WH-1: THE WHITTAKER MODEL

DARIO BERALDO

Abstract. In this talk, we define the procedures of Whittaker invariants/coinvariants. We then explain what they are good for and discuss some of their properties: a nontrivial equivalence between Whittaker coinvariants and invariants, as well as some useful consequences of techinical nature (the behaviour with respect to limits, colimits and duality).

Contents

1. The Whittaker invariant and coinvariant categories

The *(invariant) Whittaker* procedure takes as input a DG category $\mathcal{C}$ with a strong action of $\mathfrak{L}(G)$ at level κ and extracts a full subcategory, to be denoted by $\mathrm{Whit}(\mathcal{C})$ or by $\mathrm{Whit}_!(\mathcal{C})$. The remarkable feature of this subcategory is that it is expected to retain much information about $\mathcal{C}$.

1.1. *Reminder on $\mathfrak{L}(G)$-actions.* Recall that the level κ gives rise to a central extension $\widehat{\mathfrak{L}(G)}$ of $\mathfrak{L}(G)$ and to a comonoidal DG category $\text{D-mod}^!_\kappa(\mathfrak{L}(G))$, where the comonoidal structure is induced by the pullback along the group multiplication of $\widehat{\mathfrak{L}(G)}$.

By definition, we say that a DG category $\mathcal{C}$ is equipped with a *strong $\mathfrak{L}(G)$-action at level κ* if $\text{D-mod}^!_\kappa(\widehat{\mathfrak{L}(G)})$ coacts on $\mathcal{C}$.

Since $\text{D-mod}^!_\kappa(\widehat{\mathfrak{L}(G)})$ is dualizable, with dual the convolution monoidal DG category $\text{D-mod}^*_\kappa(\widehat{\mathfrak{L}(G)})$, we can rephrase the above definition in more intuitive terms: the DG categories with strong $\mathfrak{L}(G)$-action at level κ are simply the modules categories for $\text{D-mod}^*_\kappa(\widehat{\mathfrak{L}(G)})$.

1.2. Inside $\mathfrak{L}(G)$, we consider the subgroup $\mathfrak{L}(N)$. As the central extension splits over $\mathfrak{L}(N)$, there is a conservative forgetful functor

$$\mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa \longrightarrow \mathfrak{L}(N)\text{-}\mathbf{mod}.$$

In local geometric Langlands, we interested in considering the Whittaker model $\mathrm{Whit}(\mathcal{C})$ of $\mathcal{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa$; however all we need for the definition of $\mathrm{Whit}(\mathcal{C})$ is just the datum of an $\mathfrak{L}(N)$-action on $\mathcal{C}$.

1.3. *Preparing the definition.* Consider the canonical character $\chi : \mathfrak{L}(N) \to \mathbb{G}_a$ defined by

$$\chi : n \mapsto \sum_{i \in \mathcal{I}} \mathrm{Res}(\alpha_i(n)),$$

where the α_i's are the simple roots (regarded as homomorphisms $\mathfrak{L}(N) \to \mathfrak{L}(\mathbb{G}_a)$).

On $\mathbb{G}_a = \operatorname{Spec} k[t]$ lives the *multiplicative* $\mathfrak{D}$-module *exp*: the free $\mathcal{O}$-module generated by the symbol "e^t" with the relation $\frac{d}{dt}$"e^t" = "e^t". The word "multiplicative" here means that the functor Vect $\to$ D-mod$(\mathbb{G}_a)$ given by *exp* is *comonoidal*; in concrete terms, we have isomorphisms

$$m^!(exp) \simeq exp \boxtimes exp, \quad u^!(exp) \simeq k,$$

compatible in the natural way.[1]

Remark 1.4. Note that χ and *exp* together make Vect into an $\mathfrak{L}(N)$-module, to be denoted Vect_χ.

1.5. *The Whittaker invariant category.* We are finally ready to define Whit$(\mathcal{C})$, the *invariant Whittaker category of* $\mathcal{C}$. We set

$$\mathrm{Whit}(\mathcal{C}) := \mathcal{C}^{\mathfrak{L}(N),\chi} = \mathsf{Fun}_{\text{D-mod}^!(\mathfrak{L}(N))\text{-}\mathsf{comod}}(\mathrm{Vect}_\chi, \mathcal{C}).$$

This DG category can be computed as the totalization of the usual cosimplicial (cobar) construction. Tautologically, Whit$(\mathcal{C})$ comes equipped with a structure functor

$$\mathsf{oblv}^{\mathfrak{L}(N),\chi} : \mathrm{Whit}(\mathcal{C}) \longrightarrow \mathcal{C}.$$

In fact, this is an inclusion, as we remind in Corollary 2.4.

1.6. In words, we say that Whit$(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ consisting of those objects that are $\mathfrak{L}(N)$-invariant against the character χ. Thus, $c \in \mathcal{C}$ belongs to Whit$(\mathcal{C})$ if we are given an isomorphism

$$\mathsf{coact}(c) \simeq \chi^!(exp) \otimes c,$$

together with the natural system of compatibilities under further applications of coact and counit.

1.7. *Exercise.* For Vect viewed as a category with trivial $\mathfrak{L}(N)$-action, we have Whit(Vect) $\simeq 0$.

1.8. *Example.* Consider the DG category D-mod(Gr_G) together with its obvious strong $\mathfrak{L}(G)$-action at level 0. Then Whit(D-mod$(\mathsf{Gr}_G)) \simeq \mathrm{Rep}(\check{G})$.

1.9. *Example.* The DG category $\widehat{\mathfrak{g}}\text{-}\mathsf{mod}_{\kappa_c}$ admits a strong $\mathfrak{L}(G)$-action at critical level and we have Whit$(\widehat{\mathfrak{g}}\text{-}\mathsf{mod}_{\kappa_c}) \simeq \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(D^\times))$, where $\mathrm{Op}_{\check{G}}(D^\times)$ is the moduli of opers on the punctured disk. See [4].

1.10. *The Whittaker coinvariant category.* In parallel with usual representation theory, we can also define the Whittaker coinvariant DG category

$$\mathrm{Whit}_*(\mathcal{C}) := \mathrm{Vect}_\chi \underset{\text{D-mod}^*(\mathfrak{L}(N))}{\otimes} \mathcal{C},$$

where Vect_χ is now considered as a right D-mod$^*(\mathfrak{L}(N))$-module. Observe that Whit$_*(\mathcal{C})$ can be computed as a geometric realization of the usual bar construction and that it is equipped with a structure projection functor $\mathsf{pr} : \mathcal{C} \to \mathrm{Whit}_*(\mathcal{C})$.

[1] There is no homotopy theory hidden here: all formulas take place in the ordinary abelian category D-mod$(\mathbb{G}_a)^\heartsuit$.

2. Approximation by compact opens

Let us play with the above definitions and explain why $\mathsf{oblv}^{\mathfrak{L}(N),\chi} : \mathrm{Whit}(\mathcal{C}) \to \mathcal{C}$ is fully faithful.

2.1. When dealing with loop groups, one needs to face two "infinities": the ind-direction coming from poles, and the pro-direction coming from the Taylor part. A good feature of $\mathfrak{L}(N)$, not shared by $\mathfrak{L}(B)$ or $\mathfrak{L}(G)$, is that these two infinities can be "decoupled". This is a consequence of the fact that $\mathfrak{L}(N)$ is an ind-group-scheme, rather than just a group indscheme.

Explicitly, for $k \in \mathbb{Z}$, let $N_k := \mathrm{Ad}_{t^{-k\check{\rho}}}(\mathfrak{L}^+(N))$, where $\check{\rho}$ denotes half the sum of the positive coroots. These are group schemes of pro-finite type: for instance, for $G = SL_3$, we have

$$N_k = \begin{pmatrix} 1 & t^{-k}O & t^{-2k}O \\ 0 & 1 & t^{-k}O \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the obvious isomorphism

$$\mathfrak{L}(N) \simeq \operatorname*{colim}_{k \geq 0} N_k$$

exhibits $\mathfrak{L}(N)$ as an ind-group-scheme.

2.2. It follows that an $\mathfrak{L}(N)$-action on $\mathcal{C}$ consists of a compatible family of N_k-actions. Accordingly, the Whittaker category $\mathcal{C}^{\mathfrak{L}(N),\chi}$ is equivalent to the limit of the subcategories $\mathcal{C}^{N_k,\chi}$ along the relative forgetful functors $\mathsf{oblv} : \mathcal{C}^{N_{k+1},\chi} \to \mathcal{C}^{N_k,\chi}$.

Lemma 2.3. *Each* $\mathsf{oblv}^{N_k,\chi} : \mathcal{C}^{N_k,\chi} \to \mathcal{C}$ *is fully faithful.*

Proof. By changing $\mathcal{C}$ with $\mathcal{C} \otimes \mathrm{Vect}_\chi$, we may as well prove that $\mathsf{oblv}^{N_k} : \mathcal{C}^{N_k} \to \mathcal{C}$ is fully faithful. Now, the fact that N_k is a scheme (rather than an indscheme) implies that oblv^{N_k} admits a continuous right adjoint $\mathsf{Av}_*^{N_k}$. By construction, the comonad $\mathsf{oblv}^{N_k} \circ \mathsf{Av}_*^{N_k}$ is the functor of action by $k_{N_k} \in \text{D-mod}^*(N_k)$.

We need to prove that the unit of the adjunction $\mathrm{id} \to \mathsf{Av}_*^{N_k} \circ \mathsf{oblv}^{N_k}$ is an isomorphism. Since oblv^{N_k} is conservative, it suffices to show that

$$\mathsf{oblv}^{N_k} \longrightarrow \mathsf{oblv}^{N_k} \circ \mathsf{Av}_*^{N_k} \circ \mathsf{oblv}^{N_k}$$

is an equivalence. To this end, note that acting by k_{N_k} on the essential image of oblv^{N_k} is the same as tensoring with $H^*_{\mathrm{dR}}(N_k, k_{N_k})$. We conclude by the contractibility of N_k. $\square$

Corollary 2.4. *As a consequence, the functor* $\mathsf{oblv}^{\mathfrak{L}(N),\chi} : \mathrm{Whit}(\mathcal{C}) \to \mathcal{C}$ *is fully faithful, too:* $\mathrm{Whit}(\mathcal{C})$ *is the intersection, within* $\mathcal{C}$*, of the subcategories* $\mathcal{C}^{N_k,\chi}$.

2.5. Dually, we can express the coinvariant Whittaker category $\mathrm{Whit}_*(\mathcal{C})$ as a colimit of the convariant categories $\mathcal{C}_{N_k,\chi}$ along the tautological functors

$$\mathsf{pr} : \mathcal{C}_{N_k,\chi} \longrightarrow \mathcal{C}_{N_{k+1},\chi}. \tag{2.1}$$

See Section 5.7 for a more concrete description of the same colimit.

3. What is the Whittaker procedure good for?

With the main definitions in place, let us digress to give some motivation.[2] To this end, it is best to start with a finite dimensional toy model of the Whittaker procedures.

[2] This section can be certainly skipped by the reader.

3.1. Let $\mathcal{C}$ be equipped with a strong G-action. As above, we just look at the underlying N-action and consider the non-degenerate character (denoted also by χ, by abuse of notation)

$$\chi : N \to N/[N,N] \simeq \prod_i \mathbb{G}_a \xrightarrow{\text{sum}} \mathbb{G}_a.$$

Then define $\mathrm{Whit}^{fd}(\mathcal{C}) := \mathcal{C}^{N,\chi}$ and $\mathrm{Whit}^{fd}_*(\mathcal{C}) := \mathcal{C}_{N,\chi}$.

Even in this situation it is not totally clear what these constructions are good for. So, let us consider the simplest nontrivial case: $G = GL_2$. Then $N \simeq \mathbb{G}_a$ is abelian and therefore $\mathrm{Whit}^{fd}(\mathcal{C})$ can be understood via the Fourier transform, as we now recall.

3.2. For V a vector group, the Fourier-Deligne transform gives a *monoidal* equivalence $(\text{D-mod}(V), \star) \simeq (\text{D-mod}(V^*), \otimes)$, whence any $\mathcal{C}$ acted on by V can be regarded as a *crystal of categories* over V^*. Put another way, there is an equivalence of $(\infty, 2)$-categories

$$V\textbf{-mod} \simeq (\text{D-mod}(V^*), \otimes)\textbf{-mod},$$

compatible with the forgetful functors to DGCat.

3.3. *Example.* For $\mathcal{C}$ acted on by V and $\xi \in V^*$, we can take the fiber of $\mathcal{C}$ at ξ, which is by definition the DG category

$$\mathcal{C}|_\xi := \text{D-mod}(\{\xi\}) \underset{\text{D-mod}(V^*)}{\otimes} \mathcal{C}.$$

More pedantically, what we have defined above should be called the *cofiber* at ξ, the honest fiber being $\mathsf{Fun}_{\text{D-mod}(V^*)}(\text{D-mod}(\{\xi\}), \mathcal{C})$. However, the two are canonically identified since $\text{D-mod}(\{\xi\})$ is self-dual as a module for $\text{D-mod}(V^*)$.

Thus, such fiber could be called the "ξ-Fourier coefficient of $\mathcal{C}$", for it tautologically corresponds to the full subcategory $\mathcal{C}^{V,\xi} \subseteq \mathcal{C}$ of (V,ξ)-invariant objects.

Remark 3.4. As in Lemma 2.3, the fact that $\mathcal{C}^{V,\xi} \to \mathcal{C}$ is fully faithful follows from the contractibility of V. Alternatively, the same assertion in Fourier dual language is a consequence of Kashiwara's lemma.

3.5. *Example.* More generally, let $W \subseteq V$ be a vector subspace and $\eta \in W^*$. Consider the affine subspace $\widetilde{\eta} + W^\perp \subseteq V^*$ where $\widetilde{\eta}$ is a lift of η along $V^* \twoheadrightarrow W^*$. Then the restriction of $\mathcal{C}$ along $\widetilde{\eta} + W^\perp$ corresponds, under Fourier transform, to the invariant category $\mathcal{C}^{W,\eta}$.

3.6. *Example.* Let V be as above and assume that a semidirect group $H \ltimes V$ acts on $\mathcal{C}$. Then, for ξ and ξ' in the same H-orbit in V^*, the two Fourier coefficients $\mathcal{C}^{V,\xi}$ and $\mathcal{C}^{V,\xi'}$ are identified by the action of H.

3.7. *Returning to finite dimensional Whittaker.* The last example above illustrates some of the structure we have in the case of $G = GL_2$ acting on $\mathcal{C}$: indeed, the Borel subgroup $B \simeq T \ltimes N$ acts on $\mathcal{C}$. We deduce that $\mathcal{C}$ is completely determined by two Fourier coefficients: the zero one, corresponding to $\mathcal{C}^N$, and any nonzero one, corresponding to the finite dimensional Whittaker subcategory.

So, for $G = GL_2$, the subcategory $\mathcal{C}^{N,\chi}$ contains most of the information about $\mathcal{C}$.

3.8. On the other hand, as seen above, $\mathcal{C}^{N,\chi}$ is defined for general G independently of the Fourier transform. This is the reason to study $\mathcal{C} \rightsquigarrow \mathrm{Whit}^{fd}(\mathcal{C})$ in general: encouraged by the GL_2 situation, we *hope* that $\mathrm{Whit}(\mathcal{C})$ will retain most of the information about the original $\mathcal{C}$.

3.9. *Returning to Whittaker.* There is a generalization of the Fourier-Deligne transform to the loop case, that is, for $\mathfrak{L}(V)$, and the same discussion as above renders to the setting of loop groups, giving the same conclusion in the case of GL_2.

4. Properties of the Whittaker constructions

In this section, we discuss some properties of the assignments $\mathcal{C} \rightsquigarrow \mathrm{Whit}_!(\mathcal{C}) := \mathrm{Whit}(\mathcal{C})$ and $\mathcal{C} \rightsquigarrow \mathrm{Whit}_*(\mathcal{C})$. The properties are of technical nature, but essential for local geometric Langlands.

We ask: how are $\mathrm{Whit}_*(\mathcal{C})$ and $\mathrm{Whit}_!(\mathcal{C})$ related? The answer is the best possible one.

Theorem 4.1 (See [3], [1], [4] for the original sources)**.** *There is a canonical equivalence*

$$\Theta : \mathrm{Whit}_*(\mathcal{C}) \to \mathrm{Whit}_!(\mathcal{C}).$$

The construction of Θ will be explained in the next section. In this section, we explore some consequences of the theorem, while the actual proof will be the topic of Talk Wh-2.

4.2. Here is the first corollary of Theorem 4.1: the two functors

$$\mathrm{Whit}_{!\,or\,*} : \mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa \longrightarrow \mathsf{DGCat}$$

commute with both limits and colimits. Indeed, it is clear by formal nonsense that the !-one commutes with limits, while the $*$-one commutes with colimits.

4.3. Another corollary has to do with dualizability. If $\mathcal{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa$ is dualizable as a plain DG category, then its dual $\mathcal{C}^\vee$ belongs naturally to $\mathfrak{L}(G)\text{-}\mathbf{mod}_{-\kappa}$. By formal nonsense, we have the tautological equivalence

$$\mathsf{Fun}(\mathrm{Whit}_*(\mathcal{C}), \mathrm{Vect}) \simeq \mathrm{Whit}_!(\mathcal{C}^\vee), \tag{4.1}$$

which suggests that $\mathrm{Whit}_*(\mathcal{C})$ and $\mathrm{Whit}_!(\mathcal{C}^\vee)$ "want" to be mutually duals. We exploit Theorem 4.1 to show this is indeed the case.

Corollary 4.4. *If $\mathcal{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa$ is dualizable as a plain DG category with dual $\mathcal{C}^\vee$, then $\mathrm{Whit}_*(\mathcal{C})$ is dualizable with dual $\mathrm{Whit}_!(\mathcal{C}^\vee)$.*

Proof. By (4.1), we have a functor

$$\varepsilon : \mathrm{Whit}_!(\mathcal{C}^\vee) \otimes \mathrm{Whit}_*(\mathcal{C}) \longrightarrow \mathrm{Vect}$$

which is our candidate evaluation. We need to prove that, for any fixed $\mathcal{E} \in \mathsf{DGCat}$, the functor

$$\mathrm{Whit}_!(\mathcal{C}^\vee) \otimes \mathcal{E} \longrightarrow \mathsf{Fun}(\mathrm{Whit}_*(\mathcal{C}), \mathcal{E})$$

induced by ε is an equivalence. We have

$$\mathrm{Whit}_!(\mathcal{C}^\vee) \otimes \mathcal{E} \simeq \big(\lim_{[n] \in \mathbf{\Delta}} \text{D-mod}^!(\mathfrak{L}(N))^{\otimes n} \otimes \mathcal{C}^\vee \big) \otimes \mathcal{E}$$

by definition, while

$$\mathsf{Fun}(\mathrm{Whit}_*(\mathcal{C}), \mathcal{E}) \simeq \lim_{[n] \in \mathbf{\Delta}} \big(\text{D-mod}^!(\mathfrak{L}(N))^{\otimes n} \otimes \mathcal{C}^\vee \otimes \mathcal{E} \big)$$

by a formal manipulation. Under these equivalences, the functor in question goes over to the obvious functor

$$\Big(\lim_{[n] \in \mathbf{\Delta}} \text{D-mod}^!(\mathfrak{L}(N))^{\otimes n} \otimes \mathcal{C}^\vee \Big) \otimes \mathcal{E} \longrightarrow \lim_{[n] \in \mathbf{\Delta}} \big(\text{D-mod}^!(\mathfrak{L}(N))^{\otimes n} \otimes \mathcal{C}^\vee \otimes \mathcal{E} \big).$$

Hence, we must commute a limit with a tensor product. We apply Theorem 4.1 to the $\mathfrak{L}(G)$-modules $\mathcal{C}^\vee$ and $\mathcal{C}^\vee \otimes \mathcal{E}$ to convert these limits into colimits. Namely, we are now looking at a functor

$$\big(\operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \text{D-mod}^*(\mathfrak{L}(N))^{\otimes n} \otimes \mathcal{C}^\vee \big) \otimes \mathcal{E} \longrightarrow \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \big(\text{D-mod}^*(\mathfrak{L}(N))^{\otimes n} \otimes \mathcal{C}^\vee \otimes \mathcal{E} \big),$$

which is clearly an equivalence. $\square$

5. A functor from coinvariants to invariants

Let us construct the functor $\Theta : \mathrm{Whit}_*(\mathcal{C}) \to \mathrm{Whit}_!(\mathcal{C})$ of Theorem 4.1. We proceed in stages: first we consider a similar construction in the finite dimensional case, then in the pro-unipotent group case, and finally in our Whittaker case. This progression will help us explain why Theorem 4.1 is nontrivial.

5.1. *The finite dimensional case.* Let us illustrate the finite dimensional analogue first. Let H be a finite dimensional group acting on $\mathcal{C}$ (we do not assume that H be unipotent).

We leave it to the reader to see that the functor $\mathsf{Av}_*^H : \mathcal{C} \to \mathcal{C}^H$ factors through the projection $\mathsf{pr} : \mathcal{C} \to \mathcal{C}_H$, that is, we have

$$\mathsf{Av}_*^H : \mathcal{C} \xrightarrow{\mathsf{pr}} \mathcal{C}_H \xrightarrow{\theta} \mathcal{C}^{H,\chi}.$$

5.2. Similarly, in the presence of a character $\chi : H \to \mathbb{G}_a$, the functor $\mathsf{Av}_*^{H,\chi}$ descends to a functor

$$\theta : \mathcal{C}_{H,\chi} \longrightarrow \mathcal{C}^{H,\chi}.$$

Theorem 5.3. *The above functor θ is an equivalence.*

Proof for H unipotent. Indeed, thanks to the contractibility of H, the functor

$$\mathcal{C}^{H,\chi} \xrightarrow{\mathsf{oblv}} \mathcal{C} \xrightarrow{\mathsf{pr}} \mathcal{C}_{H,\chi}$$

is easily seen to be an inverse to θ. □

5.4. *Exercise.* We stress that the theorem holds true even if H is not unipotent. (We will not need this more general case in the present talk.) What changes need to be made to the proof?

5.5. *The pro-unipotent group case.* The same construction renders verbatim to the case of H a pro-unipotent group scheme acting on $\mathcal{C}$. Obviously, the examples we have in mind are the groups $H = N_k$ defined in Section 2.1.

Namely, one shows that $\mathsf{Av}_*^{H,\chi} : \mathcal{C} \to \mathcal{C}^{H,\chi}$ descends to an equivalence

$$\theta : \mathcal{C}_{H,\chi} \longrightarrow \mathcal{C}^{H,\chi}$$

with inverse $\mathcal{C}^{H,\chi} \xrightarrow{\mathsf{oblv}} \mathcal{C} \xrightarrow{\mathsf{pr}} \mathcal{C}_{H,\chi}$, as above.

5.6. *The Whittaker case.* Let us return to the setting we are interested in: G is a reductive group, $\mathcal{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}_\kappa$. We wish to construct a natural functor from $\mathrm{Whit}_*(\mathcal{C}) \to \mathrm{Whit!}(\mathcal{C})$. Inspired by the two cases above, we might be tempted to consider the functor

$$\mathsf{Av}_*^{\mathfrak{L}(N),\chi} : \mathcal{C} \longrightarrow \mathrm{Whit}_!(\mathcal{C})$$

and argue that it descends to a functor $\mathrm{Whit}_*(\mathcal{C}) \to \mathrm{Whit}(\mathcal{C})$, as wanted. This is however too naive: $*$-averaging with respect to a group ind-scheme is a discontinuous functor.[3]

5.7. To come up with a better functor, recall that $\mathrm{Whit}_!(\mathcal{C})$ is the intersection (limit) of the sequence of inclusions

$$\cdots \hookrightarrow \mathcal{C}^{N_{k+1},\chi} \hookrightarrow \mathcal{C}^{N_k,\chi} \hookrightarrow \cdots .$$

Using the equivalences θ's for all groups $H = N_k$, we see that $\mathrm{Whit}_*(\mathcal{C})$ is the colimit of the sequence of colocalizations

$$\cdots \xleftarrow{\mathsf{Av}_*^{N_{k+2},\chi}} \mathcal{C}^{N_{k+1},\chi} \xleftarrow{\mathsf{Av}_*^{N_{k+1},\chi}} \mathcal{C}^{N_k,\chi} \xleftarrow{\mathsf{Av}_*^{N_k,\chi}} \cdots .$$

[3]In harmonic analysis, this would correspond to integrating with respect to a non-compact group.

5.8. What is the relation between $\mathsf{Av}_*^{N_k,\chi}$ and $\mathsf{Av}_*^{N_{k+1},\chi}$? The answer is that there is a natural transformation

$$\mathsf{Av}_*^{N_k,\chi} \longrightarrow \mathsf{Av}_*^{N_{k+1},\chi}[2\dim(N_{k+1}/N_k)].$$

This is a general fact about the theory of D-mod*-modules on a pro-unipotent group scheme. The reader might figure out that the shift is correct by checking the finite dimensional case (that is, by looking at the same statement for an inclusion of unipotent groups).

5.9. Let us now trivialize the dimension torsor of $\mathfrak{L}(N)$ by declaring that $N_0 \simeq \mathfrak{L}^+(N)$ has dimension zero. (Any other choice would be legitimate.) Then we can consider the colimit functor

$$\Theta : \operatorname*{colim}_{k\geq 0} \mathsf{Av}_*^{N_k,\chi}[2\dim(N_k)] : \mathcal{C} \longrightarrow \mathcal{C}.$$

It is clear that the essential image of Θ is contained in $\mathrm{Whit}_!(\mathcal{C})$: indeed, by filteredness, one can index the colimit by starting at any $k_0 >> 0$, in which case the image is contained in $\mathcal{C}^{N_{k_0},\chi}$. Equally tautologically, Θ descends to a functor $\mathrm{Whit}_*(\mathcal{C}) \to \mathrm{Whit}_!(\mathcal{C})$, which we denote by Θ, too, by abuse of notation. This is the functor we were looking for.

References

[1] D. Beraldo. Loop group actions on categories and Whittaker invariants.
[2] D. Gaitsgory, Sheaves of categories and the notion of 1-affineness.
[3] D. Gaitsgory, Whittaker categories.
[4] S. Raskin. W-algebras and Whittaker categories.