e., every morphism $A^1 \rightarrow G$ is constant. This notion plays an important rôle in the structure of smooth connected unipotent groups over imperfect fields, see [20, B.2].

The notion of Albanese morphism extends to non-pointed varieties by replacing semi-abelian varieties with semi-abelian torsors. More specifically, for any variety X , there exists a semi-abelian variety Alb_X^0 , an Alb_X^0 -torsor Alb_X^1 (over $\text{Spec}(k)$) and a morphism

$$u_X : X \longrightarrow \text{Alb}_X^1$$

such that for any semi-abelian variety A^0 , any torsor A^1 under A^0 , and any morphism $f : X \rightarrow A^1$, there exists a unique morphism of varieties

$$g^1 : \text{Alb}_X^1 \rightarrow A^1$$

such that $g^1 \circ u_X = f$ and there exists a unique morphism of algebraic groups

$$g^0 : \text{Alb}_X^0 \rightarrow A^0$$

such that g_1 is g_0 -equivariant (see [62, App. A]).

The structure of anti-affine algebraic groups has been obtained in [10] and [52] independently. Our exposition follows that of [10] with some simplifications.

The notion of a pseudo-abelian variety is due to Totaro in [57], as well as Corollary 5.6.8 and further results about these varieties. In particular, it is shown that every smooth connected commutative group of exponent p occurs as the unipotent quotient of some pseudo-abelian variety, see [57, Cor. 6.5, Cor. 7.3]. This yields many more examples of pseudo-abelian varieties than those constructed in Example 4.2.7. Yet a full description of pseudo-abelian varieties is an open problem.

6. The Picard scheme

6.1. Definitions and basic properties.

DEFINITION 6.1.1. Let X be a scheme. The *relative Picard functor*, denoted by $\mathrm{Pic}_{X/k}$, is the commutative group functor that assigns to any scheme S the group $\mathrm{Pic}(X \times S)/p_2^* \mathrm{Pic}(S)$, where $p_2 : X \times S \rightarrow S$ denotes the projection, and to any morphism of schemes $f : S' \rightarrow S$, the homomorphism induced by pull-back.

If X is equipped with a k -rational point x , then for any scheme S , the map $x \times \mathrm{id} : S \rightarrow X \times S$ is a section of $p_2 : X \times S \rightarrow S$. Thus,

$$(x \times \mathrm{id})^* : \mathrm{Pic}(X \times S) \rightarrow \mathrm{Pic}(S)$$

is a retraction of

$$p_2^* : \mathrm{Pic}(S) \rightarrow \mathrm{Pic}(X \times S).$$

So we may view $\mathrm{Pic}_{X/k}(S)$ as the group of isomorphism classes of invertible sheaves on $X \times S$, trivial along $x \times S$. If in addition S is equipped with a k -rational point s , then we obtain a pull-back map $s^* : \mathrm{Pic}_{X/k}(S) \rightarrow \mathrm{Pic}_{X/k}(s) = \mathrm{Pic}(X)$ with kernel isomorphic to $\mathrm{Pic}(X \times S)/p_1^* \mathrm{Pic}(X) \times p_2^* \mathrm{Pic}(S)$. Indeed, the map

$$s^* \times x^* : \mathrm{Pic}(X \times S) \longrightarrow \mathrm{Pic}(X) \times \mathrm{Pic}(S)$$

is a retraction of

$$p_1^* \times p_2^* : \mathrm{Pic}(X) \times \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(X \times S).$$

The kernel of $s^* \times x^*$ is called the group of *divisorial correspondences*.

THEOREM 6.1.2. *Let X be a proper scheme having a k -rational point.*

- (1) $\mathrm{Pic}_{X/k}$ is represented by a locally algebraic group (that will still be denoted by $\mathrm{Pic}_{X/k}$).
- (2) $\mathrm{Lie}(\mathrm{Pic}_{X/k}) = H^1(X, \mathcal{O}_X)$.
- (3) If $H^2(X, \mathcal{O}_X) = 0$ then $\mathrm{Pic}_{X/k}$ is smooth.

PROOF. (1) This is proved in [43, II.15] via a characterization of commutative locally algebraic groups among commutative group functors, in terms of seven axioms. When X is projective, there is an alternative proof via the Hilbert scheme, see [31, Thm. 4.8].

The assertion (2) follows from [31, Thm. 5.11], and (3) from [31, Prop. 5.19]. \square

With the notation and assumptions of the above theorem, the neutral component, $\mathrm{Pic}_{X/k}^0$, is a connected algebraic group by Theorem 2.4.1. The group of connected components, $\pi_0(\mathrm{Pic}_{X/k})$, is called the *Néron-Severi group*; we denote it by $\mathrm{NS}_{X/k}$. The formation of $\mathrm{Pic}_{X/k}$ commutes with field extensions; hence the same holds for $\mathrm{Pic}_{X/k}^0$ and $\mathrm{NS}_{X/k}$. Also, the commutative group $\mathrm{NS}_{X/k}(\bar{k})$ is finitely generated in view of [SGA6, XIII.5.1].

REMARKS 6.1.3. (i) For any pointed scheme (S, s) , we have a natural isomorphism of groups

$$(6.1.1) \quad \mathrm{Hom}_{\mathrm{pt.sch.}}(S, \mathrm{Pic}_{X/k}) \cong \mathrm{Pic}(X \times S)/p_1^* \mathrm{Pic}(X) \times p_2^* \mathrm{Pic}(S),$$

where the left-hand side denotes the subgroup of $\mathrm{Hom}(S, \mathrm{Pic}_{X/k})$ consisting of those f such that $f(s) = 0$, i.e., the kernel of $s^* : \mathrm{Pic}_{X/k}(S) \rightarrow \mathrm{Pic}(X)$.

- (ii) If X is an abelian variety, then $\mathrm{Pic}_{X/k}^0$ is the dual abelian variety, \widehat{X} .

DEFINITION 6.1.4. Let X be a proper scheme over a perfect field k , having a k -rational point. The *Picard variety* $\mathrm{Pic}^0(X)$ is the reduced scheme $(\mathrm{Pic}_{X/k}^0)_{\mathrm{red}}$.

With the above notation and assumptions, $\mathrm{Pic}^0(X)$ is a smooth connected commutative algebraic group; its formation commutes with field extensions.

Returning to a proper scheme X over an arbitrary ground field k , having a k -rational point, we say that *the Picard variety of X exists* if $\mathrm{Pic}^0(X) := (\mathrm{Pic}_{X/k}^0)_{\mathrm{red}}$ is a subgroup scheme of the Picard scheme. We will see that this holds under mild assumptions on the singularities of X .

6.2. Structure of Picard varieties. Throughout this subsection, we consider a proper variety X equipped with a k -rational point x .

PROPOSITION 6.2.1. *The Albanese variety of X is canonically isomorphic to the dual of the largest abelian subvariety of $\mathrm{Pic}_{X/k}^0$.*

PROOF. Let A be an abelian variety. Then we have

$$\mathrm{Hom}_{\mathrm{gp.sch.}}(A, \mathrm{Pic}_{X/k}^0) = \mathrm{Hom}_{\mathrm{pt.sch.}}(A, \mathrm{Pic}_{X/k}^0)$$

in view of Proposition 3.3.4. Moreover,

$$\mathrm{Hom}_{\mathrm{pt.sch.}}(A, \mathrm{Pic}_{X/k}^0) = \mathrm{Hom}_{\mathrm{pt.sch.}}(A, \mathrm{Pic}_{X/k}).$$

Using the isomorphism (6.1.1), this yields an isomorphism

$$\mathrm{Hom}_{\mathrm{gp.sch.}}(A, \mathrm{Pic}_{X/k}^0) \cong \mathrm{Pic}(A \times X)/p_1^* \mathrm{Pic}(A) \times p_2^* \mathrm{Pic}(X),$$

which is contravariant in A and X . Exchanging the roles of A , X and using (6.1.1) again, we obtain an isomorphism

$$\mathrm{Hom}_{\mathrm{gp.sch.}}(A, \mathrm{Pic}_{X/k}^0) \cong \mathrm{Hom}_{\mathrm{pt.sch.}}(X, \mathrm{Pic}_A^0),$$

contravariant in A and X again. This readily yields the assertion (and reproves the existence of the Albanese morphism in this setting). \square

PROPOSITION 6.2.2. *When X is geometrically normal, its Picard variety exists and is an abelian variety.*

PROOF. By Lemma 3.3.7, it suffices to show that $\mathrm{Pic}_{X/k}^0$ is proper. For this, we may assume k algebraically closed. In view of Theorem 4.3.2, it suffices in turn to show that $\mathrm{Pic}_{X/k}^0$ contains no non-trivial smooth connected affine subgroup. As any such subgroup contains a copy of \mathbb{G}_a or \mathbb{G}_m (see e.g. [56, 3.4.9, 6.2.5, 6.3.4]), we are reduced to checking that every homomorphism from \mathbb{G}_a or \mathbb{G}_m to $\mathrm{Pic}_{X/k}^0$ is constant. But we clearly have

$$\mathrm{Hom}_{\mathrm{gp.sch.}}(\mathbb{G}_a, \mathrm{Pic}_{X/k}^0) \subseteq \mathrm{Hom}_{\mathrm{pt.sch.}}(\mathbb{A}^1, \mathrm{Pic}_{X/k}).$$

Moreover,

$$\mathrm{Hom}_{\mathrm{pt.sch.}}(\mathbb{A}^1, \mathrm{Pic}_{X/k}) \cong \mathrm{Pic}(X \times \mathbb{A}^1)/p_1^* \mathrm{Pic}(X)$$

in view of (6.1.1) and the triviality of $\mathrm{Pic}(\mathbb{A}^1)$. Since X is normal, the divisor class group $\mathrm{Cl}(X \times \mathbb{A}^1)$ is isomorphic to $\mathrm{Cl}(X)$ via p_1^* ; moreover, for any Weil divisor D on X , the pull-back $p_1^*(D)$ is Cartier if and only if D is Cartier. Thus, the map $p_1^* : \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(X \times \mathbb{A}^1)$ is an isomorphism. As a consequence, every homomorphism $\mathbb{G}_a \rightarrow \mathrm{Pic}_{X/k}^0$ is constant.

Arguing similarly with \mathbb{G}_a replaced by \mathbb{G}_m , we obtain

$$\mathrm{Hom}_{\mathrm{gp.sch.}}(\mathbb{G}_m, \mathrm{Pic}_{X/k}^0) \subseteq \mathrm{Pic}(X \times (\mathbb{A}^1 \setminus 0))/p_1^* \mathrm{Pic}(X).$$

Also, the pull-back map $\mathrm{Cl}(X \times \mathbb{A}^1) \rightarrow \mathrm{Cl}(X \times (\mathbb{A}^1 \setminus 0))$ is surjective. It follows that $p_1^* : \mathrm{Cl}(X) \rightarrow \mathrm{Cl}(X \times (\mathbb{A}^1 \setminus 0))$ is an isomorphism and restricts to an isomorphism on Picard groups. Thus, every homomorphism $\mathbb{G}_m \rightarrow \mathrm{Pic}_{X/k}^0$ is constant as well. \square

DEFINITION 6.2.3. A scheme X is *semi-normal* if X is reduced and every finite bijective morphism of schemes $f : Y \rightarrow X$ that induces an isomorphism on all residue fields is an isomorphism.

Examples of semi-normal schemes include of course normal varieties, and also divisors with smooth normal crossings. Nodal curves are semi-normal; cuspidal curves are not. By [27, Cor. 5.7], semi-normality is preserved under separable field extensions. But it is not preserved under arbitrary field extensions, as shown by Example 6.2.5 below. We say that a scheme X is *geometrically semi-normal* if $X_{\bar{k}}$ is semi-normal.

PROPOSITION 6.2.4. *When X is geometrically semi-normal, its Picard variety exists and is a semi-abelian variety.*

PROOF. Using Lemma 5.6.1, we may assume k perfect. By Proposition 5.4.5, it suffices to show that every morphism $\mathbb{A}^1 \rightarrow \mathrm{Pic}_{X/k}^0$ is constant. By (6.1.1), we have

$$\mathrm{Hom}_{\mathrm{pt.sch.}}(\mathbb{A}^1, \mathrm{Pic}_{X/k}) \subseteq \mathrm{Pic}(X \times \mathbb{A}^1)/p_1^* \mathrm{Pic}(X).$$

So it suffices in turn to show that the map

$$p_1^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \times \mathbb{A}^1)$$

is an isomorphism. When X is affine, this follows from [58, Thm. 3.6]. For an arbitrary variety X , we consider the first terms of the Leray spectral sequence associated with the morphism $p_1 : X \times \mathbb{A}^1 \rightarrow X$ and the sheaf $\mathcal{O}_{X \times \mathbb{A}^1}^*$ consisting of the units of the structure sheaf. This yields an exact sequence

$$0 \longrightarrow H^1(X, p_{1*}(\mathcal{O}_{X \times \mathbb{A}^1}^*)) \xrightarrow{p_1^*} H^1(X \times \mathbb{A}^1, \mathcal{O}_{X \times \mathbb{A}^1}^*) \longrightarrow H^0(X, R^1 p_{1*}(\mathcal{O}_{X \times \mathbb{A}^1}^*)).$$

Also, the natural map $\mathcal{O}_X^* \rightarrow p_{1*}(\mathcal{O}_{X \times \mathbb{A}^1}^*)$ is an isomorphism, since $R[t]^* = R^*$ for any integral domain R . Thus,

$$H^1(X, p_{1*}(\mathcal{O}_{X \times \mathbb{A}^1}^*)) = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X).$$

Moreover, $H^1(X \times \mathbb{A}^1, \mathcal{O}_{X \times \mathbb{A}^1}^*) = \mathrm{Pic}(X \times \mathbb{A}^1)$. Thus, it suffices to show that $R^1 p_{1*}(\mathcal{O}_{X \times \mathbb{A}^1}^*) = 0$. Recall that $R^1 p_{1*}(\mathcal{O}_{X \times \mathbb{A}^1}^*)$ is the sheaf on X associated with the presheaf $U \mapsto \mathrm{Pic}(U \times \mathbb{A}^1)$. When U is affine, we already saw that the map $p_1^* : \mathrm{Pic}(U) \rightarrow \mathrm{Pic}(U \times \mathbb{A}^1)$ is an isomorphism. Moreover, for any $x \in X$ and $L \in \mathrm{Pic}(U)$, there exists an open affine neighborhood V of x in U such that $L|_V$ is trivial. This yields the desired vanishing. \square

The above proposition does not extend to semi-normal schemes, as shown by the following:

EXAMPLE 6.2.5. Let k be an imperfect field. Set $p := \mathrm{char}(k)$ and choose $a \in k \setminus k^p$. Like in Remark 5.2.3 (iv), consider the regular projective curve X defined as the zero scheme of $y^p - xz^{p-1} - ax^p$ in \mathbb{P}^2 . Then X is not geometrically semi-normal, since $X_{\bar{k}}$ is a cuspidal curve: the zero scheme of $(y - a^{1/p}x)^p - xz^{p-1}$. We now show that $\mathrm{Pic}_{X/k}^0$ is a smooth connected unipotent group, non-trivial if $p \geq 3$.

The smoothness of $\text{Pic}_{X/k}^0$ follows from Theorem 6.1.2 (3). Also, the group $\text{Pic}_{X/k}^0(k_s)$ is non-trivial and killed by p when $p \geq 3$, see [32, Thm. 6.10.1, Lem. 6.11.1]. It follows that $\text{Pic}_{X/k}^0$ is non-trivial and killed by p as well. By using the structure of commutative algebraic groups, e.g., Lemma 5.6.2, this implies that $\text{Pic}_{X/k}^0$ is unipotent.

Next, we present a classical example of a smooth projective surface for which the Picard scheme is not smooth:

EXAMPLE 6.2.6. Assume that k is algebraically closed of characteristic 2. Let E be an ordinary elliptic curve, i.e., it has a (unique) k -rational point z_0 of order 2. Let F be another elliptic curve and consider the automorphism σ of $E \times F$ such that

$$\sigma(z, w) = (z + z_0, -w)$$

identically. Then σ has order 2 and fixes no point of $E \times F$. Thus, there exists a quotient morphism

$$q : E \times F \longrightarrow X$$

for the group $\langle \sigma \rangle$ generated by σ ; moreover, q is finite and étale, and X is a smooth projective surface. Denote by $E/\langle z_0 \rangle$ the quotient of E by the subgroup generated by z_0 . Then the projection $E \times F \rightarrow E$ descends to a morphism

$$\alpha : X \longrightarrow E/\langle z_0 \rangle.$$

We claim that α is the Albanese morphism of the pointed variety (X, x) , where $x := q(0, 0)$. Consider indeed a morphism

$$f : X \longrightarrow A,$$

where A is an abelian variety and $f(x) = 0$. Composing f with q yields a σ -invariant morphism

$$g : E \times F \longrightarrow A, \quad (0, 0) \longmapsto 0.$$

By Lemma 4.1.3, we have $g(z, w) = g(z, 0) + g(0, w)$ identically. Moreover, the map $h : F \rightarrow A$, $w \mapsto g(0, w)$ is a homomorphism by Proposition 4.1.4. In particular, $h(-w) = -h(w)$. But $h(-w) = h(w)$ by σ -invariance. Thus, h factors through the kernel of 2_F , a finite group scheme. Since F is smooth and connected, it follows that h is constant. Thus, $g(z, w) = g(z, 0) = g(z + z_0, 0)$; this yields our claim.

Combining that claim with Propositions 6.2.1 and 6.2.2, we see that

$$\text{Pic}^0(X) \cong \widehat{E/\langle z_0 \rangle} \cong E/\langle z_0 \rangle.$$

Next, we claim that *the tangent sheaf T_X is trivial*. Indeed, since q is étale, we have

$$(6.2.1) \quad q^*(T_X) \cong T_{E \times F} \cong \mathcal{O}_{E \times F} \otimes_k (\text{Lie}(E) \oplus \text{Lie}(F)).$$

These isomorphisms are equivariant for the natural action of σ on $\mathcal{O}_{E \times F}$ and its trivial action on $\text{Lie}(E) \oplus \text{Lie}(F)$ (recall that σ acts on E by a translation, and on F by -1). Thus, we obtain

$$T_X \cong q_*(q^*(T_X))^{\langle \sigma \rangle} \cong q_*(\mathcal{O}_{E \times F})^{\langle \sigma \rangle} \otimes_k (\text{Lie}(E) \oplus \text{Lie}(F)) \cong \mathcal{O}_X \otimes_k (\text{Lie}(E) \oplus \text{Lie}(F)).$$

This yields the claim.

By that claim and the Riemann-Roch theorem, we obtain $\chi(\mathcal{O}_X) = 0$. Since $h^0(\mathcal{O}_X) = 1$ and $h^2(\mathcal{O}_X) = h^0(\omega_X) = h^0(\mathcal{O}_X) = 1$, this yields $h^1(\mathcal{O}_X) = 2$. In other words, the Lie algebra of $\text{Pic}_{X/k}$ has dimension 2; thus, $\text{Pic}_{X/k}$ is not smooth.

Notes and references.

As mentioned in the introduction, a detailed reference for Picard schemes is Kleiman's article [31]; see also [9], especially Chapter 8 for general results on the Picard functor, and Chapter 9 for applications to relative curves.

Proposition 6.2.2 is well-known, see e.g. [31, Thm. 5.4].

Proposition 6.2.4 is due to Alexeev when k is algebraically closed, see [2, Thm. 4.1.7].

Example 6.2.6 is due to Igusa, see [29].

Assume that k is perfect and consider a proper scheme X having a k -rational point. Then the Picard variety of X lies in a unique exact sequence of the form (5.3.3),

$$1 \longrightarrow T_X \times U_X \longrightarrow \mathrm{Pic}^0(X) \longrightarrow A_X \longrightarrow 1,$$

where T_X is a torus, U_X a smooth connected commutative algebraic group, and A_X an abelian variety. The affine part $T_X \times U_X$ has been described by Geisser in [26]; in particular, the dual of the character group of T_X is isomorphic to the étale cohomology group $H_{\mathrm{ét}}^1(X_{\bar{k}}, \mathbb{Z})$ as a Galois module (see [26, Thm. 1], and [60, Thm. 7.1], [2, Cor. 4.2.5] for closely related results). Also, when X is reduced, U_X is the kernel of the pull-back map $f^* : \mathrm{Pic}_{X/k} \rightarrow \mathrm{Pic}_{X^+/k}$, where $f : X^+ \rightarrow X$ denotes the semi-normalization (see [26, Thm. 3]).

It is an open problem whether every smooth connected algebraic group G over a perfect field k is the Picard variety of some proper scheme X . When k is algebraically closed of characteristic 0, one can construct an appropriate scheme X by using the structure of G described in §5.3, see [12, Thm. 1.1]. In fact, X may be taken projective, and normal except at finitely many points. Yet when $\mathrm{char}(k) > 0$, the unipotent part of the Picard variety of a scheme satisfying these conditions is not arbitrary, see [12, Thm. 1.2].

In fact, it is not even known whether every torus is the largest torus of some Picard variety; equivalently, whether every free abelian group equipped with an action of the Galois group Γ is obtained as $H_{\mathrm{ét}}^1(X_{\bar{k}}, \mathbb{Z})$ for some proper scheme X (which can be assumed semi-normal).

In another direction, the structure of Picard schemes over imperfect fields is largely unknown; see [57, Ex. 3.1] for a remarkable example.

7. The automorphism group scheme

7.1. Basic results and examples. Recall from §2.2 the automorphism group functor of a scheme X , i.e., the group functor Aut_X that assigns to any scheme S the automorphism group of the S -scheme $X \times S$. We have the following representability result for Aut_X , analogous to that for the Picard scheme (Theorem 6.1.2):

THEOREM 7.1.1. *Let X be a proper scheme.*

- (1) *The group functor Aut_X is represented by a locally algebraic group (that will be still denoted by Aut_X).*
- (2) *$\mathrm{Lie}(\mathrm{Aut}_X) = H^0(X, T_X)$, where T_X denotes the sheaf of derivations of the structure sheaf \mathcal{O}_X .*
- (3) *If $H^1(X, T_X) = 0$, then Aut_X is smooth.*

PROOF. (1) This is obtained in [37, Thm. 3.7] via an axiomatic characterization of locally algebraic groups among group functors, which generalizes that of [43]. When X is projective, the result follows from the existence of the Hilbert

scheme. More specifically, the functor of endomorphisms is represented by an open subscheme End_X of the Hilbert scheme $\text{Hilb}_{X \times X}$, by sending each endomorphism u to its graph (the image of $\text{id} \times u : X \rightarrow X \times X$), see [33, Thm. I.10]. Moreover, Aut_X is represented by an open subscheme of End_X in view of [33, Lem. I.10.1].

(2) See [37, Lem. 3.4] and also [22, II.4.2.4].

(3) This follows from [SGA1, III.5.9]. \square

With the above notation and assumptions, we say that Aut_X is the *automorphism group scheme* of X ; its neutral component, Aut_X^0 , is a connected algebraic group. The formations of Aut_X and Aut_X^0 commute with field extensions. For any group scheme G , the datum of a G -action on X is equivalent to a homomorphism $G \rightarrow \text{Aut}_X$.

EXAMPLE 7.1.2. Let C be a smooth, projective, geometrically irreducible curve of genus $g \geq 2$. Then T_C is the dual of the canonical sheaf ω_C , and hence $\deg(T_C) = 2 - 2g < 0$; it follows that $H^0(C, T_C) = 0$. Thus, Aut_C is étale; equivalently, Aut_C^0 is trivial.

In fact, Aut_C is finite. To see this, it suffices to show that $\text{Aut}_C(\bar{k})$ is finite; thus, we may assume k algebraically closed. Let $f \in \text{Aut}_C(k)$ and consider its graph $\Gamma_f \subset C \times C$. Choose a point $x \in C(k)$; then $\mathcal{O}_C(x)$ is an ample invertible sheaf on C , and hence $\mathcal{L} := \mathcal{O}_C(x) \boxtimes \mathcal{O}_C(x)$ is an ample invertible sheaf on $C \times C$. Moreover, the pull-back of \mathcal{L} to $\Gamma_f \cong C$ is isomorphic to $\mathcal{O}_C(x + f(x))$; by the Riemann-Roch theorem, the Hilbert polynomial $P : n \mapsto \chi(C, \mathcal{O}_C(n(x + f(x))))$ is independent of f . As a consequence, Aut_C is equipped with an immersion into the Hilbert scheme $\text{Hilb}_{C \times C}^P$. Since the latter is projective, this yields the assertion.

By the above argument, Aut_C is an algebraic group for any geometrically irreducible curve C (take for x a smooth closed point of C).

EXAMPLE 7.1.3. Let A be an abelian variety. Then the action of A on itself by translation yields a homomorphism

$$\tau : A \longrightarrow \text{Aut}_A^0.$$

Clearly, $\text{Ker}(\tau)$ is trivial and hence τ is a closed immersion. Since A is smooth and Aut_A^0 is an irreducible scheme of finite type, of dimension at most

$$h^0(T_A) = h^0(\mathcal{O}_A \otimes_k \text{Lie}(A)) = \dim \text{Lie}(A) = \dim(A),$$

it follows that τ is an isomorphism.

Also, one readily checks that

$$\text{Aut}_A \cong \text{Aut}_{A,0} \ltimes A,$$

where $\text{Aut}_{A,0} \subseteq \text{Aut}_A$ denotes the stabilizer of the origin. It follows that $\text{Aut}_{A,0} \cong \pi_0(\text{Aut}_A)$ is étale.

If A is an elliptic curve, then $\text{Aut}_{A,0}$ is finite, as may be seen by arguing as in the preceding example. But this fails for abelian varieties of higher dimension, for example, when $A = B \times B$ for a non-trivial abelian variety B : then $\text{Aut}_{A,0}$ contains the constant group scheme $\text{GL}_2(\mathbb{Z})$ acting by linear combinations of entries.

Next, we present an application of Theorem 2 to the structure of connected automorphism group schemes. Recall that a variety X is *uniruled* if there exist an integral scheme of finite type Y and a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ such that the induced map $\mathbb{P}_y^1 \dashrightarrow X$ is non-constant for some $y \in Y$. Also, X is uniruled if and only if $X_{\bar{k}}$ is uniruled, see [33, IV.1.3].

PROPOSITION 7.1.4. *Let X be a proper variety. If X is not uniruled, then Aut_X^0 is proper.*

PROOF. We may assume k algebraically closed. If Aut_X^0 is not proper, then it contains a connected affine normal subgroup scheme N of positive dimension, as follows from Theorem 2. In turn, the reduced subgroup scheme N_{red} contains a subgroup scheme H isomorphic to \mathbb{G}_a or \mathbb{G}_m . Since H acts faithfully on X , the action morphism $H \times X \rightarrow X$ yields a uniruling. \square

EXAMPLE 7.1.5. Assume that k is algebraically closed of characteristic 2. Let X be the smooth projective surface constructed in Example 6.2.6. We claim that Aut_X is not smooth.

To see this, recall that the tangent sheaf T_X is trivial; hence $\mathrm{Lie}(\mathrm{Aut}_X)$ has dimension 2. On the other hand, X is equipped with an action of the elliptic curve E , via its action on $E \times F$ by translations on itself (which commutes with the involution σ). This yields a homomorphism $E \rightarrow \mathrm{Aut}_X$ that factors through

$$f : E \longrightarrow (\mathrm{Aut}_X^0)_{\mathrm{red}} =: G.$$

Since X contains an E -stable curve isomorphic to E (the image of $E \times y$, where $y \in F(k)$ and $y \neq -y$), the action of E on X is faithful, and hence f is a closed immersion. We now show that f is an isomorphism; this implies the claim for dimension reasons.

First, note that every morphism $\mathbb{P}^1 \rightarrow X$ is constant, since the Albanese morphism $\alpha : X \rightarrow E/\langle z_0 \rangle$ has its fibers at all closed points isomorphic to the elliptic curve F , and $E/\langle z_0 \rangle$ is an elliptic curve. In particular, X is not uniruled. By the above proposition, it follows that G is an abelian variety.

Combining this with Proposition 3.1.6, we see that $C_G(x)$ is finite for any $x \in X(k)$. Thus, the G -orbits of k -rational points of X are abelian varieties, isogenous to G . Since X is not an abelian variety (as its Albanese morphism is not an isomorphism), it follows that $\dim(G) \leq 1$. Thus, G must be the image of f .

7.2. Blanchard's lemma. Consider a group scheme G acting on a scheme X , and a morphism of schemes $f : X \rightarrow Y$. In general, the G -action on X does not descend to an action on Y ; for example, when $G = X$ is an elliptic curve acting on itself by translations, and f is the quotient by $\mathbb{Z}/2$ acting via $x \mapsto \pm x$. Yet we will obtain a descent result under the assumption that f is proper and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$; then f is surjective and its fibers are connected by [EGA, III.4.3.2, III.4.3.4]. Such a descent result was first proved by Blanchard in the setting of holomorphic transformation groups, see [6, I.1].

THEOREM 7.2.1. *Let G be a connected algebraic group, X a G -scheme of finite type, Y a scheme of finite type and $f : X \rightarrow Y$ a proper morphism such that $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is an isomorphism. Then there exists a unique action of G on Y such that f is equivariant.*

PROOF. Let $a : G \times X \rightarrow X$ denote the action. We show that there is a unique morphism $b : G \times Y \rightarrow Y$ such that the square

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \mathrm{id} \times f \downarrow & & \downarrow f \\ G \times Y & \xrightarrow{b} & Y \end{array}$$

commutes. By [EGA, II.8.11], it suffices to check that the morphism

$$\mathrm{id} \times f : G \times X \longrightarrow G \times Y$$

is proper, the map

$$(\mathrm{id} \times f)^\# : \mathcal{O}_{G \times Y} \longrightarrow (\mathrm{id} \times f)_*(\mathcal{O}_{G \times X})$$

is an isomorphism, and the composition

$$f \circ a : G \times X \longrightarrow Y$$

is constant on the fibers of $\mathrm{id} \times f$.

Since f is proper, $\mathrm{id} \times f$ is proper as well. Consider the square

$$\begin{array}{ccc} G \times X & \xrightarrow{p_X} & X \\ \mathrm{id} \times f \downarrow & & \downarrow f \\ G \times Y & \xrightarrow{p_Y} & Y, \end{array}$$

where p_X, p_Y denote the projections. As this square is cartesian and both horizontal arrows are flat, the natural map

$$p_Y^* f_*(\mathcal{O}_X) \longrightarrow (\mathrm{id} \times f)_* p_X^*(\mathcal{O}_X),$$

is an isomorphism, and hence so is the natural map $\mathcal{O}_{G \times Y} \rightarrow (\mathrm{id} \times f)_*(\mathcal{O}_{G \times X})$. We now check that $f \circ a$ is constant on the fibers of $\mathrm{id} \times f$. Let K be a field extension of k and let $g \in G(K), y \in Y(K)$. Then the fiber of $\mathrm{id} \times f$ at (g, y) is just $g \times f^{-1}(y)$. So it suffices to show that the morphism

$$h : G_K \times f^{-1}(y) \longrightarrow Y_K, \quad (g, x) \mapsto f(g \cdot x)$$

is constant on $g \times f^{-1}(y)$. But this follows e.g. from the rigidity lemma 3.3.3 applied to the irreducible components of $f^{-1}(y)$, since h is constant on $e \times f^{-1}(y)$, and $f^{-1}(y)$ is connected.

It remains to show that a is an action of the group scheme G . Note that e acts on X via the identity; moreover, the composite morphism of sheaves

$$\mathcal{O}_Y \xrightarrow{b^\#} b_*(\mathcal{O}_{G \times Y}) \xrightarrow{(e \times \mathrm{id})^\#} b_*(\mathcal{O}_{e \times Y}) \cong \mathcal{O}_Y$$

is the identity, since so is the analogous morphism

$$\mathcal{O}_X \xrightarrow{a^\#} a_*(\mathcal{O}_{G \times X}) \xrightarrow{(e \times \mathrm{id})^\#} a_*(\mathcal{O}_{e \times X}) \cong \mathcal{O}_X$$

and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Likewise, the square

$$\begin{array}{ccc} G \times G \times Y & \xrightarrow{\mathrm{id} \times b} & G \times Y \\ m \times \mathrm{id} \downarrow & & \downarrow b \\ G \times Y & \xrightarrow{b} & Y \end{array}$$

commutes on closed points, and the corresponding square of morphisms of sheaves commutes as well, since the analogous square with Y replaced by X commutes. \square

COROLLARY 7.2.2. *Let $f : X \rightarrow Y$ be a morphism of proper schemes such that $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is an isomorphism.*

(1) *f induces a homomorphism*

$$f_* : \mathrm{Aut}_X^0 \longrightarrow \mathrm{Aut}_Y^0.$$

(2) *If f is birational, then f_* is a closed immersion.*

PROOF. (1) This follows readily from the above theorem applied to the action of Aut_X^0 on X .

(2) By Proposition 2.7.1, it suffices to check that $\text{Ker}(f_*)$ is trivial. Let S be a scheme and $u \in \text{Aut}_X^0(S)$ such that $f_*(u) = \text{id}$. As f is birational, there exists a dense open subscheme $V \subseteq Y$ such that f pulls back to an isomorphism $f^{-1}(V) \rightarrow V$. Then u pulls back to the identity on $f^{-1}(V) \times S$. Since the latter is dense in $X \times S$, we obtain $u = \text{id}$. \square

The above corollary applies to a birational morphism $f : X \rightarrow Y$, where X and Y are proper varieties and Y is normal: then $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is an isomorphism by Zariski's Main Theorem.

Corollary 7.2.2(1) also applies to the two projections $p_1 : X \times Y \rightarrow X$ and $p_2 : X \times Y \rightarrow Y$, where X, Y are proper varieties: then $\mathcal{O}(X) = k = \mathcal{O}(Y)$ and hence $p_{1*}(\mathcal{O}_{X \times Y}) = \mathcal{O}_X$, $p_{2*}(\mathcal{O}_{X \times Y}) = \mathcal{O}_Y$ in view of Lemma 2.3.3. This implies readily the following:

COROLLARY 7.2.3. *Let X, Y be proper varieties. Then the homomorphism*

$$p_{1*} \times p_{2*} : \text{Aut}_{X \times Y}^0 \longrightarrow \text{Aut}_X^0 \times \text{Aut}_Y^0$$

is an isomorphism with inverse the natural homomorphism

$$\text{Aut}_X^0 \times \text{Aut}_Y^0 \longrightarrow \text{Aut}_{X \times Y}^0, \quad (u, v) \longmapsto ((x, y) \mapsto (u(x), v(y))).$$

7.3. Varieties with prescribed connected automorphism group.

THEOREM 7.3.1. *Let G be a smooth connected algebraic group of dimension n over a perfect field k . Then there exists a normal projective variety X such that $G \cong \text{Aut}_X^0$ and $\dim(X) \leq 2n + 1$.*

The proof will occupy the rest of this subsection. We will use the action of $G \times G$ on G via left and right multiplication; this identifies G with the homogeneous space $(G \times G)/\text{diag}(G)$, and e to the base point of that homogeneous space. By Theorem 5.2.2, we may choose a projective compactification Y of G which is $G \times G$ -equivariant, i.e., Y is equipped with two commuting G -actions, on the left and on the right. We may further assume that Y is normal in view of Proposition 2.5.1.

Denote by Aut_Y^G the centralizer of G in Aut_Y relative to the right G -action. Then Aut_Y^G is a closed subgroup scheme of Aut_Y by Theorem 2.2.6. Also, the left G -action on Y yields a homomorphism

$$\varphi : G \longrightarrow \text{Aut}_Y^G.$$

LEMMA 7.3.2. *The above map φ is an isomorphism.*

PROOF. Since the left G -action on itself is faithful, the kernel of φ is trivial and hence φ is a closed immersion. We show that φ is surjective on \bar{k} -points: if $u \in \text{Aut}_Y^G(\bar{k})$, then u stabilizes the open orbit of the right $G_{\bar{k}}$ -action, i.e., $G_{\bar{k}}$. As u commutes with that action, it follows that the pull-back of u to $G_{\bar{k}} \subseteq Y_{\bar{k}}$ is the left multiplication by some $g \in G(\bar{k})$. Since $G_{\bar{k}}$ is dense in $Y_{\bar{k}}$, we conclude that $u = \varphi(g)$.

As G is smooth, it suffices to check that $\text{Lie}(\varphi)$ is an isomorphism to complete the proof. We have $\text{Lie}(\text{Aut}_Y^G) = H^0(Y, T_Y)^G = \text{Der}^G(\mathcal{O}_Y)$, where $\text{Der}(\mathcal{O}_Y)$ denotes the Lie algebra of derivations of \mathcal{O}_Y , and $\text{Der}^G(\mathcal{O}_Y)$ the Lie subalgebra of invariants

under the right G -action. Moreover, the pull-back $j : \text{Der}(\mathcal{O}_Y) \rightarrow \text{Der}(\mathcal{O}_G)$ is injective by the density of G in Y . Also, recall that $\text{Der}^G(\mathcal{O}_G) \cong \text{Lie}(G)$ and this identifies the composition $j \circ \text{Lie}(\varphi)$ with the identity of $\text{Lie}(G)$. This yields the desired statement. \square

For any closed subscheme $F \subseteq G$, we denote by Aut_Y^F the centralizer of F in Aut_Y , where F is identified with $e \times F \subseteq e \times G$. Then again, Aut_Y^F is a closed subgroup scheme of Aut_Y ; we denote its neutral component by $\text{Aut}_Y^{F,0}$.

LEMMA 7.3.3. *With the above notation, there exists a finite étale subscheme $F \subset G$ such that $\text{Aut}_Y^{F,0} = \text{Aut}_Y^{G,0}$.*

PROOF. Since G is smooth, $G(\bar{k})$ is dense in $G_{\bar{k}}$ and hence the above lemma yields an isomorphism $G_{\bar{k}} \cong \text{Aut}^{G(\bar{k})}(Y_{\bar{k}})$ with an obvious notation. Consider the family of (closed) subgroup schemes $\text{Aut}_{Y_{\bar{k}}}^{F,0} \subseteq \text{Aut}_{Y_{\bar{k}}}^0$, where F runs over the finite subsets $F \subseteq G(\bar{k})$, stable by the Galois group Γ . Since $\text{Aut}_{Y_{\bar{k}}}^0$ is of finite type, we may choose F so that $\text{Aut}_{Y_{\bar{k}}}^{F,0}$ is minimal in this family. Let Ω be a Γ -orbit in $G(\bar{k})$, then $\text{Aut}_{Y_{\bar{k}}}^{F \cup \Omega,0} \subseteq \text{Aut}_{Y_{\bar{k}}}^{F,0}$ and hence equality holds. Thus, $\text{Aut}_{Y_{\bar{k}}}^{F,0} = \text{Aut}_{Y_{\bar{k}}}^{G(\bar{k}),0} = G_{\bar{k}}$. This yields the assertion by using the bijective correspondence between finite étale subschemes of G and finite Γ -stable subsets of $G(\bar{k})$. \square

Next, consider the diagonal homomorphism $\text{Aut}_Y^0 \rightarrow \text{Aut}_Y^0 \times \text{Aut}_Y^0$. By Corollary 7.2.3, we may identify the right-hand side with $\text{Aut}_{Y \times Y}^0$; this yields a closed immersion of algebraic groups

$$\Delta : \text{Aut}_Y^0 \longrightarrow \text{Aut}_{Y \times Y}^0.$$

Also, for any finite étale subscheme $F \subseteq G$, the morphism

$$\Gamma : F \times Y \longrightarrow Y \times Y, \quad (g, x) \longmapsto (x, g \cdot x)$$

is finite, and hence we may view its image Z as a closed reduced subscheme of $Y \times Y$. Note that Z is stable by the G -action via $\Delta \circ \varphi$; also, $Z_{\bar{k}}$ is the union of the graphs of the automorphisms $g \in F(\bar{k})$ of Y .

LEMMA 7.3.4. *Keep the above notation.*

- (1) Δ identifies Aut_Y^0 with $\text{Aut}_{Y \times Y, \text{diag}(Y)}^0$ (the neutral component of the stabilizer of the diagonal in $\text{Aut}_{Y \times Y}$).
- (2) If F is a finite étale subscheme of G that satisfies the assertion of Lemma 7.3.3 and contains e , then Δ identifies $\varphi(G)$ with $\text{Aut}_{Y \times Y, Z, \text{red}}^0$ (the reduced neutral component of the stabilizer of Z in $\text{Aut}_{Y \times Y}$).

PROOF. (1) Just observe that for any scheme S and for any $u, v \in \text{Aut}_X(S)$, the automorphism $u \times v$ of $(X \times S) \times_S (X \times S)$ stabilizes the diagonal if and only if $u = v$.

(2) We may assume k algebraically closed; then we may view F as a finite subset of $G(k)$. Also, by Proposition 2.5.1, the reduced subgroup $\text{Aut}_{Y \times Y, Z, \text{red}}^0$ stabilizes the graph Γ_f of any $f \in F$, since these graphs form the irreducible components of $Z_{\bar{k}}$. Now observe that for any scheme S and for any $f, g \in \text{Aut}_X(S)$, the automorphism $g \times g$ of $(X \times S) \times_S (X \times S)$ stabilizes Γ_f if and only if g commutes with f . Thus, Δ identifies $\varphi(G)$ with $\text{Aut}_{Y \times Y, Z, \text{red}}^0$. \square

We now choose a finite étale subscheme $F \subseteq G$ that satisfies the assumption of Lemma 7.3.3 and contains e . Denote by

$$f : X \longrightarrow Y \times Y$$

the normalization of the blowing-up of $Y \times Y$ along Z . Then the G -action on $Y \times Y$ via $\Delta \circ \varphi$ lifts to a unique action on X ; in other words, we have a homomorphism

$$f^* : G \longrightarrow \mathrm{Aut}_X^0.$$

As f is birational, f^* is a closed immersion by the argument of Corollary 7.2.2 (ii). Also, $Y \times Y$ is normal, since Y is normal and k is perfect. Thus, the above corollary yields a closed immersion

$$f_* : \mathrm{Aut}_X^0 \longrightarrow \mathrm{Aut}_{Y \times Y}^0.$$

Moreover, the composition $f_* \circ f^*$ is an isomorphism of G to its image $(\Delta \circ \varphi)(G)$, since f is birational. We will identify G with its image, and hence $f_* \circ f^*$ with id .

Consider first the case where $\mathrm{char}(k) = 0$. Then Aut_X^0 is smooth and hence stabilizes the exceptional locus $E \subset X$ of the birational morphism f . As a consequence, the action of Aut_X^0 on $Y \times Y$ via f_* stabilizes the image $f(E) \subset Y \times Y$. If in addition $n \geq 2$, then $f(E) = Z$ and hence f_* sends Aut_X^0 to $\mathrm{Aut}_{Y \times Y, Z}^0$, i.e. to G by Lemma 7.3.4. It follows that f^* is an isomorphism with inverse f_* .

If $n = 1$, then E is empty. We now reduce to the former case as follows: we choose a smooth, projective, geometrically integral curve C of genus ≥ 1 , equipped with a k -rational point c . Then $\mathrm{Aut}_{C, c}^0$ is trivial, as seen in Examples 7.1.2 and 7.1.3. Using Corollary 7.2.3, it follows that the natural map

$$\mathrm{Aut}_{Y \times Y, Z}^0 \longrightarrow \mathrm{Aut}_{Y \times Y \times C, Z \times c}^0$$

is an isomorphism; by Lemma 7.3.4, this yields an isomorphism

$$G \cong \mathrm{Aut}_{Y \times Y \times C, Z \times c}^0.$$

Also, note that $Y \times Y \times C$ is a normal projective variety. We now consider the morphism $f' : X' \rightarrow Y \times Y \times C$ obtained as the normalization of the blowing-up along $Z \times c$. Since the latter has codimension 2 in $Y \times Y \times C$, the exceptional locus $E' \subset X'$ satisfies $f'(E') = Z \times c$. So the above argument yields again that f'^* is an isomorphism with inverse f'_* . This completes the proof of Theorem 7.3.1 in characteristic 0.

Next, assume that $\mathrm{char}(k) = p > 0$. We will use the following additional result:

LEMMA 7.3.5. *Let $f : X \rightarrow Y \times Y$ be as above and assume that $n - 1$ is not a multiple of p (in particular, $n \geq 2$). Then the differential*

$$\mathrm{Lie}(f^*) : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(\mathrm{Aut}_X^0)$$

is an isomorphism.

PROOF. As $\mathrm{Lie}(f_*) \circ \mathrm{Lie}(f^*) = \mathrm{id}$, it suffices to show that the image of $\mathrm{Lie}(f_*)$ is contained in $\mathrm{Lie}(G)$. For this, we may assume that $k = \bar{k}$. We may thus view F as a finite subset of $G(k)$ containing e .

We will use the action of $\mathrm{Lie}(\mathrm{Aut}_X^0) = \mathrm{Der}(\mathcal{O}_X)$ on the “jacobian ideal” of f , defined as follows. Recall that the sheaf of differentials, $\Omega_X^1 = \mathcal{I}_{\mathrm{diag}(X)} / \mathcal{I}_{\mathrm{diag}(X)}^2$ is equipped with a linearization for Aut_X (see [SGA3, I.6] for background on

linearized sheaves). Likewise, $\Omega_{Y \times Y}^1$ is equipped with a linearization for $\text{Aut}_{Y \times Y}$ and hence for Aut_X^0 acting via f_* . Moreover, the canonical map

$$f^*(\Omega_{Y \times Y}^1) \longrightarrow \Omega_X^1$$

is a morphism of Aut_X^0 -linearized sheaves, since it arises from the canonical map $f^{-1}(\mathcal{I}_{\text{diag}(Y \times Y)}) \rightarrow \mathcal{I}_{\text{diag}(X)}$. Denoting by $\Omega_{Y \times Y}^{2n}$ the $2n$ th exterior power of $\Omega_{Y \times Y}^1$ and defining Ω_X^{2n} likewise, this yields a morphism of Aut_X^0 -linearized sheaves

$$f^*(\Omega_{Y \times Y}^{2n}) \longrightarrow \Omega_X^{2n}$$

and in turn, a morphism of Aut_X^0 -linearized sheaves

$$\text{Hom}(\Omega_X^{2n}, f^*(\Omega_{Y \times Y}^{2n})) \longrightarrow \text{End}(\Omega_X^{2n}).$$

The image \mathcal{J}_f of this morphism is an Aut_X^0 -linearized subsheaf of the sheaf of algebras $\text{End}(\Omega_X^{2n})$. In particular, for any open subvariety U of X , the Lie algebra $\text{Der}(\mathcal{O}_X)$ acts on the algebra $\text{End}(\Omega_U^{2n})$ by derivations that stabilize $\Gamma(U, \mathcal{J}_f)$.

Denote by $Y_{\text{reg}} \subseteq Y$ the regular (or smooth) locus and consider the open subvariety $V \subseteq Y_{\text{reg}} \times Y_{\text{reg}}$ consisting of those points that lie in at most one graph Γ_g , where $g \in F$. Then $Z \cap V$ is a disjoint union of smooth varieties of dimension n , and is dense in Z . Let $U := f^{-1}(V)$; then the pull-back

$$f_U : U \longrightarrow V$$

is the blowing-up along $Z \cap V$, and hence U is smooth. Thus, the sheaf Ω_U^{2n} is invertible and \mathcal{J}_{f_U} is just a sheaf of ideals of \mathcal{O}_U . A classical computation in local coordinates shows that $\mathcal{J}_{f_U} = \mathcal{O}_U(-(n-1)E)$, where E denotes the exceptional divisor of f_U . Hence we obtain an injective map

$$\text{Der}(\mathcal{O}_X) \rightarrow \text{Der}(\mathcal{O}_U, \mathcal{J}_{f_U}) = \text{Der}(\mathcal{O}_U, \mathcal{O}_U(-(n-1)E))$$

with an obvious notation. Since $n-1$ is not a multiple of p , we have

$$\text{Der}(\mathcal{O}_U, \mathcal{O}_U(-(n-1)E)) = \text{Der}(\mathcal{O}_U, \mathcal{O}_U(-E)).$$

(Indeed, if $D \in \text{Der}(\mathcal{O}_U, \mathcal{O}_U(-(n-1)E))$ and z is a local generator of $\mathcal{O}_U(E)$ at $x \in X$, then $D(z^{n-1}) = (n-1)z^{n-2}D(z) \in z^{n-1}\mathcal{O}_{X,x}$ and hence $D(z) \in z\mathcal{O}_{X,x}$. Also, the natural map

$$\text{Der}(\mathcal{O}_U) \longrightarrow \text{Der}(f_{U*}(\mathcal{O}_U)) = \text{Der}(\mathcal{O}_V)$$

is injective and sends $\text{Der}(\mathcal{O}_U, \mathcal{O}_U(-E))$ to $\text{Der}(\mathcal{O}_V, f_{U*}(\mathcal{O}_U(-E)))$. Moreover, $f_{U*}(\mathcal{O}_U(-E))$ is the ideal sheaf of $Z \cap V$, and hence is stable by $\text{Der}(\mathcal{O}_X)$ acting via the composition

$$\text{Der}(\mathcal{O}_X) \longrightarrow \text{Der}(f_*(\mathcal{O}_X)) = \text{Der}(\mathcal{O}_{Y \times Y}) \longrightarrow \text{Der}(\mathcal{O}_V).$$

It follows that the image of $\text{Lie}(f_*)$ stabilizes the ideal sheaf of the closure of $Z \cap V$ in $Y \times Y$, ie., of Z . By arguing as in the proof of Lemma 7.3.4 (2), we conclude that $\text{Lie}(f_*)$ sends $\text{Der}(\mathcal{O}_X)$ to $\text{Lie}(G)$. \square

As f^* is a closed immersion and G is smooth, the above lemma completes the proof of Theorem 7.3.1 when p does not divide $n-1$. Next, when p divides $n-1$, we replace $Y \times Y$ (resp. Z) with $Y \times Y \times C$ (resp. $Z \times c$) for (C, c) as above. This replaces the codimension n of Z in $Y \times Y$ with $n+1$, and hence we obtain a normal projective variety X' of dimension $2n+1$ such that $G \cong \text{Aut}_{X'}^0$.

REMARKS 7.3.6. (i) For any smooth connected *linear* algebraic group G of dimension n , there exists a normal projective *unirational* variety of dimension at most $2n+2$ such that $G \cong \mathrm{Aut}_X^0$. Indeed, the variety G is unirational (see [SGA3, XIV.6.10]) and hence so is $Y \times Y$ with the notation of the above proof. Also, in that proof, the pointed curve (C, c) may be replaced with a pair (S, C) , where S is a smooth rational projective surface such that Aut_S^0 is trivial, and $C \subset S$ is a smooth, geometrically irreducible curve. Such a pair is obtained by taking for S the blowing-up of \mathbb{P}^2 at 4 points in general position, and for C an exceptional curve.

(ii) If $\mathrm{char}(k) = 0$ then every connected algebraic group G is a connected automorphism group of some *smooth* projective variety, as follows from the existence of an equivariant resolution of singularities (see [34, Prop. 3.9.1, Thm. 3.36]).

(iii) Still assuming $\mathrm{char}(k) = 0$, consider a finite-dimensional Lie algebra \mathfrak{g} . Then \mathfrak{g} is algebraic if and only if $\mathfrak{g} \cong \mathrm{Der}(\mathcal{O}_X)$ for some proper scheme X , as follows by combining Corollary 5.1.6, Theorem 7.1.1 and Theorem 7.3.1. Moreover, X may be chosen smooth, projective and unirational by the above remarks.

Notes and references.

Most results of §7.1 are taken from [37]. Those of §7.2 are algebraic analogues of classical results about holomorphic transformation groups, see [1, 2.4].

In the setting of complex analytic varieties, the problem of realizing a given Lie group as an automorphism group has been extensively studied. It is known that every finite group is the automorphism group of a smooth projective complex curve (see [28]); moreover, every compact connected real Lie group is the automorphism group of a bounded domain (satisfying additional conditions), see [5, 51]. Also, every connected real Lie group of dimension n is the automorphism group of a Stein complete hyperbolic manifold of dimension $2n$ (see [61, 30]). Theorem 7.3.1, obtained in [11, Thm. 1], may be viewed as an algebraic analogue of the latter result. The proof presented here is a streamlined version of that in [11].

There are still many open questions about automorphism group schemes. For instance, can one realize any algebraic group over an arbitrary field as the full automorphism group scheme of a proper scheme? Also, very little is known on the group of connected components $\pi_0(\mathrm{Aut}_X)$, where X is a proper scheme, or on the analogously defined group $\pi_0(\mathrm{Aut}(M))$, where M is a compact complex manifold (then $\mathrm{Aut}(M)$ is a complex Lie group, possibly with infinitely many components). As mentioned in [15], it is not known whether there exists a compact complex manifold M for which $\pi_0(\mathrm{Aut}(M))$ is not finitely generated.

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