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Source: *Public Choice*, 1981, Vol. 36, No. 1 (1981), pp. 153-158

Published by: Springer

Stable URL: https://www.jstor.org/stable/30023422

A further characterization of Borda ranking method

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Introduction

The prime purpose of this paper is to present a characterization result for the Borda rule which is very much related to the recent axiomatization of the same rule by Young (1974) and by Hansson and Sahlquist (1976). We prove that, given a finite set of alternatives and a finite profile of connected and asymmetric individual relations which are not required to be transitive, the only ranking method which satisfies a set of four axioms, namely, Neutrality, Monotonicity, Consistency and Cancellation (A.1, A.2, A.3, A.4 below) is the so called Borda ranking method.

Our work differs from the previous studies in three respects: First, in our model each individual's list of paired comparisons of the alternatives is assumed to be connected and asymmetric and not necessarily transitive. This is in marked contrast to most of the studies in voting theory, including Young's (1974), which usually require transitive individual preference relations. Second, Young pursued his analysis in terms of a social choice function, that is, a rule which specifies a set of chosen alternatives for any specifications of voters' preference profile. Within our framework the concept of a ranking method (RM) is employed. RM is defined as a rule which assigns an ordering (transitive, complete and irreflexive relation) to any given profile of connected and asymmetric relations. This difference is insignificant as our result could still be obtained when switching to the choice sets framework. Note that the ordering of social states produced by Borda RM is not a social ordering as defined by Arrow (1963). The preference relation between a pair of alternatives in our model is determined by their total scores, whereas Arrow (1963: 6) defines the social preference ordering as based on how the alternatives fare in face-to-face competition. Third, three of our axioms are similar to three of Young's conditions except that they are stated in terms of the social preference relation instead of in

* We are indebted to the anonymous referees of this journal for their most helpful suggestions.

Public Choice 36:153-158 (1981) 0048-5829/81/0361-0153 $00.90.

terms of social choice sets. The difference is the substitution of the Monotonicity axiom for Young's Faithfulness condition.

A noteworthy aspect of the social choice construct is its analogy to a multi-round tournament set-up. In each round of the tournament each player competes only once with every other player where each contest ends in the definite victory of one of the players. The problem of ranking the players in the multi-round tournament is identical to the problem of ranking the social alternatives, given a certain preference profile, when each individual's list of paired comparisons of the alternatives is viewed as analogous to one round in the tournament. Requiring connectedness and asymmetry only is particularly natural in the tournament context as the results of the binary contests in any round of the tournament, clearly, need not be transitive. By our result, then, the scores method for ranking the players is the only method which satisfies A.1-A.4. According to this method one participant is ranked higher than another participant if and only if he has defeated a larger number of competitors during the tournament.

The model

Let A be a finite set whose elements are denoted $a, b, c, \ldots,$. The set of natural numbers is denoted N. A *profile* $\{>^i\}\ i \in M$ is a function from a finite set $M = \{1, \ldots, |M|\}$ $(N \supset M \neq \phi)$ to the set of connected and asymmetric binary relations on A.[1] A *ranking method* (RM) is a function

$$\gtrsim : \mathbf{P} \to W(A)$$

where $\mathbf{P}$ is a set of profiles and $W(A)$ is the set of orderings[2] on A.

The social choice interpretation of the model is as follows: A is a set of alternatives. M is a set of individuals. $>^i$ is a summary of individual i's paired comparisons of the alternatives, where $a >^i b$ means that individual i prefers a to b. The outcomes of the individual binary comparisons are not necessarily transitive. Finally, a RM is an extension of the social welfare function concept introduced by Arrow (1963). An alternative interpretation is that of the multi-round tournament ranking problem. Here A is viewed as a set of players participating in an $|M|$ – rounds tournament. $>^i$ is interpreted as the summary of the outcomes in round i, $a >^i b$ means that player a defeated player b in round i. The function $\gtrsim$, which is an extension of the ranking method concept introduced by Rubinstein (1980), assigns to each tournament a ranking of the players.

Given a profile P we shall occasionally simplify the notation and write $\gtrsim$ instead of $\gtrsim(P)$. For two alternatives, a, b in A we write

$$a > b \ \text{ if } a \gtrsim b \text{ and not } b \gtrsim a$$

$$a \sim b \ \text{ if } a \gtrsim b \text{ and } b \gtrsim a .$$

Similar notational convenience is obtained by omitting P from the following three terms which are also defined for a particular profile P. Let

$$\Pi_{ab} = |\{i | a >^i b\}|$$

$$S_a^i = |\{b | a >^i b\}|$$

$$S_a = \sum_{i \in M} S_a^i = \sum_{b \in A} \Pi_{ab}$$

Π_{ab} is the number of individuals preferring a to b. S_a^i is the number of alternatives which are inferior to alternative a according to individual i's preferences. S_a is the total number of pairwise wins of alternative a in the profile. Given the ordering $\gtrsim$ on A and $a, b \in A$ we denote by $\gtrsim [a, b]$ the restriction of the ordering to $\{a, b\}$.

A ranking method is called a Borda rule when

$$a \gtrsim b \text{ iff } S_a \geqslant S_b .$$

With these definitions we now turn to the main theorem.

The result

Theorem: Borda rule is the only RM that satisfies the following axioms:

(A.1) *Neutrality* – Let σ be a permutation on A, and $P = \{>^i\}_{i \in M}$. Define $\sigma(P)$ as the profile satisfying $a >^i b$ in P iff $\sigma(a) >^i \sigma(b)$ in $\sigma(P)$. Then $a \gtrsim b$ given P iff $\sigma(a) \gtrsim \sigma(b)$ given $\sigma(P)$.

(A.2) *Monotonicity* – Suppose a and b are two distinct alternatives and let $P, P' \in \mathbf{P}$. If $a \gtrsim b$ in P, and P' is identical to P except for the existence of a third alternative c and $i \in M$ such that $c >^i a$ in P, but $a >^i c$ in P', then $a > b$ in P'.

(A.3) *Consistency* – Let $P_1 = \{>_1^i\}_{i \in M_1}$, $P_2 = \{>_2^i\}_{i \in M_2}$ and denote $P_1 + P_2 = \{>^i\}_{i \in M}$ where $M = \{1, 2, \ldots, |M_1| + |M_2|\}$ and for $1 \leqslant i \leqslant |M_1|$, $>^i = >_1^i$, and for $i > |M_1|$, $>^i = >_2^{i-|M_1|}$. If $a \gtrsim b$ in P_1 and $a \gtrsim b$ *in* P_2, then $a \gtrsim b$ in $P_1 + P_2$. If, in addition, $a > b$ in one of the profiles P_1 or P_2, then $a > b$ in $P_1 + P_2$.

Following Young we introduce,

(A.4) *Cancellation* – If $\Pi_{ab} = \Pi_{ba}$ for all $a, b \in A$, then, $a \sim b$ for all $a, b \in A$.

The *Neutrality* axiom implies that no importance is attached to the labelling of alternatives.

The *Monotonicity* axiom implies an explicit strengthening of the social preference relationship between two alternatives due to the improvement in the status of the preferred alternative vis-à-vis some third alternative.

The *Consistency* axiom requires that if the RM yields similar ranking for a pair of alternatives under two profiles, then the same order will be maintained given the profile consisting of the two.

An RM satisfies the *Cancellation* axiom if whenever the number of individuals preferring one alternative to another is equal for every pair of alternatives, then all the alternatives are ranked equally.

Before proceeding with the main proof, let us demonstrate that the axioms are independent.

1. Let $\Psi : A \to \{1, 2, \ldots, |A|\}$ be a one-to-one function.
   Define $\phi_a = \Psi(a) \cdot \Sigma_{c \neq a} \Pi_{ac} - \Pi_{ca}$ and $a \gtrsim b$ if $\phi_a \geqslant \Phi_b$.
   $\gtrsim$ satisfies A.2, A.3 and A.4 but not A.1.

2. Define $a \gtrsim b$ if $S_a \leqslant S_b$.
   This RM which might be called the 'inverse Borda method' satisfies A.1, A.3 and A.4 but not A.2.

3. Define the following binary relation V on A
   $a\ V\ b$ if $\Pi_{ab} \geqslant \Pi_{ba}$ and let
   $a \gtrsim b$ if $|\{c \mid a\ V c\}| > |\{c \mid b\ V c\}|$ or
   $|\{c \mid a\ V c\}| = |\{c \mid b\ V c\}|$ and $S_a \geqslant S_b$.
   This RM satisfies A.1, A.2 and A.4, but not A.3.

4. Let $\gtrsim$ be the RM satisfying
   $a \gtrsim b$ if

   $$\sum_{i=1}^{n} (S_a^i)^2 \geqslant \sum_{i=1}^{n} (S_b^i)^2$$

   $\gtrsim$ satisfies A.1, A.2 and A.3, but not A.4.

Let P be a profile and denote by $-P$ the profile satisfying $b >^i a$ in $-P$ iff $a >^i b$ in P.

Lemma 1: If $\gtrsim$ is a RM that satisfies A.3 and A.4, then $a > b$ in P iff $b > a$ *in* $-P$.

Proof: Suppose $a > b$ in P and $a \gtrsim b$ in $-P$. By A.3 it follows that $a > b$ in $P + (-P)$. However, by A.4, $a \sim b$ in $P + (-P)$. A contradiction.

Lemma 2: If $\gtrsim$ is a RM that satisfies A.3 and A.4, then $\gtrsim$ depends only on $\{\Pi_{ab} - \Pi_{ba}\}_{a \neq b}$.

Proof: Suppose there exist two profiles P and P' such that for any pair, a, b $\in A$ $\Pi_{ab} - \Pi_{ba}$ are identical, yet there are $c, d \in A$ such that $c \gtrsim d$ in P and $d \gtrsim c$ in P'. Let $\hat{P} = P + (-P)$. In $\hat{P}$, $\Pi_{ab} = \Pi_{ba}$ for every a, b and, therefore, by A.4, $c \sim d$ in $\hat{P}$. By assumption, $c \gtrsim d$ in P and by Lemma 1 $c > d$ in $-P'$. Hence, A.3 implies $c > d$ in $\hat{P}$. A contradiction.

We now introduce a technical axiom which relates to single individual profiles:

(A.5) *Independence* – Let a, b, c, d be four distinct alternatives. Let $P = \{>^1\}$ and $\bar{P} = \{\bar{>}^1\}$ be almost identical profiles, the only difference being that $c >^1 d$ and $d \bar{>}^1 c$. Then, $\gtrsim [a, b]$ in P is identical to $\gtrsim [a, b]$ in $\bar{P}$.

The axiom implies that the relative ranking of two alternatives in a single inividual profile is independent of those comparisons in which neither is involved.

Lemma 3: If $\gtrsim$ is a RM that satisfies A.1, A.3 and A.4, then it also satisfies A.5.

Proof: Assume to the contrary, and let a, b, c and d be four distinct alternatives. Consider $P = \{>^1\}$ and $\bar{P} = \{\bar{>}^1\}$ which are almost identical profiles, where $c >^1 d$, $d \bar{>}^1 c$, $a > b$ in P and $b > a$ in $\bar{P}$. A.3 and A.4 imply Lemma 1. Hence, $a \gtrsim b$ in $-\bar{P}$ and $b > a$ in $-P$. By A.3 we therefore get $a > b$ in $\tilde{P} = P + (-\bar{P})$, and $b > a$ in $(-P) + \bar{P}$.

Denote by σ the permutation (a, b) that inverts a and b only. By A.1 $b > a$ in $\sigma(P)$ and by A.3 $b > a$ in $\sigma(P) + (-P + \bar{P})$. But in the latter profile, for every two alternatives a, b, $\Pi_{ab} = \Pi_{ba}$ and so, by A.4, $b \sim a$. A contradiction.

Lemma 4: Let $P = \{>^1\}$ be a single individual profile, and suppose A.1, A.2 and A.5 are satisfied. Then $b \gtrsim a$ in P iff $S_b \geqslant S_a$.

Proof: See Rubinstein (1980).

We now turn to the proof of the main theorem: It can be easily checked that the Borda method satisfies A.1–A.4.

Let a and b be two distinct alternatives in a profile $P = \{>^i\}_{i \in M}$ satisfying $S_a \geqslant S_b$. One can easily construct a profile $\bar{P} = \{\bar{>}_i\}_{i \in M}$ such that $S_a^i \geqslant S_b^i$ for every i in M and Π_{cd} in P is equal to Π_{cd} in $\bar{P}$ for every $c, d \in A$. By Lemma 2 the relative ranking of a and b is the same under P and under $\bar{P}$. From Lemma 3 we obtain that $\gtrsim$ satisfies A.5 and so, by Lemma 4, $a \gtrsim b$ in any single individual profile $\{>^i\}$. By A.3 we conclude that $a \gtrsim b$ in P.

NOTES

1. $>$ is connected if for all $a, b \in A$, $a \neq b$ $a > b$ or $b > a$. $>$ is asymmetric if for all $a, b \in A$ $a > b \Rightarrow b \not> a$.
2. An ordering is a connected and transitive binary relation.

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