Geometry of complex character varieties

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Contents

Chapter 1

Introduction

1.1 Version française

1.1.1 Résumé

Soit Γ un groupe de type fini et G un groupe de Lie. On peut considérer

$$\mathrm{Hom}(\Gamma, G)/G,$$

la variété de G-caractères de Γ ([66, 86]). Lorsque $\Gamma = \pi_1(\Sigma, b)$ est le groupe fondamental d'une surface, alors sa structure est très riche et a déjà été étudiée en profondeur. Par exemple, si G est un groupe unitaire et Σ est compacte, la variété de caractères est un modèle de l'espace de modules des fibrés semi-stables sur Σ, d'après le théorème de Narasimhan et Seshadri [73]. En particulier, la variété de caractères admet une structure symplectique naturelle ([5, 48]).

Par contre, si $G = \mathrm{GL}_n(\mathbb{C})$ et Σ est une courbe algébrique lisse quelconque, d'après la correspondance de Riemann-Hilbert [38] la variété de caractères paramètre les connexions singulières régulières sur des fibrés vectoriels algébriques sur Σ. Dans ce cas, les variétés de caractères possèdent une structure de Poisson naturelle. Les feuilles symplectiques sont obtenues en fixant les classes de conjugaison de la monodromie locale autour de chaque trou.

Ainsi, si on dénote les points marqués $a_1, \dots, a_m$ et si on choisit une classe de conjugaison $\mathcal{C}_i \subset \mathrm{GL}_n(\mathbb{C})$ pour tout i, alors on obtient une variété algébrique symplectique

$$\mathcal{M}_{\mathrm{B}}(\Sigma, \boldsymbol{\mathcal{C}}) \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C}),$$

où $\boldsymbol{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_m) \subset \mathrm{GL}_n(\mathbb{C})^m$.

Cette perspective suggère une généralisation vaste du concept de variété de caractères, en considérant des connexions plus générales, avec des singularités plus compliquées. La version de Deligne de la correspondance de Riemann-Hilbert a été étendue au cas irrégulier dans de nombreux travaux, notamment ceux de Sibuya

[85], Deligne [37], Balser-Jurkat-Lutz [9], Malgrange [68], Babbitt-Varadarajan [7], Martinet-Ramis [70] et Loday-Richaud [65].

Les connexions irrégulières sont classifiées par leurs données de Stokes. Elles généralisent les représentations du groupe fondamental et forment les variétés de caractères sauvages

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}}) \subset \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}.$$

Ici, $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ est l'espace de représentations de Stokes, $\mathbf{H}$ un sous-groupe de G^m et $\boldsymbol{\mathcal{C}}$ est une classe de conjugaison de $\mathbf{H}$. La définition plus détaillée sera présentée dans le Chapitre 2.

Elles partagent plusieurs propriétés avec le cas régulier : elles admettent des structures symplectiques holomorphes ([13, 15]), elles admettent des métriques hyperkahleriennes ([11]) (en relation avec les systèmes de Hitchin méromorphes), elles sont centrales dans la théorie d'isomonodromie ([56]), elles admettent des actions des groupes discrets ([16, 27]) et certains exemples sont liés avec le groupe quantique de Drinfeld—Jimbo ([14, 16]).

Par exemple, parmi les exemples qui apparaissent dans le cas simple de rang deux sur la sphère de Riemann $\mathbb{P}^1$ avec un point marqué on trouve les variétés de caractères sauvages suivantes

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}}) = \mathcal{B} \underset{q}{/\!\!/} T = \{(S_1, \ldots, S_{2k}) \in (U_+ \times U_-)^k \mid S_{2k} \cdots S_1 = q\}/T,$$

où $U_+, U_- \subset \mathrm{GL}_2(\mathbb{C})$ sont des unipotents opposés, $T = \mathbf{H}$ est le tore diagonal agissant par la conjugaison diagonale et $q \in T$ représente une classe de conjugaison $\boldsymbol{\mathcal{C}} = \{q\} \subset T$. Pour un choix de q générique, ces variétés sont lisses et symplectiques de dimension $2k - 4$.

Le but de cette thèse est d'étudier plusieurs questions concernant les variétés de caractères:

1) Puisque la théorie des variétés de caractères a été étendue au-delà des groupes linéaires, on voudrait fournir des exemples de variétés de caractères pour des groupes plus exotiques. Dans le Chapitre 3 on étudie un exemple d'une variété de caractères pour le groupe G_2. Il se trouve qu'elle est isomorphe aux surfaces de Fricke symétriques et donne une belle description des orbites de groupe de tresses comme les droites dans le plan de Fano. Ceci est un travail en commun avec P. Boalch [12] et d'après notre connaissance du sujet, c'est le premier exemple d'isomorphisme des variétés de caractères pour un groupe exotique.

2) On s'intéresse à la classification des variétés de caractères de dimension complexe deux. Ce sont des variétés hyperkahleriennes de dimension réelle quatre (instantons gravitationnels) et elles jouent un rôle important dans la physique. Dans le Chapitre 4 on construit un isomorphisme entre certaines familles infinies de tels espaces. Ceci est en accord avec la conjecture de [25], disant qu'il y a qu'un nombre fini de classes de déformation de telles variétés hyperkahlériennes.

3) On étudie les versions multiplicatives des variétés de carquois de Nakajima.

Les variétés de carquois multiplicatives «classiques» de Crawley-Boevey-Shaw et Yamakawa [36, 97] peuvent être construites à partir d'une brique élémentaire

$$\mathcal{B}_2(V_1, V_2) = \{(a, b) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \mid \det(1 + ab) \neq 0\},$$

que l'on comprend comme une arête. Elle se compose d'une paire d'applications entre deux sommets (le long de l'arête qui les connecte) avec des espaces V_1, V_2 attachés. Dans le contexte des variétés de caractères sauvages une généralisation naturelle apparaît, avec des briques élémentaires plus générales. Dans le Chapitre 5, dans le cas de k arêtes entre deux sommets, on trouvera une formule pour la structure symplectique multiplicative généralisant ainsi celle de Yamakawa. Les résultats du Chapitre 5 font partie d'un travail en cours avec P. Boalch et D. Yamakawa.

4) Finalement, on étudie les relations entre les variétés de caractères sauvages et les variétés de carquois multiplicatives classiques. Il se trouve que dans le cas de k arêtes entre deux sommets et $G = \mathrm{GL}_n(\mathbb{C})$, les variétés de carquois multiplicatives classiques se plongent sur des ouverts denses de variétés de carquois généralisées. De plus, on démontre que de tels plongements sont indéxés par les nombres de Catalan et peuvent être interprétés comme les factorisations des polynômes continuants d'Euler et les triangulations de polygones.

1.1.2 Motivations et présentation des résultats

Le but de cette thèse est d'étudier plusieurs classes d'exemples de variétés de caractères qui généralisent les variétés de caractères régulières. Ces espaces jouent un rôle fondamental dans les mathématiques et la physique. Presque tous les résultats de cette thèse concernent les variétés de caractères sauvages des groupes linéaires $\mathrm{GL}_n(\mathbb{C})$ mais certains énoncés restent vrais pour un groupe de Lie complexe réductif quelconque.

Pour chaque courbe Σ, il existe une variété algébrique affine complexe

$$\mathcal{M}_{\mathrm{B}}(\Sigma, n) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$$

de classes d'isomorphisme des représentations du groupe fondamental de Σ, que l'on appelle la variété de caractères, ou en utilisant la terminologie de Simpson [88], l'espace de modules de Betti.

La correspondance de Riemann-Hilbert fournit un isomorphisme analytique complexe entre $\mathcal{M}_{\mathrm{B}}(\Sigma, n)$ et l'espace de modules des connexions holomorphes (stables) sur les fibrés vectoriels de rang n sur Σ, l'espace de de Rham $\mathcal{M}_{\mathrm{DR}}(\Sigma, n)$, en envoyant une connexion ∇ sur sa donnée de monodromie. Cette application peut être interprétée comme une généralisation de l'application exponentielle. Elle est très transcendante et ne préserve pas les structures algébriques sur $\mathcal{M}_{\mathrm{DR}}$ et $\mathcal{M}_{\mathrm{B}}$.

Par contre, il y a le troisième espace de modules: l'espace de Dolbeault des fibrés de Higgs $\mathcal{M}_{\mathrm{Dol}}(\Sigma, n)$ qui est difféomorphe à l'espace $\mathcal{M}_{\mathrm{DR}}$ par l'isomorphisme de

“Hodge non-abélien”. Grâce aux travaux de Corlette, Donaldson, Hitchin et Simpson [51, 39, 35, 87], les trois espaces $\mathcal{M}_{\mathrm{B}}, \mathcal{M}_{\mathrm{DR}}, \mathcal{M}_{\mathrm{Dol}}$ peuvent être vus comme les réalisations d'une même variété hyperkahlerienne, vue dans deux structures complexes différentes. L'avantage principal de l'espace de Betti est son caractère explicite – il fournit des descriptions directes des espaces de modules, souvent peu évidentes quand on considère la variété comme l'espace de de Rham ou de Dolbeault.

Le cas classique de $\mathrm{GL}_n(\mathbb{C})$ pour les courbes compactes possède plusieurs généralisations. D'un côté, on peut supposer que la courbe Σ a des points marqués distincts $\{a_1, \dots, a_k\}$. Ainsi, la variété de caractères *régulière* $\mathcal{M}_{\mathrm{B}}$ de la courbe percée $\Sigma^\circ = \Sigma \setminus \{a_1, \dots, a_k\}$ admet une structure de Poisson et ses feuilles symplectiques sont obtenues en fixant les classes de conjugaison de la monodromie locale autour de chaque trou. Cette approche fournit des variétés symplectiques holomorphes. Cette structure symplectique a été construite pour la première fois de manière analytique par Atiyah et Bott [5] et plus tard par voie algébrique dans de nombreux travaux (par exemple [48, 58, 43, 2, 1]).

D'un autre côté, le groupe de structure G peut être un groupe de Lie complexe réductif connexe quelconque, et non plus seulement un groupe linéaire. Par conséquent, on peut considérer

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

l'espace de représentations du groupe fondamental de Σ dans le groupe de Lie G, ce qui donne des variétés plus générales. Au niveau des espaces de de Rham, les connexions sur des fibrés vectoriels sont remplacés par les connexions sur des G-fibrés principaux.

Dans ce cas là, l'extension de Deligne de la correspondance de Riemann-Hilbert [38] (pour les groupes linéaires) établit une bijection entre les ensembles des G-orbites dans $\mathrm{Hom}(\pi_1(\Sigma^\circ), G)$ et des classes d'isomorphisme des connexions sur les G-fibrés algébriques sur Σ° avec des singularités régulières. Une extension similaire est valable pour les groupes plus généraux. La régularité des singularités en a_i signifie qu'il existe un prolongement du fibré vectoriel en les points marqués tel que la connexion sur le fibré prolongé a au plus des pôles simples en chaque a_i.

Alors il est naturel de se demander: est-ce qu'il y a des exemples intéressants de variétés de caractères de groupes algébriques plus généraux? Dans le Chapitre 3, on étudie une variété de caractères régulière de la sphère avec quatre trous pour le groupe exotique $G_2(\mathbb{C})$. Pour $\mathrm{SL}_2(\mathbb{C})$, la variété résultante est la célèbre famille des surfaces cubiques de Fricke [45]

$$xyz + x^2 + y^2 + z^2 + b_1\, x + b_2\, y + b_3\, z + c = 0.$$

C'est un des exemples les plus simples de variétés de caractères, avec des actions de groupe de tresses intéressantes en relation étroite avec l'équation différentielle de Painlevé VI. On dit qu'une surface de Fricke est *symétrique* si $b_1 = b_2 = b_3$.

Le groupe $G_2(\mathbb{C})$ est de dimension 14 et possède une classe de conjugaison très particulière $\mathcal{C} \subset G_2(\mathbb{C})$ de dimension six, qui est un analogue complexe de la sphère de cette dimension. Si on prend trois copies de la classe $\mathcal{C}$ et on suppose la quatrième classe $\mathcal{C}_\infty \subset G_2(\mathbb{C})$ générique (alors de dimension 12), la variété de caractères sera de dimension

$$3 \times 6 + 12 - 2 \times 14 = 2.$$

On peut alors s'attendre à ce que cette variété soit liée aux surfaces de Fricke. C'est bien le cas et c'est le premier résultat principal de cette thèse:

Théorème I. *Il existe une famille à deux paramètres de variétés de caractères pour le groupe exceptionnel $G_2(\mathbb{C})$, isomorphes aux surfaces cubiques de Fricke symétriques et par conséquent aux variétés de caractères de groupe $\mathrm{SL}_2(\mathbb{C})$.*

De plus, on réexamine quelques orbites finies de groupe de tresses dans ces surfaces, trouvées dans [18, 22]. En particulier, on considère la surface cubique de Klein (l'unique surface cubique avec l'orbite de groupe de tresse de taille 7, alors un lien avec un groupe de dimension 14 n'est pas étonnant) et on démontre:

Théorème II. *Dans la surface cubique de Klein K, réalisée comme une variété de caractères de groupe G_2, l'orbite du groupe de tresses de taille 7 correspond à un triplé de générateurs d'un groupe fini simple $G_2(\mathbb{F}_2)' \subset G_2(\mathbb{C})$ d'ordre 6048. Un tel triplé de générateurs est déterminé uniquement par les trois droites passant par un point dans le plan de Fano $\mathbb{P}^2(\mathbb{F}_2)$.*

Les objets d'étude principaux de cette thèse sont des variétés de caractères *sauvages*, construites au début de manière analytique, avec leur structure symplectique, par Boalch dans [13, 15], généralisant l'approche de Atiyah et Bott, puis il a proposé une construction algébrique dans [21]. Ces variétés apparaissent lorsqu'on assouplit les conditions sur la régularité des singularités et considère les connexions avec des pôles d'ordre supérieur. Comme dans le cas régulier, les variétés de caractères sauvages encodent les données de monodromie (Stokes) des connexions méromorphes irrégulières. En revanche, puisque les données sont plus riches, ce n'est pas suffisant de considérer uniquement les représentations du groupe fondamental de Σ°.

En gros, une courbe irrégulière $\boldsymbol{\Sigma}$ est constitué d'une courbe Σ, d'un ensemble de points marqués $a_1, \ldots, a_k$, et en tout point marqué, d'un type irrégulier Q_i, qui décrit la singularité irrégulière. Ces données déterminent une nouvelle surface $\widetilde{\Sigma}$ – l'éclatement réel de tous points marqués a_i, encore avec quelques trous supplémentaires – et un sous-groupe $\mathbf{H} = H_1 \times \ldots \times H_k \subset G^k$. L'espace de représentations raffiné est l'espace

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G),$$

de "représentations de Stokes" du groupoïde fondamental Π de $\widetilde{\Sigma}$ avec des points bases sur les cercles du bord. Le groupe $\mathbf{H}$ agit sur cet espace et le quotient

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}$$

est la variété de caractères sauvage (l'espace de Betti sauvage). C'est encore une variété complexe affine qui admet une structure de Poisson. Cela signifie que le point de vue sauvage fournit en abondance des exemples de variétés de Poisson/symplectiques holomorphes. Si les types irréguliers sont tous nuls, on récupère la variété de caractères usuelle avec sa structure de Poisson.

Les variétés de caractères sauvages paramètrent les classes d'isomorphisme des connexions irrégulières et un analogue irrégulièr de la correspondance de Riemann–Hilbert a été établi il y a longtemps (au moins dans le cas de $G = \mathrm{GL}_n(\mathbb{C})$) dans de nombreux travaux, y compris ceux de Sibuya [85], Deligne [37], Balser–Jurkat–Lutz [9], Malgrange [68], Babbitt–Varadarajan [7], Martinet–Ramis [70] et Loday-Richaud [65]. Ensuite, cette correspondance a été étendue par Boalch aux groupes plus généraux dans [16]. Néanmoins, la compréhension de sa version irrégulière est beaucoup plus faible que dans le cas régulier.

Une des propriétés remarquables des variétés de caractères sauvages est le fait qu'elles viennent en familles. Si on fait varier la courbe irrégulière initiale $\boldsymbol{\Sigma}$ – la courbe Σ avec ses points marqués et les données irrégulières tous deux – dans la manière lisse (dans le sens de [27]), alors les variétés de caractères sauvages restent isomorphes et s'assemblent dans un fibré sur la base $\mathbb{B}$. Ce "système local de variétés", introduit dans le cas régulier par Simpson dans [89], admet une connexion plate non linéaire, un analogue non abélien de la connexion de Gauss–Manin. Le groupe fondamental de la base agit sur les fibres de ce fibré par les automorphismes de Poisson et fournit un analogue sauvage de l'action du mapping class group sur les variétés de caractères régulières. Cela généralise les travaux sur les déformations isomonodromiques de Jimbo–Miwa–Ueno [56] qui ont observé que dans le cas irrégulier l'espace de paramètres de déformation est plus grand, car c'est possible aussi de varier les types irréguliers.

Dans certains cas c'est possible d'écrire la connexion de Gauss–Manin explicitement en coordonnées et d'obtenir une équation différentielle non linéaire. Par exemple, pour des choix adéquats des pôles, les six équations de Painlevé apparaissent d'une telle manière, avec $\Sigma = \mathbb{P}^1$ et $G = \mathrm{SL}_2(\mathbb{C})$, mais seule l'équation de Painlevé VI admet une réalisation sans singularité irrégulière. Par conséquent, comme les espaces de phase des équations différentielles non linéaires, ces variétés de caractères sont d'un grand intérêt et ont été étudiées en détails.

Notre motivation principale provient du fait que les équations de Painlevé sont d'ordre deux, donc les espaces de phase/variétés de caractères sont de dimension complexe deux et c'est bien connu que ce sont des surfaces cubiques. D'après les résultats de Biquard et Boalch [11], qui ont généralisé les résultats de Hitchin au cas irrégulier, les variétés de caractères sauvages sont encore hyperkahleriennes et en dimension deux ce sont des exemples d'"instantons gravitationnels".

Dans [25] on trouve une liste conjecturale des 11 classes de déformation des variétés hyperkahleriennes de dimension réelle quatre émergeant dans la théorie de Hodge non abélienne, classifiées par les symboles de Dynkin. Ce sont des analogues non compacts des surfaces K3 que l'on appellera surfaces H3, en l'honneur de Higgs, Hitchin et Hodge suivant [30]. D'un point de vue mathématique, ce problème de classification s'inscrit dans la classification des "courbes quaternioniques" qui sont des analogues quaternioniques de surfaces de Riemann, comme discuté par Atiyah [4].

Surfaces H3	
Régulier	Sauvage
E_6, E_7, E_8, D_4	$(A_0), A_1, A_2, A_3, (D_0), (D_1), D_2$

Tableau 1. Classification conjecturale des surfaces H3

La correspondance avec les espaces de phase des équations de Painlevé est présentée en dessous:

Symbole	D_4	A_3	A_2	A_1	A_0	D_2	D_1	D_0
Équation de Painlevé	VI	V	IV	II	I	III	III'	III"
Ordre de pôle	1111	211	31	4	(4)	22	2(2)	(2)(2)

(les symboles exotiques E_6, E_7, E_8 correspondent aux équations aux différences de Painlevé). Les parenthèses signifient que la singularité irrégulière est tordue.

Boalch a observé que pour tout espace/symbole sur la liste il y a une liste infinie des "espaces d'écho" qui paramètrent les classes d'isomorphisme de connexions sur des fibres sur $\mathbb{P}^1$ de rangs arbitrairement grands (avec les types irréguliers bien choisis). De plus, tous les espaces d'écho sont de dimension deux et alors on peut s'attendre à ce qu'elles soient isomorphes. Le but du Chapitre 3 est d'établir les cas A_0, A_1, A_2 de cette conjecture.

Ceci est fait en utilisant la nouvelle théorie des variétés de carquois multiplicatives, introduites par Boalch dans [28]. Ce sont des analogues multiplicatifs des variétés de carquois de Nakajima [71, 72]. Les variétés de carquois multiplicatives, en relation avec l'algèbre preprojective multiplicative, ont été introduites par Crawley-Boevey et Shaw [36] et Yamakawa [97] et dans le cas de carquois étoilés elles sont isomorphes à celles de Boalch. Par contre dans d'autres cas il est effectivement possible d'obtenir des variétés qui ne sont pas isomorphes (cf. section 6. de [28]). On verra dans Chapitre 5 un exemple plus général soulignant les différences entre elles.

Certaines variétés de carquois de Nakajima ressemblent à l'espace de modules $\mathcal{M}_{\mathrm{DR}}$, elles sont isomorphes au sous-ensemble ouvert $\mathcal{M}^* \subset \mathcal{M}_{\mathrm{DR}}$ qui paramètre les classes d'isomorphisme des connexions sur des fibrés vectoriels *triviaux*. En revanche, les exemples de nouvelles variétés de carquois multiplicatives ne ressemblent pas uniquement aux espaces de Betti sauvages mais elles leur sont en fait isomorphes à eux. Le langage des variétés de carquois multiplicatives fournit une description pratique des espaces d'écho de type $A_1, A_2, A_3, D_4, E_6, E_7$ et E_8 comme les variétés de carquois multiplicatives des graphes de Dynkin associés. En particulier, cette description nous permet d'accéder directement aux cas sauvages de A_1, A_2, A_3.

Une variété de carquois multiplicative est déterminée par un graphe Γ avec k sommets, vecteur de dimension $d \in \mathbb{Z}^k_{>0}$ et un vecteur des paramètres $q \in (\mathbb{C}^*)^k$. Un type irrégulier non tordu Q détermine un graphe de fission $\Gamma(Q)$ et pour un bon choix de vecteur de dimension et de paramètres, la variété de carquois multiplicative $\mathcal{Q}(\Gamma(Q), q, d)$ est une variété de caractères.

Dans le cas des espaces d'écho A_1, A_2, A_3, les graphes qui apparaissent sont leur extensions affines $\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3$ et pour les vecteurs de dimension avec toutes ses leurs coordonnées égales à un et des paramètres génériques, les variétés de carquois sont des surfaces cubiques du Tableau 1. Nous allons dénoter abusivement l'espace d'écho de type A_0 par $\mathcal{Q}(\widetilde{A}_0, q, d)$, même si ce n'est pas une variété de carquois et le choix du paramètre q est limité aux racines primitives de l'unité. Par contre, au niveau algébrique ce cas est le plus facile et donne des bonnes intuitions pour d'autres cas. Le théorème principal du Chapitre 4 peut être énoncé comme suit:

Théorème III. *Pour $i = 0, 1, 2$, soit $\widetilde{A}_i$ le graph de Dynkin correspondant et soit n le vecteur de dimension avec toutes ses coordonnées égales à n. Alors pour un choix générique de vecteur des paramètres q_i il y a des isomorphismes*

$$\mathcal{Q}(\widetilde{A}_i, q_i, n) \simeq \mathcal{Q}(\widetilde{A}_i, q_i^n, 1).$$

Pour les carquois étoilés D_4, E_6, E_7, E_8 un résultat similaire a été établi par Etingof–Oblomkov–Rains dans [40]. Le Théorème III peut être compris comme une extension de ces résultats aux cas sauvages. De plus, on a démontré ces isomorphismes directement et on a fourni des relations explicites entre les paramètres des surfaces cubiques. Le cas de la variété de carquois multiplicative de type A_3, lié à l'équation de Painlevé V, est manquant. Nous avons recolté des preuves suggérant que les méthodes utilisées dans les trois premiers cas marchent aussi dans ce cas.

Une autre motivation pour étudier les variétés de caractères sauvages vient des travaux de Alexeev–Meinrenken–Malkin sur la géométrie quasi-Hamiltonienne [1], étendus aux groupes complexes. Leur approche mène vers une construction des structures de Poisson/symplectiques sur les variétés de caractères régulières, en utilisant des quotients symplectiques "multiplicatifs" en dimension finie. Les applications moment prennent leurs valeurs dans le groupe G, plutôt que dans le dual de l'algèbre de Lie $\mathfrak{g}$.

Supposons que Σ a k composantes du bord $\partial_1, \dots, \partial_k$ et choisissons un point base b_i sur chaque composante ∂_i. Notons par Π le groupoïde fondamental de Σ avec des points bases $\{b_1, \dots, b_k\}$. L'espace

$$\mathrm{Hom}(\Pi, G)$$

de représentations du groupoïde fondamental Π dans G est alors une variété affine lisse et possède une structure d'un G^k-espace quasi-Hamiltonien. Plus précisément, il y a une action de G^k sur $\mathrm{Hom}(\Pi, G)$ et une application moment μ à valeurs dans G^k

$$\mu : \mathrm{Hom}(\Pi, G) \to G^k$$

vérifiant des conditions similaires à celles d'action Hamiltonienne usuelle. La théorie générale des espaces quasi-Hamiltoniens garantit que le quotient

$$\mathrm{Hom}(\Pi, G)/G^k$$

hérite une structure de Poisson et est isomorphe à la variété de caractères régulière $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. Si on fixe un k-uplet des classes de conjugaison $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k) \subset G^k$, alors le quotient quasi-Hamiltonien

$$\mu^{-1}(\mathcal{C})/G^k$$

est une variété symplectique holomorphe, isomorphe aux feuilles symplectiques de la variété de caractères régulière. Cette approche admet une généralisation au cas irrégulier et la structure quasi-Hamiltonienne sur l'espace de données de Stokes a été construite dans [21]. Une des conséquences cruciales de cette construction est l'existence des structures de Poisson sur les espaces de Betti sauvages.

Les opérations de la fusion et recollement quasi-Hamiltoniens permettent de construire l'espace $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ à partir des morceaux plus simples: les "espaces de fission" $\mathcal{A}(Q)$ qui décrivent les singularités irrégulières (sur un disque avec un pôle irrégulier), les classes de conjugaison $\mathcal{C}$ qui décrivent les singularités régulières et les "doubles fusionnés par l'intérieur" qui correspondent aux anses topologiques. Alors il est important de comprendre que ce sont ces morceaux et l'espace de fission $\mathcal{A}(Q)$ qui portent le plus d'informations.

Plus précisément, on étudie les espaces de fission réduits $\mathcal{B}(Q)$ (cf. section 2.2.3 et l'équation (2.10) pour les définitions détaillées), qui paramètrent les classes d'isomorphisme des connexions sur la sphère de Riemann avec un pôle irrégulier. Un des exemples les plus simples de tel espace est l'espace de Van den Bergh $\mathcal{B}(V_1, V_2)$ [91, 92], défini pour un espace vectoriel gradué $V = V_1 \oplus V_2$:

$$\mathcal{B}(V_1, V_2) = \{(a, b) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \bigm| \det(1 + ab) \neq 0\}$$

qui est un $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-espace quasi-Hamiltonien avec l'application moment $\mu : \mathcal{B}(V_1, V_2) \to \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ donnée par

$$\mu(a, b) = ((1 + ab)^{-1}, 1 + ba) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$$

et la 2-forme quasi-Hamiltonienne

$$\omega = \frac{1}{2}\left(\mathrm{Tr}_{V_1}(1 + ab)^{-1} da \wedge db - \mathrm{Tr}_{V_2}(1 + ba)^{-1} db \wedge da\right).$$

En tant qu'ensemble, l'espace $\mathcal{B}(V_1, V_2)$ est constitué de la paire des applications (a, b) entre V_1, V_2 telles que $1+ab$ soit inversible. Il est isomorphe à l'espace de fission réduit $\mathcal{B}(V)$ et l'expression polynomiale $1 + ab$ est le deuxième polynôme continuant d'Euler [41]. En général, on peut définir le n-ième polynôme continuant par la relation de récurrence

$$(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})x_n + (x_1, \ldots, x_{n-2}).$$

Quelques premiers polynômes continuants sont alors donnés par:

$$\begin{aligned}
(\emptyset) &= 1\\
(x_1) &= x_1\\
(x_1, x_2) &= x_1x_2 + 1\\
(x_1, x_2, x_3) &= x_1x_2x_3 + x_1 + x_3\\
(x_1, x_2, x_3, x_4) &= x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1\\
(x_1, x_2, x_3, x_4, x_5) &= x_1x_2x_3x_4x_5 + x_1x_2x_5 + x_1x_2x_5 + x_1x_4x_5 +\\
&\quad + x_3x_4x_5 + x_1 + x_3 + x_5
\end{aligned}$$

Lorsque nous généralissons naturellement l'espace $\mathcal{B}(V_1, V_2)$ et nous considèrons plus que deux – mais toujours un nombre pair – d'applications entre V_1, V_2, nous obtenons l'espace de fission réduit $\mathcal{B}^k(V)$. Son application moment quasi-Hamiltonienne est encore donnée par un polynôme continuant, en à $2k$ variables (non commutatives). Si on suppose de plus que $V = W \oplus W$, alors on peut définir des analogues des espaces $\mathcal{B}^k(V)$ avec un nombre impair d'applications dans $\mathrm{End}(W)$. Il est naturel d'introduire la définition suivante:

$$\mathcal{B}_k = \{(b_1, \ldots, b_k) \in \mathrm{End}(W)^k \;\big|\; \det(b_1, \ldots, b_k) \neq 0\},$$

qui est un $\mathrm{GL}(W) \times \mathrm{GL}(W)$-espace quasi-Hamiltonien (peut-être tordu) et son application moment est le continuant. Le morceau le plus simple $\mathcal{B}_1$ est juste une copie de $\mathrm{GL}(W)$ avec la 2-forme nulle.

Dans le Chapitre 5 on étudie les factorisations des continuants, ie. les décompositions des continuants en produits des continuants plus courts. En particulier, les décompositions totales en produits des contiuants de longueur un, qui correspondent aux morceaux isomorphes à $\mathcal{B}_1 \simeq \mathrm{GL}(W)$. Le résultat principal du Chapitre 5 est:

Théorème IV. *Soit k un entier positif et soit C_k le k-ième nombre de Catalan. Alors il y a C_k factorisations totales différentes de $(x_1, \ldots, x_k)$, paramétrées par les triangulations de $(k+2)$-gone, et toute factorisation fournit un plongement*

$$\mathcal{B}_1^{\circledast k} \simeq \mathrm{GL}(W)^{\circledast k} \hookrightarrow \mathcal{B}_k$$

sur un ouvert dense de $\mathcal{B}_k$. De plus, tous ces plongements relient les structures quasi-Hamiltoniennes.

Plusieures relations entre polynômes continuants, triangulations et équations différentielles sont connues [1], mais ce n'est pas clair si la simplicité de la situation était

[1] Conway–Coxeter [32, 33] relient les triangulations et continuants (en utilisant les déterminants de Schäfli et motifs des frises), comparer aussi l'analyse dans la section 2.3.4.6. du livre de Knuth [60]. Du point de vue des équations différentielles, les configurations des droites de Stokes s'élève à une triangulation d'un polygone (cf. les travaux de Voros [95], p.271). C'est une idée bien connue (voir par exemple [96], Section 29) d'approximer (quoi?) par l'équation d'Airy sur tout triangle et cette approche est en relation avec les algèbres amassées de Fomin et Zelevinsky [44] (voir aussi [54, 47]).

observée. Notamment, l'approche quasi-Hamiltonienne donne un moyen pour coller les triangles d'Airy $\mathcal{B}_1$ que l'on peut pousser ensemble et former les parties ouvertes de $\mathcal{B}_k$. On s'attend à ce que cette approche puisse être étendue à n'importe quelle surface partagée en polygones.

Finalement, nous donnons une formule explicite pour la forme quasi-Hamiltonienne sur l'espace $\mathcal{B}_k$ qui est encore fondée sur les polynômes continuants et généralise la formule de Van den Bergh.

Théorème V. *La 2-forme quasi-Hamiltonienne sur $\mathcal{B}_k$ est donnée par*

$$\omega_k = \frac{1}{2}\big(- \operatorname{Tr}(b_1, \dots, b_k)^{-1} D^2(b_1, \dots, b_k) + \operatorname{Tr}(b_k, \dots, b_1)^{-1} D^2(b_k, \dots, b_1)\big),$$

où on pose

$$D^2(b_1, \dots, b_k) = \sum_{i<j} (b_1, \dots, b_{i-1}) db_i (b_{i+1}, \dots, b_{j-1}) db_j (b_{j+1}, \dots, b_k).$$

1.2 English version

1.2.1 Summary

Suppose Γ is a finitely generated group and G is a Lie group. Then we can consider

$$\mathrm{Hom}(\Gamma, G)/G,$$

the G-character variety of Γ ([66, 86]). In the case when $\Gamma = \pi_1(\Sigma, b)$ is a fundamental group of a Riemann surface, then the character varieties have rich structure and have been much studied. For example, when G is a unitary group and Σ is compact, then the character variety is a model of the moduli space of semistable vector bundles on Σ, after the theorem of Narasimhan and Seshadri [73]. In particular, the character variety has a natural symplectic structure ([5, 48]).

On the other hand if $G = \mathrm{GL}_n(\mathbb{C})$ and Σ is any smooth complex algebraic curve, then by Deligne's Riemann–Hilbert correspondence [38] the character variety parametrises regular singular connections on algebraic vector bundles on Σ. In this case the character varieties have a natural holomorphic Poisson structure. The symplectic leaves are obtained by fixing the conjugacy classes of local monodromy around each of the punctures.

Thus if we label the marked points $a_1, \dots, a_m$ and choose a conjugacy class $\mathcal{C}_i \subset \mathrm{GL}_n(\mathbb{C})$ for each i, then we obtain an algebraic symplectic variety

$$\mathcal{M}_{\mathrm{B}}(\Sigma, \boldsymbol{\mathcal{C}}) \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C}),$$

where $\boldsymbol{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_m) \subset \mathrm{GL}_n(\mathbb{C})^m$.

This perspective suggests a vast generalisation of the notion of character variety by considering more general connections with more complicated singularities and Deligne's Riemann–Hilbert correspondence has been extended to the irregular case. This is work of many people, including Sibuya [85], Deligne [37], Balser–Jurkat–Lutz [9], Malgrange [68], Babbitt–Varadarajan [7], Martinet–Ramis [70] and Loday-Richaud [65].

The irregular connections are classified by their Stokes data, generalising the fundamental group representations, which form the wild character varieties

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}}) \subset \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}.$$

Here $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is the space of Stokes representations, $\mathbf{H}$ is a subgroup of G^m and $\boldsymbol{\mathcal{C}}$ is a conjugacy class in $\mathbf{H}$. This will be defined later in Chapter 2.

They have similar properties to the "tame" case above: holomorphic symplectic structures ([13, 15]), hyperkähler metrics ([11]) (relating them to meromorphic Hitchin systems), they are central to the theory of isomonodromy ([56]), admit discrete group actions generalising the mapping class group actions ([16, 27]) and simple examples are known to underlie the Drinfeld–Jimbo quantum groups ([14, 16]).

For example, among the examples occurring in the simple case of rank two on the Riemann sphere $\mathbb{P}^1$ with one marked point, are the following wild character varieties

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}}) = \mathcal{B} \underset{q}{/\!\!/} T = \{(S_1, \ldots, S_{2k}) \in (U_+ \times U_-)^k \mid S_{2k} \cdots S_1 = q\}/T,$$

where $U_+, U_- \subset \mathrm{GL}_2(\mathbb{C})$ are opposite unipotents, $T = \mathbf{H}$ is the diagonal torus acting by diagonal conjugation and $q \in T$ represents a conjugacy class $\boldsymbol{\mathcal{C}} = \{q\} \subset T$. For generic q these are smooth symplectic varieties of dimension $2k - 4$.

The aim of this thesis is to study several questions about character varieties:

1) Since the theory of character varieties has been extended beyond general lineal groups, we would like to provide examples of tame character varieties for more exotic groups. In Chapter 3 we study an example a G_2 character variety. It turns out that it is isomorphic to symmetric Fricke surfaces and gives a particularly nice description of braid group orbits on such surface in terms of lines on the Fano plane. This is a joint work with P. Boalch [12] and to our knowledge it is the first example of an isomorphism of character varieties for an exotic group.

2) We are interested in classifying wild character varieties of complex dimension two. They are hyperkähler manifolds of real dimension four (gravitational instantons) and so have physical interest. In Chapter 4 we will construct isomorphisms between certain infinite families of such spaces, supporting the conjecture of [25] that there are only a finite number of deformation classes of such hyperkähler manifolds.

3) We will study multiplicative versions of the Nakajima quiver varieties. The "classical" multiplicative quiver varieties of Crawley-Boevey–Shaw and Yamakawa [36, 97] can be constructed out of a single piece

$$\mathcal{B}_2(V_1, V_2) = \{(a, b) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \mid \det(1 + ab) \neq 0\},$$

which we understand as an "edge". It consists of pairs of maps between two nodes (along an edge joining them), with vector spaces V_1, V_2 attached. In the context of wild character varieties a natural generalisation occurs, involving some more general pieces. In Chapter 5 we will find a formula generalising Yamakawa's expression for the multiplicative symplectic structure to a k-fold edge. The results of Chapter 5 are joint work in progress with P. Boalch and D. Yamakawa.

4) Also, we will study direct relations between the wild character varieties and the classical multiplicative quiver varieties. It turns out, as we will prove in Chapter 5, that in the case of such k-fold edge and $G = \mathrm{GL}_n(\mathbb{C})$, the classical multiplicative quiver varieties embed as open subsets of the generalised ones. Moreover, we show that such embeddings are counted by Catalan numbers and can be understood in terms of factorisations of Euler's continuants [41] or triangulations of a polygon.

1.2.2 Further motivations and statement of results

The goal of this thesis is to study certain classes of examples of complex wild character varieties, which are generalisations of the classical, tame character varieties. These spaces play fundamental role in both mathematics and physics and have various interesting both analytic and algebraic properties. Almost all results of this thesis concern the wild character varieties of general linear groups $\mathrm{GL}_n(\mathbb{C})$, however some statements hold true in full generality, for arbitrary complex reductive Lie groups.

Given a curve Σ, there is a complex affine algebraic variety

$$\mathcal{M}_{\mathrm{B}}(\Sigma, n) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$$

of isomorphism classes of representations of the fundamental group of Σ, the character variety, or using Simpson's terminology [88], the Betti moduli space.

The celebrated Riemann–Hilbert correspondence establishes a complex analytic isomorphism between $\mathcal{M}_{\mathrm{B}}(\Sigma, n)$ and the moduli space of (stable) holomorphic connections on rank n vector bundles on Σ, the de Rham space $\mathcal{M}_{\mathrm{DR}}(\Sigma, n)$, by sending a connection ∇ to its monodromy data. This map can be interpreted as a generalisation of the exponential map. It is highly transcendental and does not preserve the algebraic structures of $\mathcal{M}_{\mathrm{DR}}$ and $\mathcal{M}_{\mathrm{B}}$.

On the other hand, there is the third moduli space: the Dolbeault moduli space of Higgs bundles $\mathcal{M}_{\mathrm{Dol}}(\Sigma, n)$ which is in turn diffeomorphic to the space $\mathcal{M}_{\mathrm{DR}}$ via the "non-abelian Hodge" isomorphism. By the works of Corlette, Donaldson, Hitchin and Simpson [51, 39, 35, 87], the three spaces $\mathcal{M}_{\mathrm{B}}, \mathcal{M}_{\mathrm{DR}}, \mathcal{M}_{\mathrm{Dol}}$ can be seen as incarnations of the same hyperkähler manifold, viewed in two different complex structures and different points of view showcase different proprieties of the underlying manifold. The main advantage of the Betti approach is its explicitness – it provides direct descriptions of the moduli spaces, often non-obvious from different points of view.

The classical $\mathrm{GL}_n(\mathbb{C})$ story for compact curves has various generalizations. One can suppose that the curve Σ has distinct marked points $\{a_1, \dots, a_k\}$. Then the *tame* character variety $\mathcal{M}_{\mathrm{B}}$ of the punctured curve $\Sigma^\circ = \Sigma \setminus \{a_1, \dots, a_k\}$ has an algebraic Poisson structure and the symplectic leaves are obtained by fixing the conjugacy classes around the punctures. This approach provides a variety of holomorphic symplectic manifolds. These structures appeared first from the analytic perspective of Atiyah and Bott [5] and have further been understood in different, more algebraic ways in works of many people ([48, 58, 43, 2, 1] just to name a few).

Further, one can suppose that the structure group G is an arbitrary complex connected reductive Lie group, not necessarily $\mathrm{GL}_n(\mathbb{C})$ and consider the space

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

of representations of the fundamental group of Σ into the Lie group G, providing more general character varieties. On the de Rham side, the connections on vector bundles are replaced by connections on principal G-bundles.

The Deligne's extension of the Riemann–Hilbert correspondence [38] to this case (for the general linear groups) establishes a bijection between the sets of G-orbits in $\mathrm{Hom}(\pi_1(\Sigma^\circ), G)$ and isomorphism classes of connections on algebraic G-bundles on

Σ° with regular singularities and similar extension holds for more general groups. The regularity of the singularities at a_i means that the vector bundle extends through all marked points and the connection on the extended bundle has at most simple poles at each a_i.

Thus it is natural to ask: are there interesting examples of character varieties for more general algebraic groups? In Chapter 3 we study the tame character variety of the four-punctured sphere for the exotic group $G_2(\mathbb{C})$. For $\mathrm{SL}_2(\mathbb{C})$, the resulting character variety is the famous Fricke family of cubic surfaces [45]

$$xyz + x^2 + y^2 + z^2 + b_1\, x + b_2\, y + b_3\, z + c = 0.$$

This is one of the simplest nontrivial examples of character varieties, with interesting braid group actions and close relations to the Painlevé VI differential equation. We will say that a Fricke surface is *symmetric* if $b_1 = b_2 = b_3$.

On the other hand, the group $G_2(\mathbb{C})$ is of dimension 14 and has a special conjugacy class $\mathcal{C} \subset G_2(\mathbb{C})$ of dimension six, which is a complex analogue of a 6-sphere. If we take three copies of the class $\mathcal{C}$ and choose the fourth class to be a generic conjugacy class $\mathcal{C}_\infty \subset G_2(\mathbb{C})$, which is of dimension 12, the resulting character variety will be of dimension

$$3 \times 6 + 12 - 2 \times 14 = 2,$$

so one would expect that this symplectic variety will be related to the Fricke surfaces. This is indeed true and the first main result might be stated as follows:

Theorem I. *There is a two parameter family of character varieties for the exceptional group $G_2(\mathbb{C})$ which are isomorphic to smooth symmetric Fricke cubic surfaces, and thus to character varieties for the group $\mathrm{SL}_2(\mathbb{C})$.*

Moreover, we have revisited some of the finite braid group orbits in cubic surfaces, found in [18, 22]. In particular, we considered the Klein cubic surface (which is the unique cubic surface containing a braid group orbit of size 7, thus a link with group of dimension 14 is expected) and showed that:

Theorem II. *If the Klein cubic surface K is realised as a G_2 character variety then the braid orbit of size 7 in K corresponds to some triples of generators of the finite simple group $G_2(\mathbb{F}_2)' \subset G_2(\mathbb{C})$ of order 6048. One such triple of generators is uniquely determined by the three lines passing through a single point in the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$.*

The main objects of interest of this manuscript are the *wild* character varieties, first constructed analytically, together with their symplectic structure, by Boalch in [13, 15], generalising the approach of Atiyah and Bott, and later an algebraic approach was given in [21]. Such varieties arise when one relaxes the condition on the regularity of singularities and allows more general connections with higher order poles. Just like in the regular singular case, the wild character varieties encode the monodromy (Stokes) data of such irregular meromorphic connections but since the data is much more rich, it is not sufficient to consider only the representations of the fundamental group of Σ°.

Roughly speaking, an irregular curve $\mathbf{\Sigma}$ consists of a curve Σ, a set of marked points $a_1, \dots, a_k$ and at each marked point an irregular type Q_i, which describes the irregular singularity. This data determines a new surface $\widetilde{\Sigma}$, which is the real-oriented blow up of Σ at the marked points with some additional punctures, and a subgroup $\mathbf{H} = H_1 \times \dots \times H_k \subset G^k$, and the refined space of representations is the space

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G),$$

of so-called "Stokes representations" of the fundamental groupoid Π of $\widetilde{\Sigma}$ with base-points on the boundary circles. The group $\mathbf{H}$ acts on this space and one can consider the quotient

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}$$

which is the wild character variety (the wild Betti space). It is again a complex affine variety with an algebraic Poisson structure and if the irregular types are zero, one recovers the usual tame character variety with the usual Poisson structure. This means that leaving the tame world provides an abundance of examples of Poisson/symplectic holomorphic manifolds.

The wild character varieties parametrise the isomorphism classes of irregular connections and the irregular analogue of the Riemann–Hilbert correspondence has been established a long time ago (at least for $G = \mathrm{GL}_n(\mathbb{C})$) by works of many people, including Sibuya [85], Deligne [37], Balser–Jurkat–Lutz [9], Malgrange [68], Babbitt–Varadarajan [7], Martinet–Ramis [70] and Loday-Richaud [65]. Then it was extended to general groups in [16]. It is however much less understood than the regular version of this map. Moreover, there are various, although in the end equivalent, approaches to irregular connections, such as Stokes structures and Stokes local systems.

One of the remarkable properties of the wild character varieties is the fact that they come in families. If one varies the initial irregular curve $\mathbf{\Sigma}$ – both the curve Σ with marked points, and the irregular data – in a smooth way (in the sense of [27]), then the resulting wild character varieties stay isomorphic and fit together into a fibre bundle over the base $\mathbb{B}$. This "local system of varieties", as introduced in the tame case by Simpson in [89], admits a nonlinear flat connection, the nonabelian analogue of the Gauss–Manin connection, and the fundamental group of the base acts on the fibers of this bundle by Poisson automorphisms, providing the wild analogue of the classical mapping class group actions on the tame character varieties. This generalises some works on isomonodromic deformations of Jimbo–Miwa–Ueno [56] who back then observed that in the irregular case the space of deformation parameters is bigger than in the tame case, since one can vary not only the Riemann surface, but also the irregular types.

In some cases it is possible to write the Gauss–Manin connection in explicit coordinates and obtain a genuine nonlinear differential equation. For example, for appropriate choice of poles, all six Painlevé transcendents arise this way for $\Sigma = \mathbb{P}^1$ and $G = \mathrm{SL}_2(\mathbb{C})$ but only Painlevé VI admits a realisation by the means of regular singularities. Thus, being phase spaces of nonlinear ODEs, these examples of wild character varieties are of great interest and have been extensively studied. Our main motivation was the fact that the Painlevé equations are nonlinear of second order, hence the resulting phase spaces/wild character varieties are of complex dimension two and it is known that they are in fact cubic surfaces. By the work of Biquard and Boalch [11], who generalised the results of Hitchin to the irregular case, the wild character varieties are again hyperkähler manifolds and in dimension two they are examples of "gravitational instantons".

In [25] there is a conjectural list of 11 deformation classes of real four-dimensional hyperkähler manifolds arising in non-abelian Hodge theory, and these are classified by Dynkin symbols. These are noncompact analogues of K3 surfaces and we will call them H3 surfaces, after Higgs, Hitchin and Hodge as suggested in [30]. From mathematical perspective, this classification problem fits into classification of "quaternionic curves" which are quaternionic analogues of Riemann surfaces, as discussed by Atiyah [4].

H3 surfaces	
Tame	Wild
E_6, E_7, E_8, D_4	$(A_0), A_1, A_2, A_3, (D_0), (D_1), D_2$

Table 1. Conjectural classification of H3 surfaces

In turn, the correspondence with phase spaces of Painlevé equations is described in the table

Symbol	D_4	A_3	A_2	A_1	A_0	D_2	D_1	D_0
Painlevé equation	VI	V	IV	II	I	III	III'	III"
Pole orders	1111	211	31	4	(4)	22	2(2)	(2)(2)

(the exotic symbols E_6, E_7, E_8 correspond to Painlevé difference equations). The brackets mean that the irregular singularities are twisted.

Boalch observed that for each space/symbol on the list there is an infinite family of "echo spaces", parametrising isomorphism classes of certain connections on arbitrarily high rank bundles on $\mathbb{P}^1$, for a suitable choice of irregular types. Moreover, all the echo spaces are of dimension two, and hence there should be isomorphisms between them. The aim of Chapter 3 is to establish the A_0, A_1 and A_2 cases of this conjecture.

This is done with the use of theory of new multiplicative quiver varieties, introduced by Boalch in [28]. These are multiplicative analogues of Nakajima's quiver varieties [71, 72]. The multiplicative quiver varieties, in relation to the multiplicative preprojective algebra, have been first introduced by Crawley-Boevey and Shaw [36] and Yamakawa [97] and in the case of star-shaped quivers, they are isomorphic to the quiver varieties of [28]. However in other cases such isomorphisms do not need to hold (cf. Section 6. of [28]) and we will see in Chapter 5 a more general example highlighting the difference between them.

Certain Nakajima quiver varieties resemble the moduli spaces $\mathcal{M}_{\mathrm{DR}}$, in the sense that they are isomorphic to an open subset $\mathcal{M}^* \subset \mathcal{M}_{\mathrm{DR}}$ corresponding to isomorphism classes of connections on *trivial* vector bundles. On the other hand, the appropriate examples of new multiplicative quiver varieties not only resemble the wild Betti spaces, but they are in fact isomorphic. The language of multiplicative quiver varieties provides a convenient description of the $A_1, A_2, A_3, D_4, E_6, E_7$ and E_8 echo spaces as multiplicative quiver varieties of the corresponding affine Dynkin graph. In particular, it gives access to the wild echo spaces of type A_1, A_2, A_3.

A multiplicative quiver variety is determined by a graph Γ with k vertices, dimension vector $d \in \mathbb{Z}^k_{>0}$ and a parameter vector $\in (\mathbb{C}^*)^k$. In turn, an untwisted irregular type Q determines a fission graph $\Gamma(Q)$, and for the right choice of dimension vector and parameters, the resulting multiplicative quiver variety $\mathcal{Q}(\Gamma(Q), d, q)$ is the wild character variety.

In the cases of A_1, A_2, A_3 echo spaces, the graphs appearing are their affine extensions $\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3$ and for the dimension vectors with coordinates equal to one and generic parameters their multiplicative quiver varieties are the cubic surfaces from the Table 1. We will abusively denote the A_0 echo space $\mathcal{Q}(\widetilde{A}_0, q, d)$, even though it is not a quiver variety and the choice of parameter q is limited to primitive roots of unity. This case however is the easiest and gives a good sense of algebraic phenomena appearing in the remaining cases, such as similarities with Zhedanov's Askey-Wilson algebra $AW(3)$ [98]. The main theorem of Chapter 4 can be now stated as follows:

Theorem III. *For $i = 0, 1, 2$, let $\widetilde{A}_i$ denote the affine Dynkin graph and let n denote the associated dimension vector with all coordinates equal to n. Then for a choice of generic parameters q_i, there are isomorphisms*

$$\mathcal{Q}(\widetilde{A}_i, q_i, n) \simeq \mathcal{Q}(\widetilde{A}_i, q_i^n, 1).$$

For the star-shaped quivers D_4, E_6, E_7, E_8 a result of similar flavor has been established by Etingof–Oblomkov–Rains in [40]. Theorem III. can be understood as an extension of these results to some of the wild cases. Moreover, we prove these isomorphisms explicitly and give direct relations between the parameters of the cubics. The case of the affine A_3 multiplicative quiver variety, related to Painleveé V equation, is missing, but we have gathered evidence suggesting that the methods used in the first three cases should extend there as well.

Another motivation for studying wild character varieties comes from the works of Alexeev–Malkin–Meinrenken on quasi-Hamiltonian geometry [1], extended to the complex groups. Their approach yields a construction of algebraic Poisson/symplectic structures on tame character varieties by the means of finite-dimensional "multiplicative" symplectic quotients, where the moment maps take values in the Lie group G, rather than in the dual of the Lie algebra $\mathfrak{g}$. Suppose that Σ has k boundary components $\partial_1, \dots, \partial_k$ and choose a basepoint b_i at each ∂_i. Denote by Π the fundamental groupoid of Σ with basepoints $\{b_1, \dots, b_k\}$. The space

$$\mathrm{Hom}(\Pi, G)$$

of representations of the fundamental groupoid Π into G is then a smooth affine variety and it has an additional structure of a quasi-Hamiltonian G^k-space. This means that there is an action of G^k on $\mathrm{Hom}(\Pi, G)$ and a G^k-valued moment map

$$\mu : \mathrm{Hom}(\Pi, G) \to G^k$$

satisfying conditions similar to the usual Hamiltonian actions. In particular, if G is abelian (for example trivial), then the quasi-Hamiltonian spaces are complex symplectic manifolds. By the general quasi-Hamiltonian yoga, the quotient

$$\mathrm{Hom}(\Pi, G)/G^k$$

inherits a Poisson structure and is isomorphic to the usual tame character variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. If one fixes a k-tuple of conjugacy classes $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_k) \subset G^k$, then the quasi-Hamiltonian quotient

$$\mu^{-1}(\mathcal{C})/G^k$$

is a holomorphic symplectic manifold (provided it is a manifold), isomorphic to the symplectic leaves of the tame character variety. This story has a generalisation to the irregular case and in [21] the quasi-Hamiltonian structure on the space of Stokes data has been constructed. This has several important consequences, one of them being the existence of Poisson structures on the wild Betti spaces.

The operations of quasi-Hamiltonian fusion and gluing allow to build the space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ out of simpler pieces: the "fission spaces" $\mathcal{A}(Q)$, describing the irregular singularities, conjugacy classes $\mathcal{C}$ describing the tame points and "internally fused doubles" $\mathbb{D}$, which are the equivalents of topological handles. Thus it is important to understand these pieces and the one carrying the most information is the fission space $\mathcal{A}(Q)$, which parametrises isomorphism classes of connections on a disk with one irregular pole.

More precisely, we study the reduced fission spaces $\mathcal{B}(Q)$ (see Section 2.2.3 and equation (2.10) for precise definitions), which parametrise isomorphism classes of connections with one irregular pole on the Riemann sphere. One of the simplest examples of such space is the Van den Bergh space $\mathcal{B}(V_1, V_2)$ [91, 92], defined for a graded vector space $V = V_1 \oplus V_2$

$$\mathcal{B}(V_1, V_2) = \{(a, b) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \mid \det(1 + ab) \neq 0\}$$

which is a quasi-Hamiltonian $\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)$-space with moment map $\mu : \mathcal{B}(V_1, V_2) \to \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ given by

$$\mu(a, b) = ((1 + ab)^{-1}, 1 + ba) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$$

and the quasi-Hamiltonian two-form

$$\omega = \frac{1}{2}\left(\mathrm{Tr}_{V_1}(1 + ab)^{-1} da \wedge db - \mathrm{Tr}_{V_2}(1 + ba)^{-1} db \wedge da\right).$$

As a set, the space $\mathcal{B}(V_1, V_2)$ consists of pairs of maps (a, b) between V_1, V_2 such that $1 + ab$ is invertible. It is isomorphic to the reduced fission space $\mathcal{B}(V)$ and the polynomial expression $1 + ab$ is the second Euler continuant [41]. More generally, we can define the n-th continuant polynomial $(x_1, \ldots, x_n)$ by the recursive relation

$$(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})x_n + (x_1, \ldots, x_{n-2}).$$

The first few continuants are given by the following formulas:

$$\begin{aligned}
(\emptyset) &= 1\\
(x_1) &= x_1\\
(x_1, x_2) &= x_1x_2 + 1\\
(x_1, x_2, x_3) &= x_1x_2x_3 + x_1 + x_3\\
(x_1, x_2, x_3, x_4) &= x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1\\
(x_1, x_2, x_3, x_4, x_5) &= x_1x_2x_3x_4x_5 + x_1x_2x_5 + x_1x_2x_5 + x_1x_4x_5+\\
&\quad + x_3x_4x_5 + x_1 + x_3 + x_5
\end{aligned}$$

When we naturally generalise the space $\mathcal{B}(V_1, V_2)$ and consider more than two, but still even number of maps between V_1, V_2, we obtain the reduced fission space $\mathcal{B}^k(V)$. Its quasi-Hamiltonian moment map again involves a continuant polynomial, in $2k$ (non-commuting) variables. If we further suppose that $V = W \oplus W$, then we can define the analogues of the spaces $\mathcal{B}^k(V)$ with odd number of maps in $\mathrm{End}(W)$. This leads to the following definition

$$\mathcal{B}_k = \{(b_1, \ldots, b_k) \in \mathrm{End}(W)^k \;\big|\; \det(b_1, \ldots, b_k) \neq 0\},$$

which is a (possibly twisted) quasi-Hamiltonian $\mathrm{GL}(W) \times \mathrm{GL}(W)$-space and the moment map is the continuant. The simplest piece $\mathcal{B}_1$ is then just a copy of $\mathrm{GL}(W)$ with zero two-form.

In Chapter 5 we study the factorisations of continuants, ie. decompositions of continuants into product of shorter continuants and in particular the full decompositions into continuants of length one, which correspond to pieces isomorphic to $\mathcal{B}_1 \simeq \mathrm{GL}(W)$. The main result of Chapter 5 is:

Theorem IV. *Let k be a positive integer and let C_k denote the k-th Catalan number. There are C_k different factorisations of a continuant $(x_1, \dots, x_k)$, parametrised by triangulations of a $(k+2)$-gon, and each factorisation yields an embedding*

$$\mathcal{B}_1^{\circledast k} \simeq \mathrm{GL}(W)^{\circledast k} \hookrightarrow \mathcal{B}_k$$

onto a dense open subset of $\mathcal{B}_k$. Moreover, all such embeddings relate the quasi-Hamiltonian structures.

Various relations between continuants, triangulations and differential equations are known [2] but it is not clear if the simplicity of the situation has been noticed before. Namely the quasi-Hamiltonian framework gives a way to glue Airy triangles $\mathcal{B}_1$ together, which can then be pushed together to form open parts of spaces $\mathcal{B}_k$. We expect this technique approach to extend to any surface partitioned into polygons.

Finally, we give an explicit formula for the quasi-Hamiltonian form on the space $\mathcal{B}_k$, which also involves the continuants and generalises the formula of Van den Bergh.

Theorem V. *The quasi-Hamiltonian two-form on the space $\mathcal{B}_k$ is given by*

$$\omega_k = \frac{1}{2}\big(- \mathrm{Tr}(b_1, \dots, b_k)^{-1} D^2(b_1, \dots, b_k) + \mathrm{Tr}(b_k, \dots, b_1)^{-1} D^2(b_k, \dots, b_1)\big),$$

where we define

$$D^2(b_1, \dots, b_k) = \sum_{i<j} (b_1, \dots, b_{i-1}) db_i (b_{i+1}, \dots, b_{j-1}) db_j (b_{j+1}, \dots, b_k).$$

[2]Conway–Coxeter [32, 33] relate triangulations and continuants (via Schäfli determinants and frieze patterns), see also the review in section 2.3.4.6. of Knuth's book [60]. From the differential equations viewpoint the configurations of Stokes lines amount to a triangulation of a polygon (e.g. by Voros' work, [95], p.271). It is a well-known idea (see e.g. [96], Section 29) to approximate by the Airy equation on each such triangle and in turn this is related to the cluster algebras of Fomin and Zelevinsky [44] (see also e.g. [54, 47]).

Chapter 2

Background material

We will present some basic definitions and notations following [27]. Even though presented for arbitrary complex reductive groups, all the examples and applications in this manuscript will concern the general linear case, thus it is sufficient to think of (block) diagonal/triangular/unipotent matrix groups as of Levi, parabolic and unipotent groups.

2.1 Irregular curves and wild character varieties

2.1.1 Irregular types

In this section we define the untwisted and unramified irregular types and describe the Stokes data determined by such objects.

Fix a connected complex reductive group G and a maximal torus $T \subset G$, and let $\mathfrak{t} \subset \mathfrak{g}$ denote the corresponding Lie algebras. Let Δ be a complex disk and let $a \in \Delta$ be a marked point. Let $\widehat{\mathcal{O}}$ denote the formal completion at a of the ring of holomorphic functions on Δ and let $\widehat{\mathcal{K}}$ denote its field of fractions.

Definition 2.1.1. *An (unramified) irregular type at a is an element*

$$Q \in \mathfrak{t}(\widehat{\mathcal{K}})/\mathfrak{t}(\widehat{\mathcal{O}}).$$

This definition is coordinate-free. One may think of an irregular type as a $\mathfrak{t}$-valued meromorphic function germ, well defined modulo holomorphic terms. If we choose a local coordinate z on Δ vanishing at a, then $\widehat{\mathcal{O}} = \mathbb{C}[\![z]\!], \widehat{\mathcal{K}} = \mathbb{C}(\!(z)\!)$ and the irregular type Q may be written in the form

$$Q = \frac{A_r}{z^{k_r}} + \cdots + \frac{A_1}{z^{k_1}}$$

for integers $0 < k_1 < \cdots < k_r$ and elements $A_i \in \mathfrak{t} \subset \mathfrak{g}$ for $i = 1, \ldots, r$.

As noted in Remark 8.6 of [27], one can consider more general *twisted* irregular types by replacing the Cartan subalgebra $\mathfrak{t}(\!(z)\!) \subset \mathfrak{g}(\!(z)\!)$ by a nonconjugate one, which exists since $\mathbb{C}(\!(z)\!)$ is not algebraically closed. On the other hand, in the whole manuscript we will encounter only one example of a space related to a twisted irregular type, and thus we skip the technical details and we will address that case separately when necessary.

Let $\mathcal{R} \subset \mathfrak{t}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{t}$ and recall the root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{t}\}$. is the root space of $\alpha \in \mathcal{R}$. Hence for each root $\alpha \in \mathcal{R}$ we can define

$$q_\alpha = \alpha \circ Q,$$

which is a meromorphic function modulo holomorphic terms. We define the degree $\deg(q_\alpha)$ of q_α to be its pole order at a. Using the local coordinate z at a, we may identify it with an element of $z^{-1}\mathbb{C}[z^{-1}]$ and $\deg(q_\alpha)$ is the degree of the polynomial $q_\alpha(1/z)$. It is a non-negative integer, equal to zero if q_α does not have a pole at a.

A non-zero irregular type Q at a determines two special families of directions around a. It will be convenient to think about them using the real oriented blow up at a.

Definition 2.1.2. *The real oriented blow up of the origin in $\mathbb{R}^2$ is the set $\widehat{\mathbb{R}}^2 = [0, \infty) \times S^1$ with projection map $\pi : \widehat{\mathbb{R}}^2 \to \mathbb{R}^2$ given by $(r, x_1, x_2) \mapsto (rx_1, rx_2)$, where x_1, x_2 are coordinates on $\mathbb{R}^2$ such that $x_1^2 + x_2^2 = 1$.*

The map π is a diffeomorphism of $\widehat{\mathbb{R}}^2 \backslash (\{0\} \times S^1)$ onto $\mathbb{R}^2 \backslash \{0\}$ and we have $\pi^{-1}(0) \simeq S^1$. The real oriented blow up replaces the origin of $\mathbb{R}^2$ by the circle S^1 of real oriented directions. Since the construction is local, we can blow up a point on any real two-manifold M. The blowup $\widehat{M}$ of a two-manifold M at x is an oriented real two-manifold with boundary circle ∂. The points of ∂ correspond to real oriented directions at x and an interval $I \subset \partial$ determines germs of sectors at x with opening I.

Let $\widehat{\Delta} \to \Delta$ denote real oriented blow up of Δ at a. Given a root $\alpha \in \mathcal{R}$, consider the function $\exp(q_\alpha(z))$ and its behavior as z approaches zero along the rays in different directions $d \in S^1$.

Definition 2.1.3. *A direction $d \in S^1$ will be said to be a singular direction supported by α (or an anti-Stokes direction) if $\exp(q_\alpha(z))$ has maximal decay as $z \to 0$ along the direction d.*

Thus if c_α / z^k is the most singular term of q_α, these are the directions along which c_α / z^k is real and negative.

In a similar fashion, we may define the Stokes directions.

Definition 2.1.4. *A direction $d \in S^1$ will be said to be a Stokes direction supported by α if the most singular term c_α/z^k of q_α is imaginary and negative along d.*

While crossing the Stokes direction, $\exp(q_\alpha)$ changes its asymptotics (and we can not compare the asymptotics of solutions along this ray). The Stokes directions interlace the anti-Stokes directions: the family of Stokes directions is a rotation of the family of anti-Stokes directions by certain angle θ.

Let us denote the boundary of the blow up $\widehat{\Delta}$ by ∂ and two finite families of anti-Stokes and Stokes directions by $\mathbb{A} \subset \partial$ and $\mathbb{S} \subset \partial$. The Stokes directions divide ∂ into a disjoint union $I \subset \partial$ of components which are contractible subsets of the boundary. (Same holds for the anti-Stokes directions, the collection of contractible subsets of ∂ obtained this way will be a rotation of I).

Definition 2.1.5. *Two elements of I are called consecutive if they belong to the same boundary circle ∂_i and are separated by exactly one Stokes direction.*

2.1.2 Stokes data from irregular types

In this section we will describe how an irregular type determines Stokes data, which is an irregular analogue of the usual monodromy data for holomorphic connections.

Recall that a non-zero irregular type Q at $a \in \Delta$ determines a family $\mathbb{A} \subset S^1$ of singular (anti-Stokes) directions. For a singular direction $d \in \mathbb{A}$ we denote by $\mathcal{R}(d) \subset \mathcal{R}$ the subset of roots supporting d and further, for an integer k, we denote by $\mathcal{R}(d,k) \subset \mathcal{R}(d)$ the subset of roots $\alpha \in \mathcal{R}(d)$ such that $\deg(q_\alpha) = k$. Finally, we consider the root groups

$$U_\alpha = \exp(\mathfrak{g}_\alpha) \subset G$$

corresponding to $\alpha \in \mathcal{R}$. By Lemma 7.3. of [27], the collection of groups

$$\{U_\alpha \mid \alpha \in \mathcal{R}(d)\}$$

directly spans a unipotent subgroup of G. It means that the product map

$$\phi : \prod_{\alpha \in \mathcal{R}(d)} U_\alpha \to G$$

is an algebraic isomorphism onto its image and this image is a well-defined unipotent subgroup of G. Moreover, the image does not depend on the order of U_α in the product. It is worth noting that this isomorphism usually is not a homomorphism of groups, but an isomorphism of algebraic varieties. For example, if we consider a triple of unipotent subgroups og GL(3)

$$U_{12} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_{13} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

the direct product $U_{12}U_{13}U_{23}$ is isomorphic to the full upper-triangular unipotent subgroup of GL(3), but it is not a homomorphism of groups.

Definition 2.1.6. *The Stokes group* $\mathbb{S}\mathrm{to}_d$ *associated with a singular direction* $d \in \mathbb{A}$ *is the unipotent subgroup of* G *uniquely determined by the image of the product map* ϕ*:*

$$\mathbb{S}\mathrm{to}_d = \phi(\prod_{\alpha \in \mathcal{R}(d)} U_\alpha) \subset G.$$

One can further break $\mathbb{S}\mathrm{to}(d)$ into pieces, using the subset $\mathcal{R}(d,k)$. For an integer k, we can define the level k subgroup $\mathbb{S}\mathrm{to}_d(k)$, defined as the image of the product $\prod_{\alpha\in\mathcal{R}(d,k)} U_\alpha \subset G$ and the product map again gives an isomorphism

$$\prod_i \mathbb{S}\mathrm{to}_d(k_i) \simeq \mathbb{S}\mathrm{to}_d\,.$$

Definition 2.1.7. *The space of Stokes data* $\mathbb{S}\mathrm{to}(Q)$ *associated to an irregular type* Q *is the product*

$$\mathbb{S}\mathrm{to}(Q) := \prod_{d\in\mathbb{A}} \mathbb{S}\mathrm{to}_d$$

.

The product on the right is the Caertesian product of groups $\mathbb{S}\mathrm{to}_d$, not the product map of groups.

The spaces of Stokes data and their variations will be crucial objects of this manuscript. Let us introduce two more ingredients, related to Stokes data on a disk and $\mathbb{P}^1$. This will be some of the building blocks for more general spaces of Stokes data

$$\mathcal{A}(Q) = G \times \mathbb{S}\mathrm{to}(Q) \times H. \tag{2.1}$$

(where H is the centraliser of Q in G). We will see in the next section how the space $\mathcal{A}(Q)$ appears in the context of wild character varieties and later, in section 2.2 how the space $\mathcal{A}(Q)$ fits into the setup of quasi-Hamiltonian geometry.

Let $d_1, \ldots, d_m \subset \mathbb{A}$ be the singular directions determined by an irregular type Q turning in the positive sense as the index grows. We will denote by S_i the associated unipotent group $\mathbb{S}\mathrm{to}(d_i)$. There is an action of the group $G \times H$ on the space $\mathcal{A}(Q)$ given by

$$(g,k)(C,\mathbf{S},h) = (kCg^{-1}, k\mathbf{S}k^{-1}, khk^{-1}) \tag{2.2}$$

for an element $(g,k) \in G \times H$ and $\mathbf{S} = (S_1, \ldots S_m)$.

Moreover, the space $\mathcal{A}(Q)$ is a quasi-Hamiltonian $G \times H$ space (we will introduce the language of quasi-Hamiltonian spaces later in this chapter). In brief, that there is a $G \times H$-valued (and equivariant for the action (2.2)) moment map $\mu : \mathcal{A}(Q) \to G \times H$

and a two-form ω on $\mathcal{A}(Q)$ satisfying certain conditions generalising the classical Hamiltonian axioms. This allows us to introduce the space

$$\mathcal{B}(Q) = \mathcal{A}(Q) \underset{1}{/\!\!/} G, \tag{2.3}$$

reduction of $\mathcal{A}(Q)$ at the value 1 of this moment map.

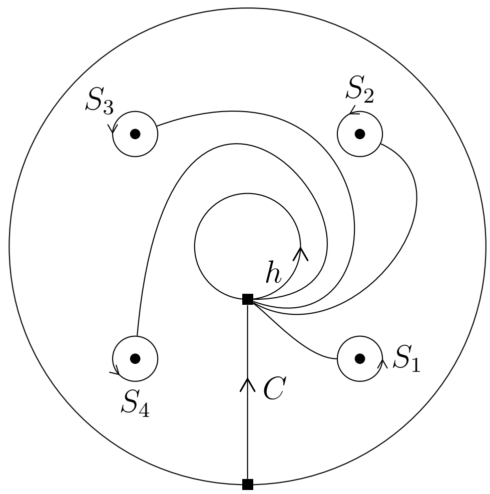

Figure 2.1: A way to picture the space $\mathcal{A}(Q)$

Finally, we can gather some of the Stokes groups together and form, by the direct spanning property, some bigger unipotent subgroups of G. Recall from Definition 2.1.3 that a singular direction $d \subset \mathbb{A}$ is supported by a root $\alpha \in \mathcal{R}$ if and only if $q_\alpha = \alpha \circ Q = \alpha(A_k)/z^k$ is real and negative, for z in the direction d. Therefore $\mathbb{A}$ is invariant under rotation by π/k and $l := s/2k = \#\mathbb{A}/2k$ is an integer. We define a "half-period" to be an ordered l-tuple of consecutive singular directions in $\mathbb{A}$.

Proposition 2.1.8 ([27], Lemma 7.13)**.** *If $\mathbf{d} \subset \mathbb{A}$ is a half-period, then the subgroups $\{\mathbb{S}\mathrm{to}_d \mid d \in \mathbf{d}\}$ directly span the unipotent radical of a parabolic subgroup of G with Levi subgroup H and rotating by π/k yields the unipotent radical of the opposite parabolic with the Levi subgroup H.*

Thus taking a half-period $\mathbf{d} = (d_1, \dots, d_l)$ yields a parabolic $P_+ \subset G$ with Levi subgroup H. Denote its unipotent radical by U_+ and by U_- the unipotent radical of the opposite parabolic, associated to $(d_{l+1}, \dots, d_{2l})$. Thus in one level case we have a direct spanning equivalence

$$\mathcal{A}(Q) = G \times H \times \prod \mathbb{S}\mathrm{to}_d \to {}_G\mathcal{A}_H^k = G \times H \times (U_+ \times U_-)^k.$$

We will call the groups U_+, U_- the *full* Stokes groups. For example, for $G = \mathrm{GL}_3(\mathbb{C})$ and irregular type

$$Q = \frac{A}{z^2}$$

with A regular semisimple, there are 12 singular directions and the first three Stokes groups $\mathbb{S}\mathrm{to}_d \subset \mathrm{GL}_3(\mathbb{C})$ can be identified with U_{12}, U_{13}, U_{23} which directly span U_+. Similarly, the next three Stokes groups directly span U_- and so on.

2.1.3 Irregular curves

Definition 2.1.9. *An "irregular curve" (or "wild Riemann surface") is a smooth compact Riemann surface, possibly with non-empty boundary, together with a finite number of points $a_1, a_2, \dots$ in the interior of Σ and an irregular type Q_i at each marked point.*

We will denote an irregular curve (Σ, a_i, Q_i) by $\mathbf{\Sigma}$. Thus an irregular curve $\mathbf{\Sigma}$ is a choice of a smooth compact Riemann surface Σ, finite number of marked points $a_1, a_2, \dots \in \Sigma$ and irregular types Q_i at each a_i.

Remark. We will say that an irregular curve is *algebraic* if its boundary is empty (it may, and will, have marked points). If m denotes the number of marked points plus the number of boundary components, we will always assume that $m > 0$.

Given an irregular curve $\mathbf{\Sigma}$, let $\widehat{\Sigma} \to \Sigma$ denote the real two-manifold with boundary obtained by taking the real oriented blow up of Σ at each marked point a_i, i.e. replacing each marked point a_i with the circle of oriented real tangent directions at a_i.

Denote the boundary of $\widehat{\Sigma}$ by ∂ and label the boundary circles of $\widehat{\Sigma}$ as $\partial_1, \dots, \partial_m$. The irregular type Q_i determines its centraliser, which is a subgroup $H_i = C_G(Q_i) \subset G$, singular directions $\mathbb{A}_i \subset \partial_i$, Stokes groups $\mathbb{S}\mathrm{to}_d \subset G$ for each $d \in \mathbb{A}_i$, Stokes directions $\mathbb{S}_i \subset \partial_i$ and a disjoint union $I \subset \partial$ of finite number of contractible subsets of ∂. We set $Q_i = 0$ if ∂_i was already a boundary component of Σ.

Let $\mathbb{H}_i$ be a small tubular neighbourhood of ∂_i and puncture $\widehat{\Sigma}$ once in its interior near each singular direction $d \in \mathbb{A}_i, i = 1, \dots, m$ such that the punctures lie on the external boundary of $\mathbb{H}_i$. Draw small cilia on the surface $\widehat{\Sigma}$ between each puncture and the corresponding singular direction $d \in \mathbb{A}_i \subset \widehat{\Sigma}$ such that they do not cross. We introduce the cilia to help keep track of the punctures, they are not an essential part of the construction. Let $\widetilde{\Sigma} \subset \widehat{\Sigma}$ denote the corresponding punctured surface.

Choose a marked point $b_i \in \partial_i$ in each boundary component of $\widetilde{\Sigma}$, and define Π to be the fundamental groupoid of $\widetilde{\Sigma}$ based at $\{b_1, \dots, b_m\}$:

$$\Pi = \Pi_1(\widetilde{\Sigma}, \{b_1, \dots, b_m\}),$$

consisting of homotopy classes of paths in $\widetilde{\Sigma}$ with endpoints in the set $\{b_1, \dots, b_m\}$.

Therefore we may consider the space $\mathrm{Hom}(\Pi, G)$ of morphisms from the groupoid Π to the group G. Explicitly, an element $\rho \in \mathrm{Hom}(\Pi, G)$ consists of a choice of an element $\rho(\gamma) \in G$ for each path in Π such that for any composable paths γ_1, γ_2 we have

$$\rho(\gamma_1 \circ \gamma_2) = \rho(\gamma_1)\rho(\gamma_2).$$

Now consider the subspace

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \subset \mathrm{Hom}(\Pi, G)$$

of *Stokes representations* ρ obeying the following conditions for any $i = 1 \dots, m$:

(SR1) If $d \in \mathbb{A}_i$ and $\widetilde{\gamma}_d$ is any loop based at b_i that goes around ∂_i (in any direction) to the direction d, and then loops once around the puncture on the cilium emanating from the direction d, without crossing any other cilia, before retracing its path back to b_i, then $\rho(\widetilde{\gamma}_d) \in \mathbb{S}\mathrm{to}_d$.

(SR2) If γ_i is a simple closed loop based at b_i going once in a positive sense around ∂_i, then $\rho(\gamma_i) \in H_i$.

There is an action of the group $\mathbf{H} := H_1 \times \cdots \times H_m \subset G^m$ on the space of Stokes representations as follows. The m-tuple $(k_1, \dots, k_m) \in \mathbf{H}$ sends ρ to the representation ρ' such that

$$\rho'(\gamma) = k_j \rho(\gamma) k_i^{-1}$$

from any path $\gamma \in \Pi$ from b_i to b_j.

Example 2.1.10. Let $\mathbf{\Sigma}$ is a disk with one marked point a and an irregular type Q at a. Then $\widetilde{\Sigma}$ has two boundary components: the external boundary of the disk ∂ and a circle ∂_a coming from the real oriented blow up at a. Thus the set of basepoints has two elements $\{b, b_a\}$, one at each boundary component, and upon choosing appropriate generating paths the space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ can be identified with the space $\mathcal{A}(Q)$ of (2.1). If $\mathbf{\Sigma}$ is the Riemann sphere $\mathbb{P}^1$, then one has just the basepoint at the boundary circle and $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ can be identified with $\mathcal{B}(Q)$.

Similarly as in the case of $\mathcal{A}(Q)$, the space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ has a quasi-Hamiltonian structure. We will state the result for completeness, but the reader might skip it and just assume that after a suitable choice of paths, there exists an explicit algebraic presentation of the space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ given by equation (2.5).

Theorem 2.1.11 ([21], Theorem 3.,[27], Theorems 7.6.,8.2.)**.** *The space* $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ *of Stokes representations of* Π *in* G *is a smooth complex affine variety and is a quasi-Hamiltonian* $\mathbf{H}$*-space, where* $\mathbf{H} = H_1 \times \cdots \times H_m \subset G^m$*.*

More generally, upon choosing appropriate generating paths of the fundamental groupoid, we can identify $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ with a reduction by G, at the identity of the moment map μ_G, of the fusion product

$$\mathbb{D}^{\circledast g} \underset{G}{\circledast} \mathcal{A}(Q_1) \underset{G}{\circledast} \cdots \underset{G}{\circledast} \mathcal{A}(Q_m), \tag{2.4}$$

where $\mathbb{D}$ is the internally fused double and g is the genus of Σ. As a set, this fusion product is just the product of spaces.

More explicitly, if we write an element of $\mathcal{A}(Q_i) = G \times H_i \times \mathbb{S}\mathrm{to}(Q_i)$ as $(C_i, h_i, \mathbf{S})$ and $\mathbb{D}^{\circledast g}$ as $\{(a_i, b_i) \,|\, a_i, b_i \in G, i = 1, \ldots, g\}$, then we may identify $\mu_G^{-1}(1)/G$ with the subvariety of (2.4) cut out by the equations $\mu_G = 1, C_1 = 1$. The equation $\mu_G = 1$ takes form

$$[a_1, b_1] \cdots [a_g, b_g]\mu_1 \cdots \mu_m = 1, \tag{2.5}$$

where $\mu_i = C_i^{-1} h_i \cdots S_2^i S_1^i C_i$ and $[a, b] = aba^{-1}b^{-1}$.

We end this section with the definition of the wild character variety.

Definition 2.1.12. *The wild character variety (the wild Betti space) $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ is the quasi-Hamiltonian reduction* $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}$.

Just like in the tame case, the symplectic leaves of $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$, obtained by fixing the conjugacy classes of (formal) monodromies in H_i at each a_i, which is equivalent to fixing a conjugacy class $\boldsymbol{\mathcal{C}} \subset \mathbf{H} = H_1 \times \ldots \times H_m$. We will denote the resulting symplectic variety by $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \boldsymbol{\mathcal{C}})$. It is the quasi-Hamiltonian reduction

$$\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \boldsymbol{\mathcal{C}}) = \mathrm{Hom}_{\mathbb{S}}(\Pi, G) \underset{\boldsymbol{\mathcal{C}}}{/\!\!/} \mathbf{H} \tag{2.6}$$

of the space of Stokes representations at the value $\boldsymbol{\mathcal{C}} \subset \mathbf{H}$ of the moment map. If it does not lead to confusion, we will also call the resulting symplectic manifold the wild character variety.

2.1.4 Covers and irregular classes

In this section we will briefly introduce an alternative point of view on the irregular types, allowing us to consider more general objects, *irregular classes*. This discussion has appeared in [31] (cf. Section 3. there) in much greater generality, here we just assemble together basic ideas. This approach is closer to original Deligne's approach using Stokes structures [37].

Definition 2.1.13. *Let G be a group. A G-torsor is a set $\mathcal{P}$ with a free transitive right action of G.*

A basic example of a torsor is the set of frames of an n-dimensional complex vector space V, which we can write as $\mathcal{P} = \mathrm{Iso}(\mathbb{C}^n, V)$. A point of $\mathcal{P}$ is an isomorphism $\Phi : \mathbb{C}^n \to V$ and the group $G = \mathrm{GL}_n(\mathbb{C})$ acts on $\mathcal{P}$ on the right $\Phi \mapsto \Phi \circ g$, changing the basis of $\mathbb{C}^n$.

A morphism of two G-torsors is a G-equivariant map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$, ie. a map satisfying $\phi(pg) = \phi(p)g$ for all $p \in \mathcal{P}_1, g \in G$. Any such map is an isomorphism of G-torsors. Let $\mathcal{F}$ denote the trivial G-torsor, which is a copy of G acting on itself by right multiplication. A *framing* of a G-torsor $\mathcal{P}$ is an isomorphism $\theta \in \mathrm{Iso}(\mathcal{F}, \mathcal{P})$ and the choice of θ is equivalent to choosing a point $p \in \mathcal{P}$.

Suppose now that Σ is a connected space (a circle, a Riemann surface etc.).

Definition 2.1.14. *A local system $\mathbb{L}$ over Σ is a covering map $\pi : \mathbb{L} \to \Sigma$. A G-local system $\mathbb{L}$ is a local system such that each fibre is a G-torsor.*

In other words, a G-local system is a sheaf which is a torsor under the constant sheaf G over Σ, which means that $\mathbb{L}(U)$ is a G-torsor for small open $U \subset \Sigma$. If we replace the constant sheaf G by a local system $\mathcal{G}$ of groups, then we can define a notion of a $\mathcal{G}$-local system $\mathbb{L}$ in a similar fashion: it is a sheaf which is a torsor under $\mathcal{G}$, so $\mathbb{L}(U)$ is a $\mathcal{G}(U)$-torsor for sufficiently small open $U \subset \Sigma$.

Let ∂ denote a circle and let $\pi : I \to \partial$ be a covering map (a local system of sets), so the fiber $I_p = \pi^{-1}(p)$ is a discrete set for any $p \in \partial$ and when p moves around the circle, the sets are locally constant. After one turn we obtain a monodromy automorphism $\sigma : I_p \to I_p$. The monodromy σ determines I up to isomorphism.

Suppose that Σ is a curve and $0 \in \Sigma$ is a smooth point. Let $\pi : \widehat{\Sigma} \to \Sigma$ denote the real oriented blow up of Σ at 0. Let $\partial = \pi^{-1}(0)$ denote the boundary circle of real directions emanating from 0 in Σ. Open intervals $U \subset \partial$ parametrise germs of open sectors $\mathrm{Sect}(U)$ at zero with opening U. Let z be a local coordinate vanishing at 0 and define $\mathcal{I}$ to be the local system of ∂ whose local sections are complex polynomials in some root $z^{-1/r}$ of z^{-1} with zero constant term. Said differently, local sections of $\mathcal{I}$ are of the form

$$q = \sum_{i=1}^{k} a_i z^{-i/r} \tag{2.7}$$

for $a_i \in \mathbb{C}, k, r \in \mathbb{N}$. We can think of those as of exponents of exponential factors appearing in formal solutions of linear differential equations. The local system $\mathcal{I}$ is the *local system of exponents*.

Each local section of $\mathcal{I}$ has a well defined ramification index, which is the minimal integer $r = \mathrm{ram}(q) \geqslant 1$ such that $q \in \mathbb{C}[z^{-1/r}]$. Therefore each local section q of $\mathcal{I}$ becomes single valued on the r-fold cyclic covering circle of ∂, where $r = \mathrm{ram}(q)$. The degree of q is the smallest integer $k = \deg(q)$ possible in (2.7).

Each connected component $I \in \pi_0(\mathcal{I})$ is a circle finitely covering ∂ and we will denote such covering by $\pi : I \to \partial$ and its ramification index by $r = \mathrm{ram}(I)$. Hence we can understand the local system $\mathcal{I}$ as a disjoint union of many circles covering ∂. If we choose an interval $U \subset \partial$, then $\pi^{-1}(U)$ has r distinct components and each component determines a function on $\mathrm{Sect}(U)$. These functions are the r branches of of the single valued function upstairs (they form a single Galois orbit). If $b \in U$ is a basepoint and we choose a point $i \in \pi^{-1}(b)$, then we get a well defined function q_i on $\mathrm{Sect}(U)$ and the analytic continuation of q_i yields the other functions q_j.

A choice of a connected component $I \subset \mathcal{I}$ is equivalent to choosing a Galois orbit of such functions. The Stokes example of Airy equation $y'' = 9xy$ [90] involves the Galois orbit $\{2z^{-3/2}, -2z^{-3/2}\}$ and determines a component $I \subset \mathcal{I}$. In general, for a positive rational number $c \in Q$, we will denote the element $I \subset \mathcal{I}$ determined by the Galois orbit of z^{-c} by $\langle z^{-c} \rangle$. We can further extend it to all functions q of the form

(2.7) and write $I = \langle q \rangle$.

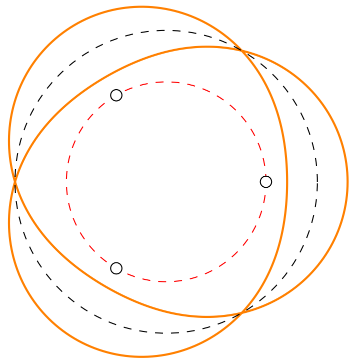

Figure 2.2: The two-fold covering determined by $\langle z^{-3/2} \rangle$

The projection of the components I, which we will call *Stokes diagrams*, represent the asymptotic behavior of the exponential factors $\exp(q_i)$, which we can compare away from the intersection points. We have already encountered them in Section 2.1.1 in the untwisted case. If the powers of z are integer, then the coverings have ramification index one and are just disjoint unions of trivial covers.

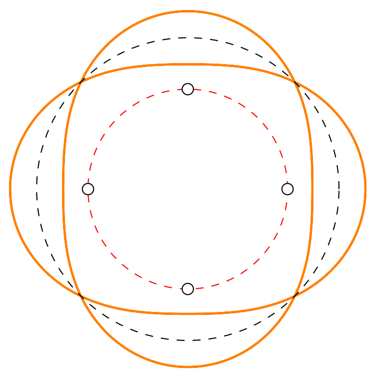

Figure 2.3: The two-fold covering determined by $q = \langle z^{-2}, -z^{-2} \rangle$ is a union of two circles

Each circle $I = \langle q \rangle \subset \mathcal{I}$ (except $\langle 0 \rangle$) has some special points on it. A point p is called a "point of maximal decay", if the corresponding function q is such that $\exp(q)$ has maximal decay as $z \to 0$ in that direction. For example, if $q = \lambda z^{-i/n}$, these are the i points where q is real and negative. They are marked on the red circle on the pictures and in the untwisted case are the same as anti-Stokes directions (and the intersection points are the Stokes directions). More generally, an untwisted irregular type Q for $\mathrm{GL}_n(\mathbb{C})$ determines such cover by comparing the asymptotics of $\exp(q_i)$ (where q_i are the elements on the diagonal of Q).

For $G = \mathrm{GL}_n(\mathbb{C})$, we can generalise the notion of an irregular type to an *irregular class* as in Proposition 8 of [31].

Definition 2.1.15. *Let $G = \mathrm{GL}_n(\mathbb{C})$. An irregular class amounts to a map $\pi_0(\mathcal{I}) \to$*

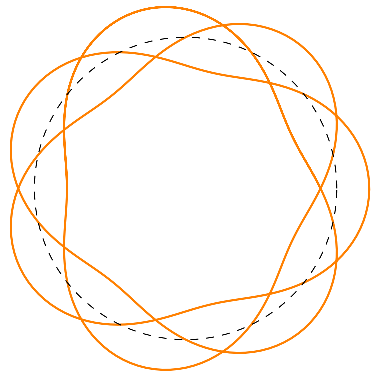

Figure 2.4: The covering determined by $\langle z^{-7/3} \rangle$

$\mathbb{N}$ *assigning a multiplicity* n_I *to each component* $I \subset \mathcal{I}$*, such that*

$$\sum n_I \operatorname{ram}(I) = n.$$

In particular, all untwisted irregular classes arise this way. We will say that an irregular class is *twisted* if any of its components has nontrivial monodromy. Such covers $I \to \partial$ can get pretty complicated and quickly get out of hand. However they give access to more general singularities and can be further generalised beyond $\mathrm{GL}_n(\mathbb{C})$ or for non-constant sheaves of groups. In this manuscript we will encounter only one kind of twisted irregular classes, which now we are going to describe.

Example 2.1.16 ([31], Example 6.2.)**.** Let k be an odd integer and set $c = k/2$. Consider the irregular class $I = \langle z^{-c} \rangle$ with multiplicity $n_I = n$, which gives the following spaces.

Let W be a complex vector space of dimension n and let $V = W \oplus W$. Thus the elements of $G := \mathrm{GL}(V)$ can be written as 2×2 block matrices and we consider the following subgroups of G:

$$U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \quad H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

and a set $H(\partial)$

$$H(\partial) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \subset G,$$

which is a *twist* of H. Let

$$U_\pm^{(i)} = U_- \times U_+ \times U_- \times \dots$$

with i unipotent groups on the right. Then we define the twisted fission space

$$\mathcal{A}^c(V) = G \times U_\pm^{(k)} \times H(\partial),$$

which is a twisted quasi-Hamiltonian $G \times H$ space with moment map

$$\mu = (\mu_G, \mu_H) : \mathcal{A}^c(V) \to G \times H(\partial),$$

$$\mu_G(C, \mathbf{S}, h) = C^{-1} h S_k \cdots S_1 C, \quad \mu_H = h^{-1},$$

where $C \in G, h \in H(\partial), \mathbf{S} = (S_1, \ldots, S_k) \in U_{\pm}^{(k)}$. We can further consider its reduction by G at the value one of the moment map and obtain the reduced twisted fission space

$$\mathcal{B}^c(V) = \{(h, \mathbf{S},) \in H(\partial) \times U_{\pm}^{(k)} \mid hS_k \cdots S_1 = 1\}, \tag{2.8}$$

which is a twisted quasi-Hamiltonian H-space with moment map h^{-1}. We will introduce the language of quasi-Hamiltonian spaces in the next section, here we simply give a direct matrix description of the spaces of Stokes data for some particular choice of a twisted irregular class.

We will relate the space $\mathcal{B}^c(V)$ to odd Euler continuants later in Chapter 5. The case of $c = -5/2$ will also appear in Chapter 4, related to the Painlevé I phase space.

2.2 Quasi-Hamiltonian geometry

In this section we will introduce the basic tools of quasi-Hamiltonian geometry of Alekseev–Malkin–Meinrenken [1] in the holomorphic setup. This theory can be understood as a multiplicative analogue of the classical Hamiltonian geometry. The moment maps take values in the Lie group G instead of the Lie algebra $\mathfrak{g}$. The axioms and interactions with group actions are more complicated but in turn the quasi-Hamiltonian approach gives a direct algebraic description of various symplectic manifolds. Another advantage of this framework is that it gives constructions via finite-dimensional means, unlike the first constructions. Finally, it happens to fit nicely into the geometry of spaces of Stokes data. We will follow [27] in the exposition, following its notations in conventions.

2.2.1 Definitions and notations

Let G be a connected complex reductive Lie group (for all the applications in this manuscript it will be enough to consider $G = \mathrm{GL}_n(\mathbb{C})$) and let $\mathfrak{g}$ denote its Lie algebra. Let $(\ ,\ ) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be a symmetric non degenerate G-invariant bilinear form on $\mathfrak{g}$. We will make this choice once and keep it throughout. Let $\theta, \overline{\theta} \in \Omega^1(G, \mathfrak{g})$ denote the Maurer-Cartan forms on G, respectively, which in any representation can be written as $\theta = g^{-1}dg, \overline{\theta} = (dg)g^{-1}$.

If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Omega^1(M, \mathfrak{g})$ are $\mathfrak{g}$-valued holomorphic one-forms on a complex manifold M, we define $(\mathcal{A}, \mathcal{B}) \in \Omega^2(M)$ and $[\mathcal{A}, \mathcal{B}] \in \Omega^2(M, \mathfrak{g})$ by pairing/braceting the Lie algebra parts and wedging the differential form parts. We define furthermore $\mathcal{A}^2 :=$

$\frac{1}{2}[\mathcal{A}, \mathcal{A}]$ which works correctly in any representation, using matrix multiplication. Then one has $d\theta = -\theta^2$ and $d\overline{\theta} = \overline{\theta}^2$. Define $(\mathcal{ABC}) := \frac{1}{2}(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$, which is invariant under permutations of $\mathcal{A}, \mathcal{B}, \mathcal{C}$. With these definitions, the canonical bi-invariant three-form on G is given by $\frac{1}{6}(\theta^3)$.

We denote the adjoint action og G on $\mathfrak{g}$ by $\mathrm{Ad}_g(X) := gXg^{-1}$ for $X \in \mathfrak{g}, g \in G$. If G acts on a manifold M, the fundamental vector field v_X of $X \in \mathfrak{g}$ is minus the tangent to the flow $(v_X)_m = -\frac{d}{dt}(e^{-tX} \cdot m)|_{t=0}$, so the map $X \mapsto v_X$ is a Lie algebra homomorphism.

Definition 2.2.1. *A complex manifold M is a complex quasi-Hamiltonian G-space if there is an action of G on M, a G-equivariant moment map $\mu : M \to G$ (G acts on itself by conjugation), and a G-invariant holomorphic two-form $\omega \in \Omega^2(M)$ such that the following axioms are satisfied:*

1) $d\omega = \frac{1}{6}\mu^*(\theta^3)$.

2) For all $X \in \mathfrak{g}$, $\omega(v_X, \cdot) = \mu^*(\theta + \overline{\theta}, X) \in \Omega^1(M)$.

3) At each point $m \in M$: $\mathrm{Ker}\, \omega_m \cap \mathrm{Ker}\, d\mu = \{0\} \subset T_m M$.

Remark 2.2.2. Observe that if G is abelian, then the quasi-Hamiltonian axioms imply that ω is a complex symplectic two-form. The quasi-Hamiltonian reduction procedure will yield an abundance of examples of complex symplectic manifolds.

2.2.2 Operations on quasi-Hamiltonian spaces

There are multiple operations one can perform on quasi-Hamiltonian spaces. In this section we will describe the fusion, gluing and the quasi-Hamiltonian reduction.

The fusion product puts a structure of a ring on the category of quasi-Hamiltonian G-spaces.

Theorem 2.2.3 (The fusion product, [1])**.** *Let M be a quasi-Hamiltonian $G \times G \times H$-space, with moment map $\mu = (\mu_1, \mu_2, \mu_3)$ and the two-form ω. Let $G \times H$ act on M by the diagonal embedding $(g, h) \to (g, g, h)$. Then M with two-form*

$$\widetilde{\omega} = \omega - \frac{1}{2}(\mu_1^*\theta, \mu_2^*\overline{\theta})$$

and moment map

$$\widetilde{\mu} = (\mu_1 \cdot \mu_2, \mu_3) \to G \times H$$

is a quasi-Hamiltonian $G \times H$-space.

Observe that the fusion does not change the underlying manifold, it changes the group, the moment map and the two-form on M. The fusion product is not commutative, however chenging the order yields isomorphic quasi-Hamiltonian spaces.

If M_1, M_2 are quasi-Hamiltonian $G \times H_1$ and $G \times H_2$-spaces, respectively, then their fusion product (or just "fusion")

$$M_1 \circledast M_2$$

is defined to be the quasi-Hamiltonian $G \times H_1 \times H_2$-space obtained from the quasi-Hamiltonian $G \times G \times H_1 \times H_2$ space $M_1 \times M_2$ by fusing two factors of G.

We define another important operation, the quasi-Hamiltonian reduction, which one can understand as a multiplicative version of the symplectic quotient.

Theorem 2.2.4 (The quasi-Hamiltonian reduction, [1])**.** *Let M be a quasi-Hamiltonian $G \times H$-space with moment map $\mu = (\mu_G, \mu_H) : M \to G \times H$ and two-form ω. Suppose that the quotient of $\mu_G^{-1}(1)$ by the action of G is a manifold. Then the restriction of the two-form ω to $\mu_G^{-1}(1)$ descends to the space*

$$M /\!\!/ G := \mu_G^{-1}(1)/G$$

and makes it into a quasi-Hamiltonian H-space.

Remark 2.2.5. Observe that if the group H in Theorem 2.2.4 is abelian (for example trivial), then the quasi-Hamiltonian reduction yields an honest complex symplectic manifold.

Theorem 2.2.6 ([27], Proposition 2.8.)**.** *Suppose M is a smooth affine variety with the structure of quasi-Hamiltonian G-space. Then the geometric invariant theory quotient M/G is a Poisson variety.*

We note here that the geometric invariant quotient parametrizes the set of closed G-orbits in M and in general is different from the set-theoretic quotient. Throughout this article, unless stated otherwise, the quotient M/G will stand for the geometric invariant theory quotient, and $M /\!\!/ G$ will denote the quasi-Hamiltonian quotient from Theorem 2.2.4, which is the geometric invariant theory quotient by the action of G of the submanifold $\mu^{-1}(1) \subset M$.

We can use the quasi-Hamiltonian reduction in order to define another operation on quasi-Hamiltonian spaces, the gluing. Given a quasi-Hamiltonian $G \times G \times H$-space, we can first fuse the two G factors and obtain a quasi-Hamiltonian $G \times H$-space. Then, if the quotient is well-defined, we can further reduce this space by G (at the identity of the G-valued moment map) which yields a quasi-Hamiltonian H space.

Definition 2.2.7. *Let M_1, M_2 be quasi-Hamiltonian $G \times H_1$ and $G \times H_2$-spaces, respectively. We define the gluing*

$$M := M_1 \underset{G}{\leftrightarrows} M_2 := (M_1 \underset{G}{\circledast} M_2) /\!\!/ G.$$

If the space obtained by gluing is a manifold, then gluing of M_1 and M_2 yields a quasi-Hamiltonian $H_1 \times H_2$-space. Usually the groups G and H_i will be clear from the context and we will abbreviate the gluing of M_1, M_2 by $M_1 \leftrightarrows M_2$. Unlike the fusion product, the gluing is actually commutative.

We remark here that in all applications the action of G will be free with a global slice, thus there will be no obstruction to performing the reductions/gluings.

2.2.3 Examples: the spaces $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathbb{D}$

In this section we will present different examples of quasi-Hamiltonian space. Even though the basic examples of such spaces are called $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathbb{D}$, this is not the order of their complexity – for example the space $\mathcal{A}$ involves the most data, whereas the space $\mathcal{C}$ is very simple. Thus we will treat them in a non-alphabetical order.

The first example, the conjugacy class $\mathcal{C} \subset G$ is an analogue of coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ and is one of the building blocks of the moduli spaces we will consider.

Example 2.2.8 (Conjugacy class $\mathcal{C}$, [1])**.** Let $\mathcal{C} \subset G$ be a conjugacy class with G acting on $\mathcal{C}$ by conjugation and the moment map defined by inclusion. Then $\mathcal{C}$ is a quasi-Hamiltonian G-space with two-form

$$\omega_g(v_X, v_Y) = \frac{1}{2}\left((X, \mathrm{Ad}_g Y) - (Y, \mathrm{Ad}_g X)\right)$$

for all $X, Y \in \mathfrak{g}, g \in \mathcal{C}$.

More generally, other examples appear as moduli spaces of holomorphic connections (equivalently: Stokes local systems) framed at one point at each boundary component. For example, an annulus with framing at each of its two boundary components is a quasi-Hamiltonian $G \times G$ space, which can be described explicitly.

Example 2.2.9 (The double (annulus) $\mathbf{D}$, [1])**.** The space $\mathbf{D} = G \times G$ is a quasi-Hamiltonian $G \times G$-space with $(k, g) \in G \times G$ acting

$$(k, g) \cdot (C, h) = (kCg^{-1}, khk^{-1}),$$

with moment map

$$\mu(C, h) = (C^{-1}hC, h^{-1}) \in G \times G$$

and the quasi-Hamiltonian two-form given by

$$\omega = \frac{1}{2}\left((\overline{\gamma}, \mathrm{Ad}_h \overline{\gamma}) + (\overline{\gamma}, \overline{\eta} + \eta)\right)$$

where $\overline{\gamma} = C^*(\overline{\theta}), \eta = h^*(\theta), \overline{\eta} = h^*(\overline{\theta})$.

We view C as a map from $\mathbf{D}$ to G and pull-back the right-invariant Maurer-Cartan form $\overline{\theta}$, obtaining a $\mathfrak{g}$-valued one-form on $\mathbf{D}$. Another example can be associated with a one-holed torus.

Example 2.2.10 (Internally fused double $\mathbb{D}$, [1])**.** The space $\mathbb{D} = G \times G$ is a quasi-Hamiltonian G-space with G acting by diagonal conjugation $g(a, b) = (aga^{-1}, gbg^{-1})$ and the moment map given by the commutator

$$\mu(a, b) = aba^{-1}b^{-1}$$

and the quasi-Hamiltonian two-form given by

$$\omega_{\mathbb{D}} = \frac{1}{2}\left(-(a^*\theta, b^*\overline{\theta}) - (a^*\overline{\theta}, b^*\theta) - ((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta})\right).$$

All these examples are special cases of a more general theorem. Let Σ be a compact connected Riemann surface with $n \geqslant 1$ boundary components. Choose a basepoint b_i at the ith boundary component for each i. We introduce the groupoid

$$\Pi = \Pi_1(\Sigma, (b_1, \dots, b_n)),$$

the fundamental groupoid of Σ with basepoints $\{b_i\}$, which is, by definition, the groupoid of homotopy classes of paths in Σ having endpoints in the set of basepoints $(b_1, \dots, b_n)$.

Theorem 2.2.11 ([1, 27])**.** *The space* $\mathrm{Hom}(\Pi, G)$ *of homomorphisms from* Π *to the group* G *is a smooth quasi-Hamiltonian* G^n*-space.*

Proof. Upon choosing the suitable generating paths of Π, one can identify $Hom(\Pi, G)$ with $G^{2(g+m-1)}$ and then with the reduction of the fusion product $\mathbb{D}^{\circledast g} \circledast \mathbf{D}^{\circledast m}$ by the diagonal action of g. $\square$

One can (and should) understand the fusion of two quasi-Hamiltonian $G \times H_1$ and $G \times H_2$-spaces as gluing their G-boundaries onto two holes of a three-holed sphere (with G-framings at the boundary). Fusing with $\mathbb{D}$ corresponds topologically to adding a genus one handle. The theorem above can be interpreted as follows: tame character varieties can be obtained from the quasi-Hamiltonian fusions/gluings of tori and annuli. We also see that the full reduction by all copies of G yields the tame character variety which is a Poisson manifold.

Now we pass to a more complicated example which will be essentially the object of study of this thesis (and variations of thereof). Choose a parabolic subgroup $P_+ \subset G$ and a Levi subgroup $H \subset P_+$ and let P_- be the opposite parabolic with the same Levi subgroup $H \subset P_-$, so that P_-P_+ is dense in G. We denote by $U_\pm$ the corresponding unipotent radicals.

Example 2.2.12 (The fission space $\mathcal{A}$, [21, 27])**.** Let $r \geqslant 1$ be an integer. We define the fission space

$${}_G\mathcal{A}^r_H := G \times (U_+ \times U_-)^r \times H.$$

There is an action of $G \times H$ on ${}_G\mathcal{A}^r_H$ given by

$$(g, k)(C, \mathbf{S}, h) = (kCg^{-1}, k\mathbf{S}k^{-1}, khk^{-1}),$$

where $(g, k) \in G \times H$ and $\mathbf{S} = (S_1, \dots, S_{2r})$. The conjugation of $\mathbf{S}$ means conjugating every term S_i. The space ${}_G\mathcal{A}^r_H$ is then a quasi-Hamiltonian $G \times H$-space with moment map

$$\mu(C, \mathbf{S}, h) = (C^{-1}hS_{2r} \cdots S_1C, h^{-1}) \in G \times H.$$

The quasi-Hamiltonian form on ${}_G\mathcal{A}^r_H$ is defined as follows. Define $C_i : {}_G\mathcal{A}^r_H \to G$ by

$$C_i = S_i \cdots S_2S_1C,$$

with $C_0 = C$, and define $b = hS_{2r} \cdots S_2 S_1$. The G-component of the moment map is then $C^{-1}bC$. We further define the following $\mathfrak{g}$-valued one-forms on ${}_G\mathcal{A}^r_H$:

$$\gamma_i = C_i^*(\theta), \quad \overline{\gamma}_i = C_i^*(\overline{\theta}), \quad \eta = h^*(\theta_H), \quad \overline{\beta} = b^*(\overline{\theta}),$$

where $\theta, \overline{\theta}, \theta_H \overline{\theta}_H$ are the Maurer–Cartan forms on G and H, respectively. The quasi-Hamiltonian two-form is then defined as

$$2\omega = (\overline{\gamma}, \mathrm{Ad}_b \overline{\gamma}) + (\overline{\gamma}, \overline{\beta}) + (\overline{\gamma}_m, \eta) - \sum_{i=1}^{2r} (\gamma_i, \gamma_{i-1}), \tag{2.9}$$

where $\overline{\gamma} = \overline{\gamma}_0$ and the brackets denote the bilinear form on $\mathfrak{g}$.

If $G = \mathrm{GL}_n(\mathbb{C})$, then the fission spaces have a convenient description. Let V be a complex vector space of dimension n with an ordered grading

$$V = \bigoplus_{i=1}^{k} V_i.$$

Such grading determines a flag F_+:

$$F_1 \subset F_2 \subset \ldots \subset F_k = V,$$

where $F_i = V_1 \oplus \ldots \oplus V_i$ and a flag F_-:

$$F'_k \subset F'_{k-1} \subset \ldots \subset F'_1 = V$$

where $F'_i = V_i \oplus \ldots \oplus V_k$.

We will consider the groups $G = \mathrm{GL}(V), H = \prod \mathrm{GL}(V_i) \subset G$ and the opposite parabolic subgroups $P_\pm$ stabilising the flags $F_\pm$ together with their unipotent radicals $U_\pm \subset P_\pm$. In an adapted basis, the group G is the group of $n \times n$ matrices, H is the block diagonal group with sizes of block determined by the dimensions of the graded pieces V_i. Similarly, the parabolics $P_\pm$ are the upper/ lower block triangular matrices, and $U_\pm$ are upper/ lower block unipotent (ie. unipotent with identity blocks on the diagonal).

Therefore for an ordered graded vector space V and a fixed r, which determine $G, H, P_\pm, U_\pm$ we define $\mathcal{A}^r(V)$ as before:

$$\mathcal{A}^r(V) := G \times (U_+ \times U_-)^r \times H.$$

An important property of $\mathcal{A}(V)$ is that, up to isomorphism, it depends only on the grading, not on the ordering of the spaces V_i.

Proposition 2.2.13 ([28], Proposition 4.4.). *Let V be an ordered graded space*

$$V = \bigoplus_{i=1}^{k} V_i$$

and V' the same graded vector space but with different ordering of pieces. Then the spaces $\mathcal{A}^r(V)$ and $\mathcal{A}^r(V')$ are isomorphic as quasi-Hamiltonian $G \times H$-spaces.

Example 2.2.14 (The reduced fission space $\mathcal{B}$, [21, 27])**.** The space ${}_G\mathcal{A}^r_H$ is a quasi-Hamiltonian $G \times H$-space, thus if $r > 1$ we can consider its quasi-Hamiltonian reduction by G (at the value one of the moment map). We denote this reduction by $\mathcal{B}^r$ and call it the *reduced* fission space $\mathcal{B}^r$

$$\mathcal{B}^r = {}_G\mathcal{A}^r_H /\!\!/ G,$$

which is a quasi-Hamiltonian H-space. More explicitly, we have

$$\mathcal{B}^r = \{(h, S_1, \ldots, S_{2r}) \in H \times (U_+ \times U_-)^r \mid hS_{2r} \cdots S_1 = 1\} \tag{2.10}$$

with moment map $h \in H$ and the quasi-Hamiltonian two-form

$$2\omega = -\sum_{i=1}^{2r} (\gamma_i, \gamma_{i-1}), \tag{2.11}$$

obtained from the two-form on ${}_G\mathcal{A}^r_H$ by restricting it to the subset $C = b = 1$. If $r = 2$, we will denote $\mathcal{B}^2$ by $\mathcal{B}$.

For the general linear group $\mathrm{GL}_n(\mathbb{C})$ and a vector space V of dimension n with ordered grading

$$V = \bigoplus_{i=1}^{k} V_i,$$

we define the space $\mathcal{B}^r(V)$ as a reduction of $\mathcal{A}^r(V)$ at the value one of the G-component of the moment map.

An important example arises when one considers the grading with two pieces $V = V_1 \oplus V_2$. Namely, the space $\mathcal{B}(V)$ can be now described as

$$\mathcal{B}^r(V) = \{(h, S_1, \ldots, S_{2r}) \in H \times (U_+ \times U_-)^r \mid hS_{2r} \cdots S_1 = 1\}$$

with

$$H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad U_+ = \begin{pmatrix} 1_{V_1} & * \\ 0 & 1_{V_2} \end{pmatrix}, \quad U_- = \begin{pmatrix} 1_{V_1} & 0 \\ * & 1_{V_2} \end{pmatrix},$$

where the asterisks denote the blocks belonging to $\mathrm{Hom}(V_2, V_1)$ for U_+, $\mathrm{Hom}(V_1, V_2)$ for U_- and $\mathrm{End}(V_1), \mathrm{End}(V_2)$ for H. The moment map is h^{-1}, which we can compute explicitly. Denote by s_i the block off-diagonal entries of the Stokes multipliers S_i. Then we have

$$S_{2r-1}\cdots S_2 = S_{2r}^{-1}h^{-1}S_1^{-1},$$

so the product $S_{2r-1}\cdots S_1$ is in the big Gauss cell $G^\circ = U_-HU_+$. The multipliers $S_2,\dots,S_{2r-1}$ uniquely determine h, S_1, S_{2r}.

In particular, for $r=2$, we get

$$S_3S_2 = \begin{pmatrix} 1+s_3s_2 & s_3 \\ s_2 & 1 \end{pmatrix}$$

and the diagonal entries of h^{-1} are $1+s_3s_2$ and $1-s_2(1+s_3s_2)^{-1}s_3 = (1+s_2s_3)^{-1}$. Hence if we set $b_1 = s_3, b_2 = s_2$, we can identify $\mathcal{B}(V)$ with

$$\mathcal{B}(V) = \{(b_1,b_2) \in \operatorname{Hom}(V_2,V_1)\oplus\operatorname{Hom}(V_1,V_2) \mid \det(1+b_1b_2)\neq 0\} \tag{2.12}$$

with moment map

$$\mu(b_1,b_2) = (1+b_1b_2,(1+b_2b_1)^{-1}).$$

On the other hand, a fundamental example of a quasi-Hamiltonian space is the Van den Bergh space $\mathcal{B}(V_1,V_2)$ [91, 92].

$$B(V_1,V_2) = \{(a,b) \in \operatorname{Hom}(V_2,V_1)\oplus\operatorname{Hom}(V_1,V_2) \mid \det(1+ab)\neq 0\} \tag{2.13}$$

and we have the following.

Theorem 2.2.15 ([91, 92, 97])**.** *The space $\mathcal{B}(V_1,V_2)$ is a quasi-Hamiltonian* $\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)$*-space with moment map*

$$\mu(a,b) = \big((1+ab)^{-1},(1+ba)\big) \in \mathrm{GL}(V_1)\times\mathrm{GL}(V_2)$$

and the quasi-Hamiltonian two-form

$$\omega = \frac{1}{2}\big(\operatorname{Tr}_{V_1}(1+ab)^{-1}da\wedge db - \operatorname{Tr}_{V_2}(1+ba)^{-1}db\wedge da\big). \tag{2.14}$$

The component $1+ab$ of the moment map is one of the Euler's continuant polynomials [41] which we will study in detail in Chapter 5.

One can consider the two-form on $\mathcal{B}(V)$, given by restriction of the form (2.9) defining the quasi-Hamiltonian structure on $\mathcal{A}^2(V)$. It comes without surprise that the space $\mathcal{B}(V_1,V_2)$ is not only isomorphic to $\mathcal{B}(V)$, but the quasi-Hamiltonian structures match up as well.

Theorem 2.2.16 ([27], Lemma 4.1.)**.** *The quasi-Hamiltonian two-form on the space $\mathcal{B}(V)$ is given by*

$$\omega = \frac{1}{2}\big(-\operatorname{Tr}(b_1,b_2)^{-1}db_1\wedge db_2) + \operatorname{Tr}(b_2,b_1)^{-1}db_2\wedge db_1)\big) \tag{2.15}$$

and thus the map $\mathcal{B}(V) \to \mathcal{B}(V_1, V_2)$ *given by*

$$(b_1, b_2) \mapsto (-a(1+ba)^{-1}, b)$$

is an isomorphism of quasi-Hamiltonian $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$*-spaces.*

One of the results of Chapter 5 generalises the formula (2.15) to $\mathcal{B}^k(V)$ for $k > 2$. The space $\mathcal{B}(V)$ fits naturally into the language of quivers. Given two vector spaces V_1, V_2 and two arrows between them, one in each direction, the map b_1 goes from V_2 to V_1 and the map b_2 goes from V_1 to V_2. We will sometimes call the space $\mathcal{B}(V)$ an *edge*. Such quasi-Hamiltonian edges are building blocks of the classical multiplicative quiver varieties of Crawley-Boevey–Shaw [36] and Yamakawa [97].

2.2.4 Twisted quasi-Hamiltonian spaces

In this section we will introduce a generalization of quasi-Hamiltonian spaces, the *twisted* quasi-Hamiltonian spaces. This generalization is necessary since the building blocks for the moduli spaces appearing later in this chapter are twisted. In general, the twisted quasi-Hamiltonian spaces appear when one considers more general Stokes local systems, when the structure group G varies along the curve. The reference for this section is [31].

Let G be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$, together with an automorphism $\phi \in \mathrm{Aut}(G)$, which we will call the *twist*. If $\phi \in \mathrm{Aut}(G)$, let Γ denote the subgroup generated by ϕ and write $\widetilde{G} = \widetilde{G}(\phi) = G \ltimes \Gamma \subset G \ltimes \mathrm{Aut}(G)$. Fix a symmetric non-degenerate bilinear form $(\ ,\ )$ on $\mathfrak{g}$, invariant under the adjoint action and under the action of ϕ. Let

$$G(\phi) = \{(g, \phi) \mid g \in G\} \subset \widetilde{G}(\phi)$$

be the component of $\widetilde{G}$ lying over ϕ. Then the natural left and right actions of G are free and transitive and $G(\phi)$ is a G-bitorsor. Explicitly, $(g_1, g_2) \in G \times G$ sends (g, ϕ) to $(g_1 g \phi(g_2), \phi)$ and in particular the conjugation action of G becomes now ϕ-conjugation. We will say that $G(\phi)$ is a twist of G.

Definition 2.2.17. *[[31], Definition 20] A twisted quasi-Hamiltonian G-space is a complex manifold M with an action of G, an invariant holomorphic two-form ω and a G-equivariant moment map $\mu : M \to G(\phi)$ to a twist of G (with the twisted conjugation action),*

Such twisted quasi-Hamiltonian space is almost a quasi-Hamiltonian space for $\widetilde{G}$, but we only consider the action of the identity component of $\widetilde{G}$. The whole machinery of twisted quasi-Hamiltonian spaces has been designed to fit into the standard framework of quasi-Hamiltonian geometry. In particular, if $M_i, i = 1, 2$ are twisted quasi-Hamiltonian G-spaces with moment maps $\mu_i \to G(\phi_i)$, then the fusion $M := M_1 \circledast M_2$

is a twisted quasi-Hamiltonian G-space with moment map $\mu_1 \cdot \mu_2 : M \to G(\phi_1 \phi_2)$. Similarly, if ϕ_1 is the inverse of ϕ_2, then the gluing $M_1 \leftrightarrows M_2$ is well defined.

The only twisted quasi-Hamiltonian spaces we are going to encounter are the spaces $\mathcal{B}^c(V)$ from Example 2.1.16, where all the groups and the twist are explicit.

2.3 Stokes local systems

Recall that we have defined an irregular curve $\mathbf{\Sigma} = (\Sigma, a_i, Q_i)$ as a complex curve Σ (possibly with boundary) with some marked points $a_1, \dots, a_m$ and irregular types $Q_1, \dots, Q_m$ at the marked points. This, in turn, determines a real surface with boundary $\widehat{\Sigma}$, obtained by performing real oriented blow ups at the marked points, and a surface $\widetilde{\Sigma}$ which is $\widehat{\Sigma}$ with some extra punctures, determined by the irregular types. Moreover, we have introduced the halos $\mathbb{H}_i$ around each a_i, and cilia in each halo to keep track of the additional punctures.

Definition 2.3.1. *A Stokes G-local system for $\mathbf{\Sigma}$ is a local system $\mathbb{L}$ on $\widetilde{\Sigma}$ together with a flat reduction of structure group to H_i inside of $\mathbb{H}_i$ for each $i = 1, \dots, m$ (i.e. an H_i-local system $\mathbb{L}_i$ defined inside $\mathbb{H}_i$ such that $\mathbb{L} = \mathbb{L}_i \times_{H_i} G$ there) such that, for any basepoint inside $\mathbb{H}_i$, the local monodromy around the puncture corresponding to $d \in \mathbb{A}_i$ lies in $\mathbb{S}\mathrm{to}_d(Q_i)$.*

Thus Stokes local systems are topological objects which generalise the usual local systems by adding some extra conditions (structure group reduction and monodromy around extra punctures in $\widetilde{\Sigma}$) on their behavior at the marked points a_i. It is known that for fixed irregular types Q_i at the marked points a_i, the Stokes local systems on the resulting surface $\widetilde{\Sigma}$ classify isomorphism classes of meromorphic connections with this irregular data (it is an equivalence of categories). Thus classification of irregular connections is equivalent to the classification of Stokes local systems. It is convenient to think about Stokes local systems in terms of pictures, which locally look like on the figure below.

The gray area indicates the halo $\mathbb{H}$, where the structure group changes from G to H, the circles on the boundary of the red circle represent the singular directions. The picture is local at each marked point: the structure group changes in each halo $\mathbb{H}_i$ around a_i and outside the group is G.

The notion of framing naturally extends to the case of Stokes local systems. Framings are an auxiliary tool and there is a freedom of choice of framings. However, certain framings will be preferred since they fit into the quasi-Hamiltonian perspective.

Definition 2.3.2. *Let $\mathbb{L}$ be a Stokes local system on $\mathbf{\Sigma}$ and $p \in \widetilde{\Sigma}$. If p does not belong to any halo $\mathbb{H}_i$, we define the G-framing of $\mathbb{L}$ at p to be a choice of isomorphism between a trivial G-torsor and the fiber of $\mathbb{L}$ at p. If p is in a halo $\mathbb{H}_i$, then we define*

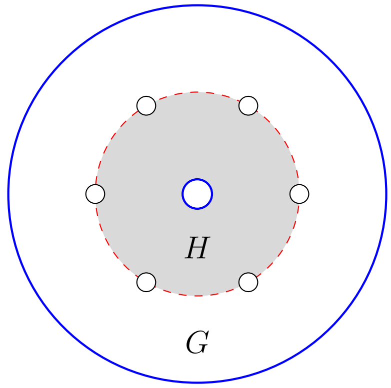

Figure 2.5: Stokes local system on a disk

a H_i-framing of $\mathbb{L}$ at p to be a choice of an isomorphism between the trivial H_i-torsor and the fiber of $\mathbb{L}$ at p. We will denote a framing of $\mathbb{L}$ at p by φ_p.

Thus if $G = \mathrm{GL}_n(\mathbb{C})$ (and the Stokes local system is a locally constant sheaf of vector spaces), then a framing at p outside the halo is a choice of an isomorphism between $\mathbb{C}^n$ and the fiber of $\mathbb{L}$ at p. A framing at p inside the halo is a choice of a H_i-graded isomorphism between $\mathbb{C}^n$ and the fiber at p.

The idea behind the framings is to give an explicit approach to abstract objects like moduli spaces of Stokes local systems. Note that given a framed Stokes local system $(\mathbb{L}, \Phi)$, where Φ simplistically denotes the collective framings at chosen points, we can simply forget the framings and recover the Stokes local system $\mathbb{L}$.

Observe that framing a Stokes local system yields additional group actions. A G-framing adds a G-action on the set of framed Stokes local systems and an H_i-framing adds an H_i-action. Thus by framing we enrich the data but also add some extra group actions.

In general, the preferred framing of a Stokes local system will consist of framing once in each halo $\mathbb{H}_i$ and once near each boundary component. The quasi-Hamiltonian spaces appearing throughout the text (fission spaces, fusions and gluings of thereof) parametrise isomorphism classes of Stokes local systems *framed* at the boundary components.

Definition 2.3.3. *Let $\mathbb{L}$ be a Stokes local system on $\boldsymbol{\Sigma}$ framed once at each boundary component of $\widetilde{\Sigma}$. We will call such Stokes local system a minimally framed Stokes local system.*

Corollary 2.3.4. *Let $\boldsymbol{\Sigma}$ be an irregular curve. The space* $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ *classifies minimally framed Stokes G-local systems on $\boldsymbol{\Sigma}$.*

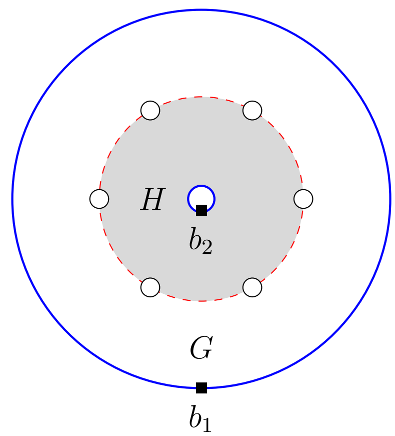

Figure 2.6: A minimally framed Stokes local system on a disk

The framed Stokes local system can be pictured as above. The figure represents a framed Stokes local system on a disk Δ with one marked point at zero and irregular type Q. The inner blue circle is the boundary circle obtained after the real oriented blow up at zero, the outer circle is the boundary of the disk. The number of extra punctures is determined by the irregular type Q (we schematically put six, but the number can be arbitrary). The framing at b_1 is the G-framing and the framing at b_2 is the H-framing.

The space corresponding to the picture is

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \simeq \mathcal{A}(Q),$$

with $\Pi = \Pi_1(\widetilde{\Delta}, \{b_1, b_2\})$ (where $\widetilde{\Delta}$ stands for the real oriented blow up of Δ with extra punctures corresponding to the singular directions, determined by Q). This fits into the quasi-Hamiltonian picture: the G-fusion of two surfaces with G-boundaries corresponds to gluing them onto a three-holed sphere (with three G-boundaries) and the gluing corresponds to simply gluing the G-boundaries together.

Given an irregular curve $\mathbf{\Sigma}$, cutting small disks around the marked points and then preforming the pants decomposition of Σ yields a decomposition of the space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ into simple pieces.

Chapter 3

Symmetric cubic surfaces and G_2 character varieties

This chapter contains the article *Symmetric cubic surfaces and G_2 character varieties* [12], which is a joint work with P. Boalch.

3.1 Abstract

We will consider a two dimensional "symmetric" subfamily of the four dimensional family of Fricke cubic surfaces. The main result is that such symmetric cubic surfaces arise as character varieties for the exceptional group of type G_2. Further, this symmetric family will be related to the fixed points of the triality automorphism of Spin(8), and an example involving the finite simple group of order 6048 inside G_2 will be considered.

3.2 Introduction

The Fricke family of cubic surfaces is the family

$$x\,y\,z + x^2 + y^2 + z^2 + b_1\,x + b_2\,y + b_3\,z + c = 0 \tag{3.1}$$

of affine cubic surfaces parameterised by constants $(b_1, b_2, b_3, c) \in \mathbb{C}^4$. Note that the moduli space of (projective) cubic surfaces is four dimensional and a generic member will have an affine piece of this form, so this family includes an open subset of all cubic surfaces. We will say a Fricke surface is *symmetric* if $b_1 = b_2 = b_3$.

The full family (3.1) is known to be a semiuniversal deformation of a D_4 singularity (which occurs at the symmetric surface $b_i = -8, c = 28$). Many other examples of cubic surfaces are isomorphic to symmetric Fricke cubics: the Markov cubic surface

($b_i = c = 0$), Cayley's nodal cubic surface ($b_i = 0, c = -4$), Clebsch's diagonal cubic surface ($b_i = 0, c = -20$) and the Klein cubic surface ($b_i = -1, c = 0$), see §3.8 below.

The Fricke surfaces are interesting since they are some of the simplest nontrivial examples of complex character varieties (and as such they are amongst the simplest examples of complete hyperkähler four-manifolds for which we do not know how to construct the metric by finite dimensional means, cf. [25] §3.2). Namely if $\Sigma = \mathbb{P}^1 \setminus \{a_1, a_2, a_3, a_4\}$ is a four punctured sphere, then the moduli space

$$\mathcal{M}_{\mathrm{B}}(\Sigma, \mathrm{SL}_2(\mathbb{C})) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$$

of $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group of Σ is a (complex) six-dimensional algebraic Poisson variety and its symplectic leaves are Fricke cubic surfaces. Indeed, choosing generators of the fundamental group leads to the identification of the character variety

$$\mathcal{M}_{\mathrm{B}}(\Sigma, \mathrm{SL}_2(\mathbb{C})) \cong \mathrm{SL}_2(\mathbb{C})^3/\mathrm{SL}_2(\mathbb{C})$$

with the quotient of three copies of $\mathrm{SL}_2(\mathbb{C})$ by diagonal conjugation. (All our quotients are affine geometric invariant theory quotients, taking the variety associated to the ring of invariant functions.) In this case the ring of invariant functions is generated by the seven functions

$$x = \mathrm{Tr}(M_2M_3), \quad y = \mathrm{Tr}(M_1M_3), \quad z = \mathrm{Tr}(M_1M_2),$$

$$m_1 = \mathrm{Tr}(M_1), \quad m_2 = \mathrm{Tr}(M_2), \quad m_3 = \mathrm{Tr}(M_3), \quad m_4 = \mathrm{Tr}(M_1M_2M_3) \tag{3.2}$$

where $M_i \in \mathrm{SL}_2(\mathbb{C})$. These generators satisfy just one relation, given by equation (3.1), with:

$$b_1 = -(m_1m_4 + m_2m_3),\ b_2 = -(m_2m_4 + m_1m_3),\ b_3 = -(m_3m_4 + m_1m_2),$$

$$c = \prod m_i - 4 + \sum m_i^2. \tag{3.3}$$

This relation amongst the generators is known as the Fricke relation[1]. The symplectic leaves are obtained by fixing the conjugacy classes of the monodromy around the four punctures and in general this amounts to fixing the values of the four invariants m_1, m_2, m_3, m_4, and thus the symplectic leaves are Fricke cubics. Note that from this point of view the D_4 singularity occurs at the trivial representation of $\pi_1(\Sigma)$.

The aim of this article is to consider some simple examples of character varieties for the exceptional simple group $G_2(\mathbb{C})$ of dimension 14. Our main result (Corollary 3.5.6) may be summarised as:

Theorem 3.2.1. *There is a two parameter family of character varieties for the exceptional group $G_2(\mathbb{C})$ which are isomorphic to smooth symmetric Fricke cubic surfaces, and thus to character varieties for the group* $\mathrm{SL}_2(\mathbb{C})$.

[1] apparently ([67]) it was discovered by Vogt ([94] eq. (11)) in 1889, and repeatedly rediscovered by many others, including Fricke ([45] p.366).

Note that the character varieties for complex reductive groups (and the naturally diffeomorphic Higgs bundle moduli spaces) are crucial to the geometric version of the Langlands program [10], and, in the case of compact curves Σ, the geometric Langlands story for G_2 shows significant qualitative differences to the case of SL_n, involving a nontrivial involution of the Hitchin base (see e.g. [53]). Thus it is surprising that there are G_2 character varieties which are in fact isomorphic to SL_2 character varieties[2]. As far as we know this is the first example of an isomorphism between nontrivial (symplectic) Betti moduli spaces involving an exceptional group. Some further motivation is described at the end of section 3.3.

Philosophically we would like to separate (or distance) the choice of the group from the "abstract" moduli space, as in the theory of Lie groups, where it is useful to consider the abstract group independently of a given representation, or embedding in another group. In this language our result says that an abstract (nonabelian Hodge) moduli space has a "G_2 representation" as well as an SL_2 representation/realisation.

In the later sections of the paper we will also consider the following topics. In §3.6 the natural braid group actions on the spaces (coming from the mapping class group of the curve Σ) will be made explicit and we will show that the isomorphisms of Theorem 3.2.1 are braid group equivariant. In §3.7 we will recall that the $\mathbb{C}^4$ parameter space of Fricke cubics is naturally related to the Cartan subalgebra of $\mathrm{Spin}_8(\mathbb{C})$ (i.e. the simply connected group of type D_4) and show that the subspace $\mathbb{C}^2 \subset \mathbb{C}^4$ of symmetric Fricke cubics corresponds to the inclusion of Cartan subalgebras

$$\mathfrak{t}_{G_2} \subset \mathfrak{t}_{\mathrm{Spin}(8)}$$

coming from the inclusion $G_2(\mathbb{C}) \subset \mathrm{Spin}_8(\mathbb{C})$ identifying $G_2(\mathbb{C})$ as the fixed point subgroup of the triality automorphism of $\mathrm{Spin}_8(\mathbb{C})$.

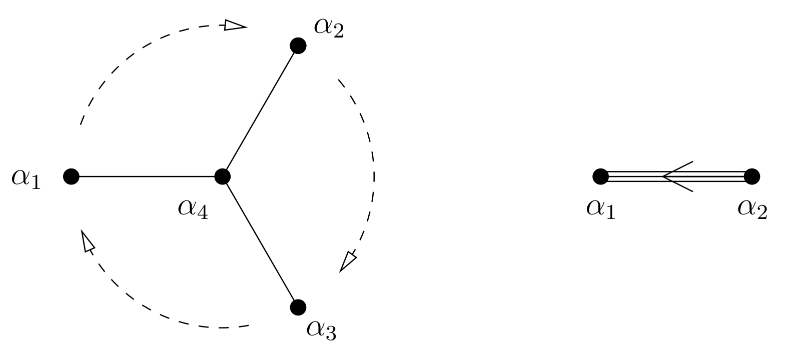

Figure 3.1: Triality automorphism of D_4 and the resulting G_2 Dynkin diagram

Finally in section §3.8 we will revisit some of the finite braid group orbits on cubic surfaces found in [18, 22]. In particular we will consider the Klein cubic surface (the unique cubic surface containing a braid group orbit of size 7) and show that:

[2] We expect there to be analogous isomorphisms also in the Dolbeault/Higgs and De Rham algebraic structures corresponding to the Betti version considered here.

Theorem 3.2.2. *If the Klein cubic surface K is realised as a G_2 character variety (via Theorem 3.2.1) then the braid orbit of size 7 in K corresponds to some triples of generators of the finite simple group $G_2(\mathbb{F}_2)' \subset G_2(\mathbb{C})$ of order 6048. One such triple of generators is uniquely determined by the three lines passing through a single point in the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$.*

3.3 Tame character varieties

Let Σ be a smooth complex algebraic curve and let G be a connected complex reductive group. Then we may consider the space

$$\mathrm{Hom}(\pi_1(\Sigma, p), G)$$

of representations of the fundamental group of Σ into the group G (where $p \in \Sigma$ is a base point). This is an affine variety with an action of G given by conjugating representations. The character variety

$$\mathcal{M}_{\mathrm{B}}(\Sigma, G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

is defined to be the resulting affine geometric invariant theory quotient (the variety associated to the ring of G invariant functions on $\mathrm{Hom}(\pi_1(\Sigma, p), G)$). It is independent of the choice of basepoint so we supress p from the notation. Set-theoretically the points of $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ correspond bijectively to the *closed* G-orbits in $\mathrm{Hom}(\pi_1(\Sigma, p), G)$.

It is known that $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ has a natural algebraic Poisson structure [5, 6]. The symplectic leaves of $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ are obtained as follows. Suppose $\Sigma = \overline{\Sigma} \setminus \{a_1, \dots a_m\}$ is obtained by removing m points from a smooth compact curve $\overline{\Sigma}$. Choose a conjugacy class

$$\mathcal{C}_i \subset G$$

for $i = 1 \dots, m$ and let $\boldsymbol{\mathcal{C}}$ denote this m-tuple of conjugacy classes. Consider the subvariety

$$\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}}) \subset \mathcal{M}_{\mathrm{B}}(\Sigma, G)$$

consisting of the representations taking a simple loop around a_i into the class $\mathcal{C}_i$ for each i. Then the symplectic leaves of $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ are the connected components of the subvarieties $\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}})$. We will also refer to these symplectic varieties $\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}})$ (and their connected components) as character varieties. Note that in general $\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}})$ is not an affine variety (i.e. it is not a closed subvariety of $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$), although it will be if all the conjugacy classes $\mathcal{C}_i \subset G$ are semisimple (since this implies that each $\mathcal{C}_i$ is itself an affine variety), and this will be the case in the examples we will focus on here.

3.3.1 Ansatz

Consider the following class of examples of character varieties. Let G be as above and choose n distinct complex numbers $a_1, \dots, a_n$. Let

$$\Sigma = \mathbb{C} \setminus \{a_1, \dots, a_n\} = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_n, \infty\}$$

be the n-punctured affine line (i.e. an $n+1$-punctured Riemann sphere), and suppose conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_n, \mathcal{C}_\infty \subset G$ are chosen so that:

a) $\mathcal{C}_1, \dots, \mathcal{C}_n \subset G$ are semisimple conjugacy classes of minimal possible (positive) dimension, and

b) $\mathcal{C}_\infty \subset G$ is a regular semisimple conjugacy class.

For example:

1) If $G = \mathrm{SL}_2(\mathbb{C})$ then these conditions just say that all the conjugacy classes are semisimple of dimension two,

2) If $G = \mathrm{GL}_n(\mathbb{C})$ then the minimal dimensional semisimple conjugacy classes are those with an eigenvalue of multiplicity $n-1$; after multiplying by an invertible scalar to make the corresponding eigenvalue 1, any such class contains complex reflections (i.e. linear automorphisms of the form "one plus rank one"), and the monodromy group will be generated by n complex reflections (note that the term "complex reflection group" is often used to refer to such a group which is also *finite*),

3) If $G = G_2(\mathbb{C})$ then this means all the classes $\mathcal{C}_1, \dots, \mathcal{C}_n$ are equal to the unique semisimple orbit $\mathcal{C} \subset G$ of dimension 6, and that $\mathcal{C}_\infty$ is one of the twelve dimensional semisimple conjugacy classes.

Suppose we now (and for the rest of the article) specialise to the case $n = 3$, so that Σ is a four punctured sphere. This is the simplest genus zero case such that Σ has an interesting mapping class group. Then it turns out that the above ansatz yields two-dimensional character varieties in all of the above cases.

Lemma 3.3.1. *Suppose $n = 3$ so that Σ is a four punctured sphere. Then in each case $1), 2), 3)$ listed above the corresponding character variety $\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}})$ is of complex dimension two, provided $\mathcal{C}_\infty$ is sufficiently generic.*

Proof. We will just sketch the idea: assuming $\mathcal{C}_\infty$ is generic the action of G on

$$\{(g_1, g_2, g_3, g_\infty) \mid g_i \in \mathcal{C}_i,\ g_1 g_2 g_3 g_\infty = 1\}$$

will have stabiliser of dimension $\dim(Z)$ where $Z \subset G$ is the centre of G. The expected dimension of $\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}})$ is then $\sum \dim(\mathcal{C}_i) - 2\dim(G/Z)$.

For 1) each orbit is dimension two so it comes down to the sum

$$4 \times 2 - 2 \times 3 = 2.$$

The resulting surfaces are the Fricke surfaces.

For 2) if $G = \mathrm{GL}_3(\mathbb{C})$ the first three orbits are of dimension four and the generic orbit is of dimension six, so the dimension of the character variety is

$$3 \times 4 + 6 - 2 \times (9 - 1) = 2.$$

This case was first analysed in [17, 18] and the resulting character varieties were explicitly related to the full four parameter family of Fricke surfaces (using the Fourier–Laplace transform). In brief, the analogue of the Fricke relation that arises is given in [18] (16) and this relation is related to the Fricke relation in [18] Theorem 1, using an explicit algebraic map that is derived from the Fourier–Laplace transform.

For 3), the group $G = G_2(\mathbb{C})$ has dimension 14 (with trivial centre), the first three orbits are of dimension six and the generic orbit is of dimension twelve, so the dimension of the character variety is

$$3 \times 6 + 12 - 2 \times 14 = 2$$

so is again a complex surface. Note that since G_2 has rank two, there is a two-parameter family of choices for the orbit $\mathcal{C}_\infty$ and so we expect to obtain a two parameter family of surfaces in this way. $\square$

This yields our main question: what are the complex surfaces arising in the G_2 case?

Some further motivation for this question is as follows:

1) In [25] §3.2 there is a conjectural classification of the hyperkähler manifolds of real dimension four that arise in nonabelian Hodge theory. If this conjecture is true we should be able to locate the G_2 character varieties of Lemma 3.3.1 on the list of [25]. We do this here. (These complete hyperkähler manifolds are noncompact analogues of K3 surfaces.)

2) The article [24] introduced the notion of parahoric bundles (by defining the notion of weight for parahoric torsors), the notion of logarithmic connection on a parahoric bundle (aka "logahoric connections"), and established a precise Riemann–Hilbert correspondence for them. At first glance these seem to be quite exotic objects, and the G_2 example considered here is one of the simplest contexts where this Riemann–Hilbert correspondence is necessary. Our main result shows that the corresponding character varieties are in fact not at all exotic. The moduli spaces of such connections will be considered elsewhere.

3.4 Octonions and G_2

The compact exceptional simple Lie group G_2 arises as the group of automorphisms of the octonions. In this section we will describe the corresponding complex simple group $G_2(\mathbb{C})$, as the group of automorphisms of the complex octonions (or complex

Cayley algebra) and describe its unique semisimple conjugacy class $\mathcal{C} \subset G_2(\mathbb{C})$ of dimension six.

This conjugacy class is somewhat remarkable since there is not a six dimensional semisimple adjoint orbit $\mathcal{O} \subset \mathfrak{g}_2 = \mathrm{Lie}(G_2(\mathbb{C}))$.

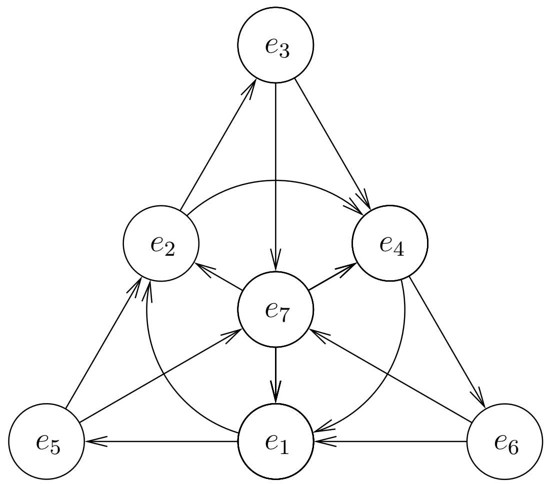

Figure 3.2: Points and lines in the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$

3.4.1 Complex octonions

Let $\mathbb{O} = \mathbb{O}(\mathbb{C})$ denote the 8 dimensional non-associative algebra with $\mathbb{C}$-basis the symbols

$$1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$$

and with multiplication table determined by the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ (and the fact that 1 is central), see Figure 3.2. Namely each triple i, j, k of vertices of an oriented line in the Fano plane (for example e_1, e_2, e_4) forms a quaternionic triple:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Thus for example $e_1 e_2 = e_4$. (Note, of course, that any two points lie on a unique line and so this determines the multiplication.) This multiplication table is symmetric under both of the two permutations

$$e_n \mapsto e_{n+1}$$

and

$$e_n \mapsto e_{2n}$$

of the indices (where all the indices are read modulo 7, and we prefer to write e_7 rather than e_0). This immediately implies that the triples of basis vectors with indices

$$124, 235, 346, 457, 561, 672, 713$$

form quaternionic triples, so it is easy to remember the labelling on the Fano plane. (Beware that some authors use a less symmetric multiplication table such that e_1, e_2, e_3 is a quaternionic triple; we follow [8, 34].)

Let $V \cong \mathbb{C}^7$ denote the complex span of the $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and let

$$\mathrm{Tr} : \mathbb{O} \to \mathbb{C}$$

denote the linear map with kernel V taking $1 \in \mathbb{O}$ to $1 \in \mathbb{C}$. Define an involution $q \mapsto \overline{q}$ of $\mathbb{O}$ to be the $\mathbb{C}$-linear map fixing 1 and acting as -1 on V. Then there is a nondegenerate symmetric $\mathbb{C}$-bilinear form on $\mathbb{O}$

$$\langle q_1, q_2 \rangle = \mathrm{Tr}(q_1 \cdot \overline{q}_2) \in \mathbb{C}.$$

Let $n(q) = \langle q, q \rangle \in \mathbb{C}$ denote the corresponding quadratic form and we will say that $n(q)$ is the norm of q. Note that if $v_1, v_2 \in V$ then $\langle v_1, v_2 \rangle = -\mathrm{Tr}(v_1 \cdot v_2)$.

The group $G_2(\mathbb{C})$ is the group of algebra automorphisms of $\mathbb{O}$. As such it acts trivially on 1 and faithfully on V, preserving the quadratic form, so there is an embedding

$$G_2(\mathbb{C}) \subset \mathrm{SO}(V, n) \cong \mathrm{SO}_7(\mathbb{C}).$$

Henceforth we will write $G = G_2(\mathbb{C})$ and think of it in this seven dimensional representation.

3.4.2 The six dimensional semisimple conjugacy class

Let $T \subset G$ be a maximal torus, so that $T \cong (\mathbb{C}^*)^2$. Any semisimple conjugacy class in G contains an element of T, and so to determine the dimension of the possible semisimple conjugacy classes it is sufficient to study the centralisers in G of the elements of T.

Of course any element $t \in T$ is the exponential of some element $X \in \mathfrak{t} = \mathrm{Lie}(T)$, but it is not always true that the centralisers of X and t are the same and so care is needed (this is essentially the phenomenon of resonance in the theory of linear differential equations). This has been analysed in detail by Kac and it is possible to determine the centraliser of t in terms of X (see Serre [84]). The result is that there is a unique semisimple conjugacy class of dimension six containing certain (special) order three elements of G. The centraliser of such elements is a copy of $\mathrm{SL}_3(\mathbb{C}) \subset G$, and so we see immediately that $\dim \mathcal{C} = \dim G/\mathrm{SL}_3(\mathbb{C}) = 14 - 8 = 6$.

In fact once we know (by the theorem of Borel–de Siebenthal or otherwise) that there is such an embedding of groups it is clear that the desired element of G generates the centre of $\mathrm{SL}_3(\mathbb{C})$, and so is of order 3.

We will skip the details of the above discussion since in terms of octonions we can be more explicit, as follows. First we will recall Zorn's proposition. Suppose $a \in \mathbb{O}$ is a nonzero octonion and let $T_a : \mathbb{O} \to \mathbb{O}$ denote the linear map

$$T_a(q) = a.q.a^{-1}$$

given by conjugation by a. (This is well-defined since $\mathbb{O}$ is associative on subalgebras generated by pairs of elements.) However non-associativity implies such maps are not always automorphisms of $\mathbb{O}$. Zorn characterised which maps T_a are automorphisms:

Proposition 3.4.1 (Zorn, see [34] p.98)**.** *$T_a \in G$ if and only if $a^3 \in \mathbb{C}.1$*

Thus suppose we take an "imaginary" octonion $v \in V$ of norm $n(v) = 3$. This means that v is a square root of -3, i.e. $v^2 = -3$. Then we can consider the element

$$a(v) = \frac{1+v}{2} \in \mathbb{O}. \tag{3.4}$$

By construction $a(v)^3 = -1$, (cf. the fact that $(1+\sqrt{-3})/2 = \exp(\pi\sqrt{-1}/3)$). Hence we have constructed an element

$$T_{a(v)} \in G = G_2(\mathbb{C})$$

and it is clear that it is of order three in G. The eigenvalues of $T_{a(v)} \in \mathrm{Aut}(V)$ are one (with multiplicity one) and the two nontrivial cube roots of one (each with multiplicity three).

Let $\mathcal{O} \subset V$ denote the set of elements of norm 3. The group G acts transitively on $\mathcal{O}$, so $\mathcal{O}$ is a single orbit of G (in the representation V).

Proposition 3.4.2. *The map $\mathcal{O} \to G = G_2(\mathbb{C})$ taking an element $v \in \mathcal{O}$ to $T_{a(v)} \in G$ is a G-equivariant isomorphism of $\mathcal{O}$ onto the six dimensional semisimple conjugacy class $\mathcal{C}$ in G.*

Proof. The G-equivariance is straightforward (where G acts on itself by conjugation). Thus $\mathcal{O}$ is mapped onto a single conjugacy class. To see the map is injective note that the eigenspace of $T_{a(v)} \in \mathrm{Aut}(V)$ with eigenvalue 1 is one-dimensional and spanned by v, so it is sufficient to check that $T_{a(v)} \neq T_{a(-v)}$, but this is clear since they are inverse to each other (and of order three). Note that $\mathcal{O}$ is the quadric hypersurface $n(v) = 3$ in V, so has dimension 6 □

Note in particular that if i, j, k is a quaternionic triple in $\mathbb{O}$ then $i + j + k \in \mathcal{O}$ and so we get an element $\frac{1}{2}(1 + i + j + k)$ of $\mathcal{C}$ for any line in the Fano plane. We will return to this in §3.8.

3.5 Some invariant theory for G_2

Our basic aim is to consider affine varieties obtained from the ring of $G = G_2(\mathbb{C})$ invariant functions on affine varieties of the form

$$\{(g_1, g_2, g_3, g_\infty) \bigm| g_1, g_2, g_3 \in \mathcal{C}, g_\infty \in \mathcal{C}_\infty, g_1 g_2 g_3 g_\infty = 1 \in G\} \tag{3.5}$$

where $\mathcal{C} \subset G$ is the six dimensional semisimple conjugacy class and $\mathcal{C}_\infty$ is one of the twelve dimensional semisimple classes, and G acts by diagonal conjugation.

Now a generic element t of the maximal torus $T \subset G$ will be a member of such a conjugacy class $\mathcal{C}_\infty$, and two such elements $t_1, t_2 \in T$ are in the same class if and only if they are in an orbit of the action of the Weyl group $W = N_G(T)/T$ on T.

The Weyl group W of G is a dihedral group of order 12 and its action on $T \cong (\mathbb{C}^*)^2$ is well understood. In brief there is a basis of V such that T is represented by diagonal matrices of the form

$$t = \mathrm{diag}(1, a_1, a_2, (a_1a_2)^{-1}, a_1^{-1}, a_2^{-1}, a_1a_2) \in \mathrm{GL}(V)$$

for elements $a_1, a_2 \in \mathbb{C}^*$. The action of W on T is generated by the two reflections

$$r_1(a_1, a_2) = (a_1^{-1}, a_1a_2), \qquad r_2(a_1, a_2) = (a_2, a_1)$$

fixing the hypertori $a_1 = 1$ and $a_1 = a_2$ respectively.

Lemma 3.5.1 ([?] p.60)**.** *The ring of W-invariant functions on T is generated by the two functions*

$$\alpha = a_1 + 1/a_1 + a_2 + 1/a_2 + a_1a_2 + 1/(a_1a_2)$$

$$\beta = a_1/a_2 + a_2/a_1 + a_1^2a_2 + a_1a_2^2 + 1/(a_1^2a_2) + 1/(a_1a_2^2).$$

Note that if $t \in T$ is represented as a diagonal matrix as above then

$$\alpha = \mathrm{Tr}_V(t) - 1, \qquad 2\beta = \alpha^2 - 2\alpha - \mathrm{Tr}_V(t^2) - 5$$

so that specifying the values of α, β is equivalent to specifying the values of $\mathrm{Tr}_V(t)$ and $\mathrm{Tr}_V(t^2)$. Of course the functions $\mathrm{Tr}_V(t), \mathrm{Tr}_V(t^2)$ are just the restrictions of the functions $\mathrm{Tr}_V(g), \mathrm{Tr}_V(g^2)$ defined on $G \subset \mathrm{GL}(V)$, and so (in this way) we can just as well view α and β as invariant functions on G. (The resulting formulae are simpler if we work with β rather than $\mathrm{Tr}_V(g^2)$.)

Thus we may rephrase our main question differently: suppose we consider the affine variety

$$\mathbb{M} := \mathcal{C}^3/G$$

associated to the ring of G-invariant functions on $\mathcal{C}^3$ (where G acts by diagonal conjugation). Then the affine varieties

$$\{(g_1, g_2, g_3, g_\infty) \bigm| g_1, g_2, g_3 \in \mathcal{C}, g_\infty \in \mathcal{C}_\infty, g_1g_2g_3g_\infty = 1 \in G\}/G$$

(associated to the invariant functions on (3.5)) arise as fibres of the map

$$\pi : \mathbb{M} \to \mathbb{C}^2; \quad [(g_1, g_2, g_3)] \mapsto (\alpha(g_1g_2g_3), \beta(g_1g_2g_3)),$$

since fixing (sufficiently generic) values of the map π will fix the conjugacy class of the product $g_1g_2g_3$, as required. Thus as a first step we need to understand the affine variety $\mathbb{M} = \mathcal{C}^3/G$ and secondly we need to understand the map π from $\mathbb{M}$ to $\mathbb{C}^2$.

Proposition 3.5.2. *The affine variety $\mathbb{M} = \mathcal{C}^3/G$ is isomorphic to an affine space of dimension four (more precisely the ring of G invariant functions on $\mathcal{C}^3$ is a polynomial algebra in four variables).*

Proof. Via Proposition 3.4.2 the affine variety $\mathcal{C}^3$ is G-equivariantly isomorphic to $\mathcal{O}^3$ where $\mathcal{O} \subset V$ is the affine quadric of "imaginary" octonions of norm 3.

In other words $\mathcal{O}^3$ is the subset of $(v_1, v_2, v_3) \in V^3$ such that $n(v_i) = 3$ for $i = 1, 2, 3$. Thus we can first consider the quotient V^3/G.

Now G. Schwarz has classified all the representations of simple algebraic groups such that the ring of invariants is a polynomial algebra [82]. Row 1 of Table 5a of [82] (see also [83]) says that the ring $\mathbb{C}[V^3]^G$ is a polynomial algebra with 7 generators, and the generators can be taken to be the invariant functions:

$$\begin{gathered} n(v_1), \quad n(v_2), \quad n(v_3) \\ p_1 = \langle v_2, v_3 \rangle, \quad p_2 = \langle v_1, v_3 \rangle, \quad p_3 = \langle v_1, v_2 \rangle, \\ p_4 = \langle v_1, v_2.v_3 \rangle. \end{gathered} \tag{3.6}$$

We are interested in the algebra obtained by fixing the values of the first three invariants to be 3, which is thus a polynomial algebra generated by the remaining four functions p_1, p_2, p_3, p_4. □

In other words the map $p = (p_1, p_2, p_3, p_4) : \mathbb{M} \to \mathbb{C}^4$, with components the four functions p_i, identifies $\mathbb{M}$ with $\mathbb{C}^4$. Thus the question now is to understand the map $\pi : \mathbb{M} \to \mathbb{C}^2$ in terms of the functions p_i. The key formulae are the following, which may be verified symbolically.

Theorem 3.5.3. *Suppose $v_1, v_2, v_3 \in \mathcal{O} \subset V \subset \mathbb{O}$ are octonions in V with norm 3, and $g_i = T_{a(v_i)} \in \mathcal{C} \subset G$ $(i = 1, 2, 3)$ are the corresponding elements in the six dimensional semisimple conjugacy class in $G = G_2(\mathbb{C})$. Then the invariant functions*

$$\alpha = \alpha(g_1 g_2 g_3), \qquad \beta = \beta(g_1 g_2 g_3)$$

of the product $g_1 g_2 g_3 \in G$ are related to the invariants

$$p_1 = \langle v_2, v_3 \rangle, \quad p_2 = \langle v_1, v_3 \rangle, \quad p_3 = \langle v_1, v_2 \rangle, \quad p_4 = \langle v_1, v_2.v_3 \rangle$$

of the octonions v_1, v_2, v_3 by the following formulae:

$$8\,\alpha = p_4\, s_1 - {s_1}^2 + 3\, p_4 + 3\, s_1 + 3\, s_2 + s_3 - 6,$$

$$\begin{aligned} 64\,\beta = &-{p_4}^3 + 3\,{p_4}^2 s_1 - 3\, p_4\, {s_1}^2 - 7\,{s_1}^3 + 9\,{p_4}^2 - 12\, p_4\, s_1 + 18\, p_4\, s_2 + 6\, p_4\, s_3 \\ &+ 39\,{s_1}^2 + 18\, s_1\, s_2 + 6\, s_1\, s_3 - 9\, p_4 - 9\, s_1 - 90\, s_2 - 30\, s_3 - 183, \end{aligned}$$

where $s_1 = p_1 + p_2 + p_3,\ s_2 = p_1 p_2 + p_2 p_3 + p_3 p_1,\ s_3 = p_1 p_2 p_3$.

Now we will explain how the symmetric Fricke cubics arise. Fix constants $k_\alpha, k_\beta \in \mathbb{C}$ and consider the subvariety of $\mathbb{M} \cong \mathbb{C}^4$ cut out by the equations

$$\alpha = k_\alpha, \qquad \beta = k_\beta$$

where α, β are viewed as functions on $\mathbb{M}$ via the above formulae.

First suppose we change coordinates on $\mathbb{M}$ by replacing p_4 by the function b defined so that

$$p_1 + p_2 + p_3 + p_4 = 4b + 5. \tag{3.7}$$

Then, in the presence of the relation $\alpha = k_\alpha$, the equation $\beta = k_\beta$ simplifies to the equation:

$$b^3 + 6\,b^2 - 3\,(k_\alpha - 1)\,b + k_\beta + 2 = 0. \tag{3.8}$$

This is a cubic equation for b and so specifying k_α, k_β determines b up to three choices.

Now we can reconsider the remaining equation $\alpha = k_\alpha$, which is easily seen to be a symmetric Fricke cubic: expanding the symmetric functions and making the substitutions

$$p_1 = 1 - 2x, \quad p_2 = 1 - 2y, \quad p_3 = 1 - 2z \tag{3.9}$$

converts the equation $\alpha = k_\alpha$ into the Fricke cubic

$$x\,y\,z + x^2 + y^2 + z^2 + b\,(x + y + z) + c = 0 \tag{3.10}$$

where

$$c = k_\alpha - 2 - 3\,b. \tag{3.11}$$

The situation may be summarised in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{B}}(\Sigma, G_2(\mathbb{C}), \boldsymbol{\mathcal{C}}) \subset \mathbb{M} \arrow[r, leftrightarrow, "\varphi", "\cong"'] \arrow[d, "\pi"'] \arrow[dr, dashed] & \mathbb{C}^4 \ni (x, y, z, b) \arrow[d, "\widehat{\pi}"] \\
\underset{\alpha, \beta}{\mathbb{C}^2} & \underset{b, c}{\mathbb{C}^2} \arrow[l, "\mathrm{pr}"', "3:1"]
\end{tikzcd}$$

Figure 3.3: Main diagram

Here, in the top left corner, $\mathbb{M} = \mathcal{C}^3/G_2(\mathbb{C})$ which we identified with $\mathbb{C}^4$ using the functions p_1, p_2, p_3, p_4 in Proposition 3.5.2. The map π on the left is given by the invariant functions α, β, expressed as explicit functions on $\mathbb{M}$ as in Theorem 3.5.3. The fibre of the map π over a generic point $(k_\alpha, k_\beta) \in \mathbb{C}^2$ is the $G_2(\mathbb{C})$ character variety $\mathcal{M}_{\mathrm{B}}(\Sigma, G_2(\mathbb{C}), \boldsymbol{\mathcal{C}})$ where $\boldsymbol{\mathcal{C}} = (\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C}_\infty)$ with $\mathcal{C}_\infty$ the twelve dimensional conjugacy class in $G_2(\mathbb{C})$ with eigenvalues having invariants $\alpha = k_\alpha, \beta = k_\beta$.

On the right we consider a universal family of symmetric Fricke cubics: we take a copy of $\mathbb{C}^4$ with coordinates x, y, z, b and consider the map $\widehat{\pi}$ to $\mathbb{C}^2$ given by (b, c), where c is viewed as a function of x, y, z, b via the Fricke equation (3.10). The fibre of $\widehat{\pi}$ over a point $(b, c) \in \mathbb{C}^2$ is the symmetric Fricke cubic (3.10) with these values of b, c.

The isomorphism φ along the top is defined by the equations (3.7),(3.9) expressing x, y, z, b in terms of p_1, p_2, p_3, p_4.

The map pr along the bottom is given by the equations (3.8),(3.11) expressing the values k_α, k_β of α, β in terms of b, c:

$$\alpha = c + 2 + 3\,b, \qquad \beta = -b^3 + 3\,b^2 + 3\,bc + 3\,b - 2. \tag{3.12}$$

This is a finite surjective map, with generic fibres consisting of three points. In fact by examining the discriminants we can be more precise.

3.5.1 Discriminants

First we can consider the discriminant $\mathcal{D}_{\mathrm{pr}} \subset \mathbb{C}^2_{\alpha,\beta}$ of the cubic polynomial (3.8), characterising when the fibres of pr have less than 3 elements. This discriminant locus (with α, β replacing k_α, k_β resp.) is:

$$4\alpha^3 - 12\alpha\beta - \beta^2 - 36\alpha - 24\beta - 36 = 0. \tag{3.13}$$

Secondly we can consider the discriminant $\mathcal{D}_W \subset \mathbb{C}^2_{\alpha,\beta}$ of the quotient map $(\alpha, \beta) : T \to T/W \cong \mathbb{C}^2$. This discriminant is the image of the 6 "reflection hypertori" in T and is the subvariety of $\mathbb{C}^2$ for which the fibres of this quotient map have less than $12 = \#W$ points. Explicitly it is cut out by the square of the Weyl denominator function, which ([46] p.413) is the following W-invariant function on T:

$$\big((a_1 - a_2)(a_2 - a_3)(a_3 - a_1)\big)^2 \times (a_1a_2 + a_2a_3 + a_3a_1 - a_1 - a_2 - a_3)^2$$

where $a_3 = 1/(a_1a_2)$. The two factors here correspond to the long and the short roots of G_2. If we now rewrite these two factors in terms of the basic invariants α, β we find that $\mathcal{D}_W = \mathcal{D}_1 \cup \mathcal{D}_2$ has two irreducible components corresponding to the two factors, and that the first component is equal to (3.13), the discriminant locus $\mathcal{D}_{\mathrm{pr}}$ of the map pr. The other irreducible component $\mathcal{D}_2$ of $\mathcal{D}_W$ is given by

$$\alpha^2 - 4\,\beta - 12 = 0. \tag{3.14}$$

Since $\mathcal{D}_{\mathrm{pr}} \subset \mathcal{D}_W$ this implies that: *if $\mathcal{C}_\infty$ has dimension twelve then the corresponding values k_α, k_β of the invariants α, β are not on the discriminant* (3.13)*, and so the fibre* $\mathrm{pr}^{-1}(k_\alpha, k_\beta)$ *has exactly three points.*

In turn we can relate this to the locus of *singular* symmetric Fricke cubics surfaces.

3.5.2 Singular points

Given $b, c \in \mathbb{C}$ the corresponding symmetric Fricke surface (3.10) is singular if and only if the derivatives of the defining equation all vanish, i.e. if there is a simultaneous solution (x, y, z) of (3.10) and the three equations

$$xy + 2z + b = 0, \qquad yz + 2x + b = 0, \qquad xz + 2y + b = 0.$$

Eliminating two variables and considering the resultant of the remaining two equations we find:

Lemma 3.5.4. *The symmetric Fricke surface* (3.10) *determined by* $b, c \in \mathbb{C}$ *is singular if and only if*

$$\left(b^2 - 8\,b - 4\,c - 16\right)\left(4\,b^3 - 3\,b^2 - 6\,bc + c^2 + 4\,c\right) = 0. \tag{3.15}$$

Now we can consider the image of this singular locus $\mathcal{D}_{sing} \subset \mathbb{C}^2_{b,c}$ under the map $\mathrm{pr} : \mathbb{C}^2_{b,c} \to \mathbb{C}^2_{\alpha,\beta}$.

Proposition 3.5.5. *The singular locus* $\mathcal{D}_{sing} \subset \mathbb{C}^2_{b,c}$ *maps onto the discriminant locus* $\mathcal{D}_W \subset \mathbb{C}^2_{b,c}$. *Specifically the first (left-hand) irreducible component of* $\mathcal{D}_{sing}$ *maps onto the first component* $\mathcal{D}_1 = \mathcal{D}_{\mathrm{pr}}$ *of* $\mathcal{D}_W$ *given in* (3.13)*, and the second component of* $\mathcal{D}_{sing}$ *maps onto the second component* $\mathcal{D}_2$ *of* $\mathcal{D}_W$*, given in* (3.14)*.*

Note however that the inverse image $\mathrm{pr}^{-1}(\mathcal{D}_1)$ has another irreducible component

$$b^2 + b - c - 1 = 0 \tag{3.16}$$

(with multiplicity two), besides the first component of the singular locus (3.15).

Consequently we see that:

Corollary 3.5.6. *Let* $\mathcal{C} \subset G_2(\mathbb{C})$ *be the six dimensional semisimple conjugacy class. Then for any regular semisimple conjugacy class* $\mathcal{C}_\infty \subset G_2(\mathbb{C})$ *the character variety*

$$\mathcal{M}_{\mathrm{B}}(\Sigma, G_2(\mathbb{C}), \boldsymbol{\mathcal{C}})$$

with $\boldsymbol{\mathcal{C}} = (\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C}_\infty)$*, has three connected components, each of which is isomorphic to a smooth symmetric Fricke cubic surface.*

Proof. Such a character variety is the fibre of the map π over a point $(k_\alpha, k_\beta) \in \mathbb{C}^2 \setminus \mathcal{D}_W$. Since π factors through pr (going around the square, i.e. via the dashed diagonal map), and $\mathcal{D}_{\mathrm{pr}} \subset \mathcal{D}_W$, such fibres consist of three fibres of $\widehat{\pi}$ over the smooth locus $\mathbb{C}^2_{b,c} \setminus \mathcal{D}_{sing}$. These fibres are smooth symmetric Fricke cubic surfaces. $\square$

Note in particular this implies we can canonically associate two other smooth cubic surfaces to any smooth symmetric Fricke cubic surface with parameters b, c not on the conic (3.16), namely the other two components of $\pi^{-1}(\mathrm{pr}(b,c)) \subset \mathbb{M}$.

Note also that in [18] the map relating the $\mathrm{GL}_3(\mathbb{C})$ character varieties to the Fricke cubics (as in part 2 of Lemma 3.3.1 above) was derived from earlier work on the Fourier–Laplace transform, whereas here we have identified the character varieties directly; we do not (yet) understand if there is a G_2 analogue/extension of Fourier–Laplace.

In the next section we will consider the natural braid group actions and show that the isomorphism φ is braid group equivariant.

Remark 3.5.7. Suppose instead we replace $\mathcal{C}_\infty$ by the closure of the regular unipotent conjugacy class in the definition of $\mathcal{M}_{\mathrm{B}}(\Sigma, G_2(\mathbb{C}), \boldsymbol{\mathcal{C}})$. The resulting variety is the fibre of π over the point $\alpha = \beta = 6$, and we readily see it has two components: the Fricke cubic $(b,c) = (-8, 28)$ with the D_4 singularity, and (with multiplicity two) the Fricke cubic with $(b,c) = (1,1)$. This surface is also singular; in fact these parameters lie at the cusp of the cuspidal cubic on the right in the singular locus (3.15).

Remark 3.5.8. The first irreducible component of the singular locus (3.15) could be called the "very symmetric locus" of Fricke cubics, since in the original Fricke–Vogt $\mathrm{SL}_2(\mathbb{C})$ picture it corresponds to the case where *all* the local monodromies are conjugate: $m_1 = m_2 = m_3 = m_4$ so that, by the formulae (3.3):

$$b = -2m^2, \qquad c = m^4 - 4 + 4m^2$$

and thus $b^2 - 8b = 4c + 16$. Although all these surfaces are singular this case has many applications, for example to anti-self-dual four-manifolds ([52] Theorem 3).

Remark 3.5.9. This very symmetric locus is closely related to the one-parameter family $b = 0$ (which does admit smooth members, so is related to G_2 via Corollary 3.5.6). This family is the family of $\mathrm{SL}_2(\mathbb{C})$-character varieties for the once-punctured torus ([49] p.584, or [94, 67]). The basic statement relating this case to the very symmetric case is that there is a (degree four) ramified covering map between the two families, as follows (cf. [18] Remark 14). Suppose $b \in \mathbb{C}$ and $c = b^2/4 - 2b - 4$ so that the polynomial

$$f = xyz + x^2 + y^2 + z^2 + b(x + y + z) + c$$

defines a very symmetric Fricke cubic. If we define $d = -4 - b/2$ and consider the polynomial

$$g = XYZ + X^2 + Y^2 + Z^2 + d$$

then, if the variables are related by $x = 2 - X^2, y = 2 - Y^2, z = 2 - Z^2$, the relation

$$f(x,y,z) = g(X,Y,Z)g(-X,Y,Z)$$

holds, so the surface $g = 0$ maps to the surface $f = 0$, and the generic fibre contains 4 points. Note that if $b = 0$ then $c = d = -4$ so this is an endomorphism of the Cayley cubic surface.

3.6 Braid group actions

Since the symmetric Fricke cubics are character varieties they admit braid group actions, coming from the mapping class group of the four-punctured sphere Σ. In explicit terms (cf. [19] §4) this can be expressed in terms of changing the choice of loops generating the fundamental group of Σ, whence it becomes the classical Hurwitz braid group action:

Lemma 3.6.1. *Let G be a group. Then there is an action of the three string Artin braid group B_3 on G^3, generated by the two operations β_1, β_2 where:*

$$\beta_1(g_1, g_2, g_3) = (g_2, g_2^{-1} g_1 g_2, g_3) \tag{3.17}$$

$$\beta_2(g_1, g_2, g_3) = (g_1, g_3, g_3^{-1} g_2 g_3). \tag{3.18}$$

Taking G to be a complex reductive group, this describes the B_3 action on the character variety $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$, and given four conjugacy classes $\boldsymbol{\mathcal{C}} \subset G^4$ this action restricts to the symplectic leaves

$$\mathcal{M}_{\mathrm{B}}(\Sigma, G, \boldsymbol{\mathcal{C}}) \cong \{(g_1, g_2, g_3, g_\infty) \bigm| g_i \in \mathcal{C}_i, g_1 g_2 g_3 g_\infty = 1\}/G$$

provided the first three conjugacy classes are equal.

In the case $G = \mathrm{SL}_2(\mathbb{C})$ it is easy to compute the resulting action on the G-invariant functions on $\mathrm{Hom}(\pi_1(\Sigma), G)$ and in turn on the family of Fricke surfaces [55, 18]. In the symmetric case, for fixed constants $b, c \in \mathbb{C}$ the formula is as follows:

$$\beta_1(x, y, z) = (x, -b - z - xy, y), \qquad \beta_2(x, y, z) = (z, y, -b - x - yz). \tag{3.19}$$

The main aim of this section is to compute directly the action in the case $G = G_2(\mathbb{C})$ we have been studying. Let $\mathcal{C} \subset G_2(\mathbb{C})$ denote the six dimensional semisimple conjugacy class, and let $\mathcal{O} \subset V$ denote the orbit of elements of norm 3.

Proposition 3.6.2. *1) The braid group action* (3.17),(3.18) *on triples $(g_1, g_2, g_3) \in \mathcal{C}^3$ of elements of $\mathcal{C}$ corresponds to the action*

$$\beta_1(v_1, v_2, v_3) = (v_2, \overline{w}_2 \cdot v_1 \cdot w_2, v_3),$$

$$\beta_2(v_1, v_2, v_3) = (v_1, v_3, \overline{w}_3 \cdot v_2 \cdot w_3)$$

on triples of elements $(v_1, v_2, v_3) \in \mathcal{O}^3 \subset V^3 \subset \mathbb{O}^3$, via the isomorphism $\mathcal{C}^3 \cong \mathcal{O}^3$ of Proposition 3.4.2, where $w_i = (1 + v_i)/2 \in \mathbb{O}$ (so that $g_i = T_{w_i}$).

2) The resulting B_3 action on $\mathbb{M} = \mathcal{C}^3/G \cong \mathbb{C}^4$ is given by the formulae:

$$\beta_1(p_1, p_2, p_3, p_4) = ((p_4 + p_1 p_3 - p_2)/2, p_1, p_3, (p_4 + 3p_2 - p_1 p_3)/2),$$

$$\beta_2(p_1, p_2, p_3, p_4) = (p_1, (p_4 + p_1 p_2 - p_3)/2, p_2, (p_4 + 3p_3 - p_1 p_2)/2)$$

in terms of the invariant functions p_1, p_2, p_3, p_4 defined in (3.6).

Proof. 1) is straightforward (taking care due to the non-associativity of $\mathbb{O}$), and can be verified symbolically.

2) is now a nice exercise in expanding G-invariant functions on $\mathcal{O}^3$ in terms of the basic invariants p_i. To illustrate this we will show $\langle v_3, (v_2v_1)v_2\rangle = 3p_2 - 2p_1p_3$. If $q \in \mathbb{O}$ we will write $q = \mathrm{Tr}(q) + \mathrm{Im}(q)$ with $\mathrm{Im}(q) \in V$ and $\mathrm{Tr}(q) \in \mathbb{C}.1$, and note that $\langle v_1, v_2v_3\rangle$ is a skew-symmetric 3-form if $v_1, v_2, v_3 \in V$. Then we have

$$\begin{aligned}\langle v_3, (v_2v_1)v_2\rangle &= \langle v_3, \mathrm{Tr}(v_2v_1)v_2\rangle + \langle v_3, \mathrm{Im}(v_2v_1)v_2\rangle \\ &= -p_1p_3 - \langle v_3, v_2\,\mathrm{Im}(v_2v_1)\rangle \\ &= -p_1p_3 - \langle v_3, v_2(v_2v_1)\rangle + \langle v_3, v_2\mathrm{Tr}(v_2v_1)\rangle \\ &= -p_1p_3 + 3p_2 - p_1p_3.\end{aligned}$$

The stated formulae can all be derived in this way. □

Observe that the sum $p_1 + p_2 + p_3 + p_4$ is preserved by this braid group action. This corresponds to the fact that b is preserved by the action on Fricke surfaces, and in fact there is a complete correspondence:

Corollary 3.6.3. *The isomorphism φ in Figure 3.3 identifying $\mathbb{M} = \mathcal{C}^3/G$ with the universal family of symmetric Fricke cubic surfaces, is braid group equivariant.*

Proof. Given the explicit formulae (in (3.19) and Proposition 3.6.2) for the braid group actions, this follows directly from the formulae (3.7),(3.9) defining φ. □

3.7 Triality

The fact that the full family of Fricke cubics is a semiuniversal deformation of a D_4 singularity is not just a coincidence, and has deep moduli theoretic meaning[3]. In brief the whole family of these Fricke moduli spaces (in fact the whole "hyperkähler nonabelian Hodge structure") is naturally attached to the D_4 root system (or more precisely to the affine D_4 root system, but that is determined by the finite root system). Note that the D_4 Dynkin diagram is the most symmetric (finite) Dynkin diagram since it has an automorphism of order three, the triality automorphism, indicated in Figure 3.1.

There are various ways to see the appearance of D_4 from the moduli space of rank 2 representations of the fundamental group of the four-punctured sphere Σ:

1) Translating to our notation, Okamoto [77] found that if we label the eigenvalues of $M_i \in \mathrm{SL}_2(\mathbb{C})$ as $\exp(\pm\pi\sqrt{-1}\theta_i)$ where $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{C}^4$, then this copy of $\mathbb{C}^4$

[3] here we mean cubic surfaces as moduli spaces of local systems—see [74] for more on the direct relation between D_4 and the moduli of cubic surfaces themselves.

should be viewed as the Cartan subalgebra of type D_4: he showed there is a natural action of the affine D_4 Weyl group on this $\mathbb{C}^4$, and it lifts to automorphisms of the corresponding moduli spaces of rank two logarithmic connections on Σ fibred over $\mathbb{C}^4$ (at least off of the affine root hyperplanes—cf. [3]). Further Okamoto showed one can add in the automorphisms of the affine D_4 Dynkin diagram, to obtain an action of $\mathrm{Sym}_4 \ltimes W_{\mathrm{aff}}(D_4)$ (which is isomorphic to the affine F_4 Weyl group).

2) The moduli spaces of rank two logarithmic connections on Σ mentioned in 1) above have simple open pieces $\mathcal{M}^*$ (where the underlying vector bundle on $\mathbb{P}^1$ is holomorphically trivial) and these open pieces are isomorphic to the D_4 asymptotically locally Euclidean hyperkähler four-manifolds constructed by Kronheimer [61]. In fact Kronheimer constructs these spaces as a finite dimensional hyperkähler quotient starting with a vector space of linear maps in both directions along the edges of the affine D_4 Dynkin graph (an early example of a "quiver variety").

3) The space of representations of the fundamental group of Σ is closely related to the perverse sheaves on $\mathbb{P}^1$ with singularities at the marked points, and such perverse sheaves have a well-known quiver description, again in terms of linear maps in both directions along the edges of an affine D_4 Dynkin graph. This leads to the statement that the Fricke cubic surfaces are affine D_4 "multiplicative quiver varieties", in the sense of [36, 97].

Note that each of these points of view has an extension to many other moduli spaces, often of higher dimensions (cf. [23, 26]).

In any case, whichever is the reader's preferred viewpoint, it is natural to consider the action of the triality automorphism on the space $\mathbb{C}^4$ of parameters θ (and the induced action on the Fricke coefficients b_1, b_2, b_3, c). It is well-known that the root system of G_2 arises by 'folding' the D_4 root system via the triality automorphism in this way, as indicated in Figure 3.1. It turns out that the fixed locus is the space of symmetric Fricke cubics, so there is another (a priori different) link between symmetric Fricke cubics and G_2:

Proposition 3.7.1. *The action of the triality automorphism on $\theta \in \mathbb{C}^4$ permutes $\theta_1, \theta_2, \theta_3$ cyclically and fixes θ_4. The fixed locus $\theta_1 = \theta_2 = \theta_3$ maps onto the parameter space $b_1 = b_2 = b_3$ of the symmetric Fricke cubic surfaces.*

Proof. Note that the triality automorphism is only well defined upto conjugation by an inner automorphism, so we have some freedom. The key point is that, in Okamoto's setup, one should not use the standard (Bourbaki) convention for the D_4 root system, rather, as explained in [19] Remark 5, it naturally appears as the set of *short* F_4 roots, i.e. as the set of 24 norm one vectors:

$$\theta \in D_4^- = \left\{ \pm\varepsilon_1, \quad \pm\varepsilon_2, \quad \pm\varepsilon_3, \quad \pm\varepsilon_4, \quad \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \right\} \subset \mathbb{C}^4, \tag{3.20}$$

where ε_i are the standard (orthonormal) basis vectors of $\mathbb{C}^4$ (i.e. as the group of unit

Hurwitz integral quaternions). Then we just note that the vectors

$$\alpha_1 = \varepsilon_1, \quad \alpha_2 = \varepsilon_2, \quad \alpha_3 = \varepsilon_3, \quad \alpha_4 = \frac{1}{2}(\varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$$

form a basis of simple roots, and the longest root is $\varepsilon_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$ and the mutual inner products of these simple roots are as indicated in the D_4 Dynkin diagram in Figure 3.1. Thus the triality automorphism shown in Figure 3.1 acts as

$$\tau(\theta_1, \theta_2, \theta_3, \theta_4) \ = \ (\theta_3, \theta_1, \theta_2, \theta_4)$$

and the fixed point locus is indeed $\theta_1 = \theta_2 = \theta_3$. Using the formulae (3.2) it is clear that this maps to the locus of symmetric Fricke cubics. □

Remark 3.7.2. In fact the Fricke functions b_1, b_2, b_3, c of θ have a direct Lie-theoretic interpretation, as follows (this a minor modification of [75] p.888-9, adjusting the weight, and thus coroot, lattice): Let $\mathfrak{t} \cong \mathbb{C}^4$ be the D_4^- Cartan subalgebra. Then the "Fricke map" (3.2),(3.3):

$$\mathfrak{t} \to \mathbb{C}^4; \qquad \theta = \sum \theta_i \varepsilon_i \mapsto (b_1, b_2, b_3, c)$$

is the quotient by the D_4^- affine Weyl group (the semidirect product of the finite Weyl group W and the coroot lattice Γ_R). Indeed with these conventions (cf. also [20]) $\Gamma_R = \langle 2D_4^- \rangle \subset \mathfrak{t}$ and the quotient $\mathfrak{t}/\Gamma_R$ is a maximal torus $T_{\mathrm{Spin}(8)}$ of the simply connected group $\mathrm{Spin}_8(\mathbb{C})$ of type D_4. In turn it is well known that the W-invariant functions on $T_{\mathrm{Spin}(8)}$ form a polynomial algebra generated by the fundamental weights ([46] p.376), and these weights D_1, D_2, D^+, D^- are described in [46] (23.30): a straightforward computation then shows that

$$D_1 = -b_1, \quad D_2 = c, \quad D^+ = -b_3, \quad D^- = -b_2$$

so it is clear that permuting the b_i corresponds to triality, permuting the standard representation and the two half-spin representations of $\mathrm{Spin}_8(\mathbb{C})$.

Note that Manin [69] §1.6 considered "Landin transforms" in this context. They are defined on a distinguished two-dimensional subspace of the full space $\mathbb{C}^4$ of parameters: this looks to be different (and inequivalent) to the symmetric subspace we are considering (it looks to be the fixed locus arising from the involution of the affine D_4 Dynkin diagram swapping two pairs of feet, rather than the triality automorphism).

3.8 The Klein Cubic Surface

The article [18] found that there was a Fricke cubic surface containing a braid group orbit of size seven. This arose by considering Klein's simple group of order 168 =

$2^3 \cdot 3 \cdot 7$, the group of automorphisms of Klein's quartic curve (the modular curve $X(7)$):

$$X^3Y + Y^3Z + Z^3X = 0,$$

and then taking the corresponding complex reflection group in $\mathrm{GL}_3(\mathbb{C})$, of order 336. The braid group orbit of the conjugacy class of the standard triple of generating reflections of this complex reflection group has size 7 and lives in a character variety of dimension two (as in part 2 of Lemma 3.3.1 above). Then using the Fourier–Laplace transform it was shown how to relate this $\mathrm{GL}_3(\mathbb{C})$ character variety to the usual $\mathrm{SL}_2(\mathbb{C})$ Fricke–Vogt story, i.e. to show that it is a Fricke cubic surface. The resulting surface has

$$m_1 = m_2 = m_3 = 2\cos(2\pi/7), \quad m_4 = 2\cos(4\pi/7)$$

([18] p.177) and the corresponding $\mathrm{SL}_2(\mathbb{C})$ monodromy group is a lift to $\mathrm{SL}_2(\mathbb{C})$ of the (infinite) $2,3,7$ triangle group $\Delta_{237} \subset \mathrm{PSL}_2(\mathbb{C})$ (cf. [22] Appendix B)[4]. Thus from the formulae (3.2), and the fact that $4\cos(2\pi/7)\cos(4\pi/7) + 4\cos(2\pi/7)^2 = 1$, we see that the *Klein cubic surface* is:

$$x\,y\,z + x^2 + y^2 + z^2 = x + y + z \tag{3.21}$$

i.e. it is the (smooth) symmetric Fricke cubic with $b = -1, c = 0$. The braid orbit of size seven consists of the points:

$$(x, y, z) = (0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (1,1,0),\ (1,0,1),\ (0,1,1).$$

Thus, even though we started with a group involving lots of seventh roots of unity, the resulting cubic surface and braid orbit only involve the integers 0 and 1.

This braid orbit turns out to have a very nice realisation from the G_2 point of view (which is perhaps not surprising given the strong link to the number 7):

Suppose we take three lines passing through a single point in the Fano plane (see Figure 3.2). For example the three lines:

$$(e_1, e_3, e_7), \qquad (e_2, e_6, e_7) \qquad (e_4, e_5, e_7)$$

through e_7. Then we obtain three imaginary octonions of norm 3:

$$v_1 = e_1 + e_3 + e_7, \qquad v_2 = e_2 + e_6 + e_7, \qquad v_3 = e_4 + e_5 + e_7$$

and, via Proposition 3.4.2, we can consider the corresponding elements of the six dimensional semisimple conjugacy class $\mathcal{C} \subset G_2$, obtained by conjugating by $a(v_i) = (1 + v_i)/2$ respectively:

$$g_1 = T_{a(v_1)}, \qquad g_2 = T_{a(v_2)}, \qquad g_3 = T_{a(v_3)} \quad \in \mathcal{C} \subset G_2.$$

[4] in fact there are three inequivalent choices of seventh root of unity that one can make: for two choices the projective monodromy group is a subgroup of PSU_2 isomorphic to Δ_{237} and for the other choice one obtains the usual $\Delta_{237} \subset \mathrm{PSL}_2(\mathbb{R})$.

Note that the fact that we have chosen three lines passing through a single point implies that the three elements $(1+v_i)/2$ are all *octavian integers* (i.e. they lie in the same maximal order, the 7-integers, cf. [34] §9.2). The octavian integers form a copy of the E_8 root lattice (rescaled so roots have norm 1) and our three elements $(1+v_i)/2$ are amongst the 240 units and so correspond to E_8 roots. This implies that with respect to a $\mathbb{Z}$-basis of the octavian integers the elements g_1, g_2, g_3 are represented by matrices with $\mathbb{Z}$ entries. In fact they generate a finite simple group, and braid to give the Klein orbit of size seven:

Theorem 3.8.1. *1) The elements $g_1, g_2, g_3 \in G_2(\mathbb{C})$ obtained from three lines passing through a single point in the Fano plane, generate a finite subgroup of $G_2(\mathbb{C})$ isomorphic to the finite simple group $G_2(2)' \cong U_3(3)$ of order 6048,*

2) the braid group orbit of the conjugacy class of (g_1, g_2, g_3) in $\mathcal{C}^3/G = \mathbb{M}$ is of size seven and lives in a $G_2(\mathbb{C})$ character variety isomorphic to the Klein cubic surface.

Proof. For 1), by construction we obtain a subgroup of the full automorphism group ($\cong G_2(2)$) of the ring of octavian integers. Computing the order shows it is the index two subgroup $\cong G_2(2)'$. (This was actually our starting point, using Griess's tables [50] to see that $G_2(2)'$ contains elements of the six dimensional class $\mathcal{C} \subset G_2(\mathbb{C})$.)

2) By construction $\langle v_i, v_j \rangle = 1$ if $i \neq j$ so that $p_1 = p_2 = p_3 = 1$ and also we compute $p_4 = \langle v_1, v_2 \cdot v_3 \rangle = -2$. Thus from (3.7) and (3.9) $b = -1$ and $x = y = z = 0$ so that in turn (by (3.10)) $c = 0$. Since these parameters are off of all the discriminants we get an isomorphism from the corresponding component of the G_2 character variety to the Klein cubic surface. By Corollary 3.6.3 the braid group orbits match up. □

Thus the Klein cubic surface (3.21) is related to both the simple group of order $168 = 2^3 \cdot 3 \cdot 7$ and the simple group of order $6048 = 2^5 \cdot 3^3 \cdot 7$.

Remark 3.8.2. If we consider, from the G_2 point of view, the symmetric Fricke cubic containing the size 18 braid orbit of [22] p.104 (also related to the 237 triangle group), then we find $p_1 = p_2 = -1, p_3 = 1 - 4\cos(\pi/7), p_4 = 1 + p_3$. Any corresponding triple of elements $g_1, g_2, g_3 \in \mathcal{C} \subset G_2$ generate an infinite group: this value of p_3 implies $g_1^2 g_2$ has an eigenvalue x with minimal polynomial $x^6 - 2x^5 + 2x^4 - 3x^3 + 2x^2 - 2x + 1$, and this polynomial has a real root > 1, so x is not a root of unity. Thus the speculation/conjecture (of [22] p.104) that there is a realisation of this Fricke surface relating this braid orbit to a finite group, remains open.

Chapter 4

Echo spaces

In this chapter we will define and study three infinite families of algebraic symplectic varieties. Recall that for a fixed structure group G and irregular curve $\mathbf{\Sigma} = (\Sigma, a_i, Q_i)$ we have defined the wild character variety

$$\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}) = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H},$$

the quasi-Hamiltonian reduction of the space of Stokes representations of the fundamental groupoid by the action of $\mathbf{H}$. It is an algebraic Poisson variety and its symplectic leaves $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, \boldsymbol{\mathcal{C}})$ are obtained by fixing the conjugacy classes in H_i of formal monodromies at each puncture, thus a conjugacy class $\boldsymbol{\mathcal{C}} \subset \mathbf{H} = H_1 \times \ldots \times H_m$.

The symplectic varieties we are going to look at come in families $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}^n, \boldsymbol{\mathcal{C}}^n)$, for suitable choices of the irregular curves $\mathbf{\Sigma}^n$, the conjugacy classes $\boldsymbol{\mathcal{C}}^n$ and the structure group G, which will be a general linear group. The rank of G will not be bounded and grows together with n.

The spaces $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}^n, \boldsymbol{\mathcal{C}}^n)$ are examples of so-called *echo* spaces and in all three cases considered the curve is $\mathbb{P}^1$ with zero as marked point, and we vary the irregular type Q and the conjugacy class $\boldsymbol{\mathcal{C}}$ (choosing G accordingly). Hence the echo spaces parametrise isomorphism classes of irregular connections on vector bundles on $\mathbb{P}^1$ with fixed irregular type Q at zero, defined by $\mathbf{\Sigma}$, and local monodromy in conjugacy class $\boldsymbol{\mathcal{C}}$. They will be related to phase spaces of equations Painlevé I, II and IV, respectively.

In each case we will show that the resulting moduli spaces $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}^n, \boldsymbol{\mathcal{C}}^n)$ are of complex dimension two and thus by results of [11] are in fact hyperkähler and fit into the classification problem of hyperkähler manifolds of real dimension four, as mentioned in the introduction. The main result of this chapter establishes the following isomorphisms.

Theorem 1. Let n be a positive integer. For $i = I, II, IV$, denote by $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_i^n, \boldsymbol{\mathcal{C}}_i^n)$ the n-th Painlevé echo space. Then there are isomorphisms

$$\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_i^n, \boldsymbol{\mathcal{C}}_i^n) \simeq \mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_i, \boldsymbol{\mathcal{C}}_i).$$

In other words, for each of three Painlevé families of echo spaces, all its members are isomorphic to the first one, which is an affine cubic surface. The aforementioned equations for phase spaces of Painlevé I and Painlevé II have been first written down by Kapaev–Kitaev [57] and Flaschka–Newell [42]. More recently, a uniform list of such spaces (for bundles of rank two) has been presented by Van der Put and Saito [93]. We will also relate the parameters of these affine cubic surfaces and see how they transform as n grows.

The echo spaces considered in this manuscript have a convenient description as multiplicative quiver varieties of [28] (there are echo spaces which do not admit, for now, such description), with a small adjustment for the Painlevé I echo space. Thus it is possible to replace the study of the quotient

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \underset{\boldsymbol{\mathcal{C}}}{/\!\!/} \mathbf{H}$$

by a more convenient geometric invariant theory quotient of the spaces of representations of certain quivers. We will analyse the rings of invariant functions and compute the quotients, providing explicit isomorphisms, which are of similar form in all three cases.

For the sake of completeness, we remark that there is a fourth family of wild character varieties fitting into the quiver perspective. The base space $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}})$ is the Painlevé V cubic, and we have gathered evidence that the similar algebraic methods should work as well.

4.1 Definition of echo spaces

Let $\Sigma = \mathbb{P}^1$ with one marked point at zero and consider an irregular type

$$Q = \frac{A_r}{z^r} + \ldots + \frac{A_1}{z}.$$

These choices determine the wild character variety $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma})$, which is a Poisson variety. For suitable choices of the group G, irregular type Q and the conjugacy class $\boldsymbol{\mathcal{C}}$ (and thus specifying a symplectic leaf in $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma})$), the resulting symplectic wild character varieties are of complex dimension two and are related to be phase spaces of Painlevé equations. We will consider the following three examples.

1. The irregular curve $\boldsymbol{\Sigma}_I = (\mathbb{P}_1, 0, Q_I)$ for $G = \mathrm{GL}_2(\mathbb{C})$ and irregular *class* (cf. Section 2.1.4 and Example 2.1.16)

$$Q_I = \langle z^{-5/2} \rangle$$

with multiplicity one so that $n_I \cdot \mathrm{ram}(Q_I) = 2$.

The irregular class Q_I is twisted and the twisted conjugacy class $\mathcal{C}_I \subset H(\partial)$ is the class of the matrix $\left(\begin{smallmatrix} 0 & a \\ -a^{-1} & 0 \end{smallmatrix}\right)$. The wild character variety $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_I, \mathcal{C}_I)$ is isomorphic to the phase space of Painlevé I equation and is isomorphic to the affine cubic surface

$$xyz + x + z - 1 = 0.$$

It parametrises connections on rank two vector bundles on $\mathbb{P}^1$ with one twisted irregular pole of order four and local monodromy around the marked point in the twisted conjugacy class $\mathcal{C}_I$.

Remark 4.1.1. Consider

$$\tau = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$$

and the Cartan subalgebra $\mathbb{C}(\!(\tau)\!) \subset \mathfrak{g}(\!(z)\!)$. The element τ satisfies $\tau^2 = \left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right)$, so we can understand it as a square root of z in $\mathfrak{g}(\!(z)\!)$. Consider the pull back of τ to the double cover π given by $t^2 \mapsto z$:

$$\pi^*(\tau) = \begin{pmatrix} 0 & t^2 \\ 1 & 0 \end{pmatrix},$$

which is in turn conjugate in $\mathfrak{g}(\!(t)\!)$ to $\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right)$, thus the lift of $Q_I = \langle z^{-5/2} \rangle$ to the cover becomes

$$Q_I = \begin{pmatrix} t^{-5} & 0 \\ 0 & -t^{-5} \end{pmatrix} \subset \mathfrak{t}(\!(t)\!)$$

with $t^2 = z$ (the diagonal entries form a Galois orbit). The twisted irregular class "straightens out" when lifted to the cover.

2. The irregular curve $\mathbf{\Sigma}_{II} = (\mathbb{P}^1, 0, Q_{II})$ for $G = \mathrm{GL}_2(\mathbb{C})$ and irregular type

$$Q_{II} = \frac{A_3}{z^3} + \frac{A_2}{z^2} + \frac{A_1}{z}$$

with A_3 regular semisimple. The conjugacy class is $\mathcal{C}_{II}$ is a generic conjugacy class in $\mathrm{GL}_2(\mathbb{C})$ and the wild character variety $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_{II}, \mathcal{C}_{II})$ is isomorphic to the phase space of Painlevé II equation and is isomorphic to the Flaschka–Newell cubic surface

$$xyz + x + y + z = b - b^{-1}.$$

It parametrises connections on rank two vector bundles on $\mathbb{P}^1$ with one irregular pole of order four and local monodromy around the marked point in a generic conjugacy class in $\mathrm{GL}_2(\mathbb{C})$.

3. The irregular curve $\mathbf{\Sigma}_{IV} = (\mathbb{P}^1, 0, Q_{IV})$ for $G = \mathrm{GL}_3(\mathbb{C})$ and irregular type

$$Q_{IV} = \frac{A_2}{z^2} + \frac{A_1}{z}$$

with A_2 regular semisimple. The conjugacy class $\boldsymbol{\mathcal{C}}_{IV}$ is the generic conjugacy class in $\mathrm{GL}_3(\mathbb{C})$ and the wild character variety $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_{IV}, \boldsymbol{\mathcal{C}}_{IV})$ is isomorphic to the phase space of Painlevé IV equation and isomorphic to the cubic surface

$$xyz + x^2 + c_1x + c_2y + c_3z + c_4 = 0$$

for suitable constants c_1, c_2, c_3, c_4. It parametrises connections on rank three vector bundles on $\mathbb{P}^1$ with one irregular pole of order three, and local monodromy around the marked point in a generic conjugacy class in $\mathrm{GL}_3(\mathbb{C})$.

The standard (minimal) representation of this wild character variety is for $G = \mathrm{GL}_2(\mathbb{C})$ and two poles: an irregular pole of order 3 and a simple pole but it will be convenient for us to work in rank three where there is only one pole.

Now we will define three families of irregular curves $\boldsymbol{\Sigma}_I^n, \boldsymbol{\Sigma}_{II}^n, \boldsymbol{\Sigma}_{IV}^n$ and conjugacy classes $\boldsymbol{\mathcal{C}}_I^n, \boldsymbol{\mathcal{C}}_{II}^n, \boldsymbol{\mathcal{C}}_{IV}^n$ as follows.

1. The irregular curves $\boldsymbol{\Sigma}_I^n = (\mathbb{P}_1, 0, Q_I^n)$ for $G = \mathrm{GL}_{2n}(\mathbb{C})$ and irregular class

$$Q_I^n = \langle z^{-5/2} \rangle = \begin{pmatrix} t^{-5} & 0 \\ 0 & -t^{-5} \end{pmatrix}$$

with multiplicity n, so that $n_I \cdot \mathrm{ram}(Q_I^n) = 2n$. The conjugacy class $\boldsymbol{\mathcal{C}}_I^n$ is the twisted conjugacy class

$$\begin{pmatrix} 0 & a \\ \varepsilon a^{-1} & 0 \end{pmatrix} \subset H(\partial) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

where ε is an n-th primitive root of unity and $a \subset \mathrm{GL}_n(\mathbb{C})$ is an $n \times n$ matrix.

2. The irregular curves $\boldsymbol{\Sigma}_{II}^n = (\mathbb{P}^1, 0, Q_{II}^n)$ for $G = \mathrm{GL}_{2n}(\mathbb{C})$ and irregular type

$$Q_{II}^n = \frac{A_3}{z^3} + \frac{A_2}{z^2} + \frac{A_1}{z}$$

with A_3 semisimple with two distinct eigenvalues, both of multiplicity n, and $A_2, A_1 \in \mathfrak{t}$ are any elements whose centralisers in G contain that of A_3, so that the centraliser $C_G(Q_{II}^n) \subset G$ is the block diagonal group $H = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. The central conjugacy class $\boldsymbol{\mathcal{C}}_{II}^n$ is a conjugacy class of a semisimple element with two distinct eigenvalues q_1, q_2

$$\mathrm{diag}(q_1 \mathrm{Id}_n, q_2 \mathrm{Id}_n),$$

both of multiplicity n, such that:

i) $q_1 q_2 = \varepsilon$, where ε is an n-th primitive root of unity,
ii) q_1, q_2 are not n-th roots of unity.

3. The irregular curves $\boldsymbol{\Sigma}_{IV}^n = (\mathbb{P}^1, 0, Q_{IV}^n)$ for $G = \mathrm{GL}_{3n}(\mathbb{C})$ and irregular type

$$Q_{IV}^n = \frac{A_2}{z^2} + \frac{A_1}{z}$$

with A_2 semisimple with three distinct eigenvalues, each of multiplicity n, and $A_1 \in \mathfrak{t}$ is any element whose centraliser in G contains that of A_2, so that the centraliser $C_G(Q^n_{IV}) \subset G$ is the block diagonal group $H = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. The central conjugacy class $\boldsymbol{\mathcal{C}}^n_{IV}$ is a conjugacy class of a semisimple element with three distinct eigenvalues q_1, q_2, q_3

$$\mathrm{diag}(q_1\mathrm{Id}_n, q_2\mathrm{Id}_n, q_3\mathrm{Id}_n),$$

each of multiplicity n, such that:

i) $q_1q_2q_3 = \varepsilon$, where ε is an n-th primitive root of unity,
ii) q_1, q_2, q_3 are not n-th roots of unity.

The three families $(\boldsymbol{\Sigma}^n_I, \boldsymbol{\mathcal{C}}^n_I), (\boldsymbol{\Sigma}^n_{II}, \boldsymbol{\mathcal{C}}^n_{II}), (\boldsymbol{\Sigma}^n_{IV}, \boldsymbol{\mathcal{C}}^n_{IV})$ determine three families of wild character varieties $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_I, \boldsymbol{\mathcal{C}}^n_I), \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{II}, \boldsymbol{\mathcal{C}}^n_{II}), \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{IV}, \boldsymbol{\mathcal{C}}^n_{IV})$ which we will show to be nonempty and of dimension two.

Definition 4.1.2. *For $i = I, II, IV$, we call the wild character variety $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_i, \boldsymbol{\mathcal{C}}^n_i)$ the n-th echo space of $\boldsymbol{\Sigma}_i$.*

We will also refer to them as Painlevé I, Painlevé II and Painlevé IV echo spaces, or $\widetilde{A}_0, \widetilde{A}_1$ and $\widetilde{A}_2$ echo spaces, respectively.

If $n = 1$, the first echo space of the family is the wild character variety $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_i, \boldsymbol{\mathcal{C}}_i)$. The goal of this chapter is to establish the following theorem.

Theorem 1. Let n be a positive integer. For $i = I, II, IV$, there are isomorphisms of echo spaces

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_i, \boldsymbol{\mathcal{C}}^n_i) \simeq \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_i, \boldsymbol{\mathcal{C}}_i).$$

4.2 Graphs, quivers and representations

In this section we will introduce the basic notions of graphs, quivers and their representations. We follow [28] in the exposition and notations, although we will use only a part of the general machinery of multiplicative quiver varieties developed in that article.

Let Γ be a graph with nodes I and edges Γ. We will suppose that both of these sets are finite and, if not stated otherwise, that each edge connects two distinct nodes. We denote by $\overline{\Gamma}$ the set of *oriented* edges of Γ, thus the set of pairs (e, o) where $e \in E$ and o is one of the two possible orientations of e. Each oriented edge $a \in \overline{\Gamma}$ has a well defined head $h(e) \in I$ and tail $t(e) \in I$.

In other words, the oriented graph (a *quiver*) $\overline{\Gamma}$ is the double of Γ, which can be obtained by arbitrarily orienting the edges of Γ and then extending the set of its (now oriented) edges by adding the edges going in opposite directions. The cardinality of $\overline{\Gamma}$ is the double of the cardinality of Γ.

Definition 4.2.1. *Let Q be a quiver (an oriented graph) with nodes I and oriented edges E. A representation of Q consists of:*
- *A finite dimensional I-graded vector space $V = \bigoplus_{i \in I} V_i$*
- *For each oriented edge $e \in E$ a linear map*

$$v_e : V_{t(e)} \to V_{h(e)}.$$

Thus a representation of a quiver Q is a choice of a vector space V_i for each vertex $i \in I$ and a linear map $v_e : V_{t(e)} \to V_{h(e)}$ for each arrow $e \in E$. A subrepresentation of a representation V of Q consists of an I-graded subspace $V' \subset V$ which is preserved by the linear maps, that is $v_e(V'_{t(e)}) \subset V'_{h(e)}$. A representation is irreducible if it does not have proper nontrivial subrepresentations.

Definition 4.2.2. *A representation of a graph Γ is a representation of its double $\overline{\Gamma}$.*

Given a quiver Q with nodes I and an I-graded vector space V, we can consider the space of all representations of Q

$$\mathrm{Rep}(Q, V) = \bigoplus_{e \in E} \mathrm{Hom}(V_{t(e)}, V_{h(e)}).$$

The group $H = \prod \mathrm{GL}(V_i)$ acts naturally on $\mathrm{Rep}(Q, V)$.

Hence it makes sense to define the space of representations of an arbitrary unoriented graph Γ as the space of representations of its double $\overline{\Gamma}$

$$\mathrm{Rep}(\Gamma, V) := \bigoplus_{e \in \overline{\Gamma}} \mathrm{Hom}(V_{t(e)}, V_{h(e)}).$$

4.2.1 The fission graph $\Gamma(Q)$

Let V be a finite dimensional complex vector space and let $\mathfrak{t} = \mathrm{End}(V)$ be a Cartan subalgebra of the Lie algebra $\mathfrak{gl}(V)$, such as the diagonal matrices. Let Q be an irregular type, which we can identify with a polynomial in variable $1/z$

$$Q = \frac{A_r}{z^r} + \ldots + \frac{A_1}{z} \in \mathfrak{t}[1/z].$$

An irregular type determines a direct sum decomposition $V = \oplus_{i \in I} V_i$ into the eigenspaces of Q (ie. common eigenspaces of all coefficients A_i), so we have

$$Q = \sum q_i(1/z) \mathrm{Id}_i$$

where Id_i is the idempotent of V_i and $q_i \in \mathbb{C}[1/z]$ are distinct polynomials. In other words, we can identify the irregular type with a diagonal matrix with polynomials in variable $1/z$ on the diagonal. Therefore we can define a graph $\Gamma(Q)$ with set of nodes I and

$$\deg(q_i - q_j) - 1$$

edges between each pair of distinct nodes $i, j \in I$.

Definition 4.2.3. *The graph $\Gamma(Q)$ is the fission graph of Q.*

It is easy to see that if Q is of degree two, then the fission graph has no multiple edges.

For $\mathbf{\Sigma} = (\mathbb{P}^1, 0, Q)$ the space $\mathrm{Rep}(\Gamma(Q), V)$ of representations of the graph $\Gamma(Q)$ will be closely related to the wild character variety $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$.

Having defined the fission graph, we can now attach graphs to irregular types $Q^n_I, Q^n_{II}, Q^n_{IV}$. The irregular type Q^n_I is twisted thus the definition above does not fit into this case, so we need to treat this case separately. We will use the following definition/theorem.

Definition 4.2.4. *The "fission graph" $\Gamma(Q_I)$ of the irregular class Q^n_I is the affine Dynkin graph $\widetilde{A}_0$, which has one node and a loop.*

Corollary 4.2.5. *The fission graph $\Gamma(Q^n_{II})$ of the irregular type Q^n_{II} is the affine Dynkin graph $\widetilde{A}_1$, which has two nodes with a double edge between them. Its set of nodes I has two elements, thus the grading $V = \bigoplus_{i\in I} V_i$ of $V = \mathbb{C}^{2n}$ determined by Q^n_{II} has two pieces, each of dimension n.*

Corollary 4.2.6. *The fission graph $\Gamma(Q^n_{IV})$ of the irregular type Q^n_{IV} is the affine Dynkin graph $\widetilde{A}_2$, which is a complete graph with three nodes (a triangle). Its set of nodes I has three elements, thus the grading $V = \bigoplus_{i\in I} V_i$ of $V = \mathbb{C}^{3n}$ determined by Q^n_{IV} has three pieces, each of dimension n.*

Observe that the irregular types $Q^n_I, Q^n_{II}, Q^n_{IV}$ are chosen in such way that the fission graph does not change as n grows. Moreover, the choice of multiplicites of eigenvalues and condition on centralisers assures that grading changes homogeneously with n. The grading by elements of I is unordered.

The choice of affine A_0 in Definition 4.2.4 comes from Okamoto's ([76, 77, 78, 79]) and Sakai's ([81]) work on Painlevé equations, so that it is coherent with the untwisted cases. For the Painlevé I equation, the symmetry group is the affine Weyl group of type A_0.

More generally, any untwisted irregular type Q of degree two determines a complete k-partite graph $\Gamma(Q)$. A graph Γ with nodes I is a complete k-partite graph if there exists a partition of $I = \bigsqcup_{j\in J} I_j$ of its nodes into k nonempty parts labeled by a set J of cardinality k, such that any two nodes are connected by an edge if and only if they are in different parts. The irregular type Q determines a grading of V by its eigenspaces and the fission graph has nodes I labeled by these spaces. We group them into parts I_j of a k-partite graph by assembling together the eigenspaces of Q forming eigenspaces of A_2.

The simplest non-trivial example of a complete k-partite graph consists of one edge connecting two nodes, which we can denote by $\Gamma(1,1)$. It is a complete 2-partite graph and is related to space $\mathcal{B}(V)$ for $V = V_1 \oplus V_2$, introduced in Section 2.2.3. Another example is a complete 3-partite graph, the triangle $\Gamma(1,1,1)$. All complete k-partite graphs without multiple edges can be obtained as fission graphs of irregular types.

Even more generally, one can consider *supernova* graphs, consisting of a core, which is a complete k-partite graph, together with some extra legs glued onto each node of the core. These graphs and their relations with wild character varieties have been studied in great detail in [28] but the most general approach is beyond the scope of this manuscript.

4.2.2 Graph representations and quasi-Hamiltonian geometry

In this section we will introduce the space $\mathrm{Rep}^*(\Gamma, V)$ which is an open subset of the space of representations of Γ on V. The space Rep^* has a quasi-Hamiltonian structure and in the case of complete k-partite graph with one node in each part, it will be isomorphic to the reduced fission space $\mathcal{B}(W)$, where W is V with suitably adapted grading.

Let Γ be a complete k-partite graph with nodes I and V a fixed I-graded vector space. In the previous section we have defined the space $\mathrm{Rep}(\Gamma, V)$ of representations of Γ, as the space of representations of its double $\overline{\Gamma}$, which is a quiver. Fix an ordering of the graph Γ, which consists of a total ordering the nodes in each part I_j and a total ordering of parts $I_1, \ldots, I_k$.

An ordering of Γ determines ordered graded vector spaces

$$W_j = \bigoplus_{i \in I_j} V_j$$

and an ordered grading $W = \bigoplus W_j$. The vector spaces W and V are the same space with different gradings (the grading of V refines the grading of W). If each of k parts of Γ has exactly one node, which will be the case later, then the spaces W and V are isomorphic as ordered graded vector spaces.

Therefore there are sequences of groups determined by the ordered gradings:

$$H := \prod \mathrm{GL}(V_i) \subset K := \prod \mathrm{GL}(W_j) \subset G := \mathrm{GL}(V)$$

and we can consider the fission spaces $\mathcal{A}(W_j)$ and the reduction

$$\mathcal{B}(W) = \mathcal{A}^2(W) /\!\!/ G$$

which is a quasi-Hamiltonian K-space.

The following important theorem establishes a connection between representations of graphs and the fission spaces.

Theorem 4.2.7 ([28], Proposition 5.3.). *Let Γ be an ordered complete k-partite graph with nodes I and V an I-graded vector space. Then there is a canonical nonempty open subset*

$$\mathrm{Rep}^*(\Gamma, V) \subset \mathrm{Rep}(\Gamma, V)$$

of the space of representations of the graph Γ on V, which is a smooth affine variety and a quasi-Hamiltonian H-space, canonically isomorphic to

$$\mathcal{B}(W) \underset{K}{\circledast} \prod_j \mathcal{A}(W_j).$$

Proof. The proof comes from [28] with some minor changes. We include it since it explains how to describe the subset $\mathrm{Rep}^*(\Gamma, V)$ in terms of Stokes multipliers and this will be useful later.

A representation (v_{ij}) of Γ on V and the ordering of I determine the following unipotent elements in $\mathrm{GL}(V)$:

$$v_+ = \mathrm{Id} + \sum_{i<j} v_{ij}, \quad v_- = \mathrm{Id} + \sum_{i>j} v_{ij},$$

where we set $v_{ij} \in \mathrm{Hom}(V_i, V_j)$ to be zero if i, j are in the same part of I. We define $\mathrm{Rep}^*(\Gamma, V)$ to be the subset of $\mathrm{Rep}(\Gamma, V)$ such that v_-v_+ is in the opposite big cell $U_+HU_- \subset \mathrm{GL}(V)$ determined by the ordered grading of V, so we can write

$$v_+v_- = w_-hw_+$$

for some $h \in H$ and unipotent elements

$$w_+ = 1 + \sum_{i<j} w_{ij}, \quad w_- = 1 + \sum_{i>j} w_{ij}$$

for some $w_{ij} \in \mathrm{Hom}(V_i, V_j)$, which are allowed to be nonzero even for i, j in the same part. This is indeed an open subset of $\mathrm{Rep}(\Gamma, V)$, since it is defined by nonvanishing of the function $f = \prod \Delta_i$, where $\Delta_i : \mathrm{Rep}(\Gamma, V) \to \mathbb{C}$ is the top left minor of v_+, v_- corresponding to the sum $\oplus_{j \leqslant i} V_j$. This subset is nonempty (it contains the zero representation) and isomorphic to the affine variety

$$\{z \cdot f = 1\} \subset \mathbb{C} \times \mathrm{Rep}(\Gamma, V).$$

On the other hand a point in the space $\mathcal{B}(W) \underset{K}{\circledast} \prod_i \mathcal{A}(W_j)$ is given by solving

$$\kappa S_4 S_3 S_2 S_1 = 1 \in \mathrm{GL}(W), \quad \kappa_j = h_j s_{2j} s_{1j} \in \mathrm{GL}(W_j),$$

where S_i, s_{kj} are the Stokes multipliers of $\mathcal{B}(W)$ and $\mathcal{A}(W_j)$, $\kappa \in K$ has components κ_j and $h_j \in \prod_{i \in I_j}$. Setting $v_+ = S_3, v_- = S_2$ translates these equations to the form $v_+ v_- = w_- h w_+$. Moreover, the element $h \in H$ has components h_j^{-1} thus

$$h : \mathrm{Rep}^*(\Gamma, V) \to H$$

is the moment map for the H-action. $\square$

Example 4.2.8. Suppose that each part of a k-partite graph Γ has exactly one node. Then $H = K$ and the space $\mathcal{B}(W) \underset{K}{\circledast} \mathcal{A}(W_j)$ is just $\mathcal{B}(W)$, and thus $\mathcal{B}(W)$ is an open subset of the space of representations of the complete graph with k nodes on W.

Remark 4.2.9. Theorem 4.2.7 says that for an irregular type $Q = \frac{A_2}{z^2} + \frac{A_1}{z}$, the space $\mathrm{Rep}^*(\Gamma(Q), V)$ is in fact isomorphic to $\mathcal{B}(Q)$. This theorem remains true for any irregular type Q with similar proof (cf. Remark 5.4. of [28]). This will be useful later, when treating the affine A_1 case, which has a double edge.

In order to define the subset $\mathrm{Rep}^*(\Gamma, V)$, we used the ordering of I. This choice, however, always yields an isomorphic quasi-Hamiltonian space.

Proposition 4.2.10. *The space* $\mathrm{Rep}^*(\Gamma, V)$ *is independent of the choice of ordering of* I.

Proof. This is a consequence of isomonodromy isomorphisms of Section 5 of [27]. In this case, it follows from the fact that for V, V' being the same graded vector space with different ordering of pieces, the quasi-Hamiltonian spaces $\mathcal{A}^r(V)$ and $\mathcal{A}^r(V')$ are isomorphic, as discussed in Section 2.2.3. $\square$

The variety $\mathrm{Rep}^*(\Gamma, V)$ is an affine variety with an action of a reductive group $H = \prod_{i \in I} \mathrm{GL}(V_i)$, thus its stable points are defined as the points whose H-orbits are closed and of dimension $\dim(H) - \dim(\mathrm{Ker})$, where Ker is the kernel of the action (subgroup of H whose action is trivial). Here this kernel is one-dimensional, thus the required dimension of orbits is $\dim(H) - 1$.

Proposition 4.2.11 ([28], Lemma 5.8.)**.** *A representation* $\rho \in \mathrm{Rep}^*(\Gamma, V)$ *is stable if and only it is irreducible.*

4.2.3 Multiplicative quiver varieties

In this section we will introduce the main object of this chapter, the multiplicative quiver varieties. These are algebraic symplectic varieties attached to graphs and generalise the classical multiplicative quiver varieties of Crawley-Boevey and Shaw [36] and Yamakawa [97]. By results of [28], for a large class of graphs these varieties are isomorphic to the (symplectic leaves of) wild Betti spaces and therefore have hyperkähler metrics.

Let Γ be a complete k-partite graph with nodes I and a fix an ordering of I. By Proposition 4.2.10 the resulting spaces will be independent of the latter choice. A dimension vector is an element

$$d \in \mathbb{Z}^I$$

with nonnegative integer coordinates $(d_i)_{i\in I}$. Similarly, a parameter vector (or simply a parameter) is an element

$$q \in (\mathbb{C}^*)^I$$

with complex coordinates $(q_i)_{i\in I}$.

Let $V_i = \mathbb{C}^{d_i}$ and consequently $V = \bigoplus V_i$ be the corresponding I-graded vector space determined by d. Recall that in Theorem 4.2.7 we have defined the space $\mathrm{Rep}^*(\Gamma, V)$ which is a quasi-Hamiltonian H-space for $H = \prod \mathrm{GL}(V_i)$. We will identify the parameter q with the point $(q_i \mathrm{Id}_{V_i})_{i\in I} \in H$.

Definition 4.2.12. *The multiplicative quiver variety of* Γ, d, q *is the quasi-Hamiltonian reduction of* $\mathrm{Rep}^*(\Gamma, V)$ *at the value* q *of the moment map:*

$$\mathcal{Q} = \mathcal{Q}(\Gamma, q, d) = \mathrm{Rep}^*(\Gamma, V) \underset{q}{/\!\!/} H = \mu^{-1}(q).$$

The quotient by H on the right is the affine goemetric invariant theory quotient, taking the affine variety associated with the ring of H-invariant functions on $\mu^{-1}(q) \subset \mathrm{Rep}^*(\Gamma, V)$. Observe that if $q^d \neq 1$, then the quiver variety $\mathcal{Q}$ is empty. The set of stable points $\mathcal{Q}^{st} \subset \mathcal{Q}$ by definition consists of the points whose orbits in $\mu^{-1}(q)$ are closed and of dimension $\dim(H) - 1$ (since the scalars in H act trivially).

A graph Γ without loops and with n nodes determines a bilinear form on the root lattice $\mathbb{Z}^I = \bigoplus_{i\in I} \mathbb{Z}\varepsilon_i$. Let A be the adjacency matrix of Γ (the i, j entry of A is the number of edges between the nodes i and j) and define a $n \times n$ matrix

$$C = 2\,\mathrm{Id} - A.$$

Then there is a bilinear form on $\mathbb{Z}^I$ defined by

$$(\varepsilon_i, \varepsilon_j) = C_{ij}$$

.

Theorem 4.2.13 ([28], Theorem 6.3.)**.** $\mathcal{Q}^{st}(\Gamma, q, d)$ *is a smooth algebraic symplectic manifold which is either empty of of dimension* $2 - (\ ,\ )$*, where* $(\ ,\ )$ *is the bilinear form on the root lattice of* Γ*. The points of* $\mathcal{Q}^{st}$ *correspond to the* H*-orbits in* $\mu^{-1}(q)$ *of irreducible representations of* Γ*.*

Definition 4.2.14. *For a fixed dimension vector d, we will say that the parameters q are generic if they obey the condition*

$$q^{\alpha} \neq 1$$

for any α in the finite set

$$R_{\oplus}(d) := \{\alpha \in \mathbb{Z}^I \mid (\alpha, \alpha) \leqslant 2 \text{ and } 0 \leqslant \alpha_i \leqslant d_i \text{ for all } i\} \setminus \{0, d\}.$$

Here the meaning of "generic" is a bit different from the usual, since it may happen that the set $R_{\oplus}(d)$ is not dense in $\{q \mid q^d = 1\}$. For example then the dimension vector is a multiplicity of another dimension vector, which will be the case.

Proposition 4.2.15 ([28], Proposition 6.5.)**.** *If parameters q are generic, then all the points of the multiplicative quiver variety are stable, $\mathcal{Q}^{st}(\Gamma, q, d) = \mathcal{Q}(\Gamma, q, d)$, and so it is smooth.*

As we have seen, for a vector space V and $G = \mathrm{GL}(V)$, the irregular type Q determines a fission graph $\Gamma(Q)$. It also determines a grading of V by its eigenspaces (which are nodes of $\Gamma(Q)$), which in turn is the gives a dimension vector d. The fundamental correspondence between the multiplicative quiver varieties and wild character varieties can be now stated as follows (taking into account the remark after Theorem 4.2.7).

Theorem 4.2.16 (cf. [28], Proposition 9.2.)**.** *Let $G = \mathrm{GL}(V)$ and $\boldsymbol{\Sigma} = (\mathbb{P}^1, 0, Q)$. Denote by $H = C_G(Q) \subset G$ the centraliser of Q in G, $\Gamma(Q)$ the associated fission graph and by d the associated dimension vector, all determined by Q. Let q be a generic parameter, which we can also identify with a conjugacy class $\boldsymbol{\mathcal{C}} \subset H$. Then there is an isomorphism*

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{C}}) \simeq \mathcal{Q}(\Gamma(Q), q, d).$$

We can now explain why we chose the non-standard representation of the Painlevé IV echo spaces. The $\mathrm{GL}_{3n}(\mathbb{C})$ representation involves only one pole, so we can conveniently use the theorem above to realise this wild character variety as a multiplicative quiver variety (and $\Gamma(Q_{IV}^n)$ is a triangle).

4.3 Linear algebra

In this section we will prove a sequence of simple algebraic lemmas which will be useful later in the chapter.

Definition 4.3.1. *Let A, B be two complex $n \times n$ matrices, $k \in \mathbb{C}^*$ and let ε be a primitive root of unity of degree n. We say that matrices A, B quasi-commute with parameter k if they satisfy the identity*

$$AB - \varepsilon BA = (1 - \varepsilon)k \cdot \mathrm{Id}.$$

We will omit the term Id later in the text supposing that the scalars appearing in matrix equations are scalar matrices of the appropriate size. The right hand side is just a scalar matrix, but we keep $(1-\varepsilon)$ for convenience. In this setup one immediately has $\mathrm{Tr}(AB) = nk$, without the necessity of dividing the traces by $1-\varepsilon$.

Proposition 4.3.2. *Suppose that A, B quasi-commute with parameter k and let λ be a non zero eigenvalue of A. Then if v is not also an eigenvector of B, then $\varepsilon\lambda$ is an eigenvalue of A. Similarly, if w is an eigenvector of B for an eigenvalue λ and not an eigenvector of A, then λ/ε is an eigenvalue of B.*

Proof. Suppose that λ is a non zero eigenvalue of A and v the eigenvector of A for this eigenvalue.

$$(AB - \varepsilon BA)v = (A - \varepsilon\lambda)Bv = (1-\varepsilon)kv.$$

Thus the matrix $A - \varepsilon\lambda$ sends both v (since it is an eigenvector of A) and Bv to the one dimensional subspace $\mathbb{C}v$. Since v is not an eigenvector of B, v, Bv are linearly independent hence $A - \varepsilon\lambda$ has non trivial kernel and $\varepsilon\lambda$ is an eigenvalue of A. The proof of the second part of the statement is the same. $\square$

Corollary 4.3.3. *Suppose that A, B quasi-commute with parameter k and do not have common eigenvectors. Then if A has a nonzero eigenvalue λ, then A is diagonalisable with n distinct eigenvalues $\lambda, \varepsilon\lambda, \dots, \varepsilon^{n-1}\lambda$ and the same holds for B.*

Proposition 4.3.4. *Suppose that A is an $n \times n$ matrix such that for $0 < k < n$ we have $\mathrm{Tr}(A^k) = 0$. Then $\det(A) = \frac{(-1)^{n+1}}{n}\mathrm{Tr}(A^n)$.*

Proof. The Cayley–Hamilton Theorem for M gives the following equation:

$$A^n + f_1(A)A^{n-1} + \ldots + f_{n-1}(A)A + (-1)^n \det(A)\mathrm{Id} = 0$$

where $f_1, f_2, \dots, f_{n-1}$ are certain functions which can be expressed as polynomials in $\mathrm{Tr}(A), \mathrm{Tr}(A^2), \dots, \mathrm{Tr}(A^{n-1})$ (these are just expressions for symmetric polynomials in terms of power sums). Since we have $\mathrm{Tr}(A^k) = 0$ for $0 < k < n$, the functions $f_1, \dots, f_{n-1}$ vanish evaluated on A. Thus we can write

$$A^n + (-1)^n \det(A)\mathrm{Id} = 0$$

and see that A^n is in fact a scalar matrix. Taking the trace of both sides finishes the proof. $\square$

Proposition 4.3.5. *Suppose that A is an $n \times n$ matrix such that for $0 < k < n$ we have $\mathrm{Tr}(A^k) = (-1)^k n$. Then $\det(A + \mathrm{Id}) = \frac{(-1)^{n+1}}{n}\mathrm{Tr}(A^n) + 1$ and $\det(A + \mathrm{Id}) = \det(A) + (-1)^{n+1}$.*

Proof. The assertion $\mathrm{Tr}(A^k) = (-1)^k n$ implies that the matrix $A + \mathrm{Id}$ satisfies the conditions of the previous proposition. Thus we can write

$$(A + \mathrm{Id})^n + (-1)^n \det(A + \mathrm{Id})\mathrm{Id} = 0.$$

Taking the trace, we get $\mathrm{Tr}((A+\mathrm{Id})^n) = n(-1)^{n+1}\det(A+Id)$. On the other hand, developing $(A+\mathrm{Id}_n)^n$ and using the fact that $\mathrm{Tr}(A^k) = (-1)^k n$, we get

$$\mathrm{Tr}((A+\mathrm{Id})^n) = \mathrm{Tr}(A^n) + (-1)^{n+1}n,$$

so we obtain

$$n(-1)^{n+1}\det(A+\mathrm{Id}) = \mathrm{Tr}(A^n) + (-1)^{n+1}n$$

and dividing both sides by $n(-1)^{n+1}$ completes the proof of the first part.

For the second part, consider the expression $\det((A+\mathrm{Id}) - \mathrm{Id})$. It is equal to $\det(A)$, but it is also equal to the characteristic polynomial, multiplied by $(-1)^n$, of $N+\mathrm{Id})$ evaluated at 1. Again, the coefficients of the characteristic polynomial vanish, to we obtain

$$(-1)^n(1+(-1)^n\det(A+\mathrm{Id})) = \det(A),$$

$$\det(A+\mathrm{Id}) = \det(A) + (-1)^{n+1}.$$

The proof is complete. □

Proposition 4.3.6. *Let A, B be $n\times n$ matrices satisfying $\mathrm{Tr}(A^i) = \mathrm{Tr}(B^i) = 0$ for $0 < i < n$, and quasi-commuting with parameter k . Then we have*

$$\det(A+B+k+1) = \det(A) + \det(B) + k^n + 1.$$

We will split the proof of this statement into few steps. Suppose that A, B satisfy the hypotheses of Proposition 4.3.6.

Lemma 4.3.7. *Let $0 < a, b < n$ be two distinct integers. Then the trace of any word W build from a copies of A and b copies of B (in any order) is zero.*

Proof. We proceed by induction by the length of the word. By assumption, the powers a, b is not divisible by n. Suppose that A is the first letter of W. We can swap in the right direction right with all elements in the word – swapping with other A does not change anything, and every time we encounter B, the trace gets multiplied by ε (we also add a constant $k(1-\varepsilon)$, but by inductive hypothesis the trace of the extra word of length $a+b-2$ is zero, since it has $a-1$ copies of A and $b-1$ copies of B). Thus after swapping with all the letters in W, we will return to the initial word, but multiplied by ε^b. This yields $\mathrm{Tr}(W) = 0$ since $n \nmid b$ and thus $\varepsilon^b \neq 1$. □

Corollary 4.3.8. *Let i be an odd integer $0 < i < n$. Then $\mathrm{Tr}((A+B)^i) = 0$.*

Lemma 4.3.9. *For $0 < i < n$, we have $\mathrm{Tr}((AB)^i) = nk^i$.*

Proof. By hypothesis, we have $AB - \varepsilon BA = (1-\varepsilon)k$. This implies $\mathrm{Tr}(AB) = nk$. We can also write $AB = (1-\varepsilon)k + \varepsilon BA$, take the i-th power of both sides, and proceed by induction, using the fact that $\mathrm{Tr}((AB)^i) = \mathrm{Tr}((BA)^i)$. □

Lemma 4.3.10. *Let $0 < a < n$ be an integer. Suppose that A and B appear exactly a times in the word W, then* $\mathrm{Tr}(W) = nk^a$.

Proof. In other words, we need to show that if a word W consists of a copies of A and a copies of B, then its trace does not depend on the order of A and B. Let us start with the word $(AB)^a$. We know that its trace is equal to nk^a. We will prove by induction that if the trace of a word W consisting of a copies of A and B is k^a, then the trace of the word W' obtained by a swap of A and B in W is also nk^a.

Suppose that we have replaced AB in W by $\varepsilon BA + (1-\varepsilon)k$ (the proof is the same when we swap BA), obtaining a new word $\varepsilon W'$ and a shorter word $(1-\varepsilon)kW''$ with $a-1$ copies of A and B. By inductive hypothesis, $\mathrm{Tr}(W'') = nk^{a-1}$ and thus

$$\varepsilon \mathrm{Tr}(W') = \mathrm{Tr}(W) - n(1-\varepsilon)k \cdot k^{a-1} = n\varepsilon k^a.$$

Hence $\mathrm{Tr}(W') = nk^a$ and since we can obtain and such word by swapping letters in $(AB)^a$ (its trace is nk^a), the lemma is proved. $\square$

Corollary 4.3.11. *Let i be an even integer $0 < i < n$. Then* $\mathrm{Tr}((A+B)^i) = nk^{\frac{i}{2}}\binom{i}{\frac{i}{2}}$.

Proof of Proposition 4.3.6. Consider the expression $\det(A+B+k+1)$. We can express this quantity as a polynomial in traces of powers not greater than n of $A+B+k+1$. On the other hand, it follows from the previous lemmas that the only traces which are not already determined are $\mathrm{Tr}(A^n), \mathrm{Tr}(B^n)$. From Cayley-Hamilton Theorem, it follows that

$$\det(A+B+k+1) = (-1)^{n+1}\left(\frac{\mathrm{Tr}(A^n)}{n} + \frac{\mathrm{Tr}(B^n)}{n}\right) + c,$$

where c is a constant, independent of choice of A, B. Thus one can suppose that A, B are diagonal (hence with n distinct eigenvalues of the form $q, \varepsilon q, \ldots, \varepsilon^{n-1}q$ for some constant q) and determine the constant c. A direct check shows that it is indeed $k^n + 1$. On the other hand, by Proposition 4.3.4 we have

$$(-1)^{n+1}\frac{\mathrm{Tr}(A^n)}{n} = \det(A), \quad (-1)^n\frac{\mathrm{Tr}(B^n)}{n} = \det(B).$$

$\square$

4.4 The Painlevé I echo spaces

In this section we will study the Painlevé I echo space, defined in Section 4.1 as the wild Betti space $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_I^n, \boldsymbol{\mathcal{C}}_I^n)$. We are going to prove the following theorem.

Theorem. Let n be a positive integer. There are isomorphisms of echo spaces

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_I^n, \boldsymbol{\mathcal{C}}_I^n) \simeq \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_I, \boldsymbol{\mathcal{C}}_I).$$

It is known that the first member of this family, the phase space of Painlevé I equation, is a cubic surface cut out by equation (cf. [57], p. 244)

$$xyz + x + z = 1$$

and the theorem states that all members of the Painlevé I echo family are isomorphic to this cubic.

Recall the Example 2.1.16. Let W be a complex vector space of dimension n. Suppose $V = W \oplus W$. Let $G = \mathrm{GL}(V), H = \mathrm{GL}(W) \times \mathrm{GL}(W)$ and U_+, U_- upper/lower triangular with off-diagonal block of size n. We can identify:

$$G = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

where an asterisk denotes an $n \times n$ block and one stands for an identity matrix of size n.

For $G = \mathrm{GL}(V)$ and the irregular class $\langle z^{-5/2} \rangle$ there are five Stokes groups which can be identified with U_- and U_+ in alternating order. Let ∂ denote the associated twist exchanging the columns $1, \dots n$ with $n+1, \dots, 2n$, so we have

$$H(\partial) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Proposition 4.4.1 ([31], Example 6.2.)**.** *The space*

$$\mathcal{B}(Q_I^n) = \mathcal{B}^{-5/2}(V) = \{(h, S_1, S_2, S_3, S_4, S_5) \in H(\partial) \times U_\pm^5 \mid hS_5S_4S_3S_2S_1 = 1\}$$

is a twisted quasi-Hamiltonian H-space with moment map

$$\mu(h, \mathbf{S}) = \begin{pmatrix} 0 & s_4s_3s_2 + s_2 + s_4 \\ -(s_2s_3s_4 + s_2 + s_4)^{-1} & 0 \end{pmatrix} \in H(\partial),$$

where $\mathbf{S} = (S_1, \dots, S_5)$ and s_i denotes the $n \times n$ block off-diagonal entry of S_i.

The matrices S_2, S_3, S_4 and the condition $hS_5 \cdots S_1 = 1$ uniquely determine h, S_5, S_1. In fact, any consecutive triple S_i, S_{i+1}, S_{i+2} does determine the remaining two Stokes multipliers and formal monodromy h. The expression $s_4s_3s_2 + s_2 + s_4$ is the third Euler's continuant. They will be studied in detail in Chapter 5.

The Painlevé I echo space is obtained by performing the quasi-Hamiltonian reduction $\mathcal{B}(Q_I^n) \mathbin{/\!\!/}_{\boldsymbol{\mathcal{C}}_I^n} H$ at the twisted conjugacy class

$$\boldsymbol{\mathcal{C}}_I^n = \begin{pmatrix} 0 & a \\ \varepsilon a^{-1} & 0 \end{pmatrix} \subset H(\partial).$$

4.4.1 The space $\mathrm{Rep}^*(\widetilde{A}_0, V)$

The *twisted* case of Painlevé I does not fit into the picture of multiplicative quiver varieties, but we can nonetheless conveniently define a similar space. This has first appeared in the talk [29].

Let V be a complex vector space of dimension n, $X, Y, Z \in \mathrm{End}(V)$ and suppose that $\mathrm{GL}(V)$ acts on the triple (X, Y, Z) diagonally by simultaneous conjugation. We define the graph $\Gamma(Q_I^n)$ to be the affine A_0 graph, which consists of one node with a loop. We will use the following theorem as the definition of the space $\mathrm{Rep}^*(\widetilde{A}_0, V)$.

Theorem 4.4.2 ([29], Section 3)**.** *The space*

$$\mathrm{Rep}^*(\widetilde{A}_0, V) := \{X, Y, Z \in \mathrm{End}(V) \mid XYZ + X + Z = 1\}$$

is a quasi-Hamiltonian $\mathrm{GL}(V)$*-space of dimension* $2n^2$ *with moment map* $\mu(X, Y, Z) = ZYX + X + Z$.

In particular, the image of the moment map takes values in $\mathrm{GL}(V)$. If $XYZ + X + Z = 1$, then the determinant of $ZYX + X + Z$ is one as well and thus $\mu(X, Y, Z)$ is an invertible matrix.

Moreover, if X, Z are invertible, then

$$\mu(X, Y, Z) = ZYX + X + Z = ZXZ^{-1}X^{-1}$$

and the moment map is the multiplicative commutator of Z and X. On the other hand, there is a clear analogy here with the additive picture, where the Nakajima quiver variety of the affine A_0 graph is well-defined, as $A, B \in \mathrm{End}(V)$ with moment map $AB - BA$, which is the additive commutator.

Remark 4.4.3. Unlike in the untwisted case of Theorem 4.2.7, the space $\mathrm{Rep}^*(\widetilde{A}_0, V)$ is not isomorphic to $\mathcal{B}(Q_I^n)$ (the groups acting do not match up). As we have seen, the irregular type Q_I^n determines the group H isomorphic to $\mathrm{GL}(V) \times \mathrm{GL}(V)$ and $\mathcal{B}(Q_I^n)$ is a twisted quasi-Hamiltonian H-space. Then $\mathrm{Rep}^*(\widetilde{A}_0, V)$ is the intermediate space between $\mathcal{B}(Q_I^n)$ and the quotient

$$\mathcal{B}(Q_I^n) \underset{\boldsymbol{\mathcal{C}}_I^n}{/\!\!/} H = \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_I^n, \boldsymbol{\mathcal{C}}_I^n),$$

obtained by quotienting $\mathcal{B}(Q_I^n)$ by one copy of $\mathrm{GL}(V)$. More precisely, the conjugation action of H on $H(\partial)$ is

$$\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & h_1 a h_2^{-1} \\ h_2 b h_1^{-1} & 0 \end{pmatrix}$$

and we can use one action to set $a = 1$.

Proposition 4.4.1 implies the following.

Proposition 4.4.4. *The space* $\mathrm{Rep}^*(\widetilde{A}_0, V)$ *is isomorphic to the quotient of* $\mathcal{B}(Q_I^n)$ *by one copy of* $\mathrm{GL}(V)$.

Corollary 4.4.5. *Let* ε *be a primitive root of unity of degree* n. *The quasi-Hamiltonian reduction*

$$\mathrm{Rep}^*(\widetilde{A}_0, V) \mathbin{/\!\!/}_{\varepsilon} \mathrm{GL}(V)$$

is isomorphic to the n*-th Painlevé I echo space.*

In other words, one can simply use the description from Corollary 4.4.5 as a definition of the Painlevé I echo space and forget the twisted irregular types.

We will abusively denote

$$\mathcal{Q}(\widetilde{A}_0, \varepsilon, n) := \mathrm{Rep}^*(\widetilde{A}_0, V) \mathbin{/\!\!/}_{\varepsilon} \mathrm{GL}(V).$$

Remark 4.4.6. If ε is a primitive root of unity, then all points in $\mu^{-1}(\varepsilon) \subset \mathrm{Rep}^*(\widetilde{A}_0, V)$ are stable. In this case the matrices $X, Y, Z \in \mu^{-1}(\varepsilon)$ cannot share a common subspace $U \subset V$, since the restriction to U would yield a representation of $\widetilde{A}_0$ on a space of dimension $k < n$ with the same ε and $\varepsilon^k \neq 1$ (since ε is primitive). This means that a point in $\mu^{-1}(\varepsilon)$ seen as a representation of a quiver with three loops is irreducible and thus stable by theorem of King [59]. So it is also stable in the subvariety $\mathrm{Rep}^*(\widetilde{A}_0, V)$.

4.4.2 Functions on $\mu^{-1}(\varepsilon)$

We are going to compute the GIT quotient

$$\mathrm{Rep}^*(\widetilde{A}_0, V) \mathbin{/\!\!/}_{\varepsilon} \mathrm{GL}(V) = \mu^{-1}(\varepsilon)/\mathrm{GL}(V),$$

at the value ε of the moment map μ. By Theorem 4.4.2 and Proposition 4.4.4 we have the following explicit description of the space $\mu^{-1}(\varepsilon)$ in terms of $n \times n$ matrices X, Y, Z:

Corollary 4.4.7. *The space* $\mu^{-1}(\varepsilon)$ *is the affine subvariety of* $\mathrm{Rep}(\widetilde{A}_0, V)$ *cut out by equations:*

$$\begin{cases} XYZ + X + Z = 1 \\ ZYX + Z + X = \varepsilon. \end{cases} \tag{4.1}$$

In order to compute the quotient

$$\mathrm{Rep}^*(\widetilde{A}_0, V) \mathbin{/\!\!/}_{\varepsilon} \mathrm{GL}(V)$$

we need to understand the ring of $\mathrm{GL}(V)$-invariant functions on $\mu^{-1}(\varepsilon)$. By the classical theorem of Procesi [80], the ring on invariants on this variety (which is an affine subvariety of $\mathrm{End}(V)^3$) is finitely generated and the generating functions are traces of words in X, Y, Z of bounded length, and the bound depends on n. However, the result is not constructive and we do not know a priori which traces to consider nor what the relations are.

Recall that we have defined (Definition 4.3.1) the notion of quasi-commutation of matrices. For A, B two $n \times n$ matrices and ε a primitive root of unity of degree n, A and B quasi-commute with parameter q if they satisfy

$$AB - \varepsilon BA = (1-\varepsilon)q.$$

From now on we suppose that the matrices $X, Y, Z \in \mu^{-1}(\varepsilon)$ are as in Corollary 4.4.7.

Proposition 4.4.8. *The pairs of matrices* $(Y, X), (Z, Y), (X, Z)$ *quasi-commute with parameters* $-1, -1$ *and* 0*, respectively:*

$$\begin{cases} YX - \varepsilon XY = -(1-\varepsilon) \\ ZY - \varepsilon YZ = -(1-\varepsilon) \\ XZ - \varepsilon ZX = 0. \end{cases} \tag{4.2}$$

Proof. Start by multiplying the first equation of (4.1) by YX on the right and the second by XY on the left. The system then reads

$$\begin{cases} XYZYX + XYX + ZYX = YX \\ XYZYX + XYZ + XYX = \varepsilon XY. \end{cases}$$

Now, if we substract the fist equation from the second one, we get

$$\varepsilon XY - YX = XYZ - ZYX = 1 - \varepsilon.$$

Multiplying the first equation of (4.1) by ZY on the left and the second one by YZ on the right and performing the same operations leads to similar identity

$$\varepsilon YZ - ZY = 1 - \varepsilon.$$

Finally, we can use the relation $\varepsilon XY - YX = 1 - \varepsilon$. in order to transform the second equation of (4.1)

$$ZYX + Z + X - \varepsilon = Z(\varepsilon XY - 1 + \varepsilon) + Z + X - \varepsilon = \varepsilon ZXY + \varepsilon Z + X - \varepsilon,$$

thus we can write

$$\varepsilon ZXY + \varepsilon Z + X = \varepsilon.$$

If we multiply this equation by Z on the right and the first equation of (4.1) by Z on the left, we obtain

$$\begin{cases} ZXYZ + ZX + Z^2 = Z \\ \varepsilon ZXYZ + XZ + \varepsilon Z^2 = \varepsilon Z. \end{cases}$$

which leads to

$$\varepsilon ZX - XZ = 0.$$

$\square$

Proposition 4.4.9. *Suppose that v is a vector such that $Xv = \lambda v$ and $Zv = 0$. Then $\lambda = 1$.*

Proof. Follows immediately from the fact that

$$(XYZ + X + Z)v = 1.v.$$

$\square$

Proposition 4.4.10. *Suppose that X is not diagonalisable and not nilpotent. Then ε is an eigenvalue of X.*

Proof. Since Y, X quasi-commute, and X is not diagonalisable, then there exists a common eigenvector for X and Y and $XYw = -w$.

Recall that we have a relation

$$\varepsilon ZXY + \varepsilon Z + X = \varepsilon.$$

Thus, since $XYw = -w$

$$(\varepsilon ZXY + \varepsilon Z + X)w = \varepsilon w,$$

which implies $Xw = \varepsilon w$, so ε is an eigenvalue of X. $\square$

Proposition 4.4.11. *The matrix X is either nilpotent or diagonalisable with n distinct eigenvalues of the form $t, \varepsilon t, \ldots, \varepsilon^{n-1}t$.*

Proof. Suppose that X is not nilpotent. Then from Corollary 4.3.3 if X, Z do not have a common eigenvector (ie. v such that $Xv = \lambda v, Zv = 0$), then X is of desired form. Thus there is v such that $Xv = \lambda v, Zv = 0$ which means, by Proposition 4.4.9 that $\lambda = 1$.

Similarly, if X, Y do not share an eigenvector, then X is of desired form, so by Proposition 4.4.10 there is a common eigenvector w for X, Y and ε is an eigenvalue of X. Now, again from Proposition 4.4.9 it follows that w is not in the kernel of Z (if it were, it would be an eigenvector for eigenvalue one but $\varepsilon \neq 1$). This means, by Proposition 4.3.2, that ε^2 is an eigenvalue of X and by induction the eigenvalues of X are $1, \varepsilon, \ldots, \varepsilon^{n-1}$. $\square$

The proof works verbatim for Z, so we can write.

Corollary 4.4.12. *The matrix Z is either nilpotent or diagonalisable with n distinct eigenvalues of the form $t, \varepsilon t, \ldots, \varepsilon^{n-1}t$.*

For completeness, we write the proof for Y as well.

Proposition 4.4.13. *The matrix Y is either nilpotent or diagonalisable with n distinct eigenvalues of the form $t, \varepsilon t, \ldots, \varepsilon^{n-1}t$.*

Proof. Suppose that Y is not nilpotent. Then by Proposition 4.3.2 Y is either of the desired form, or shares an eigenvector v (for eigenvalue λ_1) with X such that $XYv = -v$. Similarly, there is a common eigenvector w (for eigenvalue λ_2) for Y and Z such that $ZYw = -w$. We have the relation

$$XYZ + X + Z = 1,$$

which by quasi-commutation relations implies

$$XZY + X + \varepsilon Z = \varepsilon$$

and hence if w is the common eigenvector for Y, Z, it implies $\varepsilon Zw = \varepsilon w$ and $\lambda_2 = -1$ is an eigenvalue of Y. On the other hand, we also have

$$\varepsilon ZXY + X + \varepsilon Z = 1,$$

so if v is the common eigenvector for Y, X, then $Xv = \varepsilon v$ and $\lambda_1 = -1/\varepsilon$ is an eigenvalue of Y. The proof is now finished by induction. Since $\lambda_2 = -1$ is an eigenvalue of Y, then the eigenvector w of Y for this eigenvalue cannot be an eigenvector of X (this would imply the eigenvalue equal to $-1/\varepsilon$) and the eigenvalues of Y are of the form $\varepsilon^k \cdot (-1)$. $\square$

Therefore we can state two important corollaries.

Corollary 4.4.14. *The matrices X^n, Y^n, Z^n are scalar.*

Corollary 4.4.15. *For $0 < k < n$, the matrices X, Y, Z satisfy*

$$\mathrm{Tr}(X^k) = \mathrm{Tr}(Y^k) = \mathrm{Tr}(Z^k) = 0.$$

We note that the relation $XYZ + X + Z = 1$ is essential – one would be tempted to try to deduce the two corollaries above solely from the quasi-commuting relations (4.2) but it is not enough since X, Y, Z might a priori have nontrivial Jordan forms.

Let us introduce the following three functions

$$x = \frac{\mathrm{Tr}(X^n)}{n}, \quad y = (-1)^{n+1}\frac{\mathrm{Tr}(Y^n)}{n}, \quad z = \frac{\mathrm{Tr}(Z^n)}{n}.$$

Proposition 4.4.16. *The functions x, y, z generate the ring of invariant functions on $\mu^{-1}(\varepsilon)$.*

Proof. We will prove the statement by induction on the length of the word in X, Y, Z. We have $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$ and suppose that the traces of all words of length shorter than k are known. Let W be a word in X, Y, Z of length k. If W involves all three elements X, Y, Z, then we can use the quasi-commuting relations and swap the letters as many times as necessary, so that we obtain a new word W' with X, Y, Z appearing next to each other in this order plus possibly some shorter words. Each swap multiplies the word by $\varepsilon^{\pm 1}$ and possibly adds a word of length $k-2$ (since the quasi-commuting relations between X, Y and Y, Z involve constants), whose trace is known by the inductive hypothesis. Using the relation

$$XYZ + X + Z = 1$$

we can shorten W' by replacing XYZ by $1 - X - Z$ and thus we know its trace.

Suppose we have a word W in two letters, say X, Y (it does not matter which two letters we choose). We can swap all X's to the left (possibly adding words of length $k-2$ whose traces are known) and thus the trace of W will be determined if we know the trace of $X^a Y^b$ with a and b being the number of X's and Y's in W. If at least one of the numbers a and b is at least n, then the proof follows from the fact that X^n and Y^n are scalar matrices of the form $\mathrm{Id} \cdot (\mathrm{Tr}(X^n)/n)$ and $\mathrm{Id} \cdot (\mathrm{Tr}(Y^n)/n)$.

Suppose then that $0 < a, b < n$. We can take the most-left Y in the word and, using qiuasi-commutation of Y, X, swap it a times with all X, obtaining $X^a Y^b = \varepsilon^{-a} Y X^a Y^{b-1} + P$ with P being a shorter word of known trace. Now, since $\varepsilon^a \neq 1$, the trace of $X^a Y^b$ is determined by the cyclic invariance of the trace.

Traces of all powers of X, Y, Z are determined by Corollary 4.4.15. □

Proposition 4.4.17. *The functions x, y, z satisfy the relation*

$$xyz + x + z = 1.$$

Proof. We can write the relation $XYZ + X + Z = 1$ as

$$X(YZ + 1) = 1 - Z$$

and take the determinant of both sides, obtaining

$$\det(X)\det(YZ + 1) = \det(1 - Z).$$

By Lemma 4.3.9, we have$\mathrm{Tr}((YZ)^k) = n(-1)^k$, so from Propositions 4.3.4,4.3.5 we get

$$(-1)^{n+1}\frac{\mathrm{Tr}(X^n)}{n}(\det(YZ) + (-1)^{n+1}) = 1 + (-1)^n \det(Z),$$

$$(-1)^{n+1}\frac{\mathrm{Tr}(X^n)}{n}((-1)^{n+1}\frac{\mathrm{Tr}(Y^n)}{n}(-1)^{n+1}\frac{\mathrm{Tr}(Z^n)}{n} + (-1)^{n+1}) =$$
$$= 1 + (-1)^n(-1)^{n+1}\frac{\mathrm{Tr}(Z^n)}{n} = 1 - \frac{\mathrm{Tr}(Z^n)}{n}.$$

And finally, after substituting x, y, z:

$$xyz + x + z = 1.$$

□

Proposition 4.4.18. *The space* $\mu^{-1}(\varepsilon) \subset \operatorname{Rep}^*(\widetilde{A}_0, V)$ *is nonempty.*

Proof. Suppose that X and Z are invertible. We have seen that in this case the moment map becomes $ZXZ^{-1}X^{-1} = \varepsilon$. The set of matrices X, Z satisfying this equation is nonempty. Take for example Z diagonal, which implies that X has n nonzero terms, only on the first subdiagonal and in the top right corner. This provides a point in $\mu^{-1}(\varepsilon)$. □

Corollary 4.4.19. *The variety* $\mathcal{Q}(\widetilde{A}_0, \varepsilon, n)$ *is of dimension two.*

Theorem 4.4.20. *The affine geometric quotient*

$$\operatorname{Rep}^*(\widetilde{A}_0, V) \mathbin{/\!/}_{\varepsilon} \mathrm{GL}(V)$$

is isomorphic to the cubic surface

$$xyz + x + z = 1.$$

In other words, there is an isomorphism

$$\mathcal{Q}(\widetilde{A}_0, \varepsilon, n) \simeq \mathcal{Q}(\widetilde{A}_0, 1, 1).$$

Proof. All the work has already been done. The functions x, y, z generate the ring of invariants and satisfy the desired relation. Since $\mu^{-1}(\varepsilon)$ is nonempty, the quotient is of dimension two. There are no other relations between x, y, z since the affine cubic surface cut out by $xyz + x + z = 1$ is smooth and irreducible and its subvarieties are of positive codimension. □

4.5 The Painlevé II echo spaces

In this section we will study the Painlevé II echo space, defined in Section 4.1 as the wild Betti space $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{II}, \boldsymbol{\mathcal{C}}^n_{II})$. We are going to prove the following theorem.

Theorem. Let n be a positive integer. There are isomorphisms of echo spaces

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{II}, \boldsymbol{\mathcal{C}}^n_{II}) \simeq \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_{II}, \boldsymbol{\mathcal{C}}_{II}).$$

It is known that the first member of this family, the phase space of Painlevé II equation, is the Flaschka–Newell affine cubic surface cut out by equation (cf. Section 3C of [42])

$$xyz + x + y + z = b - b^{-1}$$

and the theorem states that all members of the Painlevé II echo family are isomorphic to this cubic (with different values of b).

Let V be a complex vector space of dimension $2n$. Suppose $V = W \oplus W$ with graded piece W of dimension n. Let $G = \mathrm{GL}(V), H = \mathrm{GL}(W) \times \mathrm{GL}(W)$ and U_+, U_- upper/lower triangular with off-diagonal block of size n. As before, we can identify:

$$G = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

where an asterisk denotes an $n \times n$ block and one stands for an identity matrix of size n.

For $G = \mathrm{GL}(V)$, the irregular type

$$Q_{II}^n = \frac{A_3}{z^3} + \frac{A_2}{z^2} + \frac{A_1}{z}$$

grades the space V into its eigenspaces. We have chosen the multiplicities of eigenvalues of A_3 in such way that there are only two eigenspaces, both of dimension n. The condition on centralisers of A_2, A_1 containing that of A_3 means that the two eigenspaces of A_3 do not split and the grading determined by Q_{II}^n is $V = W \oplus W$. Moreover, the centraliser of Q_{II}^n in G is H.

Lemma 4.5.1. *The irregular type Q_{II}^n determines six Stokes groups, which are U_+, U_- in alternating order.*

Proof. Since the leading term A_3 has two eigenvalues of multiplicity n and the degree of Q_{II}^n is three, there are six singular directions. The condition on centralisers implies that there is only one level thus the Stokes group at each direction does not break up and we obtain the full groups U_+ and U_-. □

Proposition 4.5.2. *The space*

$$\mathcal{B}(Q_{II}^n) = \{(h, S_1, \ldots, S_6) \in H \times (U_+ \times U_-) \mid hS_6 \cdots S_1 = 1\}$$

is a quasi-Hamiltonian H-space with moment map

$$\mu(h, \mathbf{S}) = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix},$$

where

$$\begin{aligned} f_1 &= 1 + s_5 s_2 + s_5 s_4 + s_4 s_3 + s_5 s_4 s_3 s_2 \\ f_2 &= 1 + s_4 s_3 - (s_4 s_3 s_2 + s_4 + s_2) f_1^{-1} (s_3 + s_5 + s_5 s_4 s_3) \end{aligned} \tag{4.3}$$

and s_i denotes the off-diagonal entry of S_i.

Proof. We know that $\mathcal{B}(Q_{II}^n)$ is a quasi-Hamiltonian H-space with moment map h^{-1}. The matrices S_2, S_3, S_4, S_5 uniquely determine h, S_1, S_6, since the condition $hS_6 \cdots S_1 = 1$ implies that $S_5 \cdots S_2$ is in the big cell $U_- H U_+$. The formula for the moment map is thus the block version of the usual LDU decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

applied to the entries of the matrix $M = S_5 S_4 S_3 S_2$. □

4.5.1 The space $\mathrm{Rep}^*(\widetilde{A}_1, V)$

Suppose that $V = W \oplus W$ is a graded vector space of dimension $2n$ and $G = \mathrm{GL}(V), H = \mathrm{GL}(W) \times \mathrm{GL}(W)$. By Corollary 4.2.5, the fission graph $\Gamma(Q^n_{II})$ is the affine A_1 graph with set of nodes $I = \{v_1, v_2\}$ and a double edge between them. Moreover, the irregular type Q^n_{II} determines an I-grading $V = W \oplus W$ and thus the dimension vector $n = (n, n)$ (the vector spaces at v_1, v_2 are both W).

A representation of the graph $\widetilde{A}_1$ on V is a choice of four maps $x_{12}, x_{21}, y_{12}, y_{21}$, where x_{ij}, y_{ij} go from the vector space at the node v_i to the vector space at the node v_j, both isomorphic to W. By Theorem 4.2.7 the space $\mathrm{Rep}^*(\widetilde{A}_1, V)$ is a quasi-Hamiltonian H-space isomorphic to $\mathcal{B}(Q^n_{II})$. The isomorphism between $\mathrm{Rep}^*(\widetilde{A}_1, V)$ and $\mathcal{B}(Q^n_{II})$ is given by sending the four maps x_{ij}, y_{ij} to the four entries of S_2, S_3, S_4, S_5:

$$(x_{12}, x_{21}, y_{12}, y_{21}) \mapsto (s_5, s_4, s_3, s_2).$$

We are going go study the multiplicative quiver variety

$$\mathcal{Q}(\widetilde{A}_1, q, n) = \mathrm{Rep}^*(\widetilde{A}_1, V) \mathbin{/\!\!/}_{q} H$$

with generic parameter q. Recall from Definition 4.2.14 that a parameter is generic if it obeys the condition

$$q^\alpha \neq 1$$

for any α in the finite set

$$R_\oplus(d) := \{\alpha \in \mathbb{Z}^I \mid (\alpha, \alpha) \leqslant 2 \text{ and } 0 \leqslant \alpha_i \leqslant d_i \text{ for all } i\} \setminus \{0, d\}.$$

Proposition 4.5.3. *A parameter $q = (q_1, q_2)$ is generic if $q_1 q_2 = \varepsilon$ is a primitive root of unity of degree n and both q_1, q_2 are not roots of unity of degree n (not necessarily primitive).*

Proof. The bilinear form on the lattice of $\widetilde{A}_1$ has the matrix

$$C = 2\mathrm{Id} - A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

hence $C(n_1, n_2) = 2(n_1 - n_2)^2$ and it is less or equal than two if $|n_1 - n_2| \leqslant 1$. This means that $R_\oplus(d)$ consists of elements $\alpha \in \mathbb{Z}^2$ such that both coordinates are equal or they differ by one. Therefore it implies that $(q_1 q_2)^k$ is not equal to one for $0 < k < n$ and that q_1, q_2 are not roots of unity of degree n. $\square$

4.5.2 Functions on $\mu^{-1}(q)$

For $V = W \oplus W$ of dimension $2n$, we will study the reduction of $\mathrm{Rep}^*(\widetilde{A}_1, V)$ by $H = \mathrm{GL}(W) \times \mathrm{GL}(W)$ at a generic value of the parameter q. Let v_1, v_2 denote the two nodes of the affine A_1 graph.

By Proposition 4.5.2 the space $\mathrm{Rep}^*(\widetilde{A}_1, V)$ is a quasi-Hamiltonian H-space and for a representation

$$\rho = (x_{12}, x_{21}, y_{12}, y_{21}) \in \mathrm{Rep}^*(\widetilde{A}_1, V)$$

the moment map (4.3), rewritten in the coordinates, x_{ij}, y_{ij}, is given by (μ_1, μ_2) where

$$\begin{aligned}
\mu_1 &= 1 + x_{12}x_{21} + x_{12}y_{21} + y_{12}y_{21} + x_{12}x_{21}y_{12}y_{21} \\
\mu_2 &= 1 + x_{21}y_{12} - (x_{21}y_{12}y_{21} + x_{21} + y_{21})\mu_1^{-1}(x_{12}x_{21}y_{12} + x_{12} + y_{12}).
\end{aligned}$$

Corollary 4.5.4. *For a parameter $q = (q_1, q_2)$ we have the following description of the space $\mu^{-1}(q)$:*

$$\mu^{-1}(q) = \begin{cases} 1 + x_{12}x_{21} + x_{12}y_{21} + y_{12}y_{21} + x_{12}x_{21}y_{12}y_{21} = q_1 \\ 1 + x_{21}y_{12} - (x_{21}y_{12}y_{21} + x_{21} + y_{21})q_1^{-1}(x_{12}x_{21}y_{12} + x_{12} + y_{12}) = q_2. \end{cases} \tag{4.4}$$

In order to compute the GIT quotient

$$\mathrm{Rep}^*(\widetilde{A}_1, V) \mathbin{/\!\!/}_q H$$

we need to understand the ring of $\mathrm{GL}(W) \times \mathrm{GL}(W)$-invariant functions on $\mu^{-1}(q)$. By the result of Le Bruyn and Procesi [62], the ring on invariants on this variety (which is an affine subvariety of $\mathrm{Rep}(\widetilde{A}_1, V)$) is finitely generated and the generating functions are traces of oriented cycles in the double of $\widetilde{A}_1$ of bounded length, and the bound depends on n. Again, the result is not constructive and we do not know a priori which traces to consider nor what the relations are.

Consider the following three cycles in the double of $\widetilde{A}_1$

$$X = 1 + y_{12}y_{21}, \quad Y = 1 + x_{12}x_{21}, \quad Z = 1 + y_{12}x_{21}.$$

Lemma 4.5.5. *The elements X, Y, Z satisfy the following identities:*

$$\begin{aligned}
(x_{12}x_{21}y_{12} + x_{12} + y_{12})y_{21} &= q_1 - Y \\
(x_{12}x_{21}y_{12} + x_{12} + y_{12})x_{21} &= YZ - 1 \\
x_{12}(x_{21}y_{12}y_{21} + x_{21} + y_{21}) &= q_1 - X \\
y_{12}(x_{21}y_{12}y_{21} + x_{21} + y_{21}) &= ZX - 1
\end{aligned}$$

Proof. The first identity follows from the fact that we have

$$1 + x_{12}x_{21} + x_{12}y_{21} + y_{12}y_{21} + x_{12}x_{21}y_{12}y_{21} = q_1,$$

and thus

$$x_{12}y_{21} + y_{12}y_{21} + x_{12}x_{21}y_{12}y_{21} = q_1 - (1 + x_{12}x_{21}) = q_1 - Y$$

and similarly for the third one. The second and fourth identities are just the consequences of definitions of X, Y, Z. □

Proposition 4.5.6. *The elements X, Y, Z obey the following relations:*

$$\begin{cases} ZXY - \varepsilon X - Y - q_1 Z + q_1 + \varepsilon = 0 \\ XYZ - X - \varepsilon Y - q_1 Z + q_1 + \varepsilon = 0 \\ ZXYZ - ZX - YZ - q_1 Z^2 + q_1 Z + \varepsilon Z - \varepsilon + 1 = 0 \\ XY + \varepsilon x_{12}y_{21} = q_1, \end{cases} \tag{4.5}$$

where $\varepsilon = q_1 q_2$ is a primitive root of unity of degree n.

Proof. The second equation of the moment map (4.4) reads:

$$1 + x_{21}y_{12} - (x_{21}y_{12}y_{21} + x_{21} + y_{21})q_1^{-1}(x_{12}x_{21}y_{12} + x_{12} + y_{12}) = q_2$$

so we can multiply it y_{12} on the left, y_{21} on the right and by Lemma 4.5.5 obtain, after substituting $q_1 q_2 = \varepsilon$

$$ZXY - \varepsilon X - Y - q_1 Z + q_1 + \varepsilon = 0.$$

Similarly, multiplication by x_{12} on the left and x_{21} on the right yields

$$XYZ - X - \varepsilon Y - q_1 Z + q_1 + \varepsilon = 0.$$

The third identity is obtained after multiplying the second equation of the moment map equation by y_{12} on the left and x_{21} on the right, and the last one after multiplying the equation by x_{12} on the left and y_{21} on the right. □

Corollary 4.5.7. *The pairs of matrices $(X, Y), (Y, Z), (Z, X)$ quasi-commute with parameters $q_1, 1$ and 1, respectively:*

$$\begin{cases} XY - \varepsilon YX = q_1(1 - \varepsilon) \\ YZ - \varepsilon ZY = (1 - \varepsilon) \\ ZX - \varepsilon XZ = (1 - \varepsilon). \end{cases} \tag{4.6}$$

Proof. The first equation of the moment map (4.4) can be written as

$$YX + x_{12}y_{21} = q_1,$$

which together with the fourth equation of (4.5)

$$XY + \varepsilon x_{12}y_{21} = q_1$$

implies the quasi-commutation for X, Y with parameter q_1. Multiplying the first equation of (4.5) by Z on the right and substracting the third equation yields the quasi-commutation (Z, X) and multiplying the second equation by Z on the left and substracting the third gives the desired relation for the pair (Y, Z). $\square$

The matrices X, Y, Z have similar properties as matrices X, Y, Z from the case of Painlevé I echo spaces. The proofs are almost the same as in that case as well.

Proposition 4.5.8. *Each of matrices X, Y, Z is either nilpotent or diagonalisable with n distinct eigenvalues of the form $t, \varepsilon t, \ldots, \varepsilon^{n-1}t$.*

Proof. Suppose that X is not nilpotent. Then by Proposition 4.3.2 either X is of desired form or X and Y share an eigenvector v for an eigenvalue λ_1 of X and q_1/λ_1 for Y. Similarly, if X is not of desired form, then X and Z share an eigenvector w for an eigenvalue λ_2 of X and $1/\lambda_2$ for Z.

From the first equation of (4.5) it follows that

$$(ZXY - \varepsilon X - Y - q_1 Z + q_1 + \varepsilon)v = 0$$

which implies $-\varepsilon\lambda - q_1/\lambda = -q_1 - \varepsilon$ and thus $\lambda_1 = 1$ or $\lambda_1 = q_1/\varepsilon$. The second equation of (4.5) is

$$XYZ - X - \varepsilon Y - q_1 Z + q_1 + \varepsilon = 0$$

which after substitution $XY = \varepsilon YX + q_1(1-\varepsilon)$ yields

$$\varepsilon YXZ - X - \varepsilon Y - q_1\varepsilon Z + q_1 + \varepsilon = 0.$$

Evaluation on w gives

$$-\lambda_2 - q_1\varepsilon/\lambda_2 = -q_1 - \varepsilon$$

and thus $\lambda_2 = q_1$ or ε. In any case, if $\lambda_2 = q_1$ or $\lambda_2 = \varepsilon$, w is not an eigenvector of Y (since sharing an eigenvector between X and Y implies that the eigenvalue λ of X for which it is shared is equal to 1 or q_1/ε). By Proposition 4.3.2 it implies that $\varepsilon\lambda_2$ is also an eigenvalue of X. By genericity of parameters, $\varepsilon^k\lambda_2 \neq \lambda_1$ for $0 \leqslant k < n-1$ and any of the four possible choices of λ_1, λ_2, so we can iterate Proposition 4.3.2 $n-1$ times and produce n distinct eigenvalues of X of desired form. Similar proof works for Y and Z. $\square$

Corollary 4.5.9. *The matrices X^n, Y^n, Z^n are scalar.*

Corollary 4.5.10. *For $0 < k < n$, the matrices X, Y, Z satisfy*

$$\mathrm{Tr}(X^k) = \mathrm{Tr}(Y^k) = \mathrm{Tr}(Z^k) = 0.$$

Let us introduce the following three functions

$$x = \frac{\mathrm{Tr}(X^n)}{n}, \quad y = \frac{\mathrm{Tr}(Y^n)}{n}, \quad z = \frac{\mathrm{Tr}(Z^n)}{n}.$$

Proposition 4.5.11. *The functions x, y, z generate the ring of H-invariant functions on $\mu^{-1}(q)$.*

Proof. Let W be a cycle in the double of $\widetilde{A}_1$, which is a word in x_{ij}, y_{ij}. By cyclic invariance of trace, we can suppose that the cycle starts and ends at the node v_1. W is a composition of four primitive cycles $x_{12}x_{21}, x_{12}y_{21}, y_{12}y_{21}, y_{12}x_{21}$ and $x_{12}x_{21}, y_{12}y_{21}, y_{12}x_{21}$ can be expressed directly by $Y-1, X-1, Z-1$. The fourth one, $x_{12}y_{21}$ can be expressed in terms of X, Y using the last equation of (4.5). Thus the trace of every word in the quiver is equal to the trace of a word in X, Y, Z.

The remaining part of the proof is exactly the same as the proof of Proposition 4.4.16 with the only difference being the relation used to shorten the word involving all three letters X, Y, Z, which now is

$$ZXY - \varepsilon X - Y - q_1 Z + q_1 + \varepsilon = 0.$$

$\square$

Proposition 4.5.12. *The functions x, y, z satisfy the relation*

$$xyz - x - y - q_1^n z + q_1^n + 1 = 0.$$

Proof. The equation $ZXY - \varepsilon X - Y - Y - q_1 Z + q_1 + \varepsilon = 0$ can be written as

$$(1-Z)(XY - q_1) = (1-X)(\varepsilon - Y)$$

and taking the determinant of both sides gives, by Propositions 4.3.4,4.3.5 and Lemma 4.3.9

$$\left(1 - \frac{\mathrm{Tr}(Z^n)}{n}\right)\left(\frac{\mathrm{Tr}(X^n)}{n}\frac{\mathrm{Tr}(Y^n)}{n} - q_1^n\right) = \left(1 - \frac{\mathrm{Tr}(X^n)}{n}\right)\left(1 - \frac{\mathrm{Tr}(Y^n)}{n}\right).$$

After substituting x, y, z, the relation becomes

$$xyz - x - y - q_1^n z + q_1^n + 1 = 1.$$

$\square$

Proposition 4.5.13. *The space $\mu^{-1}(q) \subset \mathrm{Rep}^*(\widetilde{A}_1, V)$ is nonempty.*

Proof. This will follow from one of the results of Chapter 3. Namely, there is an open subset of $\mu^{-1}(q)$ which is isomorphic to

$$\begin{cases} ABCD = q_1 \\ A^{-1}B^{-1}C^{-1}D^{-1} = q_2 \end{cases}$$

for $A, B, C, D \in \mathrm{GL}_n(\mathbb{C})$, which is in turn isomorphic to

$$ABCA^{-1}B^{-1}C^{-1} = \varepsilon,$$

which is nonempty. $\square$

Theorem 4.5.14. *Let $n = (n, n)$ be a dimension vector and $q = (q_1, q_2)$ a generic parameter. The affine geometric quotient*

$$\mathcal{Q}(\widetilde{A}_1, q, n) = \mathrm{Rep}^*(\widetilde{A}_1, V) /\!\!/_q H$$

is isomorphic to the affine cubic surface

$$xyz - x - y - q_1^n z + q_1^n + 1 = 0.$$

Moreover, there is an isomorphism

$$\mathcal{Q}(\widetilde{A}_1, q, n) \simeq \mathcal{Q}(\widetilde{A}_1, q^n, 1).$$

Proof. All the work has already been done. The functions x, y, z generate the ring of invariants and satisfy the desired relation. Since $\mu^{-1}(q)$ is nonempty, the quotient is of dimension two. There are no other relations between x, y, z since the affine cubic surface cut out by $xyz - x - y - q_1^n z + q_1^n + 1$ is smooth and irreducible and its subvarieties are of positive codimension. $\square$

Proposition 4.5.15. *The cubic surface $xyz - x - y - q_1^n z + q_1^n + 1 = 0$ is isomorphic to the Flashka–Newell cubic surface*

$$xyz + x + y + z = b - b^{-1}$$

Proof. The substitution

$$q_1^n = -1/b^2, \quad x = \widetilde{x}/b, \quad y = \widetilde{y}/b, \quad z = -\widetilde{z}b$$

Transforms the equation into

$$\widetilde{x}\widetilde{y}\widetilde{z} + \widetilde{x} + \widetilde{z} + \widetilde{z} = b - b^{-1}.$$

$\square$

Corollary 4.5.16. *The wild character variety $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{II}, \boldsymbol{\mathcal{C}}^n_{II})$ is of complex dimension two and is isomorphic to the Flashka–Newell cubic surface*

$$xyz + x + y + z = b - b^{-1}$$

with parameter $b = -q_1^{-n/2}$.

4.6 The Painlevé IV echo spaces

In this section we will study the Painlevé IV echo space, defined in Section 4.1 as the wild Betti space $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{IV}, \boldsymbol{\mathcal{C}}^n_{IV})$. We are going to prove the following theorem.

Theorem. Let n be a positive integer. There are isomorphisms of echo spaces

$$\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}^n_{IV}, \boldsymbol{\mathcal{C}}^n_{IV}) \simeq \mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma}_{IV}, \boldsymbol{\mathcal{C}}_{IV}).$$

It is known that the first member of this family, the phase space of Painlevé IV equation, is the cubic surface cut out by equation

$$xyz + x^2 + c_1 x + c_2 y + c_3 z + c_4 = 0$$

for suitable constants $c_1, c_2, c_3, c_4 \in \mathbb{C}$ and the theorem states that all members of the Painlevé IV echo family are isomorphic to this cubic (so the constants match up as well).

Let V be a complex vector space of dimension $3n$. Suppose $V = W \oplus W \oplus W$ with graded piece W of dimension n. Let $G = \mathrm{GL}(V), H = \mathrm{GL}(W) \times \mathrm{GL}(W) \times \mathrm{GL}(W)$ and U_+, U_- upper/lower triangular with off-diagonal blocks of size n. As before, we can identify:

$$G = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, H = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, U_+ = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, U_- = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix},$$

where an asterisk denotes an $n \times n$ block and one stands for an identity matrix of size n. For two distinct positive integers i, j not greater than three, we define the six groups U_{ij} to be the unipotent subgroup of G with off-diagonal entries in the block (i, j) of size $n \times n$.

For $G = \mathrm{GL}(V)$, the irregular type

$$Q^n_{IV} = \frac{A_2}{z^2} + \frac{A_1}{z}$$

grades the space V by its eigenspaces. We have chosen the multiplicities of eigenvalues of A_2 in such way that there are only three eigenspaces, all of dimension n. The condition on centraliser of A_1 containing the that of A_2 means that the three eigenspaces of A_2 do not split and the grading determined by Q^n_{IV} is $V = W \oplus W \oplus W$. Moreover, the centraliser of Q^n_{IV} in G is H.

Lemma 4.6.1. *The irregular type Q^n_{IV} determines four full Stokes groups, which may be identified U_+, U_- in alternating order.*

Proof. This is the block version of the commutative $\mathrm{GL}_3(\mathbb{C})$ picture. The leading term A_2 has three eigenvalues of multiplicity n and the degree k of Q^n_{IV} is two. If

q_1, q_2, q_3 are the eigenvalues of A_2, then the roots $\alpha_{ij} = q_i - q_j$ have at most six possible values, all in the set $\{\pm(q_1 - q_2), \pm(q_2 - q_3), \pm(q_3 - q_1)\}$ and if all three differences are different, then each singular direction is supported by exactly n^2 roots (it is possible to have two differences of eigenvalues equal; then there are less singular directions, but in turn two of them are supported by $2n^2$ roots). Then the six Stokes groups $\mathbb{S}\mathrm{to}_d$ can be identified with six groups U_{ij}.

Since the degree of Q is two, the half-periods $d_i \subset \mathbb{A}$ is of length $\#\mathbb{A}/2k = 3$ (of length two if the differences between eigenvalues are not pairwise distinct). Dividing the twelve (eight) singular directions into four consecutive half-periods d_1, d_2, d_3, d_4 of length three (resp. two) gives four full Stokes groups, directly spanned by the groups $\mathbb{S}\mathrm{to}_d$ for $d \in d_i$. We can choose the half-periods d_1, d_2, d_3, d_4 so that they contain the three groups U_{ij} for $i < j$ and thus directly span the group U_+ or the groups U_{ij} for $i > j$, so they directly span the opposite group U_-. $\square$

Thus the we can describe $\mathcal{B}(Q^n_{IV})$ as

$$\mathcal{B}(Q^n_{IV}) = \{(h, S_1, \dots, S_4) \in H \times (U_+ \times U_-)^2 \mid hS_4 \cdots S_1 = 1\}.$$

We will use the following notations

$$S_3 = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix}.$$

Proposition 4.6.2. *The space*

$$\mathcal{B}(Q^n_{IV}) = \{(h, S_1, \dots, S_4) \in H \times (U_+ \times U_-)^2 \mid hS_4 \cdots S_1 = 1\}$$

is a quasi-Hamiltonian H-space with moment map

$$\mu(h, \mathbf{S}) = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix},$$

where

$$\begin{aligned} f_1 &= 1 + x_{12}x_{21} + x_{13}x_{31} \\ f_2 &= 1 + x_{23}x_{32} - X_{21}f_1^{-1}X_{12} \\ f_3 &= 1 - x_{31}f_1^{-1}x_{13} - (x_{32} - x_{31}f_1^{-1}X_{12})f_2^{-1}(x_{23} - X_{21}f_1^{-1}x_{13}), \end{aligned} \tag{4.7}$$

where $X_{21} = x_{21} + x_{23}x_{31}, X_{12} = x_{12} + x_{13}x_{32}$.

Proof. We know that $\mathcal{B}(Q^n_{IV})$ is a quasi-Hamiltonian H-space with moment map h^{-1}. The matrices S_3, S_2 uniquely determine h, S_1, S_4, since the condition $hS_4 \cdots S_1 = 1$ implies that the product S_3S_2 is in the big cell U_-HU_+. The formula for the moment map is thus the block version of the usual LDU decomposition for a 3×3 matrix applied for the entries of

$$M = S_3 S_2 = \begin{pmatrix} 1 + x_{12}x_{21} + x_{13}x_{31} & x_{12} + x_{13}x_{32} & x_{13} \\ x_{21} + x_{23}x_{31} & 1 + x_{23}x_{32} & x_{23} \\ x_{31} & x_{32} & 1 \end{pmatrix}$$

and can be checked by a direct computation. □

4.6.1 The space $\mathrm{Rep}^*(\widetilde{A}_2, V)$

Suppose that $V = W \oplus W \oplus W$ is a graded vector space of dimension $3n$ and $G = \mathrm{GL}(V)$, $H = \mathrm{GL}(W) \times \mathrm{GL}(W) \times \mathrm{GL}(W)$. By Corollary 4.2.6, the fission graph $\Gamma(Q^n_{IV})$ is the affine A_2 graph which is the full 3-partite graph with three nodes. In other words, it is a triangle with set of nodes $I = \{v_1, v_2, v_3\}$. Moreover, the irregular type Q^n_{IV} determines an I-grading $V = W \oplus W \oplus W$ and thus the dimension vector $n = (n, n, n)$ (the vector spaces at v_1, v_2, v_3 are all W).

A representation of the graph $\widetilde{A}_2$ on V is a choice of six maps

$$x_{12}, x_{21}, x_{13}, x_{31}, x_{23}, x_{32},$$

where x_{ij} go from the vector space at the node v_i to the vector space at the node v_j, both isomorphic to W. By Theorem 4.2.7 the space $\mathrm{Rep}^*(\widetilde{A}_2, V)$ is a quasi-Hamiltonian H-space isomorphic to $\mathcal{B}(Q^n_{IV})$. The isomorphism between $\mathrm{Rep}^*(\widetilde{A}_1, V)$ and $\mathcal{B}(Q^n_{IV})$ is given by sending the six maps x_{ij} six entries of S_2, S_3 (which we have labeled in the same way).

We are going go study the multiplicative quiver variety

$$\mathcal{Q}(\widetilde{A}_2, q, n) = \mathrm{Rep}^*(\widetilde{A}_2, V) \mathbin{/\!\!/}_{q} H$$

with generic parameter q.

Proposition 4.6.3. *A parameter $q = (q_1, q_2, q_3)$ is generic if $q_1 q_2 q_3 = \varepsilon$ is a primitive root of unity of degree n and both q_1, q_2, q_3 are not roots of unity of degree n (not necessarily primitive).*

Proof. The bilinear form on the lattice of $\widetilde{A}_2$ has the matrix

$$C = 2\mathrm{Id} - A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

hence $C(n_1, n_2, n_3) = (n_1 - n_2)^2 + (n_2 - n_3)^2 + (n_3 - n_1)^2$ and it is less or equal than two if $|n_i - n_j| \leqslant 1$, which implies that at least two coordinates n_i are equal. This means that $R_\oplus(d)$ consists of elements $\alpha \in \mathbb{Z}^3$ such that two coordinates n_i are equal and the third differs by one. Therefore it implies that $(q_1 q_2 q_3)^k$ is not equal to one for $0 < k < n$ and that q_1, q_2, q_3 are not roots of unity of degree n. □

4.6.2 Functions on $\mu^{-1}(q)$

For $V = W \oplus W \oplus W$ of dimension $3n$, we will study the reduction of $\mathrm{Rep}^*(\widetilde{A}_2, V)$ by $H = \mathrm{GL}(W) \times \mathrm{GL}(W) \times \mathrm{GL}(W)$ at a generic value of the parameter q. Let v_1, v_2, v_3 denote the three nodes of the affine A_2 graph, which is a triangle.

By Proposition 4.6.2 the space $\mathrm{Rep}^*(\widetilde{A}_2, V)$ is a quasi-Hamiltonian H-space and for a representation

$$\rho = (x_{12}, x_{21}, x_{13}, y_{31}, x_{23}, x_{32}) \in \mathrm{Rep}^*(\widetilde{A}_1, V)$$

the moment map is given by equation (4.7)

Corollary 4.6.4. *For a parameter $q = (q_1, q_2, q_3)$ we have the following description of the space $\mu^{-1}(q)$:*

$$\mu^{-1}(q) = \begin{cases} 1 + x_{12}x_{21} + x_{13}x_{31} = q_1 \\ 1 + x_{23}x_{32} - q_1^{-1}X_{21}X_{12} = q_2 \\ 1 - q_1^{-1}x_{31}x_{13} - q_2^{-1}(x_{32} - q_1^{-1}x_{31}X_{12})(x_{23} - q_1^{-1}X_{21}x_{13}) = q_3, \end{cases} \tag{4.8}$$

where $X_{21} = x_{21} + x_{23}x_{31}, X_{12} = x_{12} + x_{13}x_{32}$.

In order to compute the GIT quotient

$$\mathrm{Rep}^*(\widetilde{A}_2, V) /\!\!/_q H$$

we need to understand the ring of H-invariant functions on $\mu^{-1}(q)$, which is again, by the result of Le Bruyn and Procesi [62], generated by traces of oriented cycles in the double of $\widetilde{A}_2$ of bounded length, and the bound depends on n.

For any generic parameter $q = (q_1, q_2, q_3)$ and a representation

$$\rho = (x_{12}, x_{13}, x_{23}, x_{21}, x_{31}, x_{32}) \in \mu^{-1}(q) \subset \mathrm{Rep}^*(\widetilde{A}_2, V)$$

define the following four cycles in the affine quiver $\widetilde{A}_2$:

$$\begin{aligned} A(\rho) &= 1 + x_{21}x_{12}, B(\rho) = 1 + x_{23}x_{32}, \\ X(\rho) &= x_{21}x_{13}x_{32} + x_{21}x_{12} - q_1 \\ Y(\rho) &= x_{23}x_{31}x_{12} + x_{21}x_{12} - q_1. \end{aligned} \tag{4.9}$$

(We denote the cycles by X, Y instead of C, D to highlight the fact that they are of different nature and involve a term making a full turn around the triangle.)

Fix $q = (q_1, q_2, q_3)$ a generic parameter and $\rho \in \mu^{-1}(q)$. For simplicity, denote $A(\rho), B(\rho), X(\rho), Y(\rho)$ by A, B, X, Y.

Lemma 4.6.5. *The elements A, X, Y satisfy the following identities:*

$$q_1x_{23}(x_{32} - q_1^{-1}x_{31}X_{12}) = X + q_1q_2$$
$$q_1(x_{23} - q_1^{-1}X_{21}x_{13})x_{32} = Y + q_1q_2$$
$$q_1x_{21}x_{13}(x_{32} - q_1^{-1}x_{31}X_{12}) = AX + q_1$$
$$q_1(x_{23} - q_1^{-1}X_{21}x_{13})x_{31}x_{12} = YA + q_1$$

Proof. For the first identity, observe that the term $x_{23}x_{31}X_{12}$ is equal to $(X_{21}-x_{21})X_{12}$ and we can substitute $X_{21}X_{12}$ using the second equation of the moment map (4.8) and $x_{21}X_{12} = X+q_1$. Hence we can expand the formula and obtain the identity. The same proof works for the second one, since $x_{13}x_{32} = X_{12} - x_{12}$ and $X_{21}x_{12} = Y + q_1$.

For the third, replace $x_{13}x_{31}$ by $q_1 - 1 - x_{12}x_{21}$ in the term $x_{21}x_{13}x_{31}X_{12}$, using the first equation of (4.8). Then again using the fact that $x_{21}X_{12} = X + q_1$ we can simplify and expand the formula, obtaining the identity. We get the last identity in a similar manner, by replacing $x_{13}x_{31}$ in $X_{21}x_{13}x_{31}x_{12}$ and using the fact that $X_{21}x_{12} = Y + q_1$. $\square$

Proposition 4.6.6. *The cycles A, B, X, Y obey the following relations.*

$$\begin{cases} -XY - q_1q_2A - \varepsilon q_1B + k_1 = 0 \\ -XYA - q_1X - \varepsilon q_1Y - q_1q_2A^2 + k_1A - k_2 = 0 \\ -AXY - \varepsilon q_1X - q_1Y - q_1q_2A^2 + k_1A - k_2 = 0 \\ -(AX+q_1)(YA+q_1) - q_1q_2A^3 + k_1A^2 - k_2A + \varepsilon q_1^2 = 0 \end{cases} \tag{4.10}$$

with $k_1 = q_1^2q_2 + q_1q_2 + \varepsilon q_1$, $k_2 = q_1^2q_2 + \varepsilon q_1^2 + \varepsilon q_1$ and $\varepsilon = q_1q_2q_3$ is a primitive root of unity of degree n.

Proof. First, multiply the last equation of the moment map (4.8) by $q_1^2q_2$ and replace $q_1q_2q_3$ by ε. Then multiply it by x_{23} on the left and x_{32} on the right. Using Lemma 4.6.5 the big product involving X_{12} and X_{21} simplifies to $(X + q_1q_2)(Y + q_1q_2)$.

The term $q_1q_2x_{23}x_{31}x_{13}x_{32}$ is equal to $q_1q_2(X_{21} - x_{21})(X_{12} - x_{12})$ and thus can be expressed in X, Y, A, B. Expanding the formula yields the first identity.

The second identity is obtained by multiplying the last equation of the moment map by x_{23} on the left and $x_{31}x_{12}$ on the right. Again, by Lemma 4.6.5 the big product involving X_{12} and X_{21} will simplify. We expand the remaining terms by replacing $x_{23}x_{31}x_{13}x_{32}$ by $(X_{21} - x_{21})(X_{12} - x_{12})$ and $x_{13}x_{31}$ by $q_1 - 1 - x_{12}x_{21}$. Gathering all the terms together and expressing them in terms of A, B, X, Y yields the identity.

The third identity is obtained in a similar fashion, by multiplying the last equation of the moment map by $x_{21}x_{13}$ on the left and x_{32} on the right. The last one is proved in the same manner after multiplying the equation by $x_{21}x_{13}$ on the left and $x_{31}x_{12}$ on the right. $\square$

Corollary 4.6.7. *The pairs of matrices (Y, A) and (A, X) quasi-commute with parameter $-q_1$:*

$$\begin{cases} YA - \varepsilon AY = -q_1(1-\varepsilon) \\ AX - \varepsilon XA = -q_1(1-\varepsilon). \end{cases} \tag{4.11}$$

Proof. Multiplying the second equation of (4.10) by A on the left and substracting the fourth yields

$$YA - \varepsilon AY = -q_1(1-\varepsilon)$$

and multiplying the third equation of (4.10) by A on the right and substracting the fourth yields

$$AX - \varepsilon XA = -q_1(1-\varepsilon).$$

□

Given the two Stokes multipliers S_2, S_3, the remaining elements h, S_1, S_4 are uniquely determined. Hence if S_3S_2 is in the big cell U_-HU_+ and can be written as

$$S_3S_2 = S_4^{-1}h^{-1}S_1^{-1},$$

then we also have

$$\begin{aligned} S_3^{-1}S_4^{-1} &= S_2h(h^{-1}S_1h) \in U_-HU_+ \\ S_1^{-1}S_2^{-1} &= (hS_4h^{-1})hS_3 \in U_-HU_+. \end{aligned}$$

Observe that the block diagonal element h of the LDU decomposition for the pair S_2, S_3 becomes h^{-1} for the pairs S_3^{-1}, S_4^{-1} and S_1^{-1}, S_2^{-1}.

Corollary 4.6.8. *A representation $\rho \in \mu^{-1}(h)$ determines two representations $\rho', \rho'' \in \mu^{-1}(h^{-1})$, corresponding to the LDU decomposition of $S_3^{-1}S_4^{-1}$ and $S_1^{-1}S_2^{-1}$, respectively. In other words, for each $h \in H$ there are two maps*

$$p_1^h, p_2^h : \mu^{-1}(h) \to \mu^{-1}(h^{-1})$$

given by $p_1^q(\rho) = \rho'$ and $p_2^q(\rho) = \rho''$. These maps satisfy

$$p_1^{h^{-1}} \circ p_1^h = p_2^{h^{-1}} \circ p_2^h = \mathrm{Id}.$$

Proposition 4.6.9. *Let $q = (q_1, q_2, q_3) \in H$ be a generic parameter and denote $\tau = p_1^q(\rho) \in \mu^{-1}(q^{-1})$. The following relations are satisfied:*

$$\begin{aligned} A(\tau) = q_1^{-1}Y(\rho), &\quad B(\tau) = -q_1^{-1}q_2^{-1}X(\rho), \\ X(\tau) = -q_1^{-1}q_2^{-1}B(\rho), &\quad Y(\tau) = -q_1^{-1}A(\rho). \end{aligned}$$

Proof. By direct computation, which is completely algebraic. Determine the entries of matrices S_3^{-1}, S_4^{-1} in terms of the elements x_{ij} of ρ and thus the representation ρ'. Then compute the expressions for $A(\rho'), B(\rho'), X(\rho'), Y(\rho')$, which will be (non-commutative) polynomials in symbols x_{ij} and use the first two moment map equations (4.7) to eliminate the elements $x_{13}x_{31}$ and $x_{23}x_{31}x_{13}x_{32}$ appearing there. After all simplifications, the resulting formulas are as above. □

Proposition 4.6.10. *Let $q = (q_1, q_2, q_3) \in H$ be a generic parameter and denote $\tau = p_2^q(\rho) \in \mu^{-1}(q^{-1})$. The following relations are satisfied:*

$$A(\tau) = q_1^{-1}X(\rho), \quad B(\tau) = -q_1^{-1}q_2^{-1}Y(\rho),$$
$$X(\tau') = -q_1^{-1}A(\rho), \quad Y(\tau) = -q_1^{-1}q_2^{-1}B(\rho).$$

Hence given a representation $\rho \in \mu^{-1}(q)$, we can send it to $\rho' \in \mu^{-1}(q)$, using p_1^q and then to $\tau \in \mu^{-1}(q)$, using $p_2^{q^{-1}}$. As a direct consequence of Propositions 4.6.9,4.6.10, we get the following corollary.

Corollary 4.6.11. *Let $\tau = p_2^{q^{-1}} \circ p_1^q(\rho) \in \mu^{-1}(q)$. The following relations are satisfied:*

$$A(\tau) = q_2^{-1}B(\rho), B(\tau) = q_2A(\rho), X(\tau) = Y(\rho), Y(\tau) = X(\rho).$$

Remark 4.6.12. These maps are simple examples of isomonodromy isomorphisms from Section 3. of [16].

Corollary 4.6.13. *The pairs of matrices (X, B) and (B, Y) quasi-commute with parameter $-q_1q_2$:*

$$\begin{cases} XB - \varepsilon BX = -q_1q_2(1-\varepsilon) \\ BY - \varepsilon YB = -q_1q_2(1-\varepsilon). \end{cases} \tag{4.12}$$

Proof. This follows from Corollary 4.6.11, since the elements $q_2^{-1}B, X, Y$ for a representation ρ become A, Y, X for the representation $\tau = p_2^{q^{-1}} \circ p_1^q(\rho)$ and since $\tau \in \mu^{-1}(q)$, they obey the quasi-commuting relations. □

Corollary 4.6.14. *The matrices A, B, X, Y satisfy the following relations*

$$\begin{aligned} \varepsilon AB - Y - \varepsilon X - q_1^{-1}k_2 = 0, \\ \varepsilon BA - \varepsilon Y - X - q_1^{-1}k_2 = 0. \end{aligned} \tag{4.13}$$

And thus

$$BA - \varepsilon AB = (\varepsilon - \varepsilon^{-1})X - (\varepsilon^{-1} - 1)(q_1q_2 + \varepsilon q_1 + \varepsilon).$$

Proof. Relations (4.13) are the images of the relation

$$-XY - q_1q_2A - \varepsilon q_1B + k_1 = 0$$

under the maps p_1^q and p_2^q. The third relation follows immediately. □

We have established multiple relations between A, B, X, Y and we will gather the necessary ones together.

Proposition 4.6.15. *Let $q = (q_1, q_2, q_3)$ be a generic parameter and let $\rho \in \mu^{-1}(q)$. Denote $A(\rho), B(\rho), Y(\rho)$ by A, B, Y. Then the following relations are satisfied:*

$$\begin{cases} YA - \varepsilon AY = -q_1(1-\varepsilon) \\ BY - \varepsilon YB = -q_1q_2(1-\varepsilon) \\ AB - \varepsilon BA = (\varepsilon - \varepsilon^{-1})Y - (\varepsilon^{-1} - 1)q_1^{-1}k_2 \\ -BYA + \varepsilon Y^2 + q_1^{-1}k_2Y - q_1q_2A - q_1B + k_1 = 0, \end{cases} \tag{4.14}$$

with $k_1 = q_1^2q_2 + q_1q_2 + \varepsilon q_1, k_2 = q_1^2q_2 + \varepsilon q_1^2 + \varepsilon q_1$.

Proof. The first three relations have already beed established. The last one is the image of

$$-XYA - q_1X - \varepsilon q_1Y - q_1q_2A^2 + k_1A - k_2 = 0$$

under the map p_1^q, combined with the first relation used to change the order from BAY to BYA. $\square$

Proposition 4.6.16. *Each of matrices A, B, Y is either nilpotent or diagonalisable with n distinct eigenvalues $t, \varepsilon t, \ldots, \varepsilon^{n-1}t$*

Proof. Suppose that $n > 2$. By Propositions 4.6.9,4.6.10 it is enough to prove it for one matrix, since it can be further mapped to the remaining ones via the transformations p_1 and p_2. The proof is similar to the previous cases. Suppose that Y is not nilpotent. Then by Proposition 4.3.2 Y is either of the desired form, or it shares an eigenvector v with A for an eigenvalue λ_1 of Y and $-q_1/\lambda_1$ of A. We have $YAv = -q_1v$, thus the evaluation

$$(-BYA - \varepsilon Y^2 + q_1^{-1}k_2Y - q_1q_2A - q_1B - k_1)v = 0$$

yields

$$-\varepsilon\lambda_1^2 + q_1^{-1}k_2\lambda_1 + q_1^2q_2/\lambda_1 - k_1 = 0$$

which has three roots: $1, q_1, q_1q_2/\varepsilon$, giving three possible values of λ_1. Similarly, Y is either of the desired form, or it shares an eigenvector w with B for an eigenvalue λ_2 of Y and $-q_1q_2/\lambda_2$ of B.

Using the first and third relation of (4.14) to swap A with Y and then with B, we can transform the fourth equation of (4.14) into

$$-ABY - \varepsilon^{-1}Y^2 + \varepsilon^{-1}q_1^{-1}k_2Y - q_1q_2A - \varepsilon q_1B - k_1 = 0$$

Evaluation on w yields

$$-\varepsilon^{-1}\lambda_2^2 - \varepsilon^{-1}q_1^{-1}k_2\lambda_2 + \varepsilon q_1^2q_2/\lambda_2 - k_1 = 0,$$

which has three roots: $\varepsilon, \varepsilon q_1, q_1 q_2$, giving three possible values for λ_2. The proof is now the same as in the previous cases. By genericity of parameters, $\varepsilon^k \lambda_2 \neq \lambda_1$ for $0 \leqslant k < n-1$ for any of the nine choices of λ_1, λ_2, so we can iterate proposition 4.3.2 $n-1$ times and produce n distinct eigenvalues of Y of desired form.

If $n = 2$, then $q_1 q_2 = \varepsilon = -1$ and there are two possibilities for eigenvalues λ_1, λ_2 instead of three. The proof remains the same. □

Corollary 4.6.17. *The matrices A^n, B^n, X^n, Y^n are scalar.*

Corollary 4.6.18. *For $0 < k < n$, the matrices A, B, X, Y satisfy*

$$\operatorname{Tr}(A^k) = \operatorname{Tr}(B^k) = \operatorname{Tr}(X^k) = \operatorname{Tr}(Y^k) = 0.$$

Let us introduce the following three functions:

$$a = \frac{\operatorname{Tr}(A^n)}{n}, \quad b = \frac{\operatorname{Tr}(B^n)}{n}, \quad y = (-1)^n \frac{\operatorname{Tr}(Y^n)}{n}$$

Proposition 4.6.19. *The functions a, b, y generate the ring of H-invariant functions on $\mu^{-1}(q)$.*

Proof. Let W be a cycle in the double of $\widetilde{A}_2$. If it does not pass through the vertex v_2, then it only passes through vertices v_1 and v_3 and by cyclic invariance of trace we can suppose that it starts and ends in v_1 and bounces between v_1 and v_3. Now we can use the first equation of the moment map and replace each copy of $x_{13}x_{31}$ by $q_1 - 1 - x_{12}x_{21}$, so its trace is equal to the trace of a cycle that bounces between v_1 and v_2. Again by cyclic invariance of the trace, we can now suppose that it starts and ends at v_2, so the trace of W is equal to the trace of a cycle W' starting and ending at v_2.

Consider a cycle C satisfying this property. We will show by induction on its length that it can be written as a word in A, B, Y. First few cases need to be addressed by hand. We have dealt with $x_{13}x_{31}$ and $x_{21}x_{12} = A - 1, x_{23}x_{32} = B - 1$, so cycles of length two are done. There are only two cycles of length three: $x_{21}x_{13}x_{32}$ and $x_{23}x_{31}x_{12}$, which can be written as words in X, Y, A. It follows from equation (4.13) that we can eliminate X and work in Y, A, B. The non-obious cycles of length four are $x_{23}x_{31}x_{13}x_{32}$, which is dealt with using the second equation of the moment map (4.7), and $x_{21}x_{13}x_{31}x_{13}$, where we can replace $x_{13}x_{31}$ by $q_1 - 1 - x_{12}x_{21}$ and it becomes a word in A (a polynomial of second degree).

Suppose that C is of length n, starting and ending at v_2. If it passes through this vertex more than once, then it splits into a product of two cycles of smaller length so we can use the inductive hypothesis and write them both as words in A, BY. If it does not, it means that it leaves v_2, bounces between v_1 and v_3 least twice (we have dealt with the case if it goes between v_1 and v_3 only once), and then comes back . However, this implies that in its expression there is $x_{13}x_{31}$ which we can replace by

$q_1 - 1 - x_{12}x_{21}$ and obtain a sum of a cycle of length $n-2$ and a cycle that passes through v_2 more than once. By the inductive hypothesis, the claim is proved.

The remaining part of the proof is exactly the same as the proof of Proposition 4.4.16 with two minor differences. The relation used to shorten the word involving all three letters A, B, Y is different and we can choose it to be for example

$$-BYA + \varepsilon Y^2 + q_1^{-1}k_2Y - q_1q_2A - q_1B + k_1 = 0.$$

The second difference is that A, B do not quasi-commute, but every time we swap them using the relation

$$AB - \varepsilon BA = (\varepsilon - \varepsilon^{-1})Y - (\varepsilon^{-1} - 1)q_1^{-1}k_2$$

the extra term involving Y is of length one, hence we can still apply the inductive hypothesis. □

Proposition 4.6.20. *The functions a, b, y satisfy the relation*

$$aby + y^2 - (q_1^nq_2^n + q_1^n + 1)y - q_1^nq_2^na - q_1^nb + q_1^{2n}q_2^n + q_1^nq_2^n + q_1^n = 0.$$

Proof. Observe that the relation

$$-BYA + \varepsilon Y^2 + q_1^{-1}k_2Y - q_1q_2A - q_1B + k_1 = 0.$$

factorises as follows

$$(BY + q_1q_2)(1 - A) = (B - \varepsilon Y - \varepsilon - q_1q_2)(Y + q_1)$$
$$(BY + q_1q_2)(1 - A) = -\varepsilon\left(Y - \frac{B}{\varepsilon} + 1 + \frac{q_1q_2}{\varepsilon}\right)(Y + q_1).$$

We can thus take the determinant of both sides and obtain an expression in a, b, y. This is the same as in the proof of Propositions 4.5.12, 4.4.17 and we use Proposition 4.3.6 to deal with the term $\det(Y - \frac{B}{\varepsilon} + 1 + \frac{q_1q_2}{\varepsilon})$ □

Proposition 4.6.21. *The space $\mu^{-1}(q) \subset \mathrm{Rep}^*(\widetilde{A}_2, V)$ is nonempty.*

Proof. Denote by $H(\partial)$ the following subset of $\mathrm{GL}(V)$, which we identify with $3n \times 3n$ matrices:

$$H(\partial) = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix},$$

where asterisks denote $n \times n$ blocks and consider a matrix

$$U = \begin{pmatrix} 1 & u_1 & u_2 \\ 0 & 1 & u_3 \\ 0 & 0 & 1 \end{pmatrix} \in U_+.$$

A direct computation shows that if $u_1, 1 - u_3 u_2^{-1} u_1$ are invertible, then there exist $L_1, L_2 \in U_-$ such that

$$L_1 U L_2 = h = \begin{pmatrix} 0 & 0 & h_1 \\ 0 & h_2 & 0 \\ h_3 & 0 & 0 \end{pmatrix} \in H(\partial)$$

with

$$h_1 = u_2, \quad h_2 = 1 - u_3 u_2^{-1} u_1, \quad h_3 = -u_2^{-1}(1 - u_1 u_3 u_2^{-1})^{-1}.$$

In other words, with these conditions we have $M \in U_- H(\partial) U_-$. If we futher suppose that u_3 is invertible and set $X_1 = u_2, X_2 = u_3 u_2^{-1}, X_3 = u_1 - u_2 u_3^{-1}$, then h becomes

$$h = \begin{pmatrix} 0 & 0 & X_1 \\ 0 & X_2 X_3 & 0 \\ -X_1^{-1} X_2^{-1} X_3^{-1} & 0 & 0 \end{pmatrix}.$$

Similarly, for a matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & l_3 & 1 \end{pmatrix} \in U_-$$

and $l_2, 1 - l_1 l_2^{-1} l_3, l_1$ invertible, we have $L \in U_+ H(\partial) U_+$ and the element $h \in H(\partial)$ such that $L = U_1 h U_2$ can be written as

$$h = \begin{pmatrix} 0 & 0 & -Y_1^{-1} Y_2^{-1} Y_3^{-1} \\ 0 & Y_2 Y_3 & 0 \\ Y_1 & 0 & 0 \end{pmatrix}$$

with $Y_1 = l_2, Y_2 = l_1 l_2^{-1}, Y_3 = l_3 - l_2 l_1^{-1}$.

Suppose that matrices $A \in U_+, B \in U_-$ satisfy

$$A \in U_- H(\partial) U_-, \quad B \in U_+ H(\partial) U_+,$$

so we can write

$$h_1 A_1 A A_2 = 1, \quad h_2 B_1 B B_2 = 1$$

for some $h_1, h_2 \in H(\partial), A_1, A_2 \in U_-, B_1, B_2 \in U_+$ Then this gives a point

$$(h, S_1, S_2, S_3, S_4) \in H \times (U_+ \times U_-)^2$$

satisfying $h S_4 S_3 S_2 S_1 = 1$, thus a point in $\mathcal{B}(Q_{IV})$. More explicitly, we set

$$h = h_2 h_1, \quad S_4 = A_1, \quad S_3 = AB_1, \quad S_2 = B, \quad S_1 = B_2 h_2 A_2 h_2^{-1}.$$

Therefore to provide a point in $\mu^{-1}(q)$ it is enough to find two unipotents $A \in U_+, B \in U_-$ such that $A = U_- h_1 U_-, B = U_+ h_2 U_+$ and $h_2 h_1 = q$. This is equivalent to showing that the following set of matrices is nonempty

$$\begin{cases} X_1 Y_1 = q_1 \\ X_2 X_3 Y_2 Y_3 = q_2 \\ X_1^{-1} X_2^{-1} X_3^{-1} Y_1^{-1} Y_2^{-1} Y_3^{-1} = q_3 \end{cases}$$

which upon eliminating Y_1 using the first equation and Y_3 using the second becomes

$$X_1^{-1} X_2^{-1} X_3^{-1} X_1 Y_2^{-1} X_2 X_3 Y_2 = \varepsilon$$

which is nonempty, since for example after setting $X_1 = Y_2 = 1$ the equation becomes $X_2^{-1} X_3^{-1} X_2 X_3 = \varepsilon$. □

Remark 4.6.22. The proof above is an application of Proposition 5.3.1 of Chapter 5 to the $3n \times 3n$ case. We have neglected the quasi-Hamiltonian aspect of this factorisation, since we have not provided a twisted irregular type for a marked point on $\mathbb{P}^1$ giving Stokes data of the form

$$h S_3 S_2 S_1 = 1$$

with $h \in H(\partial), S_3, S_1 \in U_+, S_2 \in U_-$ in rank $3n$ case.

Theorem 4.6.23. *Let $n = (n, n, n)$ be a dimension vector and $q = (q_1, q_2, q_3)$ a generic parameter. The affine geometric quotient*

$$\mathcal{Q}(\widetilde{A}_2, q, n) = \operatorname{Rep}^*(\widetilde{A}_2, V) \mathbin{/\!\!/}_q H$$

is isomorphic to the affine cubic surface

$$aby + y^2 - (q_1^n q_2^n + q_1^n + 1)y - q_1^n q_2^n a - q_1^n b + q_1^{2n} q_2^n + q_1^n q_2^n + q_1^n = 0.$$

Moreover, there is an isomorphism

$$\mathcal{Q}(\widetilde{A}_2, q, n) \simeq \mathcal{Q}(\widetilde{A}_2, q^n, 1).$$

Proof. All the work has already been done. The functions a, b, y generate the ring of invariants and satisfy the desired relation. Since $\mu^{-1}(q)$ is nonempty, the quotient is of dimension two. There are no other relations between a, b, y since the affine cubic surface cut out by this equation is smooth and irreducible and its subvarieties are of positive codimension. □

Chapter 5

Continuants, fission spaces and quasi-Hamiltonian geometry

Let G be a complex connected reductive Lie group and choose a parabolic subgroup $P_+ \subset G$ and a Levi subgroup $H \subset P_+$ and let P_- be the opposite parabolic with the same Levi subgroup $H \subset P_-$, and denote by $U_\pm$ the corresponding unipotent radicals.

Recall that in Section 2.2.3 we introduced the fission space

$$_G\mathcal{A}^r_H = G \times (U_+ \times U_-)^r \times H,$$

which is a quasi-Hamiltonian $G \times H$-space, and its reduction $\mathcal{B}^r$ at the value one of the G-component of the moment map, which is a quasi-Hamiltonian H-space. In particular, when V is an ordered graded vector space, we have defined the general linear fission space $\mathcal{A}^r(V)$ and subsequently $\mathcal{B}^r(V)$. In this chapter we will study the case $V = V_1 \oplus V_2$ with the graded pieces of the same dimension n, which we will sometimes denote as $V = W \oplus W$.

As we have seen, for $V = V_1 \oplus V_2$ (not necessarily of the same dimension), the space $\mathcal{B}(V)$ was a quasi-Hamiltonian $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-space which can be described as in (2.12) :

$$\mathcal{B}(V) = \{(b_1, b_2) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \mid \det(1 + b_1 b_2) \neq 0\}$$

with moment map

$$\mu(b_1, b_2) = (1 + b_1 b_2, (1 + b_2 b_1)^{-1})$$

The polynomial $1 + b_1 b_2$ is the second Euler's continuant polynomial. We will generalise this approach for higher numbers of Stokes matrices.

Definition 5.0.24. *Set* $(\emptyset) = 1$ *and* $(x_1) = x_1$. *We define the* n*-th continuant polynomial* $(x_1, \ldots, x_n)$ *by the recursive relation*

$$(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})x_n + (x_1, \ldots, x_{n-2}). \tag{5.1}$$

We do not suppose that the variables $x_1, x_2 \dots$ commute nor that they are invertible. In particular, we will consider continuants in non-commutative variables such as matrices. The first few continuants are given by the following formulas:

$$
\begin{aligned}
(\emptyset) &= 1 \\
(x_1) &= x_1 \\
(x_1, x_2) &= x_1 x_2 + 1 \\
(x_1, x_2, x_3) &= x_1 x_2 x_3 + x_1 + x_3 \\
(x_1, x_2, x_3, x_4) &= x_1 x_2 x_3 x_4 + x_1 x_2 + x_1 x_4 + x_3 x_4 + 1 \\
(x_1, x_2, x_3, x_4, x_5) &= x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_5 + x_1 x_2 x_5 + x_1 x_4 x_5 + \\
&\quad + x_3 x_4 x_5 + x_1 + x_3 + x_5
\end{aligned}
$$

and in general the monomials of $(x_1, \dots, x_n)$ are obtained by erasing disjoint pairs of consecutive elements on the list.

There are multiple (equivalent) definitions of continuant polynomials, another one is as follows, and will appear naturally in the context of Stokes data. Consider matrices B_k given by

$$B_k = \begin{pmatrix} x_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Then one can easily show that the entries of the product $B_1 B_2 \cdots B_n$ are given by

$$\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (x_1, \dots, x_n) & (x_1, \dots, x_{n-1}) \\ (x_2, \dots, x_n) & (x_2, \dots, x_{n-1}) \end{pmatrix}. \tag{5.2}$$

The other definitions involve for example the continued fractions or determinants of tridiagonal matrices.

5.1 The spaces $\mathcal{B}_k^+$ and $\mathcal{B}_k^-$

Let $V = V_1 \oplus V_2$ be an ordered graded vector space of dimension $2n$ with two graded pieces V_1, V_2 such that $\dim(V_1) = \dim(V_2) = n$. We can identify both pieces with a vector space W of dimension n. Recall that we have defined for any ordered graded vector space V the spaces $\mathcal{A}^r(V), \mathcal{B}^r(V)$ and there are groups

$$G = \mathrm{GL}(V), \quad H = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2),$$

the opposite parabolics $P_\pm$ stabilising the (isomorphic) flags $F_\pm$

$$F_+ = V_1 \subset V, \quad F_- = V_2 \subset V$$

together with their unipotent radicals $U_\pm$. Moreover, we can identify the groups H, U_+, U_- with explicit block matrix subgroups of $\mathrm{GL}_{2n}(\mathbb{C})$:

$$H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

with all blocks of size n. The space $\mathcal{B}^r(V)$ has then a description

$$\mathcal{B}^r(V) = \{(h, S_1, \ldots, S_{2r}) \in H \times (U_+ \times U_-)^r \mid hS_{2r} \cdots S_1 = 1\}.$$

The condition $hS_{2r} \cdots S_1 = 1$ implies that the product $S_{2r-1} \cdots S_2$ is in the big Gauss cell $G^\circ = U_- H U_+$ and the Stokes multipliers $S_2, \ldots, S_{2r-1}$ uniquely determine h, S_1, S_{2r}. Let s_i denote the off diagonal entry of S_i and introduce the block matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with off-diagonal identity blocks of size n. We have $P^2 = \mathrm{Id}_{2n}$ and

$$\begin{pmatrix} 1 & s_{2i-1} \\ 0 & 1 \end{pmatrix} \cdot P = \begin{pmatrix} s_{2i-1} & 1 \\ 1 & 0 \end{pmatrix}, \quad P \cdot \begin{pmatrix} 1 & 0 \\ s_{2i} & 1 \end{pmatrix} = \begin{pmatrix} s_{2i} & 1 \\ 1 & 0 \end{pmatrix},$$

so we can write

$$S_{2r-1} \cdots S_2 = (S_{2r-1} \cdot P) \cdot (P \cdot S_{2r-2}) \cdots (S_3 \cdot P) \cdot (P \cdot S_2) =$$

$$\begin{pmatrix} s_{2r-1} & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} s_{2r-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} s_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (s_{2r-1}, \cdots, s_2) & (s_{2r-1}, \ldots, s_3) \\ (s_{2r-2}, \ldots, s_2) & (s_{2r-2}, \ldots, s_3) \end{pmatrix},$$

as in equation (5.2).

Proposition 5.1.1. *The H-moment map for the space $\mathcal{B}^r(V)$ is given by*

$$\mu(h, S_1, \ldots, S_{2r}) = \mathrm{diag}((s_{2r-1}, \ldots, s_2), (s_2, \ldots, s_{2r-1})^{-1}) \in H$$

Proof. We have

$$S_{2r-1} \cdots S_2 = S_{2r}^{-1} h^{-1} S_1^{-1} \tag{5.3}$$

so the top left entry on the diagonal of h^{-1} is just the top left block of the product $S_{2r-1} \cdots S_2$, which is the desired continuant. To find the second entry, consider the inverses of both sides in the equation (5.3) above, which gives

$$S_2^{-1} \cdots S_{2r-1}^{-1} = S_1 h S_{2r}.$$

The inverse of S_i is in the same unipotent subgroup of G, with off-diagonal entry $-s_i$ and the entry we are looking for is the inverse of the right bottom entry of h. On the other hand, taking the inverse exchanges the big cell $U_- H U_+$ with the opposite one $U_+ H U_-$, so the bottom right entry of h is again a continuant, this time in the inverted order $(-s_2, \ldots, -s_{2r-1})$. The number of terms is $2r - 2$, so the minus ones cancel out. □

Therefore, by setting $b_i = s_{2r-i}$, we can introduce the following descritpion of the space $\mathcal{B}^r(V)$ using continuants. Let k be an even positive integer. Define the space $\mathcal{B}_k^+$

$$\mathcal{B}_k^+ := \{b_1, \dots, b_k \mid \det(b_1, \dots, b_k) \neq 0\}, \tag{5.4}$$

such that $b_{\text{odd}} \in \operatorname{Hom}(V_2, V_1), b_{\text{even}} \in \operatorname{Hom}(V_1, V_2)$, so the continuant $(b_1, \dots, b_k)$ is an element of $\operatorname{End}(V_2)$. The space $\mathcal{B}_k^+$ is isomorphic to the space $\mathcal{B}^{(k+2)/2}$ and has a natural H-action coming from the H-action on $(S_1, \dots, S_{k+2})$.

Corollary 5.1.2. *Let k be an even positive integer. The space $\mathcal{B}_k^+$ is a quasi-Hamiltonian H-space with moment map*

$$\mu(b_1 \dots, b_k) = \begin{pmatrix} (b_1, \dots, b_k) & 0 \\ 0 & (b_k, \dots, b_1)^{-1} \end{pmatrix} \in H \tag{5.5}$$

and the two-form inherited from the space $\mathcal{B}^{(k+2)/2}$.

Exchanging the roles of the two graded pieces (ie. exchanging the order in the grading of V) exchanges the unipotents and gives the "opposite" of the space $\mathcal{B}_k^+$

$$\mathcal{B}_k^- = \{b_1, \dots, b_k) \mid \det(b_1, \dots, b_k) \neq 0\},$$

such that $b_{\text{odd}} \in \operatorname{Hom}(V_1, V_2), b_{\text{even}} \in \operatorname{Hom}(V_2, V_1)$, so the continuant $(b_1, \dots, b_k)$ is an element of $\operatorname{End}(V_1)$. Since $V_1 \simeq V_2$, it is isomorphic to $\mathcal{B}_k^+$ but the moment map is different

$$\mu(b_1, \dots, b_k) = \begin{pmatrix} (b_k, \dots, b_1)^{-1} & 0 \\ 0 & (b_1, \dots, b_k) \end{pmatrix} \in H.$$

It is again a quasi-Hamiltonian H-space. The formula for the moment map is analogous to the (5.5) but now we consider the opposite big Gauss cell $U_+ H U_-$.

Remark 5.1.3. For the case of even k it is not necessary to suppose that the two graded pieces of V are of the same dimension. Everything works precisely the same for any ordered grading $V = V_1 \oplus V_2$, the even length continuants belong either to $\operatorname{End}(V_1)$ or $\operatorname{End}(V_2)$.

We will now define the spaces $\mathcal{B}_k^+, \mathcal{B}_k^-$ for odd k. This corresponds to having an odd number of Stokes multipliers S_i and a twisted irregular class $\langle z^{k/2} \rangle$. This comes with a price – the spaces $\mathcal{B}_k^+, \mathcal{B}_k^-$ will no longer be quasi-Hamiltonian H-spaces but rather *twisted* quasi-Hamiltonian H-spaces with moment map taking values in a H-bitorsor $H(\partial)$ for a twist ∂. In the case of $V = V_1 \oplus V_2 \simeq W \oplus W$ this is however fairly simple and explicit.

Let us introduce the following set $H(\partial)$, with off-diagonal block entries of size n

$$H(\partial) := \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \subset \operatorname{GL}(V).$$

The group H acts on $H(\partial)$ by left and right multiplication. Both of these actions are simply transitive, making $H(\partial)$ into a H-bitorsor. For an odd k, set $c = k/2$ and define

$$U_{\pm}^{(k)} = U_- \times U_+ \times U_- \times \dots,$$

$$U_{\mp}^{(k)} = U_+ \times U_- \times U_+ \times \dots,$$

where there are k unipotent groups on the right.

Proposition 5.1.4 ([31], Example 6.2.)**.** *The space*

$$\mathcal{A}^c(V) := G \times U_{\pm}^{(k)} \times H(\partial)$$

is a twisted quasi-Hamiltonian $G \times H$-space with moment map

$$\mu = (\mu_G, \mu_H) : \mathcal{A}^c(V) \to G \times H(\partial)$$

given by

$$\mu_G(C, \mathbf{S}, h) = C^{-1} h S_k \cdots S_1 C, \quad \mu_H(C, \mathbf{S}, h) = h^{-1},$$

where $C \in G, \mathbf{S} = (S_1, \dots, S_k) \in U_{\pm}^{(k)}, h \in H(\partial)$.

Corollary 5.1.5. *The space*

$$\mathcal{B}^c(V) = \{(h, S_1, \dots, S_k) \in H(\partial) \times U_{\pm}^k \mid hS_k \cdots S_1 = 1\}$$

is a twisted quasi-Hamiltonian H-space with moment map

$$\mu(h, \mathbf{S}) = h^{-1} \in H(\partial).$$

Proposition 5.1.6. *The $H(\partial)$-moment map for the space $\mathcal{B}^c(V)$ is given by*

$$\mu(h, S_1, \dots, S_k) = \begin{pmatrix} 0 & (s_{k-1}, \dots, s_2) \\ -(s_2, \dots, s_{k-1})^{-1} & 0 \end{pmatrix} \in H(\partial)$$

Proof. Since this time the number of Stokes multipliers is odd, S_1 and S_k are both in U_-.

Recall what we have denoted $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so $P \left(\begin{smallmatrix} 1 & 0 \\ s_{2i-1} & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} s_{2i-1} & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & s_{2i} \\ 0 & 1 \end{smallmatrix}\right) P = \left(\begin{smallmatrix} s_{2i} & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Write $h \in H(\partial)$ as $P \cdot \left(\begin{smallmatrix} h_1 & 0 \\ 0 & h_2 \end{smallmatrix}\right)$. Then we have

$$P \cdot \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_k & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} = 1,$$

so the product

$$\begin{pmatrix} s_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} s_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (s_{k-1}, \dots, s_2) & (s_{k-1}, \dots, s_3) \\ (s_{k-2}, \dots, s_2) & (s_{k-2}, \dots, s_3) \end{pmatrix} =$$

$$= S_k^{-1} \begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix} (PS_1P)^{-1}$$

is in the big cell U_-HU_+. We determine h_1, h_2 as Proposition 5.1.1, obtaining

$$\begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix} = \begin{pmatrix} (s_{k-1}, \ldots, s_2) & 0 \\ 0 & -(s_2, \ldots, s_{k-1})^{-1} \end{pmatrix}.$$

The number of elements in the continuant is now odd, thus the minus ones in the bottom right entry do not cancel out and hence an extra minus sign. The moment map $h^{-1} \in H(\partial)$ is thus

$$h^{-1} = \begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix} \cdot P = \begin{pmatrix} 0 & (s_{k-1}, \ldots, s_2) \\ -(s_2, \ldots, s_{k-1})^{-1} & 0 \end{pmatrix}.$$

□

Therefore, by setting $b_i = s_{k-i}$, we can introduce the following description of the space $\mathcal{B}^c(V)$ using continuants. Let k be an odd positive integer. Define the space $\mathcal{B}_k^+$

$$\mathcal{B}_k^+ := \{b_1, \ldots, b_k \mid \det(b_1, \ldots, b_k) \neq 0\}, \tag{5.6}$$

such that $b_{\text{odd}} \in \operatorname{Hom}(V_2, V_1), b_{\text{even}} \in \operatorname{Hom}(V_1, V_2)$, so the continuant $(b_1, \ldots, b_k)$ is an element of $\operatorname{Hom}(V_2, V_1)$. It explains why we supposed that the graded pieces are equidimensional: otherwise the determinant would not be defined. The space $\mathcal{B}_k^+$ is isomorphic to the space $\mathcal{B}^c$ and has a natural H-action coming from the H-action on $(S_2, \ldots, S_{k-1})$.

Corollary 5.1.7. *Let k be an odd positive integer. The space $\mathcal{B}_k^+$ is a twisted quasi-Hamiltonian H-space with moment map*

$$\mu(b_1 \ldots, b_k) = \begin{pmatrix} 0 & (b_1, \ldots, b_k) \\ -(b_k, \ldots, b_1)^{-1} & 0 \end{pmatrix} \tag{5.7}$$

and the two-form inherited from the space $\mathcal{B}^c$, for $c = (k+2)/2$.

If we exchange the role of the graded components in the decomposition $V = V_1 \oplus V_2$, we obtain an isomorphic space $\mathcal{B}_k^-$, defined as

$$\mathcal{B}_k^- := \{(b_1, \ldots, b_k) \in \operatorname{End}(W)^k \mid \det(b_1, \ldots, b_k) \neq 0\},$$

such that $b_{\text{odd}} \in \operatorname{Hom}(V_1, V_2), b_{\text{even}} \in \operatorname{Hom}(V_2, V_1)$. It is a twisted quasi-Hamiltonian H-space, isomorphic to $\mathcal{B}^c(V)$ (with inverted order of graded pieces), with the moment map

$$\mu(b_1 \ldots, b_k) = \begin{pmatrix} 0 & -(b_k, \ldots, b_1)^{-1} \\ (b_1, \ldots, b_k) & 0 \end{pmatrix}.$$

5.2 Continuants and factorisations

Recall that we have defined the continuant polynomials by recursive relation (5.1):

$$(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})x_n + (x_1, \dots, x_{n-2})$$

with $(\emptyset) = 1, (x_1) = x_1$.

Definition 5.2.1. *Let $m < n$ be two positive integers. We define the length of a continuant $(x_m, \dots, x_n)$ to be the number of variables x_i between the brackets (equal to $n-m+1$). It will be denoted by $l(x_m, \dots, x_n)$. If the continuant has only one term (x_i), we define the length to be one, and the length of an empty continuant is zero.*

We are going to discuss the factorisations of continuants. For simplicity, throughout the whole section we will suppose that all entries x_i of the continuants belong to some fixed $\mathrm{End}(V)$ for a finite-dimensional vector space V. Therefore we can think of them as of square matrices which do not necessarily commute.

Let us look at an easy example. Consider the continuants

$$(x_1, x_2) = x_1x_2 + 1,$$
$$(x_2, x_1) = x_2x_1 + 1.$$

If the continuant $(x_1) = x_1$ is invertible, we can set $\widehat{x}_2 = x_2 + x_1^{-1}$ and factorise the continuants, both at the same time

$$(x_1, x_2) = (x_1)(\widehat{x}_2),$$
$$(x_2, x_1) = (\widehat{x}_2)(x_1).$$

Similarly, if $(x_2) = x_2$ is invertible, we can set $\widehat{x}_1 = x_1 + x_2^{-1}$ and obtain

$$(x_1, x_2) = (\widehat{x}_1)(x_2),$$
$$(x_2, x_1) = (x_2)(\widehat{x}_1).–$$

The following theorem generalizes this observation to continuants of arbitrary length.

Proposition 5.2.2. *Suppose that the continuants $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ are invertible and set*

$$\widehat{x}_m = x_m + (y_2, \dots, y_n)(y_1, \dots, y_n)^{-1}, \quad \widehat{y}_1 = y_1 + (x_1, \dots x_m)^{-1}(x_1, \dots, x_{m-1}).$$

Then there are the following factorisations of continuants:

$$(x_1, \dots, x_m, y_1, \dots, y_n) = (x_1, \dots, \widehat{x}_m)(y_1, \dots, y_n) = (x_1, \dots, x_m)(\widehat{y}_1, \dots, y_n),$$
$$(y_n, \dots, y_1, x_m, \dots, x_1) = (y_n, \dots, \widehat{y}_1)(x_m, \dots, x_1) = (y_n, \dots, y_1)(\widehat{x}_m, \dots, x_1).$$

Before proving Proposition 5.2.2, we will establish a lemma.

Lemma 5.2.3. *For any pair of positive integers $j < k$ there is an identity*

$$(x_1, \ldots, x_k) = (x_1, \ldots, x_j)(x_{j+1}, \ldots, x_k) + (x_1, \ldots, x_{j-1})(x_{j+2}, \ldots, x_k),$$

where we set $(x_{j+2}, \ldots, x_k) = (\emptyset)$ if $j+2 > k$ and $(x_{j+2}, \ldots, x_k) = (x_k)$ if $j+2 = k$.

Proof. We will proceed by induction on k. If $k = 3$, the identity holds. Suppose that it holds for all $m < k$ and write

$$\begin{aligned}
(x_1, \ldots, x_k) &= (x_1, \ldots, x_{k-1})x_k + (x_1, \ldots, x_{k-2}) \\
&= \big((x_1, \ldots, x_j)(x_{j+1}, \ldots, x_{k-1}) + (x_1, \ldots, x_{j-1})(x_{j+2}, \ldots, x_{k-1})\big)x_k \\
&\quad + (x_1, \ldots, x_{k-2}) \\
&= (x_1, \ldots, x_j)\big((x_{j+1}, \ldots, x_k) - (x_{j+1}, \ldots, x_{k-2})\big) \\
&\quad + (x_1, \ldots, x_{j-1})\big((x_{j+2}, \ldots, x_k) - (x_{j+2}, \ldots, x_{k-2})\big) + (x_1, \ldots, x_{k-2}) \\
&= (x_1, \ldots, x_j)(x_{j+1}, \ldots, x_k) + (x_1, \ldots, x_{j-1})(x_{j+2}, \ldots, x_k) \\
&\quad - (x_1, \ldots, x_j)(x_{j+1}, \ldots, x_{k-2}) - (x_1, \ldots, x_{j-1})(x_{j+2}, \ldots, x_{k-2}) \\
&\quad + (x_1, \ldots, x_{k-2}).
\end{aligned}$$

And we have

$$(x_1, \ldots, x_{k-2}) = (x_1, \ldots, x_j)(x_{j+1}, \ldots, x_{k-2}) + (x_1, \ldots, x_{j-1})(x_{j+2}, \ldots, x_{k-2})$$

by the inductive hypothesis for $k-2$. We have also used the hypothesis for $k-1$ when passing from first to the second line. □

Proof of Proposition 5.2.2. This is a direct application of Lemma 5.2.3. We have

$$\begin{aligned}
(x_1, \ldots, \widehat{x}_m)(y_1, \ldots, y_n) &= \big((x_1, \ldots, x_{m-1})\widehat{x}_m + (x_1, \ldots, x_{m-2})\big)(y_1, \ldots, y_n) \\
&= \big((x_1, \ldots, x_{m-1})x_m + (x_1, \ldots, x_{m-2})\big)(y_1, \ldots, y_n) \\
&\quad + (x_1, \ldots, x_{m-1})(y_2, \ldots, y_n) \\
&= (x_1, \ldots, x_m)(y_1, \ldots, y_n) + (x_1, \ldots, x_{m-1})(y_2, \ldots, y_n) \\
&= (x_1, \ldots, x_m, y_1, \ldots, y_n).
\end{aligned}$$

The proof works verbatim for the remaining three identities. □

Equivalently, we can rewrite Proposition 5.2.2 as follows.

Proposition 5.2.4. *Suppose that the continuants $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ are invertible and set*

$$\widehat{x}_m = x_m - (y_2, \ldots, y_n)(y_1, \ldots, y_n)^{-1}, \quad \widehat{y}_1 = y_1 - (x_1, \ldots x_m)^{-1}(x_1, \ldots, x_{m-1}).$$

Then there are the following factorisations of continuants:

$$\begin{aligned}
(x_1, \ldots, \widehat{x}_m, y_1, \ldots, y_n) &= (x_1, \ldots, x_m, \widehat{y}_1, \ldots, y_n) = (x_1, \ldots, x_m)(y_1, \ldots, y_n), \\
(y_n, \ldots, y_1, \widehat{x}_m, \ldots, x_1) &= (y_n, \ldots, \widehat{y}_1, x_m, \ldots, x_1) = (y_n, \ldots, y_1)(x_m, \ldots, x_1).
\end{aligned}$$

5.2.1 Counting the factorisations

By Proposition 5.2.2 we know how to factorise a continuant into two pieces, given that it has an invertible subcontinuant. We can inductively continue this process until we obtain a *full* factorisation

$$(x_1, \dots, x_n) = \widetilde{x}_1 \cdots \widetilde{x}_n$$
$$(x_n, \dots, x_1) = \widetilde{x}_n \cdots \widetilde{x}_1$$

into a product of $\widetilde{x}_i$'s. At each step we split a chosen continuant into two pieces of smaller length, supposing that some subcountinuant is invertible and the final factorisation will depend on the choice of certain subset of invertible subcountinuants. However, splitting in different order might yield the same full factorisations in the end, thus it is natural to ask how many different full factorisations of a continuant $(x_1, \dots, x_n)$ there are. The goal of this section is to establish the following fact (cf. also Remark 5.2.15 at the end of the section, discussing a connection with "free duplicial algebras" of Loday [64]).

Theorem. *Let n be a positive integer. There are C_n different full factorisations of $(x_1, \dots, x_n)$, where C_n denotes the n-th Catalan number.*

Let us describe the naive count. At each step we split a continuant into a product of two subcontinuants, making a choice whether the left subcontinuant is invertible or the right one. We can axiomatize this procedure in the language of trees: we start at the root which is labelled n and it has two descendants: left subtree with n_1 leaves and the right subtree with n_2 leaves and $n_1 + n_2 = n$. Furthermore, we label the root by L or R, depending on whether the left subcontinuant (of length n_1) is supposed to be invertible, or the right one (of length n_2). We can then continue inductively and see that a factorisation of a continuant of length n determines a rooted binary tree with n leaves and a labelling L, R of each internal node, giving in total $2^{n-1}C_{n-1}$ factorisations.

This is obviously incorrect since when counting the full factorisations this way we count the same full factorisations multiple times. In other words, different sequences of intermediate splittings might lead to the same full factorisation $(x_1, \dots, x_n) = \widetilde{x}_1 \cdots \widetilde{x}_n$. In order to perform the proper count, we need some supplementary definitions and combinatorial lemmas.

Proposition 5.2.5. *Let $\widetilde{x}_1 \cdots \widetilde{x}_n$ be a full factorisation of $(x_1, \dots, x_n)$. Then the terms $\widetilde{x}_i$ are all of the form (with some of the inverted continuants possibly empty and thus on length zero):*

$$(x_m, \dots, x_n)^{-1}(x_m \dots, x_k)(x_l, \dots, x_k)^{-1}, \tag{5.8}$$

for some integers k, l, m, n such that

$$l(x_m, \dots, x_k) = l(x_m, \dots, x_n) + l(x_l, \dots, x_k) + 1.$$

Proof. We will prove the statement by induction. For $i = 2$ the statement is clear. Consider the continuant $(x_1, \ldots, x_n)$ and the first step of the factorisation, say at x_k, which yields by Proposition 5.2.2

$$(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x}_k)(x_{k+1}, \ldots, x_n)$$

with

$$\widehat{x}_k = x_k + (x_{k+2}, \ldots, x_n)(x_{k+1}, \ldots, x_n)^{-1} = (x_k, \ldots, x_n)(x_{k+1}, \ldots, x_n)^{-1}.$$

(the proof works verbatim if we split the continuant into $(x_1, \ldots, x_k)(\widehat{x}_{k+1}, \ldots, x_n)$).

Now, by the inductive hypothesis, both continuants $(x_1, \ldots, \widehat{x}_k)$ and $(x_{k+1}, \ldots, x_n)$ have the terms of desired form (5.8) after fully factorising them. The right one, $(x_{k+1}, \ldots, x_n)$ yields no problems since we have not changed its entries. However the left one does factorise as desired but in the terms of $\widetilde{x}_k$, not x_k, and we need to prove that it still is of the form (5.8) after switching back to x_k.

Suppose that we have an expression

$$(x_m, \ldots, x_p)^{-1}(x_m, \ldots, \widehat{x}_k)(x_l, \ldots, \widehat{x}_k)^{-1} \tag{5.9}$$

appearing in the full factorisation of $(x_1, \ldots, \widetilde{x}_k)$, such that the lengths match up. The index k is the greatest so the continuant does not surpass $\widetilde{x}_k$. Observe that there is the following identity:

$$(x_i, \ldots, \widehat{x}_k) = (x_i, \ldots, x_n)(x_{k+1}, \ldots, x_n)^{-1} \tag{5.10}$$

and thus we can write

$$\begin{aligned}
&(x_m, \ldots, x_p)^{-1}(x_m, \ldots, \widehat{x}_k)(x_l, \ldots, \widetilde{x}_k)^{-1} \\
&= (x_m, \ldots, x_p)^{-1}(x_m, \ldots, x_n)(x_{k+1}, \ldots, x_n)^{-1}(x_{k+1}, \ldots, x_n)(x_l \ldots, x_n)^{-1} \\
&= (x_m, \ldots, x_p)^{-1}(x_m, \ldots, x_n)(x_l \ldots, x_n)^{-1}
\end{aligned}$$

and the sum of lengths matches up. If the term $(x_l, \ldots, \widetilde{x}_k)$ on the right in (5.9) is absent, then $(x_m, \ldots, \widetilde{x}_k)$ transforms into $(x_m, \ldots, x_n)(x_{k+1}, \ldots, x_n)^{-1}$ and the lengths still match up. □

Definition 5.2.6. *We will call the terms $(x_m, \ldots, x_p)^{-1}$ and $(x_l, \ldots, x_k)^{-1}$ in an expression of the form (5.8) the left and right companion of $(x_m, \ldots, x_k)$.*

Thus we have shown that each term $\widetilde{x}_i$ in the full factorisation $(x_1, \ldots, x_n)$ consists of a continuant and its companions and sum of lengths of companions is one less that the length of the term itself. A term $\widetilde{x}_i$ has no companions if and only if it is of length one itself, equal to x_i.

We leave the following two lemmas as exercises. The inductive proofs are similar to the one of Proposition 5.2.5, one performs the first splitting of the continuant into two pieces and applies the inductive hypotheses for both pieces, eliminating the new coordinate $\widetilde{x}_k$ with identity (5.10).

Lemma 5.2.7. *Let $\widetilde{x}_1 \cdots \widetilde{x}_n$ be a full factorisation of $(x_1, \ldots, x_n)$ and suppose that $i \neq n$. If the uninverted continuant in the expression for $\widetilde{x}_i$ ends with x_n, then $\widetilde{x}_i$ has a right companion. Similarly if $i \neq 1$ and the uninverted continuant in its expression starts with x_1, then $\widetilde{x}_i$ has a left companion.*

Lemma 5.2.8. *Let $\widetilde{x}_1 \cdots \widetilde{x}_n$ be a full factorisation of $(x_1, \ldots, x_n)$. Then $\widetilde{x}_1$ has no left companion and the uninverted continuant in its expression starts with x_1. Similarly, $\widetilde{x}_n$ has no right companion and the uninverted continuant in its expression ends with x_n.*

Definition 5.2.9. *A factorisation list, or in short an f-list, is a list of integers of length $2n-1$ satisfying the following conditions:*

1) There are $2n-1$ integers, between n and $-n$, alternating in sign.
2) The integer n appears exactly once and $-n$ does not appear.
3) It admits a partition into n subintervals such that the sum of elements in each interval is 1.
4) If we repeatedly cancel the elements $a, -a$ appearing next to each other, then the only remaining term will be n.

The examples of f-lists (of length 5 and 7) are $[1, -1, 3, -1, 1]$ or $[2, -1, 1, -2, 3, -3, 4]$. Examples of full factorisations (corresponding to these lists) are

$$(x_1, x_2, x_3) = x_1[x_1^{-1}(x_1, x_2, x_3)x_3^{-1}]x_3.$$

$$(x_1, x_2, x_3, x_4) = [(x_1, x_2)x_2^{-1}][x_2][(x_1, x_2)^{-1}(x_1, x_2, x_3)][(x_1, x_2, x_3)^{-1}(x_1, x_2, x_3, x_4)]$$

Proposition 5.2.10. *A full factorisation $\widetilde{x}_1 \cdots \widetilde{x}_n$ of $(x_1, \ldots, x_n)$ determines an f-list.*

Proof. Given a full factorisation, each term $\widetilde{x}_i$ is of the form (5.8) and we can list the lengths of all continuants appearing in $\widetilde{x}_i$'s for $i = 1, \ldots, n$, giving the negative sign to the companions. We will show that such an assignment yields an f-list.

We will proceed by induction. Factorise the continuant

$$(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x}_k)(x_{k+1}, \ldots, x_n),$$

(the proof woks the same if we factorise on the right). By inductive hypothesis, the further factorisations of the two subcontinuants $(x_1, \ldots, \widehat{x}_k), (x_{k+1}, \ldots, x_n)$ yield two f-lists, of length $2k-1$ and $2(n-k)-1$, respectively, which we will merge into an f-list of length $2n-1$. This is done as follows. The left continuant $(x_1, \ldots, \widehat{x}_k)$ splits further into $\widetilde{x}_1 \cdots \widetilde{x}_k$ but it involves the variable $\widehat{x}_k$ which we need to eliminate and obtain expressions involving the original variable x_k.

However, $\widetilde{x}_i$ is of the form (5.8) and if $i \neq k$ and if the uninverted continuant involves $\widehat{x}_k$ (that is, ends with $\widehat{x}_k$ since k is the greatest index), then by Lemma 5.2.7 it has a right companion and using identity (5.10) one can eliminate $\widehat{x}_k$, obtaining longer continuants, going up to n. We then correct the list by adding $n-k$ to the number corresponding to univerted continuant in each $\widetilde{x}_i$ involving $\widehat{x}_k$ (except of $\widetilde{x}_k$) and substracting $n-k$ from its right companion. By Lemma 5.2.8, the only term involving $\widehat{x}_k$ without right companion is $\widetilde{x}_k$, where we simply replace

$$c_L(x_i,\ldots,\widehat{x}_k) = c_L(x_i,\ldots,x_n)(x_{k+1},\ldots,x_n)^{-1},$$

with c_L denoting the left companion (possibly $c_L = 1$). Observe that the whole replacing procedure adds one extra element to the list, $-n+k$ on the right, corresponding to $(x_{k+1},\ldots,x_n)^{-1}$. We then merge the new list of length $2k$ with the one coming from factorisation of $(x_{k+1},\ldots,x_n)$ and it is a straightforward to check that the list obtained this way is an f-list of length $2n-1$. □

The property 2) of an f-list means that for each factorisation $\widetilde{x}_1\cdots\widetilde{x}_n$ there is a unique element $\widetilde{x}_i$ such that its uninverted continuant is $(x_1,\ldots,x_n)$.

Proposition 5.2.11. *Let L be an f-list of length $2n-1$. Then removing the unique interval containing n yields two new f-lists if $i \neq 1, n$ and a single new f-list otherwise.*

Proof. This follows from the properties of f-lists. If the interval containing n is of length 2, it means that it is of the form $[n, n-1]$ or $[n-1, n]$ which implies it is one the far left or far right of the list and the claim is obvious. It the interval containing n is of the form $[-a, n, -b]$, then $a+b = n-1$. Removing this interval separates L into two lists L_1, L_2. The sum of all numbers on the left list L_1 is a and thus there are a intervals on the left side, giving the length $2a-1$. The same holds for L_2, which is of length $2b-1$ and it is clear that both L_1 and L_2 satisfy the conditions od Definition 5.2.9. □

Corollary 5.2.12. *A full factorisation uniquely determines an f-list.*

Proof. If two full factorisations give the same list L, then they have the same interval containing n, corresponding to unique $\widetilde{x}_i$ having $(x_1,\ldots,x_n)$ uninverted. We can then remove it and obtain by Proposition 5.2.11 one or two new sub f-lists. The claim follows then by induction. □

Thus for any factorisation of the continuant we obtain a unique f-list. The inverse is true as well and we shall prove the following.

Proposition 5.2.13. *An f-list of length $2n-1$ determines a factorisation of a continuant of length n.*

Proof. We will proceed by induction on the length of the f-list. Given an f-list L of length $2n-1$, by Proposition 5.2.11 we can remove the interval $[-a, n, -b]$ containing n and obtain two shorter f-lists L_1, L_2. The left f-list determines a factorisation of

$(x_1, \ldots, x_a)$ and the right one a factorisation of $(x_{n-b+1}, \ldots, x_b)$. Therefore sequence of splittings of $(x_1, \ldots, x_n)$ leading to L might be for example

$$(x_1, \ldots, x_n) = (x_1, \ldots, x_a)(\widehat{x}_{a+1}, \ldots, x_n)$$

and then

$$(x_1, \ldots, x_a)(\widehat{x}_{a+1}, \ldots, x_n) = (x_1, \ldots, x_a)\overline{x}_{a+1}(x_{n-b+1}, \ldots, x_n),$$

where $\overline{x}_{a+1}$ is obtained by splitting $(\widehat{x}_{a+1}, \ldots, x_n)$ at $\widehat{x}_{a+1}$ to the left. Since $a + b = n-1$, the indices $a+1$ and $n-b+1$ are in fact consecutive. Then we use the inductive hypothesis and fully factorise $(x_1, \ldots, x_a)$ and $(x_{n-b+1}, \ldots, x_n)$ as prescribed by L_1, L_2, obtaining the list L. □

We are now ready to count the number of factorisations using the f-lists.

Theorem 5.2.14. *There are C_n distinct full factorisations of the continuant $(x_1, \ldots, x_n)$, where C_n denotes the n-th Catalan number.*

Proof. Denote by L_k the set of f-lists of length $2k - 1$. As we have shown, there is a bijection between the set of full factorisations of $(x_1, \ldots, x_n)$ and L_n.

Suppose that $a, b \neq 0$ and consider two f-lists l_a, l_b of length $2a - 1, 2b - 1$. Then we can produce an f-list l_{a+b+1} of length $2a + 2b + 1$ as follows. Take an interval $[-a, a + b + 1, -b]$ and glue l_a on the left an l_b on the right of this interval. If $a = 0$, then we can glue the interval $[b+1, -b]$ onto the left end of L_b and similarly, if $b = 0$, we can glue $[-a, a + 1]$ onto the right end of l_a. It is clear that different lists l_a, l_b yield different lists l_{a+b+1}. Hence we obtain an injective map

$$L_a \times L_b \to L_{a+b+1}. \tag{5.11}$$

On the other hand, by Proposition 5.2.11 given an f-list l_n of length $2n - 1$ we can remove the unique interval containing n and obtain one or two new f-lists, depending on its placement and then merge them back (or glue on an interval $[-a, a + 1]$ or $[b + 1, b]$ if l_n did not split into two lists). Hence every element of L_n is in the image of the map (5.11).

The proof is finished since the cardinalities of L_k and Catalan numbers obey the recursive relation

$$C_n = \sum_{a+b=n-1} C_a C_b.$$

□

Remark 5.2.15. For a positive integer $k < n$ Proposition 5.2.2 defines two operations: the *right* and *left* splittings s_R^k and s_L^k of the continuant $(x_1, \ldots, x_n)$. For a pair of positive integers $l < k < n$ one has

$$s_R^k * s_L^l = s_L^l * s_R^k$$

(but not the other way). The operations s_R, s_L form a “free duplicial algebra" $\mathcal{D}$ of Loday [64] and the splittings correspond to products “over” and “under” of binary trees of [64]. In particular, it is known that for a free duplicial algebra $\mathcal{D}$ with one generator, the dimensions of homogeneous components are counted by the Catalan numbers and thus our count can be explained by Loday’s. The f-lists (without the negative terms, which are uniquely determined) seem to appear in [63], again in the context of counting binary trees.

On the other hand, the explicit count of factorisations of the continuant gives a direct link between factorisations and triangulations of polygons. The Catalan number C_k counts the triangulations of the $(k+2)$-gon. Thus it is possible to read a factorisation of the continuant off a triangulation – the diagonals of a triangulation tell us which continuants will be invertible. More precisely, let P_{k+2} be a $(k+2)$-gon and label its vertices by $0, \dots, k+1$ in counter clockwise order. Each diagonal $[i,j]$ joining two vertices i,j with $i<j$ divides the set of vertices of P_{k+2} into two segments $(j+1, \dots, i-1)$ and $(i+1, \dots, j-1)$ (modulo $k+2$) and the segment $(i+1, \dots, j-1)$ does not contain 0 nor $k+1$. The invertible continuant corresponding to this diagonal is thus $(x_{i+1}, \dots, x_{j-1})$.

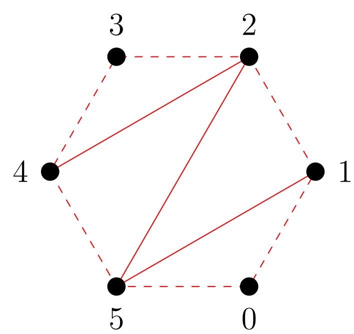

Figure 5.1: A triangulation of a hexagon

To illustrate the correspondence, consider the triangulation as on the figure above, given by diagonals $[1,5],[2,4],[2,5]$. The diagonal $[1,5]$ divides the set of vertices into two segments (0) and $(2,3,4)$, thus the invertible continuant is (x_2,x_3,x_4). Similarly, we obtain the invertible continuants (x_3) for $[2,4]$ and (x_3,x_4) for $[2,5]$. The factorisation is then

$$(x_1,x_2,x_3,x_4) = [(x_1,x_2,x_3,x_4)(x_2,x_3,x_4)^{-1}][(x_2,x_3,x_4)(x_3,x_4)^{-1}][(x_3)][(x_3)^{-1}(x_3,x_4)].$$

5.3 The factorisation map

In this section we will interpret the factorisations of continuants in terms of moment maps for the spaces $\mathcal{B}_k^{\pm}$. This will lead to the notion of “factorisation map” and we will see that the maps induced by factorisations of continuants are in fact quasi-Hamiltonian embeddings.

Suppose that $V = W \oplus W$ with $\dim(V) = 2n, \dim(W) = n$. As before, we have the group $G = \mathrm{GL}(V)$ and the following identifications with block subgroups of $\mathrm{GL}_{2n}(\mathbb{C})$

$$G = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, \quad H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

and a H-bitorsor $H(\partial)$:

$$H(\partial) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Recall that we also have the (twisted) quasi-Hamiltonian H-spaces $\mathcal{B}_k^+, \mathcal{B}_k^-$, which as a set are defined as

$$\mathcal{B}_k^+ = \{(b_1, \dots, b_k) \in \mathrm{End}(W)^k \mid \det(b_1, \dots, b_k) \neq 0\} = \mathcal{B}_k^-$$

but have different moment maps. If k is even, these are honest quasi-Hamiltonian H-spaces, if k is odd, then these are twisted quasi-Hamiltonian H-spaces with moment map taking values in the twist $H(\partial)$.

By choosing appropriate loops, for even k we can identify

$$\begin{aligned} \mathcal{B}_k^+ &= \{(h, \mathbf{S}) \in H \times U_\pm^{(k+2)} \mid hS_{k+2} \cdots S_2 S_1 = 1\}, \\ \mathcal{B}_k^- &= \{(h, \mathbf{S}) \in H \times U_\mp^{(k+2)} \mid hS_{k+2} \cdots S_2 S_1 = 1\}, \end{aligned} \tag{5.12}$$

and for odd k

$$\begin{aligned} \mathcal{B}_k^+ &= \{(h, \mathbf{S}) \in H(\partial) \times U_\pm^{(k+2)} \mid hS_{k+2} \cdots S_2 S_1 = 1\}, \\ \mathcal{B}_k^- &= \{(h, \mathbf{S}) \in H(\partial) \times U_\mp^{(k+2)} \mid hS_{k+2} \cdots S_2 S_1 = 1\}, \end{aligned} \tag{5.13}$$

where $\mathbf{S} = (S_1, \dots, S_{k+2})$ and

$$U_\pm^{(i)} = U_- \times U_+ \times U_- \times \dots, \quad U_\mp^{(i)} = U_+ \times U_- \times U_- \times \dots$$

with i alternating unipotent groups. If i is even then $U_\mp^{(i)} = (U_+ \times U_-)^{i/2}$, as in the usual definition of the reduced fission space $\mathcal{B}^k(V)$.

We will simplistically write

$$\mathcal{B}_k^\pm = \{(h, \mathbf{S}) \mid hS_{k+2} \cdots S_2 S_1 = 1\},$$

where $h \in H$ or $h \in H(\partial)$ depending in the parity of k and $\mathbf{S} \in U_\pm^{(k+2)}$ or $\mathbf{S} \in U_\mp^{(k+2)}$, depending on whether we consider $\mathcal{B}_k^+$ or $\mathcal{B}_k^-$. It is a (twisted) quasi-Hamiltonian space with moment map $\mu = h^{-1}$.

Choose two positive integers m, n and consider the four spaces $\mathcal{B}_m^\pm, \mathcal{B}_n^\pm$, which might be twisted or untwisted quasi-Hamiltonian H-spaces, depending on the parity of m, n

$$\mathcal{B}_m^\pm = \{(h_1, \mathbf{A}) \mid h_1 A_{m+2} \cdots A_2 A_1 = 1\},$$

$$\mathcal{B}_n^{\pm} = \{(h_2, \mathbf{B}) \mid h_2 B_{n+2} \cdots B_2 B_1 = 1\}.$$

Now choose one space from each pair $\mathcal{B}_m^{\pm}$ and $\mathcal{B}_n^{\pm}$ so that A_2 and B_{n+2} are in the same unipotent subgroup of G, and denote them by $\mathcal{B}_m, \mathcal{B}_n$. We will call such pair of spaces $\mathcal{B}_m, \mathcal{B}_n$ *compatible.* This yields four possible choices of compatible pairs:

- $\mathcal{B}_m = \mathcal{B}_m^+, \mathcal{B}_n = \mathcal{B}_n^+$ or $\mathcal{B}_m = \mathcal{B}_m^-, \mathcal{B}_n = \mathcal{B}_n^-$ if m is even,
- $\mathcal{B}_m = \mathcal{B}_m^+, \mathcal{B}_n = \mathcal{B}_n^-$ or $\mathcal{B}_m = \mathcal{B}_m^-, \mathcal{B}_n = \mathcal{B}_n^+$ if m is odd.

So for example $\mathcal{B}_2^+$ and $\mathcal{B}_2^+$ or $\mathcal{B}_1^+$ and $\mathcal{B}_1^-$ are compatible but $\mathcal{B}_1^+$ and $\mathcal{B}_1^+$ are not. Observe that for a fixed choice of positive integers m, n, the of sign for $\mathcal{B}_m$ uniquely determines the sign for $\mathcal{B}_n$.

For every compatible pair $\mathcal{B}_m, \mathcal{B}_n$ we consider its fusion product

$$\mathcal{B}_m \underset{H}{\circledast} \mathcal{B}_n,$$

which is again a (twisted) quasi-Hamiltonian H-space with moment map $\mu_1 \cdot \mu_2$. As a set, it is just a product $\mathcal{B}_m \times \mathcal{B}_n$.

Now for each compatible pair define the following map f_R

$$f_R((h_1, \mathbf{A}), (h_2, \mathbf{B})) = (h_2 h_1, A_{m+2}, \ldots, A_3, A_2 B_{n+2}, B_{n+1}, \ldots, B_2, \widetilde{B}_1), \tag{5.14}$$

where $\widetilde{B}_1 = B_1 h_2 A_1 h_2^{-1}$. Observe that the element $\widetilde{B}_1$ is still unipotent. If A_1 and B_1 are in opposite unipotent subgroups of G, then one must have n odd and thus $h_2 \in H(\partial)$. Then the conjugation by h_2 sends A_1 to the opposite unipotent subgroup of G containing B_1.

If we denote

$$h_2 h_1 := h,$$

$$(A_{m+2}, \ldots, A_3, A_2 B_{n+2}, B_{n+1}, \ldots, B_2, \widetilde{B}_1) := (C_{m+n+2}, \ldots, C_2, C_1) = \mathbf{C},$$

then one has

$$h C_{m+n+2} \cdots C_2 C_1 = 1,$$

since $h_1 A_{m+2} \cdots A_1 = h_2 B_{n+2} \cdots B_1 = 1$. Therefore the image of f_R is in $\mathcal{B}_{m+n}^{\pm}$, depending on the initial choice of signs for $\mathcal{B}_m^{\pm}, \mathcal{B}_n^{\pm}$. More precisely, we have the following.

Proposition 5.3.1. *The map f_R gives the following injections*

$$\mathcal{B}_m^+ \circledast \mathcal{B}_n^+ \hookrightarrow \mathcal{B}_{m+n}^+$$
$$\mathcal{B}_m^- \circledast \mathcal{B}_n^- \hookrightarrow \mathcal{B}_{m+n}^-$$

if m is even and

$$\mathcal{B}_m^+ \circledast \mathcal{B}_n^- \hookrightarrow \mathcal{B}_{m+n}^+$$
$$\mathcal{B}_m^- \circledast \mathcal{B}_n^+ \hookrightarrow \mathcal{B}_{m+n}^-$$

if m is odd. Moreover, the images of f_R are dense open subsets of $\mathcal{B}^{\pm}_{m+n}$.

Analogously, we can consider the map f_L. Suppose that $\mathcal{B}_m, \mathcal{B}_n$ are compatible, ie. A_2 and B_{n+2} are in the same unipotent. This implies that A_1, B_{n+1} are in the same unipotent as well, so we can consider

$$f_L((h_1, \mathbf{A}), (h_2, \mathbf{B})) = (h_2 h_1, \widetilde{A}_{m+2}, A_{m+1}, \dots, , A_2, A_1 B_{n+1}, B_n, \dots, B_1), \qquad (5.15)$$

where $\widetilde{A}_{m+2} = h_1^{-1} B_{n+2} h_1 A_{m+2}$. Similarly as for the map f_R, we can denote

$$h_2 h_1 := h,$$

$$(\widetilde{A}_{m+2}, A_{m+1}, \dots, , A_2, A_1 B_{n+1}, B_n, \dots, B_1) := (C_{m+n+2}, \dots, C_2, C_1) = \mathbf{C},$$

and one has

$$h C_{m+n+2} \cdots C_2 C_1 = 1.$$

Therefore the image of f_L is again in $\mathcal{B}^{\pm}_{m+n}$, depending on the initial choice of signs for $\mathcal{B}^{\pm}_m, \mathcal{B}^{\pm}_n$. More precisely, we have the following.

Proposition 5.3.2. *The map f_L gives the following injections*

$$\mathcal{B}^+_m \circledast \mathcal{B}^+_n \hookrightarrow \mathcal{B}^+_{m+n}$$
$$\mathcal{B}^-_m \circledast \mathcal{B}^-_n \hookrightarrow \mathcal{B}^-_{m+n}$$

if m is even and

$$\mathcal{B}^+_m \circledast \mathcal{B}^-_n \hookrightarrow \mathcal{B}^+_{m+n}$$
$$\mathcal{B}^-_m \circledast \mathcal{B}^+_n \hookrightarrow \mathcal{B}^-_{m+n}$$

if m is odd. Moreover, the images of f_L are dense open subsets of $\mathcal{B}^{\pm}_{m+n}$.

Definition 5.3.3. *We call the maps f_R, f_L the right and left factorisation maps.*

For a compatible pair $\mathcal{B}_m, \mathcal{B}_n$, we will denote the target space of both factorisation maps by $\mathcal{B}_{m+n}$, which is $\mathcal{B}^+_{m+n}$ or $\mathcal{B}^-_{m+n}$, depending on the choice of the compatible spaces $\mathcal{B}_m, \mathcal{B}_n$.

5.3.1 The geometry of the factorisation maps

For a compatible pair $\mathcal{B}_m, \mathcal{B}_n$, we have defined by formulas (5.14), (5.15) two maps f_R, f_L

$$f_R, f_L : \mathcal{B}_m \circledast \mathcal{B}_n \hookrightarrow \mathcal{B}_{m+n}$$

whose images are dense open subsets of $\mathcal{B}^{\pm}_{m+n}$. The formulas for these injections are not accidental and before proving in the next section that f_R, f_L are in fact quasi-Hamiltonian maps, we will explain their geometric origin.

The spaces $\mathcal{B}_k^{\pm}$ classify isomorphism classes of connections on $\mathbb{P}^1$ with one irregular pole at zero and a particular fixed irregular type Q. It is an intrinsic object, isomorphic to $\mathrm{Hom}(\Pi, \{b\})$ and a choice of paths generating the fundamental groupoid identifies it with an explicit product of groups. As we have noticed, we can think of the space $\mathcal{B}_k^{\pm}$ as a $(k+2)$-gon floating in a disk, with marked point b on the boundary.

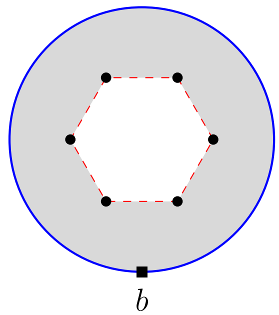

Figure 5.2: The spaces $\mathcal{B}_4^{\pm}$, isomorphic to $\mathcal{B}^3(V)$.

The black dots are identified with Stokes groups $U_{\pm}$ and the choice of generating paths gives explicit identification with description from (5.12).

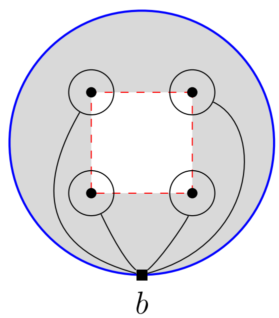

Figure 5.3: A choice of paths

If we consider the product $\mathcal{B}_m \times \mathcal{B}_n$, it corresponds to $(m+2)$-gon and $(n+2)$-gon next to each other with two H-boundaries.

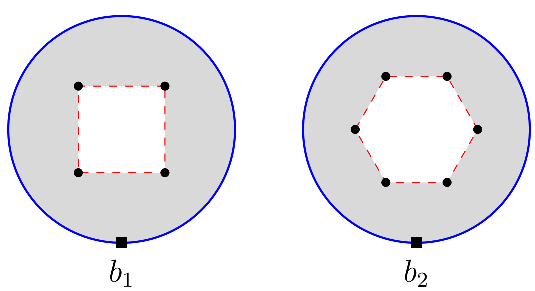

Figure 5.4: The product $\mathcal{B}_2^{+} \times \mathcal{B}_4^{+}$.

However, the fusion

$$\mathcal{B}_m \underset{H}{\circledast} \mathcal{B}_n$$

can be pictured as $(m+2)$-gon and $(n+2)$-gon floating next to each other in one big disk with a single H-boundary and one basepoint. The fusion of the two quasi-Hamiltonian H-spaces reduces one copy of H.

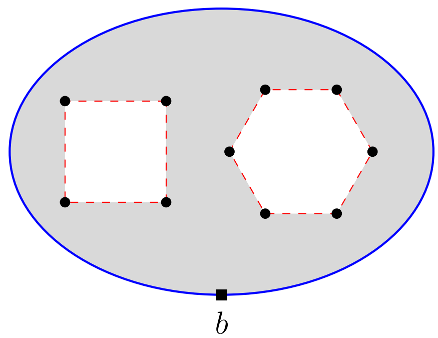

Figure 5.5: The fusion $\mathcal{B}_2^+ \circledast \mathcal{B}_4^+$

Now, the factorisation map can be understood as gluing two such polygons along an edge. However, in order to glue A_1 and B_1, we need to move the path around A_1 away from the edges we are gluing. This is done as on the picture below and explains how the term $\widetilde{B}_1$ arises, since the monodromy around the right polygon is h_2.

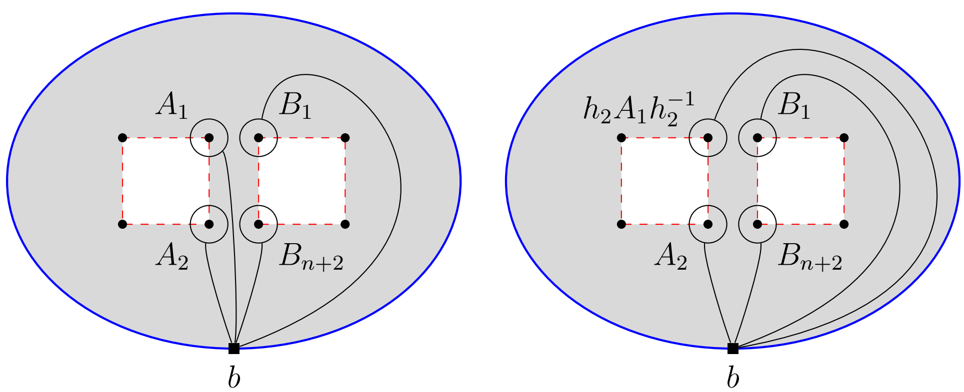

Figure 5.6: Moving the path

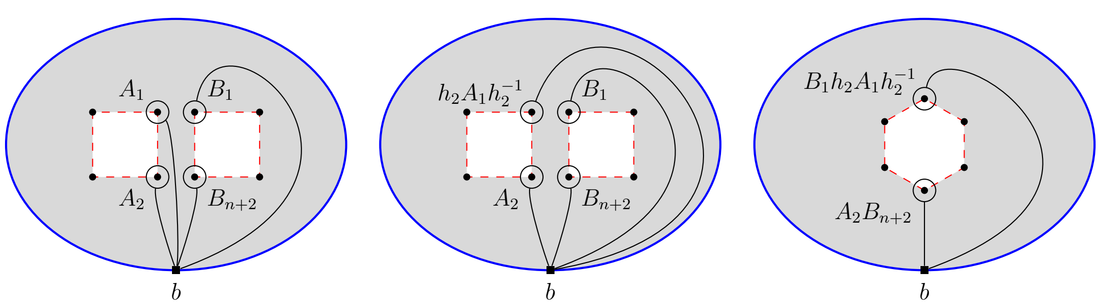

Figure 5.7: The factorisation map in two steps

5.4 Quasi-Hamiltonian embeddings

The right and left factorisation maps provide injective maps

$$f_R, f_L : \mathcal{B}_m \circledast \mathcal{B}_n \hookrightarrow \mathcal{B}_{m+n}$$

for a compatible pair $\mathcal{B}_m, \mathcal{B}_n$. Thus on the image of these maps there are a priori two quasi-Hamiltonian forms: the restriction of the form on $\mathcal{B}_{m+n}$ and the form coming from the fusion $\mathcal{B}_m \circledast \mathcal{B}_n$. The goal of this chapter is to prove that the factorisation maps relate the quasi-Hamiltonian structures.

Theorem 5.4.1. *Let $\mathcal{B}_m, \mathcal{B}_n$ be a compatible pair. Under the factorisation maps f_R, f_L, the standard quasi-Hamiltonian two-form on $\mathcal{B}_{m+n}$ restricts to the two form on the fusion $\mathcal{B}_m \circledast \mathcal{B}_n$.*

Proof. We will establish the theorem for the map f_R. The proof for f_L is similar and we leave it is as an exercise.

Set

$$\begin{aligned}
\mathcal{B}_m &= \{(h_1, \mathbf{A}) \bigm| h_1 A_{m+2} \cdots A_2 A_1 = 1\}, \\
\mathcal{B}_n &= \{(h_1, \mathbf{B}) \bigm| h_2 B_{n+2} \cdots B_2 B_1 = 1\}, \\
\mathcal{B}_{m+n} &= \{(h_2 h_1, \mathbf{C}) \bigm| h_2 h_1 C_{m+n+2} \cdots C_2 C_1 = 1\},
\end{aligned}$$

where h_1, h_2 belong to H or $H(\partial)$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collections of unipotent elements, in the order determined by the choice of parity and sign of $\mathcal{B}_m, \mathcal{B}_n$.

Set $P_i = A_i \dots A_1, Q_i = B_i \dots B_1$ and denote

$$\alpha_i = P_i^*(\theta), \quad \overline{\alpha}_i = P_i^*(\overline{\theta}), \quad \beta_i = Q_i^*(\theta), \quad \overline{\beta}_i = Q_i^*(\overline{\theta}).$$

The two-forms ω_m, ω_n on $\mathcal{B}_m, \mathcal{B}_n$ are then given by the formula (2.11) (also in the twisted odd case, cf. Theorem 24. of [31]):

$$2\omega_m = \sum_{i=1}^{m} (\alpha_i, \alpha_{i+1}), \quad 2\omega_n = \sum_{i=1}^{n} (\beta_i, \beta_{i+1}).$$

If we set $R_i = C_i \dots C_1, \gamma_i = C_i^*(\theta), \overline{\gamma}_i = C_i^*(\overline{\theta})$, then we have

$$2\omega_{m+n} = \sum_{i+1}^{m+n} (\gamma_i, \gamma_{i+1}). \tag{5.16}$$

We will compute $\eta_{m+n} = f_R^* \omega_{m+n}$. Since $\mu_2 = h_2^{-1}$, for $i \geqslant 2$

$$\begin{aligned}
C_{n+i} \dots C_1 &= A_i \dots A_1 h_2^{-1}, \\
C_{n+1} \dots C_1 &= B_{n+2}^{-1} A_1 h_2^{-1}.
\end{aligned}$$

we will simplify the terms in (5.16) as follows, using the fact that $(ab)^*(\theta) = \mathrm{Ad}_b^{-1} a^*(\theta) + b^*(\theta)$. Set $\kappa = \mu_2^*(\theta)$ so that, for $i \geqslant 2$

$$\gamma_{n+i} = (A_i \dots A_1 \mu_2)^*(\theta) = \mathrm{Ad}_{\mu_2}^{-1} \alpha_i + \kappa,$$

$$\gamma_{n+1} = (B_{n+2}^{-1} A_1 \mu_2)^*(\theta) = \mathrm{Ad}_{A_1\mu_2}^{-1} (B_{n+2}^{-1})^*(\theta) + \mathrm{Ad}_{\mu_2}^{-1} \alpha_1 + \kappa.$$

Thus consequently we get for $i \geqslant 2$

$$\begin{aligned}(\gamma_{n+i}, \gamma_{n+i+1}) &= (\mathrm{Ad}_{\mu_2}^{-1} \alpha_i + \kappa, \mathrm{Ad}_{\mu_2}^{-1} \alpha_{i+1} + \kappa) \\ &= (\alpha_i, \alpha_{i+1}) + (\alpha_i, \overline{\kappa}) + (\overline{\kappa}, \alpha_{i+1})\end{aligned}$$

and for $i = 1$

$$(\gamma_{n+1}, \gamma_{n+2}) = (\mathrm{Ad}_{A_1\mu_2}^{-1} (B_{n+2}^{-1})^*(\theta), (A_2 A_1 \mu_2)^*(\theta)) + (\alpha_1, \alpha_2) + (\alpha_1, \overline{\kappa}) + (\overline{\kappa}, \alpha_2).$$

Observe that for $2 \leqslant i \leqslant m$ the term $(\mathrm{Ad}_{h_2^{-1}}^{-1} \alpha_i, \kappa)$ appears both in (γ_{i-1}, γ_i) and (γ_i, γ_{i+1}), with opposite signs, so most of the terms in the sum of (γ_i, γ_{i+1}) will cancel out. Moreover, $(\mathrm{Ad}_{h_2^{-1}}^{-1} \alpha_1, \kappa) = 0$ since A_1 is unipotent and κ takes values in $\mathfrak{h} = \mathrm{Lie}(H)$. Therefore we have

$$\begin{aligned}\sum_{i=1}^{m+n} (\gamma_i, \gamma_{i+1}) = &\sum_{i=1}^{m} (\alpha_i, \alpha_{i+1}) + \sum_{i=1}^{n} (\gamma_i, \gamma_{i+1}) + (\overline{\kappa}, \alpha_{n+1}) \\ &+ (\mathrm{Ad}_{A_1\mu_2}^{-1} (B_{n+2}^{-1})^*(\theta), (A_2 A_1 \mu_2)^*(\theta))\end{aligned} \tag{5.17}$$

In order to eliminate the remaining expressions (γ_i, γ_{i+1}), set $s = h_2 A_1 h_2^{-1}$ and $\sigma = s^*(\theta)$. Now for $1 \leqslant i \leqslant n$ we can write

$$C_i \dots C_1 = B_i \dots B_1 s$$

and it follows that $\gamma_i = \mathrm{Ad}_s^{-1} \beta_i + \sigma$. The same simplification as before, combined with the fact that $(\beta_1, \sigma) = 0$ gives

$$\sum_{i=1}^{n} (\gamma_i, \gamma_{i+1}) = \sum_{i=1}^{n} (\beta_i, \beta_{i+1}) + (\overline{\sigma}, \beta_{n+1})$$

and the formula for $2\eta_{m+n}$ takes the form

$$\begin{aligned}\sum_{i=1}^{m+n} (\gamma_i, \gamma_{i+1}) = &\sum_{i=1}^{m} (\alpha_i, \alpha_{i+1}) + \sum_{i=1}^{n} (\beta_i, \beta_{i+1}) + (\overline{\kappa}, \alpha_{m+1}) \\ &+ (\overline{\sigma}, \beta_{n+1}) + (\mathrm{Ad}_{A_1\mu_2}^{-1} (B_{n+2}^{-1})^*(\theta), (A_2 A_1 \mu_2)^*(\theta)).\end{aligned}$$

We will show that the last two terms here sum to zero. Since $B_{n+1}\dots B_1 = B_{n+2}^{-1}h_2^{-1}$, we see that

$$\beta_{n+1} = \mathrm{Ad}_{A_1\mu_2}^{-1}(B_{n+2}^{-1})^*(\theta) + \kappa,$$

which implies

$$(\overline{\sigma}, \beta_{n+1}) = (\overline{\sigma}, \mathrm{Ad}_{A_1\mu_2}^{-1}(B_{n+2}^{-1})^*(\theta)) + (\overline{\sigma}, \kappa)$$

and $(\overline{\sigma}, \kappa) = 0$ since s is unipotent and κ is $\mathfrak{h}$-valued. Hence it remains to prove

$$(\mathrm{Ad}_{A_1\mu_2}^{-1}(B_{n+2}^{-1})^*(\theta), (A_2A_1h_2^{-1})^*(\theta) - \tau) = 0. \tag{5.18}$$

However, we have

$$\begin{aligned}(A_2A_1h_2^{-1})^*(\theta) - \tau &= \mathrm{Ad}_{A_1\mu_2}^{-1}A_2^*(\theta) + (A_1\mu_2)^*(\theta) - \mathrm{Ad}_{A_1\mu_2}^{-1}h_2^*(\theta) - (A_1\mu_2)^*(\theta)\\ &= \mathrm{Ad}_{A_1\mu_2}^{-1}(A_2^*(\theta) - h_2^*(\theta)),\end{aligned}$$

and so the left side of (5.18) simplifies to $((B_{n+2}^{-1})^*(\theta), A_2^*(\theta) - h_2^*(\theta))$ which is zero since B_{n+2} and A_2 belong to the same unipotent, and $h_2^*(\theta)$ is $\mathfrak{h}$-valued. As a consequence we have

$$\sum_{i=1}^{m+n}(\gamma_i, \gamma_{i+1}) = \sum_{i=1}^{m}(\alpha_i, \alpha_{i+1}) + \sum_{i=1}^{n}(\beta_i, \beta_{i+1}) + (\overline{\kappa}, \alpha_{m+1}).$$

On the other hand, the (twisted) quasi-Hamiltonian form $\omega_{\circledast}$ on the fusion $\mathcal{B}_m \circledast \mathcal{B}_n$ is given by

$$2\omega_{\circledast} = 2\omega_m + 2\omega_n - (\mu_1^*(\theta), \mu_2^*(\overline{\theta}))$$

so the proof will be complete if we prove that

$$(\overline{\kappa}, \alpha_{m+1}) = -(\mu_1^*(\theta), \mu_2^*(\overline{\theta}))$$

i.e. that $(\alpha_{m+1}, \overline{\kappa}) = (\mu_1^*(\theta), \overline{\kappa}))$. This holds since the $\mathfrak{h}$ component of α_{m+1} equals that of $\alpha_{m+2} = \mu_1^*(\theta)$.

□

5.5 Factorisations and quasi-Hamiltonian embeddings

Recall that for a compatible pair $\mathcal{B}_m, \mathcal{B}_n$ we have define two maps, the right and the left factorisation maps

$$f_R, f_L : \mathcal{B}_m \circledast \mathcal{B}_n \to \mathcal{B}_{m+n}$$

given by

$$\begin{aligned}f_R((h_1, \mathbf{A}), (h_2, \mathbf{B})) &= (h_2h_1, A_{m+2}, \dots, A_3, A_2B_{n+2}, B_{n+1}, \cdots, B_2, \widetilde{B}_1),\\ f_L((h_1, \mathbf{A}), (h_2, \mathbf{B})) &= (h_2h_1, \widetilde{A}_{m+2}, \dots, A_2, A_1B_{n+1}, B_n, \cdots, B_2, B_1),\end{aligned}$$

where $\mathbf{A} = (A_{n+2}, \dots, A_1), \mathbf{B} = (B_{m+2}, \dots, B_1)$ and

$$\widetilde{B}_1 = B_1 h_2 A_1 h_2^{-1}, \quad \widetilde{A}_{m+2} = h_1^{-1} B_{n+2} h_1 A_{n+2}.$$

We have shown in Theorem 5.4.1 that these are not only injections on dense open subsets of $\mathcal{B}_{m+n}$, but also (possibly twisted) quasi-Hamiltonian embeddings. We can summarise Theorem 5.4.1 for all four possible choices of compatible pairs.

Corollary 5.5.1. *The right and left factorisation maps yield the following (twisted) quasi-Hamiltonian inclusions:*

$$\mathcal{B}_m^+ \circledast \mathcal{B}_n^+ \hookrightarrow \mathcal{B}_{m+n}^+$$
$$\mathcal{B}_m^- \circledast \mathcal{B}_n^- \hookrightarrow \mathcal{B}_{m+n}^-$$

if m is even and

$$\mathcal{B}_m^+ \circledast \mathcal{B}_n^- \hookrightarrow \mathcal{B}_{m+n}^+$$
$$\mathcal{B}_m^- \circledast \mathcal{B}_n^+ \hookrightarrow \mathcal{B}_{m+n}^-$$

if m is odd.

Proposition 5.5.2. *The images of the four inclusions from Corollary 5.5.1 in $\mathcal{B}_{n+m}^{\pm}$ are open dense subsets given by*

$$\det(b_{m+1}, \dots, b_{m+n}) \neq 0 \text{ for the image of the right factorisation map,}$$
$$\det(b_1, \dots, b_m) \neq 0 \text{ for the image of the left factorisation map.}$$

Proof. Suppose that m is even. The proof works verbatim in all four cases. Consider a point $a_1, \dots, a_m \in \mathcal{B}_m^+, b_1, \dots, b_n \in \mathcal{B}_n^+$. Recall that a point in $\mathcal{B}_k^{\pm}$ corresponds to off-diagonal entries $s_2, \dots, s_{k+1}$ in the Stokes multipliers of the reduced fission space $\mathcal{B}^c(V)$ and it uniquely determines the matrices h, S_1, S_{k+2}. Thus the matrices A_1, B_{n+2} appearing in the factorisation map are uniquely determined and a direct computation shows that

$$A_1 = \begin{pmatrix} 1 & -(a_1, \dots, a_n)^{-1}(a_1, \dots, a_{n-1}) \\ 0 & 1 \end{pmatrix},$$

$$B_{n+2} = \begin{pmatrix} 1 & 0 \\ -(b_2, \dots, b_m)(b_1, \dots, b_m)^{-1} & 1 \end{pmatrix}.$$

Therefore we have

$$A_2 B_{n+2} = \begin{pmatrix} 1 & 0 \\ a_n - (b_2, \dots, b_m)(b_1, \dots, b_m)^{-1} & 1 \end{pmatrix},$$

$$A_1 B_{n+1} = \begin{pmatrix} 1 & b_1 - (a_1, \dots, a_n)^{-1}(a_1, \dots, a_{n-1}) \\ 0 & 1 \end{pmatrix}$$

and the proof follows from Proposition 5.2.2. The two factorisation maps f_R, f_L correspond the two factorisations of the continuant. The image is open and dense since it is defined as a subset where the appropriate determinant does not vanish. □

Therefore each full factorisation of the continuant determines a twisted quasi-Hamiltonian embedding of copies of $\mathcal{B}_1^+, \mathcal{B}_1^-$ into $\mathcal{B}_k^+$ and $\mathcal{B}_k^-$, We can further identify these copies with $\mathrm{GL}(W)$ with zero quasi-Hamiltonian form.

Corollary 5.5.3. *Let k be a positive integer. Each of C_k factorisations of the continuant $(b_1, \ldots, b_k)$ gives a twisted quasi-Hamiltonian inclusion*

$$\mathrm{GL}(W)^{\circledast k} \hookrightarrow \mathcal{B}_k^+$$

onto a dense open subset of $\mathcal{B}_k^+$.

Recall that we have also established a correspondence between factorisations of continuants and triangulations of polygons. Therefore, as in Remark 5.2.15, for each triangulation of a $(k+2)$-gon we obtain a factorisation of the continuant $(x_1, \ldots, x_k)$ and hence a quasi-Hamiltonian embedding, so we can restate the above corollary as follows.

Corollary 5.5.4. *Every triangulation of a $(k+2)$-gon determines a quasi-Hamiltonian embedding*

$$\mathrm{GL}(W)^{\circledast k} \hookrightarrow \mathcal{B}_k^+$$

onto a dense open subset of $\mathcal{B}_k^+$.

5.6 Quasi-Hamiltonian structure on $\mathcal{B}_k^+$

Recall that in section 2.2.3 we have defined the Van den Bergh space $\mathcal{B}(V_1, V_2)$ as follows. Let V_1, V_2 be two finite-dimensional complex vector spaces and let $V = V_1 \oplus V_2$. Define

$$\mathcal{B}(V_1, V_2) = \{(a, b) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \mid \det(1 + ab) \neq 0\}$$

and the natural actions of $\mathrm{GL}(V_1)$ on V_1 and $\mathrm{GL}(V_2)$ on V_2 induce an action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ on $B(V_1, V_2)$. The space $\mathcal{B}(V_1, V_2)$ is a quasi-Hamiltonian $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-space with moment map

$$\mu(a, b) = \big((a, b)^{-1}, (b, a)\big) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$$

and the quasi-Hamiltonian two-form given by equation (2.14):

$$\omega = \frac{1}{2}\big(\mathrm{Tr}_{V_1}(a, b)^{-1} da \wedge db - \mathrm{Tr}_{V_2}(b, a)^{-1} db \wedge da\big). \tag{5.19}$$

On the other had, for the grading $V = V_1 \oplus V_2$ there is the reduced fission space $\mathcal{B}(V)$, which is identified with

$$\mathcal{B}(V) = \{(h, S_1, S_2, S_3, S_4) \in H \times (U_+ \times U_-)^2 \mid hS_4S_3S_2S_1 = 1\}$$

and it is a quasi-Hamiltonian H-space with moment map h^{-1}. As we have seen, the quasi-Hamiltonian form on $\mathcal{B}(V)$ is obtained by restricting the form on the fission space $\mathcal{A}^2(V)$, given by equation (2.9), to the subset $C = b = 1$ and it is given explicitly by

$$\omega = \frac{1}{2}\big((\gamma_1, \gamma_2) + (\gamma_2, \gamma_3)\big)$$

where $\gamma_i = C_i^*(\theta)$ and $C_i = S_i \cdots S_1$ since $C = 1$. Moreover, Theorem 2.2.16 stated that the two-forms on spaces $\mathcal{B}(V_1, V_2)$ and $\mathcal{B}(V)$ match up and they are isomorphic as quasi-Hamiltonian spaces, so the spaces $\mathcal{B}^k(V)$ can be understood as quasi-Hamiltonian generalisations of the Van den Bergh space.

We will generalise the formula (5.19) for the two form on the space $\mathcal{B}(V)$ – which is isomorphic to $\mathcal{B}_2^+$ – to spaces $\mathcal{B}_k^+$, for any positive integer k, using the continuants.

Suppose that $V = W \oplus W$ so that the coordinates b_i on $\mathcal{B}_k^+$ are in $\mathrm{End}(W)$ and let us in introduce the following notation

$$D^2(b_1, \ldots, b_k) = \sum_{i<j} (b_1, \ldots, b_{i-1}) db_i (b_{i+1}, \ldots, b_{j-1}) db_j (b_{j+1}, \ldots, b_k), \tag{5.20}$$

which is a naive equivalent of the second derivative d^2. It is not zero since we do not factor in the signs of permutations when differentiating the expressions. The quasi-Hamiltonian two-form on the space $\mathcal{B}_2^+$ is then given by

$$\omega = \frac{1}{2}\big(- \mathrm{Tr}(b_1, b_2)^{-1} D^2(b_1, b_2) + \mathrm{Tr}(b_2, b_1)^{-1} D^2(b_2, b_1)\big).$$

Theorem 5.6.1. *The quasi-Hamiltonian two-form on the space $\mathcal{B}_k^+$ is given by*

$$\omega_k = \frac{1}{2}\big(- \mathrm{Tr}(b_1, \ldots, b_k)^{-1} D^2(b_1, \ldots, b_k) + \mathrm{Tr}(b_k, \ldots, b_1)^{-1} D^2(b_k, \ldots, b_1)\big). \tag{5.21}$$

Proof. We will proceed by induction, using the factorisation map. For simplicity, let us introduce the following notation, for $i \leqslant j$:

$$C_{i,j} := (b_i, \ldots, b_j), \quad C_{j,i} := (b_j, \ldots, b_i).$$

Since $k > 2$, by Proposition 5.5.1 we can find m, n with even m such that $m + n = k$ and a (twisted) quasi-Hamiltonian embedding given by the right factorisation map

$$\mathcal{B}_m^+ \circledast \mathcal{B}_n^+ \hookrightarrow \mathcal{B}_k^+.$$

Its image is equal to the subset of $\mathcal{B}_k^+$, where $C_{m+1,m+n}$ is invertible, which is open and dense in $\mathcal{B}_k^+$. Let $\omega_m, \omega_n, \omega_k$ denote the form ω_i from equation (5.21) for the chosen index i. Note that $\omega_{m+n} = \omega_k$. We want to show that ω_{m+n} is a quasi-Hamiltonian form on $\mathcal{B}_k^+$.

We set

$$f_1 = \mathrm{Tr}((C_{1,m+n})^{-1}D^2(C_{1,m+n})), \quad f_2 = \mathrm{Tr}((C_{m+n,1})^{-1}D^2(C_{m+n,1}))$$

so $2\omega_{m+n} = -f_1 + f_2$.

Let us further denote

$$\widehat{C}_{1,m} = (b_1, \ldots, \widehat{b}_m), \quad \widehat{C}_{m,1} = (\widehat{b}_m, \ldots, b_1),$$

where $\widehat{b}_m = b_m + C_{m+2,m+n}C^{-1}_{m+1,m+n}$, as in Proposition 5.2.2. Since

$$C_{1,m+n} = \widehat{C}_{1,m}C_{m+1,m+n}, \quad C_{m+n,1} = C_{m+n,m+1}\widehat{C}_{m,1}$$

we have

$$\begin{aligned} f_1 &= \mathrm{Tr}(C^{-1}_{m+1,m+n}\widehat{C}^{-1}_{1,m}[D^2(\widehat{C}_{1,m})C_{m+1,m+n} + \widehat{C}_{1,m}D^2(C_{m+1,m+n}) + d(\widehat{C}_{1,m})d(C_{m+1,m+n})]) \\ &= \mathrm{Tr}(\widehat{C}^{-1}_{1,m}D^2(\widehat{C}_{1,m}) + C^{-1}_{m+1,m+n}D^2(C_{m+1,m+n}) + C^{-1}_{m+1,m+n}\widehat{C}^{-1}_{1,m}d(\widehat{C}_{1,m})d(C_{m+1,m+n})) \end{aligned}$$

and in the same way we obtain

$$f_2 = \mathrm{Tr}(\widehat{C}^{-1}_{m,1}D^2(\widehat{C}_{m,1}) + C^{-1}_{m+n,m+1}D^2(C_{m+n,m+1}) + \widehat{C}^{-1}_{m,1}C^{-1}_{m+n,m+1}d(C_{m+n,m+1})d(\widehat{C}_{m,1})).$$

Thus

$$\begin{aligned} -f_1 + f_2 = {} & \mathrm{Tr}(\widehat{C}^{-1}_{m,1}C^{-1}_{m+n,m+1}d(C_{m+n,m+1})d(\widehat{C}_{m,1}) - C^{-1}_{m+1,m+n}\widehat{C}^{-1}_{1,m}d(\widehat{C}_{1,m})d(C_{m+1,m+n})) \\ & + 2\omega_m + 2\omega_n. \end{aligned}$$

Now, using the face that $(g^{-1})^*(\theta) = g^*(\overline{\theta})$, we have

$$\begin{aligned} \mathrm{Tr}(\widehat{C}^{-1}_{m,1}C^{-1}_{m+n,m+1}d(C_{m+n,m+1})d(\widehat{C}_{m,1})) &= -((\widehat{C}^{-1}_{m,1})^*(\theta), (C^{-1}_{m+n,m+1})^*(\overline{\theta})), \\ -\mathrm{Tr}(C^{-1}_{m+1,m+n}\widehat{C}^{-1}_{1,m}d(\widehat{C}_{1,m})d(C_{m+1,m+n})) &= -(\widehat{C}^*_{1,m}(\theta), C^*_{m+1,m+n}(\overline{\theta})). \end{aligned} \tag{5.22}$$

On the other hand, the quasi-Hamiltonian form $\omega_{\circledast}$ on the fusion $\mathcal{B}^+_m \circledast \mathcal{B}^+_n$ is given by

$$2\omega_{\circledast} = 2\omega_m + 2\omega_n - (\mu^*_{\mathcal{B}^+_m}(\theta), \mu^*_{\mathcal{B}^+_n}(\overline{\theta}))$$

and we will set $r = -(\mu^*_{\mathcal{B}^+_m}(\theta), \mu^*_{\mathcal{B}^+_n}(\overline{\theta}))$. The theorem will follow if the sum of right hand sides of (5.22) is equal to r.

This is an easy check that needs to be done case by case since the moment maps take values in H and $H(\partial)$, depending on the parity of m, n.

If both m, n are even, then the moment maps are

$$\mu_{\mathcal{B}_m^+} = \begin{pmatrix} \widehat{C}_{1,m} & 0 \\ 0 & \widehat{C}_{m,1}^{-1} \end{pmatrix}, \quad \mu_{\mathcal{B}_n^+} = \begin{pmatrix} C_{m+1,m+n} & 0 \\ 0 & C_{m+n,m+1}^{-1} \end{pmatrix}$$

we see that

$$-(\widehat{C}_{1,m}^*(\theta), C_{m+1,m+n}^*(\overline{\theta})) - ((\widehat{C}_{m,1}^{-1})^*(\theta), (C_{m+n,m+1}^{-1})^*(\overline{\theta})) = -(\mu_{\mathcal{B}_m^+}^*(\theta), \mu_{\mathcal{B}_n^+}^*(\overline{\theta})).$$

If n is odd, the moment maps are

$$\mu_{\mathcal{B}_m^+} = \begin{pmatrix} \widehat{C}_{1,m} & 0 \\ 0 & \widehat{C}_{m,1}^{-1} \end{pmatrix}, \quad \mu_{\mathcal{B}_n^+} = \begin{pmatrix} 0 & C_{m+1,m+n} \\ -C_{m+n,m+1}^{-1} & 0 \end{pmatrix}$$

and again the equality holds since $g^*(\theta) = -g^*(\theta)$.

Thus we have shown that on the image of the inclusion $\mathcal{B}_m^+ \circledast \mathcal{B}_n^+ \hookrightarrow \mathcal{B}_k^+$ the formula for the two form matches up with (5.21). Since the image is open and dense and the form is holomorphic, it extends to the whole $\mathcal{B}_k^+$. $\square$

Chapter 6

Bibliography

[1] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.

[2] J. E. Andersen, J. Mattes, and N. Reshetikhin, *The Poisson structure on the moduli space of flat connections and chord diagrams*, Topology **35** (1996), no. 4, 1069–1083.

[3] D. Arinkin and S. Lysenko, *Isomorphisms between moduli spaces of* SL(2)*-bundles with connections on* $\mathbf{P}^1 \setminus \{x_1, \cdots, x_4\}$, Math. Res. Lett. **4** (1997), no. 2-3, 181–190.

[4] M. F. Atiyah, *Hyper-Kähler manifolds*, Complex geometry and analysis (Pisa, 1988), Lecture Notes in Math., vol. 1422, Springer, Berlin, 1990, pp. 1–13.

[5] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615.

[6] M. Audin, *Lectures on gauge theory and integrable systems*, Gauge theory and symplectic geometry (Montreal, PQ, 1995), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 488, Kluwer Acad. Publ., Dordrecht, 1997, pp. 1–48.

[7] D. G. Babbitt and V. S. Varadarajan, *Local moduli for meromorphic differential equations*, Astérisque (1989), no. 169-170, 217.

[8] J. C. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 145–205.

[9] W. Balser, W. B. Jurkat, and D. A. Lutz, *Birkhoff invariants and Stokes' multipliers for meromorphic linear differential equations*, J. Math. Anal. Appl. **71** (1979), no. 1, 48–94.

[10] A. A. Beilinson and V. G. Drinfeld, *Quantization of Hitchin's fibration and Langlands' program*, Algebraic and geometric methods in mathematical physics

(Kaciveli, 1993), Math. Phys. Stud., vol. 19, Kluwer Acad. Publ., Dordrecht, 1996, pp. 3–7.

[11] O. Biquard and P. Boalch, *Wild non-abelian Hodge theory on curves*, Compos. Math. **140** (2004), no. 1, 179–204.

[12] P. Boalch and R. Paluba, *Symmetric cubic surfaces and G_2 character varieties*, J. Algebraic Geom. **25** (2016), no. 4, 607–631.

[13] P. P. Boalch, *Symplectic geometry and isomonodromic deformations*, D. Phil. Thesis, Oxford, 1999.

[14] ______, *Stokes matrices, Poisson Lie groups and Frobenius manifolds*, Invent. Math. **146** (2001), no. 3, 479–506.

[15] ______, *Symplectic manifolds and isomonodromic deformations*, Adv. Math. **163** (2001), no. 2, 137–205.

[16] ______, *G-bundles, isomonodromy, and quantum Weyl groups*, Int. Math. Res. Not. (2002), no. 22, 1129–1166.

[17] ______, *Painlevé equations and complex reflections*, Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002), vol. 53, 2003, pp. 1009–1022.

[18] ______, *From Klein to Painlevé via Fourier, Laplace and Jimbo*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 167–208.

[19] ______, *The fifty-two icosahedral solutions to Painlevé VI*, J. Reine Angew. Math. **596** (2006), 183–214.

[20] ______, *Six results on Painlevé VI*, Théories asymptotiques et équations de Painlevé, Sémin. Congr., vol. 14, Soc. Math. France, Paris, 2006, pp. 1–20.

[21] ______, *Quasi-Hamiltonian geometry of meromorphic connections*, Duke Math. J. **139** (2007), no. 2, 369–405, see also arXiv:0203161.

[22] ______, *Some explicit solutions to the Riemann-Hilbert problem*, Differential equations and quantum groups, IRMA Lect. Math. Theor. Phys., vol. 9, Eur. Math. Soc., Zürich, 2007, pp. 85–112.

[23] ______, *Irregular connections and Kac–Moody root systems*, 2008, arXiv:0806.1050.

[24] ______, *Riemann-Hilbert for tame complex parahoric connections*, Transform. Groups **16** (2011), no. 1, 27–50.

[25] ______, *Hyperkähler manifolds and nonabelian hodge theory of (irregular) curves*, 2012, text of a talk at Institut Henri Poincaré, arXiv:1203.6607.

[26] ———, *Simply-laced isomonodromy systems*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 1–68.

[27] ———, *Geometry and braiding of Stokes data; fission and wild character varieties*, Ann. of Math. (2) **179** (2014), no. 1, 301–365.

[28] ———, *Global Weyl groups and a new theory of multiplicative quiver varieties*, Geom. Topol. **19** (2015), no. 6, 3467–3536.

[29] ———, *Non-perturbative symplectic manifolds and non-commutative algebras*, 2015, talk at the University of Kyoto, slides available on conference website: http://www2.kobe-u.ac.jp/~mhsaito/rims1503/program.html.

[30] ———, *Wild character varieties, Hitchin systems and Dynkin diagrams*, 2017, arxiv:1703.10376.

[31] P. P. Boalch and D. Yamakawa, *Twisted wild character varieties*, 2015, arXiv:1512.08091.

[32] J. H. Conway and H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. **57** (1973), no. 401, 175–183.

[33] ———, *Triangulated polygons and frieze patterns*, Math. Gaz. **57** (1973), no. 400, 87–94.

[34] J. H. Conway and D. A. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, A K Peters, Ltd., Natick, MA, 2003.

[35] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), no. 3, 361–382.

[36] W. Crawley-Boevey and P. Shaw, *Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem*, Adv. Math. **201** (2006), no. 1, 180–208.

[37] P. Deligne, *letter to B. Malgrange*, 19 April, 1978, 2pp.

[38] ———, *équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[39] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 127–131.

[40] P. Etingof, A. Oblomkov, and E. Rains, *Generalized double affine Hecke algebras of rank 1 and quantized del Pezzo surfaces*, Adv. Math. **212** (2007), no. 2, 749–796.

[41] L. Euler, *Specimen algorithmi singularis*, Novi Commentarii academiae scientiarum Petropolitanae (1764), no. 9, 53–69.

[42] H. Flaschka and A. C. Newell, *Monodromy- and spectrum-preserving deformations. I*, Comm. Math. Phys. **76** (1980), no. 1, 65–116.

[43] V. V. Fock and A. A. Rosly, *Flat connections and polyubles*, Teoret. Mat. Fiz. **95** (1993), no. 2, 228–238.

[44] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[45] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen. I*, Druck und Verlag von B. G. Teubner, Leipzig, 1897.

[46] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.

[47] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, Adv. Math. **234** (2013), 239–403.

[48] W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. **54** (1984), no. 2, 200–225.

[49] ———, *Topological components of spaces of representations*, Invent. Math. **93** (1988), no. 3, 557–607.

[50] R. L. Griess, Jr., *Basic conjugacy theorems for G_2*, Invent. Math. **121** (1995), no. 2, 257–277.

[51] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** (1987), no. 1, 59–126.

[52] ———, *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42** (1995), no. 1, 30–112.

[53] ———, *Langlands duality and G_2 spectral curves*, Q. J. Math. **58** (2007), no. 3, 319–344.

[54] K. Iwaki and T. Nakanishi, *Exact WKB analysis and cluster algebras*, J. Phys. A **47** (2014), no. 47, 474009, 98.

[55] K. Iwasaki, *A modular group action on cubic surfaces and the monodromy of the Painlevé VI equation*, Proc. Japan Acad. Ser. A Math. Sci. **78** (2002), no. 7, 131–135.

[56] M. Jimbo, T. Miwa, and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function*, Phys. D **2** (1981), no. 2, 306–352.

[57] A. A. Kapaev and A. V. Kitaev, *Connection formulae for the first Painlevé transcendent in the complex domain*, Lett. Math. Phys. **27** (1993), no. 4, 243–252.

[58] Y. Karshon, *An algebraic proof for the symplectic structure of moduli space*, Proc. Amer. Math. Soc. **116** (1992), no. 3, 591–605.

[59] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530.

[60] Donald E. Knuth, *The art of computer programming*, Addison-Wesley, Reading, MA, 1998, Third edition.

[61] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. **29** (1989), no. 3, 665–683.

[62] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598.

[63] J.-L. Loday, *Arithmetree*, J. Algebra **258** (2002), no. 1, 275–309, Special issue in celebration of Claudio Procesi's 60th birthday.

[64] ———, *Generalized bialgebras and triples of operads*, Astérisque (2008), no. 320, x+116.

[65] M. Loday-Richaud, *Stokes phenomenon, multisummability and differential Galois groups*, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 3, 849–906.

[66] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), no. 336, xi+117.

[67] W. Magnus, *Rings of Fricke characters and automorphism groups of free groups*, Math. Z. **170** (1980), no. 1, 91–103.

[68] B. Malgrange, *La classification des connexions irrégulières à une variable*, Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, pp. 381–399.

[69] Yu. I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbf{P}^2$, Geometry of differential equations, Amer. Math. Soc. Transl. Ser. 2, vol. 186, Amer. Math. Soc., Providence, RI, 1998, pp. 131–151.

[70] J. Martinet and J.-P. Ramis, *Elementary acceleration and multisummability. I*, Ann. Inst. H. Poincaré Phys. Théor. **54** (1991), no. 4, 331–401.

[71] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[72] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[73] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Annals of Math. **82** (1965), 540–567.

[74] I. Naruki, *Cross ratio variety as a moduli space of cubic surfaces*, Proc. London Math. Soc. (3) **45** (1982), no. 1, 1–30, With an appendix by Eduard Looijenga.

[75] A. Oblomkov, *Double affine Hecke algebras of rank 1 and affine cubic surfaces*, Int. Math. Res. Not. (2004), no. 18, 877–912.

[76] K. Okamoto, *Studies on the Painlevé equations. III. Second and fourth Painlevé equations, P_{II} and P_{IV}*, Math. Ann. **275** (1986), no. 2, 221–255.

[77] ———, *Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}*, Ann. Mat. Pura Appl. (4) **146** (1987), 337–381.

[78] ———, *Studies on the Painlevé equations. II. Fifth Painlevé equation P_{V}*, Japan. J. Math. (N.S.) **13** (1987), no. 1, 47–76.

[79] ———, *The Painlevé equations and the Dynkin diagrams*, Painlevé transcendents (Sainte-Adèle, PQ, 1990), NATO Adv. Sci. Inst. Ser. B Phys., vol. 278, Plenum, New York, 1992, pp. 299–313.

[80] C. Procesi, *The invariant theory of $n \times n$ matrices*, Advances in Math. **19** (1976), no. 3, 306–381.

[81] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229.

[82] G. W. Schwarz, *Representations of simple Lie groups with regular rings of invariants*, Invent. Math. **49** (1978), no. 2, 167–191.

[83] ———, *Invariant theory of G_2*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), no. 3, 335–338.

[84] J.-P. Serre, *Coordinées de Kac*, 2006.

[85] Y. Sibuya, *Global theory of a second order linear ordinary differential equation with a polynomial coefficient*, North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975, North-Holland Mathematics Studies, Vol. 18.

[86] A. S. Sikora, *Character varieties*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5173–5208.

[87] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95.

[88] ———, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 47–129.

[89] ———, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 80, 5–79 (1995).

[90] G. G. Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments*, Cam. Phil. Trans. **X** (1857), no. 1, 105–128.

[91] M. Van den Bergh, *Double Poisson algebras*, Trans. Amer. Math. Soc. **360** (2008), no. 11, 5711–5769.

[92] ______, *Non-commutative quasi-Hamiltonian spaces*, Poisson geometry in mathematics and physics, Contemp. Math., vol. 450, Amer. Math. Soc., Providence, RI, 2008, pp. 273–299.

[93] M. van der Put and M.-H. Saito, *Moduli spaces for linear differential equations and the Painlevé equations*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 7, 2611–2667.

[94] H. Vogt, *Sur les invariants fondamentaux des équations différentielles linéaires du second ordre*, Ann. Sci. École Norm. Sup. (3) **6** (1889), 3–71.

[95] A. Voros, *The return of the quartic oscillator: the complex WKB method*, Ann. Inst. H. Poincaré Sect. A (N.S.) **39** (1983), no. 3, 211–338.

[96] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1976, Reprint of the 1965 edition.

[97] D. Yamakawa, *Geometry of multiplicative preprojective algebra*, Int. Math. Res. Pap. IMRP (2008), Art. ID rpn008, 77pp.

[98] A. S. Zhedanov, *"Hidden symmetry" of Askey-Wilson polynomials*, Teoret. Mat. Fiz. **89** (1991), no. 2, 190–204.

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École doctorale de mathématiques Hadamard (EDMH)

Titre : Géométrie des variétés de caractères complexes

Mots Clefs : variété de caractères, données de Stokes, connexion irregulière

Résumé : Le but de cette thèse est d'étudier différents exemples des variétés de caractères régulières et sauvages des courbes complexes.

La première partie est consacrée à l'étude d'un exemple de variété de caractères de la sphère avec quatre trous et groupe exotique G_2 comme son groupe de structure. On démontre que pour un choix particulier de classes de conjugaison du groupe G_2, la variété obtenue est de dimension complexe deux et isomorphe à la surface cubique de Fricke—Klein. Cette surface apparaît déjà dans le cas classique comme la variété de caractères de cette surface avec le groupe de structure $\mathrm{SL}_2(\mathbb{C})$. De plus, on interprète les orbites de groupe de tresses de taille 7 dans cette surface comme les droites passant par les triplés de points dans le plan de Fano $\mathbb{P}^2(\mathbb{F}_2)$.

Dans la deuxième partie, on établit plusieurs cas de la „conjecture d'écho", correspondant aux équations différentielles de Painlevé I, II et IV. On montre que sur la sphère de Riemann avec un point singulier, pour des choix particuliers de la singularité il y a trois familles infinies de variétés de caractères sauvages de dimension complexe deux. Dans ces familles, le rang du groupe de structure n'est pas borné et augmente jusqu'à l'infini. Le résultat principal de cette partie démontre que tous les membres de ces trois familles de variétés sont isomorphes aux espaces de phase des équations de Painlevé associées. En calculant les quotients de la théorie géométrique des invariants, on fournit des isomorphismes explicites entre les anneaux de fonctions des variétés affines qui apparaissent et relie les paramètres des surfaces cubiques.

Dans la dernière partie, avec des outils de la géométrie quasi-Hamiltonienne, on étudie une famille des espaces généralisant les hiérarchies de Painlevé I et II pour les groupes linéaires de rang supérieur. En particulier, pour toute variété B_k dans la hiérarchie il y a une application moment, prenant ses valeurs dans un groupe, qui s'avère être un polynôme continuant d'Euler. Ces polynômes admettent des factorisations en continuants plus courts et on montre que les factorisations d'un polynôme continuant de longueur k en termes de longueur un sont énumérées par le nombre de Catalan C_k. De plus, chaque factorisation fournit un plongement du produit de fusion de k copies de $\mathrm{GL}_n(\mathbb{C})$ sur un ouvert dense de B_k et on démontre que ces plongements relient les structures quasi-Hamiltoniennes. Finalement, on utilise ce résultat pour dériver une formule explicite pour la 2-forme quasi-Hamiltonienne sur B_k, généralisant la formule connue dans le cas de B_2.

Title : Geometry of complex character varieties

Keys words : character variety, Stokes data, irregular connection

Abstract : The aim of this thesis is to study various examples of tame and wild character varieties of complex curves.

In the first part, we study an example of a tame character variety of the four-holed sphere with simple poles and exotic group G_2 as the structure group. We show that for a particular choice of conjugacy classes in G_2, the resulting affine symplectic variety of complex dimension two is isomorphic to the Fricke-Klein cubic surface, known from the classical case of the character variety for the group $\mathrm{SL}_2(\mathbb{C})$. Furthermore, we interpret the braid group orbits of size 7 in this affine surface as lines passing through triples of points in the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$.

In the second part, we establish multiple cases of the so-called „echo conjecture", corresponding to the cases of Painleve I, II and IV differential equations. We show that for the Riemann sphere with one singular point and suitably chosen behavior at the singularity, there are three infinite families of wild character varieties of complex dimension two. In these families, the rank of the structure group is not bounded and goes to infinity. The main result of this part shows that in each family all the members are affine cubic surfaces, isomorphic to the phase spaces of the aforementioned Painleve equations. By computing the geometric invariat theory quotients, we provide explicit isomorphisms between the rings of functions of the arising affine varieties and relate the coefficients of the affine surfaces.

The last part is dedicated to the study of a family of spaces generalizing the Painleve I and II hierarchies for higher rank linear groups, which is done by the means of quasi-Hamiltonian geometry. In particular, for each variety B_k in the hierarchy there is a group-valued moment map and they turn out to be the Euler's continuant polynomials. These in turn admit factorisations into products of shorter continuants and we show that for a continuant of length k, the distinct factorisations into continuants of length one are counted by the Catalan number C_k. Moreover, each such factorisation provides an embedding of the fusion product of k copies of $\mathrm{GL}_n(\mathbb{C})$ onto a dense open subset of B_k and the quasi-Hamiltonian structures do match up. Finally, using this result we derive the formula for the quasi-Hamiltonian two form on the space B_k, which generalises the formula known for the case of B_2.