

# SOME RESULTS ABOUT GEOMETRIC WHITTAKER MODEL

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**ABSTRACT.** Let  $G$  be an algebraic reductive group over a field of positive characteristic. Choose a parabolic subgroup  $P$  in  $G$  and denote by  $U$  its unipotent radical. Let  $X$  be a  $G$ -variety. The purpose of this paper is to give two examples of a situation in which the functor of averaging of  $\ell$ -adic sheaves on  $X$  with respect to a character  $\chi : U \rightarrow \mathbb{G}_a$  commutes with Verdier duality. In the first example we take  $X$  to be an arbitrary  $G$ -variety and we prove the above property for all  $\overline{P}$ -equivariant sheaves on  $X$  where  $\overline{P}$  is an opposite parabolic subgroup assuming  $\chi$  satisfies a strong nondegeneracy condition (such a  $\chi$  exists for some but not all choices of  $P$ ). In the case when  $P$  is a Borel subgroup it is enough to require that the sheaf in question is  $\overline{U}$  equivariant where  $\overline{U}$  is the unipotent radical of  $\overline{P}$ . In the second example we take  $X = G$  where  $G$  acts by left translations and we prove the corresponding result when  $P$  is a Borel subgroup for sheaves equivariant under the adjoint action of  $G$  (the latter result was conjectured by B. C. Ngo who proved it for  $G = GL(n)$ ). As an application we reprove a theorem of N. Katz and G. Laumon about local acyclicity of the kernel of the Fourier-Deligne transform.

## 1. INTRODUCTION

1.1. In this paper  $k$  will be an algebraically closed field of characteristic  $p > 0$ . We choose a prime number  $\ell$  which is different from  $p$ . By a sheaf on a  $k$ -scheme  $S$  we mean an  $\ell$ -adic etale sheaf. We denote by  $\mathcal{D}^b(S)$  the bounded derived category of such sheaves. For a complex  $\mathcal{F} \in \mathcal{D}^b(S)$  we denote by  ${}^p H^i(\mathcal{F})$  its  $i$ -th perverse cohomology. Recall that for any finite subfield  $k' \subset k$  and any non-trivial character  $\psi : k' \rightarrow \overline{\mathbb{Q}}_\ell^\times$  we can construct the Artin-Schreier sheaf  $\mathcal{L}_\psi$  on  $\mathbb{G}_{a,k}$ .

Let  $G$  denote a connected reductive group over  $k$ . We shall assume that  $p > h$  where  $h$  is the Coxeter number, so that for every unipotent subgroup  $U \subset G$  with Lie algebra  $\mathfrak{u}$  the exponential map  $\mathfrak{u} \rightarrow U$  is well defined and is an isomorphism. Under this condition  $\mathfrak{g}$  carries an invariant bilinear form  $(\ , \ )$ .

Let  $m : G \times G \rightarrow G$  be the multiplication map. For every  $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b(G)$  we shall denote by  $\mathcal{F} \star \mathcal{G}$  their "!"-convolution; in other words

$$\mathcal{F} \star \mathcal{G} = m_!(\mathcal{F} \boxtimes \mathcal{G}). \quad (1.1)$$

Similarly, we shall denote by  $\mathcal{F} * \mathcal{G}$  the "\*" -convolution of  $\mathcal{F}$  and  $\mathcal{G}$ , i.e.

$$\mathcal{F} * \mathcal{G} = m_*(\mathcal{F} \boxtimes \mathcal{G}). \quad (1.2)$$

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**1.2. Generic characters.** Let  $P \subset G$  be a parabolic subgroup of  $G$  with a Levi decomposition  $U_P \cdot L$ . Let  $\overline{P} = U_{\overline{P}} \cdot L$  be an opposite parabolic.

The quotient of  $U$  modulo commutant is isomorphic to a vector group  $\mathbb{G}_a^r$  canonically up to a linear automorphism: a choice of a maximal torus in  $L$  yields an isomorphism  $\prod_{\alpha \in \Delta_T^{\min}(U_P)} \mathbb{G}_a \cong U/[U, U]$ ; since any two maximal tori in  $L$  are conjugate

while the action of  $L$  on  $\mathbb{G}_a^r$  is easily seen to be linear, the linear structure does not depend on the maximal torus. All homomorphisms  $U \rightarrow \mathbb{G}_a^r$  considered in the paper will be assumed to come from a *linear* map  $\mathbb{G}_a^r \rightarrow \mathbb{G}_a$ .

We say that a homomorphism  $\chi : U_P \rightarrow \mathbb{G}_a$  is non-degenerate if for any parabolic  $Q$  conjugate to  $\overline{P}$  and such that  $Q \cap U_P \neq \{1\}$  the restriction  $\chi|_{Q \cap U_P}$  is nontrivial.

We have a (uniquely defined) element  $e_\chi \in \mathfrak{u}_{\overline{P}} = \text{Lie}(U_{\overline{P}})$  such that  $(e_\chi, x) = d\chi(x)$  for  $x \in \mathfrak{u} = \text{Lie}(U_P)$ .

Recall that an element in the radical  $\mathfrak{u}_Q$  of a parabolic subalgebra  $\mathfrak{q} = \text{Lie}(Q)$  is called *Richardson* if its  $Q$ -orbit is open in  $\mathfrak{u}_Q$ . It is well known [7, Theorem 1.3], [4, Theorem 7.1.1], that in this case the neutral component of the centralizer  $Z(e)^0$  is contained in  $Q$  and also that the preimage of  $e$  under the moment map  $\mu_Q : T^*(G/Q) \rightarrow \mathfrak{g}^* = \mathfrak{g}$  is finite. We say that  $e \in \mathfrak{u}_Q$  is *birationally Richardson* if  $Z(e) \subset Q$ . This is equivalent to the map  $\mu_Q$  being birational onto its image and also to the condition  $|\mu_Q^{-1}(e)| = 1$ .

**Lemma 1.3.** *The character  $\chi$  is non-degenerate if and only if the element  $e_\chi \in \mathfrak{u}_{\overline{P}}$  is birationally Richardson.*

*Proof.* Suppose that  $e_\chi$  is birationally Richardson and let  $Q$  be a parabolic conjugate to  $\overline{P}$  such that  $Q \cap U_P \neq \{1\}$ . We need to show that  $\chi|_{Q \cap U_P}$  is nontrivial. This property is invariant under conjugating  $Q$  by an element of  $U_P$ . It is well known that every  $U_P$ -orbit on a partial flag manifold contains a  $C_L$  fixed point where  $C_L$  is the center of  $L$ . Thus we can assume without loss of generality that  $Q \supset C_L$ . Then we have  $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{u}_P \oplus \mathfrak{g} \cap \mathfrak{p}$ . If  $\chi|_{Q \cap U}$  is trivial then  $e_\chi$  is orthogonal to  $\mathfrak{q} \cap \mathfrak{u}$ . It is also orthogonal to  $\mathfrak{q} \cap \mathfrak{p}$  by virtue of lying in  $\mathfrak{u}_{\overline{P}}$ . Thus  $e_\chi$  is orthogonal to  $\mathfrak{q}$ , i.e. it is contained in its radical which contradicts the assumption that it does not lie in any subalgebra conjugate to but different from  $\mathfrak{u}_{\overline{P}}$ .

Suppose now that  $e_\chi$  is not birationally Richardson. If it is not Richardson then parabolic subalgebras conjugate to  $\mathfrak{u}_{\overline{P}}$  and containing  $e_\chi$  correspond to points of a closed subvariety  $(G/\overline{P})_\chi$  in  $G/\overline{P}$  of positive dimension. Being projective, such a variety can not be contained in the affine open orbit of  $U$  on  $G/\overline{P}$ . If  $x \in (G/\overline{P})_\chi$  is not contained in the open orbit then the stabilizer  $Q$  of  $x$  satisfies  $\chi|_{Q \cap U_P} = 0$ .

Assume now that  $e_\chi$  is Richardson but not birationally Richardson. Then there exists a parabolic  $Q \neq P^-$  conjugate to  $P^-$  and such that  $e_\chi$  is in the radical of  $\mathfrak{q}$ . It remains to show that  $Q \cap U$  is nontrivial. If that was not the case then there exists  $u \in U_P$  such that  $Q = u\overline{P}u^{-1}$ . Then we claim that  $\text{Ad}(u)e_\chi = e_\chi$ . To check this it suffices to check that  $(e_\chi, x) = (\text{Ad}(u)e_\chi, x)$  for every  $x \in \mathfrak{g}$ . If  $x \in \mathfrak{u}_P$  this is clear

since  $e_\chi$  is orthogonal to the commutator of  $\mathfrak{u}_P$ . If  $x \in \mathfrak{q}$  then both sides vanish since  $e_\chi$  is in the radical of  $\mathfrak{q}$ . Since  $\mathfrak{g} = \mathfrak{u}_P \oplus \mathfrak{q}$  we see that  $e_\chi$  is invariant under  $Ad(u)$ . On the other hand, since  $e_\chi$  is Richardson its stabilizer in  $U$  is trivial, which is a contradiction.  $\square$

*Remark.* We now briefly discuss examples and non examples of the above situation.

0. If  $P = B$  is a Borel subgroup and  $\chi$  does not vanish on each simple root subgroup then  $\chi$  is clearly nondegenerate.

1. Let  $(e, f, h)$  be an  $sl(2)$  triple in  $\mathfrak{g}$  where the nilpotent  $e$  is assumed to be even. Suppose that  $\mathfrak{p} = \mathfrak{g}_{\leq 0}$ , the sum of eigenspaces of  $h$  with nonpositive eigenvalues (we assume that the characteristic  $p$  is large enough so that the  $sl(2)$  triple comes from a homomorphism  $SL(2) \rightarrow G$  and all weights of  $SL(2)$  in the adjoint representation are less than  $p$ ). Then it is clear that there exists a character  $\chi$  of  $U_P$  such that  $e = e_\chi$ . It is well known that  $e$  is birationally Richardson in this case.

2. The nilpotent radical of any parabolic contains Richardson elements, however, Richardson elements coming from a character of the radical of opposite parabolic (i.e. orthogonal to the commutant of the radical of an opposite parabolic) exist in some but not all cases. For example,<sup>1</sup> let  $G = SL(n)$  and let  $P$  be the group of block upper triangular matrices with blocks of sizes  $n_1, \dots, n_d$ . A Richardson element of this sort exists if and only if we have  $n_1 \leq n_2 \leq \dots \leq n_i \geq n_{i+1} \geq \dots \geq n_d$  for some  $i \in [1..d]$ . To see this notice that an element  $e$  is Richardson if and only if the Young diagram  $\lambda$  corresponding to its Jordan type is transposed to the diagram  $\mu$  of the partition  $n = n_1 + \dots + n_d$ . In particular, this implies that  $\dim(Ker(e)) = \max\{n_i\}$ . However,  $\dim(Ker(e)) \geq n_d + \sum_{i, n_i > n_{i+1}} n_i - n_{i+1}$  which is greater than  $\max\{n_i\}$  unless  $n_i$  satisfy

the above inequalities. This shows the inequalities are necessary for existence of a Richardson element of the specified form. To see they are sufficient consider the homogeneous ideal  $J$  in  $k[x, y]$  corresponding to the partition  $m_1 = \dots = m_i = n_i, m_j = n_j$  for  $j > i$ ; thus monomials  $x^i y^j, j \leq m_i$  form a basis in  $k[x, y]/J$ . The subspace  $V \subset k[x, y]/J$  spanned by monomials  $x^i y^j, j \leq n_i$  is clearly invariant under multiplication by  $x$ . The operator of multiplication by  $x$  is easily seen to have the Jordan form given by the diagram  $\lambda$ , while decomposing  $V = \oplus V_s$  where  $V_s$  is spanned by monomials  $x^s y^j$  we see that multiplication by  $x$  corresponds to a matrix with nonzero blocks right above block diagonal as required.

Recall the well known fact that if  $G$  is of type  $A$  then every Richardson element is birationally Richardson, thus the previous paragraph provides examples of nondegenerate characters of  $U_P \subset SL(n)$ .

3. Let  $G = Sp(6)$  and let  $P$  be the stabilizer of a line in the tautological representation. Then  $U_P$  is the 5-dimensional Heisenberg group. For any character  $\chi$  not vanishing on the 1-dimensional center of  $U_P$  the element  $e_\chi$  is Richardson but not birationally Richardson.

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<sup>1</sup>The example was explained to us by Zhiwei Yun.

1.4. Let  $X$  be a  $G$ -variety. Assume that  $U$  is a subgroup of  $G$  and  $\chi : U \rightarrow \mathbb{G}_a$  is a homomorphism. Let  $a : U \times X \rightarrow X$  denote the action map and let  $p : U \times X \rightarrow X$  be the projection to the second multiple. Let  $\mathrm{Av}_{U,\chi,*} : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)$  be the functor sending  $\mathcal{F} \in \mathcal{D}^b(X)$  to  $a_*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F}) (\overline{\mathbb{Q}}_l[1](\frac{1}{2}))^{\otimes \dim U}$ . Similarly we define the functor  $\mathrm{Av}_{U,\chi,!}$  by replacing  $a_*$  by  $a_!$ . We have the natural morphism  $\mathrm{Av}_{U,\chi,!} \rightarrow \mathrm{Av}_{U,\chi,*}$ . The main result of this paper is the following:

**Theorem 1.5.** *Let  $U \subset G$  be the unipotent radical of a parabolic subgroup  $P \subset G$  and let  $\chi : U \rightarrow \mathbb{G}_a$  be non-degenerate.*

- (1) *Let  $\overline{P}$  denote a parabolic subgroup of  $G$  opposite to  $P$  and let  $\overline{U}$  denote its unipotent radical. Let  $\mathcal{F} \in \mathcal{D}^b(X)$  be  $\overline{P}$ -equivariant. Then the natural morphism*

$$\mathrm{Av}_{U,\chi,!} \mathcal{F} \rightarrow \mathrm{Av}_{U,\chi,*} \mathcal{F}$$

*is an isomorphism.*

- (2) *Assume that  $P$  is a Borel subgroup. Then the above assertion is true for all  $\overline{U}$ -equivariant sheaves  $\mathcal{F}$  (not just  $\overline{P}$ -equivariant ones).*
- (3) *Let  $\mathcal{F} \in \mathcal{D}^b(G)$  be equivariant with respect to the adjoint action and assume that  $P$  is a Borel subgroup. Then the natural morphism*

$$\mathrm{Av}_{U,\chi,!} \mathcal{F} \rightarrow \mathrm{Av}_{U,\chi,*} \mathcal{F}$$

*is an isomorphism.*

*Remarks.*

0. In the above cases the averaging functors preserve perversity: the functor is defined as a direct image under an affine morphism, thus if  $\mathcal{F}$  is perverse then  $\mathrm{Av}_{U,\chi,!} \mathcal{F}$  is in perverse degrees  $\geq 0$ , while  $\mathrm{Av}_{U,\chi,*} \mathcal{F}$  is in perverse degrees  $\leq 0$ . Since the two complexes are isomorphic, they are perverse.

1. The second statement of Theorem 1.5 was communicated to the first author as a conjecture by B. C. Ngo who also proved it for  $G = GL(n)$ .

2. In the case  $G = GL(n)$  a (much more involved) analogue of Theorem 1.5(3) is used in [5] (Theorem 5.1) in order to complete the proof of the geometric Langlands conjecture for  $GL(n)$ .

3. In the next section we also explain how the main step in the proof of Theorem 1.5(1) allows to reprove one of the main results of [6].

4. Theorem 1.5 also holds, with a parallel proof, when  $k$  is an algebraically closed field of characteristic 0 and  $\ell$ -adic sheaves are replaced by holonomic  $\mathcal{D}$ -modules (in this case one has to replace  $\mathcal{L}_\psi$  by the  $\mathcal{D}$ -module corresponding to the function  $e^x$ ).

5. In the previous version of this paper [2] the statement of Theorem 1.5(1) was formulated for an arbitrary parabolic subgroup  $P$  of  $G$  under the weaker assumption that  $\mathcal{F}$  is  $\overline{U}$ -equivariant and with a different definition of a generic character. However, our proof of that assertion was wrong (unless  $P$  is a Borel subgroup) and the definition of a generic character irrelevant as was recently pointed out to us by Z. Yun. In

fact, the statement itself is wrong for any choice of the character as the following counterexample shows. Suppose that  $H \subset G$  is a subgroup such that

- (1)  $\dim(H \cap U) > 0$ .
- (2)  $H \cap U \subset \text{Ker}(\chi)$ .
- (3)  $\overline{U} \subset H$ .

If  $X = G$  and  $\mathcal{F}$  is the constant sheaf on  $H \subset G$  then the stalks of  $\text{Av}_{U,\chi,*}(\mathcal{F})$  and  $\text{Av}_{U,\chi,!}(\mathcal{F})$  at  $1 \in G$  are isomorphic to  $H^*(H \cap U)[\dim U] \left(\frac{\dim U}{2}\right)$  and  $H_c^*(H \cap U)[\dim U] \left(\frac{\dim U}{2}\right)$  respectively, thus the two averaging functors produce non-isomorphic complexes. An example of such a subgroup is as follows. Consider  $G = SL(3)$  and let  $P$  and  $\overline{P}$  be the groups of block upper triangular and block lower triangular matrices respectively with blocks of sizes  $(2, 1)$ . Let  $H$  be the group of block lower triangular matrices with blocks of sizes  $(1, 2)$ . Then conditions (1) and (3) are clear. Also notice that in this case  $U$  is abelian and all nontrivial characters of  $U$  are conjugate under the action of  $L = P/U$ , thus for any choice of the character (2) also holds possibly after replacing  $H$  by its conjugate by an element of  $L$ .

6. On the other hand, the present formulation of Theorem 1.5(1) may not be optimal: it was pointed out to us by Z. Yun that the statement may hold under a weaker assumption of the character. However, proving the more general statement requires new ideas and is beyond the scope of this document.

7. Originally Theorem 1.5(3) was formulated for any  $P$ , it was deduced from the above (stronger and incorrect) formulation of Theorem 1.5(1).

We do not know if the latter assertion is true. On the other hand, we believe that the following generalization of Theorem 1.5(3) holds. Assume for simplicity that  $p$  is sufficiently large, let  $U$  be a (unipotent) Premet group associated to a nilpotent orbit in  $\mathfrak{g}^*$  (cf. e.g. [9]). It comes with a natural character  $\chi : U \rightarrow \mathbb{G}_m$ .

**Conjecture 1.6.** (1) *Then the assertion of Theorem 1.5(3) holds for any Premet pair  $(U, \chi)$ .*

- (2) *Let  $U$  and  $\chi$  be as above. Then for any irreducible perverse sheaf  $\mathcal{F} \in \mathcal{D}^b(G)$  equivariant with respect to the adjoint action,  $\text{Av}_{U,\chi,!}\mathcal{F}$  is an irreducible perverse sheaf or zero.*

## 2. PROOF OF THEOREM 1.5(1)

**2.1. Cleanness.** Let  $Z$  be an algebraic variety over  $k$  and let  $j : Z_0 \rightarrow Z$  be an open embedding. We shall say that  $\mathcal{G} \in \mathcal{D}^b(Z_0)$  is *clean with respect to  $j$*  if the natural map  $j_!\mathcal{G} \rightarrow j_*\mathcal{G}$  is an isomorphism.

Let  $X$  be any  $\overline{P}$ -variety. Consider the variety  $G \times_{\overline{P}} X$ . We have the natural open embedding  $j : U \times X \rightarrow G \times_{\overline{P}} X$ . We will prove Theorem 1.5(1) by a series of reductions. We claim that Theorem 1.5(1) follows from

**Theorem 2.2.** *Let  $X$  be a  $\overline{P}$ -variety and let  $\mathcal{F} \in \mathcal{D}^b(X)$  be  $\overline{P}$ -equivariant. Then the sheaf  $\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F}$  is clean with respect to  $j$ . In other words the natural morphism*

$$j_!(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F}) \rightarrow j_*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F}) \quad (2.1)$$

*is an isomorphism. The same thing is true under the assumption that  $\mathcal{F}$  is  $\overline{U}$ -equivariant if  $P$  is a Borel subgroup.*

**2.3. Theorem 2.2 implies Theorem 1.5(1).** Indeed if  $X$  is a  $G$ -variety then we have the natural proper map  $b : G \times_{\overline{P}} X \rightarrow X$  sending every  $(g, x) \bmod \overline{P}$  to  $g(x)$ . Moreover, we have  $b \circ j = a$  (recall that  $a : U \times X \rightarrow X$  denotes the action map). Hence Theorem 2.2 and the fact that  $b$  is proper imply that

$$\mathrm{Av}_{U, \chi, !} \mathcal{F} = b_!(j_!(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F})) = b_*(j_*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F})) = \mathrm{Av}_{U, \chi, *} \mathcal{F}.$$

It remains to prove Theorem 2.2. Note that in the formulation of Theorem 2.2 we do not need  $X$  to be a  $G$ -variety but only a  $\overline{P}$ -variety.

**2.4. A reformulation of the Theorem 2.2.** Let  $\pi : G \times X \rightarrow G \times_{\overline{P}} X$  be the natural projection. Also let  $\tilde{j} : U \cdot \overline{P} \times X \rightarrow G \times X$  be the natural embedding. It follows from the smooth base change theorem that it is enough to show that the natural map

$$\tilde{j}_! \pi^*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F}) \rightarrow \tilde{j}_* \pi^*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F})$$

is an isomorphism (note that we have the natural identification  $U \cdot \overline{P} \times X$  with  $\pi^{-1}(U \times X)$ ).

The sheaf  $\pi^*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F})$  is obviously  $(U, \chi)$ -equivariant with respect to the  $U$ -action by multiplication on the left. We claim that it is also  $\overline{P}$ -equivariant with respect to multiplication on the right, i.e. with respect to the  $\overline{P}$ -action on  $U \cdot \overline{P} \times X$  given by  $\overline{q} : (u, \overline{p}, x) \mapsto (u, \overline{p}\overline{q}, x)$ . Indeed, the map  $\pi$  from  $U \cdot \overline{P} \times X$  to  $U \times X$  is given by  $\pi : (u, \overline{p}, x) \mapsto (u, \overline{p}(x))$  (since the action of  $\overline{P}$  on  $G \times X$  is given by  $\overline{p} : (g, x) \mapsto (g\overline{p}^{-1}, \overline{p}x)$ ). Thus

$$\pi(u, \overline{p}\overline{q}, x) = (u, \overline{p}\overline{q}(x)) = (u, \overline{p}\overline{q}\overline{p}^{-1}(\overline{p}(x)))$$

and our statement follows from  $\overline{P}$ -equivariance of  $\mathcal{F}$ .

Similar argument shows that if  $\mathcal{F}$  is only  $\overline{U}$ -equivariant, then  $\pi^*(\chi^* \mathcal{L}_\psi \boxtimes \mathcal{F})$  is  $\overline{U}$ -equivariant with respect to the right multiplication.

Hence we see that Theorem 2.2 follows from the following lemma.

**Lemma 2.5.** *Consider the action of  $U \times \overline{P}$  on  $U \cdot \overline{P} \subset G$  given by left and right multiplications. For any variety  $X$ , if  $\mathcal{G} \in \mathcal{D}^b(U \cdot \overline{P} \times X)$  is  $(U, \chi)$ -equivariant on the left and  $\overline{P}$ -equivariant on the right, then the natural map given by the inclusion  $\tilde{j} : U \cdot \overline{P} \times X \rightarrow G \times X$ ,*

$$\tilde{j}_! \mathcal{G} \rightarrow \tilde{j}_* \mathcal{G},$$

is an isomorphism. Similar assertion holds if  $\mathcal{F}$  is only  $U \times \overline{U}$  equivariant if  $P$  is a Borel subgroup.

*Proof.* Let  $Z$  denote the complement of  $U \cdot \overline{P}$  in  $G$  and let  $i$  be the natural embedding of  $Z \times X$  to  $G \times X$ . Since  $\tilde{j}_* \mathcal{G}$  is also  $(U, \chi)$ -equivariant on the left and  $\overline{U}$ -equivariant on the right it is enough to show that for every complex  $\mathcal{H}$  on  $G \times X$  with the above equivariance properties we have  $i^* \mathcal{H} = 0$ . However, it is clear that this follows from:

**Lemma 2.6.** (1) Let  $g \in Z$ . Let  $S_g \subset U \times \overline{P}$  denote the set of all pairs  $(u, \overline{p})$  such that  $ug\overline{p} = g$ . Let also  $U_g$  be the projection of  $S_g$  to  $U$ . Then the restriction of  $\chi$  to  $U_g$  is non-trivial.  
(2) Similarly, assume that  $P$  is a Borel subgroup. Let  $g \in Z$ . Let  $S'_g \subset U \times \overline{U}$  denote the set of all pairs  $(u, \overline{u})$  such that  $ug\overline{u} = g$ . Let also  $U'_g$  be the projection of  $S'_g$  to  $U$ . Then the restriction of  $\chi$  to  $U'_g$  is non-trivial.

*Proof.* The first assertion is clear from the definition of a generic character.

If  $P = B$  then Theorem ??(1) applies, see Remark 0 after Lemma ??. Also, in this case  $\overline{U}$  coincides with the set of unipotent elements in  $\overline{P}$ , thus we have:  $S_g = S'_g$  and  $U_g = U'_g$ . Thus (2) follows from (1).  $\square$

$\square$

**Corollary 2.7.** Let  $U$  be the unipotent radical of a Borel subgroup  $B$ . Let  $j$  denote the open embedding of  $B$  into  $G/\overline{U}$ . Let  $\mathcal{F}$  be any  $(U, \chi)$ -equivariant sheaf on  $B$  (with respect to the left multiplication action). Then the natural morphism

$$j_! \mathcal{F} \rightarrow j_* \mathcal{F}$$

is an isomorphism. In other words, every  $(U, \chi)$ -equivariant sheaf on  $B$  is clean with respect to  $j$ .

*Proof.* Let  $L$  be the Levi factor of  $B$  (since  $B$  is now assumed to be a Borel subgroup,  $L$  is a maximal torus of  $G$ ). The isomorphism  $L \simeq \overline{B}/\overline{U}$  gives rise to a natural action of  $\overline{B}$  on  $L$ . Since the action of  $\overline{U}$  on  $L$  is trivial it follows that every  $\mathcal{F} \in \mathcal{D}^b(L)$  is automatically  $\overline{U}$ -equivariant.

We have the natural identifications  $U \times L \simeq B$  (by multiplication map) and  $G \times L \simeq \overline{B}$  (sending every  $(g, l) \bmod \overline{B}$  to  $gl \bmod \overline{U}$ ). Under these identification the embedding  $j : B \rightarrow G/\overline{U}$  becomes equal to the natural embedding  $U \times L \rightarrow G \times L$  considered in Theorem 2.2 (for  $X = L$ ). Also the fact that  $\mathcal{F}$  is  $(U, \chi)$ -equivariant implies that as a sheaf on  $U \times L$  it can be decomposed as  $\mathcal{F} = \chi^* \mathcal{L}_\psi \boxtimes \mathcal{F}'$  for some  $\mathcal{F}' \in \mathcal{D}^b(L)$ . Hence Corollary 2.7 is a particular case of Theorem 2.2.  $\square$

**2.8. Application to Katz-Laumon theorem.** Consider the variety  $\mathbb{A}^1 \times \mathbb{G}_m$  with coordinates  $(x, y)$ . Let  $f : \mathbb{A}^1 \times \mathbb{G}_m \rightarrow \mathbb{A}^1$  be given by  $f(x, y) = \frac{x}{y}$ . Let also

$i : \mathbb{A}^1 \times \mathbb{G}_m \rightarrow \mathbb{A}^2$  denote the natural embedding and let  $\pi : \mathbb{A}^1 \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  be the projection to the second variable. The following theorem is proved in [6].

**Theorem 2.9.** *For every  $\mathcal{F} \in \mathcal{D}(\mathbb{G}_m)$  the natural map*

$$i_!(f^* \mathcal{L}_\psi \otimes \pi^* \mathcal{F}) \rightarrow i_*(f^* \mathcal{L}_\psi \otimes \pi^* \mathcal{F}) \quad (2.2)$$

*is an isomorphism.*

Below we explain that Theorem 2.9 may be viewed as a particular case of Corollary 2.7.

*Proof.* Take now  $G = SL(2)$  and let  $B$  and  $\overline{B}$  be respectively the subgroups of lower-triangular and upper-triangular matrices, with unipotent radicals  $U$  and  $\overline{U}$ . We denote the natural isomorphism between  $U$  and  $\mathbb{G}_a$  by  $\chi$ .

Let us identify  $G/\overline{U}$  with  $\mathbb{A}^2 \setminus \{0\}$  by  $g\overline{U} \mapsto g(e_1)$  for the first standard basis vector  $e_1$  of  $\mathbb{A}^2$ . Then  $B \subset G/\overline{U}$  is identified with  $\mathbb{A}^1 \times \mathbb{G}_m \subset \mathbb{A}^2 \setminus \{0\}$  by  $\begin{pmatrix} \lambda & 0 \\ t & \lambda^{-1} \end{pmatrix} \leftrightarrow (t, \lambda^{-1})$ . The sheaf  $f^* \mathcal{L}_\psi \otimes \pi^* \mathcal{F}$  is  $(U, \chi)$ -equivariant, so by Corollary 2.7 this sheaf is clean for the embedding  $\mathbb{A}^1 \times \mathbb{G}_m \subset \mathbb{A}^2 \setminus \{0\}$ . It remains to observe that the resulting sheaf on  $\mathbb{A}^2 \setminus \{0\}$  is clean for the embedding into  $\mathbb{A}^2$  since the cone of the canonical map between the shriek and star direct images is zero – it is a  $(U, \chi)$ -equivariant sheaf supported at a point  $\{0\}$ .  $\square$

### 3. PROOF OF THEOREM 1.5(3)

In this Section we will eventually need to assume that  $P$  is a Borel subgroup. However, let us start working with arbitrary  $P$  and make the above assumption later.

**3.1. Horocycle transform.** Let  $P$  be a parabolic subgroup in  $G$  and let  $Y_P$  denote the variety of all parabolic subgroups of  $G$  which are conjugate to  $P$ . We also denote by  $W_P$  the variety of  $P$ -horocycles, i.e., the pairs  $(Q \in Y_P, x \in G/U_Q)$  where  $U_Q$  denotes the unipotent radical of  $Q$  (see section Section 3.3 below for a more direct definition of  $W_P$ ). We have the natural map  $p : W_P \rightarrow Y_P$ .

We also have the natural morphisms  $\alpha : G \times Y_P \rightarrow G$  and  $\beta : G \times Y_P \rightarrow W_P$  where  $\alpha$  is just the projection to the first multiple and  $\beta$  sends  $(g, Q)$  to  $(Q, g \bmod U_Q)$ . We define two functors  $\mathcal{R}_P : \mathcal{D}^b(G) \rightarrow \mathcal{D}^b(W_P)$  and  $\mathcal{S}_P : \mathcal{D}^b(W_P) \rightarrow \mathcal{D}^b(G)$  by setting

$$\mathcal{R}_P(\mathcal{F}) = \beta_! \alpha^*(\mathcal{F}) \otimes \left( \overline{\mathbb{Q}}_l[1](\frac{1}{2}) \right)^{\otimes \dim U_P}$$

and

$$\mathcal{S}_P(\mathcal{G}) = \alpha_! \beta^*(\mathcal{G}) \otimes \left( \overline{\mathbb{Q}}_l[1](\frac{1}{2}) \right)^{\otimes \dim U_P}.$$

The following lemma is proved in [8] when  $Q$  is a Borel subgroup in  $G$ .

**Lemma 3.2.** *The identity functor is a direct summand of  $\mathcal{S}_P \circ \mathcal{R}_P$ .*



*Proof.* Let  $\mathcal{T}_Q = \{(Q \in Y_P, u \in U_Q)\}$ . We have the natural map  $p_P : \mathcal{T}_Q \rightarrow G$  sending every  $(Q, u)$  to  $u$  (clearly the image of  $p_P$  lies in the set of unipotent elements in  $G$ ). Let  $\mathbf{Spr}_P = (p_P)_! \overline{\mathbb{Q}}_l[2 \dim Y_P](\dim Y_P)$ . It is known (cf. [3]) that  $\mathbf{Spr}_P$  is perverse and that it contains the skyscraper sheaf  $\delta_e$  at the unit element  $e \in G$  as a direct summand. We set  $\mathbf{Spr}_P = \delta_e \oplus \mathbf{Spr}'_P$ .

On the other hand, arguing as in [8] we can show that for every  $\mathcal{F} \in \mathcal{D}^b(G)$  we have a canonical isomorphism

$$\mathcal{S}_P \circ \mathcal{R}_P(\mathcal{F}) = \mathcal{F} \star \mathbf{Spr}_P. \quad (3.1)$$

Hence

$$\mathcal{S}_P \circ \mathcal{R}_P(\mathcal{F}) = \mathcal{F} \oplus (\mathcal{F} \star \mathbf{Spr}'_P) \quad (3.2)$$

which finishes the proof.  $\square$

**3.3. Another definition of  $W_P$ .** One can identify  $W_P$  with  $(G/U_P \times G/U_P)/M$  where  $M = P/U$  acts on  $G/U_P \times G/U_P$  diagonally. The identification is given by the map

$$(x_1 \bmod U_P, x_2 \bmod U_P) \mapsto (x_2 P x_2^{-1}, x_1 x_2^{-1} \bmod U_P).$$

Under this identification the natural left and right  $G$ -actions on  $(G/U_P \times G/U_P)/M$  give two actions of  $G$  on  $W_P$ , which we still call the "left" and "right" action. The left action is just the natural  $G$ -action in the fibers of  $p$ . The right action is given by

$$g : (Q, x) \mapsto (gQg^{-1}, xg^{-1} \bmod gU_Qg^{-1}).$$

The corresponding adjoint action is given by

$$g : (Q, x) \mapsto (gQg^{-1}, gxg^{-1} \bmod gU_Qg^{-1}).$$

We now claim the following

Assume now that we could prove that for any  $\mathcal{G} \in \mathcal{D}^b(W_{\overline{P}})$  which is equivariant with respect to the adjoint action and for every non-degenerate character  $\chi : U \rightarrow \mathbb{G}_a$  the natural map

$$\mathrm{Av}_{U, \chi, !} \mathcal{G} \rightarrow \mathrm{Av}_{U, \chi, *} \mathcal{G} \quad (3.3)$$

is an isomorphism. Then the same would be true for any  $\mathcal{F} \in \mathcal{D}^b(G)$  which is equivariant with respect to the adjoint action. the map

$$\mathrm{Av}_{U, \chi, !} \mathcal{F} \rightarrow \mathrm{Av}_{U, \chi, *} \mathcal{F} \quad (3.4)$$

would be an isomorphism. Indeed, since by Lemma 3.2  $\mathcal{F}$  is a direct summand of  $\mathcal{S}_{\overline{P}} \circ \mathcal{R}_{\overline{P}}(\mathcal{F})$  it is enough to show that (3.4) holds for the latter. It follows from the fact that  $\alpha$  is a proper morphism that we have the natural isomorphisms of functors

$$\mathrm{Av}_{U, \chi, !} \circ \mathcal{S}_{\overline{P}} \simeq \mathcal{S}_{\overline{P}} \circ \mathrm{Av}_{U, \chi, !} \text{ and } \mathrm{Av}_{U, \chi, *} \circ \mathcal{S}_{\overline{P}} \simeq \mathcal{S}_{\overline{P}} \circ \mathrm{Av}_{U, \chi, *}.$$

Hence it is enough to show that (3.4) holds for  $\mathcal{R}_{\overline{P}}(\mathcal{F})$ . However, it is clear that  $\mathcal{R}_{\overline{P}}$  maps ad-equivariant complexes to ad-equivariant ones which finishes the proof by (3.3).

We do not know if (3.3) is indeed true for arbitrary  $P$ . However, we claim the following:

**Theorem 3.4.** *Let  $P$  be a Borel subgroup in  $G$  and let  $U$  be its unipotent radical. Let  $\mathcal{G} \in \mathcal{D}^b(W_{\overline{P}})$  be equivariant with respect to the adjoint action. Then for every non-degenerate character  $\chi : U \rightarrow \mathbb{G}_a$  the natural map*

$$\mathrm{Av}_{U,\chi,!}\mathcal{G} \rightarrow \mathrm{Av}_{U,\chi,*}\mathcal{G}$$

*is an isomorphism (here averaging is performed with respect to the left action).*

3.5. The rest of this section is occupied by the proof of Theorem 3.4.

Let  $Y_{\overline{P}}^0$  denote the open  $U$ -orbit on  $Y_{\overline{P}}$  and let  $W_{\overline{P}}^0$  denote its preimage in  $W_{\overline{P}}$ .

First of all we claim that both  $\mathrm{Av}_{U,\chi,*}\mathcal{G}$  and  $\mathrm{Av}_{U,\chi,!}\mathcal{G}$  are equal to the extension by zero of their restriction to  $W_{\overline{P}}^0$ . Indeed we must show that the  $*$ -restriction of either of these sheaves to the fiber of  $p : W_{\overline{P}} \rightarrow Y_{\overline{P}}$  over any parabolic  $Q$  which is not opposite to  $P$  is equal to zero. Let us denote this restriction by  $\mathcal{H}$ . This is a complex of sheaves on  $p^{-1}(Q) = G/U_Q$ . The fact that  $\mathcal{G}$  is equivariant with respect to the adjoint action implies that both  $\mathrm{Av}_{U,\chi,*}\mathcal{G}$  and  $\mathrm{Av}_{U,\chi,!}\mathcal{G}$  are equivariant with respect to the adjoint action of  $U$ . Hence  $\mathcal{H}$  is equivariant with respect to the left action of  $U \cap U_Q$ . On the other hand, it is clear that  $\mathcal{H}$  is  $(U, \chi)$ -equivariant with respect to the left action of  $U$ . Thus our statement follows from the following result which is equivalent to Lemma 2.6: let  $Q$  be as above (i.e.  $Q$  is conjugate to  $\overline{P}$  but it is not in the generic position with respect to  $P$ ); then the restriction of  $\chi$  to  $U \cap U_Q$  is non-trivial.

It remains to show that the map  $\mathrm{Av}_{U,\chi,!}\mathcal{G} \rightarrow \mathrm{Av}_{U,\chi,*}\mathcal{G}$  is an isomorphism when restricted to  $W_{\overline{P}}^0$ .

The map  $u \mapsto u\overline{P}u^{-1}$  is an isomorphism between  $U$  and  $Y_{\overline{P}}^0$ . Let  $\kappa : W_{\overline{P}}^0 \rightarrow U$  be the composition of the natural projection  $W_{\overline{P}}^0 \rightarrow Y_{\overline{P}}^0$  with this isomorphism. Define now a new  $G$ -action on  $W_{\overline{P}}^0$  (denoted by  $(g, w) \mapsto g \times w$ ) by

$$g \times w = \kappa(w)g\kappa(w)^{-1}(w)$$

(in the right hand side we use the standard left action of  $G$  on  $W_{\overline{P}}$ ).

To finish the argument we need the following general (and basically tautological) result:

**Lemma 3.6.** *a) Let  $H$  be an algebraic group, and  $X$  be an algebraic variety equipped with two actions  $\phi_1, \phi_2$  of  $H$ . Suppose that the two actions differ by a conjugation, i.e. there exists a morphism of algebraic varieties  $c : X \rightarrow H$ , such that*

$$\phi_1(g)(x) = \phi_2(c(x) \cdot g \cdot c(x)^{-1})(x)$$

*for all  $g \in H, x \in X$ . Then for any character  $\chi : H \rightarrow \mathbb{G}_a$  we have canonical isomorphisms of the averaging functors corresponding to the two actions:*

$$\mathrm{Av}_{H,\chi,!}^{\phi_1} = \mathrm{Av}_{H,\chi,!}^{\phi_2},$$

$$Av_{H,\chi,*}^{\phi_1} = Av_{H,\chi,*}^{\phi_2}.$$

b) Let  $H_1, H_2$  be two algebraic groups,  $\phi_i$  be an action of  $H_i$  on an algebraic variety  $X_i$  (where  $i = 1, 2$ ). Let  $f : X_1 \rightarrow X_2$  be a morphism, and assume that there exists a morphism  $s : H_1 \times X_1 \rightarrow H_2$ , such that

$$f(\phi_1(h_1)(x_1)) = \phi_2(s(h_1, x_1))(f(x))$$

for  $x_1 \in X_1, h_1 \in H_1$ . Then for any  $H_2$ -equivariant complex of constructible sheaves on  $X_2$  the complex  $f^*(X)$  is also  $H_1$  equivariant.  $\square$

Part (a) of Lemma 3.6 shows that both averaging functors  $Av_{U,\chi,!}$  and  $Av_{U,\chi,*}$  do not change when we replace the old action by the new one. Also, since our  $\mathcal{G}$  is equivariant with respect to the adjoint action it follows that  $\mathcal{G}|_{W_{\overline{P}}^0}$  is also equivariant with respect to the new action of  $\overline{U}$  by part (b) of Lemma 3.6. The statement now follows from Theorem 1.5(2).

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