INTEGRAL ASPECTS OF FOURIER TRANFORM AND BEAUVILLE DECOMPOSITION

Contents

[1]

1. Introduction

In this notes, we discuss Ben Moonen's "project" idea on the integral aspects of Fourier transform and Beauville decompsition. Let $X \in \mathrm{AV}_k^{\dim = g}$, k of characteristic 0, $X^\vee$ the dual abelian variety, and $\mathcal{P} \in \mathrm{Pic}(X \times X^\vee)$ the *Poincaré bundle*, see Section 5. The classical Fourier-Mukai transform [Muk81], was put into the context of Chow groups [Bea83] for abelian varieties.

Theorem 1.1. *The Fourier transform consists of a pair of ring homomorphisns*

$$(CH(X)_{\mathbb{Q}}, \cap) \underset{\mathcal{F}}{\overset{\mathcal{F}^\vee}{\rightleftarrows}} (CH(X^\vee)_{\mathbb{Q}}, \star)$$

$$\begin{tikzcd} & X \times X^\vee \arrow[dl, "p_X"'] \arrow[dr, "p_{X^\vee}"] & \\ X & & X^\vee \end{tikzcd}$$

$$\mathcal{F}(\alpha) := p_{X^\vee,*}\left(p_X^*(\alpha) \cap \mathrm{ch}(\mathcal{P})\right)$$

$$\mathcal{F}^\vee(\beta) := p_{X,*}\left(p_{X^\vee}^*\beta \cap \mathrm{ch}(\mathcal{P}^\vee)\right)$$

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[1] *Milton: I thank Ben for the "literal" solutions inside project guideline, I thank my amazing group Lily - for patiently listening to mistake, Junaid - for the matcha milk tea and being my boss, Hazan - for his explanation in a crucial step of Proposition 4.5 that I was confused for a long time, Marcella - for her mental support(?) :) . What an enjoyable trip!*

(1)
$$\mathcal{F}^\vee \circ \mathcal{F} = (-1)^g \cdot [-1]_X^*$$
$$\mathcal{F} \circ \mathcal{F}^\vee = (-1)^g \cdot [-1]_{X^\vee}^*$$

(2) If $f : X \to Y$ is a morphism of abelian varieties, with dual $f^\vee : Y^\vee \to X^\vee$. The following diagram commutes
$$\begin{array}{ccc} CH(X) & \xrightarrow{\mathcal{F}} & CH(X^\vee) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f^{\vee,*}} \\ CH(Y) & \xrightarrow{\mathcal{F}} & CH(Y^\vee) \end{array}$$

Question 1.2. *Is there a way to define to Fourier transform with integral coefficients?*

This is an application of G. Pappas' work [Pap07], Theorem 3.3.

Rationally, the Fourier transform induces the Beauville decomposition.

Question 1.3. *Given a Fourier transform with coefficients in Λ, can we get a Beauville decomposition accordingly?*

One crucial lemma is how elements decompose into weight components after Fourier transform Lemma 4.4. This still holds after inverting $\frac{1}{(2g)!}$. Provided this and the integral Fourier transform, one obatins the integral Beauville, decomposition Theorem 4.2.

2. Chern class, Chern character and Todd character

Chern and Todd class are examples of symmetric polynomials, associated to the dataum of $(E, X : E \in K^0(X))$.

Proposition 2.1. *Let $X \in SmProj_k^{\dim = g}$ Let $E \in K^0(X)$, with chern roots $\alpha_1, \dots, \alpha_r$. Then we have the following commutatie diagram*

$$\begin{tikzcd}
\mathbb{Z}[[\alpha_1, \dots, \alpha_r]] \arrow[r] \arrow[d, hook] & \mathbb{Z}[\alpha_1, \dots, \alpha_r]_g^{\mathrm{Sym}_r} \arrow[r] \arrow[d, hook] & CH(X) \arrow[d] \\
\mathbb{Q}[[\alpha_1, \dots \alpha_r]]^{\mathrm{Sym}_r} \arrow[r] & \mathbb{Q}[\alpha_1, \dots, \alpha_r]_g^{\mathrm{Sym}_r} \arrow[r] & CH(X)_{\mathbb{Q}}
\end{tikzcd}$$

$$\mathbb{Z}[[\alpha_1, \dots, \alpha_r]]_d^{\mathrm{Sym}_r} := \mathbb{Z}[\alpha_1, \dots, \alpha_r]/(f : \text{homogeneous of degree } d > 0)$$

we have a factorization simply because $\mathrm{CH}^i(X)$ vanishes for $i > g$.

Example 2.2. If $l \geq g$, then $l! \cdot \mathrm{ch}(E) \in \mathbb{Z}[\alpha_1, \dots, \alpha_r]_g^{\mathrm{Sym}_r}$.

2.2.1. *Chern classes.* Associated to a $E \in \mathrm{Vect}(X)$. To compute the higher chern character. We consider the following:

- Choose any flat morphism $\mathrm{Fl} \to X$ such that $\mathrm{CH}^*(X) \hookrightarrow \mathrm{CH}^*(\mathrm{Fl})$.
- such that the chern polynomial
$$c_t(E) = 1 + \sum_{i=1}^{r} c_i(E) t^r \in \mathrm{CH}^*(\mathrm{Fl})[[t]]$$
factors as
$$\prod_{i=1}^{r} (1 + a_i t)$$

Proposition 2.3. *Properties of chern class.*

(1) If $f : X \to Y$ then $f_* : CH(X) \to CH(Y)$ respect
$$f_*(c_i(f^*(E) \cap x) = c_i(E) \cap f_*(x)$$
(2) $f : X \to Y$ is a flat morphism $l \geq \{\dim X, \dim Y\}$
$$l! \cdot \mathrm{ch}(f^* E) = f^*(l! \cdot \mathrm{ch}(E))$$

The *Chern character* of a vector bundle $E \in K^0(X)$ be can identified

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{\alpha_i} = \sum_{i=1}^{r} \sum_{m=0}^{\infty} \frac{\alpha_i^m}{m!} \in \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\alpha_i := c_1(\mathcal{L}_i) \in \mathrm{CH}^1(X)$ are the *Chern roots.*

Lemma 2.4. *If E is the trivial bundle of rank r. Then*

- $\alpha_i = c_i(E) = 0$ *for all* $i = 1, \ldots, r$.

Hence, $\mathrm{ch}(E) = rank(E) \in CH(X)_{\mathbb{Q}}$.

Proof. Let us compute $E \to \mathrm{pt} \in \mathrm{Vect}(\mathrm{pt})$. Then

$$c_i(E) \in \mathrm{CH}^{2i}(\mathrm{pt}) \simeq 0$$

Now any trivial vector bundle, is given by the pullback

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & \mathrm{pt} \end{tikzcd}$$

which is a flat morphism. Hence,

$$f^*(c_i(E) \cap y) = c_i(f^*(E)) \cap f^*(y) \quad y \in \mathrm{CH}(X)$$

by pullback formula, Proposition 2.3 thus,

$$c_i(f^*(E)) \cap f^*(y) = 0$$

□

2.4.1. *Integral polynomials.* For giving integral statements, it will be useful to consider the following definitions.

Definition 2.5. Let $m \in \mathbb{Z}$,

$$\mathfrak{s}_m := m! \cdot \mathrm{ch}_m(E)$$

Definition 2.6. Let the nth integral part of Todd class be

$$\mathfrak{T}_n(E) := T_n \cdot \mathrm{Td}(E)$$

Then $T_l \cdot \mathrm{Td}(E) = \sum_{m=0}^{\dim X} \frac{T_l}{T_m} \mathfrak{T}_m$. Thus, to make this in $\mathbb{Z}[\alpha_1, \cdots \alpha_r]^{\mathrm{Sym}_r}$, we require $l \geq \dim X$.

Definition 2.7. Let $E \in K^0(X)$, T_X be tangent bundle of $X \in \mathrm{SmProj}_k$.

$$\mathfrak{C}\mathfrak{T}_m(E) := T_m \cdot (\mathrm{ch}(E) \cap \mathrm{Td}(T_X))_m$$

Proposition 2.8. *[Pap07, Lem 2.1].*[2] *This is a homogeneous polynomial in* $\mathbb{Z}[\alpha_1, \ldots, \alpha_r]$

[2] A prior this is unclear.

3. Integral Fourier transform

Definition 3.1.

$$F : \mathrm{CH}(X) \to \mathrm{CH}(X^\vee)$$
$$F^\vee : \mathrm{CH}(X^\vee) \to \mathrm{CH}(X)$$

is (M, N) integral if

(1)
$$F = M \cdot \mathcal{F} : \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^\vee)_{\mathbb{Q}}$$
where $\mathcal{F}$ is the usual Fourier transform, [3]

(2) $N \cdot (F^\vee \circ F) = M^2 N(-1)^g \cdot [-1]^*_X$

(3) For $x, y \in \mathrm{CH}(X)$
$$M \cdot F(x * y) = F(x) \cdot F(y)$$

Our goal is to find a $(M, 1)$ integral Fourier transform for M smallest. We would need chern character functoriality against proper maps: Let $\pi : X \to Y$ be a proper morphism in SmProj_k. Then the following diagram commute: [4]

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}\cdot\mathrm{Td}"] \arrow[d, "\pi_*"] & \mathrm{CH}(X) \otimes \mathbb{Q} \arrow[d, "\pi_*"] \\ K(Y) \arrow[r, "\mathrm{ch}\cdot\mathrm{Td}"] & \mathrm{CH}(Y) \otimes \mathbb{Q} \end{tikzcd}$$

An integral version is given in [Pap07, Thm 2.2].

Theorem 3.2. *Suppose k is a field, with char $k = 0$. Let $R, S \in SmProj_k$. Suppose $f : R \to S$ is a projective morphism* [5]

(1) If $d \geq 0$,

$$\begin{tikzcd} K(X) \arrow[r, "\mathfrak{CT}"] \arrow[d, "f_*"] & CH(X) \otimes \mathbb{Q} \arrow[d, "f_*"] \\ K(Y) \arrow[r, "\frac{T_{d+n}}{T_n}\mathfrak{CT}_n"] & CH(Y) \otimes \mathbb{Q} \end{tikzcd}$$

(2) If $d < 0$, then

$$\begin{tikzcd} K(X) \arrow[r, "\mathfrak{CT}"] \arrow[d, "f_*"] & CH(X) \otimes \mathbb{Q} \arrow[d, "\frac{T_n}{T_{n+d}}f_*"] \\ K(Y) \arrow[r, "\mathfrak{CT}_n"] & CH(Y) \otimes \mathbb{Q} \end{tikzcd}$$

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[3] This means that $\frac{1}{M}F : \mathrm{CH}(X)_{\mathbb{Z}[1/M]} \to \mathrm{CH}(Z)_{\mathbb{Z}[1/M]}$ base change the desired Fourier transform $\mathcal{F}$ in $\mathbb{Q}$.

[4] all push forward in this notes are derived.

[5] We can identify X as a closed embedding, $X \hookrightarrow_{\mathrm{cl}} \mathbb{P}(E)$, for $E \in \mathrm{QCoh}(S)$.

Theorem 3.3. *Set*

$$F : (CH(X), \star) \xrightarrow{\simeq} (CH(X^\vee), \cap)$$
$$F(\alpha) := p_{X^\vee,*}\left(p_X^*\alpha \cap (2g)!\,\mathrm{ch}(\mathcal{P})\right) \qquad \gamma := (2g)!\,\mathrm{ch}\,\mathcal{P}$$

$$\begin{tikzcd} & X \times X^\vee \arrow[dl, "p_X"'] \arrow[dr, "p_{X^\vee}"] & \\ X & & X^\vee \end{tikzcd}$$

F is (M, N)-integral, where $M = (2g)!$ and N is from Lemma 3.4.

Proof. By chasing the integral version [6] and Lemma 3.4. ∎

Lemma 3.4. *The smallest integer N such that*

$$N \cdot \left(\frac{(2g)!^2}{T_{2g}}\right) \in \mathbb{Z}$$

occurs when

$$N = \begin{cases} 2g+1 & \text{if } 2g+1 \text{ is prime} \\ 1 & \text{otherwise} \end{cases}$$

Proof. Note that $T_n \sim n!$. ∎

Proposition 3.5.

$$\begin{tikzcd} K(\mathrm{Spec}\,k) \arrow[r, "\mathfrak{CT}_n"] \arrow[d, "p_{1*}"] & CH(X) \arrow[d, "\frac{T_n}{T_{n-g}} p_{1*}"] \\ K(\mathrm{Spec}\,k) \arrow[r, "\mathfrak{CT}_{n-g}"] & CH(X) \end{tikzcd}$$

we have

$$\frac{T_n}{n!} \cdot \mathfrak{s}_n\left(e_*[\mathcal{O}_{\mathrm{Spec}\,k}]\right) = \begin{cases} T_g \cdot e_*[\mathrm{Spec}\,k] & n = g \\ 0 & n \neq g \end{cases}$$

Proof. Similar as Proposition 3.6. [8] ∎

[6]For intuition, let us compute the nth component of $(F^\vee \circ F)_n$. [7] The red parts are some fractions introduced and should be corrected.

$$\begin{aligned} (F^\vee \circ F)_n &= ((2g)!)^2 m^* p_{1*} \frac{\mathfrak{s}_n(\mathcal{P})}{n!} \\ &= \frac{(2g)!^2}{n!} m^* p_{1*} \mathfrak{s}_n(\mathcal{P}) \\ &= \frac{((2g)!)^2}{T_n} \frac{T_n}{n!} m^* p_{1*} \mathfrak{s}_n(\mathcal{P}) \\ &= \frac{((2g)!)^2}{T_n} \frac{T_n}{(n-g)!} m^* \mathfrak{s}_{n-g}(p_{1*}\mathcal{P}) \end{aligned}$$

[8]check subscripts when $n = g$! (inspired from Lily)

Proposition 3.6.

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathfrak{CT}_n} & CH(X \times X^\vee) \\ \Big\downarrow{\scriptstyle p_{1*}} & & \Big\downarrow{\scriptstyle p_{1*}} \\ K(X) & \xrightarrow{\frac{T_{g+n}}{T_n}\mathfrak{CT}_{n+g}} & CH(X \times X^\vee) \end{array}$$

we have

$$\frac{T_n}{n!} p_{1*} \mathfrak{s}_n(\mathcal{P}) = \frac{T_n}{(n-g)!} \mathfrak{s}_{n-g}(p_{1*}\mathcal{P}) n \in \mathbb{Z}$$

Proof. We use that Theorem 3.2, where noting that $\mathrm{Td}(T_{\mathcal{P}}) = 1$, since $T_{\mathcal{P}}$ is trivial,

$$\begin{aligned} \frac{T_{g+n}}{T_n} \frac{T_n}{n!} \mathfrak{s}_n(p_{1*}\mathcal{P}) &= \frac{T_{g+n}}{T_n} \cdot T_n \operatorname{ch}_n(p_{1*}\mathcal{P}) \\ &= p_{1*} T_{g+n} \operatorname{ch}_{g+n}(\mathcal{P}) \\ &= \frac{T_{g+n}}{(g+n)!} p_{1*} \mathfrak{s}_{g+n}(\mathcal{P}) \end{aligned}$$

$\square$

4. Integral Beauville Decomposition

Let k be a field of characteristic 0. Let $X \in \mathrm{AV}_k^{\dim=g}$. Let us suppose $2g+1$ is not prime. So that we have integral Fourier transform $(N,1)$. Let Λ be some coefficient ring.

Definition 4.1.

$$\mathrm{CH}^i_{(s)}(X)_\Lambda := \left\{x \in \mathrm{CH}^i(X)_\Lambda \,:\, [n]_X^*(x) = n^{2i-s}x, \quad n \in \mathbb{Z}\right\}$$

$$\mathrm{CH}_{i,(s)}(X)_\Lambda = \left\{x \in \mathrm{CH}_i(X)_\Lambda \,:\, [n]_{X,*}(x) = n^{2i+s} \cdot x, \quad n \in \mathbb{Z}\right\}$$

Theorem 4.2. *Let X be an abelian variety of dimension g.*

(1) We have decomposition

$$CH(X)_\Lambda \simeq \bigoplus_{i,s\in\mathbb{Z}} CH^i_{(s)}(X)_\Lambda$$

(2) The ring structure respects the weight grading.

Proof. Let $x \in \mathrm{CH}^i(X)_\Lambda$. We will use the integral Fourier transform F we have defined. If $F(x) = \sum y_j$, where $y_j \in \mathrm{CH}^j(X)_\Lambda$. For each j , by 1 of Definition 3.1,

$$F^\vee(y_j) \in \mathrm{CH}^i(X)$$

These will give the weight decomposition of x. Now as $F \circ F^\vee y_j = (-1)^g(2g)!^2[-1]^*y_j \in \mathrm{CH}^j(X)_\Lambda$. Setting $j = g-i+s$, we deduce from Proposition 4.5, that

$$n^*F^\vee(y_j) = n^{2i-s}x = n^{2g-2j+s}x$$

$$\begin{aligned}(2g)!(-1)^g[-1]^*x &= F^\vee \circ F(x)\\ &= \sum_{y_j} F^\vee(y_j)\end{aligned}$$

□

4.3. **Lemmas for Beauville decomposition.** The following is lemma from [Bea86, F3][9]: which says that the Fourier transform of any element has a weight decomposition.

Lemma 4.4. *Let $x \in CH^i(X)_\Lambda$ and $F(x) = \sum_{j\geq 0} y_j$, $y_j \in CH^j(X)_\Lambda$. Let $n \in \mathbb{Z}$, then*

$$n^*F(x) = \sum_j n^{g-i+j}y_j$$

[9] In the paper, most results are based upon F1-F3.

Proof. Note that $p_*\left(\frac{P^k}{k!}\cdot p^*x\right) \in \mathrm{CH}^{i+k-g}(X)$. Thus, set $j = i+k-g$, so $k = g+j-i$.

$$\begin{aligned} n^*F(x) &= n^*p_{X^\vee,*}((2g)!\,\mathrm{ch}(P)\cdot \pi_X^* x) \\ &= p_{X^\vee,*}((2g)!(\mathrm{id},n)^*\,\mathrm{ch}(P)\cdot(\mathrm{id},n)^* p_X^* x) \\ &= p_{X^\vee,*}((\mathrm{id},n)^*\,\mathrm{ch}(P)\cdot p_X^* x) \\ &= p_{X^\vee,*}(\mathrm{ch}\,P^n)\cdot p^*x) \\ &= p_{X^\vee,*}\left(\sum_{k\geq 0}\frac{n^kP^k}{k!}\cdot p^*x\right) \\ &= \sum_{k\geq 0} n^k p_{X^\vee,*}\left(\frac{P^k}{k!}\cdot p^*x\right) \\ &= \sum_j n^{g+j-i}y_j \end{aligned}$$

Where the second equality follows first from [Bea86, F2], and the diagram,

$$\begin{tikzcd} X\times X^\vee \arrow[r, "\mathrm{id}\times n"] \arrow[d, "p_{X^\vee}"] & X\times X^\vee \arrow[d, "p_{X^\vee}"] \\ X^\vee \arrow[r, "n"] & X^\vee \end{tikzcd}$$

□

We recall the following conditions classically presented in Beauville, [Bea86, Prop 1]. The following five conditions are equivalent.

Proposition 4.5. *Let $x \in CH^i(X)_\Lambda$. $n \notin \{0,1,-1\}$. In the following statements:*

(1) $F(x) \in CH_s^{g-i+s}(X)_\Lambda$

(2) $F(x) \in CH^{g-i+s}(X^\vee)_\Lambda$.

(3) $n^*x = n^{2i-s}x$.

(4) $n_*x = n^{2g-2i+s}x$.

(5) $x \in CH_s^i(X)_\Lambda$, i.e. $m^*x = m^{2i-s}x$ for all $m \in \mathbb{Z}$.

Proof. $3 \Rightarrow 1$. We will first argue $3 \Rightarrow 2$. We will compute $n^*F(x)$ in two ways. The first way uses Lemma 4.4. The second way is computed as follows: first observe $n^*x = n^{2i-s}x$ and that $n_*n^* = n^{2g}\mathrm{id}_{\mathrm{CH}(X)}$. Thus

$$n_*n^{2i-s}x = n^{2g}x$$

We compute

$$\begin{aligned} n^{2i-s}n^*F(x) &= n^{2i-s}F(n_*x) \\ &= F(n^{2g}x) \\ &= n^{2g}F(x) \end{aligned}$$

Therefore, by combining with Lemma 4.4 we have

$$n^{2i-s}\sum_j n^{g-i+j}y_j = n^{2g}\sum_j y_j$$

This is equivalent to

$$\sum_j \left(n^{g+i-s+j} - n^{2g}\right) y_j = 0$$

As this is true for all n, we must have $y_j = 0$ when $j \neq g-i+s$, by 4.6. This shows $3 \Rightarrow 2$. But again, we observe that by Lemma 4.4

$$n^*F(x) = n^{g-i+j}y_j \quad j = g-i+s$$

So we have $3 \Rightarrow 1$. To prove $3 \Rightarrow 4$ apply $F^\vee$. [10]

$$\begin{aligned} (-1)^g(2g)![-1]^*n_*x &= F^\vee F n_*x \\ &= F^\vee(n^*F(x)) \\ &= F^\vee(n^{2g-2i+s}F(x)) \\ &= (-1)^g(2g)!n^{2g-2i+s}[-1]^*x \end{aligned}$$

over Λ, this gives the desired inequality. □

Lemma 4.6. *Let $R = \mathbb{Z}/p^m\mathbb{Z}$ a prime p and $m \in \mathbb{Z}$. and G a monoid. Suppose that*

$$\chi_1, \ldots, \chi_n$$

Then $\chi_1, \ldots, \chi_n$ are R-linearly independent.

Proof. to complete We prove by induction. Fix $h \in G$. Then

$$0 = \sum_{i=1}^{n-1} \lambda_i \left(\chi_n(h) - \chi_i(h)\right) \cdot \chi_i(g)$$

Then by inductive hypothesis

$$\lambda_i \left(\chi_n(h) - \chi_i(h)\right) = 0 \quad i = 1, \ldots, n-1$$

Now using Nakayama lemma's, i.e. that one can lift generators after quotient by an ideal, we see that the result holds true. □

[10]argument explained by Hazan!

5. Appendix: Abelian varieties

In this appendix, we discuss facts on abelian varieties that we require. Let $e \in X(k)$.

Definition 5.1. Let $Y \in \mathrm{SmProj}_k^{\mathrm{irr}}$, $e \in Y(k)$. Let T be a scheme, and denote the map

$$e_T : T \to Y \times T$$

$$t \mapsto (e, t)$$

A *rigidified* line bundle on $T \times X$, is a pair $(\mathcal{L}, \alpha_T)$ where $\mathcal{L}$ is a line bundle on $Y \times T$ and

$$\alpha_T : e_T^* \mathcal{L} \simeq \mathcal{O}_T$$

Definition 5.2.

$$\mathrm{Pic}_{Y/k,e} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Ab}$$

the moduli problem of rigidified line bundle.

Theorem 5.3. *Let $Y \in SmProj_k^{irr}$, $e \in Y(k)$.*

(1) $Pic_{Y/k,e}$ is representable.

(2) $X^\vee := Pic^0_{X/k,e}$ is an abelian variety of the same dimension as X. [11] *$X^\vee$ is referred to as the dual abelian vareity.*

In particular, there is a canonical (rigidified) *universal line bundle*, $\mathcal{P}_X$ on $X \times X^\vee$, whose first chern class we denote by $P := c_1(\mathcal{P}_X) = [D]$, this is the class corresponding the divisors D, such $\mathcal{P}_X \simeq \mathcal{O}(D)$.

5.4. **Properties of line bundles.** We will let $\mathcal{P}$ denote the Poincaré line bundle.

Lemma 5.5. *Let $\mathcal{P}$ be Poincaré bundle on $X \times X^\vee$. Then for all $n \in \mathbb{Z}$,*

$$([n]_X, id)^* \mathcal{P} \simeq \mathcal{P}^{\otimes n} \simeq (id, [n]_{X^\vee})^* \mathcal{P}$$

Now we consider the following diagram

Proof. We have that

$$\mathcal{P}\Big|_{X \times \{\xi\}} \xi \in X^\vee$$

are all algebraically trivially. □

Corollary 5.6. *Here $P = c_1(\mathcal{P})$.*

- $(id, [n])^* P^j = n^j P^j$.
- $(id, [n])_* n^j P^j = n^{2g} P^j$.

[11] Note when char $k \neq 0$, the structure map is not necessarily smooth.

Proof. We have that

$$\begin{aligned}
(\mathrm{id},[n])^*(c_1(P))^j &= [(\mathrm{id},[n])^*(c_1(P)]^j \\
&= c_1((\mathrm{id},[n])^*P)^j \\
&= c_1(P^{\otimes n})^j \\
&= (nc_1(P))^j \\
&= n^j c_1(P)^j \\
&= n^j \mathcal{P}^j
\end{aligned}$$

Now we know that

$$(\mathrm{id},[n])_* (\mathrm{id},[n])^* \mathcal{P}^j = n^{2g}\mathcal{P}^j$$

which yields that

$$(\mathrm{id},[n])_* n^j \mathcal{P}^j = n^{2g}\mathcal{P}^j$$

□

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