# ON LOCAL GALOIS DEFORMATION RINGS

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Abstract. We show that framed deformation rings of mod $p$ representations of the absolute Galois group of a $p$-adic local field are complete intersections of expected dimension. We determine their irreducible components and show that they and their special fibres are normal and complete intersection. As an application we prove density results of loci with prescribed $p$-adic Hodge theoretic properties.

Contents



## 1. Introduction

Let $F$ be a finite extension of $\mathbb{Q}_p$ and denote by $G_F$ its absolute Galois group. Let $L$ be a another finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer $\varpi$ and residue field $k = \mathcal{O}/\varpi$. Fix a continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ and denote by $D^{\square}_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to$ Sets the functor from the category $\mathfrak{A}_{\mathcal{O}}$ of local Artinian $\mathcal{O}$-algebras with residue field $k$ to the category of sets, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D^{\square}_{\overline{\rho}}(A)$ is the set of continuous representations $\rho_A : G_F \to \mathrm{GL}_d(A)$, such that $\rho_A \pmod{\mathfrak{m}_A} = \overline{\rho}$. The functor $D^{\square}_{\overline{\rho}}$ of framed deformations of $\overline{\rho}$ is pro-represented by a complete local Noetherian $\mathcal{O}$-algebra $R^{\square}_{\overline{\rho}}$ (with residue field $k$).

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Our first main result completely settles a folklore conjecture on ring-theoretic properties of $R^{\square}_{\overline{\rho}}$ that can be traced back to the foundational work of Mazur [35, Section 1.10]:

**Theorem 1.1** (Corollary 3.37)**.** *The ring $R^{\square}_{\overline{\rho}}$ is a local complete intersection, flat over $\mathcal{O}$ and of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$. In particular, every continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ has a lift to characteristic zero.*

Obstruction theory provides a presentation $R^{\square}_{\overline{\rho}} = \mathcal{O}[\![x_1, \ldots, x_r]\!]/(f_1, \ldots, f_s)$ with $r$ equal to the dimension of the tangent space and $s$ equal to $\dim H^2(G_F, \mathrm{ad}\,\overline{\rho})$. The Euler characteristic formula from local class field theory gives

$$r - s = d^2 + d^2[F : \mathbb{Q}_p].$$

Our theorem proves that $\dim R^{\square}_{\overline{\rho}}/\varpi$ is given by this cohomological quantity, the *expected dimension* in the spirit of the *Dimension Conjecture* of Gouvêa from [26, Lecture 4]. Having the expected dimension implies that $\varpi, f_1, \ldots, f_s$ is a regular sequence and that $R^{\square}_{\overline{\rho}}$ is a local complete intersection. It also implies (see [25, Lemma 7.5]) that the derived deformation ring of $\overline{\rho}$ as introduced by Galatius and Venkatesh in [25], see also [11], is homotopy discrete, which means the derived deformation theory of $\overline{\rho}$ does not contain more information than the usual deformation theory of $\overline{\rho}$. Theorem 1.1 is used in the forthcoming work of Matthew Emerton, Toby Gee and Xinwen Zhu on derived stacks of global Galois representations.

Our second main result describes completely the connected components of the space $\mathrm{Spec}\, R^{\square}_{\overline{\rho}}[1/p]$ as envisioned in [7]. Let $\mu := \mu_{p^\infty}(F) \subset F^\times$ be the $p$-power torsion subgroup and suppose that $L$ is sufficiently large. Let $R_{\det \overline{\rho}}$ denote the universal deformation ring of the one dimensional representation $\det \overline{\rho}$.

**Theorem 1.2** (Corollaries 4.5, 4.15, 4.19, 4.21, Proposition 5.12)**.** *The natural map $R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$ is flat and induces a bijection of connected components*

$$\pi_0(\mathrm{Spec}\, R^{\square}_{\overline{\rho}}[1/p]) \to \pi_0(\mathrm{Spec}\, R_{\det \overline{\rho}}[1/p]). \tag{1}$$

*Labeling these components in a natural way by characters $\chi : \mu \to \mathcal{O}^\times$, the connected components of $R^{\square}_{\overline{\rho}}[1/p]$ are in a natural bijection with the irreducible components $R^{\square,\chi}_{\overline{\rho}}$ of $R^{\square}_{\overline{\rho}}$, and the rings $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are normal domains and complete intersections.*

We would like to highlight the following result for the amusement of the reader.

**Theorem 1.3** (Corollary 4.25)**.** *If $\overline{\rho}$ is absolutely irreducible then $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are factorial, except in the case $d = 2$, $F = \mathbb{Q}_3$ and $\overline{\rho} \cong \overline{\rho}(1)$.*

Let $\psi : G_F \to \mathcal{O}^\times$ be a continuous character lifting $\det \overline{\rho}$. Let $R^{\square,\psi}_{\overline{\rho}}$ be the quotient of $R^{\square}_{\overline{\rho}}$ parameterizing deformations with determinant equal to $\psi$.

**Theorem 1.4** (Corollary 5.4, Theorem 5.6)**.** *The rings $R^{\square,\psi}_{\overline{\rho}}$, $R^{\square,\psi}_{\overline{\rho}}/\varpi$ are normal domains and complete intersections of dimension $\dim R^{\square}_{\overline{\rho}} - \dim R_{\det \overline{\rho}} + 1$ and $\dim R^{\square}_{\overline{\rho}} - \dim R_{\det \overline{\rho}}$, respectively. Moreover, $R^{\square,\psi}_{\overline{\rho}}$ is $\mathcal{O}$-flat.*

Our work builds in an essential way on the work of GB–Juschka [8] on the special fibres of the deformation rings of pseudo-characters (i.e. pseudo-representations) of $G_F$. The paper [8] draws its inspiration from the work of Chenevier [15], who studied rigid analytic generic fibres of these rings. Our results in turn imply that the rigid analytic spaces appearing in [15] are normal (Corollaries 4.27, 5.10).

The knowledge of irreducible components of $R^{\square}_{\overline{\rho}}$ allows us to refine the existing results on the Zariski density of the locus with prescribed $p$-adic Hodge theoretic properties.

**Theorem 1.5** (Theorem 6.1)**.** *Suppose that $p$ does not divide $2d$. Let $\Sigma$ be a subset of the maximal spectrum of $R^{\square}_{\overline{\rho}}[1/p]$ parameterizing any of the following sets of lifts of $\overline{\rho}$ to characteristic zero:*

1. *crystalline lifts with regular Hodge–Tate weights;*
2. *potentially crystabelline lifts with fixed regular Hodge–Tate weights;*
3. *potentially crystalline supercuspidal lifts with fixed regular Hodge–Tate weights.*

*Then $\Sigma$ is Zariski dense in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$.

The assumption $p \nmid 2d$ enters via our use of the patched module $M_\infty$ constructed in [13]. The paper [13] is applicable whenever $\overline{\rho}$ has a potentially diagonalisable lift. It has been proved recently by Emerton–Gee [23], using the Emerton–Gee stack, that this holds for all $\overline{\rho}$. The rest of our paper is independent of [23]. We show that the action of $R^{\square}_{\overline{\rho}}$ on $M_\infty$ is faithful (Theorem 6.8), which allows us to deduce Theorem 1.5 from [24].

Partial results towards Theorem 1.1 and also towards the more recent question solved by Theorem 1.2 appear in many places, e.g. [3], [7], [6], [19], [29], [38], in special cases. However, these papers either compute with equations defining the rings, or impose assumptions on $\overline{\rho}$ so that the deformation theory of $\overline{\rho}$ is essentially unobstructed which leads to only one irreducible component. Although there is some overlap in ideas with [29], the argument in our paper is rather different as we don't compute with equations. We refer the reader to Section 6 for a more detailed discussion on the previous results on Zariski density of specific loci in $\operatorname{Spec} R^{\square}_{\overline{\rho}}$.

*Remark* 1.6. In the theorems above we work with framed deformation rings. Our results also carry over to the versal deformation rings (which coincide with the universal deformation rings if $\overline{\rho}$ has only scalar endomorphisms), by exploiting the fact that framed deformation rings are formally smooth over versal deformation rings (see e.g. [29, Lemma 2.1]) and using [10, Theorem 2.3.6, Corollary 2.2.23 (a)].

1.1. **Overview of the proof.** To give more of an overview of our work, let us introduce two further key players. The first are determinant laws, which we refer to as *pseudo-characters* throughout the paper, and their deformations. Let $\overline{D} : k[G_F] \to k$ be the pseudo-character attached to $\overline{\rho}$ as defined in [17]. Let $D^{\mathrm{ps}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Sets}$ be the functor mapping $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$ to the set $D^{\mathrm{ps}}(A)$ of continuous $A$-valued $d$-dimensional pseudo-characters $D : A[G_F] \to A$ with $\overline{D} = D \pmod{\mathfrak{m}_A}$. The functor $D^{\mathrm{ps}}$ is pro-representable by a complete local Noetherian $\mathcal{O}$-algebra $(R^{\mathrm{ps}}, \mathfrak{m}_{R^{\mathrm{ps}}})$, see [17, Section 3.1].

Mapping a lifting of $\overline{\rho}$ to its associated pseudo-character induces a natural transformation $D^{\square}_{\overline{\rho}} \to D^{\mathrm{ps}}$ and thus a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}} \to R^{\square}_{\overline{\rho}}$. The ring $R^{\mathrm{ps}}$ has been well understood in the recent work of GB–Juschka [8]. Our basic idea is to study $R^{\square}_{\overline{\rho}}$ by studying the fibres of this map. In fact it is technically more convenient to introduce an intermediate ring $R^{\mathrm{ps}} \to A^{\mathrm{gen}} \to R^{\square}_{\overline{\rho}}$, depending on $\overline{D}$ and not on $\overline{\rho}$ itself, such that $A^{\mathrm{gen}}$ is of finite type over $R^{\mathrm{ps}}$ and $R^{\square}_{\overline{\rho}}$ is a completion of $A^{\mathrm{gen}}$ at a maximal ideal. This is our second key player.

To describe $A^{\mathrm{gen}}$, let $D^u : R^{\mathrm{ps}}[\![G_F]\!] \to R^{\mathrm{ps}}$ be the universal pseudo-character lifting $\overline{D}$ and let $\mathrm{CH}(D^u)$ be the closed two-sided ideal of $R^{\mathrm{ps}}[\![G_F]\!]$ defined in [17, §1.17], so that $E := R^{\mathrm{ps}}[\![G_F]\!]/\mathrm{CH}(D^u)$ is the largest quotient of $R^{\mathrm{ps}}[\![G_F]\!]$ for which the Cayley–Hamilton theorem for $D^u$ holds. Following [17, Section 1.17] we will call such algebras Cayley–Hamilton $R^{\mathrm{ps}}$-algebra of degree $d$. By [47, Proposition 3.6] the ring $E$ is a finitely generated $R^{\mathrm{ps}}$-module. Now a construction of Procesi [41] gives a commutative $R^{\mathrm{ps}}$-algebra $A^{\mathrm{gen}}$ together with a homomorphism $j : E \to M_d(A^{\mathrm{gen}})$ of Cayley–Hamilton $R^{\mathrm{ps}}$-algebras satisfying the following *universal property*: if $f : E \to M_d(B)$ is a map of Cayley–Hamilton $R^{\mathrm{ps}}$-algebras for a commutative $R^{\mathrm{ps}}$-algebra $B$, then there is a unique map $\tilde{f} : A^{\mathrm{gen}} \to B$ of $R^{\mathrm{ps}}$-algebras such that $f = M_d(\tilde{f}) \circ j$. We give further details in Lemma 3.1 in the main text. The superscript *gen* in $A^{\mathrm{gen}}$ stands for *generic matrices*.

Since $E$ is finitely generated as $R^{\mathrm{ps}}$-module, the construction of Procesi shows that $A^{\mathrm{gen}}$ is of finite type over $R^{\mathrm{ps}}$. Moreover one has an algebraic action of $\mathrm{GL}_d$ on $X^{\mathrm{gen}} := \operatorname{Spec} A^{\mathrm{gen}}$, which for every a $R^{\mathrm{ps}}$-algebra $B$ and point $f : E \to M_d(B)$ in $X^{\mathrm{gen}}(B)$ is simply given by conjugation of matrices. Note also that $X^{\mathrm{gen}}$ is isomorphic to $\mathrm{Rep}^{\square}_{\overline{D}} = \mathrm{Rep}^{\square}_{E,D^u}$ as defined in [47, Theorem 3.8]. It is an important observation that to $\pi : X^{\mathrm{gen}} \to X^{\mathrm{ps}} := \operatorname{Spec} R^{\mathrm{ps}}$ we can apply geometric invariant theory (GIT). As shown in [47, Theorem 2.20], the induced morphism $X^{\mathrm{gen}} /\!\!/ G \to X^{\mathrm{ps}}$ is an adequate homeomorphism in the sense of [1, Definition 3.3.1].

Our first important result on dimensions is for $\overline{X}^{\mathrm{gen}} := \operatorname{Spec} A^{\mathrm{gen}}/\varpi$.

**Theorem 1.7** (Theorem 3.30, Lemma 3.22)**.** *We have*

$$\dim X^{\mathrm{gen}}[1/p] \le \dim \overline{X}^{\mathrm{gen}} \le d^2 + d^2[F : \mathbb{Q}_p].$$

To prove the second inequality of Theorem 1.7 we decompose the base of the finite type morphism $\bar{\pi} : \overline{X}^{\mathrm{gen}} \to \overline{X}^{\mathrm{ps}} = \operatorname{Spec} R^{\mathrm{ps}}/\varpi$ as a finite union $\overline{X}^{\mathrm{ps}} = \bigcup_i U_i$ of locally closed subschemes $U_i$. The points of the $U_i$ correspond to semi-simple degree $d$ representations of $G_F$ with certain (degree) conditions on the irreducible constituents. The work [8] gives dimension estimates on the $U_i$. We combine them with bounds on the dimensions of the fibres at the closed points of $U_i$, obtained using GIT, and with results on $\bar{\pi}^{-1}(U_i) \to U_i$ from commutative algebra. In Subsection 3.1 we analyze in detail the dimensions of the fibres of $\pi$ at points $y$ of $X^{\mathrm{ps}}$ valued either in finite fields containing $k$ or local fields whose residue fields contain $k$. The analysis at such points suffices for all results in this paper. The commutative algebra results, used to analyze $\bar{\pi}^{-1}(U_i) \to U_i$ and to give the first inequality, are proved in Subsection 3.2. The key technical improvement working with $X^{\mathrm{gen}}$ instead of $\operatorname{Spec} R^{\square}_{\bar{\rho}}$ directly, is that the fibres are of finite type over $\kappa(y)$.

We apply the bounds from Theorem 1.7 to the study of lifting rings of continuous residual representations $\rho_x : G_F \to \mathrm{GL}_d(\kappa(x))$ where $x$ is a point of $X^{\mathrm{gen}}$ whose residue field $\kappa(x)$ is a finite or a local field. We distinguish three cases: (1) the field $\kappa(x)$ is a finite extension $k'$ of $k$, and then we set $\Lambda$ to be the ring of integers $\mathcal{O}'$ of the unramified extension $L'$ of $L$ with residue field $k'$; (2) the field $\kappa(x)$ is a finite extension of $L$ and then we set $\Lambda$ to be $\kappa(x)$; (3) The field $\kappa(x)$ is a local field that contains $k$. If $k'$ denotes its residue field of $\kappa(x)$, then we take as $\Lambda$ a Cohen ring of $\kappa(x)$ (with a natural topology) tensored over the Witt vector ring $W(k')$ with $\mathcal{O}'$.

Let $\mathfrak{A}_\Lambda$ be the category of local Artinian $\Lambda$-algebras $(A, \mathfrak{m}_A)$ with residue field $\kappa(x)$. We equip the rings $A$ with a natural topology, and we consider the functor $D^{\square}_{\rho_x} : \mathfrak{A}_\Lambda \to \mathrm{Sets}$ such that $D^{\square}_{\rho_x}(A)$ is the set of continuous group homomorphisms

$\rho : G_F \to \mathrm{GL}_d(A)$, such that $\rho \pmod{\mathfrak{m}_A} = \rho_x$. In cases (1) and (2) such functors occur in work of Mazur and Kisin, respectively. The formulation in case (3) appears to be new. In all cases, the functor $D^{\square}_{\rho_x}$ is pro-represented by a complete local Noetherian $\Lambda$-algebra $R^{\square}_{\rho_x}$ with residue field $\kappa(x)$. The arguments of Mazur and Kisin carry over to the case when $\kappa(x)$ is a local field of characteristic $p$ and yield a presentation

$$R^{\square}_{\rho_x} \cong \Lambda[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s) \tag{2}$$

with $r = \dim_{\kappa(x)} Z^1(G_F, \operatorname{ad}\rho_x)$ and $s = \dim_{\kappa(x)} H^2(G_F, \operatorname{ad}\rho_x)$; here $\operatorname{ad}\rho_x$ is the adjoint representation of $G_F$ on $\mathrm{End}_{\kappa(x)}(\rho_x)$ by conjugation. By a suitable version of Tate local duality results, one finds $r - s = d^2 + d^2[F : \mathbb{Q}_p]$. From this, Theorem 1.7 and some commutative algebra results that relate the completion of $A^{\mathrm{gen}}$ at $x$ to the ring $R^{\square}_{\rho_x}$, we deduce the following result.

**Corollary 1.8** (Corollaries 3.37 and 3.43)**.** *For $x$ as above the following hold:*

1. *$R^{\square}_{\rho_x}$ is a flat $\Lambda$-algebra of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is complete intersection;*
2. *if $\mathrm{char}(\kappa(x)) = p$ then $R^{\square}_{\rho_x}/\varpi$ is complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p]$.*

At first glance one might expect that for closed points $x$ of $X^{\mathrm{gen}}$ the residue field $\kappa(x)$ is always finite. However, as we show in Example 3.21, $\kappa(x)$ can also be a local field of characteristic 0 or $p$. In Subsection 3.3 we show that this exhausts all possibilities.

Corollary 1.8 gives us a handle on the completions of the local rings $\mathcal{O}_{X^{\mathrm{gen}},x}$ (resp. $\mathcal{O}_{\overline{X}^{\mathrm{gen}},x}$) at closed points $x \in X^{\mathrm{gen}}$ (resp. $x \in \overline{X}^{\mathrm{gen}}$), which allows us to deduce the following result.

**Corollary 1.9** (Corollaries 3.39 and 3.44)**.** *The following hold:*

1. *$A^{\mathrm{gen}}$ is $\mathcal{O}$-torsion free, equi-dimensional of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection;*
2. *$A^{\mathrm{gen}}/\varpi$ is equi-dimensional of dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection.*

We end Section 3 with a result on the density of (certain) absolutely irreducible points in $R^{\square}_{\overline{\rho}}$ and in $R^{\square}_{\overline{\rho}}/\varpi$. This is motivated by and relies on similar results for $R^{\mathrm{ps}}$. A point $x$ in $X^{\mathrm{ps}} = \operatorname{Spec} R^{\mathrm{ps}}$ is called *absolutely irreducible* if the associated semisimple representation $\rho_x : G_F \to \mathrm{GL}_d(\overline{\kappa(x)})$ (which is unique up to isomorphism) is irreducible. It follows from [15] that the locus of absolutely irreducible points is dense open in the generic fibre $X^{\mathrm{ps}}[1/p] = \operatorname{Spec} R^{\mathrm{ps}}[1/p]$, and this is extremely useful because such points are regular on $X^{\mathrm{ps}}[1/p]$.

A key role in the study of the regular locus in the special fibre $\overline{X}^{\mathrm{ps}} = \operatorname{Spec} R^{\mathrm{ps}}/\varpi$ in [8] is played by a class of absolutely irreducible points, which are called *non-special*. We extend this notion slightly in Appendix A. We say that an absolutely irreducible point $x$ in $\overline{X}^{\mathrm{ps}}$ with finite or local residue field is *Kummer-reducible* if there exists a degree $p$ Galois extension $F'$ of $F(\zeta_p)$ such that $\rho_x|_{G_{F'}}$ is reducible, and *Kummer-irreducible* if not. If $\zeta_p \in F$ then $x \in \overline{X}^{\mathrm{ps}}$ is Kummer-irreducible if and only if it is *non-special* in the sense of [8]. We show that if $x$ is Kummer-irreducible then $H^2(G_F, \operatorname{ad}^0\rho_x) = 0$ where $\operatorname{ad}^0\rho_x$ is the subrepresentation of $\operatorname{ad}\rho_x$ of trace zero matrices. Much more importantly for us, we also show that the locus of

Kummer-irreducible $x \in \overline{X}^{\mathrm{ps}}$ is dense open. At these points $\overline{X}^{\mathrm{ps}}$ is not necessarily smooth but it is relatively smooth over $\operatorname{Spec} R_{\det \overline{\rho}}$. Here we prove the following:

**Proposition 1.10** (Proposition 3.54 and Corollaries 3.58 and 3.60)**.** *We have:*

(1) *The set of absolutely irreducible points $x \in \operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ with $\kappa(x)$ finite over $L$ is dense in $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$.*

(2) *The set of Kummer-irreducible points $x \in \operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ with $\kappa(x)$ a local field is dense in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$.*

*In particular, every continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ has an absolutely irreducible lift to characteristic zero.*

From here on, we assume that $L$ contains $F$, so that in particular $L$ contains all roots of unity contained in $F$. We now give a more detailed overview of Theorem 1.2 on components of $R^{\square}_{\overline{\rho}}$. The homomorphism $R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$ from that theorem is induced by the natural transformation $D^{\square}_{\overline{\rho}} \to D_{\det \overline{\rho}}$ that to a representation assigns its determinant, and it clearly induces the map (1) on components.

Via the Artin map $F^{\times} \to G_F^{\mathrm{ab}}$ from local class field theory, the inclusion $\mu \subset F^{\times}$ and the identification of $R_{\det \overline{\rho}}$ with the completed group ring of the pro-$p$ completion of $G_F^{\mathrm{ab}}$, the ring $R_{\det \overline{\rho}}$ becomes an $\mathcal{O}[\mu]$-algebra. It is well-known that $R_{\det \overline{\rho}}$ is a power series ring over $\mathcal{O}[\mu]$ in $[F : \mathbb{Q}_p] + 1$ formal variables. The components of the étale $L$-algebra $\mathcal{O}[\mu][1/p] = L[\mu]$ are in bijection with the characters $\chi : \mu \to \mathcal{O}^{\times}$. Setting $R^{\square,\chi}_{\overline{\rho}} = R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ we obtain a decomposition $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p] = \coprod_{\chi} \operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$, where $\chi$ ranges over the characters $\mu \to \mathcal{O}^{\times}$.

The main step in the proof of the bijectivity of the map (1) in Theorem 1.2 is to show that the rings $R^{\square,\chi}_{\overline{\rho}}$ are normal by verifying Serre's criterion for normality. We first present $R^{\square}_{\overline{\rho}}$ over $R_{\det \overline{\rho}}$ (Proposition 4.3) by imitating Kisin's method of presenting global rings over local rings. Since $R^{\chi}_{\det \overline{\rho}} := R_{\det \overline{\rho}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ is formally smooth, by applying $\otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ we obtain a presentation of $R^{\square,\chi}_{\overline{\rho}}$ over $R^{\chi}_{\det \overline{\rho}}$ analogous to the presentation (2). Since $R^{\square,\chi}_{\overline{\rho}}$ has the same dimension as $R^{\square}_{\overline{\rho}}$, the presentation yields that $R^{\square,\chi}_{\overline{\rho}}$ is complete intersection of expected dimension, and hence satisfies Serre's condition (S2). We then show that $X^{\mathrm{gen},\chi} := \operatorname{Spec}(A^{\mathrm{gen}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O})$ and its special fibre $\overline{X}^{\mathrm{gen},\chi}$ are regular in codimension 1 by showing that the Kummer-irreducible locus in $\overline{X}^{\mathrm{gen},\chi}$ (resp. absolutely irreducible locus in $X^{\mathrm{gen},\chi}[1/p]$) is regular, and its complement has codimension at least 2 if either $F \neq \mathbb{Q}_p$, or $d > 2$ or $\overline{D}$ is absolutely irreducible. The case $F = \mathbb{Q}_p$, $d = 2$ and $\overline{D}$ reducible requires an extra analysis of the reducible locus. Since $R^{\square,\chi}_{\overline{\rho}}$ is a completion of a local ring at a closed point of $X^{\mathrm{gen},\chi}$, we deduce that $R^{\square,\chi}_{\overline{\rho}}$ is regular in codimension 1. We thus deduce that $R^{\square,\chi}_{\overline{\rho}}$ is normal. Since $R^{\square,\chi}_{\overline{\rho}}$ is a local ring it is an integral domain. A similar argument works for the special fibre.

Theorem 1.4 on $R^{\square,\psi}_{\overline{\rho}}$ is proved by reduction to the results on $R^{\square,\chi}_{\overline{\rho}}$ where $\chi : \mathcal{O}[\mu] \to \mathcal{O}^{\times}$ is the restriction of $\psi$ to $\mu$ via the Artin map. To give an idea of the proof, let $\mathcal{X} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Sets}$ be the functor, which sends $(A, \mathfrak{m}_A)$ to the group $\mathcal{X}(A)$ of continuous characters $\theta : G_F \to 1 + \mathfrak{m}_A$ such that $\theta$ restricted to $\mu$ is trivial, and let $\mathcal{O}(\mathcal{X})$ be the complete local Noetherian $\mathcal{O}$-algebra pro-representing $\mathcal{X}$. Local class field theory gives an isomorphism $\mathcal{O}(\mathcal{X}) \cong \mathcal{O}[\![y_1, \dots, y_{[F:\mathbb{Q}_p]+1}]\!]$. Let $\varphi_d : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$ be the morphism induced by the $d$-power map $\mathcal{X} \to \mathcal{X}$, $\theta \mapsto \theta^d$.

Our key technical result is Proposition 5.1 which yields a natural isomorphism

$$R^{\square,\chi}_{\overline{\rho}} \otimes_{\mathcal{O}(\mathcal{X}),\varphi_d} \mathcal{O}(\mathcal{X}) \cong R^{\square,\psi}_{\overline{\rho}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X})$$

that comes from an analogous isomorphism of functors. It allows us to compare the sets of points $x$ with $x$ finite or local at which $H^2(G_F, \mathrm{ad}^0 \rho_x)$ is non-zero on both sides. We also show that the map $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}} \otimes_{\mathcal{O}(\mathcal{X}),\varphi_d} \mathcal{O}(\mathcal{X}) \to \operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}$ induces a homeomorphism on special fibres, and a finite covering on generic fibres. Then we use topological arguments to obtain the dimension of $R^{\square,\psi}_{\overline{\rho}}$ and bound the codimension of its singular locus from the analogous results on $R^{\square,\chi}_{\overline{\rho}}$.

We also prove analogs of Theorem 1.2 (resp. Theorem 1.5) for spaces $X^{\mathrm{gen},\chi}$ and $\overline{X}^{\mathrm{gen},\chi}$ for characters $\chi : \mu \to \mathcal{O}^\times$ (resp. $X^{\mathrm{gen},\psi}$, $\overline{X}^{\mathrm{gen},\psi}$ for characters $\psi : G_F \to \mathcal{O}^\times$ lifting $\det \overline{\rho}$), see Corollaries 4.6, 4.18, 4.26 (resp. Corollaries 5.8, 5.9). We expect that our results will be useful in the study of geometry of Emerton–Gee stack and its derived versions.

It is natural to ask whether our results generalize to deformations valued in reductive groups other than $\mathrm{GL}_d$. A large part of Section 3 works in this generality, however we lack the results of GB–Juschka on the deformation rings of pseudo-characters in this more general setting; this is the subject of the ongoing PhD thesis of Julian Quast at Heidelberg University.

1.2. **Overview by section.** In Section 2 we briefly review GIT. A key result that gets used later on is Lemma 2.1. In Section 3 we introduce $X^{\mathrm{gen}}$ and its special fibre $\overline{X}^{\mathrm{gen}}$. In Subsection 3.1 we bound the dimensions of the fibres of the map $X^{\mathrm{gen}} \to X^{\mathrm{ps}}$. In Subsection 3.2 we combine this with results of [8] to bound the dimension of $X^{\mathrm{gen}}$ and $\overline{X}^{\mathrm{gen}}$. In Subsection 3.3 we relate the completions of local rings at closed points $x$ of $X^{\mathrm{gen}}$, $\overline{X}^{\mathrm{gen}}$ to the deformation theory of Galois representations $\rho_x : G_F \to \mathrm{GL}_d(\kappa(x))$ and prove Theorem 1.1. In Section 3.4 we bound the maximally reducible semi-simple locus in $X^{\mathrm{gen}}$ and $\overline{X}^{\mathrm{gen}}$. This computation later on gets used only in the case $d = 2$, $F = \mathbb{Q}_2$ and $\overline{D}$ is reducible. In Subsection 3.5 we prove the Zariski density of the Kummer-irreducible locus in $\overline{X}^{\mathrm{gen}}$ and absolutely irreducible locus in $X^{\mathrm{gen}}[1/p]$ and also establish lower bounds for the dimension of their complements. These bounds are used to establish normality later on. In Section 4 we present $R^{\square}_{\overline{\rho}}$ over $R_{\det \overline{\rho}}$ and prove Theorem 1.2. In Section 5 we prove Theorem 1.4. In Section 6 we prove Theorem 1.5. In Appendix A we introduce the notion of Kummer-irreducible points in $\operatorname{Spec} R^{\mathrm{ps}}/\varpi$, which slightly generalizes the notion of non-special points defined in [8]. This technical improvement is needed in Section 5 when $\zeta_p \notin F$.

1.3. **Notation.** Let $F$ be a finite extension of $\mathbb{Q}_p$ and let $G_F$ be its absolute Galois group. Let $L$ be another finite extension of $\mathbb{Q}_p$, such that $\mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg}}(F, L)$ has cardinality $[F : \mathbb{Q}_p]$. Let $\mathcal{O}$ be the ring of integers in $L$, $\varpi$ a uniformiser, and $k$ the residue field. We will denote by $\zeta_p$ a primitive $p$-th root of unity in a fixed algebraic closure of $F$. For a commutative ring $R$ we let $P_1 R = \{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = 1\}$.

We fix a representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ and assume that all its irreducible subquotients are absolutely irreducible. We note that we may always achieve that after enlarging $k$, since the image of $\overline{\rho}$ is a finite group. Let $\mathrm{ad}\,\overline{\rho}$ be the adjoint representation of $G_F$ and $\mathrm{ad}^0\,\overline{\rho}$ the subspace of trace zero endomorphisms, so that $G_F$ acts on $\mathrm{End}_k(\overline{\rho})$ by conjugation. We will denote the dimension as a $k$-vector space of cohomology groups $H^i(G_F, \mathrm{ad}\,\overline{\rho})$ by $h^i$.

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## 2. Geometric invariant theory

We assume the set up of [44]. Let $R$ be a Noetherian ring and let $S = \operatorname{Spec} R$. Let $G$ be a reductive group scheme over $S$, so that $G$ is an affine group scheme over $S$, $G \to S$ is smooth and the geometric fibres are connected reductive groups. In the application $G = S \times_{\operatorname{Spec}\mathbb{Z}} \mathrm{GL}_d$ and $G = S \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{G}_m^r$ so that these conditions hold.

Let $V$ be a free $R$-module of finite rank $r$ endowed with a $G$-module structure, let $\check{V} = \operatorname{Hom}_R(V, R)$ and let $\operatorname{Sym}(\check{V})$ be the symmetric algebra. The $G$-module structure on $V$ induces an action of $G$ on $\operatorname{Spec}(\operatorname{Sym}(\check{V})) = \mathbb{A}_S^r$. Let $X$ be a closed $G$-invariant subscheme of $\operatorname{Spec}(\operatorname{Sym}(\check{V}))$. The $G$-action on $X$ induces an action on $B$, the ring of functions on $X$. The GIT quotient $X /\!\!/ G$ is represented by the ring of invariants $B^G$.

Let $y = \operatorname{Spec}\kappa$ be a geometric point of $X /\!\!/ G$. We may identify the fibre $X_y$ with a closed $G$-invariant subscheme of $X$.

**Lemma 2.1.** *Let $x \in X_y(\kappa)$ be such that the orbit $G \cdot x$ is closed in $X_y$ then*
$$\dim X_y \leq \dim_\kappa T_x(X_y).$$

*Proof.* Since $X_y$ is Noetherian it has finitely many irreducible components and the $G_y$-action permutes them, but since $G_y$ is connected every irreducible component is $G_y$-invariant and thus the permutation is the identity. This can be seen as follows: let $U$ be the open subscheme of $X_y$ obtained by removing all the intersections of irreducible components. Then it is enough to show that the connected components of $U$ are $G_y$-invariant. If $U'$ is a connected component of $U$ then the image of $G \times_S U'$ in $U$ under the action map is connected and contains $U'$, so is equal to $U'$.

Let $Z$ be an irreducible component of $X_y$ such that $\dim Z = \dim X_y$. As explained above $Z$ is both closed in $X_y$ and $G$-invariant. Then by [44, Theorem 3] both $Z$ and $X_y$ have a unique closed $G$-orbit, hence those orbits must be equal. Therefore $x \in Z$ so since $Z$ is irreducible,
$$\dim X_y = \dim Z \leq \dim_\kappa T_x(Z) \leq \dim_\kappa T_x(X_y).$$
□

## 3. $R^{\square}_{\overline{\rho}}$ is complete intersection

Let $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ be a continuous representation. Let $D^{\square}_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to \text{Sets}$ be the functor from the category of local Artinian $\mathcal{O}$-algebras with residue field $k$ to the category of sets, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D^{\square}_{\overline{\rho}}(A)$ is the set of continuous representations $\rho_A : G_F \to \mathrm{GL}_d(A)$, such that $\rho_A \pmod{\mathfrak{m}_A} = \overline{\rho}$. The functor $D^{\square}_{\overline{\rho}}$ is pro-represented by a complete local Noetherian $\mathcal{O}$-algebra $R^{\square}_{\overline{\rho}}$. The proof of [36, Proposition 21.1] shows that the tangent space to $D^{\square}_{\overline{\rho}}$ is $Z^1(G_F, \operatorname{ad}\overline{\rho})$ and it follows from the proof of [35, Sec. 1.6, Proposition 2] that there is a presentation

$$R^{\square}_{\overline{\rho}} \cong \mathcal{O}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s), \tag{3}$$

where $r = \dim_k Z^1(G_F, \operatorname{ad}\overline{\rho}) = d^2 - h^0 + h^1$ and $s = h^2$ and $h^i = \dim_k H^i(G_F, \operatorname{ad}\overline{\rho})$. To show that $R^{\square}_{\overline{\rho}}$ is a complete intersection, it is enough to show that the Krull dimension of $R^{\square}_{\overline{\rho}}$ is equal to $1 + r - s$, which is equal to $1 + d^2 + d^2[F : \mathbb{Q}_p]$ by the Euler characteristic formula.

Let $\overline{D} : k[G_F] \to k$ be the determinant law attached to $\overline{\rho}$, see [17], so that $\overline{D}$ is equal to the composition of the polynomial laws induced by $k[G_F] \xrightarrow{\overline{\rho}} M_d(k)$ and $M_d(k) \xrightarrow{\det} k$. In this paper, we will refer to determinant laws as pseudo-characters. Let $D^{\mathrm{ps}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Sets}$ be the functor, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D^{\mathrm{ps}}(A)$ is the set of continuous $A$-valued $d$-dimensional pseudo-characters of $A[G_F]$, which reduce to $\overline{D}$ modulo $\mathfrak{m}_A$. The functor $D^{\mathrm{ps}}$ is pro-representable by a complete local Noetherian $\mathcal{O}$-algebra $(R^{\mathrm{ps}}, \mathfrak{m}_{R^{\mathrm{ps}}})$, see [17, Section 3.1].

Mapping a deformation of $\overline{\rho}$ to its determinant induces a natural transformation $D^{\square}_{\overline{\rho}} \to D^{\mathrm{ps}}$ and thus a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}} \to R^{\square}_{\overline{\rho}}$. The ring $R^{\mathrm{ps}}$ has been well understood in the recent work of GB–Juschka [8]. Our basic idea is to study $R^{\square}_{\overline{\rho}}$ by studying the fibres of this map. In fact it is technically more convenient to introduce an intermediate ring $R^{\mathrm{ps}} \to A^{\mathrm{gen}} \to R^{\square}_{\overline{\rho}}$, depending on $\overline{D}$ and not on $\overline{\rho}$ itself, such that $A^{\mathrm{gen}}$ is of finite type over $R^{\mathrm{ps}}$ and $R^{\square}_{\overline{\rho}}$ is a completion of $A^{\mathrm{gen}}$ at a maximal ideal. Since $\dim R^{\square}_{\overline{\rho}} \le \dim A^{\mathrm{gen}}$, it is enough to bound the dimension of $A^{\mathrm{gen}}$.

Let $D^u : R^{\mathrm{ps}}[\![G_F]\!] \to R^{\mathrm{ps}}$ be the universal pseudo-character lifting $\overline{D}$. Let $\mathrm{CH}(D^u)$ be the closed two-sided ideal of $R^{\mathrm{ps}}[\![G_F]\!]$ defined in [17, §1.17], so that $E := R^{\mathrm{ps}}[\![G_F]\!]/\mathrm{CH}(D^u)$ is the largest quotient of $R^{\mathrm{ps}}[\![G_F]\!]$ for which the Cayley–Hamilton theorem for $D^u$ holds. Following [17, §1.17] we will call such algebras Cayley–Hamilton $R^{\mathrm{ps}}$-algebra of degree $d$. Then $E$ is a finitely generated $R^{\mathrm{ps}}$-module, [47, Proposition 3.6]. If $f : E \to M_d(B)$ is a homomorphism of $R^{\mathrm{ps}}$-algebras for a commutative $R^{\mathrm{ps}}$-algebra $B$ then we say $f$ is a homomorphism of Cayley–Hamilton algebras if $\det \circ f : E \to B$ is equal to the specialization of $D^u$ along $R^{\mathrm{ps}} \to B$.

The superscript *gen* in $A^{\mathrm{gen}}$ stands for *generic matrices*, and the following construction appears in the work of Procesi [41]; Lemmas 3.1, 3.2, 3.3 are contained in [47, Theorem 3.8], but one needs to unpack all the groupoid stuff to get to them.

**Lemma 3.1.** *There is a finitely generated commutative $R^{\mathrm{ps}}$-algebra $A^{\mathrm{gen}}$ together together with a homomorphism of Cayley–Hamilton $R^{\mathrm{ps}}$-algebras $j : E \to M_d(A^{\mathrm{gen}})$, satisfying the following universal property: if $f : E \to M_d(B)$ is a map of Cayley–Hamilton $R^{\mathrm{ps}}$-algebras for a commutative $R^{\mathrm{ps}}$-algebra $B$ then there is a unique map $\tilde{f} : A^{\mathrm{gen}} \to B$ of $R^{\mathrm{ps}}$-algebras such that $f = M_d(\tilde{f}) \circ j$.*

*Proof.* By writing down a generic $d \times d$-matrix for each $R^{\mathrm{ps}}$-generator of $E$ and quotienting out by the relations the generators satisfy in $E$, one obtains a commutative $R^{\mathrm{ps}}$-algebra $C$ and a homomorphism of $R^{\mathrm{ps}}$-algebras $j : E \to M_d(C)$. More formally, $C$ is a quotient of $R^{\mathrm{ps}} \otimes_{\mathbb{Z}} \mathrm{Sym}(W)$, where $W$ is a direct sum of

$n$ copies of $\mathrm{End}(\mathrm{Std})^*$, where Std is the standard representation of $\mathrm{GL}_d$ over $\mathbb{Z}$, $n$ is the size of a generating set of $E$ as an $R^{\mathrm{ps}}$-module and $\mathrm{Sym}(W)$ is the symmetric algebra over $\mathbb{Z}$. If we would only require the maps to be maps of $R^{\mathrm{ps}}$-algebras (i.e., if we would not impose the Cayley–Hamilton condition) then the triple $j : E \to M_d(C)$ would have the required universal property. To ensure that the Cayley–Hamilton condition is satisfied we have to consider the quotient of $C$ constructed as follows. Let $\Lambda_i : E \to R^{\mathrm{ps}}$, $0 \le i \le d$ be the coefficients of the characteristic polynomial of $D^u$ - these are homogeneous polynomial laws satisfying $D^u(t-a) = \sum_{i=0}^{n}(-1)^i\Lambda_i(a)t^{d-i}$ in $R^{\mathrm{ps}}[t]$, see [17, Section 1.10]. For each $a \in E$ let $c_i(j(a))$ be the $i$-th coefficient of the characteristic polynomial of the matrix $j(a) \in M_d(C)$. Let $I$ be the ideal of $C$ generated by $\Lambda_i(a) - c_i(j(a))$ for all $a \in E$ and $0 \le i \le d$ and let $A^{\mathrm{gen}} := C/I$. Since the coefficients of the characteristic polynomial determine pseudo-characters uniquely, [17, Corollary 1.14], [46, 1.1.9.15], the composition $E \to M_d(C) \to M_d(A^{\mathrm{gen}})$ is a map of Cayley–Hamilton algebras, and the universal property of $j : E \to M_d(C)$ implies the universal property for $j : E \to M_d(A^{\mathrm{gen}})$. Since $E$ is finitely generated as $R^{\mathrm{ps}}$-module, $C$ and hence $A^{\mathrm{gen}}$ are of finite type over $R^{\mathrm{ps}}$. $\square$

Let us make a connection to GIT in Section 2. If $E$ is generated by $n$ generators as an $R^{\mathrm{ps}}$-module, then, as explained in the proof of Lemma 3.1, $A^{\mathrm{gen}}$ is a quotient of $R^{\mathrm{ps}} \otimes_{\mathbb{Z}} \mathrm{Sym}(W)$. The group $G := \mathrm{GL}_d$ acts on $W$ by conjugation, and this induces an action of $\mathrm{GL}_d$ on $X^{\mathrm{gen}} := \mathrm{Spec}\, A^{\mathrm{gen}}$. For every a $R^{\mathrm{ps}}$-algebra $B$, a point in $X^{\mathrm{gen}}(B)$ corresponds to an $n$-tuple of $d \times d$-matrices with entries in $B$ satisfying certain relations, and $\mathrm{GL}_d(B)$ acts on $X^{\mathrm{gen}}(B)$ by conjugating the matrices. The scheme $X^{\mathrm{gen}}$ is isomorphic to $\mathrm{Rep}^{\square}_{D} = \mathrm{Rep}^{\square}_{E,D^u}$ as defined in [47, Theorem 3.8].

The GIT quotient $X^{\mathrm{gen}} /\!\!/ G$ is represented by the ring of invariants $(A^{\mathrm{gen}})^G$. The map $R^{\mathrm{ps}} \to A^{\mathrm{gen}}$ is $G$-invariant and induces a homomorphism $R^{\mathrm{ps}} \to (A^{\mathrm{gen}})^G$. It follows from [47, Theorem 2.20] that the induced map

$$X^{\mathrm{gen}} /\!\!/ G \to X^{\mathrm{ps}} := \mathrm{Spec}\, R^{\mathrm{ps}} \tag{4}$$

is an adequate homeomorphism, i.e. an integral, universal homeomorphism which is a local isomorphism around points with characteristic zero residue field, see [1, Definition 3.3.1]. We denote by $\overline{X}^{\mathrm{gen}}$ and $\overline{X}^{\mathrm{ps}}$ the special fibres of $X^{\mathrm{gen}}$ and $X^{\mathrm{ps}}$, respectively. The same argument shows that

$$\overline{X}^{\mathrm{gen}} /\!\!/ G \to \overline{X}^{\mathrm{ps}}$$

is an adequate homeomorphism.

We equip $R^{\mathrm{ps}}$ with the $\mathfrak{m}_{R^{\mathrm{ps}}}$-adic topology. Since the residue field is finite $R^{\mathrm{ps}}$ is a compact ring.

**Lemma 3.2.** *Let $B$ be a topological $R^{\mathrm{ps}}$-algebra. If $f : E \to M_d(B)$ is any homomorphism of $R^{\mathrm{ps}}$-algebras then the composition $G_F \to E^\times \xrightarrow{f} \mathrm{GL}_d(B)$ defines a continuous representation of $G_F$.*

*Proof.* Since $R^{\mathrm{ps}}$ is a compact ring for every finitely generated $R^{\mathrm{ps}}$-module $M$ there is a unique topology on $M$ making $M$ into a topological $R^{\mathrm{ps}}$-module, see [2, Corollary 1.10].

We equip $R^{\mathrm{ps}}[\![G_F]\!]$ with its projective limit topology, $E$ with the quotient topology, and its group of units $E^\times$ with the subspace topology via the embedding

$E^\times \hookrightarrow E \times E$, $x \mapsto (x, x^{-1})$. Since the map $G_F \to R^{\mathrm{ps}}[\![G_F]\!]$ is continuous, the map $G_F \to E^\times$ is also continuous.

Since $E$ is a finitely generated $R^{\mathrm{ps}}$-module, its topology coincides with $\mathfrak{m}_{R^{\mathrm{ps}}}$-adic topology. Let $M = f(E) \subset M_d(B)$. Then $M$ is again a finitely generated $R^{\mathrm{ps}}$-module, thus the quotient topology on $M$ coincides with the subspace topology induced by the topology on $B$. Thus $f : E \to M_d(B)$ is continuous and hence induces a continuous group homomorphism $E^\times \to M_d(B)^\times = \mathrm{GL}_d(B)$. □

The representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ induces a map of $R^{\mathrm{ps}}$-algebras $E \to M_d(k)$ and thus a homomorphism of $R^{\mathrm{ps}}$-algebras $A^{\mathrm{gen}} \to k$. It follows from the universal property of $A^{\mathrm{gen}}$ that $R^{\square}_{\overline{\rho}}$ is isomorphic to the completion of $A^{\mathrm{gen}}$ with respect to the kernel of this map, see Proposition 3.33. Conversely, we have the following Lemma.

**Lemma 3.3.** *Let $x \in X^{\mathrm{gen}}$ be a closed point above the closed point of $X^{\mathrm{ps}}$ and let $\rho_x : G_F \to \mathrm{GL}_d(\kappa(x))$ be the representation obtained by composing*

$$G_F \to R^{\mathrm{ps}}[\![G_F]\!] \to E \xrightarrow{j} M_d(A^{\mathrm{gen}}) \to M_d(\kappa(x)).$$

*Then the pseudo-character associated to $\rho_x$ is equal to $\overline{D} \otimes_k \kappa(x)$. In particular, $\rho_x$ and $\overline{\rho} \otimes_k \kappa(x)$ have the same semi-simplification.*

*Proof.* Since $D^u \otimes_{R^{\mathrm{ps}}} k = \overline{D}$ the first part follows immediately from the definition of $A^{\mathrm{gen}}$. The second part follows from [17, Theorem 2.12]. Note that since we have assumed that all irreducible subquotients of $\overline{\rho}$ are absolutely irreducible, it is enough to prove that $\rho_x$ and $\overline{\rho}$ have the same semi-simplification after extending scalars to the algebraic closure of $k$. □

*Remark* 3.4. We note that one needs to impose the Cayley–Hamilton condition in the definition of $A^{\mathrm{gen}}$ for Lemma 3.3 to hold. For example, if $\overline{D} = \chi_1 + \chi_2$, where $\chi_1, \chi_2 : G_F \to k^\times$ are distinct characters, then $E \otimes_{R^{\mathrm{ps}}} k \cong k \times k$ by Equation (8) in the proof of [5, Lemma 1.4.3], let $\pi_1 : E \to k$ be the map obtained by projecting to the first component. Then the map $E \to M_2(k)$, $a \mapsto \mathrm{diag}(\pi_1(a), \pi_1(a))$ is a map of $R^{\mathrm{ps}}$-algebras, and hence induces a map of $R^{\mathrm{ps}}$-algebras $x : C \to k$, where $C$ is the algebra introduced in the proof of Lemma 3.1. The representation $\rho_x$ obtained by specializing $j : E \to M_2(C)$ at $x$ is isomorphic to $\chi_1 + \chi_1$; hence $\rho_x$ is not equal to $\chi_1 + \chi_2$.

3.1. **Bounding the dimension of the fibres.** Let $\mathfrak{p}$ be a prime ideal of $R^{\mathrm{ps}}$ such that $\dim R^{\mathrm{ps}}/\mathfrak{p} \leq 1$. Let $\kappa$ be an algebraic closure of the residue field of $\mathfrak{p}$ and let $y : R^{\mathrm{ps}} \to \kappa$ denote the corresponding homomorphism of $R^{\mathrm{ps}}$-algebras. The goal of this subsection (Proposition 3.14) is to bound the dimension of the fibre

$$X^{\mathrm{gen}}_y := X^{\mathrm{gen}} \times_{X^{\mathrm{ps}}, y} \operatorname{Spec} \kappa.$$

We let $E_y := E \otimes_{R^{\mathrm{ps}}, y} \kappa$ and let $D_y$ be the specialisation of the universal pseudo-character along $y : R^{\mathrm{ps}} \to \kappa$. Since $\kappa$ is algebraically closed we may write

$$D_y = \prod_{i=1}^{r} D_i,$$

where each $D_i$ is an irreducible pseudo-character of dimension $d_i$. (We follow the convention of [17], so that a pseudo-character of a direct sum of representations is a product of their pseudo-characters; the papers [8] and [47] refer to a direct sum instead.) We define an equivalence relation on the set $\{D_i : 1 \le i \le r\}$ by $D_i \sim D_j$ if $D_i = D_j(m)$ for some $m \in \mathbb{Z}$. Let $k$ be the number of the equivalence classes, $n_i$ the number of elements in the $i$-th equivalence class.

Moreover, for $1 \le i \le r$ we fix representations $\rho_i : G_F \to \mathrm{GL}_{d_i}(\kappa)$ such that $D_i$ is the pseudo-character associated to $\rho_i$. These representations are uniquely determined up to an isomorphism, but by $\rho_i$ we really mean a group homomorphism into $\mathrm{GL}_{d_i}(\kappa)$ and not the equivalence class.

**Lemma 3.5.** *If $i \neq j$ then*

$$\dim_\kappa \mathrm{Ext}^1_{E_y}(\rho_i, \rho_j) = \dim_\kappa \mathrm{Ext}^1_{G_F}(\rho_i, \rho_j).$$

*Proof.* Given an extension $0 \to \rho_j \to W \to \rho_i \to 0$ of $G_F$-representations, we let $V = W \oplus \bigoplus_{l \neq i,j} \rho_l$. Then the $G_F$-action on $V$ will factor through the action of $E_y$. Hence, $W$ is a representation of $E_y$, which implies that $\mathrm{Ext}^1_{E_y}(\rho_i, \rho_j) = \mathrm{Ext}^1_{G_F}(\rho_i, \rho_j)$. □

Since (4) is an adequate homeomorphism there is a unique point $y' \in X^{\mathrm{gen}} /\!\!/ G$ above $y$ and $X^{\mathrm{gen}}_{y'} \to X^{\mathrm{gen}}_y$ is a homeomorphism. The group $G$ acts on $X^{\mathrm{gen}}_y$. Moreover, $X^{\mathrm{gen}}_y$ is of finite type over $\kappa$ and $X^{\mathrm{gen}}_y(\kappa)$ is in bijection with the set of representations $\rho : G_F \to \mathrm{GL}_d(\kappa)$ such that the semi-simplification of $\rho$ is isomorphic to $\rho_1 \oplus \ldots \oplus \rho_r$.

**Lemma 3.6.** *The fibre $X^{\mathrm{gen}}_y$ is connected and the unique closed $G$-orbit in $X^{\mathrm{gen}}_y$ corresponds to semi-simple representations. If the $\rho_i$ are pairwise non-isomorphic, then its dimension is equal to $d^2 - r$.*

*Proof.* It follows from [44, Theorem 3] that $X^{\mathrm{gen}}_{y'}$ (and hence $X^{\mathrm{gen}}_y$, by the remark in the paragraph above) contains a unique closed $G$-orbit. Thus it is enough to show that the closure of every $G$-orbit will contain a semi-simple representation. If $x \in X^{\mathrm{gen}}_y(\kappa)$ then after conjugation we may assume that $x$ corresponds to a representation $\rho : G_F \to \mathrm{GL}_d(\kappa)$ such that the image of $\rho$ is block-upper-triangular, and the blocks on the diagonal are given by $\mathrm{diag}(\rho_{\sigma(1)}(g), \ldots, \rho_{\sigma(r)}(g))$ for some permutation $\sigma \in S_r$. By extending scalars to $\kappa[T]$, conjugating $\rho$ by $\mathrm{diag}(T^{r-1}\,\mathrm{id}_{d_{\sigma(1)}}, T^{r-2}\,\mathrm{id}_{d_{\sigma(2)}}, \ldots, \mathrm{id}_{d_{\sigma(r)}})$ and specializing at $T = 0$ we see that the closure of the $G$-orbit will contain a semi-simple representation. The action of $G$ on $X^{\mathrm{gen}}_y$ leaves the connected components invariant, see the proof of Lemma 2.1. Hence, every connected component of $X^{\mathrm{gen}}_y$ will contain the closed point corresponding to the representation $g \mapsto \mathrm{diag}(\rho_1(g), \ldots, \rho_r(g))$. Thus $X^{\mathrm{gen}}_y$ is connected.

The stabilizer of a semi-simple representation with distinct irreducible factors in $\mathrm{GL}_d$ is isomorphic to $\mathbb{G}_m^r$: a copy of $\mathbb{G}_m$ is embedded as scalar matrices inside of each block. Hence, the dimension of the closed $G$-orbit is given by $\dim \mathrm{GL}_d - \dim \mathbb{G}_m^r = d^2 - r$. □

We fix a permutation $\sigma \in S_r$ and let $P$ be the block-upper-triangular parabolic of $\mathrm{GL}_d$ with the $i$-th diagonal block of the size $d_{\sigma(i)} \times d_{\sigma(i)}$, let $N$ be its unipotent radical and $L$ be Levi subgroup consisting of block diagonal matrices and let $Z_L \cong \mathbb{G}_m^r$ denote the centre of $L$. We denote their Lie algebras by $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{l}$ and $\mathfrak{z}_L$, respectively. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GL}_d$. We have

$$\dim \mathfrak{g} = d^2, \quad \dim \mathfrak{l} = \sum_{i=1}^r d_i^2, \quad \dim \mathfrak{z}_L = r, \tag{5}$$

$$\dim \mathfrak{n} = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l}) = \sum_{1 \le i < j \le r} d_i d_j. \tag{6}$$

*Remark* 3.7. We note that although $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{l}$ and $\mathfrak{z}_L$ depend on $\sigma$, their dimensions do not.

Let $\rho_\sigma : G_F \to \mathrm{GL}_d(\kappa)$ be the representation $g \mapsto \mathrm{diag}(\rho_{\sigma(1)}(g), \dots, \rho_{\sigma(r)}(g))$.

**Lemma 3.8.** *There exists a closed subscheme $X^{\mathrm{gen}}_{y,\sigma} \subset X^{\mathrm{gen}}_y$ representing the functor sending a $\kappa$-algebra $B$ to the set of homomorphisms of Cayley–Hamilton $\kappa$-algebra $\varphi : E_y \to \mathfrak{p} \otimes_\kappa B$ such that projection onto the $i$th diagonal block is $\rho_{\sigma(i)}$ for $1 \le i \le r$.*

*Proof.* The universal map $j : E \to M_d(A^{\mathrm{gen}})$ induces a map

$$j_y : E_y \to M_d(A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa).$$

Let $I_{\rho,\sigma}$ be the ideal of $A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa$ generated by the matrix entries of $j_y(a)$ for all $a \in E_y$, which lie below the diagonal blocks of $P$, and by all the elements on the block diagonal of the matrices $(j_y(a) - \rho_\sigma(a))$ for all $a \in E_y$. Let

$$X^{\mathrm{gen}}_{y,\sigma} := \mathrm{Spec}((A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa)/I_{\rho,\sigma}).$$

Then $X^{\mathrm{gen}}_{y,\sigma}$ is a closed subscheme of $X^{\mathrm{gen}}_y$, and its defining ideal $I_{\rho,\sigma}$ was constructed precisely so that a $B$-point of $X^{\mathrm{gen}}_y$ factors through $X^{\mathrm{gen}}_{y,\sigma}$ if and only if it lands in $\mathfrak{p} \otimes_\kappa B$ and matches the $\rho_i$ on the diagonals for $1 \le i \le r$. □

The adjoint action (i.e. via conjugation) of $Z_L N$ on $\mathfrak{p}$ induces an action of $Z_L N$ on $X^{\mathrm{gen}}_{y,\sigma}$.

**Lemma 3.9.** *The unique closed $Z_L$-orbit in $X^{\mathrm{gen}}_{y,\sigma}$ is the singleton $\{\rho_\sigma\}$.*

*Proof.* This is the same proof as in Lemma 3.6, just use the same diagonal matrix trick to kill off the unipotent part. □

**Proposition 3.10.** *Let $x \in X^{\mathrm{gen}}_{y,\sigma}$ be the point corresponding to the representation $\rho_\sigma$. Then*

$$\begin{aligned} \dim T_x(X^{\mathrm{gen}}_{y,\sigma}) &= \dim \mathfrak{n} + (\dim \mathfrak{n})[F:\mathbb{Q}_p] + \sum_{1 \le i < j \le r} \dim \mathrm{Hom}_{G_F}(\rho_{\sigma(i)}, \rho_{\sigma(j)}(1)) \\ &\le \dim \mathfrak{n} + (\dim \mathfrak{n})[F:\mathbb{Q}_p] + \sum_{i=1}^{k} \binom{n_i}{2}. \end{aligned} \tag{7}$$

*Proof.* Using Lemma 3.8 and the decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ we may identify $T_x(X^{\mathrm{gen}}_{y,\sigma})$ with the space of $\kappa$-algebra homomorphisms $\varphi : E_y \to M_d(\kappa[\varepsilon])$, which can be written as $\varphi = \rho_\sigma + \varepsilon\beta$, where $\beta$ is a $\kappa$-linear map $\beta : E_y \to \mathfrak{n}$. If $\beta : E_y \to \mathfrak{n}$ is any $\kappa$-linear map then $\varphi := \rho_\sigma + \varepsilon\beta$ is a homomorphism of $\kappa$-algebras if and only if

$$\beta(aa') = \rho_\sigma(a)\beta(a') + \beta(a)\rho_\sigma(a'), \quad \forall a, a' \in E_y \tag{8}$$

For $1 \le i \le r$ we let $\mathbf{1}_i \in M_d(\kappa)$ be the block diagonal matrix with the identity matrix on the $i$-th block and zeros everywhere else. Since $\rho_\sigma(g)$ commutes with $\mathbf{1}_i$ for all $i$, we have an isomorphism

$$T_x(X^{\mathrm{gen}}_{y,\sigma}) \cong \bigoplus_{1 \le i < j \le r} V_{ij},$$

where $V_{ij}$ is the space of functions $\beta : E_y \to \mathbf{1}_i \mathfrak{n} \mathbf{1}_j$ satisfying (8). We may identify $\mathbf{1}_i \mathfrak{n} \mathbf{1}_j$ with $\mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})$. Then $V_{ij}$ is precisely the space of 1-cocycles for the Hochschild cohomology of $E_y$ with values in $\mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})$. Thus

$$\begin{aligned}\dim_\kappa V_{ij} &= \dim_\kappa HH^1(E_y, \mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})) + \dim_\kappa \mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)}) \\ &\quad - \dim_\kappa HH^0(E_y, \mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})) \\ &= \dim_\kappa \mathrm{Ext}^1_{E_y}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) + d_i d_j - \dim_\kappa \mathrm{Hom}_{E_y}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) \\ &= d_i d_j + [F : \mathbb{Q}_p] d_i d_j + \dim_\kappa \mathrm{Ext}^2_{G_F}(\rho_{\sigma(j)}, \rho_{\sigma(i)}),\end{aligned} \tag{9}$$

where the first equality follows from [14, Proposition IX.4.4.1], the second from [14, Corollary IX.4.4.4], the third from Lemma 3.5 together with the local Euler characteristic formula in this context, see [8, Theorem 3.4.1 (c)]. Thus

$$\dim_\kappa T_x(X^{\mathrm{gen}}_{y,\sigma}) = \dim \mathfrak{n} + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{1 \le i < j \le r} \dim_\kappa \mathrm{Ext}^2_{G_F}(\rho_{\sigma(j)}, \rho_{\sigma(i)}).$$

It follows from the local duality, see [8, Theorem 3.4.1 (b)], that

$$\dim_\kappa \mathrm{Ext}^2_{G_F}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) = \dim_\kappa \mathrm{Hom}_{G_F}(\rho_{\sigma(i)}, \rho_{\sigma(j)}(1)).$$

Thus if this term is non-zero then it is equal to 1 and $\rho_{\sigma(i)}$ and $\rho_{\sigma(j)}$ belong to the same equivalence class. □

*Remark* 3.11. If $\mathrm{char}(\kappa) = p$ and $\zeta_p \in F$ then $D_i \sim D_j$ if and only if $D_i = D_j$ and the bound is sharp in this case.

**Corollary 3.12.** $\dim X^{\mathrm{gen}}_{y,\sigma} \le \dim_\kappa T_x(X^{\mathrm{gen}}_{y,\sigma}) \le \dim \mathfrak{n} + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{i=1}^k \binom{n_i}{2}$.

*Proof.* This follows from Lemma 3.9 and Lemma 2.1 applied with $G = Z_L$ and $X = X^{\mathrm{gen}}_{y,\sigma}$, noting that $X^{\mathrm{gen}}_{y,\sigma} /\!\!/ Z_L$ is a singleton. □

**Lemma 3.13.** *If $f : X \to Y$ is a finite type and dominant morphism of Noetherian Jacobson universally catenary schemes, then* $\dim Y \le \dim X$.

*Proof.* Passing to reduced subschemes does not affect Krull dimension, so we may assume that $X$ and $Y$ are both reduced.

First assume $X$ and $Y$ are irreducible. Pick dense open affines $U \subset Y$, $V \subset X$ such that $f(V) \subset U$. It follows from [45, Tag 0CC1] that $A := \mathcal{O}_Y(U) \hookrightarrow B := \mathcal{O}_X(V)$ is injective. Since $A$ is an integral domain, Noether normalization [45, Tag 07NA] implies that the map factors as

$$A \hookrightarrow A[x_1, \dots, x_m] \hookrightarrow B' \hookrightarrow B,$$

with $B'$ finite over $A[x_1, \dots, x_m]$ and $B'_g \cong B_g$ for some non-zero $g \in A$. Then [45, Tag 0DRT] and [33, 13.C, Theorem 20] imply that

$$\dim X = \dim B = \dim B_g = \dim B'_g = \dim B' = \dim A + m = \dim Y + m$$

so $\dim Y \le \dim X$.

For the general case we argue as in the proof of [45, Tag 01RM]. Write $X = \bigcup_j Z_j$ as the union of its irreducible components. Because $f$ is dominant, we have $Y = \bigcup_j \overline{f(Z_j)}$. Clearly the $\overline{f(Z_j)}$ have to be irreducible, and so the irreducible components of $Y$ have to be among them. The $Z_j$ and $\overline{f(Z_j)}$ are again Noetherian, Jacobson and universally catenary, and hence by the case already treated we have

$$\dim Y = \max_j \dim \overline{f(Z_j)} \leq \max_j \dim Z_j = \dim X.$$

□

**Proposition 3.14.** $\dim X_y^{\mathrm{gen}} \leq \dim \mathfrak{g} - r + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{i=1}^k \binom{n_i}{2}$.

*Proof.* We want to apply Lemma 3.13 to

$$\coprod_{\sigma \in S_r} G \times^{Z_{L_\sigma} N_\sigma} X_{y,\sigma}^{\mathrm{gen}} \to X_y^{\mathrm{gen}}. \tag{10}$$

If $x \in X_y^{\mathrm{gen}}(\kappa)$ and $\varphi : E_y \to M_d(\kappa)$ is the corresponding $\kappa$-algebra homomorphism then there will exist $\sigma \in S_r$ such that $\kappa^d$ will admit a filtration by subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_r = V$, which is invariant under the action of $E_y$ via $\varphi$, satisfying $V_i/V_{i-1} \cong \rho_{\sigma(i)}$ for $1 \leq i \leq r$. Thus there is $g \in G(\kappa)$ such that $g\varphi g^{-1}$ will lie in $X_{y,\sigma}^{\mathrm{gen}}(\kappa)$, and hence (10) induces a surjection on $\kappa$-points. But (10) is also a map of finite type $\kappa$-schemes, and therefore is a dominant map of Noetherian Jacobson universally catenary schemes, so we can apply Lemma 3.13.

The fibre bundles $G \times^{Z_{L_\sigma} N_\sigma} X_{y,\sigma}^{\mathrm{gen}}$ have dimension equal to

$$\dim G + \dim X_{y,\sigma}^{\mathrm{gen}} - \dim(Z_{L_\sigma} N_\sigma) = \dim \mathfrak{g} - r + \dim X_{y,\sigma}^{\mathrm{gen}} - \dim \mathfrak{n}.$$

The bound in Corollary 3.12 gives the required assertion. □

**Corollary 3.15.** *If $r = 1$ then $X_y^{\mathrm{gen}}$ is smooth of dimension $\dim \mathfrak{g} - 1$.*

*Proof.* If $r = 1$ then $E_y \cong M_d(\kappa)$ and thus has a unique irreducible representation $\rho$ (up to isomorphism). Thus all the points in $X_y^{\mathrm{gen}}(\kappa)$ lie in the same $G$-orbit. Fix such a point $x$. Since the $G$-stabiliser of $x$ is equal to $Z_G$ we obtain $\dim X_y^{\mathrm{gen}} = \dim G - \dim Z_G = \dim \mathfrak{g} - 1$.

Since $E_y$ is semi-simple we have $\mathrm{Ext}^1_{E_y}(\rho, \rho) = 0$ and thus an argument as in the first paragraph of the proof of Proposition 3.10 gives us

$$\dim_\kappa T_x(X_y^{\mathrm{gen}}) = \dim_\kappa \mathrm{End}_\kappa(\rho) - \dim_\kappa \mathrm{End}_{E_y}(\rho) = \dim X_y^{\mathrm{gen}}.$$

Thus $x$ is a smooth point of $X_y^{\mathrm{gen}}$, and since $G$ acts transitively on $X_y^{\mathrm{gen}}(\kappa)$ all the points in $X_y^{\mathrm{gen}}(\kappa)$ are smooth. Since $X_y^{\mathrm{gen}}$ is of finite type over $\kappa$, we deduce that $X_y^{\mathrm{gen}}$ is smooth. □

3.2. **Bounding the dimension of the space.** In this subsection we will bound the dimension of $\overline{X}^{\mathrm{gen}}$, see Theorem 3.30. We will start with general commutative algebra lemmas. For a ring $R$ we set $P_1 R = \{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = 1\}$.

**Lemma 3.16.** *Let $(R, \mathfrak{m}_R)$ be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field $k'$. If $\mathfrak{p} \in P_1 R$ then $\kappa(\mathfrak{p})$ is either a finite extension of $L$ or a local field of characteristic $p$. Moreover, $R/\mathfrak{p}$ is contained in the ring of integers $\mathcal{O}_{\kappa(\mathfrak{p})}$ of $\kappa(\mathfrak{p})$ and the quotient topology on $R/\mathfrak{p}$ induced by the $\mathfrak{m}_R$-adic topology on $R$ coincides with the subspace topology induced by the topology on $\mathcal{O}_{\kappa(\mathfrak{p})}$.*

*Proof.* It follows from Cohen's structure theorem that if $\mathrm{char}(R/\mathfrak{p}) = 0$ then $\mathcal{O} \subset R/\mathfrak{p}$ and $R/\mathfrak{p}$ is a finitely generated $\mathcal{O}$-module. Thus $\kappa(\mathfrak{p})$ is a finite extension of $L$ and $R/\mathfrak{p}$ is contained in the integral closure of $\mathcal{O}$ in $\kappa(\mathfrak{p})$, which is equal to $\mathcal{O}_{\kappa(\mathfrak{p})}$. If $\mathrm{char}(R/\mathfrak{p}) = p$ then $R/\mathfrak{p}$ is finite over a subring isomorphic to $k'[\![t]\!]$ and the same argument carries over. Moreover, $\mathcal{O}_{\kappa(\mathfrak{p})}$ is a finitely generated $R/\mathfrak{p}$-module, and this implies that the topologies coincide. □

**Lemma 3.17.** *Let $(R, \mathfrak{m}_R)$ be a complete local Noetherian ring and $\varphi : R \to S$ a ring map of finite type. Let $U$ be a non-empty open subscheme of $U_{\max} := (\operatorname{Spec} R) \setminus \{\mathfrak{m}_R\}$, let $V$ (resp. $V_{\max}$) be the preimage of $U$ (resp. $U_{\max}$) in $\operatorname{Spec} S$, let $Z$ (resp. $Z_{\max}$) be the closure of $V$ (resp. $V_{\max}$) in $\operatorname{Spec} S$ and let $Y$ be the preimage of $\{\mathfrak{m}_R\}$ in $\operatorname{Spec} S$. Then*

(1) *$V$ is Jacobson;*
(2) *the set of closed points of $V$ is $V \cap \{$closed points of $V_{\max}\}$;*
(3) *if $x$ is a closed point of $V$ then its image $y$ in $\operatorname{Spec} R$ is a closed point of $U$ and the field extension $\kappa(x)/\kappa(y)$ is finite;*
(4) *the set of closed points of $U$ is $U \cap P_1 R$;*
(5) *if every irreducible component of $\operatorname{Spec} S$ meets $Y$ non-trivially then $\dim Z = \dim V + 1$;*
(6) *$\dim V \leq \dim U + \max_{y \in U \cap P_1 R} \dim \varphi^{-1}(\{y\})$.*

*Proof.* We will first prove parts (1), (2) and (3). If $R = S$ and if $U = U_{\max}$ then (1) follows from [45, Tag 02IM] and both (2) and (3) hold trivially. If $R = S$ and if $U$ is arbitrary then $U = V$ and (1), (2) follow from the previous case together with [45, Tag 005W] and (3) holds trivially. The case of general $\varphi$ now follows from [45, Tag 00GB] together with [45, Tag 01P4], because the map $V \to U$ induced from $\varphi$ is of finite type.

Part (4) follows from (2) applied with $S = R$, using that $\mathfrak{m}_R$ is the unique maximal ideal of $R$, so that the set of closed points of $U_{\max}$ is equal to $P_1 R$.

For (5) note first that since $V$ is open in $\operatorname{Spec} S$ the set of generic points of $V$ is a subset of the set of generic points of $\operatorname{Spec} S$. Thus $Z$ is union of irreducible components of $\operatorname{Spec} S$. Let $Z' = \operatorname{Spec} S'$ be an irreducible component of $Z$ with the induced reduced subscheme structure so that $S'$ is a domain, let $V' = Z' \cap V$, let $R'$ be the image of $R$ in $S'$. The rings $R'$ and $S'$ are excellent and hence universally catenary by [45, Tag 07QW]. If $\mathfrak{q} \in \operatorname{Spec} S'$ and $\mathfrak{p} = \mathfrak{q} \cap R'$ then

$$\begin{aligned} \dim S'_{\mathfrak{q}} &= \dim R'_{\mathfrak{p}} + \operatorname{trdeg}_{R'} S' - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \\ &= \dim R' + \operatorname{trdeg}_{R'} S' - \dim R'/\mathfrak{p} - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}), \end{aligned} \tag{11}$$

where trdeg stands for transcendence degree, the first equality is [45, Tag 02IJ], and the second is [34, Theorem 31.4]. It follows from (11) that

$$\dim S'_{\mathfrak{q}} \leq \dim R' + \operatorname{trdeg}_{R'} S' \tag{12}$$

and the equality in (12) holds if and only if $\mathfrak{q}$ maps to the maximal ideal of $R'$ and $\mathfrak{q}$ is a maximal ideal of $S'$. Since $Z' \cap Y$ is non-empty by assumption, such $\mathfrak{q}$ exists and so

$$\dim S' = \dim R' + \operatorname{trdeg}_{R'} S'.$$

Let $\mathfrak{q}$ be a closed point of $V'$ and let $\mathfrak{p} = \mathfrak{q} \cap R'$. Since $V'$ is open in $Z'$ we have $\mathcal{O}_{V',\mathfrak{q}} = S'_{\mathfrak{q}'}$. It follows from (3) that $\operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$ and $\dim R/\mathfrak{p} = 1$. Thus (11) gives us

$$\dim \mathcal{O}_{V',\mathfrak{q}} = \dim R' + \operatorname{trdeg}_{R'} S' - 1.$$

Since this holds for all closed points of $V'$ we deduce that

$$\dim V' = \dim R' + \operatorname{trdeg}_{R'} S' - 1.$$

This implies part (5).

Let $x$ be a closed point of $V$ and let $y$ be its image in $U$. Then $y$ is also a closed point of $U$. We have

$$\dim \mathcal{O}_{V,x} \leq \dim \mathcal{O}_{U,y} + \dim(\mathcal{O}_{V,x} \otimes_{\mathcal{O}_{U,y}} \kappa(y)) \leq \dim U + \dim \varphi^{-1}(\{y\}),$$

where the first inequality is given by [34, Theorem 15.1 (i)]. Since

$$\dim V = \max_x \dim \mathcal{O}_{V,x},$$

where the maximum is taken over all closed points $x$ of $V$ we get (6). □

*Remark* 3.18. We caution the reader that the equality $\dim Z = \dim V + 1$ might fail if one drops the assumption that $Y$ meets every irreducible component non-trivially. For example, if $R = \mathbb{Z}_p$ and $S = \mathbb{Z}_p[x]/(px-1) = \mathbb{Q}_p$ then $Y$ is empty and $V_{\max} = Z_{\max} = \operatorname{Spec} S$.

*Remark* 3.19. Here is another cautionary example. If $R$ and $S$ are as in Lemma 3.17, $\mathfrak{q}$ is a prime of $S$ and $S$ is a domain then it need not be true that $\dim S_{\mathfrak{q}} + \dim S/\mathfrak{q} = \dim S$. For example, if $R = \mathbb{Z}_p$, $S = \mathbb{Z}_p[x]$ and $\mathfrak{q} = (px-1)$ then $S/\mathfrak{q} = \mathbb{Q}_p$ and $S_{\mathfrak{q}}$ is a DVR, so that $\dim S_{\mathfrak{q}} + \dim S/\mathfrak{q} = 1$ and $\dim S = 2$. We also note that $\mathfrak{q}$ is a closed point of $\operatorname{Spec} S$ but it does not map to a closed point of $\operatorname{Spec} R$. Further, if $\mathfrak{q}' = (p, x)$ then $S/\mathfrak{q}' = \mathbb{F}_p$ and $p, x$ is a regular sequence of parameters in $S_{\mathfrak{q}'}$, and thus $\dim S_{\mathfrak{q}'} = 2$. Thus $\mathfrak{q}$ and $\mathfrak{q}'$ are closed points of an irreducible scheme, but their local rings have different dimensions.

**Lemma 3.20.** *Let $Y$ be the preimage of $\{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ in $X^{\mathrm{gen}}$, let $W$ be a closed non-empty $\mathrm{GL}_d$-invariant subscheme of $X^{\mathrm{gen}}$ and let $Z$ be an irreducible component of $W$. Then $Y \cap Z$ is non-empty. Moreover, if $x$ is a closed point of $Z$ then the following hold:*

1. *if $x \in Y$ then $\dim \mathcal{O}_{Z,x} = \dim Z$;*
2. *if $x \notin Y$ then $\dim \mathcal{O}_{Z,x} = \dim Z - 1$.*

*Proof.* Let $x$ be a closed point of $Z$ and let $y$ be its image in $\operatorname{Spec} R^{\mathrm{ps}}$. If $y$ is not the maximal ideal of $R^{\mathrm{ps}}$ then $x \in Z \setminus Y$.

If $Z \cap Y$ is empty then Lemma 3.17 (3) applied with $R = R^{\mathrm{ps}}$, $U = U_{\max}$ and $Z = \operatorname{Spec} S$ implies that $V_{\max} = Z$ and hence $Z$ is Jacobson. Let $W'$ be the union of irreducible components of $W$, different from $Z$. Then $Z \setminus W'$ is a non-empty open subscheme of $Z$. Let $x$ be a closed point of $Z \setminus W'$ then $x$ is also a closed point of $Z$ by [45, Tag 005W]. By construction $Z$ is the unique irreducible component of $W$ containing $x$.

It follows from Lemma 3.17 (4) that $y \in P_1 R^{\mathrm{ps}}$ and $\kappa(x)$ is a finite extension of $\kappa(y)$. Thus $\kappa(x)$ is either a finite extension of $L$ or a local field of characteristic $p$. Let $\rho_x : G_F \to \mathrm{GL}_d(\kappa(x))$ be the corresponding Galois representation. Since $G_F$ is compact, there is a matrix $M \in \mathrm{GL}_d(\kappa(x))$, such that $M\rho_x(g)M^{-1} \in \mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$ for all $g \in G_F$. Since conjugation does not change the characteristic polynomial there is an $R^{\mathrm{ps}}$-algebra homomorphism $x' : A^{\mathrm{gen}} \to \kappa(x)$, such that $\rho_{x'}(g) = M\rho_x(g)M^{-1}$ for all $g \in G_F$.

We claim that $x$ and $x'$ lie on the same irreducible component of $W_y := W \times_{R^{\mathrm{ps}}} \kappa(y)$. If $B$ is a $\kappa(x)$-algebra then the map $\mathrm{GL}_d(B) \to X^{\mathrm{gen}}(B)$, $N \mapsto [g \mapsto N\rho_x(g)N^{-1}]$ defines a map of schemes over $\operatorname{Spec} \kappa(x)$, $\mathrm{GL}_d \to W_y \times_{\kappa(y)} \kappa(x)$ and both $x$ and $x'$ are contained in its scheme theoretic image. Note that $W$ (and hence the fibre $W_y$) are $\mathrm{GL}_d$-invariant by assumption. Since $\mathrm{GL}_d$ over $\kappa(x)$ is irreducible we obtain the claim. We conclude that both $x$ and $x'$ lie on the same irreducible component of $W$, which is $Z$.

Let $\mathfrak{q}$ be the kernel of $x' : A^{\mathrm{gen}} \to \kappa(x)$. Since the image of $\rho_{x'}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$, $A^{\mathrm{gen}}/\mathfrak{q}$ is a subring of $\mathcal{O}_{\kappa(x)}$. Let $k'$ be the residue field of $\mathcal{O}_{\kappa(x)}$ and let $z \in X^{\mathrm{gen}}(k')$ be the composition $z : A^{\mathrm{gen}} \to A^{\mathrm{gen}}/\mathfrak{q} \to \mathcal{O}_{\kappa(x)} \to k'$. Then $z$ maps to the closed point in $\operatorname{Spec} R^{\mathrm{ps}}$ (i.e. $z \in Y$), and is contained in the closure of $x'$ in $X^{\mathrm{gen}}$. Since $Z$ is closed in $X^{\mathrm{gen}}$ and contains $x'$ we deduce that $z \in Z$, contradiction.

The claims about $\dim \mathcal{O}_{Z,x}$ follows from the proof of part (5) in Lemma 3.17. □

*Example* 3.21. Let us illustrate Lemma 3.20 with a concrete example. Let $\overline{D}$ be the pseudo-character of the 2-dimensional trivial representation of the group $\Gamma := \mathbb{Z}_p$. It follows from [17, Theorem 1.15] that $R^{\mathrm{ps}} \cong \mathcal{O}[\![t, d]\!]$ and

$$E \cong \frac{R^{\mathrm{ps}}[\![T]\!]}{((1+T)^2 - (2+t)(1+T) + 1 + d)},$$

where the map $\Gamma \to R^{\mathrm{ps}}[\![\Gamma]\!] \twoheadrightarrow E$ sends a fixed topological generator $\gamma$ of $\Gamma$ to $1+T$. Then $E$ is a free $R^{\mathrm{ps}}$-module with basis $1+T, 1$ and so

$$A^{\mathrm{gen}} = \frac{R^{\mathrm{ps}}[x_{11}, x_{12}, x_{21}, x_{22}]}{(x_{11} + x_{22} - (2+t), x_{11}x_{22} - x_{12}x_{21} - (1+d))},$$

and $j : E \to M_2(A^{\mathrm{gen}})$ sends $1+T$ to the matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$. Let $x : A^{\mathrm{gen}} \to L$ be the homomorphism corresponding to the representation $\rho : E \to M_2(L)$, such that $\rho(\gamma) = \left(\begin{smallmatrix} 1 & p^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. Then $x$ is a closed point of $X^{\mathrm{gen}}$ with residue field $L$, thus it does not map to the closed point in $X^{\mathrm{ps}}$. Indeed, $A^{\mathrm{gen}}/(x_{11} - 1, x_{21}, x_{22} - 1) \cong \mathcal{O}[x_{12}]$, so we are in the situation considered in Remark 3.19.

**Lemma 3.22.** *Let $W$ be a closed non-empty $\mathrm{GL}_d$-invariant subscheme of $X^{\mathrm{gen}}$ and write $W[1/p]$ and $\overline{W}$ for the generic and special fibre. Then $\dim W[1/p] \le \dim \overline{W}$. In particular, $\dim X^{\mathrm{gen}}[1/p] \le \dim \overline{X}^{\mathrm{gen}}$.*

*Proof.* We may assume that $W[1/p]$ is non-empty. Let $Z = \operatorname{Spec} A^{\mathrm{gen}}/\mathfrak{p}$ be an irreducible component of $W$ such that $Z[1/p]$ is non-empty and let $\overline{Z} = \operatorname{Spec} A^{\mathrm{gen}}/(\mathfrak{p}, \varpi)$. Lemma 3.20 implies that there is a closed point $x \in Z$, which maps to the closed point in $X^{\mathrm{ps}}$. Lemma 3.17 (5) implies that $\dim Z[1/p] = \dim Z - 1$.

Since $Z$ is irreducible and $Z[1/p] \neq \emptyset$ the local ring $\mathcal{O}_{Z,x}$ is a domain and multiplication by $\varpi$ is injective. Since $\mathrm{char}(\kappa(x)) = p$, $\varpi$ cannot be a unit in $\mathcal{O}_{Z,x}$. Thus $\dim \mathcal{O}_{\overline{Z},x} = \dim \mathcal{O}_{Z,x} - 1$. It follows from Lemma 3.20 that $\dim \overline{Z} = \dim Z - 1$. Since $\overline{Z}$ is a closed subset of $\overline{W}$ we have $\dim \overline{W} \ge \dim \overline{Z} = \dim Z - 1 = \dim Z[1/p]$, and hence $\dim \overline{W} \ge \dim W[1/p]$. □

Recall that $\overline{D} : G_F \to k$ is the specialization of the universal pseudo-character $D^u : G_F \to R^{\mathrm{ps}}$ at the maximal ideal of $R^{\mathrm{ps}}$. We may write $\overline{D} = \prod_{i=1}^m \overline{D}_i$, where $\overline{D}_i$ are absolutely irreducible pseudo-characters. Let $\mathcal{P}$ be an (unordered) partition of the set $\{1, \dots, m\}$ into $r$ disjoint subsets $\Sigma_j$, and let $\underline{\Sigma} = (\Sigma_1, \dots, \Sigma_r)$ be an ordering of the subsets in $\mathcal{P}$. For each $1 \le j \le r$ let $\overline{D}'_j = \prod_{i \in \Sigma_j} \overline{D}_i$, and let $d_j$ be the dimension of $\overline{D}'_j$. We define an equivalence relation on the set of pseudo-characters $\{\overline{D}'_j : 1 \le j \le r\}$ by $\overline{D}'_j \sim \overline{D}'_{j'}$ if $\overline{D}'_j = \overline{D}'_{j'}(t)$ for some $t \in \mathbb{Z}$. Let $k'$ be the number of the equivalence classes, $n'_i$ be the number of elements in the

$i$-th equivalence class, $c_i$ be the dimension of the pseudo-characters in the $i$-th equivalence class. We have

$$\sum_{i=1}^{k'} n'_i = r, \quad \sum_{i=1}^{k'} c_i n'_i = d.$$

We define

$$l_{\mathcal{P}} := \sum_{j=1}^{r} d_j^2 = \sum_{i=1}^{k'} n'_i c_i^2, \quad p_{\mathcal{P}} := l_{\mathcal{P}} + n_{\mathcal{P}} = \sum_{j=1}^{r} d_j^2 + \sum_{1\le j<j'\le r} d_j d_{j'}, \tag{13}$$

where

$$n_{\mathcal{P}} = \frac{1}{2}(d^2 - l_{\mathcal{P}}) = \sum_{1\le j<j'\le r} d_j d_{j'} = \sum_{1\le i<i'\le k'} c_i c_{i'} n'_i n'_{i'} + \sum_{i=1}^{k'} c_i^2 \binom{n'_i}{2}. \tag{14}$$

The notation is motivated by (5) and (6), see also Remark 3.7.

For each $1 \le j \le r$ let $R_j^{\mathrm{ps}}$ be the universal deformation ring of $\overline{D}'_j$ and let $X_j^{\mathrm{ps}} := R_j^{\mathrm{ps}}$. The functor $\mathcal{F}_{\underline{\Sigma}}$, which sends a local Artinian $\mathcal{O}$-algebra $(A, \mathfrak{m}_A)$ with residue field $k$ to the set of ordered $r$-tuples $(D_1, \dots, D_r)$ of pseudo-characters with each $D_i$ a deformation of $\overline{D}'_i$ to $A$ is represented by the completed tensor product

$$R_{\underline{\Sigma}}^{\mathrm{ps}} := R_1^{\mathrm{ps}} \widehat{\otimes}_{\mathcal{O}} \dots \widehat{\otimes}_{\mathcal{O}} R_r^{\mathrm{ps}}.$$

We let $X_{\underline{\Sigma}}^{\mathrm{ps}} := \operatorname{Spec} R_{\underline{\Sigma}}^{\mathrm{ps}}$ and denote by $\overline{X}_{\underline{\Sigma}}^{\mathrm{ps}} := \operatorname{Spec} R_{\underline{\Sigma}}^{\mathrm{ps}}/\varpi$ to its special fibre. By mapping an $r$-tuple of pseudo-characters to their product we obtain a map

$$\iota_{\underline{\Sigma}} : \overline{X}_{\underline{\Sigma}}^{\mathrm{ps}} \to \overline{X}^{\mathrm{ps}}.$$

**Lemma 3.23.** *The map $R^{\mathrm{ps}} \to R_{\underline{\Sigma}}^{\mathrm{ps}}$ is finite.*

*Proof.* By topological Nakayama's lemma it is enough to show that the fibre ring $C := k \otimes_{R^{\mathrm{ps}}} R_{\underline{\Sigma}}^{\mathrm{ps}}$ is a finite dimensional $k$-vector space. Let $\mathcal{F}$ be the closed subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ defined by $C$. If $(A, \mathfrak{m}_A)$ is a local Artinian $k$-algebra then $\mathcal{F}(A)$ is in bijection with the set of $r$-tuples $(D_1, \dots, D_r)$, each $D_i$ lifting $\overline{D}'_i$ to $A$ such that $\prod_{i=1}^r D_i = (\prod_{i=1}^r \overline{D}'_i) \otimes_k A$.

Since $C$ is a complete local Noetherian ring, it is enough to show that its Krull dimension is 0. If this is not the case then there is $\mathfrak{p} \in \operatorname{Spec} C$ such that $\dim C/\mathfrak{p} = 1$. Let $(D_{1,y}, \dots, D_{r,y})$ be the specialization of the universal object of $\mathcal{F}_{\underline{\Sigma}}$ along $y : R_{\underline{\Sigma}}^{\mathrm{ps}} \to \kappa(\mathfrak{p})$. It follows from [17, Corollary 1.14] that the coefficients of the polynomials $D_{i,y}(t-a)$, for all $a \in E$ and $1 \le i \le r$ will generate a dense subring of $R_{\underline{\Sigma}}^{\mathrm{ps}}/\mathfrak{p}$. Since $R_{\underline{\Sigma}}^{\mathrm{ps}}/\mathfrak{p}$ is a complete local $k$-algebra of dimension 1, there will exist $a \in E$ and index $i$ such that the coefficients of $D_{i,y}(t-a)$ will generate a transcendental extension of $k$ inside $\kappa(\mathfrak{p})$. Since $\mathfrak{p} \in \operatorname{Spec} C$ we have

$$\prod_{i=1}^{r} D_{i,y}(t-a) = \prod_{i=1}^{r} \overline{D}'_i(t-a).$$

Thus all the roots of $D_{i,y}(t-a)$ in the algebraic closure of $\kappa(\mathfrak{p})$ are algebraic over $k$. Since $D_{i,y}(t-a)$ is a monic polynomial, we conclude that all the coefficients are also algebraic over $k$, giving a contradiction. $\square$

Let $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ be the scheme theoretic image of $\iota_{\underline{\Sigma}}$. We note that $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ depends only on $\mathcal{P}$ and not on the chosen ordering $\underline{\Sigma}$. It follows from Lemma 3.23 that

$$\dim \overline{X}^{\mathrm{ps}}_{\mathcal{P}} = \dim \overline{X}^{\mathrm{ps}}_{\underline{\Sigma}} = \sum_{i=1}^{r} \dim \overline{X}^{\mathrm{ps}}_{i} = r + l_{\mathcal{P}}[F:\mathbb{Q}_p],$$

where the last equality is given by [8, Theorem 5.4.1(i)].

We define a partial order on the set of partitions of $\{1,\dots,m\}$ by $\mathcal{P} \le \mathcal{P}'$ if $\mathcal{P}'$ is a refinement of $\mathcal{P}$. The partition $\mathcal{P}_{\min}$ consisting of 1 part is the minimal element and the partition $\mathcal{P}_{\max}$ consisting of $m$ parts is the maximal element with respect to this partial ordering. If $\mathcal{P} \le \mathcal{P}'$ then $\overline{X}^{\mathrm{ps}}_{\mathcal{P}'}$ is a closed subscheme of $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ and $\overline{X}^{\mathrm{ps}}_{\mathcal{P}_{\min}} = \overline{X}^{\mathrm{ps}}$. Let

$$U_{\mathcal{P}} := \overline{X}^{\mathrm{ps}}_{\mathcal{P}} \setminus (\{\mathfrak{m}_{R^{\mathrm{ps}}}\} \cup \bigcup_{\mathcal{P}<\mathcal{P}'} \overline{X}^{\mathrm{ps}}_{\mathcal{P}'})$$

and let $V_{\mathcal{P}}$ be the preimage of $U_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z_{\mathcal{P}}$ be the closure of $V_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$. Let $\overline{X}^{\mathrm{gen}}_{\mathcal{P}}$ be the preimage of $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$. Then $\overline{X}^{\mathrm{gen}}_{\mathcal{P}}$ is closed in $\overline{X}^{\mathrm{gen}}$ and contains $V_{\mathcal{P}}$, hence we are in the situation of Lemma 3.17 with $\operatorname{Spec} R = \overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ and $\operatorname{Spec} S = Z_{\mathcal{P}}$. Note that Lemma 3.20 implies that every irreducible component of $\overline{X}^{\mathrm{gen}}_{\mathcal{P}}$ contains a closed point mapping to $\mathfrak{m}_{R^{\mathrm{ps}}}$. Thus the condition in part (5) of Lemma 3.17 is satisfied and hence $\dim Z_{\mathcal{P}} = \dim V_{\mathcal{P}} + 1$; the same conclusion applies to closures of various loci considered below. Moreover, we have

$$\overline{X}^{\mathrm{gen}}_{\mathcal{P}} = Y \cup \bigcup_{\mathcal{P}\le\mathcal{P}'} Z_{\mathcal{P}'}, \tag{15}$$

where $Y$ is the preimage of $\{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ in $\overline{X}^{\mathrm{gen}}$.

We will also need a variant of the situation above. Let us assume that $r > 1$ and let $i$ and $j$ be distinct indices with $1 \le i, j \le r$. Let $\mathcal{F}^{ij}_{\underline{\Sigma}}$ be a subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ parameterizing the deformations $(D_1,\dots,D_r)$ of the ordered $r$-tuple $(\overline{D}'_1,\dots,\overline{D}'_r)$ such that $D_i = D_j(1)$. Then $\mathcal{F}^{ij}_{\underline{\Sigma}}$ is a closed subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ and we let $R^{\mathrm{ps},ij}_{\underline{\Sigma}}$ be the quotient of $R^{\mathrm{ps}}_{\underline{\Sigma}}$ representing it. If $\overline{D}'_i \ne \overline{D}'_j(1)$ then $R^{\mathrm{ps},ij}_{\underline{\Sigma}}$ is the zero ring, otherwise

$$\dim R^{\mathrm{ps},ij}_{\underline{\Sigma}}/\varpi = \dim R^{\mathrm{ps}}_{\underline{\Sigma}}/\varpi - \dim R^{\mathrm{ps}}_{i}/\varpi \le r + l_{\mathcal{P}}[F:\mathbb{Q}_p] - (1 + [F:\mathbb{Q}_p]).$$

Let $\overline{X}^{\mathrm{ps},ij}_{\mathcal{P}}$ be the scheme theoretic image of $\operatorname{Spec} R^{\mathrm{ps},ij}_{\underline{\Sigma}}$ in $\overline{X}^{\mathrm{ps}}$ under $\iota_{\underline{\Sigma}}$. Then

$$\dim \overline{X}^{\mathrm{ps},ij}_{\mathcal{P}} \le r + l_{\mathcal{P}}[F:\mathbb{Q}_p] - (1 + [F:\mathbb{Q}_p]). \tag{16}$$

Let $U^{ij}_{\mathcal{P}} := U_{\mathcal{P}} \cap \overline{X}^{\mathrm{ps},ij}_{\mathcal{P}}$, let $V^{ij}_{\mathcal{P}}$ be the preimage of $U^{ij}_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z^{ij}_{\mathcal{P}}$ be the closure of $V^{ij}_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$.

**Lemma 3.24.** *If $y$ is a geometric point over a closed point in $U_{\mathcal{P}}$ then*

$$\dim X^{\mathrm{gen}}_{y} \le d^2 - r + n_{\mathcal{P}}[F:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}.$$

*If we additionally assume that $y \notin U^{ij}_{\mathcal{P}}$ for any $i \ne j$ then*

$$\dim X^{\mathrm{gen}}_{y} \le d^2 - r + n_{\mathcal{P}}[F:\mathbb{Q}_p].$$

*Proof.* We may write $D_y = D_1 + \ldots + D_r$ with $D_i$ lifting $\overline{D}'_i$. We note that all the $D_i$ are absolutely irreducible, since otherwise $y \in X^{\mathrm{ps}}_{\mathcal{P}'}$ for some $\mathcal{P}' > \mathcal{P}$. Let $k$ and $n_i$ be the numbers defined in Section 3.1. Proposition 3.14 implies that

$$\dim X^{\mathrm{gen}}_y \le d^2 - r + n_{\mathcal{P}}[F:\mathbb{Q}_p] + \sum_{i=1}^{k} \binom{n_i}{2}.$$

If $D_i = D_j(m)$ for some $m \in \mathbb{Z}$ then also $\overline{D}'_i = \overline{D}'_j(m)$. This implies that

$$\sum_{i=1}^{k} \binom{n_i}{2} \le \sum_{i=1}^{k'} \binom{n'_i}{2},$$

which implies the first assertion. We note that if $a_i, \ldots, a_s$ are positive integers then $\sum_{i=1}^s \binom{a_i}{2} \le \binom{\sum_{i=1}^s a_i}{2}$.

If $y \notin U^{ij}_{\mathcal{P}}$ for any $i \neq j$ then $D_i \neq D_j(1)$ for any $i \neq j$ and the Hom terms in (7) vanish. The assertion follows from Proposition 3.14 using this improved bound. □

**Proposition 3.25.** $\dim Z^{ij}_{\mathcal{P}} \le d^2 + p_{\mathcal{P}}[F:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2} - (1 + [F:\mathbb{Q}_p])$.

*Proof.* It follows from Lemma 3.17 (5) that the closure of $U^{ij}_{\mathcal{P}}$ has dimension $\dim U^{ij}_{\mathcal{P}} + 1$. Thus

$$\dim U^{ij}_{\mathcal{P}} + 1 \le \dim X^{\mathrm{ps},ij}_{\mathcal{P}} \le r + l_{\mathcal{P}}[F:\mathbb{Q}_p] - (1 + [F:\mathbb{Q}_p]),$$

where the last inequality is (16). Parts (5) and (6) of Lemma 3.17 together with Lemma 3.24 imply that

$$\dim Z^{ij}_{\mathcal{P}} \le (r + l_{\mathcal{P}}[F:\mathbb{Q}_p] - (1 + [F:\mathbb{Q}_p])) + (d^2 - r + n_{\mathcal{P}}[F:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}),$$

which imply the assertion. □

**Proposition 3.26.** *Let* $\delta_{\mathcal{P}} = \max\{0, \sum_{i=1}^{k'} \binom{n'_i}{2} - (1 + [F:\mathbb{Q}_p])\}$. *Then*

$$\dim Z_{\mathcal{P}} \le d^2 + p_{\mathcal{P}}[F:\mathbb{Q}_p] + \delta_{\mathcal{P}}.$$

*Proof.* Let $U'_{\mathcal{P}} := U_{\mathcal{P}} \setminus \bigcup_{i\neq j} U^{ij}_{\mathcal{P}}$, let $V'_{\mathcal{P}}$ be the preimage of $U'_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z'_P$ denote the closure of $V'_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$. If $y$ is a closed point of $U'_{\mathcal{P}}$ then $\dim \overline{X}^{\mathrm{gen}}_y \le d^2 - r + n_{\mathcal{P}}[F:\mathbb{Q}_p]$ by Lemma 3.24. Thus Lemma 3.17 implies that

$$\dim Z'_{\mathcal{P}} \le \dim \overline{X}^{\mathrm{ps}}_{\mathcal{P}} + (d^2 - r + n_{\mathcal{P}}[F:\mathbb{Q}_p]) = d^2 + p_{\mathcal{P}}[F:\mathbb{Q}_p]. \tag{17}$$

Since $Z_{\mathcal{P}} = Z'_{\mathcal{P}} \cup \bigcup_{i\neq j} Z^{ij}_{\mathcal{P}}$ we have $\dim Z_{\mathcal{P}} = \max_{i\neq j}\{\dim Z'_{\mathcal{P}}, \dim Z^{ij}_{\mathcal{P}}\}$ and the assertion follows from Proposition 3.25. □

**Proposition 3.27.** $\dim Z_{\mathcal{P}_{\min}} \le d^2 + d^2[F:\mathbb{Q}_p]$.

*Proof.* In this case $r = 1$ so $Z_{\mathcal{P}} = Z'_{\mathcal{P}}$ and the assertion follows from (17). □

**Lemma 3.28.** *Assume that $\mathcal{P} \neq \mathcal{P}_{\min}$. If $d = 2$ then*
$$d^2 + d^2[F : \mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq [F : \mathbb{Q}_p],$$
*and*
$$d^2 + d^2[F : \mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq 1 + [F : \mathbb{Q}_p],$$
*otherwise.*

*Proof.* Proposition 3.26 implies that
$$d^2 + d^2[F : \mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq n_{\mathcal{P}}[F : \mathbb{Q}_p] - \delta_{\mathcal{P}}.$$
If $d > 2$ then $n_{\mathcal{P}} \geq 2$ and if $d = 2$ then $n_{\mathcal{P}} = 1$, which implies the assertion if $\delta_{\mathcal{P}} = 0$. Let us assume that $\delta_{\mathcal{P}} \neq 0$. Then using (14) we may write
$$n_{\mathcal{P}}[F : \mathbb{Q}_p] - \delta_{\mathcal{P}} = \sum_{1 \leq i < j \leq k'} c_i c_j n'_i n'_j [F : \mathbb{Q}_p] + \sum_{i=1}^{k'} (c_i^2 [F : \mathbb{Q}_p] - 1) \binom{n'_i}{2} + 1 + [F : \mathbb{Q}_p],$$
which implies the assertion. □

**Lemma 3.29.** *Let $Y$ be the preimage of $\{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ in $\overline{X}^{\mathrm{gen}}$. Then*
$$\dim Y \leq d^2 + n_{\mathcal{P}_{\max}}[F : \mathbb{Q}_p] + n_{\mathcal{P}_{\max}} - 1.$$
*In particular, $d^2 + d^2[F : \mathbb{Q}_p] - \dim Y \geq 1 + l_{\mathcal{P}_{\max}}[F : \mathbb{Q}_p] \geq 1 + 2[F : \mathbb{Q}_p]$.*

*Proof.* Proposition 3.14 implies that
$$\dim Y \leq d^2 - m + n_{\mathcal{P}_{\max}}[F : \mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}.$$
As already explained in the proof of Lemma 3.28 we have $\sum_{i=1}^{k'} \binom{n'_i}{2} \leq n_{\mathcal{P}_{\max}}$. This implies the assertion. □

**Theorem 3.30.** $\dim \overline{X}^{\mathrm{gen}} \leq d^2 + d^2[F : \mathbb{Q}_p]$.

*Proof.* Since $\overline{X}^{\mathrm{ps}} = \{\mathfrak{m}_{R^{\mathrm{ps}}}\} \cup \bigcup_{\mathcal{P}} U_{\mathcal{P}}$ we have $\overline{X}^{\mathrm{gen}} = Y \cup \bigcup_{\mathcal{P}} Z_{\mathcal{P}}$. Since these are closed in $\overline{X}^{\mathrm{gen}}$ we have
$$\dim \overline{X}^{\mathrm{gen}} = \max_{\mathcal{P}} \{\dim Y, \dim Z_{\mathcal{P}}\} \leq d^2 + d^2[F : \mathbb{Q}_p],$$
by Proposition 3.27 and Lemmas 3.28 and 3.29. □

3.3. **Completions at maximal ideals and deformation problems.** Let $x$ be a closed point of $X^{\mathrm{gen}}$ and let $y$ be its image in $\operatorname{Spec} R^{\mathrm{ps}}$. It follows from Lemmas 3.16 and 3.17 that $\kappa(x)$ is a finite extension of $\kappa(y)$ and there are the following possibilities:

(1) $\kappa(x)$ is a finite extension of $k$;
(2) $\kappa(x)$ is a finite extension of $L$;
(3) $\kappa(x)$ is a local field of characteristic $p$.

The universal property of $A^{\mathrm{gen}}$ gives us a continuous Galois representation
$$\rho_x : G_F \to \mathrm{GL}_d(\kappa(x)).$$
In this section we want to relate the completion of the local ring $\mathcal{O}_{X^{\mathrm{gen}},x}$ to a deformation problem for $\rho_x$.

We will introduce some notation to formulate the deformation problem for $\rho_x$. More generally, let $\rho : G_F \to \mathrm{GL}_d(\kappa)$ be a continuous representation, where $\kappa$ is as above. In case (1) let $L'$ be an unramified extension of $L$ with residue field $\kappa$, let $\Lambda := \mathcal{O}_{L'}$ be the ring of integers in $L'$. In case (2) let $\Lambda = \kappa$, let $\Lambda^0$ be the ring of integers in $\Lambda$ and let $t = \varpi$. In case (3), let $\mathcal{O}_\kappa$ be the ring of integers in $\kappa$ and let $k'$ be its residue field. Since $\mathrm{char}(\kappa) = p$ by choosing a uniformizer we obtain an isomorphism $\mathcal{O}_\kappa \cong k'[\![t]\!]$. Let $L'$ be an unramified extension of $L$ with residue field $k'$, let $\Lambda^0 := \mathcal{O}_{L'}[\![t]\!]$ and let $\Lambda$ be the $p$-adic completion of $\Lambda^0[1/t]$. Then $\Lambda$ is a complete DVR with uniformiser $\varpi$ and residue field $\kappa$. We equip $\Lambda^0$ with its $(\varpi, t)$-adic topology, this induces a topology on $\Lambda^0[1/t]$ and $\Lambda^0[1/t]/p^n\Lambda^0[1/t]$ for all $n \geq 1$. We equip $\Lambda = \varprojlim_n \Lambda^0[1/t]/p^n\Lambda^0[1/t]$ with the projective limit topology.

*Remark* 3.31. In case (3), if $\Lambda'$ is an $\mathcal{O}$-algebra, which is a complete DVR with uniformiser $\varpi$ and residue field $\kappa$ then it follows from [9, Ch. IX, §2.3, Prop. 4] that $\Lambda'$ is non-canonically isomorphic to $\Lambda$. We will refer to $\Lambda'$ (and $\Lambda$) as an $\mathcal{O}$-Cohen ring of $\kappa$.

Let $\mathfrak{A}_\Lambda$ be the category of local Artinian $\Lambda$-algebras with residue field $\kappa$. Let $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$. In case (1) $A$ is a finite $\mathcal{O}/\varpi^n$-module for some $n \gg 0$, and we just put discrete topology on $A$, in case (2) $A$ is a finite dimensional $L$-vector space and we put the $p$-adic topology on $A$, in case (3) $A$ is a $\Lambda^0[1/t]/\varpi^n\Lambda^0[1/t]$ -module of finite length, for some $n \gg 0$ and we put the induced topology on $A$.

Let $D^\square_\rho(A)$ be the set of continuous group homomorphisms $\rho_A : G_F \to \mathrm{GL}_d(A)$, such that $\rho_A \pmod{\mathfrak{m}_A} = \rho$.

**Proposition 3.32.** *The functor $D^\square_\rho : \mathfrak{A}_\Lambda \to \mathrm{Sets}$ is represented by a complete local Noetherian $\Lambda$-algebra $R^\square_\rho$. Moreover, there is a presentation*

$$R^\square_\rho \cong \Lambda[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s) \tag{18}$$

*with $r = \dim_\kappa Z^1(G_F, \mathrm{ad}\,\rho)$ and $s = \dim_\kappa H^2(G_F, \mathrm{ad}\,\rho)$.*

*Proof.* Lecture 6 in [20] contains a very nice exposition of the result if $\kappa$ is a finite extension of either $k$ or $L$. The same argument works if $\kappa$ is a local field of characteristic $p$. □

If we let $h^i := \dim_\kappa H^i(G_F, \mathrm{ad}\,\rho)$ then

$$r - s = \dim_\kappa(\mathrm{ad}\,\rho) - h^0 + h^1 - h^2 = d^2 + d^2[F : \mathbb{Q}_p], \tag{19}$$

where the last equality follows from Euler characteristic formula in this setting, see [8, Theorem 3.4.1].

**Proposition 3.33.** *Let $\mathfrak{q}$ be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}} \to \kappa(x), \quad \lambda \otimes a \mapsto \bar{\lambda}\bar{a},$$

*where $\bar{\lambda}$ and $\bar{a}$ denote the images of $\lambda$ and $a$ in $\kappa(x)$. Then the completion of $(\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}})_{\mathfrak{q}}$ with respect to the maximal ideal is naturally isomorphic to $R^\square_{\rho_x}$.*

*Proof.* We will prove the proposition, when $\kappa(x)$ is a local field of characteristic $p$. The other cases are similar and are left to the reader.

Let $\widehat{B}$ be the completion of $(\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}})_{\mathfrak{q}}$. It follows from Lemma 3.35 below that $\widehat{B}/\varpi\widehat{B}$ (and hence $\widehat{B}$) is Noetherian. Thus $\widehat{B}/\mathfrak{q}^n\widehat{B} \in \mathfrak{A}_\Lambda$ for all $n \geq 1$. The composition

$$\Lambda \otimes_{\mathcal{O}} E \xrightarrow{\mathrm{id} \otimes j} \Lambda \otimes_{\mathcal{O}} M_d(A^{\mathrm{gen}}) \to M_d(\widehat{B}/\mathfrak{q}^n\widehat{B})$$

induces a continuous representation $G_F \to \mathrm{GL}_d(\widehat{B}/\mathfrak{q}^n\widehat{B})$ by Lemma 3.2, which is a deformation of $\rho_x$ to $\widehat{B}/\mathfrak{q}^n\widehat{B}$, and hence a map of local $\Lambda$-algebras $R^{\square}_{\rho_x} \to \widehat{B}/\mathfrak{q}^n\widehat{B}$. By passing to the projective limit over $n$ we obtain a continuous representation $\hat{\rho} : G_F \to \mathrm{GL}_d(\widehat{B})$ and a map of local $\Lambda$-algebras $R^{\square}_{\rho_x} \to \widehat{B}$.

Let $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$ and let $\rho : G_F \to \mathrm{GL}_d(A)$ be a continuous representation such that $\rho \pmod{\mathfrak{m}_A} = \rho_x$. We claim that there is a unique homomorphism of local $\Lambda$-algebras $\varphi : \widehat{B} \to A$, such that $\rho$ is equal to the composition $\mathrm{GL}_d(\varphi) \circ \hat{\rho}$. The claim implies that the map $R^{\square}_{\rho_x} \to \widehat{B}$ constructed above is an isomorphism.

The proof of the claim is based on [30, Proposition 9.5]. Following its proof, we may construct an ascending chain of local open $\Lambda^0$-subalgebras $A^0_n$ of $A$ for $n \geq 1$, such that for all $n$ the following hold: $A^0_n[1/t] = A$, the image of $A^0_n$ under the projection $b : A \to \kappa(x)$ is equal to $\mathcal{O}_{\kappa(x)}$ and $\bigcup_{n\geq 1} A^0_n = b^{-1}(\mathcal{O}_{\kappa(x)})$. Let $M \in \mathrm{GL}_d(\kappa(x))$ be a matrix such that the image of $G_F$ under $M\rho_x M^{-1}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$. Let $x' \in X^{\mathrm{gen}}$ correspond to the representation $M\rho_x M^{-1}$. Then $\kappa(x') = \kappa(x)$ and the image of $x' : A^{\mathrm{gen}} \to \kappa(x)$ is contained in $\mathcal{O}_{\kappa(x)}$. Let $z \in X^{\mathrm{gen}}$ be the composition $z : A^{\mathrm{gen}} \xrightarrow{x'} \mathcal{O}_{\kappa(x)} \to k'$, where $k'$ is the residue field of $\mathcal{O}_{\kappa(x)}$, let $\widetilde{M} \in \mathrm{GL}_d(A)$ be a matrix lifting $M$ and let $\rho' := \widetilde{M}\rho\widetilde{M}^{-1}$. Since $G_F$ is compact $\rho'(G_F)$ will be contained in some $\mathrm{GL}_d(A^0_n)$ for $n \gg 0$. We may consider $\rho' : G_F \to \mathrm{GL}_d(A^0_n)$ as a deformation of $\rho_z$ to $A^0_n$. Since the pseudo-character of $\rho_z$ is equal to $\overline{D}\otimes_k k'$ by Lemma 3.3, the pseudo-character of $\rho' : G_F \to \mathrm{GL}_d(A^0_n)$ is a deformation of $\overline{D} \otimes_k k'$ to $A^0_n$ and hence induces a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}} \to A^0_n$. Thus $\rho'$ factors through the map $\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[\![G_F]\!] \to M_d(A^0_n)$, which will factor through the Cayley–Hamilton quotient $(\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[\![G_F]\!])/\mathrm{CH}(\Lambda^0 \otimes_{\mathcal{O}} D^u) \to M_d(A^0_n)$. It follows from [17, Section 1.22] or [46, Lemma 1.1.8.6] that

$$\Lambda^0 \otimes_{\mathcal{O}} E \cong (\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[\![G_F]\!])/\mathrm{CH}(\Lambda^0 \otimes_{\mathcal{O}} D^u).$$

After inverting $t$ and conjugating by $\widetilde{M}^{-1}$ we obtain a map of $\Lambda^0[1/t]$-algebras $\Lambda^0[1/t] \otimes_{\mathcal{O}} E \to M_d(A)$, such that if we compose this map with the map induced by $G_F \to R^{\mathrm{ps}}[\![G_F]\!] \to E$ then we get back $\rho$. Since $A$ is an Artinian $\Lambda$-algebra, $\varpi^n\Lambda^0[1/t]$ will be mapped to zero for $n \gg 0$, and thus the map extends to a map of $\Lambda$-algebras $\alpha : \Lambda \otimes_{\mathcal{O}} E \to M_d(A)$. The universal property of $j : E \to M_d(A^{\mathrm{gen}})$ implies that there is a unique map of $\Lambda$-algebras $\varphi : \widehat{B} \to A$, such that $M_d(\varphi) \circ (\mathrm{id} \otimes j) = \alpha$.

It remains to show the uniqueness of the map $\varphi$, which is equivalent to showing that there is at most one map of $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}}$-algebras $\alpha : \Lambda \otimes_{\mathcal{O}} E \to M_d(A)$ such that the composition with $G_F \to \Lambda \otimes_{\mathcal{O}} E$ gives $\rho$. It follows from the Cayley–Hamilton theorem in $M_d(A)$ and [17, Corollary 1.14], that the map $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}} \to \Lambda \otimes_{\mathcal{O}} E \xrightarrow{\alpha} A$ is uniquely determined by $\rho$. Thus $\alpha$ is uniquely determined on the image of $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}}[G_F]$ in $\Lambda \otimes_{\mathcal{O}} E$. The map $R^{\mathrm{ps}}[G_F] \to E$ is surjective, since the image is dense and closed as $E$ is a finitely generated $R^{\mathrm{ps}}$-module, hence $\alpha$ is uniquely determined by $\rho$. □

The following Lemma is a mild generalization of [8, Lemma 3.3.3].

**Lemma 3.34.** *Let $R$ be a complete local Noetherian $k$-algebra with residue field $k$, let $A$ be a finitely generated $R$-algebra, let $\mathfrak{p} \in \operatorname{Spec} A$ such that its image in $\operatorname{Spec} R$ lies in $P_1 R$, and let $\mathfrak{q}$ be the kernel of the map*

$$B := \kappa(\mathfrak{p}) \otimes_k A \to \kappa(\mathfrak{p}), \quad x \otimes a \mapsto x(a + \mathfrak{p}).$$

*Then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}[\![T]\!]$. In particular, $A_{\mathfrak{p}}$ is regular (resp. complete intersection) if and only if $\hat{B}_{\mathfrak{q}}$ is.*

*Proof.* Let $\mathfrak{p}'$ be the image of $\mathfrak{p}$ in $\operatorname{Spec} R$. Since by assumption $\mathfrak{p}' \in P_1 R$, the residue field $\kappa(\mathfrak{p}')$ a local field of characteristic $p$. Since $A$ is finitely generated over $R$, $\kappa(\mathfrak{p})$ is a finite extension of $\kappa(\mathfrak{p}')$ and thus is also a local field of characteristic $p$. The proof of [8, Lemma 3.3.3] goes through verbatim by replacing $R$ with $A$ everywhere. $\square$

**Lemma 3.35.** *Let $R$ be a complete local Noetherian $\mathcal{O}$-algebra with residue field $k$, let $A$ be a finitely generated $R$-algebra, let $\mathfrak{p} \in \operatorname{Spec} A$ such that $\kappa(\mathfrak{p})$ is a local field of characteristic $p$, and let $\mathfrak{q}$ be the kernel of the map*

$$B := \Lambda \otimes_{\mathcal{O}} A \to \kappa(\mathfrak{p}), \quad \lambda \otimes a \mapsto \bar{\lambda}(a + \mathfrak{p}),$$

*If $\hat{B}_{\mathfrak{q}}$ or $A_{\mathfrak{p}}$ is $\varpi$-torsion free then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}[\![T]\!]$. In particular, $A_{\mathfrak{p}}$ is regular (resp. complete intersection) if and only if $\hat{B}_{\mathfrak{q}}$ is.*

*Proof.* It follows from Lemma 3.34 that the map $A \to B$, $a \mapsto 1 \otimes a$ induces a map of local rings $\hat{A}_{\mathfrak{p}} \to \hat{B}_{\mathfrak{q}}$, such that $\hat{B}_{\mathfrak{q}}/\varpi \cong (\hat{A}_{\mathfrak{p}}/\varpi)[\![T]\!]$. By choosing $b \in \hat{B}_{\mathfrak{q}}$, which maps to $T$ under this isomorphism, we obtain a map $\hat{A}_{\mathfrak{p}}[\![T]\!] \to \hat{B}_{\mathfrak{q}}$, which induces an isomorphism modulo $\varpi$. Nakayama's lemma implies that the map is surjective. If $\hat{B}_{\mathfrak{q}}$ is $\varpi$-torsion free then another application of Nakayama's lemma shows that the kernel is zero. If $A_{\mathfrak{p}}$ is $\mathcal{O}$-torsion free then $(\Lambda \otimes_{\mathcal{O}} A)_{\mathfrak{q}} = (\Lambda \otimes_{\mathcal{O}} A_{\mathfrak{p}})_{\mathfrak{q}}$ is also $\mathcal{O}$-torsion free, and the same applies to the completion $\hat{B}_{\mathfrak{q}}$. $\square$

**Lemma 3.36.** *Let $R$ be a complete local Noetherian $\mathcal{O}$-algebra with residue field $k$, let $A$ be a finitely generated $R$-algebra, let $\mathfrak{p} \in \operatorname{Spec} A$ such that $\kappa(\mathfrak{p})$ is either a finite extension of $L$, in which case we let $\Lambda = \kappa(\mathfrak{p})$, or a finite extension of $k$, in which case we let $\Lambda$ be the ring of integers in the finite unramified extension of $L$ with residue field $\kappa(\mathfrak{p})$. Let $\mathfrak{q}$ be the kernel of the map*

$$B := \Lambda \otimes_{\mathcal{O}} A \to \kappa(\mathfrak{p}), \quad \lambda \otimes a \mapsto \bar{\lambda}(a + \mathfrak{p}).$$

*Then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}$.*

*Proof.* The completion of $\Lambda \otimes_{\mathcal{O}} \Lambda$ with respect to the kernel of $\Lambda \otimes_{\mathcal{O}} \Lambda \to \Lambda$, $x \otimes y \mapsto xy$ is just $\Lambda$ (and that is why we don't get an extra variable $T$ like in Lemma 3.34, see [8, Lemma 3.3.4].) The rest of the proof is the same as the proof of Lemma 3.34. $\square$

**Corollary 3.37.** *Let $x$ be a closed point of $X^{\mathrm{gen}}$. Then the following hold:*

(1) *$R^{\square}_{\rho_x}$ is a flat $\Lambda$-algebra of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is complete intersection;*
(2) *if $\operatorname{char}(\kappa(x)) = p$ then $R^{\square}_{\rho_x}/\varpi$ is complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p]$.*

*Proof.* Let us assume that $\kappa(x)$ is a finite extension of $k$. It follows from Lemma 3.36 that $R^{\square}_{\rho_x}/\varpi \cong \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x}$, the completion of the local ring of $\overline{X}^{\mathrm{gen}}$ at $x$ with respect to the maximal ideal. We have $\dim \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x} = \dim \mathcal{O}_{\overline{X}^{\mathrm{gen}},x} \leq \dim \overline{X}^{\mathrm{gen}}$, and thus

$$\dim R^{\square}_{\rho_x}/\varpi \leq \dim \overline{X}^{\mathrm{gen}} \leq d^2 + d^2[F : \mathbb{Q}_p] = r - s,$$

where the last equality is (19). It follows from (18) that $\dim R^{\square}_{\rho_x}/\varpi \geq r-s$ and $\dim R^{\square}_{\rho_x} \geq 1+r-s$. Thus the lower bounds of the dimensions are equalities, and $\varpi, f_1, \ldots, f_s$ are a part of system of parameters in $\Lambda[\![x_1,\ldots,x_r]\!]$. Thus they form a regular sequence in $\Lambda[\![x_1,\ldots,x_r]\!]$ and so $R^{\square}_{\rho_x}$ and $R^{\square}_{\rho_x}/\varpi$ are complete intersections of the claimed dimensions. Moreover, since $\Lambda$ is a DVR with uniformiser $\varpi$, flatness is equivalent to $\varpi$-torsion equal to zero, and hence $R^{\square}_{\rho_x}$ if flat over $\Lambda$.

Let us assume that $\kappa(x)$ is a local field of characteristic $p$. Lemma 3.34 implies that $R^{\square}_{\rho_x}/\varpi \cong \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x}[\![T]\!]$ and Lemma 3.20 implies that $\dim \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x} \leq \dim \overline{X}^{\mathrm{gen}} - 1$. Thus $\dim R^{\square}_{\rho_x} \leq \dim \overline{X}^{\mathrm{gen}}$ and the same argument as above goes through.

If $\kappa(x)$ is a finite extension of $L$ then Lemma 3.36 implies that $R^{\square}_{\rho_x} \cong \widehat{\mathcal{O}}_{X^{\mathrm{gen}},x} = \widehat{\mathcal{O}}_{X^{\mathrm{gen}}[1/p],x}$. Corollary 3.22 implies that $\dim R^{\square}_{\rho_x} \leq \dim X^{\mathrm{gen}}[1/p] \leq \dim \overline{X}^{\mathrm{gen}}$. Then the same argument goes through. □

**Corollary 3.38.** *Let $x$ be a closed point in $X^{\mathrm{gen}}$ and let $\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}$ be the completion with respect to the maximal ideal of the local ring at $x$. If $\kappa(x)$ is a finite extension of $k$ or $L$ then $\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x} \cong R^{\square}_{\rho_x}$. If $\kappa(x)$ is a local field of characteristic $p$ then $R^{\square}_{\rho_x} \cong \widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}[\![T]\!]$.*

*Proof.* If $\kappa(x)$ is a finite extension of $k$ or $L$ then the assertion follows from Proposition 3.33 and Lemma 3.36. If $\kappa(x)$ is a local field of characteristic $p$ then $R^{\square}_{\rho_x}$ is $\mathcal{O}$-torsion free by Corollary 3.37, and the assertion follows from Proposition 3.33 and Lemma 3.35. □

**Corollary 3.39.** *The following hold:*

(1) *$A^{\mathrm{gen}}$ is $\mathcal{O}$-torsion free, equi-dimensional of dimension $1+d^2+d^2[F:\mathbb{Q}_p]$ and is locally complete intersection;*
(2) *$A^{\mathrm{gen}}/\varpi$ is equi-dimensional of dimension $d^2+d^2[F:\mathbb{Q}_p]$ and is locally complete intersection.*

*Proof.* Let us prove (1) as the proof of (2) is identical. Corollary 3.38 together with Corollary 3.37 implies that the local rings at closed points of $X^{\mathrm{gen}}$ are $\mathcal{O}$-torsion free and complete intersection. This implies that $A^{\mathrm{gen}}$ is $\mathcal{O}$-torsion free and $A^{\mathrm{gen}}$ is locally complete intersection by [45, Tag 09Q5].

Let $Z$ be an irreducible component of $X^{\mathrm{gen}}$. Lemma 3.20 implies that there is a closed point $x \in Z$ such that $x$ maps to the closed point of $X^{\mathrm{ps}}$. Moreover, $\dim Z = \dim \mathcal{O}_{Z,x}$. Since $\mathcal{O}_{X^{\mathrm{gen}},x}$ is complete intersection, it is equi-dimensional and thus $\dim \mathcal{O}_{Z,x} = \dim \mathcal{O}_{X^{\mathrm{gen}},x} = d^2+d^2[F:\mathbb{Q}_p]+1$, where the last equality follows from Corollaries 3.37 and 3.38. □

**Proposition 3.40.** *Let $x \in P_1 R^{\square}_{\overline{\rho}}$, where $R^{\square}_{\overline{\rho}}$ is the framed deformation ring of $\overline{\rho}: G_F \to \mathrm{GL}_d(k')$, where $k'$ is finite extension of $k$. Let $\rho_x: G_F \to \mathrm{GL}_d(\kappa(x))$ be the representation obtained by specializing the universal framed deformation of $\overline{\rho}$ at $x$. Let $\mathfrak{q}$ be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} R^{\square}_{\overline{\rho}} \to \kappa(x), \quad \lambda \otimes a \mapsto \bar{\lambda}\bar{a},$$

*where $\Lambda$ is the ring defined at the beginning of the subsection. Then the completion of $(\Lambda \otimes_{\mathcal{O}} R^{\square}_{\overline{\rho}})_{\mathfrak{q}}$ with respect to the maximal ideal is naturally isomorphic to $R^{\square}_{\rho_x}$.*

*Proof.* The proof is similar to the proof of Proposition 3.33, but easier, since the setting is much closer to the setting of [30, Proposition 9.5], [8, Theorem 3.3.1], where an analogous result is proved for versal deformation rings. We leave the details to the reader. $\square$

Let $x$ be a closed point of $X^{\mathrm{gen}}$ such that $\kappa(x)$ is a local field. Since $G_F$ is compact there is a matrix $M \in \mathrm{GL}_d(\kappa(x))$, such that the image of $M\rho_x M^{-1}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$. Let $x' : A^{\mathrm{gen}} \to \mathcal{O}_{\kappa(x)}$ be the $R^{\mathrm{ps}}$-algebra homomorphism corresponding to the representation $E \to M_d(\mathcal{O}_{\kappa(x)})$, $a \mapsto M\rho_x(a)M^{-1}$. We will denote the corresponding Galois representation by $\rho^0_{x'} : G_F \to \mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$ and let $\rho_{x'}$ be the composition $\rho_{x'} : G_F \xrightarrow{\rho^0_{x'}} \mathrm{GL}_d(\mathcal{O}_{\kappa(x)}) \to \mathrm{GL}_d(\kappa(x))$. We note that $\kappa(x') = \kappa(x)$ and let $\Lambda$ be the coefficient ring defined at the beginning of the subsection. Let $k'$ be the residue field of $\mathcal{O}_{\kappa(x)}$ and let $\rho_z : G_F \to \mathrm{GL}_d(k')$ be the representation corresponding to $z : A^{\mathrm{gen}} \xrightarrow{x'} \mathcal{O}_{\kappa(x)} \to k'$. Then $\rho^0_{x'}$ is a deformation of $\rho_z$ to $\mathcal{O}_{\kappa(x)}$, thus the map $x' : A^{\mathrm{gen}} \to \mathcal{O}_{\kappa(x)}$ factors through $x' : R^{\square}_{\rho_z} \to \mathcal{O}_{\kappa(x)}$.

**Corollary 3.41.** *There is an isomorphism of local $\Lambda$-algebras between $R^{\square}_{\rho_x}$, $R^{\square}_{\rho_{x'}}$ and the completion of $(\Lambda \otimes_{\mathcal{O}} R^{\square}_{\rho_z})_{\mathfrak{q}}$ with respect to the maximal ideal, where $\mathfrak{q}$ is as in Proposition 3.40 with respect to $x' : R^{\square}_{\rho_z} \to \mathcal{O}_{\kappa(x)}$.*

*Proof.* Let $\widetilde{M}$ be any lift of $M$ to $M_d(\Lambda)$. Since $\Lambda$ is a local ring $\det \widetilde{M}$ is a unit in $\Lambda$ and hence $\widetilde{M} \in \mathrm{GL}_d(\Lambda)$. Conjugation by $\widetilde{M}$ induces an isomorphism between the deformation problems for $\rho_x$ and $\rho_{x'}$ and hence between the deformation rings. Proposition 3.40 implies that these rings are also isomorphic to the completion of $(\Lambda \otimes_{\mathcal{O}} R^{\square}_{\rho_z})_{\mathfrak{q}}$. $\square$

*Remark* 3.42. Corollary 3.41 enables us to study local properties of $X^{\mathrm{gen}}$, by studying the completions of local rings at closed points above $\mathfrak{m}_{R^{\mathrm{ps}}}$. For example, if we could show that $R^{\square}_{\rho_z}$ is regular, we could conclude that the local ring at $x'$, $(R^{\square}_{\rho_z})_{x'}$ is regular, and hence that the completion $\widehat{(R^{\square}_{\rho_z})}_{x'}$ is regular. If $\kappa(x)$ is a local field of characteristic $p$ then Proposition 3.33, Corollary 3.41 and Lemma 3.35 imply that

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}[\![T]\!] \cong R^{\square}_{\rho_x} \cong R^{\square}_{\rho_{x'}} \cong \widehat{(R^{\square}_{\rho_z})}_{x'}[\![T]\!].$$

If $\kappa(x)$ is a finite extension of $L$ then Proposition 3.33, Corollary 3.41 and Lemma 3.36 imply that

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x} \cong R^{\square}_{\rho_x} \cong R^{\square}_{\rho_{x'}} \cong \widehat{(R^{\square}_{\rho_z})}_{x'}.$$

Thus in both cases we can deduce that $\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}$ and hence $\mathcal{O}_{X^{\mathrm{gen}},x}$ are regular. Thus if we can show that $R^{\square}_{\rho_z}$ is regular for all closed points $z \in X^{\mathrm{gen}}$ above $\mathfrak{m}_{R^{\mathrm{ps}}}$ then we can conclude that $\mathcal{O}_{X^{\mathrm{gen}},x}$ is regular for all closed points $x \in X^{\mathrm{gen}}$ and thus $X^{\mathrm{gen}}$ is regular.

Of course, one may also reverse the logic of this argument: if $X^{\mathrm{gen}}$ is regular then all its local rings and their completions are regular and hence $R^{\square}_{\rho_z}$ is regular for all closed points $z \in X^{\mathrm{gen}}$ above $\mathfrak{m}_{R^{\mathrm{ps}}}$.

**Corollary 3.43.** *Let $\rho : G_F \to \mathrm{GL}_d(\kappa)$ be a continuous representation with $\kappa$ a local field. Then the conclusion of Corollary 3.37 holds for $R^{\square}_{\rho}$.*

*Proof.* After conjugation we may assume that $\rho(G_F) \subset \mathrm{GL}_d(\mathcal{O}_\kappa)$. Let $\overline{\rho}$ be the representation obtained by reducing the matrix entries modulo a uniformizer of $\mathcal{O}_\kappa$ and let $\overline{D}$ be the associated pseudo-character. Corollary 3.37 applies to $R^{\square}_{\overline{\rho}}$. Since

$\rho$ corresponds to an $x \in P_1 R^{\square}_{\overline{\rho}}$, Proposition 3.40 together Lemma 3.36, 3.35 allows us to bound the dimension of $R^{\square}_{\rho}$ from above. Then the proof of Corollary 3.37 carries over. $\square$

**Corollary 3.44.** *Every representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ can be lifted to characteristic zero.*

*Proof.* It follows from Corollary 3.37 that $R^{\square}_{\overline{\rho}}[1/p]$ is non-zero. We may obtain a lift by specializing the universal framed deformation along any $\mathcal{O}$-algebra homomorphism $x : R^{\square} \to \overline{\mathbb{Q}}_p$. $\square$

3.4. **Bounding the maximally reducible semi-simple locus.** Writing $\overline{D} = \prod_{i=1}^m \overline{D}_i$ with $\overline{D}_i$ absolutely irreducible pseudo-characters, we now take $\mathcal{P} = \mathcal{P}_{\max}$ and consider the finite (by Lemma 3.23) $R^{\mathrm{ps}}$-algebra $R^{\mathrm{ps}}_{\underline{\Sigma}}$, where $\underline{\Sigma}$ amounts to some choice of ordering of $\{1, \ldots, m\}$. Note that if $\overline{\rho}_i : G_F \to \mathrm{GL}_d(k)$ is an (absolutely irreducible) representation with pseudo-character $\overline{D}_i$ then

$$R^{\mathrm{ps}}_{\underline{\Sigma}} \cong R_{\overline{\rho}_1} \widehat{\otimes}_{\mathcal{O}} \cdots \widehat{\otimes}_{\mathcal{O}} R_{\overline{\rho}_m}$$

where $R_{\overline{\rho}_i}$ denotes the universal deformation ring of $\overline{\rho}_i$. So let $\rho_i^{\mathrm{univ}} : G_F \to \mathrm{GL}_{d_i}(R_{\overline{\rho}_i})$ denote a representative of the strict equivalence class of the universal representation for each $i = 1, \ldots, m$. If we let $M$ denote the universal invertible matrix in $\mathrm{GL}_d(\mathcal{O}_{\mathrm{GL}_d}(\mathrm{GL}_d))$, then the representation

$$M \times \mathrm{diag}(\rho_1^{\mathrm{univ}}, \ldots, \rho_m^{\mathrm{univ}}) \times M^{-1} : G_F \to \mathrm{GL}_d(R^{\mathrm{ps}}_{\underline{\Sigma}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{GL}_d}(\mathrm{GL}_d))$$

gives rise to a map of Cayley–Hamilton algebras $E \to M_d(R^{\mathrm{ps}}_{\underline{\Sigma}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{GL}_d}(\mathrm{GL}_d))$ which satisfies the universal property of $A^{\mathrm{gen}}$ and so defines a map of $R^{\mathrm{ps}}$-schemes

$$\mathrm{GL}_d \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{gen}}$$

which descends to a map of $R^{\mathrm{ps}}$-schemes

$$\eta_{\underline{\Sigma}} : \mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{gen}}$$

where $L := L_{\underline{\Sigma}}$ denotes the standard Levi subgroup of $\mathrm{GL}_d$ with blocks corresponding to $\underline{\Sigma}$ and $Z_L$ denotes its center.

**Definition 3.45.** The *maximally reducible semi-simple locus* $X^{\mathrm{mrs}} \subset X^{\mathrm{gen}}$ is the scheme-theoretic image of $\eta_{\underline{\Sigma}} : \mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{gen}}$.

**Lemma 3.46.** *Let $x \in X^{\mathrm{gen}}$ and let $y$ be the image of $x$ in $X^{\mathrm{ps}}$. If $y$ lies in $X^{\mathrm{ps}}_{\mathcal{P}_{\max}}$ and $\rho_x$ is semi-simple then $x \in X^{\mathrm{mrs}}$. Moreover, such points are dense in $X^{\mathrm{mrs}}$.*

*Proof.* We first note that if $x \in X^{\mathrm{gen}}$ maps to $X^{\mathrm{ps}}_{\mathcal{P}_{\max}}$ and $\rho_x$ is semi-simple then $\rho_x \cong \rho_1 \oplus \ldots \oplus \rho_m$, with each $\rho_i$ an irreducible representation of $G_F$ lifting $\overline{\rho}_i$. By conjugating by an element of $h \in \mathrm{GL}_d(\kappa(x))$ we may ensure that $h^{-1}\rho_x(g)h = \mathrm{diag}(\rho_1(g), \ldots, \rho_m(g))$ for all $g \in G_F$ and this implies that $x \in X^{\mathrm{mrs}}$.

Since $\eta_{\underline{\Sigma}}$ is a map of affine schemes, it is affine and hence quasi-compact, see [45, Tag 01S5]. It follows from [45, Tag 01R8] that the set theoretic image of $\eta_{\underline{\Sigma}}$ is dense in $X^{\mathrm{mrs}}$. $\square$

**Proposition 3.47.** $\dim X^{\mathrm{mrs}} \le 1 + d^2 + [F : \mathbb{Q}_p] \sum_{i=1}^m d_i^2$.

*Proof.* The open subscheme $U_{\max} = X^{\mathrm{ps}} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}}\} \subset \overline{X}^{\mathrm{ps}}$ is Jacobson by Lemma 3.17, as is $V_{\max} := X^{\mathrm{mrs}} \times_{X^{\mathrm{ps}}} U_{\max}$. Let $Z_{\max}$ denote the closure of $V_{\max}$ in $X^{\mathrm{mrs}}$. The formation of scheme-theoretic images commutes with restriction to opens, so the map

$$(\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}) \times_{X^{\mathrm{ps}}} U_{\max} \to V_{\max}$$

is a dominant map of Jacobson Noetherian excellent schemes. Applying Lemma 3.13 we see that

$$\dim V_{\max} \leq \dim((\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}) \times_{X^{\mathrm{ps}}} U_{\max}).$$

Since $X^{\mathrm{mrs}}$ is by definition a non-empty closed $\mathrm{GL}_d$-invariant subscheme of $X^{\mathrm{gen}}$, Lemma 3.20 implies that every irreducible component of $X^{\mathrm{mrs}}$ has a point in common with the preimage of $\mathfrak{m}_{R^{\mathrm{ps}}}$ in $X^{\mathrm{mrs}}$. Therefore, Lemma 3.17 (5) implies that

$$\dim Z_{\max} = \dim V_{\max} + 1$$

Furthermore, $\mathrm{GL}_d / Z_L$ is flat over $\operatorname{Spec} \mathcal{O}$ with geometrically irreducible fibres, so the projection $\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{ps}}_{\underline{\Sigma}}$ is a flat (and hence open) map with irreducible fibres. It follows from [45, Tag 037A] that this map induces a bijection between the sets of irreducible components. Since $R^{\mathrm{ps}}_{\underline{\Sigma}}$ is a local ring, we deduce that $\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}$ satisfies the assumptions of Lemma 3.17 (5) and thus Lemma 3.17 (5) implies that

$$\dim \mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} = \dim((\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}) \times_{X^{\mathrm{ps}}} U_{\max}) + 1$$

Since $\dim X^{\mathrm{ps}}_{\underline{\Sigma}} = 1 + \sum_{i=1}^{m} (1 + d_i^2 [F : \mathbb{Q}_p])$ and the relative dimension of $\mathrm{GL}_d / Z_L$ over $\mathcal{O}$ is $d^2 - m$ we get that

$$\dim Z_{\max} \leq \dim \mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} = 1 + d^2 + [F : \mathbb{Q}_p] \sum_{i=1}^{m} d_i^2.$$

Let $Y^{\mathrm{mrs}}$ be the scheme theoretic image of $\mathrm{GL}_d / Z_L \times_{\mathcal{O}} \{\mathfrak{m}_{R^{\mathrm{ps}}}\} \to Y$. Since $Y$ is of finite type over $k$ the same argument as above shows that

$$\dim Y^{\mathrm{mrs}} \leq \dim(\mathrm{GL}_d / Z_L \times_{\mathcal{O}} \{\mathfrak{m}_{R^{\mathrm{ps}}}\}) = d^2 - m.$$

Now $Z_{\max} \cup Y^{\mathrm{mrs}}$ is a closed subscheme of $X^{\mathrm{gen}}$ containing the image of $\eta_{\underline{\Sigma}}$. It follows from Lemma 3.46 that $Z_{\max} \cup Y^{\mathrm{mrs}}$ will contain $X^{\mathrm{mrs}}$. Hence,

$$\dim X^{\mathrm{mrs}} \leq \max\{\dim Z_{\max}, \dim Y^{\mathrm{mrs}}\} = \dim Z_{\max}.$$

□

**Corollary 3.48.** $\dim \overline{X}^{\mathrm{mrs}} = \dim X^{\mathrm{mrs}} - 1 \leq d^2 + [F : \mathbb{Q}_p] \sum_{i=1}^{m} d_i^2$.

*Proof.* It follows from Corollary 3.37 that $R^{\mathrm{ps}}_{\underline{\Sigma}}$ is $\mathcal{O}$-torsion free, which implies that $\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}$ is flat over $\operatorname{Spec} \mathcal{O}$ and the same applies for $X^{\mathrm{mrs}}$. (Here we are simply saying that a subring of $\mathcal{O}$-torsion free ring is $\mathcal{O}$-torsion free.) Thus for all $x \in \overline{X}^{\mathrm{mrs}}$, $\varpi$ is a regular element in $\mathcal{O}_{X^{\mathrm{mrs}},x}$ and so $\dim \mathcal{O}_{\overline{X}^{\mathrm{mrs}},x} = \dim \mathcal{O}_{X^{\mathrm{mrs}},x} - 1$. This implies $\dim \overline{X}^{\mathrm{mrs}} = \dim X^{\mathrm{mrs}} - 1$ and the inequality follows from the Proposition 3.47. □

*Remark* 3.49. One could study the closure of the reducible semi-simple locus corresponding to more general partitions using a similar argument. We don't pursue this here, since we need the bound only for $d = 2$ and $F = \mathbb{Q}_2$, see Case 3 of 4.13 below.

3.5. **Density of the irreducible locus.** Let us first unravel the definitions of $U_{\mathcal{P}_{\min}}$ and $V_{\mathcal{P}_{\min}}$ in Section 3.2. We have that $U_{\mathcal{P}_{\min}}$ is an open subscheme of $\overline{X}^{\mathrm{ps}}$ such that the closed points of $U_{\mathcal{P}_{\min}}$ are in bijection with $\mathfrak{p} \in P_1(R^{\mathrm{ps}}/\varpi)$, such that the specialization of the universal pseudo-character along $R^{\mathrm{ps}} \to \kappa(\mathfrak{p})$ is absolutely irreducible. Now $V_{\mathcal{P}_{\min}}$ is the preimage of $U_{\mathcal{P}_{\min}}$ in $\overline{X}^{\mathrm{gen}}$, so that it is an open subscheme of $\overline{X}^{\mathrm{gen}}$ and its closed points are in bijection $\mathfrak{q} \in \overline{X}^{\mathrm{gen}}$, which map to $P_1(R^{\mathrm{ps}}/\varpi)$ in $\overline{X}^{\mathrm{ps}}$, such that the representation

$$E \xrightarrow{j} M_d(A^{\mathrm{gen}}) \to M_d(\kappa(\mathfrak{q}))$$

is absolutely irreducible.

**Proposition 3.50.** *$V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\mathrm{gen}}$.*

*Proof.* We have

$$\overline{X}^{\mathrm{gen}} \setminus V_{\mathcal{P}_{\min}} = Y \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} Z_{\mathcal{P}}$$

and it follows from Lemmas 3.28, 3.29 that $\overline{X}^{\mathrm{gen}} \setminus V_{\mathcal{P}_{\min}}$ has positive codimension in $\overline{X}^{\mathrm{gen}}$. Since $\overline{X}$ is equi-dimensional by Corollary 3.39 we conclude that $V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\mathrm{gen}}$. In particular, the inequality in Proposition 3.27 is an equality. □

We will now prove a stronger version of the above result. Following [8, Definition 5.1.2] we call $y \in U_{\mathcal{P}_{\min}}$ *special* if either $\zeta_p \notin F$ and $D_y = D_y(1)$ or $\zeta_p \in F$ and the restriction $D_y$ to $G_{F'}$ is reducible for some degree $p$ Galois extension $F'$ of $F$. Otherwise, $y$ is called *non-special*. According to [8, Lemma 5.1.3.] there is a closed subscheme $U^{\mathrm{spcl}}$ of $U_{\mathcal{P}_{\min}}$ such that the closed points of $U^{\mathrm{spcl}}$ are precisely the closed special points of $U_{\mathcal{P}_{\min}}$. Let $V^{\mathrm{spcl}}$ denote the preimage of $U^{\mathrm{spcl}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z^{\mathrm{spcl}}$ denote the closure of $V^{\mathrm{spcl}}$.

Similarly let $U^{\mathrm{Kirr}} \subset U_{\mathcal{P}_{\min}}$ be the Kummer-irreducible locus defined in Appendix A. Let $U^{\mathrm{Kred}}$ denote its complement in $U_{\mathcal{P}_{\min}}$, let $V^{\mathrm{Kred}}$ be the preimage of $U^{\mathrm{Kred}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z^{\mathrm{Kred}}$ denote the closure of $V^{\mathrm{Kred}}$. We have $Z^{\mathrm{Kred}} \subseteq Z^{\mathrm{spcl}}$ with equality if $\zeta_p \in F$.

**Lemma 3.51.** *We have*

$$\dim \overline{X}^{\mathrm{gen}} - \dim Z^{\mathrm{spcl}} \geq \frac{1}{2}[F:\mathbb{Q}_p]d^2, \quad \dim \overline{X}^{\mathrm{gen}} - \dim Z^{\mathrm{Kred}} \geq [F:\mathbb{Q}_p]d.$$

*Proof.* It follows from [8, Theorem 5.3.1 (i)] that the dimension of the Zariski closure of $U^{\mathrm{spcl}}$ in $\overline{X}^{\mathrm{ps}}$ is at most $1 + \frac{1}{2}[F:\mathbb{Q}_p]d^2$. If $y \in U^{\mathrm{spcl}}$ then its fibre $X_y^{\mathrm{gen}}$ has dimension $d^2-1$ by Corollary 3.15. Thus Lemma 3.17 implies that

$$\dim Z^{\mathrm{spcl}} \leq d^2 + \frac{1}{2}[F:\mathbb{Q}_p]d^2.$$

Since $\dim \overline{X}^{\mathrm{gen}} = d^2 + d^2[F:\mathbb{Q}_p]$ by Corollary 3.39 the assertion follows. Similarly Proposition A.8 implies that the dimension of the closure of $U^{\mathrm{Kred}}$ in $\overline{X}^{\mathrm{ps}}$ is at most $1 + (d^2-d)[F:\mathbb{Q}_p]$. The same argument gives the required bound for the codimension of $Z^{\mathrm{Kred}}$. □

Let $U^{\text{n-spcl}} := U_{\mathcal{P}_{\min}} \setminus U^{\mathrm{spcl}}$ and let $V^{\text{n-spcl}}$ the preimage of $U^{\text{n-spcl}}$ in $\overline{X}^{\mathrm{gen}}$. Let $V^{\mathrm{Kirr}}$ be the preimage of $U^{\mathrm{Kirr}}$ in $\overline{X}^{\mathrm{gen}}$. We have an inclusion $V^{\mathrm{Kirr}} \subset V^{\text{n-spcl}}$ and the subschemes coincide if $\zeta_p \in F$.

**Proposition 3.52.** *$V^{\mathrm{Kirr}}$ is Zariski dense in $\overline{X}^{\mathrm{gen}}$. Moreover, the following hold:*

(1) *if* $d = 2$ *then* $\dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \geq [F : \mathbb{Q}_p]$*;*
(2) *if* $d > 2$ *then* $\dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \geq 1 + [F : \mathbb{Q}_p]$.
(3) *if* $d > 1$ *is arbitrary but* $\overline{D}$ *is absolutely irreducible (i.e.* $m = 1$*) then* $\dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \geq d[F : \mathbb{Q}_p]$.

*Proof.* Since $V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\mathrm{gen}}$ by Proposition 3.50, we have $\overline{X}^{\mathrm{gen}} = Z_{\mathcal{P}_{\min}} = Z^{\mathrm{Kred}} \cup Z^{\mathrm{Kirr}}$, where $Z^{\mathrm{Kirr}}$ is the closure of $V^{\mathrm{Kirr}}$. Since $\dim Z^{\mathrm{Kred}} < \dim \overline{X}^{\mathrm{gen}}$ by Lemma 3.51 and $\overline{X}^{\mathrm{gen}}$ is equi-dimensional we get that $\overline{X}^{\mathrm{gen}} = Z^{\mathrm{Kirr}}$. Moreover,

$$\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}} = Y \cup Z^{\mathrm{Kred}} \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} Z_{\mathcal{P}}$$

and the assertion follows from Lemmas 3.29, 3.28, Lemma 3.51, noting that if $\overline{\rho}$ is absolutely irreducible then $\{\mathcal{P} : \mathcal{P}_{\min} < \mathcal{P}\} = \emptyset$. $\square$

We now want to transfer the density results from $\overline{X}^{\mathrm{gen}}$ to $R^{\square}_{\overline{\rho}}/\varpi$.

**Lemma 3.53.** *Let* $A \to B$ *be a flat ring homomorphism and let* $U$ *be an open subscheme of* $\operatorname{Spec} A$ *and let* $V$ *be the preimage of* $U$ *in* $\operatorname{Spec} B$*. If* $U$ *is dense in* $\operatorname{Spec} A$ *then* $V$ *is dense in* $\operatorname{Spec} B$*.*

*Proof.* Let $\mathfrak{q}$ be a minimal prime of $B$ and let $\mathfrak{p}$ be its image in $\operatorname{Spec} A$. Since the map is flat, it satisfies going down, and so $\mathfrak{p}$ is a minimal prime of $A$. Since $U$ is dense, it will contain $\mathfrak{p}$ and hence $V$ will contain $\mathfrak{q}$. Thus $V$ contains all the minimal primes of $B$ and so is dense in $\operatorname{Spec} B$. $\square$

**Proposition 3.54.** *Let* $(\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi))^{\mathrm{Kirr}}$ *be the preimage of* $V^{\mathrm{Kirr}}$ *in* $\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi)$*. Then* $(\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi))^{\mathrm{Kirr}}$ *is dense in* $\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi)$*.*

*Proof.* The map $A^{\mathrm{gen}}/\varpi \to R^{\square}_{\overline{\rho}}/\varpi$ is flat, since it is a localization followed by a completion. The assertion follows from Lemma 3.53 and Proposition 3.52. $\square$

*Remark* 3.55. Since $(\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi))^{\mathrm{Kirr}}$ is also the preimage of $U^{\mathrm{Kirr}}$ in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ we may characterise it as an open subscheme of $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$, such that its closed points are in bijection with $x \in P_1(R^{\square}_{\overline{\rho}}/\varpi)$, which map to $P_1(R^{\mathrm{ps}}/\varpi)$ in $\operatorname{Spec} R^{\mathrm{ps}}$ and for which the representation

$$\rho_x : G_F \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}}/\varpi) \to \mathrm{GL}_d(\kappa(x))$$

remains absolutely irreducible after restriction to $G_{F'}$ for all degree $p$ Galois extensions $F'$ of $F(\zeta_p)$. Lemma A.2 implies that $H^2(G_F, \mathrm{ad}^0 \rho_x) = 0$ for such $x$.

We will now prove similar results for the generic fibres. For each partition $\mathcal{P}$ as in Section 3.2 let $X^{\mathrm{ps}}_{\mathcal{P}}$ be the scheme theoretic image of $X^{\mathrm{ps}}$ inside $X^{\mathrm{ps}}_{\underline{\Sigma}}$ and let $X^{\mathrm{gen}}_{\mathcal{P}}$ be the preimage of $X^{\mathrm{ps}}_{\mathcal{P}}$ in $X^{\mathrm{gen}}$. We warn the reader that contrary to our usual notational conventions it is not clear that $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ considered in Section 3.2 is the special fibre of $X^{\mathrm{ps}}_{\mathcal{P}}$. However, the following still holds.

**Lemma 3.56.** $\dim X^{\mathrm{gen}}_{\mathcal{P}}[1/p] \leq \dim \overline{X}^{\mathrm{gen}}_{\mathcal{P}}$.

*Proof.* Let $\mathfrak{a}_{\mathcal{P}}$ be the $R^{\mathrm{ps}}$-annihilator of $R^{\mathrm{ps}}_{\underline{\Sigma}}$, and let $\mathfrak{b}_{\mathcal{P}}$ be the $R^{\mathrm{ps}}$-annihilator of $R^{\mathrm{ps}}_{\underline{\Sigma}}/\varpi$. We may write

$$X^{\mathrm{gen}}_{\mathcal{P}} = \operatorname{Spec} A^{\mathrm{gen}}/\mathfrak{a}_{\mathcal{P}} A^{\mathrm{gen}}, \quad \overline{X}^{\mathrm{gen}}_{\mathcal{P}} = \operatorname{Spec} A^{\mathrm{gen}}/\mathfrak{b}_{\mathcal{P}} A^{\mathrm{gen}}.$$

Since $R^{\mathrm{ps}}_{\underline{\Sigma}}$ is a finite $R^{\mathrm{ps}}$-module by Lemma 3.23, we have $\sqrt{\mathfrak{b}_{\mathcal{P}}} = \sqrt{(\mathfrak{a}_{\mathcal{P}}, \varpi)}$. In particular, the special fibre of $X^{\mathrm{gen}}_{\mathcal{P}}$ has dimension equal to $\dim \overline{X}^{\mathrm{gen}}_{\mathcal{P}}$. The assertion follows from Lemma 3.22. □

**Proposition 3.57.** *Let*

$$V^{\mathrm{irr}} := X^{\mathrm{gen}}[1/p] \setminus \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} X^{\mathrm{gen}}_{\mathcal{P}}[1/p].$$

*Then $V^{\mathrm{irr}}$ is an open dense subset of $X^{\mathrm{gen}}[1/p]$. Moreover, the following hold:*

1. *if $d = 2$ then $\dim X^{\mathrm{gen}}[1/p] - \dim(X^{\mathrm{gen}}[1/p] \setminus V^{\mathrm{irr}}) \geq [F : \mathbb{Q}_p]$;*
2. *if $d > 2$ then $\dim X^{\mathrm{gen}}[1/p] - \dim(X^{\mathrm{gen}}[1/p] \setminus V^{\mathrm{irr}}) \geq 1 + [F : \mathbb{Q}_p]$;*
3. *if $d > 1$ is arbitrary but $\overline{D}$ is absolutely irreducible (i.e. $m = 1$) then $X^{\mathrm{gen}}[1/p] = V^{\mathrm{irr}}$.*

*Proof.* It follows from Corollary 3.39 that $\dim X^{\mathrm{gen}}[1/p] = d^2 + d^2[F : \mathbb{Q}_p] = \dim \overline{X}^{\mathrm{gen}}$. Lemmas 3.56 and 3.28 together with (15) imply that for $\mathcal{P} > \mathcal{P}_{\min}$ we have

$$\dim X^{\mathrm{gen}}[1/p] - \dim X^{\mathrm{gen}}_{\mathcal{P}}[1/p] \geq \dim \overline{X}^{\mathrm{gen}} - \dim \overline{X}^{\mathrm{gen}}_{\mathcal{P}}. \tag{20}$$

It follows from (15) that $\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}} = Y \cup Z^{\mathrm{Kred}} \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} \overline{X}^{\mathrm{gen}}_{\mathcal{P}}$. Thus it follows from (20) and the definition of $V^{\mathrm{irr}}$ that

$$\dim X^{\mathrm{gen}}[1/p] - \dim(X^{\mathrm{gen}}[1/p] \setminus V^{\mathrm{irr}}) \geq \dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \tag{21}$$

and the lower bounds for the codimension of $X^{\mathrm{gen}}[1/p] \setminus V^{\mathrm{irr}}$ follow from Proposition 3.52.

Thus the dimension of the closure of $V^{\mathrm{irr}}$ is equal to $\dim X^{\mathrm{gen}}[1/p]$. Since $A^{\mathrm{gen}}$ is $\mathcal{O}$-torsion free and equi-dimensional, $X^{\mathrm{gen}}[1/p]$ is equi-dimensional, and so $V^{\mathrm{irr}}$ is dense in $X^{\mathrm{gen}}[1/p]$.

If $\overline{D}$ is absolutely irreducible then $\rho_x$ is absolutely irreducible for all closed points $x \in X^{\mathrm{gen}}[1/p]$ and so $X^{\mathrm{gen}}[1/p] = V^{\mathrm{irr}}$. □

**Corollary 3.58.** *Let* $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ *be the preimage of $V^{\mathrm{irr}}$ in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$*. Then* $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ *is dense in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$*.*

*Proof.* As explained in the proof of Proposition 3.54 the map $A^{\mathrm{gen}} \to R^{\square}_{\overline{\rho}}$ is flat. Hence, the localization $A^{\mathrm{gen}}[1/p] \to R^{\square}_{\overline{\rho}}[1/p]$ is also flat. The assertion follows from Lemma 3.53 and Proposition 3.57. □

*Remark* 3.59. Similarly to Remark 3.55 we may characterize $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ as an open subscheme of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ such that its closed points correspond to maximal ideals $\mathfrak{p}$ of $R^{\square}_{\overline{\rho}}[1/p]$ for which the representation

$$\rho_{\mathfrak{p}} : G_F \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}}[1/p]) \to \mathrm{GL}_d(\kappa(\mathfrak{p}))$$

is absolutely irreducible.

**Corollary 3.60.** *The characteristic zero lift of $\overline{\rho}$ in Corollary 3.44 maybe chosen to be absolutely irreducible.*

*Proof.* It follows from Corollary 3.44 that $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ is non-empty, and Corollary 3.58 implies that $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ is non-empty. A closed point in $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ gives the desired lift of $\overline{\rho}$ to characteristic zero. □

**Corollary 3.61.** *Let* $\Sigma \subset \text{m-Spec}\, R^{\square}_{\overline{\rho}}[1/p]$ *be dense in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$. *Then*

$$\Sigma^{\mathrm{irr}} := \Sigma \cap (\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$$

*is also dense in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$.

*Proof.* It follows from the proof of Proposition 3.57 that $\Sigma \setminus \Sigma^{\mathrm{irr}}$ is contained in a closed subset of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ of positive codimension. Since $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ is equi-dimensional $\Sigma^{\mathrm{irr}}$ is dense. $\square$

## 4. Irreducible components

Let $\mu := \mu_{p^\infty}(F)$ be the subgroup of $p$-power roots of unity in $F$. We note that $\mu$ is a finite group of $p$-power order. Let $\bar{\psi} : G_F \to k^\times$ be a character and let $\psi^{\mathrm{univ}} : G_F \to \mathrm{GL}_1(R_{\bar{\psi}})$ be its universal deformation. Local class field theory gives a group homomorphism

$$\mu \to F^\times \xrightarrow{\mathrm{Art}_F} G_F^{\mathrm{ab}} \xrightarrow{\psi^{\mathrm{univ}}} \mathrm{GL}_1(R_{\bar{\psi}})$$

and the map induces a homomorphism of $\mathcal{O}$-algebras $\mathcal{O}[\mu] \to R_{\bar{\psi}}$, where $\mathcal{O}[\mu]$ is the group algebra of $\mu$ over $\mathcal{O}$.

**Lemma 4.1.** $R_{\bar{\psi}} \cong \mathcal{O}[\mu][\![y_1, \dots, y_{[F:\mathbb{Q}_p]+1}]\!]$.

*Proof.* It follows from local class field theory that the pro-$p$ completion of $G_F^{\mathrm{ab}}$ is isomorphic to $\mu_{p^\infty}(F) \times \mathbb{Z}_p^{[F:\mathbb{Q}_p]+1}$ and the assertion follows from [26, Proposition 3.13]. $\square$

Let $R^{\square}_{\overline{\rho}}$ be the framed deformation ring of $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ and let $R_{\det \overline{\rho}}$ denote the universal deformation ring of the one dimensional representation $\det \overline{\rho}$. Mapping a deformation of $\overline{\rho}$ to its determinant induces a natural map $R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$, which makes $R^{\square}_{\overline{\rho}}$ into an $\mathcal{O}[\mu]$-algebra by applying the above discussion to $\bar{\psi} = \det \overline{\rho}$. The algebra $\mathcal{O}[\mu][1/p]$ is semi-simple and its maximal ideals are in bijection with characters $\chi : \mu \to \mathcal{O}^\times$. We thus have

$$R^{\square}_{\overline{\rho}}[1/p] \cong \prod_{\chi:\mu\to\mathcal{O}^\times} R^{\square,\chi}_{\overline{\rho}}[1/p], \tag{22}$$

where $R^{\square,\chi}_{\overline{\rho}} := R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$.

*Warning* 4.2. We emphasize that $R^{\square,\chi}_{\overline{\rho}}$ is *not* the fixed determinant deformation ring, but is rather constructed by fixing the value of the determinant only on the subgroup $\mathrm{Art}_F(\mu) \subset G_F^{\mathrm{ab}}$.

The aim of this section is to show that the rings $R^{\square,\chi}_{\overline{\rho}}$ are $\mathcal{O}$-torsion free integral domains by showing that they are normal. Since we already know that $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free by Corollary 3.37, this implies that the map $R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$ induces a bijection between the sets of irreducible components, which answers affirmatively a question raised by GB–Juschka in [7]. We also determine irreducible components of $A^{\mathrm{gen}}$ and $R^{\mathrm{ps}}$.

**Proposition 4.3.** *There is an isomorphism*

$$R_{\det \overline{\rho}}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_t) \xrightarrow{\simeq} R^{\square}_{\overline{\rho}}$$

*for* $r := \dim\ker(Z^1(\mathrm{tr}) : Z^1(G_F, \mathrm{ad}\,\overline{\rho}) \to Z^1(G_F, k))$ *(where* $G_F$ *acts trivially on* $k$*), and* $t := h^2(G_F, \mathrm{ad}^0\,\overline{\rho})$ *such that* $(f_1, \dots, f_t)$ *forms a regular sequence in* $R_{\det\overline{\rho}}[\![x_1, \dots, x_r]\!]$*. Moreover,* $r - t = (d^2-1)([F:\mathbb{Q}_p]+1)$*.*

*Proof.* In this proof, the maximal ideal of a ring decorated by sub- or superscripts or a tilde is decorated in the same way. This argument is a minor modification of Kisin's method of presenting global deformation rings over global ones. Kisin's is an important refinement of a similar argument of Mazur.

The map $Z^1(\mathrm{tr})$ is the induced map on Zariski tangent spaces of the map of deformation rings $R_{\det\overline{\rho}} \to R^{\square}_{\overline{\rho}}$, and lifts to a surjection

$$\widetilde{\phi} : \widetilde{R} := R_{\det\overline{\rho}}[\![x_1, \dots, x_r]\!] \twoheadrightarrow R^{\square}_{\overline{\rho}}.$$

We set $J := \ker\widetilde{\phi}$. By Nakayama's lemma, we need to show that $\dim_k J/\widetilde{\mathfrak{m}}J \le t$.

The module $J/\widetilde{\mathfrak{m}}J$ appears as the kernel in the sequence

$$0 \to J/\widetilde{\mathfrak{m}}J \to \widetilde{R}/\widetilde{\mathfrak{m}}J \to \widetilde{R}/J \cong R^{\square}_{\overline{\rho}} \to 0. \tag{23}$$

In view of the above sequence, we shall construct a homomorphism

$$\alpha : \mathrm{Hom}(J/\widetilde{\mathfrak{m}}J, k) \to \ker\left(H^2(\mathrm{tr})\colon H^2(G_F, \mathrm{ad}\,\overline{\rho}) \to H^2(G_F, k)\right)$$

and show that the kernel of $\alpha$ injects into $\mathrm{coker}(H^1(\mathrm{tr}))$. This will imply the existence of the presentation in the statement of the Proposition, since then

$$\dim_k J/\widetilde{\mathfrak{m}}J \le \dim_k \ker(H^2(\mathrm{tr})) + \dim_k \mathrm{coker}(H^1(\mathrm{tr})) = h^2(G_F, \mathrm{ad}^0\,\overline{\rho}),$$

where the last equality comes from the long exact cohomology sequence that arises from $0 \to \mathrm{ad}^0\,\overline{\rho} \to \mathrm{ad}\,\overline{\rho} \to k \to 0$.

Fix $u \in \mathrm{Hom}_k(J/\widetilde{\mathfrak{m}}J, k)$. The pushout under $u$ of the sequence (23) yields

$$0 \to I_u \to R_u \xrightarrow{\phi_u} R^{\square}_{\overline{\rho}} \to 0,$$

where $I_u = k$. The surjection of profinite groups $\mathrm{GL}_d(R_u) \twoheadrightarrow \mathrm{GL}_d(R^{\square}_{\overline{\rho}})$ has a continuous section by [42, Proposition 2.2.2]. Thus there is a continuous map $\widetilde{\rho}_u : G_F \to \mathrm{GL}_d(R_u)$ such that the diagram

$$\begin{tikzcd} G_F \arrow[r, "\widetilde{\rho}_u"] \arrow[dr, "\rho^{\square}"'] & \mathrm{GL}_d(R_u) \arrow[d, "\mathrm{GL}_d(\phi_u)"] \\ & \mathrm{GL}_d(R^{\square}_{\overline{\rho}}). \end{tikzcd}$$

commutes. The kernel $1 + M_d(I_u)$ of $\mathrm{GL}_d(\phi_u)$ can be identified with $\mathrm{ad}\,\overline{\rho} \otimes_k I_u$, and so the set-theoretic lift yields a continuous 2-cocycle

$$c_u \in Z^2(G_F, \mathrm{ad}\,\overline{\rho}) \otimes_k I_u$$

given by $1 + c_u(g_1, g_2) = \widetilde{\rho}_u(g_1g_2)\widetilde{\rho}_u(g_2)^{-1}\widetilde{\rho}_u(g_1)^{-1}$. The class

$$[c_u] \in H^2(G_F, \mathrm{ad}\,\overline{\rho}) \otimes_k I_u$$

is independent of the chosen lifting. The representation $\rho^{\square}_{\overline{\rho}}$ can be lifted to a homomorphism $G_F \to \mathrm{GL}_d(R_u)$ precisely if $[c_u] = 0$. The existence of the homomorphisms $R_{\det\overline{\rho}} \to R_u \to R^{\square}_{\overline{\rho}}$ together with the universality of $R_{\det\overline{\rho}}$ imply that the image of $[c_u]$ in $H^2(G_F, k)$ is zero. We define $\alpha$ as the homomorphism $u \mapsto [c_u]$.

To analyze the kernel of $\alpha$, let $u$ be such that $[c_u]=0$, so that $\rho^\square$ can be lifted to $R_u$. By the universality of $R^\square_{\overline{\rho}}$ we obtain a splitting $s_u$ of $\phi_u$. One deduces that the map from $I_u$ to the kernel of the surjective map

$$t_u : \mathfrak{m}_{R_u}/(\mathfrak{m}_{R_u}^2+\varpi R_u) \to \mathfrak{m}^\square/((\mathfrak{m}^\square)^2+\varpi R^\square_{\overline{\rho}})$$

of mod $\varpi$ cotangent spaces is an isomorphism.

The map $t_u$ in turn is induced from the homomorphism $\widetilde{R}/(J\widetilde{\mathfrak{m}}) \to R^\square_{\overline{\rho}}$ by pushout and from the analogous surjection

$$\widetilde{t} : \widetilde{\mathfrak{m}}/(\widetilde{\mathfrak{m}}^2+\varpi\widetilde{R}) \to \mathfrak{m}^\square/((\mathfrak{m}^\square)^2+\varpi R^\square_{\overline{\rho}})$$

Via our identification $I_u \cong \ker t_u$, the pushout along $u$ induces a surjective homomorphism $\gamma_u : \ker(\widetilde{t}) \to I_u \cong k$ of $k$-vector spaces. One easily verifies that $u \mapsto \gamma_u$ induces an injective $k$-linear monomorphism

$$\ker(\alpha) \hookrightarrow \mathrm{Hom}_k(\ker(\widetilde{t}), k)$$

Upon identifying $\ker(\widetilde{t})^*$ with $\mathrm{coker}(H^1(\mathrm{tr}))$, the proof of the bound is complete.

It remains to show that $f_1,\dots,f_t$ is a regular sequence. We may write $\mathcal{O}[\mu] = \mathcal{O}[\![z]\!]/((1+z)^m-1)$, where $m$ is the order of $\mu$. By Lemma 4.1, we get a presentation

$$\frac{\mathcal{O}[\![z,y_1,\dots,y_{[F:\mathbb{Q}_p]+1},x_1,\dots,x_r]\!]}{((1+z)^m-1,f_1,\dots,f_t)} \xrightarrow{\cong} R^\square_{\overline{\rho}}.$$

We claim that

$$\dim R^\square_{\overline{\rho}} = [F:\mathbb{Q}_p]+2+r-t. \tag{24}$$

The claim implies that $((1+z)^m-1,f_1,\dots,f_t)$ can be extended to a system of parameters in a regular ring $S := \mathcal{O}[\![z,x_1,\dots,x_{[F:\mathbb{Q}_p]+1},y_1,\dots,y_r]\!]$. Thus $((1+z)^m-1,f_1,\dots,f_t)$ is a regular sequence in $S$ and so $(f_1,\dots,f_t)$ is a regular sequence in $R_{\det\overline{\rho}}[\![x_1,\dots,x_r]\!] = S/((1+z)^m-1)$.

By Corollary 3.37 the relative dimension of $R^\square_{\overline{\rho}}$ over $\mathcal{O}$ is $\dim Z^1(G_F,\mathrm{ad}\,\overline{\rho}) - h^2(G_F,\mathrm{ad}\,\overline{\rho})$, so to verify the claim it is enough to show that

$$r+[F:\mathbb{Q}_p]+1-t = \dim Z^1(G_F,\mathrm{ad}\,\overline{\rho}) - h^2(G_F,\mathrm{ad}\,\overline{\rho}) \overset{(19)}{=} d^2+d^2[F:\mathbb{Q}_p]. \tag{25}$$

We now deduce (25) in a straightforward manner:

$$\begin{aligned}
&r+[F:\mathbb{Q}_p]+1-t \\
&\overset{(+)}{=} \quad [F:\mathbb{Q}_p]+1+\dim\ker(Z^1(\mathrm{tr})) - \dim\mathrm{coker}(Z^1(\mathrm{tr})) - \dim\ker(H^2(\mathrm{tr})) \\
&\overset{(\mathrm{EP})}{=} \quad h^1(G_F,k)-h^2(G_F,k)+\dim Z^1(G_F,\mathrm{ad}\,\overline{\rho}) - h^1(G_F,k) - \dim\ker(H^2(\mathrm{tr})) \\
&= \quad \dim Z^1(G_F,\mathrm{ad}\,\overline{\rho}) - \dim\ker(H^2(\mathrm{tr})) - h^2(G_F,k) \\
&\overset{(+)}{=} \quad \dim Z^1(G_F,\mathrm{ad}\,\overline{\rho}) - h^2(G_F,\mathrm{ad}\,\overline{\rho}),
\end{aligned}$$

where (EP) stands for the Euler-Poincaré formula for $k$, and where $(+)$ arises from counting dimensions in the exact sequences

$$0 \to \ker(Z^1(\mathrm{tr})) \to Z^1(G_F,\mathrm{ad}\,\overline{\rho}) \xrightarrow{Z^1(\mathrm{tr})} Z^1(G_F,k) \to \mathrm{coker}(Z^1(\mathrm{tr})) \to 0$$

and

$$0 \to \ker(H^2(\mathrm{tr})) \to H^2(G_F,\mathrm{ad}\,\overline{\rho}) \xrightarrow{H^2(\mathrm{tr})} H^2(G_F,k) \to 0.$$

It follows from (25) that $r-t = (d^2-1)([F:\mathbb{Q}_p]+1)$. $\square$

*Remark* 4.4. The Proposition also holds for continuous representations $\overline{\rho}: G_F \to \mathrm{GL}_d(\kappa)$, where $\kappa$ is a local field. Showing that the 2-cocycle is a continuous becomes a bit more subtle, but this is well explained in [20, Lecture 6].

**Corollary 4.5.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ and each closed point $x \in X^{\mathrm{gen}}$ above $\mathfrak{m}_{R^{\mathrm{ps}}}$ the following hold:*

1. *$R^{\square,\chi}_{\rho_x}$ is $\mathcal{O}$-torsion free of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is complete intersection;*
2. *$R^{\square,\chi}_{\rho_x}/\varpi$ is complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p]$.*

*Proof.* Without loss of generality we may assume that the residue field of $x$ is equal to $k$. Proposition 4.3 gives the presentation

$$R^{\chi}_{\det\rho_x}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_t) \xrightarrow{\cong} R^{\square,\chi}_{\rho_x},$$

where $R^{\chi}_{\det\rho_x} := R_{\det\rho_x} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$. Since $R^{\chi}_{\det\rho_x}$ is formally smooth over $\mathcal{O}$ of dimension $[F : \mathbb{Q}_p] + 2$ by Lemma 4.1, it is enough to show that

$$\dim R^{\square,\chi}_{\rho_x}/\varpi \le [F : \mathbb{Q}_p] + 1 + r - t.$$

Then the same argument as in the proof of Proposition 4.3 shows that the sequence $\varpi, f_1, \dots, f_t$ is regular in $R^{\chi}_{\det\rho_x}[\![x_1, \dots, x_r]\!]$. Since $R^{\square,\chi}_{\rho_x}$ is a quotient of $R^{\square}_{\rho_x}$ and $R^{\square}_{\rho_x}$ is $\mathcal{O}$-torsion free by Corollary 3.37, we have $\dim R^{\square,\chi}_{\rho_x}/\varpi \le \dim R^{\square}_{\rho_x}/\varpi = \dim R^{\square}_{\rho_x} - 1$ and the desired inequality follows from (24). $\square$

The restriction of a pseudo-character $D : A[G_F] \to A$ to $G$ defines a continuous group homomorphism $\det D : G_F \to A^\times$, see [8, Definition 4.1.5]. Moreover, if $D$ is associated to a representation $\rho : G_F \to \mathrm{GL}_d(A)$ then $\det D = \det \rho$. This induces a map of deformation rings $R_{\det \overline{D}} \to R^{\mathrm{ps}}$ and makes $R^{\mathrm{ps}}$ into an $\mathcal{O}[\mu]$-algebra.

Since $A^{\mathrm{gen}}$ is an $R^{\mathrm{ps}}$-algebra, we may define

$$A^{\mathrm{gen},\chi} := A^{\mathrm{gen}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}, \quad X^{\mathrm{gen},\chi} := \operatorname{Spec} A^{\mathrm{gen},\chi}$$

and we let $\overline{X}^{\mathrm{gen},\chi}$ denote its special fibre. Note that since a character of $G_F^{\mathrm{ab}}$ valued in a characteristic $p$ field is trivial after pulling back to $\mu_{p^\infty}(F)$, we have that $\overline{X}^{\mathrm{gen},\chi} = \overline{X}^{\mathrm{gen},\mathbf{1}}$ for all $\chi$, where $\mathbf{1}$ is the trivial character. Moreover, the reduced subschemes of $\overline{X}^{\mathrm{gen}}$ and $\overline{X}^{\mathrm{gen},\chi}$ coincide and so

$$\dim \overline{X}^{\mathrm{gen},\chi} = \dim \overline{X}^{\mathrm{gen}} = d^2 + d^2[F : \mathbb{Q}_p],$$

where the last equality is given by Corollary 3.39.

**Corollary 4.6.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ the following hold:*

1. *$A^{\mathrm{gen},\chi}$ is $\mathcal{O}$-torsion free, equi-dimensional of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection;*
2. *$A^{\mathrm{gen},\chi}/\varpi$ is equi-dimensional of dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection.*

*Proof.* We claim that the local rings at closed points of $X^{\mathrm{gen},\chi}$ are $\mathcal{O}$-torsion free and complete intersection. Given the claim the proof is the same as in Corollary 3.39.

We will prove the claim using the strategy outlined in Remark 3.42. We already know from Corollary 4.5 that $R^{\square,\chi}_{\rho_x}$ is $\mathcal{O}$-torsion free and complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p] + 1$ whenever $x \in X^{\mathrm{gen},\chi}$ is a closed point with $\kappa(x)/k$

a finite extension. By applying $\otimes_{\mathcal{O}[\mu],\chi}\mathcal{O}$ we obtain the $\chi$-versions of Propositions 3.33 and 3.40 and Corollary 3.41.

Let $x$ be a closed point of $X^{\mathrm{gen},\chi}$. If $\kappa(x)$ is a finite extension of $k$ then $\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x} \cong R^{\square,\chi}_{\rho_x}$ by Proposition 3.33 and hence $\mathcal{O}_{X^{\mathrm{gen},\chi},x}$ is complete intersection. Otherwise, let $x'$ and $z$ be as in Corollary 3.41. In particular, $z$ is a closed point of $X^{\mathrm{gen},\chi}$ and $\kappa(z)$ is a finite extension of $k$. It follows from the argument explained in Remark 3.42 that if $\kappa(x)$ is a local field of characteristic $p$ then

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x}[\![T]\!] \cong R^{\square,\chi}_{\rho_x} \cong R^{\square,\chi}_{\rho_{x'}} \cong \widehat{(R^{\square,\chi}_{\rho_z})_{x'}}[\![T]\!],$$

and if $\kappa(x)$ is a finite extension of $L$ then

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x} \cong R^{\square,\chi}_{\rho_x} \cong R^{\square,\chi}_{\rho_{x'}} \cong \widehat{(R^{\square,\chi}_{\rho_z})_{x'}}.$$

Since $R^{\square,\chi}_{\rho_z}$ is complete intersection, it follows from [45, Tag 09Q4] that the local ring $(R^{\square,\chi}_{\rho_z})_{x'}$ (and hence its completion) is also complete intersection. The isomorphisms above imply that $\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x}$ is complete intersection. Hence, $\mathcal{O}_{X^{\mathrm{gen},\chi},x}$ is complete intersection, see [45, Tag 09Q3]. $\square$

*Remark* 4.7. Alternatively, one could first prove a version of Proposition 4.3 for deformation rings of $\overline{\rho} : G_F \to \mathrm{GL}_d(\kappa(x))$ to Artinian $\Lambda$-algebra as in Section 3.3 for any closed point of $x \in X^{\mathrm{gen}}$, by changing $\mathcal{O}$ to $\Lambda$ and $k$ to $\kappa(x)$ everywhere. The Euler characteristic formula still holds in this setting, see [8, Theorem 3.4.1]. Then deduce Corollary 4.5 in this more general setting using the same proof and then obtain Corollary 4.6 by repeating verbatim the proof of Corollary 3.39.

In the Lemmas below, $\kappa$ is either a finite field extension of $k$, or of $L$ or a local field of characteristic $p$ containing $k$, and $\Lambda$ is defined in Section 3.3. If $\mathrm{char}(\kappa) = 0$ then $\Lambda = \kappa$, if $\mathrm{char}(\kappa) = p$ then $\Lambda$ is an $\mathcal{O}$-algebra, which is a complete DVR with uniformiser $\varpi$ and residue field $\kappa$. As in Section 3.3 we consider deformation problems of $\rho : G_F \to \mathrm{GL}_d(\kappa)$ to local Artinian $\Lambda$-algebras with residue field $\kappa$.

**Lemma 4.8.** *Let $\kappa$ be a finite or local field of characteristic $p$ or a finite extension of $L$ and let $\rho : G_F \to \mathrm{GL}_d(\kappa)$ be a representation, such that $H^2(G_F, \mathrm{ad}^0 \rho) = 0$, where $\mathrm{ad}^0 \rho$ is the kernel of the trace map. Then for all characters $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ the ring $R^{\square,\chi}_\rho$ is formally smooth over $\Lambda$.*

*Proof.* It follows from the proof of [8, Lemma 3.4.2], where an analogous statement is proved for the deformation functors without the framing and for Artinian $\kappa$-algebras, that the map

$$R_{\det \rho} \to R^{\square}_\rho,$$

induced by sending a deformation of $\rho$ to an Artinian $\Lambda$-algebra to its determinant, is formally smooth. By applying $\otimes_{\mathcal{O}[\mu],\chi}\mathcal{O}$ we deduce that the map

$$R^{\chi}_{\det \rho} \to R^{\square,\chi}_\rho$$

is formally smooth.

Since the group $G_F^{\mathrm{ab}}/\mathrm{Art}_F(\mu_{p^\infty}(F))$ is $p$-torsion free, the ring $R^{\chi}_{\det \rho}$ is formally smooth over $\Lambda$. Hence, $R^{\square,\chi}_\rho$ is formally smooth over $\Lambda$. (Alternatively, one could prove Proposition 4.3 for $\rho$, see Remark 4.4, and then obtain the Lemma as a Corollary.) $\square$

Recall that in Section 3.5 we have defined an open subscheme $U^{\text{n-spcl}}$ of $\overline{X}^{\text{ps}} \setminus \{\mathfrak{m}_{R^{\text{ps}}}\}$ and defined $V^{\text{n-spcl}}$ to be a preimage of $U^{\text{n-spcl}}$ in $\overline{X}^{\text{gen}}$. We will refer to $V^{\text{n-spcl}}$ as the *absolutely irreducible non-special locus*.

**Proposition 4.9.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ the absolutely irreducible non-special locus in $\overline{X}^{\text{gen},\chi}$ is regular.*

*Proof.* It is enough to show that localization of $A^{\text{gen},\chi}/\varpi$ at $x$ is a regular ring for every closed point $x$ in $V^{\text{n-spcl}} \cap \overline{X}^{\text{gen},\chi}$. It follows from Lemma 3.34 applied with $R = R^{\text{ps},\chi}/\varpi$ and $A = A^{\text{gen},\chi}/\varpi$ that it is enough to show that the completion of $\kappa(x) \otimes_{\mathcal{O}} A^{\text{gen},\chi}$ at the kernel of the map of $\kappa(x)$-algebras $\kappa(x) \otimes_{\mathcal{O}} A^{\text{gen},\chi} \to \kappa(x)$ is regular. Proposition 3.33 implies that we may identify this ring with deformation ring $R^{\square,\chi}_{\rho_x}/\varpi$. If $\zeta_p \in F$ then since $x$ is non-special $H^2(G_F, \operatorname{ad}^0 \rho_x) = 0$, see [8, Lemma 5.1.1], Lemma 4.8 implies that $R^{\square,\chi}_{\rho_x}/\varpi$ is formally smooth over $\kappa(x)$. If $\zeta_p \notin F$ then $\mu$ is trivial, so that $R^{\square,\chi}_{\rho_x} = R^{\square}_{\rho_x}$, and $H^2(G_F, \operatorname{ad} \rho_x) = 0$, see [8, Lemma 5.1.1]. It follows from (18) that $R^{\square}_{\rho_x}/\varpi$ is formally smooth over $\kappa(x)$. □

**Proposition 4.10.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ the absolutely irreducible locus in $X^{\text{gen},\chi}[1/p]$ is regular.*

*Proof.* Let $x$ be a closed point in $X^{\text{gen},\chi}[1/p]$ and let $\rho_x : G_F \to \mathrm{GL}_d(\kappa(x))$ be the corresponding Galois representation. We claim that if $\rho_x$ is absolutely irreducible then $H^2(G_F, \operatorname{ad}^0 \rho_x) = 0$. Since $\kappa(x)$ is a finite extension of $L$, $\operatorname{ad}^0 \rho_x$ is a direct summand of $\operatorname{ad} \rho_x$, and thus it is enough to show that $H^2(G_F, \operatorname{ad} \rho_x) = 0$. By local Tate duality, it is enough to show that $H^0(G_F, \operatorname{ad} \rho_x(1)) = 0$. Since $\rho_x$ is absolutely irreducible, non-vanishing of this group is equivalent to $\rho_x \cong \rho_x(1)$. By considering determinants we would obtain that the $d$-th power of the cyclotomic character is trivial, yielding a contradiction.

Given the claim the rest of the proof is the same as the proof of Proposition 4.9, since Lemma 3.36 implies that $\widehat{\mathcal{O}}_{X^{\text{gen},\chi},x} \cong R^{\square,\chi}_{\rho_x}$. □

**Lemma 4.11.** *Assume that $F = \mathbb{Q}_p$ and $d = 2$. Let $\kappa$ be either a finite extension of $L$ or a finite or local field of characteristic $p$, in which case we further assume that $p > 2$. Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\kappa)$ be a continuous representation, which is an extension of characters $\psi_1$ and $\psi_2$, such that $\psi_1 \neq \psi_2(1)$ and $\psi_2 \neq \psi_1(1)$. Then $H^2(G_{\mathbb{Q}_p}, \operatorname{ad} \rho) = H^2(G_F, \operatorname{ad}^0 \rho) = 0$. In particular, the ring $R^{\square,\chi}_\rho$ is formally smooth over $\Lambda$.*

*Proof.* Since $\operatorname{char}(\kappa) \neq 2$, $\operatorname{ad}^0 \rho$ is a direct summand $\operatorname{ad} \rho$, and thus it is enough to show that $H^2(G_{\mathbb{Q}_p}, \operatorname{ad} \rho) = 0$. By local Tate duality, see [8, Section 3.4], it is enough to show that $H^0(G_{\mathbb{Q}_p}, \operatorname{ad} \rho(1)) = 0$. Non-vanishing of this group would imply that $\psi_i \psi_j^{-1}(1)$ is a trivial character for some $i, j \in \{1, 2\}$. If $i = j$ then this would imply $\chi_{\text{cyc}} \otimes_{\mathbb{Z}_p} \kappa$ is trivial, which is not the case as $\operatorname{char}(\kappa) \neq 2$. If $i \neq j$ then this does not hold by assumption.

The last assertion follows from Lemma 4.8. □

**Lemma 4.12.** *Assume that $p = 2$, $F = \mathbb{Q}_2$ and $d = 2$. Let $\kappa$ be a finite or local field of characteristic 2 and let $\rho : G_{\mathbb{Q}_2} \to \mathrm{GL}_2(\kappa)$ be a continuous representation, which is a non-split extension of distinct characters. Then $H^2(G_{\mathbb{Q}_2}, \operatorname{ad}^0 \rho) = 0$. In particular, the ring $R^{\square,\chi}_\rho$ is formally smooth over $\Lambda$.*

*Proof.* After twisting we may assume that we can choose a basis of the underlying vector space of $\rho$, such that with respect to that basis

$$\rho(g) = \begin{pmatrix} 1 & b(g) \\ 0 & \psi(g) \end{pmatrix}, \quad \forall g \in G_{\mathbb{Q}_2}.$$

We use the same basis to identify $\operatorname{ad}\rho$ with $M_2(k)$ with the $G_{\mathbb{Q}_2}$-action given by

$$g \cdot M := \rho(g) M \rho(g)^{-1}.$$

For $i, j \in \{1, 2\}$ let $e_{ij} \in M_2(k)$ be the matrix with the $ij$-entry equal to 1 and all the other entries equal to zero. Let $\overline{\operatorname{ad}}\,\rho$ be the quotient $\operatorname{ad}\rho$ by the scalar matrices and let $\overline{e}_{ij}$ be the image of $e_{ij}$ in $\overline{\operatorname{ad}}\,\rho$. A direct computation shows that

$$g\cdot\overline{e}_{12} = \psi(g)^{-1}\overline{e}_{12}, \quad g\cdot\overline{e}_{11} = \overline{e}_{11} - \psi(g)^{-1}b(g)\overline{e}_{12}, \quad g\cdot\overline{e}_{21} = \psi(g)\overline{e}_{21} - \psi(g)^{-1}b(g)^2\overline{e}_{12}.$$

Since $\rho$ is non-split, $b(g) \neq 0$ for some $g \in G_{\mathbb{Q}_2}$. Thus $\kappa\overline{e}_{12}$ is the unique irreducible subrepresentation of $\overline{\operatorname{ad}}\,\rho$. Since $G_{\mathbb{Q}_2}$ acts on $\overline{e}_{12}$ by a non-trivial character, we deduce that $H^0(G_{\mathbb{Q}_2}, \overline{\operatorname{ad}}\,\rho) = 0$.

It follows from local Tate duality, see [8, Section 3.4], that $H^2(G_{\mathbb{Q}_2}, \operatorname{ad}^0\rho) = 0$. Note that the cyclotomic character is trivial modulo 2.

The last assertion follows from Lemma 4.8. □

**Proposition 4.13.** *There is an open subscheme $V^{0,\chi} \subset \overline{X}^{\mathrm{gen},\chi}$ such that*

(1) $H^2(G_F, \operatorname{ad}^0\rho_x) = 0$ *for all closed points* $x \in V^{0,\chi}$*;*
(2) $\dim \overline{X}^{\mathrm{gen},\chi} - \dim(\overline{X}^{\mathrm{gen},\chi} \setminus V^{0,\chi}) \geq 2$.

*In particular,* $\overline{X}^{\mathrm{gen},\chi}$ *is regular in codimension* 1.

*Proof.* We first note that if $V \subset \overline{X}^{\mathrm{gen},\chi}$ is open and satisfies part (1) then $V$ is regular by the argument explained in the proof of Proposition 4.9. Thus if (1) and (2) hold then $\overline{X}^{\mathrm{gen},\chi}$ is regular in codimension 1. We also note that Lemma A.2 implies that part (1) holds for $V^{\mathrm{Kirr},\chi} := V^{\mathrm{Kirr}} \cap \overline{X}^{\mathrm{gen},\chi}$. We consider three separate cases.

**Case 1: $d > 2$ or $F \neq \mathbb{Q}_p$ or $\overline{D}$ is (absolutely) irreducible.** These three conditions correspond to parts (1), (2), and (3) of Proposition 3.52 respectively, and indeed Proposition 3.52 implies that the complement of $V^{\mathrm{Kirr},\chi}$ in $\overline{X}^{\mathrm{gen},\chi}$ has dimension at most $\dim \overline{X}^{\mathrm{gen},\chi} - 2$. Hence, we may take $V^{0,\chi} = V^{\mathrm{Kirr},\chi}$.

**Case 2: $d = 2$ and $F = \mathbb{Q}_p$ and $p > 2$ and $\overline{D}$ is reducible.** In this case, $\mu = \{1\}$ so $\chi = \mathbf{1}$ and thus $\overline{X}^{\mathrm{gen},\mathbf{1}} = \overline{X}^{\mathrm{gen}}$. It follows from Proposition 3.25, Lemma 3.29, Lemma 3.51 that

$$V^{0,\chi} := \overline{X}^{\mathrm{gen}} \setminus (Y \cup Z^{12}_{\mathcal{P}_{\max}} \cup Z^{21}_{\mathcal{P}_{\max}} \cup Z^{\mathrm{Kred}})$$

satisfies part (2). We may also write $V^{0,\chi} = V^{\mathrm{Kirr}} \cup V'_{\mathcal{P}_{\max}}$, where we use the notation introduced in the proof of Proposition 3.26. Since part (1) holds for $V^{\mathrm{Kirr}}$ it is enough to consider closed points $x \in V'_{\mathcal{P}_{\max}}$. The definition of $V'_{\mathcal{P}_{\max}}$ implies firstly that $\rho_x$ is reducible and secondly that if we let $\psi_1$ and $\psi_2$ denote its irreducible Jordan-Hölder constituents then $\psi_1 \neq \psi_2(1)$ and $\psi_2 \neq \psi_1(1)$. Therefore, $H^2(G_F, \operatorname{ad}^0\rho_x) = 0$ by Lemma 4.11.

**Case 3: $d = 2$ and $F = \mathbb{Q}_2$ and $\overline{D}$ is reducible.** The proof is the same as in Case 2, using Lemma 4.12 instead of Lemma 4.11. However, one additionally has to remove the reducible semi-simple locus in $\overline{X}^{\mathrm{gen},\chi}$. Its dimension is at most

$4+2=6$ by Corollary 3.48 and the dimension of $\overline{X}^{\mathrm{gen},\chi}$ is 8. Thus the codimension is at least 2. $\square$

**Proposition 4.14.** *$\overline{X}^{\mathrm{gen},\chi}$ is normal.*

*Proof.* Since $\overline{X}^{\mathrm{gen},\chi}$ is a local complete intersection by Corollary 4.6, it is Cohen–Macaulay and satisfies property (S2), and Proposition 4.13 says that it satisfies property (R1). Hence, $\overline{X}^{\mathrm{gen},\chi}$ is normal by Serre's criterion for normality. $\square$

**Corollary 4.15.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ and $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ the ring $R^{\square,\chi}_{\overline{\rho}}/\varpi$ is a normal integral domain.*

*Proof.* Since $\overline{X}^{\mathrm{gen},\chi}$ is normal and excellent the completions of its local rings are normal, [34, Theorem 32.2 (i)]. Thus by formally completing along the maximal ideal corresponding to $\overline{\rho}$ we see that $R^{\square,\chi}_{\overline{\rho}}/\varpi$ is normal, and thus an integral domain since it is a local ring. $\square$

**Lemma 4.16.** *Let $\hat{Y}$ be the preimage of $\mathfrak{m}_{R^{\mathrm{ps}}}$ in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$. Let $W$ be a closed subscheme of $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ such that $H^2(G_F, \mathrm{ad}^0 \rho_x) \neq 0$ for all closed points $x \in W \setminus \hat{Y}$. Then $\dim R^{\square,\chi}_{\overline{\rho}}/\varpi - \dim W \geq 2$.*

*Proof.* The assumptions imply that $W$ is contained in $\hat{Z} \cup \hat{Y}$, where $Z = \overline{X}^{\mathrm{gen},\chi} \setminus V^{0,\chi}$ and $\hat{Z}$ is a formal completion of $Z$ at the point corresponding to $\overline{\rho}$. In terms of commutative algebra the ring of functions of $\hat{Z}$ corresponds to the completion of the ring of functions of $Z$ with respect to the maximal ideal corresponding to $\overline{\rho}$. Hence, $\dim \hat{Z} = \dim Z$, and Proposition 4.13 implies that $\hat{Z}$ has codimension at least 2 in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$. Similarly $\hat{Y}$ is a formal completion of $Y$ at the point corresponding to $\overline{\rho}$, and using Lemma 3.29 we conclude that $\hat{Y}$ also has codimension of at least 2 in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$. $\square$

**Proposition 4.17.** *$X^{\mathrm{gen},\chi}[1/p]$ is normal.*

*Proof.* The proof is essentially the same as the proof of Proposition 4.14. It follows from Corollary 4.6 that $X^{\mathrm{gen},\chi}[1/p]$ is Cohen–Macaulay and we have to check that the codimension of the singular locus is at least 2. Since $X^{\mathrm{gen},\chi}[1/p]$ is a preimage of $\operatorname{Spec} R^{\mathrm{ps},\chi}[1/p]$ in $X^{\mathrm{gen}}$, Lemma 3.17 implies that $X^{\mathrm{gen},\chi}[1/p]$ is Jacobson and we may argue with closed points.

We have already shown in Proposition 4.10 that the absolutely irreducible locus $V^{\mathrm{irr},\chi}$ in $X^{\mathrm{gen},\chi}[1/p]$ is regular. Thus the singular locus is contained in

$$\bigcup_{\mathcal{P}_{\min} < \mathcal{P}} X^{\mathrm{gen},\chi}_{\mathcal{P}}[1/p],$$

where $X^{\mathrm{gen},\chi}_{\mathcal{P}} := X^{\mathrm{gen},\chi} \cap X^{\mathrm{gen}}_{\mathcal{P}}$.

If either $\overline{\rho}$ is absolutely irreducible, $F \neq \mathbb{Q}_p$ or $d > 2$ then it follows from Proposition 3.57 that $X^{\mathrm{gen},\chi}[1/p]$ is regular in codimension 1.

If $\overline{\rho}$ is reducible, $F = \mathbb{Q}_p$ and $d = 2$ then there are two partitions $\mathcal{P}_{\min}$ and $\mathcal{P}_{\max}$ and $\dim X^{\mathrm{gen},\chi}[1/p] - \dim X^{\mathrm{gen},\chi}_{\mathcal{P}_{\max}}[1/p] = 1$, so the previous argument does not work. If $x \in X^{\mathrm{gen},\chi}[1/p]$ is a closed singular point then it follows from Proposition 4.10 and Lemma 4.11 that $\rho_x$ is reducible and its semi-simplification has the form $\psi \oplus \psi(1)$ for some character $\psi : G_F \to \kappa(x)^\times$. It follows from Lemma 3.22 that the dimension of this locus is at most $\dim Z^{12}_{\mathcal{P}_{\max}}$, which is at most 6 by Proposition

3.25. It follows from Corollary 4.6 that $\dim X^{\mathrm{gen},\chi}[1/p] = \dim \overline{X}^{\mathrm{gen},\chi} = 8$. Thus the codimension of the singular locus in $X^{\mathrm{gen},\chi}[1/p]$ is at least 2. $\square$

**Corollary 4.18.** *$X^{\mathrm{gen},\chi}$ is normal.*

*Proof.* Since $A^{\mathrm{gen},\chi}$ is $\mathcal{O}$-torsion free by Corollary 4.6, the map $\mathcal{O} \to A^{\mathrm{gen},\chi}$ is flat. We have shown in Propositions 4.14 and 4.17 that the fibre rings $L \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}$ and $k \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}$ are normal. Since $\mathcal{O}$ is a regular ring [10, Corollary 2.2.23] implies that $A^{\mathrm{gen},\chi}$ is normal. $\square$

**Corollary 4.19.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ and $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ the ring $R^{\square,\chi}_{\overline{\rho}}$ is a normal integral domain.*

*Proof.* The proof is the same as the proof of Corollary 4.15 using Corollary 4.18. $\square$

**Lemma 4.20.** *Let $W$ be a closed subscheme of $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$ with the property that $H^2(G_F, \mathrm{ad}^0 \rho_x) \neq 0$ for all closed points $x \in W$. Then $\dim R^{\square,\chi}_{\overline{\rho}}[1/p] - \dim W \geq 2$.*

*Proof.* Since in characteristic zero $\mathrm{ad}^0 \rho_x$ is a direct summand of $\mathrm{ad}\, \rho_x$ we obtain that $H^2(G_F, \mathrm{ad}\, \rho_x) \neq 0$ for all $x \in W$. This implies that $W$ is contained in the singular locus of $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$. Since $R^{\square,\chi}_{\overline{\rho}}[1/p]$ is normal, the singular locus has codimension of at least 2. $\square$

The next result answers affirmatively a question raised by GB–Juschka in [7].

**Corollary 4.21.** *The map $R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$ induces a bijection between the sets of irreducible components.*

*Proof.* Since $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free by Corollary 3.37, the irreducible components of $R^{\square}_{\overline{\rho}}$ and $R^{\square}_{\overline{\rho}}[1/p]$ coincide. Since the algebra $\mathcal{O}[\mu_{p^\infty}(F)][1/p]$ is semi-simple, we have

$$R^{\square}_{\overline{\rho}}[1/p] \cong \prod_{\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times} R^{\square,\chi}_{\overline{\rho}}[1/p]. \tag{26}$$

It follows from Corollaries 4.5 and 4.19 that $R^{\square,\chi}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free integral domain. We note that the special fibres of these rings are non-zero, thus the rings themselves are non-zero. Hence, the localization $R^{\square,\chi}_{\overline{\rho}}[1/p]$ is non-zero and is an integral domain. $\square$

**Corollary 4.22.** *$R^{\square}_{\overline{\rho}}[1/p]$ is normal.*

*Proof.* This follows from (26) and Corollary 4.19. $\square$

**Corollary 4.23.** *If either $d = 2$ and $[F : \mathbb{Q}_p] \geq 4$ or $d \geq 3$ and $[F : \mathbb{Q}_p] \geq 3$ then for each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ and $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ the rings $R^{\square,\chi}_{\overline{\rho}}$, $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are regular in codimension 3. In particular, $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are factorial.*

*Proof.* The assumptions together with Propositions 3.52, 3.57 imply that the complement of the absolutely irreducible non-special locus in $\overline{X}^{\mathrm{gen},\chi}$ (resp. absolutely irreducible locus in $X^{\mathrm{gen},\chi}[1/p]$) has codimension at least 4. It follows from Propositions 4.9, 4.10 that it contains the singular locus in $\overline{X}^{\mathrm{gen},\chi}$ (resp. $X^{\mathrm{gen},\chi}[1/p]$). Hence, $X^{\mathrm{gen},\chi}$ and $\overline{X}^{\mathrm{gen},\chi}$ are regular in codimension 3, which implies that $R^{\square,\chi}_{\overline{\rho}}$

and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are regular in codimension 3. Since both rings are also complete intersection by Corollary 4.5, they are factorial by a theorem of Grothendieck, see [12] for a short proof. $\square$

*Remark* 4.24. The assumptions in Corollary 4.23 are not optimal as the next Corollary shows. To find the optimal assumptions one would have to further study the reducible locus and we don't want to pursue this here. We note that if $F = \mathbb{Q}_p$, $p \geq 5$ and $\overline{\rho} = \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right)$ is non-split, where $\omega$ is the cyclotomic character modulo $p$, then it follows from [39, Corollary B.5] that $R^{\square,\chi}_{\overline{\rho}} \cong \mathcal{O}[\![x_1, \dots, x_9]\!]/(x_1x_2 - x_3x_4)$ and hence is not factorial. Hence some assumptions in Corollary 4.23 have to be made.

**Corollary 4.25.** *If $\overline{\rho}$ is absolutely irreducible then $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are factorial, except in the case $d = 2$, $F = \mathbb{Q}_3$ and $\overline{\rho} \cong \overline{\rho}(1)$.*

*Proof.* Since $\overline{\rho}$ is absolutely irreducible $X^{\mathrm{gen},\chi}[1/p]$ is regular by Proposition 4.10, and the singular locus of $\overline{X}^{\mathrm{gen},\chi}$ is contained in $Z^{\mathrm{spcl}}$, which has codimension at least $\frac{1}{2}[F : \mathbb{Q}_p]d^2$ by Lemma 3.51. Thus if either $d > 2$ or $F \neq \mathbb{Q}_p$ then we can conclude that $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are regular in codimension 3 and hence factorial.

If $\overline{\rho} \not\cong \overline{\rho}(1)$ then $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}^0\,\overline{\rho}) = 0$ and it follows from Lemma 4.8 that $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are formally smooth, hence regular and hence factorial.

If $d = 2$ then $\overline{\rho} \cong \overline{\rho}(1)$ implies that $\det \overline{\rho} = (\det \overline{\rho})\omega^2$. This leaves us with two cases $F = \mathbb{Q}_2$ or $F = \mathbb{Q}_3$ and $d = 2$. If $p = 2$ then $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are regular by [15, Proposition 4.5].

We claim that if $F = \mathbb{Q}_3$, $d = 2$ and $\overline{\rho} \cong \overline{\rho}(1)$ then the ring $R^{\square,\chi}_{\overline{\rho}}$ is not factorial. It follows from [6, Theorem 5.1] that in this case $R^{\square,\chi}_{\overline{\rho}}$ is formally smooth over $\mathcal{O}[\![b, c, d]\!]/(r)$, where $r = (1 + d)^6(1 + bcu) - (1 + bcv)$ and $u, v$ are units in $\mathcal{O}[\![b, c]\!]$. The ideal $\mathfrak{p} = (b, d)$ is prime of height 1. If $R^{\square,\chi}_{\overline{\rho}}$ were factorial then $\mathfrak{p}$ would have to be principal, [45, Tag 0AFT], and thus there would exist $\pi \in \mathcal{O}[\![b, c, d]\!]$ such that we have an equality of ideals $(b, d) = (r, \pi)$ in $\mathcal{O}[\![b, c, d]\!]$. By considering this modulo $(\varpi, c)$, we would conclude that $(d^3 - d^6, \bar{\pi})$ is the maximal ideal in $k[\![b, d]\!]$. Since $(d^3 - d^6, \bar{\pi}) \to (b, d)/(b, d)^2$ is not surjective, we obtain a contradiction. The same argument shows that $R^{\square,\chi}_{\overline{\rho}}/\varpi$ is also not factorial. $\square$

**Proposition 4.26.** *For each character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ the rings $A^{\mathrm{gen},\chi}$ and $A^{\mathrm{gen},\chi}/\varpi$ are integral domains.*

*Proof.* Since $A^{\mathrm{gen},\chi}$ is normal by Corollary 4.18 it is a product of normal domains $A^{\mathrm{gen},\chi} \cong A_1 \times \ldots \times A_m$. The action of $G$ on $X^{\mathrm{gen},\chi}$ leaves the connected components invariant, see the proof of Lemma 2.1. It follows from Lemma 3.20 that each $\operatorname{Spec} A_i$ contains a closed point over the closed point $X^{\mathrm{ps}}$. Thus $A_i \otimes_{R^{\mathrm{ps}}} k$ are non-zero for $1 \leq i \leq m$. If $m > 1$ then this would imply that the fibre at the closed point of $X^{\mathrm{ps}}$ is not connected contradicting Lemma 3.6. The same proof works also for the special fibre. $\square$

Define $R^{\mathrm{ps},\chi} := R^{\mathrm{ps}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ for a character $\chi : \mu \to \mathcal{O}^\times$ and using the isomorphism from Lemma 4.1. We let $X^{\mathrm{ps},\chi} = \operatorname{Spec} R^{\mathrm{ps},\chi}$ and let $\overline{X}^{\mathrm{ps},\chi}$ be its special fibre.

**Corollary 4.27.** *The rings $R^{\mathrm{ps}}[1/p]$, $R^{\mathrm{ps},\chi}[1/p]$ and the rigid spaces $(\operatorname{Spf} R^{\mathrm{ps}})^{\mathrm{rig}}$, $(\operatorname{Spf} R^{\mathrm{ps},\chi})^{\mathrm{rig}}$ are normal. Moreover, $R^{\mathrm{ps},\chi}[1/p]$ is an integral domain and thus the map $R_{\det \overline{\rho}}[1/p] \to R^{\mathrm{ps}}[1/p]$ induces a bijection between the sets of irreducible components.*

*Proof.* The assertion follows from [40, Theorem A.1] using Corollary 4.22. As part of the proof one obtains $R^{\mathrm{ps}}[1/p] = (A^{\mathrm{gen}}[1/p])^G$. This yields $R^{\mathrm{ps},\chi}[1/p] = (A^{\mathrm{gen},\chi}[1/p])^G$. Proposition 4.26 implies that $A^{\mathrm{gen},\chi}[1/p]$ is an integral domain. Hence $R^{\mathrm{ps},\chi}[1/p]$ is an integral domain, and the assertion about irreducible components is proved in the same manner as Corollary 4.21. □

**Corollary 4.28.** *The image of $R^{\mathrm{ps}}$ in $A^{\mathrm{gen}}$ is the maximal $\mathcal{O}$-torsion free quotient of $R^{\mathrm{ps}}$ and is also the maximal reduced quotient of $R^{\mathrm{ps}}$. In particular, the map $R_{\det \overline{D}} \to R^{\mathrm{ps}} \to R^{\mathrm{ps}}[1/p]$ induces a bijection between the sets of irreducible components. Moreover, if $\overline{D}$ is multiplicity free then $R^{\mathrm{ps}}$ is reduced and $\mathcal{O}$-torsion free.*

*Proof.* By [47, Theorem 2.20] the map $X^{\mathrm{gen}} /\!\!/ \mathrm{GL}_d \to X^{\mathrm{ps}}$ is an adequate homeomorphism. It follows from [1, Proposition 3.3.5] that the kernel of $R^{\mathrm{ps}} \to (A^{\mathrm{gen}})^{\mathrm{GL}_d}$ is nilpotent and vanishes after inverting $p$. Since $A^{\mathrm{gen}}$ is $\mathcal{O}$-torsion free and reduced, this implies that both quotients coincide and are equal to the image of $R^{\mathrm{ps}}$ in $A^{\mathrm{gen}}$. This together with the last part of Corollary 4.27 implies the assertion about the irreducible components.

If $\overline{D}$ is multiplicity free then $E$ is a generalized matrix algebra by [17, Theorem 2.22], and it follows from [47, Theorem 3.8 (4)] that $R^{\mathrm{ps}} = (A^{\mathrm{gen}})^{\mathrm{GL}_d}$, and so $R^{\mathrm{ps}}$ is $\mathcal{O}$-torsion free and reduced. □

**Corollary 4.29.** *The image of $R^{\mathrm{ps},\chi}/\varpi$ in $A^{\mathrm{gen},\chi}/\varpi$ is the maximal reduced quotient of $R^{\mathrm{ps},\chi}/\varpi$. The image of $R^{\mathrm{ps},\chi}$ in $A^{\mathrm{gen},\chi}$ is the maximal reduced quotient of $R^{\mathrm{ps},\chi}$ and is also the maximal $\mathcal{O}$-torsion free quotient of $R^{\mathrm{ps},\chi}$. Moreover, if $\overline{D}$ is multiplicity free then both $R^{\mathrm{ps},\chi}/\varpi$ and $R^{\mathrm{ps},\chi}$ are integral domains.*

*Proof.* If we work with the algebra $E^{\chi} := E \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ instead of $E$ then the argument in the proof of Corollary 4.28 gives adequate homeomorphisms

$$X^{\mathrm{gen},\chi} /\!\!/ \mathrm{GL}_d \to X^{\mathrm{ps},\chi}, \quad \overline{X}^{\mathrm{gen},\chi} /\!\!/ \mathrm{GL}_d \to \overline{X}^{\mathrm{ps},\chi}.$$

In particular, the kernel of $R^{\mathrm{ps},\chi}/\varpi \to A^{\mathrm{gen},\chi}/\varpi$ is nilpotent. Since $A^{\mathrm{gen},\chi}/\varpi$ is an integral domain by Proposition 4.26 we obtain the first assertion. The argument with $R^{\mathrm{ps},\chi}$ is the same as in Corollary 4.28 using that $A^{\mathrm{gen},\chi}$ is an integral domain.

If $\overline{D}$ is multiplicity free then $E^{\chi}$ and $E^{\chi}/\varpi$ are generalized matrix algebras, and the argument in Corollary 4.28 carries over. □

**Lemma 4.30.** *If $R^{\mathrm{ps},\chi}/\varpi$ satisfies Serre's condition (S1) then $R^{\mathrm{ps},\chi}/\varpi$ and $R^{\mathrm{ps},\chi}$ are integral domains.*

*Proof.* We first note that $R^{\mathrm{ps},\chi}/\varpi$ satisfies Serre's condition (R0). Since the underlying reduced subschemes of $\overline{X}^{\mathrm{ps}}$ and $\overline{X}^{\mathrm{ps},\chi}$ coincide, Proposition A.8 implies that the Kummer-irreducible locus $(\overline{X}^{\mathrm{ps},\chi})^{\mathrm{Kirr}}$ in $\overline{X}^{\mathrm{ps},\chi}$ is open dense. If $x \in (\overline{X}^{\mathrm{ps},\chi})^{\mathrm{Kirr}}$ is a closed point then the pseudo-character $D_x$ is absolutely irreducible, and hence is associated to an absolutely irreducible representation which we denote by $\rho_x$. Let $R_{\rho_x}$ be the universal deformation ring of $R_{\rho_x}$ and let $R^{\chi}_{\rho_x}$ be the quotient of $R_{\rho_x}$ parameterizing deformations such that the restriction of the determinant to $\mathrm{Art}_F(\mu) \subset G_F^{\mathrm{ab}}$ is equal to $\chi$. Since $R^{\square,\chi}_{\rho_x}$ is formally smooth over $R^{\chi}_{\rho_x}$, the Kummer-irreducibility of $x$ implies that $R^{\chi}_{\rho_x}$ is regular. The proof of [8, Lemma 5.1.6] shows that $x$ is a regular point in $\overline{X}^{\mathrm{ps},\chi}$. Hence $\overline{X}^{\mathrm{ps},\chi}$ contains an open dense regular subscheme, which implies that $R^{\mathrm{ps},\chi}/\varpi$ satisfies (R0). Since $R^{\mathrm{ps},\chi}/\varpi$ satisfies (S1) by assumption we conclude that $R^{\mathrm{ps},\chi}/\varpi$ is reduced. It follows from Lemma 4.29 and Proposition 4.26 that $R^{\mathrm{ps},\chi}/\varpi$ is an integral domain.

Let $R^{\mathrm{ps},\chi} \twoheadrightarrow R^{\mathrm{ps},\chi}_{\mathrm{tf}}$ be the maximal $\mathcal{O}$-torsion free quotient quotient, and let $\mathfrak{a}$ be the kernel of this map. We have an exact sequence $0 \to \mathfrak{a}/\varpi \to R^{\mathrm{ps},\chi}/\varpi \to R^{\mathrm{ps},\chi}_{\mathrm{tf}}/\varpi \to 0$. It follows from Corollary 4.29 that $\mathfrak{a}$ is nilpotent. Since $R^{\mathrm{ps},\chi}/\varpi$ is reduced we deduce from the exact sequence that $\mathfrak{a}/\varpi$ is zero. Nakayama's lemma implies that $\mathfrak{a} = 0$. Thus $R^{\mathrm{ps},\chi}$ is $\mathcal{O}$-torsion free, and hence is a subring of $A^{\mathrm{gen},\chi}$ by Corollary 4.29. Since $A^{\mathrm{gen},\chi}$ is domain we conclude that $R^{\mathrm{ps},\chi}$ is an integral domain. □

*Remark* 4.31. We expect that the rings $R^{\mathrm{ps},\chi}$ and $R^{\mathrm{ps},\chi}/\varpi$ are integral domains. Although we know the dimension of $R^{\mathrm{ps},\chi}/\varpi$ by [8, Theorem 5.4.1] we cannot conclude that the ring is complete intersection (which would imply that (S1) holds) as we lack a presentation analogous to (18). Since $A^{\mathrm{gen},\chi}$ and $A^{\mathrm{gen},\chi}/\varpi$ are integral domains this question is closely related to the embedding problem discussed in [5, Section 1.3.4].

## 5. Deformation rings with fixed determinant

Let $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ be a representation with pseudo-character $\overline{D}$ and let $\psi : G_F \to \mathcal{O}^\times$ be a character lifting $\det \overline{\rho} = \det \overline{D}$. Let

$$R^{\square,\psi}_{\overline{\rho}} := R^{\square}_{\overline{\rho}} \otimes_{R_{\det \overline{\rho}},\psi} \mathcal{O}.$$

Let $\mu := \mu_{p^\infty}(F)$ and let $\chi : \mu \to \mathcal{O}^\times$ be a character such that the restriction of $\psi$ to $\mu$ under the Artin map $\mu \to G_F^{\mathrm{ab}}$ from local class field theory is equal to $\chi$. Then $R^{\square,\psi}_{\overline{\rho}}$ is a quotient of the ring $R^{\square,\chi}_{\overline{\rho}}$ considered in the previous section. We let $X^{\square,\chi} = \mathrm{Spec}\, R^{\square,\chi}_{\overline{\rho}}$, $X^{\square,\psi} = \mathrm{Spec}\, R^{\square,\psi}_{\overline{\rho}}$ and denote by $\overline{X}^{\square,\chi}$ and $\overline{X}^{\square,\psi}$ their special fibres.

Let $\mathcal{X} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Sets}$ be the functor, which sends $(A, \mathfrak{m}_A)$ to the group $\mathcal{X}(A)$ of continuous characters $\theta : G_F \to 1+\mathfrak{m}_A$ whose restriction to $\mu$ under the Artin map is trivial. It follows from Lemma 4.1 that the functor $\mathcal{X}$ is pro-represented by

$$\mathcal{O}(\mathcal{X}) \cong R_{\mathbf{1}} \otimes_{\mathcal{O}[\mu]} \mathcal{O} \cong \mathcal{O}[\![y_1, \ldots, y_{[F:\mathbb{Q}_p]+1}]\!]. \tag{27}$$

For $e \in \mathbb{N}$ let $\varphi_e : \mathcal{X} \to \mathcal{X}$ be the natural transformation that sends $\theta \in \mathcal{X}(A)$ to $\theta^e$. We also write $\varphi_e$ for the induced maps $\mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$ and $\mathrm{Spec}\,\mathcal{O}(\mathcal{X}) \to \mathrm{Spec}\,\mathcal{O}(\mathcal{X})$. The natural transformation $D^{\square,\chi}_{\overline{\rho}} \to \mathcal{X}$, $\rho \mapsto (\det \rho)\psi^{-1}$ induces a homomorphism of local $\mathcal{O}$-algebras $\mathcal{O}(\mathcal{X}) \to R^{\square,\chi}_{\overline{\rho}}$; we will consider $R^{\square,\chi}_{\overline{\rho}}$ as $\mathcal{O}(\mathcal{X})$-algebra via this map in the statements below.

**Proposition 5.1.** *One has a natural isomorphism of functors*

$$D^{\square,\chi}_{\overline{\rho}} \times_{\mathcal{X},\varphi_d} \mathcal{X} \cong D^{\square,\psi}_{\overline{\rho}} \times \mathcal{X}.$$

*Proof.* Let $(A, \mathfrak{m}_A)$ be in $\mathfrak{A}_{\mathcal{O}}$. An element in $(D^{\square,\chi}_{\overline{\rho}} \times_{\mathcal{X},\varphi_d} \mathcal{X})(A)$ is a pair $(\rho, \theta)$ such that $\theta : G_F \to 1 + \mathfrak{m}_A$ is a continuous homomorphism that is trivial on $\mu$, $\rho : G_F \to \mathrm{GL}_d(A)$ is a continuous homomorphism such that $\det \rho$ and $\chi$ agree when restricted to $\mu$, and one has $(\det\rho)\psi^{-1} = \theta^d$. An element in $(D^{\square,\psi}_{\overline{\rho}} \times \mathcal{X})(A)$ is a pair $(\rho_1, \theta_1)$ where $\theta_1 : G_F \to 1 + \mathfrak{m}_A$ is a continuous homomorphism that is trivial on $\mu$ and $\rho_1 : G_F \to \mathrm{GL}_d(A)$ is a continuous homomorphism such that $\det \rho_1 = \psi$. One verifies that the map

$$(\rho, \theta) \mapsto (\rho \cdot \theta^{-1}, \theta)$$

defines a bijection that is natural in $A$. □

**Corollary 5.2.** *Proposition 5.1 induces a natural isomorphism*

$$R^{\square,\chi}_{\overline{\rho}} \otimes_{\mathcal{O}(\mathcal{X}),\varphi_d} \mathcal{O}(\mathcal{X}) \cong R^{\square,\psi}_{\overline{\rho}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X}).$$

We now clarify some properties of the map $\varphi_d : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$.

**Lemma 5.3.** *The map $\varphi_d$ is finite and flat and becomes étale after inverting $p$. Moreover, it induces a universal homeomorphism on the special fibres.*

*Proof.* We may write $d = ep^m$, such that $p$ does not divide $e$. Then $\varphi_d = \varphi_{p^m} \circ \varphi_e$. Since $e$ is prime to $p$, elements in $1 + \mathfrak{m}_A$ for $(A, \mathfrak{m}_A)$ in $\mathfrak{A}_{\mathcal{O}}$ possess a unique $e$-th root in $1 + \mathfrak{m}_A$ by the binomial theorem, and it follows that $\varphi_e$ is an isomorphism. We thus may assume that $d$ is a power of $p$.

The map $\varphi_d : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$ sends $y_i$ to $(1 + y_i)^d - 1$. One checks that the monomials $\prod_{i=1}^{[F:\mathbb{Q}_p]+1} y_i^{m_i}$ with $0 \le m_i \le d - 1$ form a basis of $\mathcal{O}(\mathcal{X})$ as $\mathcal{O}(\mathcal{X})$-module via $\varphi_d$, by checking the assertion modulo $\varpi$ and using Nakayama's lemma. A (standard) calculation shows that the discriminant is a power of $p$ up to a sign. Thus $\varphi_d$ becomes étale after inverting $p$.

The map $\overline{\varphi}_d : \mathcal{O}(\mathcal{X})/\varpi \to \mathcal{O}(\mathcal{X})/\varpi$ is a power of the relative Frobenius of $\mathrm{Spec}(\mathcal{O}(\mathcal{X})/\varpi)/\operatorname{Spec} k$. The last assertion follows from [45, Tag 0CCB]. □

In the following results we deduce properties of the ring $R^{\square,\psi}_{\overline{\rho}}$.

**Corollary 5.4.** *The following hold:*

1. *$R^{\square,\psi}_{\overline{\rho}}$ is a local complete intersection, flat over $\mathcal{O}$ and of relative dimension $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$.*
2. *$R^{\square,\psi}_{\overline{\rho}}/\varpi$ is a local complete intersection of dimension $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$.*

*Proof.* The pushout of the isomorphism from Proposition 4.3 under $R_{\det \overline{\rho}} \to \mathcal{O}$, which corresponds to $\psi$, gives an isomorphism

$$\mathcal{O}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_t) \xrightarrow{\simeq} R^{\square,\psi}_{\overline{\rho}}$$

with $r - t = (d^2 - 1)([F : \mathbb{Q}_p] + 1)$. To prove (1) and (2) it thus suffices to show that the dimension of $R^{\square,\psi}_{\overline{\rho}}/\varpi$ is at most $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$, or equivalently, see (27), it suffices to show that

$$\dim\bigl((R^{\square,\psi}_{\overline{\rho}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X}))/\varpi\bigr) \le d^2([F : \mathbb{Q}_p] + 1). \tag{28}$$

Let us write $\overline{\mathcal{X}} := \operatorname{Spec} \mathcal{O}(\mathcal{X})/\varpi$. Since $\overline{\varphi}_d : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ is a universal homeomorphism by Lemma 5.3 the map

$$\overline{X}^{\square,\chi} \times_{\overline{\mathcal{X}},\overline{\varphi}_d} \overline{\mathcal{X}} \to \overline{X}^{\square,\chi} \tag{29}$$

is a homeomorphism. In particular, the spaces have the same dimension, which is equal to $d^2([F : \mathbb{Q}_p] + 1)$ by Corollary 4.5. We conclude using Corollary 5.2 that (28) is an equality. □

**Lemma 5.5.** *Let $\kappa$ be either a finite or local field of characteristic $p$ or a finite extension of $L$ and let $\rho : G_F \to \mathrm{GL}_d(\kappa)$ be a representation, such that $\det\rho = \psi$ and $H^2(G_F, \mathrm{ad}^0\rho) = 0$, where $\mathrm{ad}^0\rho$ is the kernel of the trace map. Then the ring $R^{\square,\psi}_{\rho}$ is formally smooth over $\Lambda$ with $\Lambda$ as in Subsection 3.3.*

*Proof.* This is the same proof as the proof of Lemma 4.8. □

**Theorem 5.6.** *The rings $R^{\square,\psi}_{\overline{\rho}}$ and $R^{\square,\psi}_{\overline{\rho}}/\varpi$ are normal integral domains.*

*Proof.* We will first prove that $R^{\square,\psi}_{\overline{\rho}}/\varpi$ is normal. Since $R^{\square,\psi}_{\overline{\rho}}/\varpi$ is complete intersection by Corollary 5.4, it suffices to show that $R^{\square,\psi}_{\overline{\rho}}/\varpi$ satisfies Serre's condition (R1). Let $\mathfrak{p} \in \overline{X}^{\square,\psi} := \operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}/\varpi$ be a point of height at most 1 and assume that the local ring at $\mathfrak{p}$ is not regular. Then by Lemma 5.5 there is a closed irreducible subset $Z$ of $\overline{X}^{\square,\psi}$ of codimension at most 1, the closure of $\mathfrak{p}$, such that for all $z \in Z$ with finite or local residue field the space $H^2(G_F, \mathrm{ad}^0\rho_z)$ is non-zero. Using the explicit bijection from the proof of Proposition 5.1, and the isomorphism of Corollary 5.2 modulo $\varpi$ it follows that there is a closed irreducible subset $W \subset \overline{X}^{\square,\chi} \times_{\overline{\mathcal{X}},\overline{\varphi}_d} \overline{\mathcal{X}}$ of codimension at most 1, such that for all $w \in W$ with finite or local residue field the space $H^2(G_F, \mathrm{ad}^0\rho_w)$ is non-zero, where, as in the proof of Proposition 5.1, the point $w$ corresponds to a pair $(\rho_w, \theta_w)$. Since the map (29) is a homeomorphism and sends $(\rho_w, \theta_w)$ to $\rho_w$, the image of $W$ in $\overline{X}^{\square,\chi}$, which we denote by $W'$, is closed irreducible of codimension at most 1 in $\overline{X}^{\square,\chi}$ and all $x \in W'$ with finite or local residue field have non-vanishing $H^2(G_F, \mathrm{ad}^0\rho_x)$. Lemma 4.16 implies that the codimension of $W'$ is at least 2 yielding a contradiction.

Let us prove that $R^{\square,\psi}_{\overline{\rho}}$ is normal. Since $R^{\square,\psi}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free by Corollary 5.4 and we know that the special fibre is normal, it is enough to prove that $R^{\square,\psi}_{\overline{\rho}}[1/p]$ is normal, see the proof of Proposition 4.18. Lemma 5.3 implies that the map

$$X^{\square,\chi}[1/p] \times_{\mathcal{X}[1/p],\varphi_d} \mathcal{X}[1/p] \to X^{\square,\chi}[1/p] \tag{30}$$

is finite étale. We proceed exactly as in the proof for the special fibre, using (30) instead of (29) and Lemma 4.20 instead of Lemma 4.16. □

**Corollary 5.7.** *The absolutely irreducible locus is dense in* $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}[1/p]$ *and the Kummer-irreducible locus is dense in* $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}/\varpi$.

*Proof.* By Proposition 3.54 and Corollary 3.58 the absolutely irreducible locus is dense open in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ and in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$. Arguing as in the proof of Theorem 5.6 one deduces that the absolutely irreducible locus is dense open in the spaces $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}/\varpi$ and $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}[1/p]$. For absolutely irreducible $x \in \operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ Kummer-irreducibility implies $H^2(G_F, \mathrm{ad}^0\rho_x) = 0$, so the assertion on the density of the Kummer-irreducible locus in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ follows from the proof of Theorem 5.6. □

As explained in Section 4 both $R^{\mathrm{ps}}$ and $A^{\mathrm{gen}}$ are naturally $R_{\det\overline{D}}$-algebras. Moreover, $\det\overline{D} = \det\overline{\rho}$. We let

$$R^{\mathrm{ps},\psi} := R^{\mathrm{ps}} \otimes_{R_{\det\overline{D}},\psi} \mathcal{O}, \quad A^{\mathrm{gen},\psi} := A^{\mathrm{gen}} \otimes_{R_{\det\overline{D}},\psi} \mathcal{O}.$$

**Corollary 5.8.** *The following hold:*

(1) $A^{\mathrm{gen},\psi}$ *is* $\mathcal{O}$*-flat, equi-dimensional of dimension* $1+(d^2-1)([F:\mathbb{Q}_p]+1)$, *normal and is locally complete intersection;*
(2) $A^{\mathrm{gen},\psi}/\varpi$ *is equi-dimensional of dimension* $(d^2-1)([F:\mathbb{Q}_p]+1)$, *normal, and is locally complete intersection.*

*Proof.* The ring $A^{\mathrm{gen}}$ is excellent, since it is a finitely generated over a complete local Noetherian ring. Thus its local rings are also excellent. An excellent local ring is normal if and only if its completion with respect to the maximal ideal is normal, [34, Theorem 32.2 (i)]. Given this the proof is the same as the proof of Corollary 4.6 using Theorem 5.6. □

**Corollary 5.9.** *The rings* $A^{\mathrm{gen},\psi}$ *and* $A^{\mathrm{gen},\psi}/\varpi$ *are integral domains.*

*Proof.* The proof is the same as the proof of Proposition 4.26. □

**Corollary 5.10.** *The ring* $R^{\mathrm{ps},\psi}[1/p]$ *and the rigid space* $(\mathrm{Spf}\, R^{\mathrm{ps},\psi})^{\mathrm{rig}}$ *are normal. The ring* $R^{\mathrm{ps},\psi}[1/p]$ *is an integral domain.*

*Proof.* This follows from [40, Corollary A.1]. The last part is proved in the same way as Corollary 4.27 using Corollary 5.9. □

**Corollary 5.11.** *The maximal reduced quotient of* $R^{\mathrm{ps},\psi}$ *is equal to the maximal* $\mathcal{O}$*-torsion free quotient of* $R^{\mathrm{ps},\psi}$ *and is an integral domain. Moreover, if* $\overline{D}$ *is multiplicity free then* $R^{\mathrm{ps},\psi}$ *is an* $\mathcal{O}$*-torsion free integral domain.*

*Proof.* This is proved in the same way as Corollary 4.28. □

**Proposition 5.12.** *The map*

$$R_{\det\overline{\rho}} \to R^{\square}_{\overline{\rho}} \tag{31}$$

*is flat.*

*Proof.* As in the proof of Proposition 4.3 let $S := \mathcal{O}[\![z, x_1, \dots, x_{1+[F:\mathbb{Q}_p]}]\!]$ then we may choose presentations

$$R_{\det\overline{\rho}} \cong S/((1+z)^m-1), \quad R^{\square}_{\overline{\rho}} \cong S[\![y_1,\dots,y_r]\!]/((1+z)^m-1, f_1,\dots,f_t),$$

such that (31) is a map of $S$-algebras and $(1+z)^m-1, f_1, \dots, f_t$ is a regular sequence in $S[\![y_1,\dots,y_r]\!]$. Let $S' := S[\![y_1,\dots,y_r]\!]/(f_1,\dots,f_t)$. Then $S'$ is complete intersection, and hence Cohen–Macaulay, and the fibre ring $k\otimes_S S'$ is isomorphic to $R^{\square,\psi}_{\overline{\rho}}/\varpi$, which has dimension equal to $\dim R^{\square}_{\overline{\rho}} - \dim R_{\det\rho} = \dim S' - \dim S$, by Corollary 5.4. Since $S$ is regular, the fibre-wise criterion for flatness, [34, Theorem 23.1], implies that $S'$ is flat over $S$. Hence, $R^{\square}_{\overline{\rho}} \cong S'/((1+z)^m-1)$ is flat over $R_{\det\overline{\rho}} \cong S/((1+z)^m-1)$. □

## 6. Density of points with prescribed $p$-adic Hodge theoretic properties

We fix $\overline{\rho}: G_F \to \mathrm{GL}_d(k)$, let $\rho^{\square}: G_F \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}})$ be its universal framed deformation ring and let $X^{\square} = \mathrm{Spec}\, R^{\square}_{\overline{\rho}}$. If $x: R^{\square}_{\overline{\rho}} \to \overline{\mathbb{Q}}_p$ is an $\mathcal{O}$-algebra homomorphism we denote by $\rho^{\square}_x: G_F \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ the specialization of $\rho^{\square}$ at $x$. In this section we will study Zariski closures of subsets $\Sigma \subset X^{\square}(\overline{\mathbb{Q}}_p)$, such that $\rho^{\square}_x$ is potentially semi-stable for all $x\in\Sigma$ and satisfies additional conditions imposed on either the Hodge–Tate weights or the inertial type of $\rho^{\square}_x$. Recall that the Hodge–Tate weights $\mathrm{HT}(\rho)$ of a potentially semi-stable representation $\rho$ is a $d$-tuple of integers $\underline{k} = (k_{\sigma,1} \geq k_{\sigma,2} \geq \ldots \geq k_{\sigma,d})$ for each embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$ and we say that $\underline{k}$ is regular if all the inequalities are strict. Let

$$\Sigma^{\mathrm{cris}} := \{x \in X^{\square}(\overline{\mathbb{Q}}_p) : \rho_x^{\square} \text{ is crystalline with regular Hodge–Tate weights}\}.$$

For a fixed regular Hodge–Tate weight $\underline{k}$ we let

$$\Sigma_{\underline{k}} := \{x \in X^{\square}(\overline{\mathbb{Q}}_p) : \rho_x^{\square} \text{ is potentially semi-stable with } \mathrm{HT}(\rho_x^{\square}) = \underline{k}\}.$$

If $\rho$ is potentially semi-stable then to it we may attach a Weil–Deligne representation $\mathrm{WD}(\rho)$ and to it via the classical Langlands correspondence we may attach a smooth irreducible representation of $\mathrm{GL}_d(F)$, which we denote by $\mathrm{LL}(\mathrm{WD}(\rho))$.

Let $\Sigma_{\underline{k}}^{\mathrm{prnc}}$ be the subset of $\Sigma_{\underline{k}}$, such that $x \in \Sigma_{\underline{k}}$ lies in $\Sigma_{\underline{k}}^{\mathrm{prnc}}$ if and only if $\mathrm{LL}(\mathrm{WD}(\rho_x^{\square}))$ is a principal series representation. In terms of the Galois side $\Sigma_{\underline{k}}^{\mathrm{prnc}}$ may be characterised as the set of $x \in \Sigma_{\underline{k}}$ such that the restriction of $\rho_x^{\square}$ to the Galois group of some finite abelian extension of $F$ is crystalline.

Let $\Sigma_{\underline{k}}^{\mathrm{spcd}}$ be the subset of $\Sigma_{\underline{k}}$ such that $x$ lies in $\Sigma_{\underline{k}}^{\mathrm{spcd}}$ if and only if $\mathrm{WD}(\rho_x)$ is irreducible as a representation of the Weil group $W_F$ of $F$, and is induced from a 1-dimensional representation of $W_E$, where $E$ is an unramified extension of $F$ of degree $d$. In this case, $\mathrm{LL}(\mathrm{WD}(\rho_x))$ is a supercuspidal representation of $\mathrm{GL}_d(F)$.

The goal of this section is the following theorem.

**Theorem 6.1.** *Assume that $p \nmid 2d$. Let $\Sigma$ be any of the sets $\Sigma^{\mathrm{cris}}$, $\Sigma_{\underline{k}}^{\mathrm{prnc}}$, $\Sigma_{\underline{k}}^{\mathrm{spcd}}$ defined above. Then $\Sigma$ is Zariski dense in $X^{\square}$.*

*Remark* 6.2. We could additionally require the representations in $\Sigma^{\mathrm{cris}}$ to be benign in the sense of [24, Definition 6.8] or instead of considering crystalline representations fix an inertial type.

One could also change the definition of $\Sigma_{\underline{k}}^{\mathrm{spcd}}$ to allow $E$ to be a ramified extension of $F$, see [24, Section 5.3].

The problem for $\Sigma^{\mathrm{cris}}$ has been studied by Colmez [18], Kisin [32], Chenevier [16], Nakamura [37], [38]. Hellmann and Schraen have studied the problem for $\Sigma_{\underline{k}}^{\mathrm{prnc}}$ and $\Sigma^{\mathrm{cris}}$ in [27]. Emerton and VP have studied the problem for $\Sigma^{\mathrm{cris}}$, $\Sigma_{\underline{k}}^{\mathrm{prnc}}$ and $\Sigma_{\underline{k}}^{\mathrm{spcd}}$ in [24]. A common feature of these papers is that they show that the closure of $\Sigma$ is a union of irreducible components of $X^{\square}$ and density is equivalent to showing that $\Sigma$ meets each irreducible component. If one knows the irreducible components then one might hope to show density this way. This strategy has been carried out for $\Sigma^{\mathrm{cris}}$ by Colmez–Dospinescu–VP in [19] for $p = d = 2$ and $F = \mathbb{Q}_p$ and by AI in [29] for $p > d$ and $F$ arbitrary, when $\overline{\rho}$ is the trivial representation, where after determining irreducible components one can write down the lifts explicitly. We note that using Corollary 4.21 one may remove the assumption $p > d$ in [29, Theorem 5.11]. It seems impossible to carry this out for arbitrary $\underline{k}$ and $\overline{\rho}$ directly, even if one knows that the irreducible components of $X^{\square}$ are in bijection with irreducible components of $\operatorname{Spec} R_{\det \overline{\rho}}$. Instead we combine our knowledge of irreducible components with results of [24].

The paper [24] builds on the global patching arguments carried out in [13], which assumes that $p \nmid 2d$ and $\overline{\rho}$ has a potentially diagonalisable lift. This last condition can be easily verified if $\overline{\rho}$ is semi-simple, see [13, Lemma 2.2]; it has been shown to always be satisfied in [23, Theorem 1.2.2]. The output of [13] is a complete local Noetherian $\mathcal{O}$-algebra $R_\infty$ with residue field $k$ and a linearly compact $R_\infty$-module

$M_\infty$, which carries a continuous, $R_\infty$-linear action of $G := \mathrm{GL}_d(F)$. Moreover, the action of $R_\infty[K]$ on $M_\infty$ extends (uniquely) to a continuous action of the completed group algebra $R_\infty[\![K]\!]$, where $K := \mathrm{GL}_d(\mathcal{O}_F)$, so that $M_\infty$ is a finitely generated $R_\infty[\![K]\!]$-module.

**Lemma 6.3.** *We have an isomorphism of $R^{\square}_{\bar\rho}$-algebras:*

$$R_\infty \cong R^{\square}_{\bar\rho} \widehat{\otimes}_{\mathcal{O}} A,$$

*where $A$ is a complete local Noetherian $\mathcal{O}$-algebra, which is $\mathcal{O}$-torsion free, reduced and equi-dimensional. Thus the ring $R_\infty$ is a reduced, $\mathcal{O}$-torsion free and flat $R^{\square}_{\bar\rho}$-algebra.*

*After replacing $L$ by a finite extension, the irreducible components of $\operatorname{Spec} R_\infty$ are of the form $\operatorname{Spec}(R^{\square,\chi}_{\bar\rho} \widehat{\otimes}_{\mathcal{O}} A/\mathfrak{p})$, for a character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ and a minimal prime $\mathfrak{p}$ of $A$. Moreover, distinct pairs $(\chi, \mathfrak{p})$ give rise to distinct irreducible components of $\operatorname{Spec} R_\infty$.*

*Proof.* The ring $R_\infty$ is defined in [13, Section 2.8] and is formally smooth over the ring denoted by $R^{\mathrm{loc}}$ in [13, Section 2.6]. The ring $R^{\mathrm{loc}}$ is a completed tensor product over $\mathcal{O}$ of $R^{\square}_{\bar\rho}$, the ring $R^{\square}_{\tilde v_1}$, which is formally smooth over $\mathcal{O}$ by [13, Lemma 2.5], and potentially semi-stable rings at other places above $p$, denoted by $R^{\square,\xi,\tau}_{\tilde v}$ in [13, Section 2.4]. These are $\mathcal{O}$-torsion free, reduced and equi-dimensional by [31, Theorem 3.3.8]. Thus $R_\infty \cong R^{\square}_{\bar\rho} \widehat{\otimes} A$, where $A$ is formally smooth over the ring $\widehat{\otimes}_{v\in S_p\setminus\mathfrak{p}} R^{\square,\xi,\tau}_{\tilde v}$ in the notation of [13]. Since the rings $R^{\square,\xi,\tau}_{\tilde v}$ are $\mathcal{O}$-torsion free, reduced and equi-dimensional, so is the ring $A$ by [13, Corollary A.2] and [28, Lemma A.1]. Since $R^{\square}_{\bar\rho}$ is also $\mathcal{O}$-torsion free, reduced and equi-dimensional we obtain that the same holds for $R_\infty$. Since $A$ is $\mathcal{O}$-torsion free, $R_\infty$ is a flat $R^{\square}_{\bar\rho}$-algebra.

It follows from [28, Lemma A.5] that after replacing $L$ with a finite extension, we may assume that for all minimal primes $\mathfrak{p}$ of $A$, the quotient $A/\mathfrak{p}$ is geometrically integral, by which we mean that $(A/\mathfrak{p}) \otimes_{\mathcal{O}} \mathcal{O}_{L'}$ is integral domain for all finite extensions $L'/L$. If $\mathfrak{p}'$ is a minimal prime of $R^{\square}$ then $R^{\square}_{\bar\rho}/\mathfrak{p}' = R^{\square,\chi}_{\bar\rho}$ for a unique character $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$. The moduli interpretation of $R^{\square,\chi}_{\bar\rho}$ shows that the ring is geometrically integral. It follows from [4, Lemma 3.3 (5)] that the minimal primes $\mathfrak{q}$ of $R_\infty$ are of the form $\mathfrak{p}'(R^{\square}_{\bar\rho} \widehat{\otimes}_{\mathcal{O}} A) + \mathfrak{p}(R^{\square}_{\bar\rho} \widehat{\otimes}_{\mathcal{O}} A)$, where $\mathfrak{p}'$ is the image of $\mathfrak{q}$ in $\operatorname{Spec} R^{\square}_{\bar\rho}$ and $\mathfrak{p}$ is the image of $\mathfrak{q}$ in $\operatorname{Spec} A$. This implies the last assertion. $\square$

In our arguments we will not invoke the assumption $p \nmid 2d$, since eventually this restriction used in construction of $M_\infty$ should become redundant. In particular, the next two Lemmas do not use this assumption.

**Lemma 6.4.** *Let $\psi : G_F \to \mathcal{O}^\times$ be a character such that $\psi$ is trivial on the torsion subgroup of $G_F^{\mathrm{ab}}$. Then after replacing L by a finite extension we may find a character $\eta : G_F \to \mathcal{O}^\times$ such that $\eta^d = \psi$.*

*Proof.* It follows from local class field theory that the maximal torsion-free quotient of $G_F^{\mathrm{ab}}$ is isomorphic to $\widehat{\mathbb{Z}} \times \mathbb{Z}_p^m$, where $m = [F : \mathbb{Q}_p]$. We choose topological generators $\gamma_1, \dots, \gamma_{m+1}$, where $\gamma_1$ is a generator of $\widehat{\mathbb{Z}}$. Let $\overline{\psi(\gamma_1)}$ be the image of $\psi(\gamma_1)$ in $k$. If it is not equal to 1 then choose $\lambda \in \bar k$ such that $\lambda^d = \overline{\psi(\gamma_1)}$. We enlarge $L$, so that the residue field contains $\lambda$ and let $\mu : \widehat{\mathbb{Z}} \times \mathbb{Z}_p^m \to \widehat{\mathbb{Z}} \to \mathcal{O}^\times$ be the unramified character, such that $\mu(\gamma_1)$ is equal to the Teichmüller lift of $\lambda$. After replacing $\psi$ with $\psi\mu^{-d}$ we may assume that $\psi(\gamma_1) \equiv 1 \pmod{\varpi}$. Thus we may view $\psi$ as a character on $\mathbb{Z}_p^{m+1}$ and $\psi(\gamma_i) \equiv 1 \pmod{\varpi}$ for $1 \le i \le m+1$. After enlarging $L$ we may find $y_i \in (\varpi)$ such that $(1+y_i)^d = \psi(\gamma_i)$. Since the series $(1+y_i)^x := \sum_{n=0}^{\infty} \binom{x}{n} y_i^n$ converges for all $x \in \mathbb{Z}_p$, we may define $\eta$ on $\mathbb{Z}_p^{m+1}$ by sending $\gamma_i$ to $1+y_i$ and then inflate it to $G_F$. □

**Lemma 6.5.** *Let $\kappa : G_F \to \mathcal{O}^\times$ be a character. Then there is a crystalline character $\psi : G_F \to \mathcal{O}^\times$ such that $\psi\kappa^{-1}$ is trivial on the torsion part of $G_F^{\mathrm{ab}}$.*

*In particular, given characters $\overline{\kappa} : G_F \to k^\times$ and $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ there exists a crystalline character $\psi : G_F \to \mathcal{O}^\times$ lifting $\overline{\kappa}$ such that $\psi(\mathrm{Art}_F(z)) = \chi(z)$ for all $z \in \mu_{p^\infty}(F)$.*

*Proof.* The Artin map $\mathrm{Art}_F : F^\times \to G_F^{\mathrm{ab}}$ of local class field theory allows us to identify characters $\psi : G_F \to \mathcal{O}^\times$ with characters $\psi : F^\times \to \mathcal{O}^\times$. Under this identification $\psi$ is crystalline if and only if $\psi(x) = \prod_{\sigma : F \hookrightarrow L} \sigma(x)^{n_\sigma}$ for some integers $n_\sigma$ and for all $x \in \mathcal{O}_F^\times$ by [21, Proposition B.4].

Let $\zeta$ be a generator of the torsion subgroup of $F^\times$ and let $m$ be the multiplicative order of $\zeta$. Let $\xi$ be a primitive $m$-th root of unity in $L$. Then $\kappa(\zeta) = \xi^a$ for some integer $a$. Let $\sigma : F \hookrightarrow L$ be an embedding such that $\sigma(\zeta) = \xi$. Let $\psi : F^\times \to \mathcal{O}^\times$ be the character $\psi(x) = \sigma(x\varpi_F^{-v(x)})^a$ for all $x \in F^\times$, where $v$ is a valuation on $F$ normalized so that $v(\varpi_F) = 1$. Then $\psi\kappa^{-1}(\zeta) = 1$ and hence $\psi\kappa^{-1}$ is trivial on the torsion subgroup of $F^\times$. Moreover, $\psi$ is crystalline by the above. Note that $\psi\kappa^{-1} \equiv 1 \pmod{\varpi}$.

For the last part, we may choose any $\kappa : G_F \to \mathcal{O}^\times$ lifting $\overline{\kappa}$ and satisfying $\kappa(\mathrm{Art}_F(z)) = \chi(z)$ for all $z \in \mu_{p^\infty}(F)$ and apply the previous part. □

**Lemma 6.6.** *Let $\psi : G_F \to \mathcal{O}^\times$ be a character lifting $\det\overline{\rho}$ and let $x : R_{\det\overline{\rho}} \to \mathcal{O}$ be the corresponding $\mathcal{O}$-algebra homomorphism. Then the centre $Z$ of $G$ acts on $M_\infty \otimes_{R_{\det\overline{\rho}},x} \mathcal{O}$ via the character $\delta^{-1}$, where $\delta : Z \to \mathcal{O}^\times$ is the composition*

$$Z \xrightarrow{\cong} F^\times \xrightarrow{\mathrm{Art}_F} G_F^{\mathrm{ab}} \xrightarrow{\varepsilon^{d(d-1)/2}\psi} \mathcal{O}^\times,$$

*where $\varepsilon$ is the $p$-adic cyclotomic character.*

*Moreover, $M_\infty \otimes_{R_{\det\overline{\rho}},x} \mathcal{O}$ is non-zero and projective in the category of linearly compact $\mathcal{O}[\![K]\!]$-modules on which $Z \cap K$ acts by $\delta^{-1}$.*

*Further, if $\psi$ is crystalline then there is an algebraic character $\theta : \mathrm{Res}^F_{\mathbb{Q}_p} \mathbb{G}_m \to \mathbb{G}_m$ defined over $L$ such that $\delta|_{K\cap Z}$ is equal to the composition*

$$\mathcal{O}_F^\times \hookrightarrow (\mathrm{Res}^F_{\mathbb{Q}_p} \mathbb{G}_m)(\mathbb{Q}_p) \to (\mathrm{Res}^F_{\mathbb{Q}_p} \mathbb{G}_m)(L) \xrightarrow{\theta} \mathbb{G}_m(L),$$

*where $\mathrm{Res}^F_{\mathbb{Q}_p}$ denotes the restriction of scalars.*

*Proof.* It follows from the discussion at the beginning of Section 4.22 in [13] that $Z$ acts via $\delta$ on the Pontryagin dual of $M_\infty \otimes_{R_{\det\overline{\rho}},x} \mathcal{O}$. Hence it acts on $M_\infty \otimes_{R_{\det\overline{\rho}},x} \mathcal{O}$ via $\delta^{-1}$. The second part follows from [13, Corollary 4.26]. The last part follows from [21, Proposition B.4] as explained in the proof of Lemma 6.5. □

If $V$ is a continuous representation of $K$ on a finite dimensional $L$-vector space then we define a finitely generated $R_\infty[1/p]$-module $M_\infty(V)$ as follows. Since $K$ is compact it stabilizes a bounded $\mathcal{O}$-lattice $\Theta$ in $V$. Let

$$M_\infty(\Theta) := (\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M_\infty, \Theta^d))^d,$$

where $(\cdot)^d := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\cdot, \mathcal{O})$. Then $M_\infty(\Theta)$ is a finitely generated $R_\infty$-module. The module $M_\infty(V) := M_\infty(\Theta) \otimes_{\mathcal{O}} L$ does not depend on the choice of a lattice $\Theta$.

We will denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible algebraic representation of $(\mathrm{Res}^F_{\mathbb{Q}_p} \mathrm{GL}_d)_L$ defined over $L$. If $\xi \in \mathrm{Irr}(G)$ then we will consider it as a representation of $K$ by evaluating at $L$ and letting $K$ act via the composition

$$K \hookrightarrow (\mathrm{Res}^F_{\mathbb{Q}_p} \mathrm{GL}_d)(\mathbb{Q}_p) \to (\mathrm{Res}^F_{\mathbb{Q}_p} \mathrm{GL}_d)(L).$$

If $M$ is a compact $\mathcal{O}$-module then we define an $L$-Banach space

$$\Pi(M) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L),$$

equipped with supremum norm. If $M$ is a compact $\mathcal{O}[\![K]\!]$-module, then the action of $K$ on $M$ makes $\Pi(M)$ into a unitary $L$-Banach space representation of $K$. For example, the map $K \to \mathcal{O}[\![K]\!]$ induces an isomorphism of unitary $L$-Banach space representations $\Pi(\mathcal{O}[\![K]\!]) \cong \mathcal{C}(K, L)$, the space of continuous functions from $K$ to $L$, with $K$-action given by left translations, [43, Corollary 2.2].

**Lemma 6.7.** *Let $\theta : \mathrm{Res}^F_{\mathbb{Q}_p} \mathbb{G}_m \to \mathbb{G}_m$ be an algebraic character defined over $L$, and let $\delta : Z \cap K \to \mathcal{O}^\times$ be the character associated to $\theta$ in Lemma 6.6. Let $M$ be non-zero and projective in the category of linearly compact $\mathcal{O}[\![K]\!]$-modules on which $Z \cap K$ acts by $\delta^{-1}$. Then there is $\xi \in \mathrm{Irr}(G)$ such that $\mathrm{Hom}^{\mathrm{cont}}_K(M, \xi^*) \neq 0$.*

*Proof.* We may assume that $M$ is a direct summand of $\mathcal{O}[\![K]\!] \widehat{\otimes}_{\mathcal{O}[\![K\cap Z]\!], \delta^{-1}} \mathcal{O}$, since an arbitrary projective module is isomorphic to a product of indecomposable projectives, and these are direct summands of $\mathcal{O}[\![K]\!] \widehat{\otimes}_{\mathcal{O}[\![K\cap Z]\!], \delta^{-1}} \mathcal{O}$. Then the Banach space $\Pi(M)$ is a non-zero direct summand $\mathcal{C}_\delta(K, L)$, the subspace of $\mathcal{C}(K, L)$ on which $Z \cap K$ acts by $\delta$.

Using the theory of highest weight we may find $\tau \in \mathrm{Irr}(G)$ such that the central character of $\tau$ is equal to $\theta$. It follows from [22, Corollary 7.8] that the evaluation map

$$\bigoplus_{\xi' \in \mathrm{Irr}(G/Z)} \tau \otimes \xi' \otimes \mathrm{Hom}^{\mathrm{cont}}_K(\tau \otimes \xi', \Pi(M)) \to \Pi(M)$$

has dense image. Thus there is $\xi' \in \mathrm{Irr}(G/Z)$ and an irreducible summand $\xi$ of $\tau \otimes \xi'$ such that $\mathrm{Hom}^{\mathrm{cont}}_K(\xi, \Pi(M)) \neq 0$. Dually, this implies $\mathrm{Hom}^{\mathrm{cont}}_K(M, \xi^*) \neq 0$. $\square$

**Theorem 6.8.** *The action of $R^\square_{\overline{\rho}}$ on $M_\infty$ is faithful.*

*Proof.* Let $\mathfrak{p}$ be a minimal prime of $R^\square_{\overline{\rho}}$. We have shown in Corollary 4.21 that there is a character $\chi : \mu_{p^\infty}(F) \to L^\times$ such that $R^\square_{\overline{\rho}}/\mathfrak{p} = R^{\square,\chi}_{\overline{\rho}}$. It follows from Lemma 6.5 that there is a crystalline character $\psi : G_F \to \mathcal{O}^\times$ lifting $\det \overline{\rho}$ such that $\psi(\mathrm{Art}_F(z)) = \chi(z)$ for all $z \in \mu_{p^\infty}(F)$. Let $x : R_{\det \overline{\rho}} \to \mathcal{O}$ be the corresponding $\mathcal{O}$-algebra homomorphism. It follows from Lemmas 6.6, 6.7 that there is $\xi \in \mathrm{Irr}(G)$ such that

$$\mathrm{Hom}^{\mathrm{cont}}_K(M_\infty \otimes_{R_{\det \overline{\rho}}, x} \mathcal{O}, \xi^*) \neq 0.$$

This implies that $M_\infty(\xi) \otimes_{R_{\det \overline{\rho}}, x} \mathcal{O} \neq 0$.

Let $\mathfrak{a}$ be the $R_\infty$ annihilator of $M_\infty$. In [24, Theorem 6.12] it is shown, following the approach of Chenevier [16] and Nakamura [38], that the closure in $\mathrm{Spec}\, R_\infty$

of the union of the supports of $M_\infty(\xi')$ for all $\xi' \in \mathrm{Irr}(G)$ is a union of irreducible components of $\operatorname{Spec} R_\infty$. Thus there is a minimal prime $\mathfrak{q}$ of $R_\infty$ such that $\operatorname{Supp} M_\infty(\xi) \subset V(\mathfrak{q}) \subset V(\mathfrak{a})$.

Since $M_\infty(\xi) \otimes_{R_{\det \overline{\rho}, x}} \mathcal{O} \neq 0$, Lemma 6.3 implies that the image of $\mathfrak{q}$ in $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ is equal to $\mathfrak{p}$. Thus $\mathfrak{p}$ contains $\mathfrak{a} \cap R^{\square}_{\overline{\rho}}$, which is the $R^{\square}_{\overline{\rho}}$-annihilator of $M_\infty$. Since $R^{\square}_{\overline{\rho}}$ is reduced, the intersection of all minimal prime ideals is zero and hence $R^{\square}_{\overline{\rho}}$ acts faithfully on $M_\infty$. $\square$

*Proof of Theorem 6.1.* This is proved in the same way as [24, Theorems 5.1, 5.3]. Let us sketch the proof in the case of $\Sigma^{\mathrm{cris}}$ for the convenience of the reader. For each $\xi \in \mathrm{Irr}(G)$ let $\mathfrak{a}_\xi$ be the $R^{\square}_{\overline{\rho}}$-annihilator of $M_\infty(\xi)$. It follows from [13, Lemma 4.18] that $R^{\square}_{\overline{\rho}}/\mathfrak{a}_\xi$ is a quotient of the crystalline deformation ring of $\overline{\rho}$ with Hodge–Tate weights corresponding to the highest weight of $\xi$, see [13, Section 1.8], [22, Remark 5.14]. Moreover, $R^{\square}_{\overline{\rho}}/\mathfrak{a}_\xi$ is a union of irreducible components of that ring. This implies that $R^{\square}_{\overline{\rho}}/\mathfrak{a}_\xi$ is reduced and $\mathcal{O}$-torsion free. The set $\Sigma^{\mathrm{cris}}$ contains the set of maximal ideals of $(R^{\square}_{\overline{\rho}}/\mathfrak{a}_\xi)[1/p]$. Since $(R^{\square}_{\overline{\rho}}/\mathfrak{a}_\xi)[1/p]$ is Jacobson, if $a \in R^{\square}_{\overline{\rho}}$ is contained in the intersection of all maximal ideals in $\Sigma^{\mathrm{cris}}$ then $a$ will annihilate $M_\infty(\xi)$ for all $\xi \in \mathrm{Irr}(G)$. The continuous $L$-linear dual of $M_\infty(\xi)$ can be identified with $\operatorname{Hom}_K(\xi, \Pi(M_\infty))$. The key point is that the image of the evaluation map

$$\bigoplus_{\xi \in \mathrm{Irr}(G)} \xi \otimes_L \operatorname{Hom}_K(\xi, \Pi(M_\infty)) \to \Pi(M_\infty) \tag{32}$$

is dense. Thus $a$ will annihilate the left hand side of (32), and by density it will annihilate $\Pi(M_\infty)$. The continuous $L$-linear dual of $\Pi(M_\infty)$ can be identified with $M_\infty[1/p]$. Since $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free and $R^{\square}_{\overline{\rho}}$ acts faithfully on $M_\infty$ by Theorem 6.8 we deduce that $a = 0$.

If $\Sigma = \Sigma^{\mathrm{prnc}}_{\underline{k}}$ or $\Sigma^{\mathrm{spcd}}_{\underline{k}}$ then the argument is the same, except that instead of considering all $\xi \in \mathrm{Irr}(G)$ one fixes $\xi \in \mathrm{Irr}(G)$, such that the highest weight of $\xi$ corresponds to the Hodge–Tate weights $\underline{k}$ and one considers the family $\xi \otimes_L V$, where $V$ are principal series or appropriate supercuspidal types, see the proof of Theorems 5.1, 5.3 in [24] for more details. $\square$

*Remark* 6.9. It is natural to ask whether the ring $R_\infty$ acts faithfully on $M_\infty$. We cannot answer this question in general, since it boils down to whether every irreducible component of the potentially semi-stable rings $R^{\square,\xi,\tau}_{\tilde{v}}$ (see the proof of Lemma 6.3, where $v \in S_p \setminus \mathfrak{p}$ is a place above $p$, different from the place at which the patching construction is carried out) has a point corresponding to an automorphic Galois representation. These questions are connected with modularity lifting theorems and the Fontaine–Mazur conjecture, see [13, Remark 4.20].

However, if all $R^{\square,\xi,\tau}_{\tilde{v}}$ were integral domains then the ring $A$ in Lemma 6.3 would also be an integral domain, and we would deduce from the proof of Theorem 6.8 that $R_\infty$ acts faithfully on $M_\infty$. A further possibility is to avoid the modularity lifting related issues by carrying out the patching construction of [13] at all places above $p$ at once. Then the proof of Theorem 6.8 would carry over in this new setting to show that $R_\infty$ acts faithfully on $M_\infty$.

## Appendix A. Kummer-irreducible points

The purpose of the appendix is to slightly generalize the notion of non-special points in $\overline{X}^{\mathrm{ps}} = \operatorname{Spec} R^{\mathrm{ps}}/\varpi$ in [8, Definition 5.1.2]. We use the notation of the main text. If $x \in \overline{X}^{\mathrm{ps}}$ then we let $D_x = D^u \otimes_{R^{\mathrm{ps}}} \overline{\kappa(x)}$, where $\overline{\kappa(x)}$ is an algebraic closure of the residue field at $x$, and we let $\rho_x : G_F \to \mathrm{GL}_d(\overline{\kappa(x)})$ be the semisimple representation whose determinant is $D_x$.

**Definition A.1.** We say that $x \in P_1(R^{\mathrm{ps}}/\varpi)$ is *Kummer-irreducible* if the restriction of $D_x$ to $G_{F'}$ is absolutely irreducible for all degree $p$ Galois extensions $F'$ of $F(\zeta_p)$. Otherwise, we say that $x$ is *Kummer-reducible.*

Thus $x$ is Kummer-irreducible if and only if $\rho_x|_{G_{F(\zeta_p)}}$ is non-special in the sense [8, Definition 5.1.2]. In particular, if $\zeta_p \in F$ then both notions coincide. Our main interest in this notion is the following Lemma.

**Lemma A.2.** *If $x$ is Kummer-irreducible then $H^2(G_F, \operatorname{ad}^0 \rho_x) = 0$.*

*Proof.* Since the order of $\mathrm{Gal}(F(\zeta_p)/F)$ is prime to $p$ we have

$$H^2(G_F, \operatorname{ad}^0 \rho_x) \cong H^2(G_{F(\zeta_p)}, \operatorname{ad}^0 \rho_x)^{\mathrm{Gal}(F(\zeta_p)/F)}.$$

Since $x$ is Kummer-irreducible, the restriction of $\rho_x$ to $G_{F(\zeta_p)}$ is non-special, and it follows from [8, Lemma 5.1.1] that $H^2(G_{F(\zeta_p)}, \operatorname{ad}^0 \rho_x) = 0$. $\square$

We need the following result from Clifford theory.

**Lemma A.3.** *Let $G$ be a group and $H \subset G$ a normal subgroup such that $G/H$ is cyclic of order $m$. Let $\kappa$ be an algebraically closed field with $\mathrm{char}(\kappa)$ not dividing $m$. Let $V$ be a finite dimensional irreducible representation of $G$ over $\kappa$. Suppose the restriction $\operatorname{Res}^G_H V$ of $V$ to $H$ is reducible. Then there exists a proper subgroup $H^*$ of $G$ that contains $H$ and an irreducible representation $W$ of $H^*$ such that $V \cong \operatorname{Ind}^G_{H^*} W$.*

*Proof.* Let $X$ be the group of characters $\chi : G/H \to \kappa^\times$. We have

$$\operatorname{Ind}^G_H \operatorname{Res}^G_H V \cong V \otimes_\kappa \operatorname{Ind}^G_H \mathbf{1} \cong \bigoplus_{\chi \in X} V \otimes \chi.$$

Since $\operatorname{Res}^G_H V$ is semi-simple by [8, Lemma 2.1.4 (d)] and reducible by assumption

$$\dim_\kappa \operatorname{Hom}_H(\operatorname{Res}^G_H V, \operatorname{Res}^G_H V) > 1.$$

Frobenius reciprocity implies that $V \cong V \otimes \chi$ for some non-trivial $\chi \in X$. The lemma now follows from [8, Theorem 2.2.1] with $H^* = \ker(\chi)$. $\square$

Let $\overline{X}^{\mathrm{ps,irr}} \subset \overline{X}^{\mathrm{ps}} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ be the absolutely irreducible locus; it is denoted by $U_{\mathcal{P}_{\min}}$ in Section 3.5.

**Lemma A.4.** *Let $X^{\mathrm{Kred}}_{F(\zeta_p)} \subset \overline{X}^{\mathrm{ps,irr}}$ be the closure of all $x \in \overline{X}^{\mathrm{ps,irr}} \cap P_1(R^{\mathrm{ps}}/\varpi)$ such that $D_x|_{G_{F(\zeta_p)}}$ is reducible. Then $\dim \overline{X}^{\mathrm{ps,irr}} - \dim X^{\mathrm{Kred}}_{F(\zeta_p)} \geq \frac{1}{2} d^2 [F : \mathbb{Q}_p] \geq 2$.*

*Proof.* Let $y \in X^{\mathrm{Kred}}_{F(\zeta_p)}$ and let $\mathfrak{p}_y$ be the kernel of $R^{\mathrm{ps}} \to \kappa(y)$. As a byproduct of the proof of Lemma 3.23 we obtain a description of the universal object of over $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$, which shows that $D_y|_{G_{F(\zeta_p)}}$ is reducible. Since $y$ lies in the absolutely irreducible locus $D_y$ and $\rho_y$ are irreducible. Now Lemma A.3 yields a subextension $F'$ of

$F(\zeta_p)/F$ such that $\rho_y$ is induced from $G_{F'}$ and $[F':F]>1$. It follows from [8, Lemma 5.3.2] and the proof of [8, Theorem 5.3.1] that

$$\dim R^{\mathrm{ps}}/\mathfrak{p}_y \leq \frac{1}{[F':F]} d^2[F:\mathbb{Q}_p]+1.$$

This yields $\dim X^{\mathrm{Kred}}_{F(\zeta_p)} \leq d^2[F:\mathbb{Q}_p]$ by applying the above to the generic points $y$ of $X^{\mathrm{Kred}}_{F(\zeta_p)}$ and observing that $\mathfrak{m}_{R^{\mathrm{ps}}} \notin X^{\mathrm{Kred}}_{F(\zeta_p)}$. We also have $\dim \overline{X}^{\mathrm{ps,irr}} = [F:\mathbb{Q}_p]d^2$ by [8, Theorem 5.4.1]. Since $[F':F]\geq 2$ we obtain the assertion. □

*Remark* A.5. Let $F'\supset F$ be finite. Arguing as in the proof of [8, Lemma 5.1.3], one finds that $Z_{F'}=\{x\in \overline{X}^{\mathrm{ps,irr}} : D_x|_{G_{F'}} \text{ is reducible}\}$ is a closed subset of $\overline{X}^{\mathrm{ps,irr}}$. As $\overline{X}^{\mathrm{ps}}\setminus\{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ is Jacobson, one has $X^{\mathrm{Kred}}_{F(\zeta_p)} = Z_{F(\zeta_p)}$. This gives a direct definition of $X^{\mathrm{Kred}}_{F(\zeta_p)}$ avoiding a closure operation.

**Lemma A.6.** *Let $E\subset \overline{F}$ be a finite extension of $F$, denote by $R^{\mathrm{ps}}_E$ the universal ring for deformations of the pseudo-character $\overline{D}|_{G_E}$, and let $r: R^{\mathrm{ps}}_E\to R^{\mathrm{ps}}$ be the ring map induced from the morphism of deformation functors that restricts a pseudo-character of $G_F$ to one of $G_E$. Then $r: R^{\mathrm{ps}}_E\to R^{\mathrm{ps}}$ is finite.*

*Proof.* The map $r$ is a local homomorphism of local rings with residue field $k$. So we need to show that $S=R^{\mathrm{ps}}/\mathfrak{m}_{R^{\mathrm{ps}}_E}R^{\mathrm{ps}}$ has finite $k$-dimension, which amounts to showing $\operatorname{Spec} S=\{\mathfrak{m}_S\}$. We note that $S$ represents the functor of deformations $D: G_F\to A$ of $\overline{D}$ to $k$-algebras $A$ such that $D|_{G_E}=\overline{D}|_{G_E}\otimes_k A$.

Let $y$ be any point of $\operatorname{Spec} S$ with associated pseudo-character $D_y$ and semisimple representation $\rho_y: G_F\to \mathrm{GL}_d(\overline{\kappa(y)})$. The restriction $\rho_y|_{G_E}$ is semisimple, cf. [8, Lemma 2.1.4], and its associated pseudo-character is $\overline{D}|_{G_E}\otimes_k \overline{\kappa(y)}$, so that $\rho_y(G_E)$ is finite. Hence $\rho_y(G_F)$ is finite, and therefore $D_y$ is defined over a finite field $k'\supset k$. This shows that the corresponding ring map $S\to\kappa(y)$ factors via $k'$, and thus its kernel $y$ is the maximal ideal $\mathfrak{m}_S$. □

**Lemma A.7.** *For a degree $p$ Galois extension $F'$ of $F(\zeta_p)$ let $X^{\mathrm{Kred}}_{F'}\subset \overline{X}^{\mathrm{ps,irr}}$ be the closure of the set of $x\in \overline{X}^{\mathrm{ps,irr}}\cap P_1(R^{\mathrm{ps}}/\varpi)$ such that $D_x|_{G_{F'}}$ is reducible. Then*

$$\dim \overline{X}^{\mathrm{ps,irr}} - \dim\Big(\bigcup\nolimits_{F'} X^{\mathrm{Kred}}_{F'}\Big) \geq d[F:\mathbb{Q}_p]\geq 2,$$

*where the union is over all degree $p$ Galois extension $F'$ of $F(\zeta_p)$.*

*Proof.* Let $r: R^{\mathrm{ps}}_{F(\zeta_p)}\to R^{\mathrm{ps}}$ be the ring map from Lemma A.6 with $E=F(\zeta_p)$ and denote by $\overline{X}^{\mathrm{spcl}}_{F(\zeta_p)}\subset \operatorname{Spec}\big(R^{\mathrm{ps}}_{F(\zeta_p)}/\varpi\big)\setminus\{\mathfrak{m}_{R^{\mathrm{ps}}_{F(\zeta_p)}}\}$ the locus of special points as defined in [8, Definition 5.1.2]. Then by Lemma A.6 the morphism

$$\operatorname{Spec}(r/\varpi): \operatorname{Spec} R^{\mathrm{ps}}/\varpi \to \operatorname{Spec} R^{\mathrm{ps}}_{F(\zeta_p)}/\varpi$$

is finite and hence so is the induced morphism $\Big(\bigcup_{F'} X^{\mathrm{Kred}}_{F'}\Big)\setminus X^{\mathrm{Kred}}_{F(\zeta_p)}\to \overline{X}^{\mathrm{spcl}}_{F(\zeta_p)}$. We deduce

$$\dim\Big(\big(\bigcup\nolimits_{F'} X^{\mathrm{Kred}}_{F'}\big)\setminus X^{\mathrm{Kred}}_{F(\zeta_p)}\Big)\leq \dim \overline{X}^{\mathrm{spcl}}_{F(\zeta_p)},$$

from [45, Tag 01WG]. Moreover by the proof of [8, Theorem 5.3.1] we have

$$\dim \overline{X}^{\mathrm{spcl}}_{F(\zeta_p)} \leq (d/p)^2[F':\mathbb{Q}_p] = \frac{[F(\zeta_p):F]}{p} d^2[F:\mathbb{Q}_p],$$

and [8, Theorem 5.4.1] gives $\dim \overline{X}^{\mathrm{ps,irr}} = d^2[F:\mathbb{Q}_p]$. Since $p$ divides $d$ and $[F(\zeta_p):F] \leq p-1$, using the bound from Lemma A.4 we conclude that

$$\dim \overline{X}^{\mathrm{ps,irr}} - \dim\left(\bigcup\nolimits_{F'} X_{F'}^{\mathrm{Kred}}\right) \geq \frac{d}{p} d[F:\mathbb{Q}_p] \geq d[F:\mathbb{Q}_p] \geq 2.$$

□

**Proposition A.8.** *There exists an open dense subscheme $U^{\mathrm{Kirr}} \subset \overline{X}^{\mathrm{ps,irr}}$ such that $x \in P_1(R^{\mathrm{ps}}/\varpi)$ is Kummer-irreducible if and only if $x$ is a closed point in $U^{\mathrm{Kirr}}$. Moreover,* $\dim \overline{X}^{\mathrm{ps,irr}} - \dim(\overline{X}^{\mathrm{ps,irr}} \setminus U^{\mathrm{Kirr}}) \geq d[F:\mathbb{Q}_p] \geq 2$.

*Proof.* Let $U^{\mathrm{Kirr}} = \overline{X}^{\mathrm{ps,irr}} \setminus (\bigcup_{F'} X_{F'}^{\mathrm{Kred}})$, where the union is taken over all degree $p$ Galois extensions $F'$ of $F(\zeta_p)$. Since there are only finitely many such extensions, $U^{\mathrm{Kirr}}$ is open. Remark A.5 implies that its closed points are precisely the Kummer-irreducible points. Its complement in $\overline{X}^{\mathrm{ps,irr}}$ has codimension at least $d[F:\mathbb{Q}_p]$ by Lemma A.7. Since $\overline{X}^{\mathrm{ps,irr}}$ is equi-dimensional by [8, Theorem 5.4.1] this implies density. □

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