THE LOCAL LANGLANDS CONJECTURE

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1. Introduction.

Suppose G is a reductive algebraic group defined over a local field F. The local Langlands conjecture as formulated in [8] describes the set $\Pi(G(F))$ of equivalence classes of irreducible admissible representations of $G(F)$ in terms of the (purely arithmetic) Weil-Deligne group W'_F and the (purely algebraic) complex dual group ${}^\vee G$. The conjecture has been proved by Langlands in [24] when $F = \mathbb{R}$ or $F = \mathbb{C}$, and in a few other cases. The conjecture asks for a partition of the set of equivalence classes of irreducible representations into finite sets, called L-packets. Perhaps the main point of the conjecture is to understand the local factors of automorphic representations; Labesse-Langlands showed in [22] that for this it is important to consider simultaneously all the representations in an L-packet. (The simplest example of an L-packet is the set of all discrete series representations of a real semisimple group with a fixed infinitesimal character. If the occurrence of these representations in a space of automorphic forms is investigated by index-theoretic techniques, what emerges most easily is the not the multiplicity of a single discrete series but the sum of all their multiplicities.)

Since Langlands' original work, there has been some progress in refining his predictions for special classes of groups or representations. In this direction there was first of all the work of Knapp-Zuckerman and Shelstad (see [27]) describing the L-packets when $F = \mathbb{R}$. Kazhdan-Lusztig, Beilinson-Bernstein, and Brylinski-Kashiwara (see [3]) calculated the irreducible characters when $F = \mathbb{C}$ or $\mathbb{R}$; this extends Langlands' work in the sense that it describes the irreducible representations in somewhat more detail. Meanwhile Bernstein-Zelevinsky in [7] made a deep study of the reducibility of induced representations of p-adic groups, especially $GL(n)$; this led to Zelevinsky's formulation in [32] of a conjectural description of irreducible characters of $GL(n)$ when F is p-adic. Later Kazhdan-Lusztig classified precisely the representations with a vector fixed by an Iwahori subgroup when F is p-adic (see [17]); their work was motivated in part by a version of Zelevinsky's conjecture (due to Lusztig) that applies more or less to any G, but only to representations with an Iwahori-fixed vector.

At the same time, Arthur was examining the question of how non-tempered representations should fit into the trace formula, and how to generalize the ideas of [22]. What emerged from his work (see [2]) was that to have a chance at nice multiplicity formulas for non-tempered automorphic representations, one needs to consider even larger sets of representations than Langlands' L-packets. (One can get a hint of this in the example of discrete series mentioned in the first paragraph. If the infinitesimal character is close to a wall, some non-tempered automorphic

1991 *Mathematics Subject Classification*. Primary 22E50; Secondary 12B27.
Supported in part by NSF grant DMS-9011483

representations may also contribute to the index formula; what one computes easily is the sum of all the discrete series multiplicities, plus an alternating sum of multiplicities of various non-tempered representations. The set of representations appearing here is larger than the sets Arthur considers, but it has a little of the same flavor.) Since Arthur's sets are not simply unions of L-packets, Langlands' original formulation of his conjecture does not lend itself easily to describing them.

All of this work suggests in one way or another the possibility and usefulness of a slight reformulation of Langlands' conjectures. The purpose of this paper is to outline such a reformulation. In the p-adic case there is little to do; what is presented here is only a mild extrapolation of the conjectures of Zelevinsky and Lusztig, and is undoubtedly familiar to the experts. In the archimedean case the changes are more drastic; they are taken from [1].

MacPherson has suggested that the reason Langlands' approach appears to be so successful in the p-adic case is that the Weil-Deligne group of a p-adic field is exactly the right thing to use. The Weil groups of the real and complex fields, in contrast, are less compellingly natural objects. In fact the version of the Langlands conjecture given here in the archimedean case does not use the Weil group at all (except as motivation), and it is by no means clear how to divide it into an arithmetic and an algebraic aspect. (Nevertheless it may be worthwhile to try to do so.)

Since the whole purpose of this paper is the formulation of some conjectures, we will not try to present them completely in the introduction. In order to offer the reader some hint of what is to come, we will describe only the form of the main conjectures, under some simplifying assumptions. So suppose F is a local field of characteristic zero, $\overline{F}$ is an algebraic closure of F, and

$$\Gamma = \mathrm{Gal}(\overline{F}/F) \tag{1.1)(a}$$

is the Galois group. Let G be a connected reductive algebraic group defined over $\overline{F}$. An *F-rational structure on G* (which we often call simply a *rational form*) may be regarded as a homomorphism

$$\sigma : \Gamma \to \mathrm{Aut}(G(\overline{F})) \tag{1.1)(b}$$

compatible in a natural sense with the action of Γ on $\overline{F}$. (If $F = \mathbb{R}$, the compatibility condition is that complex conjugation must act by an anti-holomorphic automorphism. In general the condition is that if f is a regular function on $G(\overline{F})$ (in the sense of algebraic geometry, so that f takes values in $\overline{F}$) and $\gamma \in \Gamma$, then the function $\gamma \cdot_\sigma f$ on $G(\overline{F})$ defined by

$$(\gamma \cdot_\sigma f)(x) = \gamma \cdot (f(\sigma(\gamma^{-1}) \cdot x)) \tag{1.1)(c}$$

is again a regular function on $G(\overline{F})$.) In this case the group of F-rational points may be identified with the common fixed points of all the automorphisms (1.1)(b):

$$G(F,\sigma) = (G(\overline{F}))^{\sigma(\Gamma)}. \tag{1.1)(d}$$

Notice that $G(\overline{F})$ acts by conjugation on the set of all F-rational forms of G; the orbits, of which there are finitely many, are called *equivalence classes of rational forms*. We have

$$G(F, g\cdot\sigma) = gG(F,\sigma)g^{-1}. \tag{1.1)(e}$$

Suppose now that σ and σ' are rational forms of $G(\overline{F})$. We say that σ *is inner to* σ' if for each $\gamma \in \Gamma$ the automorphism $\sigma(\gamma)\sigma'(\gamma^{-1})$ is inner; that is, it is given by conjugation by an element g_γ of $G(\overline{F})$. This is an equivalence relation, and equivalent rational forms are necessarily inner to each other. The relation of being inner therefore partitions the equivalence classes of rational forms into a finite number of pieces; each piece is called an *inner class* of rational forms. (For examples, we refer to section 2.) We assume from now on that G is endowed with an inner class $\mathcal{C}$ of rational forms.

A *representation of a rational form of* G is a pair (π, σ), with σ a rational form of G (generally required to be in the fixed inner class $\mathcal{C}$) and π an admissible representation of $G(F, \sigma)$. The group $G(\overline{F})$ acts on representations of rational forms by

$$g \cdot (\pi, \sigma) = (\pi \circ \mathrm{Ad}(g^{-1}), g \cdot \sigma); \tag{1.2)(a}$$

this makes sense because of (1.1)(e). To go further without a closer examination of the notion of rational form, we will assume for the rest of the introduction that G *has trivial center.* We write $\Pi(G/F)$ for the set of equivalence classes of irreducible representations of rational forms in the class $\mathcal{C}$. This set may be described more traditionally as follows. Choose a representative σ_i $(i = 1, \ldots, r)$ for each rational form in the inner class $\mathcal{C}$. Then $\Pi(G/F)$ may be identified with the disjoint union of the sets of irreducible admissible representations of each of the rational forms $G(F, \sigma_i)$:

$$\Pi(G/F) \simeq \bigcup_{i=1}^{r} \Pi(G(F, \sigma_i)). \tag{1.2)(b}$$

(The surjectivity of the map from right to left here is a formality, but for the injectivity we use the assumption that G is adjoint. To treat more general groups we will need a more subtle notion of rational form.) The fundamental problem addressed by the local Langlands conjecture is

Problem A. *Parametrize the set* $\Pi(G/F)$*.*

Before turning to Langlands' proposed solution, we formulate several related problems. Recall first of all ([10], Chapters IV and XI) that each irreducible representation π of $G(F, \sigma)$ may be realized as the unique irreducible quotient of a certain *standard representation* $M(\pi)$:

$$M(\pi) \to \pi \to 0. \tag{1.3)(a}$$

(A standard representation is one induced from a tempered representation of a Levi subgroup, with a "strictly positive" parameter on the split part.) Each standard representation M has a finite composition series, so we may write in an appropriate Grothendieck group

$$M = \sum_{\pi \in \Pi(G(F,\sigma))} m(\pi, M)\pi, \tag{1.3)(b}$$

with $m(\pi, M)$ a non-negative integer (the *multiplicity of the irreducible representation* π *in the standard representation* M). The second problem we consider is

Problem B. *Calculate the multiplicities* $m(\pi, M)$.

Because of the bijection (1.3)(a) between irreducible and standard representations, we can regard $m(\pi, M)$ as a square matrix indexed by $\Pi(G/F)$; all the entries are zero except in certain finite blocks near the diagonal. It is not difficult to order $\Pi(G/F)$ so as to make m upper triangular with 1's on the diagonal; it is therefore invertible. The inverse matrix expresses irreducible representations as formal integer combinations of standard ones; this formal expression is an identity on the level of characters, for example.

Although we will have very little to say about it, there is another problem that merits inclusion here because of its fundamental importance. An irreducible admissible representation π of $G(F,\sigma)$ is said to be *unitary* if (in the archimedean case) it is infinitesimally equivalent to a unitary representation, or (in the p-adic case) it is equivalent to the set of smooth vectors of a unitary representation. This abuse of terminology identifies the unitary dual $\Pi_{unit}(G(F,\sigma))$ as a subset of $\Pi(G(F,\sigma))$; we write $\Pi_{unit}(G/F)$ for the set of unitary representations in $\Pi(G/F)$.

Problem C. *Identify the subset* $\Pi_{unit}(G/F) \subset \Pi(G/F)$.

Finally, there is the problem of stable characters and endoscopy. In this context we need to consider virtual representations, so we introduce

(1.4)(a) $K\Pi(G(F,\sigma)) =$ lattice of virtual admissible representations of $G(F,\sigma)$.

This is the Grothendieck group of an appropriate category of finite-length admissible representations of $G(F,\sigma)$; it is a free $\mathbb{Z}$-module with basis $\Pi(G(F,\sigma))$. A virtual representation $\eta \in K\Pi(G(F,\sigma))$ has a well-defined *character* $\Theta(\eta)$, which is a generalized function on $G(F,\sigma)$. The value of this generalized function on the (compactly supported smooth) test density f is

(1.4)(b)
$$\Theta(\eta)(f) = \operatorname{tr}\eta(f).$$

A little more precisely, the formula (1.4)(b) makes sense only if η is actually a representation; in that case $\eta(f)$ is the (trace class) operator

(1.4)(c)
$$\eta(f) = \int_{G(F,\sigma)} f(g)\eta(g)dg;$$

we have written the density f as a smooth function $f(g)$ times a Haar measure dg for the sake of familiarity. Now (1.4)(b) makes sense when η is a representation, and we extend it to virtual representations by linearity. Write $G(F,\sigma)_{SR}$ for the dense open subset of strongly regular elements (those for which the centralizer is a Cartan subgroup). Harish-Chandra's regularity theorem guarantees that $\Theta(\eta)$ is determined by its restriction to $G(F,\sigma)_{SR}$, and that this restriction is a smooth function (invariant under conjugation by $G(F,\sigma)$)). Hence

(1.4)(d)
$$\Theta_{SR} : K\Pi(G(F,\sigma)) \hookrightarrow C^\infty(G(F,\sigma)_{SR}).$$

A little more abstractly, we define

(1.5)(a)
$$K\Pi(G/F) = \text{ lattice with basis } \Pi(G/F),$$

the *lattice of virtual admissible representations of rational forms of* G. This may be identified with the sum of the corresponding lattices over representatives for the equivalence classes of rational forms. In particular, there is for each $\sigma \in \mathcal{C}$ a well-defined restriction homomorphism

$$K\Pi(G/F) \to K\Pi(G(F,\sigma)), \qquad \eta \mapsto \eta(\sigma). \tag{1.5)(b}$$

We would like to define characters for these virtual representations as well. They will be defined on

$$G(F,*)_{SR} = \{\, (g,\sigma) \mid \sigma \in \mathcal{C}, g \in G(F,\sigma)_{SR} \,\}. \tag{1.5)(c}$$

Explicitly, we define

$$\Theta_{SR}(\eta)(g,\sigma) = \Theta_{SR}(\eta(\sigma))(g) \tag{1.5)(d}$$

(notation (1.4)(d), (1.5)(b)). The group G acts by conjugation on $G(F,*)_{SR}$, and $\Theta_{SR}(\eta)$ is constant on orbits.

The virtual representation η is said to be *strongly stable* if the function $\Theta_{SR}(\eta)$ is constant along the fibers of the first projection

$$p_1 : G(F,*)_{SR} \to G_{SR}, \qquad p_1(g,\sigma) = g. \tag{1.6)(a}$$

(The relationship between the notion of strongly stable and Langlands' notion of *stable* is explained in [1], section 18; modulo an important conjecture (proved by Shelstad in the archimedean case) it is very simple.) The lattice of strongly stable virtual representations is written

$$K\Pi(G/F)^{st} \subset K\Pi(G/F). \tag{1.6)(b}$$

Problem D. *Identify the sublattice of strongly stable virtual representations* $K\Pi(G/F)^{st} \subset K\Pi(G/F)$*.*

More generally, the problem of endoscopic lifting asks for a description of all virtual representations of rational forms of G in terms of strongly stable virtual representations of G and certain smaller reductive groups H (the *endoscopic groups* for G — see (9.6) below). We will not formulate it more precisely here.

Because the equation (1.3)(b) lives in the Grothendieck group, two of these four basic problems involve not so much the set $\Pi(G/F)$ as the lattice $K\Pi(G/F)$. This change in emphasis is the main aspect of our reformulation of Langlands' conjectures.

So let ${}^\vee G$ be a complex dual group for G. Because we assumed that G was adjoint, ${}^\vee G$ is the simply connected complex group having as root system the system of coroots for G. The inner class $\mathcal{C}$ of rational forms of G determines an *L-group for* G. This is a pro-algebraic group ${}^\vee G^\Gamma$ endowed with a short exact sequence

$$1 \to {}^\vee G \to {}^\vee G^\Gamma \to \Gamma \to 1 \tag{1.7)(a}$$

and a ${}^\vee G$-conjugacy class $\mathcal{D}$ of distinguished splittings

$${}^\vee\delta : \Gamma \to {}^\vee G^\Gamma \qquad ({}^\vee\delta \in \mathcal{D}). \tag{1.7)(b}$$

If $\mathcal{C}$ includes the split form of G, then ${}^\vee G^\Gamma$ is just the direct product ${}^\vee G \times \Gamma$. We will discuss this definition in a little more detail in section 3; see also [8].

Write W_F' for the Weil-Deligne group of the field F (see [29]). Recall that this locally compact group is equipped with a homomorphism

$$W_F' \to \Gamma \tag{1.8)(a}$$

having dense image. A *Langlands parameter for* G is a continuous homomorphism

$$\phi : W_F' \to {}^\vee G^\Gamma \tag{1.8)(b}$$

respecting the homomorphisms (1.7)(a) and (1.8)(a) to Γ, and respecting the Jordan decompositions in W_F' and ${}^\vee G^\Gamma$ ([8], 8.1). We summarize these hypotheses by saying that ϕ is *admissible*. The set of admissible Langlands parameters is denoted $P(G/F)$. Two such parameters are called *equivalent* if they are conjugate by ${}^\vee G$. The set of equivalence classes of Langlands parameters is denoted $\Phi(G/F)$.

Conjecture 1.9. *(Langlands — see [24], [8]). To each $\phi \in \Phi(G/F)$ there is associated a finite subset (the L-packet of ϕ)*

$$\Pi_\phi \subset \Pi(G/F).$$

These subsets partition $\Pi(G/F)$.

To get a complete solution of Problem A, one must supplement this conjecture with a parametrization of each L-packet Π_ϕ. What is clear from the work of Langlands and Shelstad is that such a parametrization ought to involve the representations of the finite group A_ϕ (of connected components of the centralizer in ${}^\vee G$ of a representative of ϕ). For Problem D, Shelstad's work suggests that $K\Pi(G/F)^{st}$ ought to have a basis parametrized precisely by $\Phi(G/F)$.

To say more, it is convenient first to partition $\Phi(G/F)$ into finite subsets. (The reason is that Problems B and D involve interactions between inequivalent Langlands parameters; we need to isolate those subsets where such interaction is possible.)

Definition 1.10. *Suppose F is p-adic. An infinitesimal character for G (more precisely, for G/F) is a ${}^\vee G$-conjugacy class $\mathcal{O}_F$ of admissible homomorphisms $\lambda : W_F \to {}^\vee G^\Gamma$. (We will explain the reason for the terminology in section 7.) Here W_F is the Weil group of F, a subgroup of W_F'. A Langlands parameter ϕ is said to have infinitesimal character $\mathcal{O}_F$ if the restriction of ϕ to W_F belongs to $\mathcal{O}_F$. Define $X(\mathcal{O}_F) \subset P(G/F)$ to be the set of Langlands parameters (not equivalence classes) of infinitesimal character $\mathcal{O}_F$. We will see (Corollary 4.6) that $X(\mathcal{O}_F)$ is a smooth complex algebraic variety. The (finite) set of equivalence classes — that is, the set of orbits of ${}^\vee G$ on $X(\mathcal{O})$ — is written $\Phi(\mathcal{O}, G/F)$.*

Suppose F is complex. In this case the Weil-Deligne group $W_{\mathbb{C}}'$ is just $\mathbb{C}^\times$. An infinitesimal character for G (more precisely, for $G/\mathbb{C}$) is a pair $\mathcal{O}_{\mathbb{C}} = (\mathcal{O}_1, \mathcal{O}_2)$ of ${}^\vee G$-conjugacy classes of semisimple elements in the Lie algebra ${}^\vee\mathfrak{g}$ of the dual group. (Actually we require a little more of the pair — see (5.6) below.) A Langlands parameter ϕ is said to have infinitesimal character $\mathcal{O}_{\mathbb{C}}$ if the holomorphic part of the differential of ϕ belongs to $\mathcal{O}_1$, and the antiholomorphic part to $\mathcal{O}_2$. (These holomorphic and antiholomorphic parts are the elements λ and μ attached to ϕ by

Lemma 5.5 of [1].) We write $\Phi(\mathcal{O}_{\mathbb{C}}, G/\mathbb{C})$ *for the finite set of equivalence classes of Langlands parameters of infinitesimal character* $\mathcal{O}_{\mathbb{C}}$*. Using a simplified version of the arguments of [1], we will define in section 5 a smooth algebraic variety* $X(\mathcal{O}_{\mathbb{C}})$ *on which* ${}^\vee G$ *acts with finitely many orbits parametrized by* $\Phi(\mathcal{O}, G/\mathbb{C})$.

Suppose finally that F *is real. A classical infinitesimal character for* G *(more precisely, for* $G/\mathbb{R}$*) is a* ${}^\vee G$*-conjugacy class* $\mathcal{O}$ *of semisimple elements in the Lie algebra* ${}^\vee\mathfrak{g}$ *of the dual group. (We will define an infinitesimal character in (6.6) to include more data.) A Langlands parameter* ϕ *is said to have classical infinitesimal character* $\mathcal{O}$ *if the holomorphic part of the differential of* ϕ *belongs to* $\mathcal{O}$*. (This holomorphic part is the element* λ *attached to* ϕ *by Proposition 5.6 of [1].) Again we write* $\Phi(\mathcal{O}, G/\mathbb{R})$ *for the finite set of equivalence classes of Langlands parameters of classical infinitesimal character* $\mathcal{O}$*. Section 6 of [1] shows how to define a smooth algebraic variety* $X(\mathcal{O})$ *on which* ${}^\vee G$ *acts with orbits parametrized by* $\Phi(\mathcal{O}, G/\mathbb{R})$*. (The set* $P(\mathcal{O}, G/\mathbb{R})$ *has these properties, but is geometrically uninteresting: the orbits are all open and closed.) We recall the construction in section 6 below.*

The heart of this version of Langlands' conjectures is the idea that irreducible representations of rational forms of G should correspond to irreducible ${}^\vee G$-equivariant perverse sheaves on the varieties $X(\mathcal{O}_F)$. Any such sheaf has as its support the closure of a single orbit of ${}^\vee G$, and therefore corresponds to a unique Langlands parameter (as required by Conjecture 1.9). What is new is that this orbit closure will in general be singular. The behavior of these perverse sheaves on the smaller orbits in the boundary will (conjecturally) correspond to the interaction of different L-packets relevant to Problems B and D above.

Definition 1.11. *Suppose* $\mathcal{O}_F$ *is an infinitesimal character for* G/F *(Definition 1.10). Consider the category* $\mathcal{P}(\mathcal{O}_F, G/F)$ *of* ${}^\vee G$*-equivariant perverse sheaves on* $X(\mathcal{O}_F)$*, and write* $\Xi(\mathcal{O}_F, G/F)$ *for the (finite) set of irreducible objects in this category. (Actually it will be convenient later to write* Ξ *for a certain more elementary set parametrizing the irreducible perverse sheaves (Definition 8.2); we write* $P(\xi)$ *for the irreducible perverse sheaf corresponding to the parameter* $\xi \in \Xi(\mathcal{O}_F, G/F)$*.) Define* $K\mathcal{P}(\mathcal{O}_F, G/F)$ *to be the Grothendieck group of* $\mathcal{P}(\mathcal{O}_F, G/F)$*; this may be identified as the lattice with basis* $\Xi(\mathcal{O}_F, G/F)$*.*

Now the idea that there should be relationships between categories of representations and categories of perverse sheaves is due essentially to Kazhdan and Lusztig ([16]); such relationships were first established in [3] and [11]. An important aspect of this case, however, is that we seek not an equivalence of categories but a relationship of duality. To be precise, we require that the Grothendieck groups of the two categories should be naturally dual lattices. Here is a more complete statement.

Conjecture 1.12. *Suppose* $\mathcal{O}_F$ *is an infinitesimal character for* G/F *(Definition 1.10). "Define"* $\Pi(\mathcal{O}_F, G/F) \subset \Pi(G/F)$ *(cf. (1.2)) to be the set of irreducible representations of rational forms of* G *having infinitesimal character* $\mathcal{O}_F$*. (The quotation marks mean that the definition rests on a previous conjecture.) That is, we use Conjecture 1.9 to "define"*

$$\Pi(\mathcal{O}_F, G/F) = \bigcup_{\phi \in \Phi(\mathcal{O}_F, G/F)} \Pi_\phi.$$

Finally, in analogy with (1.4) and (1.5), we "define"

$$K\Pi(\mathcal{O}_F, G/F) = \text{ lattice with basis } \Pi(\mathcal{O}_F, G/F),$$

the lattice of virtual representations with infinitesimal character $\mathcal{O}_F$.

Then there is a natural perfect pairing

$$\langle , \rangle : K\Pi(\mathcal{O}_F, G/F) \times K\mathcal{P}(\mathcal{O}_F, G/F) \to \mathbb{Z}$$

(cf. Definition 1.11) identifying each of the lattices as the dual of the other. In this pairing, the bases of irreducible representations and irreducible perverse sheaves are essentially dual to each other. A little more precisely, suppose $(\pi, \sigma) \in \Pi(\mathcal{O}_F, G/F)$, so that π is an irreducible admissible representation of $G(F, \sigma)$ of infinitesimal character $\mathcal{O}_F$. Then there is a unique $\xi \in \Xi(\mathcal{O}_F, G/F)$ with the property that

$$\langle \pi, P(\xi') \rangle = e(\sigma)(-1)^{d(\xi)} \delta_{\xi, \xi'}.$$

Here $e(\sigma) = \pm 1$ is the Kottwitz invariant of the rational form σ (see [20]), $d(\xi)$ is the dimension of the support of the perverse sheaf $P(\xi)$, and $\delta_{\xi,\xi'}$ is a Kronecker delta.

This conjecture means that a (virtual) representation should be regarded as a $\mathbb{Z}$-linear map from the Grothendieck group of equivariant perverse sheaves to $\mathbb{Z}$. Equivalently, it is a map r from perverse sheaves to $\mathbb{Z}$ that is additive for short exact sequences. We will use this conjectural identification rather freely below, saying that a map r from perverse sheaves to $\mathbb{Z}$ "is" a virtual representation.

There are several more or less obvious classes of maps with this additivity property. For one, fix a point $x \in X(\mathcal{O}_F)$. Then we can associate to a perverse sheaf P the alternating sum $\chi_x^{loc}(P)$ of the dimensions of the stalks at x of the cohomology sheaves H^iP. This integer is called the *geometric local Euler characteristic.* (The superscript *loc* stands for "local," and is included to distinguish this map from a microlocal one to be defined in a moment.) Because we are considering equivariant sheaves, χ_x^{loc} depends only on the orbit of ${}^\vee G$ to which x belongs. We may therefore regard χ_x^{loc} as associated to an equivalence class $\phi \in \Phi(\mathcal{O}_F, G/F)$ of Langlands parameters; when we do this, we can call it χ_ϕ^{loc}.

Conjecture 1.13. *The virtual representations χ_ϕ^{loc} (Conjecture 1.12) form a basis for the lattice of strongly stable virtual representations.*

This conjecture would provide a solution to Problem D above, although not a very good one: the virtual representations in question are rather dull, as we will see in Conjecture 1.15 below. For something more interesting, fix again an equivalence class ϕ of Langlands parameters, and let $S_\phi \subset X(\mathcal{O}_F)$ be the corresponding orbit of ${}^\vee G$. To any ${}^\vee G$-equivariant perverse sheaf P on $X(\mathcal{O}_F)$ one can associate a *characteristic cycle* $\mathrm{Ch}(P)$; this is a formal sum, with non-negative integer coefficients, of conormal bundles of orbits of ${}^\vee G$ on $X(\mathcal{O}_F)$. Define

$$\chi_\phi^{mic}(P) = \text{coefficient of } T^*_{S_\phi}(X(\mathcal{O}_F)) \text{ in } \mathrm{Ch}(P); \tag{1.14}$$

this map is additive for short exact sequences, and so passes to the Grothendieck group. Kashiwara's index theorem [15] for holonomic systems says that the sets of maps $\{\chi_\phi^{mic} | \phi \in \Phi(G/F)\}$ and $\{\chi_\phi^{loc} | \phi \in \Phi(G/F)\}$ have the same $\mathbb{Z}$-span. (For a more complete discussion see [1].) It follows that Conjecture 1.13 is equivalent to

Conjecture 1.13′. *The virtual representations χ_ϕ^{mic} (cf. (1.14) and Conjecture 1.12) form a basis for the lattice of strongly stable virtual representations.*

In the case of $PGL(n)$, the virtual representation χ_ϕ^{mic} should contain at most one irreducible representation of each rational form. For other groups χ_ϕ^{mic} should be almost as near to irreducibility as problems of L-indistinguishability allow. It is in this sense that χ_ϕ^{mic} is more interesting than χ_ϕ^{loc}. There are also applications to Problem C. The virtual representation χ_ϕ^{loc} can be unitary only if it is very close to being tempered. For groups with Kazhdan's property T, for example, the trivial representation can never appear in a unitary χ_ϕ^{loc}. On the other hand, Arthur's conjectures suggest that χ_ϕ^{mic} can be unitary in a wide variety of non-tempered cases. We refer to [1], especially section 27, for more details.

The map χ_x^{loc} can be further decomposed. To do that, consider the action of the isotropy group ${}^\vee G_x$ on the cohomology stalks, and look at the isotypic subspaces for an irreducible representation τ of ${}^\vee G_x/({}^\vee G_x)_0$. The alternating sum of their dimensions is called $\chi_{x,\tau}^{loc}(P)$; this number is an *equivariant local Euler characteristic.*

Conjecture 1.15. *The virtual representation $\chi_{x,\tau}^{loc}$ (Conjecture 1.12) is equal to a single standard representation of a single rational form σ of G, multiplied by the Kottwitz sign $e(\sigma)$.*

Conjecture 1.15 provides a (conjectural) solution to Problem B. Suppose $\xi \in \Xi(\mathcal{O}_F, G/F)$ (Definition 1.11), $P(\xi)$ is the corresponding irreducible perverse sheaf, and $\pi(\xi)$ the (conjecturally) corresponding irreducible representation (Conjecture 1.12). Then the multiplicity of $\pi(\xi)$ in the standard representation corresponding to (x, τ) should be equal (up to sign) to $\chi_{x,\tau}^{loc}(P(\xi))$. Calculating multiplicities therefore amounts to calculating equivariant local Euler characteristics for intersection cohomology. For more details we refer to section 8.

In case F is archimedean, Conjecture 1.9 was established by Langlands in [24]. Conjectures 1.12 and 1.15 were proved in that case in [1]. Conjecture 1.13 is a consequence of Shelstad's work in [26].

2. Rational forms.

In this section we consider the problem of extending the definitions in (1.1) and (1.2) to groups with non-trivial center. As in the introduction, we begin with a local field F of characteristic zero, an algebraic closure $\overline{F}$ of F, and the pro-finite group

$$\Gamma = \mathrm{Gal}(\overline{F}/F) \tag{2.1}$$

Let G be a connected reductive algebraic group defined over $\overline{F}$. What we seek is a definition of $\Pi(G/F)$ that makes (1.2)(b) true. To see that it is not true with the definition of the introduction, take $F = \mathbb{R}$, $G = SL(2)$, and σ the standard real form (in which complex conjugation acts by complex conjugation of matrices). Let π^+ and π^- be two inequivalent discrete series representations of $SL(2)(\mathbb{R}, \sigma)$ of the same infinitesimal character. Set

$$g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.2)(a}$$

Then $\mathrm{Ad}(g)$ changes the signs of the off-diagonal entries of a matrix $x \in SL(2)$. This operation commutes with complex conjugation, so

$$g \cdot \sigma = \sigma. \tag{2.2)(b}$$

On the other hand, $\mathrm{Ad}(g)(x) = x^{-1}$ for $x \in SO(2)$. It follows that the weights of $SO(2)$ in a representation π are the negatives of the weights of $SO(2)$ in $\pi \circ \mathrm{Ad}(g^{-1})$. Consequently

$$\pi^+ \circ \mathrm{Ad}(g^{-1}) \simeq \pi^-. \tag{2.2)(c}$$

According to (1.2)(a), we therefore have

$$g \cdot (\pi^+, \sigma) \simeq (\pi^-, \sigma), \tag{2.2)(d}$$

precluding (1.2)(b).

The difficulty here is that g preserves the rational form σ even though it does not belong to the group of rational points. (The image of g in the adjoint group $PGL(2)$ *does* belong to $PGL(2, \mathbb{R})$.) We need to include with the rational form some additional structure that will be preserved only by the group of rational points. In order to describe it, we need a preliminary definition.

Definition 2.3. *([1], Definition 2.13). Suppose G is a connected reductive algebraic group over $\overline{F}$. A weak extended group for G is a group G^Γ endowed with a short exact sequence*

$$1 \to G(\overline{F}) \to G^\Gamma \to \Gamma \to 1, \tag{2.3)(a}$$

subject to the following conditions.

(1) For every element $\gamma \in \Gamma$, and every pre-image g_γ of of γ in G^Γ, the conjugation action of g_γ on $G(\overline{F})$ is compatible with the action of γ on $\overline{F}$. (Recall from (1.1) that this compatibility means that the automorphism of $\overline{F}$-valued functions on $G(\overline{F})$ defined by

$$(g_\gamma \cdot f)(x) = \gamma \cdot (f(g_\gamma^{-1} x g_\gamma)) \tag{2.3)(b}$$

preserves the regular functions.)

(2) Fix a Borel subgroup B of G, a maximal torus $T \subset B$, and basis vectors X_α for the simple root spaces of T in the Lie algebra $\mathfrak{b}$. Then there is an open subgroup Γ_1 of Γ and a homomorphism

$$\delta_1 : \Gamma_1 \to G^\Gamma \tag{2.3)(c}$$

so that conjugation by elements of $\delta_1(\Gamma_1)$ preserves $(B, T, \{X_\alpha\})$, and the diagram

$$\begin{tikzcd} \Gamma_1 \arrow[rr, "\delta_1"] \arrow[dr] & & {}^\vee G^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd} \tag{2.3)(d}$$

commutes. (Here the first downward arrow is the inclusion, and the second comes from (2.3)(a).)

The second condition does not depend on the choices of $(B, T, \{X_\alpha\})$, since any two choices are conjugate by some element $g \in G(\overline{F})$. In fact we do not even need to conjugate δ_1; we can simply replace it by its restriction to appropriate open subgroups. A little more precisely, (2) is equivalent to

(2′) *There is an open subgroup Γ_0 of Γ, and a homomorphism*

$$\delta_0 : \Gamma_0 \to G^\Gamma \tag{2.3)(e}$$

with the following two properties:

(a) *The diagram*

$$\begin{tikzcd} \Gamma_0 \arrow[rr, "\delta_0"] \arrow[dr] & & {}^\vee G^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd} \tag{2.3)(f}$$

commutes. (Here the first downward arrow is the inclusion, and the second comes from (2.3)(a).)

(b) *Fix a Borel subgroup B of G, a maximal torus $T \subset B$, and basis vectors X_α for the simple root spaces of T in the Lie algebra $\mathfrak{b}$. Then there is an open subgroup Γ_1 of Γ_0 so that conjugation by elements of $\delta_0(\Gamma_1)$ preserves $(B, T, \{X_\alpha\})$.*

To construct a weak extended group, one can start with any rational form σ_0 of G, and form the semidirect product

$$G^\Gamma = G(\overline{F}) \rtimes \Gamma \tag{2.4)(a}$$

using the action of Γ on $G(\overline{F})$ given by the rational form. In this setting we have a natural section

$$\delta_0 : \Gamma \to G^\Gamma, \qquad \delta_0(\gamma) = 1 \cdot \gamma. \tag{2.4)(b}$$

This construction does not produce all possible weak extended groups (cf. [1], Corollary 2.16), but we will not need more.

The study of rational forms is intimately connected with Galois cohomology. In that context one generally considers modules for the Galois group to be equipped with the discrete topology; then cohomology may be computed using continuous cochains. Here we are mostly interested in $G(\overline{F})$ as a module for Γ, so we put the discrete topology on $G(\overline{F})$. (This is unrelated to the locally compact topology one considers on the various subgroups of rational points.) It is often convenient to topologize the whole weak extended group. In the setting (2.4), it is natural to use the product of the discrete topology on $G(\overline{F})$ with the usual (compact) topology on Γ. We can achieve the same effect for any weak extended group: there is a unique topological group structure on G^Γ with the property that

(2.5) the homomorphism δ_0 of (2.3)(e) is a homeomorphism with open image.

Since Γ_0 is compact, the image of δ_0 will then be open and closed.

Definition 2.6. *([1], Definition 2.13). Suppose G^Γ is a weak extended group, topologized as in (2.5). A rigid rational form of G is a continuous map (not necessarily a homomorphism)*

$$\delta : \Gamma \to G^\Gamma \tag{2.6)(a}$$

subject to the following two conditions.

(1) The diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\delta} & G^\Gamma \\ & \searrow \quad \swarrow & \\ & \Gamma & \end{array} \tag{2.6)(b}$$

commutes. Here the first downward arrow is the identity, and the second comes from (2.3)(a).

(2) The composition of δ with the quotient map to $G^\Gamma/Z(G)$ is a homomorphism. More precisely, for every pair γ, γ' of elements of Γ there is an element $z(\gamma,\gamma') \in Z(G)$ of finite order so that

$$z(\gamma,\gamma') = \delta(\gamma\gamma')^{-1}\delta(\gamma)\delta(\gamma'). \tag{2.6)(c}$$

(Of course it would be more precise to call δ a rigid rational form of G^Γ rather than just of G.) We say that δ is a pure rational form if in addition δ is a group homomorphism; that is, if the function z of (2) is trivial.

The rational form attached to δ is the homomorphism

$$\sigma(\delta) : \Gamma \to Aut(G(\overline{F})), \qquad \sigma(\delta)(\gamma) = \text{conjugation by } \delta(\gamma). \tag{2.6)(d}$$

We will sometimes write simply $G(F,\delta)$ instead of $G(F,\sigma(\delta))$.

Two rigid rational forms δ and δ' are said to be equivalent if they are conjugate by an element $g \in G(\overline{F})$. In this case the attached rational forms $\sigma(\delta)$ and $\sigma(\delta')$ are also equivalent (cf. (1.1)). The converse is not true, however: it may happen that $\sigma(\delta)$ and $\sigma(\delta')$ are equivalent (or even equal), but that δ and δ' are inequivalent.

Definition 2.6 must be taken as provisional: we will see that it does not allow a complete extension of the conjectures of the introduction to groups with non-trivial center. What is required is a notion of "strong rational form" intermediate between pure and rigid. We will discuss the requirements on such a definition in section 9; unfortunately I do not know how to satisfy them in the p-adic case. At any rate Definition 2.6 will suffice for the formulation of many further definitions, results, and conjectures.

Here are some elementary facts about rigid rational forms.

Proposition 2.7. *Suppose G^Γ is a weak extended group (Definition 2.3).*

a) The set of rational forms associated to rigid rational forms of G constitutes an inner class (cf. (1.1)).

b) In the action of $G(\overline{F})$ on rigid rational forms, the stabilizer of δ is the group of rational points $G(F,\sigma(\delta))$.

c) Suppose G^Γ is constructed as in (2.4) from a rational form of G. Then the set of equivalence classes of pure rational forms of G may be identified with *the non-commutative cohomology set* $H^1(\Gamma, G(\overline{F}))$. *In this identification, the homomorphism* δ_0 *of (2.4)(b) corresponds to the base point of the cohomology.*

Part (a) is proved in the real case in [1], Proposition 2.14. The general case is equally easy. Part (b) is immediate from the definitions. For (c), a discussion of non-commutative cohomology may be found in [25]. There is actually an identification of pure rational forms δ with non-commutative 1-cocycles a, given by

$$\delta(\gamma) = a(\gamma)\delta_0(\gamma).$$

The mapping

$$z : \Gamma \times \Gamma \to Z(G) \tag{2.8)(a}$$

attached to a rigid rational form δ by (2.6)(c) depends only on the equivalence class of δ. For formal reasons it is necessarily a two-cocycle, and therefore defines a class

$$\zeta(\delta) \in H^2(\Gamma, Z(G)). \tag{2.8)(b}$$

In fact it is easy to see that this class depends only on the rational form $\sigma(\delta)$. (If $\sigma(\delta) = \sigma(\delta)'$, then δ differs from δ' by a map j from Γ into $Z(G)$. A calculation shows that the corresponding maps z and z' differ by the coboundary of j.) We may therefore write

$$\zeta(\sigma, G^\Gamma) \in H^2(\Gamma, Z(G)) \tag{2.8)(c}$$

for any rational form σ in the inner class defined by G^Γ (cf. Proposition 2.7(a)).

Here is a more traditional way to think of $\zeta(\sigma, G^\Gamma)$. If $Z(G)$ is trivial, the notions of pure rational form, rigid rational form, and rational form all coincide; this is what allowed us to make the simple formulations in the introduction. In particular, Proposition 2.7(c) reduces in that case to the well-known fact that there is a bijection

(2.9)(a)

$$\text{equivalence classes of rational forms of } G \text{ inner to } \sigma_0 \leftrightarrow H^1(\Gamma, G/Z(G)).$$

(Here the cohomology is defined using the action of Γ on G given by a fixed rational form σ_0.) The long exact sequence in (non-commutative) cohomology attached to the short exact sequence

$$1 \to Z(G) \to G \to G/Z(G) \to 1$$

includes a map $H^1(\Gamma, G/Z(G)) \to H^2(\Gamma, Z(G))$. Combining this with (2.9)(a), we find that every rational form σ determines a cohomology class

$$\zeta(\sigma, \sigma_0) \in H^2(\Gamma, Z(G)), \tag{2.9)(b}$$

depending only on the equivalence class of σ (and the fixed rational form σ_0). Examining the definitions, one finds that

$$\zeta(\sigma, \sigma_0) = \zeta(\sigma, G^\Gamma) - \zeta(\sigma_0, G^\Gamma). \tag{2.9)(c}$$

It follows that if σ_1 is a third rational form, we have

$$\zeta(\sigma, \sigma_1) = \zeta(\sigma, \sigma_0) - \zeta(\sigma_1, \sigma_0). \tag{2.9)(d}$$

(Of course this may also be proved directly.)

Here is an easy consequence of this formalism.

Lemma 2.10. *Suppose G^Γ is a weak extended group, and σ is a rational form in the inner class defined by G^Γ. Then σ is associated to a pure rational form of G (Definition 2.6) if and only if the cohomology class $\zeta(\sigma, G^\Gamma) \in H^2(\Gamma, Z(G))$ (cf. (2.8)(c)) is trivial.*

Example 2.11. Suppose $F = \mathbb{R}$ and $G = SL(2, \mathbb{C})$. Make the (non-trivial element of the) Galois group $\Gamma = \{1, \sigma\}$ act on G by complex conjugation of matrices, and define G^Γ to be the semidirect product

$$G^\Gamma = G \rtimes \Gamma.$$

For g a complex two by two matrix, we compute in G^Γ

$$(g\sigma)^2 = g\overline{g}.$$

A pure rational form of G^Γ is a homomorphism δ of Γ into G^Γ, satisfying

$$\delta(\sigma) = g_\delta \sigma.$$

Clearly δ is determined by g_δ, which may in turn be any element of G satisfying

$$g_\delta \overline{g}_\delta = 1.$$

Conjugating δ by $x \in G$ replaces g_δ by

$$g'_\delta = x g_\delta \overline{x}^{-1}.$$

The semidirect product structure provides a distinguished pure rational form δ_0, with $g_{\delta_0} = 1$. Its equivalence class corresponds to all the elements g_δ of the form

$$g_\delta = x\overline{x}^{-1} \qquad (x \in G).$$

The rational form associated to δ_0 is of course the standard one, with $G(\mathbb{R}, \sigma(\delta_0)) = SL(2, \mathbb{R})$. Notice that exactly the same rational form is associated to the pure rational form δ'_0 defined by

$$g_{\delta'_0} = -I.$$

These two pure rational forms are conjugate, for example by the element $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. It is easy to see that δ_0 and δ'_0 are the only two pure rational forms to which the standard split rational form is associated. It follows that all the pure rational forms to which a split rational form is associated must be equivalent.

A rigid rational form δ has

$$\delta(1) = z_\delta \in Z(G), \qquad \delta(\sigma) = g_\delta \sigma,$$

with g_δ any element of G satisfying

$$g_\delta \overline{g}_\delta \in Z(G).$$

An example is

$$\delta_1(1) = 1, \qquad \delta_1(\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The associated rational form is the compact group $SU(2)$. The corresponding 2-cocycle (cf. (2.6)(c)) is

$$z(1,1) = z(1,\sigma) = z(\sigma,1) = 1, \quad z(\sigma,\sigma) = -I.$$

It is easy to check that this represents the unique non-trivial class in $H^2(\Gamma, Z(G))$. By Lemma 2.10, we deduce that $SU(2)$ is not associated to any pure rational form of G. It is evidently associated to exactly four rigid rational forms, however: if ϵ_1 and ϵ_σ are each ± 1, then we can set

$$\delta_\epsilon(1) = \epsilon_1 \qquad \delta_\epsilon(\sigma) = \epsilon_\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It turns out that these are all inequivalent.

Example 2.12. Suppose F is a p-adic field, G is any simply connected semisimple group over $\overline{F}$, and G^Γ is a weak extended group. Fix a rigid rational form δ_0 of G; this provides (through the associated rational form σ_0) an action of Γ on G. By a theorem of Kneser ([19], Satz 2), the map

$$\zeta(*, \sigma_0) : \text{equivalence classes of rational forms of } G^\Gamma \to H^2(\Gamma, Z(G)),$$

is bijective. Because of (2.9)(c), it follows that $\zeta(*, G^\Gamma)$ is bijective. In particular there is exactly one equivalence class of rational forms that is represented by pure rational forms.

Consider for example the case $G = SL(n)$, with the inner class of rational forms including $SL(n,F)$. The center of G is naturally isomorphic to the group μ_n of nth roots of 1 in $\overline{F}$, and the action of Γ on $Z(G)$ is just the restriction of the usual action on $\overline{F}$. The cohomology group $H^2(\Gamma, Z(G))$ may be computed (for example) using the exact sequence

$$1 \to \mu_n \to \overline{F}^\times \to \overline{F}^\times \to 1,$$

the third map being the nth power map. The conclusion is that

$$H^2(\Gamma, Z(G)) \simeq \mathbb{Z}/n\mathbb{Z},$$

the isomorphism being natural. So we want to describe rational forms of G parametrized by $\mathbb{Z}/n\mathbb{Z}$. To do that, fix an integer r modulo n, and write $m = (n,r)$ for their greatest common divisor (a well-defined positive divisor of n). Write $n = md$. Then r/m is an integer modulo d, and is relatively prime to d. As such it determines a division algebra $D_{d,r/m}$ with center F and dimension d^2 over F ([31], Proposition I.4.5 and the following remark). Define

$$SL(m, D_{d,r/m})$$

to be the set of elements of norm one in the (n^2-dimensional) central simple F-algebra of m by m matrices over $D_{d,r/m}$. This is the desired F-rational form of

$SL(n)$. We omit a more detailed description of a corresponding rigid rational form; necessary auxiliary information for making such calculations may be found in [31]. We make only the following observation. The Galois group Γ has a natural quotient

$$1 \to \Gamma_0 \to \Gamma \to \mu_n \to 1$$

corresponding to the unramified extension of F of degree n. Suppose that $G^\Gamma \simeq SL(n, \overline{F}) \rtimes \Gamma$ as in (2.4). Then every rational form in the split inner class is represented by a rigid rational form δ such that $\delta|_{\Gamma_0}$ is a homomorphism.

Example 2.13. Suppose T is a torus defined over F; form the semidirect product

$$T^\Gamma = T \rtimes \Gamma$$

as in (2.4). Proposition 2.7(c) says that the set of equivalence classes of pure rational forms of T may be identified with the cohomology group $H^1(\Gamma, T)$; we want to compute this group. Hilbert's Theorem 90 says that $H^1(\Gamma, \overline{F}^\times) = 0$, from which it follows that $H^1(\Gamma, T) = 0$ whenever T is split. Therefore a split torus has only one pure rational form up to equivalence. In general, write $X_*(T)$ for the lattice of one-parameter subgroups of T. This is a module for Γ, and $T \simeq X_*(T) \otimes \overline{F}^\times$ as a module for Γ. By the fundamental theorem of local class field theory ([12] page 139, Thm VI.2.1) and a theorem in [23] (page 130, Thm 13), we have

$$H^1(\Gamma, X_*(T) \otimes \overline{F}^\times) \simeq \hat{H}^{-1}(\Gamma, X_*(T)). \tag{2.13)(a}$$

By the definition of the Tate cohomology group $\hat{H}^{-1}$, the right side is a certain finite subquotient of the lattice $X_*(T)$. More explicitly, write $\mathfrak{t}_{\mathbb{Q}}$ for the rational vector space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The action of Γ on $X_*(T)$ extends to a representation on $\mathfrak{t}_{\mathbb{Q}}$. This representation factors through some finite quotient of Γ (corresponding to an extension field over which T splits) and is therefore completely reducible. In particular, the subspace $\mathfrak{t}_{\mathbb{Q}}^\Gamma$ of Γ-invariant vectors has a unique Γ-invariant complement V. There are two natural lattices in V: $X^1 = X_*(T) \cap V$, and the lattice X^0 generated by elements of the form $x - \gamma \cdot x$, with $x \in X_*(T)$ and $\gamma \in \Gamma$. These two lattices correspond to two natural descriptions of V: as the kernel of a norm map to $\mathfrak{t}_{\mathbb{Q}}^\Gamma$, and as the span of elements $x - \gamma \cdot x$. Evidently $X^0 \subset X^1$, and by definition

$$\hat{H}^{-1}(\Gamma, X_*(T)) \simeq X^1/X^0, \tag{2.13)(b}$$

a finite group. We have shown that this group parametrizes the pure rational forms of T. It has another description in terms of the dual torus, to which we will return in the course of our discussion of L-groups (Example 4.13).

Definition 2.14. *Suppose G^Γ is a weak extended group (Definition 2.3). A representation of a rigid rational form of G is a pair (π, δ), with δ a rigid rational form of G (Definition 2.6), and π an admissible representation of $G(F, \sigma(\delta))$ (Definition 2.6(d)). (If F is archimedean, this means that π is a representation on a nice complete topological vector space having finite K-multiplicities, with K any maximal compact subgroup of $G(F, \sigma(\delta))$. If F is p-adic, it means that every vector in π is fixed by an open compact subgroup, and that the subspace of π fixed by any open compact subgroup is finite-dimensional.) Similarly we define a representation of a* *pure rational form, an irreducible representation of a rigid rational form, etc. Just as in (1.2)(a), the group* $G = G(\overline{F})$ *acts on representations of rigid rational forms. We say that* (π, δ) *is equivalent to* (π', δ') *if there is an element* $g \in G$ *so that*

(2.14)(a) $$g \cdot \delta = \delta'; \textit{ and}$$

(2.14)(b) $$\pi' \textit{ is equivalent to } \pi \circ Ad(g^{-1}).$$

Here equivalence of admissible representations has the obvious meaning (existence of an invertible intertwining operator) if F *is p-adic, and means infinitesimal equivalence if* F *is archimedean. Define* $\Pi(G^\Gamma)$ *to be the set of equivalence classes of irreducible representations of rigid rational forms of* G*. Similarly we write* $\Pi_{pure}(G^\Gamma)$ *for the set of equivalence classes of irreducible representations of pure rational forms of* G*.*

One technical problem remains to be addressed. The version of Conjecture 1.12 we are aiming for says that the irreducible representations of pure rational forms within a single L-packet are parametrized by irreducible equivariant local systems on a certain homogeneous space. The latter set has a natural distinguished element (the trivial local system), but the former does not appear to in general. Langlands' program suggests that one should use as base points representations of quasisplit rational forms admitting Whittaker models. This appears to work perfectly for adjoint groups, but in general there may be several possible notions of Whittaker model, leading to different base points in L-packets. We therefore include in our definition of extended group a preferred Whittaker model. This does not affect the definition of $\Pi(G^\Gamma)$, but it will affect the bijection to be established between $\Pi(G^\Gamma)$ and L-group data.

Definition 2.15. *([1], Definition 1.12). Suppose* G^Γ *is a weak extended group (Definition 2.3). A set of Whittaker data for* G^Γ *is a triple* (δ, N, χ)*, so that*

(1) δ *is a rigid rational form of* G^Γ *(Definition 2.6);*
(2) N *is a maximal unipotent subgroup of* G*, and the rational form* $\sigma(\delta)$ *preserves* N*; and*
(3) χ *is a non-degenerate one-dimensional unitary character of* $N(F, \delta)$*.*

(The condition of non-degeneracy in (3) means that the restriction of χ *to each simple restricted root subgroup is non-trivial.) This set is called pure if* δ *is a pure rational form. Notice that (2) forces* $\sigma(\delta)$ *to be quasisplit.*

An extended group for G *is a pair* $(G^\Gamma, \mathcal{W})$*, with* G^Γ *a weak extended group, and* $\mathcal{W}$ *a* G*-conjugacy class of sets of Whittaker data for* G^Γ*. It is called pure if the elements of* $\mathcal{W}$ *are.*

Every weak extended group admits an extended group structure. This structure may be chosen to be pure if and only if G^Γ is constructed as in (2.4) from a quasisplit rational form. It is not particularly difficult to work out the complete classification of extended groups. This is done over $\mathbb{R}$ in section 3 of [1]. The result is analogous to the classification of E-groups in Proposition 3.28 below. We will use only the following partial result.

Proposition 2.16. *(cf. [1], Proposition 3.6). Suppose F is a local field of characteristic zero as in (2.1), and G is a connected reductive algebraic group over $\overline{F}$. Assume that G is endowed with an inner class of F-rational forms. Then there is a pure extended group $(G^\Gamma, \mathcal{W})$ for G (Definition 2.15). If $((G^\Gamma)', \mathcal{W}')$ is another pure extended group for G, then the identity map from G to G extends to an isomorphism from G^Γ to $(G^\Gamma)'$ carrying $\mathcal{W}$ to $\mathcal{W}'$. This extension is unique up to an inner automorphism of G^Γ coming from $Z(G)$.*

The proposition provides a canonical bijection from $\Pi(G^\Gamma)$ onto $\Pi((G^\Gamma)')$. We may therefore define

$$\Pi(G/F) = \Pi(G^\Gamma) \tag{2.17}$$

with $(G^\Gamma, \mathcal{W})$ any pure extended group for G. Similarly we define $\Pi_{pure}(G/F)$. Notice that the quasisplit rational form is represented by a pure rational form of a pure extended group.

Example 2.18. We continue with Example 2.11. Recall the pure rational form δ_0. We can construct a set of pure Whittaker data (δ_0, N, χ) by defining N to be the group of strictly upper triangular matrices, and

$$\chi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(it).$$

The conjugacy class $\mathcal{W}$ of (δ_0, N, χ) is a pure extended group structure on G^Γ. A second pure extended group structure $\mathcal{W}'$ may be constructed in the same way, replacing the character χ by its complex conjugate χ'. We want to compute the isomorphism $(G^\Gamma, \mathcal{W}) \to (G^\Gamma, \mathcal{W}')$ whose existence is guaranteed by Proposition 2.17. To do this, first conjugate (δ_0, N, χ') by the element x of Example 2.11, obtaining $(\delta_0', N, \chi) \in \mathcal{W}'$. The required map from G^Γ to itself is the identity on G, and sends $g\sigma$ to $(-g)\sigma$.

Suppose $(G^\Gamma, \mathcal{W})$ is a pure extended group for G. According to Proposition 2.7, there is a natural one-to-one correspondence

$$\text{equivalence classes of pure rational forms of } G \leftrightarrow H^1(\Gamma, G); \tag{2.19}$$

the (quasisplit) pure rational forms in $\mathcal{W}$ correspond to the base point of the cohomology set. We therefore conclude this section with a calculation (due to Kneser and Kottwitz) of $H^1(\Gamma, G)$ for p-adic groups.

Proposition 2.20. *([19] and [21], Proposition 6.4.) Suppose F is a p-adic field, and σ is an F-rational form of the connected reductive group G. Let $X^*(G)^\Gamma$ be the lattice of F-rational one-dimensional characters of G. Let T be any maximal torus of G defined over F. Define*

$$X_*(T) = \textit{lattice of one-parameter subgroups of } T.$$

In $X_*(T)$ we define four sublattices:

$$R_*(G, T) = \textit{coroot lattice of } T \textit{ in } G;$$

$$X_*^0(T) = \textit{lattice generated by elements } \mu - \gamma\cdot\mu \ (\mu \in X_*(T), \gamma \in \Gamma);$$

$$X_*^1(T) = \textit{lattice of one-parameter subgroups annihilated by } X^*(T)^\Gamma; \textit{and}$$

$$X_*^1(G,T) = \textit{lattice of one-parameter subgroups annihilated by } X^*(G)^\Gamma.$$

a) The quotient $X_*^1(G,T)/(R_*(G,T)+X_*^0(T))$ is a finite abelian group. It is independent of the F-rational torus T (up to canonical isomorphism).

b) The Galois cohomology of T is

$$H^1(\Gamma,T) \simeq X_*^1(T)/X_*^0(T).$$

c) Suppose that T is maximally anisotropic in G; that is, that the rank of $X^*(T)^\Gamma$ is as small as possible. Then

i) $X^*(T)^\Gamma = X^*(G)^\Gamma$, so that $X_*^1(G,T) = X_*^1(T)$.

ii) The natural map $H^1(\Gamma,T) \to H^1(\Gamma,G)$ is surjective.

iii) The fibers of the map in (ii) are exactly the cosets of the subgroup

$$R_*(G,T)/(R_*(G,T)\cap X_*^0(T),$$

so there is an isomorphism

$$H^1(\Gamma,G) \simeq X_*^1(G,T)/(R_*(G,T)+X_*^0(T)).$$

Sketch of proof. For (a), the finiteness of the quotient is elementary (compare Example 2.13). If T' is another F-rational torus, then there is an element of G conjugating T to T'. The resulting isomorphism of $X_*(T)$ with $X_*(T')$ is uniquely determined up to the action of the Weyl group. It carries $X_*^1(G,T)$ onto $X_*^1(G,T')$, and $R_*(G,T)$ onto $R_*(G,T')$. Of course it need not respect the action of Γ; but the action $a(T,\gamma)$ of γ on $X_*(T)$ will be sent to $w(\gamma)\cdot a(T',\gamma)$, with $w(\gamma)$ in the Weyl group. Now the action of W on $X_*(T)/R_*(G,T)$ is trivial; and it follows at once that our isomorphism must carry $(R_*(G,T)+X_*^0(T))$ onto $(R_*(G,T')+X_*^0(T'))$. Now (a) follows.

Part (b) is the p-adic case of Example 2.13. In (c), part (i) is essentially the fact that a semisimple p-adic group always admits an anisotropic torus. Part (ii) (which is really the main point) is [19], Hilfsatz 6.2. Because the assertion in (iii) is to some extent functorial in G, it may be reduced to the cases of tori (where it is trivial) and semisimple groups. (Compare the proof of Hilfsatz 6.2 in [19].) For semisimple groups, (iii) is a reformulation of Satz 2 of [19]. Q.E.D.

Just as in Example 2.13, the set of pure rational forms has been interpreted as a quotient of two lattices. Here again we will be able to give Kottwitz's interpretation of the answer in terms of L-groups (Proposition 4.16).

3. L-groups.

In this section we recall from [8] the construction of dual groups and L-groups. Suppose at first that $\overline{F}$ is any algebraically closed field, and that B is a connected solvable algebraic group over $\overline{F}$. Write N for the unipotent radical of B, and $\overline{T} = B/N$ for the quotient group. This is an algebraic torus, and therefore isomorphic to a product of l copies of the multiplicative group of $\overline{F}$; the number l is called the *rank of B*. If $T \subset B$ is any maximal torus, then the quotient map from B to B/N

restricts to an isomorphism of T onto $\overline{T}$. The *lattice of rational characters of* B is by definition the group

$$X^*(B) = \mathrm{Hom}_{alg}(B, \overline{F}^\times) \tag{3.1)(a}$$

of algebraic group homomorphisms from B to $\overline{F}^\times$. Such a homomorphism is automatically trivial on N, so there are natural identifications

$$X^*(B) \simeq X^*(\overline{T}) \simeq X^*(T). \tag{3.1)(b}$$

It follows that $X^*(B)$ is a lattice of rank l.

Dually, a *rational one-parameter subgroup of* B is an algebraic group homomorphism

$$\mu : \overline{F}^\times \to B. \tag{3.2)(a}$$

Two such homomorphisms are called *equivalent* if they are conjugate by an element of B. The set of equivalence classes of one-parameter subgroups of B is denoted $X_*(B)$. The inclusion of T in B and the quotient map from B to B/N induce bijections on X_*, so we get

$$X_*(B) \simeq X_*(\overline{T}) \simeq X_*(T). \tag{3.2)(b}$$

These isomorphisms allow us to define a group structure on $X_*(B)$; it is a lattice of rank l. (The addition may be described a little more directly. Given one parameter subgroups μ_1 and μ_2 in $X_*(B)$, we may find an element $b \in B$ so that the images of μ_1 and $\mathrm{Ad}(b) \circ \mu_2$ commute. The sum of the equivalence classes of μ_1 and μ_2 is the class of the product homomorphism $(\mu_1)(\mathrm{Ad}(b) \circ \mu_2)$.)

If $\lambda \in X^*(B)$ and $\mu \in X_*(B)$, then the composition $\lambda \circ \mu$ is an algebraic homomorphism from $\overline{F}^\times$ to itself. Such a map is necessarily of the form $z \mapsto z^n$ for a unique $n \in \mathbb{Z}$, and we define

$$\langle \lambda, \mu \rangle = n. \tag{3.3)(a}$$

This defines a bilinear pairing on $X^*(B) \times X_*(B)$ with values in $\mathbb{Z}$. The pairing is perfect; that is, it identifies each lattice as the dual of the other. Explicitly,

$$X^*(B) \simeq \mathrm{Hom}(X_*(B), \mathbb{Z}), \qquad X_*(B) \simeq \mathrm{Hom}(X^*(B), \mathbb{Z}). \tag{3.3)(b}$$

We can recover $\overline{T}$ from either of these two lattices using natural isomorphisms

$$\overline{T} \simeq X_*(B) \otimes_{\mathbb{Z}} \overline{F}^\times, \qquad \overline{T} \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(B), \overline{F}^\times). \tag{3.3)(c}$$

The second isomorphism, for example, sends the class $tN \in B/N$ to the map $\lambda \mapsto \lambda(t)$ from $X^*(B)$ to $\overline{F}^\times$. Of course we cannot recover B from these lattices in general; if B is unipotent, the lattices are zero.

Example 3.4 Suppose B_0 is the group of n by n upper triangular matrices. A rational character of B_0 is parametrized by a sequence $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$:

$$\lambda(b) = b_{11}^{\lambda_1} b_{22}^{\lambda_2} \dots b_{nn}^{\lambda_n}. \tag{3.4)(a}$$

Here we write b_{ij} for the (i,j) entry of the (upper triangular) matrix b. Consequently

$$X^*(B_0) \simeq \mathbb{Z}^n. \tag{3.4)(b}$$

(This is a canonical isomorphism.) The rational one-parameter subgroups of B_0 are more complicated; if $n = 2$, for example, the most general one is of the form

$$\mu(z) = \begin{pmatrix} z^{\mu_1} & (z^{\mu_1} - z^{\mu_2})x \\ 0 & z^{\mu_2} \end{pmatrix}, \tag{3.4)(c}$$

with μ_1 and μ_2 integers and $x \in \overline{F}$. In general the diagonal entries of $\mu(z)$ must be of the form

$$\mu(z)_{ii} = z^{\mu_i}, \tag{3.4)(d}$$

with μ_i an integer. The n-tuple of integers (μ_i) determines the equivalence class of μ, so

$$X_*(B_0) \simeq \mathbb{Z}^n. \tag{3.4)(e}$$

The composition $\lambda \circ \mu$ is

$$\begin{aligned} \lambda(\mu(z)) &= (\mu(z)_{11})^{\lambda_1}(\mu(z)_{22})^{\lambda_2} \dots (\mu(z)_{nn})^{\lambda_n} \\ &= (z^{\mu_1})^{\lambda_1}(z^{\mu_2})^{\lambda_2} \dots (z^{\mu_n})^{\lambda_n} \\ &= z^{(\mu_1\lambda_1 + \dots + \mu_n\lambda_n)}. \end{aligned} \tag{3.4)(f}$$

It follows that the pairing between $X^*(B_0)$ and $X_*(B_0)$ is just the usual one between $\mathbb{Z}^n$ and $\mathbb{Z}^n$,

$$\langle \lambda, \mu \rangle = \sum_{i=i}^{n} \lambda_i \mu_i. \tag{3.4)(g}$$

We want to extend these constructions to non-solvable groups. So suppose now that G is any connected algebraic group over $\overline{F}$. A *based rational character for G* is a pair (B, λ) with B a Borel subgroup of G and λ a rational character of B. (We may drop the B from the notation sometimes.) Two such characters are said to be *equivalent* if they are conjugate by an element of G. We write $X^*_b(G)$ for the collection of based rational characters for G. Now any two Borel subgroups of G are conjugate, and the normalizer in G of a Borel subgroup B_0 is B_0. It follows at once that the inclusion of B_0 in G defines a bijection

$$X^*(B_0) \simeq X^*_b(G). \tag{3.5}$$

This bijection provides a lattice structure on $X^*_b(G)$, which is independent of the choice of B_0. Here is a more direct definition. Suppose (λ, B) and (λ', B') are based rational characters for G. Choose an element g of G conjugating B' to B, so that $(\lambda' \circ \mathrm{Ad}(g), B)$ is a based rational character for G equivalent to λ'. Then the class of $\lambda + \lambda'$ is represented by $(\lambda + (\lambda' \circ \mathrm{Ad}(g)), B)$. (Perhaps the best way to think of $X_b^*(G)$ is as the set of equivalence classes of equivariant line bundles on the complete flag variety of Borel subgroups of G. In this interpretation, the lattice structure is tensor product of line bundles. The lattice $X_b^*(G)$ is therefore just an equivariant Picard group.)

Dually, a *based rational one-parameter subgroup of* G is a pair (B, μ) with B a Borel subgroup of G and

$$\mu : \overline{F}^\times \to B \tag{3.6)(a}$$

an algebraic group homomorphism. Two such pairs are called *equivalent* if they are conjugate by an element of G. The set of equivalence classes of based rational one-parameter subgroups of G is denoted $X_*^b(G)$. If B_0 is any Borel subgroup of G, then the inclusion provides a bijection

$$X_*(B_0) \simeq X_*^b(G). \tag{3.6)(b}$$

It follows that $X_*^b(G)$ is a lattice, and that there is a natural perfect pairing

$$X_b^*(G) \times X_*^b(G) \to \mathbb{Z}. \tag{3.6)(c}$$

The dual lattices $X_b^*(G)$ and $X_*^b(G)$ carry some additional structure in the non-solvable case. Recall that a *minimal parabolic subgroup of* G is a subgroup P properly containing a Borel subgroup B of G, with the property that B is maximal in P. In this case the quotient space P/B is necessarily isomorphic to $\mathbb{P}^1$. In particular, B acts by a rational character $\alpha = \alpha(P/B)$ on $(\mathfrak{p}/\mathfrak{b})^*$. (Here as usual we denote Lie algebras by Gothic letters.) We call the based rational character $(B, \alpha(P/B))$ the *simple root corresponding to* P. The set of equivalence classes of simple roots forms a finite subset

$$\Delta(G) \subset X_b^*(G), \tag{3.7)(a}$$

in one-to-one correspondence with the conjugacy classes of minimal parabolic subgroups. Dually, a *simple coroot corresponding to* P is a based rational one-parameter subgroup $(B, \alpha^\vee)$ subject to two conditions. First,

$$\alpha^\vee(\overline{F}^\times) \subset B \cap [P, P]. \tag{3.7)(b}$$

Second,

$$\alpha(P/B)(\alpha^\vee(z)) = z^2 \qquad (z \in \overline{F}^x). \tag{3.7)(c}$$

These conditions determine $\alpha^\vee$ up to conjugacy by B; we may sometimes overlook this indeterminacy and write $\alpha^\vee(P/B)$. In any case the set of equivalence classes of simple coroots forms a finite subset

$$\Delta^\vee(G) \subset X_*^b(G), \tag{3.7)(d}$$

in one-to-one correspondence with the conjugacy classes of minimal parabolic subgroups (and therefore also with $\Delta(G)$).

Definition 3.8. *(Grothendieck — see [28], 1.9. Suppose G is a connected algebraic group over the algebraically closed field $\overline{F}$. The based root datum of G is the quadruple*

$$\Psi(G) = (X_b^*(G), \Delta(G), X_*^b(G), \Delta^\vee(G))$$

defined in (3.5)–(3.7) above. A little more explicitly, we include in the based root datum the lattice structures and the perfect pairing $\langle , \rangle$ of (3.6)(c), and the bijection $\Delta(G) \leftrightarrow \Delta^\vee(G)$ described after (3.7)(d).

The definition given here of $X_b^*(G)$ seems to be a reasonably good one (particularly in the "Picard group" interpretation). That of $X_*^b(G)$ is much less satisfactory. (The difficulty of defining good Langlands parameters in the archimedean case, already mentioned in the introduction, has a similar flavor.) The reader is therefore encouraged to look for improvements.

Example 3.9 Suppose $G = GL(n)$. We can choose as a Borel subgroup the group B_0 of upper triangular matrices considered in Example 3.4. It follows at once that

$$X_b^*(G) \simeq X^*(B_0) \simeq \mathbb{Z}^n, \qquad X_*^b(G) \simeq X_*(B_0) \simeq \mathbb{Z}^n; \tag{3.9)(a}$$

the isomorphisms are canonical. A minimal parabolic subgroup P containing B_0 is parametrized by a pair $(r, r+1)$, with $1 \leq r \leq n-1$. It consists of matrices (p_{ij}) with $p_{ij} = 0$ unless $i \leq j$ or $(i,j) = (r+1, r)$. The quotient of Lie algebras $\mathfrak{p}/\mathfrak{b}$ is spanned by the image of the matrix unit $e_{r+1,r}$. It follows easily that the corresponding simple root is

$$\alpha = (0, 0, \ldots, 1, -1, 0, \ldots, 0), \tag{3.9)(b}$$

the non-zero entries being in the rth and $(r+1)$st places. The commutator subgroup of P is

$$[P,P] = \{(p_{ij} \in P \mid p_{ii} = 1 \text{ for } (i \neq r, r+1), \text{ and } \det p = 1\}. \tag{3.9)(c}$$

From this one finds easily that a simple coroot corresponding to P must have

$$\mu_{rr}(z) = z, \qquad \mu_{r+1,r+1}(z) = z^{-1}, \qquad \mu_{ii}(z) = 1 \quad (i \neq r, r+1). \tag{3.9)(d}$$

Consequently

$$\begin{aligned} \Delta(G) &= \{e_r - e_{r+1} \mid 1 \leq r \leq n-1\} \subset \mathbb{Z}^n, \\ \Delta^\vee(G) &= \{e_r - e_{r+1} \mid 1 \leq r \leq n-1\} \subset \mathbb{Z}^n. \end{aligned} \tag{3.9)(e}$$

Example 3.10. Suppose $G = SL(n)$. We can choose as a Borel subgroup the intersection B_1 of B_0 with $SL(n)$. Every rational character of B_1 is the restriction of a character of B_0; the characters restricting to the trivial character on B_1 are precisely the powers of the determinant character on B_0. In the parametrization of Example 3.4, these correspond to the multiples of $\delta = (1, 1, \ldots, 1)$. Hence

$$X_b^*(G) \simeq X^*(B_1) \simeq \mathbb{Z}^n/\mathbb{Z}\delta, \tag{3.10)(a}$$

a lattice of rank $n-1$. Dually, the one parameter subgroup μ of B_0 takes values in B_1 if and only if the sum of the integers μ_i (cf. (3.4)(d)) is zero. Write

$$S\mathbb{Z}^n = \{\mu \in \mathbb{Z}^n \mid \sum \mu_i = 0\}, \tag{3.10)(b}$$

a lattice of rank $n-1$. Then it follows that

$$X_*^b(G) \simeq X_*(B_1) \simeq S\mathbb{Z}^n. \tag{3.10)(c}$$

The usual pairing on $\mathbb{Z}^n \times \mathbb{Z}^n$ descends to

$$\langle , \rangle : \mathbb{Z}^n/\mathbb{Z}\delta \times S\mathbb{Z}^n \to \mathbb{Z}; \tag{3.10)(d}$$

this is the pairing of (3.3)(a) and (3.6)(c). The simple roots of $SL(n)$ are the images of those for $GL(n)$ under $\mathbb{Z}^n \to \mathbb{Z}^n/\delta\mathbb{Z}$. The simple coroots of $GL(n)$ actually belong to $S\mathbb{Z}^n \subset \mathbb{Z}^n$, and as such give the simple coroots of $SL(n)$.

This example provides a special case of the functoriality of the based root datum with respect to algebraic group homomorphisms that are isomorphisms modulo solvable radicals. Here is another.

Example 3.11. Suppose $G = PGL(n)$, the quotient of $GL(n)$ by its center. A Borel subgroup of G is the quotient B_2 of B_0 by its center. A character λ of B_0 factors to B_2 if and only if the sum of the λ_i is zero, so

$$X_b^*(G) \simeq X^*(B_2) \simeq S\mathbb{Z}^n \tag{3.11)(a}$$

(notation as in (3.10)(b)). Similarly

$$X_*^b(G) \simeq X_*(B_2) \simeq \mathbb{Z}^n/\delta\mathbb{Z}. \tag{3.11)(b}$$

The simple roots and coroots are the obvious ones.

Example 3.12. Suppose n is a positive integer, and G is a connected classical group of rank n. By this we mean that G is either $GL(n)$ (type A_{n-1}), $SO(2n+1)$ (type B_n), $Sp(2n)$ (type C_n), or $SO(2n)$ (type D_n). (Such closely related groups as $SL(n)$ and $Spin(2n)$ are excluded.) Then the based root datum for G is very close to that for $GL(n)$. More precisely, there are canonical isomorphisms

$$X_b^*(G) \simeq \mathbb{Z}^n, \qquad X_*^b(G) \simeq \mathbb{Z}^n, \tag{3.12)(a}$$

and the pairing is the usual one. The simple roots and coroots are those for $GL(n)$ (Example 3.9) together with at most one additional pair $(\alpha \in \Delta(G), \alpha^\vee \in \Delta^\vee(G))$. Here are the extra pairs. In type B_n:

$$\alpha = e_n, \qquad \alpha^\vee = 2e_n. \tag{3.12)(b}$$

In type C_n:

$$\alpha = 2e_n, \qquad \alpha^\vee = e_n. \tag{3.12)(c}$$

In type D_n $(n \geq 2)$:

$$\alpha = e_{n-1} + e_n, \qquad \alpha^\vee = e_{n-1} + e_n. \tag{3.12)(d}$$

In the remaining cases ($GL(n)$ and $SO(2)$) no pair is added; the based root datum is that of $GL(n)$.

Here is the first main theorem about root data.

Theorem 3.13. *(see [28], Theorem 2.9). Suppose G and G' are connected reductive algebraic groups over an algebraically closed field $\overline{F}$, and suppose $j : \Psi(G) \to \Psi(G')$ is an isomorphism (that is, a collection of four bijective maps preserving the lattice structures, pairings, inclusions $\Delta \hookrightarrow X_b^*$ and $\Delta^\vee \hookrightarrow X_*^b$, and bijections $\Delta \leftrightarrow \Delta^\vee$). Then j is induced by an isomorphism of algebraic groups*

$$J : G \to G'.$$

This isomorphism is unique up to inner automorphisms of G or G'.

The theorem says that the structure of a reductive group is completely encoded by the discrete information in the based root datum. It is therefore natural to ask what the possibilities for that discrete information are. To explain the answer, we need a little notation. Suppose X^* is any lattice (that is, a finitely generated free abelian group), and

(3.14)(a) $$X_* = \mathrm{Hom}_{\mathbb{Z}}(X^*, \mathbb{Z})$$

is the dual lattice. We write $\langle , \rangle : X^* \times X_* \to \mathbb{Z}$ for the natural pairing between them. Suppose that we are given

(3.14)(b) $$(\alpha, \alpha^\vee) \in X^* \times X_*, \qquad \langle \alpha, \alpha^\vee \rangle = 2.$$

Then we can define endomorphisms s_α of X^* and $s_{\alpha^\vee}$ of X_* by the formulas

(3.14)(c) $$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \qquad s_{\alpha^\vee}(\mu) = \mu - \langle \alpha, \mu \rangle \alpha^\vee.$$

They are transposes of each other:

(3.14)(d) $$\langle s_\alpha(\lambda), \mu \rangle = \langle \lambda, s_{\alpha^\vee}(\mu) \rangle.$$

In addition, each has order two.

Suppose now that we are given a subset Δ of X^*, a subset $\Delta^\vee$ of X_*, and a bijection $\alpha \mapsto \alpha^\vee$ from Δ to $\Delta^\vee$. Assume that every pair $(\alpha, \alpha^\vee)$ satisfies (3.14)(b). Write $W = W(\Delta, \Delta^\vee)$ for the group of automorphisms of X^* generated by the various s_α, and $W^\vee = W(\Delta^\vee, \Delta)$ for the group of automorphisms of X_* generated by the $s_{\alpha^\vee}$. Then

(3.15) $$W \simeq W^\vee, \qquad w \mapsto (w^{-1})^t.$$

(Here the superscript t denotes the transpose of an endomorphism of X^*). This isomorphism carries s_α to $s_{\alpha^\vee}$.

Definition 3.16. *(cf. [28], section 1.) An abstract based root datum is a quadruple $\Psi = (X^*, \Delta, X_*, \Delta^\vee)$, together with a bijection $\alpha \mapsto \alpha^\vee$ from Δ to $\Delta^\vee$. Here X^* is a lattice, X_* is the dual lattice, Δ is a subset of X^*, and $\Delta^\vee$ is a subset of X_*. These data are subject to the following conditions:*

(BRD1) Each pair $(\alpha, \alpha^\vee)$ from $\Delta \times \Delta^\vee$ satisfies $\langle \alpha, \alpha^\vee \rangle = 2$.
(BRD2) The group $W(\Delta, \Delta^\vee)$ generated by the s_α (cf. (3.14) and (3.15)) is finite.
(BRD3) If α and β are distinct elements of Δ, then $\langle \alpha, \beta^\vee \rangle \leq 0$.
(BRD4) The sets Δ and $\Delta^\vee$ are linearly independent.

(It follows from the structure theory of algebraic groups that the based root datum of an algebraic group satisfies these conditions.) We call $W = W(\Delta, \Delta^\vee)$ the Weyl group of the based root datum. Define

$$\Phi = W \cdot \Delta \subset X^*, \qquad \Phi^\vee = W^\vee \cdot \Delta^\vee \subset X_*,$$

the root system and coroot system. The quadruple $(X^*, \Phi, X_*, \Phi^\vee)$ is a root datum in the sense considered in [28], and Δ is a set of simple roots.

The axioms given in [28] for a root datum are somewhat simpler than these for a based root datum. We prefer to work with based root data to get the strong uniqueness statement at the end of Theorem 3.13. It would certainly be nice to have cleaner axioms. Of course (BRD1) might almost be absorbed in the preamble, along with the assumption that X^* is a lattice and so on. The main assumption is (BRD2), and (BRD3) is more or less in the nature of fixing an "orientation." These first three conditions almost guarantee (BRD4), but not quite; for example, one can add to a based root datum the pair $(-\beta, -\beta^\vee)$, with β a highest root, to get a system satisfying (BRD1–3) but not (BRD4). A variety of mild assumptions could be imposed in place of (BRD4) to eliminate such possibilities.

Here is the main existence theorem for reductive groups.

Theorem 3.17. *(see [28], Theorem 2.9). Suppose Ψ is an abstract based root datum (Definition 3.16), and $\overline{F}$ is an algebraically closed field. Then there is a connected reductive algebraic group G over $\overline{F}$ such that $\Psi(G) \simeq \Psi$ (Definition 3.8).*

Our main interest in this theorem arises from the symmetry of the axioms for a based root datum.

Definition 3.18. *([8], 2.1) Suppose $\Psi = (X^*, \Delta, X_*, \Delta^\vee)$ is an abstract based root datum. The dual based root datum is the quadruple*

$$\Psi^\vee = (X_*, \Delta^\vee, X^*, \Delta).$$

It is an abstract based root datum (with a canonically isomorphic Weyl group) because of (3.15).

Suppose G is a connected reductive algebraic group over an algebraically closed field $\overline{F}$. A (complex) dual group for G is a complex connected reductive algebraic group ${}^\vee G$, together with an isomorphism of $\Psi(G)^\vee$ with $\Psi({}^\vee G)$:

(3.18)(a)

$$X^*(G) \simeq X_*({}^\vee G), \quad \Delta(G) \simeq \Delta^\vee({}^\vee G), \quad X_*(G) \simeq X^*({}^\vee G), \quad \Delta^\vee(G) \simeq \Delta({}^\vee G)$$

Theorem 3.17 guarantees that a complex dual group always exists. Theorem 3.13 implies that any two are isomorphic, and that the isomorphism is canonical up to inner automorphisms.

Example 3.19. The based root data described in Examples 3.9–3.12 provide examples of dual groups. A dual group for $GL(n, \overline{F})$ is $GL(n, \mathbb{C})$ (since the based root datum is evidently self-dual). A dual group for $SL(n, \overline{F})$ is $PGL(n, \mathbb{C})$ (Examples 3.10 and 3.11), and a dual group for $PGL(n, \overline{F})$ is $SL(n, \mathbb{C})$. A dual group for $SO(2n+1, \overline{F})$ is $Sp(2n, \mathbb{C})$ (cf. (3.12)(b) and (3.12)(c)), and a dual group for $SO(2n, \overline{F})$ is $SO(2n, \mathbb{C})$ (cf. (3.12)(d)).

Langlands' notion of L-group extends this duality to include information about the Galois group. So let F be a field with algebraic closure $\overline{F}$ and Galois group

$$\Gamma = \mathrm{Gal}(\overline{F}/F). \tag{3.20)(a}$$

(The field F may be local of characteristic zero as in (2.1), but we do not really need this at the moment.) Fix a connected reductive algebraic group G over $\overline{F}$. Because of the naturality of the based root datum, every (algebraic) automorphism of G defines an automorphism of $\Psi(G)$. In this way we get a short exact sequence ([8], (1.2))

$$1 \to \mathrm{Int}\, G \to \mathrm{Aut}\, G \to \mathrm{Aut}(\Psi(G)) \to 1. \tag{3.20)(b}$$

(This is an immediate consequence of Theorem 3.13.) Suppose now that G is defined over F, so that the Galois group Γ acts on G. We want to construct a homomorphism

$$\mu(G/F) : \Gamma \to \mathrm{Aut}(\Psi(G)). \tag{3.20)(c}$$

The Galois action is not by algebraic automorphisms, so $\mu(G/F)$ does not arise trivially from (3.20)(b). There is no difficulty in constructing it, however. Suppose for example that $\gamma \in \Gamma$, and (B, λ) is a based rational character of G. Write $a(\gamma)$ for the (non-algebraic) automorphism of $G(\overline{F})$ defined by γ. Then $B' = a(\gamma)(B)$ is a Borel subgroup of G, and $\lambda \circ a(\gamma)^{-1}$ is a group homomorphism from B' to $\overline{F}^{\times}$. It is not a rational character of B', however, because it is not a regular function. This we can cure as in (2.3)(b): the homomorphism λ' defined by

$$\lambda'(b') = \gamma \cdot \lambda(a(\gamma)^{-1} b') \tag{3.20)(d}$$

is a rational character. We have therefore defined an action of Γ on based rational characters, by $\gamma \cdot (\lambda, B) = (\lambda', B')$. This is well defined on equivalence classes and respects the lattice structure, and so defines an action of Γ on $X_b^*(G)$. The rest of the construction of $\mu(G/F)$ is similar.

If G is split over F, then the homomorphism $\mu(G/F)$ is trivial (but not conversely). Because any reductive group splits over a finite extension field, the kernel of $\mu(G/F)$ is necessarily an open subgroup. Because $\mathrm{Aut}(\Psi(G))$ is discrete, this is equivalent to saying that $\mu(G/F)$ is continuous.

Proposition 3.21. *([8], 1.3). In the setting (3.20), the homomorphism $\mu(G/F)$ depends only on the inner class of the F-rational form of G. This establishes a bijection between inner classes of F-rational forms of G and continuous homomorphisms*

$$\mu : \Gamma \to Aut(\Psi(G)).$$

Example 3.22. Suppose $F = \mathbb{R}$ and $G(\mathbb{R})$ is the unitary group $U(n)$. In this case $G = GL(n, \mathbb{C})$, and the non-trivial element σ of Γ acts by

$$a(\sigma)(g) = {}^t\overline{g}^{-1},$$

the inverse of the conjugate transpose of g. Suppose (B_0, λ) is as in Example 3.4. Then $a(\sigma)(B_0) = B'$ is the group of lower triangular matrices, and

$$\lambda \circ a(\sigma)^{-1}(b') = \lambda({}^t\overline{b'}^{-1}) = \overline{b}_{11}^{-\lambda_1} \dots \overline{b'}_{nn}^{-\lambda_n}.$$

The character λ' is obtained by composition with complex conjugation:

$$\lambda'(b') = b'^{-\lambda_1}_{11} \dots b'^{-\lambda_n}_{nn}.$$

To see which element of $X_b^*(G)$ this represents, we must choose an element of G conjugating B' back to B. This can be accomplished by the permutation matrix h reversing the order of $1, \dots, n$; we find

$$\lambda'(h^{-1}bh) = b_{11}^{-\lambda_n} \dots b_{nn}^{-\lambda_1}.$$

So finally we find that σ acts on $X_b^*(G) \simeq \mathbb{Z}^n$ by

$$\mu(\sigma)(\lambda_1, \dots, \lambda_n) = (-\lambda_n, \dots, -\lambda_1).$$

Notice that this action preserves Δ (as it must).

Example 3.23. Suppose F is arbitrary, and $G = GL(1) = \overline{F}^\times$. Since G is abelian, inner classes of rational forms are the same as rational forms. The based root datum of G is $(\mathbb{Z}, \emptyset, \mathbb{Z}, \emptyset)$. This has only two automorphisms, the identity and -1. A non-trivial homomorphism of Γ into $\mathrm{Aut}(\Psi)$ is therefore determined by its kernel, which can be any (closed) subgroup of Γ of index two. This means that non-split F-rational forms of G correspond to separable quadratic extensions of F. If E is such an extension, then the corresponding rational form has $G(F)$ isomorphic to the kernel of the norm map from $E^\times$ to $F^\times$. The Galois action on $G(\overline{F}) = \overline{F}^\times$ is $a(\gamma)z = \gamma \cdot z$ (if $\gamma \in \mathrm{Gal}(\overline{F}/E)$), $a(\gamma) \cdot z = (\gamma \cdot z)^{-1}$ (otherwise).

Definition 3.24. *Suppose F is a field with algebraic closure $\overline{F}$ and Galois group Γ, and G is a connected reductive algebraic group over $\overline{F}$. Fix an inner class of F-rational forms of G, corresponding to a homomorphism $\mu : \Gamma \to Aut(\Psi(G))$. The dual root datum has exactly the same automorphisms; write $\mu^\vee : \Gamma \to Aut(\Psi(G)^\vee)$ accordingly.*

Fix a complex dual group ${}^\vee G$ for G (Definition 3.18). A weak E-group for G (and the specified inner class of F-rational forms) is a complex reductive pro-algebraic group ${}^\vee G$ having the following structure.

1) *There is a short exact sequence*

$$1 \to {}^\vee G \to {}^\vee G^\Gamma \to \Gamma \to 1.$$

2) *If $\gamma \in \Gamma$, then any preimage g_γ of γ in ${}^\vee G^\Gamma$ acts by conjugation on ${}^\vee G$ according to an automorphism defining $\mu^\vee(\gamma) \in Aut(\Psi(G)^\vee)$. (Here we use the mapping (3.20)(b) for the group ${}^\vee G$, and the identification $\Psi({}^\vee G) \simeq \Psi(G)^\vee$ of Definition 3.18.)*
3) *There is an open subgroup Γ_0 of Γ and an isomorphism of ${}^\vee G \times \Gamma_0$ with an open subgroup of ${}^\vee G^\Gamma$, respecting the maps of (1).*

The homomorphism $\mu^\vee : \Gamma \to Aut(\Psi(G)^\vee)$ is called the first invariant of ${}^\vee G^\Gamma$.

Condition (3) of this definition is analogous to condition (2) of Definition 2.3. It is not quite as compelling as the first two conditions; my understanding of the Langlands conjectures is not deep enough to decide whether it should really be imposed. In any case, it clarifies the structure of pro-algebraic group on ${}^\vee G^\Gamma$.

Definition 3.25. *(Langlands; see [8], 2.3). In the setting of Definition 3.24, a distinguished splitting is a quadruple* $({}^\vee\delta, {}^dB, {}^dT, \{X_{\alpha^\vee}\})$. *Here* ${}^\vee\delta : \Gamma \to {}^\vee G^\Gamma$ *is a homomorphism splitting the exact sequence (1) of Definition 3.24;* ${}^dB \supset {}^dT$ *are a Borel subgroup and a maximal torus in* ${}^\vee G$; $\{X_{\alpha^\vee}\}$ *is a collection of basis vectors for the simple root spaces of* ${}^dT$ *in* ${}^d\mathfrak{b}$; *and the automorphisms* ${}^\vee\delta(\gamma)$ *of* ${}^\vee G$ *(defined by conjugation) preserve* $({}^dB, {}^dT, \{X_{\alpha^\vee}\})$.

An L-group for G/F *is a pair* $({}^\vee G^\Gamma, \mathcal{D})$ *consisting of a weak E-group* ${}^\vee G^\Gamma$ *and a* ${}^\vee G$*-conjugacy class* $\mathcal{D}$ *of distinguished splittings.*

Proposition 3.26. *(Langlands; see [8], 2.3) Suppose* G *is a connected reductive algebraic group over* $\overline{F}$, *endowed with an inner class of* F*-rational forms. Then there is an L-group* $({}^\vee G^\Gamma, \mathcal{D})$ *for* G/F. *If* $(({}^\vee G^\Gamma)', \mathcal{D}')$ *is another, then there is an isomorphism from* ${}^\vee G^\Gamma$ *to* $({}^\vee G^\Gamma)'$ *preserving the weak E-group structure and carrying* $\mathcal{D}$ *to* $\mathcal{D}'$. *This isomorphism is unique up to an inner automorphism from* ${}^\vee G$.

Not every weak E-group admits an L-group structure, and a discussion of endoscopy requires considering some that do not. With those applications in mind, we make one more definition.

Definition 3.27. *In the setting of Definition 3.24, a distinguished splitting modulo center is a quadruple* $({}^\vee\delta, {}^dB, {}^dT, \{X_{\alpha^\vee}\})$. *Here* ${}^\vee\delta : \Gamma \to {}^\vee G^\Gamma$ *is a continuous map (not necessarily a homomorphism) splitting the exact sequence (1) of Definition 3.24;* ${}^dB \supset {}^dT$ *are a Borel subgroup and a maximal torus in* ${}^\vee G$; $\{X_{\alpha^\vee}\}$ *is a collection of basis vectors for the simple root spaces of* ${}^dT$ *in* ${}^d\mathfrak{b}$; *and the automorphisms* ${}^\vee\delta(\gamma)$ *of* ${}^\vee G$ *(defined by conjugation) preserve* $({}^dB, {}^dT, \{X_{\alpha^\vee}\})$. *It follows from these assumptions that the composition of* ${}^\vee\delta$ *with the quotient map into* ${}^\vee G^\Gamma/Z({}^\vee G)$ *is a homomorphism; so for every pair* γ, γ' *of elements of* Γ *there is an element* ${}^\vee z(\gamma, \gamma') \in Z({}^\vee G)$ *so that*

$$ {}^\vee z(\gamma, \gamma') = {}^\vee\delta(\gamma\gamma')^{-1}\,{}^\vee\delta(\gamma)\,{}^\vee\delta(\gamma'). $$

This mapping is automatically a 2-cocycle of Γ *with values in* $Z({}^\vee G)$.

An E-group for G/F *is a pair* $({}^\vee G^\Gamma, \mathcal{D})$ *consisting of a weak E-group* ${}^\vee G^\Gamma$ *and a* ${}^\vee G$*-conjugacy class* $\mathcal{D}$ *of distinguished splittings modulo center. The 2-cocycle* ${}^\vee z$ *is the same for all the elements of* $\mathcal{D}$, *and is therefore an invariant of the E-group structure. It is called the second invariant of the E-group.*

Proposition 3.28. *Suppose* G *is a connected reductive algebraic group over* $\overline{F}$, *endowed with an inner class of* F*-rational forms.*

a) Every weak E-group ${}^\vee G^\Gamma$ *for* G/F *admits a distinguished splitting modulo center (Definition 3.27), and therefore an E-group structure* $({}^\vee G^\Gamma, \mathcal{D})$.

b) Suppose $({}^\vee G^\Gamma, \mathcal{D})$ *is as in (a), and* ${}^\vee z$ *is the corresponding 2-cocycle (Definition 3.27). Then the cohomology class* ${}^\vee\zeta \in H^2(\Gamma, Z({}^\vee G))$ *of* ${}^\vee z$ *depends only on the weak E-group, and not on the choice of splitting.*

c) Suppose ${}^\vee\zeta \in H^2(\Gamma, Z({}^\vee G))$, *and* ${}^\vee z$ *is a representative 2-cocycle. Then there is an E-group* $({}^\vee G^\Gamma, \mathcal{D})$ *for* G/F *having* ${}^\vee z$ *as second invariant.*

d) Suppose $({}^\vee G^\Gamma, \mathcal{D})$ *and* $(({}^\vee G^\Gamma)', \mathcal{D}')$ *are E-groups for* G/F *having the same second invariant. Then there is an isomorphism from* ${}^\vee G^\Gamma$ *to* $({}^\vee G^\Gamma)'$ *preserving the weak E-group structure and carrying* $\mathcal{D}$ *to* $\mathcal{D}'$. *This isomorphism is unique up to an inner automorphism from* ${}^\vee G$.

This is a straightforward exercise in the relationship between group cohomology and the classification of extensions; it is essentially proved for $F = \mathbb{R}$ in section 4 of [1]. Two remarks may be helpful, however. In part (d), the reader may wonder what became of the first invariant $\mu^\vee$ introduced in Definition 3.24. The point is that the phrase "E-group for G/F" means "E-group with first invariant $\mu^\vee$." The assumption that these invariants coincide for ${}^\vee G^\Gamma$ and $({}^\vee G^\Gamma)'$ is therefore implicit in the formulation of (d). A second subtlety is the meaning of the assumption that the 2-cocycles ${}^\vee z$ and $({}^\vee z)'$ are "the same," when one takes values in $Z({}^\vee G)$ and the other in $Z({}^\vee G')$. The point is that ${}^\vee G$ and ${}^\vee G'$ are isomorphic, and the isomorphism is canonical up to inner automorphism. The centers $Z({}^\vee G)$ and $Z({}^\vee G')$ are therefore canonically isomorphic.

4. Langlands parameters: p-adic case.

Suppose throughout this section that F is a non-archimedean local field of characteristic zero. As usual we write Γ for the Galois group of F. We recall from [29] the construction of the Weil group. Write k_F for the residue field of of F; this is a finite field, say with q_F elements. Our choice of algebraic closure $\overline{F}$ provides an algebraic closure $\overline{k}_F$, and therefore a (surjective) homomorphism from Γ onto $\mathrm{Gal}(\overline{k}_F/k_F)$. The kernel of this map is the *inertia group* $I_F \subset \Gamma$. The Galois group of a finite field is canonically isomorphic to the inverse limit $\widehat{\mathbb{Z}}$ of the finite quotients of $\mathbb{Z}$, so we have a short exact sequence

$$1 \to I_F \to \Gamma \to \widehat{\mathbb{Z}} \to 1. \tag{4.1)(a}$$

The compact group $\widehat{\mathbb{Z}}$ contains $\mathbb{Z}$ as a dense subgroup; here $1 \in \mathbb{Z}$ corresponds to the automorphism $x \mapsto x^{q_F}$ of $\overline{k}_F/k_F$. The preimage of $\mathbb{Z}$ in Γ is therefore a dense subgroup W_F:

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1. \tag{4.1)(b}$$

We topologize W_F by making the compact group I_F (a closed subgroup of the pro-finite group Γ) an open subgroup of W_F. The *norm homomorphism* from W_F to $\mathbb{R}^\times$ is

$$\|w\| = q_F^n \tag{4.1)(c}$$

whenever w maps to n in $\mathbb{Z}$. (As an element of Γ, this means that w acts on $\overline{k}_F$ by raising elements to the power $\|w\|$, or by extracting the corresponding root.) Finally, we define the *Weil-Deligne group* W_F' *of* F as the semidirect product

$$W_F' = \mathbb{C} \rtimes W_F, \tag{4.1)(d}$$

in which $w \in W_F$ acts on $\mathbb{C}$ by multiplication by $\|w\|$. This means that W_F' is generated by a copy of $\mathbb{C}$ and a copy of W_F, subject to the relation

$$wxw^{-1} = \|w\|x \qquad (x \in \mathbb{C}, w \in W_F). \tag{4.1)(e}$$

The product topology makes W_F' a locally compact group, and the inclusion of W_F in Γ induces a natural homomorphism

$$W_F' \to \Gamma \tag{4.1)(f}$$

(trivial on $\mathbb{C}$) with dense image.

Before we can define Langlands parameters, we need a notion of semisimple elements in an E-group ${}^\vee G^\Gamma$ (Definition 3.24). Choose a subgroup Γ_0 of Γ as in Definition 3.24(3), and a positive integer N so that every element of the finite group Γ/Γ_0 has order dividing N. If $x \in {}^\vee G^\Gamma$, then it follows that $x^N = (x_g(N), x_\gamma(N)) \in {}^\vee G \times \Gamma_0$. We say that x is *semisimple* if $x_g(N)$ is a semisimple element of ${}^\vee G$. Because an element of a complex algebraic group is semsimple if and only if its powers are, this definition is independent of choices.

Definition 4.2. *(cf. [8], 8.1). Suppose ${}^\vee G^\Gamma$ is a weak E-group (Definition 3.24). A Langlands parameter is a continuous homomorphism $\phi : W_F' \to {}^\vee G^\Gamma$ with the following additional properties:*

1) ϕ respects the homomorphisms (3.24)(1) and (4.1)(f) to Γ;
2) the restriction of ϕ to $\mathbb{C} \subset W_F'$ is algebraic (and therefore defines a unipotent one-parameter subgroup of ${}^\vee G$); and
3) $\phi(W_F)$ consists of semisimple elements of ${}^\vee G^\Gamma$.

Here in (3) semisimplicity is as defined after (4.1) above. The collection of all Langlands parameters is written $P({}^\vee G^\Gamma)$. The group ${}^\vee G$ acts on Langlands parameters by conjugation on the range; parameters in the same orbit are called equivalent, and the set of equivalence classes is denoted $\Phi({}^\vee G^\Gamma)$. If ${}^\vee G^\Gamma$ is an L-group for G/F (Definition 3.25), then we write instead $P(G/F)$ and $\Phi(G/F)$; the omission of the L-group from the notation is justified by Proposition 3.26.

Here is the basic Langlands conjecture.

Conjecture 4.3. *(Langlands — see [24] and [8], Chapter III.) Suppose F is a p-adic field, and G is a connected reductive algebraic group over $\overline{F}$ endowed with an inner class of F-rational forms. Fix a pure extended group $(G^\Gamma, \mathcal{W})$, and an L-group $({}^\vee G^\Gamma, \mathcal{D})$ for G/F (Definitions 2.15 and 3.25). Then to each equivalence class $\phi \in \Phi(G/F)$ of Langlands parameters is associated a set of representations $\Pi_\phi \subset \Pi(G/F)$ (cf. (2.17)), called the L-packet of ϕ. This correspondence should have the following properties.*

1) The sets Π_ϕ partition $\Pi(G/F)$.
2) If δ is a rigid rational form of G^Γ, then the set

$$\Pi_\phi(\delta) = \{\pi \in \Pi(G(F,\delta) \mid (\pi,\delta) \in \Pi_\phi\}$$

is finite. If δ is quasisplit, it is non-empty.

3) The following three conditions on ϕ are equivalent:
   a) some representation in Π_ϕ is square-integrable modulo center;
   b) all representations in Π_ϕ are square-integrable modulo center;
   c) the image of ϕ is not contained in any proper Levi subgroup of ${}^\vee G^\Gamma$ ([8], 3.4).

One can find in [8] many additional requirements on the sets Π_ϕ.

We want to say a little more about the structure of the set $P({}^\vee G^\Gamma)$. The most interesting and difficult part of this problem is understanding the restriction of Langlands parameters to W_F. This is an arithmetic problem, and we will ignore it entirely. In the setting of Definition 4.2, we therefore fix a continuous homomorphism

(4.4)(a) $$\lambda : W_F \to {}^\vee G^\Gamma$$

satisfying conditions (1) and (3) of Definition 4.2. (We may regard λ as a Langlands parameter trivial on the "unipotent radical" $\mathbb{C}$ of W_F'.) Define

$$P(\lambda, {}^\vee G^\Gamma) = \{\phi \in P({}^\vee G^\Gamma) \mid \phi|_{W_F} = \lambda\}. \tag{4.4)(b}$$

The image $\lambda(I_F)$ is a compact subgroup of ${}^\vee G^\Gamma$, so the centralizer in ${}^\vee G$ is a (possibly disconnected) reductive algebraic subgroup:

$${}^\vee G^{\lambda(I_F)} = \{g \in {}^\vee G \mid g\lambda(x) = \lambda(x)g, \text{all } x \in I_F\}. \tag{4.4)(c}$$

Fix a Frobenius element Fr of Γ (that is, one acting in the residue field $\overline{k}_F$ by raising elements to the power q_F.) This means that Fr belongs to W_F, and generates W_F/I_F. Because I_F is a normal subgroup of W_F, $\lambda(I_F)$ is normalized by $\lambda(\mathrm{Fr})$. It follows that $\lambda(\mathrm{Fr})$ normalizes ${}^\vee G^{\lambda(I_F)}$. By hypothesis (4.2)(3), $\lambda(\mathrm{Fr})$ is a semisimple element of ${}^\vee G^\Gamma$, so in particular

$$\mathrm{Ad}(\lambda(\mathrm{Fr}))$$

is a semisimple automorphism of the reductive Lie algebra

$${}^\vee\mathfrak{g}^{\lambda(I_F)}. \tag{4.4)(d}$$

The Lie algebra is therefore a direct sum of eigenspaces of $\mathrm{Ad}(\lambda(\mathrm{Fr}))$, and we may define

$${}^\vee\mathfrak{g}^{\lambda(I_F)}_{q_F} = q_F\text{-eigenspace of } \mathrm{Ad}(\lambda(\mathrm{Fr})). \tag{4.4)(e}$$

Because eigenvalues of an automorphism multiply under Lie bracket, it is immediate that ${}^\vee\mathfrak{g}^{\lambda(I_F)}_{q_F}$ consists of ad-nilpotent elements of ${}^\vee\mathfrak{g}$. Finally, define

$${}^\vee G^{\lambda} = \{g \in {}^\vee G \mid g\lambda(x) = \lambda(x)g, \text{all } x \in W_F\}. \tag{4.4)(f}$$

This is the group of fixed points of the semisimple automorphism $\mathrm{Ad}(\lambda(\mathrm{Fr}))$ of the reductive algebraic group ${}^\vee G^{\lambda(I_F)}$, and is therefore itself a reductive algebraic group. It acts by the adjoint representation on the vector space ${}^\vee\mathfrak{g}^{\lambda(I_F)}_{q_F}$.

Proposition 4.5. *Suppose we are in the setting (4.4)(a); use the notation of (4.4).*

a) *An element of $P(\lambda, {}^\vee G^\Gamma)$ is completely determined by its restriction to $\mathbb{C} \subset W_F'$. This restriction is a one-parameter unipotent subgroup*

$$n_\phi : \mathbb{C} \to {}^\vee G.$$

b) *The homomorphism n_ϕ is determined by a single nilpotent element $N_\phi \in {}^\vee\mathfrak{g}^{\lambda(I_F)}_{q_F}$, by the formula*

$$n_\phi(z) = \exp(zN_\phi).$$

c) *The correspondences in (a) and (b) establish a bijection between $P(\lambda, {}^\vee G^\Gamma)$ and the vector space ${}^\vee\mathfrak{g}^{\lambda(I_F)}_{q_F}$. This correspondence respects the natural actions of ${}^\vee G^\lambda$ on the two sets.*

d) *The vector space ${}^\vee\mathfrak{g}^{\lambda(I_F)}_{q_F}$ is prehomogeneous for the group ${}^\vee G^\lambda$. That is, there are only finitely many orbits.*

Corollary 4.6. *In the setting of (4.4)(a), write $\mathcal{O}_F$ for the collection of all conjugates of λ under ${}^\vee G$. Define*

$$P(\mathcal{O}_F, {}^\vee G^\Gamma) = \{\phi \in P({}^\vee G^\Gamma) \mid \phi|_{W_F} \in \mathcal{O}_F\}.$$

Then there is a ${}^\vee G$-equivariant bijection between $P(\mathcal{O}_F, {}^\vee G^\Gamma)$ and the (closed points of the) complex affine algebraic variety

$$ {}^\vee G \times_{{}^\vee G^\lambda} \left({}^\vee \mathfrak{g}_{q_F}^{\lambda(I_F)} \right).$$

In particular, the orbits of ${}^\vee G$ on $P(\mathcal{O}_F, {}^\vee G^\Gamma)$ are in one-to-one correspondence with the orbits of ${}^\vee G^\lambda$ on ${}^\vee \mathfrak{g}_{q_F}^{\lambda(I_F)}$. They are finite in number

In the terminology of Definition 1.10, $\mathcal{O}_F$ is an infinitesimal character for G/F. The corollary is therefore describing the set of Langlands parameters of a fixed infinitesimal character. We write

$$X(\mathcal{O}_F, {}^\vee G^\Gamma) = {}^\vee G \times_{{}^\vee G^\lambda} \left({}^\vee \mathfrak{g}_{q_F}^{\lambda(I_F)} \right) \tag{4.7}$$

for this affine algebraic variety. Of course it would be harmless to abuse notation and call it simply $P(\mathcal{O}_F, {}^\vee G^\Gamma)$. The reason we make a distinction is that in the archimedean case the analogous construction of an algebraic variety from Langlands parameters (Definition 5.12) will change even the underlying point set.

Proof of Proposition 4.5. Part (a) is clear from (4.1) and Definition 4.2. For (b), the homomorphism n_ϕ is evidently determined by a single nilpotent element $N_\phi \in {}^\vee\mathfrak{g}$, which (in light of (4.1)(e)) may be anything satisfying the requirement

$$\mathrm{Ad}(\lambda(w))(N_\phi) = \|w\| N_\phi.$$

Taking $w \in I_F$, we see first of all that this means that N_ϕ must belong to ${}^\vee\mathfrak{g}^{\lambda(I_F)}$. Taking $w = \mathrm{Fr}$, we find that N_ϕ must belong to the q_F-eigenspace of $\mathrm{Ad}(\lambda(\mathrm{Fr}))$, as required.

For (c), the argument for (b) may easily be reversed to show that any $N \in {}^\vee\mathfrak{g}_{q_F}^{\lambda(I_F)}$ gives rise to a continuous homomorphism from W_F' to ${}^\vee G^\Gamma$. To see that ϕ is a Langlands parameter, we only need to check that N is nilpotent. By the remark before (4.4)(f), N is certainly ad-nilpotent. It therefore remains only to show that ${}^\vee\mathfrak{g}_{q_F}^{\lambda(I_F)}$ does not meet the center ${}^\vee\mathfrak{z}$ of the Lie algebra ${}^\vee\mathfrak{g}$. For this, notice that the adjoint action of $\lambda(W_F)$ on ${}^\vee\mathfrak{z}$ is the restriction of the Galois group action $\mu^\vee$ appearing in the definition of an E-group. This action factors to some finite quotient of Γ, so the eigenvalues of $\mathrm{Ad}(\lambda(\mathrm{Fr}))$ on ${}^\vee\mathfrak{z}$ are all roots of unity. In particular, q_F does not occur.

Part (d) is a special case of the following well-known fact (applied to the reductive group ${}^\vee G^{\lambda(I_F)}$ and its semisimple automorphism $\mathrm{Ad}(\lambda(\mathrm{Fr}))$.)

Lemma 4.8. *Suppose G is a complex semisimple algebraic group, and s is a semisimple automorphism of G. Let H be the group of fixed points of s (a reductive subgroup of G). For each complex number t that is not a root of unity, define*

$$\mathfrak{g}_t = \{\, X \in \mathfrak{g} \mid s(X) = tX \,\}.$$

Then H acts on the vector space $\mathfrak{g}_t$ with finitely many orbits.

We omit the proof. This completes the proof of Proposition 4.5. Corollary 4.6 is a formal consequence. (That the fiber product is an affine variety follows from the fact that the isotropy subgroup ${}^\vee G^\lambda$ is reductive.) Q.E.D.

Example 4.9. Unramified representations. In the setting of Conjecture 4.3, suppose for simplicity that our fixed inner class of F-rational forms of G includes the split form. This means that the action μ of Γ on the based root datum (cf. (3.20)) is trivial; so $\mu^\vee$ is also trivial, and the L-group is

$$ {}^\vee G^\Gamma = {}^\vee G \times \Gamma. \tag{4.9)(a} $$

A Langlands parameter ϕ may therefore be identified with a continuous homomorphism

$$ \phi_0 : W_F' \to {}^\vee G \tag{4.9)(b} $$

carrying W_F to semisimple elements and $\mathbb{C}$ algebraically to unipotent elements. The parameter is called *unramified* if ϕ_0 is trivial on I_F. An unramified Langlands parameter may be identified with a pair (y, N). Here $y = \phi_0(\mathrm{Fr})$ is a semisimple element of ${}^\vee G$; N is a nilpotent element of ${}^\vee \mathfrak{g}$; and

$$ \mathrm{Ad}(y)(N) = q_F N. \tag{4.9)(c} $$

The conjugacy class of y corresponds to an unramified principal series representation, as follows. Fix a split rigid rational form δ for G^Γ, and $B \supset T$ a Borel subgroup and maximal torus defined over F. Principal series representations correspond to continuous complex characters of the group $B(F, \delta)$. Because T is split, the characters of $B(F, \delta)$ may be identified as

$$ B(F, \delta)^\wedge \simeq X^*(B) \otimes_{\mathbb{Z}} \mathrm{Hom}(F^\times, \mathbb{C}^\times). \tag{4.10)(a} $$

Here the last Hom is the group of continuous complex characters of the locally compact group $F^\times$. A character of $F^\times$ is called *unramified* if it is trivial on the group U_F of units of the ring of integers of F; these are the elements of $F^\times$ of norm 1. A character of $B(F, \delta)$ is called *unramified* if it is trivial on the image $\mu(U_F)$, with μ any rational one-parameter subgroup of B. These characters may be identified as

$$ B(F, \delta)^\wedge_{unramified} \simeq X^*(B) \otimes_{\mathbb{Z}} \mathrm{Hom}(F^\times / U_F, \mathbb{C}^\times). $$

Because $F^\times / U_F$ is naturally isomorphic to $\mathbb{Z}$, this last Hom may be identified with $\mathbb{C}^\times$:

$$ B(F, \delta)^\wedge_{unramified} \simeq X^*(B) \otimes_{\mathbb{Z}} \mathbb{C}^\times. \tag{4.10)(c} $$

On the other hand, fix ${}^d B \supset {}^d T$ a Borel subgroup and a maximal torus in ${}^\vee G$. Once these choices are made, the definition of the dual group provides a natural identification

$$ X_*({}^d T) \simeq X^*(B). \tag{4.10)(d} $$

Combining this with (4.10)(c), we find

$$B(F,\delta)^{\wedge}_{unramified} \simeq X_*({}^dT) \otimes_{\mathbb{Z}} \mathbb{C}^{\times} \simeq {}^dT. \tag{4.10)(e}$$

That is, given our fixed choices of Borel subgroups and maximal tori, there is a natural bijection

$$\text{unramified principal series representations} \leftrightarrow \text{elements of } {}^dT. \tag{4.10)(f}$$

Now every semisimple conjugacy class in ${}^\vee G$ meets ${}^dT$ in a single Weyl group orbit. The corresponding characters of $B(F,\delta)$ also differ by the action of the Weyl group, so the principal series representations all have the same irreducible composition factors.

Fix now an element $y \in {}^dT$, and write

$$\lambda : W_F \to {}^\vee G^\Gamma \tag{4.11)(a}$$

for the corresponding unramified Langlands parameter. Write η for the character of $B(\delta, F)$ or $T(\delta, F)$ defined by y, and $I(\eta,\delta)$ for the corresponding principal series representation of $G(F,\delta)$. The Langlands conjecture may be made a bit more explicit in this setting. It says first of all that the collection $\Pi(\eta,\delta)$ of irreducible composition factors of $I(\eta,\delta)$ is a union of L-packets:

$$\Pi(\eta,\delta) = \bigcup_{\phi \in P(\lambda, {}^\vee G^\Gamma)} \Pi_\phi(\delta). \tag{4.11)(b}$$

Now the representation $I(\eta,\delta)$ can be reducible in two rather different ways. The more subtle of these involves unitary induction, tempered representations, and L-indistinguishability (that is, sets $\Pi_\phi(\delta)$ having more than one element.) We do not wish to consider this at the moment. The other possibility is that there is a root subgroup M_α of G containing T, and that the corresponding induced representation $I_{M_\alpha}(\eta,\delta)$ contains a one-dimensional composition factor. (By a root subgroup, I mean one generated by T and the one-dimensional unipotent subgroups for some roots $\pm\alpha$. Thus M_α is essentially $SL(2)$.) From this we can get reducibility of $I(\eta,\delta)$ by induction by stages. On the other hand, the condition for $I_{M_\alpha}(\eta,\delta)$ to contain a one-dimensional composition factor may be found by calculation in $SL(2)$. In terms of the rational coroot $\alpha^\vee : F^\times \to T(F,\delta)$, it is that

$$\eta(\alpha^\vee(z)) = \|z\|^{\pm 1}. \tag{4.11)(c}$$

Of course $\alpha^\vee$ may be regarded as a root of ${}^dT$ in the dual group ${}^\vee G$, and (4.11)(c) may be rewritten on the dual group as

$$\alpha^\vee(y) = q_F^{\pm 1}. \tag{4.11)(d}$$

It follows that $I(\eta,\delta)$ exhibits reducibility of the second kind if and only if the q_F-eigenspace of $\mathrm{Ad}(y)$ on ${}^\vee\mathfrak{g}$ is non-zero. Because of Proposition 4.5, this is precisely the condition under which $P(\lambda, {}^\vee G^\Gamma)$ has more than one Langlands parameter in it. This is encouraging for (4.11)(b). We conclude this example with a special case in which the evidence is even stronger.

Suppose in the setting of (4.11)(a) that y satisfies

$$\alpha^\vee(y) = q_F \tag{4.12)(a}$$

for every simple root $\alpha^\vee$ of ${}^dT$ in ${}^\vee G$. Then the centralizer ${}^\vee G^\lambda$ of y in ${}^\vee G$ is precisely ${}^dT$, and the q_F-eigenspace of y is precisely the sum of the simple root spaces:

$$ {}^\vee\mathfrak{g}_{q_F} = \sum_{\alpha^\vee \in \Delta({}^\vee G)} \mathfrak{g}_{\alpha^\vee}. \tag{4.12)(b}$$

Write l for the cardinality of $\Delta({}^\vee G)$. Two elements of this sum are conjugate by ${}^\vee G^\lambda = {}^dT$ if and only if their non-zero components occur at exactly the same simple roots. There are exactly 2^l orbits. By Proposition 4.5, there are exactly 2^l inequivalent Langlands parameters in $P(\lambda, {}^\vee G^\Gamma)$, parametrized by subsets of $\Delta({}^\vee G)$.

On the representation-theoretic side, (4.12)(a) means that $I(\eta, \delta)$ has a one-dimensional irreducible quotient. The composition factors of such an induced representation have been determined by Casselman (see [10], section X.4). There are exactly 2^l of them (each occurring with multiplicity one), parametrized naturally by subsets of $\Delta(G)$. In this case we therefore have a bijection between equivalence classes of representations in $\Pi(\eta, \delta)$ and equivalence classes of Langlands parameters in $P(\lambda, {}^\vee G^\Gamma)$.

Example 4.13. Pure rational forms of tori. We continue here with Example 2.13. Suppose therefore that $(T^\Gamma, \mathcal{W})$ is a pure extended group for the F-rational torus T (Definition 2.15). This means that $\mathcal{W}$ is a T-conjugacy class of pure rational forms of T. (All of the rational forms in the inner class defined by T^Γ are identical; it is only the rigid rational forms that differ.) In Example 2.13, we found that the set of all T-conjugacy classs of pure F-rational forms was in one-to-one correspondence with a certain subquotient X^1/X^0 of the lattice $X_*(T)$. We want to identify that subquotient in terms of the L-group ${}^\vee T^\Gamma$. According to a result of Langlands (see [8], section 9) there is a natural bijection between the group of continuous complex characters of $T(F)$, and the set $\Phi(T/F)$ of equivalence classes of Langlands parameters. So fix a character π of $T(F)$, and a corresponding Langlands parameter $\phi \in P(G/F)$. Because ${}^\vee T$ has no nilpotent elements, ϕ is trivial on $\mathbb{C} \subset W_F'$. Write ${}^\vee T^\phi$ for the centralizer in ${}^\vee T$ of the image of ϕ. Because ${}^\vee T$ is abelian, and ϕ is a section (on the dense subgroup $W_F \subset \Gamma$) of the map (3.24)(1) from ${}^\vee T^\Gamma$ to Γ, we find that

$$ {}^\vee T^\phi \text{ is the set of fixed points of } \Gamma \text{ on } {}^\vee T. \tag{4.13)(a}$$

In particular, it is independent of ϕ.

Now the definition of the dual group identifies $X_*(T)$ with the lattice $X^*({}^\vee T)$ of rational characters of ${}^\vee T$. The action of Γ on this lattice induced by the action of Γ on ${}^\vee T$ is the same as the action used in Example 2.13 to define $X^0 \subset X^1$. The lattice X^0 is the span of the characters of the form $\tau - \gamma \cdot \tau$ (with τ a rational character and γ in Γ). Any such character is obviously trivial on the fixed points of Γ. It is an elementary fact that the converse is also true:

$$X^0 = \text{lattice of rational characters of } {}^\vee T \text{ trivial on } {}^\vee T^\phi. \tag{4.13)(b}$$

(Here one should remember that the action of Γ on ${}^\vee T$ factors through a finite quotient of Γ.) Similarly, one checks that X^1 consists of the characters whose differentials are trivial on ${}^\vee\mathfrak{t}^\phi$. Consequently

$$X^1 = \text{lattice of rational characters of } {}^\vee T \text{ trivial on } {}^\vee T_0^\phi. \tag{4.13)(c}$$

Combining these two observations, we find a natural identification

$$X^1/X^0 = \text{group of characters of } {}^\vee T^\phi/{}^\vee T_0^\phi. \tag{4.13)(d}$$

That is, the set of T-conjugacy classes of pure rational forms of T^Γ is in a natural one-to-correspondence with the group of characters of the component group of the stabilizer in ${}^\vee T$ of any Langlands parameter ϕ. (Notice that this bijection depends on the choice of the distinguished conjugacy class $\mathcal{W}$ of pure rational forms.)

With this example in mind, we can sharpen Conjecture 4.3 to include a parametrization of the (pure) L-packets.

Definition 4.14. *In the setting of Definition 4.2, suppose ϕ is a Langlands parameter. Define ${}^\vee G^\phi$ to be the centralizer in ${}^\vee G$ of the image of ϕ (an algebraic subgroup of ${}^\vee G$). Define*

$$A_\phi^{loc} = {}^\vee G^\phi/{}^\vee G_0^\phi,$$

the (pure) Langlands component group for ϕ. A complete pure Langlands parameter is a pair (ϕ,τ), with ϕ a Langlands parameter and τ an irreducible representation of A_ϕ^{loc}. The group ${}^\vee G$ acts by conjugation on the complete pure Langlands parameters. Conjugate parameters will be called equivalent, and we write $\Xi_{pure}({}^\vee G^\Gamma)$ for the set of equivalence classes. If $\mathcal{O}_F$ is a conjugacy class of admissible homomorphisms of W_F into ${}^\vee G^\Gamma$ as in Corollary 4.6, then we write $\Xi_{pure}(\mathcal{O}_F, {}^\vee G^\Gamma)$ for the set of equivalence classes of pairs (ϕ,τ) with $\phi|_{W_F} \in \mathcal{O}_F$. If ${}^\vee G^\Gamma$ is an L-group, then we may write G/F instead of ${}^\vee G^\Gamma$ in this notation.

Conjecture 4.15. *In the setting of Conjecture 4.3, there is a natural bijection*

$$\Pi_{pure}(G/F) \leftrightarrow \Xi_{pure}(G/F).$$

That is, the set of G-conjugacy classes of pairs (π,δ) (with δ a pure rational form of G^Γ (Definition 2.6) and π an irreducible admissible representation of $G(F,\delta)$) is in one-to-one correspondence with the set of ${}^\vee G$-conjugacy classes of pairs (ϕ,τ) (Definition 4.14). In the setting of Conjecture 4.3, this means that the "pure" part $\Pi_{pure,\phi}$ of the L-packet Π_ϕ (consisting of the equivalence classes of pairs (π,δ) in Π_ϕ with δ pure) is parametrized by the irreducible representations of the group of connected components of ${}^\vee G^\phi$.

For the case of tori, Conjecture 4.15 is contained in Example 4.13; of course the main point is Langlands' proof of Conjecture 4.3 for tori. The idea that the component group of ${}^\vee G^\phi$ should control the structure of the L-packets is due to Langlands; the case of $SL(2)$ was developed by Labesse-Langlands. The most general theorems of this nature in the archimedean case were proved by Shelstad.

We can interpret a result of Kottwitz as saying precisely how the complete Langlands parameter should determine the pure rational form.

Proposition 4.16. *([21], Proposition 6.4). Suppose ${}^\vee G^\Gamma$ is a weak E-group (Definition 3.24). Write $X_*({}^\vee G)^\Gamma$ for the lattice of rational one-parameter central subgroups of ${}^\vee G$ fixed by Γ-fixed; that is, the lattice of one-parameter subgroups of $Z({}^\vee G)^\Gamma$. Fix a Γ-stable maximal torus ${}^dT \subset {}^\vee G$. Define*

$$X^*({}^dT) = \textit{lattice of rational characters of } {}^dT.$$

In $X^*({}^dT)$ we define four sublattices:

$$R^*({}^\vee G, {}^dT) = \textit{root lattice of } {}^dT \textit{ in } {}^\vee G;$$

$$X^{*,0}({}^dT) = \textit{lattice generated by elements } \lambda - \gamma\cdot\lambda \ (\lambda \in X^*({}^dT), \gamma \in \Gamma);$$

$$X^{*,1}({}^dT) = \textit{lattice of rational characters trivial on } X_*(T)^\Gamma; \textit{and}$$

$$X^{*,1}({}^\vee G, {}^dT) = \textit{lattice of rational characters trivial on } X_*({}^\vee G)^\Gamma.$$

a) $X^{*,1}({}^\vee G, {}^dT)$ is the lattice of rational characters of ${}^dT$ trivial on the subgroup $(Z({}^\vee G)^\Gamma)_0$.
b) $R^*({}^\vee G, {}^dT) + X^{*,0}({}^dT)$ is the lattice of rational characters of ${}^dT$ trivial on $Z({}^\vee G)^\Gamma$.
c) The quotient $X^{*,1}({}^\vee G, {}^dT)/(R^*({}^\vee G, {}^dT)+X^{*,0}({}^dT)$ is canonically isomorphic to the finite abelian group of characters of $Z({}^\vee G)^\Gamma/(Z({}^\vee G)^\Gamma)_0$. In particular, it is independent of the choice of ${}^\vee\delta$ and T (up to canonical isomorphism).
d) Suppose ${}^\vee G^\Gamma$ is a weak E-group for G, and that σ is a fixed rational form of G in the inner class corresponding to ${}^\vee G^\Gamma$. Then the finite abelian group $X_*^1(G,T)/(R_*(G,T) + X_*^0(T))$ of Proposition 2.20 is canonically isomorphic to $X^{*,1}({}^\vee G, {}^dT)/(R^*({}^\vee G, {}^dT) + X^{*,0}({}^dT))$. Consequently there is a canonical isomorphism

$$H^1(\Gamma, G) \simeq (Z({}^\vee G)^\Gamma/(Z({}^\vee G)^\Gamma)_0)^\wedge.$$

e) Suppose $G^\Gamma, \mathcal{W}$ is a pure extended group, and ${}^\vee G^\Gamma$ is a weak E-group for G^Γ. Then there is a canonical bijection

$$\{\textit{equivalence classes of pure rational forms of } G\} \leftrightarrow (Z({}^\vee G)^\Gamma/(Z({}^\vee G)^\Gamma)_0)^\wedge.$$

Actually there is no need for ${}^\vee\delta$ to be distinguished. If it is not, the existence of a Γ-stable T must be imposed as an additional assumption, however.

Sketch of proof. The center $Z({}^\vee G)$ is necessarily contained in the maximal torus ${}^dT$. Part (a) is obvious. The lattice of characters trivial on $Z({}^\vee G)$ is $R^*({}^\vee G, {}^dT)$; the lattice trivial on ${}^dT^\Gamma$ is $X^{*,0}({}^dT)$ (cf. (4.13)(b). It follows that the lattice trivial on the intersection $Z({}^\vee G)^\Gamma$ is $R^*({}^\vee G, {}^dT) + X^{*,0}({}^dT)$. Now (c) is immediate from (a) and (b). For (d), we know that the group from Proposition 2.20 depends only on the inner class of σ; so we may calculate it using a quasisplit form and a maximally split torus T. Similarly, we may calculate the group for ${}^\vee G^\Gamma$ using the torus ${}^dT$ appearing in Definition 3.27. In this setting, the definition of dual group provides Γ-equivariant isomorphisms

$$X^{*,1}({}^\vee G, {}^dT) \simeq X_*^1(G,T), \quad R^*({}^\vee G, {}^dT) \simeq R_*(G,T), \quad X^{*,0}({}^dT) \simeq X_*^0(T).$$

This gives (d), and (e) follows from (2.19). Q.E.D.

In the setting of Definition 4.14, we always have

$$Z({}^\vee G)^\Gamma \subset {}^\vee G^\phi, \qquad (Z({}^\vee G)^\Gamma)_0) \subset {}^\vee G_0^\phi. \tag{4.17)(a}$$

Accordingly there are always natural maps

$$Z({}^\vee G)^\Gamma/(Z({}^\vee G)^\Gamma)_0 \to A_\phi^{loc}, \qquad (A_\phi^{loc})^\wedge \to (Z({}^\vee G)^\Gamma/(Z({}^\vee G)^\Gamma)_0)^\wedge. \tag{4.17)(b}$$

The first of these need not be injective, since elements of $Z({}^\vee G)^\Gamma$ not in the identity component may nevertheless lie in the identity component of ${}^\vee G^\phi$. Consequently the second map need not be surjective. Suppose now that we are in the setting of Conjecture 4.15. Composing the second map of (4.17)(b) with the bijection of Proposition 4.16(e), we get a natural map

$$(A_\phi^{loc})^\wedge \to \text{equivalence classes of pure rational forms of } G. \tag{4.17)(c}$$

These may be assembled over various ϕ to give

$$\Xi_{pure}(G/F) \to \text{equivalence classes of pure rational forms of } G. \tag{4.17)(d}$$

Of course we ask that this map should be the one implicit in the bijection of Conjecture 4.15.

Example 4.18. We return to the setting of Example 4.9, particularly (4.12). Fix $\phi \in P(\lambda, {}^\vee G)$ corresponding to a subset J of $\Delta({}^\vee G)$ (cf. (4.12)(b)) Because ${}^\vee G^\lambda$ is the maximal torus ${}^dT$, we see that ${}^\vee G^\phi$ is the subgroup of ${}^dT$ acting trivially on the nilpotent element $N_\phi \in {}^\vee\mathfrak{g}_{q_F}$. Consequently

$${}^\vee G^\phi = \{\, t \in {}^dT \mid \alpha^\vee(t) = 1, \text{all } \alpha^\vee \in J \,\}. \tag{4.18)(a}$$

We concentrate on the two extreme possibilities. First, if J is empty (so that $\phi = \lambda$ factors to W_F), then ${}^\vee G^\phi = {}^dT$ is connected, so there is only one complete pure Langlands parameter. The corresponding representation is the one-dimensional Langlands quotient representation of $I(\eta, \delta)$.

Next, suppose $J = \Delta({}^\vee G)$. Then ${}^\vee G^\phi = Z({}^\vee G)$, so the equivalence classes of complete pure Langlands parameters are parametrized by $(Z({}^\vee G)/Z({}^\vee G_0))^\wedge$. The corresponding composition factor of $I(\eta, \delta)$ is the Steinberg representation. Now the Steinberg representation can be defined for any rational form of G, and all the Steinberg representations of all the rational forms should constitute a single L-packet. Conjecture 4.15 therefore says that the pure rational forms of G^Γ should be parametrized precisely by $(Z({}^\vee G)/Z({}^\vee G)_0)^\wedge$. Of course this is a consequence of Proposition 4.16.

It is an entertaining exercise to analyze the intermediate cases. These involve Steinberg representations of parabolic subgroups of G; the size of the corresponding L-packet depends on the rational forms over which these parabolic subgroups are defined. The size of the finite group A_ϕ^{loc} exhibits a parallel behavior. We leave the details to the reader.

5. Langlands parameters: complex case.

Suppose throughout this section that $F = \mathbb{C}$, so that the Galois group Γ is trivial. A weak extended group $G = G^\Gamma$ is then just a complex connected reductive algebraic group. Such a group has a unique pure rational form, and a unique pure extended group structure. A weak E-group is a dual group ${}^\vee G = {}^\vee G^\Gamma$. There is a unique L-group structure on ${}^\vee G^\Gamma$, consisting of the unique homomorphism ${}^\vee\delta$ of Γ into ${}^\vee G$. (An E-group structure is a central element of finite order in ${}^\vee G$.) The Weil group $W_\mathbb{C}$ is the multiplicative group $\mathbb{C}^\times$.

Definition 5.1. *Suppose ${}^\vee G^\Gamma$ is a weak E-group. A Langlands parameter is a continuous homomorphism $\phi : W_\mathbb{C} \to {}^\vee G^\Gamma$ (that is, from $\mathbb{C}^\times$ to ${}^\vee G$) with semisimple image. The collection of all Langlands parameters is written $P({}^\vee G^\Gamma)$. The group ${}^\vee G$ acts on Langlands parameters by conjugation; parameters in the same orbit are called equivalent, and the set of equivalence classes is written $\Phi({}^\vee G^\Gamma)$. If ${}^\vee G^\Gamma$ is an L-group for $G/\mathbb{C}$, then we write instead $P(G/\mathbb{C})$ and $\Phi(G/\mathbb{C})$. (Of course in the complex case a weak E-group is an L-group as soon as we decide to think of it as one.)*

In analogy with Definition 4.14, we can define ${}^\vee G^\phi$ to be the centralizer in ${}^\vee G$ of the image of ϕ. This is always a Levi subgroup of G (see Corollary 5.5 below), so the *(pure) Langlands component group for ϕ,*

$$A_\phi^{loc} = {}^\vee G^\phi / {}^\vee G_0^\phi \tag{5.2)(a}$$

is always trivial. We may therefore write in analogy with Definition 4.14

$$\Xi_{pure}({}^\vee G^\Gamma) = \Phi({}^\vee G^\Gamma), \qquad \Xi_{pure}(G/\mathbb{C}) = \Phi(G/\mathbb{C}) \tag{5.2)(b}$$

We can combine analogues of Conjectures 4.3 and 4.15 into one simple statement. Langlands observed that this statement amounts to a reformulation of Zhelobenko's classification of the representations of complex groups. That is,

Theorem 5.3. *(Zhelobenko — see [13] and [24].) Suppose G is a complex connected reductive algebraic group. Fix an L-group ${}^\vee G^\Gamma$ for G. Then to each equivalence class $\phi \in \Phi(G/\mathbb{C})$ of Langlands parameters is associated an irreducible admissible representation π_ϕ of G. Every irreducible representation arises in this way exactly once, so that there is a bijection*

$$\Pi_{pure}(G/\mathbb{C}) \leftrightarrow \Xi_{pure}(G/\mathbb{C}).$$

The requirement (3) of Conjecture 4.3 is trivially satisfied: all three conditions are equivalent to the requirement that G be abelian.

We want to analyze the geometry of the set of Langlands parameters, in analogy with (4.4)–(4.8).

Lemma 5.4.. *Suppose H is a complex Lie group, with Lie algebra $\mathfrak{h}$. Then the set of continuous homomorphisms ϕ from $\mathbb{C}^\times$ into H may be identified with the set of pairs $(\lambda_1, \lambda_2) \in \mathfrak{h} \times \mathfrak{h}$, subject to the following requirements:*

i) $[\lambda_1, \lambda_2] = 0$*; and* *ii)* $\exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2)$.

The homomorphism ϕ *is then given by*

$$\phi(e^t e^{i\theta}) = \exp(t(\lambda_1 + \lambda_2))\exp(i\theta(\lambda_1 - \lambda_2)) \qquad (t, \theta \in \mathbb{R}).$$

This is elementary and well-known. The last formula is often written as

$$\phi(z) = z^{\lambda_1}\overline{z}^{\lambda_2}.$$

Corollary 5.5. *Suppose* ${}^\vee G^\Gamma$ *is a weak E-group over* $\mathbb{C}$*. Then the set of Langlands parameters* ϕ *for* ${}^\vee G^\Gamma$ *may be identified with the set of pairs* $(\lambda_1, \lambda_2) \in {}^\vee\mathfrak{g} \times {}^\vee\mathfrak{g}$*, subject to the following requirements:*

i) $[\lambda_1, \lambda_2] = 0$*;*

ii) $\exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2)$*; and*

iii) λ_1 *and* λ_2 *are semisimple.*

The isotropy group ${}^\vee G^\phi$ *is therefore the intersection of the Levi subgroups* L_i *centralizing* λ_i*; it contains a maximal torus of* ${}^\vee G$*.*

When ϕ corresponds to (λ_1, λ_2), we sometimes write $\lambda_1(\phi)$, $\lambda_2(\phi)$ or $\phi(\lambda_1, \lambda_2)$ to emphasize the relationship.

Although we do not want to use the most obvious analogue of (4.4)–(4.8) here, a brief outline of it will help to set the stage for the slightly more complicated construction to follow. Recall from Definition 1.10 that an infinitesimal character is defined in the complex case to be a pair

$$\mathcal{O}_{\mathbb{C}} = (\mathcal{O}_1, \mathcal{O}_2) \tag{5.6)(a}$$

of semisimple conjugacy classes in ${}^\vee\mathfrak{g}$. We require in addition that $\exp(2\pi i\lambda_1)$ be conjugate to $\exp(2\pi i\lambda_2)$ whenever $\lambda_i \in \mathcal{O}_i$. Write

$$\mathcal{C}(\mathcal{O}_{\mathbb{C}}) = \exp(2\pi i\mathcal{O}_1) = \exp(2\pi i\mathcal{O}_2) \subset {}^\vee G \tag{5.6)(b}$$

for the corresponding semisimple conjugacy class. Define

$$P(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma) = \{\, \phi \in P({}^\vee G^\Gamma) \mid \lambda_1(\phi) \in \mathcal{O}_1, \lambda_2(\phi) \in \mathcal{O}_2 \,\}, \tag{5.6)(c}$$

the set of *Langlands parameters of infinitesimal character* $\mathcal{O}_{\mathbb{C}}$*.* Fix $c \in \mathcal{C}(\mathcal{O}_{\mathbb{C}})$. This element plays the rôle of the Frobenius element in (4.4). Put

$$P(c, \mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma) = \{\, \phi \in P(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma) \mid \exp(2\pi i\lambda_1(\phi)) = c \,\}. \tag{5.6)(d}$$

Evidently $P(c, \mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$ meets every ${}^\vee G$-orbit on $P(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$. Define

$${}^\vee G(c) = \text{centralizer in } {}^\vee G \text{ of } c, \tag{5.6)(e}$$

a reductive subgroup of G. This group plays the rôle of ${}^\vee G^\lambda$ in (4.4). Finally, define

$$\mathcal{O}_i(c) = \{\, \lambda_i \in \mathcal{O}_i \mid \exp(2\pi i\lambda_i) = c \,\} \subset {}^\vee\mathfrak{g}(c), \tag{5.6)(f}$$

and

$$\mathcal{O}_{\mathbb{C}}^{comm}(c) = \{\, (\lambda_1, \lambda_2) \mid \lambda_i \in \mathcal{O}_i(c), [\lambda_1, \lambda_2] = 0 \,\}. \tag{5.6)(g}$$

Proposition 5.7. *Suppose we are in the setting (5.6); use the notation there.*

a) Corollary 5.5 identifies $P(c, \mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$ with $\mathcal{O}_{\mathbb{C}}^{comm}(c)$. This identification respects the actions of ${}^\vee G(c)$ on the two sets.

b) Each set $\mathcal{O}_i(c)$ is a semisimple orbit of ${}^\vee G(c)$ on ${}^\vee\mathfrak{g}(c)$.

c) The group ${}^\vee G(c)$ acts on $\mathcal{O}_{\mathbb{C}}^{comm}(c)$ with finitely many orbits.

d) There is a ${}^\vee G$-equivariant bijection between $P(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$ and

$$ {}^\vee G \times_{{}^\vee G(c)} \mathcal{O}_{\mathbb{C}}^{comm}(c). $$

In particular, the orbits of ${}^\vee G$ on $P(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$ are in one-to-one correspondence with the orbits of ${}^\vee G(c)$ on $\mathcal{O}_{\mathbb{C}}^{comm}(c)$.

This is elementary. The space in (d) is an affine algebraic variety, and appears at first to be a natural analogue of the variety $X(\mathcal{O}_F, {}^\vee G^\Gamma)$ of (4.7). The difference is that in the present case all the isotropy groups for the action of ${}^\vee G$ are Levi subgroups, and therefore reductive. It follows that the orbits of ${}^\vee G$ are (open and) closed. Our goal is Conjecture 1.12 of the introduction, which says that interesting aspects of the representation theory should be encoded by the closure relationships among the orbits. To achieve that, we must therefore use a different approach.

This new approach requires some elementary but rather convoluted constructions from [1]. The main point is this: in the p-adic case only the q_F-eigenspace of the Frobenius element matters. Here all the integral eigenspaces of λ_1 and λ_2 have a similar importance. Suppose therefore that $\lambda \in {}^\vee\mathfrak{g}$ is semisimple element. Define

$$ {}^\vee\mathfrak{g}(\lambda)_n = \{\, \mu \in {}^\vee\mathfrak{g} \mid [\lambda, \mu] = n\mu \,\} \qquad (n \in \mathbb{Z}) \tag{5.8)(a} $$

$$ {}^\vee\mathfrak{g}(\lambda) = \sum_{n\in\mathbb{Z}} {}^\vee\mathfrak{g}(\lambda)_n \tag{5.8)(b} $$

$$ \mathfrak{l}(\lambda) = {}^\vee\mathfrak{g}(\lambda)_0 = \text{centralizer of } \lambda \text{ in } {}^\vee\mathfrak{g} \tag{5.8)(c} $$

$$ \mathfrak{n}(\lambda) = \sum_{n=1,2,\ldots} {}^\vee\mathfrak{g}(\lambda)_n \tag{5.8)(d} $$

$$ \mathfrak{p}(\lambda) = \mathfrak{l}(\lambda) + \mathfrak{n}(\lambda) \tag{5.8)(e} $$

Then ${}^\vee\mathfrak{g}(\lambda)$ is a reductive subalgebra of ${}^\vee\mathfrak{g}$, and (5.8)(e) is a Levi decomposition of a parabolic subalgebra of ${}^\vee\mathfrak{g}(\lambda)$. There are analogous objects on the group level. Put

$$ e(\lambda) = \exp(2\pi i\lambda) \in {}^\vee G \tag{5.9)(a} $$

$$ {}^\vee G(\lambda) = \text{centralizer in } {}^\vee G \text{ of } e(\lambda) \tag{5.9)(b} $$

$$ L(\lambda) = \text{centralizer in } {}^\vee G \text{ of } \lambda \tag{5.9)(c} $$

$$ N(\lambda) = \text{connected unipotent subgroup with Lie algebra } \mathfrak{n}(\lambda) \tag{5.9)(d} $$

$$ P(\lambda) = L(\lambda)N(\lambda). $$

The Lie algebras of these algebraic groups are as indicated by the notation; (5.9)(d) gives a Levi decomposition of a parabolic subgroup of the reductive group ${}^\vee G(\lambda)$. The *canonical flat through* λ is the affine subspace

$$ \mathcal{F}(\lambda) = \mathrm{Ad}(P(\lambda)) \cdot \lambda = \mathrm{Ad}(N(\lambda)) \cdot \lambda = \lambda + \mathfrak{n}(\lambda). \tag{5.10} $$

(For the equality of the last three spaces, see [1], Lemma 6.3). The next result collects some easy general facts about canonical flats.

Proposition 5.11. *([1], chapter 6). Suppose $\mathcal{O} \subset {}^\vee\mathfrak{g}$ is a semisimple orbit of the adjoint action. Write $\mathcal{F}(\mathcal{O})$ for the set of canonical flats in $\mathcal{O}$, a set of affine subspaces of ${}^\vee\mathfrak{g}$. Set $\mathcal{C}(\mathcal{O}) = e(\mathcal{O})$ (cf. (5.9)(a)), a semisimple conjugacy class in ${}^\vee G$. Fix $\lambda \in \mathcal{O}$, and define $\Lambda = \mathcal{F}(\lambda) \in \mathcal{F}(\mathcal{O})$ to be the corresponding canonical flat (cf. (5.10)), and $c = e(\lambda)$.*

a) The canonical flats partition $\mathcal{O}$.

b) The normalized exponential map e of (5.9) is constant on canonical flats.

Because of (b), we may write

$$e : \mathcal{F}(\mathcal{O}) \to \mathcal{C}(\mathcal{O}).$$

Define

$$\mathcal{F}(\mathcal{O})(c) = \{\, \Lambda' \in \mathcal{F}(\mathcal{O}) \mid e(\Lambda') = c\,\}.$$

c) The groups $P(\lambda)$, $N(\lambda)$, and ${}^\vee G(\lambda)$ all depend only on the canonical flat Λ containing λ. The group ${}^\vee G(\lambda)$ depends only on $c = e(\lambda) = e(\Lambda)$.

d) The sets $\mathcal{O}$, $\mathcal{F}(\mathcal{O})$, and $\mathcal{C}(\mathcal{O})$ are all homogeneous spaces for ${}^\vee G$, with isotropy groups

$$L(\lambda) \subset P(\lambda) \subset {}^\vee G(\lambda)$$

respectively. The maps

$$\mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{F}(\mathcal{O}) \xrightarrow{e} \mathcal{C}(\mathcal{O})$$

correspond to the natural projections of homogeneous spaces

$${}^\vee G/L(\lambda) \to {}^\vee G/P(\lambda) \to {}^\vee G/{}^\vee G(\lambda).$$

The first map is an affine bundle, and the second is a projective morphism.

e) The set $\mathcal{F}(\mathcal{O})(c)$ is a projective homogeneous space for ${}^\vee G(c)$. It may be identified with the variety of parabolic subgroups of ${}^\vee G(c)$ conjugate to $P(\Lambda)$.

Because of (c), we may write $P(\Lambda)$, $N(\Lambda)$, and ${}^\vee G(\Lambda)$ or ${}^\vee G(c)$. Of course this last notation is consistent with (5.6)(e).

The main idea now is to imitate the definition of Langlands parameters as closely as possible, but to use canonical flats instead of individual elements of ${}^\vee\mathfrak{g}$. This implements the suggestion made before (5.8) that all the integral eigenspaces of λ_1 and λ_2 should be treated like the q_F-eigenspace of the Frobenius element in the p-adic case. It is the description of Langlands parameters in Corollary 5.5 that we will follow. Condition (iii) there is assured by confining our attention to the semisimple orbits $\mathcal{O}_i$; and Proposition 5.11(b) will allow us to formulate a condition like Corollary 5.5(ii). What is interesting is condition (i) in Corollary 5.5. We cannot hope to impose it on all pairs of elements from two canonical flats; in that sense a canonical flat does not usually "commute" even with itself. The alternative is to require it only for some pair of elements taken from the two canonical flats. In that form the requirement turns out to be vacuous; so we are led to a definition that is in some sense actually simpler than that of Definition 5.1.

Definition 5.12. *Suppose ${}^\vee G^\Gamma$ is a weak E-group, and $\mathcal{O}_\mathbb{C} = (\mathcal{O}_1, \mathcal{O}_2)$ is an infinitesimal character. A geometric parameter of infinitesimal character $\mathcal{O}_\mathbb{C}$ is a pair (Λ_1, Λ_2) with*

$$\Lambda_i \in \mathcal{F}(\mathcal{O}_i), \qquad e(\Lambda_1) = e(\Lambda_2)\text{''}$$

(cf. Proposition 5.11). A little more geometrically, we define the geometric parameter space of infinitesimal character $\mathcal{O}_{\mathbb{C}}$ as the fiber product

$$X(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma) = \mathcal{F}(\mathcal{O}_1) \times_{\mathcal{C}(\mathcal{O}_{\mathbb{C}})} \mathcal{F}(\mathcal{O}_2);$$

the fiber product is formed using the morphisms $e : \mathcal{F}(\mathcal{O}_i) \to \mathcal{C}(\mathcal{O}_{\mathbb{C}})$ of Proposition 5.11(b). The group ${}^\vee G$ acts on $X(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$ by conjugation; conjugate geometric parameters are called equivalent. We do not introduce any notation for the set of equivalence classes of geometric parameters, because we will see in a moment that they may be naturally identified with $\Phi({}^\vee G^\Gamma)$ (Definition 5.1).

Proposition 5.11 provides a good description of the geometric parameter space.

Proposition 5.13. *In the setting of Definition 5.12, fix $c \in \mathcal{C}(\mathcal{O}_{\mathbb{C}})$, and $\Lambda_i \in \mathcal{F}(\mathcal{O}_i)(c)$ (Proposition 5.11); put $P_i = P(\Lambda_i)$ (a parabolic subgroup of ${}^\vee G(c)$), and write $\mathcal{P}_i$ for the projective variety of parabolic subgroups of ${}^\vee G(c)$ conjugate to P_i (Proposition 5.11). Then there is a ${}^\vee G$-equivariant isomorphism*

$$X(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma) \simeq {}^\vee G \times_{{}^\vee G(c)} (\mathcal{P}_1 \times \mathcal{P}_2).$$

In particular, the orbits of ${}^\vee G$ on $X(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma)$ are in one-to-one correspondence with the orbits of ${}^\vee G(c)$ on $\mathcal{P}_1 \times \mathcal{P}_2$. They are finite in number. The isotropy group ${}^\vee G^x$ of the action at the point $x = (\Lambda_1, \Lambda_2)$ is the intersection $P_1 \cap P_2$ of two parabolic subgroups of ${}^\vee G(c)$.

This is immediate from Proposition 5.11 and the definitions. (The finiteness assertion follows from the Bruhat decomposition of ${}^\vee G(c)$.)

Finally, we need to relate the geometric parameter space to Langlands parameters.

Proposition 5.14. *(cf. [1], Proposition 6.17). In the setting of Definition 5.12, there is a natural ${}^\vee G$-equivariant map*

$$p : P(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma) \to X(\mathcal{O}_{\mathbb{C}}, {}^\vee G^\Gamma), \qquad p(\phi(\lambda_1, \lambda_2)) = (\mathcal{F}(\lambda_1), \mathcal{F}(\lambda_2)).$$

This map induces a bijection from equivalence classes of Langlands parameters to equivalence classes of geometric parameters. If $x = p(\phi)$, then the isotropy group ${}^\vee G^\phi$ is a Levi subgroup of ${}^\vee G^x$. In particular, the fiber $p^{-1}(x)$ is a principal homogeneous space for the unipotent radical of ${}^\vee G^x$.

This result follows from Corollary 5.5, Proposition 5.12, and standard structure theory. Perhaps the main point is the fact that two parabolic subgroups of a reductive group must contain a common maximal torus; this of course is more or less equivalent to the Bruhat decomposition.

6. Langlands parameters: real case.

Suppose throughout this section that $F = \mathbb{R}$, so that the Galois group Γ is $\mathbb{Z}/2\mathbb{Z}$. The Weil group $W_{\mathbb{R}}$ is generated by a copy of $\mathbb{C}^\times$ and an element j, subject to the relations

$$jzj^{-1} = \overline{z}, \qquad j^2 = -1 \in \mathbb{C}^\times. \tag{6.1)(a}$$

Accordingly there is a short exact sequence

$$1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma \to 1, \tag{6.1)(b}$$

sending j to the non-trivial element of Γ. The Weil-Deligne group $W'_{\mathbb{R}}$ is just $W_{\mathbb{R}}$.

Definition 6.2. *Suppose ${}^\vee G^\Gamma$ is a weak E-group (Definition 3.24). A Langlands parameter is a continuous homomorphism $\phi : W_{\mathbb{R}} \to {}^\vee G^\Gamma$ with the following additional properties:*

1) ϕ respects the homomorphisms (3.24)(1) and (6.1)(b) to Γ; and
2) the image of ϕ consists of semisimple elements of ${}^\vee G^\Gamma$.

Here in (2) semisimplicity is defined in analogy with the discussion before Definition 4.2. It is therefore equivalent to require

2′) *$\phi(\mathbb{C}^\times)$ consists of semisimple elements of ${}^\vee G$.*

The collection of all Langlands parameters is written $P({}^\vee G^\Gamma)$. The group ${}^\vee G$ acts by conjugation on the range. Parameters in the same orbit are called equivalent, and the set of equivalence classes is denoted $\Phi({}^\vee G^\Gamma)$. If ${}^\vee G^\Gamma$ is an L-group for $G/\mathbb{R}$, then we write instead $P(G/\mathbb{R})$ and $\Phi(G/\mathbb{R})$.

Suppose ϕ is a Langlands parameter. Define ${}^\vee G^\phi$ to be the centralizer in ${}^\vee G$ of the image of ϕ (an algebraic subgroup of ${}^\vee G$). Set

$$A_\phi^{loc} = {}^\vee G^\phi / {}^\vee G_0^\phi,$$

the (pure) Langlands component group for ϕ. A complete pure Langlands parameter is a pair (ϕ, τ), with τ an irreducible representation of A_ϕ^{loc}. The group ${}^\vee G$ acts by conjugation on the set of complete pure Langlands paramters. Conjugate parameters are said to be equivalent, and we write $\Xi_{pure}({}^\vee G^\Gamma)$ (or $\Xi_{pure}(G/\mathbb{R})$ in the case of an L-group) for the set of equivalence classes.

In this case again Langlands has proved the analogue of Conjecture 4.3. The analogue of Conjecture 4.15 is a consequence of Langlands' results and the Knapp-Zuckerman classification of tempered representations; various forms of it may be found in [27] and in unpublished work of Langlands. The form given here is from [1].

Theorem 6.3. *([24]; see also [1]). Suppose G is a complex connected reductive algebraic group endowed with an inner class of real forms. Fix a pure extended group $(G^\Gamma, \mathcal{W})$, and an L-group $({}^\vee G^\Gamma, \mathcal{D})$ for $G/\mathbb{R}$ (Definitions 2.15 and 3.25). Then to each equivalence class $\phi \in \Phi(G/\mathbb{R})$ there is a associated a collection $\Pi_\phi \subset \Pi(G/\mathbb{R})$ (called an L-packet) of equivalence classes of representations of rigid real forms of G (Definition 2.14 and (2.17)). These sets have the following properties.*

a) The L-packets Π_ϕ partition $\Pi(G/\mathbb{R})$.
b) If δ is a rigid real form of G^Γ, then the set

$$\Pi_\phi(\delta) = \{\pi \in \Pi(G(F,\delta) \mid (\pi,\delta) \in \Pi_\phi\}$$

is finite. If δ is quasisplit, it is non-empty.

c) The following three conditions on ϕ are equivalent:
1) some representation in Π_ϕ is square-integrable modulo center;
2) all representations in Π_ϕ are square-integrable modulo center;
3) the image of ϕ is not contained in any proper Levi subgroup of ${}^\vee G^\Gamma$ ([8], 3.4).

Write $\Pi_{pure,\phi}$ for the classes of pairs $(\pi,\delta) \in \Pi_\phi$ with δ a pure real form (a pure L-packet).

d) *There is a natural bijection between* $\Pi_{pure,\phi}$ *and the set of irreducible representations of* A^{loc}_{ϕ}*. Equivalently, there is a natural bijection*

$$\Pi_{pure}(G/\mathbb{R}) \leftrightarrow \Xi_{pure}(G/\mathbb{R})$$

between equivalence classes of representations of pure rational forms of G *(Definition 2.14) and equivalence classes of complete pure Langlands parameters.*

As in section 5, we now wish to analyze and modify the geometry of the set of Langlands parameters.

Lemma 6.4. *([1], Proposition 5.6). Suppose* H *is a complex Lie group, with Lie algebra* $\mathfrak{h}$*. Then the set of continuous homomorphisms from* $W_{\mathbb{R}}$ *into* H *may be identified with the set of pairs* $(y, \lambda) \in H \times \mathfrak{h}$*, subject to the following requirements:*

i) $[\lambda, Ad(y)(\lambda)] = 0$*; and*
ii) $y^2 = \exp(2\pi i \lambda)$.

The corresponding homomorphism ϕ *is given by*

$$\phi(z) = z^{\lambda}\overline{z}^{y\cdot\lambda} \quad (z \in \mathbb{C}^{\times}), \qquad \phi(j) = \exp(-\pi i \lambda) y$$

(notation after Lemma 5.4).

We omit the elementary argument (based on Lemma 5.4).

Corollary 6.5. *Suppose* ${}^{\vee}G^{\Gamma}$ *is a weak E-group over* $\mathbb{R}$*. Then the set of Langlands parameters* ϕ *for* ${}^{\vee}G^{\Gamma}$ *may be identified with the set of pairs* $(y, \lambda) \in {}^{\vee}G \times {}^{\vee}\mathfrak{g}$*, subject to the following conditions:*

i) $[\lambda, Ad(y)(\lambda)] = 0$*;*
ii) $y^2 = \exp(2\pi i \lambda)$*; and*
iii) λ *is semisimple; and*
iv) $y \in {}^{\vee}G^{\Gamma} - {}^{\vee}G$.

When ϕ corresponds to (y, λ), we may write $(y(\phi), \lambda(\phi))$ or $\phi(y, \lambda)$ to emphasize the relationship.

By an *infinitesimal character for* $G/\mathbb{R}$, we mean a pair

(6.6)(a) $$\mathcal{O}_{\mathbb{R}} = (\mathcal{Y}, \mathcal{O})$$

with $\mathcal{O}$ a semisimple conjugacy class of ${}^{\vee}G$ in ${}^{\vee}\mathfrak{g}$ and $\mathcal{Y}$ a semisimple conjugacy class of ${}^{\vee}G$ in ${}^{\vee}G^{\Gamma} - {}^{\vee}G$. We require also that y^2 be conjugate to $e(\lambda)$ for every $y \in \mathcal{Y}$ and $\lambda \in \mathcal{O}$; we write

(6.6)(b) $$\mathcal{C}(\mathcal{O}_{\mathbb{R}}) = \mathcal{Y}^2 = e(\mathcal{O})$$

for the corresponding semisimple conjugacy class in ${}^{\vee}G$. Define

(6.6)(c) $$P(\mathcal{O}_{\mathbb{R}}, {}^{\vee}G^{\Gamma}) = \{\, \phi \in P({}^{\vee}G^{\Gamma}) \mid y(\phi) \in \mathcal{Y}, \lambda(\phi) \in \mathcal{O} \,\},$$

the set of *Langlands parameters of infinitesimal character* $\mathcal{O}_{\mathbb{R}}$. For $y \in \mathcal{Y}$, define

(6.6)(d) $$K(y) = \text{centralizer of } y \text{ in } {}^{\vee}G.$$

Now fix $c \in \mathcal{C}(\mathcal{O}_{\mathbb{R}})$, and define

(6.7)(a) $$\mathcal{O}(c) = \{\, \lambda \in \mathcal{O} \mid e(\lambda) = c \,\}, \qquad \mathcal{Y}(c) = \{\, y \in \mathcal{Y} \mid y^2 = c \,\}.$$

Each element $y \in \mathcal{Y}(c)$ acts by conjugation on ${}^{\vee}G(c)$ as an involutive automorphism $\theta(y)$, with fixed point group $K(y)$. For $y \in \mathcal{Y}(c)$, define

(6.7)(b) $$\mathcal{O}^{comm}_{\mathbb{R}}(c, y) = \{\, \lambda \in \mathcal{O}(c) \mid [\lambda, \theta(y)\lambda] = 0 \,\}$$

(6.7)(c) $$P(c, y, \mathcal{O}_{\mathbb{R}}) = \{\, \phi \in P(\mathcal{O}_{\mathbb{R}}, {}^{\vee}G^{\Gamma}) \mid y(\phi) = y \,\}.$$

Proposition 6.8. *Suppose we are in the setting of (6.6) and (6.7).*

a) Corollary 6.5 identifies $P(c, y, \mathcal{O}_{\mathbb{R}})$ with $\mathcal{O}_{\mathbb{R}}^{comm}(c, y)$. This identification respects the actions of $K(y)$ on the two sets.

b) The set $\mathcal{O}(c)$ is a semisimple orbit of ${}^\vee G(c)$ on ${}^\vee\mathfrak{g}(c)$.

c) The set $\mathcal{Y}(c)$ is a semisimple orbit of ${}^\vee G(c)$.

d) The group $K(y)$ acts on $\mathcal{O}_{\mathbb{R}}^{comm}(c, y)$ with finitely many orbits.

e) Fix $y \in \mathcal{Y}(c)$. There is a ${}^\vee G$-equivariant bijection between $P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$ and the fiber product

$$ {}^\vee G \times_{K(y)} \mathcal{O}_{\mathbb{R}}^{comm}(c, y). $$

In particular, the orbits of ${}^\vee G$ on $P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$ are in one-to-one correspondence with the orbits of $K(y)$ on $\mathcal{O}_{\mathbb{R}}^{comm}(c, y)$.

The only assertion here with any content is the finiteness in (d). This amounts to the fact that there are finitely many K-conjugacy classes of θ-stable maximal tori in a reductive group H with involution θ and $K = H^\theta$.

Proposition 6.8 also shows what an isotropy group ${}^\vee G^\phi$ can look like (Definition 6.2). It is the intersection with $K(y)$ of a connected $\theta(y)$-stable Levi subgroup L of ${}^\vee G(c)$. (Here $L = L(\lambda) \cap L(\theta(y)\lambda)$.) In particular, ${}^\vee G^\phi$ is itself the fixed point group of an involution on a connected reductive group. From this (and the structure theory of reductive groups with involutions) it follows that the pure component group A_ϕ^{loc} is a product of copies of $\mathbb{Z}/2\mathbb{Z}$; the number of copies is bounded by the rank of ${}^\vee G$.

Just as in Definition 5.12, we can now formulate an analogue of Langlands parameters using canonical flats instead of Lie algebra elements.

Definition 6.9. *([1], Definition 6.9). Suppose ${}^\vee G^\Gamma$ is a weak E-group over $\mathbb{R}$, and $\mathcal{O}_{\mathbb{R}} = (\mathcal{Y}, \mathcal{O})$ is an infinitesimal character (cf. (6.6)). A geometric parameter of infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is a pair (y, Λ) with*

$$ y \in \mathcal{Y}, \qquad \Lambda \in \mathcal{F}(\mathcal{O}), \qquad y^2 = e(\Lambda) $$

(cf. Proposition 5.11). The geometric parameter space of infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is the fiber product

$$ X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) = \mathcal{Y} \times_{\mathcal{C}(\mathcal{O}_{\mathbb{R}})} \mathcal{F}(\mathcal{O}); $$

the fiber product is formed using the projective morphism $e : \mathcal{F}(\mathcal{O}) \to \mathcal{C}(\mathcal{O}_{\mathbb{R}})$ and the squaring map from $\mathcal{Y}$ to $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$. The group ${}^\vee G$ acts on $X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$ by conjugation; conjugate geometric parameters are called equivalent. We will see in a moment that the equivalence classes are naturally parametrized by $\Phi(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$, so we need no separate notation for them.

Suppose x is a geometric parameter. Define ${}^\vee G^x \subset {}^\vee G$ to be the isotropy subgroup for the action of ${}^\vee G$ at x. Set

$$ A_x^{loc} = {}^\vee G^x / {}^\vee G_0^x, $$

the (pure) geometric component group for x. A complete pure geometric parameter is a pair (x, τ), with τ an irreducible representation of A_x^{loc}. The group ${}^\vee G$ acts by conjugation on the set of complete pure Langlands paramters. Conjugate parameters are said to be equivalent. We will see in a moment (Proposition 6.11) that the equivalence classes are parametrized by $\Xi_{pure}({}^\vee G^\Gamma)$.

Proposition 6.10. *In the setting of Definition 6.9, fix $c \in \mathcal{C}(\mathcal{O}_{\mathbb{R}})$ and $\Lambda \in \mathcal{F}(\mathcal{O})$; put $P = P(\Lambda)$ (a parabolic subgroup of ${}^\vee G(c)$), and write $\mathcal{P}$ for the projective variety of parabolic subgroups of ${}^\vee G(c)$ conjugate to P. Finally, choose $y \in \mathcal{Y}(c)$ (cf. (6.7)(a)). Then there is a ${}^\vee G$-equivariant isomorphism*

$$X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) \simeq {}^\vee G \times_{K(y)} \mathcal{P}.$$

In particular, the orbits of ${}^\vee G$ on $X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$ are in one-to-one correspondence with the orbits of $K(y)$ on $\mathcal{P}$. They are finite in number. The isotropy group ${}^\vee G^x$ of the action at the point (y, Λ) is the intersection $K(y) \cap P(\Lambda)$ of the fixed points of an involution and a parabolic subgroup in the reductive group ${}^\vee G(c)$.

This is a consequence of Proposition 6.8 and the definitions. The finiteness assertion comes from the fact that the group of fixed points of an involution of a reductive group acts with finitely many orbits on a flag variety.

Proposition 6.11. *([1], Proposition 6.17). In the setting of Definition 6.9, there is a natural ${}^\vee G$-equivariant map*

$$p : P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) \to X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma), \qquad p(\phi(y, \lambda)) = (y, \mathcal{F}(\lambda)).$$

This map induces a bijection from equivalence classes of Langlands parameters to equivalence classes of geometric parameters. If $x = p(\phi)$, then the isotropy group ${}^\vee G^\phi$ is a Levi subgroup of ${}^\vee G^x$. In particular, the fiber $p^{-1}(x)$ is a principal homogeneous space for the unipotent radical of ${}^\vee G^x$; and the inclusion induces an isomorphism of component groups

$$A^{loc}_\phi \simeq A^{loc}_x.$$

This is somewhat more complicated to prove than Proposition 5.14, but it is a consequence of known structural results for fixed points of involutions on reductive groups.

7. Infinitesimal characters.

In this section we will outline the representation-theoretic ideas that correspond to the notion of infinitesimal character introduced for Langlands parameters in sections 4–6. They are due to Harish-Chandra in the archimedean case and to Bernstein-Zelevinsky and Casselman in the p-adic case. Over each field, we will recall the "classical" notion of infinitesimal character. In the complex case, this notion coincides precisely with the infinitesimal character of section 5. In the real case, there is a finite-to-one map from the infinitesimal characters of section 6 to classical infinitesimal characters. In the p-adic case, there is conjecturally a finite-to-one map from the classical infinitesimal characters to those of section 4.

Suppose first that $\mathfrak{g}$ is a complex reductive Lie algebra. Write $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{Z}(\mathfrak{g})$ for the center of $U(\mathfrak{g})$. A module V for $U(\mathfrak{g})$ is called *quasisimple* if any $z \in \mathcal{Z}(\mathfrak{g})$ acts on V by a scalar $\chi_V(z)$. (An irreducible module is automatically quasisimple.) At least if V is not zero, these scalars define an algebra homomorphism

$$\chi_V : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C} \tag{7.1)(a}$$

called the *(classical) infinitesimal character of* V. We want to study the possibilities for such homomorphisms. Algebra homomorphisms from $\mathcal{Z}(\mathfrak{g})$ to $\mathbb{C}$ may be identified with maximal ideals in $\mathcal{Z}(\mathfrak{g})$, so we write

(7.1)(b) $$\operatorname{Max}\mathcal{Z}(\mathfrak{g}) \simeq \operatorname{Hom}_{alg}(\mathcal{Z}(\mathfrak{g}), \mathbb{C}).$$

Suppose $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$; write W for the Weyl group of $\mathfrak{t}$ in $\mathfrak{g}$. The *Harish-Chandra isomorphism* is an algebra isomorphism

(7.1)(c) $$\xi : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{t})^W$$

(see for example [14]). The construction of ξ requires a choice of a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$, but the map is independent of the choice. Now every algebra homomorphism from $S(\mathfrak{t})$ to $\mathbb{C}$ is given by evaluation of polynomial at a point $\lambda \in \mathfrak{t}^*$, so

(7.1)(d) $$\operatorname{Hom}_{alg}(S(\mathfrak{t}), \mathbb{C}) \simeq \mathfrak{t}^*.$$

Because the inclusion of $S(\mathfrak{t})^W$ in $S(\mathfrak{t})$ is an integral ring extension (since W is a finite group), it follows that any algebra homomorphism from $S(\mathfrak{t})^W$ to $\mathbb{C}$ extends to $S(\mathfrak{t})$; the corresponding element $\lambda \in \mathfrak{t}^*$ is uniquely determined up to the action of W. Combining these observations, we get a bijection

(7.1)(e) $$\operatorname{Hom}_{alg}(\mathcal{Z}(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{t}^*/W;$$

the quotient on the right denotes the set of orbits of W on $\mathfrak{t}^*$.

Suppose now that $\mathfrak{g}$ is the Lie algebra of a connected reductive complex algebraic group G. The right side in (7.1)(e) may be described in terms of the based root datum of G (Definition 3.8) as

(7.2)(a) $$\mathfrak{t}^*/W \simeq (X_b^*(G) \otimes_{\mathbb{Z}} \mathbb{C})/W_b;$$

here we write W_b for the Weyl group of the based root datum. (To see this, fix a Borel subgroup $B \supset T$. The choice of B provides a bijection between $X_b^*(G)$ and the lattice of rational characters of T. By taking differentials, we identify this last lattice with a lattice in $\mathfrak{t}^*$, and so get an isomorphism

$$\mathfrak{t}^* \simeq X_b^*(G) \otimes_{\mathbb{Z}} \mathbb{C}.$$

This isomorphism identifies the two Weyl groups, and the quotient identification (7.2)(a) is independent of the choice of B.) Now fix a dual group ${}^\vee G$ for G (Definition 3.18), and a maximal torus ${}^dT \subset {}^\vee G$. Write ${}^dW$ for the Weyl group of ${}^dT$ in ${}^\vee G$. Any semisimple orbit of ${}^\vee G$ on ${}^\vee\mathfrak{g}$ meets ${}^d\mathfrak{t}$ in a single orbit of ${}^dW$. This provides a natural bijection

(7.2)(b) $$\{\text{semisimple conjugacy classes in } {}^\vee\mathfrak{g}\ \} \simeq {}^d\mathfrak{t}/{}^dW.$$

We can describe the right side in terms of the based root datum for ${}^\vee G$, in analogy with (7.2)(a). The conclusion is that there is a natural bijection

(7.2)(c) $${}^d\mathfrak{t}/{}^dW \simeq (X_*^b({}^\vee G) \otimes_{\mathbb{Z}} \mathbb{C})/W_b.$$

Now the definition of dual group identifies the right sides of (7.2)(a) and (7.2)(c). From (7.1) and (7.2), we find

Lemma 7.3. *(see [1], Lemma 15.4). Suppose G and ${}^\vee G$ are complex dual groups. Then there is a natural one-to-one correspondence between classical infinitesimal characters for the Lie algebra $\mathfrak{g}$ (cf. (7.1)(a)) and semisimple orbits of ${}^\vee G$ on ${}^\vee\mathfrak{g}$.*

Notice that this lemma attaches to *any* irreducible representation of the Lie algebra $\mathfrak{g}$ a semisimple orbit of the dual group on its Lie algebra ${}^\vee\mathfrak{g}$. This correspondence may be regarded as a primitive form of the Langlands classification (over $\mathbb{R}$). It is compatible with the full classification of Theorem 6.3, as the next result shows.

Proposition 7.4. *In the setting of Theorem 6.3, suppose $\phi \in P(G/\mathbb{R})$ is a Langlands parameter, and $(\pi, \delta) \in \Pi_\phi$ is a representation of a rigid real form of G in the L-packet of ϕ. Then the space of smooth vectors in π is a quasisimple representation of $U(\mathfrak{g})$; write χ_π for its classical infinitesimal character (cf. (7.1)(a)), and $\mathcal{O}_\pi \subset {}^\vee\mathfrak{g}$ for the corresponding semisimple orbit (Lemma 7.3). The element $\lambda(\phi)$ (Corollary 6.5) belongs to $\mathcal{O}_\pi$. If ϕ has infinitesimal character $\mathcal{O}_\mathbb{R} = (\mathcal{Y}, \mathcal{O})$ (cf. (6.6)), then $\mathcal{O}_\pi = \mathcal{O}$.*

Sketch of proof. That π (an irreducible admissible representation) is quasisimple is a classical result of Harish-Chandra ([18], Corollary 8.14). The relation between χ_π and $\lambda(\phi)$ is clear from an inspection of Langlands' (rather complicated) construction of the L-packet Π_ϕ; this is the main point of the proposition. The last assertion is just a reformulation of it. Q.E.D.

The connection between classical infinitesimal characters and Problem B of the introduction is provided by the following elementary and well-known fact. (This fact is at the heart of the omitted "inspection" step in the proof of Proposition 7.4.)

Proposition 7.5. *([18], Proposition 8.22). Suppose σ is a real form of the complex connected reductive algebraic group G, $P = MN$ is a Levi decomposition defined over $\mathbb{R}$ of a parabolic subgroup of G, and π_M is a quasisimple representation of $M(\mathbb{R}, \sigma)$. Define*

$$\pi_G = Ind_{P(\mathbb{R},\sigma)}^{G(\mathbb{R},\sigma)}(\pi_M)"$$

(normalized induction). Then π_G is a quasisimple representation of $G(\mathbb{R}, \sigma)$.

Corollary 7.6. *In the setting of Proposition 7.5, suppose π_1 and π_2 are irreducible admissible representations of $G(\mathbb{R}, \sigma)$. Write $M_1 = M(\pi_1)$ for the standard representation of which π_1 is a quotient (cf. (1.3)(a)). If π_2 occurs as a composition factor of M_1 — equivalently, if $m(\pi_2, \pi_1) \neq 0$ (cf. (1.3)(b)) — then π_1 and π_2 must have the same classical infinitesimal character.*

Proof. The standard representation M_1 is of the form $\mathrm{Ind}_{P(\mathbb{R},\sigma)}^{G(\mathbb{R},\sigma)}(\rho_1)$, with ρ_1 an irreducible representation of $M(\mathbb{R}, \sigma)$. By Proposition 7.4, ρ_1 is quasisimple; so by Proposition 7.5, M_1 is quasisimple as well. It follows that any $z \in \mathcal{Z}(\mathfrak{g})$ acts by a scalar $\chi_{M_1}(z)$ on any subquotient of M_1. By (1.3)(a), $\chi_{\pi_1} = \chi_{M_1}$. If π_2 is also a composition factor, then $\chi_{\pi_2} = \chi_{M_1}$, and the result follows. Q.E.D.

The full "infinitesimal character" defined in (6.6) is not quite so easy to explain on G, and we will confine ourselves to some general comments. In terms of the distribution character of π, the classical infinitesimal character provides some differential equations that guarantee good behavior along each connected component of a Cartan subgroup of $G(\mathbb{R}, \sigma)$. Roughly speaking, the conjugacy class $\mathcal{Y}$ provides some control on the relationship between different connected components. Part of this information is encoded in the invariant $c(\pi)$ (the "compactness" of π) introduced in [30], Definition 9.2.17. In terms of the Langlands parameters (if $\pi \in \Pi_\phi$, $y(\phi) = y$, and $y^2 = c$), then $c(\pi)$ is the rank of the symmetric space ${}^\vee G(c)/K(y)$ (notation (6.6)). Obviously this rank depends only on the conjugacy class of y, and so on the infinitesimal character $\mathcal{O}_{\mathbb{R}}$. Proposition 9.2.12 of [30] shows that $c(\pi)$ is the same for all composition factors of a standard representation. The proof of that proposition actually shows

Proposition 7.7. *In the setting of Corollary 7.6, suppose that $\pi_i \in \Pi_{\phi_i}$, and that π_2 occurs as a composition factor of M_1. Then ϕ_1 and ϕ_2 have the same infinitesimal character (cf. (6.6)).*

In the case of complex groups, matters are somewhat simpler. If π is an irreducible admissible representation, then its space of smooth vectors is (by Proposition 7.4) a quasisimple representation of the *real* Lie algebra $\mathfrak{g}|_{\mathbb{R}}$. It is therefore a quasisimple representation of the complexified Lie algebra

(7.8)(a) $$\mathfrak{g}_{\mathbb{C}} = (\mathfrak{g}|_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Now $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to a product of two copies of $\mathfrak{g}$, so

(7.8)(b) $$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \simeq \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{g}).$$

Now Lemma 7.3 implies

Lemma 7.9. *Suppose G and ${}^\vee G$ are complex dual groups. Then there is a natural one-to-one correspondence between classical infinitesimal characters for the underlying real Lie algebra $\mathfrak{g}|_{\mathbb{R}}$ (cf. (7.8)) and pairs of semisimple orbits of ${}^\vee G$ on ${}^\vee\mathfrak{g}$.*

Proposition 7.10. *In the setting of Theorem 5.3, suppose $\phi \in P(G/\mathbb{C})$ is a Langlands parameter, and π_ϕ is the corresponding irreducible representation of G. Then the space of smooth vectors in π_ϕ is a quasisimple representation of $\mathfrak{g}|_{\mathbb{R}}$; write χ_ϕ for its classical infinitesimal character (cf. (7.1)(a) and (7.8)), and $\mathcal{O}_{\mathbb{C}} = (\mathcal{O}_1, \mathcal{O}_2)$ for the corresponding pair of semisimple orbits (Lemma 7.9). Then ϕ has infinitesimal character $\mathcal{O}_{\mathbb{C}}$ (cf. (5.6)).*

This may be proved in the same way as Proposition 7.4 (or even deduced from Proposition 7.4 and a trivial "base change" argument).

We turn now to the p-adic case. The difficulty over $\mathbb{R}$ was that the center of the enveloping algebra was a little too small to reflect completely the notion of infinitesimal character defined for Langlands parameters. In the p-adic case we have the opposite problem: the "Bernstein center" is a little too big for our purposes (since it distinguishes some L-indistinguishable representations).

Definition 7.11. *(see [5] or [6]). Suppose that G is a connected reductive algebraic group defined over an algebraic closure $\overline{F}$ of a p-adic field F of characteristic zero, and that σ is an F-rational form of G. A cuspidal pair for $G(F,\sigma)$ is a pair (M,ρ) with M an F-rational Levi subgroup of an F-rational parabolic subgroup $P = MN$ of G, and ρ a supercuspidal irreducible representation of $M(F,\sigma)$. (This means that the matrix coefficients of ρ are compactly supported modulo the center of $M(F,\sigma)$.) Two such pairs are called equivalent if they are conjugate by $G(F,\sigma)$. An equivalence class of cuspidal pairs is called a classical infinitesimal character for $G(F,\sigma)$.*

The set of equivalence classes is written $\Omega(G(F,\sigma))$. This set carries a natural structure of a complex affine algebraic variety with infinitely many components; each component is a quotient of a complex torus by a finite group action.

The Bernstein center $Z(G(F,\sigma))$ is the algebra of regular functions on the variety $\Omega(G(F,\sigma))$. Obviously

$$Max(Z(G(F,\sigma))) \simeq Hom_{alg}(Z(G(F,\sigma)),\mathbb{C}) \simeq \Omega(G(F,\sigma)).$$

Write $\mathcal{H}(G(F,\sigma))$ for the convolution algebra of compactly supported smooth densities on $G(F,\sigma)$; this is the *Hecke algebra of* $G(F,\sigma)$. If $h \in \mathcal{H}(G(F,\sigma))$ and π is an admissible representation of $G(F,\sigma)$, then we can define a finite rank operator

$$\pi(h) = \int_{G(F,\sigma)} h(g)\pi(g) \tag{7.12}$$

on the space of π. This defines a representation of the Hecke algebra.

Theorem 7.13. *([5] or [6]). There is a unique structure of $Z(G(F,\sigma))$-module on the Hecke algebra with the following property. Suppose $P = MN$ is a parabolic subgroup, and (M,ρ) is a cuspidal pair. Write $\omega \in \Omega(G(F,\sigma))$ for the equivalence class represented by (M,ρ). Define*

$$\pi = Ind_{P(F,\sigma)}^{G(F,\sigma)}(\rho),$$

an admissible representation of $G(F,\sigma)$. Then for any $h \in \mathcal{H}(G(F,\sigma))$ and $z \in Z(G(F,\sigma))$, we have

$$\pi(z\cdot h) = z(\omega)\pi(h).$$

This action of the Bernstein center on the Hecke algebra is analogous to the action of $\mathcal{Z}(\mathfrak{g})$ on test densities in the real case. One could formulate and prove a (much easier) parallel theorem in that case, with the difference that $\mathcal{Z}(\mathfrak{g})$ is a rather small part of the algebra of regular functions on the analogue of $\Omega(G(\mathbb{R},\sigma))$. (If one tries to replace $\mathcal{Z}(\mathfrak{g})$ by the full algebra of regular functions, then the analogue of Theorem 7.13 is false in the archimedean cases.)

Following the discussion at (7.1), we now say that an admissible representation π of $G(F,\sigma)$ is *quasisimple* if for every $z \in Z(G(F,\sigma))$, we have

$$\pi(z\cdot h) = \chi_\pi(z)\pi(h). \tag{7.14)(a}$$

In this case Definition 7.11 guarantees that there must be an $\omega_\pi \in \Omega(G(F,\sigma))$ so that

$$\chi_\pi(z) = z(\omega_\pi) \tag{7.14)(b}$$

for all z. We call ω_π the *classical infinitesimal character of* π.

Every irreducible representation is quasisimple. This is a consequence of the following much stronger result of Casselman and Bernstein-Zelevinsky.

Theorem 7.15. *([7]). In the setting of Definition 7.11, suppose π is an irreducible admissible representation of $G(F,\sigma)$. Then there is a parabolic subgroup $P = MN$ of G defined over F, and a cuspidal pair (M,ρ), so that π is a subrepresentation of $Ind_{P(F,\sigma)}^{G(F,\sigma)}(\rho)$. The equivalence class of (M,ρ) is unique.*

Theorem 7.13 implies that (just as in Proposition 7.5) any representation parabolically induced from an irreducible representation must be quasisimple. Arguing as for Corollary 7.6, we find

Corollary 7.16. *In the setting of Theorem 7.15, suppose π_1 and π_2 are irreducible admissible representations of $G(F,\sigma)$. Write $M_1 = M(\pi_1)$ for the standard representation of which π_1 is a quotient (cf. (1.3)(a)). If π_2 occurs as a composition factor of M_1 — equivalently, if $m(\pi_2, \pi_1) \neq 0$ (cf. (1.3)(b))) — then π_1 and π_2 must have the same classical infinitesimal character.*

Here the conclusion just means that π_1 and π_2 are subquotients of a common induced-from-supercuspidal representation.

We cannot prove an analogue of Proposition 7.4 in the p-adic case, but we would like to formulate a crude version of what it would say. It will be in the form of a desideratum for the correspondence of Conjecture 4.3. To begin with, we need to understand the set of infinitesimal characters (in the sense of Definition 1.10) in the p-adic case. Suppose that λ and λ' are as in (4.4)(a). We say that λ and λ' are *connected* if there is a homomorphism

$$\zeta_{\lambda,\lambda'} : W_F/I_F \to Z({}^\vee G^\lambda)_0, \qquad \lambda'(x) = \lambda(x)\zeta_{\lambda,\lambda'}(x). \tag{7.17)(a}$$

This assumption implies immediately that ${}^\vee G^\lambda = {}^\vee G^{\lambda'}$, and we deduce that connectedness is an equivalence relation. Each equivalence class is a principal homogeneous space for the complex torus

$$\mathrm{Hom}(\mathbb{Z}, Z({}^\vee G^\lambda)_0) \simeq Z({}^\vee G^\lambda)_0 \tag{7.17)(b}$$

(with λ any representative of the class), and so is an affine algebraic variety. Two infinitesimal characters $\mathcal{O}_F$ and $\mathcal{O}'_F$ are said to be *connected* if there is a connected pair $(\lambda, \lambda') \in \mathcal{O}_F \times \mathcal{O}'_F$. Connectedness is an equivalence relation on infinitesimal characters as well. The equivalence classes are called *connected components* of the set of infinitesimal characters. To see what they look like, fix $\lambda \in \mathcal{O}_F$. Every infinitesimal character in the connected component $\mathcal{C}(\mathcal{O}_F)$ of $\mathcal{O}_F$ has a representative in the connected component $\mathcal{C}(\lambda)$ of λ; we only need to understand which define the same infinitesimal characters. Define

$$\begin{aligned} {}^\vee G^{\mathcal{C}(\lambda)} &= \{\, g \in {}^\vee G \mid g \cdot \mathcal{C}(\lambda) = \mathcal{C}(\lambda) \,\} \\ &= \{\, g \in {}^\vee G \mid g \cdot \lambda \in \mathcal{C}(\lambda) \,\} \end{aligned} \tag{7.17)(c}$$

The connected component $\mathcal{C}(\mathcal{O}_F)$ may be identified with the set of orbits of ${}^\vee G^{\mathcal{C}(\lambda)}$ on $\mathcal{C}(\lambda)$. The identity component of ${}^\vee G^{\mathcal{C}(\lambda)}$ is ${}^\vee G_0^\lambda$, which acts trivially on $\mathcal{C}(\lambda)$; so we see finally that $\mathcal{C}(\mathcal{O}_F)$ is the quotient of a complex torus by a finite group action. We may therefore regard the set of all infinitesimal characters for ${}^\vee G^\Gamma$ as an affine algebraic variety having infinitely many components. In analogy with Definition 7.11, we write

$$\Omega({}^\vee G^\Gamma) = \text{variety of infinitesimal characters for } {}^\vee G^\Gamma \tag{7.17)(d}$$

$$\mathcal{Z}({}^\vee G^\Gamma) = \text{algebra of regular functions on } \Omega({}^\vee G^\Gamma). \tag{7.17)(e}$$

If ${}^\vee G^\Gamma$ is an L-group, we write $\Omega(G/F)$ and $\mathcal{Z}(G/F)$. We call $\mathcal{Z}(G/F)$ the *stable Bernstein center.*

Conjecture 7.18. *In the setting of Conjecture 4.3, suppose σ is a rational form of G^Γ (Definition 2.6). Then there is a natural finite map of algebraic varieties*

$$f : \Omega(G(F,\sigma)) \to \Omega(G/F),$$

having the following property. Suppose π is an irreducible representation of $G(F,\sigma)$ of classical infinitesimal character ω (cf. (7.14)), and that π belongs to an L-packet Π_ϕ (Conjecture 4.3). Then ϕ must have infinitesimal character $f(\omega)$.

If G is quasisplit, then the map f is surjective.

Equivalently, there should be a map from the stable Bernstein center $\mathcal{Z}(G/F)$ to the Bernstein center $\mathcal{Z}(G(F,\sigma))$. The map should be injective if $G(F,\sigma)$ is quasisplit, and should be an integral ring homomorphism in general. This immediately suggests the problem of characterizing the image. It is more or less the same as asking when two supercuspidal representations should belong to the same L-packet.

8. Kazhdan-Lusztig conjectures.

In this section we formulate again the main conjectures of the introduction, with the additional precision allowed by the discussion in section 2 and sections 4–6. For more detailed accounts of almost everything, we refer to [1]. We begin with the geometric category in terms of which everything will be formulated. We will be applying this definition with Y a geometric Langlands parameter space $X(\mathcal{O}_F, G/F)$ and H the dual group ${}^\vee G$.

Definition 8.1. *Suppose Y is a smooth complex algebraic variety on which the complex algebraic group H acts with finitely many orbits. Write $\mathcal{D}_Y$ for the sheaf of algebraic differential operators on Y ([9], VI.1). Define*

$\mathcal{D}(Y,H) =$ *category of H-equivariant coherent sheaves of $\mathcal{D}_Y$-modules on Y.*

The sheaves are taken in the Zariski topology. By [9], Theorem VII.12.11, the modules in this category are automatically regular holonomic. Sometimes it is convenient to consider instead

$\mathcal{P}(Y,H) =$ *category of H-equivariant perverse sheaves of complex vector spaces on Y,*

for which we refer to [4]. In this case it is necessary to work in the analytic topology. (A serious technical problem is that there is no complete treatment of H-equivariant perverse sheaves in the published literature. Because of the use of derived categories in the theory, this is not a routine generalization. We will ignore this difficulty, however.) The Riemann-Hilbert correspondence ([9], Theorem VIII.14.4) provides an equivalence of categories between $\mathcal{D}(Y,H)$ and $\mathcal{P}(Y,H)$.

These categories are abelian, and every object has finite length; in fact there are only finitely many irreducible objects. Consequently each category has a nice Grothendieck group, isomorphic to a lattice with basis corresponding to the irreducible objects. We write $K\mathcal{P}(Y,H)$ for either Grothendieck group, using the Riemann-Hilbert correspondence to identify them.

The next definition introduces the notation we need to describe $K\mathcal{P}(Y,H)$ more explicitly. Some of the teminology is chosen to fit with established usage for the Langlands classification.

Definition 8.2. *Suppose Y is a smooth complex algebraic variety on which the algebraic group H acts with finitely many orbits. A local geometric parameter for H acting on Y is a point of Y. Two such points are called equivalent if they differ by the action of H. The set of equivalence classes of local geometric parameters — that is, the set of orbits of H on Y — is written $\Phi(Y, H)$.*

Suppose $y \in Y$; write H_y for the isotropy group of the action of H at y. The local equivariant fundamental group at y is

$$A_y^{loc} = H_y/(H_y)_0; \tag{8.2)(a}$$

it is finite because H_y is algebraic. If y' is another point equivalent to y, then any element of H carrying y to y' induces an isomorphism

$$A_y^{loc} \simeq A_{y'}^{loc}. \tag{8.2)(b}$$

This isomorphism is unique up to inner automorphism of A_y^{loc}. If $S \in \Phi(Y, H)$ is an H-orbit, we may therefore define the equivariant fundamental group of S by

$$A_S^{loc} = A_y^{loc} \qquad (y \in S); \tag{8.2)(c}$$

this group is well-defined up to inner automorphism.

A local complete geometric parameter for H acting on Y is a pair (y, τ) with $y \in Y$ and $\tau \in (A_y^{loc})^\wedge$ an irreducible representation. Two such parameters (y, τ) and (y', τ') are called equivalent if y and y' belong to the same orbit and τ and τ' are identified by the isomorphisms (8.2)(c). The set of equivalence classes of local complete geometric parameters is written $\Xi(Y, H)$. We may therefore also think of an element of $\Xi(Y, H)$ as a pair $(S(\xi), \tau(\xi))$, with $S(\xi) \in \Phi(Y, H)$ an H-orbit and $\tau(\xi) \in (A_S^{loc})^\wedge$ an irreducible representation. We write

$$d(\xi) = \dim S(\xi) \tag{8.2)(d}$$

for the dimension of the H-orbit corresponding to $\xi \in \Xi(Y, H)$. The isotropy representations τ define an irreducible H-equivariant local system $\mathcal{V}(\xi)$ on $S(\xi)$. We may call the pair $(S(\xi), \mathcal{V}(\xi))$ a complete geometric parameter.

We will describe two examples of these definitions. The first is related to representations of $PGL(n)$ over a p-adic field, in the block of the trivial representation (Example 4.18). The second is related to representations of $U(p, q)$ over $\mathbb{R}$, again in the block of the trivial representation.

Example 8.3. Suppose H is the group of (complex) diagonal matrices of determinant one:

$$H \simeq \{\, h = (h_1, \dots, h_n) \in (\mathbb{C}^\times)^n \mid \prod z_i = 1 \,\}.$$

We take for Y the linear space of matrices with zeros everywhere except immediately above the main diagonal:

$$Y \simeq \{\, (y_1, \dots, y_{n-1}) \in \mathbb{C}^{n-1} \,\}.$$

The group H acts on Y by conjugation. In coordinates,

$$h \cdot y = ((h_1 h_2^{-1}) y_1, (h_2 h_3^{-1}) y_2, \dots, (h_{n-1} h_n^{-1}) y_{n-1}).$$

The orbits of H on Y are parametrized by subsets of $\{1,\dots,n-1\}$: if A is such a subset, then

$$S_A = \{\, y \in Y \mid y_i \neq 0 \Leftrightarrow i \in A \,\}.$$

If $y \in S_A$, then

$$H_y = \{\, h \in H \mid h_i = h_{i+1},\ \text{all } i \in A \,\}.$$

To calculate the component group of H_y, it is convenient to identify the subsets A of $\{1,\dots,n\}$ with partitions P of the sequence $(1,\dots,n)$ into "connected" subsequences, according to the rule that i and $i+1$ belong to the same subsequence if and only if $i \in A$. (Thus for example the subset $\{2,3\}$ of $\{1,2,3,4\}$ corresponds to the partition $((1)(2,3,4)(5))$ of the sequence $(1,2,3,4,5)$.) Now if y belongs to S_A, and A corresponds to P, then

$$H_y = \{\, h \in H \mid h \text{ is constant on the parts of } P \,\}.$$

Now suppose that the subsequences in P have lengths $\pi_1,\dots,\pi_r$. Then

$$H_y = \{\, h \in H \mid h = (z_1,\dots,z_1,z_2,\dots,z_2,\dots,z_r,\dots,z_r) \,\};$$

the repeated sequences have length π_1, π_2, and so on. To belong to H, such an element must have determinant one. Accordingly

$$H_y \simeq \{\, z \in (\mathbb{C}^\times)^r \mid z_1^{\pi_1}\cdots z_r^{\pi_r} = 1 \,\}.$$

Let d be the greatest common divisor of the π_j. By an elementary argument, the group

$$\{\, z \in (\mathbb{C}^\times)^r \mid z_1^{\pi_1/d}\cdots z_r^{\pi_r/d} = 1 \,\}$$

is connected. It must therefore be the identity component of H_y, and we get

$$A_y^{loc} \simeq \text{group of } d\text{th roots of unity in } \mathbb{C}^\times.$$

From this we can easily calculate the total number of complete geometric parameters in this example: it is

$$\sum_{d|n} 2^{(n/d)-1}\phi(d),$$

with ϕ the Euler phi-function.

Example 8.4. Suppose $H = O(n)$ is the group of complex orthogonal matrices, and Y is the variety of Borel subgroups of $GL(n)$. (Following the usual fear of regarding subgroups as points, we will actually write B_y for the Borel subgroup corresponding to a point $y \in Y$.) The relative position of any two Borel subgroups B and B' of $GL(n)$ is measured by a permutation $\sigma(B,B')$ of $\{1,\dots,n\}$. We have $\sigma(B,B') = 1$ if and only if $B = B'$, and $\sigma(B,B') = \sigma(B',B)^{-1}$. Write w_0 for the permutation that reverses $\{1,\dots,n\}$. Then the transpose antiautomorphism on G acts by conjugation by w_0 on relative positions:

$$(*) \qquad \sigma({}^tB, {}^tB') = w_0\sigma(B,B')w_0.$$

If $y \in Y$, write

$$\phi(y) = \sigma(B_y, {}^tB_y)w_0.$$

Because ${}^t k = k^{-1}$ for k in $O(n)$, $\phi(y)$ is an invariant of the $O(n)$ orbit of y. Because the transpose is an anti-automorphism, $(*)$ shows that $\phi(y)$ is an element of order two in the symmetric group. It turns out that this establishes a bijection

$$\text{(8.4)(a)} \qquad \Phi(Y, H) \leftrightarrow \text{permutations of order 2.}$$

Now a permutation of order 2 must be a product of disjoint 2-cycles; write $r(y)$ for the number of 2-cycles in $\phi(y)$, and $l(y)$ for the length of $\phi(y)$. Then it turns out that

$$\text{(8.4)(b)} \qquad H_y \simeq (O(1)^{n-2r(y)} \times SO(2)^{r(y)})N_y,$$

with N_y the unipotent radical, a group of dimension $(l(y) - r(y))/2$. It follows that the codimension of the orbit $H \cdot y$ in Y is $(l(y) + r(y))/2$, and that

$$\text{(8.4)(c)} \qquad A_y^{loc} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-2r(y)}.$$

(I am grateful to Roger Howe for enlightening discussions of this example.)

Definition 8.5. *Suppose Y is a smooth complex algebraic variety on which the complex algebraic group H acts with finitely many orbits. Suppose P is an H-equivariant perverse sheaf on Y, and $y \in Y$. Then the cohomology sheaves H^iP are H-equivariant constructible sheaves on Y, so the stalks $(H^iP)_y$ are finite-dimensional vector spaces carrying representations of A_y^{loc}. The equivariant local Euler characteristic of P at y is the formal alternating sum*

$$\text{(8.5)(a)} \qquad h_y^{loc}(P) = \sum_i (-1)^i (H^iP)_y,$$

a virtual representation of A_y^{loc}. The character of this virtual representation is a class function on A_y^{loc}, denoted $\chi_y^{loc}(P)$:

$$\text{(8.5)(b)} \qquad \chi_y^{loc}(P)(\sigma) = \sum_i (-1)^i tr(\sigma\, on\ (H^iP)_y) \qquad (\sigma \in A_y^{loc}).$$

In particular,

$$\text{(8.5)(c)} \qquad \chi_y^{loc}(P) = \chi_y^{loc}(P)(1) = \sum_i (-1)^i \dim(H^iP)_y$$

is the geometric local Euler characteristic considered in the introduction. If $\tau \in (A_y^{loc})^\wedge$, we will also write

$$\text{(8.5)(d)} \qquad \chi_{y,\tau}^{loc}(P) = \textit{multiplicity of } \tau \textit{ in } h_y^{loc}(P).$$

We may regard these as defined on A_S^{loc} (Definition 8.2(c)), writing in that case h_S^{loc}, χ_S^{loc}, and $\chi_{S,\tau}^{loc}$.

The local Euler characteristics can be described directly in terms of $\mathcal{D}_Y$-modules as well. This is perhaps most simply explained if we replace P by its Verdier dual DP. If then DP corresponds to an equivariant $\mathcal{D}_Y$-module $\mathcal{M}$ (Definition

8.1), then $(H^iP)_y$ *is the local solution space* $Ext^{-i}_{\mathcal{D}_Y}(\mathcal{M}, \hat{\mathcal{O}}_{Y,y})$*, which also carries a natural representation of* A^{loc}_y*. (If* $i = 0$*, this is the space of formal power series solutions at* y *of the system of differential equations represented by* $\mathcal{M}$*.) We omit the details.*

Finally, we will need a "microlocal Euler characteristic," which we will discuss only in its geometric form. (For the equivariant version, see [1], chapter 24.) Suppose P *is an* H*-equivariant perverse sheaf on* Y*. The characteristic cycle of* P *is a formal sum, with non-negative integer coefficients, of conormal bundles of orbits of* H *on* Y*. We write it as*

$$Ch(P) = \sum_{S \in \Phi(Y,H)} \chi^{mic}_S(P) T^*_S(Y).$$

(Actually this invariant is much easier to define on the level of $\mathcal{D}_Y$*-modules; we speak of perverse sheaves only for consistency with the discussion of* χ^{loc}_S*.*

Using the local Euler characteristics, it is easy to characterize the irreducible objects in $\mathcal{P}(Y, H)$.

Theorem 8.6. *(see [4], Théorème 4.3.1, or [9], Theorem VII.10.6). Suppose* Y *is a smooth complex algebraic variety on which the algebraic group* H *acts with finitely many orbits. Then the equivalence classes of irreducible objects in the category* $\mathcal{P}(Y, H)$ *(Definition 8.1) are in one-to-one correspondence with the set* $\Xi(Y, H)$ *of equivalence classes of local complete geometric parameters (Definition 8.2). The irreducible perverse sheaf* $P(\xi)$ *corresponding to* ξ *may be characterized (among all irreducible* H*-equivariant perverse sheaves) as follows.*

i) *The cohomology sheaves* H^iP *are supported on* $\overline{S}(\xi)$*; and*
ii) *the restriction of* H^iP *to* $S(\xi)$ *is the local system* $\mathcal{V}(\xi)$ *if* $i = -d(\xi)$*, and zero otherwise.*

In the language of Definition 8.6, this is equivalent to the following conditions.

i′) *The local Euler characteristic* $h^{loc}_{S'}(P(\xi)) = 0$ *unless* $S' \subset \overline{S(\xi)}$*; and*
ii′) $h^{loc}_{S(\xi)}(P(\xi)) = (-1)^{d(\xi)}\tau(\xi)$.

These conditions in turn are equivalent (still for P *irreducible) to*

i″) *The local Euler characteristic* $\chi^{loc}_{S',\tau'}(P(\xi)) = 0$ *unless* $S' \subset \overline{S(\xi)}$*;*
ii″) $\chi^{loc}_{S,\tau'}(P(\xi)) = 0$ *unless* $\tau = \tau'$*; and*
iii″) $\chi^{loc}_{S,\tau}(P(\xi)) = (-1)^{d(\xi)}$.

Actually it is probably easier to prove this theorem directly for $\mathcal{D}_Y$ modules than for perverse sheaves; we have used the perverse sheaf formulation only because the result stated in [4] is slightly closer to what we need than the one in [9].

For the purposes of Kazhdan-Lusztig theory, "understanding" a perverse sheaf P generally means (at least) calculating all the equivariant local Euler characteristics $h^{loc}_y(P)$. All such information is encoded in the *geometric character matrix for* H *acting on* Y. This is an array of integers indexed by $\Xi(Y, H) \times \Xi(Y, H)$, given by

(8.7)(a) $$c_g(\xi, \gamma) = (-1)^{d(\xi)} \chi^{loc}_{S(\xi),\tau(\xi)}(P(\gamma)) \qquad \xi, \gamma \in \Xi(Y, H).$$

According to Theorem 8.6,

$$c_g(\xi,\xi) = 1, \qquad c_g(\xi,\gamma) \neq 0 \text{only if } d(\xi) < d(\gamma) \qquad (\xi \neq \gamma). \tag{8.7)(b}$$

The matrix c_g is therefore upper triangular with ones on the diagonal.

Finally, it is helpful to have available some easy general nonsense about fiber products.

Proposition 8.8. *([1], Proposition 7.14). Suppose Y is a smooth complex algebraic variety on which the complex algebraic group H acts with finitely many orbits, and suppose G is an algebraic group containing H. Form the fiber product*

$$X = G \times_H Y.$$

a) X is a smooth complex algebraic variety on which G acts with finitely many orbits.

b) Every orbit of G on X meets Y in a single H orbit. Accordingly the inclusion of $i : Y \to X$ induces a bijection

$$i : \Phi(Y, H) \to \Phi(X, G).$$

c) For every $y \in Y$, the isotropy group of G at $i(y)$ is $G_{i(y)} = H_y$. Consequently

$$A^{loc}_{i(y)} = A^{loc}_y, \qquad \Xi(X, G) \simeq \Xi(Y, H).$$

d) There are natural equivalences of categories

$$\mathcal{D}(X, G) \simeq \mathcal{D}(Y, H), \qquad \mathcal{P}(X, G) \simeq \mathcal{P}(Y, H).$$

These are compatible with the parametrizations of irreducibles given by Theorem 8.6 and the bijections of (c).

e) The geometric character matrices for G acting on X and H acting on Y coincide: $c_g(i(\xi), i(\gamma)) = c_g(\xi, \gamma)$.

We turn now to the Kazhdan-Lusztig conjectures. Suppose F is a local field of characteristic zero, $\overline{F}$ is an algebraic closure of F, and G is a connected reductive algebraic group over $\overline{F}$. Fix an inner class of F-rational forms of G, a corresponding pure extended group $(G^\Gamma, \mathcal{W})$ (Definition 2.15), and an L-group $({}^\vee G^\Gamma, \mathcal{D})$ (Definition 3.25). Fix an infinitesimal character $\mathcal{O}_F$ for G/F (described in Definition 1.10 and section 4 in the p-adic case; in (5.6) in the complex case; and in (6.6) in the real case. "Define"

$$\Pi(\mathcal{O}_F, G/F) \tag{8.9)(a}$$

to be the set of equivalence classes of irreducible representations of rigid rational forms of infinitesimal character $\mathcal{O}_F$. (In the p-adic case, to say that a pair (π, δ) has infinitesimal character $\mathcal{O}_F$ means that the image of the Bernstein infinitesimal character $\omega_\pi \in \Omega(G(F, \delta))$ under the map f of Conjecture 7.18 is $\mathcal{O}_F$. As in the introduction, we use the quotation marks to signal the dependence of the definition on Conjecture 7.18. In the archimedean cases, the definition is complete because of Theorems 5.3 and 6.3.) Similarly, we define $\Pi_{pure}(\mathcal{O}_F, G/F)$ (Definition 2.14).

We want to define a *representation-theoretic multiplicity matrix* m_r indexed by $\Pi(\mathcal{O}_F, G/F)$. Suppose $\Theta, \Lambda \in \Pi(\mathcal{O}_F, G/F)$; choose representatives $(\pi(\Theta), \delta(\Theta))$ and $(\pi(\Lambda), \delta(\Lambda))$ for the equivalence classes. As in (1.3), we write $M(\Lambda)$ for the standard representation of $G(F, \delta(\Lambda))$ of which $\pi(\Lambda)$ is the Langlands quotient. We want $m_r(\Theta, \Lambda)$ to be in some sense the multiplicity of $(\pi(\Theta), \delta(\Theta))$ in $M(\Lambda)$. Of course the difficulty is that these are representations of different groups. First, if $\delta(\Theta)$ and $\delta(\Lambda)$ are inequivalent, we define $m_r(\Theta, \Lambda)$ to be zero. Next, suppose that $g \in G$ conjugates $\delta(\Theta)$ to $\delta(\Lambda)$. Then $\pi(\Theta) \circ \mathrm{Ad}(g^{-1})$ is a representation of $G(F, \delta(\Lambda))$, and we define

(8.9)(b) $$m_r(\Theta, \Lambda) = \text{multiplicity of } \pi(\Theta) \circ \mathrm{Ad}(g^{-1}) \text{ in } M(\Lambda).$$

It is easy to check that this definition is independent of the choice of g, because we are using rigid rational forms and not merely rational forms.

Define

(8.9)(c) $$K\Pi_{pure}(\mathcal{O}_F, G/F) = \text{ lattice with basis } \Pi_{pure}(\mathcal{O}_F, G/F).$$

This may be interpreted as a Grothendieck group in the following way. Pick representatives $\delta_1, \ldots, \delta_r$ for the equivalence classes of pure rational forms of G. Whenever σ is a rational form of G in our inner class, we can define

(8.9)(d) $$\mathcal{R}(\mathcal{O}_F, \sigma) = \text{category of finite length representations of } G(F, \sigma) \text{ of infinitesimal character } \mathcal{O}_F.$$

(In the p-adic case, we use smooth representations. In the archimedean case some care is required to get a nice abelian category — see [1], chapter 15.) Put

(8.9)(e) $$\mathcal{R}_{pure}(\mathcal{O}_F, G/F) = \sum_{i=1}^{r} \mathcal{R}(\mathcal{O}_F, \delta_i).$$

This is an abelian category in which every object has finite length. The irreducibles are parametrized by $\Pi_{pure}(\mathcal{O}_F, G/F)$, so there is a natural isomorphism

(8.9)(f) $$K\Pi_{pure}(\mathcal{O}_F, G/F) = \text{ Grothendieck group of } \mathcal{R}_{pure}(\mathcal{O}_F, G/F).$$

(The problem of finding a better description of the category $\mathcal{R}_{pure}(\mathcal{O}_F, G/F)$ is an interesting one. The analogy with Langlands parameters suggests that one should consider the space of all pure rational forms as some kind of variety with a G action; objects in $\mathcal{R}_{pure}(\mathcal{O}_F, G/F)$ should be regarded as infinite-dimensional equivariant vector bundles on that variety.)

Still in the setting of (8.9), recall from (4.7), Definition 5.12, and Definition 6.9 the geometric parameter space $X(\mathcal{O}_F, G/F)$ of infinitesimal character $\mathcal{O}_F$. This is a smooth complex algebraic variety on which the algebraic group ${}^\vee G$ acts with finitely many orbits. We may therefore apply the ideas of Definitions 8.1, 8.2, 8.5, and 8.7. We write

(8.10)(a) $$\mathcal{P}_{pure}(\mathcal{O}_F, G/F) = \mathcal{P}(X(\mathcal{O}_F, G/F), {}^\vee G)$$

for the category of equivariant perverse sheaves, and $K\mathcal{P}_{pure}(\mathcal{O}_F, G/F)$ for its Grothendieck group. Write

$$\Xi_{pure}(\mathcal{O}_F, G/F) = \Xi(X(\mathcal{O}_F, G/F), {}^\vee G) \tag{8.10)(b}$$

for the set of complete geometric parameters for ${}^\vee G$ acting on the geometric parameter space. Comparing Definitions 4.14, 5.1, and 6.2 with Definition 8.2, and applying Propositions 5.14 and 6.11, we find that $\Xi_{pure}(\mathcal{O}_F, G/F)$ may be identified with the set of equivalence classes of complete pure Langlands parameters of infinitesimal character $\mathcal{O}_F$. The basic Langlands conjecture (Conjecture 4.15, Theorem 5.3, and Theorem 6.3) therefore asks for a bijection

$$\Xi_{pure}(\mathcal{O}_F, G/F) \leftrightarrow \Pi_{pure}(\mathcal{O}_F, G/F), \qquad \xi \leftrightarrow \Lambda(\xi). \tag{8.10)(c}$$

We will write $(\pi(\xi), \delta(\xi))$ for a representative of the (conjectural) equivalence class $\Lambda(\xi)$, and $M(\xi)$ for the standard representation of $G(F, \delta)$ of which $\pi(\xi)$ is a quotient. Using this conjectural bijection, we may therefore transfer the representation-theoretic multiplicity matrix to $\Xi_{pure}(\mathcal{O}_F, G/F)$, defining

$$m_r(\xi, \gamma) = m_r(\Lambda(\xi), \Lambda(\gamma)) = \text{multiplicity of } \pi(\xi) \text{ in } M(\gamma) \tag{8.10)(d}$$

(see (8.9)(b)).

Now Theorem 8.6 says that each complete geometric parameter ξ also defines an irreducible ${}^\vee G$-equivariant perverse sheaf $P(\xi)$. Here is the simplest form of the Kazhdan-Lusztig conjecture.

Conjecture 8.11. *(cf. [32]). In the setting of (8.9)–(8.10), suppose ξ and γ are complete geometric parameters in $\Xi_{pure}(\mathcal{O}_F, G/F)$. Then the representation-theoretic multiplicity matrix (8.9)(b) and the geometric character matrix (8.7)(b) are related by*

$$m_r(\xi, \gamma) = (-1)^{d(\xi)} c_g(\gamma, \xi). \tag{8.11}$$

That is, the multiplicity of the irreducible representation $\pi(\xi)$ in the standard representation $M(\gamma)$ is (up to sign) the multiplicity of the representation $\tau(\gamma)$ in the local Euler characteristic of the perverse sheaf $P(\xi)$.

Even over $\mathbb{C}$ this conjecture is somewhat cleaner than the original Kazhdan-Lusztig conjecture, in that it provides a direct geometric interpretation of the multiplicity matrix even at singular infinitesimal character. (The original conjecture computes not multiplicities but irreducible characters, and the formula at a singular infinitesimal character has a sum with extensive cancellations.)

The varieties $X(\mathcal{O}_F, G/F)$ are very nice from the point of view of naturality, but they are unnecessarily large and cumbersome for calculations. Corollary 4.6, Proposition 5.7, and Proposition 6.10 provide fiber product decompositions of this variety. By Proposition 8.8, this reduces the calculation of c_g to the smaller variety (${}^\vee G^\lambda$ acting on ${}^\vee \mathfrak{g}_{q_F}^{\lambda(I_F)}$, or ${}^\vee G(c)$ acting on $\mathcal{P}_1 \times \mathcal{P}_2$, or $K(y)$ acting on $\mathcal{P}$, respectively). In the last two cases (corresponding to archimedean F) the matrices c_g (that is, the local Euler characteristics of intersection homology) are known. In the first case (corresponding to p-adic F) they are not; we refer to [32] for some examples.

As explained in the introduction, there is another formulation of Conjecture 8.11. The equivalence of the two forms is elementary.

Conjecture 8.11′. *In the setting of (8.9)–(8.10), "define" a perfect pairing between the lattices $K\Pi_{pure}(\mathcal{O}_F, G/F)$ and $K\mathcal{P}_{pure}(\mathcal{O}_F, G/F)$ by*

$$\langle \pi(\xi), P(\gamma)\rangle = e(\delta(\xi))(-1)^{d(\xi)}\delta_{\xi,\gamma}. \tag{8.11′(a)}$$

Here $e(\delta(\xi)) = \pm 1$ is the Kottwitz invariant of the pure rational form $\delta(\xi)$ (see [20]), $\delta_{\xi,\gamma}$ is a Kronecker delta, and the other terms are defined in (8.10). (The quotation marks indicate the dependence of this definition on the basic Langlands conjecture.) In this way every virtual representation of pure rational forms is identified with an additive (for short exact sequences) $\mathbb{Z}$-valued map on equivariant perverse sheaves. Then the standard representation $M(\xi)$ corresponds (up to sign) to the additive function sending a perverse sheaf P to the multiplicity of $\tau(\xi)$ in the local Euler characteristic $h^{loc}_{S(\xi)}(P)$. Explicitly,

$$\langle M(\xi), P\rangle = e(\delta(\xi))\chi^{loc}_{S(\xi),\tau(\xi)}(P). \tag{8.11′(b)}$$

From this point of view, the irreducible representation $P(\xi)$ corresponds (up to sign) to the additive function sending a perverse sheaf P to the multiplicity of $P(\xi)$ in the composition series of P. That is, the definition $(8.11')(a)$ may be rewritten as

$$\langle \pi(\xi), P\rangle = e(\delta(\xi))(-1)^{d(\xi)}(\text{multiplicity of } P(\xi) \text{ in } P). \tag{8.12}$$

One of the main theorems of [1] is that Conjecture 8.11 is true in the archimedean case. (Over $\mathbb{C}$ this is equivalent to the original Kazhdan-Lusztig conjecture, if one uses in addition the inversion formula for the Kazhdan-Lusztig polynomials.)

Although it is not strictly speaking Kazhdan-Lusztig theory, we will conclude this section with a restatement of Conjecture 1.13 (on strongly stable representations). Just as in (1.6), we can define the lattice of strongly stable virtual representations of infinitesimal character $\mathcal{O}_F$ of pure rational forms of G,

$$K\Pi_{pure}(\mathcal{O}_F, G/F)^{st} \tag{8.13}$$

Here is the standard conjecture on stable characters.

Conjecture 8.14. *(Langlands, Shelstad). Fix a Langlands parameter ϕ of infinitesimal character $\mathcal{O}_F$. For each $\tau \in (A^{loc}_\phi)^\wedge$, the pair (ϕ, τ) is a complete pure Langlands parameter, and so corresponds (conjecturally) to an irreducible representation $\pi(\phi,\tau)$ of a pure rational form $\delta(\tau)$. This in turn is a quotient of a standard representation $M(\phi,\tau)$. Then the formal sum of standard representations "defined" by*

$$M(\phi) = \sum_{\tau\in(A^{loc}_\phi)^\wedge} e(\delta(\phi,\tau))\dim(\tau)M(\phi,\tau) \tag{8.14}$$

is strongly stable (cf. (1.6)). As ϕ varies over $\Phi(\mathcal{O}_F, G/F)$, the virtual representations $M(\phi)$ form a basis for the lattice $K\Pi_{pure}(\mathcal{O}_F, G/F)^{st}$.

This has been proved by Shelstad in the archimedean case. Now Conjecture 8.11′ suggests that we should try to understand $M(\phi)$ as an additive map from

$K\mathcal{P}_{pure}(\mathcal{O}_F, G/F)$ to $\mathbb{Z}$. Write S for the orbit on $X(\mathcal{O}_F, G/F)$ corresponding to ϕ. Then (8.14) and $(8.11')(b)$ give

$$\langle M(\phi), P\rangle = \sum_{\tau\in(A_S^{loc})^\wedge} \dim(\tau)\chi_{S,\tau}^{loc}(P).$$

The sum on the left is just a geometric local Euler characteristic:

$$\langle M(\phi), P\rangle = \chi_S^{loc}(P)$$

(cf. (8.5)). This leads to the formulation of Conjecture 8.14 given in the introduction.

Conjecture 8.14$'$**.** *Suppose S is an orbit of ${}^\vee G$ on the geometric parameter space $X(\mathcal{O}_F, G/F)$. "Define" $M^{loc}(S)$ to be the virtual representation corresponding (in the pairing of Conjecture* 8.11$'$*) to the function χ_S^{loc} on equivariant perverse sheaves:*

$$\langle M(S), P\rangle = \chi_S^{loc}(P).$$

Then $M^{loc}(S)$ is strongly stable. As S varies over the orbits of ${}^\vee G$, these virtual representations form a basis for the lattice $K\Pi_{pure}(\mathcal{O}_F, G/F)^{st}$.

The lattice of strongly stable virtual representations is a sublattice of all virtual representations, and we know (conjecturally) the dual lattice of the larger lattice. It is therefore natural to try to characterize strongly stable representations as the annihilator of a sublattice of the Grothendieck group of perverse sheaves. Conjecture 8.14$'$ essentially does that. In the setting of Definition 8.5, let us say that a virtual perverse sheaf $P \in K\mathcal{P}(Y, H)$ is *stably trivial* if all the geometric local Euler characteristics $\chi_y^{loc}(P)$ are zero. Now Conjecture 8.14 may be reformulated as

Conjecture 8.15**.** *In the pairing of Conjecture* 8.11$'$*, the lattices of strongly stable virtual representations and of stably trivial virtual perverse sheaves are each other's annihilators. That is, M is a strongly stable virtual representation if and only if*

$$\langle M, P\rangle = 0$$

for every stably trivial virtual perverse sheaf P (and similarly with the rôles of M and P exchanged).

Finally, we recall from Definition 8.5 the $\mathbb{Z}$-valued additive function χ_S^{mic} on $K\mathcal{P}_{pure}(\mathcal{O}_F, G/F)$. Define M_S^{mic} (in terms of the conjectural pairing $(8.11')(a)$ to be the corresponding virtual representation. That is, M_S^{mic} is defined by the requirement that

$$\langle M_S^{mic}, P\rangle = \chi_S^{mic}(P). \tag{8.16)(a}$$

Using $(8.11')(a)$, we can calculate

$$M_S^{mic} = \sum_{\xi\in\Xi_{pure}(\mathcal{O}_F, G/F)} e(\delta(\xi)(-1)^{d(\xi)-\dim S}(\chi_S^{mic}(P(\xi))\pi(\xi). \tag{8.16)(b}$$

That is, M_S^{mic} is a sum of all the irreducible representations for which the corresponding irreducible perverse sheaf has the conormal bundle of S in its characteristic cycle. This collection of representations is called the *Arthur packet attached to the orbit S*. It contains the L-packet attached to S. As explained in the introduction, the index theorem in [15] establishes the equivalence of Conjecture 8.15 and

Conjecture 8.15′. *The virtual representations M_S^{mic} of (8.16) form a basis of the lattice of strongly stable virtual representations $K\mathcal{P}_{pure}(\mathcal{O}_F, G/F)^{st}$.*

9. Strong rational forms.

A serious shortcoming of the conjectures in section 8 is that they refer only to pure rational forms of G. As we saw in section 2 (Example 2.12, for instance) this is a significant restriction. If $G = SL(n)$ in the p-adic case, the only pure rational form is the split one. Now we also saw in Lemma 2.10 that the difference between pure rational forms and arbitrary ones is measured by the center $Z(G)$. This suggests that on the L-group side, the corresponding phenomenon is the failure of ${}^\vee G$ to be simply connected. If H is any connected complex algebraic group, we define

(9.1)(a) $$H^{alg} = \text{algebraic universal cover of } H;$$

this is the inverse limit of the (connected) algebraic covering groups of H. There is an exact sequence

(9.1)(b) $$1 \to \pi_1^{alg}(H) \to H^{alg} \to H \to 1,$$

with π_1^{alg} a pro-finite group (the *algebraic fundamental group of* H). The group H^{alg} is pro-algebraic, and there is no problem in extending the discussion in 8.1–8.8 to such groups. (For example, an action of H^{alg} on an algebraic variety just means an action of some finite cover of H.) Whenever ${}^\vee G^\Gamma$ is an E-group, we may therefore define

(9.1)(c) $$\mathcal{P}(\mathcal{O}_F, {}^\vee G^\Gamma) = \mathcal{P}(X(\mathcal{O}_F, {}^\vee G^\Gamma), {}^\vee G^{alg}),$$

the category of ${}^\vee G^{alg}$-equivariant perverse sheaves on the geometric parameter space. Irreducible objects in this category are parametrized by

(9.1)(d) $$\Xi(\mathcal{O}_F, {}^\vee G^\Gamma) = \Xi(X(\mathcal{O}_F, {}^\vee G^\Gamma), {}^\vee G^{alg}).$$

If we are in the setting of (8.9) and (8.10), we may write instead $\mathcal{P}(\mathcal{O}_F, G/F)$ and $\Xi(\mathcal{O}_F, G/F)$. Just as in (8.7), we get a *geometric character matrix*

(9.1)(e) $$c_g(\xi, \gamma) = (-1)^{d(\xi)} \chi^{loc}_{S(\xi),\tau(\xi)}(P(\gamma)) \qquad (\xi, \gamma \in \Xi(\mathcal{O}_F, {}^\vee G^\Gamma)).$$

This is an upper triangular matrix of integers with ones on the diagonal.

To identify these parameters in more classical terms, fix a Langlands parameter ϕ, and write as usual ${}^\vee G^\phi$ for the centralizer in ${}^\vee G$ of its image (Definitions 4.14, 5.1, and 6.2). Then we write

(9.2)(a) $$({}^\vee G^{alg})^\phi = \text{ preimage of } {}^\vee G^\phi \text{ in } {}^\vee G^{alg}$$

(9.2)(b) $$A_\phi^{loc,alg} = ({}^\vee G^{alg})^\phi / ({}^\vee G^{alg})^\phi_0,$$

the *universal component group for* ϕ. Evidently there is an exact sequence

(9.2)(c) $$\pi_1^{alg}({}^\vee G) \to A_\phi^{loc,alg} \to A_\phi^{loc} \to 1;$$

the first map need not be injective. A *complete Langlands parameter* is a pair (ϕ, τ) with ϕ a Langlands parameter and τ an irreducible representation of $A_\phi^{loc,alg}$. The group ${}^\vee G$ acts by conjugation on complete Langlands parameters; conjugate parameters are called *equivalent*. We write $\Xi({}^\vee G^\Gamma)$ for the set of equivalence classes. Again this is partitioned by infinitesimal characters. Suppose ϕ corresponds to a point $p(\phi)$ of a geometric parameter space. Because of Propositions 5.14 and 6.11 (and trivially in the p-adic case) there is a natural isomorphism

(9.2)(d) $$A_\phi^{loc,alg} \simeq \text{local equivariant fundamental group of } {}^\vee G^{alg} \text{ at } p(\phi).$$

It follows that the geometric parameters $\Xi(\mathcal{O}_F, {}^\vee G^\Gamma)$ of (9.1)(d) are in one-to-one correspondence with equivalence classes of complete Langlands parameters.

Now we have the L-group half of an extension of Conjecture 8.11. Here is what is wanted for the representation-theoretic half.

Problem 9.3. *Suppose G^Γ is a weak extended group. Find a natural class of rigid rational forms of G^Γ (Definition 2.6), to be called strong rational forms, and having the following properties.*

1) *Every pure rational form (Definition 2.6) is strong.*
2) *Every rational form in the inner class defined by G^Γ is represented by at least one strong rational form.*
3) *If δ and δ' are equivalent rigid rational forms, then both are strong or neither is.*
4) *Suppose Z is a Γ-stable algebraic central subgroup of G, and δ is a strong rational form of G^Γ. Then the natural quotient rigid rational form $\overline{\delta}$ of $(G^\Gamma)/Z$ is strong.*
5) *Suppose Γ_0 is an open subgroup of Γ of finite index, and δ is a strong rational form of G^Γ. Then the restriction δ_0 of δ to Γ_0 is a strong rational form of G^{Γ_0}.*
6) *Suppose F is archimedean. Then δ is strong if and only if $\delta(1) = 1$.*
7) *Suppose F is p-adic, $(G^\Gamma, \mathcal{W})$ is a pure extended group (Definition 2.15), and ${}^\vee G^\Gamma$ is a weak E-group for G (Definition 3.24). Then the equivalence classes of strong rational forms of G are naturally parametrized by the rational characters of the pro-finite group*
$$[Z({}^\vee G)^\Gamma]^{alg}/[Z({}^\vee G)^\Gamma]_0^{alg}.$$
Here $Z({}^\vee G)^\Gamma$ denotes the subgroup of $Z({}^\vee G)$ on which Γ acts trivially, and $[Z({}^\vee G)^\Gamma]^{alg}$ is its preimage in the algebraic universal cover ${}^\vee G^{alg}$.

Condition (6) defines strong rational forms in the archimedean case, and it is easy to check that the other requirements are satisfied. The condition in (7) is of course a natural extension of Kottwitz's formulation of Kneser's results (Proposition 4.16 above). Despite this very precise desideratum, in the p-adic case I have not been able to find a satisfactory definition of strong rational form. Assuming that this problem can be resolved, let us complete the extension of Conjecture 8.11 to strong rational forms.

Following Definition 2.14, we define a *representation of a strong rational form of* G^Γ to be a pair (π, δ), with δ a strong rational form (Problem 9.3), and π an admissible representation of $G(F, \sigma(\delta))$. Equivalence is defined as for representations of rigid rational forms (Definition 2.14), and we write $\Pi_{strong}(G^\Gamma)$ for the set of equivalence classes of irreducible representations of strong rational forms. Hence

(9.4)(a) $$\Pi_{pure}(G^\Gamma) \subset \Pi_{strong}(G^\Gamma) \subset \Pi(G^\Gamma).$$

In particular, we can now speak of a representation-theoretic multiplicity matrix indexed by $\Pi_{strong}(G^\Gamma)$ (cf. (8.9)). If we choose a collection of representatives $\{\delta_i \mid i \in I\}$ for the equivalence classes of strong rational forms, then

$$\Pi_{strong}(G^\Gamma) \simeq \bigcup_{i\in I} \Pi(G(F, \sigma(\delta_i))) \tag{9.4)(b}$$

(cf. (1.2)(b).) If $(G^\Gamma, \mathcal{W})$ is a pure extended group, we write $\Pi_{strong}(G/F)$, etc. In the setting of (8.9), we "define" $\Pi_{strong}(\mathcal{O}_F, G/F)$ to be the classes of representations of infinitesimal character $\mathcal{O}_F$. Set

$$K\Pi_{strong}(\mathcal{O}_F, G/F) = \text{free abelian group with basis } \Pi_{strong}(\mathcal{O}_F, G/F). \tag{9.4)(c}$$

(We say free abelian group instead of lattice because there may be infinitely many inequivalent strong rational forms. Even though each equivalence class contributes only finitely many irreducible representations of infinitesimal character $\mathcal{O}_F$, the complete set $\Pi_{strong}(\mathcal{O}_F, G/F)$ may be infinite.)

Conjecture 9.5. *Suppose we are in the setting of (8.9) and (9.4). Then there is a natural bijection*

$$\Xi(\mathcal{O}_F, G/F) \leftrightarrow \Pi_{strong}(\mathcal{O}_F, G/F), \qquad \xi \leftrightarrow \Lambda(\xi) \tag{9.5)(a}$$

between complete geometric parameters (for ${}^\vee G^{alg}$-equivariant irreducible perverse sheaves on the geometric parameter space) and equivalence classes of irreducible representations of strong rational forms of G^Γ. Write $(\pi(\xi), \delta(\xi))$ for a representative of the class $\Lambda(\xi)$. The group $[Z({}^\vee G)^\Gamma]^{alg}/[Z({}^\vee G)^\Gamma]_0^{alg}$ acts by scalars on any complete geometric parameter ξ (or on the corresponding irreducible equivariant $\mathcal{D}$-module or perverse sheaf). Write $\zeta(\xi)$ for the corresponding character. If F is p-adic, then the equivalence class of $\delta(\xi)$ should correspond to $\zeta(\xi)$ by condition (7) of Problem 9.3.

Use the conjectural bijection (9.5)(a) to transfer the representation-theoretic multiplicity matrix to $\Xi(\mathcal{O}_F, G/F)$ as in (8.10)(d). Then we should have

$$m_r(\xi, \gamma) = (-1)^{d(\xi)} c_g(\gamma, \xi) \tag{9.5)(b}$$

(cf. (8.11) and (9.1)(e)). Similarly, the analogues of Conjectures 8.11′, *8.14, and 8.15 should hold.*

In section 8 we discussed endoscopy only in its simplest aspect, the structure of stable characters. One would like to give a more complete discussion, along the lines of section 26 of [1]. (This refers only to the problems peculiar to non-tempered irreducible representations, and not to the more fundamental and harder problems associated with lifting standard representations. The results in [1] are complete only because Shelstad has solved those fundamental problems in the archimedean case.) Here is a very brief sketch of what is involved. In the setting of (8.9), an endoscopic group arises in the following way. Fix a semisimple element $s \in {}^\vee G$. Assume that the centralizer ${}^\vee G^\Gamma(s)$ of s in ${}^\vee G^\Gamma$ maps surjectively to Γ in the map of (3.24)(1):

$$1 \to {}^\vee G(s) \to {}^\vee G^\Gamma(s) \to \Gamma \to 1. \tag{9.6)(a}$$

We want to make a weak E-group out of ${}^\vee G^\Gamma(s)$ (Definition 3.24); the problem is that the reductive group ${}^\vee G(s)$ need not be connected. Write ${}^\vee H = {}^\vee G(s)_0$ for its identity component. Fix a subgroup ${}^\vee H^\Gamma \subset {}^\vee G^\Gamma(s)$ with the property that ${}^\vee H^\Gamma$ still maps surjectively to Γ, and the kernel of the map is ${}^\vee H$:

$$1 \to {}^\vee H \to {}^\vee H^\Gamma \to \Gamma \to 1. \tag{9.6)(b}$$

The choice of such a group ${}^\vee H^\Gamma$ amounts to the choice of a splitting of the finite extension

$$1 \to {}^\vee G(s)/{}^\vee H \to {}^\vee G^\Gamma(s)/{}^\vee H \to \Gamma \to 1 \tag{9.6)(c}$$

of Γ arising from (9.6)(a). (There may not be such a splitting in general.) The group ${}^\vee H^\Gamma$ is by construction a weak E-group; it is therefore a weak E-group for some pure extended group $(H^\Gamma, \mathcal{W}_H)$. The group H, with the inner class of rational forms defined by H^Γ, is an *endoscopic group* for G.

At any rate, the pair $(s, {}^\vee H^\Gamma)$ is part of what is called an *endoscopic datum*. Now the geometric formalism of section 8 does not require an L-group structure at all; we can introduce an infinitesimal character $\mathcal{O}_{F,H}$ and a geometric parameter space $X(\mathcal{O}_{F,H}, {}^\vee H^\Gamma)$. Applying fixed point formulas for the action of s (as in [1]) one can relate ${}^\vee G$-equivariant perverse sheaves on $X(\mathcal{O}_F, G/F)$ to ${}^\vee H$-equivariant perverse sheaves on $X(\mathcal{O}_{F,H}, {}^\vee H^\Gamma)$.

In order to give such relationships representation-theoretic content, we would like to invoke (8.10)(c). To do that, we need an L-group structure on ${}^\vee H^\Gamma$ (Definition 3.25). Unfortunately such a structure need not exist; so we must content ourselves with fixing an E-group structure $\mathcal{D}_H$ (Definition 3.27). The question then is what to put on the right side of (8.10)(c) when we have an E-group instead of an L-group on the left. The answer (suggested by Langlands) is representations of certain coverings of rational forms of H: more precisely, of the preimages of rational forms in algebraic covers of H. We therefore need a way to associate to an E-group a complex character of $\pi_1^{alg}(H)$. For the real case this is provided by [1], Theorem 10.4. In the p-adic case it leads to a problem parallel to Problem 9.3. I hope to return to these issues in a future paper.

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