Scheme theoretic vector bundles

The best reference for this material is the first chapter of [Gro61]. What can be found below is a less complete treatment of the same material.

1. Introduction

Let's start with a definition, to fix ideas. Notation will be defined at a later point in this note.

(1.1) **Definition**: Let $S = (S, \mathcal{O}_S)$ be a scheme. An S-scheme X is said to be a *scheme-theoretic vector bundle over* S if there exists an $\mathcal{O}_S$-algebra $\mathcal{A}$, quasi-coherent as an $\mathcal{O}_S$-module, and an S-isomorphism $X \cong \mathrm{Spec}_S(\mathcal{A})$, where $\mathrm{Spec}_S(\mathcal{A})$ is considered an S scheme via its canonical structure map.

This could be summarized by saying there is an isomorphism $X \to \mathrm{Spec}_S(\mathcal{A})$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Spec}_S(\mathcal{A}) \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

commutative. The point of this note is to elucidate the following:

(P1) there is a fully-faithful functor $\mathsf{Mod}(\mathcal{O}_S) \to \mathsf{Vec}_S$ between the category of quasi-coherent $\mathcal{O}_S$-modules and scheme-theoretic vector bundles over S

(P2) there is an autoequivalence of the category $\mathsf{Loc}(\mathcal{O}_S)$ of finite locally free $\mathcal{O}_S$-modules

(P3) the full subcategory $\mathsf{Loc}(\mathcal{O}_S)$ of $\mathsf{Mod}(\mathcal{O}_S)$ is equivalent, under the functor of (P1), to the category of classical vector bundles "admitting a cover of S trivializing them".

The plan is to start by describing all of the pieces of the functor in (P1). This should take up essentially three sections: to describe the symmetric algebra sheaf, to describe the relative spectrum construction, and to show the functor is fully-faithful. From this work (P3) will follow completely formally from the definition of a finite locally free $\mathcal{O}_S$-module. (P2) is independent of (P1) and (P3) and will be completed afterwards.

Throughout the first part of this note I hope to clarify the following: (P1) is contravariant and (P2) is a contravariant equivalence. Composing the two functors defines a sequence equivalence between the categories of (P3) but this time *covariant*. This has been a source of much of my confusion regarding the definition of the tautological and hyperplane bundles on projective space. This will form the content of sections 2-5.

Section 6 will be an analysis of the functor assigning to a scheme-theoretic vector bundle its sheaf of sections.

As an aside, and not to be taken completely seriously, I want to criticize the use of the term vector bundle when one truly means finite locally free $\mathcal{O}_S$-module. Although the habit won't die, it really does cause a source of confusion for the learner, especially when one works with both concepts, goes between the two, and doesn't mention the choice of covariant or contravariant equivalence between the categories. While the expert may follow, the length of time I have put off learning the definition of the tautological line bundle is remarkable, and (almost) completely due to my confusion with the above concepts.

2. The symmetric algebra

The purpose of the symmetric algebra is to functorially construct a commutative algebra starting from just a module. We'll first do this locally, where we work with rings and modules instead of sheaves of rings and sheaves of modules.

(2.1) **Lemma**: Let A be a ring and M an A-module. There exists a commutative A-algebra $S(M)$, and an A-linear map $f : M \to S(M)$, unique up to isomorphism satisfying the universal property

(2.2) *Universal Property*: for any A-algebra B and any A-linear map $M \to B$ there exists a unique factorization $M \xrightarrow{f} S(M) \xrightarrow{\rho} B$ with ρ a morphism of A-algebras.

Remark. That this A-algebra is unique up to isomorphism follows from standard arguments. Namely, given another pair satisfying (2.2), say $(U, g : M \to U)$ we obtain unique factorizations

$$\begin{tikzcd} & M \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S(M) \arrow[rr, shift left] & & U \arrow[ll, shift left] \end{tikzcd}$$

which must compose to the identity. Hence $S(M)$ and U are uniquely isomorphic as A-algebras.

Proof of (2.1) and (2.2). Let $T(M)$ be the A-algebra defined by $T(M) = \bigoplus_{n\geq 0} M^{\otimes n}$ where the tensor product is taken over A. The multiplication on $T(M)$ is defined $(m_1 \otimes \cdots \otimes m_s) * (n_1 \otimes \cdots \otimes n_t) := m_1 \otimes \cdots \otimes m_s \otimes n_1 \cdots \otimes n_t$. It's not necessarily commutative but, it is generated by its degree 1 component.

Quotient by the ideal generated by elements $x \otimes y - y \otimes x$ for any x, y in M. Call this quotient $S(M)$. This should universally impose commutativity relations.

Now suppose we are given an A-linear map $\phi : M \to B$ with B an A-algebra. We start by defining a map $M \to T(M)$ taking M to the degree 1 component of $T(M)$. Using this map we can factor ϕ as $M \to T(M) \xrightarrow{\psi} B$ where in the second arrow we send $m_1 \otimes \cdots \otimes m_s \to \phi(m_1) \cdots \phi(m_s)$. Since B is commutative, the ideal generated by tensors $x \otimes y - y \otimes x$ in $T(M)$ gets sent to zero by ψ, and we obtain a well-defined induced map $\tilde{\psi} : S(M) \to B$. Both the

map f and the factorization of the lemma statement now have potential candidates: since the degree 1 component was unaffected by this quotient we have constructed, by the above, a factorization of ϕ as $M \to S(M) \to B$; we take the leftmost arrow to be $f : M \to S(M)$ sending $m \mapsto m$ in the degree 1 component.

All that's left is to check the unicity of the factorization. But this follows from the fact $S(M)$ is generated by its degree 1 component, isomorphic to M, and the image of this component is determined by requiring the unnamed arrow in

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\phi"'] & & S(M) \arrow[dl] \\ & B & \end{tikzcd}$$

to make the diagram commute. $\square$

The desired functorality we obtain from this construction is summarized by:

(2.3) **Lemma**: Given a map of A-modules, $p : M \to N$ we obtain a uniquely determined map $S(p) : S(M) \to S(N)$. This construction respects composition in the sense for any other A-linear map $q : N \to L$ $S(q \circ p) = S(q) \circ S(p)$, and $S(\mathrm{id}) = \mathrm{id}$. Hence, S is a functor from the category of A-modules to the category of A-algebras.

Proof. Given $p : M \to N$ as above, we may compose with $N \to S(N)$ described as in the proof of (2.1) to obtain a uniquely determined map of A-algebras $S(M) \to S(N)$. That $S(\mathrm{id}) = \mathrm{id}$ follows from unicity of the factorization in (2.2). Functorality comes from the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d] & N \arrow[r, "q"] \arrow[d] & L \arrow[d] \\ S(M) \arrow[r, "S(p)"] & S(N) \arrow[r, "S(q)"] & S(L) \end{tikzcd}$$

combined with the unicity saying $S(q) \circ S(p)$ must now equal $S(q \circ p)$. $\square$

We also get for free properties of commuting with colimits.

(2.4) **Lemma**: For a ring A, an A-module M, and an A-algebra B there are canonical isomorphisms of A-algebras

$$S(M \oplus N) \simeq S(M) \otimes_A S(N)$$

and of B-algebras

$$S(M \otimes_A B) \simeq S(M) \otimes_A B.$$

Proof. In each of the above isomorphisms, the proof proceeds by checking both objects have the universal property of (2.2). To see the first isomorphism, given an A-linear map from $M \oplus N$ to an A-algebra B is the same as giving an A-linear map from M to B and from N to B. Hence, we get a bilinear map from $S(M)$ and $S(N)$ to $S(M \oplus N)$ or, equivalently, an A-linear map from $S(M) \otimes_A S(N)$ to $S(M \oplus N)$. By unicity we get the first isomorphism. The second argument is similar. A B-linear map from $M \otimes_A B$ to a B-algebra E gives also an A-linear map from $M \to M \otimes_A B \to E$. Factoring this map we get $M \to S(M) \to E$. Tensoring this with B gives $M \otimes_A B \to S(M) \otimes_A B \to E \otimes_A B$. Composing with the multiplication map $E \otimes_A B \to E$ defined by $e \otimes b \mapsto eb$ shows we have factored $M \otimes_A B \to E$ through $S(M) \otimes_A B$. Going through everything carefully we see the second isomorphism is also a result of the unicity of (2.1). $\square$

Now suppose we are given a sheaf of rings $\mathcal{O}_S$ on a topological space S (or an arbitrary site, it won't make a difference). Let $\mathcal{M}$ be an $\mathcal{O}_S$-module.

(2.5) **Definition**: The symmetric algebra $\mathrm{Sym}(\mathcal{M})$ associated to an $\mathcal{O}_S$-module $\mathcal{M}$ is defined to be the sheaf associated to the presheaf $\mathcal{S}(\mathcal{M})$:

$$U \mapsto S(\mathcal{M}(U)).$$

It is an $\mathcal{O}_S$-algebra.

(2.6) **Lemma**: The presheaf $\mathcal{S}(\mathcal{M})$ defined above really is a presheaf.

Proof. We should define and check the naturality of restriction maps. This comes down to using the universal property as we did before. For any $V \subset U$ we have the restriction map of $\mathcal{O}_S(U)$-modules $\mathcal{M}(U) \to \mathcal{M}(V)$. Composing on the right by the canonical map $\mathcal{M}(V) \to S(\mathcal{M}(V))$ we are now in the situation of the universal property (2.2), hence, we obtain a map of $\mathcal{O}_S(U)$-algebras $S(\mathcal{M}(U)) \to S(\mathcal{M}(V))$. We take this to be our restriction morphism. Naturality is checked in the same manner as the proof of (2.3). $\square$

(2.7) **Proposition**: Let $\mathcal{O}_S$ be a sheaf of rings, $\mathcal{M}$ a sheaf of $\mathcal{O}_S$-modules. Suppose $g : \mathcal{M} \to \mathcal{B}$ is a $\mathcal{O}_S$-module morphism with $\mathcal{B}$ an $\mathcal{O}_S$-algebra. Then there exists an $\mathcal{O}_S$-module morphism $f : \mathcal{M} \to \mathrm{Sym}(\mathcal{M})$ and a unique factorization of g as $\mathcal{M} \xrightarrow{f} \mathrm{Sym}(\mathcal{M}) \xrightarrow{\rho} \mathcal{B}$ with ρ a morphism of $\mathcal{O}_S$-algebras.

Proof. We define f to be the composition of $\mathcal{M} \to \mathcal{S}(\mathcal{M}) \to \mathrm{Sym}(\mathcal{M})$ with the right arrow being the sheafification morphism.

For any open set U we get a unique factorization $\mathcal{M}(U) \to S(\mathcal{M}(U)) \to \mathcal{B}(U)$ with the right arrow a morphism of $\mathcal{O}_S(U)$-algebras. For this to induce a morphism of presheaves

$\mathcal{S}(\mathcal{M}) \to \mathcal{B}$ we need to check the commutativity of the square

$$\begin{array}{ccc} S(\mathcal{M}(U)) & \longrightarrow & \mathcal{B}(U) \\ \downarrow & & \downarrow \\ S(\mathcal{M}(V)) & \longrightarrow & \mathcal{B}(V) \end{array}$$

for any restriction between open sets $V \subset U$. This can be checked directly on elements since $m \mapsto \rho(U)(m) \mapsto \rho(U)(m)|_V = \rho(V)(m|_V)$ with the last equality a result of the naturality of ρ and this equals $m \mapsto m|_V \mapsto \rho(V)(m|_V)$.

Therefore, we do get a morphism $\mathcal{M} \to \mathcal{S}(\mathcal{M}) \to \mathcal{B}$ with the rightmost arrow a morphism of presheaves of algebras. Since $\mathcal{B}$ is a sheaf this morphism factors through f uniquely as claimed. $\square$

When S is a scheme, there are some properties of $\mathrm{Sym}(\mathcal{M})$ that are inherited from properties of $\mathcal{M}$. This will be useful in our precise formulation of the functors in (P1) and (P3). We specialize to the case S is a scheme below.

(2.8) **Proposition**: If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module then $\mathrm{Sym}(\mathcal{M})$ is quasi-coherent. Similarly, if $\mathcal{M}$ is finite locally free, then $\mathrm{Sym}(\mathcal{M})$ is locally free and locally finite type as an $\mathcal{O}_S$-algebra.

Proof. We first show $\mathcal{M}$ quasi-coherent implies $\mathrm{Sym}(\mathcal{M})$ quasi-coherent. Suppose U is an open subset of S, isomorphic with $\mathrm{Spec}(A)$ and $\mathcal{M}|_U$ is isomorphic to $\widetilde{M}$ for an A-module M. From (2.4) we see, for any $f \in A$, $S(M_f) = S(M \otimes_A A_f) \simeq S(M) \otimes_A A_f$. Hence $\mathcal{S}(\mathcal{M})|_U \simeq \mathcal{S}(\mathcal{M}|_U) \simeq \widetilde{S(M)}$. This rightmost object is already a sheaf. Since the sheafification functor, call it a_{Zar}, commutes with restriction to open sets we find

$$\mathrm{Sym}(\mathcal{M})|_U = a_{Zar}(\mathcal{S}(\mathcal{M})) \simeq a_{Zar}(\mathcal{S}(\mathcal{M})|_U) \simeq a_{Zar}(\widetilde{S(M)}) = \widetilde{S(M)}$$

proving $\mathrm{Sym}(\mathcal{M})$ is quasi-coherent.

Next, suppose $\mathcal{M}$ is finite locally free. Then on a cover $\{U_i\}_i$ we have isomorphisms $\mathcal{M}|_{U_i} \cong \mathcal{O}|_{U_i}^{\oplus n}$. We can take the U_i to be affine by choosing smaller open sets if needed. Then U_i is isomorphic to some affine space, say $\mathrm{Spec}(A_i)$. Similar to before, the presheaf $\mathcal{S}(\mathcal{M})$ then becomes the presheaf associated to $U \mapsto S(A_i^{\oplus n}) \simeq S(A_i)^{\otimes n}$ where the last isomorphism is from (2.4). On choosing a basis of the free A_i-module A_i, we can identify $S(A_i)$ with $A_i[x_1]$. Thus $S(A_i)^{\otimes n} \cong A_i[x_1, ..., x_n]$. By restricting to open sets we conclude

$$\mathrm{Sym}(\mathcal{M})|_U = a_{Zar}(\widetilde{S(A_i)^{\otimes n}}) \simeq \widetilde{S(A_i)^{\otimes n}} \cong A_i[\widetilde{x_1, ..., x_n}].$$

Hence, if $\mathcal{M}$ is finite locally free then $\mathrm{Sym}(\mathcal{M})$ is locally free and locally finite type as an $\mathcal{O}_S$-algebra. $\square$

3. The relative spectrum

The relative spectrum construction is really a generalization of the spectrum of a ring. To explain what I mean, note for any ring A we have a morphism of $\mathbf{Z}$-algebras $\mathbf{Z} \to A$ given by $1 \mapsto 1$. This gives $\mathrm{Spec}(A)$ the structure of a $\mathrm{Spec}(\mathbf{Z})$ scheme in a unique way. The construction we'll see below is meant to generalize this construction of schemes over the base $\mathrm{Spec}(\mathbf{Z})$ to the case of an arbitrary base scheme. This is incredibly useful in the study of a classical vector bundles considered as schemes, as these are always given over a specific base scheme. That being said, suppose we are given a scheme S and we'd like to construct a scheme over S.

(3.1) **Construction**: For any $\mathcal{O}_S$-algebra $\mathcal{A}$ which is quasi-coherent as an $\mathcal{O}_S$-module, there exists a scheme $\mathrm{Spec}_S(\mathcal{A})$ together with a uniquely determined map $\pi : \mathrm{Spec}_S(\mathcal{A}) \to S$. Moreover, the construction $\mathrm{Spec}_S(\mathcal{A})$ is functorial in the following sense:

1. there is a canonical isomorphism $\pi^\# : \mathcal{A} \xrightarrow{\sim} \pi_*(\mathcal{O}_{\mathrm{Spec}_S(\mathcal{A})})$ considered as $\mathcal{O}_S$-algebras
2. $\mathrm{Spec}_S(-)$ is a contravariant fully-faithful functor from the category of $\mathcal{O}_S$-algebras which are quasi-coherent as $\mathcal{O}_S$-modules to the category of schemes over S

Construction by gluing. First, we single out what our scheme will look like locally, over affine open sets $\{U_i\}_i$ of S. Over each of these open sets we have a scheme, namely $\mathrm{Spec}(\mathcal{A}(U_i))$ and our task will be to show these glue together. The structure map to S will come for free, as over each U_i it is defined by the structure map of algebras $\mathcal{O}_S(U_i) \to \mathcal{A}(U_i)$.

To construct the scheme $\mathrm{Spec}_S(\mathcal{A})$ we need to show we have the following data:

i) For any two indices i, j we have an isomorphism

$$\varphi_{ij} : \mathcal{O}_{\mathrm{Spec}(\mathcal{A}(U_i))}|_{U_i \cap U_j} \to \mathcal{O}_{\mathrm{Spec}(\mathcal{A}(U_j))}|_{U_i \cap U_j}$$

with $\varphi_{ij} = \varphi_{ji}^{-1}$.

ii) For any three indices i, j, k, we have a commutative triangle

$$(\varphi_{jk}|_{U_i \cap U_j \cap U_k}) \circ (\varphi_{ij}|_{U_i \cap U_j \cap U_k}) = \varphi_{ik}|_{U_i \cap U_j \cap U_k}.$$

Note, when defining φ_{ij} we can assume $\mathcal{O}_{\mathrm{Spec}(\mathcal{A}(U_i))} \cong \widetilde{\mathcal{A}(U_i)}$, respectively with j, since $\mathcal{A}$ is quasi-coherent. So fix two indices i, j. We can cover $U_i \cap U_j$ by a basis of affine open sets, basic in both U_i, U_j. To define a morphism $\widetilde{\mathcal{A}(U_i)}|_{U_i \cap U_j} \to \widetilde{\mathcal{A}(U_j)}|_{U_i \cap U_j}$ it suffices to do it on this basis. But for basic open affines $D(f_i) \subset U_i$, $D(f_j) \subset U_j$, we have

$$\widetilde{\mathcal{A}(U_i)}|_{D(f_i)} \cong \widetilde{\mathcal{A}(D(f_i))}$$

and

$$\widetilde{\mathcal{A}(U_j)}|_{D(f_j)} \cong \widetilde{\mathcal{A}(D(f_j))}.$$

Since $D(f_j) \cong D(f_i)$, we can define $\varphi_{ij}(D(f_j))$ to be the identity on these sheaves. Thus, we have a collection of morphisms φ_{ij} as in condition i), and we want to see they satisfy condition ii). But, this is immediate from the local description of φ_{ij}. $\square$

Proof of 1. On a basis this is an isomorphism on the level of sections, hence the result. $\square$

Proof of 2. Given two $\mathcal{O}_S$-algebras $\mathcal{A}, \mathcal{B}$ and a morphism of $\mathcal{O}_S$-algebras $f : \mathcal{A} \to \mathcal{B}$ we can define a contravariant morphism $\mathrm{Spec}_S(f) : \mathrm{Spec}_S(\mathcal{B}) \to \mathrm{Spec}_S(\mathcal{A})$ by defining it locally. That is to say, on any small affine open $U \subset S$ we define $\mathrm{Spec}_S(f)|_U := \mathrm{Spec}(f(U)) : \mathrm{Spec}(\mathcal{B}(U)) \to \mathrm{Spec}(\mathcal{A}(U))$ which is a morphism of S-schemes.

Given any two U, V where we have locally defined $\mathrm{Spec}_S(f)$, we can, in a similar spirit to the construction, show these restrict to the same map $(\mathrm{Spec}_S(f)|_U)|_{U\cap V} = (\mathrm{Spec}_S(f)|_V)|_{U\cap V}$. Thus we get a morphism of schemes $\mathrm{Spec}_S(f)$ over S from this data and gluing.

To see it is full, we work in reverse. Given a map $f : \mathrm{Spec}_S(\mathcal{B}) \to \mathrm{Spec}_S(\mathcal{A})$, we obtain, on a basis for S, morphisms $\mathrm{Spec}(\mathcal{B}(U)) \to \mathrm{Spec}(\mathcal{A}(U))$ over S. By the fully-faithfulness of $\mathrm{Spec}(-)$, we get morphisms $\mathcal{A}(U) \to \mathcal{B}(U)$ which respect the $\mathcal{O}_S(U)$-algebra structure. These define an $\mathcal{O}_S$-algebra morphism $\mathcal{A} \to \mathcal{B}$, showing fullness. Faithfulness comes from the following observation: if two morphisms from $\mathcal{A}$ to $\mathcal{B}$ differ, then they differ on some affine open set U of S. Then on U the maps $\mathrm{Spec}_S(\mathcal{B})|_U = \mathrm{Spec}(\mathcal{B}(U)) \to \mathrm{Spec}(\mathcal{A}(U)) = \mathrm{Spec}_S(\mathcal{A})|_U$ are also different, by the faithfulness of $\mathrm{Spec}(-)$. $\square$

Remark. Throughout the above we should pay attention to: we either fix a basis for every scheme S where the above construction holds, or we show the choice of a basis doesn't matter. But note, the latter holds since over any two bases for S, say $\mathcal{B}_1, \mathcal{B}_2$, we can find open sets covering the intersection of opens of $\mathcal{B}_1, \mathcal{B}_2$. On restricting constructions to this finer cover we get the same result.

(3.2) **Corollary**: The functor $\mathrm{Spec}_S(-)$ is an equivalence of categories between $\mathcal{O}_S$-algebras quasi-coherent as $\mathcal{O}_S$-modules and schemes which are affine over S.

Proof. We've shown that $\mathrm{Spec}_S(-)$ is fully faithful. It remains to show the essential image is as stated. Suppose then we have an S-scheme $\pi : X \to S$, which is affine over $\mathcal{S}$.

We'll show $X \cong \mathrm{Spec}_S(\pi_*\mathcal{O}_X)$. To define this isomorphism we can work locally. That is, over an affine open $U \subset S$ we have $X|_U$ is affine and, $(\pi_*\mathcal{O}_X)|_U \cong \pi_*(\mathcal{O}_X|_U)$ by either checking this by hand or by flat base change along the open immersion $U \to S$. This gives us the gluing data needed. $\square$

4. The functors in (P1) and (P3)

(4.1) **Theorem**: There is a fully-faithful contravariant functor $\mathsf{Mod}(\mathcal{O}_S) \to \mathsf{Vec}_S$ given by the composition $\mathrm{Spec}_S(-) \circ \mathrm{Sym}(-)$.

Proof. Recall, $\mathsf{Mod}(\mathcal{O}_S)$ is the category whose objects are quasi-coherent $\mathcal{O}_S$-modules and whose morphisms are morphisms of $\mathcal{O}_S$-modules. The category Vec_S is the category of S-schemes which are isomorphic to a scheme of the form $\mathrm{Spec}_S(-)$. Morphisms in the latter category are morphisms of S-schemes.

We want to show the functor $\mathrm{Spec}_S(-) \circ \mathrm{Sym}(-) = \mathrm{Spec}_S(\mathrm{Sym}(-))$ is fully-faithful. However, the chain of maps

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_S)}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} &\mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_S)}(\mathcal{E}, \mathrm{Sym}(\mathcal{F})) \\ \simeq &\mathrm{Hom}_{\mathsf{Alg}(\mathcal{O}_S)}(\mathrm{Sym}(\mathcal{E}), \mathrm{Sym}(\mathcal{F})) \\ \simeq &\mathrm{Hom}_{\mathsf{Sch}/S}(\mathrm{Spec}_S(\mathrm{Sym}(\mathcal{F})), \mathrm{Spec}_S(\mathrm{Sym}(\mathcal{E})))\end{aligned}$$

given by first composing with $\mathcal{F} \to \mathrm{Sym}(\mathcal{F})$, then applying the universal property of Sym, and then applying the fully-faithfulness of $\mathrm{Spec}_S(-)$ shows that $\mathrm{Spec}_S(-) \circ \mathrm{Sym}(-)$ is fully-faithful. □

(4.2) **Corollary**: the functor of (4.1) induces, on restriction, an equivalence between the full subcategory of Vec_S formed by scheme-theoretic vector bundles X such that there exists a cover of S by open neighborhoods $\{U_i\}_i$ with U_i-isomorphisms $g_i : X|_{U_i} \xrightarrow{\sim} U_i \times \mathbf{A}^n_{U_i}$ for some n and finite locally free $\mathcal{O}_S$-modules.

Remark. Such a cover as $\{U_i\}_i$ is said to trivialize X. Sometimes covers with this property are also called local trivializations.

Proof of (4.2). Any finite locally free sheaf looks finite free, locally. Hence locally, the symmetric algebra looks like a polynomial algebra. From the construction of $\mathrm{Spec}_S(-)$ this means, for any finite locally free $\mathcal{O}_S$-module $\mathcal{A}$, $\mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A}))$ is isomorphic to $\mathbf{A}_{\mathcal{O}_S(U)} := U_i \times_S \mathbf{A}$ as desired. □

(4.3) **Definition**: a *classical vector bundle over* S is an object in the essential image of the restriction of $\mathrm{Spec}_S(-) \circ \mathrm{Sym}(-)$ to the category $\mathsf{Loc}(\mathcal{O}_S)$.

5. The equivalence in (P2)

Let $S = (S, \mathcal{O}_S)$ be a ringed space. Recall, for any $\mathcal{O}_S$-modules $\mathcal{F}, \mathcal{E}$ there exists an internal Hom sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{E})$.

(5.1) **Lemma**: The presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{E})$ defined by $U \mapsto Hom_{\mathsf{Mod}(\mathcal{O}_U)}(\mathcal{F}|_U, \mathcal{E}|_U)$ is actually a sheaf, called the internal Hom of the $\mathcal{O}_S$-modules $\mathcal{F}, \mathcal{E}$. Moreover, it is an $\mathcal{O}_S$-module.

Proof. To check this, we need to verify the sheaf condition, i.e. for any open cover $\{U_i\}_i$ of S we need to check the exactness of the sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{E})(S) \to \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{E})(U_i) \to \prod_{i,j} \mathcal{H}om(\mathcal{F}, \mathcal{E})(U_i \cap U_j)$$

given by $s \mapsto (s|_{U_i})_i$ and $(s_i)_i \mapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i,j}$ for the two nontrivial arrows.

It's injective: Given any element $f \in \mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_S)}(\mathcal{F}, \mathcal{E})$ that restricts on the cover U_i to the 0-morphism also induces the 0-morphism on stalks. Hence f is the 0-morphism since $\mathcal{E}$ is a sheaf.

It's a complex: Take any $f : \mathcal{F} \to \mathcal{E}$ which is a morphism of $\mathcal{O}_S$-modules. We'll show $f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}$ is zero by determining that it's zero on the stalks. But this is true since, as morphisms $(f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j})_x = (f_i|_{U_i \cap U_j})_x - (f_j|_{U_i \cap U_j})_x = f_x - f_x = 0$ for any $x \in S$.

It's exact: Suppose $(s_i)_i$ is some element which maps to 0 in the right most product of modules. These are maps $s_i : \mathcal{F}|_{U_i} \to \mathcal{E}|_{U_i}$ which then satisfy the condition that they agree on the overlaps. This means we can glue them together to a morphism of sheaves $s : \mathcal{F} \to \mathcal{E}$. The explicit definition would be $s(U) : \mathcal{F}(U) \to \mathcal{E}(U)$ $t \mapsto \tilde{t}$ where $\tilde{t}$ is the section of $\mathcal{E}(U)$ that restricts on U_i to $s_i(t|_{U_i})$.

For the moreover part, we define $f + g$ to be the morphism $(f + g)(U)(s) := f(U)(s) + g(U)(s)$ and $a \cdot f$ to be the morphism $(a \cdot f)(U)(s) = a(u)(s) \cdot f(u)(s)$. Since $\mathcal{E}$ is an $\mathcal{O}_S$-module, we don't have to check any of these conditions because they are determined by the ones already apparent in $\mathcal{E}$. □

(5.2) **Proposition**: $\mathcal{H}om(-, \mathcal{O}_S)$ is a contravariant functor from $\mathsf{Mod}(\mathcal{O}_S)$ to itself.

Proof. Given a morphism $f : \mathcal{F} \to \mathcal{E}$ we may functorially construct a morphism $\mathcal{H}om(f, \mathcal{O}_S)$. That is, for any open set U we define

$$\mathcal{H}om(f, \mathcal{O}_S)(U) : \mathcal{H}om(\mathcal{E}, \mathcal{O}_S)(U) \to \mathcal{H}om(\mathcal{F}, \mathcal{O}_S)(U)$$

by

$$(g : \mathcal{E}|_U \to \mathcal{O}_U) \mapsto (g \circ f|_U : \mathcal{F}|_U \to \mathcal{O}_U).$$

Since f is a morphism of sheaves, this definition commutes with the restriction of open sets $V \subset U$ and so defines a morphism of sheaves. Functorality follows from the definition of the morphism. □

It's worth noting, and we will use in Theorem (5.6), the following properties of the restriction of the internal Hom sheaf.

(5.3) **Lemma**: Assume $\mathcal{F}$ is a finitely presented $\mathcal{O}_S$-module, and $\mathcal{E}$ is an arbitrary $\mathcal{O}_S$-module. Then, there is a canonical isomorphism

$$\mathcal{H}om(\mathcal{F}, \mathcal{E})_x \simeq \mathrm{Hom}_{\mathcal{O}_{S,x}}(\mathcal{F}_x, \mathcal{E}_x).$$

Reference. [Gro60, Chapter 0, (5.2.6)], [Sta17, Tag 01CM].

(5.4) **Lemma**: Suppose $f : X \to S$ is a flat map of ringed spaces. Suppose further $\mathcal{F}$ is a finitely presented $\mathcal{O}_S$-module. Then there is a canonical isomorphism

$$f^*\mathcal{H}om(\mathcal{F}, \mathcal{E}) \simeq \mathcal{H}om(f^*\mathcal{F}, f^*\mathcal{E}).$$

Reference. [Gro60, Chapter 0, (6.7.6.1)], [Sta17, Tag 01CM]. □

(5.5) **Lemma**: If $\mathcal{F}$ is a finite locally free $\mathcal{O}_S$-module, then so is $\mathcal{H}om(\mathcal{F}, \mathcal{O}_S)$.

Proof. Suppose we are given a collection of open sets $\{U_i\}_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. Then on the same U_i we have the isomorphisms

$$\begin{aligned}\mathcal{H}om(\mathcal{F}, \mathcal{O}_S)(U_i) \cong& \mathrm{Hom}(\mathcal{F}|_{U_i}, \mathcal{O}_{U_i}) \\ \cong& \mathrm{Hom}(\mathcal{O}_{U_i}^{\oplus n}, \mathcal{O}_{U_i}) \\ \cong& \bigoplus_{j=0}^{n} \mathrm{Hom}(\mathcal{O}_{U_i}, \mathcal{O}_{U_i}) \\ \cong& \bigoplus_{j=0}^{n} \mathcal{O}_{U_i}(U_i)\end{aligned}$$

where the first is definition, the second is by assumption, the third is since $\mathrm{Hom}(-, \mathcal{O}_{U_i})$ commutes with finite direct sums, and the last since for all $\mathcal{O}_{U_i}$-modules $\mathcal{F}$ we have a canonical isomorphism $\mathrm{Hom}(\mathcal{O}_{U_i}, \mathcal{F}) = \mathcal{F}(U_i)$ determined by where 1 gets mapped to. □

With all of the machinery above in place, we are finally ready to prove the equivalence in (P2). We have to specialize to the case S is a scheme for this proof. This is because we will use that a finite locally free $\mathcal{O}_S$-module, where S is a scheme, is finitely presented [Sta17, Tag 05P1]. This allows us to use lemma (5.4) in the case f is an open immersion and $\mathcal{F}$ finite locally free.

(5.6) **Theorem**: For any locally free sheaf of finite rank on S there is a canonical isomorphism $\mathcal{F} \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_S), \mathcal{O}_S)$. Consequently, $\mathcal{H}om(-, \mathcal{O}_S)$ is a fully-faithful functor from $\mathsf{Loc}(\mathcal{O}_S)$ to itself.

Proof. There is always a canonical map $\mathcal{F} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_S), \mathcal{O}_S)$ whose map of sections on an open set U

$$\begin{aligned}\mathcal{F}(U) \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_S), \mathcal{O}_S)(U) =& \mathrm{Hom}(\mathcal{H}om(\mathcal{F}, \mathcal{O}_S)|_U, \mathcal{O}_U) \\ \simeq& \mathrm{Hom}(\mathcal{H}om(\mathcal{F}|_U, \mathcal{O}_U), \mathcal{O}_U)\end{aligned}$$

is given by $f \mapsto (g \mapsto g(U)(f))$ for any $g : \mathcal{F}|_U \to \mathcal{O}_U$.

To see this is an isomorphism we need to check that it is an isomorphism on stalks. But on stalks we get a morphism, using (5.3) twice,

$$\begin{aligned}\mathcal{F}_x \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_S), \mathcal{O}_S)_x \simeq& \mathrm{Hom}_{\mathcal{O}_{S,x}}(\mathcal{H}om(\mathcal{F}, \mathcal{O}_S)_x, \mathcal{O}_{S,x}) \\ \simeq& \mathrm{Hom}_{\mathcal{O}_{S,x}}(\mathrm{Hom}_{\mathcal{O}_{S,x}}(\mathcal{F}_x, \mathcal{O}_{S,x}), \mathcal{O}_{S,x})\end{aligned}$$

which reduces the problem to the situation of modules. The first of the two statements in the theorem then follows, since the induced morphism on stalks canonically induces an isomorphism between free $\mathcal{O}_{S,x}$-modules.

For the second of the two statements in the theorem, we proceed by applying $\mathcal{H}om(-, \mathcal{O}_S)$ twice to get maps

$$\mathrm{Hom}(\mathcal{F}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{H}om(\mathcal{E}, \mathcal{O}_S), \mathcal{H}om(\mathcal{F}, \mathcal{O}_S)) \to \mathrm{Hom}(\mathcal{F}, \mathcal{E})$$

composing to the identity. The leftmost map is then injective (showing faithfulness) and the rightmost map is surjective. The same argument applies to the maps

$$\mathrm{Hom}(\mathcal{H}om(\mathcal{E}, \mathcal{O}_S), \mathcal{H}om(\mathcal{F}, \mathcal{O}_S)) \to \mathrm{Hom}(\mathcal{F}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{H}om(\mathcal{E}, \mathcal{O}_S), \mathcal{H}om(\mathcal{F}, \mathcal{O}_S))$$

and so the rightmost map is again surjective (showing fullness). □

Remark. The isomorphism of the previous lemma really depends on the canonical isomorphism of modules induced on the stalks (which is not fully proved above). To see why this is an absolute necessity: for every point x of X we could define an isomorphism $\mathcal{F}_x \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_S)_x$ by choosing a basis of the left and right hand side respectively. However, this is not induced from a morphism of sheaves $\mathcal{F} \to \mathcal{H}om(\mathcal{F}, \mathcal{O}_S)$ and so does not define an isomorphism of sheaves $\mathcal{F} \simeq \mathcal{H}om(\mathcal{F}, \mathcal{O}_S)$.

6. The sheaf of sections

Throughout the above, we described a fully faithful functor $\mathrm{Spec}_S(\mathrm{Sym}(-))$ from $\mathsf{Mod}(\mathcal{O}_S)$, the category of quasi-coherent $\mathcal{O}_S$-modules, to Vec_S, the category of scheme-theoretic vector bundles over S. We also described an autoequivalence of the category $\mathsf{Loc}(\mathcal{O}_S)$ consisting of locally free finite rank $\mathcal{O}_S$-modules. In this section, we describe a functor which goes in the opposite direction of the above constructions. That is, we describe a functor $\mathsf{Vec}_S \to \mathsf{Mod}(\mathcal{O}_S)$.

The functor we construct below is not an inverse to the functor $\mathrm{Spec}_S(\mathrm{Sym}(-))$. However, when we restrict to the category of classical vector bundles the functor we construct will be an equivalence. This has been a source of confusion, as the equivalence described by the functor below is *not* an inverse to the functor $\mathrm{Spec}_S(\mathrm{Sym}(-))$ restricted to $\mathsf{Loc}(\mathcal{O}_S)$. If one really wants, it will be clear how to remedy this after we prove some basic properties of the functor.

(6.1) **Definition**: Let X be a scheme-theoretic vector bundle over S with structural map π. We define the *sheaf of sections associated to* X via the following assignment

$$U \rightsquigarrow \{U \xrightarrow{\sigma} X | \pi \circ \sigma = \mathrm{id}_U\}$$

for every open set $U \subset S$ and

$$(V \hookrightarrow U) \rightsquigarrow (\sigma \mapsto \sigma|_V)$$

for any inclusion of open sets $V \subset U$. For any scheme-theoretic vector bundle X over S we will write $\mathcal{L}_S(X)$ for the associated sheaf of sections.

(6.2) **Lemma**: The sheaf of sections of a scheme-theoretic vector bundle X over S is a sheaf of sets. If X is isomorphic as S-schemes to $\mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A}))$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{A}$, then $\mathcal{L}_S(X)$ is a sheaf of $\mathcal{O}_S$-modules.

Proof. First, we consider the functor underlying $\mathcal{L}_S(X)$ but considering it only on a basis of open affines of S (i.e. considering the restriction to a basis). This is required to show $\mathcal{L}_S(X)$ is a $\mathcal{O}_S$-module when $X \cong \mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A}))$. Next, we show the assignment given in (6.1) defines a sheaf of sets. Since there is at most one sheaf associated to its values on a basis for S, this will show both claims of the lemma.

$\mathcal{L}_S(X)$ defines a presheaf, since restriction of functions is functorial. For any scheme-theoretic vector bundle it is at least a sheaf of sets. To be precise, given an open cover $\{U_i\}_i$ of an open subscheme U of S and a collection of sections $\sigma_i : U_i \to X$ which agree on the overlaps $U_i \cap U_j$, then we can glue these together to get a section $\sigma : U \to X$ by defining $\sigma(U_i) := \sigma_i(U_i)$. This shows the gluing condition. To show the separated condition, suppose we are given two sections $s, t : U \to X$ that agree locally everywhere on the collection U_i. Then they are the same section, since morphisms of schemes are determined locally.

Now suppose $X \cong \mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A}))$ with $\mathcal{A}$ a quasi-coherent $\mathcal{O}_S$-module. We want to show $\mathcal{L}_S(X)$ is a sheaf of modules so we consider the underlying functor restricted to a basis of S consisting of open affines. For any open affine U of S, the restriction $X|_U$ is affine since X is a scheme-theoretic vector bundle.

To give things names say $U \cong \mathrm{Spec}(A)$ and $X|_U \cong \mathrm{Spec}(S(B))$ with B an A-module. Then to give a morphism $\sigma : U \to X|_U$ is to give a ring map $S(B) \to A$. Since composing σ with the structure map π is required to be the identity on U, we must also have the composition $A \to S(B) \to A$ obtained by precomposing with the A-algebra structure map is the identity.

Suppose σ_1, σ_2 are two ring maps corresponding to sections $\mathcal{L}_S(X)(U)$ over U, and let α be an element of $\mathcal{O}_S(U)$. Then we define $\alpha\sigma_1 + \sigma_2 : S(B) \to A$ to be the section corresponding to the ring map defined by

$$S(B) \to S(B) \otimes_A S(B) \to S(B) \otimes_A S(B) \otimes_A A[t] \to A$$

$$x \mapsto x \otimes 1 + 1 \otimes x, \;\; m \mapsto m \otimes 1, \;\; s \mapsto \sigma_1 \otimes \sigma_2 \otimes (t \mapsto \alpha)(s)$$

To be fair, I should check the conditions necessary to show this is a module. I'm not quite sure if these maps are the ones I want, but the rest is elaborated on below. $\square$

Remark. The above module structure should seem totally mysterious to the first time viewer (or the seasoned viewer if I've defined them incorrectly). To see what's really going on we should apply $\mathrm{Spec}(-)$. Then, the ring map defined for $\alpha\sigma_1 + \sigma_2$ above is the composite of the maps

$$\mathrm{Spec}(S(B)) \times_{\mathrm{Spec}(A)} \mathbf{A}^1_A \to \mathrm{Spec}(S(B))$$

and

$$\mathrm{Spec}(S(B)) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(S(B)) \to \mathrm{Spec}(S(B)).$$

The first map gives $\mathrm{Spec}(S(B))$ the structure of a multiplication by scalars from A, and the second gives $\mathrm{Spec}(S(B))$ the structure of a commutative group.

(6.3) *Example.* Suppose $\mathcal{A}$ is a locally free $\mathcal{O}_S$-module of finite rank n for some scheme S. Locally about an open affine $\mathrm{Spec}(A)$ of S the maps given above are isomorphic with, after choosing a basis for the symmetric algebra, maps

$$\mathbf{A}^n_A \times_{\mathrm{Spec}(A)} \mathbf{A}^1_A \to \mathbf{A}^n_A$$

and

$$\mathbf{A}^n_A \times_{\mathrm{Spec}(A)} \mathbf{A}^n_A \to \mathbf{A}^n_A.$$

On rational points the first map can be described

$$((t_1 - a_1, ..., t_n - a_n), (t - \alpha)) \mapsto (t_1 - \alpha a_1, ..., t_n - \alpha a_n)$$

and the second map can be described

$$((t_1 - a_1, ..., t_1 - a_n), (t_1 - b_1, ..., t_n - b_n)) \mapsto (t_1 - a_1 - b_1, ..., t_1 - a_n - b_n).$$

(6.4) **Corollary**: If $\mathcal{A}$ is a locally free $\mathcal{O}_S$-module of finite rank n and $X = \mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A}))$, then $\mathcal{L}_S(X)$ is locally free of finite rank n.

Proof. Given an open affine $U = \mathrm{Spec}(A)$ of S, the set of sections $\mathrm{Spec}(A) \to \mathrm{Spec}(S(\mathcal{A}(U)))$ is in bijection with the A-algebra morphisms of $\mathrm{Hom}_{A-alg}(S(\mathcal{A}(U)), A)$. This is finite locally free as any such map is determined by the images of a basis for $S(\mathcal{A}(U))$ and these are allowed to be arbitrary.

(6.5) **Corollary**: Given a locally free $\mathcal{O}_S$-module $\mathcal{A}$ of finite rank n, there is a canonical isomorphism $\mathcal{L}_S(\mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A})) \simeq \mathcal{H}om(\mathcal{A}, \mathcal{O}_S)$.

Proof. For any small open affine subset $U \cong \mathrm{Spec}(A)$ of S we have isomorphisms

$$\begin{aligned}\mathcal{H}om(\mathcal{A}, \mathcal{O}_S)(U) =& \mathrm{Hom}(\mathcal{A}|_U, \mathcal{O}_U)\\ \simeq& \mathrm{Hom}_{A-mod}(\mathcal{A}(U), A)\\ \simeq& \mathrm{Hom}_{A-alg}(S(\mathcal{A}(U), A))\\ \simeq& \mathcal{L}_S(\mathrm{Spec}_S(\mathrm{Sym}(\mathcal{A}))(U)\end{aligned}$$

with the last isomorphism apparent from the proof of (6.4) $\square$

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