NOTES ON THE SUMMER SCHOOL

ZHAOLIN LI

Contents

1. Introduction

These are the notes that I took when I was a graduate student at the University of Minnesota during the summer school of the Relative Langlands Program in the summer of 2024.

2. June 3

2.1. **Yiannis Sakellaridis.** Let F be a global field or a local field of the definition of a reductive group G, k the filed of coefficients, and usually $k = \mathbb{C}$.

Let $\pi = \otimes' \pi_\nu \hookrightarrow C^\infty([G])$ be an irreducible automorphic representation of $G(\mathbb{A}) = \otimes' G(F_\nu)$, where $[G] = G(F)\backslash G(\mathbb{A})$. Fix a large enough finite set of places S of F, outside of which π is unramified, i.e., $\pi^{G(\mathcal{O}_\nu)} \neq 0$. Then the Hecke algebra $\mathcal{H}(G(F_\nu), G(\mathcal{O}_\nu))$ acts on $\pi^{G(\mathcal{O}_\nu)}$ through a character $\chi : \mathcal{H}(G(F_\nu), G(\mathcal{O}_\nu)) \to \mathbb{C}$, which corresponds to a Langlands parameter up to $G^\vee$-conjugacy via the Satake isomorphism:

$$\varphi_\nu : W_{F_\nu} \to {}^L G = G^\vee \rtimes W_{F_\nu},$$

or

$$\varphi_\nu : W_{F_\nu}/I_\nu \cong \langle \mathrm{Frob}_\nu \rangle \to G^\vee \rtimes \langle \mathrm{Frob}_\nu \rangle : \mathrm{Frob}_\nu \mapsto g \cdot \mathrm{Frob}_\nu,$$

Date: June 8, 2024.

hence an element in $G^{\vee,\mathrm{s.s}}$ up to conjugacy. More generally, at every ν, the LLC associates to π_ν a parameter φ_μ, which is not necessarily unramified. Then $\varphi = (\varphi_\nu)_\nu$ will be used to define L-functions of π.

Another important input will be a representation $r : {}^LG \to \mathrm{GL}(V)$.

Definition 2.1.

$$L(\pi, r, s) := \prod_\nu L(\pi_\nu, r, s),$$

where

$$L(\pi_\nu, r, s) := \frac{1}{\det(1 - q_\nu^{-s} \cdot r \circ \varphi_\nu|_{V^{I_\nu}})}$$

at non-Archimedean places.

Example 2.2. *Let $G = \mathrm{GL}_2$. At each unramified place, write $\varphi_\nu : \mathrm{Frob}_\nu \mapsto \begin{pmatrix} \alpha_\nu & \\ & \beta_\nu \end{pmatrix}$, and $r = \mathrm{Sym}^n\mathrm{Std}$, then*

$$L(\pi_\nu, r, s) = \frac{1}{\prod_{i=0}^n (1 - q_\nu \alpha_\nu^i \beta_\nu^{n-i})}.$$

Remark 2.3. *Fix $r : {}^LG \to \mathrm{GL}(V)$, and consider the diagram*

$$\begin{array}{ccc} {}^LG & \xrightarrow{r} & \mathrm{GL}(V) \\ \downarrow & & \uparrow \\ W_{F_\nu} & \xrightarrow{|\cdot|^s} & \mathbb{R}_+^\times \subset \mathbb{C}^\times \end{array},$$

and we can replace $|\cdot|^s$ with the cyclotomic character if $k = \overline{\mathbb{Q}_l}$. Then we have

$$L(\pi, r, s) = L(\pi, r_s, 0),$$

where $r_s = r \otimes |\cdot|^s$.

Remark 2.4. *In the whole series, we may take $s \in \frac{1}{2}\mathbb{Z}$, and choose $q_\nu^{1/2} \in k$. In fact, no choices are really made if we use the arithmetic version of ${}^LG$, the C-group, see [Buzzard-Gee].*

Recall that when $F = \mathbb{Q}$, and fix $N \in \mathbb{Z}$, and a Dirichlet character χ of $(\mathbb{Z}/N\mathbb{Z})^\times$, we have

$$\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_+^\times \prod_p (\mathbb{Z}_p^\times \cap (1 + N\mathbb{Z}_p)) \cong (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times,$$

and when χ is trivial, we have essentially the Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

Riemann proved that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty y^{s/2} \sum_{n=1}^\infty e^{-n^2\pi y}\, \mathrm{d}y.$$

Write

$$\vartheta(y) := \sum_{n=1}^\infty e^{-n^2\pi y}$$

to be the Jacobi theta series, then the Poisson summation formula tells us

$$\vartheta(y) = y^{-1/2}\vartheta(y^{-1}),$$

which gives the functional equation of $\zeta(s)$ relating $\zeta(s)$ and $\zeta(1-s)$.

The following is the Iwasawa-Tate reformulation. Write $z = y^{1/2} \in \mathbb{R}_+^\times = \mathbb{Q}^\times\backslash\mathbb{A}^\times/\prod_p \mathbb{Z}_p^\times$, the above integral can be written as

$$\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} |z|^s \sum_{\gamma\in\mathbb{Q}} \Phi(\gamma z)\,\mathrm{d}^\times z, \tag{2.1}$$

where $\Phi \in \mathcal{S}(\mathbb{A}^1)$, the space of Schwartz functions on the affine line $\mathbb{A}^1$, and $\Phi = \prod_{p\le\infty}\Phi_p$, where

$$\Phi_p(x) = \begin{cases} 1_{\mathbb{Z}_p}(x) & \text{if} \quad p<\infty \\ e^{-\pi x^2} & \text{if} \quad p=\infty \end{cases}.$$

Then (2.1) is

$$\int_{\mathbb{A}^\times} \Phi(z)|z|^s\,\mathrm{d}^\times z = \prod_{p\le\infty}\int_{\mathbb{Q}_p^\times} \Phi_p(z)|z|^s\,\mathrm{d}^\times z.$$

Note that we have the multiplicative group $G = \mathbb{G}_m$ acting on $\mathbb{A}^1$, and hence on $\mathcal{S}(\mathbb{A}^1)$ with unramified factors

$$\int_{\mathbb{Q}_p^\times} 1_{\mathbb{Z}_p}(z)|z|^s\,\mathrm{d}^\times z = 1 + p^{-s} + p^{-2s} + \cdots = \frac{1}{1-p^{-s}}$$

if we assume $\mathrm{vol}(\mathbb{Z}_p^\times) = 1$.

We have the Hecke algebra $\mathcal{H}(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times) = \mathbb{C}[\mathbb{Q}_p^\times/\mathbb{Z}_p^\times] \overset{\mathrm{val}}{=} \mathbb{C}[z]$ acting on $\mathcal{S}(\mathbb{Q}_p)^{\mathbb{Z}_p^\times}$, which is freely generated by $1_{\mathbb{Z}_p}$. And the same is true with $1_{\mathbb{Z}_p^\times}$. The Zeta functions appear when we think of these as modules with $\langle\cdot,\cdot\rangle$.

Indeed, for $h \in \mathcal{H}(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times) = \mathbb{C}[z]$, since the unitary dual of $\mathbb{Z}$ is $\mathbb{S}^1$, using the Parserval identity, we have

$$\langle h * 1_{\mathbb{Z}_p}(x)|\,\mathrm{d}x|^{1/2}, 1_{\mathbb{Z}_p}(x)|\,\mathrm{d}x|^{1/2}\rangle = \int_{\mathbb{S}^1} \frac{\widehat{h}(z)\,\mathrm{d}^\times z}{(1-p^{-1/2}z)(1-p^{-1/2}z^{-1})}.$$

Then let us talk about more general periods of automorphic forms. The name **period** came from the consideration of the embedding of the associated Shimura varieties $\mathrm{Sh}_H \hookrightarrow \mathrm{Sh}_G$ of a reductive subgroup H of a reductive group G. In the following, the formulas to be presented depend on both choices of Haar measures and choices of automorphic forms $f_\pi \in \pi$.

Assume $F = \mathbb{F}_q(\Sigma)$ for some smooth projective curve Σ over $\mathbb{F}_q$, and the representation π is unramified everywhere, and as tempered or generic as meaningful. For $f_\pi \in \pi^{G(\widehat{\mathcal{O}})}$, even if we do not specify which $k^\times$-multiple, the answer of no-vanishing questions is still meaningful, and the same choice is expected to work for all formulas.

Example 2.5 (Hecke Case)**.** *Consider $f_\pi \in \pi \hookrightarrow \mathcal{C}^\infty([\mathrm{GL}_2])$, and the following integral*

$$\int_{F^\times\backslash\mathbb{A}^\times} f_\pi\begin{pmatrix} a & \\ & 1\end{pmatrix}|a|^s\,\mathrm{d}^\times a = L\left(\pi, \mathrm{std}, \frac{1}{2}+\mathrm{s}\right).$$

We may let $G = \mathbb{G}_m \times \mathrm{GL}_2$ and $H = \mathbb{G}_m^\Delta \subset G$, and view $f_\pi \otimes |\cdot|^s \in \mathcal{C}^\infty([\mathbb{G}_m \times \mathrm{GL}_2])$.

Example 2.6 (Rankin-Selberg Case). *Let* $H = \mathrm{GL}_n \overset{\Delta}{\hookrightarrow} G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, *and*

$$\int_{[H]} f_\pi = L(\pi, \mathrm{std} \otimes \mathrm{std}, 1/2).$$

The proof uses the Whittaker/Fourier normalization:

$$\int_{[N]} f_\pi(n)\psi^{-1}(n)\,\mathrm{d}n = 1,$$

where N *is the subgroup of* GL_n *consisting of all strictly upper triangular matrices, and* $\psi : N(F)\backslash N(\mathbb{A}) \to F\backslash\mathbb{A} \to \mathbb{C}^\times$ *is a generic character.*

Example 2.7 (Gross-Prasad-Ichino-Ikeda Case). *Let* $H = \mathrm{SO}(V) \overset{\Delta}{\hookrightarrow} G = \mathrm{SO}(V) \times \mathrm{SO}(V \oplus F)$, *then conjecturally we have*

$$\left|\int_{[H]} f_\pi\right|^2 = L(\pi, \mathrm{std} \otimes \mathrm{std}, 1/2),$$

under the normalization that

$$\|f_\pi\|^2 = \int_{[G]} f_\pi \overline{f_\pi} = |G_\varphi| L(\pi, \mathrm{Ad}, 1),$$

where φ *is the Langlands parameter of* π, *and* G_φ *is its centralizer. A more common version of this conjecture is*

$$\left|\int_{[H]} f_\pi\right|^2 = |G_\varphi|^{-1} \frac{L(\pi, \mathrm{std} \otimes \mathrm{std}, 1/2)}{L(\pi, \mathrm{Ad}, 1)}$$

under the normalization that $\|f_\pi\|^2 = 1$.

All these forms are expeceted in the Langlands dual of $X = H\backslash G$, or rather of $M = T^*X$, which is some symplectic variety $M^\vee$ with ${}^LG$.

- For $X = \mathrm{GL}_n\backslash\mathrm{GL}_n \times \mathrm{GL}_{n+1}$, $M^\vee = T^*(\mathrm{std} \otimes \mathrm{std})$,
$$\left|\int_{[H]} f_\pi\right|^2 = L(\pi, T_0 M^\vee \cong M^\vee, 1/2).$$
- For $X = \mathrm{SO}_n\backslash\mathrm{SO}_n \times \mathrm{SO}_{n+1}$, $M^\vee = \mathrm{std} \otimes \mathrm{std}$,
$$\left|\int_{[H]} f_\pi\right|^2 = L(\pi, \mathrm{std} \otimes \mathrm{std}, 1/2).$$
- For $X = (N, \psi)\backslash G$, $M^\vee = 0$,
$$\left|\int_{[H]} f_\pi\right|^2 = 1 = L(\pi, 0, 1/2).$$
- For $X = H$ with $G = H \times H$, $M^\vee = T^*H^\vee$ but with twisted $H^\vee \times H^\vee$-action with Chevalley involution on the second factor. For simplicity, assume H is semisimple, $\pi = \tau \otimes \overline{\tau}$
$$|f_\tau|^2 = \left|\int_{[H]} f_\pi\right| = \sum_x \sqrt{L(\pi, T_x M^\vee)},$$
where the sum is over all fixed points of φ_π on $M^\vee$, and has cardinality $|G_\varphi|$. It makes sense as $L(\pi, \mathrm{Ad}, 1)$.

2.2. **David Ben-Zvi.** The goal of the Relative Langlands Program is to study the functoriality of the Langlands correspondence.

Given a reductive group G over a field F, we have both the automorphic theory $\mathcal{A}_G$ and the spectral theory $\mathcal{B}_{G^\vee}$, where $G^\vee$ is the Langlands dual group of G over the coefficient field k.

Usually we view $\mathcal{A}_G$ as functions on G and want to upgrade information of G to $\mathcal{A}_G$. We hope there will be certain relation $\mathcal{A}_M$ between $\mathcal{A}_H$ and $\mathcal{A}_G$ if there is some relation M between two reductive groups H and G. Dually we veiw $\mathcal{B}_{G^\vee}$ as the algebraic geometry of the Langlands parameters, and hope similar things between $\mathcal{B}_{H^\vee}$ and $\mathcal{B}_{G^\vee}$. Moreover, we want some compatibility like

$$\begin{array}{ccc} \mathcal{B}_{H^\vee} & \overset{\mathcal{B}_{M^\vee}}{\to} & \mathcal{B}_{G^\vee} \\ \wr\Vert & & \wr\Vert \\ \mathcal{A}_H & \overset{\mathcal{A}_M}{\to} & \mathcal{A}_G \end{array}$$

A natural question is what kind of things are $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$ and a suitable model is the $4d$ Topological Quantum Field Theory, which is basically a linear representation of topology of manifolds of dimension ≤ 4. Let us use the notation $\mathcal{Z}$ to denote it.

Roughly speaking,

- for a 4-manifold M, $\mathcal{Z}(M) \in k$ will be a scalar, for a 3-manifold Ξ, $\mathcal{Z}(\Xi)$ will be a vector space over k, and for a 2-manifold Σ, $\mathcal{Z}(\Sigma)$ will be a k-linear category and finally 1-manifolds correspond to 2-categories. The disjoint union of 4-manifolds with 3-manifolds transfers to scalar multiplications, and the disjoint union of two 3-manifolds corresponds to the tensor product.
- $\mathcal{Z}$ is functorial under bordisms, cut and paste.
- It is locally constant under deformations.

There are some examples of relations among them. For instance, let Σ be a manifold, which is not necessarily 2 dimensional, then $\mathcal{Z}(\Sigma \times \mathbb{S}^1)$ is determined by $\mathcal{Z}(\Sigma)$ as the dimension or the cocenter. And if we have certain map f on Σ, then $\mathcal{Z}(f)$ acts on $\mathcal{Z}(\Sigma)$, and we can recover the trace of $\mathcal{Z}(f)$ on $\mathcal{Z}(\Sigma)$ as $\mathcal{Z}$ of the mapping torus of f.

There are some sources of 2-manifolds. Let Σ be a smooth projective curve over $\overline{\mathbb{F}_q}$, then it behaves like a Riemann surfaces, so we may view it as a 2-manifold. If Ξ is a projective curve over $\mathbb{F}_q$, we may view it as a projective curve over $\overline{\mathbb{F}_q}$ with the Frobenius action, hence a 3-manifold according to the above paragraph. Another way to justify this is that $\Sigma := \Xi \times_{\mathrm{Spec}\mathbb{F}_q} \mathrm{Spec}\overline{\mathbb{F}_q}$ can be viewed as the special fiber of Ξ, or Ξ is the mapping torus of the Frob on Σ. Here we view $\mathrm{Spec}\mathbb{F}_q$ as the torus.

But when we go to the arithmetic world of the TQFT, there are some problems due to [M. Kim]:

- We do not have a precise category of manifolds. We may view number fields and function fields as 3-manifolds, and local fields or curves over $\overline{\mathbb{F}_q}$ as 2-manifolds, and think the **geometric local field** $\mathbb{C}((t))$ as a 1-manifold.
- There is no good theory of bordisms.

There are three key structures:

- Tr(Frob).

- Given Ξ and $x \in \Xi$, write $\Xi := \Xi \setminus \{x\} \bigsqcup_{D^\times} D$, this corresponds to the local-global principal. For example, we have
$$\mathrm{Spec}\mathbb{Z} = \mathrm{Spec}\mathbb{Z}[1/p] \bigsqcup_{\mathrm{Spec}\mathbb{Q}_p} \mathrm{Spec}\mathbb{Z}_p$$
at every prime p.
- Given Σ, we can consider some $x \in \Sigma$ and $\Sigma \times I$, where $I = [0,1]$. We may choose for example certain small ball $\mathbb{S}^2$ such that $(x, 1/2) \in \mathbb{S}^2 \subset \Sigma \times I$, and consider the disjoint union
$$\Sigma \bigsqcup \mathbb{S}^2 \to \Sigma,$$
which gives
$$\mathcal{Z}(\Sigma) \otimes \mathcal{Z}(\mathbb{S}^2) \to \mathcal{Z}(\Sigma),$$
certain operators on $\mathcal{Z}(\Sigma)$. Moreover, we can collapse $\Sigma \bigsqcup \mathbb{S}^2$ to obtain the doubled Σ with doubled x, which we may denote by Σ_x. Then we have two maps
$$\begin{tikzcd} \Sigma \arrow[dr] & & \Sigma \arrow[dl] \\ & \Sigma_x = \Sigma \bigsqcup_{\Sigma\setminus\{x\}} \Sigma = \Sigma \bigsqcup_{D_x} (D \bigsqcup_{D^\times} D) & \end{tikzcd}.$$
This corresponds to the Hecke functions. Moreover, since we may put many balls $\mathbb{S}^2$ inside $\Sigma \times I$, and move around them in three directions, using the locally constant property above, $\mathcal{Z}(\mathbb{S}^2)$ is an associate algebra, and $\mathcal{Z}(\Sigma)$ is a $\mathcal{Z}(\mathbb{S}^2)$-module. We may view $\mathbb{Z}(\mathbb{S}^2)$ arising for a particular x as the local Hecke algebra, and the tensor products over all x as the global version, which may be further viewed as the **observables**, with $\mathcal{Z}(\Sigma)$ being **States**.

From the above three structures, for a manifold Σ, we get observables $\mathrm{Obs}_{\mathcal{Z}(\Sigma)}$ acting on $\mathcal{Z}(\Sigma)$, plus the local-global principal and the trace relation given by Frob.

From the TQFT point of view, we have a diamond

$$\begin{tikzcd} & \text{curves}/\mathbb{F}_q \arrow[dr, dash] & \\ \text{local fields} \arrow[ur, dash] \arrow[dr, dash] & & \text{curves}/\overline{\mathbb{F}_q} \\ & \text{geometric loca fields } \mathbb{C}((t)) \arrow[ur, dash] & \end{tikzcd}. \tag{2.2}$$

At each corner, we expect there is certain isomorphism
$$\mathcal{A}_G(\Sigma) \cong \mathcal{B}_{G^\vee}(\Sigma),$$
that is compatible with the actions of observables. The $\nwarrow$-direction is from geometry to arithmetic, and the $\nearrow$-direction is from local to global. The dimensions are $1, 2$ and 3 from bottom to the top. In more details, the $\mathcal{B}$-theory $\mathcal{B}_{G^\vee}$ is the algebraic geometry of $\mathrm{Loc}_{G^\vee}\Sigma$, the local systems. We may view it as maps $\pi_1\Sigma \to G^\vee$, where $\pi_1\Sigma$ can be viewed as certain Galois group, or (etale) locally constant maps from Σ to $\cdot/G^\vee$. And for each Σ, we may associate the volume form $\omega(\mathrm{Loc}_{G^\vee}\Sigma)$, or its derived version $\mathrm{R}\Gamma(\omega(\mathrm{Loc}_{G^\vee}\Sigma))$. For the $\mathcal{A}$-theory, it is referred to the topology of spaces of G-bundles on Σ. We may associate Σ things like $\mathrm{Bun}_G\Sigma$, $[G]/K$, $H^*([G]/K)$ for some compact K, or $H^*(\mathrm{Bun}_G\Sigma(\mathbb{F}_q))$, serving as automorphic functions on Σ. In fact, The bottom theory is the geometric local Langlands, the left one is the local Langlands correspondence and the right one is the geometric Langlands.

2.3. Yiannis Sakellaridis.

Remark 2.8. *In the duality $(G, M) \leftrightarrow (G^\vee, M^\vee)$, there are two things that are not very clear:*

- *there is no combinatorial description of this duality, which is expected to be a version over* $\mathrm{Spec}\mathbb{Z}$,
- *and the hyperspherical condition is mysterious. There are examples that (G, M) fail to be coisotropic, and correspondingly $M^\vee$ is not smooth affine. We need to have a closer look at this condition.*

Let $G = \mathbb{G}_m$ act on $X = \mathbb{A}^1$, then we have the theta series

$$\Theta : \mathcal{S}(\mathbb{A}^1) \to \mathcal{C}^\infty([G]) : \Phi \mapsto \left(g \overset{\Theta_\Phi}{\mapsto} \sum_{\gamma \in F} \Phi(\gamma g) \right).$$

Let $\chi = |\cdot|^s \in \pi$, integration over $[G]$ gives a functional

$$\pi \otimes \mathcal{S}(\mathbb{A}^1) \to \mathbb{C} : \chi \otimes \Phi \mapsto \int_{[G]} \chi(g) \Theta_\Phi(g)\, \mathrm{d}g.$$

More generally, consider $H \subset G$ being a subgroup, and for $f_\pi \in \pi$ of G, we can also consider the H-period

$$\int_{[H]} f_\pi \in \mathbb{C}.$$

We can rewrite it in the following way. Let $X = H\backslash G$, which is assumed to be smooth affine. We also assume $X(F) = H(F)\backslash G(F)$. We have similar theta series

$$\Theta : \mathcal{S}(X(\mathbb{A})) = \otimes' \mathcal{S}(X(F_\nu)) \to \mathcal{C}^\infty([G]) : \Phi \mapsto \left(g \mapsto \sum_{\gamma \in X(F)} \Phi(\gamma g) \right),$$

where the restricted tensor product is taking with respect to $1_{X(\mathcal{O}_\nu)}$. We claim that there is some $f'_\pi \in \pi$ such that

$$\int_{[G]} \Theta_\Phi(g) f_\pi(g)\, \mathrm{d}g = \int_{[H]} f'_\pi(h)\, \mathrm{d}h.$$

This can be shown by assuming $\Phi(hg) = \int_H \varphi(hg)$ for some $\varphi \in \mathcal{S}(G(\mathbb{A}))$. There is a mixer of

- vector spaces with reductive group actions as in the Riemann case,
- reductive subgroups, which is the homogeneous case,
- characters of unipotent groups and more generally Heisenberg representaitons.

Most examples up to this point are of the form $X = (HU, \psi)\backslash G$. But the duality also includes non-polarizable Hamiltonian G spaces M.

Example 2.9. *Let $G = \mathrm{SO}(V) \times \mathrm{Sp}(W)$ act on $M = V \otimes W = T^*X$ for any Lagrangian X, but there is no X that is G-invariant.*

$\widetilde{G}$ acts on $\mathcal{S}(X(\mathbb{A}))$, and if we assume the anomaly free condition, which is the case when $\dim V$ is even, there is a lift of the covering map $\widetilde{G}(\mathbb{A}) \to G(\mathbb{A})$, and then $G(\mathbb{A})$ acts on $\mathcal{S}(X(\mathbb{A}))$. The automorphic theory in this case is just the Howe duality. And the dual Hamiltonian is $T^*(\mathrm{SO}(V)\backslash \mathrm{SO}(V) \times \mathrm{SO}(V \oplus F))$, where we have the Rallis Inner Product.

Next let us talk about the theta series in the geometric setting. Back to Iwasawa and Tate, consider $G = \mathbb{G}_m$. Let $F = \mathbb{F}_q(\Sigma)$, then we have

$$F^\times \backslash \mathbb{A}^\times / \widehat{\mathcal{O}^\times} \cong \mathrm{Bun}_G(\mathbb{F}_q).$$

Note that $\mathbb{A}^\times / \widehat{\mathcal{O}^\times}$ is the divisor group $\mathrm{Div}(\Sigma)$ of Σ, and for each $D \in \mathrm{Div}(\Sigma)$, we have the line bundle $\mathcal{O}_{(D)}$, whose rational sections are in bijection with $F^\times$, under which the regular sections are

$$\{f \in F^\times \mid (f) + D \geq 0\}.$$

For $\Phi = 1_{\widehat{\mathcal{O}}} \in \mathcal{S}(\mathbb{A})$, it turns out $\Theta_\Phi([g])$ is the number of rational sections of $\mathcal{O}_{([g])}$. Eventually we will have a geometrization of this calculation:

Let Bun_G^X be the parametrization space of pairs $(\mathcal{L}^\times, \sigma)$, where $\mathcal{L}^\times$ is a G-bundle and σ is a section of $X \times^G \mathcal{L}^\times =: \mathcal{L}$, and we have a projection $p : \mathrm{Bun}_G^X \to \mathrm{Bun}_G = \mathrm{Pic}(\Sigma)$. Then the pushforwd of the constant sheaf $\underline{k}$ is the period sheaf $\mathcal{P}_X := p_* \underline{k}$ in the derived category over Bun_G.

manifolds (dim)	G-theory (TQFT)	(G, M)-theory (TQFT w/ bd)
$\Sigma_{\mathbb{F}_q}$, number fields (3)	Vect Sp of unram auto functions	$\Theta_{1_{X(\widehat{\mathcal{O}})}}$
$\Sigma_{\overline{\mathbb{F}_q}}$ (2)	$D(\mathrm{Bun}_G)$	$\mathcal{P}_X$-period sheaf
$F = \mathbb{F}_q((t))$, $\mathbb{Q}_p$ (2)	$G(F)$-representations	$\mathcal{S}(X(F))$ or $\mathcal{L}^2(X(F))$

Then we are going to talk about the local periods and Plancherel densities. The unramified Ichino-Ikeda conjecture is the following: Let $H = \mathrm{SO}_n \overset{\Delta}{\hookrightarrow} \mathrm{SO}_n \times \mathrm{SO}_{n+1}$

$$\left| \int_{[H]} f_\pi \right|^2 = L\left(\pi, \otimes, \frac{1}{2}\right)$$

More generally, write $\pi = \otimes'_\nu \pi_\nu$, $f_\pi = \otimes'_\nu f_\nu$, then we have

$$\left| \int_{[H]} f_\pi \right|^2 = |G_\phi|^{-1} \prod_\nu{}' \int_{H_\nu} \langle \pi_\nu(h) f_\nu, f_\nu \rangle \,\mathrm{d}h,$$

where the prime means the product is not necessarily convergent, and we need to regularize it, and $H_\nu = H(F_\nu)$. The local integral

$$\int_{H_\nu} \langle \pi_\nu(h) f_\nu, f_\nu \rangle \,\mathrm{d}h$$

is called the local Ichino-Ikeda period. This period gives an $H_\nu \times H_\nu$-equivariant map

$$\pi_\nu \otimes \overline{\pi_\nu} \to \mathbb{C},$$

which by Frobenius gives a $G_\nu \times G_\nu$ map

$$\pi_\nu \otimes \overline{\pi_\nu} \to \mathcal{C}^\infty(X_\nu \times X_\nu),$$

where $X = H \backslash G$. This has an interpretation in terms of Plancherel formula for $\mathcal{L}^2(X_\nu)$. Dually, we have

$$J_{\pi_\nu} : \mathcal{S}(X_\nu \times X_\nu) \to \overline{\pi_\nu} \otimes \pi_\nu \overset{\langle \cdot, \cdot \rangle}{\to} \mathbb{C},$$

which is G^Δ-invariant, and we have

$$\int_{X_\nu} \Phi_1(x) \Phi_2(x) \,\mathrm{d}x = \int_{\widehat{G_\nu}} J_{\pi_\nu}(\Phi_1 \otimes \Phi_2) \,\mathrm{d}\mu_G(\pi_\nu),$$

where $\widehat{G_\nu}$ is the unitary dual, the measure $\mathrm{d}\mu_G$ is the Plancherel measure, and J_{π_ν} is called the relative character.

For the group case, let $\varphi_1, \varphi_2 \in \mathcal{S}(G)$, and write $\varphi_2^*(g) := \overline{\varphi_2(g^{-1})}$, then we have

$$\langle \varphi_1, \varphi_2 \rangle = \varphi_1 * \varphi_2^*(1) = \int_{\widehat{G}} \mathrm{tr}(\pi(\varphi_1 * \varphi_2^*))\, \mathrm{d}\mu_G(\pi).$$

Remark 2.10. *Let $\mathcal{F}$ and $\mathcal{G}$ be Weil sheaves on the $\mathbb{F}_q$-variety X with the associated functions f and g on $X(\mathbb{F}_q)$, then*

$$\sum_{x \in X(\mathbb{F}_q)} f(x) g(x)$$

is the geometric Frobenius trace on $\mathrm{Ext}^{\cdot}(\mathcal{F}, \mathbb{D}\mathcal{G})$, where $\mathbb{D}$ is the Verdier duality.

Then let us go to Satake and Macdonald. Recall that for the action of $\mathbb{G}_m$ on $\mathbb{A}^1$, we have

$$\langle h * 1_{\mathbb{Z}_p} |\, \mathrm{d}x|^{1/2}, 1_{\mathbb{Z}_p} |\, \mathrm{d}x|^{1/2} \rangle = \int_{\mathbb{S}^1} \frac{\widehat{h}(z)\, \mathrm{d}^\times z}{(1 - p^{-1/2} z)(1 - p^{-1/2} z^{-1})}.$$

For the group case, let $X = H$ with the action of $G = H \times H$, for simplicity, we will write H for the F-points as well if no confusion. Then the Hecke algebra $\mathcal{H}(G, G(\mathcal{O}))$ acts on $\mathcal{S}(X)^{G(\mathcal{O})}$, i.e., $\mathcal{H}(H, H(\mathcal{O}))$ is a $\mathcal{H}(H, H(\mathcal{O}))$-bimodule. We may assume H is split and $k = \mathbb{C}$. Then the Satake isomorphism tells us

$$\mathcal{H}(H, H(\mathcal{O})) \cong \mathcal{H}(A, A(\mathcal{O}))^W \cong \mathbb{C}[X_*(A)]^W = \mathbb{C}[A^\vee]^W = \mathbb{C}[H^\vee]^{H^\vee} = \mathbb{C}[\mathrm{Rep} H^\vee],$$

where $A = B/N$ is the universal Catan of H, W is the Weyl group, $X_*(A)$ is the co-character group, $A^\vee$ is the Langlands dual of A, and $H^\vee$ is the Langlands dual of H. Then the actions of A and H in $N\backslash B$ give actions of $\mathcal{H}(A, A(\mathcal{O}))$ and $\mathcal{H}(H, H(\mathcal{O}))$ on $\mathcal{S}(N\backslash H)^{H(\mathcal{O})}$. Using the Iwasawa decompostion

$$H = \bigsqcup_{\lambda \in X^*(A)} N \varpi^\lambda H(\mathcal{O}),$$

we know the above is a free module under $\mathcal{H}(A, A(\mathcal{O}))$ generated by $1_{N\backslash H(\mathcal{O})}$.

On $\mathbb{C}[\mathrm{Rep} H^\vee]$, there is a canonical basis $\{s_\lambda\}$, where λ runs over all anti-dominant weights $X^*(A^\vee)^-$, which indexes the classes of irreducible representations with lowest weight λ. Then we have $h_\lambda \in \mathcal{H}(H, H(\mathcal{O}))$. But this is not compatible with inner products since it is natural to think of s_λ's as an orthogonal basis, while

$$\langle h_\lambda, h_\mu \rangle \neq 0$$

in general.

3. June 4

3.1. **David Ben-Zvi.** We will first explain the $\mathcal{A}$-theory of (2.2) is

$$\begin{tikzcd}[column sep=small] & C(\mathrm{Bun}_G \Sigma(\mathbb{F}_q)) \arrow[dr, dash] & \\ \mathrm{Rep} G(F) \arrow[ur, dash] \arrow[dr, dash] & & \mathrm{Shv}(\mathrm{Bun}_G \Sigma) \\ & G(F) - \text{category} \arrow[ur, dash] & \end{tikzcd} \tag{3.1}$$

and the $\mathcal{B}$-theory is

$$\begin{tikzcd} & \omega(\mathrm{Loc}_{G^\vee}\Sigma_{\mathbb{F}_q}) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{QC}^!(\mathrm{Loc}_{G^\vee}D^\times_{\mathbb{F}_q}) & & \mathrm{QC}^!(\mathrm{Loc}_{G^\vee}\Sigma_{\overline{\mathbb{F}_q}}) \\ & \mathrm{ShvCat}(\mathrm{Loc}_{G^\vee}D^\times) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} . \tag{3.2}$$

The TQFT usually factors through

$$\begin{tikzcd} \text{manifolds, bordsims} \arrow[rr] \arrow[dr, "\text{spaces of fields}"'] & & \text{numbers, vect, cat} \\ & \text{stacks, correspondences} \arrow[ur, "\text{sheaf theory}"'] & \end{tikzcd} .$$

$\mathcal{A}_G$ is about the topology of spaces of bundles, and we have

$$\Sigma \mapsto \mathrm{Bun}_G(\Sigma) \mapsto \mathrm{Shv}(\mathrm{Bun}_G\Sigma),$$

and can be thought as $\mathrm{Maps}_{\mathrm{alg}}(\Sigma, \cdot/G)$. On the other side, $\mathcal{B}_{G^\vee}$ is about the algebraic geometry of stacks of local systems, and we have

$$\Sigma \mapsto \mathrm{Loc}_{G^\vee}(\Sigma) = \mathrm{Maps}_{l.c.}(\Sigma, \cdot/G^\vee),$$

then we can take $\mathrm{QC}^!(\mathrm{Loc}_{G^\vee}(\Sigma))$.

As for the functoriality from $\mathcal{A}_G(\Sigma_{\mathbb{F}_q}) \to \mathcal{A}_H(\Sigma_{\mathbb{F}_q})$, if $H = \{1\}$, then it becomes $\mathcal{A}_G(\Sigma_{\mathbb{F}_q}) \to k$. Morphisms in field theory is **interface**, which is an analog of bimodule. An interface between two field theory $\mathcal{Z}$ and $\mathcal{Z}'$ is basically about the extension of $\mathcal{Z}(M)$ and $\mathcal{Z}'(M)$ to $M \times I$, where we veiw $M = M \times \{0\}$ and $M = M \times \{1\}$. When it comes from morphisms $H \to G$, we may also view the interface as the graph of the morphism as a special case.

When $\mathcal{Z}'$ is the trivial theory, then it becomes the boundary theory for $\mathcal{Z}$, and when both $\mathcal{Z}$ and $\mathcal{Z}'$ are trivial theories, the interface $\mathcal{P}$ between them is just the $3d$ TQFT. In particular, when $H^\vee = \{1\} \hookrightarrow G^\vee$, it is called the Dirichlet boundary condition, and is the skyscript at the trivial local system. Since we consider correspondences in stacks, a better picture will be considering Morita theory or the integral transforms of certain

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl] \arrow[dr] & \\ \cdot/H & & \cdot/G \end{tikzcd} ,$$

which arises when X admits actions of H and G, and we may take $\mathcal{Y} = X/(H \times G)$. Similarly, if we have a diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl] \arrow[dr] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd} ,$$

then glueing maps to $\mathcal{X}$ and $\mathcal{Y}$ is like considering the compatibility of maps to $\mathcal{Z}$. For example, if $\mathcal{X} = \cdot/G$, $\mathcal{Y} = \cdot/T$, then we may consider the action of G and T on $N\backslash G$, and take $\mathcal{Z} = \cdot/B$.

The boundary theory for $\mathcal{A}_G$ is the theory of periods, and a source comes from G-spaces X and the diagram looks like

$$\begin{array}{ccc} & X/G & \\ \swarrow & & \searrow \\ \cdot/G & \overset{\mathcal{P}_X}{\to} & \cdot \end{array},$$

and then we may consider $\mathrm{Maps}(\Sigma, X/G) \to \mathrm{Maps}(\Sigma, \cdot/G)$, where the later one is Bun_G, and the formal one classifies the sections of the associated X-bundles. In the case that $G = \mathbb{G}_m$, $X = \mathbb{A}^1$, then we can view

$$\mathrm{Maps}(\Sigma, X/G) = \mathrm{Bun}_G^X \to \mathrm{Maps}(\Sigma, \cdot/G) = \mathrm{Bun}_G.$$

When one theory is the trivial theory, we can view trivial $\to \mathcal{A}_G$ as objects in $\mathcal{A}_G(\cdot)$, and $\mathcal{A}_G \to$ trivial as functionals on $\mathcal{A}_G(\cdot)$. In the case that we have a subgroup $H \subset G$, then the above is the inclusion $\mathrm{Bun}_G^X = \mathrm{Bun}_H \to \mathrm{Bun}_G$, and the pushforward of the constant sheaf the sheaf representing the period integral.

On the dual side, if we have $G^\vee$ acting on some $X^\vee$, it gives

$$\mathrm{Loc}_{G^\vee}^{X^\vee} = \mathrm{Maps}_{l.c.}(\Sigma, X^\vee/G^\vee) \to \mathrm{Loc}_{G^\vee},$$

where $\mathrm{Loc}_{G^\vee}^{X^\vee}$ classifies the the local systems together with twisted locally constant maps to $X^\vee$. And the pushforwd of $1 \in \mathrm{Loc}_{G^\vee}^{X^\vee}$ is $\omega(\mathrm{Loc}_{G^\vee}\Sigma(\mathbb{F}_q))$, which is related to L-functions. In particular when $X^\vee$ is a representation V of $G^\vee$, it is the associated L-functions.

3.2. **Hiraku Nakajima.** The goal of the two lectures is to understand the identity (3.13) of Gaiotto-Witter in 0807.3720:

$$\mathcal{T}^\vee = (\mathcal{T} \times \mathcal{T}[G] \mathbin{/\!/\!/} G)^*,$$

where

- G is a reductive group over $k = \mathbb{C}$,
- $\mathcal{T}$ is $3d$ $N = 4$ SQFT with G-symmetry,
- $\mathcal{T}[G]$ is the kernel of $3d$ $N = 4$ $SQFT$,
- $\mathbin{/\!/\!/}$ is the (supersymmetric) gauging,
- * is the $3d$ mirror,
- $\mathcal{T}^\vee$ is another $3d$ $N = 4$ $SQFT$ with $G^\vee$-symmetry, the Langlands dual group of G.

In this talk, $F = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$.

Remark 3.1. *If we start with an Hamiltonian G-variety M, then we can associate it with a $\mathcal{T} = \mathcal{T}_{(G,M)}$, then **sometimes** $T^\vee$ arises as $T^\vee_{(G^\vee,M^\vee)}$, we do not know if $M^\vee$ is the dual in the sense of BZSV.*

*Moreover, suppose $M = T^*N$ for some affine smooth algebraic G-variety N, then according to Braverman-Finkelberg, we have an affine symplectic $G^\vee$-variety, which is in general singular, and we do not know if the above **sometimes** is exactly when $M^\vee$ is smooth.*

§0 **Geometric Satake.**

Let G be a reductive variety, $T \subset G$ the maximal torus, and W be its Weyl group. Let $\mathrm{Gr}_G = G(F)/G(\mathcal{O})$ be the affine Grassmannian. We have the Schubert decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in X_*(T)^+} \mathrm{Gr}_G^\lambda,$$

where $\mathrm{Gr}_G^\lambda = G(\mathcal{O})[z^\lambda]$. Let $D_{G(\mathcal{O})}(\mathrm{Gr}_G)$ be the $G(\mathcal{O})$-equivariant derived category of k-constructible sheaves on Gr_G, which is a monoidal category under the convolution, and let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ be the subcategory of perverse sheaves, which is commutative.

We know $G(\mathcal{O})\backslash \mathrm{Gr}/G(\mathcal{O})$ can be viewed as the moduli of G-bundles over $\Sigma = D \bigsqcup_{D^\times} D$. The Geometric Satake tells us

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \cong \mathrm{Rep} G^\vee.$$

§1 **Definition of Coulomb Branch**

Let $M = T^*N$ or a symplectic representation of G, which is assumed to be anomaly free, i.e.,

$$\pi_4(G) \to \pi_4(\mathrm{Sp}(M)) = \mathbb{Z}/2\mathbb{Z}$$

is trivial. Let us construct the coulom branch in the first case. Let $\mathcal{T} = G(F) \times^{G(\mathcal{O})} N(\mathcal{O})$ with natural projections $\pi : \mathcal{T} \to \mathrm{Gr}_G$ and $\mathcal{T} \xrightarrow{\Pi} N(F) : [g(z), s(z)] \mapsto g(z)s(z)$. Let $\mathcal{R} := \Pi^{-1}(N(\mathcal{O}))$. Note that $[G(\mathcal{O})\backslash \mathcal{R}] = \mathrm{Bun}_G^N$, which is the moduli stack of G-bundles together with N-valued sections.

Theorem 3.2. (1) *The equivariant Borel-Moore homology group $H_*^{G(\mathcal{O})}(\mathcal{R})$ has a product given by the convolution.*
(2) *The product is commutative, which is the same reason as the commutativity as in the geometric Satake, for example, we can consider the Beilinson-Drinfeld construction.*
(3) *The loop rotation $\mathbb{C}^\times \times D : (\lambda, z) \mapsto \lambda z$ induces $\mathbb{C}^\times$-actions on the spaces above, and then $H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})$ is a non-commutative algebra, which can be viewed as the deformation of $H_*^{G(\mathcal{O})}$ parametrized by $H^*_{\mathbb{C}^\times}(\{\mathrm{pt}\}) = \mathbb{C}[\hbar]$. Then $H_*^{G(\mathcal{O})}(\mathcal{R})$ has a Poisson bracket.*

Definition 3.3 (Coulomb branch of $3d$ $N = 4$ SUSY gauge theory $\mathcal{T} = \mathcal{T}_{M /\!\!/\!\!/ G}$)**.**

$$\mathcal{M}_C = \mathrm{Spec} H_*^{G(\mathcal{O})}\mathcal{R}.$$

This is an affine normal algebraic variety, possibly with singularities.

Proposition 3.4.
- *$\mathcal{M}$ is independent of the choice of $M = N \oplus N^*$.*
- *The Poisson structure is induced from the symplectic form on $\mathcal{M}_C^{\mathrm{reg}}$.*
- *$\mathcal{M}_C$ has only symplectic singularities in the sense of Beauville (Bellamy).*
- *$\pi_0(\mathcal{R}) = \pi_0(\mathrm{Gr}_G) = \pi_1(G)$, so we have*

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{\gamma \in \pi_1(G)} H_*^{G(\mathcal{O})}(\mathcal{R}_\gamma),$$

and then $\widehat{\pi_1(G)}$ acts on $\mathcal{M}_C$, where $\widehat{\pi_1(G)}$ is the Pontryagin dual of $\pi_1(G)$.

Example 3.5. *For $G = \mathbb{G}_m$, $N = M = 0$, we have $\mathrm{Gr}_G = \{[z^n] \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$. Let r_n be the fundamental class of $[z^n]$, then we have*

$$r_n * r_m = r_{n+m}.$$

Since $H_G^*(\{\mathrm{pt}\}) = \mathbb{C}[w]$, we have

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[w, \{r_n\}_{n\in\mathbb{Z}}]/\langle r_n * r_m = r_{n+m} \mid n, m \in \mathbb{Z}\rangle = \mathbb{C}[w, r_1, r_{-1}]/\langle r_1 r_{-1} = 1\rangle,$$

then $\mathcal{M}_C = \mathbb{C} \times \mathbb{C}^\times = T^*(\mathbb{C}^\times)$.

Example 3.6. *$G=\mathbb{G}_m$, $N=\mathbb{C}$, the weight 1 representation of $\mathbb{G}_m$. Let $M=T^*N$. Then*
$$\mathcal{R}=\{([z^n],s(z))\mid s(z)\in\mathbb{C}[[z]]\cap z^{-n}\mathbb{C}[[z]],n\in\mathbb{Z}\}=\begin{cases} z^n\mathbb{C}[[z]] & \text{if } n\geq 0\\ \mathbb{C}[[z]] & \text{if } n<0\end{cases}.$$
Let r'_n be the fundamental class, which is $w^n r_n$ when $n\geq 0$, and r_n when $n<0$. Then we have
$$H_*^{G(\mathcal{O})}(\mathcal{R})=\mathbb{C}[w,r'_1,r'_{-1}]/\langle r'_1r'_{-1}=w\rangle=\mathbb{C}[x,y]:r'_1\mapsto x,r'_{-1}\mapsto y,$$
hence $\mathcal{M}_C=\mathbb{C}^2$. This is a ***self-dual*** *example.*

Example 3.7. *Let $G=\mathbb{G}_m$, $N=\mathbb{G}_m=\mathbb{C}^\times$, $M=T^*N$, then we have*
$$\mathcal{R}=\{([z^n],s(z))\mid s(z)\in s_0+z\mathbb{C}[[z]],z^ns(z)\in s'_0+z\mathbb{C}[[z]],s_0,s'_0\in\mathbb{C}^\times\},$$
which implies we must have $n=0$ and $s(z)\in N(\mathcal{O})$. Then
$$H_*^{G(\mathcal{O})}(G(\mathcal{O}))=H_*(\{\mathrm{pt}\})=\mathbb{C},$$
hence $\mathcal{M}_C=\{\mathrm{pt}\}$.

Example 3.8. *$G=\mathbb{G}_m$, and $N=\mathbb{C}^l$, all of which are weight 1 representations. We have $r''_n=w^{nl}r_n$, and we have*
$$H_*^{G(\mathcal{O})}(\mathcal{R})=\mathbb{C}[w,r''_1,r''_{-1}]/\langle r''_1r''_{-1}=w^l\rangle,$$
which has the type A_{l-1}-singularity.

3.3. **Chen Wan.** Let $\Delta=(G,H,\rho_H,\iota)$ be a BZSV quadruple, and $\widehat{\Delta}=(\widehat{G},\widehat{H'},\rho_{\widehat{H'}},\widehat{\iota'})$ be its dual quadruple. Decompose
$$\mathfrak{g}=\oplus_{k\geq 0}\rho_k\otimes\mathrm{Sym}^k$$
according to the adjoint action of $H\times\mathrm{SL}_2$, and correspondingly
$$\widehat{\mathfrak{g}}=\oplus_k\widehat{\rho_k}\otimes\mathrm{Sym}^k.$$
We have the following conjecture due to [BZSV]

Conjecture 3.9. (1) *$\mathcal{P}_\Delta(\phi)\neq 0$ only if the Arthur parameter of ϕ factors through $\widehat{\iota'}$.*

(1)′ *$\mathcal{P}_{\widehat{\Delta}}(\phi)\neq 0$ only if the Arthur parameter of ϕ factors through ι.*

(2) *If ϕ is a lifting of a tempered Π of $H'(\mathbb{A})$, then*
$$\frac{|\mathcal{P}_\Delta(\phi)|^2}{\langle\phi,\phi\rangle}=\frac{L\left(\frac{1}{2},\Pi,\rho_{\widehat{H'}}\right)\prod_k L(\frac{k}{2}+1,\Pi,\widehat{\rho_k})}{L(1,\Pi,\mathrm{Ad})^2}$$

Let us consider the special case that $\widehat{\Delta}=(\widehat{G},\widehat{H'},0,1)$, then $\mathcal{P}_\Delta(\phi)\neq 0$ only if ϕ comes from $\Pi\in\mathcal{A}(H'(\mathbb{A}))$. Then for the dual side in this case, H is the dual group of the spherical variety $\widehat{H'}\backslash\widehat{G}$, and ι is the Arthus SL_2. Then in this case the requirement is that there is no type N-root.

Example 3.10. *Let $\widehat{G}=\mathrm{GL}_{2n}$, $\widehat{H'}=\mathrm{Sp}_{2n}$, this is the case of Jacquet-Rallis. In this case, we have $G=\mathrm{GL}_{2n}$, $H=\mathrm{GL}_n$, $\iota=[2^n]$, $\rho_H=0$, and the associated period is*
$$\mathcal{P}_\Delta(\phi)=\iint\phi\left(\begin{pmatrix}\mathrm{I}_n & X\\ & \mathrm{I}_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\right)\psi(\mathrm{tr}X)\,\mathrm{d}x\,\mathrm{d}h.$$

Remark 3.11. *If we take $\widehat{G}=\mathrm{GL}_n$ and $\widehat{H'}=\mathrm{SO}_n$, it will not fit into the framework because of type N-root.*

Example 3.12. *If $\widehat{G} = \mathrm{Sp}_{2n+2m}$, $\widehat{H'} = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ with $n \geq m$. Then we have $G = \mathrm{SO}_{2n+2m+1}$, $H = \mathrm{Sp}_{2m}$, ι is the principal nilpotent orbit in $\mathrm{GL}_2^n \times \mathrm{SO}(2n-2m+1)$, and $\rho_H = \mathrm{std}$.*

In particular if $n = m$, then

$$P = MN = \left\{ \begin{pmatrix} g & & \\ & 1 & \\ & & g^* \end{pmatrix} \mid g \in \mathrm{GL}_{2n} \right\} \cdot \left\{ \begin{pmatrix} \mathrm{I}_{2n} & X & Y \\ & 1 & X^* \\ & & \mathrm{I}_{2n} \end{pmatrix} \right\}$$

Let $\Theta_N : [N] \to \mathbb{C}$, then the period is

$$\mathcal{P}_\Delta(\phi) = \iint \phi(hn)\Theta_N(h)\Theta_H(h)\,\mathrm{d}n\,\mathrm{d}h.$$

This is used to detect the functoriality of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1} \to \mathrm{SO}_{4n+1}$.

The second case if when $\widehat{\Delta} = (\widehat{G}, \widehat{G}, 1, \widehat{\rho})$. Then in this case we have

$$\frac{|\mathcal{P}_\Delta(\phi)|^2}{\langle \phi, \phi \rangle} = \frac{L(1/2, \pi, \widehat{\rho})}{L(1, \pi, \mathrm{Ad})}.$$

In such cases, the conditions are

- $\widehat{\rho}$ is symplectic, and anomaly free,
- geometric stabilizer is connected,
- and multiplicity free.

For such cases, we may look at the table by [Loseu, Knop]. For example, if $\widehat{\Delta} = (E_6, E_5, 1, T^*(\mathrm{std}))$, then we have $\Delta = (E_6, A_2, \iota, T^*(\mathrm{std}))$, where ι is given by the principal nilpotent orbit in D_4. And this period is the Ginzburg integral. The limitation is that in such cases, $\widehat{\rho}$ need to be multiplicity free, which is not the case for the adjoint L-functions of $\mathrm{SL}_3, \mathrm{SL}_4, \mathrm{SL}_5$, whose integrals have already appeared in the literature.

4. June 5

4.1. **Yiannis Sakellaridis.** Let us continue with the group case that $G = H \times H$ and $X = H$ for some split reductive group H over a non-Archimedean local field F, with ϖ being the uniformizer in the ring of integers $\mathcal{O}$, and $k = \mathbb{C}$. For simplicity, write $\mathcal{H}_H$ for the unramfied Hecke algebra $\mathcal{H}(H(F), H(\mathcal{O}))$. For $\lambda \in X*(A^\vee) = X_*(A)$, write $e^\lambda \in \mathbb{C}[A^\vee]$.

Recall that for

$$s_\lambda = \sum_{w \in W} {}^w\left(\prod_{\alpha > 0} \frac{1}{1 - e^{\alpha^\vee}} e^\lambda \right),$$

we have

$$\frac{1}{|W|} \int_{A_c^\vee} s_\lambda(t)\overline{s_\mu(t)}\,\mathrm{d}_{\mathrm{Weyl}}(t) = \delta_{\lambda,\mu},$$

where

$$\mathrm{d}_{\mathrm{Weyl}}(t) = \prod_{\alpha \in \Phi} (1 - e^{\alpha^\vee}(t))\,\mathrm{d}t.$$

Then a natural question is that is the inner product coming from $\mathcal{H}_H$? In fact, there is an orthogonal basis of $\mathcal{H}_H$ indexed by $\lambda \in X_*(A)^-$. Since

$$H = \bigsqcup_\lambda K_H \varpi^\lambda K_H,$$

then we may take

$$f_\lambda := 1_{K_H \varpi^\lambda K_H} q^{\langle \rho, \lambda \rangle}.$$

Then according to the Macdonald's formula, the Satake transform of f_λ is

$$\widehat{f_\lambda} = p_\lambda = \sum_{w \in W} {}^w\left(\prod_{\alpha > 0} \frac{1 - q^{-1} e^{\alpha^\vee}}{1 - e^{\alpha^\vee}} e^\lambda \right),$$

which gives the Plancherel formula for $\mathcal{H}_H$:

$$\begin{aligned}
\langle h_1, h_2 \rangle &= \frac{1}{(1-q^{-1})^{\mathrm{rk}\, H}} \frac{1}{|W|} \int_{A_c^\vee} \frac{\widehat{h_1}(t)\widehat{h_2}(t)}{\prod_{\alpha \in \Phi}(1 - q^{-1} e^{\alpha^\vee})} \, \mathrm{d}_{\mathrm{Weyl}}(t) \\
&= \frac{1}{|W|} \int_{A_c^\vee} \widehat{h_1}(t)\widehat{h_2}(t^{-1}) L(t, \mathfrak{h}^\vee, 1) \, \mathrm{d}_{\mathrm{Weyl}}(t) \\
&= \frac{1}{|W|} \int_{A_c^\vee} \widehat{h_1}(t)\widehat{h_2}(t) \sum_{i \geq 0} q^{-i} \mathrm{tr}(t | S^i \mathfrak{h}^\vee) \, \mathrm{d}_{\mathrm{Weyl}}(t),
\end{aligned}$$

where $\mathfrak{H}^\vee$ is the Lie algebra of $H^\vee$, the L-functions is the adjoint L-function, and $S^i\cdot$ means the symmetric i-th power. For simplicity, assume $h_2 = 1_{H(\mathcal{O})}$, and $h_1 = h_V$, the trace character for some irreducible representation V of $H^\vee$. Then the above is

$$\sum_{i \geq 0} q^{-i} \dim \mathrm{Hom}(V, S^i \mathfrak{h}^\vee).$$

Observe that this is the **trace of Frobenious on the derived geometric Satake**. Now let F be a function field, and we will use $H_F = LH$ to denote the loop space, and $H_{\mathcal{O}} = L^+H$ for the are space. Then $H_F/H_{\mathcal{O}}$ is the affine Grassmannian. Write $H_{\mathcal{O}} \backslash H_F / H_{\mathcal{O}}$ for $H_{\mathcal{O}}$-equivariant objects. Then the geometric Satake tells us

$$D^b(H_{\mathcal{O}} \backslash H_F / H_{\mathcal{O}})^\heartsuit = \mathrm{Perv}(H_{\mathcal{O}} \backslash H_F / H_{\mathcal{O}}) \cong \mathrm{Rep}(H^\vee)$$

Theorem 4.1. *$D^b(H_{\mathcal{O}} \backslash H_F / H_{\mathcal{O}}) \cong \mathrm{Perf}^{H^\vee}(k[\mathfrak{h}^{\vee,*}])$, with proper shearing on the right hand side.*

Remark 4.2. *$k[\mathfrak{h}^{\vee,*}]$ is the symmetric algebra of $\mathfrak{h}^\vee$, and the right-hand-side can be viewed as quasi-coherent sheaves on $\mathfrak{h}^{\vee,*}/H^\vee$.*

$\mathbb{G}_m$ acts on $\mathfrak{H}^{\vee,*}$ by square of the usual action, then we get an even grading on $k[\mathfrak{h}^{\vee,*}]$. We can think of this as a DG-algebra with trivial differentials.

For the translations, we may think of degree n part as graded by $q^{-n/2}$, then

$$\begin{aligned}
\langle \mathrm{tr\ Frob\ IC}_V, \mathrm{tr\ Frob}\ \underline{k}_{L^+H} \rangle &= \langle h_V, 1 \rangle = \mathrm{tr}(\mathrm{Frob}_q, \mathrm{Hom}^\cdot(\mathrm{IC}_V, \underline{k}_{L^+H})) \\
&= \mathrm{tr}(\mathrm{Frob}_q, \mathrm{Hom}^\cdot(V \otimes k[\mathfrak{h}^{\vee,*}]), k[\mathfrak{h}^{\vee,*}]) \\
&= \mathrm{tr}(\mathrm{Frob}_q, \mathrm{Hom}^\cdot_{H^\vee-\mathrm{Rep}}(V, k[\mathfrak{h}^\vee, *])),
\end{aligned}$$

with proper shearing being understood. We expect similar things for groups G acting on spherical smooth affine varieties X. Then $\mathcal{H}_G$ acts on $\mathcal{S}(X)^{G(\mathcal{O})}$. We expect, which are theorems in many cases, that there is a reductive subgroup $G_X^\vee \subset G^\vee$, and a graded representation V_X of $G_X^\vee$, together with a $\mathbb{G}_m$ action giving the grading, such that for any $h_V \in \mathcal{H}_G$, we have

$$\langle h * 1_{X(\mathcal{O})}, 1_{X(\mathcal{O})} \rangle = \int_{(G_X^\vee)_c} \widehat{h_V}(t) L(t, V_X) \, \mathrm{d}_{\mathrm{Weyl}}(t),$$

where the value $L(t, V_X)$ depends on the grading, and on i-th grading part, we put $L(t, V_X, i/2)$. Set $M^\vee = V_X \times^{G_X^\vee} G^\vee$, which turns out to be the dual Hamiltonian space. The above integral is $\mathrm{tr}(\mathrm{Frob}_q, k[M^\vee])$, with suitable shearing. We have the local geometric conjecture

Conjecture 4.3 (Local Geometric Conjecture)**.** *There is an equivalent*

$$D^b(X_F/G_\mathcal{O}) \cong \mathrm{Perf}^{G^\vee}(k[M^\vee]),$$

with proper shearing on the right-hand-side, compitable with the $D^b(G_\mathcal{O}\backslash G_F/G_\mathcal{O})$*-action and the corresponding* $\mathrm{Perf}^{G^\vee}(k[\mathfrak{g}^{\vee,*}])$ *action on the right-hand-side, with shearing again. And the right-hand-side action is via the moment map* $M^\vee \to \mathfrak{g}^{\vee,*}$*.*

We can read off many things from this conjecture. For example,

$$\mathrm{Hom}^\cdot(\underline{k}_{X_\mathcal{O}}, \underline{k}_{X_\mathcal{O}}) = \mathrm{Hom}^\cdot_{k[M^\vee]}(k[M^\vee], k[M^\vee])^{G^\vee} = k[M^\vee]^{G^\vee} = k[M^\vee//G^\vee].$$

Then how to obtain the entire $k[M^\vee]$? Since $M^\vee = (G^\vee \times M^\vee)/G^\vee$, we have

$$k[M^\vee] = \mathrm{Hom}^\cdot(\mathcal{R}_{\mathrm{reg}} \otimes k[M^\vee], k[M^\vee]) = \mathrm{Hom}(\mathcal{R}_{\mathrm{reg}} * \underline{k}_{X_\mathcal{O}}, \underline{k}_{X_\mathcal{O}}),$$

where $\mathcal{R}_{\mathrm{reg}}$ is the ind-object in the Hecke category corresponding to the regular representation. This will give an $\mathcal{A}$, which can be used to construct the Coulomb branch

$$R = H^\cdot(G_\mathcal{O}\backslash G_F/G_\mathcal{O}, \mathcal{A}) = k[M^\vee|_{\text{Kostant section}}],$$

where the Kostant section is the distinguished section of $\mathfrak{g}^{\vee,*} \to \mathfrak{g}^{\vee,*}//G^\vee = \mathfrak{c}$ due to the pinning.

4.2. **David Ben-Zvi.** Let F be a function field of a smooth projective curve Σ over $\mathbb{F}_q$, and let $\rho : \mathrm{Gal}(\overline{F}/F) \to G^\vee \to \mathrm{GL}(V)$, then we have defined the L-function $L(\rho, V, t)$ as Euler products.

Let us have a look at Grothendieck's point of view using Lefschetz fixed point theory. $L(\rho, V, t)$ is the super characteristic polynomial of the Frobenious action on $H^\cdot_{et}(\Sigma, V_\rho)$, where V_ρ is the associated local system. Then we have

$$L(\rho, V, t) = \prod_{i=0}^{2} \det(1 - t\rho(\mathrm{Frob})|_{H^i_{et}(\Sigma, V_\rho)})^{(-1)^{i+1}}.$$

If we have an operator A on some vector space W of dimension n, then we have

$$\det(1 - tA) = \sum_{i=0}^{n} (-1)^i \cdot t^i \mathrm{tr}(A|_{\wedge^i W}) = \mathrm{tr}_{\mathrm{gr}}(A, \Lambda^\cdot W),$$

hence

$$\frac{1}{\det(1 - tA)} = \sum_{i=0}^{\infty} t^i \mathrm{tr}(A, \mathrm{Sym}^i W) = \mathrm{tr}_{\mathrm{gr}}(A, \mathrm{Sym}^\cdot W).$$

In particular,

$$L(\rho, V, t) = \mathrm{tr}_{\mathrm{gr}}(\mathrm{Frob}, \mathrm{Sym}^\cdot H^\cdot_{et}(\Sigma, \rho_V)).$$

Then what is this cohomology? It is the derived version of the Galois invariants on V, or derived global sections of ρ_V, which is a ρ-twisted map of $\Sigma \to V$, i.e., the linearization of derived fixed points of the Galois action on V.

We may view the set of sections of the associated V-bundles of ρ_V as a subset of $\mathrm{Maps}(\Sigma, V/G^\vee) = \mathrm{Loc}^V_{G^\vee}(\Sigma)$, which is the preimage of ρ under the map to $\mathrm{Maps}(\Sigma, \cdot/G^\vee) = \mathrm{Loc}_{G^\vee}(\Sigma)$.

We observe that if V has a trivial Galois representation, which is equivalent to $H^0(\rho_V) \neq 0$, and is also equivalent to $H^2(\rho_V) \neq 0$, then the L-function has a pole. Otherwise $0 \in V$ is an isolated fixed point.

Proposition 4.4. *Away from the poles, the L-function is the Frobenius trace on the $\mathcal{L}$-sheaves.*

For the relative setting, if we have an action of $G^\vee$ on $X^\vee$, then we have the boundary theory $\mathcal{B}_{(G^\vee, X^\vee)}$ for $\mathcal{B}_{G^\vee}$. Assume we have a curve Σ over $\overline{\mathbb{F}_q}$, then we have $\mathcal{L}_{X^\vee} = \pi_*(\omega) \in \mathrm{QC}^!(\mathrm{Loc}_{G^\vee}\Sigma)$, where ω is the volume form on π_1-fixed points on $X^\vee$, where $\pi : \mathrm{Loc}_{G^\vee}^{X^\vee} \to \mathrm{Loc}_{G^\vee}$. And we have similarly things for $\mathcal{L}_{X^\vee}$ for $\Sigma/\mathbb{F}_q$.

Now view $\mathcal{A}$ and $\mathcal{B}$ as functors from some **relative group actions** to arithmetic TQFTs. Assume we have G and H both acting on X, then we expect boundary theory $\mathcal{A}_G$ and $\mathcal{A}_G$ coming from $\mathcal{A}_X$. Then we need to think of the compositions. Assume G and H acts on X, H and K acts on Y, then we may consider the G and K action on $X \times^H Y$. For example, $\mathcal{A}_{(G,X)}$ is the theory of period sheaf in $\mathrm{Shv}(\mathrm{Bun}_G\Sigma)$, and the $\mathcal{B}$-theory is the theory of L-functions. And we have some examples of the duality

- For the Tate case, the dual side of the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ is $\mathbb{G}_m$ on $\mathbb{A}^1$.
- The dual side of $(G, X, H) = (G, G/N, T)$ is $(G^\vee, G^\vee/N^\vee, T^\vee)$.
- The dual of (G, G, G) is $(G^\vee, G^\vee, G^\vee)$.
- For the group case, the dual side of $(G, X) = (G \times G, G)$ is $(G^\vee \times G^\vee, G^\vee$ up to the Chavelley twist.
- When considering the group case with H being the trivial group, then $\mathcal{A}$-side can be thought as the period theory for G, and $\mathcal{B}$-side is the period theory for $G^\vee$.

Remark 4.5. *From the point view of physics, the action of both G and H on X is equivalent to the action of $G \times H$ on X.*

The theory of $\mathcal{A}_{(G,X)}$ has more symmetries of $T^*X = M$, not just X itself. For example, the Fourier transform can be viewed as some operation on $T^*\mathbb{A}^1$ in the case of Tate. On the other hand, we cannot observe the dual of X, and only can observe $M^\vee$ from the $\mathcal{A}$-side of the theory. Then from this point of view, we may think of $\mathcal{A}$ and $\mathcal{B}$ from **Reductive Hamiltonian actions** to TQFT's, with compositions given similarly by the G-K-action on $M \times^H_{\mathfrak{h}^*} N$ for the G-H-variety on M, and the H-K-variety N.

Example 4.6. *The group action on a symplectic representation is a Hamiltonian variety.*

Example 4.7 (Whittaker induction)**.** *If we have the $\{1\}$-H acting on M, and H-G acts on T^*G, then the composition gives the Hamiltonian induction.*

More generally, if we have an additional $\iota : \mathrm{SL}_2 \to G$, we may consider the G-H-action on $T^*G//_\Psi U$, where all the notations and details are explained in [BZSV].

Theorem 4.8. *Any hyperspherical G-Hamiltonian variety is of the form of $\mathrm{Ind}^G_{H \times \mathrm{SL}_2} W$.*

Then we hope there is a duality between **reductive Hamiltonion actions** such that the $\mathcal{A}$ and $\mathcal{B}$-theory to TQFT's commute.

4.3. **Hiraku Nakajima.**

Remark 4.9. *Let $\mathcal{T} = \mathcal{T}_{(G,M)}$, then $\mathcal{T}$ contains the fields of morphisms from 3-manifolds to M: $\{f : \Xi \to M\}$, and $\mathcal{T} \Vvdash G$ contains the fields of the above morphisms, together with* *connections on Ξ, module the gauge transfers. Hence for $\mathcal{M}_C = \mathrm{Spec} H_*^{G(\mathcal{O})}(\mathcal{R})$, it already integrates over G-connections, so this is defined for $\mathcal{T} /\!/\!/ G$, $\mathcal{M}_C = \mathcal{M}(\mathcal{T} /\!/\!/ G)$.*

On the other hand $\mathcal{M}_C(\mathcal{T}) = \{\mathrm{pt}\}$. The point is that contributions to $\mathcal{M}_C$ is G-conenctions.

In Examples (3.5, 3.6),we see the two $\mathcal{M}_C$ are birational. In fact, if N is a representation, then $\mathcal{M}_C$ is always birational to $T^*T^\vee/W$, the contanget bundle of the dual torus $T^\vee$, quotient by the Weyl group, which is independent of N.

Definition 4.10. *If $\mathcal{T} = \mathcal{T}_{(G,M)}$, we define the Higgs branches*

$$\mathcal{M}_H(\mathcal{T}) := M,\ \mathcal{M}_H(\mathcal{T} /\!/\!/ G) := M/\!/\!/G.$$

The physical intuition is the approximation of $\mathcal{T} /\!/\!/ G$ by a G-model of maps from Ξ to spaces.

We have a naive hope that $\mathcal{M}_C$ is smooth if and only if $\mathcal{M}_H$ is a point. In Examples (3.5, 3.6, 3.7), we have $\mathcal{M}_H = \{\mathrm{pt}\}$, and in Example (3.8), $\mathcal{M}_C$ is singular, in which case

$$\mathcal{M}_H = T^*N/\!/\!/\mathbb{C}^\times = \overline{\mathcal{N}_{\min}(\mathfrak{sl}_l)}.$$

Definition 4.11. *Let $\mathcal{T}$ be a 3d $N = 4$ SQFT, and T^* another 3d $N = 4$ SQFT (3d mirror), then*

$$\mathcal{M}_C(\mathcal{T}^*) := \mathcal{M}_H(\mathcal{T}),\ \mathcal{M}_H(\mathcal{T}^*) := \mathcal{M}_C(\mathcal{T}).$$

Remark 4.12. *We expect $\mathcal{T}^{**} = \mathcal{T}$.*

Example 4.13. *Consider $1 \to T \to (\mathbb{C}^\times)^n \to T_F \to 1$, and $(\mathbb{C}^\times)^n$ acts on $T^*(\mathbb{C}^n)$. Then we have*

$$\mathcal{M}_C(\mathcal{T}_{((\mathbb{C}^\times)^n, T^*(\mathbb{C}^n))} /\!/\!/ (\mathbb{C}^\times)^n) = T^*\mathbb{C}^n,$$

and

$$(\mathcal{T}_{((\mathbb{C}^\times)^n, T^*(\mathbb{C}^n))} /\!/\!/ (\mathbb{C}^\times)^n)^* = \mathcal{T}_{((\mathbb{C}^\times)^n, T^*(\mathbb{C}^n))} /\!/\!/ (T_F)^\vee,$$

with

$$1 \to T_F^\vee \to (\mathbb{C}^\times)^n \to T^\vee \to 1.$$

§2 **Ring Objects**

Let notations be as before, and $\omega_{\mathcal{R}}$ be the dualizing sheaf on $\mathcal{R}$, and $\mathcal{A} := \pi_*\omega_{\mathcal{R}} \in D_{(G(\mathcal{O}))}(\mathrm{Gr}_G)$, where $\pi : \mathcal{R} =: \mathrm{Gr}_G^N \to \mathrm{Gr}_G$ is the canonical quotient map. Then $\mathcal{A}$ is a ring object. And if $\mathcal{A}$ is a ring object, we know $H^*_{G(\mathcal{O})}(\mathcal{A})$ is a commutative algebra.

Let $\varphi : G_1 \to G_2$ be a group homomorphism, then we have $\mathrm{Gr}_\varphi : \mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$. If $\mathcal{A}_1 \in D_{G_1(\mathcal{O})}(\mathrm{Gr}_{G_1})$ us a ring object, then $(\mathrm{Gr}_{G_1}\varphi)_*(\mathcal{A}_1)$ is also a ring object. If $\mathcal{A}_2 \in D_{G_2(\mathcal{O})}(\mathrm{Gr}_{G_2})$ us a ring object, then $(\mathrm{Gr}_{G_2}\varphi)^!(\mathcal{A}_2)$ is still a ring object.

Example 4.14. *Consider the Geometric Satake isomorphism $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \cong \mathrm{Rep}(G^\vee)$, and write $\mathcal{A}_R$ be the preimage of*

$$\mathbb{C}[G^\vee] = \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$$

in a suitable sense, then we have

$$\mathcal{A}_R = \bigoplus_\lambda V_\lambda^* \otimes \mathrm{IC}_\lambda.$$

Example 4.15. *If $G = \mathbb{G}_m$, then $\mathcal{A}_R = \underline{\mathbb{C}}_{\mathrm{Gr}_G}$.*

We expect if $\mathcal{T}$ is a $3d$ $N = 4$ SUSY with G-symmetry, then there should be some $\mathcal{A}_{\mathcal{T}} \in D_{G(\mathcal{O})}(\mathrm{Gr}_G)$, a ring object such that

$$i^! \mathcal{A}_{\mathcal{T}} = \mathbb{C}[\mathcal{M}_C(\mathcal{T})],$$

where $i : G_1 := \{1\} \hookrightarrow \mathrm{Gr}_G$. We also expect there should be some $\mathcal{A}_{\mathcal{T}^*} \in D_{G'(\mathcal{O})}(\mathrm{Gr}_{G'})$ for a possible different G'.

For $\mathcal{T} = \mathcal{T}_{(G,M)}$ with $M = T^*N$ and N a representation of G, then we have $\mathcal{A}_{\mathcal{T}} = \pi_* \omega_{\mathcal{R}}$, and $i^! \mathcal{A}_{\mathcal{T}} = \mathbb{C} = \mathbb{C}[\{\mathrm{pt}\}]$ since $\mathcal{M}_C(\mathcal{T}) = \{\mathrm{pt}\}$. If we consider $\mathcal{T} \Vvdash G$, then

$$\mathcal{A}(\mathcal{T} \Vvdash G) = (\mathrm{Gr}_G \to \mathrm{Gr}_{\{1\}})_*(\mathcal{A}_{\mathcal{T}}) \in D(\{\mathrm{pt}\}).$$

Example 4.16. *Let $\mathcal{A}_R$ be the preimage of the regular presentation of $G^\vee$ under the Satake inverse, then we have [Artc Bez Gin]*

$$i^! \mathcal{A}_R = \mathbb{C}[\mathcal{N}_G],$$

which should be $\mathcal{A}_{\mathcal{T}[G]}$. The Coulomb branch of $\mathcal{T}[G]$ is $\mathcal{N}_G$ and $\mathcal{T}[G]^* = \mathcal{T}[G^\vee]$.

Let us try to understand $\mathcal{T}^\vee = ((\mathcal{T} \times \mathcal{T}[G]) \Vvdash G)^*$. Assume $\mathcal{T} = \mathcal{T}_{(G,M)}$, and $\mathcal{T}^\vee = \mathcal{T}_{(G^\vee, M^\vee)}$. Then we have $\mathcal{M}_H(\mathcal{T}^\vee) = \mathcal{M}^\vee$ and $\mathcal{M}_H(\mathcal{T}^\vee \Vvdash G^\vee) = M^\vee///G^\vee$. On the other hand,

$$\mathbb{C}[\mathcal{M}_H(\mathcal{T}^\vee)] = \mathbb{C}[\mathcal{M}_C((\mathcal{T} \times \mathcal{T}[G]) \Vvdash G)] = H^*_{G(\mathcal{O})}(\mathcal{A}_{\mathcal{T} \times \mathcal{T}[G]}) = H^*_{G(\mathcal{O})}(\mathcal{A}_{\mathcal{T}} \otimes^! \mathcal{A}_{\mathcal{T}[G]}).$$

Since $\mathcal{A}_{\mathcal{T}} = \pi_* \omega_{\mathcal{R}}$, the above is $H^*_{G(\mathcal{O})}(\pi_* \omega_{\mathcal{R}} \otimes^! \mathcal{A}_R)$.

Remark 4.17. *The dg-refinement of $H^*_{G(\mathcal{O})}(\cdot \otimes^! \mathcal{A}_R)$ realizes derived Satake of [Bez-Fin].*

Example 4.18. *Consider the dual of G acting on $\{\mathrm{pt}\}$, we get*

$$\mathrm{Spec}\ H^*_{G(\mathcal{O})}(\omega_{G \times G} \otimes^! \mathcal{A}_R) = \mathrm{Spec}\ H^*_{G(\mathcal{O})}(\mathcal{A}_R) = G^\vee \times \Sigma^\vee,$$

where the last equality is due to derived Satake, and $\Sigma^\vee$ is the Kostant slice for principal nilpotent in $\mathfrak{g}^\vee$.

If instead we calculate the Coulomb branch, we have

$$\mathcal{M}_C(\mathcal{T}^\vee) = \mathcal{M}_H(\mathcal{T} \times \mathcal{T}[G] \Vvdash G) = \mathcal{M}_H(T \times \mathcal{T}[G])//G = (\mathcal{M}_H(\mathcal{T}) \times \mathcal{M}_H(\mathcal{T}[G]))//G,$$

which is $\mathcal{M} \times \mathcal{N}_G//G$. Then we may guess the hyperspherical condition is equivalent to $M \times \mathcal{N}_G//G$ is a point?

4.4. **Chen Wan.** Let $\widehat{\iota'} : \mathrm{SL}_2 \to G^\vee$, then using the BV-duality we get a nilpotent orbit, hence $\iota' : \mathrm{SL}_2 \to G$. Let $\mathcal{P}_{\iota'}(\phi)$ be the associated degenerate Whittaker period. Assume ϕ is a lifting from a tempered L-packet Π of $G_{\widehat{\iota'}}(\mathbb{A})$, where $\widehat{G}_{\widehat{\iota'}}$ is the connected component of the centralizer of the image of $\widehat{\iota'}$ in $\widehat{G}$. Similarly we may decompose

$$\widehat{\mathfrak{g}} = \bigoplus_k \widehat{\rho_k} \otimes \mathrm{Sym}^k,$$

and write

$$\bigoplus_{k \text{ odd}} \widehat{\rho_k} = (\oplus_i \tau_i \oplus \tau_i^\vee) \bigoplus (\oplus_j \sigma_j),$$

where σ_j's consist of those distinct symplectic representations appearing odd times, and write $\widehat{\rho_{\widehat{\iota'}}} = \oplus_j \sigma_j$. Then

Conjecture 4.19 (Mao-Zhang-Wan)**.**

$$\frac{\mathcal{P}_{\iota'}(\phi)}{\langle \phi, \phi \rangle} = \frac{L\left(\frac{1}{2}, \Pi, \widehat{\rho_{\widehat{\iota'}}}\right)}{\prod_k L\left(\frac{k}{2}+1, \Pi, \widehat{\rho}_k\right)}.$$

In the special case that $\widehat{\iota'} = 0$, then $\mathcal{P}_{\iota'}$ is the Whittaker period, and we recover the Lapid-Mao conjecture

$$\frac{\mathcal{P}_{\iota'}(\phi)}{\langle \phi, \phi \rangle} = \frac{1}{L(1, \pi, \mathrm{Ad})}.$$

Consider the special case of Conjectures (3.9) and (4.19) when $\rho_{\widehat{H'}} = 0$ and $\rho_{\widehat{\iota'}} = 0$. Then (2) of Conjecture (3.9) is

$$\frac{|\mathcal{P}_{\Delta}(\phi)|^2}{\langle \phi, \phi \rangle} = \frac{\prod_k L(\frac{k}{2}+1, \Pi, \widehat{\rho}_k)}{L(1, \Pi, \mathrm{Ad})^2},$$

and Conjecture (4.19) is

$$\frac{\mathcal{P}_{\iota'}(\phi)}{\langle \phi, \phi \rangle} = \frac{1}{\prod_k L\left(\frac{k}{2}+1, \Pi, \widehat{\rho}_k\right)}.$$

Therefore

$$\frac{|\mathcal{P}_{\iota'}(\phi)\mathcal{P}_{\iota'}(\phi)|}{\langle \phi, \phi \rangle} = \frac{1}{L(1, \Pi, \mathrm{Ad})}.$$

This suggests that there should be a RTF comparison between

- KFT on H': for $f' \in \mathcal{S}(H'(\mathbb{A}))$, let $K_{f'}(\cdot,\cdot)$ be the ussal kernel function, and
$$J(f') := \int_{[N']} \int_{[N']} K_{f'}(x, y)\xi(x^{-1}y)\,\mathrm{d}x\,\mathrm{d}y,$$
- and RTF on G: for $f \in \mathcal{S}(G(\mathbb{A}))$, with usual kernel function $K_f(\cdot,\cdot)$, and
$$I(f) : \mathcal{P}_{\iota'}(\mathcal{P}_{H,\iota,\rho_H,1}(K_f)).$$

Conjecture 4.20. *There should be a comparison between $I(f)$ and $J(f')$.*

Example 4.21. *Consider $\Delta = (\mathrm{GL}_{2n}, \mathrm{GL}_n, [2^n], 0)$ and $\widehat{\Delta} = (\mathrm{GL}_{2n}, \mathrm{Sp}_{2n}, 1, 0)$, then the case $n = 2$ is due to Friedberg-Jacquet for the fundamental lemma, and later Mao gave another easier proof for the fundamental lemma.*

Theorem 4.22 (Mao-Wan-Zhang)**.** *Smooth transfers and fundamental lemma hold for the 6 cases in [MWZ] over p-adic fields.*

Remark 4.23. *The cases of $\mathrm{SL}_6, \mathrm{Spin}_{12}$ and E_7 are due to Rallis-Mao. One also notes that in these cases, $\widehat{H'} = \mathrm{PGL}_2$, and hence $H' = \mathrm{SL}_2$, so the KFT on SL_2 is not that complicated. Moreover, $\widehat{\iota'}$ all have even orbits only, and $\rho_{\widehat{H'}}$'s are all trivial.*

If we do not assume $\rho_{\widehat{H'}} = \rho_{\widehat{\iota'}} = 0$, then what we expect is

$$\frac{|\mathcal{P}_{\iota'}(\phi)\mathcal{P}_{\iota'}(\phi)|}{\langle \phi, \phi \rangle} = \frac{\sqrt{L\left(\frac{1}{2}, \Pi, \rho_{\widehat{H'}} \otimes \rho_{\widehat{\iota'}}\right)}}{L(1, \Pi, \mathrm{Ad})}.$$

If we consider $\widehat{\Delta'} := (\widehat{H'}, \widehat{H'}, \rho_{\widehat{H'}} \oplus \widehat{\rho_{\iota'}}, 1)$, which is strongly tempered. Then the BZSV conjecture predicts that there is some Δ' dual to it. Then the above is the product of the Whittaker period and $\mathcal{P}_{\Delta'}$ on $H'(\mathbb{A})$, so we need to change the KTF to

$$J'(f') := \mathcal{P}_{\Delta'}(\mathcal{P}_{\mathrm{Whittaker}}(K_{f'}(\cdot,\cdot))).$$

5. June 6

5.1. Chen Wan.

Definition 5.1. *Δ is strongly tempered if $\widehat{\Delta} = (\widehat{G}, \widehat{G}, 1, \widehat{\rho})$ up to some central elements.*

In this case, conjecturally,

$$\frac{|\mathcal{P}_\Delta(\phi)|^2}{\langle \phi, \phi \rangle} = \frac{L\left(\frac{1}{2}, \pi, \widehat{\rho}\right)}{L(1, \pi, \mathrm{Ad})}.$$

Local relative characters are easy to define in this case. For example, if $\Delta = (G, H, 1, 0)$, then we can define

$$I_\nu(\phi_\nu) := \int_{H_\nu} \langle \pi(h)\phi_\nu, \phi_\nu \rangle \,\mathrm{d}h.$$

To classify strongly tempered cases, it suffices to classify $\widehat{\Delta}$ satisfying the hyperspherical condition. This can be done by checking the tables of [Knop]. Then the question is how to write down Δ.

Definition 5.2. *$\Delta = (G, H, \iota, \rho_H)$ is called reductive if ι is trivial.*

Theorem 5.3. *For all quadruples in table (21)-(24) in [MWZ24] except the quadruple $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), 1, \Lambda^2 \otimes \mathrm{std}_{\mathrm{GL}_2})$, the local relative characters of the periods are equal to the expected L-values.*

Theorem 5.4. *For quadruples in (21), (23) and (25) in [MWZ24], Conjecture 3.9 $(1)'$ with expected L-values follow from Rallis inner product and the GGP conjecture.*

As for the Δ-side, we can also read it from [Knop]. Bascailly,

- W_V seems to be the root type of H,
- ι is the principal $\mathfrak{sl}_2$ there,
- and we determine ρ_H in an ad hoc way.

Next let us discuss how to use Whittaker induction to reduce to the above cases. Let $\Delta = (G, H, \iota, \rho_H)$ be a quadruple. From ι we can construct a parabolic subgroup $P = MN$, where M is the centralizer of $\iota\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\right)$ in G, and the Lie algebra of N is the positive root space.

Definition 5.5. *We call (G, H, ι, ρ_H) the Whittaker induction of $\Delta_0 := (M, H, 1, \rho')$, where*

$$\rho' := \rho \bigoplus (\oplus_{k \text{ odd}} \rho_k).$$

Assume we know the dual quadruple $\widehat{\Delta_0} = (\widehat{M}, \widehat{M}, 1, \rho_{\widehat{M}})$, which is strongly tempered and reductive.

Definition 5.6. *If $\rho_{\widehat{M}}$ is an irreducible representation of $\widehat{M}$ with highest weight $\varpi_{\widehat{M}}$. Let $\rho_{\widehat{M}}^{\widehat{G}}$ be the irreducible representation of $\widehat{G}$ with highest weight $\varpi_{\widehat{G}}$ such that $\varpi_{\widehat{G}} = w\varpi_{\widehat{M}}$ for some $w \in W(G)$. If*

$$\rho_{\widehat{M}} = \oplus \rho_{i,\widehat{M}}$$

for irreducible representations, we define

$$\rho_{\widehat{M}}^{\widehat{G}} = \oplus \rho_{i,\widehat{M}}^{\widehat{G}}.$$

Conjecture 5.7.

$$\widehat{\Delta} = (\widehat{G}, \widehat{G}, 1, \rho_{\widehat{M}}^{\widehat{G}}).$$

Theorem 5.8. *Each Δ in Table* (23)-(26) *in [MWZ24] is a Whittaker induction of some reductive quadruple Δ_0 in Table* (21)-(22) *in [MWZ24].*

Assume the duality holds for Δ_0, then the Conjecture 5.7 holds for Δ if and only if the duality holds for Δ.

School of Mathematics, University of Minnesota, 206 Church St. S.E., Minneapolis, MN 55455, USA.

June 8 Nakajima.

Let $F = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]]$, $k = \mathbb{C}$, $G \curvearrowright M = T^*N$ for some smooth affine G-variety

S-dual. $G^\vee \curvearrowright M^\vee := \operatorname{Spec} H^*_{G(\mathcal{O})}(\mathcal{A}_{G,N} \otimes \mathcal{A}_{\mathbb{C}[\check{G}]})$, where

$$\mathcal{A}_{\mathbb{C}[G^\vee]} := (\text{Satake})^{-1}(\mathbb{C}[G^\vee]) \in \operatorname{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G),$$

and $\mathcal{A}_{G,N} := \pi_* \omega_{\mathcal{R}_{G,N}}$, with $\mathcal{R}_{G,N} = \left\{ [(g(z), s(z))] \in G(\mathcal{K}) \times^{G(\mathcal{O})} N(\mathcal{O}) \;\middle|\; g(z)s(z) \in N(\mathcal{O}) \right\}$

$$\downarrow \pi$$

$$\mathrm{Gr}$$

and $\omega_{\mathcal{R}_{G,N}}$ is the dualizing sheaf on $\mathcal{R}_{G,N}$. We have

$$M^\vee = \operatorname{Spec}(H^*(\text{der Satake } \mathcal{A}_{G,N}))$$

More generally, we consider interfaces $G_1 \curvearrowright M \curvearrowleft G_2$.

$\Longleftrightarrow$ $G_1 \times \underline{G_2} \curvearrowright M$ up to a Chevalley twist

Composition:

$$(G_1 \curvearrowright M_{12} \circlearrowleft G_2) \circ (G_2 \curvearrowright M_{23} \circlearrowleft G_3) = \left(G_1 \curvearrowright M_{12} \underset{\mathfrak{g}_2^*}{\overset{G_2}{\times}} M_{23} \circlearrowleft G_3\right)$$

For the ring objects:

$$\mathcal{A}_{12} \circ \mathcal{A}_{23} = \pi_*(\mathcal{A}_{12} \otimes^! \mathcal{A}_{23}) \in D_{G_1(\mathcal{O}) \times G_3(\mathcal{O})}(Gr_{G_1} \times Gr_{G_3})$$

For the theories: $\mathcal{T}_{12} \circ \mathcal{T}_{23} = \mathcal{T}_{12} \underset{G_2}{\overset{\times}{/\!\!/\!\!/}} \mathcal{T}_{23}$.

[BFN, Remark 5.22]: S-dual $(\cdot)^\vee$ respects "$\circ$";

$$(\mathcal{T}_{12} \circ \mathcal{T}_{23})^\vee = \mathcal{T}_{12}^\vee \circ \mathcal{T}_{23}^\vee \quad \cdots$$

$$H^*(\text{der Sat } \mathcal{A}_{12} \circ \mathcal{A}_{23}) = H^*(\text{der Sat } \mathcal{A}_{12}) \circ H^*(\text{der Sat } \mathcal{A}_{23}).$$

Thm: For $G = GL_n$, framed vertex

$$\mathcal{T}[G] \cong \boxed{n} - (n-1) - (n-2) - \cdots - (1)$$

Let $N = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n-1}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^1)$

$\mathbb{G} = GL_n \times GL_{n-1} \times \cdots$

Then RHS $= \mathcal{T}_{(\mathbb{G}, T^*N)} /\!\!/\!\!/ G . \circlearrowleft GL_n$.

$$\mathcal{A}_{\mathcal{T}[G^\vee]} = (\pi_{Gr_{\mathbb{G}} \to Gr_{GL_n}})_* (\omega_{\mathcal{R}_{\mathbb{G}, N}}) \in D_{GL_n(\mathcal{O})}(Gr_{GL_n})$$

Therefore if $GL_n \curvearrowright M$, then

$$M^\vee = \text{Coulomb branch of } M \times T^*N /\!\!/ _{\Delta GL_n \times GL_{n-1} \times \cdots}$$

Eg:

$$\left(GL_n \curvearrowright \boxed{n} - (n-1) - (n-2) - \cdots - (1) \right)^\vee = M_C \left((1) - (2) - \cdots - (n-1) - (n) - (n-1) - \cdots - (1) \right)$$

Write $\underset{\substack{\circlearrowleft \\ GL_m}}{\overset{m}{—}} \!\!\circ\!\! \underset{\substack{\circlearrowright \\ GL_n}}{\overset{n}{—}} = T^* \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ $\qquad (m) - (n)$

$\underset{\substack{\circlearrowleft \\ GL_m}}{\overset{m}{—}} \!\times\! \underset{\substack{\circlearrowright \\ GL_n}}{\overset{n}{—}}$ interface in 1-dim gauge theory

= moduli space of solutions of Nahm's eq. on $[-1, 1]$ with Nahm pole at $t=0$ up to gauge transform γ with $\gamma(0) = \gamma(1) = \mathrm{id}$.

$$\cong GL_m \curvearrowright \left(GL_m \times \sum_{m-n} \right)^{\circlearrowright GL_n} \quad \text{if } m > n$$

$\longleftrightarrow$ change m & n. $\qquad$ if $m < n$.

$T^*GL_n \times \underbrace{T^*\mathbb{C}^n}_{\substack{\| \\ T^*\mathrm{Hom}(\underline{\mathbb{C}}, \mathbb{C}^n)}}$ $\quad$ framing $\qquad$ if $m = n$

$\overset{\ell}{-}\circ\overset{m}{-}\times\overset{n}{-} \qquad \overset{\ell}{-}\circ\overset{m}{-} \underset{\mathfrak{gl}_m^*}{\overset{GL_m}{\times}} \overset{m}{-}\times\overset{n}{-}$

$-\circ-\circ-\times-\times-\times-$

<u>Eg</u>: (1) $\overset{0}{-}\times\overset{1}{-}\times\overset{2}{-}\times- \cdots -\times\overset{n}{-} = GL_n \times \Sigma_{\boxed{1 \cdots 1}} = T^*GL_n$

$\overset{0}{-}\circ\overset{1}{-}\circ\overset{2}{-}\circ- \cdots -\circ\overset{n}{-} = \mathcal{M}_H\left(\boxed{n} - (n-1) - \cdots - (1) \right)$

$\boxed{\begin{array}{ll} \text{Write} & \dot{\times} \;\; \text{R-L} \\ & \dot{\circ} \;\; \text{L-R} \end{array}}$

$= \mathcal{N}il_{GL_n}.$

(2) $\underbrace{\overset{\circ}{-\times}\overset{\circ}{-\times} \cdots}_{\ell} \overset{\circ}{-\times}\overset{n}{-} = T^*GL_n \times T^*\mathrm{Hom}(\mathbb{C}^\ell, \mathbb{C}^n)$

$\uparrow$ ℓ-dim framing

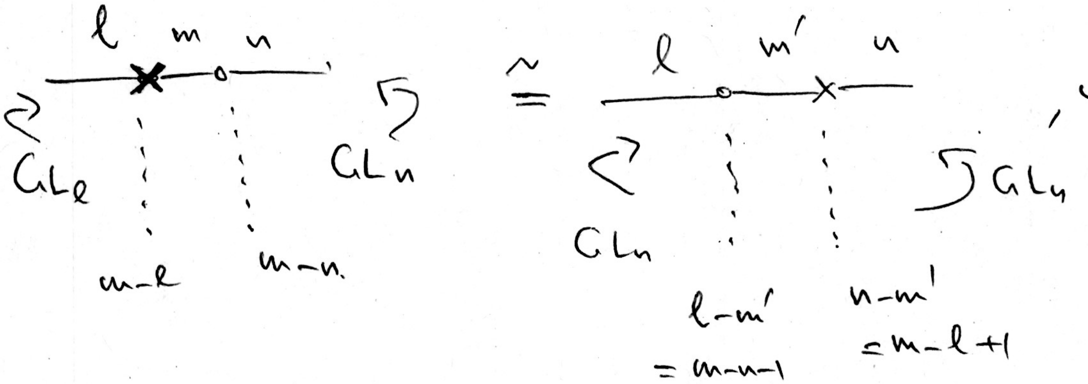
<u>"Thm"</u> $\left(\overset{m}{-}\circ\overset{n}{-} \right)^\vee = \overset{m}{-}\times\overset{n}{-}$

$\circlearrowleft_{GL_m} \;\; \circlearrowright_{GL_n} \qquad \circlearrowleft_{GL_m} \;\; \circlearrowright_{GL_n}.$

[composition, $\vee$] = 0.

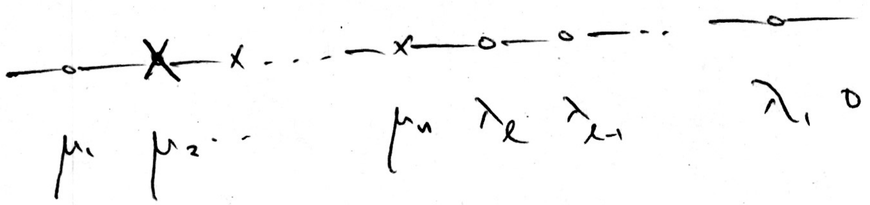
$\left(-\times-\circ-\circ-\circ-\times-\times- \right)^\vee = -\circ-\times-\times-\times-\circ-\circ-$

Hanany-Witten transition:



, where $m+m' = l+n-1$

So we are reduced to



, $|\lambda| = |\mu|$

$\simeq \overline{\mathcal{O}_{\lambda^T}} \cap \Sigma_\mu$ if $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_l$, λ: partition

$\mu_1 \le \mu_2 \le \cdots \le \mu_n$, μ = partition.

<u>Eg</u> (2405, 17699)

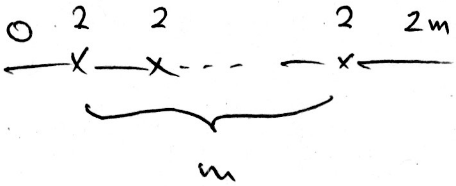
[Table 6, 2^{nd} row] $\check{G} = GL_{2m} \curvearrowright \check{M} = T^* \Lambda^2$

[Table 8] $(G, H, \iota, \rho_H) = (GL_{2m}, GL_m, (A_1)^m = [2^m], T^* \mathbb{C}^m)$

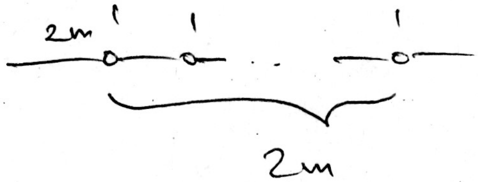
$$M = \left(GL_{2m} \times \Sigma_L \times T^*\mathbb{C}^m\right) /\!\!/\!\!/ GL_m$$

$\circlearrowleft$

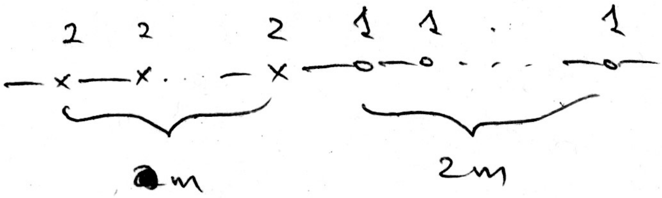
GL_{2m}

$=$ 

$\cdot \ (T^*\mathbb{C}^m) /\!\!/\!\!/ GL_m$

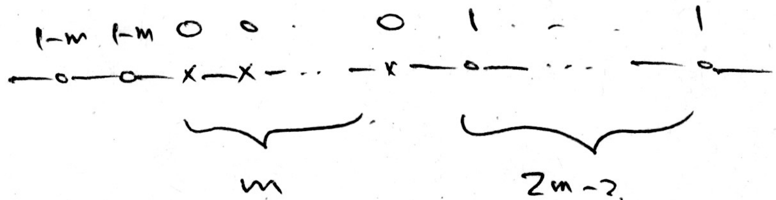
$\mathcal{J}[GL_{2m}] =$ 

$$M^\vee \overset{?}{=} M_c\left(M \times \mathcal{J}[GL_{2m}] /\!\!/\!\!/ GL_{2m}\right)$$

$= M_c\Big($ 

$\times$

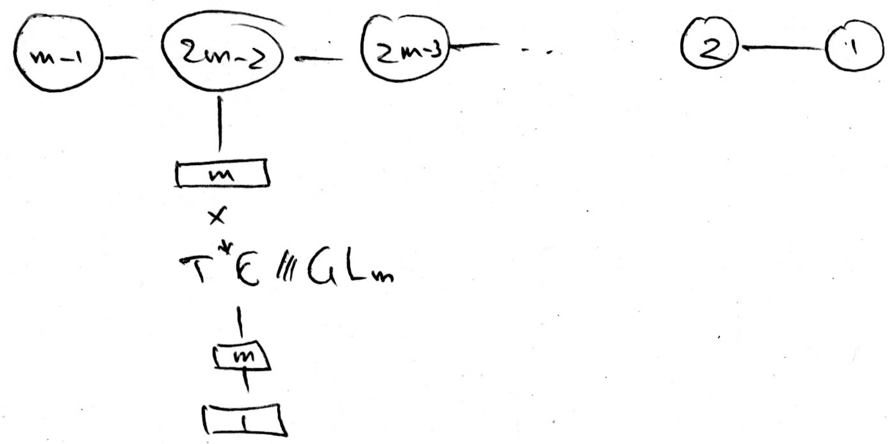
$T^*\mathbb{C}^m /\!\!/\!\!/ GL_m. \Big)$

$\overset{HW}{=} M_c\Big($ 

$\times$

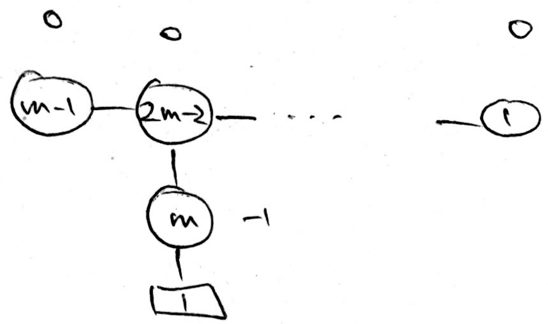
$T^*\mathbb{C}^m /\!\!/\!\!/ GL_m \Big)$

$= \mathcal{M}_C \Big($



$\Big)$

$= \mathcal{M}_C \Big($



$\Big)$, type D_m quiver gauge theory

$Gr_G \supset \overline{Gr^\lambda} \supset Gr_\mu$, $\mathcal{M}_C$ = slice to Gr_μ in $\overline{Gr^\lambda}$ if

μ is dominant. $\lambda = \varpi_{2m}$, $\mu = -\varpi_{2m} = w_0(\lambda)$

[Krylov - Perunov (1903.08277)]: $\mathcal{M}_C \cong \mathbb{C}^{2\langle \rho, \lambda - \mu \rangle}$ if

λ is minuscule and $\mu = w\lambda$, $w \in W$.