Sheaf Frobenius, Lefschetz trace formula, and purity

Niccolò Ronchetti

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1 Frobenius morphisms and action on cohomology

We start by recalling various morphisms of $\mathbb{F}_p$-schemes that deserve to be called Frobenius, and the actions that they induce on étale cohomology.

Let X be an $\mathbb{F}_p$-scheme. Let $k = \mathbb{F}_q$ be the finite field of size $q = p^r$.

Definition 1 (Absolute Frobenius morphism)**.** We define $F_X : X \longrightarrow X$ to be the identity map on the underlying topological space, and the p-power map on the structure sheaf $\mathcal{O}_X$.

That this is a well-defined morphism obviously relies on the fact that $\mathcal{O}_X$ is an $\mathbb{F}_p$-algebra, and hence the p-power map is a ring homomorphism, and it is compatible with the identity map on the topological space.

Example 1. Let $X = \operatorname{Spec} A$ be a affine scheme, so A is an $\mathbb{F}_p$-algebra. Then $F_X^* : A \longrightarrow A$ is $f \mapsto f^p$, and the preimage of a prime ideal $\mathfrak{p} \subset A$ is itself (due to the prime-ideal property).

Proposition 1. *Let $f : X \longrightarrow Y$ be a morphism of $\mathbb{F}_p$-schemes, then $F_Y \circ f = f \circ F_X$.*

Proof. Obviously the two sides coincide on the underlying topological spaces, while the induced maps $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ coincide since the p-power map is a ring homomorphism for every $\mathbb{F}_p$-algebra, and hence commutes with the map induced by f on any affine chart. $\square$

Definition 2 (q-absolute Frobenius morphism)**.** If X is a k-scheme, we call *absolute q-Frobenius morphism* and denote ϕ_r the map $F_X^r : X \longrightarrow X$ (r-fold iterate of F_X), which is a morphism of k-schemes.

Let now $X \longrightarrow \operatorname{Spec} k$ be a k-scheme, and consider its base change $X_{\overline{k}}$ to the algebraic closure. Then the pair of maps $\big(\mathrm{id} : \operatorname{Spec} \overline{k} \longrightarrow \operatorname{Spec} \overline{k}, \phi_r : X \longrightarrow X\big)$ determines a morphism $X_{\overline{k}} \longrightarrow X_{\overline{k}}$ which we denote by $\overline{\phi_r}$.

This corresponds to 'raising coordinates', as the following example show.

Example 2. Let $E = \{y^2 = x^3 - x\} \subset \mathbb{A}^2_{\mathbb{F}_q}$ be your favorite affine elliptic curve. Writing $E = \operatorname{Spec} \mathbb{F}_q[x, y]/(y^2 - x^3 + x)$, we get that $\phi_r : E \longrightarrow E$ is induced by

$$\phi_r^* : \mathbb{F}_q[x, y]/(y^2 - x^3 + x) \longrightarrow \mathbb{F}_q[x, y]/(y^2 - x^3 + x) \qquad f \mapsto f^q.$$

On the other hand, $\overline{\phi_r}$ is induced by

$$\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x, y]/(y^2 - x^3 + x) \xrightarrow{1 \otimes \phi_r^*} \overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x, y]/(y^2 - x^3 + x).$$

In particular, $x - a = 1 \otimes x - a \otimes 1 \mapsto 1 \otimes x^q - a \otimes 1 = x^q - a$ and then $x - a^q \mapsto x^q - a^q = (x - a)^q$. The preimage of the prime ideal $(x - a, y - b) \subset \mathbb{F}_q[x, y]/(y^2 - x^3 + x)$ is then $(x - a^q, y - b^q)$. Passing to affine coordinates, this means that $\overline{\phi_r}(a, b) = (a^q, b^q)$.

We can also consider the q-Frobenius element $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{k}/k)$ sending $\alpha \in \overline{k}$ to α^q, which is an automorphism of $\mathrm{Spec}\,\overline{k}$ as a k-scheme. Hence, the pair of maps $\left(\mathrm{Frob}_q : \mathrm{Spec}\,\overline{k} \longrightarrow \mathrm{Spec}\,\overline{k}, \mathrm{id} : X \longrightarrow X\right)$ determines a morphism $\mathrm{Frob}_{q,X} : X_{\overline{k}} \longrightarrow X_{\overline{k}}$. This commutes with the map $\overline{\phi}_r$, because the two compositions are induced by the same pair of maps $\left(\mathrm{Frob}_q : \mathrm{Spec}\,\overline{k} \longrightarrow \mathrm{Spec}\,\overline{k}, \phi_r : X \longrightarrow X\right)$.

Example 3. On our favorite example $E = \{y^2 = x^3 - x\}$, the map $\mathrm{Frob}_{q,E}$ is induced by

$$\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x,y]/(y^2 - x^3 + x) \xrightarrow{\mathrm{Frob}_q \otimes \mathrm{id}} \overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x,y]/(y^2 - x^3 + x) \qquad a \otimes f(x,y) \mapsto a^q \otimes f(x,y)$$

In particular, $x - a = 1 \otimes x - a \otimes 1 \mapsto 1 \otimes x - a^q \otimes 1 = x - a^q$, so that $x - a^{1/q} \mapsto x - a$ and hence the preimage of the prime ideal $(x - a, y - b) \subset \mathbb{F}_q[x,y]/(y^2 - x^3 + x)$ is $(x - a^{1/q}, y - b^{1/q})$. Passing to affine coordinates, this means that $(a,b) \mapsto (a^{1/q}, b^{1/q})$.

Fact 2. *The composition of $\overline{\phi_r}$ and $\mathrm{Frob}_{q,X}$ is the absolute q-Frobenius of $X_{\overline{k}}$ as a k-scheme.*

Proof. Check on an affine chart, and see that the map induced by the composition at the level of rings is

$$\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} A \longrightarrow \overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} A \qquad x \otimes a \mapsto x^q \otimes a^q = (x \otimes a)^q.$$

It is also clear, as in the previous example, that the map is the identity on the underlying topological space of $X_{\overline{k}}$. ☐

Remark. In proposition 5.2, we will see that the for any constant sheaf $\underline{M}$ on a k-scheme X, the pullback-action of the absolute q-Frobenius morphism $F_{X_{\overline{k}}}$ on étale cohomology $H^*_{\text{ét}}(X_{\overline{k}}, M)$ is the identity map. Therefore, the 'geometric' action by pullback of $\overline{\phi}_r$ (an endomorphism of $\overline{X}$) coincide with the 'arithmetic' action by pullback of $(\mathrm{Frob}_{q,X})^{-1}$ which is induced by the pair of maps $\left(\mathrm{Frob}_q^{-1} : \mathrm{Spec}\,\overline{k} \longrightarrow \mathrm{Spec}\,\overline{k}, \mathrm{id} : X \longrightarrow X\right)$.

More generally, the action of the geometric endomorphism ϕ_r coincides with that induced by $\mathrm{Frob}_q^{-1} \in \mathrm{Gal}\left(\overline{k}/k\right)$ on the étale sheaf $\mathrm{R}^i f_* \underline{M}$. This explains why we call the Galois automorphism $\mathrm{Frob}_q^{-1} : \alpha \mapsto \alpha^{1/q}$ a *geometric* Frobenius.

We now turn to the relative situation. Let U be an X-scheme, and consider the diagram

$$\begin{tikzcd}
U \arrow[dr, dotted, "F_{U/X}"] \arrow[drr, bend left, "F_U"] \arrow[ddr, bend right] & & \\
& F_X^{-1}(U) \arrow[r] \arrow[d] & U \arrow[d] \\
& X \arrow[r, "F_X"'] & X
\end{tikzcd}$$

Definition 3 (Relative Frobenius)**.** The map $F_{U/X} : U \longrightarrow F_X^{-1}(U)$ induced by the universal property of the fiber product in the diagram above is called *relative Frobenius* (of U with respect to X).

Proposition 3. *$F_{U/X}$ is a radicial surjection. Moreover, if U is étale over X then it is an isomorphism.*

Proof. Recall (see e.g. EGA [4], section I.3.5) that a map of schemes $f : X \longrightarrow Y$ is radicial iff it satisfies any of the following four equivalent conditions:

1. For any field K for which it makes sense, the map $X(K) \xrightarrow{f(K)} Y(K)$ is injective.
2. As above, but for $K = \overline{K}$ only.
3. f is injective on the underlying topological spaces, and for each $x \in X$ the induced map on residue fields $k(f(x)) \hookrightarrow k(x)$ is purely inseparable.
4. Every base change of f is injective on the underlying topological spaces (universally injective).

Two more useful facts are:

- If $g \circ f$ is radicial, then so is f (EGA, [4], I.3.5.6).
- Being radicial is preserved under base change (Stacks project, [6], lemma 28.10.4).

Notice that every absolute Frobenius morphism is radicial (since it is the identity on topological spaces, and it is easy to check on an affine cover that the map induced on the residue fields is an extension $K^q \hookrightarrow K$ which is purely inseperable for every characteristic p field K - so it satisfies the third equivalent condition), so F_X and F_U are radicial. The first fact yields then that $F_{U/X}$ is radicial. Moreover, both F_X and F_U are surjective, and hence so is the base change of F_X and a fortiori $F_{U/X}$.

If $U \longrightarrow X$ is étale, then so is $F_X^{-1}(U) \longrightarrow X$ and hence $F_{U/X}$ is a morphism between étale X-schemes and hence an étale radicial surjection. By [3], theorem 17.9.1 an étale radicial surjection is an open embedding, so being also surjective it is an isomorphism in $X_{\text{ét}}$. □

Example 4. Suppose $X = \operatorname{Spec} L$ with $\operatorname{char}(L) = p$ and $U = \operatorname{Spec} K$ is a finite separable extension. Then $F_X^{-1}(U) = \operatorname{Spec}\left(L \otimes_{\mathrm{Frob}_p, L} K\right)$ is the spectrum of the Frobenius twist of K: it is still K as a set, but the L-algebra action is via Frob_p^{-1}, since $a \otimes x = 1 \otimes a^{1/p} x$ for $a \otimes x \in L \otimes_{\mathrm{Frob}_p, L} K$. The commutative diagram

$$\begin{tikzcd}
K & & \\
& L \otimes_{\mathrm{Frob}_p,L} K \arrow[ul, dotted, "F_{K/L}"'] & K \arrow[l] \arrow[ull, bend right, "\mathrm{Frob}_p"'] \\
& L \arrow[u] \arrow[uul, bend left] & L \arrow[l, "\mathrm{Frob}_p"] \arrow[u]
\end{tikzcd}$$

shows that for the general element $a \otimes x = 1 \otimes a^{1/p} x$ of $L \otimes_{\mathrm{Frob}_p, L} K$ we have $F_{K/L}(a \otimes x) = \mathrm{Frob}_p\left(a^{1/p} x\right) = a x^p$. We conclude that the image of $F_{K/L}$ is LK^p. The statement of the proposition boils then down to $K = LK^p$, which is familiar in this case.

Let now $\mathcal{F}$ be an étale sheaf of sets on X: for every étale $U \longrightarrow X$ we have then an isomorphism

$$\mathcal{F}\left(F_{U/X}\right) : \mathcal{F}\left(F_X^{-1}(U)\right) \longrightarrow \mathcal{F}(U) \tag{1}$$

thanks to proposition 3. Noticing that $(F_X)_* \mathcal{F}(U) := \mathcal{F}\left(F_X^{-1}(U)\right)$ by definition allows the one to define

$$\mathcal{F} \cong (F_X)_* \mathcal{F}$$

as the inverse of the isomorphism given by formula 1. We also denote its adjoint:

$$\mathrm{Frob}_{\mathcal{F}} : F_X^* \mathcal{F} \longrightarrow \mathcal{F}.$$

Proposition 4. $\mathrm{Frob}_{\mathcal{F}}$ *is compatible with tensor constructions, and moreover it is an isomorphism.*

Proof. We prove the second part. As $\mathcal{F} \cong (F_X)_*\mathcal{F}$ is an isomorphism we have

$$F_X^*\mathcal{F} \cong F_X^*\left((F_X)_*\mathcal{F}\right) \longrightarrow \mathcal{F}$$

where the last map is the adjunction (the morphism corresponding to $\mathrm{id} : (F_X)_*\mathcal{F} \longrightarrow (F_X)_*\mathcal{F}$). Since F_X is an integral, radicial surjection, it is a universal homeomorphism (we have taken those three properties to be the definition of the latter notion); a previous remark [1.1.6.4 in [1]] covered in the introductory lectures by Brian C. shows that under this assumption, the adjunction maps $f^*f_* \longrightarrow \mathrm{id}$ and $\mathrm{id} \longrightarrow f_*f^*$ are isomorphisms. □

Let's sum up a few facts on the map $\mathrm{Frob}_{\mathcal{F}}$ and its induced action on étale cohomology.

Proposition 5. 1. *Let $\mathcal{F} = \underline{\Sigma}$ be the constant sheaf for a set Σ. Then* $\mathrm{Frob}_{\mathcal{F}}$ *is the inverse of the canonical isomorphism $\underline{\Sigma} \cong F_X^*\underline{\Sigma}$.*

2. *Let X be an $\mathbb{F}_p$-scheme and M any abelian group. Then the composition*

$$H^i_{\text{ét}}(X, M) \xrightarrow{F_X^*} H^i_{\text{ét}}(X, F_X^*M) \xrightarrow{\mathrm{Frob}_M} H^i_{\text{ét}}(X, M)$$

is the identity.

3. *(compatibility of* $\mathrm{Frob}_{\mathcal{F}}$ *with higher direct images) Let $f : X \longrightarrow S$ be a map of $\mathbb{F}_p$-schemes so that by proposition 1 we have the commutative diagram*

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

The composition

$$F_S^*\mathrm{R}^i f_*\mathcal{F} \longrightarrow \mathrm{R}^i f_*\left(F_X^*\mathcal{F}\right) \xrightarrow{\mathrm{R}^i f_*(\mathrm{Frob}_{\mathcal{F}})} \mathrm{R}^i f_*\mathcal{F}$$

equals $\mathrm{Frob}_{\mathrm{R}^i f_*\mathcal{F}}$ *which is an isomorphism, so that in particular the first map is also an isomorphism.*

Proof. 1. This follow from representability of $\underline{\Sigma}$. See for example [5], section 2.1, pages 453-454 for a detailed discussion.

2. We sketch the proof the same statement for a general abelian étale sheaf $\mathcal{F}$ on X. See proposition 10.3.3 in [2] for the details.

   We start by reducing to degree $n = 0$ via a universal δ-functor argument. Étale cohomology is a derived functor, hence a universal δ-functor from the category of abelian étale sheaves on X to that of abelian groups.

   Notice that the family of natural transformations

$$\mathcal{F} \mapsto (T^n(\mathcal{F})) \text{ where } T^n(\mathcal{F}) = \mathrm{Frob}_{\mathcal{F}} \circ F_X^* \in \mathrm{End}\left(H^n_{\text{ét}}(X, \mathcal{F})\right)$$

   yields a morphism of δ-functors from $H^*_{\text{ét}}(X, -)$ to itself. By universality, to prove that this family of natural transformation coincides with the family of identical natural transformations, it suffices to do so in degree $n = 0$.

   Now use functoriality to reduce to the case of the trivial constant sheaf $\underline{\mathbb{Z}}$. Or see proposition 2 in [5], section 2.2.

3. Similarly as above, use a universal δ-functor argument to reduce to the case of $n = 0$. Then, consider the following commutative diagram ($\mathcal{F}$ is a sheaf on X)

$$\begin{tikzcd} f_*\mathcal{F} \arrow[r, "F_S", "\cong"'] \arrow[dr, "\cong", "f_*"'] & (F_S)_* f_*\mathcal{F} \arrow[d, "\cong"] \\ & f_*(F_X)_*\mathcal{F} \end{tikzcd}$$

where the vertical arrow is due to the diagram in the statement being commutative, the horizontal arrow due to $(F_S)_*$ being an isomorphism as proven above, and the diagonal arrow is f_* applied to the isomorphism $\mathcal{F} \cong F_*\mathcal{F}$.

As $\mathrm{Frob}_{f_*\mathcal{F}}$ is defined to be the adjoint map to $f_*\mathcal{F} \xrightarrow{F_S} (F_S)_* f_*\mathcal{F}$, it suffices to show that the composition $F_S^* f_*\mathcal{F} \longrightarrow f_*\left(F_X^*\mathcal{F}\right) \longrightarrow f_*\mathcal{F}$ is adjoint to $f_*\mathcal{F} \xrightarrow{F_S} (F_S)_* f_*\mathcal{F}$ as well.

The adjoint map to $F_S^* f_*\mathcal{F} \longrightarrow f_*\left(F_X^*\mathcal{F}\right)$ is $f_*\mathcal{F} \longrightarrow (F_S)_* f_*\left(F_X^*\mathcal{F}\right) = f_*(F_X)_*\left(F_X^*\mathcal{F}\right) \cong f_*\mathcal{F} \longrightarrow (F_S)_* f_*\mathcal{F}$ where the last isomorphism is due to $(F_X)^* F_X^* \longrightarrow \mathrm{id}$ being an isomorphism as in remark 1.1.6.4 of [1]. On the other hand, the map $f_*\left(F_X^*\mathcal{F}\right) \longrightarrow f_X\mathcal{F}$ is defined to be the adjoint of $f_*\mathcal{F} \xrightarrow{F_X} f_*(F_X)_*\mathcal{F}$.

We conclude that the composition $F_S^* f_*\mathcal{F} \longrightarrow f_*\left(F_X^*\mathcal{F}\right) \longrightarrow f_*\mathcal{F}$ is adjoint to $f_*\mathcal{F} \longrightarrow f_*(F_X)_*\mathcal{F}$, and so the above diagram concludes the proof.

$\square$

The whole discussion can be carried out for F_X^r if $|k| = p^r$ and we consider k-schemes.

Lemma 6. *Let S be a k-scheme, and $f : X \longrightarrow S$ be a finite-type separated map. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\text{ét}}$. Then the morphism*

$$F_{S,r}^* \mathrm{R}^i f_!\mathcal{F} \longrightarrow \mathrm{R}^i f_!\left(F_{X,r}^*\mathcal{F}\right) \xrightarrow{\mathrm{Frob}_{\mathcal{F},r}} \mathrm{R}^i f_!\mathcal{F}$$

of étale sheaves on $S_{\text{ét}}$ is exactly $\mathrm{Frob}_{\mathrm{R}^i f_!\mathcal{F},r}$.

The first map is the compatibility of pullback and higher direct image with compact support.

Proof. Since the equality of morphism of sheaves can be checked on an open covering of S, we can work locally on the base and thus assume that S is affine. Fix then a 'compactification', i.e. a diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

where j is an open immersion and $\overline{f}$ is proper, making $\overline{X}$ a proper S-scheme. Consider then the diagram of sheaves on $\overline{X}$:

$$\begin{tikzcd} j_! F_{X,r}^*\mathcal{F} \arrow[r, "j_!\mathrm{Frob}_{\mathcal{F},r}"] \arrow[d, "\cong"'] & j_!\mathcal{F} \\ F_{\overline{X},r}^*\left(j_!\mathcal{F}\right) \arrow[ur, "\mathrm{Frob}_{j_!\mathcal{F},r}"'] & \end{tikzcd}$$

Away from X, the stalks are zero for all three sheaves involved and moreover $\mathrm{Frob}_{\mathcal{F},r}$ respects localization for the Zariski topology, because the p^r-power map on rings does, hence the diagram commutes.

We can then replace f by $\overline{f}$ and $\mathcal{F}$ by $j_!\mathcal{F}$ and reduce to the case of higher direct images, where the statement is the content of proposition 5.3. $\square$

Example 5. Consider the case where $S = \operatorname{Spec} k$, so that $F_{S,r}$ is the identity morphism (also on the structure sheaf!). Given an étale sheaf $\mathcal{F}$ on S, the morphism

$$\mathrm{Frob}_{\mathcal{F},r} : F_{S,r}^*\mathcal{F} \longrightarrow \mathcal{F}$$

is then an endomorphism of $\mathcal{F}$, but it is not the identity! Indeed, on each étale $U \longrightarrow S$, this endomorphism is the inverse of the isomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ induced by pullback along F_U - which is not trivial.

The upshot is that in the dictionary between discrete $\mathrm{Gal}(\overline{k}/k)$-sets and étale sheaves over S, the action of the geometric Frobenius element in $\mathrm{Gal}(\overline{k}/k)$ corresponds precisely to $\mathrm{Frob}_{\mathcal{F},r}$ (see also example 4). The last lemma provides a 'higher degree' version of this equivalence of actions: explicitly, it says that the Frobenius endomorphism of $\mathrm{R}^i f_!\mathcal{F}$ corresponds to the endomorphism of $H^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}_{\overline{k}})$ induced by pullback along geometric Frobenius.

Remark. The entire content of this section can be applied term-wise to a projective system of sheaves, giving then functorial constructions in the Artin-Rees category.

2 L-functions and the Lefschetz trace formula

The setup for this section: let k be a finite field of size $q = p^r$, X a finite type k-scheme. Fix a mixed characteristic complete discrete valuation ring $(\Lambda, \mathfrak{m})$ with finite residue field of characteristic $l \neq p$ and denote K its fraction field. Let $\mathcal{F} \in X_{\text{ét}}$ be a constructible $\mathfrak{m}$-adic sheaf or a constructible K-sheaf.

We will simply say 'Frobenius' for q-Frobenius, since everything is relative to k in this section. The absolute Frobenius $F_X : X \longrightarrow X$ induces then a natural Frobenius morphism $\mathrm{Frob}_{\mathcal{F}} : F_X^*\mathcal{F} \longrightarrow \mathcal{F}$ in the Artin-Rees category.

For each closed point $x \in |X|$ denote $d_x = [k(x) : k]$ the relative residue field degree. Then the $q^{d(x)}$-Frobenius is the 'natural' Frobenius on $k(x)$-schemes: in particular the d_x-fold iterate $\left(F_X^{d_x}\right)^* \mathcal{F} \longrightarrow \mathcal{F}$ of $\mathrm{Frob}_{\mathcal{F}}$ has pullback along the fiber at x that becomes an endomorphism $\mathrm{Frob}_{\mathcal{F}_x} : \mathcal{F}_x \longrightarrow \mathcal{F}_x$ of constructible $\mathfrak{m}$-adic sheaves on $(\operatorname{Spec} k(x))_{\text{ét}}$.[1]

By the usual identification between étale shaves on $\operatorname{Spec} k(x)$ and $\mathrm{Gal}(k(x)^s/k(x))$-modules, choosing a separable closure $k(\overline{x})$ of $k(x)$ identifies $\mathcal{F}_x$ with a *finite* Λ-module (still denoted $\mathcal{F}_x$) having an action of the geometric Frobenius element ϕ_x generating the Galois group $\mathrm{Gal}\left(k(\overline{x})/k(x)\right)$.

Fact 7. *This action of ϕ_x agrees with the endomorphism* $\mathrm{Frob}_{\mathcal{F}_x}$.

Proof. This is the content of example 5, together with the construction of the endomorphism $\mathrm{Frob}_{\mathcal{F}_x}$ induced by pullback on the fiber of the appropriate power of F_X. $\square$

[1]It is clear that the stalks of $(F_X^{d_x})^*\mathcal{F}$ and $\mathcal{F}$ coincide, because $F_X^{d_x}$ is the identity on the topological space. The non-trivial part is that taking the d_x-th power carries over the $\mathrm{Gal}\left(\overline{k(x)}/k(x)\right)$-action, and this holds because $F_X^{d_x}$ acts as the identity morphism on $\operatorname{Spec} k(x)$.

Definition 4. The L-function of the constructible K-sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is defined to be the formal power series

$$L(X, \mathcal{F}, t) = \prod_{x \in |X|} \det\left(1 - \phi_x t^{d_x} | \mathcal{F}_x\right)^{-1} \in 1 + t\Lambda[[t]]$$

Notice that from the definition it is obvious that the L-function is compatible with some topological constructions, for example excision.

Theorem 8 (Lefschetz trace formula)**.** *Let $f : X \longrightarrow S$ be a separated map of finite type k-schemes. For any constructible K-sheaf $\mathcal{F}$ on $X_{\text{ét}}$ we have*

$$L(X, \mathcal{F}, t) = \prod_{n \geqslant 0} L\left(S, \mathrm{R}^n f_! \mathcal{F}, t\right)^{(-1)^n}$$

We will not prove this theorem, but we will explain the reduction to the absolute case when $S = \operatorname{Spec} k$ (which is where the beef is, anyway - Tony will prove this case on a latter talk, building on the important lemma 5), and we will draw in that case a comparison with the topological situation. Let's start with the latter.

2.1 The absolute case

Let's assume then that $S = \operatorname{Spec} k$, Then $L(\operatorname{Spec} k, \mathrm{R}^n f_! \mathcal{F}, t) = \det\left(1 - \phi t | H^n_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right)\right)^{-1}$ where ϕ is a geometric Frobenius in $\mathrm{Gal}(\overline{k}/k)$ acting on the compactly supported étale cohomology via pullback along $\mathrm{Frob}_{\mathcal{F}}$ (this is example 5 again). In this case, theorem 8 yields

$$L(X, \mathcal{F}, t) = \prod_{n \geqslant 0} \det\left(1 - \phi t | H^n_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right)\right)^{(-1)^{n+1}}. \tag{2}$$

As the compactly supported étale cohomology vanishes in high degrees (see theorem 1.3.6.3 in [1], covered in Ka Yu's lecture), we obtain that the L-function is a rational function in t, with zeros and poles corresponding to eigenvalues of the Frobenius-action on $H^*_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right)$.

To justify the 'trace formula' label, we do some formal linear algebra. Let $F \in \mathrm{End}(V)$ for a k-vector space V, then the general identity

$$\det\left(1 - Ft | V\right)^{-1} = \exp\left(\sum_{i \geqslant 1} \mathrm{Tr}(F^i) \frac{t^i}{i}\right)$$

is proven by putting F in Jordan form (after extending scalars to $\overline{k}$) and then using the formulas

$$(1 - \lambda t)^{-1} = \exp\left(-\log(1 - \lambda t)\right) = \exp\left(\sum_{i \geqslant 1} \lambda^i \frac{t^i}{i}\right)$$

eigenvalue by eigenvalue.

Applying this to our L-function yields

$$L(X, \mathcal{F}, t) = \exp\left(\sum_{n \geqslant 0} (-1)^n \sum_{i \geqslant 1} \mathrm{Tr}\left(\phi^i | H^n_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right)\right) \frac{t^i}{i}\right).$$

Therefore, taking the log-derivative of the L-function yields

$$\frac{L'(X,\mathcal{F},t)}{L(X,\mathcal{F},t)} = \sum_{i\geqslant 1}\sum_{n\geqslant 0}(-1)^n \mathrm{Tr}\left(\phi^i|H^n_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right)t^{i-1} = \sum_{i\geqslant 1}\chi\left(\phi^i|H^*_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right)t^{i-1} \tag{3}$$

where we define the Euler characteristic of the operator ϕ^i on compactly supported étale cohomology as the alternate sum of traces: $\chi\left(\phi^i|H^*_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right) = \sum_{n\geqslant 0}(-1)^n\mathrm{Tr}\left(\phi^i|H^n_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right)$.

On the other hand, we can apply the formal exponential formula to the definition of the L-function and obtain that

$$L(X,\mathcal{F},t) = \exp\left(\sum_{x\in|X|}\sum_{i\geqslant 1}\mathrm{Tr}\left(\phi_x^i|\mathcal{F}_x\right)\frac{t^{d_x i}}{i}\right)$$

In the inner summation, for each fixed $x \in |X|$ consider the change of variables $m_x = d_x i$. The summand becomes then

$$\mathrm{Tr}\left(\phi_x^{m_x/d_x}|\mathcal{F}_x\right)d_x\frac{t^{m_x}}{m_x}$$

and rather then summing over i's we are summing over m_x's divisible by the residue field degree d_x. Interchanging order of summation yields then that

$$L(X,\mathcal{F},t) = \exp\left(\sum_{m\geqslant 1}\sum_{x\in|X|,d_x|m} d_x\mathrm{Tr}\left(\phi_x^{m/d_x}|\mathcal{F}_x\right)\frac{t^m}{m}\right)$$

Taking again the log-derivative of the L-function and comparing coefficients of t^{i-1} with the previous expression 3 yields

$$\chi\left(\phi^i|H^*_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right) = \sum_{x\in|X|,d_x|i} d_x\mathrm{Tr}\left(\phi_x^{i/d_x}|\mathcal{F}_x\right),$$

where on the right hand side recall that ϕ_x was a geometric Frobenius for $\mathrm{Gal}\left(k(\overline{x})/k(x)\right)$. In particular, ϕ_x^{i/d_x} is a geometric Frobenius for $\mathrm{Gal}(\overline{k}/k_i)$ where k_i is the degree i extension of k in a fixed algebraic closure $\overline{k} = k(\overline{x})$ - let us denote it by $\widetilde{\phi}_x$.

Notice now that if $x \in |X|$ and $d_x|i$ then there are exactly d_x points in $X(k_i)$ above $x \in |X|$. Moreover, each of these points $\widetilde{x} \in X(k_i)$ above x have geometric Frobenius exactly $\widetilde{\phi}_x$, and the stalk $\mathcal{F}_x$ coincides as a Galois module with the stalk $\mathcal{F}_{\widetilde{x}}$. Substituting in the above formula yields

$$\chi\left(\phi^i|H^*_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right) = \sum_{\widetilde{x}\in X(k_i)}\mathrm{Tr}\left(\widetilde{\phi}_x|\mathcal{F}_{\widetilde{x}}\right).$$

Noticing that on the left hand side ϕ^i is precisely a geometric Frobenius for k_i, we can drop dependence on the i (and relax the notation by dropping the tilde) to write

$$\chi\left(\phi|H^*_{c,\text{ét}}\left(X_{\overline{k}},\mathcal{F}\right)\right) = \sum_{x\in X(k)}\mathrm{Tr}\left(\phi_x|\mathcal{F}_x\right). \tag{4}$$

This is visibily a formula for the Euler characteristic of a Frobenius action on cohomology as a sum of local traces at fixed-points for the Frobenius action - i.e. a Lefschetz trace formula as is known in topology.

2.2 Reduction to the absolute case

Now we reduce the general case of theorem 8 to the absolute case $S = \operatorname{Spec} k$.

For any open subscheme $U \subset S$ with closed complement Z, we have a corresponding excition of X by the open subscheme $f_U : X_U \longrightarrow U$ and the closed complement $f_Z : X_Z \longrightarrow Z$. Since the definition of the L-function is obviously compatible with excision, we obtain that

$$L(X, \mathcal{F}, t) = L(X_U, \mathcal{F}_U, t) \cdot L(X_Z, \mathcal{F}_Z, t).$$

On the other hand, in remark 1.3.6.2 and formula 1.3.6.3 of [1] (covered in Ka Yu's talk) we have the long exact sequence induced by excision on higher direct image of torsion sheaves (which holds in the Artin-Rees setup by 1.4.6 in loc. cit. covered in Brian L.'s talk):

$$\ldots \xrightarrow{\delta} \mathrm{R}^n f_{U!}(\mathcal{F}_U) \longrightarrow \mathrm{R}^n f_! \mathcal{F} \longrightarrow \mathrm{R}^n f_{Z!}(\mathcal{F}_Z) \xrightarrow{\delta} \ldots$$

For every $x \in |X|$ we can take the stalk at x in this long exact sequence $\ldots \longrightarrow \mathcal{G}_h \longrightarrow \ldots$ which then becomes a long exact sequence of $\operatorname{Gal}(k(\overline{x}), k(x))$-modules. As such, we can consider the action of $1 - \phi_x t^{d_x}$ and then the Euler characteristic formula for the determinant of this operators yields that the product of $\det\left(1 - \phi_x t^{d_x} | \mathcal{G}_h\right)$ for even h's is the same as that for odd h's.

Keeping track of the powers of (-1) in the exponents, this yields

$$\prod_{n \geqslant 0} \det\left(1 - \phi_x t^{d_x} | \left(\mathrm{R}^n f_! \mathcal{F}\right)_x\right)^{(-1)^n} = \prod_{n \geqslant 0} \det\left(1 - \phi_x t^{d_x} | \left(\mathrm{R}^n f_{U!} \mathcal{F}_U\right)_x\right)^{(-1)^n} \cdot \prod_{n \geqslant 0} \det\left(1 - \phi_x t^{d_x} | \left(\mathrm{R}^n f_{Z!} \mathcal{F}_Z\right)_x\right)^{(-1)^n}.$$

Taking the inverses of both sides and then the product over all $x \in |X|$ gives

$$\prod_{n \geqslant 0} L\left(S, \mathrm{R}^n f_! \mathcal{F}, t\right)^{(-1)^n} = \prod_{n \geqslant 0} L\left(U, \mathrm{R}^n f_{U!} \mathcal{F}_U, t\right)^{(-1)^n} \cdot \prod_{n \geqslant 0} L\left(Z, \mathrm{R}^n f_{Z!} \mathcal{F}_Z, t\right)^{(-1)^n},$$

so it suffices to prove theorem 8 separately for $f_U : X_U \longrightarrow U$ and $f_Z : X_Z \longrightarrow Z$ and in particular we can assume that both X and S are separated.

Therefore, applying the absolute case as in formula 2 to the sheaves $\mathrm{R}^n f_! \mathcal{F}$ on S yields that

$$L(S, \mathrm{R}^n f_! \mathcal{F}, t) = \prod_{m \geqslant 0} \det\left(1 - \phi t | H^m_{c,\text{ét}}\left(S_{\overline{k}}, \mathrm{R}^n f_! \mathcal{F}\right)\right)^{(-1)^{m+1}}.$$

Using again formula 2 for $(X, \mathcal{F})$ it turns out we have to show that

$$\prod_{s \geqslant 0} \det\left(1 - \phi t | H^s_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right)\right)^{(-1)^{s+1}} = \prod_{n \geqslant 0} \left(\prod_{m \geqslant 0} \det\left(1 - \phi t | H^m_{c,\text{ét}}\left(S_{\overline{k}}, \mathrm{R}^n f_! \mathcal{F}\right)\right)^{(-1)^{m+1}} \right)^{(-1)^n}$$

or more compactly

$$\prod_{s \geqslant 0} \det\left(1 - \phi t | H^s_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right)\right)^{(-1)^{s+1}} = \prod_{n,m \geqslant 0} \det\left(1 - \phi t | H^m_{c,\text{ét}}\left(S_{\overline{k}}, \mathrm{R}^n f_! \mathcal{F}\right)\right)^{(-1)^{m+n+1}}. \tag{5}$$

The Leray spectral sequence (see remark 1.3.6.2 in [1], which holds in the Artin-Rees setup by 1.4.6 of loc. cit.) for the composition $X \xrightarrow{f} S \longrightarrow \operatorname{Spec} k$ is

$$E_2^{m,n} = H^m_{c,\text{ét}}\left(S_{\overline{k}}, \mathrm{R}^n f_! \mathcal{F}\right) \Rightarrow H^{m+n}_{c,\text{ét}}\left(X_{\overline{k}}, \mathcal{F}\right).$$

Moreover, this spectral sequence is Frobenius-equivariant by lemma 6. Therefore, we can use the 'determinantal Euler characteristic' of a spectral sequence to complete the proof of formula 5. More precisely, here's a general statement:

Lemma 9. *Let*

$$E_2^{p,q} \Rightarrow E_\infty^{p,q}$$

be a cohomological spectral sequence of finite-dimensional vector spaces. Suppose that for all $r \geqslant 2$, $p, q \geqslant 0$ we have endomorphisms $T_r^{p,q}$ that commutes with the differentials of the spectral sequence, i.e.

$$d_r^{p,q} \circ T_r^{p,q} = t_r^{p+r,q-r+1} \circ d_r^{p,q}.$$

Define the T-determinantal Euler characteristic of the r-page as $\chi_r(T) = \prod_{p,q\geqslant 0} \det\left(T|E_r^{p,q}\right)^{(-1)^{p+q+1}}$. Then the T-determinantal Euler characteristic is independent of r, and in particular coincides with that of the ∞-page. In particular the T-determinantal Euler characteristic of the spectral sequence is well-defined, and we have

$$\prod_{p,q\geqslant 0} \det\left(T|E_2^{p,q}\right)^{(-1)^{p+q+1}} = \chi_2(T) = \chi_\infty(T) = \prod_{p,q\geqslant 0} \det\left(T|E_\infty^{p,q}\right)^{(-1)^{p+q+1}}.$$

Proof. (Sketch). There is a T_r-equivariant decomposition

$$E_r^{p,q} \cong \operatorname{Im}\left(d_r^{p-r,q+r-1}\right) \oplus E_{r+1}^{p,q} \oplus \operatorname{Im}\left(d_r^{p,q}\right)$$

which allows one to pull out $\chi_{r+1}(T)$ from the formula defining $\chi_r(T)$. Then one shows that the remaining terms cancel out in pairs, e.g. $\det\left(T_r^{p,q}|\operatorname{Im}(d_r^{p-r,q+r-1})\right)^{(-1)^{p+q+1}}$ cancels out with $\det\left(T_r^{p'+r,q'-r+1}|\operatorname{Im}(d_r^{p',q'})\right)^{(-1)^{p'+q'+1}}$ when we choose $p' = p - r$, $q' = q + r - 1$. □

3 Purity and Deligne's theorem

Finally, we are interested in studying the Galois action on the étale cohomology of a proper smooth k-scheme X. Since in general this is too complicated, we content ourselves with understanding the (archimedean) size of the eigenvalues of Frobenius.

Definition 5. Let E be a characteristic zero field, and $q, w \in \mathbb{R}_{\geqslant 0}$. Then $\lambda \in E$ is a q-Weil number of weight w if λ is algebraic over $\mathbb{Q}$, and its minimal polynomial has all complex roots having absolute value $q^{w/2}$.

Remark. Notice the division by 2 in the exponent, which will give as a counterpart that Tate twists do not modify the parity of an integer weight.

Definition 6. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}_l}$-sheaf on a k-scheme X. $\mathcal{F}$ is called *pure of weight w* if for every closed point $x \in X$ the $\overline{\mathbb{Q}_l}$-eigenvalues of $\mathrm{Frob}_{\mathcal{F},x}$ on $\mathcal{F}_x$ are q^{d_x}-Weil numbers of weight w.

If $\mathcal{F}$ admits a finite increasing filtration by constructible $\overline{\mathbb{Q}_l}$-subsheaves with successive quotients pure of weights $w_1, \ldots, w_n$, then $\mathcal{F}$ is said to be *mixed*, with weights $w_1, \ldots, w_n$.

One could also consider whether the eigenvalues are algebraic after applying a fixed isomorphism $\iota : \overline{\mathbb{Q}_l} \cong \mathbb{C}$, and then similarly define sheaves that are ι-pure of weight w and ι-mixed.

Remark. Beware! If one consider different isomorphisms $\iota_1, \iota_2 : \overline{\mathbb{Q}_l} \cong \mathbb{C}$, it is possible for a constructible $\overline{\mathbb{Q}_l}$-sheaf to have different weights for ι_1 and ι_2. See example 1.5.3.5 in [1].

On the other hand, if $\mathcal{F}$ is ι-pure of weight w for *all* isomorphisms $\iota : \overline{\mathbb{Q}_l} \cong \mathbb{C}$, then it is pure of weight w. To see this, notice that by definition this means that every eigenvalue of $\mathrm{Frob}_{\mathcal{F},x}$ on $\mathcal{F}_x$ has all its $\mathrm{Aut}(\mathbb{C})$-conjugate with the same absolute value, and hence must be algebraic.

Example 6. Consider the sheaf $\overline{\mathbb{Q}_l}(r)$ on X. For each $x \in |X|$ with $q_x = |k(x)|$, the action of geometric q_x-Frobenius on l-power roots of unity on $k(\overline{x})$ (i.e. on $\mathbb{Z}_l(1)$) is given by raising to the power q_x^{-1}, so the action on $\overline{\mathbb{Q}_l}(1)_{\overline{x}}$ is given by multiplication by $q_x^{-2/2}$. Taking tensor powers yield that $\overline{\mathbb{Q}_l}(r)$ is pure of weight $-2r$.

It turns out that the notion of Weil numbers and of weights for a constructible sheaf is very useful, and extremely important results can be stated in this language:

Theorem 10 (Weil's Riemann hypothesis)**.** *Let $f : X \longrightarrow S$ be smooth and projective, than the higher direct image sheaf $\mathrm{R}^w f_* \mathbb{Q}_l$ is pure of weight w.*

Proposition 11. *The notion of weight behaves well with respect to algebraic operations, in the following sense: suppose $\mathcal{F}$ is pure of weight w and $\mathcal{G}$ pure fo weight v. Then we have*

1. *$\mathcal{F}^\vee$ is pure of weight $-w$.*
2. *$\mathcal{F} \otimes \mathcal{G}$ is pure of weight $w + v$.*
3. *the Tate twist $\mathcal{F}(r)$ is pure of weight $w - 2r$.*

Obvious similar properties hold for mixed shaves.

Proof. This is an easy exercise left to the reader (just unwind the definitions!). □

Finally for the big fireworks:

Theorem 12 (Deligne's purity theorem)**.** *Let $f : X \longrightarrow S$ be a separated map of finite type k-schemes. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}_l}$-sheaf on X and $w \in \mathbb{R}$. Fix an isomorphism $\iota : \overline{\mathbb{Q}_l} \cong \mathbb{C}$. If $\mathcal{F}$ is ι-mixed (resp. mixed) of weights $\leqslant w$, then $\mathrm{R}^n f_! \mathcal{F}$ is ι-mixed (resp. mixed) of weights $\leqslant w + n$ for all n. Moreover, each weight of $\mathrm{R}^n f_! \mathcal{F}$ is congruent* $\mathrm{mod}\mathbb{Z}$ *to a weight of $\mathcal{F}$.*

We will not prove this theorem today, but instead show how to deduce from it the following very useful

Corollary 13. *Let X be a smooth, separated, finite type k-scheme, and $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}_l}$-sheaf, pure of weight $w \in \mathbb{Z}$. Then the image $\widetilde{H}^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ of $H^n_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ in $H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $w + n$.*

We also have the following immediate consequence:

Corollary 14 (Weil's Riemann hypothesis)**.** *Let X be a smooth, projective k-variety. Then $\mathrm{R}^n f_* \mathbb{Q}_l$ is pure of weight n.*

Proof of corollary 14. X satisfies the assumptions of corollary 13 with $\mathcal{F} = \mathbb{Q}_l$, which is clearly lisse, constructible and pure of weight $w = 0$. □

We can then conclude that Deligne's purity theorem generalizes Weil's Riemann hypothesis in two directions: it allows for non-proper objects and for non-constant sheaves.

Proof of corollary 13. The main idea is to use Poincaré duality to turn the upper bound given by the purity theorem on a dual cohomology group into a lower bound on $H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$, and then show that this lower bound coincide with the upper bound given by the purity theorem, and hence the sheaf is pure.

We can assume that X has pure dimension d, since it is k-smooth.By Deligne's purity theorem, $H^n_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights $\leqslant w+n$. Since $\mathcal{F}$ is lisse, X is smooth, separated of dimension d, all assumptions of the Poincaré duality theorem 1.4.6.5 in [1] (covered in Brian L.'s talk) are satisfied and hence we have the perfect pairing

$$H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F}) \times H^{2d-n}_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}^\vee(d)) \longrightarrow H^{2d}_{c,\text{ét}}\left(X_{\overline{k}}, \overline{\mathbb{Q}_l}(d)\right) \cong \overline{\mathbb{Q}_l}.$$

This pairing is Galois equivariant (not completely obvious, but follows by going back to see how we defined the pairing in sections 1.3.8 and 1.4.6 of [1], covered in the lectures by Tony and Sheela) and thus $H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is the Galois dual representation of $H^{2d-n}_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}^\vee(d))$ - in particular, the action of ϕ on $H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ goes to the action of ϕ^{-1} on $H^{2d-n}_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}^\vee(d))$. We study this latter action.

Since $\mathcal{F}^\vee(d)$ is still a lisse sheaf on X of pure weight $-w-2d$, Deligne's purity theorem gives us that $H^{2d-n}_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}^\vee(d))$ has weights $\leqslant (-w-2d)+(2d-n) = -w-n$. Therefore, on the dual space $H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ the weights are $\geqslant w+n$.

Finally, the map

$$H^n_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}) \longrightarrow H^n_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$$

is equivariant for the action of the q-Frobenius (recall that this was defined as 'q-power on the structure sheaf, identity on topological space'), and hence the image also has weights $\geqslant w+n$. This completes the squeeze and proves that the image is pure of weight $w+n$. □

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