SPHERICAL VARIETIES AND LANGLANDS DUALITY

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ABSTRACT. Let G be a connected reductive complex algebraic group. This paper is devoted to the space Z of meromorphic quasimaps from a curve into an affine spherical G-variety X. The space Z may be thought of as a finite-dimensional algebraic model for the loop space of X. The theory we develop associates to X a connected reductive complex algebraic subgroup $\check{H}$ of the dual group $\check{G}$. The construction of $\check{H}$ is via Tannakian formalism: we identify a certain tensor category $\mathbf{Q}(Z)$ of perverse sheaves on Z with the category of finite-dimensional representations of $\check{H}$. The group $\check{H}$ encodes many aspects of the geometry of X.

CONTENTS

1. Introduction

1.1. **Overview.** Let G be a connected reductive complex algebraic group, and let $S \subset G$ be a complex algebraic subgroup. The following conditions are known to be equivalent:

(1) There is an open S-orbit in the flag variety of G.
(2) There are finitely many S-orbits in the flag variety of G.
(3) For any G-equivariant line bundle $\mathcal{L} \to G/S$, the multiplicity of any irreducible G-representation in $H^0(G/S, \mathcal{L})$ is zero or one.
(4) For any G-variety X and point $x \in X$ fixed by S, there are a finite number of G-orbits in the closure of the G-orbit through x.

(See [BLV86] for references to proofs.)

A subgroup $S \subset G$ is said to be *spherical* if any of the above equivalent conditions holds. A G-variety X is said to be spherical if there is a dense G-orbit $\mathring{X} \subset X$ and the stabilizer of any point $x \in \mathring{X}$ is a spherical subgroup.

Examples of spherical varieties include flag varieties, symmetric spaces, and toric varieties. In general, spherical varieties form a class of varieties whose geometry is combinatorially tractable. They play a fundamental role in the geometric representation theory of complex Lie algebras and real and p-adic Lie groups. One can interpret the theory developed here as a generalization of the geometric Satake correspondence (developed by Lusztig [Lus83], Ginzburg [Gin96], and Mirkovic-Vilonen [MV04]) from groups to spherical varieties. In particular, this opens up the study of spherical varieties to the rich methods of the geometric Langlands program (see the papers of Beilinson-Drinfeld [BD] and Kapustin-Witten [KW06] among many other important works).

In what follows, we give a brief synopsis of the theory developed in this paper.[1] Our goal is to associate to any affine spherical G-variety X a connected reductive complex algebraic subgroup $\check{H}$ of the Langlands dual group $\check{G}$ along with a canonical maximal torus of $\check{H}$. To define $\check{H}$, we construct its category of finite-dimensional representations in the geometry of the space Z of meromorphic quasimaps from a curve into X. A meromorphic quasimap consists of a point of the curve, a G-bundle on the curve, and a meromorphic section of the associated X-bundle with a pole only at the distinguished point. The space Z may be thought of as a finite-dimensional algebraic model for the loop space of X. By construction, the category of finite-dimensional representations of $\check{H}$ is equivalent to a certain tensor category $\mathbf{Q}(Z)$ of perverse sheaves on Z.

The category of finite-dimensional representations of $\check{G}$ naturally acts on sheaves on Z via the geometric Satake correspondence. By definition, the category $\mathbf{Q}(Z)$ consists of all sheaves that arise when one acts on a certain distinguished sheaf. One of our main technical results is that objects of $\mathbf{Q}(Z)$ are direct sums of the intersection cohomology sheaves of certain subspaces of Z. The definition of these subspaces is local: it refers only to the restriction of a quasimap to the formal neighborhood of its distinguished point. To understand what this restriction may look like, we need a parametrization of the meromorphic quasimaps on a formal disk. Since all G-bundles on a formal disk are trivial, this is the same thing as a parametrization of the formal loops in the open

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[1]This paper is the combination of two earlier preprints [GN04b] and [GN04c].

G-orbit $\overset{\circ}{X} \subset X$. A result of Luna-Vust [LV83] identifies the equivalence classes of such formal loops with the cone $\mathcal{V}(X)$ of G-invariant discrete valuations of the function field of X.[2]

To identify $\mathbf{Q}(Z)$ with the category of finite-dimensional representations of $\check{H}$, we use Tannakian formalism: we endow $\mathbf{Q}(Z)$ with a tensor product, a fiber functor, and the necessary compatibility constraints so that it must be equivalent to the category of representations of such a group. Recall that a category of representations comes equipped with a forgetful functor which assigns to a representation its underlying vector space. A fiber functor is a functor to vector spaces with all of the same properties as the forgetful functor. We construct both the tensor product and fiber functor for $\mathbf{Q}(Z)$ via nearby cycles in canonically defined families. For the tensor product, we consider quasimaps with multiple distinguished points, and consider the family resulting from allowing the points to collide. This kind of fusion product is inspired by the factorization structures of conformal field theory. It is worth emphasizing that the fusion product is not derived from any group structure on X, but rather from the homotopy group structure on the loop space of X. (See the end of this overview for a brief topological field theory interpretation of the fusion product.)

For the fiber functor, we consider the family obtained by considering quasimaps into the specialization of X to its asymptotic cone X_0. The asymptotic cone X_0 belongs to a special class of G-varieties closely related to flag varieties. A subgroup $S_0 \subset G$ is said to be *horospherical* if it contains the unipotent radical of a Borel subgroup of G. A G-variety X_0 is said to be horospherical if for each point $x \in X_0$, its stabilizer is horospherical. When X_0 is an affine horospherical G-variety, the main result of [GN04a] implies that the category $\mathbf{Q}(Z_0)$ is equivalent to the category of finite-dimensional representations of a torus. For an arbitrary affine spherical G-variety X, the specialization to its asymptotic cone X_0 provides a corresponding specialization for quasimaps. By properly interpreting the nearby cycles in this family, we obtain a functor which corresponds to the restriction of representations from $\check{H}$ to a canonical maximal torus. In particular, by forgetting the torus action, we obtain the sought-after fiber functor.

Finally, one can interpret our results as a categorification of the structure theory of spherical varieties. First, let us summarize what we know about the subgroup $\check{H} \subset \check{G}$ associated to the affine spherical G-variety X. As stated above, we know that $\check{H}$ is connected and reductive, and it comes equipped with a canonical maximal torus. We know that the irreducible representations of $\check{H}$ are indexed by a subsemigroup of the cone $\mathcal{V}(X)$ of G-invariant discrete valuations of the function field of X. One expects this subset to be of finite index in the entire cone, or equivalently that the Weyl group of $\check{H}$ is the same as that associated to X in the structure theory of spherical varieties [Bri90, Kno96]. This Weyl group controls many aspects of the geometry of X. For example, by the results of [Kno94], the center of the ring of differential operators on X is isomorphic to the invariants of the Weyl group of $\check{H}$ in the ring of polynomial functions on a Cartan algebra of $\check{H}$.

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[2]We thank F. Knop for pointing out that the parametrization appears in [LV83]. Since we need some further refinements of the parametrization, we include a proof using their compactification theory.

In the special case when X is a symmetric variety, the subgroup $\check{H}$ coincides with that associated to the corresponding real group in [Nad05]. One may view this as an instance of the general Harish Chandra framework that real groups may be studied in a complex algebraic context.

In the next section of the introduction below, we give a more detailed description of the contents of this paper. Before continuing, it is worth commenting about the technical nature of quasimaps. What we have called the space Z of meromorphic quasimaps is not in fact a scheme but rather an ind-stack. To the interested but daunted reader, we recommend thinking of Z as a sophisticated version of the loop space of X. In a less complicated world, we would not need to consider Z but could work directly with the loop space of X. Unfortunately, we do not know how to deal with the infinite-dimensionality of this space. Another possible approach would be to study sheaves on the affine Grassmannian of G which are equivariant for the loop group of the generic stabilizer of X. But in general the orbits of such a group are neither finite-dimensional nor finite-codimensional, and we do not know how to make sense of sheaves on them.

Although quasimaps involve many technical challenges, they seem to be a natural model to study the kind of geometry we are interested in. For example, we do not know how to see such structures as the tensor product on $\mathbf{Q}(Z)$ without some global considerations. This is not surprising from the perspective of the geometric Langlands program as a topological field theory [KW06]. Under the geometric Satake correspondence, one can interpret the tensor product on the category of finite-dimensional representations of the dual group $\check{G}$ as coming from two distinct sources. On the one hand, there is the usual group multiplication on G, which in turn induces a group structure on the loop group of G. If we consider the geometric Satake category as morphisms between two copies of the vacuum brane on a circle, then the convolution product realizes the composition of morphisms. On the other hand, there is the fusion product on the geometric Satake category coming from a two dimensional pair-of-pants with boundary circles labelled by the vacuum brane. This is an algebraic form of the homotopy group structure on the loop space of G independent of the group structure on G. Now we can interpret sheaves on the loop space of the spherical variety X as a brane with which we can label the circle. Then we can further interpret our category of spherical sheaves $\mathbf{Q}(Z)$ as morphisms between this brane and the vacuum brane. Though such morphisms can not be composed (since they have different source and target), they can be fused: a version of a three dimensional pair-of-pants realizes the fusion product of such morphisms. In this framework, the global nature of the fusion product on $\mathbf{Q}(Z)$ is not an artifact of our technical approach but evidence of the topological field theory structures underlying our constructions.

1.2. **Summary.** We now turn to a more detailed description of the specific contents of this paper.

Let C be a smooth complete complex algebraic curve, and let Bun_G be the moduli stack of G-bundles on C. For a finite set I, we write C^I for the product of I copies of C, and for a point $c_I \in C^I$, we write $|c_I| \subset C$ for the union of the points $c_i \in C$, for $i \in I$.

Let X be an affine spherical G-variety, and let $\mathring{X} \subset X$ be the dense G-orbit. Fix a point $x \in \mathring{X}$, and let $S \subset G$ be the stabilizer of x, so we have $\mathring{X} \simeq G/S$. We define the ind-stack Z_I of *meromorphic quasimaps* to be that classifying the data

$$(c_I \in C^I, \mathcal{P}_G \in \mathrm{Bun}_G, \sigma : C \setminus |c_I| \to \mathcal{P}_G \overset{G}{\times} X|_{C \setminus |c_I|})$$

where σ is a section which factors

$$\sigma|_{C'} : C' \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C'} \to \mathcal{P}_G \overset{G}{\times} X|_{C'},$$

for some open curve $C' \subset C$. We call the subset $|c_I| \subset C_{\mathbb{S}}$ the pole points of the quasimap. We call the largest open curve $C' \subset C$ on which σ factors

$$\sigma|_{C'} : C' \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C'} \to \mathcal{P}_G \overset{G}{\times} X|_{C'}$$

the nondegeneracy locus of the quasimap, and we call its complement $C \setminus C'$ the degeneracy locus of the quasimap.

Over the nondegeneracy locus $C' \subset C$, the section σ defines a reduction of the G-bundle $\mathcal{P}_G$ to an S-bundle $\mathcal{P}_S$. We refer to $\mathcal{P}_S$ as the generic S-bundle associated to the quasimap. We refer to the $\pi_0(S)$-bundle induced from $\mathcal{P}_S$ as the generic $\pi_0(S)$-bundle associated to the quasimap. We call the quasimap untwisted if its associated generic $\pi_0(S)$-bundle is trivial.

We write $'Z_I \subset Z_I$ for the ind-closed substack of untwisted meromorphic quasimaps. When I has one element, we write Z in place of Z_I, and $'Z$ in place of $'Z_I$.

Let $\mathrm{Sh}(Z)$ be the bounded constructible derived category of sheaves of $\mathbb{C}$-modules on Z, and let $\mathbf{P}(Z)$ be the full abelian subcategory of perverse sheaves on Z. We have two natural operations on $\mathrm{Sh}(Z)$ coming from Hecke correspondences.

First, modifications of the generic S-bundle of a quasimap provide correspondences which act on $\mathrm{Sh}(Z)$. We refer to these as generic Hecke correspondences, and to their collection as the generic Hecke action. We define a *Hecke equivariant* perverse sheaf on Z to be an object of $\mathbf{P}(Z)$ equipped with a collection of isomorphisms for smooth generic Hecke correspondences. We write $\mathbf{P}_{\mathcal{H}}(Z)$ for the abelian category of Hecke equivariant perverse sheaves on Z.

Second, modifications of the G-bundle of a quasimap at its pole point also provide correspondences which act on $\mathrm{Sh}(Z)$. We refer to these as meromorphic Hecke correspondences, and to their collection as the meromorphic Hecke action. To describe the types of meromorphic Hecke correspondences which we consider, we introduce some more notation. Let $\mathcal{K} = \mathbb{C}((t))$ be the field of formal Laurent series, and let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series. Let $G(\mathcal{K})$ be the $\mathcal{K}$-valued points of G, and let $G(\mathcal{O})$ be its $\mathcal{O}$-valued points. The affine Grassmannian Gr_G is an ind-scheme whose set of $\mathbb{C}$-points is naturally the quotient $G(\mathcal{K})/G(\mathcal{O})$. The types of modifications of a G-bundle at a point are given by the $G(\mathcal{O})$-orbits in Gr_G. The geometric Satake correspondence [MV04] states that the category $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G is a tensor category equivalent to the category of finite-dimensional representations of the dual group $\check{G}$. In this paper, we consider the meromorphic Hecke correspondences whose integral kernels are given by objects of $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.

Our aim is to study a certain semisimple category $\mathbf{Q}(Z)$ of Hecke equivariant perverse sheaves on Z. To construct $\mathbf{Q}(Z)$, we define the *untwisted basic stratum* $'Z^0 \subset Z$ to consist of those untwisted quasimaps of the form

$$(c \in C, \mathcal{P}_G \in \text{Bun}_G, \sigma : C \to \mathcal{P}_G \overset{G}{\times} \overset{\circ}{X}).$$

In other words, $'Z^0 \subset Z$ consists of those quasimaps whose nondegeneracy locus is all of C, and whose associated generic $\pi_0(S)$-bundle is trivial. Note that if the nondegeneracy locus of a quasimap is all of C, then its associated generic S-bundle is defined over all of C. Thus the product of the curve C with the moduli stack Bun_{S^0} for the identity component $S^0 \subset S$ is naturally a $\pi_0(S)$-torsor over the basic stratum $'Z^0 \subset Z$.

The intersection cohomology sheaf $'\text{IC}_Z^0$ of the closure $'\overline{Z}^0 \subset Z$ of the untwisted basic stratum is naturally a Hecke equivariant perverse sheaf. By acting on it by meromorphic Hecke correspondences and taking perverse cohomology sheaves, we obtain a semisimple functor

$$\text{Conv} : \mathbf{P}_{G(\mathcal{O})}(\text{Gr}_G) \to \mathbf{P}_{\mathcal{H}}(Z)$$

which we call   convolution.   We define $\mathbf{Q}(Z)$ to be the strict full subcategory of $\mathbf{P}_{\mathcal{H}}(Z)$ whose objects are isomorphic to subquotients of those Hecke equivariant perverse sheaves arising via convolution.

We prove that the irreducible objects of $\mathbf{Q}(Z)$ are isomorphic to the intersection cohomology sheaves with constant Hecke equivariant coefficients of certain substacks of Z. To identify the substacks which may occur, let $\mathcal{V}(X)$ be the cone of G-invariant discrete valuations of the function field of X. Thanks to Luna-Vust [LV83], we have a canonical parametrization

$$G(\mathcal{O})\backslash \overset{\circ}{X}(\mathcal{K}) \simeq \mathcal{V}(X)$$

which is invariant under automorphisms of the ring $\mathcal{O}$. Since all G-bundles on the formal disk are trivial, this bijection may also be thought of as parameterizing meromorphic quasimaps on the formal disk. For a valuation $\theta \in \mathcal{V}(X)$, we write $\overset{\circ}{X}(\mathcal{K})^\theta \subset \overset{\circ}{X}(\mathcal{K})$ for the formal loops of type θ. We define the untwisted local stratum $'Z^\theta \subset Z$ to consist of untwisted quasimaps of the form

$$(c \in C, \mathcal{P}_G \in \text{Bun}_G, \sigma : C \setminus c \to \mathcal{P}_G \overset{G}{\times} \overset{\circ}{X}|_{C\setminus c})$$

which when restricted to the formal neighborhood D_c of the pole point $c \in C$ are represented by an element of $\overset{\circ}{X}(\mathcal{K})^\theta$ for any trivialization of the restriction of $\mathcal{P}_G$ to D_c and any identification of D_c with the abstract formal disk.

Using direct geometric methods, we prove the following about the convolution.

Theorem 1.2.1. *Every irreducible object of $\mathbf{Q}(Z)$ is isomorphic to the intersection cohomology sheaf of an untwisted local stratum with constant Hecke equivariant coefficients.*

The results of this paper and those of [GN04a] restrict which untwisted local strata support irreducible objects of $\mathbf{Q}(Z)$ to those indexed by a certain finite-index subsemigroup of $\mathcal{V}(X)$. One expects that all of the strata indexed by this subsemigroup in fact support objects of $\mathbf{Q}(Z)$.

The main result of this paper is the following.

Theorem 1.2.2. *The category $\mathbf{Q}(Z)$ is naturally a tensor category equivalent to the category of finite-dimensional representations of a connected reductive complex algebraic subgroup $\check{H} \subset \check{G}$.*

To prove this, we use Tannakian formalism. We first construct a fusion product on $\mathbf{Q}(Z)$ making it a tensor category. To do this, we consider the analogous category $\mathbf{Q}(Z_I)$ for the ind-stack Z_I of meromorphic quasimaps with more than a single pole point. We show that there is a canonical equivalence

$$\gamma_I : \mathbf{Q}(Z)^{\otimes I} \xrightarrow{\sim} \mathbf{Q}(Z_I).$$

The tensor product is then defined by allowing the pole points of Z_I to collide and taking nearby cycles. The convolution functor is naturally a tensor functor with respect to the fusion product on $\mathbf{Q}(Z)$ and the tensor product on $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$. Under Tannakian formalism, it corresponds to the restriction of representations from $\check{G}$.

To apply Tannakian formalism, we need a fiber functor on $\mathbf{Q}(Z)$. This is an exact faithful tensor functor from $\mathbf{Q}(Z)$ to the category of finite-dimensional vector spaces. What we construct is an exact faithful tensor functor to a category of vector spaces graded by a lattice. By forgetting the grading, we obtain the desired fiber functor. Under Tannakian formalism, the graded fiber functor corresponds to the restriction of representations from $\check{H}$ to a maximal torus.

To describe some of the geometry involved in constructing the graded fiber functor, we consider the case when X is an affine horospherical G-variety. From the results of [GN04a], we deduce that in this case the category $\mathbf{Q}(Z)$ is equivalent to a category of finite-dimensional vector spaces graded by a lattice. In other words, it is equivalent to the category of finite-dimensional representations of a complex algebraic torus.

Now for a general affine spherical G-variety X, to construct the graded fiber functor on $\mathbf{Q}(Z)$, we work with a family $\mathfrak{X}$ of G-varieties whose general fiber is canonically isomorphic to X and whose special fiber is an affine horospherical G-variety X_0. The family $\mathfrak{X}$ is defined by filtering the ring of regular functions on X according to the action of G. The ring of regular functions on X_0 is the associated graded of this filtration. From the family $\mathfrak{X}$, we obtain a family $\mathfrak{Z}$ of ind-stacks whose general fiber is canonically isomorphic to Z and whose special fiber is the ind-stack Z_0 of meromorphic quasimaps into X_0. The nearby cycles in the family $\mathfrak{Z}$ provide a functor

$$\psi : \mathbf{Q}(Z) \to \mathbf{P}_{\mathcal{H}}(Z_0).$$

Unfortunately applying ψ to objects of $\mathbf{Q}(Z)$ does not necessarily produce objects of $\mathbf{Q}(Z_0)$. Instead, it may produce a complicated object whose composition series contains an object of $\mathbf{Q}(Z_0)$ but also other "noise". The presence of this noise reflects the fact that by working with Z, rather than some local object, we have introduced global complications. To deal with this, we work in a certain quotient category of $\mathbf{P}_{\mathcal{H}}(Z_0)$ where we ignore the noise.

We call an object of $\mathbf{P}_{\mathcal{H}}(Z_0)$ a *bad* sheaf if each of its simple constituents is not an object of $\mathbf{Q}(Z_0)$. We denote by $\mathrm{Bad}(Z_0)$ the full Serre subcategory of $\mathbf{P}_{\mathcal{H}}(Z_0)$ whose objects are bad sheaves. We denote by $\mathrm{Quot}(Z_0)$ the quotient of the category $\mathbf{P}_{\mathcal{H}}(Z_0)$ by the Serre subcategory $\mathrm{Bad}(Z_0)$. We denote by $\mathrm{Subquot}(Z_0)$ the full image

of $\mathbf{Q}(Z_0)$ under the natural projection to $\mathrm{Quot}(Z_0)$. The natural projection provides an equivalence

$$\mathbf{Q}(Z_0) \xrightarrow{\sim} \mathrm{Subquot}(Z_0).$$

The main technical result needed to proceed is the following.

Theorem 1.2.3. *For any object $\mathcal{P} \in \mathbf{Q}(Z)$, the image of the nearby cycles $\psi(\mathcal{P})$ in the quotient category $\mathrm{Quot}(Z_0)$ belongs to the full subcategory $\mathrm{Subquot}(Z_0)$. Thus the nearby cycles lift to a functor*

$$\Psi : \mathbf{Q}(Z) \to \mathbf{Q}(Z_0).$$

We take the functor Ψ to be our graded fiber functor. We must show that Ψ is an exact faithful tensor functor. The only part of this which is not easily deduced from previous results is that it is a tensor functor. To see this, we must consider its interaction with the fusion product. This involves showing that the nearby cycles in a certain family over a base of dimension greater than one are independent of any choices. To see this, we exploit the fact that we know the assertion for objects which come from convolution. This allows us to reinterpret the fusion product as the fiber of a middle-extension, rather than as nearby cycles. The characterizing properties of the middle-extension imply the independence of the choices involved in working over a base of dimension greater than one.

What remains to establish in order to invoke Tannakian formalism comes for free. In [Nad05], a brief Tannakian dictionary was collected which contains the results we need. For example, at this point, the rigidity of $\mathbf{Q}(Z)$ is automatic. Furthermore, the assertions that the associated group $\check{H}$ are connected and reductive are immediate from our explicit understanding of $\mathbf{Q}(Z)$.

Part I. Main constructions

In this part, we will carry out a series of constructions that lead to the definition of the group $\check{H}$. Most of the technical assertions will be stated without proofs; the latter will be given in Parts II, III and IV.

The structure of this part is as follows:

In Sect. 2, we introduce the space Z of quasimaps from a curve C into an affine variety X on which G acts with an open orbit. The space Z may be thought of as a finite-dimensional model of the space of loops $X(\mathcal{K})$.

We then introduce the category of generic-Hecke equivariant perverse sheaves. This may be thought of as a model for the technically inaccessible "category of $G(\mathcal{O})$-equivariant perverse sheaves on $X(\mathcal{K})$".

In Sect. 3, we study the space Z of quasimaps under the assumption that the G-variety X is spherical.

We first recall some basic facts about spherical G-varieties, and discuss the stratification of $X(\mathcal{K})$ by $G(\mathcal{O})$-orbits. We then use this to describe a (partial) stratification of Z.

At the end of the section, we specialize further to the case when X is horospherical, and discuss the stratification of Z in this situation.

In Sect. 4, we introduce the convolution action of the monoidal category of spherical perverse sheaves on the affine Grassmannian of G on the category of perverse sheaves on Z.

This leads to the definition of the category $\mathbf{Q}(Z)$ that will eventually be shown to be equivalent to the category of representations of a reductive subgroup $\check{H} \subset \check{G}$.

In Sect. 5, we construct a functor from the category $\mathbf{Q}(Z)$ to a similar category on the space Z_0, where the latter is the space of quasimaps into a canonical horospherical *collapse* of X. The construction involves taking nearby cycles of objects of $\mathbf{Q}(Z)$, and then projecting to a suitable quotient category equivalent to $\mathbf{Q}(Z_0)$.

We use this functor in Sect. 7 in order to construct a fiber functor on $\mathbf{Q}(Z)$. The main idea is that the corresponding category $\mathbf{Q}(Z_0)$ is equivalent to that of representations of a torus, and hence admits a canonical forgetful functor to vector spaces.

In Sect. 6, we endow $\mathbf{Q}(Z)$ with a tensor structure using the idea of fusing pole points.

Finally, in Sect. 7, we show that the existing structures on $\mathbf{Q}(Z)$ define a tensor equivalence between it and the category of representations of a reductive subgroup $\check{H} \subset \check{G}$.

We identify the maximal torus of $\check{H}$ and state a conjecture describing the cone formed by its dominant weights.

2. Quasimaps

2.1. Definition. Let C be a smooth complete complex algebraic curve. For a scheme $\mathcal{S}$, we write $C_{\mathcal{S}}$ for the product $\mathcal{S} \times C$, and for an $\mathcal{S}$-point $c \in C(\mathcal{S})$, we also write $c \subset C_{\mathcal{S}}$ for its graph. Let Bun_G be the moduli stack of G-bundles on C. It represents the functor which assigns to a scheme $\mathcal{S}$ the category of G-bundles on $C_{\mathcal{S}}$. For a finite set I, we write C^I for the product of I copies of C. For a scheme $\mathcal{S}$, and a point $c_I \in C^I(\mathcal{S})$, we write $|c_I| \subset C_{\mathcal{S}}$ for the union of the graphs $c_i \subset C_{\mathcal{S}}$.

Let X be an affine G-variety which we assume to have a dense G-orbit $\mathring{X} \subset X$. Fix a point $x \in \mathring{X}$, and let $S \subset G$ be the stabilizer of x, so we have $\mathring{X} \simeq G/S$. We define the ind-stack Z_I of *meromorphic quasimaps* to be that representing the functor which assigns to a scheme $\mathcal{S}$ the category of data

$$(c_I \in C^I(\mathcal{S}), \mathcal{P}_G \in \mathrm{Bun}_G(\mathcal{S}), \sigma : C_{\mathcal{S}} \setminus |c_I| \to \mathcal{P}_G \overset{G}{\times} X|_{C_{\mathcal{S}} \setminus |c_I|})$$

where σ is a section which factors

$$\sigma|_{C'_{\mathcal{S}}} : C'_{\mathcal{S}} \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C'_{\mathcal{S}}} \to \mathcal{P}_G \overset{G}{\times} X|_{C'_{\mathcal{S}}},$$

for some open subscheme $C'_{\mathcal{S}} \subset C_{\mathcal{S}}$ which is the complement $C'_{\mathcal{S}} = C_{\mathcal{S}} \setminus \mathcal{D}$ of a subscheme $\mathcal{D} \subset C_{\mathcal{S}}$ which is finite and flat over $\mathcal{S}$. We call the subscheme $|c_I| \subset C_{\mathcal{S}}$ the pole points of the quasimap. We call the largest subscheme $C'_{\mathcal{S}} \subset C_{\mathcal{S}}$ on which σ factors

$$\sigma|_{C'_{\mathcal{S}}} : C'_{\mathcal{S}} \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C'_{\mathcal{S}}} \to \mathcal{P}_G \overset{G}{\times} X|_{C'_{\mathcal{S}}}$$

the nondegeneracy locus of the quasimap, and we call its complement $C_{\mathcal{S}} \setminus C'_{\mathcal{S}}$ the degeneracy locus of the quasimap.

Over the nondegeneracy locus $C'_{\mathcal{S}} \subset C_{\mathcal{S}}$, the section σ defines a reduction of the G-bundle $\mathcal{P}_G$ to an S-bundle $\mathcal{P}_S$. We refer to $\mathcal{P}_S$ as the generic S-bundle associated to the quasimap. We refer to the $\pi_0(S)$-bundle induced from $\mathcal{P}_S$ as the generic $\pi_0(S)$-bundle associated to the quasimap. We call the quasimap untwisted if for every geometric point $s \in \mathcal{S}$, the restriction of the associated generic $\pi_0(S)$-bundle to $\{s\} \times C \subset C_{\mathcal{S}}$ is trivial.

When I has one element, we write Z in place of Z_I. We write $'Z_I \subset Z_I$ for the ind-substack of untwisted meromorphic quasimaps.

The following lemma will be proved in Sect. 9.

Lemma 2.1.1. *The ind-substack $'Z_I \subset Z_I$ is closed.*

We shall also have frequent use for the following notations. In general, for an ind-stack of quasimaps, we add the mark $'$ to the notation to signify the ind-closed substack of untwisted quasimaps. We write $\mathring{C}^I \subset C^I$ for the variety of those $c_I \in C^I$ such that c_i is distinct from c_j, for distinct $i, j \in I$. We write $Z_{\mathring{I}} \subset Z_I$ for the ind-open substack of meromorphic quasimaps with distinct pole points, or in other words, the fiber product

of Z_I and C^I over $\mathring{C}^I$. In general, for an ind-stack over C^I, we add the mark $\circ$ to the notation to signify the fiber product with $\mathring{C}^I \subset C^I$.

In the remainder of this paper, when defining various schemes and stacks, we often present the moduli problem for geometric points and leave it to the reader to extend it to an arbitrary base.

2.2. Generic-Hecke equivariance.

In what follows, we write $C^{(n)}$ for the nth symmetric power of C. For a point $c_{(n)} \in C^{(n)}$, we write $|c_{(n)}| \subset C$ for its support in C, and we write $\|c_{(n)}\| \subset C$ for the finite (not necessarily reduced) subscheme it defines.

We define the ind-stack $\mathcal{H}_{Z_I,(n)}$ of *generic-Hecke modifications* to be that classifying data

$$(c_I, \mathcal{P}_G^1, \mathcal{P}_G^2, \sigma_1, \sigma_2; c_{(n)}, \alpha)$$

where $(c_I, \mathcal{P}_G^i, \sigma_i) \in Z_I$, $c_{(n)} \in C^{(n)}$, with $|c_{(n)}|$ disjoint from the degeneracy locus of $(c_I, \mathcal{P}_G^i, \sigma_i)$, and α is an isomorphism of G-bundles

$$\alpha : \mathcal{P}_G^1|_{C\backslash|c_{(n)}|} \xrightarrow{\sim} \mathcal{P}_G^2|_{C\backslash|c_{(n)}|}$$

such that the following diagram commutes

$$
\begin{array}{ccc}
C \setminus (|c_I| \cup |c_{(n)}|) & \xrightarrow{\sigma_1} & \mathcal{P}_G^1 \overset{G}{\times} X|_{C\backslash(|c_I|\cup|c_{(n)}|)} \\
\downarrow & & \downarrow \alpha \\
C \setminus (|c_I| \cup |c_{(n)}|) & \xrightarrow{\sigma_2} & \mathcal{P}_G^2 \overset{G}{\times} X|_{C\backslash(|c_I|\cup|c_{(n)}|)}
\end{array}
$$

where the left vertical map is the identity, We call a generic-Hecke modification *trivial* if the isomorphism α extends to an isomorphism over all of C. We have the natural projections

$$Z_I \xleftarrow{h^\leftarrow} \mathcal{H}_{Z_I,(n)} \xrightarrow{h^\rightarrow} Z_I,$$

and the natural projection

$$\mathcal{H}_{Z_I,(n)} \to C^I \times C^{(n)}.$$

We define a *smooth generic-Hecke correspondence* to be any stack Y equipped with smooth maps

$$Z_I \xleftarrow{h_Y^\leftarrow} Y \xrightarrow{h_Y^\rightarrow} Z_I$$

such that for some n, there exists a map

$$Y \to \mathcal{H}_{Z_I,(n)}$$

such that the following diagram commutes

$$
\begin{array}{ccccc}
Z_I & \xleftarrow{h_Y^\leftarrow} & Y & \xrightarrow{h_Y^\rightarrow} & Z_I \\
\downarrow & & \downarrow & & \downarrow \\
Z_I & \xleftarrow{h^\leftarrow} & \mathcal{H}_{Z_I,(n)} & \xrightarrow{h^\rightarrow} & Z_I
\end{array}
$$

where the outer vertical maps are the identity. We call a smooth generic-Hecke correspondence Y *trivial* if the image of such a map

$$Y \to \mathcal{H}_{Z_I,(n)}$$

consists of trivial generic-Hecke modifications. We define a morphism of smooth generic-Hecke correspondences to be a map

$$p : Y_1 \to Y_2$$

such that the following diagram commutes

$$
\begin{array}{ccccc}
Z_I & \overset{h_{Y_1}^{\leftarrow}}{\leftarrow} & Y_1 & \overset{h_{Y_1}^{\rightarrow}}{\to} & Z_I \\
\downarrow & & \downarrow & & \downarrow \\
Z_I & \overset{h_{Y_2}^{\leftarrow}}{\leftarrow} & Y_2 & \overset{h_{Y_2}^{\rightarrow}}{\to} & Z_I
\end{array}
$$

where the outer vertical maps are the identity.

2.3. Generic-Hecke equivariant sheaves.

We define a *generic-Hecke equivariant* perverse sheaf on Z_I to be an object $\mathcal{F} \in \mathbf{P}(Z_I)$ of the category of perverse sheaves on Z_I equipped with isomorphisms

$$I_Y : h_Y^{\leftarrow *}(\mathcal{F}) \overset{\sim}{\to} h_Y^{\rightarrow *}(\mathcal{F}),$$

for every smooth generic-Hecke correspondence Y, satisfying the following conditions. First, for any morphism of smooth generic-Hecke correspondences

$$p : Y_1 \to Y_2,$$

we require that

$$I_{Y_1} = p^*(I_{Y_2}).$$

Second, consider any stack Y, and for $i \in \mathbb{Z}/3\mathbb{Z}$, consider smooth generic-Hecke correspondences Y_i, and maps

$$p_i : Y \to Y_i$$

such that the compositions

$$h_{Y_i}^{\leftarrow} \circ p_i \qquad h_{Y_i}^{\rightarrow} \circ p_i$$

are smooth and the following diagrams commute

$$
\begin{array}{ccc}
Y & \overset{p_{i+1}}{\to} & Y_{i+1} \\
p_i \downarrow & & \downarrow h_{Y_{i+1}}^{\leftarrow} \\
Y_i & \overset{h_{Y_i}^{\rightarrow}}{\to} & Z_I.
\end{array}
$$

Then we require that the composite isomorphism

$$p_3^*(I_{Y_3}) \circ p_2^*(I_{Y_2}) \circ p_1^*(I_{Y_1})$$

be the identity morphism. Finally, for any trivial smooth generic-Hecke correspondence Y, we require that the isomorphism I_Y be the identity morphism.

Observe that a Hecke equivariant structure is determined by its values on substacks of the ind-stack $\mathcal{H}_{Z_I,(1)}$ of generic-Hecke modifications at a single point.

Perverse sheaves on Z_I, equipped with a generic-Hecke equivariant structure naturally form a category, which we will denote by $\mathbf{P}_{\mathcal{H}}(Z_I)$.

Observe that the pre-images in $\mathcal{H}_{Z_I,(n)}$ of the untwisted locus $'Z_I$ under the projections $h^{\leftarrow}$ and $h^{\rightarrow}$ coincide. Hence we also may introduce the category $\mathbf{P}_{\mathcal{H}}('Z_I)$.

We will denote by $\mathbf{P}_{\mathcal{H}}(Z_{\mathring{I}})$ and $\mathbf{P}_{\mathcal{H}}('Z_{\mathring{I}})$ the versions of the above categories attached to the locus of distinct pole points $Z_{\mathring{I}}$.

3. STRATIFICATIONS

3.1. **Spherical varieties.** From now on we will assume that X is a *spherical G-variety*.

Definition 3.1.1. *X is said to be* spherical *if a Borel subgroup of G acts on X with a dense orbit.*

For the remainder of this paper, with the exception of Sects. 3.2 and 8.2 we will assume that X is affine. Then the above definition can be rephrased in terms of the action of G on the ring of regular functions $\mathbb{C}[X]$. As a representation of G, it decomposes into a direct sum of isotypic components

$$\mathbb{C}[X] \simeq \sum_{\check{\lambda} \in \check{\Lambda}_G^+} \mathbb{C}[X]_\lambda.$$

Definition 3.1.2. *$\mathbb{C}[X]$ is said to be* simple *if each $\mathbb{C}[X]_\lambda$ is an irreducible G-representation.*

In other words, X is simple if the multiplicity of each irreducible V^λ in $\mathbb{C}[X]$ is not greater than 1.

Proposition 3.1.3. [Pop86, Theorem 1] *X is spherical if and only if $\mathbb{C}[X]$ is simple.*

We recall below some well-known structure theory of spherical varieties. The material here is largely borrowed from [BLV86].

3.1.4. *The associated tori.* Consider the subset $\check{\Lambda}_X^+ \subset \check{\Lambda}_T$ of dominant weights $\lambda \in \check{\Lambda}_G^+$ such that $\mathbb{C}[X]_\lambda$ is nonzero. One shows that this is in fact a sub-semigroup. Let $\check{\Lambda}_X \subset \check{\Lambda}_T$ be the lattice generated by $\check{\Lambda}_X^+$. Consider the torus

$$A := \mathrm{Spec}(\mathbb{C}[\check{\Lambda}_X]).$$

Thus the coweight lattice Λ_A of A is the dual of $\check{\Lambda}_X$. We define the subsemigroup $\Lambda_X^{\mathrm{pos}} \subset \Lambda_A$ to consist of those coweights which are nonnegative on $\check{\Lambda}_X^+ \subset \check{\Lambda}_A$.

Lemma 3.1.5. *The semigroup $\Lambda_X^{\mathrm{pos}} \subset \Lambda_A$ is strictly convex.*

Proof. By definition, the semigroup $\check{\Lambda}_X^+$ is of full rank in the lattice $\check{\Lambda}_X$. $\qquad\square$

Consider the saturation $\mathrm{Sat}(\check{\Lambda}_X) \subset \check{\Lambda}_T$ of the lattice $\check{\Lambda}_X \subset \check{\Lambda}_T$, and the torus

$$A_0 := \mathrm{Spec}(\mathbb{C}[\mathrm{Sat}(\check{\Lambda}_X)]).$$

We let F denote the kernel

$$1 \to F \to A_0 \to A \to 1,$$

or equivalently the cokernel

$$0 \to \Lambda_{A_0} \to \Lambda_A \to F \to 0.$$

Remark 3.1.6. *The tori A and A_0 can be associated to any (i.e., not necessarily affine) spherical variety, and they only depend on the homogeneous space $\mathring{X} = G/S$. This more general theory will be reviewed in Sect. 8.2.*

3.1.7. *The associated parabolic.* Choose a Borel subgroup $B^{op} \subset G$, and let $\mathring{X}^+ \subset \mathring{X} \subset X$ be the corresponding open B^{op}-orbit. Let $P^{op} \subset G$ be the parabolic, consisting of elements $g \in G$ such that $g \cdot \mathring{X}^+ \subset \mathring{X}^+$, and let $U^{op} \subset P^{op}$ be its unipotent radical.

The choice of B^{op} defines highest weight lines $\mathfrak{l}^\lambda \subset \mathbb{C}[X]_\lambda$.

Lemma 3.1.8.

(1) For $\lambda \in \check{\Lambda}_X^+$, the line $\mathfrak{l}^\lambda \subset V^\lambda$ is P^{op}-stable.

(2) The open subvariety $\mathring{X}^+ \subset X$ is the locus of non-vanishing of the lines $\mathfrak{l}^\lambda$.

Thus we see that A is naturally a quotient of P^{op}. Moreover, the projection $P^{op} \to A$ factors as surjections
$$P^{op} \twoheadrightarrow A_0 \twoheadrightarrow A,$$
where the first arrow has a connected kernel. Furthermore, we have
$$\mathbb{C}[X]^{U^{op}} \simeq \mathbb{C}[\check{\Lambda}_X^+].$$
Setting
$$\overline{A} := \operatorname{Spec}(\mathbb{C}[\check{\Lambda}_X^+]),$$
we have a Cartesian diagram

(3.1)
$$\begin{array}{ccc} \mathring{X}^+ & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\ A & \longrightarrow & \overline{A}, \end{array}$$

where the vertical maps are P^{op}-equivariant, and the horizontal ones are open embeddings.

Lemma 3.1.9. *The action of U^{op} on $\mathring{X}^+$ is free, i.e., $\mathring{X}^+$ is a principal U^{op}-bundle over A.*

3.1.10. *The finite groups.* By construction, we have a map of stacks $\mathring{X}^+/P^{op} \to \mathring{X}/G$, which induces a map on the level of the corresponding fundamental groups
$$F \simeq \pi_1(\mathring{X}^+/P^{op}) \simeq \pi_1(\mathring{X}/G) \simeq \pi_0(S).$$
Since $\mathring{X}^+ \to \mathring{X}$ is an open embedding, we obtain that the above map is surjective. In particular, we deduce that $\pi_0(S)$ is abelian.

3.2. **Stratification of loops.** As usual, let $\mathcal{K} = \mathbb{C}((t))$ be the field of formal Laurent series, and let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series. Let $G(\mathcal{K})$ be the group of $\mathcal{K}$-valued points of G, and let $G(\mathcal{O})$ be the group of $\mathcal{O}$-valued points of G. The affine Grassmannian Gr_G is an ind-scheme whose set of $\mathbb{C}$-points is naturally the quotient $G(\mathcal{K})/G(\mathcal{O})$.

The following result, which essentially follows from [LV83], will be proved in Sects. 8.2 and 8.3.

Theorem 3.2.1. *For a subgroup $S \subset G$ the following conditions are equivalent:*

(1) The quotient G/S is a spherical G-variety.

(2) The group $S(\mathcal{K})$ acts on Gr_G with countably many orbits.

(3) The group $G(\mathcal{O})$ acts on $(G/S)(\mathcal{K})$ with countably many orbits.

In what follows for $S \subset G$ satisfying the equivalent conditions of the theorem, we will denote by $\mathcal{V}(G/S)$ the set of $G(\mathcal{O})$-orbits on $(G/S)(\mathcal{K})$. (We will see during the proof of Theorem 3.3.1 below in Sect. 8.2 that $\mathcal{V}(G/S)$ coincides with the cone of G-invariant discrete valuations of the function field of X.)

Remark 3.2.2. *Condition (3) in the Theorem tautologically implies condition (2). The inverse implication is less evident, since not every $\mathcal{K}$-point of G/S lifts to a $\mathcal{K}$-point of G, because S may be disconnected.*

3.3. **Description of orbits.** For the remainder of this section we let X be an *affine, spherical G-variety*. Recall that $\mathring{X} \subset X$ denotes an open G-orbit, which can be identified with the quotient G/S. The next result that we are going to state identifies the set of $G(\mathcal{O})$ orbits on $\mathring{X}(\mathcal{K})$ with a subset of the lattice Λ_A.

Note that $X(\mathcal{K})$ is naturally (the set of $\mathbb{C}$-points) of an ind-affine ind-scheme. It contains an open subscheme $X(\mathcal{K}) \setminus (X \setminus \mathring{X})(\mathcal{K})$, whose set of $\mathbb{C}$-points identifies with the set of $\mathcal{K}$-points of $\mathring{X}$.[3]

Similarly, $\overline{A}(\mathcal{K})$ is naturally an ind-affine ind-scheme, which contains an open subscheme $\overline{A}(\mathcal{K}) \setminus (\overline{A} \setminus A)(\mathcal{K})$. The set of $A(\mathcal{O})$-orbits on $\overline{A}(\mathcal{K}) \setminus (\overline{A} \setminus A)(\mathcal{K})$ identifies naturally with Λ_A.[4]

Let $\mathbf{O} \subset X(\mathcal{K}) \setminus (X \setminus \mathring{X})(\mathcal{K})$ be a $G(\mathcal{O})$-orbit. We can view it as a (set of $\mathbb{C}$-points) of a scheme of infinite type. Let $\mathbf{O}^+ \subset \mathbf{O}$ be the open subscheme

$$\mathbf{O} \cap \left(X(\mathcal{K}) \setminus (X \setminus \mathring{X}^+)(\mathcal{K}) \right).$$

It is non-empty, since G/P^{op} is proper, and hence $\mathcal{O}$- and $\mathcal{K}$-points of G/P^{op} are in bijection. We have a Cartesian diagram

$$
\begin{array}{ccc}
\mathbf{O}^+ & \longrightarrow & \mathbf{O} \\
{\scriptstyle p}\downarrow & & {\scriptstyle \overline{p}}\downarrow \\
\overline{A}(\mathcal{K}) \setminus (\overline{A} \setminus A)(\mathcal{K}) & \longrightarrow & A(\mathcal{K}),
\end{array}
$$

induced by (3.1).

Since $\mathbf{O}^+$ is irreducible, its image under p is contained in the closure of a single $A(\mathcal{O})$-orbit on $\overline{A}(\mathcal{K}) \setminus (\overline{A} \setminus A)(\mathcal{K})$. Thus, we obtain a map

$$\upsilon : \mathcal{V}(G/S) \simeq \left(X(\mathcal{K}) \setminus (X \setminus \mathring{X})(\mathcal{K}) \right)/G(\mathcal{O}) \to \left(\overline{A}(\mathcal{K}) \setminus (\overline{A} \setminus A)(\mathcal{K}) \right)/A(\mathcal{O}) \simeq \Lambda_A.$$

The following reformulation of a result of [LV83] will be proved in Sects. 8.2 and 8.3 using their compactification theory of spherical varieties.

---

[3]Note that since $\mathring{X}$ may not be affine, the set of $\mathcal{K}$-points of $\mathring{X}$ does not a priori have an ind-scheme structure. Even when $\mathring{X}$ is affine, $\mathring{X}(\mathcal{K})$ is *not* isomorphic to $X(\mathcal{K}) \setminus (X \setminus \mathring{X})(\mathcal{K})$.

[4]An orbit corresponding to $\lambda_1 \in \Lambda_A$ is contined in the closure of the orbit corresponding to λ_2 if and only if $\check{\lambda}_1 - \check{\lambda}_2 \in \Lambda_X^{\text{pos}}$.

Theorem 3.3.1.

(1) The map υ defines a bijection between $\mathcal{V}(G/S)$ and a finitely generated saturated subsemigroup of full rank in Λ_A.

(2) An orbit $\mathbf{O}$ is contained in $X(\mathcal{O})$ if and only if $\upsilon(\mathbf{O}) \in \Lambda_X^{\mathrm{pos}}$.

Remark 3.3.2. *As was remarked above, the lattice Λ_A is naturally associated to the homogeneous space $\overset{\circ}{X} = G/S$, i.e., the additional data of the affine variety X containing $\overset{\circ}{X}$ is, in fact, redundant. As will become clear in Sect. 8.3, the map υ also depends only on $\overset{\circ}{X}$.*

By contrast, the subsemigroup $\Lambda_X^{\mathrm{pos}} \subset \Lambda_A$ does depend on X.

Let $\mathrm{Aut}(\mathcal{O})$ be the group-scheme of automorphisms of $\mathcal{O}$. It naturally acts on $X(\mathcal{K})$.

Corollary 3.3.3. *The orbits of $G(\mathcal{O})$ on $X(\mathcal{K}) \setminus (X \setminus \overset{\circ}{X})(\mathcal{K})$ are stable under the $\mathrm{Aut}(\mathcal{O})$-action.*

3.4. **Stratification of quasimaps.** We content ourselves here with defining the strata of the ind-stack Z_I of meromorphic quasimaps into an affine spherical G-variety X which play a role in what follows. The interested reader will be able to use the results of the previous sections to extend the definitions given here and give a complete stratification of Z_I. Recall that we write $\overset{\circ}{X} \simeq G/S \subset X$ for the dense G-orbit in X.

To a quasimap $(c_I, \mathcal{P}_G, \sigma) \in Z_I$, and a point $c \in C$, we may associate an element $\bar{\sigma}_c \in \mathcal{V}(G/S)$ as follows. First, if we fix an isomorphism of the restriction $\mathcal{P}_G|\mathcal{O}_c$ with the trivial bundle $\mathcal{O}_c \times G$, then the restriction $\sigma|\mathcal{K}_c$ may be thought of as a loop $\sigma_c \in \overset{\circ}{X}(\mathcal{K}_c)$. Another choice of trivialization of $\mathcal{P}_G|\mathcal{O}_c$ will lead to another loop in the $G(\mathcal{O}_c)$-orbit in $\overset{\circ}{X}(\mathcal{K}_c)$ through σ_c. Thus we have constructed a well-defined element $\bar{\sigma}_c \in \overset{\circ}{X}(\mathcal{K}_c)/G(\mathcal{O}_c)$. Now, if we fix an identification of the completed local ring $\mathcal{O}_c$ with the power series ring $\mathcal{O}$, then we obtain an element $\bar{\sigma}_c \in \overset{\circ}{X}(\mathcal{K})/G(\mathcal{O})$. By Corollary 3.3.3, the $G(\mathcal{O})$-orbits in $\overset{\circ}{X}(\mathcal{K})$ are invariant under the action of $\mathrm{Aut}(\mathcal{O})$, so this element is well-defined.

We are now ready to define what we call the *local strata* of the ind-stack Z_I of meromorphic quasimaps. For a partition $\mathfrak{p}$ of the set I, and a labelling $\Theta : \mathfrak{p} \to \mathcal{V}(G/S)$, we say that a quasimap $(c_I, \mathcal{P}_G, \sigma) \in Z_I$ is of type $(\mathfrak{p}, \Theta)$ if the following conditions hold. First, the section σ factors

$$\sigma : C \setminus |c_I| \to \mathcal{P}_G \overset{G}{\times} \overset{\circ}{X}|_{C \setminus |c_I|} \to \mathcal{P}_G \overset{G}{\times} X|_{C \setminus |c_I|}.$$

Second, the coincidences among the pole points $|c_I| \subset C$ are given by the partition $\mathfrak{p}$. And third, the $\mathcal{V}(G/S)$-valued labelling of $\mathfrak{p}$ associated to the quasimap is given by Θ. We define the local stratum

$$Z_I^{(\mathfrak{p},\Theta)} \subset Z_I$$

to consist of those quasimaps of type $(\mathfrak{p}, \Theta)$. When $\mathfrak{p}$ is the complete partition of I into singleton parts, we write $Z_{\overset{\circ}{I}}^{\Theta}$ in place of $Z_I^{(\mathfrak{p},\Theta)}$. When I has a single element, the partition is vacuous, the labelling Θ is a single element $\theta \in \Lambda_A^+$, and we write Z^{θ} in place of $Z^{(\mathfrak{p},\Theta)}$.

Note that for $\Theta = 0$, the corresponding stratum $Z_I^{(\mathfrak{p},0)}$ is isomorphic to $\mathring{C}^{\mathfrak{p}} \times \mathrm{Bun}_S$, where $\mathring{C}^{\mathfrak{p}} \subset C^I$ is the locally closed subset determined by the partition $\mathfrak{p}$.

In what follows, we shall be more interested in the untwisted local strata

$$'Z_I^{(\mathfrak{p},\Theta)} \subset {'Z}_I$$

consisting of those untwisted quasimaps of type $(\mathfrak{p}, \Theta)$. As above, when $\mathfrak{p}$ is the complete partition of I into singleton parts, we write $'Z_{\hat{I}}^{\Theta}$ in place of $'Z_I^{(\mathfrak{p},\Theta)}$. When I has a single element, the partition is vacuous, the labelling Θ is a single element $\theta \in \Lambda_A^+$, and we write $'Z^{\theta}$ in place of $'Z^{(\mathfrak{p},\Theta)}$.

We will denote by $\mathrm{IC}_I^{\mathfrak{p},\Theta}$, respectively $'\mathrm{IC}_I^{\mathfrak{p},\Theta}$, $\mathrm{IC}_{\hat{I}}^{\Theta}$, $'\mathrm{IC}_{\hat{I}}^{\Theta}$, IC^{θ}, $'\mathrm{IC}^{\theta}$ the intersection cohomology sheaves on the closures of the respective locally closed subvarieties of the space of quasimaps.

3.5. **Relation to Hecke equivariance.** Fix $c_I \in C^I$, and let $\mathfrak{p}$ be the partition of I describing the coincidences among the points $|c_I| \subset C$. Consider the ind-stack of *based* untwisted meromorphic quasimaps

$$'Z_{c_I} = {'Z}_I \underset{C^I}{\times} \{c_I\},$$

and the corresponding based untwisted local strata

$$'Z_{c_I}^{(\mathfrak{p},\Theta)} = {'Z}_I^{(\mathfrak{p},\Theta)} \underset{C^I}{\times} \{c_I\}.$$

Observe that the generic-Hecke modifications

$$Z_I \overset{h^{\leftarrow}}{\leftarrow} \mathcal{H}_{Z_I,(n)} \overset{h^{\rightarrow}}{\rightarrow} Z_I,$$

preserve the based untwisted local strata $'Z_{c_I}^{(\mathfrak{p},\Theta)}$. In other words, we obtain diagrams

$$\begin{array}{ccccc} 'Z_{c_I}^{(\mathfrak{p},\Theta)} & \overset{h^{\leftarrow}}{\leftarrow} & \mathcal{H}_{'Z_{c_I}^{(\mathfrak{p},\Theta)},(n)} & \overset{h^{\rightarrow}}{\rightarrow} & 'Z_{c_I}^{(\mathfrak{p},\Theta)} \\ \downarrow & & \downarrow & & \downarrow \\ Z_I & \overset{h^{\leftarrow}}{\leftarrow} & \mathcal{H}_{Z_I,(n)} & \overset{h^{\rightarrow}}{\rightarrow} & Z_I \end{array}$$

in which each square is Cartesian.

In particular, it makes sense to introduce the categories $\mathbf{P}_{\mathcal{H}}(Z_I^{\mathfrak{p},\Theta})$, $\mathbf{P}_{\mathcal{H}}('Z_I^{\mathfrak{p},\Theta})$, $\mathbf{P}_{\mathcal{H}}(Z_{\hat{I}}^{\Theta})$, $\mathbf{P}_{\mathcal{H}}('Z_{\hat{I}}^{\Theta})$, $\mathbf{P}_{\mathcal{H}}(Z_{c_I}^{\mathfrak{p},\Theta})$, $\mathbf{P}_{\mathcal{H}}('Z_{c_I}^{\mathfrak{p},\Theta})$, etc. And the following are naturally generic-Hecke equivariant objects:

$$\mathrm{IC}_I^{\mathfrak{p},\Theta} \in \mathbf{P}_{\mathcal{H}}(Z_I), \ \ '\mathrm{IC}_I^{\mathfrak{p},\Theta} \in \mathbf{P}_{\mathcal{H}}('Z_I), \ \ \mathrm{IC}_{\hat{I}}^{\Theta} \in \mathbf{P}_{\mathcal{H}}(Z_{\hat{I}}), \ \ '\mathrm{IC}_{\hat{I}}^{\Theta} \in \mathbf{P}_{\mathcal{H}}('Z_{\hat{I}}), \ \ \text{etc.}$$

We call two geometric points of $'Z_{c_I}^{(\mathfrak{p},\Theta)}$ *generic-Hecke equivalent* if they are equivalent under the equivalence relation on the geometric points of $'Z_{c_I}^{(\mathfrak{p},\Theta)}$ generated by the generic-Hecke correspondences. One of the reasons for working with untwisted quasimaps is the following result, which will be proved in Sect. 9. We note that a similar statement without the restriction to untwisted quasimaps would not be true.

Proposition 3.5.1. *All geometric points of the based untwisted local stratum* $'Z_{c_I}^{(\mathfrak{p},\Theta)} \subset Z_I$ *are generic-Hecke equivalent.*

This proposition is an analogue of the statement that a group H acts transitively on a space Z. The next proposition is an analogue of the following toy situation. If a group H acts transitively on a space Z, then an H-equivariant local system on Z is the same thing as a representation of the component group $\pi_0(\mathsf{H}_z)$ of the stabilizer $\mathsf{H}_z \subset \mathsf{H}$ of a point $\mathsf{z} \in \mathsf{Z}$. If the local system is constant as an ordinary local system, then the representation must factor through the map $\pi_0(\mathsf{H}_z) \to \pi_0(\mathsf{H}_{[z]})$ where the subgroup $\mathsf{H}_{[z]} \subset \mathsf{H}$ is that preserving the connected component of $\mathsf{z} \in \mathsf{Z}$. Thus if $H_{[z]}$ is connected, then any H-equivariant local system on Z whose underlying ordinary local system is constant is itself isomorphic to the standard constant H-equivariant local system of the same rank.

Proposition 3.5.2. *All generic-Hecke equivariant local systems on the untwisted local stratum* $'Z_I^{(\mathfrak{p},\Theta)} \subset Z_I$, *whose underlying ordinary local system is constant, are isomorphic to the standard constant generic-Hecke equivariant local system of the same rank.*

The proof will also be given in Sect. 9.

3.6. **The horospherical case.**

3.6.1. *Horospherical varieties.* Recall that a G-variety X is *horospherical* if for each point $x \in X$, its stabilizer $S_x \subset G$ contains the unipotent radical of a Borel subgroup of G.

We have the following well-known characterization when X is affine. Let

$$\mathbb{C}[X] \simeq \sum_{\check\lambda \in \check\Lambda_T} \mathbb{C}[X]_\lambda$$

be its ring of function broken into G-isotypic components.

Lemma 3.6.2. *X is horospherical if and only if* $\mathbb{C}[X]_\lambda \cdot \mathbb{C}[X]_\mu \subset \mathbb{C}[X]_{\lambda+\mu}$, *for all* $\lambda, \mu \in \check\Lambda_G^+$.

Until the end of this subsection, we will assume that X is a spherical and horospherical affine variety. From Lemma 3.6.2 we obtain that the torus A acts naturally on X, commuting with the action of G. The following is easy to verify.

Lemma 3.6.3. *For X horospherical, the map $F \to \pi_0(S)$ is an isomorphism.*

3.6.4. *Loops into horospherical varieties.* For X horospherical the subsemigroup $\mathcal{V}(G/S)$ coincides with the entire Λ_A. The corresponding bijection

$$\mathcal{V}(G/S) \xrightarrow{\sim} \Lambda_A$$

can be explicitly described as follows. Consider the unit $G(\mathcal{O})$-orbit

$$\mathbf{O}_0 := \mathring{X}(\mathcal{O}) \subset X(\mathcal{K}).$$

Using the A-action of X mentioned above, we can translate this orbit by an element of $A(\mathcal{K})/A(\mathcal{O}) \simeq \Lambda_A$ and obtain another $G(\mathcal{O})$-orbit. This describes the desired bijection.

4. Convolution

4.1. The convolution diagram. For a finite set I, we define the ind-stack $\mathcal{H}_{I,G}$ of *meromorphic Hecke modifications* to be that classifying data

$$(c_I, \mathcal{P}_G^1, \mathcal{P}_G^2, \alpha)$$

where $c_I \in C^I$, $\mathcal{P}_G^1, \mathcal{P}_G^2 \in \mathrm{Bun}_G$, and α is a G-equivariant isomorphism

$$\alpha : \mathcal{P}_G^1|_{C\backslash|c_I|} \xrightarrow{\sim} \mathcal{P}_G^2|_{C\backslash|c_I|}.$$

We call the subset $|c_I| \subset C$ modification points. We have the natural projection

$$\mathcal{H}_{I,G} \to C^I,$$

and also the natural projections

$$\mathrm{Bun}_G \xleftarrow{h^{\leftarrow}} \mathcal{H}_{I,G} \xrightarrow{h^{\rightarrow}} \mathrm{Bun}_G.$$

As usual, let $\mathring{C}^I \subset C^I$ consist of those $c_I \in C^I$ such that c_i is distinct from c_j, for distinct $i,j \in I$. We write $\mathcal{H}_{\mathring{I},G}$ for the fiber product

$$\mathcal{H}_{\mathring{I},G} = \mathcal{H}_{I,G} \underset{C^I}{\times} \mathring{C}^I.$$

Consider the $(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I$-torsor

$$\widehat{\mathrm{Bun}}_{\mathring{I},G} \to \mathrm{Bun}_G$$

that classifies data

$$(c_I, \mathcal{P}_G, \mu, \tau)$$

where $c_I \in \mathring{C}^I$, $\mathcal{P}_G \in \mathrm{Bun}_G$, μ is an isomorphism of G-bundles

$$\mu : D_{|c_I|} \times G \xrightarrow{\sim} \mathcal{P}_G|_{D_{|c_I|}},$$

and τ is an isomorphism

$$\tau : D \times |c_I| \xrightarrow{\sim} D_{|c_I|}.$$

We may identify $\mathcal{H}_{\mathring{I},G}$ with the twisted product

$$\mathcal{H}_{\mathring{I},G} \simeq \widehat{\mathrm{Bun}}_{\mathring{I},G} \overset{(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I}{\times} (\mathrm{Gr}_G)^I$$

so that $h^{\rightarrow}$ corresponds to the evident projection to Bun_G. Here as usual Gr_G denotes the affine Grassmannian of G.

For the ind-stack Z_I of meromorphic quasimaps, we have the diagram

$$
\begin{array}{ccccc}
Z_I & \xleftarrow{h^{\leftarrow}} & \mathcal{H}_{I,G} \underset{C^I \times \mathrm{Bun}_G}{\times} Z_I & \xrightarrow{h^{\rightarrow}} & Z_I \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \xleftarrow{h^{\leftarrow}} & \mathcal{H}_{I,G} & \xrightarrow{h^{\rightarrow}} & \mathrm{Bun}_G
\end{array}
$$

in which each square is Cartesian. Restricting the diagram to $\mathring{C}^I \subset C^I$, we obtain a similar diagram for the ind-stack $Z_{\mathring{I}}$ of meromorphic quasimaps with distinct pole points. Consider the $(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I$-torsor

$$\widehat{Z}_{\mathring{I}} \to Z_{\mathring{I}}$$

that classifies data
$$(z, \mu, \tau)$$
where $z \in Z_{\mathring{I}}$, with G-bundle $\mathcal{P}_G \in \mathrm{Bun}_G$ and pole points $c_I \in \mathring{C}^I$, μ is an isomorphism of G-bundles
$$\mu : D_{|c_I|} \times G \xrightarrow{\sim} \mathcal{P}_G|_{D_{|c_I|}},$$
and τ is an isomorphism
$$\tau : D \times |c_I| \xrightarrow{\sim} D_{|c_I|}.$$
For the twisted product
$$\widetilde{Z}_{\mathring{I}} = \widehat{Z}_{\mathring{I}} \overset{(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I}{\times} (\mathrm{Gr}_G)^I,$$
we have the obvious identification
$$\widetilde{Z}_{\mathring{I}} \simeq \mathcal{H}_{\mathring{I},G} \underset{\mathring{C}^I \times \mathrm{Bun}_G(C)}{\times} Z_{\mathring{I}}.$$
Thus we have a diagram
$$Z_{\mathring{I}} \overset{h^{\leftarrow}}{\leftarrow} \widetilde{Z}_{\mathring{I}} \overset{h^{\rightarrow}}{\rightarrow} Z_{\mathring{I}}$$
in which $h^{\rightarrow}$ is the evident projection.

4.2. **Convolution of sheaves.** The geometric Satake correspondence [MV04] states that the category $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G is a tensor category equivalent to the category of finite-dimensional representations of the dual group $\check{G}$. We shall denote the corresponding functor by
$$V \in \mathrm{Rep}(\check{G}) \mapsto \mathcal{A}^V \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G).$$
More generally, for a labelling $V_I : I \to \mathrm{Rep}(\check{G})$, we write $\mathcal{A}_G^{V_I}$ for the corresponding object of $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$.

For objects $\mathcal{P} \in \mathbf{P}(Z_{\mathring{I}})$, and $\mathcal{A} \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$, we may form the twisted product
$$\mathcal{P} \widetilde{\boxtimes} \mathcal{A} \in \mathbf{P}(\widetilde{Z}_{\mathring{I}})$$
with respect to the projection $h^{\rightarrow} : \widetilde{Z}_{\mathring{I}} \to Z_{\mathring{I}}$.

We define the functor
$$H_G^I : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \times \mathbf{P}(Z_{\mathring{I}}) \to \mathbf{P}(Z_{\mathring{I}})$$
by the formula
$$H_G^I(\mathcal{A}, \mathcal{P}) = \bigoplus_k {}^p H^k(h_!^{\leftarrow}(\mathcal{P} \widetilde{\boxtimes} \mathcal{A})).$$
In what follows, we shall only be interested in applying $H_G^I(\cdot, \cdot)$ to the intersection cohomology sheaf ${}'\mathrm{IC}_{\mathring{I}}^0$ of the closure ${}'\overline{Z}_{\mathring{I}}^0 \subset {}'Z_{\mathring{I}}$ of the untwisted basic stratum. In this way, we obtain a functor
$$\mathrm{Conv}_I : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{P}({}'Z_{\mathring{I}}),$$
given by
$$\mathrm{Conv}_I(\mathcal{A}) = H_G^I(\mathcal{A}, {}'\mathrm{IC}_{\mathring{I}}^0).$$
When I has a single element, we write Conv in place of Conv_I.

Since objects of the category $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ are direct sums of intersection cohomology sheaves, by the decomposition theorem, every object obtained as $\mathrm{Conv}_I(\mathcal{A})$ is semi-simple.

The following result will be established in Sect. 15.

Theorem 4.2.1. *For any object $\mathcal{A} \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$, each irreducible summand of the convolution $\mathrm{Conv}_I(\mathcal{A})$ is isomorphic to the intersection cohomology sheaf of the closure of a connected component of an untwisted local stratum $'Z_{\mathring{I}}^{\Theta} \subset 'Z_{\mathring{I}}$ with constant coefficients.*

4.2.2. *Image category.* Since generic-Hecke modifications commute with meromorphic Hecke modifications in the natural sense, the convolution functor descends to a well-defined functor

$$\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \times \mathbf{P}_{\mathcal{H}}(Z_{\mathring{I}}) \to \mathbf{P}_{\mathcal{H}}(Z_{\mathring{I}}).$$

Since the intersection cohomology sheaf $'\mathrm{IC}_{\mathring{I}}^0$ of the closure $'\overline{Z}_{\mathring{I}}^0 \subset 'Z_{\mathring{I}}$ of the untwisted basic stratum is naturally generic-Hecke equivariant, we obtain a functor

$$\mathrm{Conv}_I : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{P}_{\mathcal{H}}('Z_{\mathring{I}}).$$

Definition 4.2.3. *We define $\mathbf{Q}(Z_{\mathring{I}})$ to be the full subcategory of $\mathbf{P}_{\mathcal{H}}('Z_{\mathring{I}})$ whose objects are isomorphic to direct summands of perverse sheaves that belong to the image of the above functor.*

When I has a single element, we write $\mathbf{Q}(Z)$ in place of $\mathbf{Q}(Z_{\mathring{I}})$.

Putting together Theorem 4.2.1 and Proposition 3.5.2, we arrive at the following description of the image category.

Theorem 4.2.4. *Every irreducible object of $\mathbf{Q}(Z_{\mathring{I}})$ is isomorphic to the intersection cohomology sheaf of an untwisted local stratum $'Z_{\mathring{I}}^{\Theta} \subset 'Z_{\mathring{I}}$.*

4.2.5. *Varying the set I.* Our present goal is to define an equivalence

$$\gamma_I : \mathbf{Q}(Z)^{\otimes I} \overset{\sim}{\to} \mathbf{Q}(Z_{\mathring{I}}).$$

Since the categories in questions are semi-simple, it suffices to specify what this functor does on irreducible objects. We set

$$\gamma_I(\otimes_{i \in I} '\mathrm{IC}^{\theta_i}) = '\mathrm{IC}_{\mathring{I}}^{\Theta}, \text{ where } \Theta(i) = \theta_i.$$

Here all of the sheaves have the tautological generic-Hecke equivariant structure.

Along with Theorem 4.2.1, in Sect. 15, we will prove the following:

Corollary 4.2.6. *For a finite set I, there is a canonical isomorphism*

$$\mathrm{Conv}_I \simeq \gamma_I \circ \mathrm{Conv}^{\otimes I} : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{Q}(Z_{\mathring{I}}).$$

5. Specialization

5.1. **Specialization of affine varieties.** Let $\mathcal{B}$ be the quotient of the Lie algebra of the universal Cartan T by the Lie algebra of the center $Z(G)$. Observe that its ring of regular function $\mathbb{C}[\mathcal{B}]$ may be represented as a polynomial algebra with generators t^α, for simple roots $\alpha \in \check{\Delta}_G$, and relations $t^\alpha t^\beta = t^{\alpha+\beta}$, for $\alpha, \beta \in \check{\Delta}_G$. The quotient torus $T/Z(G)$ acts on $\mathcal{B}$ by linear transformations with a dense orbit isomorphic to $T/Z(G)$.

Let X be an affine variety, acted on by G, and consider the decomposition of the ring $\mathbb{C}[X]$ into isotypic components.

$$\mathbb{C}[X] \simeq \sum_{\lambda \in \check{\Lambda}_G^+} \mathbb{C}[X]_\lambda.$$

As we have seen in Proposition 3.6.2, multiplication in $\mathbb{C}[X]$ respects this grading if and only if X is horospherical. In general, multiplication in $\mathbb{C}[X]$ respects the filtration by the subspaces

$$\mathbb{C}[X]_{\leq \mu} = \sum_{\lambda \leq \mu} \mathbb{C}[X]_\lambda.$$

More precisely, we have

$$\mathbb{C}[X]_{\leq \mu} \cdot \mathbb{C}[X]_{\leq \nu} \subset \mathbb{C}[X]_{\leq \mu+\nu},$$

and $1 \in \mathbb{C}[X]_{\leq 0}$.

We define the $\mathbb{C}$-algebra $\mathrm{Filt}[X]$ to be the direct sum

$$\mathrm{Filt}[X] = \sum_{\lambda \in \check{\Lambda}_G^+} \sum_{\alpha \in \check{R}_G^{\mathrm{pos}}} \mathbb{C}[X]_\lambda t^{\lambda+\alpha},$$

with multiplication

$$f_\lambda t^{\lambda+\alpha} \cdot g_\mu t^{\mu+\beta} = f_\lambda g_\mu t^{\lambda+\mu+\alpha+\beta}.$$

In the isotypic decomposition

$$f_\lambda g_\mu = \sum_{\nu \in \check{\Lambda}_G^+} h_\nu,$$

a term h_ν is possibly nonzero only if

$$\lambda + \mu - \nu \in \check{R}_G^{\mathrm{pos}},$$

and so only if

$$\lambda + \mu + \alpha + \beta = \nu + \gamma, \text{ for some } \gamma \in \check{R}_G^{\mathrm{pos}}.$$

Thus the multiplication in $\mathrm{Filt}[X]$ is well-defined.

The group $G \times T$ naturally acts on $\mathrm{Filt}[X]$, and we have a T-equivariant inclusion of $\mathbb{C}$-algebras

$$i : \mathbb{C}[\mathcal{B}] \to \mathrm{Filt}[X]$$

given by the formula

$$t^\alpha \mapsto 1 t^\alpha, \text{ for } \alpha \in \check{\Delta}_G.$$

Let $\mathfrak{X} = \mathrm{Spec}(\mathrm{Filt}[X])$ and $\Delta = \mathrm{Spec}(i) : \mathfrak{X} \to \mathcal{B}$.

Theorem 5.1.1 ([Pop86])**.** *The projection $\Delta : \mathcal{X} \to \mathcal{B}$ is surjective and flat. The fiber $\mathcal{X}_v = \Delta^{-1}(v)$, for regular $v \in \mathcal{B}$, is canonically G-isomorphic to X, and the zero fiber $\mathcal{X}_0 = \Delta^{-1}(0)$ is a horospherical G-variety.*

We call the zero fiber $\mathcal{X}_0$ the horospherical variety associated to X, and usually denote it by X_0.

5.2. **Basic properties in spherical case.** Now assume that X is an affine spherical G-variety.

Proposition 5.2.1. *The fiber $\mathcal{X}_v = \Delta^{-1}(v)$, for $v \in \mathcal{B}$, is a spherical G-variety.*

Proof. As a representation of G, the ring of regular functions $\mathbb{C}[\mathcal{X}_v]$, is isomorphic to $\mathbb{C}[X]$. Therefore it is simple and so $\mathcal{X}_v$ is spherical by Proposition 3.1.3. $\square$

By construction, the tori A_v, associated to $\mathcal{X}_v$, depend only on $\mathbb{C}[\mathcal{X}_v]$ as G-modules. Hence, the family $v \mapsto A_v$ is constant with the fiber A, attached to initial spherical G-variety X.

For a Borel subgroup $B^{op} \subset G$, let $\mathring{\mathcal{X}}_v^+ \subset \mathcal{X}_v$, for $v \in \mathcal{B}$, be the dense B^{op}-orbit provided by Proposition 5.2.1. Evidently, there exists an open sub-variety $\mathring{\mathcal{X}}^+ \subset \mathcal{X}$, which specializes to $\mathring{\mathcal{X}}_v^+$ for each $v \in \mathcal{B}$.

The parabolics $P_v^{op} \supset B^{op}$ depend only on the sublattices $\check{\Lambda}_{\mathcal{X}_v} \subset \check{\Lambda}_T$. Since the above family of lattices is constant, so is the family of parabolics.

Proposition 5.2.2. *There is a P^{op}-equivariant isomorphism $\mathring{\mathcal{X}}^+ \simeq \mathring{X}_0^+ \times \mathcal{B}$ such that the map $\Delta : \mathring{\mathcal{X}}^+ \to \mathcal{B}$ coincides with the projection onto the second factor of the product.*

Proof. This follows from the fact that the action of U^{op} on each $\mathring{\mathcal{X}}_v^+$ is free and $\mathring{\mathcal{X}}_v^+/U^{op} \simeq A$. $\square$

Let $\mathring{\mathcal{X}}_v \subset \mathcal{X}_v$, for $v \in \mathcal{B}$, be the dense G-orbit provided by Proposition 5.2.1. Evidently, there exists an open subvariety $\mathring{\mathcal{X}} \subset \mathcal{X}$ that specializes to $\mathring{\mathcal{X}}_v$ for each v.

Corollary 5.2.3. *The open subset $\mathring{\mathcal{X}} \subset \mathcal{X}$ is smooth and the map $\Delta : \mathring{\mathcal{X}} \to \mathcal{B}$ is smooth.*

Proof. For $x \in \mathring{\mathcal{X}}$, we may choose B^{op} so that $x \in \mathring{\mathcal{X}}^+$, and then apply the previous proposition. $\square$

5.3. **Family of quasimaps.** We define the family $\mathcal{Z}_I \to \mathcal{B}$ of meromorphic quasimaps into the family $\mathcal{X} \to \mathcal{B}$ to be the ind-stack classifying the data

$$(v \in \mathcal{B}, c_I \in C^I, \mathcal{P}_G \in \mathrm{Bun}_G, \sigma : C \setminus |c_I| \to \mathcal{P}_G \overset{G}{\times} \mathcal{X}_v|_{C \setminus |c_I|})$$

where σ is a section which factors

$$\sigma|_{C'} : C'_\mathrm{s} \to \mathcal{P}_G \overset{G}{\times} \mathring{\mathcal{X}}_v|_{C'} \to \mathcal{P}_G \overset{G}{\times} \mathcal{X}_v|_{C'},$$

for some open curve $C' \subset C$.

We write $\mathcal{Z}_{v,I}$ for the fiber of $\mathcal{Z}_I$ over a point $v \in \mathcal{B}$, and usually denote the zero fiber by $Z_{0,I}$.

Proposition 5.3.1. *The fiber $\mathcal{Z}_{v,I}$, for regular $v \in \mathcal{B}$, is canonically isomorphic to Z_I.*

Proof. Immediate from Theorem 5.1.1. $\square$

5.4. **Specialization of sheaves.** In this section, we study the nearby cycles in the family $\mathcal{Z}_I \to \mathcal{B}$. The quotient torus $T/Z(G)$ acts transitively on the regular elements of $\mathcal{B}$. For any line $\ell \subset \mathcal{B}$ such that $\ell \setminus \{0\}$ consists of regular elements, we have the nearby cycles functor

$$\psi_I : \mathbf{P}(Z_I) \to \mathbf{P}(Z_{0,I}).$$

Thanks to the action of $T/Z(G)$, the nearby cycles functors for different lines are canonically isomorphic. When I consists of one element, we shall write ψ_I in place of ψ.

In Sect. 16.4 we will prove the following:

Proposition 5.4.1.

(1) For $\theta \in \mathcal{V}(G/S)$ the perverse sheaf $'\mathrm{IC}_0^\theta$ appears in the Jordan-Hölder series of $\Psi('\mathrm{IC}^\theta)$ with multiplicity one.

(2) If for some $\theta' \in \Lambda_A$, the perverse sheaf $'\mathrm{IC}_0^{\theta'}$ appears in the Jordan-Hölder series of $\Psi('\mathrm{IC}^\theta)$, then $\theta - \theta' \in \Lambda_X^{\mathrm{pos}}$.

5.4.2. *Specialization and generic-Hecke equivariance.* The following result will be established in Sect. 9:

Proposition 5.4.3. *The composition*

$$\mathbf{P}_{\mathcal{H}}(Z_I) \hookrightarrow \mathbf{P}(Z_I) \xrightarrow{\psi_I} \mathbf{P}(Z_{0,I})$$

factors canonically through a functor

$$\mathbf{P}_{\mathcal{H}}(Z_I) \to \mathbf{P}_{\mathcal{H}}(Z_{0,I}).$$

5.4.4. *Specialization and convolution.* Let $\mathcal{A}$ be an object of $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ and $\mathcal{P} \in \mathbf{P}_{\mathcal{H}}(Z_{\mathring{I}})$.

Proposition 5.4.5. *There exists a functorial isomorphism*

$$\psi_I(H_G^I(\mathcal{A}, \mathcal{P}))) \simeq H_G^I(\mathcal{A}, \psi_I(\mathcal{P}))$$

in the category $\mathbf{P}_{\mathcal{H}}(Z_{0,\mathring{I}})$.

Proof. For the family $\mathcal{Z}_{\mathring{I}} \to \mathcal{B}$, we may form the diagram

$$
\begin{array}{ccccccc}
\mathcal{Z}_{\mathring{I}} & \xleftarrow{h^{\leftarrow}} & \mathcal{H}_{I,G} \underset{C^I \times \mathrm{Bun}_G}{\times} \mathcal{Z}_{\mathring{I}} & \xrightarrow{h^{\rightarrow}} & \mathcal{Z}_{\mathring{I}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \xleftarrow{h^{\leftarrow}} & \mathcal{H}_{I,G} & \xrightarrow{h^{\rightarrow}} & \mathrm{Bun}_G
\end{array}
$$

in which each square is Cartesian. We have canonical isomorphisms

$$
\begin{aligned}
\psi_I(H_G^I(\mathcal{A}, \mathcal{P})) &= \bigoplus_k \psi_I({}^p H^k(h_!^{\leftarrow}(\mathcal{P} \widetilde{\boxtimes} \mathcal{A}))) \\
&\simeq \bigoplus_k {}^p H^k(\psi_I(h_!^{\leftarrow}(\mathcal{P} \widetilde{\boxtimes} \mathcal{A}))) && (\psi_I \text{ is exact in the perverse t-structure}) \\
&\simeq \bigoplus_k {}^p H^k(h_!^{\leftarrow}(\psi_I(\mathcal{P} \widetilde{\boxtimes} \mathcal{A}))) && (\psi_I \text{ commutes with proper pushforward}) \\
&\simeq \bigoplus_k {}^p H^k(h_!^{\leftarrow}(\psi_I(\mathcal{P}) \widetilde{\boxtimes} \mathcal{A})) && (\psi_I \text{ commutes with twisted product}) \\
&= H_G^I(\mathcal{A}, \psi_I(\mathcal{P}))
\end{aligned}
$$

$\square$

5.4.6. *Bad perverse sheaves.* By Corollary 4.2.6 and Theorem 10.4.1, the irreducible objects of $\mathbf{Q}(Z_{0,\mathring{I}})$ are isomorphic to the intersection cohomology sheaves with constant generic-Hecke equivariant coefficients of untwisted local strata

$$'Z_{0,\mathring{I}}^{\Theta} \subset Z_{0,\mathring{I}}, \text{ for } \Theta : I \to \Lambda_{A_0}.$$

We call an object of $\mathbf{P}_{\mathcal{H}}(Z_{0,I})$ a *bad* perverse sheaf if each of its simple constituents is not an object of the category $\mathbf{Q}(Z_{0,\mathring{I}})$. We denote by $\mathrm{Bad}(Z_{0,I})$ the full Serre subcategory of $\mathbf{P}_{\mathcal{H}}(Z_{0,I})$ whose objects are bad perverse sheaves.

Thus, a simple object of $\mathrm{Bad}(Z_{0,I})$ is either the intersection cohomology of a substack of $Z_{0,I}$ which is *not* an untwisted local stratum, or is the intersection cohomology of an untwisted local stratum with coefficients in a *nontrivial* irreducible generic-Hecke equivariant local system.

We denote by $\mathrm{Quot}(Z_{0,I})$ the quotient of the category $\mathbf{P}(Z_{0,I})$ by the Serre subcategory $\mathrm{Bad}(Z_{0,I})$.

We denote by $\mathrm{Subquot}(Z_{0,I})$ the full image of $\mathbf{Q}(Z_{0,\mathring{I}})$ under the natural projection to $\mathrm{Quot}(Z_{0,I})$. Since $\mathbf{Q}(Z_{0,\mathring{I}})$ is semisimple, and simple objects go to simple objects under the projection $\mathbf{P}_{\mathcal{H}}(Z_{0,I}) \to \mathrm{Quot}(Z_{0,I})$, the projection $\mathbf{Q}(Z_{0,\mathring{I}}) \to \mathrm{Subquot}(Z_{0,I})$ is clearly an equivalence.

In Sect. 16 we will prove the following:

Theorem 5.4.7.

(1) In the quotient category $\mathrm{Quot}(Z_{0,I})$, *we have an isomorphism*

$$\psi_I('\mathrm{IC}_{\mathring{I}}^0) \simeq \, '\,\mathrm{IC}_{0,\mathring{I}}^0.$$

(2) The above isomorphism induces an isomorphism

$$H_G^I(\mathcal{A}, \psi_I('\mathrm{IC}_{\mathring{I}}^0)) \simeq H_G^I(\mathcal{A}, \,'\,\mathrm{IC}_{0,\mathring{I}}^0) \in \mathrm{Quot}(Z_{0,I}),$$

functorial in $\mathcal{A} \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$

As a corollary, we obtain:

Corollary 5.4.8. *The image of the functor*

$$\mathbf{P}_{\mathcal{H}}(Z_I) \overset{\psi_I}{\to} \mathbf{P}_{\mathcal{H}}(Z_{0,I}) \to \mathrm{Quot}(Z_{0,I})$$

belongs to the full subcategory $\mathrm{Subquot}(Z_{0,I})$.

Thus, by Corollary 5.4.8, we may define the functor

$$\Psi_I : \mathbf{Q}(Z_{\mathring{I}}) \to \mathbf{Q}(Z_{0,\mathring{I}})$$

by taking the nearby cycles ψ_I, passing to the quotient category, and then lifting back. When I has a single element, we write Ψ in place of Ψ_I.

Recall now the functor $\gamma^I : \mathbf{Q}(Z)^{\otimes I} \to \mathbf{Q}(Z_{\mathring{I}})$. In Sect. 16 we will also prove the following:

Proposition 5.4.9. *There is a canonical isomorphism*

$$\Psi_I \circ \gamma_I \simeq \gamma_I \circ \Psi^{\otimes I} : \mathbf{Q}(Z)^{\otimes I} \to \mathbf{Q}(Z_{0,\check{I}}).$$

such that the following diagram commutes

$$
\begin{array}{ccc}
\Psi_I \circ \gamma_I \circ \mathrm{Conv}^{\otimes I} & \simeq & \gamma_I \circ \Psi^{\otimes I} \circ \mathrm{Conv}^{\otimes I} \\
\downarrow & & \downarrow \\
\Psi_I \circ \mathrm{Conv}_I \circ \gamma_I & & \gamma_I \circ \mathrm{Conv}_0^{\otimes I} \\
\downarrow & & \downarrow \\
\mathrm{Conv}_{0,I} \circ \gamma_I & = & \mathrm{Conv}_{0,I} \circ \gamma_I
\end{array}
$$

where the vertical arrows are previously defined isomorphisms.

6. Fusion

6.1. **Universal local acyclicity.** The constructions in this section will be based on the following general principle.

Let $\mathcal{Y}$ be a scheme (or stack), mapping to a smooth base scheme B (in practice we will take B to be C^I for various finite sets I). Let $\mathcal{F}$ be an object of $\mathrm{Sh}(\mathcal{Y})$, and let us assume that $\mathcal{F}$ is universally locaclly acyclic (ULA) over B (we refer the reader to [BG02] and [Gai04] for a review of the ULA property).

The following properties follow from the assumption:

Lemma 6.1.1. *Let $\mathcal{F}'$ be a subquotient of $^pH^n(\mathcal{F})$ for some n.*

(1) Let $B^0 \overset{\jmath}{\hookrightarrow} B$ be an open subvariety. Then $\mathcal{F}' \simeq \jmath_{!*}\jmath^*(\mathcal{F}')$.

(2) Let $B' \overset{\imath}{\hookrightarrow} B$ be a smooth locally closed subvariety of codimension k. Then $^pH^{k'}(\imath^*(\mathcal{F}')) = 0$ for $k' \neq k$, and $^pH^k(\imath^*(\mathcal{F}'))$ is ULA over B'.

Combining the two assertions of the above lemma, we obtain that it $\mathcal{Y}' \overset{\imath}{\hookrightarrow} \mathcal{Y}$ is a smooth locally closed subvariety of $\mathcal{Y}$ and $\mathcal{Y}'' \overset{\imath'}{\hookrightarrow} \mathcal{Y}'$ is a smooth closed subvariety of $\mathcal{Y}'$ with the complement $\mathcal{Y}'^{,0} \overset{\jmath'}{\hookrightarrow} \mathcal{Y}'$ and $\mathcal{F}'$ is as above, then

$$(\imath' \circ \imath)^*(\mathcal{F}')[-\mathrm{codim}(Y',Y)] \simeq \imath'^* \circ \jmath'_{!*}\big(\jmath'^* \circ \imath^*(\mathcal{F}')[-\mathrm{codim}(Y',Y)]\big).$$

Finally, let us recall that if $p : \mathcal{Y} \to \mathcal{Y}'$ is a proper map (where $\mathcal{Y}'$ is another scheme or stack over B), then the push-forward $p_!(\mathcal{F}) \simeq p_*(\mathcal{F}) \in \mathrm{Sh}(\mathcal{Y}')$ is also ULA over B.

6.2. **Fusion for the affine Grassmannian.** Recall that Gr_G denotes the affine Grassmannian of G, and $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G. We begin by recalling standard results leading to the fusion product on $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.

Recall that $\mathrm{Aut}(\mathcal{O})$ denotes the group-scheme of automorphisms of $\mathcal{O}$. It naturally acts on $G(\mathcal{K})$, $G(\mathcal{O})$, and Gr_G. Let $\mathbf{P}_{G(\mathcal{O})\rtimes\mathrm{Aut}(\mathcal{O})}(\mathrm{Gr}_G)$ be the category of $G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O})$-equivariant perverse sheaves on Gr_G, and let $\mathbf{P}_S(\mathrm{Gr}_G)$ be the category of perverse sheaves on Gr_G constructible with respect to the $G(\mathcal{O})$-orbits.

Lemma 6.2.1. *The forgetful functors are equivalences*

$$\mathbf{P}_{G(\mathcal{O})\rtimes\mathrm{Aut}(\mathcal{O})}(\mathrm{Gr}_G) \overset{\sim}{\to} \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G) \overset{\sim}{\to} \mathbf{P}_S(\mathrm{Gr}_G).$$

Fix a smooth complex algebraic curve C. For a finite set I, let Gr_G^I be the Beilinson-Drinfeld Grassmannian over C. It classifies data

$$(c_I \in C^I, \mathcal{P}_G \in \mathrm{Bun}_G(C), \alpha : C \setminus |c_I| \times G \xrightarrow{\sim} \mathcal{P}_G|_{C \setminus |c_I|})$$

where α is an isomorphism of G-bundles. When $I = \{1, \ldots, n\}$, we write $\mathrm{Gr}_G^{(n)}$ in place of Gr_G^I.

We have the natural projection

$$\mathrm{Gr}_G^I \to C^I.$$

For a subvariety $U \subset C^I$, we write $\mathrm{Gr}_G^I|_U$ for the fiber product

$$\mathrm{Gr}_G^I|_U = \mathrm{Gr}_G^I \underset{C^I}{\times} U$$

Let $\mathfrak{s} : I \to J$ be a surjection of finite sets. We have the corresponding closed embedding $\delta_{\mathfrak{s}} : C^J \to C^I$ which in turn induces a closed embedding

$$\Delta_{\mathfrak{s}} : \mathrm{Gr}_G^J \to \mathrm{Gr}_G^I.$$

Lemma 6.2.2.

(1) The above map indices an isomorphism

$$\mathrm{Gr}_G^J \simeq \mathrm{Gr}_G^I|_{C^J}.$$

(2) We have the identification

$$\mathrm{Gr}_G^I|_{\mathring{C}^I} \simeq (\mathrm{Gr}_G^{(1)})^I|_{\mathring{C}^I}.$$

Let $\hat{C} \to C$ be the $\mathrm{Aut}(\mathcal{O})$-torsor of formal parameters. It classifies data

$$(c \in C, \tau : D \xrightarrow{\sim} D_c)$$

where D denotes the formal disk, D_c denotes the formal neighborhood of $c \in C$, and τ is an isomorphism of formal disks.

Lemma 6.2.3. *We have the identification*

$$\mathrm{Gr}_G^{(1)} \simeq \hat{C} \overset{\mathrm{Aut}(\mathcal{O})}{\times} \mathrm{Gr}_G.$$

By Lemmas 6.2.1 and 6.2.3, we have the fully faithful functor

$$\rho : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G) \to \mathbf{P}(\mathrm{Gr}_G^{(1)}),$$

given by

$$\rho(\mathcal{A}) = \mathbb{C}_C \widetilde{\boxtimes} \mathcal{A}[1],$$

and, the corresponding fully faithful functor

$$\rho_I : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{P}(\mathrm{Gr}_G^I|_{\mathring{C}^I}).$$

Consider the inclusion

$$j : \mathrm{Gr}_G^I|_{\mathring{C}^I} \to \mathrm{Gr}_G^I,$$

and the corresponding middle-extension functor

$$j_{!*} : \mathbf{P}(\mathrm{Gr}_G^I|_{\mathring{C}^I}) \to \mathbf{P}(\mathrm{Gr}_G^I).$$

The following fundamental fact is directly implied by [MV04] (see also [Gai04]).

Proposition 6.2.4. *For any $\mathcal{A} \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ the perverse sheaf*

$$j_{!*}(\rho_I(\mathcal{A}))$$

is ULA with respect to C^I.

For a surjection $\mathfrak{s} : I \to J$ of finite sets, consider the functor

$$\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{P}(\mathrm{Gr}_G^J),$$

given by

$$\mathcal{A} \mapsto \Delta_{\mathfrak{s}}^*(j_{!*}(\rho_I(\mathcal{A})))[|J| - |I|].$$

It is easy to see that for any $\mathcal{A} \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ the resulting object of $\mathbf{P}(\mathrm{Gr}_G^J)$ belongs to the image of the fully faithful functor ρ_J. Hence, we obtain a well-defined functor

$$\circledast_{G,\mathfrak{s}} : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes J},$$

characterized by the property that

$$\rho_J(\circledast_{G,\mathfrak{s}}(\mathcal{A})) \simeq \Delta_{\mathfrak{s}}^*(j_{!*}(\rho_I(\mathcal{A})))[|J| - |I|].$$

From Lemma 6.1.1 it follows that for surjections $\mathfrak{s} : I \to J$ and $\mathfrak{r} : J \to K$, there is a canonical isomorphism

$$\circledast_{\mathfrak{r} \circ \mathfrak{s}} \simeq \circledast_{\mathfrak{r}} \circ \circledast_{\mathfrak{s}},$$

providing associativity and commutativity constraints for the above operation. This endows $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ with a structure of symmetric monoidal category.

6.3. **Fusion product on $\mathbf{Q}(Z)$.** We shall now define a (symmetric) monoidal product on $\mathbf{Q}(Z)$ in parallel with the fusion product of the previous subsection.

For the inclusion

$$j : Z_{\mathring{I}} \to Z_I,$$

we have the middle-extension

$$j_{!*} : \mathbf{P}_{\mathcal{H}}(Z_{\mathring{I}}) \to \mathbf{P}_{\mathcal{H}}(Z_I).$$

Lemma 6.3.1. *For any object $\mathcal{Q}_I \in \mathbf{Q}(Z_{\mathring{I}})$, the perverse sheaf $j_{!*}(\mathcal{Q}_I)$ is ULA with respect to C^I.*

Proof. This follows from Proposition 6.2.4 and the fact that the map $h^{\leftarrow}$ in the definition of convolution is proper. $\square$

For a surjection $\mathfrak{s} : I \to J$ of finite sets, we have the corresponding closed embedding $\delta_{\mathfrak{s}} : C^{I'} \to C^I$ which in turn induces a closed embedding

$$\Delta_{\mathfrak{s}} : Z_J \to Z_I.$$

By the previous lemma and Sect. 6.1, we obtain:

Corollary 6.3.2. *The functor*

$$\Delta_{\mathfrak{s}}^* \circ j_{!*} : \mathbf{P}_{\mathcal{H}}(Z_{\mathring{I}}) \to \mathbf{P}_{\mathcal{H}}(Z_J)$$

descends to a functor

$$\Delta_{\mathfrak{s}}^* \circ j_{!*} : \mathbf{Q}(Z_{\mathring{I}}) \to \mathbf{Q}(Z_{\mathring{J}}).$$

Recall that we have an equivalence

$$\gamma_I : \mathbf{Q}(Z)^{\otimes I} \xrightarrow{\sim} \mathbf{Q}(Z_{\hat{I}}).$$

We define the *fusion product*

$$\circledast_{\mathfrak{s}} : \mathbf{Q}(Z)^{\otimes I} \to \mathbf{Q}(Z)^{\otimes J}$$

to be the composite functor

$$\circledast_{\mathfrak{s}}(\mathcal{P}_I) = \gamma_J^{-1}(\Delta_{\mathfrak{s}}^*(j_{!*}(\gamma_I(\mathcal{P}_I))))[|J| - |I|].$$

By the preceding discussion, we immediately have the following.

Corollary 6.3.3. *For a surjection* $\mathfrak{s} : I \to J$ *of finite sets, There is a canonical isomorphism*

$$\circledast_{\mathfrak{s}} \circ \mathrm{Conv}^{\otimes I} \simeq \mathrm{Conv}^{\otimes J} \circ \circledast_{G,\mathfrak{s}} : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{Q}(Z)^{\otimes J}.$$

When J has one element, so that there is only one possible surjection $\mathfrak{s} : I \to J$, we write $\circledast$ in place of $\circledast_{\mathfrak{s}}$.

In Sect. 16.4 we will prove the following:

Proposition 6.3.4.
(1) For $\theta_1, \theta_2 \in \mathcal{V}(G/S)$, the fusion product $'\mathrm{IC}_1^\theta \circledast' \mathrm{IC}_2^\theta$ contains $'\mathrm{IC}^{\theta_1+\theta_2}$ as a subquotient with multiplicity 1.
(2) If for some $\theta_1 + \theta_2 \neq \theta \in \mathcal{V}(G/S)$, the perverse sheaf $'\mathrm{IC}^\theta$ is a constituent of $'\mathrm{IC}_1^\theta \circledast' \mathrm{IC}_2^\theta$, then $\theta - (\theta_1 + \theta_2) \notin \Lambda_X^{\mathrm{pos}}$.

6.4. **Fusion and specialization.** Our goal now is to establish the following compatibility property between fusion and specialization functors:

Theorem 6.4.1. *There is a canonical isomorphism*

$$\circledast_{\mathfrak{s}} \circ \Psi^{\otimes I} \simeq \Psi^{\otimes J} \circ \circledast_{\mathfrak{s}} : \mathbf{Q}(Z)^{\otimes I} \to \mathbf{Q}(Z_0)^{\otimes J}$$

such that the following diagram commutes

$$
\begin{array}{ccc}
\Psi^{\otimes J} \circ \circledast_{\mathfrak{s}} \circ \mathrm{Conv}^{\otimes I} & \simeq & \circledast_{\mathfrak{s}} \circ \Psi^{\otimes I} \circ \mathrm{Conv}^{\otimes I} \\
\downarrow & & \downarrow \\
\Psi^{\otimes J} \circ \mathrm{Conv}^{\otimes J} \circ \circledast_{G,\mathfrak{s}} & & \circledast_{\mathfrak{s}} \circ \mathrm{Conv}_0^{\otimes I} \\
\downarrow & & \downarrow \\
\mathrm{Conv}_0^{\otimes J} \circ \circledast_{G,\mathfrak{s}} & = & \mathrm{Conv}_0^{\otimes J} \circ \circledast_{G,\mathfrak{s}}
\end{array}
$$

where the vertical arrows are previously defined isomorphisms.

Proof. After a diagram chase starting from Proposition 5.4.9, it remains to exhibit a canonical isomorphism

(6.1) $$\Delta_{\mathfrak{s}}^* \circ j_{!*} \circ \psi_I \simeq \psi_J \circ \Delta_{\mathfrak{s}}^* \circ j_{!*}$$

compatible with convolution.

Lemma 6.4.2. *For* $\mathcal{P}_I \in \mathbf{Q}(Z_{\hat{I}})$, *the perverse sheaf* $\psi_I \circ j_{!*}(\mathcal{P}_I)$ *is ULA with respect to the projection* $Z_{0,I} \to C^I$.

Proof. We have to show that $\psi_I(\mathcal{P}_I) \in \mathbf{P}(Z_{0,\mathring{I}})$ can be extended to an object of $\mathrm{Sh}(Z_{0,I})$, which is ULA with respect to C^I. For that we can replace $\mathcal{P}_I$ by $H_G^I(\mathcal{A}, {}'\mathrm{IC}_{\mathring{I}}^0)$ for some $\mathcal{A} \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^I$.

We claim that

$$\psi_I\left(H_G^I(j_{!*}(\mathcal{A})), {}'\mathrm{IC}_I^0 \right).$$

provides the required extension. Indeed, as in Proposition 5.4.5, the above expression is isomorphic to

$$H_G^I(j_{!*}(\mathcal{A}), \psi_I({}'\mathrm{IC}_I^0)),$$

and the latter is the direct image under the proper map $h^{\leftarrow}$ of the perverse sheaf

$$\psi({}'\mathrm{IC}^0) \widetilde{\boxtimes} j_{!*}(\mathcal{A})$$

on

$$\mathcal{H}_{I,G} \underset{\mathrm{Bun}_G}{\times} \mathcal{Z} \subset \mathcal{H}_{I,G} \underset{C^I \times \mathrm{Bun}_G}{\times} \mathcal{Z}_I,$$

which is ULA over C^I by Proposition 6.2.4.

$\square$

From the lemma we obtain that

$$\psi_I \circ j_{!*}(\mathcal{P}_I) \simeq j_{!*} \circ \psi_I(\mathcal{P}_I).$$

Therefore,

$$\Delta_{\mathfrak{s}}^* \circ j_{!*} \circ \psi_I(\mathcal{P}_I) \simeq \Delta_{\mathfrak{s}}^* \circ \psi_I \circ j_{!*}(\mathcal{P}_I).$$

Note that there exists a functorial morphism

$$\Delta_{\mathfrak{s}}^* \circ \psi_I(\mathcal{Q}_I) \to \psi_J \circ \Delta_{\mathfrak{s}}^*(\mathcal{Q}_I)$$

for $\mathcal{Q}_I \in \mathrm{Sh}(Z_I)$. As in the above lemma, we show that this morphism is an isomorphism when applied to objects of the form $j_{!*}(\mathcal{P}_I)$, $\mathcal{P}_I \in \mathbf{Q}(Z_{\mathring{I}})$. Combining, we obtain the isomorphism of (6.1).

We leave it to the reader to unwind the isomorphisms to verify the asserted compatibility.

$\square$

7. Tannakian formalism

7.1. **Geometric Satake equivalence.** Our starting point is the following fundamental result, due to Lusztig, Ginzburg, Drinfeld, and Mirkovic-Vilonen [MV04].

Recall that the category $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G carries a natural structure of symmetric monoidal (= tensor) category.

Theorem 7.1.1 ([MV04]). *The category $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is equivalent to that of finite-dimensional representations of the Langlands dual group $\check{G}$.*

This theorem is the basis for the relation between the geometry of loop spaces of G and representation theory of $\check{G}$.

7.2. **The fiber functor.** By the main result of [GN04a], we may identify the category $\mathbf{Q}(Z_0)$ with the category of finite dimensional representations of the torus $\check{A}_0 \subset \check{G}$. The following will be established in Sect. 10:

Proposition 7.2.1.

(1) The equivalence $\mathbf{Q}(Z_0) \simeq \mathrm{Rep}(\check{A}_0)$ is naturally equipped with a monoidal structure, compatible with the commutativity constraints.

(2) The resulting tensor functor

$$\mathrm{Rep}(\check{G}) \simeq \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G) \overset{\mathrm{Conv}}{\to} \mathbf{Q}(Z_0) \simeq \mathrm{Rep}(\check{A}_0)$$

is naturally isomorphic to the restriction functor under $\check{A}_0 \hookrightarrow \check{T} \hookrightarrow \check{G}$.

Composing the functor Ψ with the forgetful functor from $\mathrm{Rep}(\check{A}_0)$ to the category of vector spaces, we obtain that $\mathbf{Q}(Z)$ is a symmetric monoidal category, equipped with a tensor functor to the category of finite-dimensional vector spaces.

7.3. **The associated subgroup.** By Theorem 7.1.1, we have a tensor functor

$$\mathrm{Rep}(\check{G}) \to \mathbf{Q}(Z),$$

and by Theorem 6.4.1 and Proposition 7.2.1, its composition with $\mathbf{Q}(Z) \to \mathrm{Vect}$ is the tautological forgetful functor on $\mathrm{Rep}(\check{G})$.

By the Tannakian dictionary collected in [Nad05, Section 9], this implies that $\mathbf{Q}(Z)$ is a tensor category equivalent to the category of finite dimensional representations of a subgroup $\check{H} \subset \check{G}$ such that $\check{A}_0 \subset \check{H}$. Under this identification, the convolution Conv corresponds to the restriction of representations from $\check{G}$ to $\check{H}$, and the nearby cycles Ψ corresponds to the restriction of representations from $\check{H}$ to $\check{A}_0$.

Proposition 7.3.1.

(1) The subgroup $\check{H} \subset \check{G}$ is connected and reductive.

(2) The torus $\check{A}_0 \subset \check{H}$ is a maximal torus.

Proof. By Proposition 6.3.4, the fusion product $('\mathrm{IC}^\theta)^{\otimes n}$ contains $'\mathrm{IC}^{n\theta}$ as a subquotient. Using [DM82, Corollary 2.22], this shows that $\check{H}$ is connected. The fact that $\check{H}$ reductive follows from the semi-simplicity of $\mathbf{Q}(Z)$.

Let us consider again the restriction functor $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{A}_0)$ that geometrically translates as the functor Ψ. By Proposition 5.4.1(1), for every irreducible object $'\mathrm{IC}^\theta \in \mathbf{Q}(Z)$, the corresponding object $\Psi('\mathrm{IC}^\theta)$ contains a distinguished irreducible constituent, namely, $'\mathrm{IC}_0^\theta$.

By Proposition 5.4.1(2) and Proposition 6.3.4(2), the resulting collection of lines in vector spaces underlying irreducible representations of $\check{H}$ defines a Borel subgroup in $\check{H}$, and $\check{A}_0$ manifestly identifies with its Cartan quotient.

$\square$

Note that Proposition 5.4.1 implies that if $'\mathrm{IC}^\theta$ happens to be an object of $\mathbf{Q}(Z)$, then $\theta \in \Lambda_{A^0} \subset \Lambda_A$.

Conjecture 7.3.2. *For every $\theta \in \Lambda_{A^0} \cap \mathcal{V}(G/S)$, the irreducible object $'\operatorname{IC}^\theta \in \mathbf{P}_{\mathcal{H}}('Z)$ belongs to $\mathbf{Q}(Z)$.*

This conjecture would imply that the Weyl group of $\check{H}$ is the same as that associated to X in the theory of spherical varieties [Bri90, Kno96].

Part II. Proofs–A

Parts II, III and IV of this paper are devoted to the proofs of various assertions stated in Part I. Here in Part II, we present proofs that do not require the use of local models. The structure of this part is as follows:

In Sect. 8, we prove the statements of Sect. 3 regarding $G(\mathcal{O})$-orbits on $\mathcal{K}$-points of spherical varieties. The main tool will be a distinguished family of partial compactifications of spherical varieties.

In Sect. 9, we prove miscellaneous results concerning spaces of quasimaps and the generic-Hecke action.

In Sect. 10, we consider quasimaps into the particular target of a horospherical G-variety X. We describe explicitly its stratification, and the behavior of convolution functors in this case.

8. SPHERICAL GEOMETRY

8.1. **Structure theory.**

8.1.1. *The Levi subgroup.* Recall the family $\mathcal{X}$ of spherical varieties, and recall that the open P^{op}-stable sub-family $\mathring{\mathcal{X}}^+$ was in fact constant. Choose a point $x \in \mathring{X}^+$ and set

$$Q := \mathrm{Stab}_{P^{op}}(x) \simeq S \cap P^{op}.$$

We can write

$$\mathring{\mathcal{X}}^+ \simeq P^{op}/M_S \times \mathcal{B},$$

and thus Q is a constant $\mathcal{B}$-family, contained in the non-constant family $v \mapsto S_v$, where $S_v \subset G$ denotes the stabilizer of $x \in \mathcal{X}_v$, for $v \in \mathcal{B}$.

By what we have seen, $Q \cap U^{op} = \{1\}$ and there is a short exact sequence

$$1 \to Q \to P^{op}/U^{op} \to A \to 1.$$

Let us now specialize to the point $0 \in \mathcal{B}$. Let S_0 denote the corresponding subgroup of G (note that it may be disconnected). By a result of Knop, the normalizer of S_0 is a parabolic subgroup, denoted $P \subset G$, which is opposite to P^{op}, i.e., $P \cap P^{op}$ is a Levi subgroup of both P and P^{op}. Moreover, $[P, P] = [S_0, S_0] \subset P$.

The above implies, in particular, that $Q \subset S_0$ is a Levi subgroup of S_0.

8.2. **Parametrization of loops.** Our aim here is to describe the equivalence classes of loops in spherical varieties, a result due to Luna-Vust. We give an independent proof using their compactification theory [LV83] since it provides further details we will need in what follows.

8.2.1. *Toroidal compactifications.* We collect here some results from the theory of compactifications of spherical G-varieties. Our basic reference is [Kno91]. In what follows, all varieties are assumed to be normal.

Let $S \subset G$ be a spherical subgroup. A *(partial) compactification* of G/S is a G-variety X^c with a point $x \in X^c$ such that the stabilizer of x is S and the G-orbit through x is dense. A compactification X^c is said to be *toroidal* (or *without color*) if for any Borel subgroup $B \subset G$, no B-stable divisor in G/S contains a G-orbit of X^c in its closure. Our first aim is to recall the classification of toroidal compactifications of G/S. There is a more general theory describing all compactifications but we have no need for it.

Let $\mathbb{C}(G/S)$ denote the function field of G/S. For a Borel subgroup $B \subset G$, let $\mathbb{C}(G/S)^{(B)}$ denote the multiplicative group of nonzero B-eigenfunctions in $\mathbb{C}(G/S)$, and let $\mathcal{Q}(G/S)$ denote the lattice dual to the lattice of B-characters occuring in $\mathbb{C}(G/S)^{(B)}$. In what follows, let $\mathcal{V}(G/S)$ be the set of G-invariant discrete valuations of $\mathbb{C}(G/S)$. (Our aim is to show that it coincides with the set of orbits it previously was defined to parametrize in Sect. 3.) There is a canonical inclusion

$$\mathcal{V}(G/S) \subset \mathcal{Q}(G/S)$$

which takes $v \in \mathcal{V}(G/S)$ to the homomorphism $\chi_f \mapsto v(f)$, where χ_f is the B-character of $f \in \mathbb{C}(G/S)^{(B)}$. (See [Kno91], Corollary 1.8.) It is known that $\mathcal{V}(G/S)$ forms a simplicial cone of full rank in $\mathcal{Q}(G/S)$.

We call a subset $\mathcal{C} \subset \mathcal{Q}(G/S)$ an *allowable strictly convex cone* if $\mathcal{C}$ is a strictly convex cone generated by finitely many elements of $\mathcal{V}(G/S)$. We call a nonempty finite set $\mathcal{F}$ of allowable strictly convex cones in $\mathcal{Q}(G/S)$ an *allowable fan* if for $\mathcal{C} \in \mathcal{F}$, each face of $\mathcal{C}$ also belongs to $\mathcal{F}$, and for each $v \in \mathcal{V}(G/S)$, there is at most one $\mathcal{C} \in \mathcal{F}$ with v in the interior of $\mathcal{C}$.

Let X^c be a compactification of G/S. To each G-orbit $Y \subset X^c$, we associate a cone $\mathcal{C}_Y(X^c) \subset \mathcal{Q}(G/S)$ as follows. Let $\mathcal{D}_Y(X^c)$ be the set of B-stable prime divisors $D \subset X^c$ such that $Y \subset D$. Each $D \in \mathcal{D}_Y(X^c)$ defines a valuation $v_D \in \mathcal{V}(G/S)$, and we write $\mathcal{B}_Y(X^c)$ for the valuations which arise in this way. (For the construction of v_D, see [Kno91], Lemma 1.4 and the sentence following it.) Now define the cone $\mathcal{C}_Y(X^c) \subset \mathcal{Q}(G/S)$ to be that generated by $\mathcal{B}_Y(X^c)$. To the compactification X^c, we assign the fan $\mathcal{F}(X^c)$ which is the union of the cones $\mathcal{C}_Y(X^c)$, for all G-orbits $Y \subset X^c$.

Theorem 8.2.2 ([Kno91], Theorem 3.3). *The map $X^c \mapsto \mathcal{F}(X^c)$ induces a bijection between isomorphism classes of toroidal compactifications of G/S and allowable strictly convex fans in $\mathcal{Q}(G/S)$.*

Let $S_1 \subset G$ be a second spherical subgroup, and let $\phi : G/S \to G/S_1$ be a dominant G-equivariant morphism. It is possible to similarly classify the maps between the compactifications of G/S and those of G/S_1. Such a morphism ϕ induces a map $\phi_* : \mathcal{Q}(G/S) \to \mathcal{Q}(G/S_1)$ such that $\phi_*(\mathcal{V}(G/S)) \subset \mathcal{V}(G/S_1)$. We say that a cone $\mathcal{C} \subset \mathcal{Q}(G/S)$ maps under ϕ to a cone $\mathcal{C}' \subset \mathcal{Q}(G/S_1)$ if we have $\phi_*(\mathcal{C}) \subset \mathcal{C}'$, and a fan $\mathcal{F}$ maps to a fan $\mathcal{F}'$ if each cone $\mathcal{C} \in \mathcal{F}$ maps to some cone $\mathcal{C}' \in \mathcal{F}'$.

Theorem 8.2.3 ([Kno91], Theorem 4.1). *Let X^c and X_1^c be toroidal compactifications of G/S and G/S_1 respectively. Then a dominant G-equivariant morphism $\phi : G/S \to$*

G/S_1 *extends to a G-equivariant morphism* $\phi^c : X^c \to X_1^c$ *if and only if* $\mathcal{F}(X^c)$ *maps under* ϕ *to* $\mathcal{F}(X_1^c)$.

Observe that when $\mathcal{V}(G/S)$ itself is strictly convex, Theorem 8.2.2 implies the existence of a canonical compactification corresponding to the fan $\mathcal{V}(G/S)$. It is called the *wonderful compactification* of G/S. In this case, since the fan $\mathcal{V}(G/S)$ contains a unique open cone, the wonderful compactification contains a unique closed G-orbit. In general, a compactification X^c is said to be *simple* if there is a unique closed G-orbit in X^c.

Theorem 8.2.4 ([BP87], 5.3, Corollaire). *The following are equivalent:*

(1) *There exists a simple complete toroidal compactification of* G/S.
(2) *The quotient* $N_G(S)/S$ *is finite.*
(3) *The cone* $\mathcal{V}(G/S)$ *is strictly convex.*

The structure of the normalizer $N_G(S) \subset G$ is well-understood.

Theorem 8.2.5 ([BP87], 5.2, Corollaire). *The quotient* $N_G(S)/S$ *is diagonalizable.*

Our next aim is to recall a fundamental result in the local structure theory of toroidal compactifications. We recall some of preliminary material. Fix a Borel subgroup $B^{op} \subset G$ such that $B^{op}S$ is open in G. Let $P^{op} \subset G$ be the parabolic subgroup of all elements $p \in G$ such that $pB^{op}S = B^{op}S$, and let U^{op} be the unipotent radical of P^{op}. The complement $G \setminus B^{op}S$ is a union of divisors, and so we may choose a function $f \in \mathbb{C}[G]$ such that $G \setminus B^{op}S$ is the set-theoretic zero locus of f. The differential df at the identity $1 \in G$ defines an element in the coadjoint representation of G. The centralizer $M \subset G$ of this element is a Levi factor of P^{op}. The quotient $M/M \cap S$ is a torus which we denote by A.

Via the embedding $A \subset G/S$, we obtain a map $\mathbb{C}(G/S)^{(B^{op})} \to \mathbb{C}(A)$, defined by $\chi_f \mapsto f|_A$, which induces an isomorphism

$$\mathcal{Q}(G/S) \simeq \Lambda_A.$$

We write $\Lambda^+_{G/S} \subset \Lambda_A$ for the image of $\mathcal{V}(G/S) \subset \mathcal{Q}(G/S)$ under this identification. By abuse of notation, we sometimes write Λ^+_A in place of $\Lambda^+_{G/S}$. When S is the stabilizer of a point x in the dense G-orbit $\mathring{X} \subset X$ of a spherical G-variety X, so that we have $\mathring{X} \simeq G/S$, we also sometimes write Λ^+_X in place of $\Lambda^+_{G/S}$.

Now for the moment, assume that the quotient $N_G(S)/S$ is finite. Let X^c be the wonderful compactification of G/S, and let $x \in X^c$ be a point with stabilizer S. Identify the M-orbit through $x \in X^c$ with the torus A. Let $Y^+ \subset X^c$ be the toric compactification of A characterized by the property that for $\lambda \in \Lambda_A$, we have

$$\lim_{t \to 0} \lambda(t) \in Y^+ \text{ if and only if } \lambda \in \Lambda^+_{G/S}.$$

In the classification of toric varieties, Y^+ corresponds to the cone $\Lambda^+_{G/S}$.

Theorem 8.2.6 ([BLV86], Théorème 3.5). *The action map* $U^{op} \times Y^+ \to X^c$ *is an open embedding and its image contains an open nonempty subset of each G-orbit in* X^c.

Now for an arbitrary spherical subgroup $S \subset G$ for which the quotient $N_G(S)/S$ is not necessarily finite, consider the spherical subgroup $S_1 \subset G$ generated by S and the connected component of the normalizer $N_G(S)$. The natural morphism $\phi : G/S \to G/S_1$ induces a map $\phi_* : \Lambda_A \to \Lambda_{A_1}$ such that $\phi_*(\Lambda^+_{G/S}) \subset \Lambda^+_{G/S_1}$. The following is easy to check.

Lemma 8.2.7. *The cone $\Lambda^+_{G/S_1} \subset \Lambda_{A_1}$ is strictly convex, and the cone $\Lambda^+_{G/S} \subset \Lambda_A$ is its inverse image under ϕ_*.*

By the first assertion of the lemma and Theorem 8.2.4, $N_G(S_1)/S_1$ is finite, and G/S_1 admits a wonderful compactification X^c_1. Let X^c be any toroidal compactification of G/S such that ϕ extends to a G-equivariant morphism $\phi^c : X^c \to X^c_1$. Let $Y^+_1 \subset X^c_1$ be the toric compactification of A_1 constructed above, and let $Y \subset X^c$ be its inverse image under ϕ^c. Applying Theorem 8.2.6 to the pair $Y^+_1 \subset X^c_1$, we conclude that the assertion of Theorem 8.2.6 holds equally well for the pair $Y^+ \subset X^c$.

8.2.8. *Statement of parametrization.* Recall from the previous section that we may identify $\mathcal{V}(G/S)$ with a subset $\Lambda^+_{G/S} \subset \Lambda_A$. Via the embedding $A \subset G/S$, we may consider Λ_A, and so also $\Lambda^+_{G/S}$, as a subset of $(G/S)(\mathcal{K})$.

Theorem 8.2.9. *Each element of $(G/S)(\mathcal{K})$ contains a unique element of $\Lambda^+_{G/S}$ in the $G(\mathcal{O})$-orbit through it.*

The following well-known cases of the theorem are worth pointing out.

Example 8.2.10. *Let G be the product $H \times H$, and let S be the diagonal copy of H. The theorem gives the Cartan decomposition*

$$H(\mathcal{O})\backslash H(\mathcal{K})/H(\mathcal{O}) \xrightarrow{\sim} \Lambda^+_H$$

where Λ^+_H is the semigroup of dominant coweights of H.

Example 8.2.11. *Let S be the unipotent radical U of a Borel subgroup of G. The theorem gives the Iwasawa decomposition*

$$G(\mathcal{O})\backslash G(\mathcal{K})/U(\mathcal{K}) \xrightarrow{\sim} \Lambda_G$$

where Λ_G is the lattice of coweights of G.

Proof of Theorem 8.2.9. The proof is a simple application of the theory of compactifications of spherical varieties discussed in the previous section.

Let $S_1 \subset G$ be the spherical subgroup generated by S and the connected component of the normalizer $N_G(S)$. Let X^c_1 be the wonderful compactification of G/S_1, and let X^c be any complete toroidal compactification of G/S which maps to X^c_1.

Via the embedding $G/S \to X^c$, we obtain an injection $(G/S)(\mathcal{K}) \to X^c(\mathcal{K})$. Since X^c is compact, each element $\gamma \in X^c(\mathcal{K})$ extends to an element $\bar{\gamma} \in X^c(\mathcal{O})$. By the slight generalization of Theorem 8.2.6 discussed at the end of the previous section, the $G(\mathcal{O})$-orbit through $\bar{\gamma}$ contains an element $\bar{\gamma}' \in X^c(\mathcal{O})$ which lies in the image of the action map $U^{op} \times Y^+ \to X^c$. Therefore we may consider $\bar{\gamma}'$ as an element of $U^{op}(\mathcal{O}) \times Y^+(\mathcal{O})$. Thus acting by an element of $U^{op}(\mathcal{O})$, we see that the $G(\mathcal{O})$-orbit through γ contains

an element in $Y^+(\mathcal{O})$. Acting by an element of $T(\mathcal{O})$, we conclude that the $G(\mathcal{O})$-orbit through γ contains an element $\lambda \in \Lambda^+_{G/S}$.

To check that two distinct elements $\lambda, \lambda' \in \Lambda^+_{G/S}$ are not in the same $G(\mathcal{O})$-orbit, it suffices to check that their interactions with the divisor at infinity of X^c may be distinguished. If neither λ or λ' is a $\mathbb{Z}_{>0}$-multiple of the other, then by Theorems 8.2.2 and 8.2.3, we may choose the compactification X^c to have the property that $\lim_{t\to 0} \lambda(t)$ and $\lim_{t\to 0} \lambda'(t)$ do not lie in the same G-orbit. If either λ or λ' is a $\mathbb{Z}_{>0}$-multiple of the other, then we may choose the compactification X^c so that $\lim_{t\to 0} \lambda(t) = \lim_{t\to 0} \lambda'(t)$ lies in a G-orbit of codimension 1. It is easy to check that in this case, the orders of intersection of the closures of $\lambda(t)$ and $\lambda'(t)$ with this G-orbit are distinct.

8.3. **Applications.** First, observe that Theorem 3.3.1 follows immediately from Theorem 8.2.9.

To establish Theorem 3.2.1, it remains to show that for a subgroup $S \subset G$, if $S(\mathcal{K})$ acts on Gr_G with countably many orbits, then $S \subset G$ is spherical.

Any one parameter subgroup $\lambda : \mathbb{C}^\times \to G$ defines a point of Gr_G which we also denote by λ. It is easy to see that the G-orbit $F_\lambda \subset \mathrm{Gr}$ through λ is a flag variety of G. If $S(\mathcal{K})$ acts on Gr_G with countably many orbits, then the number of orbits intersecting $F_\lambda \subset \mathrm{Gr}_G$ is countable. Therefore one of the orbits intersects F_λ in an open set. Let $\mu \in F_\lambda$ be a point in this open set.

We may identify the tangent space of F_λ at the point μ with the quotient $\mathfrak{g}/\mathfrak{p}_\mu$, where $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{p}_\mu$ is the Lie algebra of the parabolic subgroup $P_\mu \subset G$ which stabilizes μ. In order for an $S(\mathcal{K})$-orbit in Gr_G to intersect F_λ in an open set containing μ, we must have that the Lie algebra $\mathfrak{s}$ of S surjects onto $\mathfrak{g}/\mathfrak{p}_\mu$. Choosing λ regular, so that μ is regular as well, we conclude that $\mathfrak{s}$ must surject onto the quotient of $\mathfrak{g}$ by a Borel subalgebra. This implies that S has an open orbit in the flag variety of G, and so it is a spherical subgroup.

The following will be used in Sect. 13.2.

Proposition 8.3.1. *For $\lambda \in \mathring{X}^+(\mathcal{K})$, and $u \in U^{op}(\mathcal{K})$, suppose $u \cdot \lambda$ is in a $G(\mathcal{O})$-orbit in $X(\mathcal{K})$ in the closure of the orbit through λ. Then $u \in U^{op}(\mathcal{O})$, and hence $u \cdot \lambda$ is in the $G(\mathcal{O})$-orbit through λ.*

Proof. Consider the loops λ and $u \cdot \lambda$ as elements of $X^c(\mathcal{K})$. If $u \cdot \lambda$ lies in $X^+(\mathcal{O})$ then we are done by similar arguments as in the proof of Theorem 8.2.9 and the fact that $U^{op} \cap S$ is the identity.

Suppose $u \cdot \lambda$ lies in $X^+(\mathcal{K}) - X^+(\mathcal{O})$, and let $\eta \in \Lambda_A$ denote the projection $p(u \cdot \lambda)$. Then we may find a conjugate open set X_1^+ such that $u \cdot \lambda$ lies in $X_1^+(\mathcal{O})$. We also have the map $p_1 : X_1^+ \to Y^+$, and the projection $\eta_1 = p_1(u \cdot \lambda) \in \Lambda_A$. It follows from the fact that $u \cdot \lambda \in \left(X^+(\mathcal{K}) \cap X_1^+(\mathcal{K}) \right) - X^+(\mathcal{O})$ that η_1 must have a deeper pole than η. But then $u \cdot \lambda$ is in a deeper stratum of $\mathring{X}(\mathcal{K})$ so can not be in the closure of that containing λ. $\qquad\square$

9. Generic-Hecke action

9.1. **Proof of Lemma 2.1.1.** Let $p : C_{\mathcal{S}} \to \mathcal{S}$ be the projection, and let $\mathcal{P}'_{\pi_0(S)}$ be the generic $\pi_0(S)$-bundle of an $\mathcal{S}$-valued quasimap. We first claim that the quasimap is untwisted if and only if $\mathcal{P}'_{\pi_0(S)}$ arises as the pullback via p of a $\pi_0(S)$-bundle on $\mathcal{S}$. Clearly, the latter condition is sufficient. To see it is necessary, choose a faithful representation V of $\pi_0(S)$, and consider the associated local system

$$\mathcal{L}' = \mathcal{P}'_{\pi_0(S)} \overset{\pi_0(S)}{\times} V.$$

By assumption, $\mathcal{L}'$ is trivial on the subschemes $\{s\} \times C \subset C_{\mathcal{S}}$ for every geometric point $s \in \mathcal{S}$. Thus it extends to a local system $\mathcal{L}$ on all of $C_{\mathcal{S}}$, and, since C is connected, the adjunction morphism

$$p^* R^0 p_* \mathcal{L} \to \mathcal{L}$$

is an isomorphism. We conclude that $\mathcal{L}$, and thus $\mathcal{L}'$ as well, is the pullback of the local system $R^0 p_* \mathcal{L}$ and the claim is proved.

Now suppose $\mathcal{S}$ is a dense subscheme of $\overline{\mathcal{S}}$, and we have a $\overline{\mathcal{S}}$-valued quasimap whose restriction to $\mathcal{S}$ is untwisted. Let $\overline{\mathcal{P}}_{\pi_0(S)}$ be the generic $\pi_0(S)$-bundle of the quasimap, and let $\mathcal{P}_{\pi_0(S)}$ be that of its restriction. By the above discussion, there is a $\pi_0(S)$-local system $\mathcal{R}_{\pi_0(S)}$ on $\mathcal{S}$ such that

$$p^* \mathcal{R}_{\pi_0(S)} \simeq \mathcal{P}_{\pi_0(S)}.$$

If $\mathcal{R}_{\pi_0(S)}$ did not extend to the complement $\overline{\mathcal{S}} \setminus \mathcal{S}$, then $\mathcal{P}_{\pi_0(S)}$ would not extend to an open subset of $p^{-1}(\overline{\mathcal{S}} \setminus \mathcal{S})$. Since this is a contradiction to the existence of $\overline{\mathcal{P}}_{\pi_0(S)}$, we conclude that $\mathcal{R}_{\pi_0(S)}$ does indeed extend to a $\pi_0(S)$-local system $\overline{\mathcal{R}}_{\pi_0(S)}$ on all of $\overline{\mathcal{S}}$, and the above isomorphism extends to an isomorphism

$$p^* \overline{\mathcal{R}}_{\pi_0(S)} \simeq \overline{\mathcal{P}}_{\pi_0(S)}.$$

Thus the $\overline{\mathcal{S}}$-valued quasimap is also untwisted and the lemma is proved.

9.2. **Proof of Proposition 3.5.1.** For $k = 1, 2$, fix closed points $z_k \in {}'Z_{c_I}^{(\mathfrak{p},\Theta)}$. We may think of z_k, for $k = 1, 2$, in terms of data $(c_I, \mathcal{P}_G^k, \mathcal{P}_S^k)$, where $\mathcal{P}_G^k$ is a G-bundle on C, and $\mathcal{P}_S^k$ is an S-bundle on $C \setminus |c_I|$ equipped with an S-equivariant bundle map

$$\mathcal{P}_S^k \to \mathcal{P}_G^k|_{C \setminus |c_I|}.$$

The assertion of the proposition is that on some open curve $C' \subset C$ containing $|c_I|$, we have an isomorphism

$$\mathcal{P}_G^1 \simeq \mathcal{P}_G^2$$

which restricts on $C' \setminus |c_I|$ to give an isomorphism

$$\mathcal{P}_S^1 \simeq \mathcal{P}_S^2.$$

By choosing an appropriate open curve $C_1 \subset C$ containing $|c_I|$, we may assume that $\mathcal{P}_G^2$ is the trivial bundle

$$\mathcal{P}_G^0 = C_1 \times G.$$

Then to prove the proposition, it suffices to find a trivialization of $\mathcal{P}_G^1$ on some open curve $C' \subset C_1$ containing $|c_I|$ such that the induced isomorphism

$$\mathcal{P}_G^1|_{C'} \simeq \mathcal{P}_G^0|_{C'}$$

restricts to give an isomorphism

$$\mathcal{P}_S^1|_{C' \setminus |c_I|} \simeq \mathcal{P}_S^2|_{C' \setminus |c_I|}.$$

Since the $\pi_0(S)$-bundles induced from $\mathcal{P}_S^k$, for $k = 1, 2$, are trivial, by choosing an appropriate open curve $C_2 \subset C_1 \setminus |c_I|$, we may assume that we have an isomorphism

$$\alpha : \mathcal{P}_G^1|_{C_2} \xrightarrow{\sim} \mathcal{P}_G^0|_{C_2}$$

which restricts to give an isomorphism

$$\mathcal{P}_S^1|_{C_2} \xrightarrow{\sim} \mathcal{P}_S^2|_{C_2}.$$

Then via the standard trivialization of $\mathcal{P}_G^0$ and the isomorphism α, we obtain a trivialization τ of the restriction $\mathcal{P}_G^1|_{C_2}$. If τ extends across the points $|c_I|$, then we are done. Otherwise, we are left to try to change τ so that it extends, but so that we do not change the bundle $\mathcal{P}_S^1|_{C_2}$. More precisely, to prove the proposition, it suffices to find an open curve $C_3 \subset C_2$, and a map

$$\Gamma : C_3 \to S,$$

so that multiplying by Γ, we obtain a trivialization

$$\Gamma \cdot \tau \quad \text{of} \quad \mathcal{P}_G^1|_{C_2}$$

which does extend across the points $|c_I|$.

Now by definition, the assumption that $z_k \in {}'Z_{c_I}^{(\mathfrak{p}, \Theta)}$, for $k = 1, 2$, implies that we may prove the analogue of the proposition on the formal punctured neighborhood $D_{|c_I|}^\times$. In other words, there is a map

$$\gamma : D_{|c_I|}^\times \to S^0$$

such that the trivialization

$$\gamma \cdot (\tau|_{D_{|c_I|}^\times}) \quad \text{of} \quad \mathcal{P}_G^1|_{D_{|c_I|}^\times}$$

extends to a trivialization on the formal neighborhood $D_{|c_I|}$. Taking the class of γ in the product of affine Grassmannians of S^0 at the points $|c_I|$, we may find an open curve $C_3 \subset C_2$ and a map $\Gamma : C_3 \to S^0$ such that we have an equality of classes

$$[\Gamma] = [\gamma]$$

in the product of affine Grassmannians. In other words, we have an equality of maps

$$\Gamma|_{D_{|c_I|}^\times} = \gamma \cdot \gamma_+,$$

for some $\gamma_+ \in S^0(\mathcal{O}_{|c_I|})$. For any congruence subgroup $S_{++}^0 \subset S^0(\mathcal{O}_{|c_I|})$, we may find an open curve $C_4 \subset C_3 \cup |c_I|$ containing $|c_I|$ and a map $\Gamma_+ : C_4 \to S^0$ such that we have an equality

$$[\Gamma_+|_{D_{|c_I|}}] = [\gamma_+]$$

in the quotient group $S^0(\mathcal{O}_{|c_I|})/S^0_{++}$. In particular, we may take S^0_{++} to be contained in the stabilizer of the class of τ in the product of affine Grassmannians of G. Thus we conclude that the map

$$\Gamma \cdot \Gamma^{-1}_+ : C_4 \to S^0$$

takes τ to a trivialization which extends across $|c_I|$. This completes the proof of the proposition.

9.3. **Proof of Proposition 3.5.2.** Recall that a Hecke equivariant structure is determined by its values on substacks of the ind-stack $\mathcal{H}_{Z_I,(1)}$ of generic-Hecke modifications at a single point. We may realize $\mathcal{H}_{Z_I,(1)}$ as the twisted product of an open subset of $Z_I \times C$ with the affine Grassmannian Gr_{S^0}.

Lemma 9.3.1. *Suppose smooth generic-Hecke modifications $Y_1, Y_2 \subset \mathcal{H}_{Z_I,(1)}$ both lie in the twisted product of an open subset $U \subset Z_I \times C$ with a single component of the affine Grassmannian Gr_{S^0}. Then there is a sequence of nonempty smooth generic-Hecke modifications and morphisms of Hecke modifications*

$$Y_1 \leftarrow W_1 \to W_2 \leftarrow \cdots \to W_{k-1} \leftarrow W_k \to Y_2.$$

Proof. We may assume Y_2 is the trivial modification.

Let S^0_r be the maximal reductive quotient of S^0. Consider the projection of affine Grassmannians $\mathrm{Gr}_{S^0} \to \mathrm{Gr}_{S^0_r}$, and the induced projection on their twisted products with U. Let $W_r \subset \mathrm{Gr}_{S^0_r}$ be the largest $S^0_r(\mathcal{O})$-orbit such that its twisted product $U \tilde\times W_r$ intersects the projection of Y_1. In particular, the intersection is a nonempty open subset of the projection of Y_1. We may truncate the inverse image of W_r to obtain a smooth $S^0(\mathcal{O})$-invariant subset $W \subset \mathrm{Gr}_{S^0}$ so that its twisted product $U \tilde\times W$ intersects Y_1 in a nonempty open set. Thus we have a diagram of nonempty smooth generic-Hecke modifications

$$Y_1 \leftarrow Y_1 \cap (U \tilde\times W) \to U \tilde\times W.$$

Let U_{S^0} be a maximal unipotent subgroup of S^0. We may truncate the $U_{S^0}(\mathcal{K})$-orbit through the base point $U_{S^0}(\mathcal{K}) \cdot [S^0(\mathcal{O})] \subset \mathrm{Gr}_{S^0}$ to obtain a smooth subset $V \subset \mathrm{Gr}_{S^0}$ such that the intersection $W \cap V$ is nonempty. Let $(W \cap V)^{sm}$ denote the smooth part of the intersection. Let $\hat U$ denote the open set U equipped with large level structure at the modification point. Taking the product with $\hat U$, we obtain a diagram of nonempty smooth generic-Hecke modifications

$$U \tilde\times W \leftarrow \hat U \times (W \cap V)^{sm} \to \hat U \times V.$$

Finally, taking Y_2 to be the trivial Hecke modification corresponding to the base point $[S^0(\mathcal{O})] \subset \mathrm{Gr}_{S^0}$, we have a diagram of nonempty smooth generic-Hecke modifications

$$\hat U \times V \leftarrow \hat U \times \{[S^0(\mathcal{O})]\} \to Y_2.$$

$\square$

By the lemma, it suffices to show that if the modifications lying in a component of Gr_{S^0} preserve a component of $'Z_I^{(\mathfrak{p},\Theta)}$, then the component of Gr_{S^0} is the one containing the trivial modification. To see this, first observe that it is clearly true for the basic

stratum $'Z_I^{(\mathfrak{p},0)}$ since it is isomorphic to the product $C^{(\mathfrak{p})} \times \mathrm{Bun}_{S^0}$. Here we write $C^{(\mathfrak{p})} \subset C^I$ for the locally closed subvariety of points $c_I \in C^I$ such that the coincidences among the points $|c_I| \subset C$ are given by the partition $\mathfrak{p}$. In general, for any pair of points $z_1, z_2 \in {}'Z_I^{(\mathfrak{p},\Theta)}$ which are related by a generic-Hecke modification given by a connected subscheme of Gr_{S^0}, we may simultaneously modify their associated G-bundles at their pole points so that we obtain points $z_1^0, z_2^0 \in {}'Z_I^{(\mathfrak{p},0)}$ which are still related by the generic-Hecke modification given by the same connected subscheme of Gr_{S^0}. If the points $z_1, z_2 \in {}'Z_I^{(\mathfrak{p},\Theta)}$ are in the same connected component of $'Z_I^{(\mathfrak{p},\Theta)}$, then the same is true for the points $z_1^0, z_2^0 \in {}'Z_I^{(\mathfrak{p},0)}$. Thus we are done since we have already seen the assertion is true for $'Z_I^{(\mathfrak{p},0)}$.

9.4. Generic-Hecke Levi equivariance.
We define the ind-stack $\mathcal{H}_{Z_I,(n)}^{(m)}$ of generic-Hecke modifications with level structure to be that classifying data

$$(c_I, \mathcal{P}_G^1, \mathcal{P}_G^2, \sigma_1, \sigma_2; c_{(n)}, \alpha; \beta_1, \beta_2)$$

where $(c_I, \mathcal{P}_G^1, \mathcal{P}_G^2, \sigma_1, \sigma_2; c_{(n)}, \alpha) \in \mathcal{H}_{Z_I,(n)}$, and β_i is an isomophism of G-bundles

$$\beta_i : \|m \cdot c_{(n)}\| \times G \xrightarrow{\sim} \mathcal{P}_G^i|_{\|m \cdot c_{(n)}\|}$$

such that for $1 \in G$, the following induced diagram commutes

$$
\begin{array}{ccc}
\|m \cdot c_{(n)}\| & \xrightarrow{\beta_i} & \mathcal{P}_G^i \overset{G}{\times} \mathring{X}|_{\|m \cdot c_{(n)}\|} \\
\downarrow & & \downarrow \\
\|m \cdot c_{(n)}\| & \xrightarrow{\sigma_i} & \mathcal{P}_G^i \overset{G}{\times} \mathring{X}|_{\|m \cdot c_{(n)}\|}
\end{array}
$$

where the vertical maps are the identity.

For sufficiently small modifications α and large order m, it makes sense to ask for α to come from a subgroup of S. We take the subgroup in question to be Q (see Sect. 8.1.1).

In particular, we may define the ind-substack

$$\mathcal{H}_{Z_I,(n),Q}^{(m)} \subset \mathcal{H}_{Z_I,(n)}^{(m)}$$

of *generic Levi modifications with level structure* to consist of those modifications which come from Q.

We call a smooth generic-Hecke correspondence Y a *Levi correspondence* if its defining map factorizes

$$Y \to \mathcal{H}_{Z_I,(n),Q}^{(m)} \to \mathcal{H}_{Z_I,(n)}$$

for some m. In analogy with the notion of Hecke equivariant perverse sheaf, we have the abelian category $\mathbf{P}_{\mathcal{H}_Q}(Z_I)$ of Levi equivariant perverse sheaves on Z_I. We write $\mathbf{P}_{\mathcal{H}_Q,c}(Z_I)$ for the full subcategory of $\mathbf{P}_{\mathcal{H}_Q}(Z_I)$ whose underlying objects are constructible along the orbits of the generic-Hecke modifications.

Proposition 9.4.1. *For X horospherical, the forgetful functor*

$$\mathbf{P}_{\mathcal{H}}(Z_I) \to \mathbf{P}_{\mathcal{H}_Q,c}(Z_I)$$

is an equivalence.

Proof. Let $U(S)$ be the unipotent radical of S so that we have $S \simeq U \rtimes Q$. By the proof of Proposition 3.5.2, we see that a generic-Hecke equivariant structure is determined by its values on Levi correspondences. In other words, the forgetful functor

$$\mathbf{P}_{\mathcal{H}}(Z_I) \to \mathbf{P}_{\mathcal{H}_Q}(Z_I)$$

is a fully faithful embedding. Thus to prove the proposition, we must check that every object of $\mathbf{P}_{\mathcal{H}_Q,c}(Z_I)$ can be equipped with a generic-Hecke equivariant structure. As explained in the proof of Proposition 3.5.2, it suffices to consider Hecke modifications which are twisted products with subschemes of the affine Grassmannian Gr_{S^0}. Observe that Gr_{S^0} is exhausted by the $U(S)(\mathcal{K})$-orbits through Gr_Q, and each orbit may be written as the increasing union of smooth affine spaces. Thus there is no obstruction to lifting the Hecke equivariant structure from the Levi subgroup Q. $\qquad \square$

10. Convolution action in the horospherical case

Throughout this section, we will only consider X horospherical.

10.1. **Stratification in the horospherical case.** When X is horospherical, we will need a complete stratification of Z_I, not only what we called local strata. We provide the definitions here and refer the reader to [GN04a] for more details.

For a positive coweight $\theta^{\mathrm{pos}} \in \Lambda_A^{\mathrm{pos}}$, we write $\mathfrak{U}(\theta^{\mathrm{pos}})$ for a decomposition $\theta^{\mathrm{pos}} = \sum_m n_m \theta_m^{\mathrm{pos}}$, for $\theta_m^{\mathrm{pos}} \in \Lambda_X^{\mathrm{pos}} \setminus \{0\}$ distinct, and n_m positive integers. We say that a Λ_X^{pos}-valued divisor on C is of type $\mathfrak{U}(\theta^{\mathrm{pos}})$ if it is of the form $\sum_m \sum_{n=1}^{m_n} \theta_m^{\mathrm{pos}} \cdot c_{m,n}$, for $c_{m,n} \in C$ distinct. For a partition $\mathfrak{p}$ of the set I, a labelling $\Theta : \mathfrak{p} \to \Lambda_A$, and a decomposition $\mathfrak{U}(\theta^{\mathrm{pos}})$ of a positive coweight $\theta^{\mathrm{pos}} \in \Lambda_A^{\mathrm{pos}}$, we say that a quasimap $(c_I, \mathcal{P}_G, \sigma) \in Z_I$ is of type $(\mathfrak{p}, \Theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$ if the coincidences among the pole points $|c_I| \subset C$ are given by the partition $\mathfrak{p}$, and the Λ_A^+-valued divisor on C associated to the quasimap is equal to $\Theta \cdot c_I + \sum_m \sum_{n=1}^{n_m} \theta_m^{\mathrm{pos}} \cdot c_{m,n}$, for $c_{m,n} \in C$ distinct and disjoint from $|c_I| \subset C$. We define the stratum

$$Z_I^{(\mathfrak{p}, \Theta, \mathfrak{U}(\theta^{\mathrm{pos}}))} \subset Z_I$$

to consist of those quasimaps of type $(\mathfrak{p}, \Theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$. When I is empty, the partition $\mathfrak{p}$ and labelling Θ are vacuous, and we write $Z_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ in place of $Z_{\emptyset}^{(\mathfrak{p}, \Theta, \mathfrak{U}(\theta^{\mathrm{pos}}))}$.

10.2. **Adding an auxiliary A-bundle.** In many arguments, we will need the following generalization of Z_I. Define the ind-stack $^\star Z_I$ to be that classifying data

$$(c_I \in C^I, \mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_A \in \mathrm{Bun}_A, \sigma : \mathcal{P}_A|_{C \setminus |c_I|} \to \mathcal{P}_G \overset{G}{\times} X|_{C \setminus |c_I|})$$

where σ is an A-equivariant map (here we are using the fact that A acts on X), which factors

$$\sigma|_{C'} : \mathcal{P}_A|_{C'} \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C'} \to \mathcal{P}_G \overset{G}{\times} X|_{C'}$$

for some open curve $C' \subset C \setminus |c_I|$. We write $\mathfrak{r} : Z_I \to {}^\star Z_I$ for the obvious induction map, and have a Cartesian diagram

$$
\begin{array}{ccc}
Z_I & \xrightarrow{\mathfrak{r}} & {}^\star Z_I \\
\downarrow & & \downarrow \\
\mathrm{Bun}_{\langle 1 \rangle} & \to & \mathrm{Bun}_A .
\end{array}
$$

10.2.1. *Description of the strata.* We stratify ${}^\star Z_I$ in the same way that we stratified Z_I. For data $(\mathfrak{p}, \Theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$, we have the corresponding stratum ${}^\star Z_I^{\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}})}$, and a Cartesian diagram

$$
\begin{array}{ccc}
Z_I^{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))} & \xrightarrow{\mathfrak{r}} & {}^\star Z_I^{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))} \\
\downarrow & & \downarrow \\
\mathrm{Bun}_{\langle 1 \rangle} & \to & \mathrm{Bun}_A .
\end{array}
$$

We can describe the stratum ${}^\star Z_I^{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))}$ more explicitly in terms of the stack Bun_P as follows.

For a partition $\mathfrak{U}(\theta^{\mathrm{pos}})$ as above, consider the corresponding partially symmetrized power of the curve $C^{\mathfrak{U}(\theta^{\mathrm{pos}})} = \prod_m C^{(n_m)}$. We will write $\prod_m \prod_{n=1}^{n_m} c_{m,n}$ for an element of $C^{\mathfrak{U}(\theta^{\mathrm{pos}})}$. Let $\mathring{C}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \subset C^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ be the complement to the diagonal divisor. Similarly, for a partition $\mathfrak{p}$ of the set I into k elements let $C^{\mathfrak{p}}$ denote C^k, and let $\mathring{C}^{\mathfrak{p}} \subset C^{\mathfrak{p}}$ be the complement to the diagonal divisor. Finally, let $C^{(\mathfrak{p},\mathfrak{U}(\theta^{\mathrm{pos}}))} \subset \mathring{C}^{\mathfrak{p}} \times \mathring{C}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ be the complement to the diagonal divisor.

Observe that there a canonical *finite* map

$$
\overline{j}_{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))} : C^{\mathfrak{p}} \times C^{\mathfrak{U}(\theta^{\mathrm{pos}})} \times {}^\star Z_\emptyset \to {}^\star Z_I,
$$

given by

$$
\overline{j}_{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))}(c_I, \prod_m \prod_{n=1}^{n_m} c_{m,n}, (\mathcal{P}_G, \mathcal{P}_A, \sigma)) = (\mathcal{P}_G, \mathcal{P}_A(-\Theta \cdot c_I - \sum_m \sum_{n=1}^{n_m} \theta_m^{\mathrm{pos}} \cdot c_{m,n}), \sigma).
$$

We denote by $j_{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))}$ the composition of $\overline{j}_{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))}$ with the open embedding

$$
C^{(\mathfrak{p},\mathfrak{U}(\theta^{\mathrm{pos}}))} \times \mathrm{Bun}_P \hookrightarrow C^{\mathfrak{p}} \times C^{\mathfrak{U}(\theta^{\mathrm{pos}})} \times {}^\star Z_\emptyset.
$$

It is easy to see that $j_{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))}$ is an isomorphism onto ${}^\star Z_I^{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))}$.

10.2.2. *Translational Hecke action.* One of the primary reasons for introducing ${}^\star Z_I$ is that it comes with extra symmetries given by modifications of the auxiliary A-bundle. Namely, we have a diagram

$$
{}^\star Z_I \xleftarrow{h^\leftarrow} {}^\star \mathcal{H}_{I,A} \xrightarrow{h^\rightarrow} {}^\star Z_I
$$

where the Hecke ind-stack ${}^\star \mathcal{H}_{I,A}$ classifies data $(z, \mathcal{P}'_A, \alpha)$ where $z \in {}^\star Z_I$ is given by data $(c_I, \mathcal{P}_G, \mathcal{P}_A, \sigma)$, $\mathcal{P}'_A \in \mathrm{Bun}_A$, α is an isomorphism of A-bundles

$$
\alpha : \mathcal{P}'_A|_{C \setminus |c_I|} \xrightarrow{\sim} \mathcal{P}_A|_{C \setminus |c_I|},
$$

$h^\leftarrow$ is the obvious projection to z, and $h^\rightarrow$ is the projection to the data $(c_I, \mathcal{P}_G, \mathcal{P}'_A, \sigma \circ \alpha)$.

Suppose we fix a partition $\mathfrak{p}$ of the set I, and a labelling $\Theta' : \mathfrak{p} \to \Lambda_A$. Then we may consider the ind-substack ${}^\star \mathcal{H}_{I,A}^{(\mathfrak{p},\Theta')}$ where the coincidences among c_I are prescribed by

$\mathfrak{p}$, and the modifications by Θ'. It is easy to see that restricting the above diagram gives a diagram

$$\star Z_I^{(\mathfrak{p},\Theta,\mathfrak{U}(\theta^{\mathrm{pos}}))} \overset{h^{\leftarrow}}{\longleftarrow} \star \mathcal{H}_{I,A}^{(\mathfrak{p},\Theta')} \overset{h^{\rightarrow}}{\longrightarrow} \star Z_I^{(\mathfrak{p},\Theta+\Theta',\mathfrak{U}(\theta^{\mathrm{pos}}))}$$

in which both projections are isomorphisms.

10.3. **Proof of Proposition 7.2.1(1).** We will use the ind-stack $\star Z_I$ introduced in the previous section. Observe that the construction of the fusion product for Z_I extends to $\star Z_I$ in an obvious way. Furthermore, for the canonical map $\mathfrak{r}: Z_I \to \star Z_I$, we have a functorial identification $\mathfrak{r}^* \circ \circledast \simeq \circledast \circ \mathfrak{r}^*$. Finally, since $\mathfrak{r}$ respects strata, we conclude that we may use $\star Z_I$ to calculate the fusion product for Z_I.

The strategy of the proof is as follows. First, we will check that the monoidal structures on the subcategories generated by the trivial object agree. Then, we will use the translational symmetry of $\star Z_I$ to extend this to other objects.

Let V_i, for $i \in I$, be a finite collection of vector spaces thought of as trivial representations of $\check{A}_0$. Then we must place a monoidal structure on the functor taking it to the collection of objects $\mathrm{IC}_{\star Z}^0 \otimes V_i$, for $i \in I$.

Lemma 10.3.1. *We have a functorial identification*

$$\circledast_{i \in I}(\mathrm{IC}_{\star Z}^0 \otimes V_i) \simeq \mathrm{IC}_{\star Z}^0 \otimes (\otimes_{i \in I} V_i)$$

Proof. Recall that the left hand side is obtained by considering the intersection cohomology sheaf $\mathrm{IC}_{\star Z_{\check{I}}}^0 \otimes (\otimes_{i \in I} V_i)$ of the basic stratum closure $\overline{\star Z}_{\check{I}}^0$, taking its middle-extension to all of $\star \check{Z}_I$, then restricting the result to the locus where the pole points coincide. Observe that this only involves the substack of $\star Z_I$ consisting of quasimaps without poles. The projection to the base C^I becomes a trivial fibration when restricted to this substack. Thus the fiber of the middle-extension is canonically isomorphic to the right hand side. $\qquad\square$

Now to extend this identification to any finite collection of representations of $\check{A}_0$, we only need observe that the action of Hecke modifications of the A-bundle at the pole points is transitive on the local strata and clearly compatible with the fusion product.

10.4. **Proof of Proposition 7.2.1(2).** Let us recall the main result of [GN04a], which gave the following explicit description of the convolution when X is horospherical.

To state it, recall that in this case, the normalizer $P \subset G$ of the stabilizer $S \subset G$ is a parabolic subgroup with Levi factor $M \subset G$. We write $2\check{\rho}_M \in \check{\Lambda}_T$ for the sum of the positive roots of M, and for $\lambda \in \Lambda_T$, we write $\langle 2\check{\rho}_M, \lambda \rangle \in \mathbb{Z}$ for the natural pairing. We write q for the natural surjection $\Lambda_T \to \Lambda_{A_0}$ which may be thought of as the map of weight lattices induced by the inclusion $\check{A}_0 \to \check{T}$. For $\mu \in \Lambda_T$ and a representation V of $\check{G}$ we write $V(\mu)$ for the corresponding weight space.

Theorem 10.4.1. *When X is horospherical, for any $V \in \mathrm{Rep}(\check{G})$, we have*

$$\mathrm{Conv}(\mathcal{A}_G^V) \simeq \bigoplus_{\kappa \in \Lambda_{A_0}} \bigoplus_{\substack{\mu \in \Lambda_T \\ q(\mu)=\kappa}} {}' \mathrm{IC}_Z^\kappa \otimes V(\mu)[\langle 2\check{\rho}_M, \mu \rangle].$$

By the theorem, under the identifications $\text{Rep}(\check{G}) \simeq \mathbf{P}_{G(\mathcal{O})}(\text{Gr}_G)$ and $\text{Rep}(\check{A}_0) \simeq \mathbf{Q}(Z)$, the restriction $\text{Rep}(\check{G}) \to \text{Rep}(\check{A}_0)$ and the convolution $\mathbf{P}_{G(\mathcal{O})}(\text{Gr}_G) \to \mathbf{Q}(Z)$ are canonically isomorphic. To finish the proof of Proposition 7.2.1(2), we need only confirm that the monoidal structures on these functors agree. This follows immediately from the theorem and the explicit description of the fusion product in the proof of Proposition 7.2.1(1).

10.5. **Towards the proof of Theorem 5.4.7(2).** The reader should skip this subsection and return to it when needed in Section 16.2. The main goal is to prove Proposition 10.5.1 below about the interaction of convolution and bad sheaves on Z_I. It is included here due to the fact that its proof does not use the local model but does use $^\star Z_I$.

Proposition 10.5.1. *For a nonzero positive coweight $\theta^{\text{pos}} \in \Lambda_A^{\text{pos}}$, a decomposition $\mathfrak{U}(\theta^{\text{pos}})$, and a Hecke equivariant local system $\mathcal{L}$ on the stratum*

$$Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})} \subset Z_\emptyset,$$

we have that

$$H_G^I(\mathcal{A}, \mathbb{C}_{\check{C}^I} \boxtimes \text{IC}_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})}(\mathcal{L})) \text{ is a bad sheaf.}$$

Proof. We will prove the stronger statement that forgetting the Hecke equivariant structure on $H_G^I(\mathcal{A}, \mathbb{C}_{\check{C}^I} \boxtimes \text{IC}_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})}(\mathcal{L}))$ gives an ordinary perverse sheaf none of whose summands is isomorphic to a summand of a perverse sheaf which results from forgetting the Hecke equivariant structure on an object of $\mathbf{Q}(Z_I)$.

To use the added flexibility of the ind-stack $^\star Z_I$, we first need to confirm that the sheaf $\mathbb{C}_{\check{C}^I} \boxtimes \text{IC}_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})}(\mathcal{L})$ arises as the pullback under $\mathfrak{r}$ of a sheaf on $^\star Z_{\mathring{I}}$.

Lemma 10.5.2. *For a positive coweight $\theta^{\text{pos}} \in \Lambda_A^{\text{pos}}$, a decomposition $\mathfrak{U}(\theta^{\text{pos}})$, and a local system $\mathcal{L}$ on the stratum*

$$Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})} \subset Z_\emptyset,$$

there exists a local system $^\star \mathcal{L}$ on the stratum

$$^\star Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})} \subset {}^\star Z_\emptyset,$$

such that $\mathcal{L}$ is a direct summand of $\mathfrak{r}^*(^\star \mathcal{L})$.

Proof. As usual, we write P for the normalizer of S. We give a proof in the case when $S \subset G$ is connected so that we may choose a section $A \to P$ of the projection $P \to A$. In this case, we show that there is a local system $^\star \mathcal{L}$ and an isomorphism

$$\mathcal{L} \simeq \mathfrak{r}^*(^\star \mathcal{L}).$$

In the general case, one must work with the components of $Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})}$ separately. Otherwise the argument is the same and we leave the details to the reader.

Recall that we have a fibration $^\star Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})} \to \text{Bun}_A$ with fiber above the trivial bundle $Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})}$. Recall as well that have an isomorphism

$$j_{\mathfrak{U}(\theta^{\text{pos}})} : \mathring{C}^{\mathfrak{U}(\theta^{\text{pos}})} \times \text{Bun}_P \xrightarrow{\sim} {}^\star Z_\emptyset^{\mathfrak{U}(\theta^{\text{pos}})}.$$

Thus, choosing a point $c_{\mathfrak{U}(\theta^{\mathrm{pos}})} \in \mathring{C}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$, we may define a section $\mathrm{Bun}_A \to {}^{\star}Z_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ by the formula

$$\mathcal{P}_A \mapsto (c_{\mathfrak{U}(\theta^{\mathrm{pos}})}, \mathcal{P}_A(-c_{\mathfrak{U}(\theta^{\mathrm{pos}})}) \overset{A}{\times} P).$$

We conclude that the fundamental group of ${}^{\star}Z_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ is a product of that of the base Bun_A and that of the fiber $Z_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$. Therefore we may extend any local system from the fiber to the total space. $\square$

Observe that for any $\mathcal{P} \in \mathbf{P}({}^{\star}Z_{0,\mathring{I}})$, we have the obvious compatibilty

$$H_G^I(\mathcal{A}, \mathfrak{r}^*\mathcal{P}) \simeq \mathfrak{r}^* H_G^I(\mathcal{A}, \mathcal{P}).$$

Thus by the previous lemma, to prove the proposition, it suffices to show that for a local system ${}^{\star}\mathcal{L}$ on the horospherical stratum

$$ {}^{\star}Z_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \subset {}^{\star}Z_{\emptyset},$$

we have that

$$\mathfrak{r}^*(H_G^I(\mathcal{A}, \mathbb{C}_{\mathring{C}^I} \boxtimes {}^{\star}\mathrm{IC}_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}({}^{\star}\mathcal{L}))) \text{ is a bad sheaf.}$$

Furthermore, we may assume that ${}^{\star}\mathcal{L}$ is irreducible, so that under the isomorphism

$$j_{\mathfrak{U}(\theta^{\mathrm{pos}})} : \mathring{C}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \times \mathrm{Bun}_P \overset{\sim}{\to} {}^{\star}Z_{\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})},$$

it comes from a product of local systems

$$j_{\mathfrak{U}(\theta^{\mathrm{pos}})*}(\mathcal{L}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \boxtimes \mathcal{L}^0) \simeq {}^{\star}\mathcal{L}.$$

Clearly we have the following.

Lemma 10.5.3. *For a positive coweight $\theta^{\mathrm{pos}} \in \Lambda_A^{\mathrm{pos}}$, a decomposition $\mathfrak{U}(\theta^{\mathrm{pos}})$, and $\mathcal{P} \in \mathbf{P}({}^{\star}Z_{0,\mathring{I}})$, and $\mathcal{C} \in \mathbf{P}(C^{\mathfrak{U}(\theta^{\mathrm{pos}})})$, there is a canonical isomorphism*

$$H_G^I(\mathcal{A}, j_{\mathfrak{U}(\theta^{\mathrm{pos}})*}(\mathcal{C} \boxtimes \mathcal{P})) \simeq j_{\mathfrak{U}(\theta^{\mathrm{pos}})*}(\mathcal{C} \boxtimes H_G^I(\mathcal{A}, \mathcal{P})).$$

By the lemma, it remains to show that

$$\mathfrak{r}^* j_{\mathfrak{U}(\theta^{\mathrm{pos}})*}(\mathrm{IC}_C^{\mathfrak{U}(\theta^{\mathrm{pos}})}(\mathcal{L}^{\mathfrak{U}(\theta^{\mathrm{pos}})}) \boxtimes H_G^{\lambda_I}(\mathbb{C}_{\mathring{C}^I} \boxtimes \mathrm{IC}_{\emptyset}^0(\mathcal{L}^0))) \text{ is a bad sheaf.}$$

But clearly no irreducible summand of such a sheaf could be supported on the closure of an untwisted local stratum. This completes the proof of the proposition. $\square$

Part III. The local model

As was mentioned in the introduction, the space of quasimaps Z_I is supposed to model the wildly infinite-dimensional space $X(\mathcal{K})$. The price we pay is that although Z_I is a more manageable object (i.e., it carries a well-defined category of sheaves), it is not local in nature.

The goal of this Part is to remedy this "non-locality". Namely, we will introduce spaces W_I^η that, on the one hand, will be local with respect to the curve C, and on the other hand, will be equivalent to Z_I in the smooth topology.

In Part IV, the machinery developed here will be applied to prove results stated in Part I. The structure of this part is as follows:

In Sect. 11, we describe a general pattern, pointed out by Drinfeld, which explains in many cases why a space that is built out of maps from a curve C to a target space $\mathcal{Y}$ will have a local behavior with respect to C. Roughly speaking, this happens when $\mathcal{Y}$ contains an open substack $\mathcal{Y}^0$ isomorphic to a point pt. In our specific case this open substack will be isomorphic to pt/F, where F is a finite abelian group.

Our usual quasimaps may be thought of as maps from C to the stack X/G. We show that if we replace G by a certain subgroup R of the parabolic P^{op}, we achieve the desired locality.

In Sect. 12, we establish a version of the factorization property of the local model W_I^η which is the expression of its locality with respect to C.

In Sect. 13, we show that the fibers of the natural projection from W_I^η to a suitable configuration space can be completely described in terms of the affine Grassmannian of G.

In Sect. 14, we show that the quasimaps space Z_I and the local model W_I^η are essentially equivalent in the smooth topology. This allows us to reduce questions about the local behavior of the former to those about the latter.

11. Construction of the local model

11.1. **Base ind-scheme.** The input for the construction described below is the torus A, and the seimgroup of dominant weights $\check\Lambda_X^+$, or alternatively the semigroup of positive coweights $\Lambda_X^{\mathrm{pos}} \subset \Lambda_A$.

Note that a rational section τ of an A-bundle on C defines a Λ_A-valued divisor $\mathrm{div}(\tau)$ on C. The support of $\mathrm{div}(\tau)$ is a finite subset of C which we denote by $|\tau|$.

For a finite set I, and $\eta \in \Lambda_A$, define the ind-scheme C_I^η to be that classifying data

$$(c_I; \mathcal{P}_A, \tau),$$

where $c_I \in C^I$, $\mathcal{P}_A \in \mathrm{Bun}_A^\eta$, and τ is a rational section of $\mathcal{P}_A$ such that the following holds. For any $\lambda \in \check\Lambda_X^+$, let $\mathcal{L}^\lambda$ be the corresponding one-dimensional representation of A, and let $\mathcal{L}^\lambda$ be the line bundle induced from the A-bundle $\mathcal{P}_A$. Then we require that the meromorphic section of $\mathcal{L}^\lambda$ associated to τ be regular on $C \setminus |c_I|$. We write c_I^η for a

point of C_I^η. It is the same thing as a point $c_I \in C^I$, and a Λ_A-valued divisor on C of degree η which takes values in Λ_X^{pos} on $C \setminus |c_I|$. Note that C_I^η is indeed an ind-scheme since $\check{\Lambda}_X^+$ generates $\check{\Lambda}_A$.

For a point $c_I^\eta \in C_I^\eta$, we call the subset $|c_I| \subset C$ the pole points of c_I^η. We call the union $|c_I| \cup |\tau| \subset C$ the degeneracy locus of c_I^η, and denote it by $|c_I^\eta|$.

11.2. Base ind-stack.
The additional input for the construction described here is the torus A_0, and the short exact sequence

$$1 \to F \to A_0 \to A \to 1,$$

or equivalently the short exact sequence

$$0 \to \Lambda_{A_0} \to \Lambda_A \to F \to 0.$$

For a finite set I, and $\eta \in \Lambda_{A_0}$, define the ind-stack M_I^η to be the fiber product

$$M_I^\eta = \mathrm{Bun}_{A_0} \underset{\mathrm{Bun}_A}{\times} C_I^\eta$$

where the map $\mathrm{Bun}_{A_0} \to \mathrm{Bun}_A$ is the induction

$$\mathcal{P}_{A_0} \mapsto \mathcal{P}_{A_0} \overset{A_0}{\times} A,$$

and the map $C_I^\eta \to \mathrm{Bun}_A$ is the obvious projection. In other words, the stack M_I^η classifies data

$$(c_I; \mathcal{P}_A, \tau; \mathcal{P}_{A_0}, \tau_0)$$

where $(c_I; \mathcal{P}_A, \tau) \in C_I^\eta$, $\mathcal{P}_{A_0} \in \mathrm{Bun}_{A_0}^\eta$, and τ_0 is an A-equivariant isomorphism

$$\tau_0 : \mathcal{P}_{A_0} \overset{A_0}{\times} A \overset{\sim}{\to} \mathcal{P}_A.$$

Note that we assume $\eta \in \Lambda_{A_0}$, or else the stack M_I^η would be empty.

We have the natural projection

$$M_I^\eta \to C_I^\eta.$$

For a point $m_I^\eta \in M_I^\eta$, over a point $c_I^\eta \in C_I^\eta$, we call the subset $|c_I| \subset C$ the pole points of m_I^η, and denote it by $|m_I|$. We call the subset $|c_I^\eta| \subset C$ the degeneracy locus of m_I^η, and denote it by $|m_I^\eta|$.

To a point $m_I^\eta \in M_I^\eta$, with representative $(c_I; \mathcal{P}_A, \tau; \mathcal{P}_{A_0}, \tau_0)$, we may associate an F-bundle on $C \setminus |m_I^\eta|$ via the pullback of τ under $\tau_0|_{C \setminus |m_I^\eta|}$. Since F is finite, this is the same thing as an F-local system on $C \setminus |m_I^\eta|$.

11.3. The general pattern.

11.3.1. *A simplified version.* Let us recall the following general construction[BFG03], Sect. 2.16. Let $\mathcal{Y}$ be an algebraic stack with an open substack $\mathcal{Y}^0 \subset \mathcal{Y}$ isomorphic to a point pt.

Assume that we are given an A-bundle $\mathcal{P}_{A,\mathcal{Y}}$ over $\mathcal{Y}$, and its trivialization over $\mathcal{Y}^0$. Moreover, assume that for every $\lambda \in \check{\Lambda}_X^+$ the meromorphic section of the corresponding line bundle $\mathcal{L}_{\mathcal{Y}}^\lambda$ is regular, and that $\mathcal{Y}^0$ is the locus of non-vanishing of these sections.

For a curve C we can consider the space (i.e., functor on the category of schemes) $\mathrm{Maps}_I(C, \mathcal{Y})$ over C^I that classifies maps

$$(C \setminus |c_I|) \to \mathcal{Y}$$

such that all but finitely many points of $C \setminus |c_I|$ get mapped to $\mathcal{Y}^0$. The space $\mathrm{Maps}_I(C, \mathcal{Y})$ splits into connected components according to the degree $\eta \in \Lambda_X^{\mathrm{pos}}$ of the pull-back of $\mathcal{P}_{A,\mathcal{Y}}$.

By construction, we have a canonical map

$$\mathrm{Maps}_I(C, \mathcal{Y})^\eta \to C_I^\eta.$$

For a given map $\sigma : C \setminus |c_I| \to \mathcal{Y}$ the locus of $C \setminus |c_I|$ for which this map lands in $\mathcal{Y}^0$ equals the complement of the degeneracy locus of the resulting element of C_I^η.

11.3.2. *A generalization.* Next let $\mathcal{Y}$ be an algebraic stack equipped with an A_0-torsor $\mathcal{P}_{A_0,\mathcal{Y}}$. Let $\mathcal{P}_{A,\mathcal{Y}}$ denote the induced A-bundle, and assume that it is trivialized over an open substack $\mathcal{Y}^0 \subset \mathcal{Y}$, such that this trivialization has the same properties as in Section 11.3.1.

In particular, over $\mathcal{Y}^0$, the A_0-torsor $\mathcal{P}_{A_0,\mathcal{Y}}$ admits a canonical reduction to an F-torsor $\mathcal{P}_{F,\mathcal{Y}}$. We assume furthermore that the resulting map

$$\mathcal{Y}^0 \to pt/F$$

is an isomorphism.

We define the space $\mathrm{Maps}_I(C, \mathcal{Y})$ in the same way as above. By construction, we have a canonical map

$$\mathrm{Maps}_I^\eta(C, \mathcal{Y}) \to M_I^\eta.$$

Note that $\mathrm{Maps}_I^\eta(C, \mathcal{Y})$ is empty unless $\eta \in \Lambda_{A_0}$.

11.3.3. *The case of spherical varieties.* Let us now explain in what situation we will apply the pattern of Section 11.3.2.

Recall the canonical surjection

$$P^{op} \twoheadrightarrow P^{op}/U^{op} \simeq M \twoheadrightarrow A_0,$$

and let us choose a splitting $M \hookleftarrow A_0$. Let R denote the preimage of A_0 in P^{op}. By construction, $R \cap S \simeq F$.

We set $\mathcal{Y} := X/R$ and $\mathcal{Y}_0 := \mathring{X}^+/R$. The A_0-torsor $\mathcal{P}_{A_0,\mathcal{Y}}$ is the pull-back of the tautological A_0-torsor under

$$X/R \to pt/R \to pt/A_0.$$

The lines $l^\lambda \subset \mathbb{C}[X]$ for $\lambda \in \check{\Lambda}_X^+$ define the trivialization of the induced A-torsor over $\mathcal{Y}_0$ with the required properties.

We have the resulting space

$$W_I^\eta := \mathrm{Maps}_I^\eta(C, \mathring{X}^+/R).$$

Let us describe it in terms of bundles on C, which will in particular imply that W_I^η is an ind-algebraic stack.

Namely, W_I^η classifies the data of

$$(c_I; \mathcal{P}_A, \tau; \mathcal{P}_{A_0}, \tau_0; \mathcal{P}_R, \tau_R, \sigma)$$

where $(c_I; \mathcal{P}_A, \tau; \mathcal{P}_{A_0}, \tau_0) \in M_I^\eta$, $\mathcal{P}_R \in \mathrm{Bun}_R$, τ_R is an A_0-equivariant isomorphism

$$\tau_R : \mathcal{P}_R \overset{R}{\times} A_0 \overset{\sim}{\to} \mathcal{P}_{A_0},$$

and σ is a section

$$\sigma : C \setminus |c_I| \to \mathcal{P}_R \overset{R}{\times} X|_{C \setminus |c_I|}.$$

For some open curve $C' \subset C \setminus |c_I|$, the section σ is required to factor

$$\sigma|_{C'} : C' \to \mathcal{P}_R \overset{R}{\times} \mathring{X}^+|_{C'} \to \mathcal{P}_R \overset{R}{\times} X|_{C'},$$

and the composition

$$C' \overset{\sigma|_{C'}}{\to} \mathcal{P}_R \overset{R}{\times} \mathring{X}^+|_{C'} \to \mathcal{P}_A|_{C'}$$

is required to coincide with the rational section

$$\tau|_{C'} : C' \to \mathcal{P}_A|_{C'}.$$

Note that we assume $\eta \in \Lambda_{A_0}$, or else the stack W_I^η would be empty.

12. Factorization

12.1. "Simple" factorization.
Assume for a moment that we are in the context of Section 11.3.1. For $\eta_1, \eta_2 \in \Lambda_X^{\mathrm{pos}}$, consider the natural map

$$C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2} \to C_{I_1 \cup I_2}^{\eta_1 + \eta_2}$$

which we denote by

$$(c_{I_1}^{\eta_1}, c_{I_2}^{\eta_2}) \mapsto c_{I_1}^{\eta_1} \uplus c_{I_2}^{\eta_2}.$$

Its restriction to the open subscheme

$$(C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2})_{\mathrm{disj}} \subset C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2}$$

of points $(c_{I_1}^{\eta_1}, c_{I_2}^{\eta_2})$ such that $|c_{I_1}^{\eta_1}|$ is disjoint from $|c_{I_2}^{\eta_2}|$ is étale.

We claim

$$(12.1) \quad \left(\mathrm{Maps}_I(C, \mathcal{Y})^{\eta_1} \times \mathrm{Maps}_I(C, \mathcal{Y})^{\eta_2}\right) \underset{C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2}}{\times} (C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2})_{\mathrm{disj}}$$

$$\simeq \mathrm{Maps}_I(C, \mathcal{Y})^{\eta_1 + \eta_2} \underset{C_{I_1 \cup I_2}^{\eta_1 + \eta_2}}{\times} (C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2})_{\mathrm{disj}}.$$

We refer the reader to [BFG03] for the proof. We shall now generalize this to the context where the finite group is present.

12.2. **Factorization and the base ind-stack.** We have a natural map

$$M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2} \to M_{I_1 \cup I_2}^{\eta_1 + \eta_2}$$

defined by

$$((c_{I_1}^{\eta_1}; \mathcal{P}_{A_0}^1, \tau_0^1), (c_{I_2}^{\eta_2}; \mathcal{P}_{A_0}^2, \tau_0^2)) \mapsto (c_{I_1}^{\eta_1} \uplus c_{I_2}^{\eta_2}; \mathcal{P}_{A_0}^1 \otimes \mathcal{P}_{A_0}^2, \tau_0^1 \otimes \tau_0^2).$$

Consider the fiber products

$$(M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\mathrm{disj}} = (M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2}) \underset{C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2}}{\times} (C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2})_{\mathrm{disj}}$$

$$(M_{I_1 \cup I_2}^{\eta_1 + \eta_2})_{\mathrm{disj}} = M_{I_1 \cup I_2}^{\eta_1 + \eta_2} \underset{C_{I_1 \cup I_2}^{\eta_1 + \eta_2}}{\times} (C_{I_1}^{\eta_1} \times C_{I_2}^{\eta_2})_{\mathrm{disj}}.$$

It is easy to see that the induced map

$$(M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\mathrm{disj}} \to (M_{I_1 \cup I_2}^{\eta_1 + \eta_2})_{\mathrm{disj}}$$

is an étale cover.

To glue mapping spaces, we will also need the stack

$$(M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\widetilde{\mathrm{disj}}}$$

over $(M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\mathrm{disj}}$ whose fiber over a point $((c_{I_1}^{\eta_1}; \mathcal{P}_{A_0}^1, \tau_0^1), (c_{I_2}^{\eta_2}; \mathcal{P}_{A_0}^2, \tau_0^2))$ is the data of trivialization of the F-bundle $\mathcal{P}_F^1$ over the finite scheme $|c_{I_2}^{\eta_2}|$ and of the F-bundle $\mathcal{P}_F^2$ over the finite scheme $|c_{I_1}^{\eta_1}|$. It is clear that the forgetful map

$$(M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\widetilde{\mathrm{disj}}} \to (M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\mathrm{disj}}$$

is also an étale cover.

12.3. **The general case.** Now let $\mathcal{Y}^0 \subset \mathcal{Y}$ be as in Sect. 11.3.2.

Proposition 12.3.1. *There is a canonical isomorphism*

$$\left(\mathrm{Maps}_{I_1}(C, \mathcal{Y})^{\eta_1} \times \mathrm{Maps}_{I_2}(C, \mathcal{Y})^{\eta_2}\right) \underset{M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2}}{\times} (M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\widetilde{\mathrm{disj}}}$$

$$\simeq \mathrm{Maps}_{I_1 \cup I_2}(C, \mathcal{Y})^{\eta_1 + \eta_2} \underset{M_{I_1 \cup I_2}^{\eta_1 + \eta_2}}{\times} (M_{I_1}^{\eta_1} \times M_{I_2}^{\eta_2})_{\widetilde{\mathrm{disj}}}.$$

Proof. Let f_1, f_2 be the maps in the data of an element of the left hand side. We define the corresponding map f of the right hand side as follows. First, break up the curve C as the union of the punctured curve $C \setminus (|c_{I_1}^{\eta_1}| \cup |c_{I_2}^{\eta_2}|)$ and the disjoint completed formal neighborhoods $D_{|c_{I_1}^{\eta_1}|}, D_{|c_{I_2}^{\eta_2}|}$. Define the map f on the pieces to be

$$f_1 \otimes f_2 : C \setminus (|c_{I_1}^{\eta_1}| \cup |c_{I_2}^{\eta_2}|) \to \mathcal{Y}^0 = pt/F$$

$$f_1 : D_{|c_{I_1}^{\eta_1}|} \to \mathcal{Y} \qquad f_2 : D_{|c_{I_2}^{\eta_2}|} \to \mathcal{Y}$$

Here we have written $f_1 \otimes f_2$ to denote the map classifying the F-bundle obtained by tensoring the F-bundles $\mathcal{P}_F^1, \mathcal{P}_F^2$ classified by f_1, f_2 respectively.

Now to see that the pieces canonically glue, we use the remaining data. Namely, we have trivializations τ_1, τ_2 of the F-bundles $\mathcal{P}_F^1, \mathcal{P}_F^2$ over the finite schemes $|c_{I_2}^{\eta_2}|, |c_{I_1}^{\eta_1}|$

respectively. Observe that τ_1, τ_2 canonically extend to trivializations of $\mathcal{P}^1_F, \mathcal{P}^2_F$ over $D_{|c^{\eta_2}_{I_2}|}$, $D_{|c^{\eta_1}_{I_1}|}$ respectively. Thus τ_1, τ_2 provide isomorphisms of F-bundles over the punctured completed formal neighborhoods

$$\mathcal{P}^1_F|_{D^{\times}_{|c^{\eta_1}_{I_1}|}} \overset{\tau_2}{\simeq} (\mathcal{P}^1_F \otimes \mathcal{P}^2_F)|_{D^{\times}_{|c^{\eta_1}_{I_1}|}} \qquad \mathcal{P}^2_F|_{D^{\times}_{|c^{\eta_2}_{I_2}|}} \overset{\tau_1}{\simeq} (\mathcal{P}^1_F \otimes \mathcal{P}^2_F)|_{D^{\times}_{|c^{\eta_2}_{I_2}|}}$$

Thus we may identify the restricted maps

$$f_1|_{D^{\times}_{|c^{\eta_1}_{I_1}|}} \simeq (f_1 \otimes f_2)|_{D^{\times}_{|c^{\eta_1}_{I_1}|}} \qquad f_2|_{D^{\times}_{|c^{\eta_2}_{I_2}|}} \simeq (f_1 \otimes f_2)|_{D^{\times}_{|c^{\eta_2}_{I_2}|}}$$

and glue them to obtain a single map f. Clearly this identifies the moduli problem of the left hand side with that of the right. $\hfill\square$

12.4. **Complements: open curves.** It is useful to have generalizations of the constructions of the preceding sections for open curves. Our primary application will be the following. In the context of Sect. 11.3.2, the moduli problem $\mathrm{Maps}_I(C, \mathcal{Y})^{\eta}$ involves an F-bundle $\mathcal{P}_F$ on the curve $C \setminus |c^{\eta}_I|$. Suppose we would like to apply the factorization of Sect. 12.3. It may turn out that Proposition 12.3.1 has less content than needed. Namely, given a particular F-bundle $\mathcal{P}_F$ on $C \setminus (|c^{\eta_1}_{I_1}| \cup |c^{\eta_2}_{I_2}|)$, there may not be any F-bundles on $C \setminus |c^{\eta_1}_{I_1}|$ and $C \setminus |c^{\eta_2}_{I_2}|$ whose tensor is equal to $\mathcal{P}_F$. Thus the base changes on both sides of Proposition 12.3.1 will miss any map living over $\mathcal{P}_F$. But if we remove an auxiliary point $c \in C$, then we may always use the added flexibility of nontrivial F-monodromies around c to find F-bundles on $C \setminus (c \cup |c^{\eta_1}_{I_1}|)$ and $C \setminus (c \cup |c^{\eta_2}_{I_2}|)$ whose tensor is equal to $\mathcal{P}_F$. With this motivation, we outline below how one may generalize our constructions to the open curve $C \setminus c$.

Recall that for a finite set I, and $\eta \in \Lambda_A$, a point c^{η}_I of the ind-scheme C^{η}_I classifies the data of a point $c_I \in C^I$, and a Λ_A-valued divisor on C of degree η which takes values in Λ^{pos}_A on $C \setminus |c_I|$. Thus it makes sense to consider C^{η}_I for a not necessarily complete curve such as $C \setminus c$.

For a finite set I, and $\eta \in \Lambda_A$, define the ind-stack $M^{\eta}_{I,1}$ to be that classifying data

$$(c; c^{\eta}_I; \mathcal{P}_{A_0}, \tau_0)$$

where $c \in C$, $c^{\eta}_I \in (C \setminus c)^{\eta}_I$, $\mathcal{P}_{A_0}$ is an A_0-bundle on $C \setminus c$, and τ_0 is an A-equivariant isomorphism

$$\tau_0 : \mathcal{P}_{A_0} \overset{A_0}{\times} A \xrightarrow{\sim} \mathcal{P}^0_A(c^{\eta}_I),$$

where $\mathcal{P}^0_A(c^{\eta}_I)$ denotes the trivial A-bundle on $C \setminus c$, twisted by the Λ_A-valued divisor associated to c^{η}_I. Note that even if $\eta \in \Lambda_A$ is not in $\Lambda_{A_0} \subset \Lambda_A$, the stack $M^{\eta}_{I,1}$ still makes sense and is nonempty.

For the moment, consider the ind-stack $\overline{M}^{\eta}_{I,1}$ that classifies the same data as $M^{\eta}_{I,1}$ except that the A_0-bundle $\mathcal{P}_{A_0}$ it classifies is defined on C rather than $C \setminus c$. The following lemma confirms that this makes no difference. Its assertion is completely local and we leave its proof to the reader.

Lemma 12.4.1. *The natural restriction $\overline{M}^{\eta}_{I,1} \to M^{\eta}_{I,1}$ is an open and closed embedding.*

For a finite set I, and $\eta \in \Lambda_A$, define the open ind-substack

$$(M_I^\eta \times C)_{\mathrm{disj}} \subset M_I^\eta \times C$$

to consist of pairs (m_I^η, c) such that c is disjoint from the degeneracy locus $|m_I^\eta|$. The following is immediately implied by Lemma 12.4.1.

Lemma 12.4.2. *The natural map*

$$(M_I^\eta \times C)_{\mathrm{disj}} \to M_{I,1}^\eta$$

is an open and closed embedding.

Now for a finite set I, and $\eta \in \Lambda_A$, define the space $\mathrm{Maps}_{I,1}(C, \mathcal{Y})^\eta$ to be that classifying $c \in C$, $c_I \in (C \setminus c)^I$, and a map $C \setminus (|c_I| \cup c) \to \mathcal{Y}$ as described in Sect. 11.3.2. By construction, we have a canonical map

$$\mathrm{Maps}_{I,1}^\eta(C, \mathcal{Y}) \to M_{I,1}^\eta.$$

We also have an obvious analogue of the factorization of Proposition 12.3.1. Note that even if $\eta \in \Lambda_A$ is not in $\Lambda_{A_0} \subset \Lambda_A$, the space $\mathrm{Maps}_{I,1}(C, \mathcal{Y})^\eta$ still makes sense and is nonempty.

In the case where $\mathcal{Y} = X/R$, we set $W_{I,1}^\eta := \mathrm{Maps}_{I,1}(C, \mathcal{Y})^\eta$. It is an ind-algebraic stack classifying data

$$(c; c_I^\eta; \mathcal{P}_{A_0}, \tau_0; \mathcal{P}_R, \tau_R, \sigma)$$

where $(c; c_I^\eta; \mathcal{P}_{A_0}, \tau_0) \in M_{I,1}^\eta$, $\mathcal{P}_R$ is an R-bundle on $C \setminus c$, τ_R is an A_0-equivariant isomorphism

$$\tau_R : \mathcal{P}_R \overset{R}{\times} A_0 \overset{\sim}{\to} \mathcal{P}_{A_0},$$

and σ is a section

$$\sigma : C \setminus (c \cup |c_I|) \to \mathcal{P}_R \overset{R}{\times} X|_{C \setminus (c \cup |c_I|)}.$$

For some open curve $C' \subset C \setminus (c \cup |c_I|)$, the section σ is required to factor

$$\sigma|_{C'} : C' \to \mathcal{P}_R \overset{R}{\times} \mathring{X}^+|_{C'} \to \mathcal{P}_R \overset{R}{\times} X|_{C'},$$

and the composition

$$C' \overset{\sigma|_{C'}}{\to} \mathcal{P}_R \overset{R}{\times} \mathring{X}^+|_{C'} \to \mathcal{P}_A^0(c_I^\eta)|_{C'}$$

is required to have divisor c_I^η. As explained in Sect. 13 below, the canonical map

$$W_{I,1}^\eta \to M_{I,1}^\eta.$$

is ind-representable.

Finally, for a finite set I, and $\eta \in \Lambda_A$, define the open ind-substack

$$(W_I^\eta \times C)_{\mathrm{disj}} \subset W_I^\eta \times C$$

to consist of pairs such that c is disjoint from the degeneracy locus $|c_I^\eta|$. The following is immediately implied by Lemma 12.4.1.

Lemma 12.4.3. *The natural map*

$$(W_I^\eta \times C)_{\mathrm{disj}} \to W_{I,1}^\eta$$

is an open and closed embedding.

13. Description of fibers

13.1. **Relation to the affine Grassmannian.** From now on we will specialize to the case where $\mathcal{Y} = X/R$, and so $\mathrm{Maps}_I(C, \mathcal{Y})^\eta = W_I^\eta$. Our present goal is to describe the fibers of the morphism $W_I^\eta \to M_I^\eta$ in terms of the affine Grassmannian Gr_G. This description will imply, in particular, that the above morphism is (ind)-representable even if C is not complete. First, we will do this on a point-wise level.

Given a point $m_I^\eta \in M_I^\eta$, we have a divisor $c_I^\eta = \Sigma \eta_k \cdot c_k \in C_I^\eta$, and an F-bundle $\mathcal{P}_F$ over $C \setminus |c_I^\eta|$. Recall that by construction, η_k is arbitrary when c_k is a pole point, but η_k is constrained to lie in Λ_X^{pos} otherwise. By restriction, we obtain an F-torsor $\mathcal{P}_F^k$ on the formal neighborhood D_{c_k} of each c_k.

Recall that $F \simeq R \cap S$ is a subgroup of G. Consider the twisted version of the affine Grassmannian $\mathrm{Gr}_{G,\mathcal{P}_F^k}$ that classifies the data of a G-torsor $\mathcal{P}_G$ on D_{c_k} and an identification

$$\beta : \mathcal{P}_G \simeq G \overset{F}{\times} \mathcal{P}_F^k|_{D_{c_k}^\times}.$$

By [BL94], this is equivalent to giving $\mathcal{P}_G$ over an open subset $U \subset C$ with

$$c_k \in U \subset C \setminus (\underset{k' \neq k}{\cup} c_{k'})$$

and giving β over $U \setminus c_k$.

Let $(W_I^\eta)_{m_I^\eta}$ denote the fiber of W_I^η above m_I^η. We claim that there is natural morphism

$$(13.1) \hspace{3cm} (W_I^\eta)_{m_I^\eta} \to \underset{k}{\Pi} \, \mathrm{Gr}_{G,\mathcal{P}_F^k}.$$

Namely, consider the restriction of the data to each D_{c_k}. We obtain a G-bundle $\mathcal{P}_G$, endowed with a reduction to R, and a reduction to S over $D_{c_k}^\times$. Furthermore, the resulting map $D_{c_k}^\times \to S \backslash G / R$ hits the open substack

$$pt/F \simeq \mathring{X}^+/R \subset S \backslash G / R$$

and the induced F-torsor is equal to $\mathcal{P}_F^k$. Thus the restriction of $\mathcal{P}_G$ to each $D_{c_k}^\times$ is induced from $\mathcal{P}_F^k$.

13.1.1. *Description of the image.* We will now show that the map (13.1) is an (ind)-locally closed embedding, and describe its image.

First, let $\mathrm{Gr}_{R,\mathcal{P}_F^k}$ be the corresponding twisted version of the affine Grassmannian of the group R. The projection $R \to A$ induces a canonical map

$$\mathrm{Gr}_{R,\mathcal{P}_F^k} \to \mathrm{Gr}_A,$$

and we write $\mathrm{Gr}^{\eta_k}_{R,\mathcal{P}^k_F} \subset \mathrm{Gr}_{R,\mathcal{P}^k_F}$ for the preimage of the corresponding connected component $\mathrm{Gr}^{\eta_k}_A \subset \mathrm{Gr}_A$. The inclusion $R \subset G$ induces a canonical locally closed embedding

$$\mathrm{Gr}^{\eta_k}_{R,\mathcal{P}^k_F} \hookrightarrow \mathrm{Gr}_{G,\mathcal{P}^k_F} .$$

In fact, one can check that $\mathrm{Gr}^{\eta_k}_{R,\mathcal{P}^k_F}$ is the orbit of a connected component of a twisted version of the group $U^{op}(\mathcal{K}_{c_k}) \times A_0(\mathcal{O}_{c_k})$ acting on $\mathrm{Gr}_{G,\mathcal{P}^k_F}$.

By construction, the map (13.1) factors through $\mathrm{Gr}^{\eta_k}_{R,\mathcal{P}^k_F}$.

Next, let $\mathcal{G}_{F^k}$ be the $G(\mathcal{O}_{c_k})$-torsor over $\mathrm{Gr}_{G,\mathcal{P}^k_F}$ that classifies triples $(\mathcal{P}_G, \alpha, \beta)$, where $(\mathcal{P}_G, \beta)$ is as in the definition of $\mathrm{Gr}_{G,\mathcal{P}^k_F}$, and α is a trivialization of $\mathcal{P}_G$ over D_{c_k}. We have a natural map

(13.2) $$\mathcal{G}_{F^k} \to X(\mathcal{K}_{c_k}) \setminus (X \setminus \mathring{X})(\mathcal{K}_{c_k})$$

For an element $\theta \in \mathcal{V}(G/S)$ recall that we write

$$\mathbf{O}_\theta \subset X(\mathcal{K}_{c_k}) \setminus (X \setminus \mathring{X})(\mathcal{K}_{c_k})$$

for the corresponding $G(\mathcal{O}_{c_k})$-orbit. Let $\overline{\mathbf{O}}_\theta$ denote its closure.

The preimage of $\overline{\mathbf{O}}_\theta$ under the map (13.2) is a $G(\mathcal{O}_{c_k})$-invariant closed subscheme of $\mathcal{G}_{F^k}$. By invariance, we have the corresponding closed subscheme

$$\overline{\mathrm{Gr}}^{S,\theta}_{G,\mathcal{P}^k_F} \subset \mathrm{Gr}_{G,\mathcal{P}^k_F} .$$

Similarly, let $\mathrm{Gr}^{S,\theta}_{G,\mathcal{P}^k_F}$ be the open subset of $\overline{\mathrm{Gr}}^{S,\theta}_{G,\mathcal{P}^k_F}$ corresponding to $\mathbf{O}_\theta$.

Now, fix a labeling $\Theta : k \mapsto \theta_k \in \mathcal{V}(G/S)$ such that θ_k lies in Λ^{pos}_X when c_k is not a pole point. Let $(W^{\eta,\Theta}_I)_{m^\eta_I}$ (resp., $(\overline{W}^{\eta,\Theta}_I)_{m^\eta_I}$) be the locally closed (resp., closed) substack of $(W^\eta_I)_{m^\eta_I}$, corresponding to the condition that for each k, the map $D^\times_{c_k} \to X$ (defined up to $G(\mathcal{O}_{c_k})$-conjugacy) lies in $\mathbf{O}_{\theta_k}$ (resp., $\overline{\mathbf{O}}_{\theta_k}$).

Each piece $(\overline{W}^{\eta,\Theta}_I)_{m^\eta_I}$ is an algebraic stack, the entire $(W^\eta_I)_{m^\eta_I}$ is their union, and the various pieces $(W^{\eta,\Theta}_I)_{m^\eta_I}$ define a stratification.

By construction, we immediately have the following.

Lemma 13.1.2. *The map* (13.1) *defines isomorphisms*

$$(W^{\eta,\Theta}_I)_{m^\eta_I} \simeq \prod_k \mathrm{Gr}^{\eta_k}_{R,\mathcal{P}^k_F} \cap \mathrm{Gr}^{S,\theta_k}_{G,\mathcal{P}^k_F}$$

$$(\overline{W}^{\eta,\Theta}_I)_{m^\eta_I} \simeq \prod_k \mathrm{Gr}^{\eta_k}_{R,\mathcal{P}^k_F} \cap \overline{\mathrm{Gr}}^{S,\theta_k}_{G,\mathcal{P}^k_F}.$$

13.1.3. *Description in families.* The contents of the previous two subsections can be repeated over the base M^η_I rather than for individual fibers. Let us spell out the relevant definitions, amended slightly to meet our future needs.

We define the twisted version of the Beilinson-Drinfeld Grassmannian Gr_{G,M^η_I} over M^η_I to classify data $(m^\eta_I; \mathcal{P}_G, \beta)$ where $m^\eta_I \in M^\eta_I$ with associated generic F-bundle $\mathcal{P}_F$,

$\mathcal{P}_G$ is a G-torsor over C, and β is a reduction of $\mathcal{P}_G$ to $\mathcal{P}_F$ away from $|m_I^\eta|$. The fiber of Gr_{G,M_I^η} over a given point m_I^η is the product

$$(\mathrm{Gr}_{G,M_I^\eta})_{m_I^\eta} \simeq \prod_k \mathrm{Gr}_{G,\mathcal{P}_F^k}.$$

We similarly define Gr_{R,M_I^η} and its connected component $\mathrm{Gr}_{R,M_I^\eta}^\eta$ which are both locally closed ind-subschemes of Gr_{G,M_I^η}.

Now, for simplicity, we focus on the open locus of W_I^η where the pole points are distinct. Fix a labeling

$$\Theta : I \to \mathcal{V}(G/S) \qquad i \mapsto \theta_i.$$

We have the locally closed substack $\mathrm{Gr}_{G,M_I^\eta}^{S,\Theta}$ (resp. closed substack $\overline{\mathrm{Gr}}_{G,M_I^\eta}^{S,\Theta}$) of Gr_{G,M_I^η} whose fiber over $m_I^\eta \in M_I^\eta$ with distinct pole points is the product

$$\prod_{i\in I} \mathrm{Gr}_{G,\mathcal{P}_F^i}^{S,\theta_i} \times \prod_{k\notin I} \mathrm{Gr}_{G,\mathcal{P}_F^k}^{S,0} \quad (\text{resp. } \prod_i \overline{\mathrm{Gr}}_{G,\mathcal{P}_F^i}^{S,\theta_i} \times \prod_{k\notin I} \overline{\mathrm{Gr}}_{G,\mathcal{P}_F^k}^{S,0}).$$

We also have the locally closed substack $W_I^{\eta,\Theta}$ (resp., closed substack $\overline{W}_I^{\eta,\Theta}$) corresponding to the following conditions. For each $i \in I$, the map $D_{c_i}^\times \to X$ (defined up to $G(\mathcal{O}_{c_i})$-conjugacy) lies in $\mathbf{O}_{\theta_i}$ (resp., $\overline{\mathbf{O}}_{\theta_i}$). For each $k \notin I$, the map $D_{c_k}^\times \to X$ (defined up to $G(\mathcal{O}_{c_k})$-conjugacy) lies in $\mathbf{O}_0$ (resp., $\overline{\mathbf{O}}_0$).

By construction, we immediately have the following.

Lemma 13.1.4. *We have canonical isomorphisms*

$$W_I^{\eta,\Theta} \simeq \mathrm{Gr}_{R,M_I^\eta}^\eta \underset{M_I^\eta}{\times} \mathrm{Gr}_{G,M_I^\eta}^{S,\Theta}$$

$$\overline{W}_I^{\eta,\Theta} \simeq \mathrm{Gr}_{R,M_I^\eta}^\eta \underset{M_I^\eta}{\times} \overline{\mathrm{Gr}}_{G,M_I^\eta}^{S,\Theta}.$$

13.2. **The transverse locus.** Let η be an element of $\mathcal{V}(G/S)$, and let $\mathcal{P}_F$ be an F-bundle on the formal punctured disc $D^\times$ with monodromy around the origin equal to the image of η in Λ_A/Λ_{A_0}.

The following is a straightforward reformulation of Proposition 8.3.1.

Proposition 13.2.1.
(1) The intersection

$$\mathrm{Gr}_{R,\mathcal{P}_F}^\eta \cap \mathrm{Gr}_{G,\mathcal{P}_F}^{S,\eta}$$

is a point-scheme.
(2) The inclusion

$$\mathrm{Gr}_{R,\mathcal{P}_F}^\eta \cap \mathrm{Gr}_{G,\mathcal{P}_F}^{S,\eta} \hookrightarrow \mathrm{Gr}_{R,\mathcal{P}_F}^\eta \cap \overline{\mathrm{Gr}}_{G,\mathcal{P}_F}^{S,\eta}$$

is an isomorphism.

Let $C_I^{\eta,+}$ be the subscheme of C_I^η where we require that for $c_I^\eta = \Sigma\, \eta_k \cdot c_k$ such that all η_k belong to $\mathcal{V}(G/S)$. Let $M_I^{\eta,+}$ be the corresponding substack of M_I^η obtained by base change.

Now consider the closed substack of the base change

$$W_I^{\eta,+} \subset W_I^\eta \underset{M_I^\eta}{\times} M_I^{\eta,+}$$

corresponding at the level of fibers to

$$\underset{k}{\Pi} \ \mathrm{Gr}_{R,\mathcal{P}_F^k}^{\eta_k} \cap \overline{\mathrm{Gr}}_{G,\mathcal{P}_F^k}^{S,\eta_k}$$

in terms of the identification of Lemma 13.1.2.

Proposition 13.2.1 immediately implies the following.

Corollary 13.2.2. *The projection $W_I^{\eta,+} \to M_I^{\eta,+}$ is an isomorphism.*

In what follows, we will refer to the substack $W_I^{\eta,+} \subset W_I^\eta$ as the *transverse locus*. By construction, a point $w \in W_I^\eta$ is transverse if and only if the following holds:

(1) w projects to a divisor $c_I^\eta = \Sigma \eta_k \cdot c_k$ with each $\eta_k \in \mathcal{V}(G/S)$,
(2) the associated map $D_{c_k}^\times \to X$ (defined up to $G(\mathcal{O}_{c_k})$-conjugacy) lies in $\mathbf{O}_{\eta_k}$.

14. RELATION TO QUASIMAPS

14.1. An intermediate stack.
We shall now introduce a stack, denoted $Z_{I,P^{op}}$, that mediates between the quasimaps space Z_I and the local model W_I.

Consider first the fiber product $Z_I \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^{op}}$. By definition, it classifies data

$$(c_I \in C^I, \mathcal{P}_P \in \mathrm{Bun}_P, \sigma : C \setminus |c_I| \to \mathcal{P}_P \overset{P}{\times} X|_{C \setminus |c_I|})$$

where σ is a section which factors

$$\sigma|_{C'} : C' \to \mathcal{P}_P \overset{P}{\times} \mathring{X}|_{C'} \to \mathcal{P}_P \overset{P}{\times} X|_{C'},$$

for some open curve $C' \subset C \setminus |c_I|$.

Let $Z_{I,P^{op}}$ be the open subset of the above fiber product that corresponds to the condition that σ factors as

$$\sigma|_{C'} : C' \to \mathcal{P}_P \overset{P}{\times} \mathring{X}^+|_{C''} \to \mathcal{P}_P \overset{P}{\times} X|_{C''},$$

for some (possibly smaller) open subset $C'' \subset C'$.

14.2. Projection onto Z_I.
We have the obvious forgetful map $Z_{I,P^{op}} \to Z_I$. Of course, we do not claim that it is smooth. However, it will be smooth over a large enough open subset.

Let M denote the Levi factor of P^{op}. Define the open substack $\mathrm{Bun}_{M,r} \subset \mathrm{Bun}_M$ to be that for which

$$H^1(C, \mathcal{P}_M \overset{M}{\times} V) = 0,$$

for all M-modules V which appear as subquotients of $\mathrm{Lie}(U^{op})$. For any stack $\mathcal{Q}$ mapping to Bun_M, define the open substack $\mathcal{Q}_r \subset \mathcal{Q}$ to be the fiber product

$$\mathcal{Q}_r = \mathcal{Q} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{M,r}.$$

For $\mu \in \Lambda_{M/[M,M]}$ let Bun_P^μ be the corresponding connected component of Bun_P.

Lemma 14.2.1.

(1) The map $\mathfrak{r} : \mathrm{Bun}_{P,r} \to Bun_G$ is smooth.

(2) Any open substack of finite type $\mathrm{Bun}_G^{\mathrm{fin}} \subset \mathrm{Bun}_G$, is contained in the image of $\mathrm{Bun}_{P,r}^{\mu}$ for a a sufficiently large $\mu \in \Lambda_{M/[M,M]}$, and the restriction of $\mathfrak{r}$ to the inverse image $\mathfrak{r}^{-1}(\mathrm{Bun}_G^{\mathrm{fin}}) \subset \mathrm{Bun}_{P,r}^{\mu}$ has connected fibers.

These assertions are all well-known, except perhaps for the connectedness one, which is implied immediately by the following:

Proof. Using [BFG03], we know there is a nonempty open subset $U_0 \subset \mathrm{Bun}_G$ and coweight $\mu_0 \in \Lambda_G$ such that the restriction of the natural map $p : \mathrm{Bun}_B^{\mu_0} \to \mathrm{Bun}_G$ to the inverse image $p^{-1}(U_0) \subset \mathrm{Bun}_B^{\mu_0}$ has connected fibers. Choose $\mu \in \Lambda_G$ such that U is in the image of the projection

$$\mathrm{Bun}_G \overset{h^{\leftarrow}}{\leftarrow} \mathcal{H}_G^{\mu} \underset{\mathrm{Bun}_G}{\times} U_0.$$

Consider the diagram

$$\mathrm{Bun}_B^{\mu+\mu_0} \overset{h^{\leftarrow}}{\leftarrow} \mathcal{H}_B^{\mu} \underset{\mathrm{Bun}_B}{\times} \mathrm{Bun}_B^{\mu_0} \subset \mathcal{H}_G^{\mu} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^{\mu_0} \overset{h^{\rightarrow}}{\rightarrow} \mathrm{Bun}_B^{\mu_0}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\mathrm{Bun}_G \overset{h^{\leftarrow}}{\leftarrow} \mathcal{H}_G^{\mu} \qquad = \qquad \mathcal{H}_G^{\mu} \overset{h^{\rightarrow}}{\rightarrow} \mathrm{Bun}_G .$$

Since the map

$$\mathrm{Bun}_B^{\mu+\mu_0} \overset{h^{\leftarrow}}{\leftarrow} \mathcal{H}_B^{\mu} \underset{\mathrm{Bun}_B}{\times} \mathrm{Bun}_B^{\mu_0}$$

is a bijection, to prove the lemma, it suffices to show that for a point $x \in U \subset \mathrm{Bun}_G$, the inverse image

$$(h^{\leftarrow})^{-1}(p^{-1}(x)) \subset \mathcal{H}_G^{\mu} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^{\mu_0}$$

is connected. We know that $(h^{\leftarrow})^{-1}(x) \subset \mathcal{H}_G^{\mu}$ is connected. Furthermore, since the rightmost square of the diagram is Cartesian, we know that the restriction of the map

$$\mathcal{H}_G^{\mu} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^{\mu_0} \to \mathcal{H}_G^{\mu}$$

to the inverse image of $(h^{\rightarrow})^{-1}(U_0)$ has connected fibers. By construction, we have that the intersection

$$(h^{\leftarrow})^{-1}(x) \cap (h^{\rightarrow})^{-1}(U_0)$$

is nonempty and dense in $(h^{\leftarrow})^{-1}(x)$, and so we are done.   □

Consider the corresponding open subset

$$Z_{I,P^{op},r}^{\mu} \subset Z_{I,P^{op}}^{\mu} = Z_I \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P^{\mu} .$$

Applying base change with respect to $Z_I \to \mathrm{Bun}_G$, we obtain that the statements of the above lemma apply to the forgetful morphism

$$Z_{I,P^{op},r}^{\mu} \to Z_I.$$

Thus the space $Z_{I,P^{op}}$ can be used for the local study of the space Z_I.

14.3. **Passage to the local model.** One relationship between $Z_{I,P^{op}}$ and W_I is the following obvious one. Consider the map $\mathrm{Bun}_{A_0} \to \mathrm{Bun}_M$ corresponding to our choice of the splitting $A_0 \hookrightarrow M$. For $\eta \in \Lambda_{A_0}$, let μ be its image in $\Lambda_{M/[M,M]}$. Then we have a Cartesian diagram

$$
\begin{array}{ccc}
W_I^\eta & \longrightarrow & Z_{I,P^{op}}^\mu \\
\downarrow & & \downarrow \\
\mathrm{Bun}_{A_0}^\eta & \longrightarrow & \mathrm{Bun}_M^\mu \, .
\end{array}
$$

But as we now explain, in fact $Z_{I,P^{op}}^\mu$ is equivalent in the smooth topology to W_I^η. Recall that Q denotes the subgroup $S \cap P^{op}$, so we have a short exact sequence

$$1 \to Q \to M \to A \to 1.$$

Let Q^0 denote the connected component of Q, which equals the kernel of $M \to A_0$. Our choice of the splitting $A_0 \hookrightarrow M$ gives rise to a decomposition

$$Q \simeq Q^0 \times F.$$

Because of our intended application, we state the following for Q^0 though its assertion and proof use nothing special about Q^0.

Lemma 14.3.1. *Let $\mathrm{Bun}_{Q^0}^{\mathrm{fin}}$ be an open substack of finite type of Bun_{Q^0}, and let $\mathfrak{c} \subset C$ be a finite subset. Then there exists a scheme $\mathcal{Y}$ with a smooth surjective map $\mathcal{Y} \to \mathrm{Bun}_{Q^0}^{\mathrm{fin}}$ with connected fibers, such that the corresponding universal bundle on $C \times \mathcal{Y}$ is trivialized on a Zariski-open subset of $C \times \mathcal{Y}$ containing $\mathfrak{c} \times \mathcal{Y}$.*

Proof. Let $B \subset Q^0$ be a Borel subgroup, let $N \subset B$ be its unipotent radical, and let $B \to B/N = T$ be its Cartan quotient. As we have seen in Lemma 14.2.1, we can replace Bun_{Q^0} by Bun_B^ν for a large enough coweight $\nu \in \Lambda_T$. Namely, we can choose ν large enough so that there is an open $\mathcal{V} \subset \mathrm{Bun}_B^\nu$ that maps smoothly onto $\mathrm{Bun}_{Q^0}^{\mathrm{fin}}$ with connected fibers.

Consider the canonical map $\mathrm{Bun}_B \to \mathrm{Bun}_T$, choose a scheme $\mathcal{Y}_T$ satisfying the assertion of the lemma for Bun_T, and set

$$\mathcal{Y} = \mathcal{Y}_T \underset{\mathrm{Bun}_T}{\times} \mathcal{V}.$$

Let $\mathcal{W}$ be an open subset of $\mathcal{Y} \times C$ over which the T-bundle is trivialized. We may assume that $\mathcal{W}$ is affine over $\mathcal{Y}$. But then the whole B-bundle is trivialized, since an N-bundle over an affine is trivial. $\square$

Observe that the lemma generalizes to families. Namely, if we choose an $\mathcal{S}$-family $\mathfrak{c} \subset C \times \mathcal{S}$ finite over $\mathcal{S}$, then there is a scheme $\mathcal{Y}_\mathcal{S}$ over $\mathcal{S}$ with a smooth surjective map $\mathcal{Y}_\mathcal{S} \to \mathrm{Bun}_{Q^0}^{\mathrm{fin}}$ with connected fibers, such that the corresponding universal bundle on $C \times \mathcal{Y}_\mathcal{S}$ is trivial on a Zariski-open subset of $C \times \mathcal{Y}_\mathcal{S}$ containing the inverse image of $\mathfrak{c}$.

Note as well that the above construction of $\mathcal{Y}_\mathcal{S}$ involves the choice of a coweight $\nu \in \Lambda_{Q^0}$. We write $\mathcal{Y}_\mathcal{S}^\nu$ to denote this dependence.

Consider an $\mathcal{S}$-family $\mathfrak{c} \subset C \times \mathcal{S}$ finite over $\mathcal{S}$, and let $Z_{I,P^{op},\mathfrak{c}}^\mu$, respectively $W_{I,\mathfrak{c}}^\eta$, be the $\mathcal{S}$-family of $Z_{I,P^{op}}^\mu$, respectively W_I^η, of maps whose degeneracy locus belongs to $\mathfrak{c}$.

Lemma 14.3.2. *For $\mu = \eta + \nu$ in $\Lambda_{M/[M,M]}$, we have an identification of $\mathcal{S}$-families*

$$Z^\mu_{I,P^{op},\mathfrak{c}} \underset{\mathrm{Bun}_{Q^0} \times \mathcal{S}}{\times} \mathcal{Y}^\nu_\mathcal{S} \simeq W^\eta_{I,\mathfrak{c}} \underset{\mathcal{S}}{\times} \mathcal{Y}^\nu_\mathcal{S}.$$

Proof. We define a map from the left hand side to the right as follows.

Let $\mathcal{W}_\mathcal{S}$ be the Zariski open subset of $C \times \mathcal{Y}_\mathcal{S}$ on which the universal Q^0-bundle is trivialized. The base change of the left hand side provides a reduction of the P^{op}-bundle to an R-bundle over $\mathcal{W}_\mathcal{S}$.

On the other hand, let $\mathcal{Y}_\mathfrak{c} \subset \mathcal{Y}_\mathcal{S}$ be the inverse image of $\mathfrak{c}$, and let $\mathcal{Y}_\mathcal{S} \setminus \mathcal{Y}_\mathfrak{c}$ be its complement. The quasimap data of the left hand side equips the P^{op}-bundle with a reduction to $Q \simeq Q^0 \times F$ over $\mathcal{Y}_\mathcal{S} \setminus \mathcal{Y}_\mathfrak{c}$.

By construction, these two reductions are compatible over the intersection $\mathcal{W}_\mathcal{S} \cap (\mathcal{Y}_\mathcal{S} \setminus \mathcal{Y}_\mathfrak{c})$ in the sense that we may define a global R-bundle by gluing the R-bundle over $\mathcal{W}_\mathcal{S}$ with the R-bundle induced from the F-bundle over $\mathcal{Y}_\mathcal{S} \setminus \mathcal{Y}_\mathfrak{c}$. The quasimap data of the right hand side is simply the tautological reduction over $\mathcal{Y}_\mathcal{S} \setminus \mathcal{Y}_\mathfrak{c}$ of the induced R-bundle to the F-subbundle from which it was induced. $\qquad\square$

We conclude that the space W^η_I can be used for the local study of the space $Z^\mu_{I,P^{op}}$.

14.4. Behavior with respect to the transverse locus.
Consider the data $(\mathcal{P}_{P^{op}}, \sigma)$ of a quasimap on the formal disk: $\mathcal{P}_{P^{op}}$ is a P^{op}-bundle on D, and σ is a map

$$D^\times \to \mathcal{P}_{P^{op}} \overset{P^{op}}{\times} \mathring{X}^+.$$

Alternatively, the data can be interpreted as a point of the fiber product

$$\mathrm{Maps}(D, pt/P^{op}) \underset{\mathrm{Maps}(D^\times, pt/P^{op})}{\times} \mathrm{Maps}(D^\times, \mathring{X}^+/P^{op}).$$

On the one hand, using the projections $P^{op} \to A$ and $\mathring{X}^+ \to A$, we see that $(\mathcal{P}_{P^{op}}, \sigma)$ gives rise to a point of Gr_A. We interpret this an element $\eta \in \Lambda_A$.

On the other hand, $(\mathcal{P}_{P^{op}}, \sigma)$ defines a $P^{op}(\mathcal{O})$-orbit $\mathbf{O}'_{P^{op}}$ in $\mathring{X}^+(\mathcal{K})$, and hence a $G(\mathcal{O})$-orbit $\mathbf{O}_\lambda$ in $\mathring{X}(\mathcal{K})$, for some $\lambda \in \mathcal{V}(G/S)$.

By Sect. 8.2, the following conditions are equivalent:

(1) $\eta = \lambda \in \Lambda_A$.
(2) $\mathbf{O}'_{P^{op}}$ is open in $\mathbf{O}_\lambda$.

If these conditions are satisfied, we say that $(\mathcal{P}_{P^{op}}, \sigma)$ is *transverse*. We say that a point of the stack $Z_{I,P^{op}}$ is transverse at c (resp., globally transverse), if its restriction to the formal disk around c (resp., for every c) is transverse.

It is clear that for a given point of $z_I \in Z_I$, the locus of points in the preimage

$$\mathfrak{r}^{-1}(z_I) \subset Z_{I,P^{op}}$$

that are transverse at a given finite collection of points of C is open. Moreover, $\mathfrak{r}^{-1}(z_I) \cap Z^\mu_{I,P^{op},r}$ is non-empty for μ large enough.

14.4.1. Recall now that in Sect. 13.2 we introduced the notion of transversality for points of W_I^η. We claim that this notion matches with the above notion of global transversality under the identification of Lemma 14.3.2:

$$Z_{I,P^{op},\mathfrak{c}}^\mu \underset{\mathrm{Bun}_{Q^0} \times \mathcal{S}}{\times} \mathcal{Y}_\mathcal{S}^\nu \simeq W_{I,\mathfrak{c}}^\eta \underset{\mathcal{S}}{\times} \mathcal{Y}_\mathcal{S}^\nu$$

Indeed, suppose that the pair $(\mathcal{P}_{P^{op}}, \sigma)$ as above is such that $\mathcal{P}_{P^{op}}$ is induced from an A_0-bundle $\mathcal{P}_{A_0}$. As in Sect. 13.1.1, such data give rise to an F-bundle $\mathcal{P}_F$ on $D^\times$ and a point of $\mathrm{Gr}_{R,\mathcal{P}_F}^\eta \cap \mathrm{Gr}_{G,\mathcal{P}_F}^{S,\lambda}$. We see that in both cases, the transversality condition is that $\eta = \lambda$.

We denote by $Z_{I,P^{op}}^{\mu,+} \subset Z_{I,P^{op}}^\mu$ the locus of transverse points. By the above, this is a closed ind-substack.

14.4.2. *Transversality in the horospherical case.* Let us assume that X is horospherical. Recall that Bun_P is naturally a locally closed substack of $^\star Z_I$ and for any η we have an ind-locally closed embedding

$$j_I : C_I^\eta \times \mathrm{Bun}_P^{\mu'} \hookrightarrow {}^\star Z_I^{\mu'+\eta}.$$

Let $^\star Z_{I,P^{op}}^{\mu,\mu'}$ denote the analogue of the stack $Z_{I,P^{op}}^\mu$ when we use $^\star Z_I^{\mu'}$ instead of Z_I. Note that the fiber product

$$\mathrm{Bun}_P^{\mu'} \underset{^\star Z_I^{\mu'}}{\times} {}^\star Z_{I,P^{op}}^{\mu'} \subset \mathrm{Bun}_P^{\mu'} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^{op}}^{\mu'}$$

identifies naturally with $\mathrm{Bun}_M^{\mu'}$.

From j_I we obtain a map

$$C_I^\eta \times \mathrm{Bun}_M^\mu \to {}^\star Z_{I,P^{op}}^{\mu+\eta,\mu}.$$

The following is easy to check.

Lemma 14.4.3. *The above map is an isomorphism on the transverse locus*

$$^\star Z_{I,P^{op}}^{\mu+\eta,\mu,+} \subset {}^\star Z_{I,P^{op}}^{\mu+\eta,\mu}.$$

It induces an isomorphism

$$C_I^\eta \underset{\mathrm{Bun}_A}{\times} \mathrm{Bun}_M^\mu \simeq Z_{I,P^{op}}^{\mu+\eta,+},$$

The second map of the lemma induces a map

$$M_I^\eta \to W_I^{\eta,+},$$

which is the inverse of the one of Corollary 13.2.2.

Part IV. Proofs–B

In this part we will apply the local model W_I^η of Sect. 12 to prove two main technical assertions of Part 1, namely Theorem 4.2.1 (in Sect. 15) and Theorem 5.4.7 (in Sect. 16).

15. Convolution and local model

15.1. **Proof of Theorem 4.2.1.** Our proof is organized into several steps, each involving the use of more specialized structure than the previous. Recall that since $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ is semisimple, by the decomposition theorem, the convolution $\mathrm{Conv}_I(\mathcal{A})$ is semisimple as well. Our goal is to see that each of its summands is the intersection cohomology sheaf with constant coefficients of a connected component of an untwisted local stratum ${}'Z_{\mathring{I}}^{\Theta} \subset {}'Z_{\mathring{I}}$.

15.1.1. *Reformulation of convolution.* We begin with the trivial observation that since $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ is semisimple, it suffices to prove the theorem for simple objects. These are the intersection cohomology sheaves $\mathcal{A}_G^{\lambda_I}$ of the product of $G(\mathcal{O})$-orbit closures

$$\overline{\mathrm{Gr}}_G^{\lambda_I} = \prod_{i \in I} \overline{\mathrm{Gr}}_G^{\lambda_i} \subset (\mathrm{Gr}_G)^I, \text{ where } \lambda_I : I \to \Lambda_G^+.$$

This allows for the following slight reformulation of the convolution. By restricting the $(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I$-torsor

$$\widehat{Z}_{\mathring{I}} \to Z_{\mathring{I}}$$

to the closure ${}'\overline{Z}_{\mathring{I}}^{0} \subset Z_{\mathring{I}}$ of the untwisted basic stratum, we obtain a $(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I$-torsor

$$'\widehat{\overline{Z}}_{\mathring{I}}^{0} \to {}'\overline{Z}_{\mathring{I}}^{0}.$$

The twisted product

$$'\widetilde{\overline{Z}}_{\mathring{I}}^{\lambda_I} = {}'\widehat{\overline{Z}}_{\mathring{I}}^{0} \overset{(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I}{\times} \overline{\mathrm{Gr}}_G^{\lambda_I}$$

fits into a diagram

$$'Z_{\mathring{I}} \overset{h^{\leftarrow}}{\leftarrow} {}'\widetilde{\overline{Z}}_{\mathring{I}}^{\lambda_I} \overset{h^{\rightarrow}}{\to} {}'\overline{Z}_{\mathring{I}}^{0}$$

in which $h^{\rightarrow}$ is the evident projection, and $h^{\leftarrow}$ is the modification map. In other words, the diagram results from restricting the diagram

$$Z_{\mathring{I}} \overset{h^{\leftarrow}}{\leftarrow} \widetilde{Z}_{\mathring{I}} \overset{h^{\rightarrow}}{\to} Z_{\mathring{I}}$$

along the inclusion

$$'\widetilde{\overline{Z}}_{\mathring{I}}^{\lambda_I} \hookrightarrow \widetilde{Z}_{\mathring{I}}.$$

In the category $\mathbf{P}(\widetilde{Z}_{\mathring{I}})$, we have a canonical isomorphism

$$'\widetilde{\mathrm{IC}}_{\mathring{I}}^{\lambda_I} \simeq {}'\mathrm{IC}_{\mathring{I}}^{0} \widetilde{\boxtimes} \mathcal{A}_G^{\lambda_I}$$

identifying the intersection cohomology sheaf of $'\widetilde{\overline{Z}}{}^{\lambda_I}_{\mathring{I}}$ with the twisted product. Thus we have a canonical isomorphism

$$\mathrm{Conv}_I(\mathcal{A}_G^{\lambda_I}) \simeq \sum_k {}^p H^k(h^{\leftarrow}_{\lambda_I!}('\widetilde{\mathrm{IC}}{}^{\lambda_I}_{\mathring{I}})).$$

We will prove the theorem using this formulation of convolution.

15.1.2. *Moving the pole points.* Suppose we have established the following.

Claim 15.1.3. *For any fixed configuration of pole points $c_I \in C^I$, the restriction of each irreducible summand of the convolution $\mathrm{Conv}_I(\mathcal{A}_G^{\lambda_I})$ to the based ind-substack $'Z_{c_I} \subset 'Z_I$ is the middle-extension of the constant sheaf on a component of some untwisted based local stratum $'Z_{c_I}^{\Theta} \subset 'Z_{c_I}$.*

Then we may complete the proof of Theorem 4.2.1 as follows. Observe that the closure of the untwisted basic stratum is a product

$$'\overline{Z}{}^0_{\mathring{I}} = '\overline{Z}{}^0_{\emptyset} \times \mathring{C}^I,$$

and we have a commutative diagram

$$\begin{array}{ccc}
'Z_{\mathring{I}} & \overset{h^{\leftarrow}}{\leftarrow} & '\widetilde{Z}_{\mathring{I}} \\
\pi \downarrow & & \downarrow h^{\rightarrow} \\
\mathring{C}^I & \overset{p}{\leftarrow} & '\overline{Z}{}^0_{\emptyset} \times \mathring{C}^I
\end{array}$$

where the map p is the obvious projection. We have an isomorphism

$$h^{\rightarrow}_!('\mathrm{IC}^0 \widetilde{\boxtimes} \mathcal{A}_G^{\lambda_I}) \simeq '\mathrm{IC}^0_{\emptyset} \boxtimes V_{gr}^{\lambda_I}$$

where we write $V_{gr}^{\lambda_I}$ for the graded constant sheaf on $\mathring{C}^I$ with fiber the tensor product of the irreducible representations of $\check{G}$ of highest weights $\lambda_i \in \Lambda_G^+$, for $i \in I$, with canonical grading given by the action of the principal nilpotent. Thus we have an isomorphism

$$\pi_! h^{\leftarrow}_!('\mathrm{IC}^0 \widetilde{\boxtimes} \mathcal{A}_G^{\lambda_I}) \simeq p_!('\mathrm{IC}^0_{\emptyset} \boxtimes V_{gr}^{\lambda_I}).$$

We conclude that the pushforward via π of the convolution $\mathrm{Conv}_I(\mathcal{A}_G^{\lambda_I})$ is a constant sheaf. Therefore the top cohomology of the pushforward of each of its irreducible summands is also a constant sheaf (over the generic point of the base it is a sublocal system of a constant local system). Coupled with Claim 15.1.3, this establishes Theorem 4.2.1. Thus our aim in what follows is to prove Claim 15.1.3.

15.1.4. *Convolution for local model.* We begin by constructing a version of convolution for the local model. Consider the $(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I$-torsor

$$\widehat{W}^{\eta}_{\mathring{I}} \to W^{\eta}_{\mathring{I}}$$

that classifies data

$$(w, \mu, \tau)$$

where $w \in W^{\eta}_{\mathring{I}}$, with pole points $c_I \in \mathring{C}^I$ and R-bundle $\mathcal{P}_R \in \mathrm{Bun}_R$, μ is an isomorphism of G-bundles

$$\mu : D_{|c_I|} \times G \overset{\sim}{\to} \mathcal{P}_R \overset{R}{\times} G|_{D_{|c_I|}},$$

and τ is an isomorphism

$$\tau : D \times |c_I| \xrightarrow{\sim} D_{|c_I|}.$$

The twisted product

$$\widetilde{W}_{\mathring{I}}^{\eta} = \widehat{W}_{\mathring{I}}^{\eta} \overset{(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I}{\times} (\mathrm{Gr}_G)^I$$

classifies data

$$(w, \mathcal{P}_G, \alpha)$$

where $w \in W_{\mathring{I}}^{\eta}$, with pole points $c_I \in \mathring{C}^I$ and R-bundle $\mathcal{P}_R \in \mathrm{Bun}_R$, $\mathcal{P}_G \in \mathrm{Bun}_G$, and α is an isomorphism of G-bundles

$$\alpha : \mathcal{P}_R \overset{R}{\times} G|_{C \setminus |c_I|} \xrightarrow{\sim} \mathcal{P}_G|_{C \setminus |c_I|}.$$

To a point $c \in C$, and a point $\tilde{w} \in \widetilde{W}_{\mathring{I}}^{\eta}$, with pole points $c_I \in \mathring{C}^I$, R-bundle $\mathcal{P}_R \in \mathrm{Bun}_R$, section

$$\sigma : C \setminus |c_I| \to \mathcal{P}_R \overset{R}{\times} X|_{C \setminus |c_I|},$$

G-bundle $\mathcal{P}_G \in \mathrm{Bun}_G$, and isomorphism

$$\alpha : \mathcal{P}_R \overset{R}{\times} G|_{C \setminus |c_I|} \xrightarrow{\sim} \mathcal{P}_G|_{C \setminus |c_I|},$$

we may associate two elements $\bar{\sigma}_c^{\leftarrow}, \bar{\sigma}_c^{\rightarrow} \in \mathcal{V}(\mathring{X})$ as follows. For some open curve $C' \subset C \setminus |c_I|$, the section σ factors

$$\sigma|_{C'} : C' \to \mathcal{P}_R \overset{R}{\times} \mathring{X}|_{C'} \to \mathcal{P}_R \overset{R}{\times} X|_{C'}.$$

For any trivialization of the restriction $\mathcal{P}_R|_{D_c}$, we obtain a projection

$$\mathcal{P}_R \overset{R}{\times} \mathring{X}|_{D_c^{\times}} \to \mathring{X}.$$

For any choice of formal coordinate at c, we define $\bar{\sigma}_c^{\leftarrow}$ to be the composition of the resulting identification $D^{\times} \xrightarrow{\sim} D_c^{\times}$ with the restriction

$$\sigma|_{D_c^{\times}} : D_c^{\times} \to \mathcal{P}_R \overset{R}{\times} \mathring{X}|_{D_c^{\times}} \to \mathring{X}.$$

Similarly, for any trivialization of the restriction $\mathcal{P}_G|_{D_c}$, we obtain a projection

$$\mathcal{P}_G \overset{G}{\times} \mathring{X}|_{D_c^{\times}} \to \mathring{X}.$$

For any choice of formal coordinate at c, we define $\bar{\sigma}_c^{\rightarrow}$ to be the composition of the resulting identification $D^{\times} \xrightarrow{\sim} D_c^{\times}$ with the restriction of the composition

$$\alpha \circ \sigma|_{D_c^{\times}} : D_c^{\times} \to \mathcal{P}_R \overset{R}{\times} \mathring{X}|_{D_c^{\times}} \xrightarrow{\sim} \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{D_c^{\times}} \to \mathring{X}.$$

By Corollary 3.3.3, the resulting elements $\bar{\sigma}_c^{\leftarrow}, \bar{\sigma}_c^{\rightarrow} \in (X(\mathcal{K}) - (X(\mathcal{K}) - \mathring{X}(\mathcal{K})))/G(\mathcal{O})$ are independent of the choices.

For a labelling $\lambda_I : I \to \Lambda_G^+$, we define the opposite labelling $\lambda_I^{op} : I \to \Lambda_G^+$ to be that for which λ_i^{op} is the dominant coweight in the Weyl group orbit through $-\lambda_i$, for $i \in I$. We define the ind-substack

$$'\widetilde{W}_{\mathring{I}}^{\eta, \lambda_I} \subset {}'\widehat{W}_{\mathring{I}}^{\eta} \overset{(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I}{\times} \overline{\mathrm{Gr}}_G^{\lambda_I^{op}}$$

to consist of those points such that the associated element $\bar{\sigma}_c^{\rightarrow} \in \mathcal{V}(\mathring{X})$ is trivial for all $c \in C$, and we define the ind-substack

$$'\widetilde{\overline{W}}_{\mathring{I}}^{\eta,\lambda_I} \subset '\widehat{W}_{\mathring{I}}^{\eta} \overset{(G(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))^I}{\times} \overline{\mathrm{Gr}}_G^{\lambda_I^{op}}$$

to be its closure. We have the evident projection

$$'W_{\mathring{I}}^{\eta} \leftarrow '\widetilde{\overline{W}}_{\mathring{I}}^{\eta,\lambda_I}$$

which we denote by $h_{\lambda_I}^{\leftarrow}$.

Recall that $\mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$ is semisimple. We define the local version of the convolution functor

$$\mathrm{Conv}_I^{\eta} : \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I} \to \mathbf{P}('W_{\mathring{I}}^{\eta})$$

on simple objects by the formula

$$\mathrm{Conv}_I^{\eta}(\mathcal{A}_G^{\lambda_I}) = \sum_k {}^p H^k(h_{\lambda_I!}^{\leftarrow}('\widetilde{\mathrm{IC}}_{\mathring{I}}^{\eta,\lambda_I}))$$

where $'\widetilde{\mathrm{IC}}_{\mathring{I}}^{\eta,\lambda_I}$ denotes the intersection cohomology sheaf of $'\widetilde{\overline{W}}_{\mathring{I}}^{\eta,\lambda_I}$.

15.1.5. *Relating the convolutions.* Recall that in Sect. 14, we constructed a correspondence relating the quasimaps space and the local model

$$W_{I,\mathfrak{c}}^{\eta} \leftarrow W_{I,\mathfrak{c}}^{\eta} \underset{\mathcal{S}}{\times} \mathcal{Y}_{\mathcal{S}}^{\nu} \simeq Z_{I,P^{op},\mathfrak{c}}^{\mu} \underset{\mathrm{Bun}_{Q^0} \times \mathcal{S}}{\times} \mathcal{Y}_{\mathcal{S}}^{\nu} \to Z_{I,P^{op},\mathfrak{c}}^{\mu} \to Z_I$$

Given a a labelling $\lambda_I : I \to \Lambda_G^+$, we may extend the convolution maps of the preceding two sections to a Cartesian diagram

$$
\begin{array}{ccccccccc}
'\widetilde{\overline{W}}_{I,\mathfrak{c}}^{\eta} & \leftarrow & '\widetilde{\overline{W}}_{I,\mathfrak{c}}^{\eta} \underset{\mathcal{S}}{\times} \mathcal{Y}_{\mathcal{S}}^{\nu} & \simeq & '\widetilde{\overline{Z}}_{I,P^{op},\mathfrak{c}}^{\mu} \underset{\mathrm{Bun}_{Q^0} \times \mathcal{S}}{\times} \mathcal{Y}_{\mathcal{S}}^{\nu} & \to & '\widetilde{\overline{Z}}_{I,P^{op},\mathfrak{c}}^{\mu} & \to & '\widetilde{\overline{Z}}_I \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
'W_{I,\mathfrak{c}}^{\eta} & \leftarrow & 'W_{I,\mathfrak{c}}^{\eta} \underset{\mathcal{S}}{\times} \mathcal{Y}_{\mathcal{S}}^{\nu} & \simeq & 'Z_{I,P^{op},\mathfrak{c}}^{\mu} \underset{\mathrm{Bun}_{Q^0} \times \mathcal{S}}{\times} \mathcal{Y}_{\mathcal{S}}^{\nu} & \to & 'Z_{I,P^{op},\mathfrak{c}}^{\mu} & \to & 'Z_I
\end{array}
$$

We will use the following properties of this diagram. They all follow from the results of Sect. 14:

- For large enough parameters, the horizontal maps are all smooth.
- For fixed pole points c_I, given any based local stratum $'Z_{c_I}^{\Theta}$, we may arrange so that the corresponding stratum of $'W_{c_I}^{\eta,\Theta}$ is transverse at the pole points.
- The fibers of the rightward pointing horizontal maps are connected.

Since intersection cohomology sheaves are preserved by smooth pullback, to calculate the convolution locally in the smooth topology, it suffices to understand what happens for the local model. By the last point, to calculate how the cohomology sheaves of the convolution might twist along a locus of constructibility, again it suffices to understand what happens for the local model.

15.1.6. *Proof of Claim 15.1.3.* By the discussion immediately preceding, to prove Claim 15.1.3, it suffices to prove the following assertion for the local model.

Proposition 15.1.7. *For any fixed configuration of pole points $c_I \in C^I$, the restriction of the convolution to an untwisted based local stratum ${}'W_{c_I}^{\eta,\Theta} \subset {}'W_{c_I}^{\eta}$ which is transverse at its pole points is constant.*

If we were only attempting to prove that the restriction were *locally constant*, we could simply appeal to the factorization pattern of the local model. Namely, the factorization diagram for $W_{c_I}^{\eta}$ naturally extends to a factorization diagram for its convolution. Furthermore, the local version of the convolution and its factorization readily generalize to the local model $W_{c_I,1}^{\eta}$ over the open curve $C \backslash c$. Using this added freedom, we could reduce the assertion of the proposition to the following cases:

(1) the restriction of $\mathrm{Conv}_{i,1}^{\eta_i}(\mathcal{A}_i)$ to the untwisted based local stratum ${}'W_{c_i,1}^{\eta_i,\eta_i} \subset {}'W_{c_i,1}^{\eta_i}$ which is transverse at its single pole point c_i and has no other degeneracies,

(2) the restriction of the basic sheaf ${}'\mathrm{IC}^0$ to the smooth locus of the local model ${}'W_{\emptyset,1}^{\eta'}$ with no pole points but otherwise arbitrary degeneracies.

In the second case, the restriction of the intersection cohomology sheaf to the smooth locus is constant. In the first case, the stratum ${}'W_{c_i,1}^{\eta_i,\eta_i}$ reduces to a copy of the stack of F-local systems on $C \setminus (c_i \cup c)$ whose induced $\pi_0(S)$-local system is trivializable. Thus each of its connected component is isomorphic to the classifying space pt/F. Since all sheaves on the classifying space are locally constant, we would be done. Unfortunately, to factorize the local model, we must make base changes with disconnected fibers. Thus in the course of the above argument, we could potentially unwind local systems. Because we seek the stronger statement that our sheaves are in fact constant, we will proceed with a slightly modified approach.

Proof. First, let θ be the total degree of Θ, and let $\eta' = \eta - \theta$ be the total excess degeneracies. We base change to the open subset

$$W_{c_I}^{\theta,\eta'} = W_{c_I}^{\eta} \underset{C^{\eta}}{\times} (C^{\theta} \times C^{\eta'})_{\mathrm{disj}}$$

where we separate apart the excess degeneracies.

Next, let $C_{I,ex}^{\theta}$ be the space of pairs (c_I^{θ}, e) of disjoint divisors on C, with the first $c_I^{\theta} \in C_I^{\theta}$. Define the space $W_{c_I,ex}^{\theta}$ to be that classifying the data of a pair of disjoint divisors $(c_I^{\theta}, e) \in C_{I,ex}^{\theta}$ together with the usual local model data of total degree θ on the open curve $C \setminus e$ with associated divisor c_I^{θ}. We have the obvious projection

$$W_{c_I}^{\theta,\eta'} \to W_{c_I,ex}^{\theta}$$

where we only keep track of the fibers of the local model at the degeneracy points of the first type. Moreover, we clearly may extend the convolution maps so that we have

a Cartesian diagram

$$
\begin{array}{ccc}
\widetilde{\overline{W}}_{c_I}^{\theta,\eta',\lambda_I} & \rightarrow & \widetilde{\overline{W}}_{c_I,ex}^{\theta,\lambda_I} \\
\downarrow & & \downarrow \\
W_{c_I}^{\theta,\eta'} & \rightarrow & W_{c_I,ex}^{\theta}
\end{array}
$$

where the first vertical map is simply the base change to $(C_I^\theta \times C^{\eta'})_{\mathrm{disj}}$ of the usual local convolution map. Thus we see that to prove the proposition, it suffices to prove it for the convolution given by the second vertical map.

Fix a point $c \in C$ disjoint from the pole points, and consider the base change of the previous constructions to the open curve $C \setminus c$. Let $(C \setminus c)_{I,ex}^\theta$ be the space of pairs (c_I^θ, e) of disjoint divisors on $C \setminus c$, with the first $c_I^\theta \in (C \setminus c)_I^\theta$. Let $W_{c_I,ex,1}^\theta$ be the space classifying the data of a pair of disjoint divisors $(c_I^\theta, e) \in (C \setminus c)_{I,ex}^\theta$ together with the usual local model data of total degree θ on the open curve $C \setminus (c \cup e)$ with associated divisor c_I^θ.

Define the group scheme

$$
\mathcal{F}_{ex}^\theta \rightarrow (C \setminus c)_{I,ex}^\theta
$$

to be that classifying the data of a pair

$$
(c_I^\theta, e) \in (C \setminus c)_{I,ex}^\theta,
$$

an F-bundle $\mathcal{P}_F$ on the complement $C \setminus (c \cup e)$, and a trivialization of $\mathcal{P}_F$ above the divisor $|c_I^\theta| \subset C \setminus (c \cup e)$. The group structure is given by tensor product of F-bundles.

Observe that the relative group $\mathcal{F}_{ex}^\theta$ naturally acts on the convolution map

$$
\begin{array}{ccc}
\widetilde{\overline{W}}_{c_I,ex,1}^{\theta,\lambda_I} \underset{(C\setminus c)_{I,ex}^\theta}{\times} \mathcal{F}_{ex}^\theta & \rightarrow & \widetilde{\overline{W}}_{c_I,ex,1}^{\theta,\lambda_I} \\
\downarrow & & \downarrow \\
W_{c_I,ex,1}^\theta \underset{(C\setminus c)_{I,ex}^\theta}{\times} \mathcal{F}_{ex}^\theta & \rightarrow & W_{c_I,ex,1}^\theta
\end{array}
$$

To establish the proposition, we will prove the a priori stronger statement that the restriction of the convolution to an untwisted transverse based local stratum is constant as an equivariant object for $\mathcal{F}_{ex}^\theta$. But note that the action of $\mathcal{F}_{triv}^\theta$ is free so that the stronger assertion follows from the weaker analogue.

Let $\mathcal{F}_{triv}^\theta \subset \mathcal{F}_{ex}^\theta$ be the subgroup of those F-bundles with trivial monodromy around any point in $C \setminus c$. Similarly, let $W_{c_I,triv,1}^\theta \subset W_{c_I,ex,1}^\theta$ be the subspace of points whose associated generic F-bundle has trivial monodromy around the excess divisor in $C \setminus c$. Then it is easy to see that inclusion induces an isomorphism of quotient stacks

$$
W_{c_I,triv,1}^\theta / \mathcal{F}_{triv}^\theta \xrightarrow{\sim} W_{c_I,ex,1}^\theta / \mathcal{F}_{ex}^\theta.
$$

Thus to prove the stronger assertion, it suffices to prove it for the first quotient of the above isomorphism. So we are left to see that the restriction of the convolution to an untwisted transverse based local stratum of $W_{c_I,triv,1}^\theta$ is constant. Observe that here the excess divisor is playing no role. In other words, we may forget it completely to

obtain a Cartesian diagram

$$
\begin{array}{ccc}
\widetilde{\widetilde{W}}{}^{\theta}_{c_I,triv,1} & \rightarrow & \widetilde{\widetilde{W}}{}^{\theta,\lambda_I}_{c_I,1} \\
\downarrow & & \downarrow \\
W^{\theta}_{c_I,triv,1} & \rightarrow & W^{\theta,\lambda_I}_{c_I,1}
\end{array}
$$

with smooth horizontal maps. Now, each connected component of an untwisted transverse based local stratum of $W^{\theta}_{c_I,1}$ is nothing more than the classifying space pt/F. Thus we are left to see that the F-action on the fiber of the convolution is trivial. But the F-action factors through $F \to \pi_0(S)$ and all of our constructions can be made on the closed subspace of untwisted quasimaps, i.e. those quasimaps with trivial associated $\pi_0(S)$-bundle. Thus the F-action is trivial and we are done. $\qquad\square$

15.2. Proof of Corollary 4.2.6.
The corollary will follow from the following refinement of Theorem 4.2.1.

Proposition 15.2.1. *For any object $\mathcal{A}_I \in \mathbf{P}_{G(\mathcal{O})}(\mathrm{Gr}_G)^{\otimes I}$, if we fix isomorphisms*

$$
\mathrm{Conv}(\mathcal{A}_i) \simeq \sum_{\theta \in \Lambda^+_{\mathring{X}}} {}' \mathrm{IC}^{\theta}(V^{\theta}_i),
$$

where V^{θ}_i are vector spaces, then we have a canonical isomorphism

$$
\mathrm{Conv}_I(\mathcal{A}_I) \simeq \sum_{\Theta : I \to \Lambda^+_{\mathring{X}}} {}' \mathrm{IC}^{\Theta}_{\mathring{I}}(\otimes_{i \in I} V^{\Theta(i)}_i).
$$

Proof. Fix a labelling $\Theta : I \to \Lambda^+_{\mathring{X}}$, and a point $z \in {}'Z^{\Theta}_{\mathring{I}}$. By our previous results, to calculate the stalk of $\mathrm{Conv}_I(\mathcal{A}_I)$ at z, we may instead consider the stalk of $\mathrm{Conv}^{\eta}_I(\mathcal{A}_I)$ at a point $w \in {}'W^{\eta,\Theta}_{\mathring{I}}$ such that w is transverse at its pole points. Then via the factorization pattern for the convolution discussed in the previous section, this stalk is the tensor product of the following:

(1) the stalk of $\mathrm{Conv}^{\eta_i}_{i,1}(\mathcal{A}_i)$ at a point $w_i \in {}'W^{\eta_i}_{i,1}$ with a degeneracy of degree η_i at its single pole point c_i and no other degeneracies,

(2) the stalk of the basic sheaf ${}'\mathrm{IC}^0$ at a point $w' \in {}'W^{\eta'}_{\emptyset,1}$ with no pole points but otherwise arbitrary degeneracies.

In the second case, we are taking the stalk of the intersection cohomology sheaf at a smooth point, and so we may canonically identify the stalk with the trivial vector space $\mathbb{C}$. But as in the previous proposition, $\mathrm{Conv}^{\eta_i}_{i,1}(\mathcal{A}_i)$ is constant along the locus appearing in the first case. $\qquad\square$

16. Specialization of sheaves

16.1. Proof of Theorem 5.4.7(1).
By construction, we have a surjection of abelian groups $\pi_0(S_0) \to \pi_0(S)$. We denote the kernel of this projection by *ker*. Consider the closed substack of the basic stratum

$$
{}''Z^0_{0,\mathring{I}} \subset Z^0_{0,\mathring{I}}
$$

consisting of quasimaps whose generic $\pi_0(S_0)$-bundle is induced from a *ker*-bundle. We refer to it as the *partially twisted basic stratum*.

Lemma 16.1.1. *The restriction of the nearby cycles $\psi({}'\mathrm{IC}^0_{\mathring{I}})$ to the basic stratum $Z^0_{0,\mathring{I}} \subset Z_{0,I}$ is isomorphic to the constant sheaf of rank one on the partially twisted basic stratum ${}''Z^0_{0,\mathring{I}} \subset Z^0_{0,\mathring{I}}$.*

Proof. Choose a point in the basic horospherical stratum $Z^0_{0,\mathring{I}}$ and identify a neighborhood of it in the family $\mathcal{Z}_I$ with a neighborhood of some point in a local model $\mathcal{W}^\eta_I$. Since the family $\mathring{\mathcal{X}}^+ \to \mathcal{B}$ is constant, we see that the relevant neighborhood in $\mathcal{W}^\eta_I$ is also constant. Since the finite group F coincides with $\pi_0(S_0)$, the assertion follows from the definitions. $\square$

Observe that the nearby cycles $\psi({}'\mathrm{IC}^0_{\mathring{I}})$ are of the form

$$\psi({}'\mathrm{IC}^0_{\mathring{I}}) \simeq \mathbb{C}_{\mathring{\mathcal{C}}^I} \boxtimes \psi({}'\mathrm{IC}^0_\emptyset).$$

The only irreducible object of $\mathbf{Q}(Z_{0,I})$ which is such a product is the basic sheaf ${}'\mathrm{IC}^0_{0,\mathring{I}}$. By the lemma, this occurs exactly once as an irreducible constituent of $\psi({}'\mathrm{IC}^0_{\mathring{I}})$.

16.2. **Proof of Theorem 5.4.7(2).** To understand the nearby cycles of arbitrary objects of $\mathbf{Q}(Z_I)$, we need a version of the local model for the family $\mathcal{X} \to \mathcal{B}$. The results from Section III extend to this setting with only notational changes. Thus we keep the discussion here brief and focus on what we will need specifically for the current proof. For a finite set I, and $\eta \in \Lambda_A$, in analogy with the local model W^η_I for the ind-stack Z_I, we write $\mathcal{W}^\eta_I$ for the local model for the ind-stack $\mathcal{Z}_I$.

For convenience, we work with the local model $\mathcal{W}^\eta_{I,\mathbb{A}^1}$ with respect to the curve $\mathbb{A}^1$. By choosing local coordinates on the curve C, we have the following.

Theorem 16.2.1. *For any point $z \in \mathcal{Z}_I$, there is a finite set K, and for $k \in K$, finite sets J_k, coweights $\eta_k \in \Lambda_A$, and points $w_k \in \mathcal{W}^{\eta_k}_{J_k,\mathbb{A}^1}$ such that the following properties hold.*

(1) *In the smooth topology, there is a neighborhood of $z \in \mathcal{Z}_I$ and a neighborhood of*

$$\prod_{k\in K} w_k \in \prod_{k\in K} W^{\eta_k}_{J_k,\mathbb{A}^1}$$

   which are isomorphic.
(2) *The isomorphism identifies the restrictions of the untwisted local strata to the neighborhoods.*
(3) *The isomorphism identifies the restrictions of the horospherical strata to the neighborhoods.*
(4) *For all $k \in K$, the degeneracy locus of the point w_k is a single point of $\mathbb{A}^1$.*
(5) *If z lies in the fiber $\mathcal{Z}_{0,I} \subset \mathcal{Z}_I$, then for all $k \in K$, the point w_k is transverse at all points of $\mathbb{A}^1$.*

Proof. All of the assertions are natural generalizations of the results of Section III, except for the last. For that, what remains to be seen is the following. Consider

the substack of the family $\mathcal{W}_I^\eta$ consisting of elements with no degenerations, that is, elements whose defining section σ extends to a map from all of C to the open G-orbit $\mathring{X}$. We must check that this substack is smooth. But by the results of Section III, it is isomorphic in the smooth topology to the analogous substack of $\mathcal{Z}_I$. And this substack is smooth by Corollary 5.2.3. $\square$

We will apply the theorem to prove the following constructibility result. Recall that the stack $Z_{0,\emptyset}$ is the disjoint union of the horospherical strata

$$Z_{0,\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \subset Z_{0,\emptyset},$$

for decompositions $\mathfrak{U}(\theta^{\mathrm{pos}})$ of positive coweights $\theta^{\mathrm{pos}} \in \Lambda_{A_0}^{\mathrm{pos}}$.

Theorem 16.2.2. *For any positive coweight $\theta^{\mathrm{pos}} \in \Lambda_{A_0}^{\mathrm{pos}}$, and decomposition $\mathfrak{U}(\theta^{\mathrm{pos}})$, the restriction of the nearby cycles $\psi('\mathrm{IC}_\emptyset^0)$ to the stratum $Z_{0,\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \subset Z_{0,\emptyset}$ is locally constant.*

Proof. It suffices to prove an analogue of the statement for the local model discussed above. More precisely, for a positive coweight $\eta^{\mathrm{pos}} \in \Lambda_A^{\mathrm{pos}}$, consider the nearby cycles functor

$$\psi : \mathbf{P}(\mathcal{W}_{\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}}}) \to \mathbf{P}(\mathcal{W}_{0,\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}}}),$$

and the intersection cohomology sheaf $'\mathrm{IC}_\emptyset^{\eta^{\mathrm{pos}},0}$ of the untwisted basic stratum $'\mathcal{W}_\emptyset^{\eta^{\mathrm{pos}},0,\mathbb{A}^1} \subset \mathcal{W}_{\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}}}$. By Theorem 16.2.1, it suffices to prove that the restriction of the nearby cycles $\psi('\mathrm{IC}_{\emptyset,0}^{\eta^{\mathrm{pos}}})$ to the horospherical stratum $\mathcal{W}_{0,\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}},\mathfrak{U}_0(\eta^{\mathrm{pos}})} \subset \mathcal{W}_{0,\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}}}$, where $\mathfrak{U}_0(\eta^{\mathrm{pos}})$ is the trivial decomposition, is locally constant in a neighborhood of a point $w \in \mathcal{W}_{0,\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}},\mathfrak{U}_0(\eta^{\mathrm{pos}})}$. This assertions follows by transversality: the action of the affine line $\mathbb{A}^1$ on itself by translation lifts to an action on the family $\mathcal{W}_{\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}}}$ which is locally transitive on the horospherical stratum $\mathcal{W}_{0,\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}},\mathfrak{U}_0(\eta^{\mathrm{pos}})} \subset \mathcal{W}_{0,\emptyset,\mathbb{A}^1}^{\eta^{\mathrm{pos}}}$. $\square$

Finally, we finish the proof of Theorem 5.4.7(2) as follows. By Theorem 5.4.7(1), in the category $\mathbf{P}_\mathcal{H}(Z_{0,I})$, we have a filtration

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \psi('\mathrm{IC}_{\mathring{I}}^0)$$

such that there is an isomorphism

$$\mathcal{F}_2/\mathcal{F}_1 \simeq {}'\mathrm{IC}_{0,\mathring{I}}^0.$$

Furthermore, the restrictions of the subsheaf $\mathcal{F}_1$ and quotient sheaf $\psi('\mathrm{IC}_{\mathring{I}}^0)/\mathcal{F}_2$ to the basic untwisted stratum $'Z_{0,\mathring{I}}^0 \subset Z_{0,I}$ are zero.

To prove the theorem, we must check that $H_G^I(\mathcal{A}, \mathcal{F}_1)$ and $H_G^I(\mathcal{A}, \psi('\mathrm{IC}_{\mathring{I}}^0)/\mathcal{F}_2)$ are bad sheaves. By the definition of bad sheaves, it suffices to prove that $H_G^I(\mathcal{A}, \mathcal{R})$ is a bad sheaf for each simple constituent $\mathcal{R}$ of either sheaf. We must deal with two possible types of simple constituents $\mathcal{R}$.

First, we must consider the case when $\mathcal{R}$ is the intersection cohomology sheaf of a component of the partially twisted basic stratum and the component is not untwisted.

In this case, we may argue using the results of [GN04a] as follows. Since we explicitly know what the convolution looks like for the intersection cohomology sheaf of the entire basic stratum and of the untwisted basic stratum, we know what the convolution looks like for the intersection cohomology sheaf of our partially twisted component. Namely, it must be a sum of intersection cohomology sheaves of components of partially twisted local strata which themselves are not untwisted. Thus the convolution is a bad sheaf and the assertion in this case is verified.

Now, what remains is the case when the restriction of $\mathcal{R}$ to the entire basic stratum is zero. Recall that the nearby cycles $\psi('\mathrm{IC}^0_{\mathring{I}})$ are of the form

$$\psi('\mathrm{IC}^0_{\mathring{I}}) \simeq \mathbb{C}_{\mathring{C}^I} \boxtimes \psi('\mathrm{IC}^0_{\emptyset}).$$

By Theorem 16.2.2, the restriction of the nearby cycles $\psi('\mathrm{IC}^0_{\emptyset})$ to each stratum

$$Z_{0,\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})} \subset Z_{0,\emptyset}$$

is locally constant, and so each of its simple constituents $\mathcal{R}$ is as well. Since the restriction of $\mathcal{R}$ to the basic stratum is zero, we conclude that $\mathcal{R}$ must be of the form

$$\mathbb{C}_{\mathring{C}^I} \boxtimes \mathrm{IC}_{0,\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}(\mathcal{L}), \text{ for } \theta^{\mathrm{pos}} \neq 0,$$

where $\mathcal{L}$ is some Hecke equivariant local system on $Z_{0,\emptyset}^{\mathfrak{U}(\theta^{\mathrm{pos}})}$. Now the theorem is an immediate consequence of Proposition 10.5.1 of Sect. 10.5.

16.3. **Proof of Proposition 5.4.9.** This is very similar to the proof of Corollary 4.2.6. It is immediately implied by the following analogue of Proposition 15.2.1.

Proposition 16.3.1. *For any object $\mathcal{Q}_I \in \mathbf{Q}(Z)^{\otimes I}$, if we fix isomorphisms*

$$\Psi(\mathcal{Q}_i) \simeq \sum_{\theta \in \Lambda_{A_0}} {'\mathrm{IC}}^{\theta}(V_i^{\theta}),$$

where V_i^{θ} are vector spaces, then we have a canonical isomorphism

$$\Psi_I(\gamma_I(\mathcal{Q}_I)) \simeq \sum_{\Theta: I \to \Lambda_{A_0}} {'\mathrm{IC}}^{\Theta}_{\mathring{I}}(\otimes_{i \in I} V_i^{\Theta(i)}).$$

Proof. Fix a labelling $\Theta : I \to \Lambda_{\mathring{X}}^+$, and a point $z \in {'Z}^{\Theta}_{\mathring{I}}$. By our previous results, to calculate the stalk of $\Psi_I(\gamma_I(\mathcal{A}_I))$ at z, we may instead consider the stalk of $\Psi_I^{\eta}(\gamma(\mathcal{A}_I))$ at a point $w \in {'W}^{\eta,\Theta}_{\mathring{I}}$ such that w is transverse at its pole points. Then via factorization, this stalk is the tensor product of the following:

(1) the stalk of $\Psi_{i,1}^{\eta_i}(\mathcal{A}_i)$ at a point $w_i \in {'W}^{\eta_i}_{i,1}$ with a degeneracy of degree η_i at its single pole point c_i and no other degeneracies,

(2) the stalk of $\Psi_{\emptyset,1}^{\eta'}('\mathrm{IC}^0)$ at a point $w' \in {'W}^{\eta'}_{\emptyset,1}$ with no pole points but otherwise arbitrary degeneracies.

In the second case, by Corollary 5.2.3, we are taking the stalk of the nearby cycles of the intersection cohomology sheaf at a smooth point of the family, and so we may canonically identify the stalk with the trivial vector space $\mathbb{C}$.

Thus to prove the proposition, it remains to canonically identify the stalks of $\Psi_{i,1}^{\eta_i}(\mathcal{A}_i)$ at points $w_i, w_i' \in {}'W_{i,1}^{\eta_i}$ satisfying the conditions of the first case. Note that the condition is equivalent to the points being transverse everywhere.

For a finite set I, and $\eta \in \Lambda_A$, define the scheme $\mathrm{Bun}_{F,I}^\eta$ to be that classifying data

$$(c_I^\eta; \mathcal{P}_F, \alpha)$$

where $c_I^\eta \in C_I^\eta$, $\mathcal{P}_F \in \mathrm{Bun}_F$, and α is a trivialization

$$\alpha : |c_I^\eta| \times F \to \mathcal{P}_F|_{|c_I^\eta|}.$$

We have a diagram of étale maps

$$M_{I,1}^\eta \leftarrow M_{I,1}^\eta \underset{C_I^\eta}{\times} \mathrm{Bun}_{F,I}^\eta \to M_{I,1}^\eta$$

in which the left map is the obvious projection, and the right map is defined by

$$((c; c_I^\eta; \mathcal{P}_{A_0}, \tau_0), (c_I^\eta; \mathcal{P}_F, \alpha)) \mapsto (c; c_I^\eta; \mathcal{P}_{A_0} \otimes \mathcal{P}_F, \tau_0).$$

This extends to a diagram of étale maps

$$'\mathcal{W}_{I,1}^\eta \leftarrow{}' \mathcal{W}_{I,1}^\eta \underset{C_I^\eta}{\times} \mathrm{Bun}_{F,I}^\eta \to{}' \mathcal{W}_{I,1}^\eta$$

where the left map is the obvious projection, and the right map is defined in a similar way to the factorization map.

Now since $w_i, w_i' \in {}'W_{i,1}^{\eta_i}$ have the same degeneracies and are transverse everywhere, they are related by the above correspondence. Thus we conclude that the stalk of $\Psi_{i,1}^{\eta_i}(\mathcal{A}_i)$ is the same at w_i, w_i'. $\qquad\square$

16.4. **Proofs of Propositions 5.4.1 and 6.3.4.**

16.4.1. *The first assertions.* We simultaneously establish the first assertions of both propositions, in the process checking their compatibility.

Set $I = \{1, 2\}$ and $\Theta = (\theta_1, \theta_2)$, and consider the intersection cohomology sheaf $'\mathrm{IC}_{\mathring{I}}^\Theta$ of the stratum closure

$$'\overline{Z}_{\mathring{I}}^\Theta \subset {}'Z_{\mathring{I}}.$$

Our goal is to understand:

(1) the middle-extension of $'\mathrm{IC}_{\mathring{I}}^\Theta$ across the divisor in Z_I where the pole points collide, and specifically its structure along the stratum $'Z^{\theta_1+\theta_2}$ of the divisor;
(2) the nearby cycles of $'\mathrm{IC}_{\mathring{I}}^\Theta$ at the special fiber $Z_{0,\mathring{I}}$ in the specializing family $\mathcal{Z}_{\mathring{I}}$, and specifically its structure along the stratum $'Z_0^\Theta$ of the special fiber.

To do this, we will focus on the entire specializing family $\mathcal{Z}_I$ in a neighborhood of the stratum $'Z_0^{\theta_1+\theta_2}$ in the collision divisor of the special fiber.

Fix a point z_0 in the stratum $'Z_0^{\theta_1+\theta_2}$. In the étale topology, there is a neighborhood of z_0 in the family $\mathcal{Z}_I$ which is isomorphic to a neighborhood of a point w_0 in the local model $\mathcal{W}_I^\eta$. Furthermore, we may arrange so that w_0 is transverse at its single pole point.

Once we have removed a point from the curve C, factorization provides a neighborhood of w_0 in the local model $\mathcal{W}_I^\eta$ which is isomorphic to a product of neighborhoods of points

(1) $w_{0,1}$ in the local model $\mathcal{W}_{I,1}^{\theta_1+\theta_2}$ such that $w_{0,1}$ has a degeneracy of degree $\theta_1 + \theta_2$ at its single pole point, and no other degeneracies,

(2) $w'_{0,1}$ in the local model $\mathcal{W}_{\emptyset,1}^{\eta'}$ with no pole points but otherwise arbitrary degeneracies.

In the second case, by Corollary 5.2.3, our calculation reduces to the middle-extension and nearby cycles of the intersection cohomology sheaf along the smooth locus of the family $\mathcal{W}_{\emptyset,1}^{\eta'}$.

In the first case, the closure of the stratum ${}'W_{\mathring{I},1}^{\theta_1+\theta_2,\Theta}$ in the family $\mathcal{W}_{I,1}^{\theta_1+\theta_2}$ lies in the globally transverse locus $\mathcal{W}_{I,1}^{\theta_1+\theta_2,+}$. This is canonically isomorphic to the constant family with fiber the untwisted transverse base ${}'M_{I,1}^{\theta_1+\theta_2,+}$.

Thus we see in this case as well that our calculation reduces to the middle-extension and nearby cycles of the intersection cohomology sheaf along the smooth stack ${}'M_{I,1}^{\theta_1+\theta_2,+}$. This provides a canonical isomorphism of the stalks of these sheaves at any points. By our previous results, this identification of stalks is all that is needed to confirm the first assertions of Propositions 5.4.1 and 6.3.4.

16.4.2. *The second assertions.* The second assertion of Proposition 5.4.1 follows immediately from the description of the strata and ind-scheme structure of $X(\mathcal{K}) - (X - \mathring{X})(\mathcal{K})$.

Since Λ_X^{pos} is strictly convex, to establish the scond assertion of Proposition 6.3.4, it suffices to prove the stronger statement: if ${}'\mathrm{IC}^\theta$ is a constituent of ${}'\mathrm{IC}^{\theta_1} \circledast {}'\mathrm{IC}^{\theta_2}$, then $(\theta_1 + \theta_2) - \theta \in \Lambda_X^{\mathrm{pos}}$.

This follows immediately from the two assertions of Proposition 5.4.1.

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