FOURIER TRANSFORMS ON THE BASIC AFFINE SPACE OF A QUASI-SPLIT GROUP

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Abstract. We extend the Gelfand and Graev construction of generalized Fourier transforms on basic affine space from split groups to quasi-split groups over a local non-archimedean field F.

1. Introduction

1.0.1. *Notation.*

- Let F be a local non-archimedean field with the norm $|\cdot| = |\cdot|_F$, the ring of integers $\mathcal{O}$ and a fixed uniformizer ϖ such that $|\varpi| = q^{-1}$, where q is the cardinality of the residue field.
- We fix a non-trivial additive character ψ throughout the paper. The self-dual Haar measure dx on F with respect to ψ defines the Haar measure $d^\times x = \frac{dx}{|x|}$ on $F^\times$.
- For a quadratic extension K of F we denote by χ_K the quadratic character of $F^\times$, associated to K by class field theory. We also denote by χ_0 the trivial character of $F^\times$.
- For a space Y over F we denote by $\mathcal{S}^\infty(Y)$ (resp. $\mathcal{S}_c(Y)$) the space of locally constant (resp. locally constant of compact support) functions on Y.
- Throughout this paper we use boldface characters for group schemes over F, such as $\mathbf{H}$, and plain text characters for their group of F-points, such as H.
- Let $\mathbf{G}$ be a simply-connected quasi-split group defined over F with a maximal F-split torus $\mathbf{T}'$ and the maximal torus $\mathbf{T} = \mathbf{Z}_{\mathbf{G}}(\mathbf{T}')$. We fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$ so that $\mathbf{B} = \mathbf{T} \cdot \mathbf{U}$. We write $\mathbf{U}^{op}$ for the unipotent radical of the opposite Borel subgroup.
- The Weyl group $W = N_G(T')/T$ acts on T by conjugation and we write t^w for $w^{-1}tw$ for all $t \in T$, $w \in W$.
- The quotient $X = U\backslash G$ is called the basic affine space of G. For any $g \in G$ we write $[g]$ for the element Ug in X. The space X admits unique, up to a scalar, G-invariant measure ω_X. The precise choice of ω_X is not important for general G, but will be fixed for groups of rank 1.

1.1. **Fourier transforms on the basic affine space of a quasi-split group.** We define a unitary representation θ of the group $G\times T$ on $L^2(X, \omega_X)$

by:

$$\theta(g,t)f([h]) = \delta_B^{1/2}(t)f([t^{-1}hg]),$$

where δ_B is the modular character.

For split groups Gelfand and Graev in [GG73], see also [KL88], [Kaz95], extended the action θ of $G \times T$ to a representation of $G \times (T \rtimes W)$, so that every element w of W acts on $L^2(X, \omega_X)$ by an operator Φ_w, called a generalized Fourier transform. Our paper has two goals:

- To extend the construction by Gelfand and Graev to quasi-split groups.
- To show that the Whittaker map intertwines the action of W on a dense subspace $\mathcal{S}_0(X)$ in $L^2(X)$ with the natural action of W on the space of Whittaker vectors. We show (see Theorem 1.2) that this property characterizes uniquely the operators Φ_w.

1.1.1. *Whittaker map.* Fix a non-degenerate character Ψ of U^{op}. The map

$$\mathcal{W}_\Psi : \mathcal{S}_c(X) \to \mathcal{S}_c(T), \quad \mathcal{W}_\Psi(f)(t) = \int\limits_{U^{op}} \theta(t)f([u])\Psi^{-1}(u)du, \tag{1.1}$$

defines an isomorphism $\mathcal{S}_c(X)_{U^{op},\Psi} \simeq \mathcal{S}_c(T)$.

We define an action of W on $\mathcal{S}_c(T)$. For split groups set

$$w \cdot \varphi(t) = \varphi(t^w).$$

For quasi-split groups see Definition 5.6.

We define (see 6.1) a $G \times T$ submodule $\mathcal{S}_0(X)$ that is dense in $L^2(X)$ and put

$$\mathcal{S}_0(T) = \mathcal{W}_\Psi(\mathcal{S}_0(X)) \simeq \mathcal{S}_0(X)_{U^{op},\Psi}.$$

There is a natural map $\kappa_\Psi : \mathrm{End}_G(\mathcal{S}_0(X)) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(X)_{U^{op},\Psi}) = \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(T))$ such that for every $\mathcal{B} \in \mathrm{End}_G(\mathcal{S}_0(X))$ the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{S}_0(X) \arrow[r, "\mathcal{B}"] \arrow[d, "\mathcal{W}_\Psi"'] & \mathcal{S}_0(X) \arrow[d, "\mathcal{W}_\Psi"] \\
\mathcal{S}_0(T) \arrow[r, "\kappa_\Psi(\mathcal{B})"] & \mathcal{S}_0(T)
\end{tikzcd}$$

We prove in Proposition 6.2 that the map κ_Ψ is injective.

1.1.2. *Main Theorem.* With this notation we formulate our main result.

Theorem 1.2. *There exists unique family of unitary operators $\Phi_w, w \in W$, on $L^2(X, \omega_X)$, preserving the space $\mathcal{S}_0(X)$ and satisfying:*

$$\begin{cases} \Phi_w \circ \theta(g, t^w) = \theta(g,t) \circ \Phi_w & \forall w \in W, t \in T, g \in G \\ \Phi_{w_1}\Phi_{w_2} = \Phi_{w_1 w_2} & \forall w_1, w_2 \in W \\ \kappa_\Psi(\Phi_w)(\varphi) = w \cdot \varphi & \forall w \in W, \varphi \in \mathcal{S}_0(T) \end{cases} \tag{1.3}$$

Let us sketch the proof.

(1) First consider a quasi-split, almost simple, simply-connected group G_1 of rank one. The group G_1 is isomorphic to either $\mathrm{Res}_L SL_2$ or $\mathrm{Res}_L SU_3$ for a finite extension L of F. Without loss of generality we can assume that $L = F$. In both cases the Weyl group $W = \{e, s\}$ consists of two elements. We shall define the generalized Fourier operator Φ_s, separately for these two cases.
   - In the case $G_1 = SL_2$ the set X can be identified with $V - 0$ for a symplectic two dimensional plane V. In this case $\Phi_s \in \mathrm{Aut}(L^2(X)) = \mathrm{Aut}(L^2(V))$ is defined to be the classical Fourier transform with respect to the symplectic form on V. Theorem 1.2 in this case is proven in Section 3.
   - In the case $G_1 = SU_3$, the set X can be identified with the set of non-zero isotropic vectors in a 6 dimensional quadratic space. The treatment of this case is the crux of the paper. In [GK22] we have defined a unitary operator $\Phi \in L^2(X)$ of order 2, commuting with G_1 and anti-commuting with T', and provided an explicit formula for the restriction of Φ to the space $\mathcal{S}_c(X)$. We put $\Phi_s = \Phi$ and prove Theorem 1.2 in this case in Section 4.

(2) For a general quasi-split group G and any simple reflection s we, using the results for groups of rank 1, define a unitary involution $\Phi_s \in \mathrm{Aut}(L^2(X))$, satisfying

$$\begin{cases} \Phi_s \circ \theta(g, t^s) = \theta(g, t) \circ \Phi_s & \forall t \in T, g \in G \\ \kappa_\Psi(\Phi_s)(\varphi) = s \cdot \varphi & \forall \varphi \in \mathcal{S}_0(T) \end{cases} .$$

(3) For arbitrary $w \in W$ with a presentation $w = s_1 \cdot s_2 \cdot \ldots \cdot s_n$ as a product of simple reflections we define $\Phi_w = \Phi_{s_1} \circ \Phi_{s_2} \ldots \circ \Phi_{s_n}$. Hence the operators Φ_w are unitary and possess the desired equivariance properties. It remains to prove that Φ_w does not depend on the presentation. For every $\varphi \in \mathcal{S}_0(T)$ one has $\kappa_\Psi(\Phi_w)(\varphi) = w \cdot \varphi$ and so $\kappa_\Psi(\Phi_w)$ does not depend on the presentation of w. Since κ_Ψ is injective, the operator Φ_w does not depend on the presentation of w as well. In particular, $\Phi_{w_1} \circ \Phi_{w_2} = \Phi_{w_1 w_2}$ for $w_1, w_2 \in W$ and the operators $\{\Phi_w, w \in W\}$ satisfy 1.3.

Remark 1.4. *We expect that a similar strategy can be applied to prove Theorem 1.2 for $F = \mathbb{R}$.*

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2. On the space $\mathcal{S}_0(X)$

In [BK99] the authors have defined for split groups the spaces

$$\mathcal{S}(X) = \sum_{w \in W} \Phi_w(\mathcal{S}_c(X)), \quad \mathcal{S}^0(X) = \cap_{w \in W} \Phi_w(\mathcal{S}_c(X)).$$

In particular

$$\mathcal{S}^0(X) \subset \mathcal{S}_c(X) \subset \mathcal{S}(X) \subset L^2(X, \omega_X)$$

and the spaces $\mathcal{S}^0(X), \mathcal{S}(X)$ are preserved by the family of operators $\Phi_w, w \in W$. The space $\mathcal{S}(X)$, called Schwartz space, is potentially important for construction of integral representations of L-functions.

The description of the Schwartz space $\mathcal{S}(X)$ explicitly is a deep problem. For example for $G = SL_2$ one has $\mathcal{S}(X) = \mathcal{S}_c(V)$ and for $G = SU_3$ the space $\mathcal{S}(X)$ can be identified with the space of smooth vectors in the unitary minimal representation of a group $SO(8)$ containing SU_3 inside its Levi subgroup $GL_1 \times SO(6)$, see [GK22].

The space $\mathcal{S}_0(X)$ in this paper is contained in $\mathcal{S}^0(X)$. Let us highlight its useful properties:

- It is explicitly given as an intersection of kernels of certain partial Mellin transforms.
- The Fourier transforms corresponding to simple reflections preserve this space and can be written as integral operators with explicitly given continuous kernels.
- The family $\{\Phi_w\}$ is unique for given $\mathcal{S}_0(X)$.

On the other hand this space is not canonical and can easily be replaced by other subspaces in $\mathcal{S}_c(X)$, dense in $L^2(X, \omega_X)$ and preserved by Φ_w, for example by $\mathcal{S}^0(X)$.

The space $\mathcal{S}_0(X)$ will be defined separately for the groups of rank one, and, based on this, for general group.

The density of $\mathcal{S}_0(X)$ in $L^2(X, \omega_X)$ is the consequence of Proposition 2.1 below.

Consider a finite set

$$\mathbb{B} = \{(L_i, a_i, \chi_i), \quad 1 \le i \le k\},$$

where L_i is a finite extension of F, $a_i : L_i^\times \hookrightarrow T$ is an embedding and χ_i is a character of $L_i^\times$. For each (L_i, a_i, χ_i) consider a partial Mellin transform $P_i : \mathcal{S}_c(X) \to \mathcal{S}^\infty(X)$ defined by

$$P_i(f) = \int_{L_i^\times} \theta(a_i(y)) f \ \chi_i(y) d^\times y.$$

Define $\mathcal{S}_{\mathbb{B}}(X) = \cap_{i=1}^k \operatorname{Ker}(P_i)$. It is a $G \times T$ invariant subspace of $\mathcal{S}_c(X)$. The following proposition will be repeatedly used in the paper.

Proposition 2.1. *The space $\mathcal{S}_{\mathbb{B}}(X)$ is dense in $L^2(X, \omega_X)$.*

Proof. Let us prove this first for the case all the characters χ_i are not unitary. Precisely assume that all χ_i satisfy $|\chi_i| = |\cdot|^{b_i}$ with real $b_i \neq 0$ for all i. Let $b = \min(|b_i|) > 0$.

To show that the space $\mathcal{S}_{\mathbb{B}}(X)$ is dense, assume existence of a non-zero function $f \in \mathcal{S}_{\mathbb{B}}(X)^{\perp} \subset L^2(X, \omega_X)$. Since $\mathcal{S}_c(X)$ is dense in $L^2(X)$ there exists a function $g \in \mathcal{S}_c(X)$ such that $\langle f, g\rangle \neq 0$.

Denote by ϖ_i an uniformizer of L_i. For any $n \in \mathbb{N}$ define operators E_n^i, E_n on $\mathcal{S}_c(X)$ by $E_n = \Pi_{i=1}^k E_n^i$, where

$$E_n^i = \begin{cases} \mathrm{Id} - \theta(a_i(\varpi_i)^n)\chi_i(\varpi_i^n) & b_i > 0 \\ \mathrm{Id} - \theta(a_i(\varpi_i)^{-n})\chi_i(\varpi_i^{-n}) & b_i < 0 \end{cases}.$$

Clearly, $E_n(g) \in \mathcal{S}_B(X)$. Set $g_n = g - E_n(g)$. Note that $|\chi_i(\varpi_i)|$ (resp. $|\chi_i(\varpi_i^{-1})|$) is bounded by q^{-b} for $b_i > 0$ (resp. $b_i < 0$) for any i. Moreover the action $\theta(a_i(\varpi_i))$ is unitary. This implies $\|g_n\| \leq q^{-nb}(2^k - 1)\|g\|$. Hence

$$0 \neq |\langle g, f\rangle| = |\langle g_n, f\rangle| \leq q^{-nb}(2^k - 1)\|g\| \cdot \|f\| \to 0$$

as $n \to \infty$, which is a contradiction.

Now let us treat the general set of characters $\mathbb{B}$. For any compact subset $\mathcal{K}$ in X let $\mathcal{S}_{\mathbb{B}}(X; \mathcal{K})$ be the space of functions in $\mathcal{S}_{\mathbb{B}}(X)$ supported on $\mathcal{K}$. Obviously, $\mathcal{S}_{\mathbb{B}}(X) = \cup_{\mathcal{K}} \mathcal{S}_{\mathbb{B}}(X; \mathcal{K})$.

Since the action of T on X is free, for any character χ of T there exists a smooth function h on X, such that

$$h([t^{-1}g]) = \chi(t)h([g]), \quad h([g]) \neq 0, \quad \forall [g] \in X, t \in T.$$

Multiplication on h defines a T-equivariant isomorphism between $\mathcal{S}_{\mathbb{B}}(X)$ and $\mathcal{S}_{\mathbb{B}'}(X)$, where $\mathbb{B}' = \{(L_i, a_i, \chi_i \cdot \chi \circ a_i)\}$, which is also homeomorphism between $\mathcal{S}_{\mathbb{B}}(X; \mathcal{K})$ and $\mathcal{S}_{\mathbb{B}'}(X; \mathcal{K})$ for all compact $\mathcal{K} \subset X$. Hence $\mathcal{S}_{\mathbb{B}}(X)$ is dense if and only if $\mathcal{S}_{\mathbb{B}'}(X)$ is dense in $L^2(X, \omega_X)$. By choosing appropriate χ we can ensure that $\mathbb{B}'$ does not contain unitary characters. We are done. □

3. $G_1 = SL_2$

Let $(V, \langle\cdot,\cdot\rangle_V)$ be a two dimensional symplectic space with the standard basis e_1, e_2 such that $\langle e_1, e_2\rangle_V = 1$.

The group G_1 acts on V on the right, preserving the symplectic form. Let $B_1 = T_1 \cdot U_1$ be the Borel group, stabilizing the line Fe_2. The space $X = U_1 \backslash G_1$ is identified with $V - 0$ via $[g] \mapsto e_2 g$. The G_1-invariant measure ω_X on X is fixed to be the self-dual measure $|dv|$ on V with respect to the additive character ψ and the symplectic form on V.

The Fourier transform $\Phi \in \mathrm{Aut}(\mathcal{S}_c(V))$ is defined by the formula

$$\Phi(f)(w) = \int_V f(v)\psi(\langle w, v\rangle_V)dv.$$

The following properties of Φ are well-known:

Proposition 3.1. *(1) Φ extends to a unitary involution on $L^2(V, |dv|) = L^2(X, \omega_X)$*
(2) $\theta(t, g) \circ \Phi = \Phi \circ \theta(t^{-1}, g)$ for all $(t, g) \in T_1 \times G_1$.

For a function f on X the argument will be denoted either as a class $[g]$ or as a vector $(x, y) = xe_1 + ye_2 \in V - 0$.

We define certain typical elements of G_1 :

$$x(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One has $\alpha(t(a)) = a^2$ for the unique positive root α of G_1 with respect to T_1.

3.1. **The space $\mathcal{S}_0(X)$.** Let $\mathbb{B}$ be the set of two triples

$$\mathbb{B} = \{(F^\times, t : F^\times \to T_1, \chi_\pm(y) = |y|^{\pm 1})\}.$$

We define $\mathcal{S}_0(X)$ to be $\mathcal{S}_{\mathbb{B}}(X)$, see section 2 for the definition. It is obviously a $G_1 \times T_1$ representation and is dense in in $L^2(X, \omega_X)$ by Proposition 2.1.

Proposition 3.2. *The operator Φ preserves $\mathcal{S}_0(X)$.*

Proof. First note, that for any $f \in \mathcal{S}_0(X)$, the function $\Phi(f)$ belongs to $\mathcal{S}_c(X)$. Indeed, the germ $[\Phi(f)]_0$ of $\Phi(f)$ at zero is constant and equals

$$[\Phi(f)]_0 = \int\limits_V f(v) dv = \int\limits_F \int\limits_{F^\times} \theta(t(x)) f(1, y) |x| d^\times x dy = \int\limits_F P(\chi_+)(f)(1, y) dy = 0.$$

For any character χ of T_1 one has

$$P(\chi)(\Phi(f))(v) = \int\limits_{T_1} \theta(t) \Phi(f)(v) \chi(t) dt = \int\limits_{T_1} \Phi(\theta(t^{-1}) f)(v) \chi(t) dt.$$

Since f is of compact support, the integral defining $\Phi(f)$ is taken over a compact set in X, and hence the integral over T_1 can also be replaced by an integral over a compact set. By interchanging the order of integration we see that if $f \in \operatorname{Ker} P(\chi^{-1})$ then $\Phi(f) \in \operatorname{Ker} P(\chi)$.

Hence for $f \in \mathcal{S}_0(X)$ the function $\Phi(f)$ belongs to $\mathcal{S}_0(X)$. This proves the Lemma.

$\square$

3.1.1. *The Whittaker map.* We fix a character Ψ on U_1^{op} by $\Psi(x(r)) = \psi(r)$. The Whittaker map $\mathcal{W}_\Psi : \mathcal{S}_c(X) \to \mathcal{S}_c(T_1)$ is defined by

$$\mathcal{W}_\Psi(f)(t) = \int\limits_{U_1^{op}} \theta(t) f([u]) \Psi^{-1}(u) du.$$

The map $\mathcal{W}_\Psi$ defines an isomorphism $\mathcal{S}_0(X)_{U_1^{op}, \Psi} \simeq \mathcal{S}_0(T_1)$, where $\mathcal{S}_0(T_1) = \mathcal{W}_\Psi(\mathcal{S}_0(X))$, which induces the map

$$\kappa_\Psi : \operatorname{End}_{G_1}(\mathcal{S}_0(X)) \to \operatorname{End}_{\mathbb{C}}(\mathcal{S}_0(X)_{U^{op}, \Psi}) = \operatorname{End}_{\mathbb{C}}(\mathcal{S}_0(T_1)).$$

Lemma 3.3. *κ_Ψ is injective.*

Proof. See the proof of 6.2 for a general quasi-split G. □

Definition 3.4. *We define an action of W on $S_c(T_1)$ by*

$$s\cdot\varphi(t)=\varphi(t^s),\quad \varphi\in\mathcal{S}_c(T_1).$$

Proposition 3.5. *For any $\varphi\in\mathcal{S}_0(T_1)$ one has $\kappa_\Psi(\Phi)(\varphi)=s\cdot\varphi$.*

Proof. Any function in $\mathcal{S}_0(T_1)$ is of the form $\mathcal{W}_\Psi(f)$ for $f\in\mathcal{S}_0(X)$. It is enough to show that

$$\mathcal{W}_\Psi(\Phi(f))(1)=\mathcal{W}_\Psi(f)(1). \tag{3.6}$$

Indeed, once this is proven one has for $t\in T_1$

$$\mathcal{W}_\Psi(\Phi(f))(t)=\mathcal{W}_\Psi(\theta(t)\Phi(f))(1)=\mathcal{W}_\Psi(\Phi(\theta(t^s)f)(1)=$$
$$\mathcal{W}_\Psi(\theta(t^s)(f))(1)=\mathcal{W}_\Psi(f)(t^s).$$

There is a injective map with open dense image

$$j:T_1\times U_1^{op}\to X,\quad j(t,u)=[t^{-1}u]$$

and the push-forward of the measure $\delta_B(t)dt\,du$ on $T_1\times U_1^{op}$ equals dv.

$$\Phi(f)([g])=\int\limits_{U_1^{op}}\int\limits_{T_1}f([t^{-1}u])\psi(\langle[g],[t^{-1}u]\rangle_V)\delta_B(t)dtdu. \tag{3.7}$$

Hence

$$\mathcal{W}_\Psi(\Phi(f))(1)=\int\limits_{U_1^{op}}\Phi(f)([u])\Psi(u)^{-1}du=$$

$$\int\limits_{U_1^{op}}\int\limits_{U_1^{op}}\int\limits_{T_1}f([t^{-1}u'])\psi(\langle[u],[t^{-1}u']\rangle_V)\Psi(u^{-1})\delta_B(t)dt\,du'\,du=$$

$$\int\limits_{U_1^{op}}\int\limits_{T_1}\left(\int\limits_{U_1^{op}}f([t^{-1}u'])\Psi(u'^{-1})du'\right)\psi(\langle[1],[t^{-1}u]\rangle_V)\Psi(u)\delta_B(t)dt\,du=$$

$$\int\limits_{U_1^{op}}\int\limits_{T_1}\mathcal{W}_\Psi(f)(t)\psi(\langle[1],[t^{-1}u]\rangle_V)\Psi(u)\delta_B^{1/2}(t)dt\,du.$$

Put $t=t(b)$ and $u=x(r)$ and notice that $\langle[1],[t^{-1}u]\rangle_V=-br$. Then

$$\mathcal{W}_\Psi(\Phi(f))(1)=\int\limits_F\left(\int\limits_F\mathcal{W}_\Psi(f)(t(b))\psi(-br)db\right)\psi(r)dr=$$

$$\int\mathcal{F}_\psi(\mathcal{W}_\Psi(f))(-r)\psi(r)dr=\mathcal{W}_\Psi(f)(1),$$

where $\mathcal{W}_\Psi(f)$ is considered as a function on $\mathcal{S}_c(F^\times)$ via $b \mapsto \mathcal{W}_\Psi(f)(t(b))$ and $\mathcal{F}_\psi : \mathcal{S}_c(F) \to \mathcal{S}_c(F)$ denotes the one-dimensional Fourier transform with respect to ψ and the self-dual measure dx on F.

□

Theorem 3.8. *There exists a unique unitary operator $\Phi_s \in \mathrm{Aut}(L^2(X, \omega_X))$, that preserves the space $\mathcal{S}_0(X)$ and satisfies*

$$\begin{cases} \theta(g,t) \circ \Phi_s = \Phi_s \circ \theta(g, t^s) & g \in G_1, t \in T_1 \\ \Phi_s \circ \Phi_s = \mathrm{Id} & \\ \kappa_\Psi(\Phi_s)(\varphi) = s \cdot \varphi & \varphi \in \mathcal{S}_0(T_1) \end{cases} \tag{3.9}$$

Proof. The injectivity of κ_Ψ implies the uniqueness of the operator Φ_s, hence it is enough to construct such an operator.

We define Φ_s to be Φ. The properties follow from Propositions 3.1, 3.2, 3.5.

□

4. $G_1 = SU_3$

4.1. **The structure and compatibility of measures.**

4.1.1. *The field.* Let K be a quadratic field extension over F with the Galois involution $x \mapsto \bar{x}$, the norm Nm and the trace Tr. We write $|\cdot|_K$ for the absolute value on K, such that $|x|_K = |\,\mathrm{Nm}(x)|_F$. We fix an element $\tau \in \mathcal{O}_F$ such that $\mathcal{O}_K = \mathcal{O}_F + \sqrt{\tau}\mathcal{O}_F$.

The space K admits a quadratic form $x \mapsto \mathrm{Nm}(x)$ and the associated bilinear form on K is $(x, y) \mapsto \mathrm{Tr}(x\bar{y})$.

We fix on K a self dual measure dx with respect to ψ and Nm. The Fourier transform on K is denoted by $\mathcal{F}_{\psi,K}$, to distinguish it from the Fourier transform $\mathcal{F}_\psi$ with respect to ψ and the self-dual measure on F.

4.1.2. *The unitary group.* Let $(\mathbb{W}, h)$ be the following Hermitian space

$$\mathbb{W} = K^3, \quad h(v_1, v_2) = x_1\bar{z}_2 + y_1\bar{y}_2 + z_1\bar{x}_2, \quad v_i = (x_i, y_i, z_i).$$

The group $G_1 = SU(\mathbb{W}, h)$ is the group of automorphisms of $\mathbb{W}$, acting on the right, preserving the Hermitian form h and having determinant 1. Its elements are 3×3 matrices over K.

We denote by $B_1 = T_1 \cdot U_1$ the Borel subgroup of G_1, preserving the line $K(0, 0, 1)$ in $\mathbb{W}$. The unipotent radical U_1 is the stabilizer of the vector $(0, 0, 1)$. The space $X = U_1\backslash G_1$ is naturally identified with the set $\mathbb{W}^0$ of h-isotropic non-zero vectors in the space $\mathbb{W}$. We write T' for the maximal split torus of T_1.

4.1.3. *The measures.* The space $\mathbb{W}$ with $\dim_F(\mathbb{W}) = 6$ admits the F-bilinear form $\langle v_1, v_2\rangle = \mathrm{Tr}\, h(v_1, v_2)$ and the corresponding quadratic form q is given by

$$q(v) = \langle v, v\rangle/2 = \mathrm{Tr}(x\bar{z}) + \mathrm{Nm}(y), \quad v = (x, y, z).$$

We fix the self-dual measure dw on $\mathbb{W}$ with respect to ψ and q. It gives rise to a measure on the cone $\mathbb{W}^0$ and hence to a measure on X which we denote by ω_X.

We fix bijections

$$x : K \times \sqrt{\tau}F \to U_1^{op}, \quad t : K^\times \to T_1$$

by

$$x(r,y) = \begin{pmatrix} 1 & 0 & 0 \\ -\bar{r} & 1 & 0 \\ -\frac{\mathrm{Nm}(r)}{2} + y & r & 1 \end{pmatrix} \quad t(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1}\bar{a} & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \quad a \in K^\times.$$

We also fix a representative n_s of the Weyl element s by

$$n_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Haar measures on $K \times \sqrt{\tau}F$ and $K^\times$ define the measures on U_1^{op} and T_1 respectively.

By Bruhat decomposition for G_1, there is an embedding $j : T_1 \times U_1^{op} \to X$ with dense image, defined by $j(t,u) = [t^{-1}u]$.

It is straightforward to check that for any $f \in \mathcal{S}_c(X)$ one has

$$\int\limits_X f(v)\omega_X(v) = \int\limits_K \int\limits_{\sqrt{\tau}F} \int\limits_{K^\times} f([t(b)^{-1}x(r,y)])|\mathrm{Nm}(b)|^2 d^\times b\, dy\, dr.$$

The root system with respect to the torus T' is

$$R(G_1, T') = \{\pm\alpha, \pm 2\alpha\}, \quad \alpha(t(a)) = a, \quad \forall a \in F^\times.$$

The operator Φ_s for the group G_1 is defined using the normalized Radon transform on the cone X. Below we recall the definition and the relevant properties. We refer to [GK22] for proofs.

4.2. **Mellin transform.** Let χ be a character of $\mathcal{O}^\times$, extended to $F^\times$ by setting $\chi(\varpi) = 1$. We write χ_s for the character $\chi|\cdot|^s$ of $F^\times$. The character χ_s is lifted to the character of $T' \simeq F^\times$ via isomorphism $t(x) \mapsto x$.

Define the Mellin transform $P(\chi, s) : \mathcal{S}_c(X) \to \mathcal{S}^\infty(X)$ along T' by

$$P(\chi,s) = \int\limits_{T'} \theta(t) f \chi_s(t) dt.$$

The image $\mathcal{S}(\chi,s)$ consists of functions $f \in \mathcal{S}^\infty(X)$ satisfying $\theta(t)f = \chi_s^{-1}(t)f$.

The Mellin transform can be also computed on functions on $\mathcal{S}^\infty(X)$, not necessarily of compact support, provided the integral converges.

The following statement is obvious and will be used later.

Lemma 4.1. *Let $G : \mathcal{S}_c(X) \to \mathcal{S}(\chi,s)$, such that $G \circ \theta(t) = \theta(t^{-1}) \circ G$ for all $t \in T'$. Then* $\mathrm{Ker}\, G$ *contains* $\mathrm{Ker}\, P(\chi^{-1}, -s)$.

4.3. **The Radon transform.** Recall that X can be identified with the space $\mathbb{W}^0$ of non-zero isotropic vectors in $\mathbb{W}$. In this section elements in X will be denoted by $u, v, w \ldots$, isotropic vectors in $\mathbb{W}$.

For any vector $w \in \mathbb{W}^0 = X$ consider an algebraic map

$$p_w : \mathbb{W}^0 \to F, \quad p_w(v) = \langle v, w \rangle.$$

The measure ω_X defined above and the measure dx on F give rise to well-defined measure $\omega_{w,a}$ on the fiber $p_w^{-1}(a) = \{v \in \mathbb{W}^0, \langle v, w \rangle = a\}$ for any $a \in F$.

For any $a \in F$ we define Radon transform $\mathcal{R}(a) : \mathcal{S}_c(X) \to \mathcal{S}^\infty(X)$ by

$$\mathcal{R}(a)(f)(w) = \int\limits_{p_w^{-1}(a)} f(v)\omega_{w,a}(v).$$

The function $a \mapsto \mathcal{R}(a)(f)(w)$ is continuous, of bounded support. The normalized Radon transform on $\mathcal{S}_c(X)$ is defined by

$$\hat{\mathcal{R}}(f)(w) = \int\limits_F \mathcal{R}(a)(f)(w)\psi(a)da.$$

In addition set

$$\mathcal{R}_1(f)(w) = \int\limits_{F^\times} \theta(t(x))f(w)\chi_K(x)d^\times x$$

Below we list the properties of the operators $\mathcal{R}(a)$ and $\hat{\mathcal{R}}$, all proven in [GK22], section 3. The quadratic space (V_K, q_K) in loc. cit. is isomorphic to the quadratic space $(\mathbb{W}, q)$ and the results proven in loc.cit. hold in our setting.

For all $f \in \mathcal{S}_c(X)$, $w \in X$ one has

(1) $\mathcal{R}(xa)(f)(xw) = |x|^{-1}\mathcal{R}(a)(f)(w)$ for all $x \in F^\times$. This implies
$$\hat{\mathcal{R}}(f)(xw) = \int\limits_F \mathcal{R}(a)(f)(w)\psi(ax)da.$$
(2) $\mathcal{R}(a) \circ \theta(g,t) = \theta(g, t^{-1}) \circ \mathcal{R}(a)$ for $g \in G_1, t \in T'$ and the same is true for $\hat{\mathcal{R}}$.
(3) There exists a constant $c_{\psi,q}$ such that for $|a|$ small enough one has
$$\mathcal{R}(a)(f)(w) = \mathcal{R}(0)(f)(w) + c_{\psi,q}\chi_K(a)|a|\mathcal{R}_1(f)(w).$$
(4) The function $x \mapsto \theta(t(x))\hat{\mathcal{R}}(f)(w)$ is bounded for $x \in F$.
(5) $\hat{\mathcal{R}}(f)$ extends to a locally constant function on $\mathbb{W}^0 \cup \{0\}$ whose value at 0 is $\int\limits_X f(v)\omega_X(v)$.

Lemma 4.2. *(1) If $f \in \operatorname{Ker} P(\chi_K, z)$ then $\hat{\mathcal{R}}(f) \in \operatorname{Ker} P(\chi_K, -z)$ for $Re(z) > 0$.*

(2) *Let $f \in \operatorname{Ker} P(\chi_K,-1) \cap \operatorname{Ker} P(\chi_K,0)$. For $w \in X$ the function $a \mapsto \mathcal{R}(a)(f)(w)$ is of compact support on $F^\times$.*

Proof. (1) The transform $P(\chi_K,-z)(\hat{\mathcal{R}}(f))(w)$ is well-defined for $Re(z) > 0$ by the property (4). The Lemma 4.1 yields the result.

(2) By properties (1), (2), the map $\mathcal{R}(0)$ has image in $\mathcal{S}(\chi_0,1)$ and satisfies the condition of Lemma 4.1. Similarly, the map $\mathcal{R}_1$ has image in $\mathcal{S}(\chi_K,0)$ and satisfies the condition. Hence for $f \in \operatorname{Ker} P(\chi_K,-1) \cap \operatorname{Ker} P(\chi_K,0)$ one has $\mathcal{R}(0)(f) = \mathcal{R}_1(f) = 0$. By the property (3), the function $\mathcal{R}(a)(f)(w)$ vanishes for small $|a|$ and hence is of compact support on $F^\times$.

□

Let us fix terminology for convergence of integrals of locally constant functions, not necessary of compact support, on $F^\times$. For $f \in \mathcal{S}^\infty(F^\times)$ we say that $\int\limits_{|x|\leq 1} f(x)d^\times x$

- converges absolutely if $\int\limits_{|x|\leq 1} |f(x)|d^\times x$ converges
- converges if $\lim_{n\to\infty} \int_{|x|\geq q^{-n}} f(x)d^\times x$ exists.
- stabilizes if the sequence $\int_{|x|\geq q^{-n}} f(x)d^\times x$ becomes constant for $n > N$.

Similarly we say that the integral $\int\limits_{|x|>1} f(x)d^\times x$ converges absolutely, (resp. converges or stabilizes) the integral if $\int\limits_{|x|<1} f(x^{-1})d^\times x$ converges absolutely,(resp. converges or stabilizes).

Given an integral $I = \int_{F^\times} f(x)d^\times x$ we say that it stabilizes at zero and converges absolutely at infinity if $\int\limits_{|x|\leq 1} f(x)d^\times x$ stabilizes and $\int\limits_{|x|>1} f(x)d^\times x$ converges absolutely.

For example, for any unitary character χ and $Re(s) > 0$ the integral $\int_{F^\times} \psi(x)\chi(x)|x|^s d^\times x$ stabilizes at infinity and converges absolutely at zero.

4.4. **The operators Φ.** We define an operator $\Phi : \mathcal{S}_c(X) \to \mathcal{S}^\infty(X)$ by

$$\Phi(f) = \int\limits_{F^\times} \theta(t(x))\hat{\mathcal{R}}(f)\psi(x^{-1})\chi_K(x)|x|^{-1}d^\times x.$$

By properties (4), (5) the integral converges absolutely. By property (2) it satisfies the equivariance property for $G \times T'$.

In [GK22], using the minimal representation for the group $O(8)$ we proved

Theorem 4.3. *The operator Φ*

(1) *has its image in the space of functions of bounded support on X.*
(2) *extends to a unitary involution on $L^2(X,\omega_X)$,*
(3) *satisfies $\theta(g,t) \circ \Phi = \Phi \circ \theta(g,t^s)$ for all $g \in G_1, t \in T'$.*

The operator Φ is our candidate for Fourier transform. To prove Theorem 1.2 for G_1 it remains

- to show that Φ enjoys the equivariance property with respect to T_1,
- to define a space $\mathcal{S}_0(X) \subset \mathcal{S}_c(X)$, preserved by Φ and dense in $L^2(X, \omega_X)$ and
- to compute $\kappa_\Psi(\Phi)$ on the space $\mathcal{S}_0(T_1) = \mathcal{W}_\Psi(\mathcal{S}_0(X))$.

4.5. **The space $\mathcal{S}_0(X)$.** Define the space $\mathcal{S}_0(X) = \mathcal{S}_{\mathbb{B}}(X)$, where $\mathbb{B}$ is the following finite set of characters of $T' \simeq F^\times$:

$$\mathbb{B} = \{\chi_K, \chi_K|\cdot|^{\pm 1}, |\cdot|^{\pm 2}\}.$$

Proposition 4.4. *The operator Φ preserves $\mathcal{S}_0(X)$.*

Proof. We start by showing that for $f \in \mathcal{S}_0(X)$ one has $\Phi(f) \in \mathcal{S}_c(X)$. Since $\Phi(f)$ has bounded support, it is enough to show that the germ $[\Phi(f)]_0$ at zero vanishes.

The operator Φ can be naturally decomposed as a sum $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1(f)(w) = \gamma(\chi_K, \psi) \int\limits_{F^\times} \theta(t(x))\hat{\mathcal{R}}(f)(w)\chi_K(-x)|x|^{-1}d^\times x,$$

and

$$\Phi_2(f)(w) = \gamma(\chi_K, \psi) \int\limits_{F^\times} \theta(t(x))\hat{\mathcal{R}}(f)(w)(\psi(x^{-1}) - 1)\chi_K(-x)|x|^{-1}d^\times x.$$

For $f \in \mathcal{S}_0(X)$ one has $\Phi_1(f) = 0$ by 4.2. Let us show that the germ $[\Phi_2(f)]_0$ is zero for $f \in \mathcal{S}_0(X)$. The function

$$g(x) = \gamma(\chi_K, \psi)(\psi(x^{-1}) - 1)\chi_K(-x)|x|$$

has a bounded support, denote it by $\mathcal{B}$. For $|w|$ small enough, the function $x \mapsto \hat{\mathcal{R}}(f)(xw)$ is constant for $x \in \mathcal{B}$. Hence for $|w|$ small one has

$$\Phi_2(f)(w) = \hat{\mathcal{R}}(f)(w) \cdot \int_{F^\times} g(x)dx.$$

By property (5) $\hat{\mathcal{R}}(f)(w) = \int\limits_X f(v)\omega_X(v)$ for $|w|$ small enough. The map $f \mapsto \int\limits_X f(v)\omega_X(v)$ has its image in $\mathcal{S}(\chi_0, |\cdot|^{-2})$. Hence by Lemma 4.1 if $f \in \operatorname{Ker} P(\chi_0, |\cdot|^2)$ then $\hat{\mathcal{R}}(f)(w) = 0$ for small w and so $[\Phi_2(f)]_0 = 0$. Hence $\Phi(f)$ is of compact support.

Since $\Phi \circ \theta(t) = \theta(t^{-1}) \circ \Phi$ for $t \in T'$ the properties $f \in \operatorname{Ker} P(\chi, s)$, and $\Phi(f) \in \mathcal{S}_c(X)$ imply $\Phi(f) \in \operatorname{Ker} P(\chi^{-1}, -s)$, as in Proposition 3.2. This yields the result. $\square$

Proposition 4.5. *For $f \in \mathcal{S}_0(X)$ one has*

$$\Phi(f)(w) = \int\limits_X f(v)\mathcal{L}(\langle v, w\rangle)\omega_X(v),$$

where for $a \in F^\times$

$$\mathcal{L}(a) = \gamma(\chi_K, \psi) \int\limits_{F^\times} \psi(ax + x^{-1})\chi_K(-x)|x| d^\times x. \tag{4.6}$$

Proof. For $a \in F^\times$, the integral defining $\mathcal{L}$ stabilizes both at zero at and infinity. In particular, there exists a compact set $\mathcal{K}_1$ in $F^\times$ such that

$$\mathcal{L}(a) = \gamma(\chi_K, \psi) \int\limits_{\mathcal{K}_1} \psi(ax + x^{-1})\chi_K(-x)|x| d^\times x.$$

For $f \in \mathcal{S}_0(X)$, the function $a \mapsto \mathcal{R}(a)(f)(w)$ is of compact support on $F^\times$ by Lemma 4.2, part (2). We can assume that the support is contained in $\mathcal{K}_1$.

By the Fubini theorem

$$\int\limits_X f(v)\mathcal{L}(\langle v, w\rangle)\omega_X(v) = \int\limits_F \mathcal{R}(a)(f)(w)\mathcal{L}(a)da =$$

$$\gamma(\chi_K, \psi) \int\limits_{\mathcal{K}_1}\int\limits_{\mathcal{K}_1} \mathcal{R}(a)(f)(w)\psi(ax)\psi(x^{-1})\chi_K(-x)|x| d^\times x da.$$

We can change the order of integration over compact sets. This gives

$$\gamma(\chi_K, \psi) \int\limits_{\mathcal{K}_1} \left(\int\limits_{\mathcal{K}_1} \mathcal{R}(a)(f)(w)\psi(ax)da \right) \psi(x^{-1})\chi_K(-x)|x| d^\times x da =$$

$$\gamma(\chi_K, \psi) \int\limits_{F^\times} \theta(t(x))\hat{\mathcal{R}}(f)(w)\psi(x^{-1})\chi_K(-x)|x|^{-1} d^\times x da = \Phi(f)(w),$$

as required. □

Proposition 4.7. *One has* $\Phi \circ \theta(t^s)(f) = \theta(t) \circ \Phi(f)$ *for all* $f \in \mathcal{S}_0(X)$ *and* $t \in T_1$.

Proof. This is a straightforward computation and is very similar to the proof of the equivariance property of the classical Fourier transform.

$$\theta(t)\Phi(f)(w) = \delta_B^{1/2}(t) \int\limits_X f(v)\mathcal{L}(\langle v, tw\rangle)\omega_X(v).$$

One has $\langle v, tw\rangle = \langle (t^s)^{-1}v, w\rangle$ for all $t \in T_1$. Applying the change of variables $v \mapsto (t^s)^{-1}v$ and taking the measure into account, we get that the integral equals

$$\delta_B^{1/2}(t^s) \int\limits_X f(t_s v)\mathcal{L}(\langle v, w\rangle)\omega_X(v) = \Phi(\theta(t^s)f)(w)$$

as required. □

4.6. **The Whittaker map.** It remains to compute $\kappa_\Psi(\Phi)$.

We fix a character Ψ of U_1^{op} such that $\Psi(x(r,r'))=\psi(\mathrm{Tr}(r))$. The Whittaker map $\mathcal{W}_\Psi:\mathcal{S}_c(X)\to\mathcal{S}_c(T_1)$ is defined as in introduction.

Proposition 4.8. *Let $f\in\mathcal{S}_0(X)$.*

(1) $\mathcal{W}_\Psi(\Phi(f))(1)=\mathcal{W}_\Psi(f)(t(-1))$,
(2) $\mathcal{W}_\Psi(\Phi(f))(t)=\mathcal{W}_\Psi(f)(t(-1)t^s)$.

The proof occupies the rest of this subsection. We start with the following technical Lemmas, whose proofs are postponed to the end of this subsection.

Lemma 4.9. *For any $x\in F$ and $g\in\mathcal{S}_c(K)$ one has*

$$\int\limits_{\sqrt{\tau}F}\int\limits_K g(b)\psi(-x\,\mathrm{Tr}(b\cdot y))db\,dy=|x|^{-1}\int\limits_F g(b)db.$$

According to Weil, [Wei64] there exists a constant $\gamma(\chi_K,\psi)$, which is a fourth root of unity, satisfying

$$\int\limits_K \mathcal{F}_{\psi,K}(f)(x)\psi(\mathrm{Nm}(x))dx=\gamma(\chi_K,\psi)\int\limits_K f(x)\psi(-\mathrm{Nm}(x))dx. \tag{4.10}$$

For all $t\in F^\times$ denote by ψ_t the additive character $\psi_t(x)=\psi(tx)$. One has

- $\gamma(\chi_K,\psi_t)=\chi_K(t)\gamma(\chi_K,\psi)$,
- $\gamma(\chi_K,\psi)=1$ if K is split.

Lemma 4.11. *For any $g\in\mathcal{S}_c(F)$ one has*

$$\int\limits_K\int\limits_{F^\times} g(x)\psi\left(-\frac{\mathrm{Nm}(r)}{x}\right)\chi_K(x)d^\times x\psi(\mathrm{Tr}(r))dr=\gamma(\chi_K,\psi)\chi_K(-1)\mathcal{F}_\psi(g)(1). \tag{4.12}$$

Proof of Proposition 4.8. It is easy to see that the first part implies the second. Indeed, assuming part (1), for any $t\in T_1$ and $f\in\mathcal{S}_0(X)$ one has

$$\mathcal{W}_\Psi(\Phi)(f)(t)=\mathcal{W}_\Psi(\theta(t)\Phi(f))(1)=\mathcal{W}_\Psi(\Phi(\theta(t^s)f))(1)=$$
$$\mathcal{W}_\Psi(\theta(t^s)f)(t(-1))=\mathcal{W}_\Psi(f)(t(-1)t^s).$$

Using Bruhat decomposition for G_1 this equals.

$$\mathcal{W}_\Psi(\Phi(f))(1)=\int\limits_{U_1^{op}}\int\limits_{T_1}\int\limits_{U_1^{op}} f([t^{-1}u_1])\mathcal{L}(\langle[t^{-1}u_1],[u_2]\rangle)\Psi(u_2)^{-1}\delta_B(t)\,du_1\,dt\,du_2=$$

$$\int\limits_{U_1^{op}}\int\limits_{T_1}\int\limits_{U_1^{op}}\theta(t)f([u_2])\Psi(u_2)^{-1}du_2\mathcal{L}(\langle[t^{-1}u_1],[1]\rangle)\Psi(u_1)\delta_B^{1/2}(t)dt\,du_1=$$

$$\int\limits_{U_1^{op}}\int\limits_{T_1}\mathcal{W}_\Psi(f)(t)\mathcal{L}(\langle[t^{-1}u_1],[1]\rangle)\Psi(u_1)\delta_B^{1/2}(t)dt\,du_1.$$

We put

$$t = t(b), b \in K^\times, \quad u_1 = x(r, y), r \in K, y \in \sqrt{\tau}F.$$

To ease notation we write $\bar{f} \in S_c(F^\times)$ for the function $b \mapsto \mathcal{W}_\Psi(f)(t(b))$.

One has

$$\langle [t^{-1}(b)x(r,y)], [1] \rangle = -\operatorname{Tr}(b)\operatorname{Nm}(r)/2 - \operatorname{Tr}(by).$$

Hence the above equals

$$\int\limits_K \int\limits_{\sqrt{\tau}F} \int\limits_{K^\times} \bar{f}(b)\mathcal{L}(-\operatorname{Tr}(b)\frac{\operatorname{Nm}(r)}{2} - \operatorname{Tr}(b\sqrt{\tau}y))\psi(\operatorname{Tr}(r))|\operatorname{Nm}(b)|d^\times b dr t y.$$

Writing explicitly the expression for $\mathcal{L}$ from 4.6 and rearranging the change of integrals this equals

$$\gamma(\chi_K, \psi)\chi_K(-1) \cdot \int\limits_K \int\limits_{F^\times} \tag{4.13}$$

$$\int\limits_{\sqrt{\tau}F} \int\limits_{K^\times} \left(\bar{f}(b)\psi(-\operatorname{Tr}(bx\operatorname{Nm}(r)/2))\right) \psi(-\operatorname{Tr}(bxy))dbdy$$

$$\chi_K(x)\psi(x^{-1})|x|d^\times x \ \psi(\operatorname{Tr}(r))dr.$$

We apply Lemma 4.9 to the middle line, i.e. for

$$g(b) = \bar{f}(b)\psi(-\operatorname{Tr}(bx\operatorname{Nm}(r)/2)).$$

Notice that for $b \in F$ one has $\operatorname{Tr}(bx\operatorname{Nm}(r)/2) = bx\operatorname{Nm}(r)$. Hence the middle line equals

$$|x|^{-1} \int\limits_F \bar{f}(b)\psi(-bx\operatorname{Nm}(r))db$$

The integral becomes $\gamma(\chi_K, \psi)\chi_K(-1)$ times

$$\int\limits_{r\in K} \int\limits_{b\in F} \bar{f}(b) \int\limits_{x\in F^\times} \psi(-bx\operatorname{Nm}(r))\chi_K(x)\psi(x^{-1})d^\times x db \ \psi(\operatorname{Tr}(r))dr$$

After the change of variables $bx \mapsto x^{-1}$ this becomes $\gamma(\chi_K, \psi)\chi_K(-1)$ times

$$\int\limits_{r\in K} \int\limits_{x\in F^\times} \left(\int\limits_{b\in F} \bar{f}(b)\chi_K(b)\psi(xb)db \right) \psi(-\frac{\operatorname{Nm}(r)}{x})\chi_K(x)d^\times x \ \psi(\operatorname{Tr}(r))dr =$$

$$\gamma(\chi_K, \psi)\chi_K(-1) \int\limits_{r\in K} \int\limits_{x\in F^\times} \mathcal{F}_\psi(\bar{f}\chi_K)(x)\psi(-\frac{\operatorname{Nm}(r)}{x})\chi_K(x)d^\times x \ \psi(\operatorname{Tr}(r))dr.$$

Applying Lemma 4.11 to $g = \bar{f}\chi_K$ and the properties of $\gamma(\chi_K, \psi)$ we obtain that $\mathcal{W}_\Psi(\Phi(f))(1)$ equals

$$(\gamma(\chi_K, \psi)\chi_K(-1))^2 \mathcal{F}_\psi(\mathcal{F}_\psi(\bar{f}\chi_K))(1) = \bar{f}(-1) = \mathcal{W}_\Psi(f)(t(-1)),$$

as required.

$\square$

It remains to prove Lemmas.

Proof of Lemma 4.9. We fix the isomorphism of vector spaces

$$K \simeq F \oplus F, \quad b_1 + \sqrt{\tau} b_2 \mapsto (b_1, b_2)$$

which induces the isomorphism $S_c(K) \simeq S_c(F) \otimes S_c(F)$. The self-dual measure on K with respect to (ψ, Nm) is transported under this isomorphism to $|2||\tau|^{1/2} db_1 db_2$.

It is enough to prove Lemma for $g = g_1 \otimes g_2$, where $g_1, g_2 \in \mathcal{S}_c(F)$, so that $g(b_1 + \sqrt{\tau} b_2) = g_1(b_1) g_2(b_2)$.

Let us write $y = \sqrt{\tau} y'$ for $y' \in F$ and $dy = |\tau|^{1/2} dy'$. Then for $b = b_1 + \sqrt{\tau} b_2$ one has $\mathrm{Tr}(bx\sqrt{\tau}y') = 2\tau b_2 x y'$.

$$\int\limits_F \int\limits_K g(b)\psi(\mathrm{Tr}(bx\sqrt{\tau}y) db dy =$$

$$\int\limits_{F^3} g(b_1, b_2)\psi(2\tau b_2 x y)|2\tau| db_1 db_2 dy' = |2\tau| \int\limits_F g_1(b_1) db_1 \int\limits_F \mathcal{F}_\psi(g_2)(2\tau x y) dy =$$

$$g_2(0)|x|^{-1} \int\limits_F g_1(b_1) db_1 = |x|^{-1} \int\limits_F g(b) db.$$

$\square$

Proof of Lemma 4.11. Let $c \in F^\times \backslash \mathrm{Nm}(K^\times)$, so that $F^\times = \mathrm{Nm}(K^\times) \cup c\,\mathrm{Nm}(K^\times)$. The measures $d^\times y$ on $K^\times$ and $d^\times x$ on $\mathrm{Nm}(K^\times) \subset F^\times$ define Haar measures on the fibers of $\mathrm{Nm} : K^\times \mapsto F^\times$. All the fibers are compact and have the same measure C. By the Fubini theorem for any function $h \in L^1(F^\times)$ one has

$$\int\limits_{F^\times} h(x) d^\times x = C^{-1} \int\limits_{K^\times} h(\mathrm{Nm}(y)) + h(c\,\mathrm{Nm}(y)) d^\times y. \tag{4.14}$$

Applying this integral over $F^\times$ in the LHS of 4.12 we obtain

$$C^{-1} \int\limits_K \int\limits_{K^\times} g(\mathrm{Nm}(y))\psi(-\mathrm{Nm}(r/y)) - g(c\,\mathrm{Nm}(y))\psi(-c^{-1}\,\mathrm{Nm}(r/y)) d^\times x \psi(\mathrm{Tr}(r)) dr.$$

After the change of variables $r \mapsto r\bar{y}$ this equals

$$C^{-1} \times \int\limits_K \left(\int\limits_K g(\mathrm{Nm}(y))\psi(\mathrm{Tr}(r\bar{y}))\, dy \right) \psi(-\mathrm{Nm}(r))dr -$$
$$\int\limits_K \left(\int\limits_K g(c\,\mathrm{Nm}(y))\psi(\mathrm{Tr}(r\bar{y}))\, dy \right) \psi(-c^{-1}N(r))dr =$$

$$C^{-1} \times \int\limits_K \mathcal{F}_{K,\psi}(g \circ \mathrm{Nm})(r)\psi(-\mathrm{Nm}(r))dr - \int\limits_K \mathcal{F}_{K,\psi}(g_c \circ \mathrm{Nm})(r) \int\limits_K \psi(-c^{-1}\,\mathrm{Nm}(r))dr$$

where $g_c(x) = g(cx)$ for any x. This equals by 4.10

$$C^{-1} \times \chi_K(-1)\gamma(\chi_K, \psi) \int\limits_K (g \circ \mathrm{Nm})\psi(\mathrm{Nm}(r))dr + |c| \int\limits_K (g_c \circ \mathrm{Nm})(r)\psi(c\,\mathrm{Nm}(r))dr.$$

Applying equation 4.14 again this equals

$$\chi_K(-1)\gamma(\chi_K, \psi) \int g(x)\psi(x)dx = \gamma(\chi_K, \psi)\chi_K(-1)\mathcal{F}_\psi(g)(1).$$

□

The restriction $\mathcal{W}_\Psi : \mathcal{S}_0(X) \to \mathcal{S}_c(T)$, whose image we denote by $\mathcal{S}_0(T)$, gives rise to the homomorphism $\kappa_\Psi : \mathrm{End}_G(\mathcal{S}_0(X)) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(T))$.

Lemma 4.15. *κ_Ψ is injective.*

Proof. See the proof of 6.2 for the general case. □

Let us define the action of W on $\mathcal{S}_0(T_1)$.

Definition 4.16. *The action of W on $\mathcal{S}_0(T_1)$ is defined by*

$$s \cdot \varphi(t) = \varphi(t(-1)t^s)$$

Theorem 4.17. *There exists a unique unitary involution $\Phi_s \in \mathrm{Aut}(L^2(X, \omega_X))$ that preserves the space $\mathcal{S}_0(X)$ and satisfies*

$$\begin{cases} \theta(g,t) \circ \Phi_s = \Phi_s \circ \theta(g, t^s) & g \in G_1, t \in T_1 \\ \kappa_\Psi(\Phi_s)(\varphi) = s \cdot \varphi & \varphi \in \mathcal{S}_0(T_1) \end{cases} \tag{4.18}$$

Proof. The injectivity of κ_Ψ implies uniqueness of such operator. and hence it is enough to construct such Φ_s. We put $\Phi_s = \Phi$. The properties follow from Theorem 4.3, Propositions 4.7, 4.4 and 4.8, part (2). □

5. Quasi-split groups

We recall below the structure of reductive quasi-split groups. Our main reference is [BT84].

5.1. **Relative and absolute root systems.** Let $\mathbf{G}$ be a reductive, connected, simply-connected quasi-split group over F with a maximal split torus $\mathbf{T}'$. We denote by $Lie(G)$ the Lie algebra of $\mathbf{G}$ an by Ad the adjoint action of $\mathbf{G}$ on $Lie(G)$. Let $\mathbf{T}$ be the centralizer of $\mathbf{T}'$ and $\mathbf{N}$ be the normalizer of $\mathbf{T}'$, both defined over F.

The root datum of $\mathbf{G}$ with respect to $\mathbf{T}'$ is a quadruple $(X^*(\mathbf{T}'), R, X_*(\mathbf{T}'), R^\vee))$, where the set of roots $R \subset X^*(\mathbf{T}')$ consists of the weights that appear in the representation $Ad : \mathbf{T}' \to Aut(Lie(G))$.

The root system R is not necessarily reduced. For any root $\alpha \in R$, its root ray is defined as $1 \otimes R \cap \mathbb{R}_{>0} \otimes \alpha$, in $\mathbb{R} \otimes X^*(\mathbf{T}')$. Each root ray contains one or two elements. We denote by $\mathbf{R}$ the set of root rays.

The choice of a Borel subgroup $\mathbf{B}$, containing $\mathbf{T}$ and defined over F determines the decomposition $R = R^+ \cup R^-$ into the set of positive and negative roots and the subset $\Delta \subset R^+$ of simple roots. We call a root ray positive (resp. negative, resp. simple) if it contains a positive (resp. negative, resp. simple) root.

The groups $\mathbf{G}$ and $\mathbf{T}$ are split over the separable closure F_s of F. There exists a minimal extension $F \subset E \subset F_s$ over which $\mathbf{T}$ and hence $\mathbf{G}$ splits. Then E/F is Galois. We denote this split E-group by $\tilde{\mathbf{G}}$. It has a root datum $(X^*(\mathbf{T}), \tilde{R}, X_*(\mathbf{T}), \tilde{R}^\vee)$. Note that all root rays in $X^*(\mathbf{T}) \otimes \mathbb{R}_{>0}$ are singletons.

The Borel subgroup $\tilde{\mathbf{B}}$ containing $\mathbf{B}$ of $\tilde{\mathbf{G}}$, determines the set $\tilde{R}^+$ of positive roots and the set $\tilde{\Delta}$ of simple roots. The Galois group $\Gamma = Gal(E/F)$ acts on $X^*(\mathbf{T}), \tilde{R}, \tilde{R}^+$ and $\tilde{\Delta}$.

There is a bijection $\beta \leftrightarrow \tilde{R}_\beta$ between the set R of roots and the set of Γ orbits of $\tilde{R}$. The restriction of every root in $\tilde{R}_\beta$ to T' equals to β.

Definition 5.1. *Let $\alpha \in \tilde{R}$. The field $L_\alpha = E^{\Gamma_\alpha}$ is called the field of definition of α, where $\Gamma_\alpha \in \Gamma$ is the stabilizer of α.*

Proposition 5.2. *(1) For any $\gamma \in \Gamma$ and $\alpha \in \tilde{R}$ one has $L_{\gamma(\alpha)} = \gamma(L_\alpha)$.*

(2) For $\alpha \in \tilde{R}$, if $\alpha|_{T'}$ is a divisible root in R, then there exist roots $\alpha_1, \alpha_2 \in \tilde{R}$ such that

$$\alpha_1|_{T'} = \alpha_2|_{T'} = \alpha/2|_{T'}, \quad \alpha = \alpha_1 + \alpha_2.$$

In addition $L_{\alpha_1} = L_{\alpha_2}$ is a quadratic extension of L_α.

5.2. **The Chevalley-Steinberg pinning.** For any $a \in \mathbf{R}$ there exists a maximal connected subgroup $\mathbf{U}_a$ of $\mathbf{G}$, defined over F, such that the weights that appear in the representation $Ad : \mathbf{T}' \to \mathrm{Aut}(Lie(\mathbf{U}_a))$ belong to a. The group $\mathbf{U}_a$ is called the root subgroup corresponding to $a \in \mathbf{R}$.

For any simple root ray a in $\mathbf{R}$, let $\mathbf{G}_a$ be the group generated by $\mathbf{U}_a$ and $\mathbf{U}_{-a}$. Since the group $\mathbf{G}$ is simply-connected, the group $\mathbf{G}_a$ is a simply connected group of rank 1 over F. We denote by $\mathbf{T}_a$ and $\mathbf{T}'_a$ the maximal torus and the maximal split torus of $\mathbf{G_a}$ respectively. The group $\tilde{\mathbf{G}}_a$ in $\tilde{\mathbf{G}}$ is $\mathbf{G}_a$ considered as a group over E.

The following proposition describes G_a and $\tilde{G}_a$.

Proposition 5.3. *Let a be a root ray. There are two possible cases*

- $a = \{\alpha\}$. *In this case the group $\tilde{\mathbf{G}}_a$ is isomorphic over E to a product of copies of the group SL_2, indexed by $\tilde{R}_\alpha$.*
  There exists an isomorphism $\phi_a : SL_2(L_\alpha) \to G_a$ such that
  $$\phi_a(x(r)) \in U_{-a}, \quad \phi_a({}^t x(r)) \in U_a, \quad \phi_a(n_s) \in N$$
- $a = \{\alpha, 2\alpha\}$. *In this case the group $\tilde{\mathbf{G}}_a$ is isomorphic to a product of copies of SL_3 indexed by the set I of subsets $\{\alpha_1, \alpha_2\} \subset \tilde{R}_\alpha$, such that $\alpha_1 + \alpha_2 \in \tilde{R}$. The field $L_{\alpha_1} = L_{\alpha_2}$ is a quadratic extension of $L\alpha_1 + \alpha_2$ with a non-trivial automorphism $x \mapsto \bar{x}$. Let SU_3 be the group of automorphisms on the Hermitian space $L^3_{\alpha_1}$ preserving the form $h(x, y, z) = \mathrm{Tr}(\bar{x}z) + \mathrm{Nm}(\bar{y}y)$ and having determinant 1. It is a quasi-split group of rank 1 over $L_{\alpha_1+\alpha_2}$.*
  There exists an isomorphism $\phi_a : SU_3(L_{\alpha_1+\alpha_2}) \to G_a$ such that
  $$\phi_a(x(r, r')) \in U_{-a}, \quad \phi_a({}^t x(r, r')) \in U_a, \quad \phi_a(n_s) \in N.$$

From now on we fix a family of isomorphisms $\phi_a, a \in \mathbf{R}$ such that ϕ_a define a Steinberg-Chevalley pinning of the group $\mathbf{G}$. See [BT84], page 78.

5.3. **The Weyl group.** The Weyl group W is isomorphic to N/T. For any $a \in \mathbf{R}$ the image of the element $n_{s_a} = \phi_a(n_s)$ in W is denoted by s_a. These elements, called simple reflections, generate W.

The roots in the same W orbit have the same field of definition.

For any $w \in \tilde{W}$ we denote by $l(w)$ the length of a reduced presentation of w as a product of simple reflections.

For any $w \in W$ we define $R(w) = R^+ \cap w^{-1}(R^-)$. Then $l(w) = |R(w)|$.

We denote by w_0 the longest element of W, and by n_0 its representative in N.

5.4. **The action of W on $\mathcal{S}_c(T)$.**

Definition 5.4. *Define for any $w \in W$ the element and*
$$t_w = \Pi_{a \in \mathbf{R}(w)} t_a \in T,$$
where $t_a = \phi_a(t(-1))$ for $a = \{\alpha, 2\alpha\}$ and $t_a = 1$ otherwise.

Lemma 5.5.
$$t_{w_2} \cdot (w_2^{-1} t_{w_1} w_2) = t_{w_1 w_2}$$

Proof. The set $R(w_1 w_2)$ can be written as a disjoint union
$$R(w_1 w_2) = \left(R(w_2) \backslash - w_2^{-1} R(w_1)\right) \cup \left(w_2^{-1} R(w_1) \backslash - R(w_2)\right).$$
Indeed,
$$R(w_2) \backslash - w_2^{-1} R(w_1) = \{\alpha > 0, w_2\alpha < 0, w_1 w_2 \alpha < 0\},$$
$$w_2^{-1} R(w_1) \backslash - R(w_2) = \{\alpha > 0, w_2\alpha > 0, w_1 w_2 \alpha < 0\}.$$

and the union is $R(w_1w_2)$. Besides $R(w) = -wR(w^{-1})$.

Writing by definition

$$t_{w_2} = \Pi_{R(w_2)\setminus -w_2^{-1}R(w_1)} t_a \cdot \Pi_{R(w_2)\cap -w_2^{-1}R(w_1)} t_a$$

and

$$t_{w_1} = \Pi_{R(w_1)\setminus -w_2R(w_2)} t_a \cdot \Pi_{R(w_1)\cap -w_2R(w_2)} t_a$$

we conclude that $t_{w_2} w_2^{-1} t_{w_1} w_2 = t_{w_1w_2}$.

□

Proposition 5.6. *The map $W \times \mathcal{S}_c(T) \to \mathcal{S}_c(T)$ defined by*

$$w \cdot \varphi(t) = \varphi(t_w \cdot w^{-1}tw) \tag{5.7}$$

is an action of W on $\mathcal{S}_c(T)$.

Proof. For $w_1, w_2 \in W$ one has

$$w_1 \cdot (w_2 \cdot \varphi)(t) = (w_2 \cdot \varphi)(t_{w_1} w_1^{-1} t w_1) =$$
$$\varphi((t_{w_2} \cdot w_2^{-1} t_{w_1} w_2) \cdot (w_1w_2)^{-1} t (w_1w_2)),$$

which by Lemma 5.5 equals $w_1w_2 \cdot \varphi(t)$. □

For groups of rank 1 this action was defined in 3.4 and 4.16.

6. Generalized Fourier transforms

In this section we generalize Theorems 3.8 and 4.17 that concern the quasi-split groups of F-rank one to a general quasi-split group G. We keep the notation of Section 5.

For any root ray a of the group G we fix the isomorphisms $\phi_a : G_1 \to G_a$, where G_1 is a quasi-split group of rank 1.

To formulate the main result we introduce the spaces $\mathcal{S}_0(X)$, $\mathcal{S}_0(T)$ and the homomorphism $\kappa_\Psi : \mathrm{End}_G(\mathcal{S}_0(X)) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(T))$.

6.0.1. *The space $\mathcal{S}_0(X)$.* We define for each positive root ray a a set of triples $\mathbb{B}_a$ as in section 2 as follows.

(1) Assume that $a = \{\alpha\}$ and L_α be the field of definition of α. Then
$$\mathbb{B}_a = \{(L_\alpha, a_i(x) = \phi_a(t(x)), \chi_\pm(x) = |x|^{\pm 1}_{L_\alpha})\}$$
(2) Assume that $a = \{\alpha, 2\alpha\}$ and $L_\alpha \supset L_{2\alpha}$ are the fields of definition of α and 2α. The set $\mathbb{B}_a$ consists of the triples (L_i, a_i, χ_i) where
$$L_i = L_{2\alpha}, \quad a_i(x) = \phi_a(t(x)), \quad \chi_i \in \{\chi_{L_\alpha}, \chi_{L_\alpha}|\cdot|^{\pm 1}, |\cdot|^{\pm 2}\}.$$

Definition 6.1. *Define $\mathcal{S}_0(X) = \mathcal{S}_{\mathbb{B}}(X)$, where $\mathbb{B} = \cup_a \mathbb{B}_a$ and the union is taken over all positive root rays.*

In particular $\mathcal{S}_0(X) = \cap_a \mathcal{S}_{\mathbb{B}_a}(X)$. The Weyl group acts naturally on the set $\mathbb{B}$, by $w(L_\alpha, \phi_a \circ t, \chi_i) = (L_{w(\alpha)} = L_\alpha, \phi_{w(a)} \circ t, \chi_i)$ where $\alpha \in a$. Note that under this action $w(\mathbb{B}_a) = \mathbb{B}_{wa}$.

For groups of rank one, the definition of the space $\mathcal{S}_0(X)$ coincides with the definition given in 3.1 and 4.5.

6.0.2. *Whittaker map and the map κ_Ψ.* We define a distinguished non-degenerate character $\Psi : U^{op} \to \mathbb{C}$ that is compatible with the fixed family of isomorphisms $\{\phi_a\}$ from section 5.

For a quasi-split group G_1 of F-rank 1 with Borel subgroup $T_1 \cdot U_1$, we define a complex character Ψ_1 of U_1^{op} by

- $\Psi_1(x(r)) = \psi(\mathrm{Tr}_{L/F}(r))$ if $G_1 = \mathrm{Res}_L SL_2$
- $\Psi_1(x(r,s)) = \psi(\mathrm{Tr}_{K/F}(r))$ if $G_1 = \mathrm{Res}_L SU_3$, corresponding to a quadratic field extension K/L.

Let Ψ be the unique character of U^{op} such that for every simple root ray a the restriction Ψ to U_{-a} equals $\Psi_1^a = \Psi_1 \circ \phi_a^{-1}$.

For this Ψ the Whittaker map $\mathcal{W}_\Psi : \mathcal{S}_c(X) \to \mathcal{S}_c(T)$, defined as in the introduction,

$$\mathcal{W}_\Psi(f)(t) = \int_{U^{op}} \theta(t) f([u]) \Psi^{-1}(u) du$$

gives rise to an isomorphism $\mathcal{S}_0(X)_{U^{op},\Psi} \simeq \mathcal{S}_0(T)$, where $\mathcal{S}_0(T) = \mathcal{W}_\Psi(\mathcal{S}_0(X))$. This isomorphism induces the map

$$\kappa_\Psi : \mathrm{End}_G(\mathcal{S}_0(X)) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(X)_{U^{op},\Psi}) = \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(T))$$

Lemma 6.2. *The map κ_Ψ is injective.*

Proof. Let us show that $\mathrm{Ker}\, \mathcal{W}_\Psi$ does not contain non-zero G-modules. Indeed, assume that $V \subset \mathrm{Ker}\, \mathcal{W}_\Psi \subset \mathcal{S}_0(X)$ is a non-zero G-module. For any character χ of T the space of coinvariants $\mathcal{S}_0(X)_{T,\chi^{-1}}$ is naturally isomorphic to the normalized principal series representation $\mathrm{Ind}_B^G(\chi)$. The functor of coinvariants induces a map $V_{T,\chi^{-1}} \to \mathrm{Ind}_B^G(\chi)$. For every character χ in a Zarisky-open set one has:

- for some $f \in V$ the Mellin transform $P_\chi(f) = \int_T \theta(t) f \cdot \chi(t) dt \neq 0$,
- the representation $\mathrm{Ind}_B^G(\chi)$ is irreducible.

We pick such χ. Since f does not belong to the kernel of P_χ, so the map $V_{T,\chi^{-1}} \to \mathrm{Ind}_B^G(\chi)$ is non-zero, thus surjective. The functor of coinvariants with respect to (U^{op}, Ψ) is exact and hence there is a surjection

$$(V_{T,\chi^{-1}})_{U^{op},\Psi} \to \mathrm{Ind}_B^G(\chi)_{U^{op},\Psi}.$$

Since $V \subset \mathrm{Ker}\, \mathcal{W}_\Psi$, one has $0 = V_{U^{op},\Psi} = (V_{U^{op},\Psi})_{T,\chi^{-1}}$, while $\mathrm{Ind}_B^G(\chi)_{U^{op},\Psi} \neq 0$. This is a contradiction.

Let $\mathcal{B} \in \mathrm{End}_G(\mathcal{S}_0(X))$ such that $\kappa_\Psi(\mathcal{B}) = 0$. Then $\mathcal{W}_\Psi \circ \mathcal{B} = 0$, and $\mathrm{Im}(\mathcal{B})$ is a G-module, contained in $\mathrm{Ker}\, \mathcal{W}_\Psi$ and hence is zero. So $\mathcal{B} = 0$ and κ_Ψ is injective. $\square$

We have defined all the notation, mentioned in Theorem 1.2. It states:

There exists a unique family of unitary operators $\Phi_w \in \mathrm{Aut}(L^2(X)), w \in W$ that preserves the space $\mathcal{S}_0(X)$ and satisfies

$$\begin{cases} \theta(g,t) \circ \Phi_w = \Phi_w \circ \theta(g, t^w) & g \in G, t \in T \\ \kappa_\Psi(\Phi_w)(\varphi) = w \cdot \varphi & \varphi \in \mathcal{S}_0(T) \\ \Phi_{w_1} \circ \Phi_{w_2} = \Phi_{w_1 w_2} & w_1, w_2 \in W \end{cases} \tag{6.3}$$

We begin with the construction of the operators Φ_s for simple reflections, based on the results for the groups of rank one.

6.0.3. *The definition of* Φ_{s_a}. The space $L^2(X)$ is the unitary completion $L^2\text{-}\mathrm{ind}_U^G 1$ of the space $\mathcal{S}_c(X) = \mathrm{ind}_U^G 1$.

For a simple root ray a of G consider a parabolic subgroup $P_a = M_a \cdot U^a$, with the derived group $P'_a = M'_a U^a$, where $M'_a = G_a$ is a semisimple group of rank 1. We denote by $B_a = T_a \cdot U_a$ the Borel subgroup of G_a and put $X_a = U_a \backslash G_a$.

Consider the isomorphism, implied by the transitivity of induction,

$$\iota_a : L^2(X) \to L^2\text{-}\mathrm{ind}_{P'_a}^G L^2(X_a).$$

defined by $\iota_a(f)(g)([m]) = f([mg])$.

The isometry Φ_s on $L^2(X_a)$, defined in sections 3 and 4 gives rise to an isometry on $L^2(X)$ by functoriality of induction. We continue to denote this isometry by Φ_{s_a}.

Definition 6.4. *The operator* $\Phi_{s_a} \in \mathrm{Aut}_G(L^2(X))$ *is defined by*

$$\iota_a(\Phi_{s_a}(f))(g) = \Phi_s(\iota_a(f)(g)), \quad f \in L^2(X), g \in G.$$

Proposition 6.5. *For any simple root ray a the operator* $\Phi_{s_a} \in \mathrm{Aut}(L^2(X))$ *is a unitary involution satisfying* $\theta(g,t) \circ \Phi_{s_a} = \Phi_{s_a} \circ \theta(g, t^{s_a})$.

Proof. The only non-trivial statement is the equivariance of T which is enough to prove for $f \in \mathcal{S}_0(X)$.

Consider an embedding with dense image

$$j : T_a \times U_a \hookrightarrow X_a, \quad (t,u) \mapsto t^{-1} n_{s_a} u.$$

For $f \in \mathcal{S}_0(X)$ the Fourier transform is given by

$$\Phi_{s_a}(f)([g]) = \int_{T_a} \int_{U_a} f(t^{-1} n_{s_a} u g) \mathcal{L}(\langle [t^{-1} n_{s_a} u], [1] \rangle_{X_a}) \delta_B(t_1) dt_1 du.$$

Here $\mathcal{L} = \psi$ for $a = \{\alpha\}$ and is defined by 4.6 for $a = \{\alpha, 2\alpha\}$.

Assume that $a = \{\alpha\}$. Using 3.7 for $f \in \mathcal{S}_c(X)$ one has

$$\theta(t_1)\Phi_{s_a}(f)([g]) = \delta_B^{1/2}(t_1)\Phi_{s_a}(f)([t_1^{-1} g]) =$$

$$= \delta_B^{1/2}(t_1) \int_{T_a} \int_{U_a} f([t^{-1} n_{s_a} u t_1^{-1} g]) \mathcal{L}(\langle [1], [t^{-1} n_{s_a}] \rangle) \delta_B(t) dt du =$$

$$\delta_B^{1/2}(t_1) \delta_{B_a}^{M'_a}(t_1)^{-1} \int_{T_a} \int_{U_a} f((t_1^{s_a})^{-1} t^{-1} n_{s_a} u g) \psi(\langle [1], [t^{-1} n_{s_a}] \rangle) \delta_B(t) dt du =$$

$$\Phi_{s_a}(\theta(t_1^{s_a})f)([g]).$$

We have used the fact that the inner product is G_a invariant and that

$$\delta_B^{1/2}(t_1)\delta_{B_a}^{M'_a}(t_1)^{-1} = \delta_B^{1/2}(t_1^{s_a}).$$

□

Proposition 6.6. *For any simple root ray a the operator Φ_{s_a} preserves $\mathcal{S}_0(X)$.*

Proof. We have defined for any root ray a the set of triples $\mathbb{B}_a$ such that $\mathcal{S}_0(X) \subset \mathcal{S}_{\mathbb{B}_a}(X) \subset \mathcal{S}_c(X)$. In fact $\mathcal{S}_{\mathbb{B}_a}(X) = \mathrm{ind}_{P'_a}^G \mathcal{S}_0(X_a)$ which is preserved by Φ_{s_a} by Definition 6.1 and by Propositions 3.8, 4.17. In particular, $\Phi_{s_a}(\mathcal{S}_0(X)) \subset \mathcal{S}_c(X)$.

For $f \in \mathcal{S}_0(X)$ let us show that $\Phi_s(f) \in \mathcal{S}_0(X)$. For any triple $(L_\alpha, \phi_a \circ t, \chi) \in \mathbb{B}_a$ denote the Mellin transform by $P(\chi, \alpha)$. Then $P(\chi, s_a(\alpha))\Phi_{s_a}(f) = \Phi_{s_a}(P(\chi,\alpha)f) = 0$ by the equivariance property of Φ_{s_a}.

□

6.0.4. *The operator $\kappa_\Psi(\Phi_{s_a})$.* In this subsection we compute $\kappa_\Psi(\Phi_{s_a})$ for the character Ψ defined in 6.0.2.

Proposition 6.7. *For any $\varphi \in \mathcal{S}_0(T)$ one has*

$$\kappa_\Psi(\Phi_{s_a})(\varphi) = s_a \cdot \varphi.$$

Proof. We shall show first the statement for $t = 1$, i.e.

$$\kappa_\Psi(\Phi_{s_a})(\varphi)(1) = \theta(t_a)\varphi(1).$$

Let a be a positive root ray.

$$\mathcal{W}_\Psi(\Phi_{s_a}(f))(1) = \int\limits_{U^{op}} \Phi_{s_a}(f)([u])\Psi^{-1}(u)du.$$

We use decomposition $U^{op} = U^{-a}U_{-a}$ where U^{-a} is the product of all root subgroups corresponding to the negative root rays, except $-a$. One has

$$\mathcal{W}_\Psi(\Phi_{s_a}(f))(1) = \int\limits_{U^{-a}} \left(\int\limits_{U_{-a}} \Phi_{s_a}(f)([u_1u_2])\Psi^{-1}(u_1)du_1 \right) \cdot \Psi^{-1}(u_2)du_2.$$

The character Ψ restricted to U_{-a} equals Ψ_1^a by the definition of Ψ. The inner integral equals

$$\int\limits_{U_{-a}} \Phi_{s_a}(\iota_a(f)(u_2))([u_1])\Psi_1^a(u_1^{-1})du_1 = \mathcal{W}_{\Psi_1^a}(\Phi_{s_a}(\iota_a(f)(u_2))(1).$$

By Theorems 3.8 and 4.17 it equals to $\mathcal{W}_{\Psi_1^a}(\theta(t_a)\iota_a(f)(u_2))(1)$.

Thus $\mathcal{W}_\Psi(\Phi_{s_a}(f))(1)$ equals

$$\int_{U^{-a}} \left(\int_{U_{-a}} (\theta(t_a)\iota_a(f))(u_2)([u_3])\Psi^{-1}(u_3)du_3 \right) \cdot \Psi^{-1}(u_2)du_2 = \mathcal{W}_\Psi(\theta(t_a)f)(1).$$

For an arbitrary $t \in T$ one has

$$\kappa_\Psi(\Phi_{s_a})(\varphi)(t) = \kappa_\Psi(\theta(t)\Phi_{s_a})(\varphi)(1) =$$
$$\kappa_\Psi(\Phi_{s_a})(\theta(t^{s_a})\varphi)(1) = \theta(t_a t^{s_a})\varphi(1) =$$
$$\varphi(t_a t^{s_a}) = s_a \cdot \varphi(t)$$

as required. □

Now we are ready to prove Theorem 1.2.

Proof. The injectivity of κ_Ψ implies the uniqueness of the family $\Phi_w, w \in W$. To prove Theorem it is enough to construct the operators Φ_w. For any $w \in W$ there is a presentation $w = s_{a_1} \cdot \ldots \cdot s_{a_n}$ as a product of simple reflections. We define the operator $\Phi_w \in \mathrm{Aut}(L^2(X))$

$$\Phi_w(f) = \Phi_{s_{a_1}} \circ \ldots \circ \Phi_{s_{a_n}}.$$

The operator Φ_w is unitary, preserves $\mathcal{S}_0(X)$ and satisfies $\theta(g,t) \circ \Phi_w = \Phi_w \circ \theta(g, t^w)$ for $g \in G, t \in T$.

Clearly, κ_Ψ is a homomorphism of algebras. In particular,

$$\kappa_\Psi(\Phi_w)(\varphi) = \kappa_\Psi(\Phi_{s_{a_1}}) \circ \ldots \circ \kappa_\Psi(\Phi_{s_{a_n}})(\varphi) = s_{a_1} \cdot \ldots s_{a_n} \cdot \varphi = w \cdot \varphi,$$

and hence $\kappa_\Psi(\Phi_w)$ does not depend on the presentation of w. Since κ_Ψ is injective, the operator Φ_w does not depend on the presentation of w. The property $\Phi_{w_1 w_2} = \Phi_{w_1} \circ \Phi_{w_2}$ is obvious from the definition. □

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