The BZSV formula for Bun_G^X

Let $G \circlearrowright X$, and $\Sigma_B = \Sigma_{\mathrm{Betti}} \simeq B\pi_1$. As well as $\mathrm{Rep}_G := \underline{\mathrm{Hom}}_{\mathrm{Grp}}(\pi_1, G)$, so that

Set
$$\left.\begin{array}{l} \mathrm{Bun}_G := \underline{\mathrm{Map}}(\Sigma_{\mathrm{Betti}}, BG) \\ \mathrm{Bun}_G^X := \underline{\mathrm{Map}}(\Sigma_{\mathrm{Betti}}, X/G) \end{array}\right\} \text{mapping stacks}$$

$$\mathrm{Bun}_G \simeq (\mathrm{Rep}_G)/G.$$

Let $\rho : \pi_1 \to G$ be a group map, i.e. $\rho : * \to \mathrm{Rep}_G$

Induces $\pi_1 \circlearrowright X$ with $\gamma \in \pi_1$ acting by $\rho(\gamma) \circlearrowright X$ ..... so $\underset{\sim}{\text{set}}$ $\boxed{X^\rho := X^{\pi_1}}$ fixed-pts wrt this π_1-action.

$$\boxed{\textbf{Prop: } \text{The fiber of } \mathrm{Bun}_G^X \longrightarrow \mathrm{Bun}_G \text{ over } * \xrightarrow{\rho} \mathrm{Rep}_G \to \mathrm{Bun}_G \text{ is } X^\rho.}$$

Proof: Let us first (for notational simplicity) work this out on the level of points:

$$\left.\begin{aligned}
\mathrm{Map}_{\mathrm{Stk}}(*, \mathrm{Bun}_G^X \times_{\mathrm{Bun}_G} \{\rho\}) &\underset{\substack{\text{by def of} \\ \text{mapping stacks}}}{\simeq} \mathrm{Map}_{\mathrm{Stk}}(B\pi_1, X/G) \times_{\mathrm{Map}_{\mathrm{Stk}}(B\pi_1, BG)} \{\rho\} \\
&\simeq \mathrm{Map}_{\mathrm{Stk}/BG}(B\pi_1, X/G) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}/B\pi_1}(B\pi_1, X/\pi_1) \\
&\simeq \mathrm{Map}_{\mathrm{LMod}_{\pi_1}(\mathrm{Stk})}(*, X) \\
&\underset{\substack{\text{univ. prop} \\ \text{of fixed-pts}}}{\simeq} \mathrm{Map}_{\mathrm{Stk}}(*, X^{\pi_1}) \\
&\underset{\text{def of } X^\rho}{\simeq} \mathrm{Map}_{\mathrm{Stk}}(*, X^\rho)
\end{aligned}\right\}$$

$$\begin{tikzcd} B\pi_1 \arrow[r, dotted] \arrow[dr, "\rho"'] & X/G \arrow[d] \\ & BG \end{tikzcd}$$

both are ways of expressing the space of dotted maps in this triangle

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X/\pi_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X/G \arrow[d] \\ * \arrow[r] & B\pi_1 \arrow[r, "\rho"] & BG \end{tikzcd}$$

the π_1-action on X, induced by $\rho : \pi_1 \to G$, fits in terms of quotient stacks into a double pullback square on the left

$$\mathrm{Stk}_{/B\pi_1} \simeq \mathrm{LMod}_{\pi_1}(\mathrm{Stk})$$
$$(Y/\pi_1 \to B\pi_1) \longleftrightarrow (\pi_1 \circlearrowright Y)$$

standard equivalence btw stacks over $B\pi_1$ & stacks with a π_1-action

by replacing $\mathrm{Stk} \mapsto \mathrm{Stk}_{/S}$ for any base S (bcs the argument really works in any ∞-topos), we obtain an equiv on the level of S-points $\forall S$, thus an equivalence of stacks. $\square$

Or worked out here for your convenience; it's really the same argument as for $S = *$

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Stk}}(S, \mathrm{Bun}_G^X \times_{\mathrm{Bun}_G} \{\rho\}) &\simeq \mathrm{Map}_{\mathrm{Stk}}(S \times B\pi_1, X/G) \times_{\mathrm{Map}_{\mathrm{Stk}}(S \times B\pi_1, BG)} \{S \times B\pi_1 \xrightarrow{pr_2} B\pi_1 \xrightarrow{\rho} BG\} \\
&\simeq \mathrm{Map}_{\mathrm{Stk}/BG}(S \times B\pi_1, X/G) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}/B\pi_1}(S \times B\pi_1, X/\pi_1) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}/S \times B\pi_1}(S \times B\pi_1, S \times X/\pi_1) \\
&\simeq \mathrm{Map}_{\mathrm{LMod}_{\pi_1}(\mathrm{Stk}_{/S})}(S, S \times X) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}_S}(S, (S \times X)^{\pi_1}) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}_S}(S, S \times X^{\pi_1}) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}}(S, X^{\pi_1})
\end{aligned}$$

both $- \times S$ & $(-)^{\pi_1}$ are limits, so they commute

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Using that in the above argument, we could have taken ρ to be an A-pt, or even S-pt, of Rep_G if we wanted

More precisely, for any A-point $\rho \in \mathrm{Rep}_G(A)$, we have a natural equivalence over $\mathrm{Spec}(A)$.

$$\mathrm{Bun}_G^X \times_{\mathrm{Bun}_G} \mathrm{Spec}(A) \simeq (X \times \mathrm{Spec}(A))^{\rho_A}$$

as before, just a name for the π_1-fixed pts.

Applying this to $A = \mathcal{O}(\mathrm{Rep}_G)$, since $\mathrm{Rep}_G = \mathrm{Spec}(A)$ is affine, and the "universal repres." $\rho_{\mathrm{univ}} \in \mathrm{Rep}_G(A)$, corresponding to $\mathrm{Rep}_G \xrightarrow{\mathrm{id}} \mathrm{Rep}_G$, we get

$$\mathrm{Bun}_G^X \times_{\mathrm{Bun}_G} \mathrm{Rep}_G \simeq (X \times \mathrm{Rep}_G)^{\rho_{\mathrm{univ}}} \underset{\substack{\text{by def} \\ \text{of what } (-)^{\rho_{\mathrm{univ}}} \\ \text{means}}}{=} (X \times \mathrm{Rep}_G)^{\pi_1}$$

Now observe the diagram of pullback squares:

$$\begin{tikzcd} (X \times \mathrm{Rep}_G)^{\pi_1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Bun}_G^X \arrow[d] \\ \mathrm{Rep}_G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \text{since } \mathrm{Bun}_G \simeq (\mathrm{Rep}_G)/G", very near start] & \mathrm{Bun}_G \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

and the outer pullback square exhibits an equivalence of stacks

$$\boxed{\mathrm{Bun}_G^X \simeq (X \times \mathrm{Rep}_G)^{\pi_1}/G}$$