Cycles, Transfers and Motivic Homology Theories, by Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander, Annals of Mathematics Studies, No. 143, Princeton Univ. Press, Princeton, NJ, 2000, 254pp., ISBN 0-691-04815-0

When thinking about this book, three questions come to mind: What are motives? What is motivic cohomology? How does this book fit into these frameworks?

Let me begin with a very brief answer to these questions. The theory of motives is a branch of algebraic geometry, dealing with algebraic varieties over a fixed field k. The basic idea is simple in its audacity: enlarge the category of varieties into one which is *abelian*, meaning that it resembles the category of abelian groups: we should be able to add morphisms, take kernels and cokernels of maps, etc. The objects in this enlarged category are to be called *motives*, whence the notion of the motive associated to an algebraic variety.

Any reasonable cohomology theory on varieties should factor through this category of motives. Even better, the abelian nature of motives allows us to do homological algebra. In particular, we can use Ext groups to form a universal cohomology theory for varieties. Not only does this motivate our interest in motives, but the universal property also gives rise to the play on words "motivic cohomology."

Different ways of thinking about varieties leads to different classes of motives, and different aspects of the theory. And each aspect of the theory quickly leads us into conjectural territory.

The best understood part of the theory is the abelian category of *pure motives*. This is what we get by restricting our attention to smooth projective varieties, identifying numerically equivalent maps and taking coefficients in the rational numbers $\mathbb{Q}$. The pure motives of smooth projective varieties are the analogues of semisimple modules over a finite-dimensional $\mathbb{Q}$-algebra. The theory of pure motives is related to many deep unsolved problems in algebraic geometry, via what are known as the *standard conjectures*.

At the other extreme, we have the most general part of the theory of motives, obtained by considering all varieties and taking coefficients in the integers $\mathbb{Z}$. This is the theory of *mixed motives*, and it is where homological algebra comes into play. It is also related to deep problems, such as the behavior of Hasse-Weil ζ-functions over a number field. Sadly, I must report bad news: an actual category of mixed motives is not yet fully known to exist, even with coefficients in $\mathbb{Q}$.

Here's the good news. It turns out that in order to construct the motivic cohomology of a variety we do not need to know whether or not (mixed) motives exist. Using homological algebra, all we need is a candidate for the *derived category* of motives. Indeed, most reasonable cohomology groups of a variety may be viewed as morphisms in some derived category.

The book under review constructs a candidate for the derived category of mixed motives, and hence motivic cohomology, and investigates its main properties. Although the book is not easy reading, I think it may quite possibly become one of the very most influential books in this field since *Dix Exposés* (1968).

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Before I allow this review to become technical, let me remind you that the fundamental question about motives remains unanswered:

What should a motive be?

Attempts to answer this question have been wonderfully vague from the start of the subject. Grothendieck first described his idea of motives in a 1964 letter [G64] to Serre, as a kind of universal cohomology theory for algebraic varieties. Attempting to clarify what he meant, Grothendieck explained (in French):

> "A motive ('motif') over a field k is something like an ℓ-adic cohomology group of an algebraic scheme over k, but considered as being independent of ℓ, and with its $\mathbb{Z}$-linear (or $\mathbb{Q}$-linear) structure, deduced from the theory of algebraic cycles."

The culturally inclined reader may enjoy knowing that the word 'motif' was borrowed from Cézanne, who used it to describe his impessionist method of painting. (See [Ma].) Cézanne would first to choose his 'motif' – a person, object or view attractive to him – and study it directly, keeping in mind that his sensation of the motif was ever-changing. Cézanne's motif would then be realized by the rendering of his impressions on canvas. See [Céz] for a fuller description of Cézanne's method.

Pure Motives and Chow Motives

Grothendieck gave a course on Motives at the Institut des Hautes Études Scientifiques in Spring 1967. In it, he constructed the category M_k of "pure motives" and presented its basic properties, along with the two so-called "standard conjectures." The contents of this course were presented to the mathematical public by Demazure [Dz], Manin [Ma] and Kleiman [K], but never by Grothendieck himself. A summary of the state of the art in 1994 may be found in the book [M].

The construction of M_k is done in three steps, starting with the category $\mathcal{V}_k$ of smooth projective algebraic varieties. One first forms a category of correspondendences (of degree 0), which we shall call $\mathcal{C}$. Then we enlarge it in two steps, essentially by adding the Lefschetz motive and its inverse.

The objects X, Y, etc. of $\mathcal{C}$ are smooth projective varieties over k. For morphisms we set $\mathrm{Hom}_{\mathcal{C}}(X,Y) = A^{\dim Y}(X\times Y)$, where $A^j(T)$ denotes the group of $\mathbb{Q}$-linear algebraic cycles of codimension j on a variety T, modulo numerical equivalence. Composition of $f\in\mathrm{Hom}(X_1,X_2)$ and $g\in\mathrm{Hom}(X_2,X_3)$ is defined using pullback, intersection and push-forward of cycles: if p_{ij} denotes the projection from $X_1\times X_2\times X_3$ onto $X_i\times X_j$ then $f\circ g = p^{13}_*(p^*_{12}f\cdot p^*_{23}g)$. The category $\mathcal{C}$ is additive, with $\oplus$ being disjoint union, and has an internal tensor product: $\otimes$ is the product of varieties. Moreover, the category $\mathcal{V}_k$ of smooth projective varieties over k sits inside of $\mathcal{C}$, once we identify a morphism $f\colon X\to Y$ with the class of its graph in $A^{\dim Y}(X\times Y)$.

For example, the endomorphisms of the projective line in $\mathcal{C}$ form the semisimple algebra $\mathrm{Hom}_{\mathcal{C}}(\mathbb{P}^1,\mathbb{P}^1)\cong\mathbb{Q}\times\mathbb{Q}$. However, this calculation reveals an embarrassing drawback of the category of correspondences: there is no decomposition of $\mathbb{P}^1$ corresponding to the decomposition of this algebra. One factor may be geometrically described by the following basic construction. Fix your favorite point (say ∞) on the projective line $\mathbb{P}^1$, and consider the composition $e\colon \mathbb{P}^1 \to \text{point} \to \mathbb{P}^1$. This map is *idempotent* in $\mathcal{C}$, meaning that $e^2 = e$ holds, so the class of a point forms one factor of $\mathbb{P}^1$. However, $\mathbb{P}^1$ has no complementary factor in $\mathcal{C}$; the category of correspondences just isn't big enough.

Fortunately, there is a standard fix for this problem. The *idempotent completion* $\hat{\mathcal{C}}$ of any category $\mathcal{C}$ is defined as follows. The objects of $\hat{\mathcal{C}}$ are pairs (C, e) where e is an idempotent element of $\mathrm{Hom}(C, C)$. Morphisms from (C, e) to (C', e') are just maps $f\colon C \to C'$ in $\mathcal{C}$ with $e'f = fe$. It is easy to see that $\mathcal{C}$ is a full subcategory of $\hat{\mathcal{C}}$, and that every idempotent map in $\hat{\mathcal{C}}$ factors as a projection and an inclusion. For example, any idempotent $e\colon C \to C$ in $\mathcal{C}$ factors in $\hat{\mathbb{C}}$ as $C \twoheadrightarrow (C, e) \hookrightarrow C$. (The idempotent completion of a category $\mathcal{C}$ is sometimes called its "Karoubianization, or "pseudo-abelianization" if $\mathcal{C}$ is additive, but the general construction dates to [Fr].)

The category M_k^{eff} of *effective motives* is defined to be the idempotent completion of $\mathcal{C}$, the category of correspondences. We define the *Lefschetz motive* L to be the unique factor of $\mathbb{P}^1$ in $\hat{\mathcal{C}}$ so that e gives a direct sum decomposition in $\hat{\mathcal{C}}$:

$$\mathbb{P}^1 = \text{point} \oplus L.$$

In M_k^{eff} we also have a decomposition $\mathbb{P}^n = \text{point} \oplus L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes n}$. This category contains the other graded pieces of the ring $A^*(X)$, because $A^i(X) \cong \mathrm{Hom}_{M_k^{\text{eff}}}(L^{\otimes i}, X)$. If we write $X(i)$ for $X \otimes L^{\otimes i}$ and set $d = \dim(X)$ there is a natural isomorphism

$$A^{d+i-j}(X \times Y) \cong \mathrm{Hom}(X(i), Y(j)).$$

Finally, the category of pure motives, M_k, is obtained from the category of effective motives by formally inverting the Lefschetz motive L. That is, we add objects $M(i)$ for negative values of i, for each effective motive M. Because of the description of maps between effective motives, we may thus describe the objects of M_k as triples (X, e, i), where X is a smooth projective variety over k, e is an idempotent element of $A^{\dim X}(X, X)$, and i is an integer. Morphisms from (X, e, i) to (Y, f, j) are elements of the group fAe, where $A = A^{\dim(X)+i-j}(X \times Y)$.

Jannsen proved [J] that M_k is a semi-simple abelian category. However, there is a lot of unknown territory surrounding the facts we do know. Grothendieck's standard conjectures [G69] would yield enough extra information about M_k to yield a new proof of the Weil conjectures, and the equivalence of numerical and homogical equivalence of cycles. However, this is part of another story and not directly relevant to the book under review.

Before leaving this topic, we mention an important variation of the above construction. Suppose we let $A^j(X)$ denote the group of algebraic cycles of codimension j on X, modulo *rational* equivalence. Repeating exactly the construction above yields the bulkier category M_k^{rat} of *Chow motives.* The extra bulkiness is typified by the Jacobian $J(X)$ of a connected curve X: $\mathrm{Hom}(\text{point}, X) = A^1(X)$ is $\mathbb{Q}$ for pure motives, but equals $\mathbb{Z} \oplus J(X)$ for Chow motives. The category of Chow motives fails to be abelian, even with coefficients $\mathbb{Q}$, because of this extra bulk.

Mixed Motives

The use of extensions to describe "mixed motives," or motives for arbitrary varieties, had its beginning in Deligne's study [D] of mixed Hodge structures, which may be thought of as iterated extensions of pure Hodge structures (the analogue of pure motives). Deligne isolated a certain class of mixed Hodge structures, which he called *1-motives*, and showed that they gave a good description of the cohomology group H^1. The appropriate mixed Hodge structure for a curve was even called its *motivic* H^1 [D, 10.3.4]. Inspired by this, Grothendieck suggested the probable existence of a category of [mixed] motives in a 1973 letter to Illusie [G73], "using Deligne's motives of level 1 as a model." Skipping over another interesting story, the point for this review is that the terminology "mixed motive" and "motivic cohomology," as well as the connection to homological algebra came out of the ideas in [D].

Beilinson's Vision

Around 1980, Beilinson [Be1, Be2] observed that the Chern characters make higher algebraic K-theory (see [Q]) into a universal cohomology theory H^*_M with coefficients in $\mathbb{Q}$, provided that we re-index K-theory: $H^n_M(X, \mathbb{Q}(i))$ is the eigenspace $K^{(i)}_{2i-n}(X)$ on which the Adams operations ψ^k have eigenvalue k^i. Beilinson coined the term *Absolute motivic cohomology* for this cohomology theory.

In trying to guess what happens over $\mathbb{Z}$, Beilinson considered suitable truncations $\tau_i R\pi_*\mu_m^{\otimes i}$ of the Zariski direct images of the étale sheaves $\mu_m^{\otimes i}$, and observed that their hypercohomology groups formed a cohomology theory. This led him to make a remarkable series of conjectures about motivic cohomology in 1981–82, which form the primary inspiration for the book under review. (Similar conjectures for the étale topology were made by Lichtenbaum [Li].)

Beilinson conjectured [Be3, Be4] that the motivic cohomology $H^n(X, \mathbb{Z}(i))$ of a smooth X should arise as the Zariski hypercohomology of certain natural chain complexes of sheaves $\mathbb{Z}(i)$, and that these complexes should have the following properties.

(1) $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$, and $\mathbb{Z}(i) = 0$ for negative i.

(2) $\mathbb{Z}(1)$ is the sheaf of units, $\mathcal{O}^\times$, placed in cohomological degree one (this is written as $\mathcal{O}^\times[-1]$). This forces all the hypercohomology of $\mathbb{Z}(1)$ to vanish except $\mathbb{H}^1(X, \mathbb{Z}(1)) = \mathcal{O}^\times(X)$ (global units) and $\mathbb{H}^2(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$, the Picard group of X.

(3) The motivic cohomology groups $H^n(S, \mathbb{Z}(n))$ of the scheme S associated to a field k agree with Milnor's K-groups, $K^M_n(k)$. In particular, they have an explicit presentation by generators $\{a_1, \dots, a_n\}$, $a_i \in k^\times$, with well known relations.

(4) The Chow groups $A^*(X)$ of cycles on a smooth X, modulo rational equivalence, appear as motivic groups:

$$A^n(X) = \mathbb{H}^{2n}(X, \mathbb{Z}(n)).$$

Of course, the cases $A^0(X) = \mathbb{Z}$ and $A^1(X) = \mathrm{Pic}(X)$ are redundant.

(5) For any smooth X, motivic cohomology is related to Quillen's algebraic K-groups $K_*(X)$ in two ways. Rationally,

$$H^n(X, \mathbb{Z}(i)) \otimes \mathbb{Q} \cong K^{(i)}_{2i-n}(X).$$

Integrally, there should be a spectral sequence with $E_2^{p,q} = H^p(X, \mathbb{Z}(q))$, converging to $K_{2q-p}(X)$.

The book under review constructs motivic cohomology by constructing chain complexes $\mathbb{Z}(i)$ satisfying all of the above properties. Other constructions have been given independently by Bloch [Bl], Hanamura [H], and Levine [Le], among others.

Beilinson's original list contained two further axioms which are still open, and which should be called conjectures. The first has been proven for $m = 2^\nu$ by Voevodsky [V], starting from the ideas in this book, and his proof has far-reaching consequences in algebraic K-theory, quadratic forms and étale cohomology.

(6) $H^n(X, \mathbb{Z}/m(i)) \cong H^n_{zar}(X, \tau_i R\pi_* \mu_m^{\otimes i})$ for all m, n and i.

(7) (Vanishing Conjecture) For any smooth X, $H^n(X, \mathbb{Z}(i)) = 0$ for negative n.

The book

After all this preparation, we are ready to talk about the book under review. It is a sequence of five essays which develop motivic cohomology in Beilinson's sense, as the hypercohomology of a complex $\mathbb{Z}(i)$. As mentioned, the ideas in this book have already had a strong effect upon several other areas of mathematics.

In my view, the climax of this book is Voevodsky's essay "Triangulated Categories of Motives over a Field," which I shall refer to as [TCM]. It contains the construction of the triangulated category DM_k^{eff}, and hence DM_k, which contains Grothendieck's category M_k^{rat} of Chow motives in a natural way.

One key new idea which gives their construction such power is the notion of a *sheaf with transfers*. This is a sheaf F defined on smooth schemes which is equipped with extra maps called "transfers." Transfer maps are generalizations of the familiar norm and trace maps in Galois theory; there is a transfer map $F(Y) \to F(X)$ associated to each finite correspondence from X to Y. The terms in the complex $\mathbb{Z}(i)$ are sheaves with transfers. (The Zariski topology suffices to define $H^n(X, \mathbb{Z}(i))$, but for technical reasons the working topology for sheaves is either the Nisnevich or *cdh* topology.)

One first defines a category Cor of finite correspondences, whose objects are smooth schemes over k. The morphisms from X to Y are cycles in $X \times Y$ whose projections onto X are finite and surjective over a connected component of X. A *presheaf with transfers* is just a contravariant functor F from Cor to abelian groups; F is a sheaf with transfers if its restriction to each X is a sheaf. Actually, every essay except [TCM] uses the weaker notion of a "pretheory," but afterwards one can check that almost all results hold for sheaves with transfers.

The category Cor is clearly related to the category $\mathcal{C}$ of correspondences used by Grothendieck to define motives for smooth projective varieties, and the idempotent completion of Cor lies inside the category of sheaves with transfers. There is a canonical way to make any sheaf F *homotopy invariant*, meaning that $F(X) \cong F(X \times \mathbb{A}^1)$ for all X, by replacing F by the class of a certain chain complex in the derived category. This leads us from Cor into the derived category $\mathbf{D}^-$ of chain complexes of sheaves with transfer. By definition, DM_k^{eff} is the full subcategory of $\mathbf{D}^-$ consisting of chain complexes C whose cohomology sheaves are homotopy invariant.

In particular, to each smooth scheme X we associate a complex $[X]$ in DM_k^{eff}, called its "geometric motive." Mimicking Grothendieck's construction of M_k, we obtain the Lefschetz motive L, defined so that $[\mathbb{P}^1] = [\text{point}] \oplus L$. The complex $\mathbb{Z}(1)$ is defined to be the translate $L[-2]$ of L, and $\mathbb{Z}(i)$ is the tensor product of i copies of $\mathbb{Z}(1)$.

Another theme in this book is that the higher Chow groups defined by Bloch [Bl] are isomorphic to motivic cohomology. The main step occurs in the final essay, "Higher Chow groups and Etale Cohomology," but the final step occurs in the essay [TCM].

The other essays in the book contain foundational material. The first essay, "Relative Cycles and Chow Sheaves," is an exhaustive but elementary study of the relative cycle theory underlying this construction. The second essay, "Cohomological Theory of Presheaves with Transfers," studies the sheaf theory of homotopy invariant pretheories (and implicitly sheaves with transfers). Motivic cohomology is introduced in the third essay, "Bivariant Cycle Cohomology," along with a bivariant generalization $A_{r,i}(X,Y)$. The key technical trick in this essay is the use of the Friedlander-Lawson moving lemma from [FL] to prove a suitable duality theorem for the $A_{*,i}(X,Y)$.

This book was written as a sequence of related research papers, with all the lack of polish this implies. A case in point is the evolution from qfh sheaves to Chow sheaves (first essay), to pretheories (second and third essays), to sheaves with transfer (essay [TCM]). Nonetheless, this book is important because it lays the foundations of a new branch of algebraic geometry, namely motivic cohomology. It takes serious work to digest the mathematical ideas in this book. But the mathematical ideas are well worth reading.

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