LECTURES ON SCHUBERT VARIETIES

SARA BILLEY'S SPRING 2007 COURSE

1. INTRODUCTION(MARCH 28, 2007)

1.1. **History of Schubert Calculus.**

(1) Enumerative geometry
  - Hermann Grassmann (1809-1877)
  - Hermann Cäsar Hannibal Schubert (1848-1911)
  - Francesco Severi (1879-1961)
  - Mario Pieri
  - Giovanni Zeno Giambelli (1879-1935)

(2) Topology of homogeneous spaces
  - Charles Ehresmann (1905-1979)
  - Claude Chevalley (1907-1984)

(3) Representation theory (in 1950's)
  - A. Borel
  - R. Bott
  - B. Kostant

(4) Explicit computation
  - I. Berenstein
  - I. Gelfand
  - S. Gelfand
  - M. Demazure

(5) More concrete(combinatorial) theory
  - A. Lascoux
  - M.P. Schützenberger

1.2. **Enumerative Geometry.**

(1) Some of typical questions in enumerative geometry
  - Given two planes $P_1, P_2 \subseteq \mathbb{R}^3$ through the origin, what is $\dim(P_1 \cap P_2)$? or equivalently, given two lines in projective space $\mathbb{P}^2$, how many points are in the intersection?
  - How many lines meet four given lines in $\mathbb{R}^3$?

(2) An example of Schubert variety
  - A family of lines meeting a point and contained in a 2-dimensional space.

(3) Schubert problem
  - Count the number of points in the intersection of Schubert varieties such that the intersection is 0-dimensional.

(4) Many flavors
  - Grassmannian manifold(variety)
  - Flag manifold

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- Affine Grassmannian manifold
- Partial flag varieties
- GKM ...

(5) Common Theme
- Schubert varieties form a nice basis of $H^*(X, \mathbb{Z})$.

2. Flag Variety

Let G be the general linear group $GL_n(\mathbb{C})$, and B be the set of upper triangular matrices in G. Then for $g \in G$, the coset gB is determined by the subspaces $\langle \mathbf{c}_1, \ldots, \mathbf{c}_i \rangle$, $i = 1, \ldots, n$, where $\mathbf{c}_i$ is the ith column vector of g.

Definition 2.1. A *flag* is a nested sequence of vector subspaces $F_\bullet = (F_1 \subset F_2 \subset \cdots \subset F_k)$ of $\mathbb{C}^n$. A flag is *complete* if $\dim F_i = i$ and $k = n$.

With the above definition we have a bijection between G/B and the set of complete flags in $\mathbb{C}^n$. Observe that we inherently chose a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{C}^n$ so that $\mathbf{c}_i = \sum_{k=1}^{i} g_{ki}\, \mathbf{e}_k$. Equivalently, we chose a base (complete) flag

$$\langle \mathbf{e}_1 \rangle \subset \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \subset \cdots \subset \langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle .$$

How can we canonically represent gB? Right multiplication by B adds some multiple of the ith column of g to jth column for $i \leq j$. Therefore we can always find $b \in B$ so that gb is in its column echelon form; the lowest non-zero entry in each column is a 1 and the entries to the right of each leading one are all zeros. The leading 1's in each column form a permutation matrix. For example,

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ is the canonical form of the matrix } g = \begin{bmatrix} 9 & 5 & 9 & 7 \\ 6 & 2 & 4 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

This permutation determines the *position of g with respect to the base flag*, denoted $\mathrm{pos}(g) = w$.

We will write $w = [w_1, w_2, \cdots, w_n]$ to mean w is the bijection mapping i to w_i. This is the *one-line notation* for $w \in S_n$. In the above example, $w = [2\,4\,1\,3]$ is the position of g with respect to the base flag. In general, we read the column numbers of leading 1's in the column echelon form of g from top to bottom to get the one-line notation for w when w determines the position of g with respect to the base flag.

WARNING: there are at least 8 different names used for this permutation matrix in the literature on Schubert varieties! We have chosen this way of naming our permutation matrices to agree with the *diagram of the permutation* which is defined below.

Definition 2.2. For $w \in S_n$, the associated *Schubert cell* is defined as

$$C_w = \{g \in G \,|\, \mathrm{pos}(g) = w\} .$$

Example 2.3. The Schubert cell corresponding to $w = [2\,4\,1\,3]$ is $C_{2413} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\}$.

We now can see that invertible matrices in column echelon form determine a set of representatives of G/B. Therefore, G/B can be written as a disjoint union $\bigcup_{w \in S_n} C_w$.

Definition 2.4. For $w \in S_n$, *diagram of w* is defined as follows:

$$D(w) = \{(i, w_j) \in [n]^2 \,|\, i < j \ \text{ and } \ w_i > w_j\} .$$

Here, $[n] = \{1, 2, \cdots, n\}$.

An *inversion* in w is a pair $1 \leq i < j \leq n$ such that $w_i > w_j$. The *length of* w, denoted $\ell(w)$, is the number of inversions for w. The pairs in $D(w)$ are in bijection with the inversions for w. Therefore, we can observe that for $w \in S_n$,

$$\dim_{\mathbb{C}} C_w = \ell(w) = \text{ number of inversions of } w = |D(w)|\,.$$

Let T be the set of diagonal matrices in G. The invertible diagonal matrices form an abelian group that is isomorphic to $(\mathbb{C}^*)^n$.

How does T act on C_w under the left multiplication? A diagonal matrix acts on a matrix g by scaling the rows of g. Therefore, the column echelon form of g is unchanged, hence $T \cdot C_w = C_w$. Moreover, the fixed points of the T-action on G/B are exactly the permutation matrices; one T-fixed point in each C_w.

How does the left B-action work on C_w? It is easy to see that $BC_w \subset C_w$, and since the identity matrix is in B, we have $BC_w = C_w$, which is same as Bw.

Exercise 2.5. Prove that there are enough free variables to get any matrix in C_w by Bw.

Theorem 2.6. (Bruhat decomposition)

$$G = \dot{\bigcup}_{w \in S_n} BwB$$

and hence we have

$$G/B = \dot{\bigcup}_{w \in S_n} Bw = \dot{\bigcup}_{w \in S_n} C_w\,.$$

Remark 2.7. The Schubert cell C_w can be viewed in three ways:

(1) The set of cosets $B \cdot wB/B$.
(2) The orbit of the permutation matrix $B \cdot w$.
(3) The set of complete flags in position w with respect to the base flag.

All three points of view are useful.

If we understand elements in C_w as complete flags, what is the condition for a flag to be an element of C_w in terms of equations?

Definition 2.8. For $w \in S_n$, let $rk_{\nwarrow} w_{[i,j]}$ be the rank of the submatrix of w with top left corner at $(1, 1)$ and lower right corner at (i, j).

Note that w can be completely recovered from the matrix $rk_{\nwarrow} w$.

Then we have another description for Schubert cell:

$$C_w = \{\text{ complete flag } F_\bullet \mid \langle \mathbf{e}_1, \ldots, \mathbf{e}_i \rangle \cap F_j = rk_{\nwarrow} w_{[i,j]}\}\,.$$

Example 2.9. For $w = [2\,4\,1\,3]$, the condition for a complete flag $F_\bullet$ to be in C_w is

$$[\dim(\langle \mathbf{e}_1, \ldots, \mathbf{e}_i \rangle \cap F_j)]_{i,j} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}\,.$$

Definition 2.10. (coordinate free description of Schubert cells) For a fixed base flag $B_\bullet = (B_1 \subset B_2 \subset \cdots \subset B_n)$, and $w \in S_n$, the corresponding Schubert cell is defined by

$$C_w(B_\bullet) = \{\text{ complete flags } F_\bullet \mid \dim(B_i \cap F_j) = rk_{\nwarrow} w_{[i,j]}\}\,.$$

Summary
$G = GL_n(\mathbb{C})$, invertible matrices.
$B =$ upper triangular matrices.
$T =$ diagonal matrices, maximal torus.
$W = S_n$.

3. Schubert varieties and Bruhat order (March 30, 2007)

Definition 3.1. The *Zariski topology* on $\mathbb{A}^n = \mathbb{C}^n$ (or $\mathbb{P}^n$) has closed sets given by the vanishing sets of some collection of polynomials (homogeneous polynomials, respectively).

Note that $n \times n$ complex matrices form an affine space of dimension n^2. Our goal is to describe the closure of a Schubert cell in G/B by equations.

Example 3.2. For $w = [2\,4\,1\,3]$, an element in $C_w = Bw/B$ has the following form: $\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
Therefore, if $M = (m_{ij}) \in BwB$, then the following equations on m_{ij}'s must be satisfied:

$$m_{41} = m_{42} = 0, \ \det \begin{bmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} = 0, \ \det \begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix} = 0\,.$$

Definition 3.3. For any $n \times n$ matrix M, let $rk_{\swarrow} M_{[i,j]}$ be the rank of the smallest lower left submatrix of M containing M_{ij}. Then the rank of M is the size of largest non-vanishing minors of M.

Exercise 3.4. Let M be any $k \times l$ matrix and let $r(p,q) = rk_{\swarrow} M_{[p,q]}$. Show that there exists a $k \times l$ matrix U with all $0,1$-entries and at most one 1 n each row and column such that $r(p,q) = rk_{\swarrow} U_{[p,q]}$ for all p, q.

Lemma 3.5. *For $w \in S_n$, $M \in BwB$ if and only if $rk_{\swarrow} M_{[p,q]} = rk_{\swarrow} w_{[p,q]}$ for all p, q.*
Equivalently, $N = b_1 M b_2$ for some $b_1, b_2 \in B$ if and only if $rk_{\swarrow} M = rk_{\swarrow} N$.

Proof. Left (or right) multiplication by B adds some multiple of row (column) j to row (column) i for $i < j$. $\square$

Notation Let $r_w(p,q) = rk_{\swarrow} w_{[p,q]}$.

Definition 3.6. *Matrix Schubert variety* is defined as follows:

$$\begin{aligned} \widetilde{X}(w) =& \{M \in Mat_{n\times n}(\mathbb{C}) \mid rk_{\swarrow} M_{[p,q]} \le rk_{\swarrow} w_{[p,q]} \text{ for all } p,q\} \\ =& \left\{ M \in Mat_{n\times n}(\mathbb{C}) \mid \text{ all } r_w(p,q)+1 \text{ minors vanish on } \begin{bmatrix} m_{p1} & \cdots & m_{pq} \\ \vdots & \cdots & \vdots \\ m_{n1} & \cdots & m_{nq} \end{bmatrix} \text{ for all } p,q \right\}. \end{aligned}$$

We let I_w be the ideal generated by all $r_w(p,q)+1$ minors of $\begin{bmatrix} x_{p1} & \cdots & x_{pq} \\ \vdots & \cdots & \vdots \\ x_{n1} & \cdots & x_{nq} \end{bmatrix}$ for all p, q.

Remark 3.7. $G \subseteq M_{n\times n}$ is open in Zariski topology. Hence $\widetilde{X}(w) \cap G$ is defined by the same equations.

Since vanishing/non-vanishing minors are unchanged by right multiplication by B, I_w also defines $\overline{C_w}$ in G/B, and we have the following definition:

Definition 3.8. For $w \in S_n$, the associated *Schubert variety* X_w is defined as the closure of C_w in Zariski topology:

$$X_w = \overline{C_w}\,.$$

Note that X_w is a projective variety.

Now, our question is the following: Which Schubert cells are in X_w?

Remark 3.9. For $w \in S_n$ and $i < j$, if $w(i) > w(j)$ then $r_{w'}(p,q) = r_w(p,q) - 1$ for $i < p \le j$ and $w(j) \le q < w(i)$, where $w' = wt_{ij}$.

Proposition 3.10. *(***Ehressmann Tableaux Criterion 1934***) Let $v = [v_1, v_2, \ldots, v_n]$ and $w = [w_1, w_2, \ldots, w_n]$ be two permutations in S_n.*

(1) $C_v \subseteq X_w \Leftrightarrow X_v \subseteq X_w \Leftrightarrow \{v_i, \ldots, v_n\}_< \ge \{w_i, \ldots, w_n\}_<$ *for all* i.
(2) w *cover* v *in containing order of Schubert variety if and only if* $wt_{ij} = v$ *and* $l(w) = l(v) + 1$.

Example 3.11. $X_{531624} \subseteq X_{651342}$, but $X_{3214} \not\subseteq X_{4132}$ since $\{4,1\}_< \not\le \{3,2\}_<$.

Proof. (of Proposition 3.10)
(1)

$$\begin{aligned} C_v \subseteq X_w \Leftrightarrow & r_v(p,q) \le r_w(p,q) \text{ for all } p,q \\ \Leftrightarrow & |\{v_p, \ldots, v_n\} \cap \{1, \ldots, q\}| \le |\{w_p, \ldots, w_n\} \cap \{1, \ldots, q\}| \text{ for all } p,q \\ \Leftrightarrow & \{v_p, \ldots, v_n\}_< \ge \{w_p, \ldots, w_n\}_< \text{ for all } p. \end{aligned}$$

(2) ($\Leftarrow$) Compare r_v and r_w. Since $l(w) = l(v) + 1$, no other 1 is in boundary box for t_{ij}. Therefore, there is no $z \in S_n$ such that $r_v < r_z < r_w$.

($\Rightarrow$) Suppose that $X_v \subset X_w$ and w covers v in containing order, so no z exists such that $X_v \subset X_z \subset X_w$. Let j be the largest such that $v_j \ne w_j$, then $w_j < v_j$. Let i be the largest index such that $i < j$ and $v_j > v_i \ge w_j$.

`claim`: $X_v \subset X_{vt_{ij}} \subseteq X_w$, and hence $vt_{ij} = w$ and $l(w) = l(v) + 1$.

`proof of claim`: We need to show that $X_{vt_{ij}} \subseteq X_w$ that is $r_{vt_{ij}} \le r_w$ everywhere. Let $\mathfrak{A}$ be the boxes consisting of rows from ith to jth and columns from w_jth to v_jth. Let $\mathfrak{B}$ be the boxes consisting of rows from $(i+1)$st to the jth and columns from v_ith to $9v_j - 1)$st. Since $r_v(p,q) \le r_w(p,q)$ for all p,q and $r_v(p,q) = r_vt_{ij}(p,q)$ outside of $\mathfrak{B}$, $r_{vt_{ij}}(p,q) \le r_w(p,q)$ outside of $\mathfrak{B}$. Since $v_n = w_n, \ldots v_{j+1} = w_{j+1}$ and $v_j > w_j$, we have $r_vt_{ij}(j,q) = r_v(j,q) + 1 = r_w(j,q)$ for all $w_j \le q < v_j$. Furthermore, $r_{vt_{ij}}(p, w_j - 1) = r_v(p, w_j - 1) \le r_w(p, w_j - 1)$ for $i \le p \le j$.

By the choice of i, there are no other 1's in matrix for v in $\mathfrak{A}$, so all jumps in rank in $\mathfrak{B}$ happen because of 1's to the SW of $\mathfrak{B}$, so accounted for along left edge of $\mathfrak{B}$ or lower edge of $\mathfrak{B}$. Therefore, $r_{vt_{ij}}(p,q) \le r_w(p,q)$ for all p,q. □

Corollary 3.12. $X_v \subseteq X_w$ *if and only if there exists a sequence of transpositions* $t_{a_1b_1}, \ldots, t_{a_kb_k}$ *such that* $w = vt_{a_1b_1} \cdots t_{a_kb_k}$ *and* $l(vt_{a_1b_1} \cdots t_{a_jb_j}) = l(v) + j$ *for all* $1 \le j \le k$.

Exercise 3.13. (**Chevalley's criterion**) Let $s_i = t_{ii+1}$ and write $w = s_{a_1} \cdots s_{a_p}$ with $p = l(w)$. Then $X_v \subseteq X_w$ if and only if there exists $1 \le i_1 < i_2 < \cdots < i_k \le p$ such that $v = s_{a_{i_1}} \cdots s_{a_{i_k}}$.

Definition 3.14. $v \le w$ *in Bruhat-Chevalley order* if $X_v \le X_w$.

Remark 3.15. (1) $v \le w \Leftrightarrow \{v_1, \ldots, v_i\}_< \le \{w_1, \ldots, w_i\}_<$ for all i.
(2) (Björner-Brenti) We only need to check the condition in (1) for i's such that $vs_i < v$ or only need to check i's such that $ws_i > w$.

(3)

$$\begin{aligned} X(w) =& \{gB \in G/B \,|\, r_g \leq r_w\} \\ =& \{F_\bullet : \text{ complete flag } \mid \dim(\langle \mathbf{e_1}, \ldots, \mathbf{e_i} \rangle \cap F_j) \geq rk_{\nwarrow} w_{ij}\} \end{aligned}$$

4. Fulton's Essential Set (April 4, 2007)

Let C_w and X_w be the Schubert cell and Schubert variety, respectively, that are associated to the permutation $w \in S_n$.

Definition 4.1. Let $D'(w) = \{(i,j) \in [n]^2$ such that $w(i) > j$ and $w^{-1}(j) < i\}$. This (alternate) diagram is obtained from the entries of the permutation matrix associated to w by canceling entries that lie to the North and East of a 1 entry.

The *essential set* of w, denoted $Ess'(w)$ consists of the Northeast corners of connected components in $D'(w)$. More precisely, $Ess'(w) = \{(i,j) \in D'(w)$ such that $(i-1,j), (i,j+1)$ and $(i-1,j+1) \notin D'(w)$.

Example 4.2. We have that $D'(2413)$ consists of the entries which are circles in the matrix:

$$\begin{bmatrix} . & 1 & . & . \\ . & \bullet & . & 1 \\ 1 & . & . & . \\ \circ & \bullet & 1 & . \end{bmatrix}$$

and $Ess'(2413)$ consists of the $\bullet$ entries.

Observe that the cardinality of $D'(w)$ is the codimension of X_w, which is $\binom{n}{2} - l(w)$.

We previously found that

$$\begin{aligned} M \in X_w \iff rk_{\swarrow}(M) \leq rk_{\swarrow}(w) \iff rk_{\nwarrow}(M) \geq rk_{\nwarrow}(w) \\ \iff \text{ all } rk_{\swarrow} w_{[i,j]} + 1 \text{ minors in } X_{[i,j]} \text{ vanish on } M \end{aligned}$$

where $X_{[i,j]}$ is $\begin{bmatrix} x_{i1} & \ldots & x_{ij} \\ \ldots & & \ldots \\ x_{n1} & \ldots & x_{nj} \end{bmatrix}$.

Let I_w be the ideal generated by all $rk_{\swarrow} w_{[i,j]} + 1$ minors in $X_{[i,j]}$. Here are some observations.

(1) $Ess'(w)$ is all on one row if and only if w has at most one ascent. Here, we mean that the entries in the 1-line notation for $w = (w_1, w_2, \ldots, w_n)$ satisfy $w_1 > w_2 > \ldots w_k, w_{k+1} > w_{k+2} > \cdots > w_n$.
(2) All of the entries in $Ess'(w)$ correspond to 0 entries in the canonical matrix form for C_w.
(3) For $(i,j) \in D'(w)$, we have $rk_{\swarrow} w_{[i,j]} = \#1$ entries Southwest of $[i,j]$. This quantity is also the number of lines crossed as we look South from $[i,j]$. Equivalently, it is the number of lines crossed as we look West from $[i,j]$.
(4) $rk_{\swarrow} w$ is constant on connected components of $D'(w)$.

Proposition 4.3. (Fulton) *Let I be the ideal generated by the $rk_{\swarrow} w_{[i,j]} + 1$ minors of $X_{[i,j]}$ for all $(i,j) \in Ess'(w)$. Then, for any $w \in S_n$, the ideal $I_w = I$.*

Proof. It suffices to show that all of the defining minors of I_w are in I.

Case 1. Suppose $(p,q) \in D'(w)$. Then, (p,q) is in the same connected component as some $(i,j) \in Ess'(w)$ with $i \leq p$ and $j \geq q$. We have $rk_{\swarrow} w_{[p,q]} = rk_{\swarrow} w_{[i,j]}$ by (4) and all $rk_{\swarrow} w_{[p,q]} + 1$ minors of $X_{[p,q]}$ are also minors in $X_{[i,j]}$.

Case 2. Suppose (p, q) lies on a vertical edge of the matrix describing the diagram of w. More precisely, suppose $(p, q) \notin D'(w)$ with $w^{-1}(q) \geq p$ and $w(p) \geq q$. Then, we find the largest k such that $w^{-1}(q-i) > p$ for all $0 \leq i < k$. If $k = q$ then $rk_{\swarrow} w_{[p,q]} = q$ so there are no $rk_{\swarrow} w_{[p,q]} + 1$ minors in $X_{[p,q]}$. Otherwise, $k < q$ and $(p, q-k) \in D'(w)$. We know the defining minors for $(p, q-k)$ are in I by Case 1, and we will show the defining minors for (p, q) are in the ideal generated by these.

Note that the r-minors of a matrix M generate an ideal that contains all $(r+1)$-minors of M', where M' is obtained from M by adding a column. If $r = rk_{\swarrow} w_{[p,q-k]}$ then $rk_{\swarrow} w_{[p,q-i]} = r + (k-i)$ for $0 \leq i \leq k$. Hence, all $rk_{\swarrow} w_{[p,q]} + 1$ minors of $X_{[p,q]}$ are in I by Case 1.

Case 3. Suppose (p, q) lies on a horizontal edge of the matrix describing the diagram of w. More precisely, suppose $(p, q) \notin D'(w)$ with $w^{-1}(q) < p$ and $w(p) < q$. This case is very similar to Case 2, and can be completed by looking down instead of left.

Case 4. Suppose (p, q) lies on an intersection of edges in the matrix describing the diagram of w. More precisely, suppose $(p, q) \notin D'(w)$ with $w^{-1}(q) > p$ and $w(p) < q$. If $w^{-1}(q-1) \leq p$ then by Case 2, we have that the $rk_{\swarrow} w_{[p,q-1]} + 1$ minors are in I. By the note in Case 2, this implies that all $rk_{\swarrow} w_{[p,q]} + 1$ minors of $X_{[p,q]}$ are in I.

If $w^{-1}(q-1) > p$ then by induction we may assume these minors are in I. We may use the note in Case 2 to see that all (p, q) minors are in I.

These cases exhaust all of the possibilities, so we have completed the proof. $\square$

The following proposition shows that no proper subset of the essential set will work to define I_w.

Proposition 4.4. (Fulton) *Let $w \in S_n$ and $(p_0, q_0) \in Ess'(w)$. Then there exists an $n \times n$ matrix M such that $rk_{\swarrow} M_{[p,q]} \leq rk_{\swarrow} w_{[p,q]}$ for all $(p, q) \in Ess'(w) \setminus \{(p_0, q_0)\}$, and $rk_{\swarrow} M_{[p_0,q_0]} = rk_{\swarrow} w_{[p_0,q_0]} + 1$.*

Proof. Define $M = (m_{ij})$ by

$$m_{ij} = \begin{cases} w_{ij} & \text{if } i \geq p_0,\ j \leq q_0 \text{ but } (i,j) \neq (p_0, q_0) \\ 1 & \text{if } (i,j) = (p_0, q_0) \\ 0 & \text{otherwise.} \end{cases}$$

Then, $rk_{\swarrow} M_{[p,q]}$ is the number of 1 entries lying Southwest of (p, q), so M has the desired properties. $\square$

Warning 4.5. The set of determinantal equations corresponding to $rk_{\swarrow} w_{[p,q]} + 1$ minors for $(p, q) \in Ess'(w)$ is not necessarily minimal. A Gröbner basis is given by Knutson-Miller.

Open Question 4.6. *What is the analogue of the essential set for G/B in other types?*

Example 4.7. For $w = 2413$ we have $D'(w)$ given by

$$\begin{bmatrix} . & 1 & . & . \\ . & \bullet & . & 1 \\ 1 & . & . & . \\ \circ & \bullet & 1 & . \end{bmatrix}$$

so the essential set tells us us that I_w is generated by the 1×1 and 2×2 minors of

$$X_{[4,2]} = \begin{matrix} x_{4,1} & x_{4,2} \end{matrix} \qquad X_{[2,2]} = \begin{matrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ x_{4,1} & x_{4,2} \end{matrix}.$$

So

$$I_w = \langle x_{41}, x_{42}, det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}, det \begin{bmatrix} x_{21} & x_{22} \\ x_{41} & x_{42} \end{bmatrix}, det \begin{bmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{bmatrix} \rangle$$

which can be reduced to

$$I_w = \langle x_{41}, x_{42}, det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \rangle$$

Example 4.8. For $w = 3412$ we have $D'(w) = Ess'(w)$ given by

$$\begin{bmatrix} . & . & 1 & . \\ . & . & \bullet & 1 \\ 1 & . & . & . \\ \bullet & 1 & . & . \end{bmatrix}$$

so

$$I_w = \langle x_{41}, det \begin{bmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix} \rangle$$

Example 4.9. For $w = 4321$ we have $D'(w)$ given by

$$\begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix}$$

so $Ess'(w)$ is the empty set. In this case, $X_w = G/B$.

Definition 4.10. Let w_0 be the permutation $(n, n-1, n-2, \ldots, 2, 1)$. Then $\tilde{X}(w_0) = M_{n\times n}$ and $X_{w_0} = G/B = \cup_{v \le w_0} C_v$.

Definition 4.11. The *coordinate ring* of X_w is $\mathbb{C}[x_{11}, \ldots, x_{nn}]/I_w$, consisting of polynomial functions on X_w.

The standard monomials form a basis for the coordinate ring. This set was originally described by Lakshmibai-Musili-Seshadri. See also the paper by Reiner-Shimozono in our class library.

We claim that G/B is a smooth manifold. This means that every point has an affine neighborhood of dimension $\binom{n}{2}$ and at every point, the dimension of the tangent space is $\binom{n}{2}$. Note that C_{w_0} consists of matrices of the form $\begin{bmatrix} * & * & * & 1 \\ * & * & 1 & 0 \\ * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. In particular, the permutation matrix $w_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in C_{w_0}$, and C_{w_0} is its affine neighborhood of dimension $\binom{n}{2}$.

Since G/B has a transitive GL_n action by left multiplication, every point looks locally like any other point. Therefore, every point in G/B has an affine neighborhood of dimension $\binom{n}{2}$, and the tangent space to every point has the same dimension, hence G/B is smooth. This proves the claim.

Are all Schubert varieties smooth? No, and as n tends to ∞, we find that almost all Schubert varieties are singular. We outline a way to test for smoothness based on the defining equations for X_w.

The local properties of points in a Schubert variety X_w are determined by the permutation matrices for $v \le w$ since X_w is the union of the B-orbits of these permutation matrices: $X_w = \cup_{v\le w} C_v = \cup_{v \le w} Bv$. Smoothness is a local property so X_w is smooth at every point in C_v for $v \le w$ if and only if X_w is smooth at v.

Theorem 4.12. (Jacobian criterion) *Let Y be an affine variety in $\mathbb{A}^n$ defined by $I(Y) = \{f_1, \ldots, f_r\}$ where the f_i are polynomials in variables $x_1, \ldots, x_n$. Then, we define the Jacobian matrix $J(x_1, \ldots, x_n) = (\frac{\partial f_i}{\partial x_j})$. If $p = (p_1, \ldots, p_n) \in \mathbb{A}^n$ then the following hold:*

(1) *We always have $rk\ J(p_1, \ldots, p_n) \le codim_{\mathbb{A}^n}\ Y = n - dim\, Y$.*

(2) *The point p is a smooth point of Y if and only if $rk\ J(p_1,\dots,p_n) = codim_{\mathbb{A}^n}\ Y = n - dim\, Y$.*

Note that $\{p \in Y : rk\ J < k\}$ is a closed set in the Zariski topology, being defined by the vanishing of minors. The intersection of two closed sets is again a closed set. We said above that if X_w is singular at $v \le w$, then X_w is singular at all points in C_v. Therefore, X_w is singular on $\overline{C_v} = X_v$. Equivalently, X_w is singular at u for all $u \le v$. Hence, we have the following corollary.

Corollary 4.13. *The Schubert variety X_w is smooth everywhere if and only if X_w is smooth at $v = id$.*

5. Smooth Schubert Varieties(April 6,2007)

Recall that X_w is smooth everywhere if and only if X_w is smooth at the identity matrix by Corollary 4.13. Our goal is to test smoothness with the Jacobian criterion, which requires us to identify an affine neighborhood of the identity.

Consider $G/B = X_{w_0} = C_{w_0} \cup \cup_{v<w_0} C_v$. The Schubert cell C_{w_0} is an affine neighborhood of w_0. For example when $n = 4$

$$\begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix} \in \begin{pmatrix} *&*&*&1\\ *&*&1&0\\ *&1&0&0\\1&0&0&0 \end{pmatrix} = C_{w_0}. \tag{5.1}$$

We can move this neighborhood around to contain the identity by left multiplication by the matrix w_0

$$w_0 C_{w_0} = w_0 B w_0 = \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}\begin{pmatrix} *&*&*&1\\ *&*&1&0\\ *&1&0&0\\1&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ x_{21}&1&0&0\\ x_{31}&x_{32}&1&0\\ x_{41}&x_{42}&x_{43}&1 \end{pmatrix}. \tag{5.2}$$

In the matrix on the right, we have filled in the stars with affine coordinates.

Definition 5.1. Set $Y(w, id) \equiv X_w \cap w_0 C_{w_0}$ and $Y(w, v) \equiv X_w \cap v w_0 C_{w_0}$.

Note, $Y(w, v)$ is an affine neighborhood containing v in X_w if $v \le w$.

Example 5.2. Is $X(2413)$ smooth?

In Example 4.7, we determined that X_w is defined by $I_{2413} =< x_{41}, x_{42}, x_{21}x_{32} - x_{31}x_{22} >$. Therefore, $Y(2413, id)$ is determined by the same equations evaluated on the last matrix in (4.7): $< x_{41}, x_{42}, x_{21}x_{32} - x_{31} >$ where x_{22} was set equal to 1.

To use the Jacobian criterion, calculate

$$J = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ x_{32}&-1&x_{21}&0&0 \end{pmatrix} \tag{5.3}$$

where the columns are indexed by the ordered list of variables $x_{21}, x_{31}, x_{32}, x_{41}, x_{42}$. The matrix $J(I)$ is obtained by setting all the variables x_{ij} equal to 0. Therefore, the rank of $J(I)$ is 3, while the codimension of $X(2413)$ is $\binom{4}{2} - 3 = 3$. Hence, $X(2413)$ is smooth.

Example 5.3. Is $X(3412)$ smooth?

The affine variety $Y_{3412,id}$ is defined by I_{3412} restricted to the matrix in (5.2) again. This ideal is generated by $< f_1, f_2 >$, where

$$
\begin{aligned}
f_1 &= x_{41}\\
f_2 &= \begin{vmatrix} x_{21} & 1 & 0\\ x_{31} & x_{32} & 1\\ 0 & x_{42} & x_{43}\end{vmatrix} = x_{21}(x_{32}x_{43}-x_{42}) - x_{31}x_{43}.
\end{aligned}
$$

$$
J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ x_{32}x_{43}-x_{42} & -x_{43} & 0 & 0 & x_{21} & x_{21}x_{32}-x_{31}\end{pmatrix} \tag{5.4}
$$

where the variables are in order $x_{21}, x_{22}, x_{31}, x_{41}, x_{42}, x_{43}$. Setting the variables equal to 0 we see that the rank of $J(I)$ is 1, whereas the codimension of $X(3412)$ is $\binom{4}{2} - 4 = 2$, so $X(3412)$ is NOT smooth.

What is $Sing(X(3412)) = \displaystyle\bigcup_{v \text{ singular in } X_w} X(v)$, the singular locus of X_w?

Definition 5.4. $\mathrm{maxsing}(w) = \{v \in S_n : v \le w$ and v is a maximally singular point in $X_w\}$.

Claim 5.5. $Sing(X(3412)) = X(1324)$ or equivalently $\mathrm{maxsing}(w) = \{1324\}$.

Proof. We need to verify the following statements:

(1) $v = 1324$ is a singular point of $X(3412)$.
(2) No t_{ij} exists such that $v < vt_{ij} \le w$ with $l(vt_{ij}) = l(v) + 1$ and vt_{ij} is singular.
(3) All t_{ij} such that $l(vt_{ij}) = 1$, different from $t_{ij} = 1324$, are smooth.

To verify (1), let $v = 1324$. We have

$$
Y(3412, 1324) = \begin{pmatrix} 1 & 0 & 0 & 0\\ x_{21} & x_{22} & 1 & 0\\ x_{31} & 1 & 0 & 0\\ x_{41} & x_{42} & x_{43} & 1\end{pmatrix} \cap X_w \tag{5.5}
$$

The ideal defining $Y(3412, 1324)$ is generated by f_1, f_2 again but now we evaluate these polynomials on the matrix in (5.5). So

$$
\begin{aligned}
f_1 =& x_{41}\\
f_2 =& \begin{vmatrix} x_{21} & x_{22} & 1\\ x_{31} & 1 & 0\\ x_{41} & x_{42} & x_{43}\end{vmatrix} = -x_{31}(x_{22}x_{43} - x_{42}) + x_{21}x_{43}.
\end{aligned}
$$

$$
J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ x_{43} & -x_{31}x_{43} & -x_{22}x_{32}+x_{42} & 0 & x_{31} & -x_{31}x_{22}+x_{21}\end{pmatrix} \tag{5.6}
$$

where the variables are again in lexicographic order. Setting the variables equal to 0 we see that the rank of $J(1324)$ is $1 < 2 = \mathrm{codim}(X_{3412})$, so 1324 is a singular point of $X(3412)$. □

Exercise 5.6. Complete the above proof by verifying that all permutations with length 2 above 1324 and below 3412 in Bruhat order are smooth points of X_w, and that 2134 and 1243 are also smooth.

This sort of calculation "works", but it does not give us a good feeling for how the singularities of one Schubert variety relates to the singularities of another. Therefore, we need to study singular Schubert varieties and their tangent spaces as a family. This is the discrete analog of studying a moduli space of curves in order to see their properties in context.

Definition 5.7. For any subgroup H of $Gl_n(\mathbb{F})$, the *Lie algebra* of H, $Lie(H)$, is the space of tangent vectors to the identity matrix.

In many ways, the Lie algebra of a group is easier to work with because $Lie(H)$ is a vector space. The representation theory of the Lie algebra and its Lie group are closely related. Furthermore, the connected component of H containing the identity matrix can be recovered from $Lie(H)$.

Observe that $Gl_n/B \approx Sl_n/(\overline{B \cap Sl_n})$. Therefore, to understand the tangent spaces to G/B we want to understand $Lie(Sl_n(\mathbb{C}))$. We will discuss Lie algebras in general first and then restrict our attention to this special case by the end of the lecture.

Consider a differentiable path $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_k(t)) \in \mathbb{R}^k$ with velocity vector

$$v = \left(\frac{\partial \phi_1(x)}{\partial t}, \frac{\partial \phi_2(x)}{\partial t}, \dots, \frac{\partial \phi_k(x)}{\partial t} \right),$$

then v is tangent to ϕ at $x \in \mathbb{R}^k$.

Definition 5.8. Given any subset $S \subseteq \mathbb{R}^k$, v is *tangent* to S at $x \in S$ if there exists a differentiable path $\phi(t)$ lying in S such that $\phi(0) = x$ and $\phi'(0) = v$.

Recall the *gradient* of a function $f(x) = f(x_1, \cdots, x_k)$ is

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \dots \\ \frac{\partial f}{\partial x_k} \end{pmatrix} \tag{5.7}$$

Lemma 5.9. *If $S = V(f_1, \dots, f_r) \subseteq \mathbb{R}^k$ is a closed set in the Zariski topology, and f vanishes on S, then the tangent vectors to S at $x \in S$ are orthogonal to the gradient of f evaluated at x.*

Proof. Let $\phi(t)$ be a differentiable path in S such that $\phi(0) = x$ and $\phi'(0) = v$. Then $f(y) = 0$ for all $y \in S \Rightarrow \frac{d}{dt} f(\phi(t)) = 0$ for all $t \in \mathbb{R}$. This implies $\frac{d}{dt} f(\phi(t)) = \frac{\partial f}{\partial x_1}(x) \frac{\partial \phi_1}{\partial t}(0) + \dots + \frac{\partial f}{\partial x_k}(x) \frac{\partial \phi_k}{\partial t}(0) = (\nabla f(x).v) = 0$. □

Remark 5.10. Two paths $\phi(t)$, $\xi(t)$ have the same tangent at $x = 0$ if and only if the first two terms of their Taylor expansions agree. That is, $\phi(t) = a + bt + \dots$ and $\xi(t) = a + bt + \dots$. Therefore, we only need to consider paths with coefficients in $E = \{a + b\epsilon : a, b \in \mathbb{R}\}$, with $\epsilon^2 = 0$. Note that E is a ring with multiplication and addition defined by $(a + b\epsilon)(c + d\epsilon) = ac + (ad + bc)\epsilon$, and $(a + b\epsilon) + (c + d\epsilon) = (a + b) + (c + d)\epsilon$.

Using this map on Taylor series, the set of all paths $\phi(t)$ with $\phi(0) = x$ and $\phi'(0) = v$ map to vector $x + v\epsilon = (x_1 + v_1\epsilon, \dots, x_k + v_k\epsilon)$, with $x_i, v_i \in \mathbb{R}$.

Definition 5.11. Define v to be an *infinitesimal tangent* of $S = V(f_1, \dots, f_r)$ at x if $f(x + v\epsilon) = 0$ as a polynomial over E for all f vanishing on S.

Lemma 5.12. *Every tangent to $S = V(f_1, \dots, f_r)$ at $x \in S$ is an infinitesimal tangent.*

Proof. Given a polynomial $f(x) = f(x_1, \dots, x_k)$ over E, then by multivariate Taylor expansion

$$f(x + v\epsilon) = f(x) + (v_1\epsilon \frac{\partial f}{\partial x_1}(x) + \dots + v_k\epsilon \frac{\partial f}{\partial x_k}(x)) + (v_1\epsilon \frac{\partial}{\partial x_1} + \dots + v_k\epsilon \frac{\partial}{\partial x_k})^2 f(x) + \dots \tag{5.8}$$

All the degree 2 and higher terms in this expansion vanish over E, hence

$$f(x + v\epsilon) = f(x) + (\nabla f(x).v)\epsilon.$$

If f vanishes on S, then $f(x + v\epsilon) = 0$ if and only if $\nabla f(x).v = 0$. Apply Lemma 5.9 to conclude v is an infinitesimal tangent. □

When S is sufficiently smooth then the converse to Lemma 5.12 holds, namely every infinitesimal tangent is a tangent vector. We will state this precisely for linear algebraic groups and prove it for SL_n.

Definition 5.13. A group H is a *linear algebraic group* if it is both a subgroup of Gl_n and it is defined by the vanishing of some set of polynomials. For example, $Sl_n = V(\det -1)$ is a linear algebraic group.

Recall that for any matrix A,

$$e^{tA} = I + tA + t^2\frac{A^2}{2!} + \dots.$$

If $\{e^{tA} : t \in \mathbb{R}\}$ is a one-parameter subgroup of H, we have a parameterized path $\phi(t) = e^{tA}$ through the identity $\phi(0)$ with velocity vector $\frac{\partial\phi}{\partial t}(0) = A$. Therefore, A is a tangent vector of H at the identity so $A \in Lie(H)$.

Fact 5.14. Let H be a linear algebraic group. For every infinitesimal tangent A of H at the identity, (i.e. $f(I + At) = 0$ for all f vanishing on H) the group $\{e^{tA} : t \in \mathbb{R}\}$ is a subgroup of H.

Corollary 5.15. $Lie(H) = \{A \in M_{n\times n} : I + A\epsilon \in H\}$.

For example, let $H = SL_n = V(\det -1)$. The vector $I + A\epsilon \in H$ if and only if $\det(I + A\epsilon) = 1$. Observe that

$$I + A\epsilon = \begin{pmatrix} 1 + a_{11}\epsilon & a_{12}\epsilon & \dots & a_{1n}\epsilon \\ a_{21}\epsilon & 1 + a_{22}\epsilon & \dots & a_{2n}\epsilon \\ \dots & \dots & \ddots & \dots \\ a_{n1}\epsilon & a_{n2}\epsilon & \dots & 1 + a_{nn}\epsilon \end{pmatrix} \tag{5.9}$$

so $\det(I + A\epsilon) = 1 + (a_{11} + \dots + a_{nn})\epsilon$, which is 1 if and only if $tr(A) = 0$. Therefore,

$$Lie(Sl_n(\mathbb{F})) = \{A \in M_{n\times n} : tr(A) = 0\} = \mathfrak{sl}_n.$$

Fact 5.14 is equivalent to saying $tr(A) = 0 \Leftrightarrow \det(e^{tA}) = 1$ for all $t \in \mathbb{R}$. To prove the fact in this case we simply note that

$$\det(e^B) = e^{tr(B)}.$$

Definition 5.16. A *Lie algebra* V over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ with a law of composition $V \times V \to V$ mapping $(a, b) \mapsto [a, b]$ such that

1. $[,]$ is bilinear.
2. $[a, a] = 0$.
3. The Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Exercise 5.17. For $\mathbf{sl}_n$, the bracket operation is defined by

$$[A, B] = AB - BA$$

Verify Properties 1,2,3 hold for this law of composition on $\mathbf{sl}_n$.

6. Criteria for Smoothness of X_w (April 11,2007)

Recall, $\mathfrak{g} = Lie(Sl_n(\mathbb{F})) = \{A \in M_{n\times n} : tr(A) = 0\} = \mathfrak{sl}_n$. Observe that $\mathfrak{g}$ has a vector space basis

$$\mathcal{B} = \{E_{ij} : i \neq j\} \cup \{H_i : 1 \leq i < n\}, \tag{6.1}$$

where E_{ij} is the matrix whose ij^{th} entry is 1 and all other entries are 0 and H_i is the matrix whose ii^{th} entry is 1, $(i + 1, i + 1)^{st}$ entry is -1 and all other entries are 0.

Exercise 6.1. Compute $[E_{ij}, E_{kl}], [E_{ij}, H_k],$ and $[H_i, H_j]$ to find which elements of $\mathcal{B}$ commute.

For B the subgroup of upper triangular matrices in Sl_n, Corollary 5.15 and Equation 6.1 implies $\mathfrak{b} = Lie(B \cap Sl_n)$ has a basis $\{E_{ij} : i < j\} \cup \{H_i : 1 \leq i < n\}$. Therefore the tangent space to G/B at I is equal to the tangent space to $Sl_n/(B \cap Sl_n)$ at I which is equal to $\mathfrak{g}/\mathfrak{b}$. We have bijections

$$\mathfrak{g}/\mathfrak{b} = Span\{E_{ji} : i < j\} \longleftrightarrow \{t_{ij} \in S_n\} = R \longleftrightarrow \{e_j - e_i : i < j\} := \Delta_+. \tag{6.2}$$

More generally, for $v \in S_n$,

$$v^{-1}(\mathfrak{g}/\mathfrak{b})v = Span\{v^{-1}E_{ji}v : i < j\} = Span\{E_{v(j)v(i)} : j < i\} \tag{6.3}$$

is the tangent space to G/B at v. Note, $E_{ji}v = E_{j,v(i)}$ and $v^{-1}E_{ji} = E_{v(j),i}$ so $v^{-1}E_{ji}v = E_{v(j)v(i)}$.

Definition 6.2. Let $T_w(v)$ be the tangent space to X_w at v.

Theorem 6.3. *(Lakshmibai-Seshadri) We have that*

$$T_w(v) = Span\{E_{v(j)v(i)} : i < j \text{ and } t_{v(j)v(i)}v \leq w\} \tag{6.4}$$

$$= Span\{E_{v(j)v(i)} : i < j \text{ and } vt_{ij} \leq w\} \tag{6.5}$$

Proof. The two sets are equivalent since $t_{v(j)v(i)}v = vt_{ij}$. $X_w \subseteq X_{w_0} = G/B$ implies $T_w(v) \subseteq T_{w_0}(v)$. Thus, we only need to identify which $E_{v(j)v(i)} \in T_w(v)$ i.e. which $I + E_{v(j)v(i)}\epsilon \in X_w$ by Corollary 5.15.

If v is the identity, then

$$rk(I + E_{ji}\epsilon) = rk \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & & \\ 0 & & 1_i & & & & \\ 0 & & \vdots & \ddots & & & \\ 0 & & \epsilon & \cdots & 1_j & & \\ 0 & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} = rk \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & & \\ 0 & & 0_i & \cdots & 1 & & \\ 0 & & \vdots & \ddots & \vdots & & \\ 0 & & 1 & \cdots & 0_j & & \\ 0 & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix},$$

which implies $rk(I + E_{ji}\epsilon) = rk_{\swarrow}(t_{ji})$. Therefore $I + E_{ji}\epsilon \in X(w)$ if and only if $t_{ji} \leq w$ in Bruhat order.

Similarly, if $v \leq w$, say $v(j) = k$ and $v(i) = \ell$.

$$rk_{\swarrow}(v + E_{ji}\epsilon) = rk_{\swarrow} \begin{pmatrix} & & 1 & & & \\ & & & & & 1 \\ & 1 & & & & \\ & & \vdots & & & \\ & & \epsilon & \cdots & & 1 \\ 1 & & & & & \\ & & & & & \end{pmatrix} = \begin{Bmatrix} rk_{\swarrow}(v) & \text{if } l(v) > l(t_{kl}v) \\ rk_{\swarrow}(t_{kl}v) & \text{if } l(v) < l(t_{kl}v) \end{Bmatrix}.$$

Thus, $rk_{\swarrow}(v + E_{ji}\epsilon) = rk_{\swarrow}w$ if and only if both $v \leq w$ and $vt_{ji} \leq w$. $\square$

Corollary 6.4. *For all $v \leq w$,*

(1) $dimT_w(v) = |\{t_{ij} \in R : vt_{ij} \leq w\}|$.
(2) X_w *is smooth at* $v \Leftrightarrow dimT_w(v) = l(w) \Leftrightarrow |\{t_{ij} \in R : vt_{ij} \leq w\}| = l(w)$.
(3) $dimT_w(v) \geq l(w)$.
(4) $|\{t_{ij} \in R : v < vt_{ij} \leq w\}| \geq l(w) - l(v)$.

Exercise 6.5. Prove Property 3 in Corollary 6.4 using $|\{t_{ij} \in R : vt_{ij} \leq w\}| = dimT_w(v)$.

Is $X(4231)$ smooth? No! There are 6 transpositions in the interval from the identity to 4231 in the Hasse diagram for Bruhat order, but $l(4231) = 5$. We have that $v = 2143$ is not a smooth point of X_{4231} along with its lower order ideal in Bruhat order.

Fact 6.6. $Sing(X(4231)) = X(2143)$ and $Sing(X(3412)) = X(1324)$. (Memorize these patterns!) All other $X(w) \in Sl_4/B$ are smooth.

Definition 6.7. The *Bruhat graph* of w has vertices $v \leq w$ and edges between u and v if $u = vt_{ij}$ for some i, j.

Fact 6.8. The dimension of $T_w(v)$ is equal to the degree of v in the Bruhat graph of w.

Geometric Interpretation: Say $v < vt_{ij}$. Then

$$L_v^{(i,j)} = \left\{ \begin{pmatrix} & & & 1 & \\ & 1_i & & & \\ & \vdots & & & \\ & * & \cdots & & 1_j \\ 1 & & & & \end{pmatrix} : 1_k \text{ is on the } k^{th} \text{ row and } * \in \mathbb{C} \right\} \bigcup \left\{ \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \ddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \right\}$$
$$= \{v + E_{jv(i)}\} \cup \{vt_{ij}\}.$$

$L_v^{(i,j)}$ is fixed by left multiplication by T.

Example 6.9. Let $v = 4231$. Then left multiplication by T does the following:

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & t_4 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & 1 & & \\ & * & 1 & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} & & & t_1 \\ & t_2 & & \\ & *t_3 & t_3 & \\ t_4 & & & \end{pmatrix}.$$

Exercise 6.10. Show

$\{L_v^{(i,j)}, v \in S_n, i < j : v < vt_{ij}\} = \{1-\text{dimensional subvarieties of } X(w) \text{ fixed by } T\text{-action}\}$.

Remark 6.11. A *GKM space* (Goresky-Kac-MacPherson) is any symplectic manifold with a torus T action with a finite number of T fixed points and a finite number of T fixed 1-dimensional orbits. Then this data determines a graph just like the Bruhat graph. One asks, What can we say about the singularities and cohomology in terms of this graph?

Theorem 6.12. *(Uber Smoothness Theorem) The following are equivalent for any $w \in S_n$:*

(1) X_w *is smooth.*
(2) X_w *is smooth at* id*.*
(3) $|\{t_{ij} \leq w\}| = l(w)$.
(4) *The Bruhat graph for w is regular.*
(5) $P_{id,w}(q) = 1$.
(6) $P_{v,w}(q) = 1$ *for all* $v \leq w$.
(7) $r_w(t) = \sum_{v \leq w} t^{l(v)}$ *is palindromic.*
(8) $r_w(t) = \prod_{j=1}^{k} (1 + q + \ldots + q^{i_j})$ *for some* $\{i_1, \ldots, i_k\} \in \mathbb{N}$.
(9) w *avoids* 3412 *and* 4231.

Definition 6.13. w *avoids* 3412 and 4231 means there is no $1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq n$ such that $fl(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}) \in \{3412, 4231\}$. Here for any set $\{d_1, d_2, \ldots, d_k\}$ of distinct real numbers, $fl(d_1, \ldots, d_k) = (v_1, \ldots, v_k)$ if d_i is the v_i^{th} largest element among $\{d_1, \ldots, d_k\}$.

Open Question 6.14. *Explain why S_4 is enough to determine all singularities.*

If G is semisimple, then smoothness(1,2 of Uber Theorem) implies rational smoothness(4,5,6,7,8') implies combinatorial smoothness(3). In Type A, all three types of smoothness are equivalent. In Types D and E, smoothness is equivalent to rational smoothness (Petersen, Carell-Kuttler). All three flavors of smoothness can be characterized by pattern avoidance using root systems (Billey-Postnikov).

6.1. **Kazhdan-Lusztig Polynomials in a Nutshell.** The Kazhdan-Lusztig polynomial $P_{v,w}(q)$ is a polynomial in 1 variable with the following properties:

(1) $P_{v,w}(q)$ has constant term 1 if $v \leq w$.
(2) The degree of $P_{v,w}(q)$ is less than or equal to $\dfrac{l(w)-l(v)-1}{2}$.
(3) $P_{w,w}(q)) = 1$.
(4) $P_{v,w}(q) \neq 0 \Leftrightarrow v \leq w$.
(5) If $s = t_{i,i+1}$, $ws < w$, and $vs < v$, then
$$P_{v,w}(q)) = qP_{vs,ws}(q) + P_{v,ws}(q) - \sum_{zs<z<ws} \mu(z, ws) q^{\frac{l(w)-l(z)}{2}} P_{v,z}(q),$$
where $\mu(x, y$ is the coefficient of $q^{\frac{l(y)-l(x)-1}{2}}$ in $P_{x,y}(q)$.
(6) If $ws < w$ and $vs > v$ then $P_{v,w}(q)) = P_{vs,w}(q)$.

Theorem 6.15. *Let $IH(w)$ be the intersection cohomology sheaf of X_w with respect to middle perversity. Then*

(1) $P_{v,w}(q) = \sum dim(IH^{2j}(X_w)_v)q^j$.
(2) $\sum_{v \leq w} P_{v,w}(q)q^l(v) = \sum dim(IH^{2j}(X_w))q^j$.
(3) $P_{x,w}(q) = 1$ *for every* $x \leq w$ *if and only if* X_w *is rationally smooth.*

In Theorem 6.15, the first statement implies the coefficients of $P_{v,w)}(q)$ are nonnegative.

Open Question 6.16. *Find an all positive formula for $P_{v,w}(q)$.*

The second statement implies $\sum_{v \leq w} P_{v,w}(q)q^{l(v)}$ is palindromic. We will take the third statement to be the definition for rational smoothness.

7. The "uber" theorem (April 13, 2007)

Theorem 7.1 (Uber Theorem)**.** *The following are equivalent for $w \in S_n$:*

(1) X_w *is smooth.*
(2) X_w *is smooth at* $id \in S_n$.
(3) $\#\{t_{ij} : t_{ij} \leq w\} = \ell(w)$.
(4) *The Bruhat graph of* w *is regular of degree* $\ell(w)$.
(5) $P_{id,w}(q) = 1$.
(6) $P_{x,w}(q) = 1$ *for all* $x \leq w$.
(7) $r_w(t) = \sum_{v \leq w} t^{\ell(v)}$ *is palindromic.*
(8) $r_w(t) = \prod_j (1 + t + t^2 + \cdots + t^{i_j})$.
(9) w *avoids the patterns* 3412 *and* 4231.

Today's lecture is devoted to the proof of the Uber Theorem.

Proof of the Uber Theorem: A number of the cases follow from results we have discussed in past lectures.

$\underline{(1) \Leftrightarrow (2)}$: follows since the singular locus is closed.

$\underline{(2) \Leftrightarrow (3) \Leftrightarrow (4)}$: is a corollary of the Lakshmibai-Seshadari Theorem.

$\underline{(5) \Leftrightarrow (6)}$: follows from a theorem of Ron Irving.

Theorem 7.2 (Irving)**.** *Suppose*

$$P_{v,w}(q) = 1 + a_1 q + a_2 q^2 + \cdots + a_k q^k, \text{ and}$$
$$P_{x,w}(q) = 1 + b_1 q + b_2 q^2 + \cdots + b_k q^k.$$

If $v \leq x \leq w$, *then* $a_i \geq b_i$ *for all* $1 \leq i \leq k$.

$\underline{(4) \Leftrightarrow (6) \Leftrightarrow (7)}$: follows from the work of Carroll-Peterson.

Example 7.3. We see that $r_{4231} = 1 + 3t + 5t^2 + 6t^3 + 4t^4 + t^5$ is not palindromic.
However $r_{4213} = 1 + 3t + 4t^2 + 3t^3 + t^4$ is palindromic, and factors as $(1+t)^2(1+t+t^2)$.

Pop quiz: Which X_w are smooth among the following w? Which characterization from the uber theorem seems the easiest to apply? (Class consensus: (9).)

$w = 45678123 \in S_8$? Contains subsequence 4512, $fl(4512) = 3412$. Singular.
$w = 7432651 \in S_7$? Contains subsequence 7351, $fl(7351) = 4231$. Singular.
$w = 7654321 \in S_7$? No ascents. Avoids 3412 and 4231. Smooth.
$w = 7132645 \in S_7$? Avoids 3412 and 4231. Smooth.
$w = 7263154 \in S_7$? Contains subsequence 7231, $fl(7231) = 4231$. Singular.

We still need to show that

(1) w avoids the patterns 3412 and 4231 implies that $r_w(t)$ factors "nicely", that this then implies that $r_w(t)$ is palindromic, and that this implies that X_w is smooth.
(2) w contains one of the patterns 3412 and 4231 implies that $r_w(t)$ is not palindromic, and that therefore $r_w(t)$ does not factor "nicely".

Lemma 7.4. *For* $w \in S_n$, $r_w(t)$ *factors (nicely) if one of the following hold:*

Rule 1: *If* $w = [\ldots, n, w_{k+1}, \ldots, w_n]$ *with* $n = w_k > w_{k+1} > \cdots > w_n$, *then*

$$r_w(t) = (1 + t + t^2 + \cdots + t^{n-k}) \cdot r_{w'}(t),$$

where $w' = [w_1, \ldots, \widehat{w_k}, \ldots, w_n] \in S_{n-1}$ *(or equivalently,* $w = w' s_{n-1} s_{n-2} \cdots s_k$, *where* $s_j = t_{j,j+1}$ *is the adjacent transposition indexed by* j*).*

Rule 2: *Say that w contains a* consecutive sequence *if for some $1 \le j \le n$ the $j+1$ integers $\{n-j, n-j+1, \ldots, n\}$ appear in the one-line notation of w in decreasing order, with $w_n = n-j$. If $w = [\ldots, n, \ldots, n-1, \ldots, n-j]$ contains the consecutive sequence of length $j+1$, then*

$$r_w(t) = (1 + t + t^2 + \cdots + t^j) \cdot r_{w'}(t),$$

where $w' = s_{n-1}s_{n-2}\cdots s_{n-j}w$.

Example 7.5 (Rule 2 example:). $w = 7132645 \in S_7$ contains the consecutive sequence 765.
$r_w(t) = (1+t+t^2) \cdot r_{w_6}(t)$, where $w_6(t) = 613254$, which contains 654.
$r_w(t) = (1+t+t^2)^2 \cdot r_{w_5}(t)$, where $w_5(t) = 51324$, which contains 54.
$r_w(t) = (1+t+t^2)^2(1+t) \cdot r_{w_4}(t)$, where $w_4(t) = 4132$, which contains 432.
$r_w(t) = (1+t+t^2)^3(1+t) \cdot r_{w_3}(t)$, where $w_3(t) = 312$, which contains 32.
$r_w(t) = (1+t+t^2)^3(1+t)^2 \cdot r_{w_2}(t)$, where $w_2(t) = 21$.
$r_w(t) = (1+t+t^2)^3(1+t)^3$.

Proof of Lemma 7.4. Note that $w_j = i \Leftrightarrow w^{-1}(i) = j$. Note also that w satisfies the hypotheses for Rule 1 if and only if w^{-1} satisfies the hypotheses for Rule 2. *Why?* By Chevalley's Criterion for Bruhat order and because $[id, w]$ and $[id, w^{-1}]$ are isomorphic posets as intervals in the Bruhat order. Thus it suffices to prove Rule 1. For the next part of the proof, you are asked to do the following exercise.

Exercise 7.6. Show that if $u, v \in S_n$ such that $u_k = v_k$, then $u \le v$ if and only if

$$fl(u_1, \ldots, \widehat{u_k}, \ldots, u_n) \le fl(v_1, \ldots, \widehat{v_k}, \ldots, v_n).$$

Claim: If $n = w_k > w_{k+1} > \cdots > w_n$, then the interval $[id, w]$ in the Bruhat order can be partitioned into $n-k+1$ blocks $\{B_i\}$ such that each block B_i with the induced order is isomorphic to the interval $[id, w']$, and the smallest element in each block has length $0, 1, 2, \ldots, n-k$ respectively.

Proof: Let $B_i = \{x \le w : x_i = n\}$. By Exercise 7.6, the Bruhat order on B_i induces a poset isomorphic to the interval $[id, w']$, where $w' = [w_1, \ldots, \widehat{w_k = n}, \ldots, w_n] \in S_{n-1}$. Since $n = w_k > w_{k+1} > \cdots > w_n$, the maximum length element in B_n is w', the maximum length element in B_{n-1} is $w's_{n-1}$, etc., and in general, he maximum length element in B_k is $w's_{n-1}s_{n-2}\cdots s_k$. Similarly, the minimal length element in B_i in general is $[1, 2, \ldots, i-1, n, i, i+1, \ldots, n-1]$, which is of length $n-i$. Therefore the map $\varphi : \{0, 1, 2, \ldots, n-k\} \times [id, w'] \to [id, w]$ given by $(i, x) \mapsto [x_1, x_2, \ldots, x_{i-1}, n, x_i, x_{i+1}, \ldots, x_{n-1}]$ is a bijection. Moreover, this bijection is length-preserving in the sense that $\ell(x) + i = \ell(\varphi(x))$, and the bijection respects Bruhat order. This establishes the claim.

From the claim it follows that $r_w(t) = (1 + t + t^2 + \cdots + t^{n-k}) \cdot r_{w'}(t)$. $\square$

Exercise 7.7. Show that the interval $[id, w]$ in the Bruhat order has a symmetric chain decomposition if w is smooth.

Corollary 7.8. *The interval $[id, w]$ is rank-symmetric, rank-unimodal, and K-Sperner if w is smooth.*

Lemma 7.9. *If $w \in S_n$ avoids the patterns 3412 and 4231, then w satisfies the hypotheses of either Rule 1 or Rule 2 of Lemma 7.4. Moreover, the respective $w' \in S_{n-1}$ also avoids the patterns 3412 and 4231.*

Proof. Regarding the latter part of the statement of the lemma, if Rule 1 applies and w avoids both patterns, then w' also avoids the patterns since subsequences of w' are also subsequences of w. If Rule 2 applies, then for any $\{i_1, i_2, i_3, i_4\} \subset \{1, \ldots, n-1\}$, we have $fl(w'_{i_1}, w'_{i_2}, w'_{i_3}, w'_{i_4}) = fl(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4})$, so again, if w avoids the patterns 3412 and 4231, then so does w'. It remains to show that w satisfies the hypotheses of either Rule 1 or Rule 2.

Say w_d = n. We consider cases by the value of d.

Case $d = n$ or $n-1$. In this case, Rule 1 applies.

Case $d = n-2$. If $w_n = n-1$, then Rule 2 applies. If $w_{n-1} = n-1$, then Rule 1 applies. Otherwise $w_i = n-1$ for some $i < d$. It cannot be the case that $w_{n-1} < w_n$, for otherwise $fl(w_i = n-1, w_d =$

$n, w_{n-1}, w_n) = (3412)$ and so w would contain the pattern 3412. Thus it must be that $w_{n-1} > w_n$, and Rule 1 applies.

Case $d < n-2$. If $w_d > w_{d+1} > \cdots > w_n$, then Rule 1 applies. Otherwise $w_d > w_e < w_{e+1}$ for some $d < e < n$. Without loss of generality we may assume that e is the largest such index. If $w^{-1}(k) < d$ for any $w_{e+1} < k < n$, then $fl(k, n, w_e, w_{e+1}) = (3412)$, contradicting the assumption that w avoids this pattern. In particular, $w^{-1}(n-1) > d$, say $w_f = n-1$. If $f = n$, then Rule 2 applies. Otherwise $f \leq n-1$.

Let $w_g = n-2$. If $g < d$, then either

- $w_{e+1} \neq n-1$, when $fl(n-2, n, w_e, w_{e+1}) = (3412)$, or
- $w_{e+1} = n-1$ and $w_e < w_n$, when $fl(n-2, n, w_e, w_n) = (3412)$, or
- $w_{e+1} = n-1$ and $w_e > w_n$, when $fl(n-2, w_e, n-1, w_n) = (4231)$.

If $d < g < f$, then $fl(n, n-2, n-1, w_n) = (4231)$. All these cases contradict our assumption that w avoids these patterns. Thus we must have $g > f$, i.e. $w^{-1}(n-2) > w^{-1}(n-1)$.

Performing similar analysis for $w^{-1}(n-3)$, $w^{-1}(n-4)$, etc. successively, in general comparing $w^{-1}(k)$ against the values $\{d = w^{-1}(n), w^{-1}(n-1), \ldots, w^{-1}(k+1)\}$, and knowing that w avoids the patterns 3412 and 4231, shows that

$$d < w^{-1}(n-1) < w^{-1}(n-2) < \cdots < w^{-1}(k) < w^{-1}(k-1) < \cdots$$

until at some point when $w^{-1}(k) = n$ for some k. At this point it is seen that w contains a consecutive sequence, and Rule 2 applies.

$\square$

Lemma 7.10. *If $w \in S_n$ such that $fl(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}) = (3,4,1,2)$ or $(4,2,3,1)$, then X_w is not smooth.*

Proof. Let $w' = fl(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}) \in S_4$. Let $u' \in S_4$ be such that $Sing\ X(w') = X(u')$. Then either $u' = 1324$ or $u' = 2143$. Then $\#\{t_{ij} \in S_4 : u't_{ij} \leq w'\} > \ell(w')$ by the Lakshmibai-Seshadari Theorem.

Let $u \in S_n$ such that $u_i = w_i$ for all $i \notin \{i_1, i_2, i_3, i_4\}$, and $fl(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}) = u'$. Define $D_w(x) := \{t_{ij} \in S_n : xt_{ij} \leq w\}$, and define $D_w^k(x) := \{t_{ij} \in D_w(x) : |\{i, j, i_1, i_2, i_3, i_4\}| = k\}$ for $k = 4,5,6$. Then $D_w(x) = D_w^4(x) \sqcup D_w^5(x) \sqcup D_w^6(x)$.

By Exercise 7.6, if $v, w \in S_n$ such that $v_i = w_i$, then $v \leq w$ if and only if

$$[v_1, \ldots, \widehat{v_i}, \ldots, v_n) \leq [w_1, \ldots, \widehat{w_i}, \ldots, w_n].$$

Therefore, by repeated application of this fact, to determine if $ut_{ij} \leq w$, we only need to compare on positions in $\{i, j, i_1, i_2, i_3, i_4\}$. Computer-run verifications show that for $w'' \in S_6$ containing either of the patterns 3412 or 4231 and for u'' constructed from w'' as above,

$$\begin{aligned} |D^4_{w''}(u'')| &> |D^4_{w''}(w'')| \\ |D^5_{w''}(u'')| &\geq |D^5_{w''}(w'')| \\ |D^6_{w''}(u'')| &= |D^6_{w''}(w'')| \end{aligned}$$

and hence

$$|D_{w''}(u'')| = \sum_{k=4,5,6} |D^k_{w''}(u'')| > \sum_{k=4,5,6} |D^k_{w''}(w'')| = \ell(w'').$$

Thus u is a singular point of X_w. $\square$

This concludes our proof of the Uber Theorem.

One can generalize some conditions of the Uber Theorem for intervals in the Bruhat order poset:

(4'): The Bruhat graph of $[v, w]$ is regular of degree $\ell(w) - \ell(v)$.
(5'): $P_{v,w}(q) = 1$.
(6'): $P_{x,w}(q) = 1$ for all $v \leq x \leq w$.

The following theorem was proved around the year 2000-2001 by several teams of mathematicians: Billey-Warrington, Cortez, Kassel-Lascoux-Reutenauer, and Manivel.

Theorem 7.11. *For $w \in S_n$, we have*

$$Sing(X_w) = \bigcup_{v \in maxsing(w)} X_v.$$

Moreover, $v \in maxsing(w)$ if and only if X_v is an irreducible component of $Sing(X_w)$. Also, $v \in maxsing(w)$ is obtained from w (diagrammatically) in one of three ways: ... (diagrams to be provided).

8. Grassmannians (April 18, 2007)

Open Question 8.1. *Characterize the maximal singular set of $X_w \subset G/B$ where $G = SO(2n+1)$, $SO(2n)$, $Sp(2n)$ or any other semisimple Lie group.*

Let V be a vector space over some field $\mathbb{F}$ of dimension n, so that $V \simeq \mathbb{F}^n$.

Definition 8.2. $G(k, n) = \{k\text{-dimensional subspaces of } \mathbb{F}^n\}$.

Example 8.3. $G(1, n) = \{\text{lines through the origin}\} = \mathbb{P}(V)$.

Example 8.4. $G(2, 4) = \{\text{planes through the origin in } \mathbb{F}^4\} = \{\text{affine lines in } \mathbb{P}^3\}$.

Fix a basis for V: $e_1, \ldots, e_n$. Then a k-dimensional subspace can be represented by a matrix:

$$k\text{-dimensional subspace} \longleftrightarrow \text{span}\left\{\sum a_{i1}e_i, \ldots, \sum a_{ik}e_i\right\} \longleftrightarrow \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{pmatrix}.$$

Let $\mathcal{M}_{G(k,n)}$ be the set of $n \times k$ matrices with rank k. Note that $G(k, n) \simeq G(n - k.n)$, since $A \in \mathcal{M}_{G(k,n)}$ corresponds to a linear map $A : \mathbb{F}^n \to \mathbb{F}^n$ given by $(v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_n)A$ and the kernel of this map is an $n - k$-dimensional subspace.

Exercise 8.5. Show that $\ker A = \ker B$ if and only if columns of A span the same subspace as columns of B.

Note that rescaling columns and adding to any column a linear combination of other columns does not change the span. Therefore, we can choose a column echelon form as the canonical one:

$$\begin{pmatrix} 3 & 4 & 7 \\ 9 & 2 & 9 \\ 6 & 0 & 7 \\ 3 & 0 & 3 \end{pmatrix} \xrightarrow[\text{bottom 1's}]{\text{rescale to get}} \begin{pmatrix} 1 & 2 & 7 \\ 3 & 1 & 9 \\ 2 & 0 & 7 \\ 1 & 0 & 3 \end{pmatrix} \xrightarrow[\text{right from 1's}]{\text{clear to the}} \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{left from 1's}]{\text{clear to the}} \begin{pmatrix} -5 & 2 & 4 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow[\text{left from 1's}]{\text{clear to the}} \begin{pmatrix} -13 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{columns}]{\text{rearrange}} \begin{pmatrix} 2 & 4 & -13 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that all these operations can be realized as multiplication by some invertible matrices.

Claim 8.6. If $A \in \mathcal{M}_{G(k,n)}$, then there exists $g \in Gl_K(\mathbb{F})$ such that $Ag = B$ and B is in the canonical form, i.e. it has the form

$$\begin{pmatrix} * & * & * \\ 1 & 0 & 0 \\ 0 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & * \\ 0 & 0 & 1 \end{pmatrix}$$

with zeros below each of the ones and zeros to the left and right of each one.

Let $i_1, \dots, i_k$ be the indices of rows containing ones.

Claim 8.7. There is a bijection $G(k,n) \longleftrightarrow \mathcal{M}_{G(k,n)}/\langle A = Bg, g \in Gl_k \rangle$.

For $U \in G(k,n)$ define $\mathcal{M}(U)$ to be the corresponding matrix in the canonical form. Then columns of $\mathcal{M}(U)$ span U. For any k-subset $\{i_1 < \dots, i_k\} \subset \{1, \dots, n\} = [n]$ define the Schubert cell

$$C_{\{i_1,i_2,\dots,i_k\}} = \left\{ U \in G(k,n) \text{ such that } \mathcal{M}(U) \text{ has the form } \begin{array}{cc} \begin{pmatrix} 1 & & & \\ & & & \\ & 1 & & \\ & & 1 & \\ & & & \\ & & & 1 \\ & & & \end{pmatrix} & \begin{array}{l} \leftarrow i_1 \\ \\ \\ \\ \\ \leftarrow i_k \\ \\ \end{array} \end{array} \right\}.$$

Then $G(k,n) = \bigcup_{\{i_1,i_2,\dots,i_k\} \subset [n]} C_{\{i_1,i_2,\dots,i_k\}}$. Schubert varieties in $G(k,n)$ are $X_{\{i_1,i_2,\dots,i_k\}} = \overline{C_{\{i_1,i_2,\dots,i_k\}}}$.

Claim 8.8. There exists a map $\phi : Gl_n/B_n \to G(k,n)$ given by $A \mapsto \operatorname{span}\{\operatorname{col}_1(A), \dots, \operatorname{col}_k(A)\}$.

Proof. Since B_n adds previous columns to later ones, it does not change spans of this kind. Thus ϕ is well defined. $\square$

Define parabolic subgroups of Gl_n by

$$P_{k,n-k} = \left\{ \left(\begin{array}{c|c} * \in Gl_k & * \\ \hline 0 & * \in Gl_{n-k} \end{array} \right) \right\}.$$

Fact 8.9. $G(k,n) \simeq Gl_n/P_{k,n-k}$.

Follows from the fact that the map ϕ forgets everything but the span informations of the first k columns.

Fact 8.10. Since we have $Gl_n/B_n \twoheadrightarrow Gl_n/P_{k,n-k} \simeq G(k,n)$, we get $H^*(G(k,n)) \hookrightarrow H^*(Gl_n/B_n)$ on cohomology.

Note that the map ϕ is given by $\phi(X_w) = X_{\mathrm{sort}\left\{w_1^{-1},\ldots,w_k^{-1}\right\}}$. Starting with a matrix in $G(k,n)$ we can extend it in the following way without adding new $*$'s and thus obtaining a bijection:

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \left(\begin{array}{cccc|cccc} * & * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & \boxed{0} & 1 & 0 & 0 \\ * & * & * & * & 0 & \boxed{0} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & * & * & * & 0 & 0 & \boxed{0} & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{0} & 1 & 0 & 0 & 0 & 0 \end{array}\right),$$

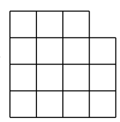
note the essential set.

Definition 8.11. Let $w\{i_1,\ldots,i_k\} = \left\{i_1,\ldots,i_k,1\ldots,\hat{i_1},\ldots,\hat{i_2},\ldots,\hat{i_k},\ldots,n\right\}$.

Warning 8.12. $X_{w\{i_1,\ldots,i_k\}}$ is not homeomorphic to $X_{\{i_1,\ldots,i_k\}}$.

In order to determine which Schubert cells $C_{\{j_1,\ldots,j_k\}}$ are in $X_{\{i_1,\ldots,i_k\}}$, it is convenient to look at partitions.

Lemma 8.13. *There is a bijection between k-subsets of $[n]$ and partitions $(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$ such that $\lambda_k \leq n-k$, given by $\{i_1,\ldots,i_k\} \longleftrightarrow (i_1-1,\ldots,i_k-k)$.*

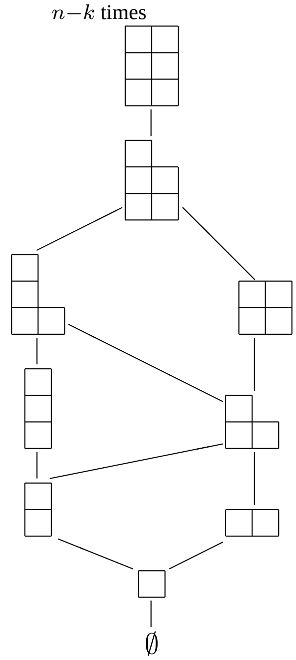
Example 8.14. $\{4,6,7,8\} \longleftrightarrow (3,4,4,4) \longleftrightarrow$ .

Let $|\lambda| = \sum \lambda_i$. Then $\dim C_\lambda = |\lambda|$. (We can index Schubert cells and varieties by k-subsets or partitions.)

Theorem 8.15. *$X_\mu \subset X_\nu$ if and only if $\mu \subset \nu$ as Ferrer's diagrams, i.e. $\mu_i \leq \nu_i$ for $i = 1,\ldots,k$.*

This partial order is isomorphic to $[\Phi, (\underbrace{k,k,\ldots,k}_{n-k \text{ times}})]$ in Young's lattice.

Example 8.16. For $G(3,5)$ we get



This is a rank symmetric self dual lattice (for any two elements there exists a unique minimal one covering both of them).

Now let's put 1's in decreasing order:

$$\left(\begin{array}{cccc|cccc} * & * & * & * & * & * & * & 1 \\ * & * & * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & \boxed{0} & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Notice that the essential set is always in the column to the left from the line.

Definition 8.17. $W\{i_1, \dots, i_k\}$ = "largest permutation in Bruhat order mapping to $\{i_1, \dots, i_k\}$" = $\Big(n, n-1, \dots, \hat{i_k}, \dots, \widehat{i_{k-1}}, \dots, \hat{i_1}, \dots, 1, i_k, i_{k-1}, \dots, i_1\Big)$.

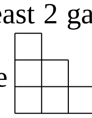
We will show that equations for $X_{\{i_1,\dots,i_k\}} \subset G(k,n)$ are the same as for $X_{W\{i_1,\dots,i_k\}} \subset G/B$.

Fact 8.18. $X_{W\{i_1,\dots,i_k\}} \twoheadrightarrow X_{\{i_1,\dots,i_k\}}$ and this is a smooth morphism.

When is $X_{W\{i_1,\dots,i_k\}}$ smooth? Note that 3412 cannot happen since $W\{i_1, \dots, i_k\}$ is a concatenation of two decreasing sequences. But our example has 4231:

$$\left(\begin{array}{cccc|cccc} * & * & * & * & * & * & * & 1 \\ * & * & * & * & * & * & 1 & 0 \\ * & * & * & \boxed{*} & * & \boxed{1} & 0 & 0 \\ * & * & * & \boxed{1} & 0 & 0 & 0 & 0 \\ * & * & \boxed{*} & 0 & \boxed{1} & 0 & 0 & 0 \\ * & * & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where $\boxed{*}$ correspond to inner corners of the partition.

Claim 8.19. If 4231 exists in $W\{i_1, \dots, i_k\}$, then there are at least 2 gaps in $\{i_1, \dots, i_k\}$ and Ferrer's diagram for the corresponding partition λ has internal corners like .

Theorem 8.20. $X_\lambda \subset G(k,n)$ *is smooth if and only if* λ *is a rectangle (i.e. does not have internal corners).*

If λ is a rectangle (i^m), then $X_\lambda = G(i, m+i)$.

Exercise 8.21. What is the set E_μ such that $\operatorname{sing}(X_\lambda) = \bigcup_{\mu \in E_\lambda} X_\mu$?

Now let's consider equations for X_λ and Plücker coordinates for $G_{\mathbb{R}}(2,4) = \{$2-planes in $\mathbb{R}^4\}$, i.e.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} /Gl_2.$$

Let

$$P_{ij}(X) = \det \begin{vmatrix} X_{i1} & X_{i2} \\ X_{j1} & X_{j2} \end{vmatrix}.$$

Let

$$P(A) = (P_{12}(A), P_{13}(A), P_{14}(A), P_{23}(A), P_{24}(A), P_{34}(A)).$$

Then $P(A) \in \mathbb{P}^5$ for any $A \in \mathcal{M}_{G(k,n)}$. Now if $A = Bg$, then $P_{ij}(A) = P_{ij}(B) \cdot \det(g)$, so $P(A) = P(B)$ in $\mathbb{P}^5$ if and only if the span of columns of A is equal to the span of columns of B.

There is the following relation between Plücker coordinates, which follows from Sylvester's Lemma: $P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14} = 0$.

Theorem 8.22. *$G(2,4) = V(P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14})$ in $\mathbb{P}^5$.*

Why they are equal? Look for the largest (i,j) in the lexicographical order such that $p \in V(P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14})$ does not vanish and rescale the coordinates so that $P_{ij} = 1$. Put 1's into $(i,1)$ and $(j,2)$. Put 0's below them and into $(i,2)$. For example, for $(i.j) = (2,4)$ we get

$$\begin{pmatrix} * & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{pmatrix}.$$

Now all a_{rs} in this matrix can be recovered: $a_{r1} = P_{ir}$ and $a_{r2} = P_{rj}$.

In general we have

$$P_{i_1\ldots i_k}(X) = \det \begin{vmatrix} X_{i_1 1} & \ldots & X_{i_1 k} \\ \vdots & \vdots & \vdots \\ X_{i_k 1} & \ldots & X_{i_k k} \end{vmatrix},$$

$$P(A) = (P_{i_1\ldots i_k}(A) : \{i_1 < \cdots < i_k\} \subset [n]) \in \mathbb{P}^{\binom{n}{k}-1}.$$

Theorem 8.23. $G(k,n) \simeq V(\text{Plücker equations}) \subset \mathbb{P}^{\binom{n}{k}-1}$*, where equations are constructed in the following way: for each $d \in [k]$*

$$P_{i_1\ldots i_k}P_{j_1\ldots j_k} - \sum P_{i'_1\ldots i'_k}P_{j'_1\ldots j'_k}$$

with the sum over all (i', j') obtained from (i,j) by picking $1 \le q_1 < q_2 < \cdots < q_d \le k$ positions and exchanging i_{q_s} with j_s for $s \in [d]$:

$$i' = (i_1, \ldots, i_{q_1-1}, j_1, i_{q_1+1}, \ldots, i_{q_2-1}, j_2, i_{q_2+1}, \ldots, i_k),$$
$$j' = (i_{q_1}, i_{q_2}, \ldots, i_{q_d}, j_{d+1}, \ldots, j_k).$$

Exercise 8.24. Compare equations for $X_{\{i_1,\ldots,i_k\}}$ to the equations for the essential set for $X_{W\{i_1,\ldots,i_k\}}$.

9. Partial Flags, Intersecting and Chow cohomology (April 20, 2007)

Definition 9.1. For $0 < d_1 < d_2 < \cdots < d_p \le n$, define the *partial flag manifold* $Fl(n, d_1, d_2, \cdots, d_p) = Gl_n/P \approx \{F_\bullet = F_{d_1} \subset F_{d_2} \subset \cdots \subset Fd_p = \mathbb{F}^n\}$, with $dimF_{d_i} = d_i$. Where P is the set of upper triangular block matrices with block widths (and heights) $d_1, d_2 - d_1, \cdots, n - d_p$.

In the Grassmannian we have the bijections:

$$\begin{aligned}\{K-sheets\} \Leftrightarrow &\{\text{partitions in a } k\times(n-k) \text{ box}\}\\ \Leftrightarrow &S_k\times S_{n-k}\setminus S_n \text{ mod on left}\\ \Leftrightarrow &\text{max length coset representative in } S_k\times S_{n-k}\setminus S_n\\ \Leftrightarrow &\text{min length coset representative in } S_k\times S_{n-k}\setminus S_n\end{aligned}$$

Thus $\{i_1,i_2,\cdots,i_k\}\Leftrightarrow W\{i_1,i_2,\cdots,i_k\}\Leftrightarrow w\{i_1,i_2,\cdots,i_k\}$. So for smoothness, $X_{\{j_1,j_2,\cdots,j_n\}}\subset X_{\{i_1,i_2,\cdots,i_n\}}$ if and only if $w\{j_1,\cdots,j_n\}\leq w\{i_1,\cdots,i_n\}$ and hence $X_{\{i_1,i_2,\cdots,i_n\}}$ is smooth if and only if $X_{W\{i_1,i_2,\cdots,i_n\}}$ is.

The T fixed points in G/P are permutations in $S_I\setminus S_n$.

Definition 9.2. For the cosets of $S_I\setminus S_n$, we define $(S_n)^I$ to be the max length coset representative and $(S_n)_I$ to be the min length coset representative.

Then the Schubert varieties in G/P are indexed by the cosets of $S_I\setminus S_n$, and thus by $(S_n)^I$ or $(S_n)_I$. So looking at the map $\varphi: G/B\to G/P$, with $X_{min(S_Iw)}\to X_{S_Iw}$ and $X_{max(S_Iw)}\to X_{S_Iw}$. So we have $min(S_Iu)\leq min(S_Iw)$ if and only if $X_{S_Iu}\subset X_{S_Iw}$ and X_{S_Iw} is smooth if and only if $X_{max(S_Iw)}$ is smooth.

Now for intersecting Schubert varieties, done in the Grassmannian but can be generalized. Consider the question, how many lines meet four given lines in $\mathbb{R}^3$? We can view $\mathbb{R}^3$ as an open subset of $\mathbb{P}^3=\mathbb{R}^4/(1,\cdots,1)$. So fix a line in $\mathbb{R}^3$, say the line, $\overline{e_1e_2}$, not containing the origin. This determines a 2 dimensional subspace of $\mathbb{R}^4$, $span\{e_1,e_2\}$. Then extend $\{e_1,e_2\}$ to a basis $\{e_1,e_2,e_3,e_4\}$ for $\mathbb{R}^4$. So lines meeting $\overline{e_1e_2}=\overline{\left\{\begin{bmatrix}* & *\\ 1 & 0\\ 0 & *\\ 0 & *\end{bmatrix}\right\}}=X_{\{2,4\}}\subset G(2,4)$. That is $X_{(1,2)}$ in partition notation, note that this is with respect to the ordered basis, $\{e_1,e_2,e_3,e_4\}$. So for lines intersecting four given lines we look at $X^G_{(1,2)}(E\bullet)\cap X^G_{(1,2)}(F\bullet)\cap X^G_{(1,2)}(G\bullet)\cap X^G_{(1,2)}(H\bullet)$. One method of doing this is to solve equations!

Another method is Intersection theory.

Definition 9.3. Let X be any variety.

(1) Two subvarieties U and V in X meet *transversally* if $U\cap V=\bigcup Z_i$, where the Z_i's are the irreducible components, then for each i and each point z in an open sets of $Z_i\cap U\bigcap V$ we have $T_z(Z_i)=T_z(U)\cap T_z(V)$.

(2) The Chow ring of X, denoted $A^\bullet(X)$, is the formal sum of $\{[V]: V \text{ is a subvariety of } X\}/\sim$, where $[U]\equiv[V]$ if they are rationally equivalent. With multiplication given by $[U][V]=[U\cap V']$, with $[V]\sim[V']$ and V' meets V transversely,

Definition 9.4. If for each z_i, $Codim(z_i)=Codim(v)+Codim(w)$ then $V\cup W$ is *proper*.

If $V\cup W$ is proper, then intersection theory says that $[U][V]=\Sigma m_i[Z_i]$ and the m_i's are non-negative integers.

Theorem 9.5. *If X has a cell decomposition, $X=\bigcup C_i$ then $A^\bullet(X)=$ (linear) span$\{\overline{[C_i]}: i\in E_x\}$.*

Theorem 9.6. $A^\bullet(X_w) = H^*(X_w, \mathbb{Z}) =$*(linear) span*$\{[X_v] : v \leq w\}$.

Definition 9.7. Define the Poincare polynomial to be $\sum_{i \in E_x} t^{codim(\overline{C_i})} = \sum_{i \in E_x} dim(A^i(X))i^i$. eg: $\sum_{\lambda \subset \ k \times (n-k) box} t^{|\lambda|}$ is the Poincare polynomial of G(k,n).

On the $G(k, n)$, we have the map $\lambda \to dual(\lambda)$. This map gives the Poincare duality on $H^*(G(k, n))$. On G/B, we have $[X_w] \to [X_{ww_0}]$. But since $X_u \subset X_w$ implies $X_{ww_0} \subset X_{uw_0}$, so we only get duality if X_w is smooth.

Open Problem (Reiner) - Which smooth varieties in G/B exhibit Poincare duality? That is, when is $\{v \leq w\}$ self dual, with a nice combinatorial map?

The ring structure on $A^\bullet(X)$ is given by $[X^G_\lambda][X^G_\mu] = [X^G_\lambda(E_\bullet) \cap X^G_\mu(F_\bullet)]$ where $F_\bullet \in X^{Fl}_{w_0}(E_\bullet)$, so we could take $F_\bullet = \{< e_1 > \subset < e_1, e_2 > \subset \cdots \subset < e_1, \ldots, e_n >\}$. Notationally, denote $X_\mu(F_\bullet) = \widetilde{X_\mu}$. Then we get the matrix form by canceling up, left and right, instead of down, left and right.

Proposition 9.8. *If* $cod(\mu) = cod(\lambda) = k(n - k) = dimG(k, n)$ *then* $[X_\lambda][X_\mu]$ *is* $1[X_\emptyset]$ *when* $\lambda = dual(\mu)$ *and 0 otherwise.*

Proof. First consider $\mu = dual(\lambda)$, then

$$\begin{aligned}
[X_\lambda][X_\mu] =& X_\lambda(E_\bullet) \cap \widetilde{X}_{dual(\lambda)} \\
=& \overline{\left\{ \begin{bmatrix} * & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{bmatrix} \right\}} \cap \overline{\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ * & 0 \\ 0 & 1 \end{bmatrix} \right\}} \\
=& \overline{\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}} \text{(as all the stars are canceled by 0's)} \\
=& [X_\emptyset] \text{for some basis.}
\end{aligned}$$

Now if $\mu \neq dual(\lambda)$ then a 1 of one of the matrices will be a 0 in the other, so there intersection will be empty. Hence $X_\lambda(E_\bullet) \cap \widetilde{X}_\mu = \emptyset$. ∎

Trick - To find the $C^\nu_{\lambda\mu}$'s in $[X_\lambda][X_\mu] = \sum_\nu C^\nu_{\lambda\mu}[X_\nu]$. Multiple both sides by $[X_{dual(\nu)}]$ and expand, getting $C^\nu_{\lambda\mu}[X_\emptyset] = [X_{dual(\nu)}][X_\lambda][X_\mu] = [X_{dual(\nu)}(E_\bullet) \cap X_\lambda(F_\bullet) \cap X_\mu(H_\bullet)]$. So $C^\nu_{\lambda\mu}$ is the number of points in $X_{dual(\nu)}(E_\bullet) \cap X_\lambda(F_\bullet) \cap X_\mu(H_\bullet)$.

Proposition 9.9. $[X_\lambda][X_\hbar] = \sum_{\nu \leq \lambda; |\nu| = |\lambda - 1|} [X_\nu]$, *where* $\hbar$ *has the shape of the* $k \times (n - k)$ *box minus the bottom left corner.*

Proof. To see this we count the number of points in $\widetilde{X}_{dual(\nu)} \cap X_\lambda(E_\bullet) \cap X_\hbar(H_\bullet)$. But $\widetilde{X}_{dual(\nu)} \cap X_\lambda = \emptyset$ if we have any gaps between ν and λ, thus $dual(\lambda) \subset dual(\nu)$ and they differ by only one box (or else $C^\nu_{\lambda\hbar} = 0$). So assume $dual(\nu) = dual(\lambda) + 1$ box. So just looking at their differences, we see $X_\lambda = \begin{pmatrix} 1 \\ \star \end{pmatrix}$ and $\widetilde{X}_{dual(\nu)} = \begin{pmatrix} \star \\ 1 \end{pmatrix}$, where $\star \neq 0$. Then $\widetilde{X}_{dual(\nu)} \cap X_\lambda = \begin{pmatrix} \star \\ \star \end{pmatrix}$. Now, taking $Y = \{V \in G(k,n) : V$contains$< e_{i_1}, e_{i_2}, \cdots, \widehat{e_{i_j}}, \cdots, e_{i_k} >$ and intersects $span < e_{i_j}, e_{i_{j+1}} >$ in 1 dimension$\}$. Then $Y \cap X_\hbar(H_\bullet) = \{V \in G(k,n) : V$meets span $< h_1, \cdots, h_{n-k} >$ in 1 dimension$\}$. Generally, span$< e_{i_1}, \cdots, \widehat{e_{i_j}}, \cdots, e_{i_k} > \cap$span$< h_1, \cdots, h_{n-k} >= \{0\}$. Thus $< e_{i_j}, \cdots, e_{i_{j+1}} > \cap < h_1, \cdots, h_{n-k} >$ is 1 dimensional. So $Y \cap X_\hbar(H_\bullet) = \{$one point$\} = X_\emptyset(G_\bullet)$ and thus $C^\nu_{\lambda\hbar} = 1$. $\square$

Wow! This is the same as the Schur functions! $S_\lambda S_{box} = \sum S_{\lambda+wx}$ and $S_\lambda(x_1, \cdots, x_k) = \sum_{T filling of \lambda} x^T$, with $S_\lambda(x_1, \cdots, x_k) = 0$ if λ has more then k rows.

Theorem 9.10. *Multiplication in $A^\bullet(G(k,n))$ (or in $A^\bullet(G/B)$) is determined by multiplication by the unique codimension 1 subvariety (or the $n-1$ codimension 1 subvarieties in $A^\bullet(G/B)$), along with stability.*

Corollary 9.11. $Pic(G(k,n)) = \mathbb{Z}$.

Theorem 9.12. *We have a bijection (as rings) between $A^\bullet(G(k,n)) \leftrightarrow span\{S_\lambda : \lambda$is contained in a $k \times (n-k)$box$\}$. Given by $[X_\lambda] \to \mathcal{S}_{dual(\lambda)}$.*

Now back to the line intersecting four lines problem. As we say the intersection of 4 lines is $[X_{(1,2)}]^4$. So we are looking for the coefficient of $[X_\emptyset] = S_{(2,2)}$ in $(S_{(1,0)})^4$. Then we see

$$\begin{aligned} S_{(1,0)} \times S_{(1,0)} \times S_{(1,0)} \times S_{(1,0)} =& S_{(1,0)} \times S_{(1,0)} \times (S_{(1,1)} + S_{(2,0)}) \\ =& S_{(1,0)} \times (S_{(2,1)} + S_{(2,1)}) \\ =& S_{(1,0)} \times 2S_{(2,1)} \\ =& 2S_{(2,2)}. \end{aligned}$$

So we find out that the answer is 2.

10. Vakil's Geometric Littlewood-Richardson Rule (April 25, 2007)

10.1. From last time.

(1) If $\mathrm{codimX}_\lambda + \mathrm{codimX}_\mu = \mathrm{dimG(k,n)}$ then

$$[X_\lambda][X_\mu] = \begin{cases} [X_{\text{pt.}}], \lambda = \mathrm{dual}(\mu) \\ \\ 0 \text{ otherwise} \end{cases}$$

(2) In general, we have

$$[X_\lambda][X_\mu] = \sum C^\nu_{\lambda\mu}[X_\nu]$$

where this sum is taken over all ν with $\mathrm{codimX}_\nu = \mathrm{codimX}_\lambda + \mathrm{codimX}_\mu$,

$$[X_{\mathrm{dual}\nu}][X_\lambda][X_\mu] = C^\nu_{\lambda\mu}[X_{\mathrm{pt.}}],$$

and

$$C^\nu_{\lambda\mu} = \#(X_{\mathrm{dual}\nu}(E_\cdot) \cap X_\lambda(F_\cdot) \cap X_\mu(G_\cdot)$$

when these are all in transverse position. Hence the coefficients $C^\nu\lambda\mu$ are nonnegative integers.

(3) There is a ring isomorphism $A^\cdot G(k,n) \to \mathbb{C}[x_1, \ldots, x_k]^{S_k}/(S_\lambda | \lambda_k > n-k)$ given by $[X_\lambda] \mapsto S_{\mathrm{dual}\lambda}$. The proof of this theorem uses formulae of Giambelli, Jacobi, and Trudi. Given the elementary symmetric functions $e_i(x_1, \ldots, x_k) = \sum_{j_1 < \ldots < j_i} x_{j_1} \cdots x_{j_i}$, we have

$$S_\lambda = \det \begin{bmatrix} e_{\lambda_1} & e_{\lambda_2} & e_{\lambda_3} & \cdots \\ 1 & e_{\lambda_2} & e_{\lambda_3} & \cdots \\ 0 & 1 & e_{\lambda_3} & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Recall also the Pieri Formula,

$$S_\lambda S_{1^j} = \sum S_{\lambda'}$$

where this sum is over all λ' containing λ such that $|\lambda'| = |\lambda| + j$ and $\lambda' - \lambda$ contains no two boxes in the same row.

(4) $S_\lambda S_\mu = \sum C^\nu_{\lambda\mu}$ similarly, where the $C^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients.

(5) These coefficients are traditionally computed via reverse lattice fillings: one places the upper leftmost corner of the tableau for μ just beneath and to the right of the lower rightmost corner of that for λ and fills the resulting array from bottom to top in all possible ways, strictly increasing as one ascends in the columns and weakly increasing in the rows.

(6) The first four results also hold for the flag manifolds Fl_n as well as the Grassmannians, where the third becomes $A^\cdot Fl_n \simeq \mathbb{C}[x_1, \ldots, x_n]/(S_\lambda | \lambda \neq 0)$ with the isomorphism being given by $[X_w] \mapsto \mathcal{S}_{ww_0}$ where these last are the Schubert polynomials. However, there does not exist a combinatorial technique like reverse lattice fillings to compute the Littlewood-Richardson coefficients in this case. Coskun is working on this currently.

10.2. **Vakil's Checkerboard.** Vakil approaches Littlewood-Richardson rules for the Grassmannian geometrically, via a series of degenerations indexed by the moves of a checker game. Index Schubert varieties by k-subsets of $[n]$. Given such subsets A, B, C, we seek a rule to compute the integer C^C_{AB}, which is the number of irreducible components of $X_A(M_\cdot) \cap X_B(F_\cdot)$ rationally equivalent to $X_C(F_\cdot)$ where $F_\cdot$ is some fixed frame and $M_\cdot$ is a moving frame. This rule will take the form of a game involving black and white checkers on an $n \times n$ board.

The game will define a family of sets $Y_{\circ\bullet} \subset G(k,n)$ indexed by arrangements $\circ$ of white checkers and $\bullet$ of black checkers on the $n \times n$ board, which will change as the checkers move. These should have the properties that:

(1) $X_A(M_\cdot) \cap X_B(F_\cdot) = \overline{Y_{\circ_{AB}\bullet_{\mathrm{init}}}}$

(2) There is a sequence of degenerations forming a tree such that

$$[\overline{Y_{\circ_{AB}\bullet}}] = \sum_{\text{leaves}} [\overline{Y_{\circ_C \bullet_{\text{final}}}}]$$

and at each stage of the degeneration we have only three possibilities,

$$[\overline{Y_{\circ\bullet}}] = \begin{cases} [\overline{Y_{\circ\bullet_{next}}}] \\ [\overline{Y_{\circ_{step}\bullet_{next}}}] \\ [\overline{Y_{\circ\bullet_{next}}}] + [\overline{Y_{\circ_{step}\bullet_{next}}}] \end{cases}$$

(3) C_{AB}^{C} equals the number of steps until $\circ_C$.

10.3. **Black Checkers.** Black checkers encode the position of the moving frame $M_.$ with respect to $F_.$ in a table of dimensions. Initially $F_. =< e_1, \ldots, e_n >$ and $M_. =< e_n, \ldots, e_1 >$ and the number of black checkers NW (inclusive) of (i, j) is $\mathrm{dim M_i \cap F_j}$.

$$\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \bullet & & \\ \hline & \bullet & \\ \hline \end{array} \quad \leftrightsquigarrow \quad \begin{array}{c|ccc} & F_1 & F_2 & F_3 \\ \hline M_1 & 0 & 0 & 1 \\ M_2 & 1 & 1 & 2 \\ M_3 & 1 & 2 & 3 \end{array}$$

This example corresponds to a flag $M_.$ whose marked line meets the marked line F_1 only at the point F_0 and whose marked point misses F_1 entirely.

10.4. **White Checkers.** Fix $M_.$ and $F_.$ and some $V \in G(k, n)$. The white checkers encode the positions of $F_.$ and $M_.$ relative to V: $\mathrm{dim V \cap M_i \cap F_j}$ equals the number of white checkers NW (inclusive) of (i, j). Thus for the same $M_.$ and $F_.$ as above, an arrangement of white checkers

$$\begin{array}{|c|c|c|} \hline & & \circ \\ \hline & & \\ \hline & \circ & \\ \hline \end{array}$$

corresponds to a V which intersects both marked lines and passes through the marked point M_0 but not F_0.

10.5. **Details.** On any checkerboard, let $w(i, j)$ denote the number of white checkers NW (inclusive) of (i, j), and similarly $b(i, j)$ for black checkers.

$$X_{\circ\bullet} = \{(V, M_., V_.) \in G(k, n) \times Fl_n \times Fl_n | \mathrm{dim M_i \cap F_j = b(i, j), dim V \cap M_i \cap F_j = w(i, j)}\}$$

These sets partition $G(k, n) \times Fl_n \times Fl_n$. We define the sets $Y_{\circ\bullet}$ to be the projections of $X_{\circ\bullet}$ onto $G(k, n)$ obtained by fixing an $M_.$ and an $F_.$.

The closures $\overline{Y_{\circ\bullet}}$ are called closed two-step Schubert varieties, Richardson varieties, or skew Schubert varieties.

(1) $G(k, n) = \bigsqcup Y_{\circ\bullet}$ for fixed $M_.$ and $F_.$.
(2) $Y_{\circ\bullet} \neq \emptyset$ only if every white checker is happy: there is one black checker due north and one due west (inclusive) of every white checker, and every row and column has at most one white checker in it.
(3) $X_A(M_.) \cap X_B(F_.) = \overline{Y_{\circ\bullet}}$ where $\bullet$ gives the position of $M_.$ with respect to $F_.$, $\circ = \{(a_1, b_k), \ldots, (a_k, b_1)\}$, $A = \{a_1 < \ldots < a_k\}$, $B = \{b_1 < \ldots < b_k\}$.

Lemma 10.1. *The "variety" $Y_{\circ\bullet}$ is irreducible and smooth, and* $\mathrm{dim Y_{\circ\bullet}} = \sum \mathrm{b(i, j) - w(i, j)}$ *where the sum is over all* (i, j) *containing white checkers.*

Example 10.2.

$$\begin{bmatrix} & & & & \bullet e_5 \\ & & \bullet e_3 & & \\ & & & \bullet e_4 & \\ \bullet e_1 & & \circ & & \\ & \bullet e_2 & & & \circ \end{bmatrix}$$

This corresponds to $F_. =< e_1, \ldots, e_5 >$, $M_. =< e_2, e_1 e_4 e_3 e_5 >$ and a V with basis $< v_1, v_2 >$. $v_1 \in \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_3\}$ but is neither of these two, so we can choose $v_1 = \star e_1 + e_3$ for $\star \neq 0$ for $2 - 1 = 1$ dimension of freedom. Likewise $v_2 \in \mathrm{span}\{\mathrm{e}_1, \ldots, \mathrm{e}_5\}$ and has nonzero projection onto e_5, so we may choose $v_2 = \star e_1 + \star e_2 + \star e_3 + \star e_4 + e_5$ and can eliminate e_3 altogether using v_1.

Corollary 10.3. $Y_{\circ \bullet_{init}} = X_A(M_.) \cap X_B(F_.) \neq \emptyset$ *if and only if*

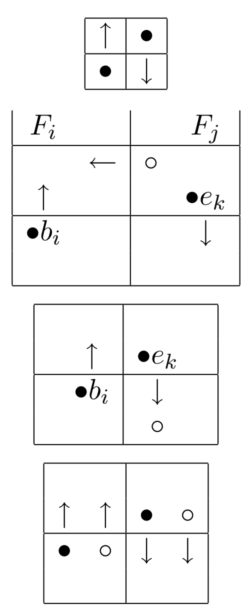
(1) *all white checkers are happy,*
(2) *the union of the partitions A and B covers all the squares, and*
(3) $\mathrm{dim Y_{\circ\bullet_{init}} = k(n-k) - codim X_A - codim X_B}$.

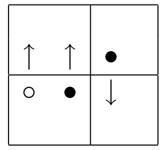
If these conditions hold, $[X_A][X_B] \neq 0$ so the degenerations will lead somewhere.

10.6. **Degeneration Rules.**

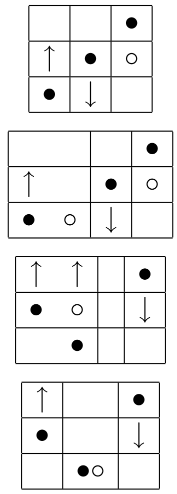
(1) Black checkers start in the configuration $\bullet_{init}$ along the antidiagonal. White checkers start in the configuration $\circ_{AB} = \{(a_1, b_k), \ldots, (a_k, b_1)\}$.
(2) Black moves: sort black checkers (corresponding to basis vectors) by transposing two adjacent rows at a time, working from left to right.
(3) White moves: after each black move, white checkers move according to nine rules on 2×2, 3×3, and 4×4 critical submatrices, obtained by considering which white checkers can be affected by the next black move.
(4) Black moves correspond to degenerations: $\circ \mapsto \circ_{next}$ where $F_. =< e_1, \ldots, e_n >$, $M_. =< b_1, \ldots, b_n >$ corresponds to taking the limit as $t \to 0$ of $b_i = te_j + (1-t)e_k$.
(5) Two key rules:
    (a) If $Y_{\circ\bullet} \mapsto Y_{\circ' \bullet_{next}}$ then $Y_{\circ' \bullet_{next}}$ has the same dimension.
    (b) All span and intersection data are preserved.

10.7. **2 x 2 Rules.**





10.8. **3 x 3 Rules.**



11. Izzet Coskun's Lecture (April 27, 2007)

Let's set up the basic definitions and notation. Let $G(k,n)$ be the Grassmannian manifold of k-planes in n space. Let

$$\Sigma_\lambda = \{[\Lambda] \in G(k,n) \,|\, \dim(\Lambda \cap F_{n-k+i-\lambda_i} \geq i\}.$$

be the Schubert variety in $G(k,n)$ indexed by the partition λ with $n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. (Note, this is written in the reverse order to the lectures above.) Let σ_λ be the corresponding Schubert cycle in the cohomology ring for $G(k,n)$. The notation follows Lecture 6 of Harris's "Algebraic Geometry" which is a good book for more details .

Example 11.1. Consider $\sigma_{(1)}\sigma_{(1)}$ in $G(2,4)$. Note $G(2,4) = \mathbb{G}(1,3)$ when we *projectivize*. The class $\sigma_{(1)} \cdot \sigma_{(1)}$ corresponds with the family of lines meeting two given lines in general position in 3-space. To find the irreducible components of this family, swing one of the lines around until it crosses the other line in some plane. Now, which lines intersect both lines. There are two components:

(1) The family of lines passing through the plane exactly at the point of intersection between the two lines which corresponds with the Schubert cycle $\sigma_{(2)}$.
(2) The family of lines in the plane defined by the two lines which corresponds with the Schubert cycle $\sigma_{(1,1)}$.

Therefore, $\sigma_{(1)} \cdot \sigma_{(1)} = \sigma_{(2)} + \sigma_{(1,1)}$.

The coefficients in the expansion of $\sigma_{(1)} \cdot \sigma_{(1)}$ are 1 because of the Pieri rule of course, but we could have figured this out another way by studying tangent spaces. We could have shown that even after swinging ℓ_2 around to meet ℓ_1 we still have a transverse intersection.

Let V be an n-dimension vector space and assume $G(k,n)$ is the set of k planes in V. If $S \in G(k,n)$ the we get a short exact sequence

$$0 \longrightarrow S \longrightarrow V \longrightarrow V/S \longrightarrow 0.$$

Fact 11.2. The tangent space to the Grassmannian at a point $[\Lambda]$ is given by

$$T_{[\Lambda]}(G(k,n)) = \mathrm{Hom}(\Lambda, V/\Lambda) \approx \{k \times (n-k) \text{ matrices}\}.$$

Therefore, $\dim T_{[\Lambda]}(G(k,n)) = k \cdot (n-k)$ as expected for a smooth manifold.

Similarly, assume $[\Lambda] \in \Sigma_\lambda$ is a smooth point. Suppose further that $\dim(\Lambda \cap F_{n-k+i-\lambda_i}) = i$ which holds on a dense open set in Σ_λ for all i. Then

$$T_{[\Lambda]}(\Sigma_\lambda) = \{\phi \in \mathrm{Hom}(\Lambda, V/\Lambda) \,|\, \phi(\Lambda \cap F_{n-k+i-\lambda_i}) \in F_{n-k+i-\lambda_i}/\Lambda \;\; \forall i\}$$

Key Fact: To show that at $[\Lambda] \in \Sigma_{(1)}(\ell_1) \cap \Sigma_{(1)}(\ell_2)$ the two Schubert varieties meet transversally, we need to show that the corresponding tangent spaces intersect transversally, i.e.

$$\dim T_{[\Lambda]}(\Sigma_{(1)}(\ell_1) \cap T_{[\Lambda]}(\Sigma_{(1)}(\ell_2) = 2.$$

(hmm, I am a little lost here. I think this intersection should have dimension 0, but the example given has dimension 1 and it says above we should have dimension 2.)

11.1. **Coskun's version of the Littlewood-Richardson Rule for $G(k,n)$.** Fix an ordered basis say $(e_1, e_2, \dots, e_n)$ for V. Fix k vector spaces $V_1, \dots, V_k$ which are spans of consecutive subsets of basis elements. To visualize these sets, put $e_1, e_2, \dots, e_n$ along the antidiagonal of a matrix with e_1 in the lower left corner. Then if V_i is spanned by $\{e_j, e_{j+1} \cdots, e_k\}$, we represent V_i by the minimal square containing all the basis elements in V_i. The collection of squares is called a *Mondrian tableau* or *tableau* for short. These tableau were named after the artist Piet Mondrian 1872-1944. Check out

`http://en.wikipedia.org/wiki/Piet_Mondrian`

for more information.

Example 11.3. If $V_1 \subset V_2 \subset \cdots \subset V_k$ then

$$S = \{[\Lambda] \in G(k,n) \,|\, \dim(\Lambda \cap V_i) \geq i\}$$

is a Schubert variety. The set S will be represented by a nested sequence of squares in the Mondrian tableau. For a more specific example, if V_j is spanned by $\{e_1, \dots, e_{i_j}\}$ then this set is the Schubert variety $X_{\{i_1,\dots,i_k\}}$ in our previous class notation. This Schubert variety will correspond with a nested sequence of squares of side lengths $i_1, \dots, i_k$ with a common lower left corner in the lower left corner of the matrix.

Example 11.4. If the V_i's have no inclusion relations and the basis elements in V_i precede the basis elements for V_{i+1} then the corresponding Mondrian tableau will have k non-overlapping squares. This tableau will correspond with the intersection of two Schubert varieties in opposite position.

We want to generalize the idea in Example 11.1 to intersecting any two Schubert varieties in the Grassmannian. More specifically, take two Schubert varieties Σ_λ and Σ_μ with respect to ordered bases $(e_1, \dots, e_n)$ and $(e_n, \dots, e_1)$ respectively. We want to find a sequence of degenerations which will keep the cohomology class of the intersection of these varieties the same but eventually leads to the union of irreducible components. The degenerations we will use are all of the form $te_i + (1-t)e_j$. At $t = 1$ is gives the basis element e_i and then as t approaches 0 this degenerates e_i into e_j. If U is the span of $(e_i, \dots, e_{j-1})$ then after the degeneration U will be the span of $(e_{i+1}, \dots, e_j)$, therefore the degeneration is visualized as moving the square for U north east along the diagonal by one unit.

Definition 11.5. A *tableau* for $G(k,n)$ is a collection of k distinct squares such that

(1) Each square corresponds to a subspace spanned by consecutive basis elements.
(2) No 2 squares share the same lower left corner.

(3) If S_1 and S_2 are any 2 squares of the tableau that share their upper right corner then every square whose lower left corner is southwest of the lower left corner of S_2 contains S_2.

Definition 11.6. A square S is *nested* if all the squares containing S are totally ordered by inclusion and for any other square S' either $S \subset S'$ or $S' \subset S$.

Definition 11.7. Let S be the square containing $< e_i, \dots, e_j >$ and S' be the square containing $< e_{i+1}, \dots, e_{j+1} >$ so S' is the result of degenerating e_i to e_{j+1}. A *neighbor* N of S is a square such that

(1) $e_{j+1} \in N$.
(2) N does not contain S.
(3) Let $\tilde{S}$ be any square whose lower left corner is between S' and N, then either $\tilde{S} \subset S$ or $N \subset \tilde{S}$.

Note that by definition the neighbors of S are all ordered by containment.

Degeneration Algorithm: Given a tableau M,

Step 1. If every square of M is nested, stop.
Step 2. If not, let S be the square whose lower left corner is southwest most among the non-nested squares. Let $N_1 \subset N_2 \subset N_r$ be the neighbors of S.
Step 3. For every neighbor define $M_1(N_i)$ to be the tableau obtained from M by replacing S and N_i by the square representing $S' \cap N$ and the square representing the span of $S \cup N_i$. Define M_0 to be the tableau obtained from M by replacing S by S' and normalizing again.
Step 4. Among the collection $M_0, M_1(N_1), \dots, M_1(N_r)$ retain the tableau with the same dimension as M and repeat Step 1 with each of these as the given tableau.

For more information and pictures see Izzet Coskun's web page. A new preprint will be coming soon. For now see

`http://math.mit.edu/~coskun/seattleoct17.pdf`

12. Divided difference operators and the Chow cohomology of the flag variety (May 2, 2007)

In previous lectures we derived a presentation for the Chow cohomology of the Grassmannian, given by:

$$A^*(G(k,n)) \simeq \mathbb{Z}[s_\lambda(x_1, \dots, x_n)]/ < s_\lambda : \lambda \subsetneq k \times (n-k) >.$$

Here the isomorphism is given by $[X_\lambda] \mapsto s_{dual(\lambda)}$. Today we will discuss a similar result, due to Borel, for the Chow cohomology of the Flag variety:

Theorem 12.1 (Borel)**.** *The Chow cohomology of the flag variety Fl_n) is given by $A^*(Fl_n) \simeq \mathbb{Z}[x_1, \dots, x_n]/I_n$, where I_n is the ideal*

$$I_n := < e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n) >.$$

Here $e_i(x_1, \dots, x_n)$ is the i^{th} *elementary symmetric function*; for instance $e_1(x_1, \dots, x_n) = x_1 + \cdots + x_n$, and $e_n = x_1 x_2 \cdots x_n$.

We will be interested in determining $?_w$, the image of $[X_w]$ under the above isomorphism $[X_w] \mapsto ?_w$. We first point out some properties that $?_w$ that must satisfy.

- The isomorphism is a ring map: If $[X_u][X_v] = \sum c_{uv}^w [X_w]$ then we must have $?_u ?_v = \sum c_{uv}^w ?_w$. The map is also graded: Since $\deg(?_u ?_v) = \deg(?_u) + \deg(?_v)$, and codim$(X_u \cap X_v) =$ codim$(X_u) +$ codim(X_v), we must have $\deg(?_w) =$ codim(X_w).
- We have seen $[X_{w_0}][X_u] = [X_{w_0}(F_\bullet) \cap X_u(G_\bullet)] = [X_u]$, and hence $[X_{w_0}] \mapsto 1$.

- If $\ell(u)+\ell(v)=\binom{n}{2}=\dim(Fl_n)$, then we have the formula

$$[X_u][X_v]=\left\{\begin{array}{ll}[X_{id}] & \text{if } v=uw_0\\ 0 & \text{otherwise}\end{array}\right\}$$

.

Hence in this case we must have

$$?_u?_v=\left\{\begin{array}{ll}?_{w_0} & \text{if } v=uw_0\\ 0 \mod I_n & \text{otherwise}\end{array}\right\}$$

Exercise 12.2. Show that $R_n:=\mathbb{Z}[x_1,\dots,x_n]/I_n$ is isomorphic as a vector space to the (linear) span of the set $\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}: i_k\le n-k\}$. For example, if $n=3$ then $R_3=span\{1,x_1,x_2,x_1^2,x_1x_2,x_1^2x_2\}$. Garcia calls this the 'Artin basis'.

Hint: Find a Gröbner basis for I_n.

The Artin basis will help us determine the Hilbert series for the ring R_n. We will need the following bijection between elements of this basis and elements of the symmetric group.

Lemma 12.3. *There exists a bijection $S_n\sim\{(i_1,\dots,i_n): i_k\le n-k\}$, sending w to $code(w)$.*

Here the *code* of an element $w\in S_n$ is defined to be the vector:

$$code(w):=(|*s \text{ on row } 1|,|*s \text{ on row } 2|,\dots,|*s \text{ on row } n|)$$

in the matrix obtained by crossing out entries below and to the right of each entry of w. For example, $code(316425)=(2,0,3,1,0,0)$.

Using this we derive the Hilbert series for R_n:

$$\sum \dim R_n^{(i)}t^i=\sum_{w\in S_n}t^{\ell(n)}=\prod_{i=1}^{n-1}(1+t+\cdots+t^i).$$

We now return to the issue of determining $?_w$, the image of $[X_w]$ in the isomorphism of Theorem 12.1. Using the properties that we so far have derived, we consider the case $n=3$ and make a naive guess for what these values might be.

Recall that the set of generators for $A^*(Fl_3)$ contains a single element $[X_{id}]$ of rank 0, two elements $[X_{213}],[X_{132}]$ of rank 1, two elements $[X_{312}],[X_{231}]$ of rank 2, and a single element $[X_{321}]$ of rank $\binom{3}{2}=3$. To define a map to R_n we choose to map these generators to elements of the Artin basis according to $[X_{id}]\mapsto x_1^2x_2$, $[X_{213}]\mapsto x_1^2$, $[X_{132}]\mapsto x_1x_2$, $[X_{312}]\mapsto x_1$, $[X_{231}]\mapsto x_2$, and $[X_{321}]\mapsto 1$.

One can check that some of the conditions on our list of requirements for $?_w$ are indeed satisfied (e.g., it is a graded group map that sends $[X_{w_0}]$ to 1).

Exercise 12.4. Show that $x_i^3=0 \mod I_3$ and $x_1x_2^2=o \mod I_3$. For instance $x_1^3=x_1^2(x_1+x_2+x_3)-x_1(x_1x_2+x_1x_3+x_2x_3)+x_1x_2x_3$.

However, in checking that our map is indeed a ring morphism, we note that $x_1\cdot x_2=x_1x_2$, and hence are asked to verify the formula:

$$[X_{312}][X_{231}]=[X_{132}].$$

However, we consider the intersection of flags in question and compute that in fact $[X_{312}][X_{231}]=[X_{132}]+[X_{312}]$. Accordingly, we tweak our map to R_n described above to now have $[X_{231}]\mapsto x_1+x_2$. One can check that this indeed defines the desired isomorphism.

We seek a method for determining $?_w$ in a uniform way for all n. This is provided by the following procedure due to Bernstein, Gelfand, and Gelfand (BGG).

Observation. We note that $\dim(R_n^{\binom{n}{2}}) = 1$, and so for $?_{w_0}$ we can choose and homogeneous polynomial σ_{w_0} of degree $\binom{n}{2}$ that is not in I_n. We follow BGG and choose $\sigma_{w_0} = \prod_{1\leq i<j\leq n} (x_i - x_j)$ to be the so-called 'Vandermonde determinant'.

To determine the other $?_w$, we will need the notion of a *divided difference operator* on $\mathbb{Z}(x_1, \ldots, x_n)$. If $f \in \mathbb{Z}(x_1, \ldots, x_n)$, define $s_i f(x_1, \ldots, x_n) := f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$. The divided difference operator ∂_i is then defined to be

$$\partial_i(f) := (f - s_i f)/(x_i - x_{i+1}).$$

Observations. We point out some properties of the ∂_i.

- $\partial_i(f) = 0$ if $f = s_i f$.
- $\partial_i(s_i f) = -\partial_i f$
- $\partial_i(x_i^r x_{i+1}^s) = x_i^r x_{i+1}^r (x_{i+1}^{s-r-1} + x_i x_{i+1}^{s-r-2} + \cdots + x_i^{s-r-1}$ if $s > r$.
- $\partial_i(x_i^r x_{i+1}^s \cdot f(x_1, \ldots, \hat{x_i}, \hat{x_{i+1}}, \ldots, x_n)) = f \cdot \partial_i(x_i^r x_{i+1}^s)$.

  These last two facts imply that if f is polynomial then so is $\partial_i(f)$. The ∂_i also satisfy some Coxeter like relations.
- $\partial_i \partial_j = \partial_j \partial_i$ if $|i - j| > 1$.
- $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$.
- $\partial_i^2 = 0$.

Exercise 12.5 (Leibnitz rule for divided difference operators)**.** Show that $\partial_i(fg) = (\partial_i f)g + (s_i f)(\delta_i g)$.

Theorem 12.6 (Bernstein, Gelfand, Gelfand)**.** *Suppose $[X_{ww_0}] \mapsto \sigma_w \in R_n$ under the isomorphism of 12.1. Then we have the following:*

$$\partial_i(\sigma_w) = \begin{cases} \sigma_{ws_i} \mod I_n & \textit{if } w_i > w_{i+1} \\ 0 \mod I_n & \textit{otherwise} \end{cases}$$

The proof of this theorem relies on the following key lemma.

Lemma 12.7. *If $s_{a_1} \cdots s_{a_p} = s_{b_1} \cdots s_{b_p} = w$ are both reduced expressions for $w \in S_n$ (so that $p = \ell(w)$), then $\partial_{a_1} \cdots \partial_{a_p} = \partial_{b_1} \cdots \partial_{b_p}$.*

To prove the lemma we introduce the notion of a string diagram associated to an element $w \in S_n$.

Definition 12.8. If $w \in S_n$, we define the following sets of inversions:

$$I(w) := \{(i, j) : i < j, w(i) > w(j)\}.$$
$$J(w) := I(w^{-1}) = \{(i, j) : \text{string } i \text{ crosses string } j\}.$$

Every element $w \in S_n$ can be written $s_{a_1} \cdots s_{a_p}$ such that $p = |I(w)| = |J(w)| = \ell(w)$. In this case, we will say that $(a_1 \cdots a_p)$ is a *reduced word* for w. The collection of all reduced words will be denoted $R(w) := \{(a_1 \cdots a_p) : s_{a_1} \cdots s_{a_p}, p = \ell(w)\}$. For example, $R(4132) = \{3213, 3231, 2321\}$.

Lemma 12.9. *Suppose $a_1 \cdots a_p$ is a reduced word for w. Then we have $J(w) = \{(a_1, a_1+1), s_{a_1}(a_2, a_2+1), \ldots, s_{a_1} \cdots s_{a_{p-1}}(a_p, a_p + 1)\}$.*

Proof. We prove this by induction on the length $\ell(w)$. We see $J(id) = \emptyset$ and also $J(s_i) = \{(i, i + 1)\}$ so that result holds for $\ell(w) = 0, 1$. We assume the result holds for all v with $\ell(v) < \ell(w)$. If $v = s_{a_1} \cdots s_{a_{p-1}}$ then all string crossings for w are string crossings for v except the last crossing $(v(a_p), v(a_p + 1))$. This proves the lemma and also gives us the following formula:

$$J(ws_r) = \begin{cases} J(w) \cup w(r, r + 1) & \text{if } \ell(ws_r) > \ell(w) \\ J(w) \backslash w(r, r + 1) & \text{if } \ell(ws_r) < \ell(w) \end{cases}$$

□

This formula provides us with an interpretation of the so-called *weak Bruhat order* on S_n. By definition these are given by:

$$\text{Right order: } v \leq w \Leftrightarrow J(v) \subseteq J(w), vs_i > w \Leftrightarrow \ell(vs_i) > \ell(w)$$
$$\text{Left order: } v \geq w \Leftrightarrow I(v) \subseteq I(w), s_i v > v \Leftrightarrow \ell(s_i v) > \ell(w).$$

Lemma 12.10 (Exchange Lemma)**.** *Suppose* $(a_1 \cdots a_p), (b_1 \cdots b_p) \in R(w)$ *are both reduced expressions for* w*. Then there exists an integer* i *such that* $(b_1 a_1 \cdots \hat{a}_i \cdots a_p) \in R(w)$*.*

Proof. From Lemma 12.9, we have $(b_i, b_i+1) \in J(w)$ and hence $(b_i, b_i+1) = s_{a_1} \cdots s_{a_{i-1}}(a_i, a_i+1)$. This implies that $s_{b_1} = s_{a_1} \cdots s_{a_{i-1}} s_{a_i} s_{a_{i-1}} \cdots s_{a_1}$ and hence $s_{b_1} s_{a_1} \cdots s_{a_{i-1}} = s_{a_1} \cdots s_{a_i}$. We then have $w = s_{a_1} \cdots s_{a_p} = s_{b_1} s_{a_1} \cdots s_{a_{i-1}} s_{a_{i+1}} \cdots s_{a_p}$, as desired. □

Definition 12.11. Given an element $w \in S(n)$, we define $G(w)$ to be the graph with vertex set $R(w)$, and with edges given by $(a_1 \cdots a_p) \sim (b_1 \cdots b_p) \Leftrightarrow$ the expressions differ by some s_i.

Theorem 12.12 (Tits)**.** *The graph* $G(w)$ *is connected.*

Proof. We prove this by induction on $\ell(w)$, the length of w. First, it's clear that $G(s_i)$ is connected. Next assume $a = (a_1 \cdots a_p)$ and $b = (b_1 \cdots b_p)$ are elements of $R(w)$. From the previous lemma we have that $(b_1 a_1 \cdots \hat{a}_i \cdots a_p) \in R(w)$, and hence by induction $(b_1 \cdots b_p)$ is connected to $(b_1 a_1 \cdots \hat{a}_i \cdots a_p)$. If $i \neq p$ then we also have $(b_1 a_1 \cdots \hat{a}_i \cdots a_p)$ adjacent to $(a_1 \cdots a_p)$. If $i = p$ then we have $(b_1 a_1 \cdots a_{p-1}) \in R(w)$, and by the Exchange Lemma we have $(a_1 b_1 a_1 \cdots \hat{a}_j \cdots a_{p-1}) \in R(w)$, which is in turn adjacent to $(b_1 a_1 b_1 \cdots \hat{a}_j \cdots a_{p-1}) \in R(w)$. By induction, the former of these is adjacent to a, while the latter is adjacent to b, and hence the claim follows. □

13. Schubert Polynomials (May 4, 2007)

Last time we saw that there is a map

$$A^* F\ell_n \to \mathbb{Z}[x_1, \ldots, x_n] / < e_1, \ldots, e_n >$$

and the Bernstein, Gelfand, Gelfand Theorem says that the classes corresponding Schubert varieties form an additive basis for the Chow ring. This map sends $[X_w] \mapsto \sigma_{ww_0}$ so that modulo the ideal $< e_1, \ldots, e_n >$, the divided difference operators act by

$$\partial_i(\sigma_w) = \begin{cases} \sigma_{ws_i} & \text{if } w_i > w_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Last time, we had the following candidates for the σ_w:

$$[X_{321}] \mapsto \sigma_{123} = 1$$
$$[X_{312}] \mapsto \sigma_{213} = x_1$$
$$[X_{231}] \mapsto \sigma_{132} = x_1 + x_2$$
$$[X_{213}] \mapsto \sigma_{312} = x_1^2$$
$$[X_{132}] \mapsto \sigma_{231} = x_1 x_2$$
$$[X_{123}] \mapsto \sigma_{321} = x_1^2 x_2.$$

Let's check that this actually works. Start at the bottom of this list with $[X_{123}] \mapsto \sigma_{321}$. Here, $3 > 2$ and $2 > 1$, so we expect that $\partial_1 \sigma_{321}$ and $\partial_2 \sigma_{321}$ are both nonzero. In particular,

$$[321] \cdot s_1 = [321][213] = [231]$$

and

$$[321] \cdot s_2 = [321][132] = [312]$$

so we expect that $\partial_1\sigma_{321} = \sigma_{231}$ and $\partial_2\sigma_{321} = \sigma_{312}$. Indeed,

$$\partial_1(x_1^2x_2) = \frac{x_1^2x_2 - x_1x_2^2}{x_1 - x_2} = x_1x_2 = \sigma_{231}$$

and

$$\partial_2(x_1^2x_2) = \frac{x_1^2x_2 - x_1^2x_3}{x_2 - x_3} = x_1^2 = \sigma_{312}.$$

Next, we look at $[X_{132}] \mapsto \sigma_{231}$. Since $1 < 3$, we expect that $\partial_1\sigma_{231} = 0$; and since $3 > 1$, we should get that $\partial_2\sigma_{213} = \sigma_{[213]\cdot s_2} = \sigma_{231} = x_1x_2$. Indeed, $\partial_1(x_1x_2) = 0$ because x_1x_2 is symmetric in x_1 and x_2, and $\partial_2(x_1x_2) = \frac{x_1x_2 - x_1x_3}{x_2 - x_3} = x_1$. The remaining cases follow similarly: $\partial_1\sigma_{312} = x_1 + x_2 = \sigma_{[312]\cdot s_1}$ and $\partial_2\sigma_{312} = 0$; $\partial_1\sigma_{132} = 0$ and $\partial_2\sigma_{132} = 1 = \sigma_{[132]\cdot s_2}$; $\partial_1\sigma_{213} = 1 = \sigma_{[213]\cdot s_1}$ and $\partial_2\sigma_{213} = 0$.

Now it might be natural to ask what would happen if we took a different choice for the polynomial σ_{321}? Suppose for example we took $\sigma_{321} = x_3^2x_2$. Then we compute the following:

$$\partial_1\partial_2\partial_1(x_3^2x_2) = \partial_1\partial_2(-x_3^2) = \partial_1(x_2 + x_3) = -1$$

and

$$\partial_2\partial_1\partial_2(x_3^2x_2) = \partial_2\partial_1(-x_2x_3) = \partial_2(x_3) = -1.$$

This doesn't quite work because we want $[X_{w_0}][X_u] = [X_u]$ for all $u \in S_n$, and in this case we get $[X_{321}] \mapsto \sigma_{123} = \partial_1\partial_2\partial_1\sigma_{321} = -1$. However, if we take $\sigma_{321} = -x_3^2x_2$, everything works. However, modulo the ideal generated by the elementary symmetric functions, $-x_3^2x_2$ and $x_1^2x_2$ are equal:

$$\begin{aligned}
x_1^2x_2 + x_2x_3^2 &= x_1x_2(x_1 + x_2 + x_3) - x_1x_2^2 - x_1x_2x_3 \\
&\quad +x_2x_3(x_1 + x_2 + x_3) - x_1x_2x_3 - x_2^2x_3 \\
&\equiv x_1x_2^2 + x_2^2x_3 \\
&= x_2^2(x_1 + x_2 + x_3) - x_2^3 \\
&\equiv x_2^3.
\end{aligned}$$

In the last lecture, we saw that $x_2^3 \equiv 0$ modulo the ideal.

However, our original choice, with $\sigma_{321} = x_1^2x_2$ had the advantage that each monomial in σ_w had nonnegative integer coefficients. In fact, we could have taken σ_{w_0} to be the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$, but the resulting system is a mess. The niceness of our original choice motivates the following definition from Lascoux and Schutzenberger.

Definition 13.1. Let $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1x_n^0$. For $w < w_0$, we define the *Schubert polymomial* $\mathfrak{S}_w$ as

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} \cdot \mathfrak{S}_{w_0}.$$

We claim that this is a "good" definition of the Schubert polynomials in the following proposition. First we need some notation: let $\delta = (n-1, n-2, \dots, 1, 0) \in [n]^n$. We can partially order elements $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in [n]^n$ by containment, decreeing that $\alpha \subset \beta$ if $\alpha_i \leq \beta_i$ for all i. For such an α, let $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$.

Proposition 13.2. *The Schubert polynomials $\mathfrak{S}_w(x)$ have the following properties:*

1. $\mathfrak{S}_{w_0} = x^\delta$ *and* $\mathfrak{S}_{id} = 1$. *Moreover,* $\mathfrak{S}_w$ *is a homogeneous polynomial of degree* $\ell(w)$ *in* $\mathbb{Z}[x_1, \dots, x_n]$.
2. *The monomial terms of* $\mathfrak{S}_w$ *are contained in the set of Artin monomials, i.e.* $\mathfrak{S}_w$ *is contained in the* $\mathbb{Z}$*-span of monomial terms of the form* x^α *for* $\alpha \subset \delta$. *This implies that the* $\mathfrak{S}_w$ *are nonzero.*

(3) *If r is the last descent of $w \in S_n$, i.e. $w_r > w_{r+1} < w_{r+2} < \ldots < w_n$, then $\mathfrak{S}_w$ is a polynomial in $x_1, \ldots, x_r$.*
(4) *The polynomial $\mathfrak{S}_w$ is symmetric in x^i and x^{i+1} if and only if $w_i < w_{i+1}$.*
(5) $\mathfrak{S}_{s_i} = x_1 + \ldots + x_i$
(6) *(Stability) Let $i : S_n \hookrightarrow S_{n+1}$ be the natural inclusion $[w_1, \ldots w_n] \mapsto [w_1, \ldots, w_n, n+1]$. Then $\mathfrak{S}_w = \mathfrak{S}_{i(w)}$.*

Proof. (1) By definition, $\mathfrak{S}_{w_0} = x^\delta$. Let $w_0^{(n)} = [n, n-1, \ldots, 1]$ and $w_0^{(n-1)} = [n-1, n-2, \ldots, 1, n]$. Certainly $w_0^{(n)} = w_0^{(n-1)} s_{n-1} s_{n-2} \cdots s_2 s_1$ so

$$\begin{aligned}
\mathfrak{S}_{id} &= \partial_{w_0^{(n)}} \cdot \mathfrak{S}_{w_0} \\
&= \partial_{w_0^{(n-1)}} \partial_{n-1} \partial_{n-2} \cdots \partial_1 (x^\delta) \\
&= \partial_{w_0^{(n-1)}} \partial_{n-1} \partial_{n-2} \cdots \partial_1 (x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1) \\
&= \partial_{w_0^{(n-1)}} \partial_{n-1} \partial_{n-2} \cdots \partial_2 (x_1^{n-2} x_2^{n-2} \cdots x_{n-1}^1) \\
&= \ldots \\
&= \partial_{w_0^{(n-1)}} (x_1^{n-2} x_2^{n-3} \cdots x_{n-1}^0).
\end{aligned}$$

But this is the Schubert polynomial indexed by the identity in S_{n-1}. Inductively, this is equal to 1.

Finally, to see that the polynomials are homogeneous, simply note that $\mathfrak{S}_{w_0}$ is homogeneous of degree $n(n-1)/2$ and that the divided difference operators preserve homogeneity and decrease degree by 1.

(2) If x^α is a monomial term with $\alpha \subseteq \delta$, then in computing $\partial_i(x^\alpha)$, we only care about the terms $x_i^{\alpha_i}$ and $x_{i+1}^{\alpha_{i+1}}$. Last time, we saw that $\partial_i(x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}})$ is an element of the span of monomial terms of the form $x_i^{\beta_i} x_{i+1}^{\beta_{i+1}}$ such that β_i and β_{i+1} are at most $\max\{\alpha_i, \alpha_{i+1}\} - 1$. In particular, if $x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \subseteq x^\delta$, then so are all terms $x_i^{\beta_i} x_{i+1}^{\beta_{i+1}}$ with the above restrictions.
(3) This will follow from (4). If r is the last descent of w, then $\mathfrak{S}_w$ is symmetric in $x_{r+1}, \ldots, x_n$. However there isn't an x_n term in $\mathfrak{S}_{w_0}$, so there can't be an x_n term in $\mathfrak{S}_w$. Thus none of the terms $x_{r+1}, \ldots, x_n$ appear.
(4) By Bernestein-Gelfand-Gelfand, $w_i < w_{i+1}$ if and only if $\partial_i(\mathfrak{S}_w)$ = 0 if and only if $\mathfrak{S}_w$ is symmetric in x_i and x_{i+1}.
(5) Since $s_i = [1, \ldots, i-1, i+1, i, i+2, \ldots, n]$, by (4) $\mathfrak{S}_{s_i}$ is symmetric in $x_1, \ldots, x_i$; and by (3), $\mathfrak{S}_{s_i}$ is a polynomial in only these variables. By (1), $\mathfrak{S}_{s_i}$ is homogeneous of degree $\ell(s_i) = 1$. Thus $\mathfrak{S}_{s_i} = c(x_1 + \ldots + x_i)$ for some $c \in \mathbb{Z}$. If we apply ∂_i to $\mathfrak{S}_{s_i}$, on the one hand, we must get $\mathfrak{S}_{s_i \cdot s_i} = \mathfrak{S}_{id} = 1$. On the other hand, by direct computation, we get
$$\partial_i(c(x_1 + \ldots + x_i)) = \frac{c(x_1 + \ldots + x_i) - c(x_1 + \ldots + x_{i+1})}{x_i - x_{i+1}} = c.$$
Thus $c = 1$.
(6) By direct computation, as in part (1).
$$\begin{aligned}
\mathfrak{S}_w &= \partial_{w^{-1}w_0}(x^\delta) \\
&= \partial_{w^{-1}w_0} \partial_n \partial_{n-1} \cdots \partial_1 (x_1^n x_2^{n-1} \cdots x_{n-1}^2 x_n) \\
&= \mathfrak{S}_{i(w)}
\end{aligned}$$

□

Remark 13.3.

- In geometry, the variables x_i are the Chern classes of certain line bundles E_i/E_{i-1}, but this is special to type A.
- Stability gives us a sequence on cohomology:

$$\dots \hookrightarrow H^*(F\ell_n) \hookrightarrow H^*(F\ell_{n+1}) \hookrightarrow \dots$$

that sends $[X_{ww_0}] \mapsto [X_{i(ww_0)}]$.

Proposition 13.4. (1) *The set $\{\mathfrak{S}_w : w \in S_n\}$ forms a basis for $H_n = \text{span}\{x^\alpha : \alpha \subseteq \delta\}$.*

(2) *If we order the Artin monomials in reverse lexicographic order, then each Schubert polynomial $\mathfrak{S}_w$ can be written as a sum whose highest monomial term is $x^{c(w)}$.*

(3) *Let $S_\infty = \bigcup_{n=1}^\infty S_n$. Then $\{\mathfrak{S}_w : w \in S_\infty\}$ form a basis for the polynomials in $x_1, x_2, \dots$ over $\mathbb{Z}$.*

Proof. (1) Since $|S_n| = n! = \dim(H_n)$, we need only show that the Schubert polynomials are independent over $\mathbb{Z}$. Suppose we have some linear combination $\sum_{w \in S_n} a_w \mathfrak{S}_w = 0$. In particular, if we look at the homogeneous terms of degree k, we must have

$$\sum_{\substack{w \in S_n: \\ l(w)=k}} a_w \mathfrak{S}_w = 0.$$

Now for any u of length k, if we apply the divided difference operator ∂_u to the above sum, we expect to get 0 because each summand $\mathfrak{S}_w$ is homogeneous of degree k, and applying ∂_u lowers the degree by k. On the other hand, $\partial_u(\sum a_w \mathfrak{S}_w) = a_u$, hence $a_u = 0$.

(2) We induct on $\ell(w)$, starting from the top. Certainly $\mathfrak{S}_{w_0} = x^{c(w_0)}$. Suppose inductively that $\mathfrak{S}_w$ can be written as $x^{\text{code}(w)}$ plus lower terms in reverse lexicographic order, and that $w_i > w_{i+1}$. In the diagram $D(w)$, there are 1's in positions (i, w_i) and $(i+1, w_{i+1})$, and the former lies above and to the right of the latter. Thus the number of free entries in the ith row is at least equal to the number of free entries in the $(i+1)$th row. In terms of the code of the permutation, $c(w) = (c_1, \dots, c_n)$, $c_i = c_{i+1} + r$ for some nonnegative r. Now we compute the following:

$$\begin{aligned}
\partial_i(x^{c(w)}) &= \partial_i(x_1^{c_1} \cdots x_i^{c_{i+1}+r} x_{i+1}^{c_{i+1}} \cdots x_n^{c_n}) \\
&= (x_1^{c_1} \cdots x_i^{c_{i+1}} x_{i+1}^{c_{i+1}} \cdots x_n^{c_n}) \cdot \partial_i(x_{i+1}^r) \\
&= (x_1^{c_1} \cdots x_i^{c_{i+1}} x_{i+1}^{c_{i+1}} \cdots x_n^{c_n}) \cdot (x_{i+1}^{r-1} + \text{ lower terms}) \\
&= (x_1^{c_1} \cdots x_i^{c_{i+1}} x_{i+1}^{c_{i+1}+r-1} \cdots x_n^{c_n}) + \text{ lower terms} \\
&= x^{c(ws_i)} + \text{ lower terms.}
\end{aligned}$$

(3) Any x^α has the property that there is a $w \in S_n$ for n large enough such that $c(w) = \alpha$. Thus x^α can be written as a linear combination of $\mathfrak{S}_w$ and a sum of terms $\mathfrak{S}_v$ where $c(v) \subset c(w)$ by (2) and induction. Since the Artin monomials form a basis for the ring of polynomials in the x_i, so too must the $\mathfrak{S}_w$.

□

Corollary 13.5. *We can write $\mathfrak{S}_u \mathfrak{S}_v == \sum b_{uv}^w \mathfrak{S}_w$ where the coefficients $b_{uv}^w = C_{uw_0, vw_0}^{ww_0}$ count intersection multiplicities. In particular, these coefficients are nonnegative integers.*

Proposition 13.6. *Let $f = \sum c_i x_i$ with $c_i \in \mathbb{Z}$. Then*

$$f \cdot \mathfrak{S}_w = \sum (c_i - c_j) \mathfrak{S}_{wt_{ij}}$$

where the sum is taken over all t_{ij} such that $\ell(w) = \ell(wt_{ij}) - 1$.

Proof. Since $f \cdot \mathfrak{S}_w$ is homogeneous of degree $\ell(w)+1$, we can write

$$f \cdot \mathfrak{S}_w = \sum_{v:\ell(v)=\ell(w)+1} \partial_v(f \cdot \mathfrak{S}_w) \cdot \mathfrak{S}_v.$$

Let $a_1 \cdots a_p$ be a reduced expression for v and compute

$$\begin{aligned}
\partial_v(f \cdot \mathfrak{S}_w) &= \partial_{a_1} \cdots \partial_{a_p}(f \cdot \mathfrak{S}_w) \\
&= \partial_{a_1} \cdots \partial_{a_{p-1}}(s_{a_p}(f)\partial_{a_p}(\mathfrak{S}_w) + \partial_{a_p}(f)\mathfrak{S}_w) \\
&= \ldots \\
&= s_{a_1} \cdots s_{a_p}(f)\partial_{a_1} \cdots \partial_{a_p}(\mathfrak{S}_w) + \sum_{r=1}^{p} s_{a_1} \cdots \partial_{a_r} \cdots s_{a_p}(f) \cdot \partial_{a_1} \cdots \widehat{\partial}_{a_r} \cdots \partial_{a_p}(\mathfrak{S}_w).
\end{aligned}$$

Since $p = \ell(v) = \ell(w)+1$, $\partial_{a_1} \cdots \partial_{a_p}(\mathfrak{S}_w) = 0$. In the second summation, the only nonzero terms will be those in which $s_{a_1} \cdots \widehat{s}_{a_r} \cdots s_{a_p} = w$. This implies that $s_{a_1} \cdots \widehat{s}_{a_r} \cdots s_{a_p} \cdot s_{a_p} \cdots s_{a_r} \cdots s_{a_p} = wt_{ij}$ for $i < j$ such that $t_{ij} = s_{a_p} \cdots s_{a_r} \cdots s_{a_p}$. Thus $(i,j) = s_{a_p} \cdots s_{a_{r+1}}(a_r, a_{r+1})$ and hence $s_{a_1} \cdots \partial_{a_r} \cdots s_{a_p}(f) = c_i - c_j$. □

Corollary 13.7. *(Monk's formula)*

$$\mathfrak{S}_{s_r} \cdot \mathfrak{S}_w = \sum_{\substack{1 \le r < j \\ \ell(w)=\ell(wt_{ij})-1}} \mathfrak{S}_{wt_{ij}}$$

Proof. We know that $\mathfrak{S}_{s_r} = x_1 + \ldots + x_r$ so we just apply the previous proposition and note that the coefficient $c_i \neq c_j$ if and only if $i \le r < j$. □

Example 13.8. Compute $\mathfrak{S}_{s_3} \cdot \mathfrak{S}_{3146527}$.

To do this, we need only determine which transpositions that interchange one of $\{3,1,4\}$ with one of $\{6,5,2,7\}$ increase the length of the permutation $[3146527]$ by 1. The only such transpositions are $4 \leftrightarrow 5$, $4 \leftrightarrow 6$, and $1 \leftrightarrow 2$. Thus

$$\mathfrak{S}_{s_3} \cdot \mathfrak{S}_{3146527} = \mathfrak{S}_{3164527} + \mathfrak{S}_{3156427} + \mathfrak{S}_{3246517}.$$

Exercise 13.9. Do any six applications of Monk's formula.

Exercise 13.10. Prove that $[X_{ww_0}][X_{s_r w_0}] = \sum_{1 \le r < j} [X_{wt_{ij}w_0}]$.

Open Question 13.11. *Expand $\mathfrak{S}_u\mathfrak{S}_v$ via some tree using iterated applications of Monk's formula.*

Corollary 13.12. *(Transition Equation) Suppose $w = vt_{rs}$ where r is the position of the last descent of w and s is the largest position of w such that $w_s < w_r$. Then*

$$\mathfrak{S}_w = x_r \cdot \mathfrak{S}_v + \sum \mathfrak{S}_{w'}$$

where the sum is taken over all $w' = vt_{ir}$ such that $i < r$ and $\ell(w) = \ell(w')$.

Proof. The polynomial x_r is the simplest linear homogeneous polynomial of degree 1 that we can ask for, so by the product formula,

$$x_r \cdot \mathfrak{S}_v = \mathfrak{S}_w - \sum \mathfrak{S}_{vt_{ir}}$$

where the latter sum is taken over all $i < r$ such that $\ell(v) = \ell(vt_{ir}) - 1$. □

Remark 13.13. This is a very efficient way to multiply out Schubert polynomials without eating up as much computer memory; it is a sort of "depth-first" multiplication.

14. On "Flags, Schubert Polynomials, Degeneracy Loci, and Determinantal Formulas" by William Fulton.

This lecture was given by Ashesh Bakshi and Steve Klee.

14.1. **Fiber Bundles.** We need to introduce some definitions and facts about fiber bundles. More detailed information can be found in Milnor and Stasheff's *Characteristic Classes*.

Definition 14.1. Let B be a topological space. A *real vector bundle* ξ of rank n (or n*-plane bundle*) over B consists of a topological space $E = E(\xi)$ and a continuous surjective map $p : E \to B$ with the following properties:

1. For each $b \in B$, the set $p^{-1}(b)$ has the structure of a real n-dimensional vector space.
2. The bundle is *locally trivial*, i.e. each point $b \in B$ has a neighborhood U and a homeomorphism $h : U \times \mathbb{R}^n \to p^{-1}(U)$ such that for each $b' \in U$, the correspondence $v \mapsto h(b', v)$ defines an isomorphism from $\mathbb{R}^n$ to $p^{-1}(b')$ and the following diagram commutes: where π_1 is projection onto the first coordinate.

We say B is the *base space* of the bundle and E is the *total space* of the bundle. The map p is called the *projection map* and the set $p^{-1}(b)$ is called the *fiber* over b, and is occasionally denoted $F_b(\xi)$.

Similarly, if F is a topological space, we can define a *fiber bundle* with fiber F to be a continuous surjection $p : E \to B$ such that for each $b \in B$, $p^{-1}(b)$ is homeomorphic to F and such that local triviality is satisfied.

Remark 14.2. For general fiber bundles, we may require additional structure such as a Lie group action on F. We omit these requirements at this point for the sake of simplicity. For example, in the case of a vector bundle, the Lie group $GL_n(\mathbb{R})$ acts transitively on each fiber.

Remark 14.3. Depending on what we want to emphasize, we may simply say that $p : E \to B$ is a fiber bundle with fiber F rather than saying ξ is a fiber bundle with total space E, base space B, projection map p, and fiber F.

Example 14.4. Any topological space F can be viewed as a fiber bundle over a space consisting of a single point $\{*\}$.

Example 14.5. If ξ is an n-plane bundle with projection $p : E \to B$ such that E is homeomorphic to $B \times \mathbb{R}^n$, we say that ξ is a *trivial bundle*.

Example 14.6. If M is a smooth n-manifold, the tangent bundle TM is a smooth n-plane bundle over M.

Definition 14.7. If $p : E \to B$ is a bundle, a *section* of the bundle is a map $s : B \to E$ such that $p \circ s$ is the identity map on B.

Sometimes it is important to construct new vector bundles from known vector bundles. The following constructions will be of particular importance:

Definition 14.8. Suppose ξ is a fiber bundle with projection $p : E \to B$. If U is a subset of B, we define the restriction of ξ to U, denoted $\xi|U$ to be the bundle with total space $E(\xi|U) = p^{-1}(U)$, base space U (both with the subspace topology), and projection map given by the restriction of p to $E(\xi|U)$. It is trivial to check that this is indeed a fiber bundle.

For our purposes, we will focus primarily on bundles whose fibers are vector spaces, projective spaces, and flag manifolds.

Definition 14.9. Suppose ξ is a fiber bundle with fiber F and projection $p : E \to B$. Let X be any space. Given a continuous map $f : X \to B$, we construct the *induced bundle* or *pullback bundle* $f^*\xi$ over X as follows. The total space $E(f^*\xi)$ is the subset of $X \times E$ consisting of all pairs (x, v) such that $f(x) = p(v)$. The projection map $p_1 : E(f^*\xi) \to X$ sends $(x, v) \mapsto x$. There is also an induced map $\hat{f} : E(f^*\xi) \to E(\xi)$ sending $(x, v) \mapsto v$ so that the following diagram commutes.

In the specific case that ξ is an n-plane bundle, notice that the map $\hat{f}$ is a linear isomorphism on each fiber of $f^*\xi$. This motivates the following definition.

Definition 14.10. Suppose $p : E \to B$ is an n-plane bundle and $p' : E' \to B'$ is an m-plane bundle. A *bundle map* is a pair of continuous maps $F : E \to E'$ and $f : B \to B'$ such that $p \circ F = f \circ p'$ and such that F is $\mathbb{R}$-linear when restricted to any fiber of p. We will usually only describe the map $F : E \to E'$ and use the requisite commutative diagram to induce the map $f : B \to B'$.

If $p : E(\xi) \to B$ is a vector bundle with fiber $\mathbb{R}^n$, it may be natural to ask questions about vector subspaces of the fibers of p. A *subbundle* $\eta \subset \xi$ is a k-plane bundle over B whose total space $E(\eta)$ is a subspace of $E(\xi)$ such that over each point $b \in B$, $F_b(\eta)$ is a k-dimensional subspace of $F_b(\xi)$. This gives rise to one additional construction:

Definition 14.11. Suppose ξ is an n-plane bundle over B and $\eta \subset \xi$ is a k-plane subbundle. The *quotient bundle* ξ/η is an $(n-k)$-plane bundle over B. Over each point $b \in B$, $F_b(\xi/\eta)$ is the vector space quotient $F_b(\xi)/F_b(\eta)$.

Remark 14.12. As before, if we do not wish to emphasize the subbundle $\eta \subset \xi$, we may simply say that $E' \subset E$ is a k-plane subbundle of E. In saying this, we mean to indicate that there is a subbundle η of ξ whose total space is E', and whose projection map is $p|E'$.

Example 14.13. Suppose $p : E \to B$ is an n-plane bundle. We form a new projective bundle $\mathbb{P}(E)$ over B such that $F_b(\mathbb{P}(E)) = \mathbb{P}(F_b(E))$, i.e. the fiber over b in $\mathbb{P}(E)$ is the real projective space on the vector space $F_b(E)$. We can also view $\mathbb{P}(E)$ as a rank $(n-1)$-vector bundle over E.

Definition 14.14. Suppose $\xi : E \xrightarrow{p} B$ and $\eta : E' \xrightarrow{p'} B'$ are n- and $m-$plane bundles over a common base space B. We obtain the total space of new bundle Hom(ξ, η) by decreeing that for each $b \in B$, $F_b(\text{Hom}(\xi, \eta))$ is the mn-dimensional vector space of linear maps from $p^{-1}(b) \to p'^{-1}(b)$. (It takes a fair amount of work to show that this space can be given a topology that gives it the structure of a vector bundle.) Similarly we can obtain bundles $\xi \otimes \eta$ (respectively $\xi \oplus \eta$) by applying the tensor product (resp. direct sum) functors to the fiber over each point. (The latter bundle is called the *Whitney sum* of ξ and η.)

14.2. **Degeneracy Loci.** In general, subbundles of vector bundles are not as easy to find as one might expect. Using cohomology theory, it is possible to show that the tangent space of real projective space $\mathbb{P}^n$ admits a line subbundle (i.e. a subbundle of rank 1) if and only if n is odd. Moreover, the tangent space to complex projective space never admits a line subbundle.

Now, however, we will bestow upon ourselves a most fortunate situation that is analogous to a decomposition of a vector space into a flag of subspaces.

Definition 14.15. Suppose $p : E \to B$ is a vector bundle of rank n. We say E is a *filtered vector bundle* (over B) if there is a chain of subbundles $E_1 \subset E_2 \subset \ldots \subset E_n = E$ such that E_i has rank i over B.

Dually, if $E = E_\bullet$ is a filtered vector bundle, there is a descending chain of surjective bundle maps

$$E = Q_n \to Q_{n-1} \to \ldots \to Q_2 \to Q_1$$

where $Q_i = E/E_{n-i}$ has rank i. As a means of notation, in this case we occasionally abuse notation and write $E = E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow E_1$ to indicate this chain of quotient bundles.

With this notation in place, suppose X is a variety, and $E_\bullet$ and $F_\bullet$ are filtered vector bundles of rank n over X. If $h : E \to F$ is a bundle map then we can form a chain of maps

$$E_1 \subset E_2 \subset \ldots \subset E_n \xrightarrow{h} F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow F_1.$$

For any $1 \leq p, q \leq n$, we will denote the composition of bundle maps

$$E_p \hookrightarrow E_{p+1} \hookrightarrow \ldots \hookrightarrow E_n \xrightarrow{h} F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow F_q$$

as $E_p \to F_q$.

Definition 14.16. For a permutation $w \in S_n$, we define the *degeneracy locus* $\Omega_w = \Omega_w(h) = \Omega_w(h, E_\bullet, F_\bullet)$ to be the subset of points $x \in X$ such that the restricted map $E_p|x \to F_q|h(x)$ has rank at most $r_w(q,p)$ for all $1 \leq p, q \leq n$. As a means of convenience we abbreviate this by saying that Ω_w is defined by the conditions that rank$(E_p \to F_q) \leq r_w(q,p)$ for all p, q.

As in the case of flag varieties, we can use Fulton's essential set to determine a smaller generating set for Ω_w.

Definition 14.17. In the theorem below we are using the definition that the Essential Set is the collection of southeast corners of connected components in the permutation matrix for w after canceling south and east from the entries $(i, w(i))$.

Proposition 14.18. *The variety Ω_w is defined by the conditions*

$$\textit{rank}(E_p \to F_q) \leq r_w(q,p)$$

for all p and q in $\{1, \ldots n\}$ such that $(q,p) \in Ess(w)$.

Proof. If we pick a point $x \in \Omega_w$, then we know the condition that the rank of the map $E_p|_x \to F_q|_x$ is bounded by $r_w(q,p)$ for all p, q is equivalent to the rank being bounded for all pairs (q,p) in the essential set of w. If $U \subset X$ is a trivializing neighborhood of x for all E_i (i.e. $E_i|U$ is homeomorphic to the trivial bundle), then the result certainly holds over $U \cap \Omega_w$ where E_p and F_q have the same behavior over all fibers. Thus the assertion holds in a neighborhood of each point in Ω_w. □

14.3. **Flag Bundles.** Suppose that $p : E \to X$ is a rank n vector bundle over a variety X. The *flag bundle* of E, denoted $Fl(E)$ is a vector bundle of rank $n(n-1)/2$ over X with projection map $\rho : Fl(E) \to X$. Since ρ is a continuous map to the base space of a bundle, using the pullback construction of definition 14.9 we can form an induced bundle $\rho^* E$. The flag bundle comes equipped with a universal flag of subbundles $U_\bullet$ of $\rho^* E$, with each U_i a vector bundle of rank i. The situation is as follows:

The flag bundle is universal in the following sense: if $f : Y \to X$ is a continuous map such that $f^* E$ has a complete flag of subbundles $V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset f^* E$, then there is a unique map $\tilde{f} : Y \to Fl(E)$ such that $\tilde{f}^* U_i = V_i$ as subbundles of $f^* E$ for all i. Finally, the universal flag of subbundles of $Fl(E)$ gives a universal flag of quotient bundles

$$\rho^* E \twoheadrightarrow Q_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow Q_1$$

where $Q_i = \rho^* E / U_{n-i}$ for all i.

We can construct $Fl(E)$ as a sequence of projective bundles (fiber bundles whose fiber is some projective space). Start with the canonical projection map $\rho_1 : \mathbb{P}(E) \to X$. The n-plane bundle $\rho_1^*(E)$ admits a universal line subbundle M_1: for $L \in \mathbb{P}(E)$, let $b = \rho_1(L)$. As a point in $\mathbb{P}(E)$, L represents a line through the origin in $F_b(E)$, and the fiber over L in $\rho_1^*(E)$ is canonically isomorphic to $F_b(E)$. Over L, take M_1 to be the collection of points $x \in F_b(E)$ such that x lies on L. Now we can take $\mathbb{P}(\rho_1^* E/M_1)$, which is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}(E)$ with projection map ρ_2. As before, the pullback bundle $\rho_2^*(\rho_1^* E)$ admits a universal line subbundle M_2. Iterating this process, we obtain $Fl(E)$.

We attempt to gain some intuition for the universal flag bundle with the following example, which will be useful later.

Example 14.19. Suppose V is an n-dimensional vector space, viewed as the trivial bundle over the one point space $\{b_0\}$ with projection $p : V \to \{b_0\}$. The flag bundle $Fl(V)$ is the flag manifold. Moreover, we can explicitly describe the universal flag of subbundles $U_\bullet$ of $\rho^* E$: for a complete flag $W_\bullet$ on V, $U_i | W_\bullet = W_i$.

Proof. We run through the above algorithm to construct the universal flag bundle $Fl(V)$. We start by taking $\mathbb{P}(V)$, the space of lines V_1 through the origin in V with projection ρ_1 onto $\{b_0\}$. In this case, $\rho_1^* V$ is the trivial bundle $\mathbb{P}(V) \times V$ with the universal line subbundle $M_1 = \{(V_1, x) \in \mathbb{P}(V) \times V : x \in V_1\}$.

Next, we look at the projective space on the quotient bundle $(\mathbb{P}(V) \times V)/M_1$, and consider the projection map $\rho_2 : \mathbb{P}(\rho^* V/M_1) \to \mathbb{P}(V)$. For $V_1 \in \mathbb{P}(V)$, we can explicitly compute $\rho_2^{-1}(V_1)$: the fiber over V_1 in $\rho^* V/M_1$ is the vector space V/V_1, so the fiber over V_1 by ρ_2 is $\mathbb{P}(V/V_1)$. But a line V_2 through the origin in V/V_1 can be identified with a plane in $V_2 \subset V$ that contains V_1. Thus we can identify $\mathbb{P}(\rho_1^* V/M_1)$ with the collection of partial flags $V_1 \subset V_2$ in V with projection map $(V_1 \subset V_2) \overset{\rho_1}{\mapsto} V_1$.

Under this identification, $\rho_2^*(\rho_1^*(V))$ can be viewed as the set of triples (V_2, V_1, x) where $V_1 \in \mathbb{P}(V)$ is a line through the origin in V and V_2 is a 2-plane containing V. This has a universal two-plane subbundle M_2 whose fiber over the pair $(V_2, V_1) \in \mathbb{P}(\rho_1^*(V)/M_1)$ is the set of points $x \in V_2$, which can be used to define a universal line subbundle $M_2/\rho_2^* M_1$ over $\mathbb{P}(\rho^* V/M_1)$.

From here, we appeal to the induction gods. □

Suppose our bundle $E \xrightarrow{p} X$ admits a complete flag of subbundles $E_\bullet = E_1 \subset E_2 \subset \ldots \subset E_{n-1} \subset E$. Then $\rho^* E$ admits a complete flag of subbundles $\rho^* E_1 \subset \ldots \subset \rho^* E_n$ is a complete flag of subbundles. Thus, over $Fl(E)$ we have a map of filtered vector bundles

$$\rho^* E_1 \subset \ldots \subset \rho^* E_{n-1} \subset \rho^* E = \rho^* E \twoheadrightarrow Q_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow Q_1.$$

For $w \in S_n$, we can form the degeneracy locus Ω_w in $Fl(E)$, denoted $\Omega_w(E_\bullet)$. We will show that if X is an irreducible variety, then $\Omega_w(E_\bullet)$ is an irreducible subvariety of $Fl(E)$ of codimension $l(w)$.

14.4. **Schubert Varieties.** We will now work in the specific case that the variety X is a single point. Here a bundle over X is an n-dimensional vector space V; and, as noted in example 14.19, the flag bundle $Fl(V) = Fl(n)$ is just the flag manifold with universal subbundle $U_\bullet$. Suppose we fix a flag $V_\bullet$ of vector spaces in V. For a permutation w in S_n, we look at the open Schubert cell

$$X_w^o = X_w^o(V_\bullet) = \{W_\bullet \in Fl(V) : \dim(W_q \cap V_p) = r_w(q,p); p, q \in [n]\}$$

and the closed Schubert variety

$$X_w = \{W_\bullet \in Fl(V) : \dim(W_q \cap V_p) \geq r_w(q,p); p, q \in [n]\}.$$

Now we consider $\rho : Fl(V) \to X = \{*\}$. As before, $\rho^* V$ is the trivial bundle $Fl(V) \times V$. Using our fixed flag $V_\bullet$, we get a flag of subbundles of $Fl(V) \times V$:

$$Fl(V) \times V_1 \subset Fl(V) \times V_2 \subset \ldots \subset Fl(V) \times V_{n-1} \subset Fl(V) \times V.$$

We also have a universal flag of quotient bundles

$$Fl(V) \times V \to Q_{n-1} \ldots \to Q_1$$

where Q_i is the quotient bundle $(Fl(V) \times V)/U_{n-i}$. As in the previous section, $Fl(V) \times V_i = \rho^* V_i$, so we can look at the degeneracy locus $\Omega_w(V_\bullet)$.

Proposition 14.20. *For any* $w \in S_n$, $\Omega_w(V_\bullet) = X_{w \cdot w_0}$.

Proof. Recall that the universal subbundle $U_\bullet$ of $Fl(V) \times V$ satisfies $U_i|W_\bullet = W_i$ for $W_\bullet \in Fl(V)$. To compute the degeneracy locus $\Omega_w(V_\bullet)$, we need to compute the ranks of the maps $(\rho^* V)_p \to Q_q$. For a complete flag $W_\bullet$, we can compute $(\rho^* V)_p | W_\bullet = V_p$ and

$$Q_q|W_\bullet = (\rho^* V/U_{n-q})|W_\bullet = \rho^* V|W_\bullet/U_{n-q}|W_\bullet = V/W_{n-q}.$$

Thus over $W_\bullet$, the map $(\rho^* V)_p \to Q_q$ is simply the restriction of the quotient map $V_p \to V/W_{n-q}$.

Now $\Omega_w(V_\bullet)$ is the subvariety of $Fl(V)$ of all flags $W_\bullet$ such that the map $V_p \to Q_q$ has rank at most $r_w(q,p)$. Over a single flag, this is just a map out of V_p, so saying its rank is at most $r_w(q,p)$ is equivalent to saying that its kernel has dimension at least $p - r_w(q,p)$. The kernel of the map $V_p \to V/W_{n-q}$ is clearly $V_p \cap W_{n-q}$. Thus $\Omega_w(V_\bullet)$ is the collection of complete flags such that for all p and q,

$$\begin{aligned}
\dim(W_q \cap V_p) &\geq p - r_w(n-q,p) \\
&= p - \#\{i \leq n-q : w(i) \leq p\} \\
&= p - \#\{i \leq p : w^{-1}(i) \leq n-q\} \\
&= \#\{i \leq p : w^{-1}(i) > n-q\} \\
&= \#\{i \leq p : w_0 w^{-1}(i) \leq q\} \\
&= \#\{i \leq q : w \cdot w_0(i) \leq p\} \\
&= r_{w \cdot w_0}(q,p).
\end{aligned}$$

But this is precisely $X_{w \cdot w_0}$.

□

14.5. **Double Schubert polynomials.** To state the main theorem in this paper we will need a slight generalization of Schubert polynomials.

Definition 14.21. Let A be a commutative ring. Recall that for each $1 \leq i \leq n-1$, the *divided difference* operator on $A[x_1, \ldots, x_n]$ takes a polynomial P to

$$\partial_i P = \frac{P(x_1, \ldots, x_n) - P(x_1, \ldots, x_{i-1}, x_{x+i}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}} \tag{14.1}$$

Definition 14.22. Let s_i denote the transposition exchanging i and $i+1$. The *double Schubert polynomial* for the permutation w is a homogeneous polynomial in $2n$ variables of degree $\ell(w)$. Write w as the product $w_0 \cdot s_{i_1} \cdot \cdots \cdot s_{i_r}$, where $r = \ell(w_0) - \ell(w)$, and set

$$\mathfrak{S}_w(x_1, \ldots, x_n, y_1, \ldots, y_n) = \partial_{i_r} \circ \cdots \circ \partial_{i_1}\Big(\prod_{i+j \leq n} (x_i - y_j)\Big) \tag{14.2}$$

Here the y_i are to be regarded as constants, i.e., the operators are defined on the ring $A[x_1 \ldots, x_n]$, where A is the ring $\mathbb{Z}[y_1, \ldots, y_n]$.

Remark 14.23. The above definition is independent of the choice of $s_{i_1}, \ldots, s_{i_j}$ and we can recover the ordinary Schubert polynomials as $\mathfrak{S}_w(x_1, \ldots, x_n) = \mathfrak{S}_w(x_1, \ldots, x_n, 0, \ldots, 0)$.

Conversely, the double Schubert polynomials can be expressed in terms of ordinary Schubert polynomials. For brevity, write x and y in place of $x_1, \ldots, x_n$ and $y_1 \ldots, y_n$ (respectively.)

Fact 14.24. For any permutation $w \in \mathbb{S}_n$,

$$\mathfrak{S}_w(x,y) = \sum (-1)^{\ell(v)} \mathfrak{S}_u(x) \mathfrak{S}_v(y)$$

where the sum is over pairs of permutations (u,v) such that $v^{-1} \cdot u = w$ and $\ell(u) + \ell(v) = \ell(w)$.

Remark 14.25. Suppose A is a ring, $c_1, \ldots, c_n$ are elements of A, and I is the ideal of $A[x]$ generated by the polynomials $e_i(x) - c_i$ where the e_i are the elementary symmetric functions. Notice that the operators ∂_w map I to itself, and hence descend to well-defined operators, also denoted by ∂_w from $A[x]/I \to A[x]/I$.

14.6. **Chern classes.** Let $\omega : E \xrightarrow{p} B$ be a complex n-plane bundle. There are certain *characteristic classes* in the singular cohomology ring of the base space. Working with $\mathbb{Z}/(2)$ coefficients, one has the Stiefel-Whitney classes, which can be used to study diverse questions: When can a smooth closed m-manifold be realized as the boundary of an $(m+1)$-manifold? For which m is the projective space $\mathbb{P}^m$ parallelizable? For which m can a manifold be immersed in $\mathbb{R}^m$?

Working instead with integer coefficients, we have for each $0 \leq i \leq n$ a *Chern class* $c_i(\omega) \in H^{2i}(B;\mathbb{Z})$. As is the case for the Stiefel-Whitney classes, they play an important role in the study of the topology of vector bundles and manifolds, and as we'll see below they crop up when we study the strucutre of the Chow ring as well. A proof of the existence of Chern classes is beyond the scope of this lecture; see Fulton's *Young Tableax*, chapter three of Fulton's *Intersection Theory*, or Milnor and Stasheff's *Characteristic Classes* for different constructions. For now we content ourselves with listing a few of their properties, not all of which will be used here:

Fact 14.26. (Vanishing) For any complex n-plane bundle ω, $c_i(\omega) = 0$ for $i > n$.

Fact 14.27. (Naturality) Given complex n-plane bundles $E \to B$ and $E' \to B'$, and maps $F : E \to E'$ and $f : B \to B'$ so that the pair (F, f) is a bundle map (cf. definition 14.10), then for each i, $c_i(\omega) = f^* c_i(\omega')$.

Definition 14.28. A *short exact sequence* of vector bundles is a sequence

$$0 \to \omega' \xrightarrow{f} \omega \xrightarrow{g} \omega'' \to 0$$

of bundles over a common base space B such that: f and g induce the identity map on the base space, and for each $b \in B$, the restriction to the fiber over b

$$0 \to \omega'|_b \xrightarrow{f|_b} \omega|_b \xrightarrow{g|_b} \omega''|_b \to 0$$

is a short exact sequence of linear maps between vector spaces.

Fact 14.29. (Whitney formula) If $0 \to \omega' \to \omega \to \omega'' \to 0$ is a short exact sequence of vector bundles over B, where B is paracompact (or, if you prefer to avoid lengthy topological definitions, where B is a variety), then $c_k(\omega) = \sum_{i+j=k} c_i(\omega') \cup c_j(\omega'')$.

It is often convenient to work in the ring $H^{\Pi}(B;\mathbb{Z})$ of all formal infinite series $a = a_0 + a_1 + \cdots$ with $a_i \in H^i(B)$, with the product operation $H^{\Pi}(B;\mathbb{Z}) \times H^{\Pi}(B;\mathbb{Z}) \xrightarrow{*} H^{\Pi}(B;\mathbb{Z})$ given by:

$$(a_0 + a_1 + \cdots) * (b_0 + b_1 + \cdots) = \sum_{l=0}^{\infty} \sum_{j+k=l} (a_j \cup b_k) = (a_0 \cup b_0) + (a_1 \cup b_0 + a_0 \cup b_1) + \cdots$$

In this ring define the *total Chern class* $c(\omega) = 1 + c_1(\omega) + c_2(\omega) + \cdots$. Two useful consequences are the following: the total Chern class of a vector bundle is a unit in the ring and the Whitney formula becomes simply $c(\omega) = c(\omega') * c(\omega'')$.

Definition 14.30. Let $G_k(\mathbb{C}^\infty)$ denote the complex Grassmannian of k-planes through the origin in $\mathbb{C}^\infty$. Notice in particular that when $k = 1$, $G_k(\mathbb{C}^\infty) = \mathbb{C}P^\infty$. The *universal k-plane bundle* γ^k over $G_k(\mathbb{C}^\infty)$ is the bundle whose total space is the subset of $G_k(\mathbb{C}^\infty) \times \mathbb{C}^\infty$ consisting of all pairs (V, x) such that V is a k-plane in $\mathbb{C}^\infty$ and $x \in V$.

Fact 14.31. If $p : E \to B$ is a complex k-plane bundle, then there is a map $f : B \to G_k(\mathbb{C}^\infty)$ such that $E = f^*\gamma^k$.

Fact 14.32. The cohomology ring $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ is a polynomial ring generated in degree 2 by $c_1(\gamma^1)$. As a corollary, $H^i(\mathbb{C}P^\infty) = 0$ for all odd integers i.

We need to introduce one final piece of notation. Suppose ξ is an n-plane bundle over X and η is an m-plane bundle over Y. The *external tensor product* of ξ and η is the bundle $\xi \hat{\otimes} \eta$ over $X \times Y$ whose fiber over (x, y) is $F_x(\xi) \otimes F_y(\eta)$.

Lemma 14.33. *Suppose ξ and η are line bundles over a common base space B. Then $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$.*

Proof. (The reader is invited to skip this proof if so inclined; only the statement will be of use below.) Recall the Kunneth formula for cohomology: if X and Y are spaces, then the cross product gives an isomorphism $H^k(X \times Y; \mathbb{Z}) \to \bigoplus_{i+j=k} H^i(X; \mathbb{Z}) \otimes H^j(Y; \mathbb{Z})$ in cases (like this one) where all the cohomology groups are free.

Let $d : B \to B \times B$ be the diagonal map $x \mapsto (x, x)$. By Fact 14.31, there are maps $f, g : B \to \mathbb{C}P^\infty$ such that $f^*\gamma^1 = \xi$ and $g^*\gamma^1 = \eta$. We can certainly map $f \times g : B \times B \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, and it is clear that $(f \times g)^*(\gamma^1 \hat{\otimes} \gamma^1) = \xi \hat{\otimes} \eta$. Moreover, $d^*(\xi \hat{\otimes} \eta) = \xi \otimes \eta$. This is illustrated in the following commutative diagram:

By the Kunneth formula, $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z})$ is isomorphic to $(H^2(\mathbb{C}P^\infty) \otimes H^0(\mathbb{C}P^\infty; \mathbb{Z})) \oplus (H^0(\mathbb{C}P^\infty; \mathbb{Z}) \otimes H^2(\mathbb{C}P^\infty; \mathbb{Z}))$; $H^1(\mathbb{C}P^\infty) = 0$ by Fact 14.32. Thus $c_1(\gamma^1 \hat{\otimes} \gamma^1)$ can be written as $ac_1(\gamma^1) \times 1 + 1 \times bc_1(\gamma^1)$ for some $a, b \in \mathbb{Z}$. Fix any point $*$ in $\mathbb{C}P^\infty$. To determine the constants a and b, we consider the maps $\iota_1 : \mathbb{C}P^\infty \times \{*\} \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ sending $(P, *) \mapsto (P, *)$ and $\iota_2 : \{*\} \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ defined similarly. We can identify $\iota_1^*(\gamma^1 \hat{\otimes} \gamma^1)$ with γ^1 and hence $\iota_1^*(c_1(\gamma^1 \hat{\otimes} \gamma^1)) = c_1(\gamma^1) \times 1$. Using the equation for $c_1(\gamma^1 \hat{\otimes} \gamma^1)$ from the Kunneth formula, $\iota_1^*(ac_1(\gamma^1) \times 1 + 1 \times bc_1(\gamma^1)) = ac_1(\gamma^1) \times 1 + 0$. Thus $a = 1$. Similarly, $b = 1$.

Now by naturality, we can compute $c_1(\xi \otimes \eta)$:

$$\begin{aligned}
c_1(\xi \otimes \eta) &= d^*((f \times g)^*(c_1(\gamma^1 \hat{\otimes} \gamma^1))) \\
&= d^*((f \times g)^*(c_1(\gamma^1) \times 1 + 1 \times c_1(\gamma^1))) \\
&= d^*(f^*(c_1(\gamma^1)) \times g^*1 + f^*1 \times g^*(c_1(\gamma^1))) \\
&= d^*(c_1(\xi) \times 1 + 1 \times c_1(\eta)) \\
&= c_1(\xi) \cup 1 + 1 \cup c_1(\eta) \\
&= c_1(\xi) + c_1(\eta))
\end{aligned}$$

$\square$

14.7. A Giambelli formula for flag bundles.

The key result in the paper is the following:

Theorem 14.34. *For any complete flag $E_\bullet$ of subbundles in a vector bundle E of rank n on a nonsingular variety X and any $w \in S_n$, the class of $\Omega_w(E_\bullet)$ in $A^{\ell(w)}(Fl(E))$ is given by the formula*

$$[\Omega_w(E_\bullet)] = \mathfrak{S}_w(x_1, \ldots, x_n.y_1, \ldots, y_n) \tag{14.3}$$

where x_i is the first Chern class of $\ker(Q_i \to Q_{i-1})$ and y_i is the first Chern class of E_i/E_{i-1}.

This proof depends on a few key facts, which we will state without proof.

First of all, let $A = A^*X$ be the Chow ring of X. There is a canonical (surjective) homomorphism from $A[x_1, \dots, x_n] \to A^*(Fl(E))$ whose kernel is the ideal I generated by the polynomials $e_i(x_1, \dots, x_n) - c_i(E)$. As noted in 14.25, this means that the divided difference operators descend to well-defined operators on $A[x_1, \dots, x_n]/I$, and hence also on $A^*(Fl(E))$. We'll also need:

Lemma 14.35. *Let $w \in S_n$, $1 \le k < n$, and set $w' = w \cdot s_k$. Then*

$$\partial_k([\Omega_w(E_\bullet)]) = \begin{cases} [\Omega_{w'}(E_\bullet)] & \text{if } w(k) > w(k+1), \\ 0 & \text{if } w(k) < w(k+1). \end{cases}$$

Lemma 14.36. *Let E be a vector bundle of rank d on a nonsingular variety X, and let $s : X \to E$ be a section. If the image of s intersects the zero-section transversally, then $[s^{-1}(0)] \in H^{2d}X$ is equal to $c_d(E)$.*

In words, the top Chern class of a complex vector bundle is Poincaré dual to the zero set of a generic section. This theorem is well known to algebraic geometers working in the area but a concise statement is hard to find. One reference containing a (somewhat) accesible proof is: W. Fulton, Equivariant Intersection Theory, Notes by Dave Anderson, Michigan University, 2005-2006, available at:

`http://www.math.lsa.umich.edu/~danderson/notes.html`

Proof. (of Theorem 14.34) Our first goal is to establish the formula for the longest permutation w_0. The essential set for this permutation is the collection of pairs $\{(p, n-p) : 1 \le p < n\}$, so the degeneracy locus of $\Omega_{w_0}(E_\bullet)$ is determined by the condition that $\mathrm{rank}(\rho^* E_p \to Q_{n-p}) \le r_{w_0}(p, n-p) = 0$ for $1 \le p < n$, i.e. all these maps have rank 0.

Let $H = \bigoplus_{p=1}^{n-1} \mathrm{Hom}(\rho^* E_p, Q_{n-p})$ and $H' = \bigoplus_{p=1}^{n-2} \mathrm{Hom}(\rho^* E_p, Q_{n-p-1})$. Over each element $W_\bullet \in Fl(E)$, the fiber $H|_{W_\bullet}$ is a direct sum of complex vector spaces $\bigoplus_{p=1}^{n-1} \mathrm{Hom}(\rho^* E_p|_{W_\bullet}, Q_{n-p}|_{W_\bullet})$. An element of the fiber can be written as an $(n-1)$-tuple $(\alpha_1, \dots \alpha_{n-1})$, with each α_p a $\mathbb{C}$-linear map from $\rho^* E_p|_{W_\bullet}$ to $Q_{n-p}|_{W_\bullet}$. Let $K_{W_\bullet}$ be the collection of $(n-1)$-tuples making the following diagram commute for each p:

I claim that $K = \bigcup_{W_\bullet \in Fl(E)} K_{W_\bullet}$ is a subbundle of H. In fact K is the kernel of a bundle map $g : H \to H'$ carrying $(\alpha_1, \dots, \alpha_{n-1})$ to $(\beta_1, \dots, \beta_{n-2})$ where, for each p, $\beta_p = \alpha_{p+1} \circ \iota_p - j_{n-p} \circ \alpha_p$. One can check that this map is surjective, so we have an exact sequence of bundles

$$0 \to K \hookrightarrow H \xrightarrow{g} H' \to 0$$

Since $\mathrm{Hom}(\rho^* E_p|_{W_\bullet}, Q_{n-p}|_{W_\bullet})$ has rank $p(n-p)$, H has rank $\sum_{p=1}^{n-1} p(n-p)$. Similarly, H' has rank $\sum_{p=1}^{n-2} p(n-p-1)$, and by dimension considerations it follows that K has rank $N = n(n-1)/2$. Define a section $s : Fl(E) \to K$, $W_\bullet \mapsto (\alpha_1, \dots, \alpha_{n-1})$, where α_p is the linear map $(\rho^* E_p \to Q_{n-p})|_{W_\bullet}$. The zero locus is an old friend: $s^{-1}(0) = \{W_\bullet \in Fl(E) : \rho^* E_p|_{W_\bullet} \to Q_{n-p}|_{W_\bullet} \equiv 0\} = \Omega_{w_0}$. Therefore by lemma 14.36, we can relate the class of the degeneracy locus to the Chern classes of K:

$$[\Omega_{w_0}(E_\bullet)] = c_N(K)$$

.

Our goal now is to compute $c_N(K)$. Recall that since we have an exact sequence of bundles

$$0 \to K \to H \to H' \to 0$$

, by the Whitney formula $c(H) = c(H')c(K)$. In particular, the top Chern class of H is the product of the top Chern class of H' with $c_N(K)$. Noting the similarity between the bundles H and H', we will study the bundle $\mathrm{Hom}(\rho^* E_k, Q_\ell)$. By some basic homological algebra,

$$\mathrm{Hom}(\rho^* E_k, Q_\ell) = (\rho^* E_k)^* \otimes Q_\ell$$

where $(\rho^* E_k)^*$ denotes the dual bundle $\mathrm{Hom}(\rho^* E_k, \mathbb{C})$. One final fact we must cite is that for a complex bundle ω, $c_i(\omega^*) = (-1)^i c_i(\omega)$.

Since we have a filtration

$$Q_\ell \twoheadrightarrow Q_{\ell-1} \twoheadrightarrow \ldots \twoheadrightarrow Q_1$$

and for all i the sequence

$$0 \to \ker(Q_i \twoheadrightarrow Q_{i-1}) \to Q_i \to Q_{i-1} \to 0$$

is exact, we can decompose Q_ℓ as a sum of line bundles:

$$Q_\ell = \bigoplus_{i=1}^{\ell} \ker(Q_i \twoheadrightarrow Q_{i-1}).$$

Similarly, by the filtration

$$\rho^* E_1 \to \rho^* E_2 \to \ldots \to \rho^* E_k,$$

for all $j \leq k$ we get a short exact sequence

$$0 \to \rho^* E_{j-1} \to \rho^* E_j \to \rho^* E_j / \rho^* E_{j-1} \to 0.$$

Hence we can decompose $\rho^* E_k$ as a sum of line bundles

$$\rho^* E_k = \bigoplus_{j=1}^{k} \rho^* E_j / \rho^* E_{j-1}.$$

Thus (since the Hom functor commutes with finite direct sums)

$$(\rho^* E_k)^* = \bigoplus_{j=1}^{k} (\rho^* E_j / \rho^* E_{j-1})^*.$$

This means we have reduced our study to

$$\mathrm{Hom}(\rho^* E_k, Q_\ell) = \bigoplus_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}} \left((\rho^* E_j / \rho^* E_{j-1})^* \otimes \ker(Q_i \twoheadrightarrow Q_{i-1}) \right).$$

Since each factor of the above sum is a tensor product of line bundles, Lemma 14.33 implies that

$$c(\rho^* E_j / \rho^* E_{j-1} \otimes \ker(Q_i \to Q_{i-1})) = 1 + x_i - y_j.$$

Thus by the Whitney product theorem,

$$c(\mathrm{Hom}(\rho^* E_k, Q_\ell)) = \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}} (1 + x_i - y_j).$$

Since the top Chern class of a product bundle is the product of the top Chern classes of each of its factors, the top Chern class of $\mathrm{Hom}(\rho^* E_k, Q_\ell)$ is

$$\prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}} (x_i - y_j).$$

In particular, this means that the top Chern class of H is the product

$$\prod_{p=1}^{n-1} \prod_{\substack{1\le i\le p \\ 1\le j\le n-p}} (x_i - y_j)$$

and the top Chern class of H' is

$$\prod_{p=1}^{n-2} \prod_{\substack{1\le i\le p \\ 1\le j\le n-p-1}} (x_i - y_j).$$

Since $c_N(K)$ is the quotient of the top Chern classes of H and H',

$$\begin{aligned} c_N(K) &= \prod_{1\le i\le n-1} (x_i - y_1) \prod_{p=1}^{n-1} \prod_{1\le i\le p} (x_i - y_{n-p}) \\ &= \prod_{i+j\le n} (x_i - y_j) \\ &= \mathfrak{S}_{w_0}(x, y). \end{aligned}$$

We have established the following identities:

$$[\Omega_{w_0}(E_\bullet)] = \prod_{i+j\le n} (x_i - y_j) = \mathfrak{S}_{w_0}(x, y).$$

Now, for any $w \in S_n$, write $w = w_0 \cdot s_{k_1} \cdot \cdots \cdot s_{k_r}$, where $r = n(n-1)/2 - \ell(w)$. Since $s_k(k) > s_k(k+1)$ for any k, applying Lemma 14.35 r times gives

$$\begin{aligned} [\Omega_w(E_\bullet)] &= \partial_{k_r} \circ \cdots \circ \partial_{k_1}([\Omega_{w_0}(E_\bullet)]) \\ &= \partial_{k_r} \circ \cdots \circ \partial_{k_1}(\mathfrak{S}_{w_0}(x, y)) \\ &= \mathfrak{S}_w(x, y). \end{aligned}$$

□

From this theorem Fulton deduces a more general statement that generalizes the above in a number of ways:

(1) Instead of applying to a flag of subbundles inside a single bundle, it applies in the setting of a morphism of filtered bundles $A_1 \subset \cdots \subset A_s \xrightarrow{h} B_t \twoheadrightarrow \cdots \twoheadrightarrow B_1$.
(2) Instead of requiring a complete flag, it allows for partial flags.
(3) It permits X to be singular.

The proof (and even the full statement) of the generalization are out of reach; they require additional tools from algebraic geometry and the theory of characteristic classes. (The idea of the proof is to set $E = A_s \oplus B_t$, embed A_s in E as the graph of h, apply the above theorem, and pull back the resulting formula for a cohomology class in $H^*(Fl(E))$ to one in $H^*(X)$.) Let's content ourselves with a weaker statement, still too hard to prove but strong enough to work an interesting example:

Corollary 14.37. *Given a generic morphism of filtered complex vector bundles $A_1 \subset \cdots \subset A_s \xrightarrow{h} B_t \twoheadrightarrow \cdots \twoheadrightarrow B_1$ on a smooth variety X, the class of the degeneracy locus in $A^*(X)$ is given by*

$$[\Omega_w(h)] = \mathfrak{S}_w(x_1, \ldots, x_n . y_1, \ldots, y_n)$$

where $x_i = c_1(\ker(B_i \to B_{i-1}))$ and $y_i = c_1(A_i/A_{i-1})$.

Example 14.38. Let us work out one explicit computation of the class of a degeneracy locus in the Chow ring. Suppose $h : E \to F$ is a generic map of 4-plane bundles on a variety, and $E_1 \subset E_2 \subset \cdots \subset E_4 = E$ and $F = F_4 \twoheadrightarrow F_3 \twoheadrightarrow \cdots \twoheadrightarrow F_1$ are complete flags of subbundles and quotient bundles. Let $w = [2\,4\,3\,1] \in S_4$. The the essential set is $\{(3,1),(2,3)\}$, and corresponding rank matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & \mathbf{1} & 2 \\ \mathbf{0} & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Therefore Ω_w consists of those points of x over which $E_1 \to F_3$ is zero and $E_3 \to F_2$ has rank ≤ 1. Since $w = w_0 \cdot s_2 \cdot s_1$,

$$\begin{aligned} \mathfrak{S}_w(x,y) &= \partial_1 \circ \partial_2((x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)(x_2-y_2)(x_3-y_1)) \\ &= \partial_1((x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)(x_3-y_1)) \\ &= (x_1-y_1)(x_2-y_1)(x_3-y_1)(x_1+x_2-y_2-y_3). \end{aligned}$$

Setting $x_i = c_1(\ker(F_i \twoheadrightarrow F_{i-1}))$ and $y_i = c_1(E_i/E_{i-1})$, our formula yields:

$$[\Omega_w(h)] = (x_1-y_1)(x_2-y_1)(x_3-y_1)(x_1+x_2-y_2-y_3).$$