

Arrow's Theorem, May's Axioms, and Borda's Rule

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Abstract

We argue that Arrow's (1951) independence of irrelevant alternatives condition (IIA) is unjustifiably stringent. Although, in elections, it has the desirable effect of ruling out spoilers and vote-splitting (Candidate A spoils the election for B if B beats C when all voters rank A low, but C beats B when some voters rank A high - - because A splits off support from B), it is stronger than necessary for this purpose. Worse, it makes a voting rule insensitive to voters' preference intensities. Accordingly, we propose a modified version of IIA to address these problems. Rather than obtaining an impossibility result, we show that a voting rule satisfies modified IIA, Arrow's other conditions, May's (1952) axioms for majority rule, and a mild consistency condition if and only if it is the Borda count (Borda 1781), i.e., rank-order voting.

1. Arrow, May, and Borda

A. Arrow's IIA Condition

In his monograph *Social Choice and Individual Values* (Arrow 1951), Kenneth Arrow introduced the concept of a *social welfare function* (SWF) – a mapping from profiles of individuals' preferences to social preferences.¹ The centerpiece of his analysis was the celebrated

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¹ Formal definitions are provided in section 2.

Impossibility Theorem, which establishes that, with three or more social alternatives, there exists no SWF satisfying four attractive conditions: *unrestricted domain* (U), the *Pareto Principle* (P), *non-dictatorship* (ND), and *independence of irrelevant alternatives* (IIA).

Condition U requires merely that a social welfare function be defined for all possible profiles of individual preferences (since ruling out preferences in advance could be difficult). P is the reasonable requirement that if all individuals (strictly) prefer alternative x to y , then x should be (strictly) preferred to y socially as well. ND is the weak assumption that there should not exist a single individual (a “dictator”) whose strict preference always determines social preference.

These first three conditions are so undemanding that virtually any SWF studied in theory or used in practice satisfies them all. For example, consider *plurality rule* (or “first-past-the-post”), in which x is preferred to y socially if the number of individuals ranking x first is bigger than the number ranking y first.² Plurality rule satisfies U because it is well-defined regardless of individuals’ preferences. It satisfies P because if all individuals strictly prefer x to y , then x must be ranked first by more individuals than y .³ Finally, it satisfies ND because if everyone else ranks x first, then even if the last individual strictly prefers y to x , y will not be ranked above x socially.

Alternatively, consider *instant-runoff voting* (called ranked-choice voting in the United States, preferential voting in Australia and the United Kingdom, and Hare’s rule or single-transferable vote in some of the voting literature), in which x is preferred to y socially if x is dropped after y in the candidate-elimination process (the candidate dropped first is the one who

² As used in elections, plurality rule is, strictly speaking, a *voting rule*, not a SWF: it merely determines the *winner* (the candidate who is ranked first by a plurality of voters). By contrast, a SWF requires that *all* candidates be ranked socially (Arrow 1951 sees this as a contingency plan: if the top choice turns out not to be feasible, society can move to the second choice, etc.). See Section 5 for further discussion of voting rules.

³ This isn’t quite accurate, because it is possible that x is *never* ranked first. But we will ignore this small qualification.

is ranked first by the fewest voters; her supporters' second choices are then elevated into first place; and the process iterates). It is easy to check that it too satisfies the three conditions.

By contrast, IIA – which requires that social preferences between x and y should depend only on individuals' preferences between x and y , and not on preferences concerning some third alternative – is satisfied by very few SWFs.⁴ Even so, it has a compelling justification: to prevent *spoilers* and *vote-splitting* in elections.⁵

To understand the issue, consider Scenario 1 (modified from Maskin and Sen 2016). There are three candidates – Donald Trump, Marco Rubio, and John Kasich (the example is inspired by the 2016 Republican primary elections) – and three groups of voters. One group (40%) ranks Trump above Kasich above Rubio; the second (25%) places Rubio over Kasich over Trump; and the third (35%) ranks Kasich above Trump above Rubio (see Figure A).

<u>40%</u>	<u>25%</u>	<u>35%</u>
Trump	Rubio	Kasich
Kasich	Kasich	Trump
Rubio	Trump	Rubio

Figure A: Scenario 1

⁴ One SWF that does satisfy IIA is *majority rule* (also called Condorcet voting), in which alternative x is socially preferred to y if a majority of individuals prefer x to y . However, unless individuals' preferences are restricted, social preferences with majority rule may cycle (i.e., x may be preferred to y , y preferred to z , and yet z preferred to x), as Condorcet (1785) discovered (see formula (1) below). In that case, majority rule is not actually a SWF (since its social preferences are intransitive). That is, majority rule violates U.

⁵ Eliminating spoilers and vote-splitting has frequently been cited in the voting literature as a rationale for IIA. See, for example the Wikipedia article on vote-splitting https://en.wikipedia.org/wiki/Vote_splitting, especially the section on “Mathematical definitions.”

Many Republican primaries in 2016 used plurality rule; so the winner was the candidate ranked first by more voters than anyone else.⁶ As applied to Scenario 1, Trump is the winner with 40% of the first-place rankings. But, in fact, a large majority of voters (60%, i.e., the second and third groups) prefer Kasich to Trump. The only reason why Trump wins in Scenario 1 is that Rubio *spoils* the election for Kasich by splitting off some of his support;⁷ Rubio and Kasich split the first-place votes that don't go to Trump.

An SWF that satisfies IIA avoids spoilers and vote-splitting. To see this, consider Scenario 2, which is the same as Scenario 1 except that voters in the middle group now prefer Kasich to Trump to Rubio (see Figure B).

<u>40%</u>	<u>25%</u>	<u>35%</u>
Trump	Kasich	Kasich
Kasich	Trump	Trump
Rubio	Rubio	Rubio

Figure B: Scenario 2

Pretty much any non-pathological SWF will lead to Kasich being ranked above Trump in Scenario 2 (Kasich is not only top-ranked by 60% of voters, but is ranked second by 40%; by contrast, Trump reverses these numbers: he is ranked first by 40% and second by 60%).

⁶ In actual plurality rule elections, citizens simply vote for a single candidate rather than rank candidates. But this leads to the same winner as long as citizens vote for their most preferred candidate.

⁷ In common parlance (arising from plurality rule and runoff elections), candidate A spoils the election for B if (i) B wins when A doesn't run, and (ii) C wins when A does run (because some citizens vote for A, and these votes would otherwise have gone to B). In Arrow's (1951) framework (which we adopt here), however, there is a *fixed* set of candidates, and so we interpret a "candidate who doesn't run" as one ranked at the bottom by all voters (since a candidate ranked at the bottom has zero effect on what happens to other candidates – just like a candidate who doesn't run). Similarly, we interpret "some citizens voting for A" as their ranking A first (i.e., above B and C). Thus, formally, A is a spoiler for B if B beats C when all voters rank A at the bottom, but C beats B when some voters switch to ranking A at the top (with no other changes to the preference profile).

However, if the SWF satisfies IIA, it must also rank Kasich over Trump in Scenario 1, since each of the three groups has the same preferences between the two candidates in both scenarios.

Hence, unlike plurality rule, a SWF satisfying IIA circumvents spoilers and vote-splitting:

Kasich will win in Scenario 1.

But imposing IIA is too demanding: It is stronger than necessary to prevent spoilers (as we will see), and makes sensitivity to preference intensities impossible.⁸ To understand this latter point, consider Scenario 3, in which there are three candidates x , y , and z and two groups of voters, one (45% of the electorate) who prefer x to z to y ; and the other (55%), who prefer y to x to z (see Figure C).

<u>45%</u>	<u>55%</u>	Under the Borda count
x	y	x gets $3 \times 45 + 2 \times 55 = 245$ points
z	x	y gets $3 \times 55 + 1 \times 45 = 210$ points
y	z	z gets $2 \times 45 + 1 \times 55 = 145$ points
		so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Figure C: Scenario 3

For this scenario, let's apply the *Borda count* (rank-order voting), in which, if there are m candidates, a candidate gets m points for every voter who ranks her first, $m - 1$ points for a

⁸ Arrow (1951) assumes that a SWF is a function only of individuals' *ordinal* preferences, which means that preference intensities cannot directly be expressed in his framework. However, this does not rule out the possibility of *inferring* intensities from ordinal data, as we argue in footnote 9. And even if one takes the view that preference intensities have no place in political elections, they are central to welfare economics, for which case we can relabel "candidates" as "social alternatives".

second-place ranking, and so on. Candidates are then ranked according to their vote totals. The calculations in Figure C show that in Scenario 3, x is socially preferred to y and y is socially preferred to z . But now consider Scenario 4, where the first group's preferences are replaced by x over y over z (see Figure D).

<u>45%</u>	<u>55%</u>	Under the Borda count, the
x	y	
y	x	
z	z	social ranking is now $\begin{matrix} y \\ x, \\ z \end{matrix}$ a
		violation of IIA as applied to x and y

Figure D: Scenario 4

As calculated in Figure D, the Borda social ranking becomes y over x over z . This violates IIA: in going from Scenario 3 to 4, no individual's ranking of x and y changes, yet the social ranking switches from x above y to y above x .

However, the anti-spoiler/anti-vote-splitting rationale for IIA doesn't apply to Scenarios 3 and 4. Notice that candidate z doesn't split first-place votes with y in Scenario 3; indeed, she is *never* ranked first. Moreover, her position in group 1 voters' preferences in Scenarios 3 and 4 provides potentially useful information about the intensity of those voters' preferences between x and y . In Scenario 3, z lies between x and y – suggesting that the preference gap between x and y may be substantial. In the second case, z lies below both x and y , implying that the difference between x and y is not as big. Thus, although z may not be a strong candidate herself (i.e., she is,

in some sense, an “irrelevant alternative”), how individuals rank her vis à vis x and y is arguably pertinent to social preferences^{9, 10} i.e., IIA should not apply to these scenarios.

Accordingly, we propose a relaxation of IIA.¹¹ Under *modified independence of irrelevant alternatives* (MIIA), if given two alternatives x and y and two profiles of individuals’ preferences, (i) each individual ranks x and y the same way in the first profile as in the second, and (ii) each individual ranks the same number of alternatives *between* x and y in the first profile as in the second, then the social ranking of x and y must be the same for both profiles.

If we imposed only requirement (i), then MIIA would be identical to IIA. Requirement (ii) is the one that permits preference intensities to figure in social rankings. Specifically, notice

⁹ We do not claim that preference intensities can always be inferred from ordinal data, but here is one setting in which the argument goes through precisely: Imagine that, from the perspective of an outside spectator (society), each of a voter’s utilities $u(x)$, $u(y)$, and $u(z)$ (where u captures preference intensity) is drawn randomly and independently from some distribution (the spectator might learn this distribution from survey data). For reasons given in footnote 10, however, the spectator cannot directly observe the utilities in any particular aggregation problem; she can observe only the voter’s ranking of alternatives. Nevertheless, the expected difference $u(x) - u(y)$ conditional on z being between x and y in the voter’s preference ordering is greater than the difference conditional on z *not* being between x and y . Thus, the spectator can infer cardinal information (expected intensity differences) from the ordinal ranking.

¹⁰ One might wonder why, instead of depending only on individuals’ ordinal rankings, a SWF is not allowed to depend directly on their *cardinal* utilities, as in Benthamite utilitarianism (Bentham, 1789) or majority judgment (Balinski and Laraki, 2010). But it is not at all clear how to ascertain these utilities, even leaving aside the question of deliberate misrepresentation by individuals. Indeed, for that reason, Lionel Robbins (1932) rejected the idea of cardinal utility altogether, and Arrow (1951) followed in that tradition. Notice that in the case of ordinal preferences, there is an experiment we can perform to verify an individual’s asserted ranking: if he says he prefers x to y , we can offer him the choice and see which he selects. But there is no known corresponding experiment for verifying cardinal utility - except in the case of risk preferences, where we can offer lotteries (in the von Neumann-Morgenstern 1944 procedure for constructing a utility function, utilities are cardinal in the sense that they can be interpreted as probabilities in a lottery). Yet, risk preferences are not the same thing as preference intensities. And introducing risk preferences in social choice situations in which the outcomes entail no uncertainty (e.g., in an election, an outcome is simply the winning candidate, not a lottery) seems of dubious moral relevance (for that reason, Harsanyi’s 1955 derivation of utilitarianism based on risk preferences is often criticized). Finally, even if there were an experiment for eliciting utilities, misrepresentation might interfere with it. Admittedly, there are circumstances when individuals have the incentive to misrepresent their rankings with the Borda count. But a cardinal SWF is subject to much greater misrepresentation because individuals have the incentive to distort even when there are only two alternatives (see Dasgupta and Maskin 2020). Thus, we are left only with the possibility of inferring preference intensities from *ordinal* preferences, as in footnote 9.

¹¹ Other authors who have considered variants of IIA include Brandl and Brandt (2020), Dhillon and Mertens (1999), Eden (2020), Fleurbaey, Hansson (1973), Mayston (1974), Suzumura, and Tadenuma (2005) and (2005a), Osborne (1976), Roberts (2009), Saari (1998), Young (1988), and Young and Levenglick (1978).

that, since z lies between x and y in group 1's preferences in Scenario 3 but not in Scenario 4, MIIA does *not* require the social rankings of x and y to be the same in the two scenarios. That is, accounting for preference intensities is permissible under MIIA.

Even so, MIIA is strong enough to rule out spoilers and vote-splitting (i.e., a SWF satisfying MIIA cannot exhibit the phenomenon of footnote 7). In particular, it rules out plurality rule: in neither Scenario 1 nor Scenario 2 do group 2 voters rank Rubio between Kasich and Trump. Therefore, MIIA implies that the social ranking of Kasich and Trump must be the *same* in the two scenarios, contradicting plurality rule.

Runoff voting is also ruled out by MIIA. Under that voting rule, a candidate wins immediately if he is ranked first by a majority of voters.¹² But failing that, the two top vote-getters go to a runoff. Notice, that if we change Scenario 1 so that the middle group constitutes 35% of the electorate and the third group constitutes 25%, then Trump (with 40% of the votes) and Rubio (with 35%) go to the runoff (and Kasich, with only 25%, is left out). Trump then wins in the runoff, because a majority of voters prefer him to Rubio. If we change Scenario 2 correspondingly (so that the 25% and 35% groups are interchanged), then Kasich wins in the first round with an outright majority. Thus, runoff voting violates MIIA (and so does instant runoff voting) for essentially the same reason that plurality rule does.

B. May's Axioms for Majority Rule

When there are just two alternatives, majority rule is far and away the most widely used democratic method for choosing between them. Indeed, almost all other commonly used voting

¹² Like plurality rule, runoff voting in *practice* is usually administered so that a voter just picks one candidate rather than ranking them all (see footnote 6).

rules – e.g., plurality rule, runoff voting, instant runoff voting, and the Borda count – reduce to majority rule in this case.

May (1952) crystallized why majority rule is so compelling in the two-alternative case by showing that it is the only voting rule satisfying *anonymity* (A), *neutrality* (N), and *positive responsiveness* (PR). Axiom A is the requirement that all individuals be treated equally, i.e., that if they exchange preferences with one another (so that individual j gets i 's preferences, individual k get j 's, and so on), social preferences remain the same. N demands that all alternatives be treated equally, i.e., that if the alternatives are permuted and individuals' preferences are changed accordingly, then social preferences are changed in the same way. And PR requires that if alternative x rises relative to y in some individual's preference ordering, then (i) x doesn't fall relative to y in the social ordering, and (ii) if x and y were previously tied socially, x is now strictly above y .¹³

C. Ranking Consistency

Young (1974) provided a well-known characterization of the Borda count in which the central axiom is a *consistency*¹⁴ condition: if the top social alternative for each of several different populations is x , then x must be the top social alternative for the union of those populations.

This is a strong condition. Indeed, Young (1975) shows that, together with anonymity and neutrality, it implies that the SWF must be a scoring rule: there are m numbers a_1, \dots, a_m such that each time an alternative is ranked first it gets a_1 points, each time it is ranked second a_2 points, etc. Alternatives are then ranked socially according to their point totals. The set of all

¹³ May (1952) expressed the A, N, and PR axioms only for the case of two alternatives. In section 2 we give formal extensions for three or more alternatives (See also Dasgupta and Maskin 2020).

¹⁴ Moulin (1988) calls this axiom "reinforcement."

scoring rules includes both the Borda count and plurality rule (for which

$$a_1 > 0 \text{ and } a_2 = \dots = a_m = 0).$$

We shall invoke a far weaker axiom called *ranking consistency* (RC), which requires only that if the *entire social ranking* is strict and the same for each of several disjoint populations then its (unique) top-ranked alternative must be socially top-ranked for the union. In fact, RC is so mild that (as far as we can tell) it is satisfied by every standard voting method used in practice and nearly every one studied in the literature (see Section 2).

D. Borda's Rule and Condorcet Cycles: A Special Case

The main result of this paper establishes that a SWF satisfies U, MIIA, A, N, PR, and RC (the other Arrow conditions – P and ND – are redundant) if and only if it is the Borda count.^{15 16} Checking that the Borda count satisfies the six axioms is straightforward.¹⁷

To illustrate a central idea of the proof in the other direction, let us focus on the case of three alternatives x , y , and z and suppose that F is a SWF satisfying the six axioms. We will show that

when F is restricted to the domain of preferences $\left\{ \begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix} \right\}$ (i.e., when we consider only

profiles with preferences drawn from this domain¹⁸), it must coincide with the Borda count.

¹⁵ Saari (2000) and (2000a) provide a vigorous defense of the Borda count based on its geometric properties.

¹⁶ The characterization shows that we can understand the Borda count from an essentially Arrovian perspective, with just a trace of Young's consistency. An earlier version of this paper did not mention the RC condition explicitly, but, on close inspection, the proof implicitly used it. And the RC condition turns out to be necessary; Gendler (2022) has constructed an ingenious example in which a non-Borda SWF satisfies all the axioms other than RC.

¹⁷ To see that the Borda count satisfies MIIA, note that if two profiles satisfy the hypotheses of the condition, then the difference between the number of points a given voter contributes to x and the number she contributes to y must be the *same* for the two profiles (because the number of alternatives ranked between x and y is the same). Thus, the differences between the total Borda scores of x and y – and hence their social rankings – are the same. To see that the Borda count satisfies RC, imagine that the Borda ranking for x and y is the same for each of several disjoint subpopulations. Because the Borda scores for the union of the subpopulations are just the sums of those for the individual subpopulations, the Borda ranking of x and y for the union population must coincide with that for the subpopulations.

¹⁸ From U, F is defined for every such profile.

Consider, first, the profile in which 1/3 of individuals have ranking $\begin{smallmatrix} x \\ z \end{smallmatrix}$; 1/3 have ranking $\begin{smallmatrix} y \\ x \end{smallmatrix}$;

and 1/3 have ranking $\begin{smallmatrix} z \\ y \end{smallmatrix}$.¹⁹ We claim that the social ranking of x and y that F assigns to this profile

is *social indifference*:

$$(1) \quad \begin{array}{ccc} \frac{1/3}{x} & \frac{1/3}{z} & \frac{1/3}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} x \sim y$$

If (1) doesn't hold, then either

$$(2) \quad \begin{array}{ccc} \frac{1/3}{x} & \frac{1/3}{z} & \frac{1/3}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}$$

or

$$(3) \quad \begin{array}{ccc} \frac{1/3}{x} & \frac{1/3}{z} & \frac{1/3}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}$$

If (2) holds, then apply permutation σ – with $\sigma(x) = y$, $\sigma(y) = z$, and $\sigma(z) = x$ – to (2). From

N, we obtain

$$(4) \quad \begin{array}{ccc} \frac{1/3}{y} & \frac{1/3}{x} & \frac{1/3}{z} \\ z & y & x \\ x & z & y \end{array} \xrightarrow{F} \begin{array}{c} y \\ z \end{array}$$

Applying σ to (4) and invoking N, we obtain

$$(5) \quad \begin{array}{ccc} \frac{1/3}{z} & \frac{1/3}{y} & \frac{1/3}{x} \\ x & z & y \\ y & x & z \end{array} \xrightarrow{F} \begin{array}{c} z \\ x \end{array}$$

¹⁹ From A, we don't need to worry about which individuals have which preferences.

But the profiles in (2), (4), and (5) are the same except for permutations of individuals' preferences, and so, from A, give rise to the same social ranking under F , which in view of (2), (4), and 5 must be

$$\begin{array}{c} x \\ y \\ z \\ x \end{array},$$

violating transitivity. The analogous contradiction arises if (3) holds. Hence, (1) must hold after all. From MIIA and (1), we have

$$(6) \quad \begin{array}{ccc} \frac{a}{x} & \frac{b}{z} & \frac{1/3}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} x \sim y, \text{ for all } a \geq 0 \text{ and } b \geq 0 \text{ such that } a + b = 2/3$$

From PR and (6), we have

$$(7) \quad \begin{array}{ccc} \frac{a}{x} & \frac{b}{z} & \frac{1-a-b}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}, \text{ where } a + b > 2/3, \text{ and } a, b, 1-a-b \geq 0,$$

and

$$(8) \quad \begin{array}{ccc} \frac{a}{x} & \frac{b}{z} & \frac{1-a-b}{y} \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}, \text{ where } a + b < 2/3, \text{ and } a, b, 1-a-b \geq 0.$$

But (6), (7), and (8) collectively imply that x is socially preferred to y if and only if x 's Borda score exceeds y 's Borda score,²⁰ i.e., F is the Borda count²¹. Q.E.D

²⁰ Alternative x 's Borda score is $3a + 2b + 1 - a - b$, and y 's Borda score is $3(1 - a - b) + 2a + b$. Hence, x is Borda-ranked above y if and only if

$$3a + 2b + 1 - a - b > 3(1 - a - b) + 2a + b,$$

which reduces to $a + b > 2/3$, i.e., we obtain formula (7).

²¹ Notice that, for this special case, we did not need to invoke axiom RC. Roughly speaking, RC allows us to conclude that a SWF coinciding with the Borda count on each several disjoint populations must be the Borda count on the union of those populations.

The domain $\begin{Bmatrix} x & z & y \\ y & x & z \\ z & y & x \end{Bmatrix}$ is called a Condorcet cycle because, as Condorcet (1785)

showed, majority rule may cycle for profiles on this domain (indeed, it cycles for the profile in (1)). This domain is the focus of much of the social choice literature, e.g., Arrow (1951) makes crucial use of Condorcet cycles in the proof of the Impossibility Theorem; Barbie et al (2006) show that it is essentially the unique domain (for three alternatives) on which the Borda count is strategy-proof; and Dasgupta and Maskin (2008) show that no voting rule can satisfy all of P, A, N, and IIA on this domain. One implication of our result in this section is that there is a sense in which the Borda count comes closer than any other voting rule to satisfying these four axioms on a Condorcet cycle domain - it satisfies P, A, and N and captures (through MIIA) the “essence” of IIA.

E. Outline

In Section 2, we lay out the model and the axioms. Section 3 introduces the critical concept of an indifference curve for a SWF. In Section 4, we show that a SWF satisfying our axioms must be the Borda count. Section 5 concludes the paper by discussing a few open questions.

2. Formal Model and Definitions

Consider a society consisting of a continuum of individuals²² (indexed by $i \in [0,1]$) and a finite set of social alternatives X , with $|X| = m$.²³ For each individual i , let \mathfrak{R}_i be a set of

²² In assuming a continuum, we are following Dasgupta and Maskin (2008) and (2020). Those earlier papers invoked this assumption primarily to ensure that ties are nongeneric. The assumption plays that role in this paper too, but more importantly, it guarantees together with Positive Responsiveness that ties actually *occur*. Indeed, our proof technique relies critically on analyzing a SWF’s indifference curves, i.e., the sets of profiles for which there are ties.

²³ $|X|$ is the number of alternatives in X .

possible *strict* rankings²⁴ of X for individual i and let \succsim_i be a typical element of \mathfrak{R}_i ($x \succsim_i y$ means that individual i prefers alternative x to y). Then, a *social welfare function* (SWF) F is a mapping

$$F : \times_{i \in [0,1]} \mathfrak{R}_i \rightarrow \mathfrak{R},$$

where \mathfrak{R} is the set of all possible social rankings (here we *do* allow for indifference and the typical element is \succsim).

With a continuum of individuals, we can't literally count the number of individuals with a particular preference; we have to work with proportions instead. For that purpose, let μ be Lebesgue measure on $[0,1]$. Given profile \succsim_\cdot , interpret $\mu(\{i \mid x \succsim_i y\})$ as the proportion of individuals who prefer x to y .²⁵

The Arrow conditions for a SWF F are:

Unrestricted Domain (U): The SWF must determine social preferences for all possible preferences that individuals might have. Formally, for all $i \in [0,1]$, \mathfrak{R}_i consists of *all* strict orderings of X .

Pareto Property (P): If all individuals (strictly) prefer x to y , then x must be strictly socially preferred. Formally, for all profiles $\succsim_\cdot \in \times \mathfrak{R}_i$ and all $x, y \in X$, if $x \succsim_i y$ for all i , then $x \succ_F y$, where $\succ_F = F(\succsim_\cdot)$.

Nondictatorship (ND): There exists no individual who always gets his way in the sense that if he prefers x to y , then x must be socially preferred to y , regardless of others' preferences. Formally,

²⁴ Thus, we rule out the possibility that an individual can be indifferent between two alternatives. However, we conjecture that our results extend to the case where she can be indifferent (see Section 5).

²⁵ To be accurate, we must restrict attention to profiles \succsim_\cdot for which $\{i \mid x \succsim_i y\}$ is a measurable set.

there does *not* exist i^* such that for all $\succsim_i \in \times \mathfrak{R}_i$ and all $x, y \in X$, if $x \succsim_{i^*} y$, then $x \succsim_F y$, where

$$\succsim_F = F(\succsim).$$

Independence of Irrelevant Alternatives (IIA): Social preferences between x and y should depend only on individuals' preferences between x and y , and not on their preferences concerning some third alternative. Formally, for all $\succsim, \succsim' \in \times \mathfrak{R}_i$ and all $x, y \in X$, if, for all i , $x \succsim_i y \Leftrightarrow x \succsim'_i y$, then \succsim_F ranks x and y the same way that \succsim'_F does, where $\succsim_F = F(\succsim)$ and $\succsim'_F = F(\succsim')$.

Because we have argued that IIA is too strong, we are interested in the following relaxation:

Modified IIA: If, given two profiles and two alternatives, each individual (i) ranks the two alternatives the same way in both profiles and (ii) ranks the same number of other alternatives *between* the two alternatives in both profiles, then the social preference between x and y should be the same for both profiles. Formally, for all $\succsim, \succsim' \in \times \mathfrak{R}_i$ and all $x, y \in X$, if, for all i , $x \succsim_i y \Leftrightarrow x \succsim'_i y$, and $|\{z \mid x \succsim_i z \succsim_i y\}| = |\{z \mid x \succsim'_i z \succsim'_i y\}|$, $|\{z \mid y \succsim_i z \succsim_i x\}| = |\{z \mid y \succsim'_i z \succsim'_i x\}|$, then \succsim_F and \succsim'_F rank x and y the same way, where $\succsim_F = F(\succsim)$ and $\succsim'_F = F(\succsim')$.²⁶

May (1952) characterizes majority rule axiomatically in the case $|X| = 2$. We will consider natural extensions of his axioms to three or more alternatives:

Anonymity (A): If we permute a preference profile so that individual j gets i 's preferences, k gets j 's preferences, etc., then the social ranking remains the same. Formally, fix any (measure-

²⁶ There is a similar condition developed in Maskin (2020).

preserving)²⁷ permutation of society $\pi : [0,1] \rightarrow [0,1]$. For any profile $\succsim \in \times \mathfrak{R}_i$, let \succsim^π be the profile such that, for all i , $\succsim_i^\pi = \succsim_{\pi(i)}$. Then $F(\succsim^\pi) = F(\succsim)$.

Neutrality (N): Suppose that we permute the alternatives so that x becomes y , y becomes z , etc., and we change individuals' preferences in the corresponding way. Then, if x was socially ranked above y originally, now y is socially ranked above z . Formally, for any permutation $\rho : X \rightarrow X$ and any profile $\succsim \in \times \mathfrak{R}_i$, let \succsim^ρ be the profile such that, for all $x, y \in X$ and all $i \in [0,1]$, $x \succsim_i y \Leftrightarrow \rho(x) \succsim_i^\rho \rho(y)$. Then, $x \succsim_F y \Leftrightarrow \rho(x) \succsim_F^\rho \rho(y)$ for all $x, y \in X$, where $\succsim_F = F(\succsim)$ and $\succsim_F^\rho = F(\succsim^\rho)$.

*Positive Responsiveness (PR)*²⁸: If we change individuals' preferences so that alternative x moves up relative to y in some individuals' rankings and doesn't move down relative to y in anyone's ranking, then, first, x moves up socially relative to y ; second, it does so continuously. (More specifically, (i) if x was previously weakly preferred to y , x is now strictly preferred; and (ii) if y was socially preferred to x for the first profile and x is socially preferred for the second, then there exists an intermediate profile for which x and y are socially indifferent²⁹). Formally, suppose \succsim and \succsim' are two profiles such that, for some $x, y \in X$ $\mu(\{i | y \succsim_i x \text{ and } x \succsim'_i y\}) > 0$ and for all $j \notin \{i | y \succsim_i x \text{ and } x \succsim'_i y\}$,

(*) $x \succsim_j z \Rightarrow x \succsim'_j z$, $w \succsim_j y \Rightarrow w \succsim'_j y$, and $r \succsim_j s \Leftrightarrow r \succsim'_j s$ for all $z \neq x$, $w \neq y$ and $r, s \in X - \{x, y\}$.

²⁷ Because we are working with a continuum of individuals, we must explicitly assume that $\mu(\{i | x \succsim_i y\}) = \mu(\{i | x \succsim_{\pi(i)} y\})$, which holds automatically with a finite number of individuals.

²⁸ For a different generalization of PR to more than two alternatives, see Horan, Osborne, and Sanver (2019).

²⁹ This part of PR is a continuity requirement.

Then, (i) $x \succsim_F y \Rightarrow x \succ'_F y$, where $\succsim_F = F(\succ)$ and $\succ'_F = F(\succ')$. Furthermore, (ii) if $y \succ_F x$ and $x \succ'_F y$, then there exists profile \succ'' satisfying (*) (with \succ''_j replacing \succ'_j) such that $x \sim''_F y$, where $\succ''_F = F(\succ'')$.

Ranking Consistency (RC): If, given a profile of individual preferences, each of a set of disjoint subpopulations has the same strict social ranking, then the (unique) top-ranked alternative for that ranking is also the (unique) top-ranked alternative for the union of those subpopulations.

Formally, consider a partition C^1, \dots, C^k of $[0,1]$ and a profile \succ . For each $h \in \{1, \dots, k\}$ let \succ^h be the restriction of \succ to the individuals in C^h and suppose that $F(\succ^1) = \dots = F(\succ^k) = \succ$, where \succ is a strict ranking. Then, if $x \succ y$ for all $y \neq x$, we have $x \succ_{F(\succ)} y$ for all $y \neq x$.

We are not aware of a SWF actually used in practice that fails to satisfy RC. Indeed, RC holds for almost any standard SWF studied in the literature. As mentioned in Section 1C, RC is satisfied by all scoring rules – including plurality rule and the Borda count – because these rules satisfy the much stronger requirement of consistency. Furthermore, it is satisfied by instant-runoff voting,³⁰ Coomb's rule (which is the same as instant-runoff voting except that instead of eliminating the candidate ranked first least often, it drops the candidate ranked last most often), and majority rule.³¹ RC also holds for a wide array of Condorcet-conforming voting methods

³⁰ Suppose that, given profile \succ , each of the subpopulations C_1, \dots, C_k eliminates alternative x_m first in an instant run-off election, then x_{m-1} , and so on until only x_1 remains (so that the social ranking for each subpopulation is $x_1 \succ x_2 \succ \dots \succ x_m$). Because x_m is eliminated first, in each subpopulation, it must be ranked first least often in the overall population $[0,1]$, and so will be eliminated first in the instant-runoff. But then the same argument applies to x_{m-1} , etc. In other words, the social ranking for $[0,1]$ is, again, $x_1 \succ \dots \succ x_m$.

³¹ Suppose that, given profile \succ , the majority social ranking in each subpopulation C_1, \dots, C_k is $x_1 \succ x_2 \succ \dots \succ x_m$. That is, for every $r < s$, a majority of individuals in each subpopulation C_k prefer x_r to x_s . But then a majority of individuals in the overall population $[0,1]$ must also prefer x_r to x_s . And so, the same majority ranking $x_1 \succ x_2 \dots \succ x_m$ holds for $[0,1]$.

(methods that elect a Condorcet winner if one exists and otherwise rank candidates some other way). For example, the Kemeny-Young method – in which the social ranking minimizes the sum of the Kendall tau distances (the Kendall tau distance between two rankings is the total number of discordant pairs) between it and the individuals’ rankings—satisfies RC,³² as do Copeland’s method,³³ Smith’s method,³⁴ Tideman’s ranked-pairs method,³⁵ and Baldwin’s method.³⁶ Finally, we note that two other popular methods, approval voting³⁷ and range voting³⁸, also satisfy RC (strictly speaking, neither is a SWF in our formal sense, since we require that the

³² See the Wikipedia article on the Kemeny-Young method https://en.wikipedia.org/wiki/Kemeny-Young_method.

³³ In Copeland’s method, alternatives are ranked socially according to how many other alternatives they defeat by a majority in a pairwise comparison. This method satisfies RC because if, for a given subpopulation, there is a strict social ranking of the m alternatives, then the top-ranked alternative must defeat each of the other $m - 1$ alternatives (i.e., it is a Condorcet winner), the second-ranked alternative must defeat all but the Condorcet winner, etc. And if this same social ranking holds for all other subpopulations, then it must also hold for the overall population.

³⁴ The Smith set is the smallest set of alternatives each of which defeats any alternative not in the set by a majority in a pairwise comparison. Smith’s method chooses the Smith set as the top indifference curve in the social ranking, the Smith set for the remaining alternatives once the top indifference curve is removed, etc. It satisfies RC because if, as RC demands, the social ranking is a strict ordering, then the method is the same as majority rule.

³⁵ In the ranked-pairs method (Tideman 1987), for each pair of alternatives, x and y , x is provisionally ranked above y socially (for a given profile) if and only a majority of individuals prefer x to y . These pairwise rankings are then sequentially locked in: first, the ranking for the pair for which the majority is largest, then the one for the second-largest majority, etc. If, however, we reach a pair for which locking in would create a Condorcet cycle, that pair is skipped. To see that the method satisfies RC, assume $X = \{x, y, z\}$, and suppose that the social ranking is $x \succ y \succ z$ for each subpopulation. Within a subpopulation, the only way that the majority ranking between some pair could differ from the social ranking of that pair is if a majority prefer z to x , but the majorities for x over y and y over z are bigger. But then the same must be true for the overall population: the only way that an overall majority could prefer z to x is if the majorities for x over y and y over z are bigger, so that the overall social ranking is still $x \succ y \succ z$. The argument is similar for $|X| > 3$.

³⁶ Baldwin’s method is a form of instant-runoff voting in which if no alternative is ranked first by a majority of votes, the alternative with the lowest Borda score is dropped, and the process iterates with this reduced set of alternatives. It is a Condorcet-conforming method because the alternative with the lowest Borda score can’t be a Condorcet winner. It satisfies RC by argument similar to that for IRV.

³⁷ In approval voting, each individual approves or disapproves each alternative, and alternatives are ranked according to their approval totals. RC is satisfied because approvals are additive across subpopulations.

³⁸ In range voting, an individual “grades” each alternative on a numerical scale, and alternatives are ranked according to their total grades. RC is satisfied because total grades are additive across subpopulations.

social ranking depend only on ordinal information about individuals' preference – and both rely on cardinal data).³⁹

We now define the Borda count formally:

Borda Count: Alternative x is socially (weakly) preferred to y if and only if x 's Borda score is (weakly) bigger than y 's Borda score (where x gets m points every time an individual ranks it first, $m-1$ points every time an individual ranks it second, etc.). Formally, for all $x, y \in X$ and all profiles $\succsim_i \in \times \mathfrak{R}_i$,

$$(**) \quad x \succsim_{Bor} y \Leftrightarrow \int r_{\succsim_i}(x) d\mu(i) \geq \int r_{\succsim_i}(y) d\mu(i),$$

where $r_{\succsim_i}(x) = |\{y \in X \mid x \succ_i y\}| + 1$ and \succsim_{Bor} is the Borda ranking corresponding to \succsim_i .

3. The Indifference Curve of a Social Welfare Function

The proof of our characterization result makes much use of a SWF F 's indifference curve. To define the concept of an indifference curve, let us consider the case of three alternatives⁴⁰ $X = \{x, y, z\}$ and fix a profile \succsim_i . Let $a_{xyz}(\succsim_i)$ be the fraction of individuals who

have ranking $\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}$. Then, if F satisfies A, the 6-tuple

$$(9) \quad \alpha = (\alpha_{xzy}, \alpha_{yzx}, \alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx}) \\ = (a_{xzy}(\succsim_i), a_{yzx}(\succsim_i), a_{xyz}(\succsim_i), a_{zxy}(\succsim_i), a_{yxz}(\succsim_i), a_{zyx}(\succsim_i))$$

³⁹ The only standard SWFs violating RC that we know of are certain hybrid methods, e.g., Black's method in which for the case $|X| = 3$, the ranking coincides with majority rule as long as there is a Condorcet winner, and otherwise amounts to Borda. This method violates RC because it's possible that the same social ranking may arise in two different subpopulations but for different reasons (e.g., majority rule in one, Borda in the other), in which case its top alternative may not carry over to the overall population.

⁴⁰ The case $|X| > 3$ is handled in the Appendix.

is a sufficient statistic for \succsim in determining social preferences \succsim_F and we can use the 6-tuple interchangeably with \succsim .

We define the *indifference curve* for x and y , I_F^{xy} , to be the set of 6-tuples for which society is indifferent between x and y : $I_F^{xy} = \left\{ \alpha \in \Delta^5 \mid x \sim_{F(\alpha)} y \text{ for } \alpha \text{ satisfying (9)} \right\}$. For example, the Borda indifference curve is given by

$$(10) \quad I_{Bor}^{xy} = \left\{ \alpha \mid \alpha_{xyz} + \alpha_{zxy} + 2\alpha_{xzy} = \alpha_{yxz} + \alpha_{zyx} + 2\alpha_{yzx} \right\}.$$

The indifference curve is useful in proving that a SWF F satisfying the axioms is the Borda count. In particular, we rely on the following simple result:

Lemma: Suppose that F satisfies U, A, N, and PR. If, for some $x, y \in X$,

$$(11) \quad I_F^{xy} = I_{Bor}^{xy},$$

then

$$(12) \quad F = \text{Borda count}.$$

In other words, to show that F and the Borda count coincide, we need show only that they have the same indifference curve. And, as the proof demonstrates, this follows largely because of PR.

Proof: Suppose that (11) holds but there exist x, y , and $\alpha = (\alpha_{xzy}, \alpha_{yzx}, \alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx})$ for which

$$(13) \quad F(\alpha) \neq \text{Bor}(\alpha).$$

Without loss of generality, we can assume that

$$(14) \quad x \succ_{\text{Bor}(\alpha)} y \text{ and } y \succ_{F(\alpha)} x$$

(if (13) holds yet we have social indifference for either SWF, then we contradict (11)).

Suppose that we (a) continuously decrease α_{xzy} while increasing α_{zyx} by the same amount; (b)

continuously decrease α_{xyz} while increasing α_{yxz} by the same amount; and (c) continuously

decrease α_{zy} while increasing α_{yx} by the same amount. If all of α_{xzy} , α_{xyz} , and α_{zxy} are reduced to zero, then $y \succ_{\text{Bor}} x$ for the corresponding profile. Thus, before then, we must reach a 6-tuple α^* for which

$$(15) \quad \alpha^* \in I_{\text{Bor}}^{xy}$$

But from PR and (14), $y \succ_{F(\alpha^*)} x$, which, in view of (15) contradicts (11).

Q.E.D.

4. The Characterization Theorem

We are now ready to establish our characterization theorem.

Theorem: SWF F satisfies U, MIIA, A, N, PR, and RC if and only if F is the Borda count.

Proof: The “if” part is clear. We shall concentrate on “only if.”

For $|X| = 2$, the result follows from May (1952).

Suppose that $X = \{x, y, z\}$.⁴¹ We will show first that if

$$(16) \quad \alpha = (\alpha_{xzy}, \alpha_{yzx}, \alpha_{xyz}, \alpha_{zxy}, \alpha_{yxz}, \alpha_{zyx}) \in I_{\text{Bor}}^{xy}$$

then

$$(17) \quad \alpha \in I_F^{xy}$$

Assume that

$$(18) \quad \alpha_{xzy} \geq \alpha_{yzx}$$

(If the inequality in (18) goes the other way, we can simply interchange x and y in what follows)

From (10) and (16),

$$(19) \quad \alpha_{xyz} + \alpha_{zxy} = (1 - 3\alpha_{xzy} + \alpha_{yzx}) / 2$$

⁴¹ Again, the case $|X| > 3$ is treated in the appendix.

and

$$(20) \quad \alpha_{yxz} + \alpha_{zyx} = (1 - 3\alpha_{yzx} + \alpha_{xzy}) / 2$$

Consider

$$(21) \quad \begin{aligned} \hat{\alpha} &= (\hat{\alpha}_{xzy}, \hat{\alpha}_{yzx}, \hat{\alpha}_{xyz}, \hat{\alpha}_{zyx}, \hat{\alpha}_{yxz}, \hat{\alpha}_{zxy}) \\ &= (\alpha_{xzy}, \alpha_{yzx}, 0, (1 - 3\alpha_{xzy} + \alpha_{yzx}) / 2, (1 - \alpha_{yzx} - \alpha_{xzy}) / 2, \alpha_{xzy} - \alpha_{yzx}) \end{aligned}$$

From (18) – (21), $\hat{\alpha} \in \Delta^5$. Because the Borda count satisfies MIIA, (16) and (19) – (21) imply that

$$\hat{\alpha} \in I_{Bor}^{xy}$$

And because F satisfies MIIA, it suffices to show that

$$(22) \quad \hat{\alpha} \in I_F^{xy}$$

in order to establish (17).

Now, as we will see, $\hat{\alpha}$ decomposes into three subprofiles for each of which we have social indifference among x, y , and z according to F . Specifically, let

$$(23) \quad \gamma_{\cdot}^{A_1} = \begin{array}{ccc} \frac{1/3}{x} & \frac{1/3}{y} & \frac{1/3}{z} \\ z & x & y \\ y & z & x \end{array} = (\frac{1}{3}, 0, 0, 0, \frac{1}{3}, \frac{1}{3})$$

and

$$(24) \quad \gamma_{\cdot}^{B_1} = \begin{array}{cc} \frac{1/2}{x} & \frac{1/2}{y} \\ z & z \\ y & x \end{array} = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)$$

and

$$(25) \quad \gamma_{\cdot}^{B_0} = \frac{1/2}{y} \frac{1/2}{z} = (0, 0, 0, 1/2, 1/2, 0)$$

$$\begin{array}{cc} x & x \\ z & y \end{array}$$

Then, from (21) and (23) – (25)

$$(26) \quad \hat{\alpha} = 3(\alpha_{xzy} - \alpha_{yzx}) \gamma_{\cdot}^{A_1} + 2\alpha_{yzx} \gamma_{\cdot}^{B_1} \\ + (1 - 3\alpha_{xzy} + \alpha_{yzx}) \gamma_{\cdot}^{B_0}$$

From the argument in Section 1D applied to (23), we have

$$(27) \quad x \sim_{F(\gamma_{\cdot}^{A_1})} y \sim_{F(\gamma_{\cdot}^{A_1})} z$$

From A, N, (25) and the permutation $\sigma(y) = z$ and $\sigma(z) = y$

$$(28) \quad y \sim_{F(\gamma_{\cdot}^{B_0})} z$$

From A, N, and permutation $\sigma'(x) = y$ and $\sigma'(y) = x$,

$$(29) \quad x \sim_{F(\gamma_{\cdot}^{B_0})} y,$$

where

$$\gamma_{\cdot}^{B'_0} = \frac{1/2}{y} \frac{1/2}{x}$$

$$\begin{array}{cc} x & y \\ z & z \end{array}$$

From (29) and MIIA,

$$(30) \quad x \sim_{F(\gamma_{\cdot}^{B_0})} y$$

Combining (28) and (30), we have

$$(31) \quad x \sim_{F(\gamma_{\cdot}^{B_0})} y \sim_{F(\gamma_{\cdot}^{B_0})} z.$$

Similarly,

$$(32) \quad x \sim_{F(\succ_{\cdot}^{B_1})} y \sim_{F(\succ_{\cdot}^{B_1})} z.$$

Now, we cannot *immediately* infer from RC and (26), (27), (31), and (32) that (22) holds, since RC applies only to strict rankings. So, we will perturb $\succ_{\cdot}^{A_1}$, $\succ_{\cdot}^{B_1}$, and $\succ_{\cdot}^{B_0}$ slightly in two different ways, first to make their corresponding social rankings all $x \succ z \succ y$, and then to make them, all $y \succ z \succ x$. In the first case, RC will imply that x is the top social alternatives for the overall population and, in the second, that y is top. We will then send the perturbations to zero and use PR(ii) to conclude that (22) holds after all.

As we showed in Section 1D, F coincides with the Borda count on $\begin{Bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{Bmatrix}$. Thus, if

we choose $\varepsilon_1, \varepsilon_2$, and ε_3 such that

$$(33) \quad \succ_{\cdot}^{A_1^{\varepsilon}} = \begin{array}{ccc} \frac{1/3 + \varepsilon_1}{x} & \frac{1/3 + \varepsilon_2}{y} & \frac{1/3 + \varepsilon_3}{z} \\ z & x & y \\ y & z & x \end{array}$$

and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, the social ranking for F will be determined by the Borda scores of x, y , and z :

score for $x = 2\varepsilon_1 + \varepsilon_2 + 1$ (where a top-ranked alternative gets 2 points, and a second-ranked alternative gets one point)

score for $y = 2\varepsilon_2 + \varepsilon_3 + 1$

score for $z = 2\varepsilon_3 + \varepsilon_1 + 1$

Thus, as long as $2\varepsilon_1 + \varepsilon_2 > 2\varepsilon_3 + \varepsilon_1 = -2\varepsilon_2 - \varepsilon_1 > 2\varepsilon_2 + \varepsilon_3 = \varepsilon_2 - \varepsilon_1$, i.e., $\varepsilon_1 > -\varepsilon_2 > 0$ we can conclude that

$$(34) \quad x \succ_{F(\succ_{\cdot}^{A_1^{\varepsilon}})} z \succ_{F(\succ_{\cdot}^{A_1^{\varepsilon}})} y$$

Next, choose small $\beta > 0$ such that

$$(35) \quad \succsim_{\cdot}^{B_0^\beta} = \frac{\beta}{x} \quad \frac{1/2 - \beta}{y} \quad \frac{1/2}{x}$$

$$\begin{array}{ccc} z & x & y \\ y & z & z \end{array}$$

From MIIA, (31), and (35), $x \sim_{F(\succsim_{\cdot}^{B_0^\beta})} z$ and from PR(i), (31), and (35), $x \succ_{F(\succsim_{\cdot}^{B_0^\beta})} y$. Hence,

$$(36) \quad x \sim_{F(\succsim_{\cdot}^{B_0^\beta})} z \succ_{F(\succsim_{\cdot}^{B_0^\beta})} y$$

Choose small $\varepsilon > 0$ such that

$$(37) \quad \succsim_{\cdot}^{B_0^{\beta\varepsilon}} = \frac{\beta + \varepsilon}{x} \quad \frac{1/2 - \beta}{y} \quad \frac{1/2 - \varepsilon}{z}$$

$$\begin{array}{ccc} z & x & x \\ y & z & y \end{array}$$

From (36), (37), and PR(i), $x \succ_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon}})} z$ and $x \succ_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon}})} y$. Now, if for some ε , $y \succ_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon}})} z$, then

from PR(ii) there exists ε^* such that

$$(38) \quad x \succ_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon^*}})} z \sim_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon^*}})} y$$

But then from (36) – (38), we have

$$(39) \quad x \succ_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon}})} z \succ_{F(\succsim_{\cdot}^{B_0^{\beta\varepsilon}})} y$$

for ε slightly less than ε^* .

Finally, choose $\gamma > 0$. Then from PR(i) and (32)

$$\succsim_{B_1^\gamma} = \frac{1/2 + \gamma}{x} \quad \frac{1/2 - \gamma}{y} \quad \xrightarrow{F} x$$

$$\begin{array}{ccc} z & z & z \\ y & x & y \end{array}$$

That is,

$$(40) \quad x \succ_{F(\gamma_{\cdot}^{B_1^y})} z \succ_{F(\gamma_{\cdot}^{B_1^x})} y.$$

From (34), (39), (40) and RC

$$(41) \quad x \succ_{F(\hat{\alpha}^\theta)} y,$$

where $\hat{\alpha}^\theta$ is the perturbation of $\hat{\alpha}$ obtained by replacing

$\gamma_{\cdot}^{A_1}$, $\gamma_{\cdot}^{B_0}$, and $\gamma_{\cdot}^{B_1}$ by $\gamma_{\cdot}^{A_1^\epsilon}$, $\gamma_{\cdot}^{B_0^{\beta_\epsilon}}$, and $\gamma_{\cdot}^{B_1^\gamma}$, respectively.

Symmetrically, we can choose perturbations of $\gamma_{\cdot}^{A_1}$, $\gamma_{\cdot}^{B_0}$, and $\gamma_{\cdot}^{B_1}$ such that the social preferences in (34), (39), and (40) are reversed and so, from RC,

$$(42) \quad y \succ_{F(\hat{\alpha}^{\theta^-})} x,$$

where $\hat{\alpha}^{\theta^-}$ is the analogous perturbation of $\hat{\alpha}$.

Now let $\theta \rightarrow 0$ and $\theta^- \rightarrow 0$. From PR(ii), we have (22), and so (17) holds, as we claimed.

To complete the proof, it remains to show the converse of the result that (16) implies (17). Specifically, we must demonstrate that if

$$(44) \quad \alpha = (\alpha_{xzy}, \alpha_{yzx}, \alpha_{xyx}, \alpha_{zyx}, \alpha_{yxz}, \alpha_{zxy}) \in I_F^{xy}$$

then

$$(45) \quad \alpha \in I_{Bor}^{xy}.$$

Suppose, to the contrary, that for $\alpha \in I_F^{xy}$,

$$(46) \quad x \succ_{Bor(\alpha)} y$$

Given α , continuously decrease α_{xzy} and correspondingly increase α_{zyx} ; continuously decrease α_{xyx} and correspondingly increase α_{yxz} ; and continuously decrease α_{yzx} and correspondingly increase α_{zxy} . Eventually, we will obtain α^* with $\alpha^* \in I_{Bor}^{xy}$. But then from the argument above,

$\alpha^* \in I_F^{xy}$, which contradicts PR. Thus (44) implies (45), and so from the Lemma, F is the Borda count.

Q.E.D.

To briefly summarize the proof: if $\alpha \in I_{Bor}^{xy}$, α decomposes into subprofiles α^{A_1} , α^{B_1} , and α^{B_0} for each of which, using A, N and MIIA, we can show that x and y are socially indifferent according to F . We can then apply RC to conclude that the overall profile α satisfies $\alpha \in I_F^{xy}$ (this isn't quite right because RC applies only to strict rankings and so we have to perturb α^{A_1} , α^{B_1} , and α^{B_0} slightly to make the argument work). It is then a simple matter to show that if $\alpha \in I_F^{xy}$, then $\alpha \in I_{Bor}^{xy}$. Putting these two implications together, we infer that F and the Borda count share the same indifference curve. Finally, the Lemma can be invoked to conclude that F and the Borda count coincide everywhere.

5. Open Questions

There are at least four questions that seem worth pursuing in follow-up work:

First, we have assumed throughout that, although society can be indifferent between a pair of alternatives, x and y , individuals are never indifferent. We conjecture that if individual indifference were allowed, we would obtain the natural extension of the Borda count, e.g., if an individual is indifferent between x and y , then instead of x getting p points and y getting $p-1$ (as would be the case if the individual ranked x immediately above y), the alternatives will split the point count $p + p-1 = 2p-1$ equally.

Second, we have made important use of the continuum of voters in our proof. Specifically, the continuum, together with the continuity provision of PR, guarantees that there will be profiles for which society is indifferent between x and y . It would be interesting to explore to what extent the characterization result extends to the case of finitely many voters.

Third, this paper studies SWFs, which rank *all* alternatives. By contrast, a voting rule simply selects the *winner* (see footnote 2). In a previous draft of this paper, we proposed a way to modify the axioms to obtain a characterization of the voting-rule version of the Borda count (i.e., the winner is the alternative with the highest Borda score). However, that draft considered only the case in which social indifference curves are linear or polynomial. Whether that characterization holds in the more general setting of the current draft is not yet known.

Finally, although the anonymity and neutrality axioms are quite natural in political elections, they don't apply universally (think, for example, of corporate elections where voters are weighted by their ownership stake and certain alternatives – e.g., the status quo – may be privileged). It is clear that certain variants of the Borda count – e.g., where different people can have different weights or some particular alternatives get extra Borda points – satisfy the remaining axioms when A and N are dropped, but we do not have a full characterization of all SWFs satisfying those axioms.

Appendix

Indifference Curves

Suppose $|X| = m$, where $m \geq 3$. Choose $x, y \in X$ and fix profile \succsim . For any $k \in \{0, \dots, m-2\}$, let $\alpha_k^{xy}(\succsim)$ be the fraction of individuals in \succsim who prefer x to y and rank k alternatives between x and y . If SWF satisfies A, N, and MIIA, then

$$(A1) \quad \alpha = (\alpha_0^{xy}, \dots, \alpha_{m-2}^{xy}, \alpha_0^{yx}, \dots, \alpha_{m-2}^{yx}) \\ = (\alpha_0^{xy}(\succsim), \dots, \alpha_{m-2}^{xy}(\succsim), \alpha_0^{yx}(\succsim), \dots, \alpha_{m-2}^{yx}(\succsim))$$

is a sufficient statistic for \succsim in determining the social ranking of x and y . F 's indifference curve for x and y is defined as

$$I_F^{xy} = \left\{ \alpha \in \Delta^{2m-5} \mid x \sim_{F(\alpha)} y \right\}$$

In particular, the Borda indifference curve is:

$$(A2) \quad I_{Bor}^{xy} = \left\{ \alpha \mid \sum_{k=0}^{m-2} (k+1)(\alpha_k^{xy} - \alpha_k^{yx}) = 0 \right\}$$

The counterpart of the Lemma in Section 3 is:

Lemma: Suppose F satisfies U, MIIA, A, N, and PR. If, for some $x, y \in X$,

$$(A3) \quad I_F^{xy} = I_{Bor}^{xy},$$

then

$$(A4) \quad F = \text{Borda count}.$$

Proof: Suppose (A3) holds but there exist x, y , and $\alpha = (\alpha_0^{xy}, \dots, \alpha_{m-2}^{xy}, \alpha_0^{yx}, \dots, \alpha_{m-2}^{yx})$ for which

$$F(\alpha) \neq \text{Bor}(\alpha).$$

Then, without loss of generality, we may also assume

$$(A5) \quad x \succ_{\text{Bor}(\alpha)} y \text{ and } y \succ_{F(\alpha)} x$$

Let us continuously decrease $\alpha_0^{xy}, \dots, \alpha_{m-2}^{xy}$ and correspondingly increase $\alpha_0^{yx}, \dots, \alpha_{m-2}^{yx}$. If all of $\alpha_0^{xy}, \dots, \alpha_{m-2}^{xy}$ are reduced to zero, then $y \succ_{\text{Bor}} x$ for the corresponding profile. Thus, before then, we must reach a profile α^* for which

$$(A6) \quad \alpha^* \in I_{\text{Bor}}^{xy}.$$

But from PR and (A5), $y \succ_{F(\alpha^*)} x$, which in view of (A6) violates (A3).

Q.E.D.

Proof of Characterization Theorem

We start with the case $|X| = 4$. Let $X = \{x, y, z, w\}$. We will show first that if

$$(A7) \quad \alpha \in I_{\text{Bor}}^{xy}$$

then

$$(A8) \quad \alpha \in I_F^{xy}$$

Assume that

$$(A9) \quad \alpha_2^{xy} \geq \alpha_2^{yx}$$

(if the inequality is reversed, we can simply interchange x and y in what follows).

Also, assume for now that

$$(A10) \quad \alpha_1^{xy} \geq \alpha_1^{yx}$$

(we shall consider the opposite inequality below). From (A2) and (A7)

$$(A11) \quad \alpha_0^{xy} = (1 - 4\alpha_2^{xy} + 2\alpha_2^{yx} - 3\alpha_1^{xy} + \alpha_1^{yx}) / 2$$

$$(A12) \quad \alpha_0^{yx} = (1 - 4\alpha_2^{yx} + 2\alpha_2^{xy} - 3\alpha_1^{yx} + \alpha_1^{xy}) / 2$$

From (A9) – (A12) α can be written as

$$(A13) \quad \alpha = 4(\alpha_2^{xy} - \alpha_2^{yx}) \gamma_{\cdot}^{A_2} + 2\alpha_2^{yx} \gamma_{\cdot}^{B_2} + 3(\alpha_1^{xy} - \alpha_1^{yx}) \gamma_{\cdot}^{A_1} + 2\alpha_1^{yx} \gamma_{\cdot}^{B_1} \\ + (1 - 4\alpha_2^{xy} + 2\alpha_2^{yx} - 3\alpha_1^{xy} + \alpha_1^{yx}) \gamma_{\cdot}^{B_0},$$

where

$$(A14) \quad \gamma_{\cdot}^{A_2} = \frac{1/4}{x} \quad \frac{1/4}{y} \quad \frac{1/4}{y} \quad \frac{1/4}{y} \\ \begin{array}{cccc} \bullet & y & y & y \\ \bullet & x & x & x \\ y & & & \end{array}$$

$$(A15) \quad \gamma_{\cdot}^{B_2} = \frac{1/2}{x} \quad \frac{1/2}{y} \\ \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ y & x \end{array}$$

$$(A16) \quad \gamma_{\cdot}^{A_1} = \frac{1/3}{x} \quad \frac{1/3}{y} \quad \frac{1/3}{y} \\ \begin{array}{ccc} \bullet & y & y \\ & x & x \\ y & & \end{array}$$

$$(A17) \quad \gamma_{\cdot}^{B_1} = \frac{1/2}{x} \quad \frac{1/2}{y} \\ \begin{array}{cc} \bullet & \bullet \\ y & x \end{array}$$

and

$$(A18) \quad \gamma_{\cdot}^{B_0} = \frac{1/2}{x} \quad \frac{1/2}{y} \\ \begin{array}{cc} y & x \end{array}$$

and each time $\begin{smallmatrix} x \\ \bullet \\ y \end{smallmatrix}$ appears in (A14) or (A15), it can be replaced by some ranking which x is three

places above y (i.e., either $\begin{smallmatrix} x & x \\ z & w \\ w & z \\ y & y \end{smallmatrix}$ or $\begin{smallmatrix} x & x \\ w & z \\ z & y \end{smallmatrix}$); each time $\begin{smallmatrix} x \\ \bullet \\ y \end{smallmatrix}$ appears in (A16) or (17), it can be replaced by

some ranking in which x is two places above y ; and each time $\begin{smallmatrix} x \\ \bullet \\ y \end{smallmatrix}$ appears, it can be replaced by

some ranking in which x is immediately above y (analogously, for $\begin{smallmatrix} y \\ \bullet \\ x \end{smallmatrix}$, $\begin{smallmatrix} y \\ \bullet \\ x \end{smallmatrix}$, and $\begin{smallmatrix} y \\ \bullet \\ x \end{smallmatrix}$).

Now, from MIIA and (A13) – (A18), we can move z and w around in profile α to obtain

$\hat{\alpha}$ such that $F(\alpha)$ and $F(\hat{\alpha})$ rank x and y the way and $\hat{\alpha}$ satisfies

$$(A19) \quad \hat{\alpha} = 4(\alpha_2^{xy} - \alpha_2^{yx}) \succ_{\bullet}^{\hat{A}_2} + 2\alpha_2^{yx} \succ_{\bullet}^{\hat{B}_2} + 3(\alpha_1^{xy} - \alpha_1^{yx}) \succ_{\bullet}^{\hat{A}_1} + 2\alpha_1^{yx} \succ_{\bullet}^{\hat{B}_1} \\ + (1 - 4\alpha_2^{xy} + 2\alpha_2^{yx} - 3\alpha_1^{xy} + \alpha_1^{yx}) \succ_{\bullet}^{\hat{B}_0}$$

where

$$(A20) \quad \succ_{\bullet}^{\hat{A}_2} = \begin{array}{cccc} \frac{1/4}{x} & \frac{1/4}{y} & \frac{1/4}{w} & \frac{1/4}{z} \\ z & x & y & w \\ w & z & x & y \\ y & w & z & x \end{array}$$

$$(A21) \quad \succ_{\bullet}^{\hat{B}_2} = \begin{array}{cc} \frac{1/2}{x} & \frac{1/2}{y} \\ z & w \\ w & z \\ y & x \end{array}$$

$$(A22) \quad \gamma_{\cdot}^{\hat{A}} = \begin{array}{ccc} \frac{1/3}{y} & \frac{1/3}{x} & \frac{1/3}{z} \\ z & y & x \\ x & z & y \\ w & w & w \end{array}$$

$$(A23) \quad \gamma_{\cdot}^{\hat{B}_1} = \begin{array}{cc} \frac{1/2}{x} & \frac{1/2}{y} \\ z & z \\ y & x \\ w & w \end{array}$$

and

$$(A24) \quad \gamma_{\cdot}^{\hat{B}_0} = \begin{array}{cc} \frac{1/2}{x} & \frac{1/2}{w} \\ y & z \\ z & y \\ w & x \end{array}$$

The argument in Section 1D generalizes immediately to apply to (A20) and (A22) and so we have

$$(A25) \quad x \sim_{F(\gamma_{\cdot}^{\hat{A}_2})} y \sim_{F(\gamma_{\cdot}^{\hat{A}_2})} z$$

$$(A26) \quad x \sim_{F(\gamma_{\cdot}^{\hat{A}_1})} y \sim_{F(\gamma_{\cdot}^{\hat{A}_1})} z$$

Similarly, from A and N,

$$(A27) \quad x \sim_{F(\gamma_{\cdot}^{\hat{B}_2})} y \sim_{F(\gamma_{\cdot}^{\hat{B}_2})} z$$

$$(A28) \quad x \sim_{F(\gamma_{\cdot}^{\hat{B}_1})} y \sim_{F(\gamma_{\cdot}^{\hat{B}_1})} z$$

and

$$(A29) \quad x \sim_{F(\gamma_{\cdot}^{\hat{B}_0})} y \sim_{F(\gamma_{\cdot}^{\hat{B}_0})} z$$

Now, the argument in Section 1D for why $F = \text{Borda count}$ on a Condorcet cycle extends

immediately to $\left\{ \begin{matrix} x & y & w & z \\ z & x & y & w \\ w & z & x & y \\ y & w & z & x \end{matrix} \right\}$. Hence, we can find small $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ such that

$$(A30) \quad \alpha^\varepsilon = \begin{array}{cccc} \frac{1/4 + \varepsilon_1}{x} & \frac{1/4 + \varepsilon_2}{y} & \frac{1/4 + \varepsilon_3}{w} & \frac{1/4 + \varepsilon_4}{z} \\ z & x & y & w \\ w & z & x & y \\ z & w & z & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ z \\ y \\ w \end{array}$$

Similarly, the argument in Section 1D and (A25) imply that we can find small $\gamma_1, \gamma_2, \gamma_3$ such that

$$(A31) \quad \alpha^\gamma = \begin{array}{ccc} \frac{1/3 + \gamma_1}{y} & \frac{1/3 + \gamma_2}{x} & \frac{1/3 + \gamma_3}{z} \\ z & y & x \\ x & z & y \\ w & w & w \end{array} \xrightarrow{F} \begin{array}{c} x \\ z \\ y \\ w \end{array}$$

From (A28) and PR,

$$(A32) \quad \alpha^\delta = \begin{array}{ccc} \frac{1/2 + \delta}{x} & \frac{1/2 - \delta}{y} & \\ z & z & \\ y & x & y \\ w & w & w \end{array} \xrightarrow{F} \begin{array}{c} x \\ z \\ y \\ w \end{array}$$

for $\delta > 0$ small.

Next, choose small $\beta > 0$. Then from (A27) and PR

$$(A33) \quad \alpha^\beta = \begin{array}{ccc} \frac{\beta}{x} & \frac{1/2}{x} & \frac{1/2 - \beta}{y} \\ z & z & w \\ y & w & z \\ w & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ z \\ y - w \end{array}$$

Now, for $\eta > 0$, (A33) and PR imply that

$$(A34) \quad \alpha^{\beta\eta} = \begin{array}{ccc} \frac{\beta+\eta}{x} & \frac{1/2-\eta}{x} & \frac{1/2-\beta}{y} \\ z & z & w \\ y & w & z \\ w & y & x \end{array} \xrightarrow{F} \begin{array}{cc} x & y \\ z & w \end{array}$$

If there exists η^* for which $z \sim_{F(\alpha^{\beta\eta^*})} y$, then for η slightly less than η^* , (A34) and PR imply

$$(A35) \quad x \succ_{F(\alpha^{\beta\eta})} z \succ_{F(\alpha^{\beta\eta})} y \succ_{F(\alpha)} w$$

Finally, for $\lambda > 0$, (A29) and PR imply that

$$\begin{array}{ccc} \frac{\lambda}{x} & \frac{1/2}{x} & \frac{1/2-\lambda}{w} \\ z & y & z \\ y & z & y \\ w & w & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y-z \\ w \end{array}$$

Thus, for μ sufficiently small

$$(A36) \quad \alpha^{\lambda\mu} = \begin{array}{ccc} \frac{\lambda+\mu}{x} & \frac{1/2-\mu}{x} & \frac{1/2-\lambda}{w} \\ z & y & z \\ y & z & y \\ w & w & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ z \\ y \\ w \end{array}$$

Thus, for each of α^ε , $\alpha^{\beta\eta}$, α^γ , α^δ and $\alpha^{\lambda\mu}$, the social ranking is

$$(A37) \quad x \succ z \succ y \succ w$$

Hence, from RC,

$$(A38) \quad x \succ_{F(\alpha^\theta)} y,$$

where α^θ is α perturbed so that $\succ_{\cdot}^{\hat{A}_2}$ is replaced by α^ε , $\succ_{\cdot}^{\hat{B}_2}$ is replaced by $\alpha^{\beta\hat{\eta}}$, $\succ_{\cdot}^{\hat{A}_1}$ is replaced by α^γ , $\succ_{\cdot}^{\hat{B}_1}$ is replaced by α^δ , and $\succ_{\cdot}^{\hat{B}_0}$ is replaced by $\alpha^{\lambda\mu}$. Symmetrically, we can choose perturbation α^{θ^-} of α so that the social ranking is

$$(A39) \quad y \succ_{F(\alpha^{\theta^-})} x$$

Letting the perturbations go to zero, we obtain from PR(ii), (A38), and (A39) that

$$x \sim_{F(\hat{\alpha})} y,$$

i.e., from MIIA, (A8) holds as claimed.

To complete the proof for $m = 4$, it remains only to show that if (A8) holds, then (A7) holds too. Suppose to the contrary that, for some $\alpha \in I_F^{xy}$,

$$(A40) \quad \alpha \in I_F^{xy}$$

and

$$(A41) \quad x \succ_{\text{Bor}(\alpha)} y.$$

Given α , continuously decrease α_0^{xy} , α_1^{xy} , and α_2^{xy} and correspondingly increase α_0^{yx} , α_1^{yx} , and α_2^{yx} . Eventually we will obtain α^* with $\alpha^* \in I_{\text{Bor}}^{xy}$. But then, from the argument above, $\alpha^* \in I_F^{xy}$, which contradicts PR.

We have been assuming that (A10) holds, but suppose instead that

$$(A42) \quad \alpha_1^{yx} \geq \alpha_1^{xy}$$

Then, α decomposes as

$$(A43) \quad \begin{aligned} \alpha = & 4(\alpha_2^{xy} - \alpha_2^{yx}) \succ_{\cdot}^{A_2} + 2\alpha_2^{yx} \succ_{\cdot}^{B_2} + 3(\alpha_1^{yx} - \alpha_1^{xy}) \succ_{\cdot}^{A_1'} + 2\alpha_1^{xy} \succ_{\cdot}^{B_1} \\ & + (1 + 2\alpha_2^{yx} - 4\alpha_2^{xy} - 3\alpha_1^{xy} + \alpha_1^{yx}) \succ_{\cdot}^{B_0}, \end{aligned}$$

where

$$(A44) \quad \gamma_{\cdot}^{A_1} = \frac{1/3}{y} \quad \frac{1/3}{x} \quad \frac{1/3}{x}$$

$$\quad \quad \quad \bullet \quad \quad y \quad y$$

$$\quad \quad \quad x \quad \quad \quad$$

Thus, the exact analogue of the previous argument applies.

The argument for $m = 4$ extends immediately to $m > 4$. Specifically, if $\alpha \in I_{Bor}^{xy}$, then the counterparts of (A11) and (A12) are

$$(A45) \quad \alpha_0^{xy} = (1 - 3\alpha_1^{xy} - \dots - m\alpha_{m-2}^{xy} + 2\alpha_1^{yx} + \dots + (m-1)\alpha_{m-2}^{yx}) / 2$$

$$(A46) \quad \alpha_0^{yx} = (1 - 3\alpha_1^{yx} - \dots - m\alpha_{m-2}^{yx} + 2\alpha_1^{xy} + \dots + (m-1)\alpha_{m-2}^{xy}) / 2$$

And the counterpart of (A13) is

$$(A47) \quad \alpha = m(\alpha_{m-2}^{xy} - \alpha_{m-2}^{yx}) \gamma_{\cdot}^{A_{m-2}} + 2\alpha_{m-2}^{yx} \gamma_{\cdot}^{B_{m-2}} + (m-1)(\alpha_{m-3}^{xy} - \alpha_{m-3}^{yx}) \gamma_{\cdot}^{A_{m-3}}$$

$$+ 2\alpha_{m-3}^{yx} \gamma_{\cdot}^{B_{m-3}} + \dots + 3(\alpha_1^{xy} - \alpha_1^{yx}) \gamma_{\cdot}^{A_1} + 2\alpha_1^{yx} \gamma_{\cdot}^{B_1}$$

$$+ (1 + \alpha_1^{yx} - 3\alpha_1^{xy} + \dots + (m-2)\alpha_{m-2}^{yx} - m\alpha_{m-2}^{xy}) \gamma_{\cdot}^{B_0},$$

where, for $k = 2, \dots, m-2$,

$$\gamma_{\cdot}^{A_k} = \frac{1/k}{x} \quad \frac{1/k \dots 1-k}{y} \quad \frac{1-k}{x}$$

$$\quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\quad \quad \quad \} k-2 \quad \quad \quad \} k-2$$

$$\quad \quad \quad y \quad \quad \quad x$$

and

$$\gamma_{\cdot}^{B_k} = \frac{1/2}{x} \quad \frac{1/2}{y} \quad \frac{1/2}{x}$$

$$\quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\quad \quad \quad \} k-2 \quad \quad \quad \} k-2$$

$$\quad \quad \quad y \quad \quad \quad x$$

and assuming that

$$(A48) \quad \alpha_k^{xy} - \alpha_k^{yx} \geq 0$$

(if any of the inequalities in (A48) is reversed then we can argue just as in (A42) – (A44)). Using the decomposition in (A47), MIIA, and RC, we can conclude that $\alpha \in I_F^{xy}$.

Q.E.D.

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