2. Complex differential forms

Some approaches to this topic obscure the difficulties using deceptively simple notation. I will explain why the equation $(\partial+\overline{\partial})^2 = 0$ is equivalent to the bracket conditions $\left[T_X^{0,1}, T_X^{0,1}\right] \subseteq T_X^{0,1}$ and $\left[T_X^{1,0}, T_X^{1,0}\right] \subseteq T_X^{1,0}$.

2.1. Definitions.

We assume that X is a complex manifold and we will use holomorphic local coordinates. But it is easier to keep track of what we are doing if we use the notation of an almost complex manifolds and we have actions of $I = I \otimes 1$ and $i = 1 \otimes i$ on $T_X \otimes \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$. In the old notes and in our book [Voisin], there is some confusion about that does Ω_X mean. We will use the notation Ω_X for the space of 1-forms on X and $\Omega_X^{p,q}$ for the space of (p,q)-forms on X (Definition 2.1.3).

Recall that a 1-*form* on X is a section of the cotangent bundle T_X^* of X. The space of 1-forms on X is denoted Ω_X^1 or simply Ω_X. In local coordinates Ω_X is generated by 1-forms $dx_1, \cdots, dx_n$ and general 1-forms are given by

$$\sum_{i=1}^{n} f_i \, dx_i$$

where $f_1, \cdots, f_n$ are elements of $C^\infty(X)$, the ring of smooth functions $X \to \mathbb{R}$.

Definition 2.1.1. Given a complex manifold X, let $\Omega_{X,\mathbb{C}}$ denote the space of $\mathbb{C}$-valued smooth (C^∞) 1-forms on X. Then $\Omega_{X,\mathbb{C}} = \Omega_X \otimes \mathbb{C}$ is the space of smooth sections of $T_X^* \otimes \mathbb{C}$. On a coordinate chart (U, φ) with holomorphic local coordinates $z_i = x_i + iy_i$, elements of $\Omega_{X,\mathbb{C}}$ are generated over $\mathbb{C}$ by the 1-forms $dx_i, dy_i = I dx_i$. Let

$$dz_j = dx_j + i dy_j.$$

Evaluation at any point $x \in U$ gives homomorphisms $dz_j : T_{U,x} \to \mathbb{C}$ given by:

$$dz_j : \frac{\partial}{\partial x_j} \mapsto 1$$

$$dz_j : \frac{\partial}{\partial y_j} \mapsto i$$

and dz_j sends the other basis elements $\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}$, $k \neq j$ to 0. (So, $dz_j \, I = i \, dz_j$.) Similarly,

$$d\overline{z}_j = dx_j - i dy_j.$$

Note that $d\overline{z}_j$ is the composition of dz_j with complex conjugation c which is an $\mathbb{R}$-linear automorphism of $\mathbb{C}$:

$$d\overline{z}_j = c \circ dz_j.$$

As in the case of $T_{X,\mathbb{C}} = T_X \otimes \mathbb{C}$, let

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

be the decomposition of $\Omega_{X,\mathbb{C}}$ into the i and $-i$ eigenspaces of I. This decomposition is independent of the choice of local coordinates.

Locally, $dz_j \in \Omega_U^{1,0}$ since $dz_j I = idz_j$ and the dz_j form a complex basic for $\Omega_U^{1,0}$ in the sense that every $\alpha \in \Omega_U^{1,0}$ can be written uniquely as

$$\alpha = \sum_{j=1}^{n} \alpha^j dz_j$$

where $\alpha^j : U \to \mathbb{C}$ are smooth maps (C^∞ over $\mathbb{R}$). Similarly, $d\overline{z}_j$ forms a basis of $\Omega_X^{0,1}$.

Proposition 2.1.2. *If $f : X \to \mathbb{C}$ is a smooth function ($f = g + ih$, $g, h : X \to \mathbb{R}$) then $df = dg + idh \in \Omega_{X,\mathbb{C}}$ is given in local coordinates by*

$$df = \sum \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

Proof. Expand the RHS:

$$\sum \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)(dx_j + idy_j) + \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)(dx_j - idy_j)$$

$$= \sum \frac{1}{2}\left(\frac{\partial f}{\partial x_j} dx_j - i\frac{\partial f}{\partial y_j} idy_j\right) + same$$

$$= \sum \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j = df.$$

□

Definition 2.1.3. Let $\Omega_X^{p,q} \subset \Omega_{X,\mathbb{C}}^{p+q}$ denote the space of $p+q$-forms on X with coefficients in $\mathbb{C}$ which are given locally by

$$\alpha = \sum_{I,J} \alpha^{I,J} \underbrace{dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}}_{dz_I} \wedge \underbrace{d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \wedge \cdots \wedge d\overline{z}_{j_q}}_{d\overline{z}_J} = \sum_{I,J} \alpha^{I,J} dz_I \wedge d\overline{z}_J$$

where the sum is over all pairs of multi-indices $I = (i_1, i_2, \cdots, i_p)$ and $J = (j_1, j_2, \cdots, j_q)$ and $\alpha^{I,J} \in C^\infty(X, \mathbb{C})$.

We will show in section 2.2 below that this is independent of the choice of local coordinates. Suppose for a moment that this is true. Then, the following are obvious.

Proposition 2.1.4.

$$\Omega_{X,\mathbb{C}}^k = \sum_{p+q=k} \Omega_X^{p,q}$$

$$d\Omega_X^{p,q} \subset \Omega_X^{p+1,q} \oplus \Omega_X^{p,q+1}$$

$d = \partial + \overline{\partial}$ where

$$\partial : \Omega_X^{p,q} \to \Omega_X^{p+1,q}$$

$$\overline{\partial} : \Omega_X^{p,q} \to \Omega_X^{p,q+1}$$

$$\partial^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \overline{\partial}^2 = 0.$$

Proof. The key part is the second line. Since $d\alpha^{I,J} = \sum \frac{\partial \alpha^{I,J}}{\partial z_j} dz_j + \frac{\partial \alpha^{I,J}}{\partial \overline{z}_j} d\overline{z}_j$,

$$\begin{aligned} d\alpha = \sum \frac{\partial \alpha^{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z}_J &+ \frac{\partial \alpha^{I,J}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J \\ \in \qquad \Omega_X^{p+1,q} \quad &\oplus \quad \Omega_X^{p,q+1}. \end{aligned}$$

□

The important properties of the boundary operators $\partial, \overline{\partial}$ have been trivialized! The key step is independence of coordinates which I need to explain.

2.2. **Coordinate invariant definitions.** We need to show that the definition of $\Omega_X^{p,q}$ is independent of the choice of local coordinates. To do this we formulate the definition intrinsically, without using local coordinates.

Suppose that X is a real n-manifold and Ω_X^k is the *space of k-forms* on X. Elements are alternating $C^\infty(X)$-multilinear maps

$$\omega : \Gamma T_X \otimes \cdots \otimes \Gamma T_X \to C^\infty(X)$$

where ΓT_X is the space of sections of T_X, i.e., vector fields on X. *Alternating* means $\omega(\sigma_1, \cdots, \sigma_k) = 0$ if the σ_i are not distinct (equivalently, the sign changes if we switch two of them). $C^\infty(X)$-*multilinear* means additive in each entry and

$$\omega(f_1\sigma_1, \cdots, f_n\sigma_k) = f_1 f_2 \cdots f_n \omega(\sigma_1, \cdots, \sigma_k)$$

for $f_i : X \to \mathbb{R}$ in $C^\infty(X)$. This second condition is equivalent to saying that ω is determined by its value at each point, i.e., ω is a section of the k-th tensor power of the cotangent bundle.

Given k vector fields $\sigma_1, \cdots, \sigma_k$ on X we have $\omega(\sigma_1, \cdots, \sigma_k) : X \to \mathbb{R}$. In local coordinates we have

$$\omega = \sum_I \omega^I dx_I$$

where the sum is over all multi-indices $I = (i_1, i_2, \cdots, i_k)$ and $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$. We also recall that, if $f : X \to \mathbb{R}$ is a smooth map and $\sigma = \sum \sigma^i \frac{\partial}{\partial x_i}$ then

$$\sigma(f) = \sum \sigma^i \frac{\partial f}{\partial x_i} \in C^\infty(X).$$

The coordinate-free description is that σ is a derivation on $C^\infty(X)$.

The differential $d : \Omega_X^k \to \Omega_X^{k+1}$ in the deRham complex of X has the following coordinate-free definition.

$$\begin{aligned} d\omega(\sigma_0, \cdots, \sigma_k) &= \sum_{i=0}^{k} (-1)^i \sigma_i \left(\omega(\sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \sigma_k)\right) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \widehat{\sigma_j}, \cdots, \sigma_k). \end{aligned} \tag{2.1}$$

For example, when $k = 1$ and $\alpha \in \Omega_X^1$ we have

$$d\alpha(\sigma, \tau) = \sigma(\alpha(\tau)) - \tau(\alpha(\sigma)) - \alpha([\sigma, \tau]).$$

Proposition 2.2.1. *Suppose that* $\omega = \sum_I \omega^I dx_I \in \Omega^k_X$. *Then*

$$d\omega = \sum_I \sum_i \frac{\partial \omega^I}{\partial x_i} dx_i \wedge dx_I.$$

Proof. $d\omega = \sum c^J dx_J$. To find c^J for $J = (j_0, \cdots, j_k)$:

$$c_j = d\omega(\sigma_0, \cdots, \sigma_k) = d\omega\left(\frac{\partial}{\partial x_{j_0}}, \cdots, \frac{\partial}{\partial x_{j_k}}\right)$$

where $\sigma_i = \frac{\partial}{\partial x_{j_i}}$. But, $[\sigma_i, \sigma_j] = 0$ for these vector fields (because their coefficients are constant functions). So, we only get the first sum in (2.1), the definition of $d\omega$. But

$$\omega(\sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \sigma_k) = \omega^I$$

where $I = (j_0, \cdots, \widehat{j_i}, \cdots, j_n)$. Call this $I = \delta_i J$. So,

$$d\omega = \sum_J c^J dx_J = \sum_J \sum_i (-1)^i \frac{\partial \omega^{\delta_i J}}{\partial x_{j_i}} dx_J$$

which is the same as the other equation since $(-1)^i dx_J = dx_{j_i} \wedge dx_{\delta_i J}$ and $I = \delta_i J$. $\square$

$\Omega^{p,q}_X$, already defined, has the following coordinate-free characterization.

Lemma 2.2.2. *A k-form on X with coefficients in $\mathbb{C}$: $\omega \in \Omega^k_{X,\mathbb{C}}$ lies in $\Omega^{p,q}_X$ where $p+q=k$ if and only if it has the following property.*

Suppose that $\sigma_1, \cdots, \sigma_k$ are sections of either $T^{1,0}_X$ or $T^{0,1}_X$. Then

$$\omega(\sigma_1, \cdots, \sigma_k) = 0$$

unless exactly p of the σ_i lie in $T^{1,0}_X$ and q of them lie in $T^{0,1}_X$.

Note that this characterization makes sense for any almost complex manifold. In other words, $\Omega^{p,q}_X$ is well-defined for any almost complex X. The following observation from [Voisin] is the crucial concept underlying the construction of the double complex $\Omega^{*,*}_X$ which was hidden in the other approach (which is also correct).

Theorem 2.2.3. *The differential $d : \Omega^k_X \to \Omega^{k+1}_X$ sends $\Omega^{p,q}_X$ into $\Omega^{p+1,q}_X \oplus \Omega^{p,q+1}_X$ if and only if the bracket conditions holds: $[T^{0,1}_X, T^{0,1}_X] \subseteq T^{0,1}_X$ and similarly for $T^{1,0}_X$.*

Proof. Look at (2.1), the definition of $d\omega$. Assume the bracket conditions. In the first sum, one of the σ_i is deleted. So, in order to have p holomorphic and q antiholomorphic sections we needed to start with $p+1, q$ or $p, q+1$.

In the second sum, if σ_i, σ_j are in $T^{1,0}_X$ then so is $[\sigma_i, \sigma_j]$. So, the number of terms in $\Gamma T^{1,0}_X$ decreases by one in that case. Similarly, if σ_i, σ_j are in $\Gamma T^{0,1}_X$ then so is $[\sigma_i, \sigma_j]$.

Finally, if one of σ_i, σ_j is in $\Gamma T^{1,0}_X$ and the other in $\Gamma T^{0,1}_X$ then $[\sigma_i, \sigma_j]$ is in $\Gamma T_{X,\mathbb{C}} = \Gamma T^{1,0}_X \oplus \Gamma T^{0,1}_X$.

Conversely, suppose the bracket condition does not hold. Then, e.g, there might be σ_i, σ_j in $\Gamma T^{1,0}_X$ so that $[\sigma_i, \sigma_j]$ has a $\Gamma T^{0,1}_X$ component. Such a component will be linearly independent from all other terms, so there is a $\omega \in \Omega^{p,q}_X$ so that $d\omega$ is nonzero on that term and zero on all other terms giving $d\omega$ a $\Omega^{p+2,q-1}_X$ component. $\square$

Remark 2.2.4. (1) The equation $d^2 = 0$ which implies that $\partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0$ follows from the definition since any differential form on X with coefficients in $\mathbb{C}$ is the sum of two real forms $\omega = \alpha + i\beta$ and $d\omega = d\alpha + id\beta$. So, $d^2\omega = d^2\alpha + id^2\beta = 0$.

(2) The bracket condition holds for $T_X^{1,0}$ if and only if it holds for $T_X^{0,1}$ because these are related by complex conjugation (which acts on the second factor of $T_X \otimes \mathbb{C}$) and

$$[\overline{\sigma}, \overline{\tau}] = \overline{[\sigma, \tau]}.$$

So, conjugating both sides of $[T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$ gives

$$\overline{[T_X^{0,1}, T_X^{0,1}]} = [\overline{T_X^{0,1}}, \overline{T_X^{0,1}}] = [T_X^{1,0}, T_X^{1,0}] \subseteq \overline{T_X^{0,1}} = T_X^{1,0}.$$

2.3. **Local exactness of the $\overline{\partial}$ complex.** Review: We have a complex manifold X. We use holomorphic local coordinates $z_1, \cdots, z_n$ where $z_j = x_j + iy_j$.

Complexified differential forms are:

$$\Omega^k_{X,\mathbb{C}} = \Omega^k_X \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega_X^{p,q}.$$

In local coordinates, elements of $\Omega_X^{p,q}$ have the form

$$\alpha = \sum_{I,J} \alpha^{I,J} dz_I \wedge d\overline{z}_J$$

where $I = (i_1, \cdots, i_p), J = (j_1, \cdots, j_q)$. We proved that the standard differential $d : \Omega_X^k \to \Omega_X^{k+1}$ only has terms in degree $(1,0)$ and $(0,1)$:

$$d = \partial + \overline{\partial} : \Omega_X^{p,q} \to \Omega_X^{p+1,q} \oplus \Omega_X^{p,q+1}$$

where

$$\partial\alpha = \sum_{I,J} \partial\alpha^{I,J} dz_I \wedge d\overline{z}_J = \sum_{I,J,i} \frac{\partial\alpha^{I,J}}{\partial dz_i} dz_i \wedge dz_I \wedge d\overline{z}_J$$

$$\overline{\partial}\alpha = \sum_{I,J} \overline{\partial}\alpha^{I,J} dz_I \wedge d\overline{z}_J = \sum_{I,J,i} \frac{\partial\alpha^{I,J}}{\partial d\overline{z}_i} d\overline{z}_i \wedge dz_I \wedge d\overline{z}_J.$$

It follows immediately that

$$\partial\partial = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \overline{\partial}\overline{\partial} = 0$$

and that ∂ and $\overline{\partial}$ are derivations.

Today, we will prove the local exactness of $\overline{\partial}$:

Theorem 2.3.1. *Suppose $\alpha \in \Omega_X^{p,q}$ with $q > 0$ so that $\overline{\partial}\alpha = 0$ in a nbh of some point $x \in X$. Then there is a $\beta \in \Omega_X^{p,q-1}$ so that $\overline{\partial}\beta = \alpha$ in a nbh of x.*

To prove this we have to go back to Chapter I.

2.4. **One complex variable.** We start with Stokes' Theorem:

$$\int_{M^n} d\alpha = \int_{\partial M} \alpha$$

if M is a (real) n-manifold $\alpha \in \Omega^{n-1}_{M,\mathbb{C}}$ (applying Stokes to real and complex parts of α separately) and either

(1) M is compact or
(2) α has compact support

We apply this to the case $n = 2$ and $M \subseteq \mathbb{C}$.

Case (1) M is compact in $\mathbb{C}$. Suppose $\alpha = f(z)dz$ where f is a complex differentiable function on M ($\frac{\partial f}{\partial \overline{z}} = 0$). Then

$$d\alpha = \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}\right) \wedge dz = 0$$

since $dz \wedge dz = 0$ and $\frac{\partial f}{\partial \overline{z}} = 0$. So,

$$0 = \int_{\partial M} f(z)dz = 0.$$

This implies:

Theorem 2.4.1 (Cauchy's formula). *If f is C^1 and $\frac{\partial f}{\partial \overline{z}} = 0$ then, for all $|z| < 1$,*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_1} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Proof. By Stokes' thm for M the closure of $D_1 - D_\varepsilon$, this integral is equal to:

$$\frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z}d\zeta \tag{2.2}$$

where D_ε is the ε-disk around z. But (as we will see in a minute) this integral gives the average value of $f(\zeta)$ on ∂D_ε. So, as $\varepsilon \to 0$, this converges to $f(z)$. Since it is independent of ε, it must be equal to $f(z)$ for all ε. □

This formula shows that $f(z)$ is analytic since the integrand is given by a converging (geometric) series which can be differentiated term by term by the dominated convergence theorem. (See Theorem 11.0.2 in Appendix.)

Case (2) Suppose f is not holomorphic but has compact support. Then (2.2) is not equal to $f(z)$, but it is still equal to the average value of $f(\zeta)$ as we now verify:

Let $\zeta' = \zeta - z$. Then $|\zeta'| = \varepsilon$. So, $\zeta' = \varepsilon e^{i\theta}$, $d\zeta' = i\zeta' d\theta$ and

$$(2.2) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+\zeta')}{\zeta'} i\zeta' d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z+\zeta')d\theta = ave\, f(\zeta).$$

Since f is continuous, this converges to $f(z)$ as $\varepsilon \to 0$.

By Stokes' thm,

$$(2.2) = -\frac{1}{2\pi i}\int_{\mathbb{C}-D_\varepsilon(z)} \frac{\partial}{\partial \overline{\zeta}'}\left(\frac{f(z+\zeta')}{\zeta'}\right) d\overline{\zeta}' \wedge d\zeta' \to f(z).$$

So the following improper integral converges:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\zeta'}\frac{\partial f(z+\zeta')}{\partial \overline{\zeta}'}\, d\zeta' \wedge d\overline{\zeta}' \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\zeta'}\frac{\partial f(z+\zeta')}{\partial \overline{z}}\, d\zeta' \wedge d\overline{\zeta}' \\ &= \frac{\partial}{\partial \overline{z}} \underbrace{\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\zeta'} f(z+\zeta')\, d\zeta' \wedge d\overline{\zeta}'}_{=g(z)}. \end{aligned}$$

So, $f(z) = \frac{\partial g}{\partial \overline{z}}$ where, substituing $\zeta = z + \zeta'$, $d\zeta = d\zeta'$, $d\overline{\zeta} = d\overline{\zeta}'$, we have

$$g(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{2.3}$$

If f is C^k then this integral can be (partially) differentiated k times under the integral sign (integral over $\mathbb{C} - D_\varepsilon$). So, g is C^k.

Theorem 2.4.2. *Any C^k function $f : \mathbb{C} \to \mathbb{C}$ is locally equal to $\frac{\partial g}{\partial \overline{z}}$ for the C^k function $g : \mathbb{C} \to \mathbb{C}$ given by* (2.3) *above.*

We need the multi-variable version of this:

Lemma 2.4.3. *Suppose that U, V are open subsets of $\mathbb{C}^p, \mathbb{C}^q$ and $f : U \times V \to \mathbb{C}$ is holomorphic in the first p variables. I.e., $\frac{\partial f}{\partial \overline{z}_i} = 0$ for $i \leq p$. Then, locally,*

$$f = \frac{\partial g}{\partial \overline{z}_{p+1}}$$

where $g : U \times V \to \mathbb{C}$ is also holomorphic in the first p variables.

Proof. Replace V with $\mathbb{C}^q$ and multiply f with a C^∞ function on V with compact support. Let

$$g(u, z_{p+1}, \cdots, z_n) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(u, \zeta, z_{p+2}, \cdots, z_n)}{\zeta - z_{p+1}} d\zeta \wedge d\overline{\zeta}.$$

Then $f(z) = \frac{\partial g(z)}{\partial \overline{z}_{p+1}}$ by Theorem 2.4.2 above. □

2.5. **Back to differential forms.** We are proving Theorem 2.3.1: Given $\alpha \in \Omega_X^{p,q}$ with $q > 0$ so that $\overline{\partial}\alpha = 0$ we need to show that, locally, $\alpha = \overline{\partial}\beta$ for some β. If

$$\alpha = \sum_{I,J} \alpha^{I,J} dz_I \wedge d\overline{z}_J.$$

then

$$\overline{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha^{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$

For this to be zero, the sum of all terms with the same dz_I must be zero. So, we may assume $p = 0$ and

$$\alpha = \sum_J \alpha^J d\overline{z}_J \in \Omega_X^{0,q}$$

where $q > 0$.

Let t be the smallest index which occurs in any J. (We do downward induction on t.) Then

- Every α^J is holomorphic in every z_i where $i < t$ since the term

$$\frac{\partial \alpha^J}{\partial \overline{z}_i} d\overline{z}_i \wedge d\overline{z}_J$$

must be zero (it cannot be cancelled with another term in $\overline{\partial}\alpha = 0$.)

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$$\alpha = d\overline{z}_t \wedge \beta + \alpha'$$

where the smallest index that occurs in either α' or β is $t' > t$ and

$$\beta = \sum_K \beta^K d\overline{z}_K$$

where each β^K is holomorphic in $z_1, \cdots, z_{t-1}$.

- By Lemma 2.4.3, there are g^K holomorphic in $z_1, \cdots, z_{t-1}$ so that

$$\frac{\partial g^K}{\partial \overline{z}_t} = \beta^K.$$

Then,

$$\overline{\partial} \sum_K g^K d\overline{z}_K = \sum_K \beta^K d\overline{z}_t \wedge d\overline{z}_K + \text{ terms with smallest index } > t.$$

So, $\alpha - \overline{\partial} \sum_K g^K d\overline{z}_K$ has smallest index $> t$ (and is thus equal to zero if t is maximal, proving the base case of the induction). By downward induction on t, $\alpha - \overline{\partial} \sum_K g^K d\overline{z}_K = \overline{\partial}\gamma$ for some $\gamma \in \Omega_X^{0,q-1}$ and thus

$$\alpha = \overline{\partial}\left(\gamma + \sum_K g^K d\overline{z}_K\right).$$

For $p > 0$ we have $\alpha = \sum \alpha^I dz_I$ and $\overline{\partial}\alpha^I = 0$ for each I. So, $\alpha^I = \overline{\partial}\gamma^I$ for some $\gamma^I \in \Omega_X^{0,q-1}$ and

$$\alpha = \overline{\partial} \sum_I \gamma^I dz_I.$$

This completes the proof of Theorem 2.3.1.

2.5.1. *$\partial\overline{\partial}$ lemma.* This is one variation of Theorem 2.3.1 that we will need later. We need the observation that complex conjugation reverses ∂ and $\overline{\partial}$:

$$\overline{(\partial\alpha)} = \overline{\partial}\,\overline{\alpha}$$

So, if $\partial\alpha = 0$ for some (p,q)-form α, then, $\overline{\partial}\,\overline{\alpha} = 0$. So, by Theorem 2.3.1, there is a locally defined β so that $\overline{\partial}\beta = \overline{\alpha}$ which means $\partial\overline{\beta} = \alpha$.

Theorem 2.5.1 ($\partial\overline{\partial}$-lemma)**.** *Suppose $\omega \in \Omega_X^{p,q}$ with $p,q > 0$ so that $d\omega = 0$ in a nbh of some point $x \in X$. Then there is a $\varphi \in \Omega_X^{p-1,q-1}$ so that $\partial\overline{\partial}\varphi = \omega$ in a nbh of x.*

Proof. We given the proof only in the case that we need which is the case $p = q = 1$.

Given that $d\omega = 0$, there is 1-form β so that $d\beta = \omega$. If we write

$$\beta = \beta^{0,1} + \beta^{1,0}$$

we get

$$\partial\beta^{0,1} + \overline{\partial}\beta^{1,0} = \omega$$

and

$$\overline{\partial}\beta^{0,1} = 0 = \partial\beta^{1,0}$$

By Theorem 2.3.1 there is a $\gamma_1 \in \Omega^{0,0}$ so that $\overline{\partial}\gamma_1 = \beta^{0,1}$. By the observation above, there is also a $\gamma_2 \in \Omega^{0,0}$ so that $\partial\gamma_2 = \beta^{1,0}$. Let $\varphi = \gamma_1 - \gamma_2$. Then

$$\begin{aligned}\partial\overline{\partial}\varphi &= \partial\overline{\partial}\gamma_1 - \partial\overline{\partial}\gamma_2 \\ &= \partial\beta^{0,1} + \overline{\partial}\partial\gamma_2 \\ &= \partial\beta^{0,1} + \overline{\partial}\beta^{1,0} = \omega.\end{aligned}$$

In the general case when $p,q > 0$, we need another argument to show that β can be chosen to be in the form

$$\beta = \beta^{p-1,q} + \beta^{p,q-1}.$$

To see this, let $k < p-1$ be minimal so that β has an unwanted term $\beta^{k,p+q-k-1}$. Then $\overline{\partial}\beta^{k,p+q-k-1} = 0$, so $\beta^{k,p+q-k-1} = \overline{\partial}\gamma$. Then, we replace β with $\beta - d\gamma$ which has no term in $\Omega^{k,p+q-k-1}$. □

2.6. **Dolbeault complex.** Although $\Omega_X^{p,q}$ is a bicomplex with vertical and horizontal boundary maps $\partial, \overline{\partial}$, we will see that, for a holomorphic bundle, only $\overline{\partial}$ gives a well-defined boundary map.

Let E be a holomorphic bundle over a complex manifold X. By definition this is locally trivial, we have local trivalizations $\varphi_i : E_{U_i} \xrightarrow{\approx} U_i \times \mathbb{C}^k$ whose transition maps $\varphi_{ij} = \varphi_j^{-1}\varphi_i$ give a $k \times k$ matrix of holomorphic maps

$$U_i \cap U_j \to M_k(\mathbb{C}) \cong \mathbb{C}^{k^2}$$

let $A^{0,q}(E) = \Omega_X^{0,q} \otimes E$ be the space of $(0,q)$-forms on X with coefficients in E. If E is a trivial line bundle, $A^{0,q}(E) = \Omega_X^{0,q}$. More generally, $A^{0,q}(E)$ looks like k copies of $\Omega_X^{0,q}$. Thus, in local coordinates an element $\alpha \in A^{0,q}(E)$ is given by

$$\alpha = (\alpha_1, \cdots, \alpha_k)$$

where each $\alpha_i \in \Omega_U^{0,q}$. Let $\overline{\partial}\alpha$ be given by

$$\overline{\partial}\alpha = (\overline{\partial}\alpha_1, \cdots, \overline{\partial}\alpha_k)$$

Given two coordinate charts $(U_i, \varphi_i), (U_j, \varphi_j)$, we need to know that $\overline{\partial}\alpha$ defined by this formula agree on $U_i \cap U_j$. Then we can conclude that $\overline{\partial}$ is a well defined operator

$$\overline{\partial} : A^{0,q}(E) \to A^{0,q+1}(E)$$

Proposition 2.6.1. *The boundary $\overline{\partial}\alpha = (\overline{\partial}\alpha_1, \cdots, \overline{\partial}\alpha_k)$ is well defined. I.e., if α is changed to $\alpha' = (\alpha_1', \cdots, \alpha_k')$ by holomorphic change of coordinates*

$$\alpha_t' = \sum (\varphi_{ij})_{st}\alpha_s$$

then $\overline{\partial}\alpha'$ is related to $\overline{\partial}\alpha$ by the same change in coordinates.

Proof. This follows from the Leibnitz rule and the fact that φ_{ij} is holomorphic. So,

$$\overline{\partial}\alpha_t' = \overline{\partial} \sum (\varphi_{ij})_{st}\alpha_s = \sum (\varphi_{ij})_{st}\overline{\partial}\alpha_s$$

□

Definition 2.6.2. The *Dolbeault complex* of a homomorphic bundle E is

$$\cdots \xrightarrow{\overline{\partial}} A^{0,q-1}(E) \xrightarrow{\overline{\partial}} A^{0,q}(E) \xrightarrow{\overline{\partial}} A^{0,q+1}(E) \xrightarrow{\overline{\partial}} \cdots$$

On of the key point is:

Proposition 2.6.3. *The kernel of*

$$\overline{\partial} : A^{0,0}(E) = \Gamma(E) \to A^{0,1}(E)$$

is the space of holomorphic sections of E.

Proof. In local coordinates, an element of $A^{0,0}(E)$ is k functions: $(f_1, \cdots, f_k)$ and

$$\overline{\partial}(f_1, \cdots, f_k) = (\overline{\partial}f_1, \cdots, \overline{\partial}f_k)$$

is zero iff every f_i is holomorphic. □