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1. Conventions

1.1. **In the mixed characteristic setting.** We recall some six functor formalisms. On the automoprhic side, we consider v-stacks or diamond X.

- We write
$$D_{\text{ét}}(X, \Lambda)$$
for derived category of étale sheaves on X.
- $\Lambda \in \left\{\bar{\mathbb{F}}_p, \mathbb{Z}_l, \mathbb{Q}_l\right\}$
- $\breve{E} := \widehat{E^{\text{nr}}}$, we can construct this via $\widehat{W(\bar{\mathbb{F}}_p) \otimes_{W(\mathbb{F}_p)}} \mathcal{O}_E$. There is a natural q-frobenius, as $\Gamma_{E^{\text{nr}}/E} \simeq \Gamma_{k_E^{\text{nr}}/k_E}$. In which case, if $E = \mathbb{Q}_p, k = \mathbb{F}_p$, the extensions of degree n of $\mathbb{F}_p$ are splitting field of $x^{p^n} - x$. This is equivalent to finding the $p^n - 1$th roots of unity. This is equivalent to adjoining pth roots of unity, for all n, so that setting $K_n := \mathbb{Q}_p(\mu_{p^n-1})$, we have
$$\mathbb{Q}_p^{\text{nr}} \simeq \varinjlim_{(n,p)=1} K_n$$
- For a Huber pair (R, R^+) a Huber ring, we denote
$$\text{Spa}(R, R^+) := \left\{| \quad | \in \text{Cts}(R) \,:\, |r| \leq 1, \quad r \in R^+\right\}$$
For an adic space $X \to \text{Spa}(\mathcal{O}_F)$, we define
$$X^\diamond : \text{Pftd}_{k_F}^{\text{op}} \to \text{Set}$$
$$X^\diamond(S) := \left\{S^\sharp \,:\, (S^\sharp)^\flat \simeq S, S^\sharp \to X\right\}$$

Remark 1.1. Algebraic closures versus separable closure. k is perfect if and only if $k^{\text{alg}} = k^{\text{sep}}$. see discussion.

On the

- We let $\Lambda = \bar{\mathbb{Q}}_l$.
- On the spectral side, we consider a derived artin stack X, and $\text{QCoh}(-), \text{Coh}(-)$.
-

1.2. **In the geometric Langlands setting.** When we discuss the de Rhan setting:

- $\Sigma \in \text{SmProj}_{\mathbb{F}}^{\text{cn}}$, where $\mathbb{F} = \mathbb{C}$.

When we discuss the Betti setting:

- $\Sigma \in \text{SmProj}_{\mathbb{F}}^{\text{cn}}$,

1.3. **Computing limits and colimits.** When given a diagram $\Phi^L : \mathcal{C}^{\text{op}} \to \text{Pr}_{\text{st},L}$, we can compute
$$\varinjlim_{\mathcal{C}^{\text{op}}} \Phi^L \simeq \varprojlim_{\mathcal{C}} \Phi^R$$
crucially, the limit here is can be computed in $\text{Pr}_{\text{st}}^R \hookrightarrow \text{Cat}_\infty$.

2. Introduction

This note analyzes the Tate period in the setting of Fargues Fontaine curve. Let

$$(G, X) = \left(\mathrm{GL}_1, \mathbb{A}^1\right), \quad \left(\check{G}, \check{X}\right) = (\mathbb{G}_m, \mathbb{A}^1) \tag{1}$$

be dual pairs under [BSV].

Remark 2.1. More generally, there are conjectured Hamiltonian dual pairs

$$(G, M) \longleftrightarrow (\check{G}, \check{M})$$

In Langlands, for such a pair, there are often equivalence of the following form:

$$\langle X\text{-Poincare series}, f_G\rangle \sim L(\check{X}, f_G)$$

where f_G is a cusp form on G. Our choice of dual pair is simple in many ways. It is *conical*, Definition 2.2, which guarantees that $0 \in X$ is the only $\mathbb{G}_m$ fix point.

Definition 2.2. Let e be a field. $X \in \mathbb{G}_m\text{-}\mathrm{Aff}_e$ an affine variety with $\mathbb{G}_m$ action. If $e[X]$ has only nonnegative $\mathbb{G}_m$-weights, and the 0th garded piece is isomorphic to e.

The local L-factors of conical space are easy to compute, which we discuss in Section 2.4.

Example 2.3. Godement-Jacquet dual pair.

Example 2.4. The Whittaker and trivial period

2.1. What we will do in mixed characteristic.

The Fargues-Fontaine curve should be a *global object* of dimension 2 under the TQFT dictionary of [BSV]. Associated to the dual datum we can define the period sheaf $\mathcal{P}_X \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ and the *L-sheaf* $\mathcal{L}_{\check{X}} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(Z^1(W_E, \check{G})/\check{G})$, which categorifies classical notion of period and L-functions respectively. Our goal is to imitate the construction in the equal characteristic setting, Section 5. Though we do not need categorical Local Langlands conjectures, Section 2.2, we review the torus case.

2.2. Short recollection on categorical local Langlands.

Recall in [FS24], they constructed a spectral action

$$\mathrm{Perf}(Z^1(W_E, \check{G})/\check{G}) \circlearrowright \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

This action is required to be satisfy various properties. [FS24, p. IX], in particular the Hecke action, Equation 4. For the Iwasawa-Tate case, one can actually avoid this action.

Conjecture 2.5. [FS24] Let l be a prime coprime to q. $\Lambda \hookrightarrow L$ ring of integers of algebraic field extension. Fix $\sqrt{q} \in \Lambda$, and a Whittaker datum: fix a borel $B \hookrightarrow G$, and a generic character $\psi : U(E) \to \Lambda^\times$. This gives rise to $\mathrm{cInd}_{U(E)}^{G(E)}\psi$, which in turn yields a sheaf $\mathcal{W}_\psi$ on Bun_G. There is an $\mathrm{Perf}(Z^1(W_E, \check{G})/\check{G})$-linear equivalence

$$c_\psi : \mathrm{Coh}^b_{\mathrm{Nilp}}([Z^1(W_E, \check{G})/\check{G}]) \simeq D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega \tag{2}$$

where we have

$$\mathcal{O} \mapsto \mathcal{W}_\psi$$

But what is conjectured in this setting?

2.3. **Some conjectures in Local Langlands.** References [Ngu24]. We briefly recall how we obtain a datum

$$\left(I, W, (\gamma_i)_{i\in I}, \alpha, \beta\right)$$

which should induce the following compatibility

Theorem 2.6. *[FS24, p. I.9.6] There is a functorial mapping*

$$\pi \mapsto \phi_\pi^{FS}$$

(1) compatibility with local class filed theory: if T is torus, then $\pi \mapsto \phi_\pi$ is compatible with local Langlands correspondence.

2.4. **What the L-sheaf is.** This section is a leisurely introduction to L-sheaf.

Example 2.7. Classically, let F be a global field. Then for $V \in \mathrm{Rep}_{\mathbb{C}}(\check{G}(F))$, every where unramified, and a map $\rho : \Gamma_F \to \check{G}$, a global parameter, we obtain

- Local parameters $\Gamma_{F_v} \to \check{G}$, where $\Gamma_{F_v} \circlearrowright V^{I_v}$.
- Local L-values

$$L_v(V) := \mathrm{tr}\left(\mathrm{Fr}_v, V^{I_v}\right)$$

Combining to give *global L-value*

$$L(V) := \prod_{v\in V} L_v(V)$$

An observation of [BSV] is that instead of attach L functions to to $\check{G}$ representations, we can attach that to $\check{G}$-spaces.

3. Recollection on The Torus case

We make the following simplification, we consider $Q = \Gamma_{F/E} = *$. In particular, we consider the case of split torus, $\check{G} = \mathbb{G}_m$. We will keep $\check{G}$ notation, however.

Example 3.1. When $\check{G}$ is a torus, $\mathrm{Coh}_{\mathrm{Nil}} = \mathrm{Perf}$, so that the spectral action pins down the equivalence, see Section 13 for nilpotent support. In this case, $\mathcal{W}_\psi$ is sheaf induced from $\mathrm{cInd}_*^{T(E)}\mathrm{triv} := \mathrm{Hom}_{\mathrm{cts,grp}}(T(E), \Lambda^\times)$.

When G is the Torus, CLL, Equation 2 was proved by Zou [Zou24]. This is reduced on the zero the component of Whittaker sheaves

$$\begin{tikzcd} \bigoplus_{\chi \in X^*(Z\check{G})} \mathrm{Perf}(\mathrm{Loc}_{\check{G}})_\chi \arrow[r] & \bigoplus_{\alpha \in \pi_1(G)} D(\mathrm{Bun}_G^{\alpha=c})^\omega \\ \mathrm{Perf}(\mathrm{Loc}_{\check{G}})_0 \arrow[u, hook] \arrow[r, "\simeq(-)*W"] & \mathrm{Mod}_{T(E)}(\Lambda)^\omega \arrow[u, hook] \end{tikzcd}$$

But let us break this down concerning the topology of the Weil group.

Proposition 3.2. *$Z^1(W_E, \check{G})/\check{G} \simeq \mathrm{Hom}_{cts}(W_E, \check{G}) \times B\check{G} \simeq \mathrm{Hom}_{cts}(W_E^{ab}, \check{G}) \times B\check{G}$.*

Proof. $\check{G}$ is split and action of $\check{G}$ on $Z^1(W_E, \check{G})$ is trivial. □

Thus, we would wish to study $\mathrm{QCoh}(\bigcup_{K \hookrightarrow_{\mathrm{cpt,open}} E^\times} Z^1_K)$, where K ranges over open pro-p subgroups of $E^\times$, thus by Corollary 6.12 we now have an explicit description.

$$c_\psi : D(\mathrm{Bun}_G, \Lambda) \simeq \prod_{n \in \mathbb{Z}} D^{\mathrm{sm}}(E^\times, \Lambda) \xrightarrow{\simeq} \prod_{n \in -\mathbb{Z}} \mathrm{QCoh}(Z^1) \simeq \mathrm{QCoh}(\mathrm{Loc}_{\check{G}})$$

There is a twist by -1 [Zou24, p. 5.3.3]. Further, the equivalence is designed so that they are compatible with the presentations

$$D^{\mathrm{sm}}(E^\times, \Lambda) \simeq \varinjlim_{K \text{open pro-}p E^\times} D^{\mathrm{sm}}(E^\times/K, \Lambda) \simeq \varinjlim_{K \hookrightarrow_{\text{open pro-}p} E^\times} \mathrm{Mod}_{\Lambda[E^\times/K]} \simeq \mathrm{QCoh}(Z^1)$$

which we have described in Lemma 6.10.

$$V = \bigcup_{K \hookrightarrow_{\text{open pro-}p} E^\times} V^K \mapsto \mathrm{colim}_{K \hookrightarrow_{\text{open pro-}p} E^\times} V^K \in \mathrm{QCoh}(Z^1)$$

where $V^K \in \mathrm{QCoh}(Z^1_K) \hookrightarrow \mathrm{QCoh}(Z^1)$. We illustrate two examples:

Example 3.3. We have

$$W_\psi := \mathrm{cInd}_*^{G(E)}\mathrm{triv} = C_c^\infty(E^\times) = \bigcup_{K \hookrightarrow E^\times} C_c^\infty(E^\times/K) = \bigcup_{K \hookrightarrow_{\text{open pro-}p} E^\times} \Lambda[E^\times/K]$$

Which justifies the fact that unique Whittaker sheaf is sent to the structure sheaf.

$$c_\psi(W_\psi) = \varinjlim \mathcal{O}_{Z^1_K} \simeq \mathcal{O}_{Z^1}$$

Example 3.4. Let $\chi : E^\times \to \Lambda^\times$ be a smooth character, which factors through $E^\times/K$. Then $c_\psi(\chi) \in \mathrm{QCoh}(Z^1_K) \simeq \mathrm{Mod}_{\bar{\mathbb{Q}}_l[E^\times/K]}$. In fact this induces a closed point of Z^1, from the ring homomorphism

$$\Lambda[E^\times/K] \twoheadrightarrow \Lambda$$

This induces $i_\chi : \mathrm{Spec}\,\bar{\mathbb{Q}}_l \hookrightarrow Z^1_K \hookrightarrow Z^1$, and that

$$c_\psi(\chi) = i_{\chi *}(\Lambda)$$

One can thus write the complete table using the formula

	$D(\mathrm{Bun}^n_G, \Lambda)$	$\mathrm{QCoh}(Z^1)_{-n}$
$n < 0$	triv	$i_{\mathrm{triv}*}\Lambda$
$n = 0$	$C_c(E)$	?
$n > 0$	norm character	$i_{\mathrm{cyc},*}\Lambda$

Now let us stratify $Z^1/\check{G} = \mathrm{Loc}_{\check{G}}$ by its χ component, $\mathrm{Loc}_{\check{G}} = \bigsqcup_\chi \mathrm{Loc}_{\check{G},\chi} = \bigsqcup_\chi \mathbb{G}_m \times B\mathbb{G}_m$. We have the following table

		$\mathrm{Loc}_{\check{G},\chi}$
χ non trivial		$\mathcal{O}_{\mathrm{Loc}_{\check{G},\chi}}$
χ trivial	$n < 0$	$(i_{\mathrm{triv}})_*\bar{\mathbb{Q}}_l$
	$n = 0$	$\mathcal{O} \to ? \to i_{\mathrm{triv}}\bar{\mathbb{Q}}_l$
	$n > 0$	$(i_{\mathrm{cyc}})_*\bar{\mathbb{Q}}_l$

3.1. **Torus CLL via the Weil group.** The same diagram below can be broken down as [Zou24, Lem 6.3.3], by ranging over open subgroups $P \hookrightarrow_{\mathrm{open}} F^\times$, refer to Section 4.2 for the nature of these subgroups.

Definition 3.5. [Zou24, Lem 4.4.2] Let $P \hookrightarrow_{\mathrm{open}} F^\times$, then

$$\Theta(P) := \ker(H_1(W_{F/E}, X_*(T) \to H_1(W_{F/E}/P, X_*(T))$$

Where we recall that by This is actually subgroup of $T(E)$, since we have by Langlands,

$$H_1(W_{F/E}, X_*(T)) \simeq H_1(F^\times, X_*(T))^Q \simeq T(E) \tag{3}$$

Example 3.6. Let us take $P \hookrightarrow_{\mathrm{open}} E^\times$, $E = \mathbb{Q}_p$. Consider $P := U^{(1)} = 1 + \mathbb{Z}_p$ the first principal unit **??**. Then

$$\Theta(P) = \ker(H_1(E^\times, X_*(T)) \to H_1(E^\times/P, X_*(T))$$

Then as $E^\times/P \simeq \varpi^{\mathbb{Z}} \times \mathbb{F}_p$. It is not as clear what this corresponds to but one may suggest that this is $\Theta(P) = N_{F/E}(P)$ [Zou24, Rem 4.4.3].

We thus obtain decomposition,

$$\begin{tikzcd}
\bigcup\limits_{P \hookrightarrow_{\mathrm{open}} F^\times} \mathrm{Perf}(Z^1(W_{F/E}/P, \check{T})) \arrow[r] & \bigcup \mathrm{Mod}_{T(E)/\Theta(P)}(\Lambda)^\omega \\
\mathrm{Perf}(Z^1(W_{F/E}/P, \check{T})/\check{T}) \arrow[r] \arrow[u, hook] & \mathrm{Mod}_{T(E)/\Theta(P)}(\Lambda)^\omega \arrow[u, hook]
\end{tikzcd}$$

decomposition down further where the Hecke action respects the decomposition up to twisting $X^*(Z(\check{G})) \simeq \pi_1(G)$ by 1, and the spectral action described in Section 5.1. Now how to identify this? Recall that we have a filtration of Gerbes, [Zou24, Lem 4.1.4]

$$\begin{tikzcd} Z^1(W_{F/E}/P, \check{T}) \arrow[r] \arrow[d] & \mathrm{Loc}_{\check{G}} = Z^1(W_E, \check{T})/\check{T} \arrow[d] \\ \mathrm{Hom}_{\text{cts-grp}}(T(E)/\Theta(P), \mathbb{G}_m) \arrow[r] & \mathrm{Hom}_{\text{cts-grp}}(T(E), \mathbb{G}_m) \end{tikzcd}$$

This reduces the study to equivalences [1]

$$\begin{tikzcd} \mathrm{Perf}(Z^1(W_{F/E}/P, \check{T})/\check{T})_0 \arrow[r] \arrow[d] & \mathrm{Mod}_{T(E)/\Theta(P)}(\Lambda)^\omega \arrow[d] \\ \mathrm{Perf}(\mathcal{O}(Z^1(W_{F/E}/P, \check{T}))^{\check{T}}) \arrow[r] & \mathrm{Perf}(\mathbb{Z}_l[T(E)/\Theta(P)]) \\ \mathrm{Perf}(\mathrm{CtsGrp}(T(E)/\Theta(P), \mathbb{G}_m)) \arrow[u, "\simeq"] \arrow[ur, "\simeq"] & \end{tikzcd}$$

Example 3.7. If $|\quad| \in \mathrm{Mod}_{T(E)/\Theta(P)}\bar{\Lambda}$. Then under local classified theory,

$$W_E \to W_E^{\mathrm{ab}} \simeq \mathbb{Q}_p^\times \to \mathbb{G}_m(\Lambda) \simeq \Lambda^\times$$

This corresponds to, when $\bar{\Lambda} = \bar{\mathbb{Q}}_l$, the cyclotomic character. Thus we have a point

$$\begin{tikzcd} x_{\mathrm{cyc}} : \mathrm{Spec}\,\bar{\Lambda} \arrow[r] \arrow[dr] & Z^1(W_E, \check{T})/\check{T} \arrow[d] \\ & \mathrm{CtsGrp}(T(E), \mathbb{G}_m) \end{tikzcd}$$

This we have

$$\begin{tikzcd} \mathrm{Perf}(\mathrm{CtsGrp}(T(E)/\Theta(P), \mathbb{G}_m) \arrow[r, "\simeq"] & \mathrm{Perf}(\bar{\Lambda}[T(E)/T(P)]) \arrow[r, "\simeq"] & \mathrm{Mod}_{T(E)/T(P)}(\bar{\Lambda})^\omega \\ \mathrm{Perf}(\mathrm{Spec}\,\bar{\Lambda}) \arrow[u, "x_{\mathrm{cyc}*}"] \arrow[r, "\simeq"] & \mathrm{Perf}(\bar{\Lambda}) \arrow[u] \arrow[r, "\simeq"] & \mathrm{Mod}(\bar{\Lambda})^\omega \arrow[u] \end{tikzcd}$$

In this diagram, we should have that $x_{\mathrm{cyc}*}\mathcal{O}$ corresponds to the trivial norm representation. But is this the same as $(-) * \mathcal{W}$?

$\mathrm{Hom}_{\text{cts-grp}}(T(E), \mathbb{G}_m(R))$, is still a stack over Λ and we equip R the discrete topology, is a gerbe banded by $B\check{T}$.

Proof. [Zou24, Lem 4.1.4]. $\square$

This is what induces the categorical decomposition.

[1] Note that the notation is a little misleading on RHS, it is $\mathrm{Perf}(R)$ for a ring.

4. Basics of local Class field theory

4.1. **On the Galois side.** We have the following diagrams

$$\begin{tikzcd} W_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma_F \arrow[d] \\ W_E \arrow[r, hook] & \Gamma_E \end{tikzcd}$$

The relatively Weil group can be thought more appropriately as the *abelinization* extension. We have that

$$W_E \xrightarrow{\simeq} \varprojlim_{F/E} W_{F/E}$$

As a corollary, we always have exact sequence

$$0 \longrightarrow F^{\times} \longrightarrow W_{F/E} \longrightarrow \Gamma_{F/E} \longrightarrow 0$$

Example 4.1. Counter intuitively $W_{E/E} \simeq E^{\times}$.

4.2. **On the automorphic side.** For a more complete discussion, see [Pie18]. We first analyze the structure of $E^{\times}$. Using

$$1 \to \mathcal{O}_E^{\times} \to E^{\times} \to \varpi^{\mathbb{Z}} \to 1$$

we deduce that

$$E^{\times} \simeq \varpi^{\mathbb{Z}} \times \mathcal{O}_E^{\times}$$

The key observation in understanding both the *quotients* of $E^{\times}$, and their topology is [**<empty citation>**] Now we define filtration on $E^{\times}$ given by

Definition 4.2. The *nth principal units of E* are the middle term:

$$1 \to U_E^{(n)} \to \mathcal{O}_E^{\times} \to (\mathcal{O}_E/\varpi_E^n)^{\times} \to 1$$

As a corollary we can deduce the following three equivalences:

$$\mathcal{O}_E \simeq \varprojlim \mathcal{O}_E/\varpi^n$$
$$\mathcal{O}_E^{\times} \simeq \varprojlim (\mathcal{O}_E/\varpi^n)^{\times}$$

In the first case, where $q := |\mathcal{O}_E/\varpi_E|$ is size of the resixdue field, and $\mu(E)$ is all root of unit contained in E.

Definition 4.3. Let A/E be an E algebra,

Theorem 4.4. *For any finite Galois extension F/E,*

$$F \to E^{\times}/N_{F/E}(F^{\times})$$

This induces

$$\mathit{FinAbExt}/E \simeq \mathit{OpenSubgroups}/E^{\times}$$

The open subgroups of $E^{\times}$ are still rather unclear, but in general there is an open subgroup of neighborhood basis.

Lemma 4.5. *Let E be a local field.*

- *Any subgroup of finite index in $E^\times$ is closed (equivalently open[2]).*
- *$U^{(n)}$ forms a basis of neighborhood.*

Proof. First bullet is standard argument. Second bullet point is a standard argument. ☐

Example 4.6.
- For $\mathbb{Q}_p$, $a = 0$, we have that the open subgroups

$$\mathbb{Z}_p^\times \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^0\mathbb{Z} \times \mathbb{Z}_p$$

If $i, j \in \mathbb{Z}_{\geq 0}$ Then

$$\left\{ (\varpi_E^i)^n (1 + \mathcal{O}_E^j) \ : \ n \in \mathbb{Z} \right\}$$

is an open subgroup of $E^\times$ of finite index. The subgroups $1 + \mathcal{O}_E^j \simeq \mathcal{O}_E^j$ are pro-p.

4.3. **Types of Galois extensions.** For a general Galois extension, we always have a break down

$$\begin{tikzcd} K \arrow[d, dash, "\text{ramified}"] \\ K^{\mathrm{nr}} \arrow[d, dash, "\text{unramified}"] \\ E \end{tikzcd}$$

Example 4.7. Let $m \in \mathbb{Z}_{\geq 0}$, then $\mathbb{Q}_p(\mu_{p^m})$ are the totally ramified extensions.

Example 4.8. If we do such a decomposition on a local field, we have

$$\begin{tikzcd} E^s \arrow[d, dash] \\ E^t = \bigcup E^{\mathrm{nr}}(\varpi^{1/n}) \arrow[d, dash] \\ E^{\mathrm{nr}} = \bigcup_{(p,n)=1} E(\mu_n) \end{tikzcd}$$

Naturally, we have

$$\Gamma_{E^s/E^t} = P_E \to \Gamma_{E^s/E^{\mathrm{nr}}} = I_E \to I_E/P_E =: I_E^t$$

This is the *tamed* part, in which we quotient out the pro-p part. To study the Galois group, it is thus natural to consider $K \hookrightarrow_{\text{open,cpt}} P_E$, we will see same diagram Proposition 15.6.

[2] Open subgroups are automatically closed, by the coset argument.

5. Strategy of proof in Equal characteristic setting

Let us now describe two parallel decompositions in the Geometric de Rham Langlands, following [FW24]. Using decomposition $\mathbb{A}^1 = 0 \bigsqcup (\mathbb{A}^1 \backslash 0)$ In the automorphic side we have

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, dash, "="'] & \mathrm{Bun}_G^X \arrow[d] & U \simeq \bigsqcup_{d \geq 0} C^{(d)} \arrow[l] & C^{(d)} \simeq \mathrm{Div}^d \arrow[l] \arrow[d] \\ & \mathrm{Bun}_G \simeq \bigsqcup_{d \in \mathbb{Z}} \mathrm{Bun}_G^d & & \mathrm{Bun}_G^d \arrow[ll] \end{tikzcd}$$

This induces a short exact sequence on $\mathrm{Dmod}(\mathrm{Bun}_G)$, with

$$\pi_! k_U \longrightarrow \pi_! k_{\mathrm{Bun}_G^X} \longrightarrow \pi_! i_* k_Z \simeq k_Z$$

But now we have $\mathrm{Shv}(\mathrm{Bun}_G^d) \xrightarrow{j_!} \mathrm{Shv}(\mathrm{Bun}_G)$. [3] We have a similar diagram in the spectral side

$$\begin{tikzcd} Z \arrow[rr, hook] \arrow[dr] & & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d] & U \simeq * \arrow[l] \\ & \mathcal{Z} \arrow[ur] & \mathrm{Loc}_{\check{G}} & \end{tikzcd}$$

Inducing short exact sequence

$$\pi_{\mathcal{Z},*}(\omega_{\mathcal{Z}}) \longrightarrow \pi_* \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}} \longrightarrow \mathcal{O}_{\mathrm{triv}}$$

where $\mathcal{O}_{\mathrm{triv}}$ is skyscraper sheaf induced from $\mathrm{IndCoh}(*)$.

Proposition 5.1. *$\mathbb{L}(\mathcal{O}_{triv}) = k_Z \langle g-1 \rangle$ under normalization in Equation 2.*

More generally, by Langlands action

$$\mathbb{L}_G \left(\prod_{n \geq 0} k_{\mathrm{Hk}^{\mathrm{std}}_{G,C^{(n)}}} \star \delta_{\mathrm{triv}} \right) = \mathrm{Loc}^{\mathrm{spec}} \mathrm{Fact}\, \mathrm{Sym}^{\bullet}(\mathrm{std}) \otimes \omega_{\mathrm{Loc}_{\check{G}}} \langle -(g-1) \rangle$$

5.1. **Spectral action.** Fix a finite set I. where Hck^I stack given by

$$\begin{tikzcd} \mathrm{Hck}^I := L^+G \backslash \mathrm{Gr}_G^I \arrow[d] \\ (\mathrm{Div}^1)^I := \mathrm{Div}^1 \times \cdots \times \mathrm{Div}^1 \end{tikzcd}$$

This defines the Hecke operator diagram via the correspondence

$$\begin{tikzcd} & \mathrm{Hck}^I \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{tikzcd}$$

[3] Note that Dmod does not see the difference between formal completions - is this the same for D_{lis}?

One can define a full subcategory

$$\mathrm{Sat}^I_G(\Lambda) \hookrightarrow D_{\text{ét}}(\mathrm{Hck}^I_G, \Lambda)^{\mathrm{bd}} \hookrightarrow D_{\text{ét}}(\mathrm{Hck}^I_G, \Lambda)$$

As a consequence, let ${}^L G := \check{G} \rtimes W_E$, then

Theorem 5.2. *[FS24, p. IX.2] For any $V \in \mathrm{Rep}_\Lambda({}^L G^I)$ there is naturally associated functor*

$$T_V : D_{lis}(Bun_G, \Lambda) \to D_{\blacksquare}(Bun_G \times (Div^1)^I)$$

$Div^1 \to [*/W_E]$, and that $\pi_1 Div^1 = W_E$, we have

$$T_V : D_{lis}(Bun_G, \Lambda) \longrightarrow D_{lis}(Bun_G, \Lambda)^{BW_E} \longrightarrow D_{lis}(Bun_G, \Lambda)$$

Further, the action satisfies the following commutative diagram

$$\begin{tikzcd}
& D_{\mathrm{lis}}(\mathrm{Hk}^I, \Lambda) \arrow[dr] & \\
\mathrm{Rep}_\Lambda({}^L G) \arrow[ur] \arrow[dr, "\Delta^*"] & & \mathrm{End}(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \\
& \mathrm{IndPerf}(Z^1(W_E, \hat{G})/\check{G}) \arrow[ur] &
\end{tikzcd} \tag{4}$$

5.2. **The classical story.** There is a difference between geometric Hecke operator and classical Hecke operators - the former being defined globally. Note that for $d \geq 1$, we have a map

$$\pi_d : \Sigma^d \to \mathrm{Div}^d_\Sigma := \Sigma^d / \Sigma_d$$

where we take the categorical quotient. Now we have almost the Abel Jacobi map in that the map from $\mathrm{Div}^d_\Sigma \to \mathrm{Pic}^d_\Sigma$ factors through the $\mathrm{Bun}_{\mathrm{GL}_1}$. Following classical notation, we Pic_Σ be the normal abelian scheme. Now we have the following diagram

$$\mathrm{Div}^d_\Sigma$$

In this case, the only interesting Hecke operator is when $\mu = 1$.

In the case of GL_1, which is discussed thoroughly in [Gai09], $\mathrm{Hk}^{\{1\}}$ is equivalent to datum of two line bundles (L, L', x) such that $L' \simeq L \otimes \mathcal{O}(x)$.

$$\mathrm{AJ} : \Sigma \times \mathcal{P}\mathrm{ic}^d(\Sigma) \to \mathcal{P}\mathrm{ic}^{d+1}(\Sigma)$$

$$(x, L) \mapsto L \otimes \mathcal{O}([x])$$

which induces

$$\mathrm{AJ} : \Sigma \times \mathrm{Pic}(\Sigma) \to \mathrm{Pic}(\Sigma)$$

Proposition 5.3. *A Hecke Eigensheaf for $G = GL_1$ with respect to $\mathcal{E} \in Dmod(X)$ is equivalent to the condition that*

$$AJ^* \mathcal{F} \simeq \mathcal{E} \boxtimes \mathcal{F}$$

- *and various second order condition.*
- *AJ corresponds to the Hecke action $Hk^{I,\mu}$ with $I = 1$, $\mu = 1$.*

But how to construct $\mathcal{E} \in \mathrm{Dmod}(X)$? In fact this uses the following observation, that

$$\pi_1(\mathcal{P}\mathrm{ic}) \simeq \pi_1(\mathrm{Pic}) = \pi_1(X)^{\mathrm{ab}}$$

where $\pi_1(X)$ is the unramified Galois group $\Gamma^{\mathrm{nr}}_{k(X)}$. [4]

But what are the Eigensheaves of AJ? This would be the eigensheave corresponding to the étale local system of the trivial representation of $D_{\mathrm{lis}}([*/W_E])$. This then correspond to a natural sheaf.

In these settings this only accounts for minimal ways to modify a vector a bundle at a point. For instance, we can also define

$$\mathrm{Hck}_x^{\Lambda^i} := \{\mathcal{M}, \mathcal{M}' : \mathrm{length}(\mathcal{M}/\mathcal{M}') = i \text{ and supported scheme theoretically at} x \in X\}$$

$$\mathrm{Hck}_x^{\mathrm{Sym}^i} := \{\mathcal{M}, \mathcal{M}' : \mathrm{length}(\mathcal{M}/\mathcal{M}') = i \text{ and supported set theoretically at} x \in X\}$$

And that consider the Hecke action with respect to this Hecke stack, by requiring that $\mathrm{Hck}^{V\otimes W} \simeq H^V \circ H^W$, we in fact induce a functor a $\otimes$-functor $\mathrm{Rep}(\check{G}) \to \mathrm{End}(\mathrm{Dmod}(\mathrm{Bun}_G))$.

Let us briefly argue the geometric Langlands. In fact, let us call the natural object in

$$D_{\mathrm{lis}}(\mathrm{Bun}_G)^\omega \simeq \bigoplus_{\alpha \in \pi_1(G)_Q} \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^{c_1=\alpha}, \Lambda)^\omega$$

We might also recall that the Hecke stack has the following description,

Proposition 5.4. *[Zou24, Lem 6.4.2]*

$$Hck_T \simeq Bun_T \times \bigsqcup_{\mu \in X_*(T)_Q} Div^1_{E_\mu}$$

where E_μ is reflex field of μwhat does this mean?

Lemma 5.5. *For $\chi \in X^*(\check{T})$ the sheaf $q^*\mathcal{S}_\chi \in D_{lis}(Hck_G)$ via geometric is constant sheaf on component $\bar{\chi}$.*

Thus the corresponding map

$$D_{\mathrm{lis}}(\mathrm{Bun}_G) \to D_{\mathrm{lis}}(\mathrm{Bun}_G \times \mathrm{Div}^1)$$

And also that in the case of GL_1, we have the constant sheaf. But how do we construct this? Let us recall local class field theory, which says that for a local field, we have an isomorphism $W_E^{\mathrm{ab}} \xrightarrow{\simeq} E^\times$. Thus, for a character ϕ_T of W_E, this induces a character of $E^\times$. We can thus define

$$\mathcal{S}_{\phi_T} := \bigoplus j_{b!}(\chi) \quad \chi \in \mathrm{Shv}(\mathrm{Bun}_T^b) \simeq \mathrm{Shv}([*/\underline{T(E)}])$$

We thus deduce that

$$T_{(\nu_{i\in I})}(\mathcal{S}_{\phi_T})$$

Example 5.6. If $\nu_{i\in I}$ all corresponds to std. These are representation of weight 1. Then $T_{(1)_{i\in I}}(\mathcal{S}_{\phi_T}) \simeq \boxtimes_{i\in I}\phi_T \otimes \mathcal{S}_{\phi_T}$. In otherwords, the action seems to be twisted by ϕ_T as a $\mathbb{Z}$-module. But how does this correspond to Hecke Eigensheaf? This is the classical story of geometric class field theory, [Ten15], [Tót11]. What happens to other characters?

[4] More generally one uses the Riemann hilbert correspondence.

$$\mathcal{L} \mapsto \mathrm{Aut}_{\mathcal{L}}$$

In which case the Hecke eigensheaf condition is required to satisfy

$$\mathrm{AJ}^* \, \mathrm{Aut}_{\mathcal{L}} \simeq \mathcal{L} \boxtimes \mathrm{Aut}_{\mathcal{L}}$$

6. Iwasawa Tate on the $\mathcal{A}$-side.

Recall, in **??**, we would hope to analyze the diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, dash, "="'] & \mathrm{Bun}_G^X \arrow[d] & \mathrm{Div} := \bigsqcup_{d \geq 0} \mathrm{Div}^d \arrow[l] & \mathrm{Div}^d \simeq \mathcal{BC}(n) \backslash \{0\} / \underline{\mathbb{Q}_p}^\times \arrow[l] \arrow[d] \\ & \mathrm{Bun}_G \simeq \bigsqcup_{d \in \mathbb{Z}} \mathrm{Bun}_G^d & & \mathrm{Bun}_G^d \arrow[ll] \end{tikzcd}$$

Definition 6.1.

$$\mathrm{Bun}_G : \mathrm{Pftd}_{\mathbb{F}_q} \to \mathcal{S}$$
$$S \mapsto \mathrm{Tors}_G(X_{S,E}) \quad X_{S,E} \in \mathrm{Adic}_E$$

Recall that for field extension L/K, we can define the norm map $N_{L/K} := (\alpha \mapsto \det m_\alpha)$ where $m_\alpha : L \to L$, is the matrix of multiplication for L regarded as a K-vector space.

Definition 6.2. For $d \geq 1$, $b \in G(\breve{\mathbb{Q}}_p)$, $* \to \mathrm{Bun}_G$, then we define functor $\mathcal{H}^*(\mathcal{E}_b)$:

- $\mathcal{H}^0(\mathcal{E}_b) : \mathrm{Pftd}_{\mathbb{F}_p} \to \mathrm{Set}$ given by $S \mapsto H^0(X_S, \mathcal{E}_{b,S})$.
- $\mathrm{Div}^1 := \mathrm{Spa}\, \breve{E}^\diamond / \varphi^{\mathbb{Z}} : \mathrm{Pftd}_{\mathbb{F}_p} \to S$. These are precisely the classical points of X_S. Indeed the $(-)^\diamond$ functor sends
$$S \mapsto \left\{ S^\#, f : S^\# \to Z \right\}$$
- $\left(\mathrm{Div}^1\right)^d \to \mathrm{Div}^d \simeq \left(\mathrm{Div}^1\right)^d / \Sigma_d$

It is apriori useful to work in the case of non algebraically closed field. Our interest would be the case for GL_1, in which $\mathcal{O}(d)$, where $d \in \mathbb{Z}$.

Proposition 6.3. *Let $\lambda = \frac{d}{r}$, and with $(d, r) = 1$.*

- *$\mathcal{H}^0(X, \mathcal{O}(\lambda))$ is a v-sheaf but fails to be a perfectoid space.*
- *There is an isomorphism*
$$\mathrm{Spd}\, k[[x_1^{1/q^\infty}, \ldots, x_d^{1/q^\infty}]]$$

The way to make such computations, particularly in the equal characteristic case is discussed in [SW13]. For $\mathcal{E}$ a isocrystal, we let $G_{\mathcal{E}} := \mathrm{Aut}(\mathcal{E})$ be the automorphism scheme of $\mathcal{E}$. Then

$$\det : G_{\mathcal{E}}(F) \to G_{\mathcal{E}}/G_{\mathcal{E}}^{\mathrm{der}} \to E^\times$$

Example 6.4. If $F := \mathcal{O}(\lambda)$, $n = 1$, then

$$\det :$$

Theorem 6.5. *Let $\lambda = \frac{d}{r}$, $(d, r) = 1$, $r \geq 1$. Consider the map*

$$f_{\lambda,n} :$$

(1) *If* $\lambda > 0$, *then* $\mathcal{H}^1(\mathcal{O}(\lambda)^n) \simeq *$

$$f_{\lambda,n!}\Lambda \simeq |Nrd_{M_n(D_{-\lambda})/F}(-)|^{-1}[-2dn]$$

(2) *If* $\lambda = 0$, *then* $\mathcal{H}^0(\mathcal{O}(\lambda)^n) \simeq \underline{E}^n$, *and* $\mathcal{H}^1(\mathcal{O}(\lambda)^n) \simeq *$.

Corollary 6.6. *The following is a full description of*

$$\mathcal{P}_X\Big|_{Bun_G^n} = \begin{cases} \underline{E} \text{ or trivial representation} & n < 0 \\ C_c^\infty(E) & n = 0 \\ |\quad|, \quad \text{norm character} & n > 0 \end{cases}$$

6.1. **Understandin the** 0**th piece.**

6.2. **Study on open piece.** Bun_G has a stratification $\bigsqcup_{n\in\mathbb{Z}} \mathrm{Bun}_G^n$, which we describe in Section 7.

Proposition 6.7. *[FS24, Ch. II]*

$$U^n \simeq \begin{cases} \mathcal{BC}(n)\setminus\{0\}/\underline{\mathbb{Q}_p^\times} & n > 0 \\ [\underline{\mathbb{Q}_p}\setminus\{0\}/\underline{\mathbb{Q}_p^\times}] & n = 0 \\ [*/\underline{\mathbb{Q}_p}^\times] & n < 0 \end{cases}$$

Now let us compute the constant sheaf via localizing sequence. Note that by proper base change, to compute $\pi_!\Lambda_{\mathrm{Div}^d}$, we can consider the diagram

$$\begin{tikzcd} \mathrm{Div}^d \arrow[d] & \mathcal{H}^0(X,\mathcal{O}(d))\setminus\{0\} \arrow[l] \arrow[d] \\ \mathrm{Bun}_G^d \simeq [*/\underline{\mathbb{Q}_p}^\times] & * \arrow[l] \end{tikzcd}$$

Thus we compute

$$R\Gamma_c(\mathcal{H}^0(X,\mathcal{O}(d))\setminus\{0\},\Lambda)$$

Note that the left hand map is [FS24, p. II.3.7] proper, and cohomologically smooth. Again, we use the localizing sequence

$$\mathcal{H}^0(X,\mathcal{O}(d))\setminus\{0\} \longrightarrow \mathcal{H}^0(X,\mathcal{O}(d)) \longleftarrow \{0\}$$

This induces a short exact sequence (using that $j^! = j^*, i_! = i_*$))

$$j_!j^*\Lambda \to \Lambda \to i_*i^*$$

Inducing that

$$\pi_{!,\mathcal{H}^0(X,\mathcal{O}(d)\setminus\{0\}}\Lambda \to \pi_{!,\mathcal{H}^0(X,\mathcal{O}(d)}\Lambda \to \pi_{!,\{0\}}\Lambda$$

Corollary 6.8. $\pi_{!,\mathcal{H}^0(X,\mathcal{O}(d)\setminus\{0\}}\Lambda \simeq \Lambda[-1] \oplus \Lambda[-2d]$

Here we used the following properties:

- That for $C/\mathbb{F}_p$ complete algebraically closed field, we have that
$$\pi_{!,\mathcal{H}^0(X,\mathcal{O}(d))_C}\Lambda \simeq \Lambda(-d)[-2d]$$
It is an interesting feature that it suffices to compute over C and as sheaves on $\mathrm{Bun}_{G,C}$ is same as that $\mathrm{Bun}_{G,\bar{\mathbb{F}}_p}$ for computational purposes this is sufficient, see comments after [Has21, Prop 4.7].
- That $f^!\Lambda \simeq \Lambda(d)[2d]$.

Example 6.9. Now we would like to track the $\mathbb{Q}_p^\times$ action. Let us compute this in the case of $d = 1$. Note the definition of $\mathcal{H}^0(X, \mathcal{O}(d))$ can be naturally define over $\mathrm{Spd}(\mathbb{F}_p)$. In fact the dualizing sheaf satisfies
$$\omega_{\mathcal{H}^0(X,\mathcal{O}(d)} \simeq \Lambda(-d)[2d]$$

6.3. Sheaves on components Bun_G^n.

Lemma 6.10. *[Zou24, p. 6.3.1], Let $K \hookrightarrow_{open\ pro\text{-}p} E^\times$ then*
$$T(E)/K\text{-}Mod_\Lambda^{\heartsuit,sm} \hookrightarrow T(E)\text{-}Mod_\Lambda^{\heartsuit,sm}$$
is fully faithful. This induces an adjunction, [HM24], on the derived category of smooth representations,
$$\begin{tikzcd} D(*/T(E),\Lambda) \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & D(*/T(E)/K,\Lambda) \arrow[l, "f^*"'] \end{tikzcd}$$
for the map $f : T(E) \to T(E)/K$.

There is an explicit description in general for such a map of locally profinite groups, [HM24, Theorem 1.4.1].

(1) f^* is the restriction map.
(2) As f is the quotient map, the map f_* is given by taking K- invariants.
(3) As $T(E)$ has finite Λ-cohomological dimension, we have $f_!$, the compact induction functor.

Corollary 6.11. $D^{sm}(E^\times, \Lambda) \simeq \varinjlim_{K\,open\ pro\text{-}pE^\times} D^{sm}(E^\times/K, \Lambda)$

One can make a similar statement for Quasi-coherent sheaves.

Corollary 6.12. $QCoh(\varinjlim \mathrm{Spec}\,\Lambda E^\times/K) = \varinjlim QCoh(\mathrm{Spec}\,\Lambda[E^\times/K]) = \varinjlim_{K\,open\ pro\text{-}pE^\times} Mod_{\Lambda[E^\times/K]}$

6.4. **Study of Hecke action.** Now let us compute the Hecke action of $j_!\Lambda \in D(\mathrm{Bun}_G, \Lambda)$. This follows from the following diagram: That we have
$$\begin{tikzcd} \mathrm{Bun}_T & \mathrm{Bun}_T \times \mathrm{Div}^1_\mu \arrow[l] \arrow[r] & \mathrm{Bun}_T \times [*/W_E] \arrow[d] \\ & * \times \mathrm{Div}^1_\mu \arrow[ul] \arrow[u] \arrow[ur] \arrow[r] & \mathrm{Bun}_T \end{tikzcd}$$
where the vertical map sends $(*, D)$ to $(\mathcal{O}(D), D)$. But recall [FS24, p. I.2.11] that we have
$$D(\mathrm{Bun}_G \times \mathrm{Div}^1) \xrightarrow{\simeq} D(\mathrm{Bun}_G \times [*/W_E]) \simeq D(\mathrm{Bun}_G)^{BW_E}$$

Corollary 6.13.
$$h^*(j_{0!}\Lambda) \simeq AJ_!(\Lambda) \boxtimes triv$$
[5]

Now we compute $\mathbb{L}(j_{0!}\Lambda)$ under the Langlands correspondence.

But now one has the following chain of equivalence that
$$\mathrm{QCoh}(\mathrm{Loc}_{\check{G}})_0 \simeq \mathrm{QCoh}(\mathrm{Hom}_{\mathrm{ctsgrp}}(T(E), \mathbb{G}_m) \simeq \mathrm{Mod}_{T(E)}(\Lambda)$$

One first pullback $\mathcal{O}$ to $\mathrm{Perf}(B\check{G})_0$, which we regard as the structure sheaf. Then we consider $\mathrm{cInd}_*^{T(E)} 1 \simeq \mathrm{Fct}_c(T(E), \Lambda^\times)$ [6]

Proposition 6.14.

- *let ϕ be a trivial representation in $Mod_{T(E)}(\Lambda)$. This induces parameter $\phi : T(E) \to \mathbb{G}_m$. $\mathcal{O}_{triv} := j_*\phi$.*
- *Let $\mathcal{O} \in QCoh(Loc_{\check{G}})_0$ then $\mathcal{O}$ is sent to $\pi_!\Lambda$.*

From six functor formalism of groups here.

[5] Thus, the underlying representation of $h^*(j_{0!}\Lambda)$ is $\mathrm{AJ}_!(\Lambda) \otimes \mathrm{triv}$.

[6] which is also given by the image of $D(*, \Lambda) \xrightarrow{\pi_!} D(*/\underline{T(E)}, \Lambda)$

6.5. **A construction of Bun_G^X.** The fiber of $\mathrm{Bun}_G^X \to \mathrm{Bun}_G$ over a G bundle L is again a vector space. In fact, let $\mathcal{E} := \mathrm{map}(\mathcal{O}_X, \mathcal{E}^{\mathrm{univ}} \otimes \Omega_C^{1/2})$ be the perfect complex, such that

$$\begin{tikzcd} \mathbb{V}(\Gamma(X_S, \mathcal{L})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{V}(\mathcal{E}) \arrow[d] \\ S \arrow[r, "\mathcal{L}"] & \mathrm{Bun}_G \end{tikzcd}$$

where we recall the construction of $\mathbb{V}$, in Section 12.1.

In the work of [JB21, Cor 3.10], there is a Fourier transform,

$$R\tau_* : D(X_S) \to D(S_v, \underline{E})$$

which satisfies a *relative Serre duality.* i.e.

$$(R\tau^* K^\vee) \simeq (R\tau^* K)^\vee[1] \quad K \in D(X_S)$$

We may allow $S = \mathrm{Bun}_G$ or any v-stack. Let $\mathcal{E}^{\mathrm{univ}}$ be the universal complex on X_{Bun_G}. Thus we study $R\tau_*\mathcal{E}^{\mathrm{univ}}$ which, supposedly, associated v-stack is , Bun_G^X, is the moduli stack of section of $\Gamma_{\mathrm{dR}}(X, \mathcal{E}^{\mathrm{univ}} \otimes \Omega_X^{1/2})$.

Let us now discuss the Fourier transform in more detail. Recall Serre Duality suppose S is a derived Artin stack, where E, E' are vector bundles, $E^\vee \simeq E'[1]$, which is used in [FW24]. The Fourier vector bundle theory has a *somewhat* acceptable theory. For a choice of $\psi : \mathbb{Q}_p \to \Lambda^\times$, we can define Fourier transform

$$\mathrm{FT}_E : D_{\text{ét}}(E, \Lambda) \to D_{\text{ét}}(E^\vee, \Lambda)$$

where the functoriality of the shift is almost the same.

In the work of [BSV] on one consider the constructed

6.6. **Bun_G^X in mixed characteristic setting.**

Definition 6.15. The *relative curve associated to* $S = \mathrm{Spa}(R, R^+) \in \mathit{Pftd}_{\mathbb{F}_q}$ is given by

$$Y_{S,E} := \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \backslash V(\pi[\varpi]) \in \mathrm{Adic}_E$$

where π is uniformizer of E, and ϖ is uniformizer of R.

$$X_{S,E} := Y_{S,E}/\varphi^{\mathbb{Z}}$$

Any point of the base, $S = \mathrm{Spa}(R, R^+) \to \mathrm{Spa}\, E^\diamond$, induces a Cartier divisor

$$S^\sharp \hookrightarrow X_S$$

The formal completion along this divisor is $\mathrm{Spf} B^+_{\mathrm{dR}}(R^\sharp)$. [7] We work over the algebraic closure. [8]

[7]Already, one sees that the various formal completions are distinct.

[8]note that $* := \mathrm{Spa}(\bar{\mathbb{F}}_q)$ is not representable.

7. The structure of Bun_G

We let $E/\mathbb{Q}_p$ a finite extension. For theory of isocrystals we refer to Section 24.

Definition 7.1. Let $\lambda = d/r \in \mathbb{Q}$, where $r \in \mathbb{N}_+$ and $\gcd(r, d) = 1$. We define an isocrystal as the vector space

$$\left(\varphi_{-\lambda}, \breve{E}^r\right)$$

$$\varpi_{-\lambda} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

So action $\varpi_{-\lambda}^d$

Example 7.2. If $\lambda = r = d = 1$, $s = 1$. Then we have φ_{-1} acting on $\breve{E}$, acting by multiplication $\varpi_{-\lambda} = \varpi$. Thus $\mathrm{End}(\breve{E}) = E$

Theorem 7.3. *Consider the v-topology on $Pftd_{\bar{\mathbb{F}}_p}$.* [9]

- *$Bun_{G,\bar{\mathbb{F}}_q}$ is an artin v-stack (l-cohomogically)*[10] *smooth of dimension 0.*
- *Let $\breve{E} := \widehat{E^{nr}}$, note that $X_{S,E} = Y_{S,E}/\varphi^{\mathbb{Z}}$, where $Y_S \to \mathrm{Spa}(\breve{E})$. Let $B(G) := G(\breve{E})/\sigma$-cjg, $b \sim gbg^{-\sigma}$, the Kottwitz sets of Iso crystal.*

$$B(G) \simeq |Bun_{G,\bar{\mathbb{F}}_q}|$$

The geometry of $\mathrm{Bun}_{G,\bar{\mathbb{F}}_q}$ is nice.

Definition 7.4. Let G_b denote the automorphism group of G-isocrystal attached to b.

Example 7.5. If G is quasisplit

- G_b is an inner form of a Levi subgroup of G
- G_b is an inner form of G iff b is basic.

Theorem 7.6.

- *$\pi_0(Bun_{G,\bar{\mathbb{F}}_q}) \simeq \pi_1(G)_\Gamma$.*
- *There is a nice Harder Narasimhan stratification. In particular, there is an open substack*

$$Bun^{ss}_{G,\bar{\mathbb{F}}_p} \hookrightarrow Bun_{G,\bar{\mathbb{F}}_p}$$

With the following stratification

$$Bun^{ss}_{G,\bar{\mathbb{F}}_p} \simeq \coprod_{[b]\ basic} [*/\underline{G_b(E)}]$$

where G_b is inner form of G, for example $G_1 = G$.

Example 7.7. When $E = \mathbb{F}_q((\varpi))$, we suppose $\mathrm{Spa}(E)^\diamond = \mathrm{Spa}\,E$. Then the B_{dR} Grassmanian is a proétale sheaf in Pftd_E. If $S = \mathrm{Spa}(R, R^+) \in \mathrm{Pftd}_E$, then

$$\mathrm{Gr}^{B_{\mathrm{dR}}}(S) = \{\mathcal{F}, \xi\} / \simeq \quad \mathcal{F} \in \mathrm{Tors}_G(\mathrm{Spec}\, B^+_{\mathrm{dR}}(R)), \xi \text{ is trivialization at } B_{\mathrm{dR}}(R)$$

Note that we may replace bundles on $\mathrm{Spa}(R, R^+)$ with proétale bundles on $\mathrm{Spec}\, R$ due to the result of Kedlaya and Liu.

[9] This is analogous to the fpqc topology of schemes, which is finer than the pro-étale topology

[10] Notions in v-stacks are to be made sense l-cohoomogically.

[FS24] has defined a 5 functor formalism of solid sheaves, with $f_\natural$ taking the place of $Rf_!$.

Proposition 7.8. *[FS24, Prop. VII.7.1]* $\mathcal{D}_{lis}(Bun_G^b.\Lambda) \simeq \mathcal{D}(G_b(E)\text{-}Mod_\Lambda)$.

Example 7.9. $G = T$ is a torus over E. Then $B(T) = X_*(T)_\Gamma$. Thus, all b are basic. We have a semi-infinite orthogonal decomposition

$$D_{\text{lis}}(\text{Bun}_T, \Lambda) \simeq \prod_{[\chi] \in X_*(T)_\Gamma} \mathcal{D}(T(E)\text{-Mod}_\Lambda)$$

Example 7.10. $G = \text{GL}_1$. Everything is semistable, so

$$\text{Pic} := \text{Bun}_{\text{GL}_1} \simeq \bigsqcup_{\mathbb{Z}} [*/\underline{E^\times}]$$

8. Iwasawa Tate on the $\mathcal{B}$-side

For $\Lambda \in \mathrm{Alg}_{\mathbb{Z}_l}$ algebra made a condensed ring via $\Lambda := \Lambda^{\mathrm{disc}} \otimes_{\mathbb{Z}_l^{\mathrm{disc}}} \mathbb{Z}_l$. Recall, we wish to analyze the following diagram

$$\begin{tikzcd} \mathrm{fib}\rho \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d] \\ * \arrow[r, "\rho"] & \mathrm{Loc}_{\check{G}} \end{tikzcd}$$

To further our discussion, we will use another interpretation of $\mathrm{Loc}_{\check{G}}^{\check{X}}$.

Theorem 8.1. *Let $Z^1_{cts}(W_E, \hat{G}) \in Fun(Aff_{\mathbb{Z}_l}, \mathcal{S})$ be the functor sending*

$$\Lambda \mapsto Z^1_{cts}(W_E, \hat{G}(\Lambda))$$

This is represented by a flat locally complete intersection scheme.

Lemma 8.2. *$Z^1(W_E, \check{G})$ has the following two presentations.*

$$\mathrm{Spec}\, Z^1 \simeq \bigcup_{K \hookrightarrow_{cpt,open} E^\times} \mathrm{Spec}\, \Lambda[E^\times/K] \simeq \coprod_{\chi:\mathcal{O}_E^\times \to \Lambda^\times, finite\ order} \mathbb{G}_{m,\Lambda} \tag{5}$$

. We will omit the full "finite order" indexing notation in the future for simplicity.

Proof.

$$\begin{aligned} Z^1 &= \bigcup_{K \hookrightarrow_{\mathrm{cpt,open}} W_E^{\mathrm{ab}}} \mathrm{Hom}(W_E^{\mathrm{ab}}/K, \mathbb{G}_m) \\ &\simeq \bigcup_{K \hookrightarrow_{\mathrm{cpt,open}} E^\times} \mathrm{Hom}_{\mathrm{cts}}(E^\times/K, \mathbb{G}_m) \\ &\simeq \bigcup_{K \hookrightarrow_{\mathrm{cpt,open}} E^\times} \mathrm{Spec}\, \Lambda[E^\times/K] \end{aligned}$$

Note that $E^\times/K$ is a discrete set, as explained in Section 4. If $E^\times/K$ has the discrete topology, then $\mathrm{Hom}_{\mathrm{cts}}(E^\times/K, \mathbb{G}_m) = \mathrm{Hom}(E^\times/K, \mathbb{G}_m)$, has A-points, for $A \in \mathrm{Alg}_\Lambda$,

$$\mathrm{Hom}_{\mathrm{Grp}}(E^\times/K, A^\times) \simeq \mathrm{Hom}_{\mathrm{Alg}_\Lambda}(\Lambda[E^\times/K], A^\times)$$

We can use the decomposition in Section 4, for $\mathcal{O}_E^\times$. For generally, the compact open subgroups $K \hookrightarrow_{\mathrm{open}} E^\times$, (e.g. take the principal units) satisfy that

$$E^\times/K \simeq \left(\varpi^{\mathbb{Z}} \times \mathcal{O}_E^\times\right)/K \simeq \varpi^{\mathbb{Z}} \times (\mathcal{O}_E^\times/K)$$

where $\mathcal{O}_E^\times/K$ is a finite group. Note that $\Lambda[X \times Y] \simeq \bigoplus_{X \times Y} \Lambda \simeq \Lambda[X] \otimes_\Lambda \Lambda[Y]$. [11] Then we have the desired decomposition from Theorem 8.3.

∎

[11] In other words the left adjoint of adjunction $(\Lambda[-], (-)^\times)$ from $(\mathrm{Alg}_\Lambda, \otimes_\Lambda)$ to $(\mathrm{Grp}, \times)$, is symmetric monoidal.

Remark 8.3. Let Λ be an appropriate coefficient[12] Let us first recall some representation theory of $\Lambda[G]$. This admits both a left and right action, for a general element $(g,h) \in G \times G$, we have

$$(g,h) \cdot f : x \mapsto f(g^{-1}xh)$$

We know that $\mathrm{Rep}^{\mathrm{fd}}_{\Lambda}(G)$ semisimple.[13] To determine the irreducible decomposition of $V \in \mathrm{Rep}_{\Lambda}(G)$, We compute

$$\hom_{\mathrm{Mod}_{\Lambda}}(V_i, V) \in \mathrm{Rep}^{\mathrm{fd}}_{\Lambda}(G)$$

with the G-action given by $g \cdot f(-) := gf(g^{-1}(-))$. Its G-fixed points is $\hom_{\mathrm{Rep}^{\mathrm{fd}}_{\Lambda}(G)}(-,-) \in \mathrm{Mod}_{\Lambda}$. This induces an isomorphism

$$\bigoplus_{V_i \in \mathrm{Simple}(\mathrm{Rep}^{\mathrm{fd}}_{\Lambda}(G)))} \hom(V_i, V) \otimes V_i \to V$$

This is in fact an isomorphism of $G \times G$ representation. If G were abelian, then the irreducible representation of G are all one dimensional, equivalent to the characters. As a left G-representation,

$$\bigoplus_{\chi \in \mathrm{Char}(G)/\sim} n_{\chi} \Lambda_{\chi} \xrightarrow{\simeq} \Lambda[G]$$

But how does one compute this n_{χ}? In fact by character theory.

$$n_{V_i} = \dim V_i$$

We have thus reduced our study to the case of a diagonalizable group scheme. We know know that any finite abelian group is the direct sum of cycle groups. Thus, we are reduced to study

$$\mu_{n,\Lambda} = \mathrm{Spec}\, \Lambda[C_n] \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m$$

Which is the Cartier dual of $\underline{\mathbb{Z}/n\mathbb{Z}}_{\Lambda}$. Note that if Λ contains all the nth roots of unity, then we have isomorphism of Hopf algebras,

$$\Lambda[x]/(x^n - 1) \simeq \prod_{i=1}^{n} \Lambda[x]/(x - \zeta_n^i) \simeq \prod_{i=1}^{n} \Lambda$$

For more see [HK20].

Definition 8.4. Let $\varphi_{\mathrm{univ}} \in \mathcal{O}(Z^1(W_E, \check{T}))$ be the universal representation.

We can define a zero dimensional lci algebraic stack over $\mathbb{Z}_l$.

Definition 8.5. The *stack of l-adically continuous L-parameters over* Λ

$$\mathrm{Loc}_{\hat{G},\Lambda} := [Z^1(W_E, \hat{G})_{\Lambda}/\hat{G}_{\Lambda}]$$

This definition works well for any reductive groups.

Example 8.6. When $\Lambda = \overline{\mathbb{Q}}_l$, $\mathrm{Loc}_{\hat{G}}$ parameterized isomorphism class of l-adically continuous L-parameters

$$\phi : W_E \to {}^L G(\bar{\mathbb{Q}}_l) := \hat{G}(\bar{\mathbb{Q}}_l) \rtimes W_E$$

[12] So far characteristic 0, seems sufficient.

[13] This is an abelian category; in fact this is an cartesian closed category. In abelian category: a simple object has precisely two quotient and two subobjects. Algebraically closed in this case guarantees $\mathrm{End}(X) \simeq \Lambda$.

8.1. **Alternative description of $\check{X}$-local system.** Let us consider the case in the Betti setting. representation stacks, $\mathrm{Rep}_{\check{G}}(\pi) := \mathrm{map}_{\mathrm{grp}}(\pi, G)$, hence $\mathrm{Loc}_{\check{G}} := \mathrm{map}_{\mathrm{Stk}_{\mathbb{C}}}(\Sigma_{\mathrm{Betti}}, B\check{G}) \simeq \mathrm{Rep}_{\check{G}}/\check{G}$.

For a S point $\rho_S : S \to \mathrm{Loc}_{\check{G}}$, corresponding to an element in $\rho : S \times B\pi \to B\check{G}$ in Stk. But now using the adjoint triplet for an arbitrary topos $\mathfrak{X}$, and map $f : X \to Y$,

$$(f_!, f^*, f_*) : \mathfrak{X}_X \to \mathfrak{X}_Y$$

where $f_!$ is the composition map. We have

$$\mathrm{Map}_{\mathrm{Stk}}(f_!(S \times B\pi, B(\check{G}) \simeq \mathrm{Map}_{\mathrm{Stk}_{/S}}(B\pi_S, B\check{G}_S)$$

hence an element $\rho_S : B(\pi_S) \to B(\check{G}_S)$ in $\mathrm{Stk}_{/S}$. This naturally induces an action on $\pi_S \circlearrowright X_S$, given by $\rho_S(\gamma_S) \circlearrowright X_S$ for $\gamma_S \in \pi_S$ Thus, we could define

$$(X_S)^{\pi_S, \rho_S} \in \mathrm{Stk}_{/S}$$

we will omit π_S for convenience.

Now $\mathrm{Rep}_{\check{G}}$ is representable by a scheme hence, admit $\rho_{\mathrm{univ}} \in \mathrm{Rep}_{\check{G}}(\mathrm{Rep}_{\check{G}})$. We will show we have the following pullback.

$$\begin{tikzcd}
(\check{X} \times \mathrm{Rep}_{\check{G}})^{\rho_{\mathrm{univ}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d] \\
\mathrm{Rep}_{\check{G}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Loc}_{\check{G}} \arrow[d] \\
* \arrow[r] & B\check{G}
\end{tikzcd}$$

As a corollary, this would imply that

$$\mathrm{Loc}_{\check{G}}^{\check{X}} \simeq (\check{X} \times \mathrm{Rep}_{\check{G}})^{\rho_{\mathrm{univ}}}/\check{G}$$

To warm up, let us do a computation for $*$ on a similar diagram to the upper square.

Example 8.7. Let $\rho : * \to \mathrm{Rep}_{\check{G}}$ be a point in $\mathrm{Stk}_{\mathbb{C}}$. Then

$$\mathrm{Loc}_{\check{G}}^{\check{X}} \times_{\mathrm{Loc}_{\check{G}}} \{\rho\} \simeq X^{\rho}$$

Indeed

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Stk}_{\mathbb{C}}}(*, \mathrm{Loc}_{\check{G}}^{\check{X}} \times_{\mathrm{Loc}_{\check{G}}} \{\rho\}) &\simeq \mathrm{Map}_{\mathrm{Stk}}(B\pi, X/G) \times_{\mathrm{Map}_{\mathrm{Stk}}(B\pi, BG)} \{\rho\} \\
&\simeq \mathrm{Map}_{\mathrm{Stk}_{/B\check{G}}}(\rho_! B\pi, X/\check{G}) \\
&\simeq^{(1)} \mathrm{Map}_{\mathrm{Stk}_{/B\pi}}(B\pi, \rho^* X/\check{G}) \\
&\simeq \mathrm{Map}_{\mathrm{Stk}_{/B\pi}}(B\pi, X/\pi) \\
&\simeq^{(2)} \mathrm{Map}_{\mathrm{LMod}_{\pi}(\mathrm{Stk}_{\mathbb{C}})}(*, X)
\end{aligned}$$

where in (1) we use the adjoint triple of ρ and the pullback square

$$\begin{tikzcd}
\check{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \check{X}/\pi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \check{X}/\check{G} \arrow[d] \\
* \arrow[r] & B\pi \arrow[r] & B\check{G}
\end{tikzcd}$$

and we used in (2) the equivalence

$$\begin{gathered}\mathfrak{X}_{/B\pi} \simeq \mathrm{LMod}_{\pi}(\mathfrak{X}) \\ (X/\pi \to B\pi) \mapsto (\pi \circlearrowright X)\end{gathered}$$

by taking pullback, for an aribtiary topos $\mathfrak{X}$.

Proposition 8.8. *For any S test scheme , $\rho_S \in \mathrm{Rep}_{\check{G}}(S)$, we have,*

$$Loc_{\check{G}}^{\check{X}} \times_{Loc_{\check{G}}} S \simeq (X_S)^{\rho_S}$$

Proof. Sketch. The argument is same as example, Example 8.1.

$$\begin{aligned}\mathrm{Map}_{\mathrm{Stk}/S}(S, \mathrm{Loc}_{\check{G}}^{\check{X}}) &\simeq \mathrm{Map}_{\mathrm{Stk}}(B\pi \times S) \\ &\simeq \mathrm{Map}_{\mathrm{Stk}/BG}(X \times B\pi, X/G) \\ &\simeq \mathrm{Map}_{\mathrm{Stk}/S}(S, (X_S)^{\pi_1, \rho})\end{aligned}$$

□

Corollary 8.9. *We have equivalence*

$$Loc_{\check{G}}^{\check{X}} \times_{Loc_{\check{G}}} \mathrm{Rep}_{\check{G}} \simeq (X \times \mathrm{Rep}_{\check{G}})^{\rho_{univ}}$$

9. $\mathcal{B}$-side Brief Recollection on the proof of Feng and Wang

Note that we would like to compute $\mathrm{Loc}_{\check{G}}$ in the *de Rham* context. In this case $\mathbb{F} = k = \mathbb{C}$. Although it is in the *étale* context $\mathbb{F} = \bar{\mathbb{F}}_q$, for which we can do function sheaf dictionary, [BSV], discussed the computation in *any* context. The first goal is to understand the diagram

$$\begin{tikzcd} \mathrm{fib}\rho \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d] \\ * \arrow[r, "\rho"] & \mathrm{Loc}_{\check{G}} \end{tikzcd}$$

Then this induces a localization sequence

$$\begin{tikzcd} Z \arrow[r] & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d, "\pi"] & U \simeq * \simeq \mathrm{Map}(C_{\mathrm{dR}}, \mathbb{G}_m/\mathbb{G}_m) \arrow[l, hook'] \arrow[d, "\pi"] \\ & \mathrm{Loc}_{\check{G}} & */\mathbb{G}_m \arrow[l, hook', "i"] \end{tikzcd}$$

This induces a short exact sequence

$$\hat{\pi}_{Z,*}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}) \to \pi_* \left(\omega_{\mathrm{Loc}^{\check{X}}\check{G}}\right) \longrightarrow \mathcal{O}_{\mathrm{triv}}$$

where $\mathcal{O}_{\mathrm{triv}} := i_*\pi_*\mathrm{triv}$. The strategy is that first one identifies the nonunital part. For a symmetric monoidal category, $(\mathcal{C}, \otimes, 1)$

$$\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \simeq \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$$

$$A \mapsto 1 \oplus A$$

$$\bar{A} := \ker \varepsilon \hookleftarrow (A, \varepsilon)$$

The first claim is then that $\overline{\hat{\pi}_{Z,*}\left(\omega_{\mathrm{Loc}^{\check{X}}}\right)}$ is identified with factorization algebra associated to $\mathrm{std} \in \mathrm{Rep}(\check{\mathrm{GL}}_1)$. We then describe how to identify extension class in.

10. Identification of graded algebra

10.1. **Factorization algebras.** The history is a little longer than I understand. We briefly discuss the topological one. The focus however would be on the holomorphic version. [BunG, Nafcha]. Let us recall vertex operator algebra (VOAs). This is the datum of

$$V \otimes V \to V[[z^{\pm 1}]]$$

Definition 10.1. A *Chiral algebra*. Let $X \in \mathrm{SmProj}_{\bar{k}}$, k is characteristic 0.

Example 10.2. $\Omega_X[-1]$. We have map

$$j_*j^!\omega_X[-1]^{\boxtimes 2} \xrightarrow{\simeq} j_*j^!\omega_{X^2}[-2]$$

10.2. **Classical definition of Ran space.** For some reason, in [FW24], it would be useful to work in the divisor version.

Definition 10.3. Let $Y \in \mathrm{Sch}$ Let $\mathrm{Ran}(Y) := \mathrm{colim}_{\mathrm{FinSurj}^{\mathrm{op}}} Y^I$.

What do étale sheaves on this stack look like? As explained in [GL]

$$\mathrm{Shv}^!(\mathrm{Ran}(\Sigma)) \simeq \mathrm{Fun}_{\mathrm{Sch}_k}(\mathrm{Fin}^{\mathrm{Surj}}, \mathrm{Shv}^!)$$

Thus we can regard an object as a family

$$\left\{ \mathcal{F}^{(T)} \in \mathrm{Shv}(C^T) \right\}_{T \in \mathrm{Fin}}$$

$(\mathrm{Ran}(\Sigma), \sqcup)$ has the structure of a commutative semigroup. In fact, we have the following diagram

$$\begin{tikzcd} C \arrow[r, "\Delta\times\Delta"] \arrow[d] & \mathrm{Ran}C \times \mathrm{Ran}C \arrow[d] \\ C \arrow[r] & \mathrm{Ran}C \end{tikzcd}$$

Yielding the following adjunction

$$\begin{tikzcd} \left(\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(C)), \otimes^!\right) \arrow[d, hook, shift left] \\ (\mathrm{CAlg}^{\mathrm{nu}}\,\mathrm{Shv}(\mathrm{Ran}(C)), \otimes^*) \arrow[u, shift left, "\Delta^!"] \end{tikzcd} \tag{6}$$

where under the functorially perspective $\otimes^*$ corresponds to the day convolution. There is another structure referred as the *chiral algebra structure*, $\otimes^{\mathrm{ch}}$.

Definition 10.4. The essential image of Equation 6, is denoted $\mathrm{CFact}_*(\mathrm{Shv}(\mathrm{Ran}(C)) \hookrightarrow \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(\mathrm{Ran}(C))$.

Example 10.5. This yields the commutative diagram

$$\begin{tikzcd} \mathrm{Mod} \arrow[r] \arrow[d] & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}) \arrow[d] \\ \mathrm{Shv}(C) \arrow[r] \arrow[d] & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(C), \otimes^!) \arrow[d, hook] \\ \mathrm{Shv}(\mathrm{Ran}(C)) \arrow[r] & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(\mathrm{Ran}(C)), \otimes^!) \end{tikzcd}$$

Let $M \in \mathrm{Shv}(C)$, then we can construct via

$$\mathrm{Sym}^*(\Delta_! M) \in \mathrm{CFact}_*(\mathrm{Shv}(\mathrm{Ran}(C))$$

There is an almost equivalent version: using divisors. But first we would like to add equivariance to the whole diagram above.

10.3. **Graded version.** Where we have an equivalence [FW24, Thm 4.3.2]

$$\begin{tikzcd}
\mathrm{Mod}^{\Lambda^+} \arrow[r] \arrow[d] & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}^{\Lambda^+}) \arrow[d] & \\
\mathrm{Shv}(C)^{\Lambda^+} \arrow[r] \arrow[d] & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(C)^{\Lambda^+}, \otimes^!) \arrow[d, hook] & \mathrm{CFact}(\mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C)) \arrow[l, "\simeq"'] \\
\mathrm{Shv}(\mathrm{Ran}(C)^{\Lambda^+}) \arrow[r] & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(\mathrm{Ran}(C)), \otimes^*) &
\end{tikzcd}$$

The construction of colored divisors appeared in [Gai16]. Where he constructed

Definition 10.6. Let $\mathrm{Ran}(X, \Lambda^+)$ be the prestack defined by

$$S \mapsto \left\{ I \subset \mathrm{Map}(S, X), \phi : I \to \Lambda^+ \right\}$$

Thus one has a natural map

$$\mathrm{Ran}(X, \Lambda^+) \to \coprod_{\lambda \in \Lambda^+} X^\lambda, X^\lambda = X^{(n_1)} \times \cdots \times X^{(n_k)}$$

for $\lambda = (n_1, \ldots, n_k) \in \Lambda^+$ now this induces a map under shefifications [Gai16, Lem 4.1.3].

Corollary 10.7. $\mathrm{Shv}(\mathit{Div}^{\Lambda^+}(C)) \simeq \prod_{\lambda \in \Lambda^+} \mathrm{Shv}(C^{(\lambda)})$.

The latter $\mathrm{Shv}(C^{(\lambda)} \xrightarrow{\Delta_\lambda^!} \mathrm{Shv}(C)$ thus inducing a map from

$$\mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C)) \to \mathrm{Shv}(C)^{\Lambda^+}$$

One deduce

$$\left(\mathrm{CFact}(\mathrm{Div}^{\Lambda^+}(C)), \otimes^{\mathrm{ch}}\right) \xrightarrow{\simeq} \mathrm{CAlg}^{\nu}(\mathrm{Shv}(C)^{\Lambda^+}, \otimes^!) \xrightarrow{\simeq} \left(\mathrm{Shv}(\mathrm{Ran}(C)^{\Lambda^+}, \otimes^*)\right)$$

Example 10.8. Let $\Lambda^+ := \mathbb{Z}_{>0}$, we will construct the standard representation. Also we will consider the.

10.4. **Ran version of a stack.** Let $\mathcal{Y} \in \mathrm{PShv}_{\mathcal{S}}(\mathrm{Aff})$. Then we can always define the *Ran* version

$$\begin{tikzcd}
\mathcal{Y}^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}_{\mathrm{Ran}(X)} \arrow[d] \\
* \arrow[r] & \mathrm{Ran}(X)
\end{tikzcd}$$

where $\mathcal{Y}_{\mathrm{Ran}(C)}(R)$, consists of $x \in \mathit{Ran}(C)$, and a map, with $S = \operatorname{Spec} R$,

$$(D_x)_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S \to \mathcal{Y}$$

where D_x is the formal completion of $S \times C$ along Γ_x, the union of the closed graphs $(S)_{\mathrm{red}} \to C$ corresponding to x. The diagram is a pullback given that $* \to \mathrm{Ran}(X)$ picks out $(x_1, \ldots, x_I) \subseteq X(R)$. Since

$$(D_x)_{\mathrm{dR}} = \bigcup_{i=1}^{I} (D_{x_i})_{\mathrm{dR}} \simeq \coprod_{i=1}^{I} *$$

Intuitively this is a stack living over $\mathrm{Ran}(X)$ whose fibers is as many copies of $\mathcal{Y}$.

10.5. **Recollection on Ran spaces.** For a $\check{G}$ eqivaraint factorization algebra $\mathcal{A} \in \mathrm{Shv}(\mathrm{pt}/\check{G})_{\mathrm{Div}^{\Lambda^+}(C)}$ it has an underlying structure. $\mathrm{oblv}(\mathcal{A}) \in \mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C))$. In particular

$$\begin{tikzcd} \mathrm{Fact}_{\check{G}}(\mathrm{Div}^+(C)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Shv}((\mathrm{pt}/\check{G})_{\mathrm{Div}^{\Lambda^+}(C)}) \arrow[d] \\ \mathrm{Fact}(\mathrm{Div}^+(C)) \arrow[r] & \mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C)) \simeq \prod_{\lambda \in \Lambda^+} \mathrm{Shv}(C^{(\lambda)}) \end{tikzcd}$$

The the notation $\mathrm{Div}^{\Lambda^+}(C) \to (*/\check{G})$, is the formal completion of the images of the divisors.

11. Evaluation morphism

Example 11.1. We construct an object in $\mathrm{Fact}\,\mathrm{Sym}^* \mathrm{std} \in \mathrm{Shv}(\mathrm{Loc}_{\check{G}})$. To do this we use the localization map

$$\mathrm{Loc}^{\mathrm{spec}} : \mathrm{Shv}(*/\check{G}_{\mathrm{Div}^{\Lambda^+}}) \to \mathrm{Shv}(\mathrm{Loc}_{\check{G}})$$

This is induced fro the evaluation map, i.e. that there is a diagram

$$\begin{tikzcd} \mathrm{Div}^{\Lambda^+}(C) \times \mathrm{Loc}_{\check{G}} \arrow[r] \arrow[d] & (\mathrm{pt}/\check{G})_{\mathrm{Div}^{\Lambda^+}(C)} \\ \mathrm{Loc}_{\check{G}} & \end{tikzcd}$$

Which intuitively - as the definition goes through $\mathrm{Ran}(C)$ - is the evaluation sends a pair,

$$D \in \mathrm{Div}^{\Lambda^+}(C) \text{ and a } \check{G}\text{-bundle } L : C_{\mathrm{dR}} \to */\check{G}$$

to the precomposition. Thus, we in fact only have to determine object in $\mathrm{Shv}(*/\check{G}_{\mathrm{Div}^{\Lambda^+}}) \to \mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C))$. Thus, we have that $\overline{\mathrm{Sym}}^*\mathrm{std}$ is a constant cocommutative coalgebra on $\mathrm{Shv}(C)^{\mathbb{Z}_{\geq 0}}$. Then we have

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^*\mathrm{std}) \in \mathrm{Fact}(\mathrm{Div}^{\mathbb{Z}_{\geq 0}}(C))$$

In fact this has a very simple description, formally by adjunction

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^*\mathrm{std}) \simeq \left\{\pi_{n!}(\mathrm{std}^{\boxtimes n})_{\Sigma_n}\right\}$$

where $\pi_n : C^n \to C^{(n)}$ is the quotient map.

11.1. **Discussion of Etale setting.** It would be useful to note that BG in fact is a 0-truncated quasi-geometric stack. By a geometric stack we mean:

- $X \simeq |X_\bullet|$ in the category $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{CAlg}^{\mathrm{cn}}) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.
- $d_0 : X_1 \to X_0$ is representable, affine, faithfully flat map. In the case of $X \times G \to X$ this is faithfully flat.

Mapping into quasi-geometric stacks implies that there is a nice Tannkian theory [Lur18, p. 9].

Theorem 11.2. *[Lur18, p. 9.2.0.2] Let $X, Y \in PShv(CAlg^{cn})$. There there is a fully faithful embedding*

$$\mathrm{Map}_{Fun(CAlg,\hat{S})}(Y, X) \hookrightarrow Fun^{\otimes}(QCoh(X), QCoh(Y))$$

We will discuss further under the existence of gentle Tannkian category.

Example 11.3. Consider $Y = \Sigma$, and $X = BG$. Then we have

$$\mathrm{Map}(\Sigma, BG) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Rep}\, G, \mathrm{QCoh}(B\pi_1)) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Rep}\, G, \mathrm{Rep}\, \pi_1)$$

But how does one make this a stack? In fac, it is a stack whose A points are $\mathrm{Map}(\Sigma \times \mathrm{Spec}\, A, BG)$.

Example 11.4. If $X = \mathbb{A}^1/\mathbb{G}_m$ this is a gain geometric. Then we can equally define a stack whose points are given by

$$\mathrm{Fun}'(\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m), \mathrm{Rep}\, \pi_1)$$

Then one can use localization of $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$ via, $\mathrm{QCoh}(*/\mathbb{G}_m)$ and $\mathrm{QCoh}(*)$. This should induce sequence.

11.2. **Identifying the extension class.** On the other we have

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(\mathcal{O}_{\mathrm{triv}}, \hat{\pi}_{Z,*}(\omega_{\mathrm{Loc}^{\check{X}}})) &\simeq \mathrm{Map}_{\mathcal{C}}(i_* q_* \mathcal{O}_{\mathrm{pt}}, -) \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(B\mathbb{G}_m)}(q_* \mathcal{O}_{\mathrm{pt}}, i^! \pi_{\mathcal{Z},*} \omega_{\mathcal{Z}}) \\ &\simeq \mathrm{Map}_{\mathrm{Rep}(\mathbb{G}_m)} \end{aligned}$$

where this $\Re(\mathbb{G}_m)$ is from the following adjunction

$$\begin{tikzcd} \mathrm{QCoh}(B\mathbb{G}_m) \arrow[r] & \mathrm{Rep}(\mathbb{G}_m) \arrow[d] & \simeq & \mathrm{coMod}_{k[t^{\pm 1}]} \\ & \mathrm{QCoh}(*) \arrow[rr] & & \mathrm{Mod}_k \end{tikzcd}$$

Now to identify the $i^!$, we have the following diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] & \mathcal{V}/\mathbb{G}_m \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ * \arrow[r] & B\mathbb{G}_m \arrow[r] & \mathrm{Loc} \end{tikzcd}$$

induced from the diagram of formal completion.

$$\begin{tikzcd} & \mathcal{V}/\mathbb{G}_m \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] & Z \arrow[l] \arrow[dl] \\ V \arrow[r] \arrow[d] & V/\mathbb{G}_m \arrow[r] \arrow[d] & \mathrm{Loc}^{\check{X}} \arrow[d] & \\ * \arrow[r] & */\mathbb{G}_m \arrow[r] & \mathrm{Loc} & \end{tikzcd}$$

where each of the squares are fiber pullbacks. We have that

$$i^! \pi_{\mathcal{Z},*} \omega_{\mathcal{Z}} \simeq \pi_{\mathcal{V}/\mathbb{G}_m}(\varpi_{\mathcal{V}/\mathbb{G}_m})$$

This allows us to What have the equivalence

$$\mathrm{Map}_{\mathrm{IndCoh}(B\mathbb{G}_m)}(\mathcal{O}(\mathbb{G}_m), \mathrm{Sym}\, \mathcal{E})) \simeq \prod_{n \in \mathbb{N}} \mathrm{Map}_{\mathrm{QCoh}(*)}(k, \mathrm{Sym}_k^n\, \mathcal{E}) \simeq \prod_{n \in \mathbb{N}} \Omega^{\infty}\, \mathrm{Sym}_k^n\, \mathcal{E}$$

Note that to compute homotopy orbits of $E \in \mathrm{Fun}(BG, \mathrm{Sp})$, for G a finite group, of a spectrum we use the Whitehead tower, giving fiber sequence

$$\tau_{\geq n+1} E \to \tau_{\geq n} E \to \Sigma^n H \pi_n E$$

these induces [Lur09, Ch.1]

$$E^2_{p,q} \simeq H_p(\Sigma_n, \pi_q E) \Rightarrow \pi_{p+q}(E_{hG})$$

As k is a projective $k[\Sigma_n]$ my module when k is of characteristic 0, where $H_p(\Sigma_n, \pi_q E) \simeq \mathrm{Ext}^p_{k[\Sigma_n]}(k, \pi_q E)$ is concentrated only for degrees $p = 0$. We deduce

$$\pi_0 \operatorname{Sym}^n \mathcal{E} \simeq \operatorname{Sym}^n_k \pi_0 \mathcal{E} \simeq H^0(\Sigma, k)$$

Definition 11.5. Let $\mathrm{Mod}^{\mathrm{free}}_A$, where $A \simeq \mathbb{Z}[x_1, \dots, x_m]$, is a free polynomial ring. Then one can define the derived symmetric powers

$$\begin{tikzcd} \mathrm{Mod}^{\mathrm{free}}_A \arrow[d] \arrow[dr, "M \mapsto \pi_0(M \otimes_A M \otimes_A \cdots \otimes_A M)_{\Sigma_n}"] & \\ \mathrm{Mod}^{\mathrm{cn}}_A \arrow[r] & \mathrm{CAlg}^{\mathrm{cn}}_A \end{tikzcd}$$

Remark 11.6. For an algebraic group $G \in \mathbb{E}^{\mathrm{grp}}_1(\mathrm{Sch}_k) \simeq \mathrm{Grp}(\mathrm{Sch}_k)$, $\mathrm{QCoh}(BG) \simeq \mathrm{Rep}(G) \in \mathrm{Pr}^L_{\mathrm{st},\omega}$,

$$\mathrm{IndCoh}(BG) \simeq \mathrm{QCoh}(BG)$$

Note that if G were a finite group, we may regard it as a group scheme via

$$\mathrm{Grp}(\mathrm{Set}) \to \mathrm{Grp}(\mathrm{Sch}_k)$$

$$X \mapsto \coprod_{x \in X} \operatorname{Spec} k =: \underline{X}$$

where we note that $\coprod_X \times \coprod_Y \simeq \coprod_{X \times Y}$, indeed the cardinality of right hand side is $|X \times Y|$, whilst that of left hand side is $|X| \cdot |Y|$. The map from (x, y) component to right hand side is to $*_x \to \coprod_x$, and $*_y \to \coprod_Y$. This induces a bijection on the level of sets. We abusively denote for a finite group G, then

$$\mathrm{QCoh}(BG) \simeq \mathrm{Fun}(|BG|, \mathrm{Mod})$$

i.e. modules with a G-action. Indeed to see this: take a Čech resolution of the land side

$$\mathrm{Tot}\left(\mathrm{QCoh}(G^\bullet)\right) := \varprojlim \left(\mathrm{QCoh}(*) \to \mathrm{QCoh}(G) \to \mathrm{QCoh}(G) \otimes \mathrm{QCoh}(G) \to \cdots\right)$$

Recall that $\mathrm{QCoh} : (\mathrm{Sch}_k, \times) \to \left(\mathrm{Pr}^L_{\mathrm{st}}, \otimes\right)$ is symmetric monoidal, whilst the forgetful functor $\mathrm{Pr}^L_{\mathrm{st}} \to \widehat{\mathrm{Cat}}$ preserves limits. Now reduces to the observation that

$$\mathrm{Fun}(G, \mathrm{Mod}) \simeq \prod_{g \in G} \mathrm{Mod} \simeq \mathrm{QCoh}\left(\coprod_{g \in G} *\right)$$

using the fact that $\mathrm{QCoh}(-)$ satisfies fpqc-descent.

Remark 11.7. The coalgebra structure of $\mathbb{G}_m$. We imagine $\mathcal{O}(G)$, as a family of function

$$\{f_R : G(R) \to R\}_{R \in \mathrm{Alg}_k}$$

Then Δf is the unique element on $\mathcal{O}(G) \otimes \mathcal{O}(G)$, such tha t

$$\Delta f(a, b) = f(a, b)$$

In the case of $\mathbb{G}_m$, this element would be that linearly extended from $\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ by $t \mapsto t \otimes t$. We can also analyze how $\mathbb{G}_m$ acts on $\mathcal{O}(G)$. Let $t^n \in \mathcal{O}(G)$.

$$r \cdot t^n(r') := (r \cdot r')^n = r^n (r')^n$$

Thus $r \in \mathbb{G}_m(R)$ acts by character n.

Remark 11.8.
- Vezossi has a cheat sheaf graded modules.

12. Linear Stacks: recollection

Let us work in the category Sch_S, where $S = \operatorname{Spec} k$. If G is reductive algebraic group, then we can construct the GIT quotient $\operatorname{Spec} A^G \simeq X/G$. Naïvely, if

Theorem 12.1. *Luna's étale slice theorme. Let $G \in AlgGrp_k^{red}$. Let $x \in X$ be a closed point such that $Gx \hookrightarrow X$ is closed.*

12.1. Relative notion of vector bundle.

By stack I mean either $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}^{\heartsuit}, \mathcal{S})$, or $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}, \mathcal{S})$.

Definition 12.2. Given $Y \in \mathrm{Stack}_X$, we can construct the stack of sections.

$$\mathrm{Sect}(X, Y)$$

In otherwords, when we mean V, G we will implicitly mean V as a vector bundle on X, and $G \in \mathbb{E}_1^{\mathrm{grp}}(\mathrm{Stk}_{/X})$.

Example 12.3. We set

$$\mathrm{Sect}(X, V/G)$$

- $\mathcal{F} \to X$ a G-bundle.
- $\mathcal{F} \to V$ a G-equivariant map. This is equivalent to the datum of associated to data of section. One way to see this is via the "cube".

Thus, initively the fiber is $H^0(X_\kappa, \mathcal{F} \times^G V)$. In fact we will argue that that this is $R\Gamma(X_k, F \times^G V)$, a perfect connective cochain complex. Thus its dual is connective perfect chain complex. Then we claim the fibre at a point $k \xrightarrow{\mathcal{F}} \mathrm{Sect}(X, X/G)$, corresponds to

$$\mathbb{V}(R\Gamma(X_k, \mathcal{F} \times^G V)^\vee) \in \mathrm{Aff}_k$$

12.2. Construction of symmetric bundle.

References, [FYZ24]. We are at the *de Rham setting.* That is our sheaves are over $\mathbb{C}$, with $\mathbb{C}$ coefficient theory. Let $\mathcal{E} \in \mathrm{QCoh}(X)$. We will construct the associated linear stack, [Mon21]. In this set up, we will consider $\mathrm{Stack}_X := \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S})$.

Definition 12.4.

$$\mathbb{V} : \mathrm{QCoh}(X) \to \mathrm{Stack}_X$$

$$\mathbb{V}(\mathcal{E})(T \xrightarrow{f} X) := \mathrm{Map}_{\mathrm{QCoh}(T)}(f^*(\mathcal{E}), \mathcal{O}_T) \in \mathcal{S}$$

Proposition 12.5. *If $\mathcal{E} \in QCoh(X)^{cn}$, then*

$$\mathbb{V}(\mathcal{E}) \simeq \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E})$$

Proof.

$$\begin{aligned}\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{E}, f_*\mathcal{O}_T) &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X))}\left(\mathrm{Sym}_{\mathcal{O}_X}\mathcal{E}, f_*\mathcal{O}_T\right)\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})}(\mathrm{Sym}_{\mathcal{O}_X}\mathcal{E}, f_*\mathcal{O}_Y)\\ &\simeq \mathrm{Map}_{\mathrm{Stack}_X}(T, \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}}\mathcal{E})\end{aligned}$$

Where we note that Spec_X is only defined for *connective* spectrum, **??**. □

Proposition 12.6. *Let $Stack_X^{aff}$ be the full subcategory of of Delign Munford Stacks spanned by affine morphisms. There is an equivalence*

$$\left(Stack_X^{aff}\right)^{op} \xrightarrow{\simeq} CAlg(QCoh(X)^{cn})$$

$$(Y \to X) \mapsto f_*\mathcal{O}_Y$$

The (left adjoint) inverse of this functor is Spec_X. *In otherwords,*

$$\mathrm{Map}_{Stack_X}(Y, \mathrm{Spec}_X \mathcal{F}) \simeq \mathrm{Map}_{CAlg(QCoh(X))}(\mathcal{F}, f_*\mathcal{O}_Y)$$

Proof. We can do descent, and assume $X = \mathrm{Spec}\, A$. Then we have

$$\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}}) \simeq \mathrm{CAlg}_A^{\mathrm{cn}} \xrightarrow{\simeq} \mathrm{Stack}_{\mathrm{Spec}\,A}^{\mathrm{aff}}$$

□

Remark 12.7. Note that $\mathbb{V}$ factors through $\mathbb{G}_m$-$\mathrm{Stk}_{/X}$. This can be seen through the points: for $\mathrm{Spec}\, B \to X = \mathrm{Spec}\, A$, then [14]

$$\mathbb{G}_{m,X}(B) \simeq \mathrm{Map}_{\mathrm{Mod}_B}^{\simeq}(B, B) \circlearrowright \mathrm{Map}_{\mathrm{Mod}_A}(\mathcal{E}, B)$$

Example 12.8. Let $\mathcal{E} = \mathcal{O}_X$, $X = \mathrm{Spec}\, A$, then for every A-algebra B,

$$\mathbb{V}(\mathcal{O}_X^\vee)(B) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(A^\vee, B) \simeq B$$

[15] And the $\mathbb{G}_m$-action is the natural action on the space.

Example 12.9. Let $S := \mathrm{Spec}\, A$ be an affine scheme, then $\mathrm{QCoh}(S) \simeq \mathrm{Mod}_A$. $E = \mathbb{V}(\mathcal{O}_S^\vee) \to S$, thus $\mathcal{O}_E = \mathrm{Sym}_{\mathcal{O}_S}\mathcal{O}_S^\vee$. This is naturally an $\mathcal{O}_S$-module, so that

$$\pi_* : \mathrm{QCoh}(E) \to \mathrm{QCoh}(S)$$

satisfies

$$\pi_*\mathcal{O}_E = \mathrm{Sym}_{\mathcal{O}_S}\mathcal{O}_S^\vee \simeq \mathbb{A}_S^1 \to S$$

Remark 12.10. Why dual? Consider $\mathrm{Sym}_k V^\vee$, regarded as an algebra.

$$\mathrm{Map}_{\mathrm{Alg}_k}(\mathrm{Sym}_k V^\vee, B) \simeq \mathrm{Map}_{\mathrm{Mod}_k^\heartsuit}(V^\vee, B)$$

but we know that

$$V \otimes_k B \simeq \mathrm{map}_{\mathrm{Mod}_k^\heartsuit}(V^\vee, B)$$

By property of dualizable objects.

Definition 12.11. $\mathcal{L} \in \mathrm{Mod}_{\mathcal{O}_X}$ is invertible if $\mathcal{L} \otimes - : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_X}$ is an equivalence.

[14] Or more generally, given that that the geometric morphism $X \to \mathrm{pt}$ is connected

[15] It should be the case that

Example 12.12. If $X = \operatorname{Spec} R$, then $\mathcal{L}$ is invertible iff it corresponds to a rank one projective module over R.

It may be slightly unintuitive but we have the equivalence

$$\mathrm{Mod}_{\mathcal{O}_X}^{\text{loc.free rk 1}} \simeq \mathrm{Mod}_{\mathcal{O}_X}^{\text{invertible}}$$

and line bundles, those locally free isomorphic that $L \times \mathbb{A}^1_X \to X$, has globalls ections which are locally free of rank 1.

Example 12.13. $\mathbb{G}_m$ bundle. Let $X = \operatorname{Spec} B$, then

$$X \to B\mathbb{G}_m$$

corresponds to a $\mathbb{G}_m$ torsor hence line bundle. The associated $\mathbb{G}_m$torsor can all be constructed as follows

$$\operatorname{Spec}_X \left(\operatorname{Sym}_B L[L^{-1}]\right)$$

For instance if we take the rank 1 module B, then we are considering $B[x][x^{-1}]$.

where Spec_X is as in **??**.

12.3. **Symmetric algebra construction.** Note we always have a functor

$$\mathrm{Mod}_{R,\geq 0} \to \mathrm{Mod}_{R,\geq 0}$$
$$M \mapsto \operatorname{Sym}^n_R(M) := \pi_0(M \otimes_A M \cdots \otimes_A M)_{\Sigma_n}$$

Example 12.14. If M were rank 1.

13. Nilpotent support condition

Let us recall the definition of nilpotent cone. Classically, one constructs the nilpotent cone as the pullback.

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \check{\mathfrak{g}} \arrow[d] \\ 0 \arrow[r] & \check{\mathfrak{g}}//\check{G} := \mathrm{Spec}(\mathrm{Sym}(\check{\mathfrak{g}})^{\check{G}}) \end{tikzcd}$$

In [AG15], they constructed global nilpotent cone, as

$$\mathrm{Arth}_{\check{G}} = \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$$

Theorem 13.1.

14. Gerbes

We will focus on $\mathbb{G}_m$ gerbes. Let $X \in \mathrm{Sch}_k^{\mathrm{qcqs}}$. Then one definition of $\mathbb{G}_m$ gerbe is the following pair of datum :

Definition 14.1. A G gerbe consists of $\mathcal{G} \in \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}^{\heartsuit}, \mathcal{S}_{\leq 1})_{/X}$, such that

- $\alpha : \mathcal{G} \to X$, étale locally equivalent to $BG \times X$.
- A *G-banding*, or isomorphism
$$G \to \alpha_* \mathcal{I}_{\mathcal{G}} = B_{\mathcal{G}}(\mathcal{G} \times_X \mathcal{G})$$

Ultimately, we wish a categorical decomposition due to the Gerbe.

Theorem 14.2. *[BS19, Thm 4.7] We have product decomposition*
$$QCoh(\mathcal{G}) \simeq \prod_{\chi \in X^*(G)} QCoh(\mathcal{G})_\chi$$
where ever G is a diagonalizable group.

The function α is induced as follows - that for any morphism of stacks there is an adjunction

$$\mathrm{Stk}_{/X} \longrightarrow \mathrm{Stk}_{/Y}$$

Example 14.3. Let $X = \mathrm{Spec}\, k$. $G = B\mathbb{G}_m$. Then

- $\mathcal{I}_{B\mathbb{G}_m} \simeq B\mathbb{G}_m \times \mathbb{G}_m \in \mathbb{E}_1^{\mathrm{grp}}(\mathrm{Stk}_{/B\mathbb{G}_m \times B\mathbb{G}_m})$, indeed this is the Čech map of
$$B\mathbb{G}_m \to B\mathbb{G}_m \times B\mathbb{G}_m$$
- The banding in this case is the identity
$$\mathbb{G}_m \to \alpha_* \mathcal{I}_{B\mathbb{G}_m} \simeq \mathbb{G}_m$$

We similarly have a product decomposition $\mathrm{QCoh}(B\mathbb{G}_m) \simeq \prod_{\chi \in X^*(\mathbb{G}_m)} \mathrm{QCoh}(B\mathbb{G}_m)_\chi$.

15. L-PARAMETERS

References: for an introduction, see [Sch21], [Che23]. The main papers of discussions are [Zhu21], [FS24]. Further notes, Bvchurch.

Local Langlands correspondence predicts [16]

$$\pi_0 \operatorname{Rep}_\Lambda^{\mathrm{Irr,sm}}(G(E)) \to \{L\text{-parameters}\}$$

More generally, in categorical local langlands (CLL), one hopes for a fully faithful embedding

$$\operatorname{Rep}_\Lambda(G(F)) \hookrightarrow \operatorname{QCoh}(Z^1/\check{G})$$

for some Z^1 parameter space. For the story of $\Lambda = \mathbb{C}$, we refer to [Dat22, Sec. 1], where they argue Definition 15.13 is the relevant one using representation theoretic arguments.

(1) How does one define the arithmetic version of L-parameters?
(2) What is the geometric version of parameters? How does it relate to the construction in [FS24].

Example 15.1 (Harris-Taylor)**.** They proved the map to be an isomorphism in the case of $G = \mathrm{GL}_n$.

15.1. Breaking down field extensions.

15.2. Discretizing the unramified part.

As a first approximation, one replaces $\Gamma_E := \mathrm{Gal}(\bar{E}/E)$ with W_E. Every finite extension K/E is still a local field with ring of integers $\mathcal{O}_K$. There is a a canonical extension of valuation $v : K \to \mathbb{Z}$ extending that of E.

Definition 15.2. An algebraic extension K/E is *unramified* if $e_{E/F} := v(\varpi_K)/v(\varpi_E)$, is 1.

These are extensions extensions of the finite field, $\mathbb{F}_q$,

Example 15.3. Unramified extensions of function field. $\mathbb{F}_{q^n}((\varpi))/\mathbb{F}_q((\varpi))$.

Note that $|\mathbb{F}_{q^n} : \mathbb{F}_q| = q^{n-1}$. From Example 15.3, we see that one obtains unramified extensions by adjoining all roots of unity of order coprime to 1, see Example 15.4.

Example 15.4. $\bar{\mathbb{F}}_p$ obtained from $\mathbb{F}_p$.

We have the following diagram

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & I_E \arrow[r] \arrow[d] & W_E \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & I_E \arrow[r] & \Gamma_E \arrow[r] & \Gamma_{\mathbb{F}_q} \simeq \widehat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

But what does this mean?

[16] Usually, work with $\mathbb{C}$ coefficients. Because you work with $\mathbb{C}$-coefficeinnts, there's a canonical square root of $\sqrt{q}$. This is actually implicit in the assignment.

15.3. **Discretizing the ramified part.** Fix a compatible system roots of unity $\{\zeta_n\}_{p\neq n}$.

$$K^t := \bigcup_{p\neq n} K^{\mathrm{nr}}(\varpi_E^{1/n})$$

This further fits in the following diagram

$$\begin{tikzcd}
1 \arrow[r] & I_E/P_E \arrow[r] & W_E/P_E \arrow[r] & W_E/I_E \arrow[r] & 1 \\
\left\langle s^{\mathbb{Z}[1/q]} \right\rangle \arrow[r] \arrow[u, hook] & s^{\mathbb{Z}[1/q]} \rtimes \mathrm{Fr} \arrow[u, hook] & & &
\end{tikzcd}$$

We have the following sequence which denote as the projection

$$I_E \longrightarrow I_E^t := I_E/P_E \simeq \prod_{l'\neq p} \mathbb{Z}_{l'} \longrightarrow \mathbb{Z}_l$$

for $l' \neq p$, we have that p is divisible in l'. This is why we have $\mathbb{Z}[1/q]$.

In fact one can understand quite well the normalizing action of $\Gamma_E \circlearrowleft I_E$ under the maps t and t_l.

Proposition 15.5. *[HC]*

Proposition 15.6. $Z^1_{cts}(W_E, \hat{G}(\bar{\mathbb{Q}}_l) \simeq Z^1_{cts}(W_E^0, \hat{G}(\bar{\mathbb{Q}}_l)_{disc})$.

15.4. **Various definition of L-parameters.** There are at less three different definition of Langlands parameter of $\overline{\mathbb{Q}}_\ell$.

(1) Paris (r, N).

(2) ${}^L\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ which are l-adically continuous, see Definition 15.9.

The approach taken by, [DHKM24] is the third, this follows by "discreteizing" the tame inertia group I_E/P_E, as explained in [..]

15.5. **Langlands dual group.** References, [GH22]. Let $G \in \mathrm{AlgGrp}_{\mathbb{F}}^{\text{spl, red}}$. If one records the data of torus, and denote $\mathrm{Spl}_{\mathbb{F}}$, then we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Spl}_{\mathbb{F}} \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & \mathrm{Spl}_{\mathbb{F}'} \arrow[dl, "\simeq"] \\ & \mathrm{RRD} & \end{tikzcd}$$

Where the map sends

$$(G, T) \mapsto \Psi(G, T)$$

From $G \in \mathrm{AlgGrp}_E^{\text{spl,red}}$ this arises the dual grp $\widehat{G}/\mathbb{Z}$. This has an action of $\Gamma_E \twoheadrightarrow Q$, which factors through a finite quotient.

Next we briefly recalling the construction of L-group. Is that there is an short exact sequence [GH22, Prop. 7.3.3]

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\Psi) \to 1$$

Example 15.7. Consider a torus over $T/\mathbb{Q}$. $\Gamma_{\mathbb{Q}}$.

Definition 15.8. ${}^L G := \widehat{G} \rtimes Q$

This seems awkward: it seems dependent with Q, but should not matter.

15.6. **Comparison of various definitions.**

Definition 15.9. An L-parameter over $\mathbb{C}$ is a continuous map

$$\begin{tikzcd} W_E \arrow[rr, "{}^L\varphi"] \arrow[dr] & & {}^L G(\mathbb{C}) = \widehat{G} \rtimes W_E \arrow[dl] \\ & W_E & \end{tikzcd}$$

The map ${}^L\varphi(w) = (\varphi(w), w)$, uniquely determines a map $\varphi : W_E \to \hat{G}$. This is equivalent to a continuous 1 cocycle [17]

$$W_E \to \widehat{G}(\mathbb{C})$$

Example 15.10. When G is split, ${}^L G(\mathbb{C}) \simeq \widehat{G} \times W_E$. These conditions are vacuous, and this is simply equivalent to a continuous homomorphism $W_E \to \hat{G}(\mathbb{C})$.

These are the kind of parameters we can attach to representations. In fact, this does not really matter if we change $\mathbb{C}$, to any Λ a $\mathbb{Z}_l$-algebra. As argued in Proposition 15.6, we can do a "discretization" argument.

Proposition 15.11. *continuity iff factours thorugh a discrete quotient W_E/I', $I' \hookrightarrow I_E$ open finite index subgroup.*

Proof. The topology of complex numbers is incompatible with the inertia subgroup. $\square$

Deligne: It is also better to keep track of a monodromy operator. N.

[17] Recall that if $G \circlearrowleft^{\varphi} A$, then a continuous cocycle f is the condition that $f(gh) = f(g) + \varphi(g) f(h)$ for $g, h \in G$.

15.12. Take 2. Definitions.

Definition 15.13. A L-parameter over $\mathbb{C}$ is a pair (φ, N) where
$$\varphi : W_E \to {}^L G(\mathbb{C})$$
cts grp homomorphism, $N \in \mathrm{Lie}\widehat{\mathfrak{g}} \otimes \mathbb{C}$ st. for all $w \in W_E$,
$$\mathrm{Ad}(\varphi(w))(N) = q^{|w|} N$$

For $G = \mathrm{GL}_n$ these are also called the Weil Deligne representations. I will discuss this later.

15.14. There is also a further refinement that does not play a role. Take 3.

Definition 15.15. A L-parameter over $\mathbb{C}$ is (φ, r) where
$$\varphi : W_E \to {}^L G(\mathbb{C})$$
cts grp homomorphism st.
$$r : \mathrm{SL}_2 \to \widehat{G}/\mathbb{C}$$
st (r, φ) commute $W_E \times \mathrm{SL}_2 \to {}^L G$.

Then
$$\varphi'(w) := \varphi(w) r(\mathrm{dia}(q^{|w|/2}, q^{-|w|/2}))$$
with $N = \mathrm{Lie}\, r \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

The reason I mentioned take 3 is this is what appears in the modern discussion. All takes give rise to a variety.

15.16. Q: Do you need to say that monodromy operator in Take 2 is nilptoent? A: good question, *No*. The condition in fact implies all ev. of N are 0.

15.17. Parameters in sense of Take 2 and 3 are up to $\widehat{G}(\mathbb{C})$ cjg, in bijection, but scheme structures are different.

- In take 2, $N \neq 0$ can degenerate to $N = 0$.
- In take 3, SL_2 has "rigid" representations.

We would *want* these degenerations. So take 2 is the correct one.

15.18. Deligne's motivation. Fix $\mathbb{C} \simeq \bar{\mathbb{Q}}_l$. Take 2'.

Definition 15.19. An L-parameter over $\bar{\mathbb{Q}}_l$ is a continuous group homomorphism

$$\begin{tikzcd} \varphi : W_E \arrow[rr] \arrow[dr] & & {}^L G(\bar{\mathbb{Q}}_l) \arrow[dl] \\ & Q & \end{tikzcd}$$

Equivlently a continuous 1-cocycle $W_E \to \widehat{G}(\mathbb{C})$.

Theorem 15.20. *Grothendieck Deligne. Take 2 and Take 2' are equivalent: Fix a trivialization of $\mathbb{Z}_l(1) \simeq \mathbb{Z}_l$ and a Frobenius element $\Phi \in W_E$. Once you made these choices, we get a retract from the short exact sequence.*

$$t_l: \; W_E \longrightarrow I_E \longrightarrow \mathbb{Z}_l(1) \simeq \mathbb{Z}_l$$

Then any continuous grp homomorphism φ_l. is of the form

$$\varphi_l(w) = \varphi(w)\exp(t_l(w) \cdot N)$$

for a *unique* L-parameter (φ, N) in the sense of Take 2. The key point:

$$W_E \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$$

need not be trivial on an open subgro $I' \hookrightarrow I_E$, can only find I' such it factors over $I' \twoheadrightarrow \mathbb{Z}_l$, then $\mathrm{Hom}(\mathbb{Z}_l, \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ Then

$$\mathrm{Hom}(\mathbb{Z}_l, \mathrm{GL}_2(\mathbb{Q}_l))$$

are, on an open open subgrp:

$$x \mapsto \exp(xN)$$

where N is uniptoent matrix. Thus, we are looking at matrices N st... [32:22]

15.21. Note: to go from l-adic parameter to a WD type parameter, is *not canonical.* Upto iso, some how not matter. For me now what is most canonical is the cts group homomorphism. We will now adopt Take 2' as the definition.

- For this reason, we are forced to work over $\mathbb{Z}_l$.

Goal: construct a moduli space of L-parameters. i.e. scheme locally of finite type.

$$Z^1(W_E, \widehat{G})/\mathbb{Z}_l$$

st. A valued points are the continuous grp homomorphisms. i.e. its 1- cocyles.

$$\begin{tikzcd} W_E \arrow[rr] \arrow[dr] & & {}^LG(A) \arrow[dl] \\ & Q & \end{tikzcd}$$

We *definitely* want: nontrivial topology. Any $\mathbb{Z}_l$-module M can be endowed with the filtered colimit topology. i.e.

$$M \simeq \varinjlim_{M' \hookrightarrow M, \text{fin.gen}} M'$$

equivalently, something is open iff its restriction to any of the M' is open. But this is a mismatch.

15.22. In the Language of condensed mathematics, there is always a fully faithful embedding [CS19, Prop 1.7]

$$\mathrm{TopSpc}^{\kappa\mathrm{cg}} \hookrightarrow \mathrm{CondSet}$$

from κ-compactly generated spaces to κ condensed sets which preserves products. In other words, this induces a map. the correspondence, condensed group:

$$\underline{M} = M_{\mathrm{disc}} \otimes_{\mathbb{Z}_{l,\mathrm{disc}}} \mathbb{Z}_l$$

why is this the same?

- All operations commutes with filtered colimits.

- What one has to check is that the map factors through.

15.23. So a priori: this might be a derived scheme. If it would be a derived scheme, then the usual topological framework is not so good to talk. You would have to mix topology and homotopy.

- If you stick a dg-algebra A result does turn out to be classical.

15.24.

Theorem 15.25. *There is $Z^1(W_E, \widehat{G}) \in \mathrm{Sch}_{\mathbb{Z}_l}$ of L-parameters for G - it is a disjoint union of affine scheme of finite type over $\mathbb{Z}_l$,that are flat, complete intersections and of dimension* $\dim G = \dim \widehat{G}$*. In the usual Langlands, this is the local systems.*

Note:

- can divide by cjg of $\widehat{G}$ and get an Artin stack $\mathrm{LS}_{\widehat{G}}$.
- The natural extension to animated $\mathbb{Z}_l$-algebras is the same moduli space.
- Usual this scheme is only affine.
- I will explain the index set. The connected components are much more subtle.

Proof. Any cts 1 -cocycle:

$$\varphi : W_E \to \widehat{G}(A)$$

is trivial on an open subgroup P of wild inertia. This implies already we have the union

$$Z^1(W_E, \widehat{G}) = \bigcup_P Z^1(W_E/P, \widehat{G})$$

- The transion maps are open and closed immersions. Why is this ? To understand $W_E/P, W_E/P'$.
- We somehow look at the locus of elements of order p in side $\widehat{G}$. [51:35]

□

Convention.

- Let E be narc. local field, G/E red. grp.
- Let $l \neq p$, $\widehat{G}/\mathbb{Z}_l$ dual grp, this is canonically split.
- This comes equipped with an action W_E, there's an algebraic one, which factors through a finite quotient, the other one related to cyclotomic twist.
- Fix $\sqrt{q}$. So for all occurences $\mathbb{Z}_l$ relplace it with $\mathbb{Z}_l[\sqrt{q}]$.

16. Recollection of stacks of local system

References. [Zhu21]. Underlying the following story, there is the Hidden smoothness philosophy. In this section we denote $\mathrm{Stk}_k := \mathrm{Shv}(\mathcal{S})$.

Example 16.1. The circle. $\mathrm{Loc}_G(S^1) = G/G$.

16.1. Local systems in geometric Langlands. [BSV, Appendix C.2]

There are three different stacks of local system:

(1) de Rham stacks
(2) Betti context: $\mathbb{F} = \mathbb{C}$, k is algebraically closed of characteristic 0.
(3) stack of restricted local system $\mathrm{Loc}^{\text{ét}}_{\hat{G}}$.

16.2. Representation schemes.

Let Γ be a grp. M an affine group scheme over a ring k. We can define teh representation stack $\mathcal{R}_{\Gamma,M}$ i.e. group homomorphisms $\Gamma \to M(A)$.

Proposition 16.2. *Assumptions: k is Noetherian.*

- *M sm. affine group scheme of dim d.*
- *Γ fg. of time $FP_\infty(k)$. Meaning, there exists a resolution of k $P^\bullet \to k$ by finite projective modules $k[\Gamma]$,*

16.3. Betti moduli stack.

If $C \in \mathrm{SmProj}_{\mathbb{C}}$, then for $c_0 \in C$, we have a presentation

Proposition 16.3. *Let C be smooth orientable genus g curve,*

$$\pi_1(C, c_0) \simeq \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \;:, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

Hence, this impose some compactness conditions on $\pi_1(C)$ - being finitely presented.

Example 16.4. Fundamental group of $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ two torus. here.

We define the derived fiber product

$$e \longrightarrow \mathrm{GL}_n$$

Example 16.5. $C = \mathbb{CP}^n$. Then we have that

$$\pi_k \mathbb{CP}^n = \begin{cases} * & k = 0 \\ 1 & k = 2 \end{cases}$$

Indeed we can use the Hopf fibration, with $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$, inducing

$$\begin{tikzcd} U(1) \simeq S^1 \arrow[r] & S^{2n+1} \arrow[d, two heads] \\ & \mathbb{CP}^n \end{tikzcd}$$

for all $n \geq 1$. THe details of computations are here. Thus we have

$$\mathbb{Z} \longrightarrow \pi_1(S^{2n+1}) \longrightarrow \pi_1(\mathbb{CP}^n) \longrightarrow 0$$

16.4. **de Rham moduli stack.**

Definition 16.6. Let $G \in \mathrm{AffGrp}_{\mathbb{F}}$. We set

$$\mathrm{LS}_G^{\mathrm{dR}}(S) := \mathrm{Map}(\Sigma_{\mathrm{dR}} \times S, BG) \simeq$$

16.5. **étale case.** This is apparently referred as the étale case. Our coefficient is $e = \bar{\mathbb{Q}}_l$. Suppose X has genus $g > 0$. Naı̈vely this should copy the definition of $\mathrm{LS}_G^{\mathrm{Betti}}$.

Example 16.7. Continuous homomorphism $\mathrm{Map}_{\mathrm{GrpCts}}(\widehat{\mathbb{Z}}^{2g}, e^{\times}) \simeq (\mathcal{O}_e^{\times})^{2g}$.

In general, if some one wish to find a scheme over e whose e points $\mathcal{O}_E^{\times}$ it is slightly hard.

17. Scholze's L-parameter stack

References. Lecture 12-02-21.

There are two sides to Langland's correspondence.

(1) Automoprhic side: $D(G(E), \mathbb{Z}_l)$, the cat of sm. $G(E)$-repns. This embeds ff.

$$D(G(E), \mathbb{Z}_l) \hookrightarrow D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_l)$$

This is a variant of $D_{\text{ét}}$ that works for $\mathbb{Z}_l$-algebra Λ, uses solid 6-functor formalism

On the Galois side, we have the Artin stack of L-parameters.

$$Z^1(W_E, \widehat{G})/\widehat{G}$$

of L-parameters.

17.1. What is classically done:

$$\text{irreducible object} \mapsto \text{point}$$
$$\pi \mapsto \varphi_\pi$$

but this should vary algebraically.

17.2.

Definition 17.3. The Bernstien center of G is the *commutative* algebra of endomorphisms of the identity functor on $\mathrm{Rep}^{\mathrm{sm}}(G(E))$.

- For each π, we give

$$f(\pi) : \pi \to \pi$$

In particular, if $f \in Z(G)$, $\pi \in \mathrm{Irr}_{\bar{\mathbb{Q}}_l}(G)$. Schur's is true in this setting, $\mathrm{End}(\pi) \simeq \bar{\mathbb{Q}}_l$.

we get scalar $f(\pi) \in \bar{\mathbb{Q}}_l$. So we get a function

$$Z(G)_{\bar{\mathbb{Q}}_l} \to \left\{\text{functions on } \mathrm{Irr}_{\bar{\mathbb{Q}}_l}(G)\right\}$$

In some sense: this should be thought as "the algebraic functions on the set $\mathrm{Irr}_{\bar{\mathbb{Q}}_l}(G)$.

- Bernstein center is not quite of finnite type.

17.4. We want: for any $f \in \mathcal{O}(Z^1(W_E, \widehat{G})^{\widehat{G}}$ the map

$$\pi \mapsto f(\varphi_\pi)$$

Definition 17.5. The *Spectral Bernstein center* is

$$Z^{\text{spec}}(G) := \mathcal{O}(Z^1(W_E, \widehat{G})^{\widehat{G}}$$

also consider

$$Z^{\text{geom}}(G) = \text{"bernstein center} D_{lis}(\text{Bun}_G, \mathbb{Z}_l) = \text{End}(\text{id}) \to Z(G)$$

Theorem 17.6 (Fargues-S)**.** *There exists a canonical map*

$$\psi : Z^{\text{Spec}}(G) \to Z^{geom}(G)/\mathbb{Z}_l$$

17.7. What does this mean concretely? For each $A \in D_{\text{lisse}}(\text{Bun}_G, L)$, $L/\mathbb{Z}_l$ ac. closed filed $\text{End}(A) = L$. There exists unique upto conjguation L-parameter

$$\varphi_A : W_E \to \widehat{G}(L)$$

ss. st. for all $f \in Z^{\text{Spec}}(G)$.

$$f(\varphi_A) = \psi(f)(A) \in L$$

...[36:44]

17.1. Construction of $\psi : Z^{\textbf{spec}}(G) \to Z^{\textbf{geom}}(G)$.

17.8. We have the following, for any ∞-cat. $C := D_{lis}(\text{Bun}_G, \mathbb{Z}_l)$, for any set I an exact monoidal functor

$$W_E \twoheadrightarrow Q \circlearrowleft \widehat{G}$$

$$\text{Rep}_{\mathbb{Z}_l}(\widehat{G} \rtimes Q)^I \to \text{End}(\mathcal{C})^{W_E^I}$$

- I.e. when ever you have a representation, you can always build an action to C, the "Hecke action", which W_E^I. This is linear over $\text{Rep}_{\mathbb{Z}_l}(Q^I)$, functorially in I.
- This comes from the Hecke action.

We will only need this abstract data. This is also the same kind of formal structure you get in Betti geometric Langlands.

17.9.

Proposition 17.10. *For any $P \in \mathcal{C}^\omega$, exists $p \hookrightarrow W_E$ open in wild inertia, st. for all I $V \in \text{Rep}(\widehat{G} \rtimes Q)^I$, the W_E^I action on $T_V(A)$ factors over W_E/P^I.*

This basically means we can replace W_E by W_E/P. Then, as last time, by discretization $W \hookrightarrow W_E/P$.

17.11. Last time: we can compute invariant functions

Theorem 17.12.

$$\varinjlim_{n, F_n \to W} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \xrightarrow{\simeq} \mathcal{O}(Z^1(W, \widehat{G})^{\widehat{G}} \dashrightarrow Z^{geom}(G) = \text{End}(id_G)$$

The theorem tells us that to establish our goal $\dashrightarrow$ it is suffiicent to construct the map from the colimit.

Definition 17.13. An <u>excursion datum</u> is a tuple $(I, V \in \mathrm{Rep}(\widehat{G} \rtimes Q)^I, \alpha : 1 \to V|_{\Delta\widehat{G}}, \beta : V|_{\Delta\widehat{G}} \to 1, (\gamma_i \in W)_{i\in I}$.

Given excursion data, the excursion operators is the following element of $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$. For any $A \in \mathcal{C}$.

$$A = T_1(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_1(A) = A$$

Note: $T_V(A)$ has *a lot of endomorphisms.* A priori A does not. This is because we have this equivariance result. There's a bit of translation to do here, but

Proposition 17.14. *The excursion operators define a map*

$$colim_{n,F_n\to W}\mathcal{O}(\widehat{G}^n)^{\widehat{G}} \to \mathrm{End}(id_{\mathcal{C}})$$

17.15. Corollary: the L-parameters are characterized as follows: for all excursion data: the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{\varphi_A(\gamma_i)} V \xrightarrow{\beta} L$$

Agrees with the scalar - Here we assume Schur irreducibility.

$$A \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} A$$

18. Spectral Action

18.1.

Theorem 18.2 (Nadler Yun, GKRV)**.** *The data above is equivalent to an actin of*

$$Perf(Z^1(W_E, \check{G}/\check{G}) \circlearrowleft D_{lis}(Bun, \mathbb{Z}_l)$$

The previous works over $\mathbb{Q}_l$.

18.3. Let us assume you have no Q. One has a map

$$\begin{array}{ccc} \mathrm{Rep}(\widehat{G} \rtimes Q)^I & \longrightarrow & \mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})^{W_E^I} \\ & & \downarrow \\ & & \mathrm{End}(\mathcal{C})^{W_E^I} \end{array}$$

18.4. What does this mean for "elliptic L-parameters? Assume for simplicity G ss. coefficient $\bar{\mathbb{Q}}_l$. We say that φ is elliptic if it defines an isolated component of $Z^1(W_E, \widehat{G})/\widehat{G}$. [1:04:00]

24th June 2021 04-06-21

This is joint work with David Hansen, for a finite type map $f : X \to S$. There is a good notion of *pervisity over S*

Definition 18.5. $A \in D_{\text{ét}}(X, \Lambda)$ is *perverse over* S iff for all geometric fibers $X_{\bar{s}}$, $\bar{s} \to S$ a geometric point $A\Big|_{X_{\bar{s}}} \in D_{\text{ét}}(X_{\bar{s}}, \Lambda)$ is perverse.

This interacts very will *ULA* sheaves[18]. That is

[18] For coherent sheaves there's a notion of flat family. This is what ULA sheaves roughly corresponds to.

Proposition 18.6. *$A \in D_{ét}(X, \Lambda)$ is ULA, iff ${}^{p/S}\mathcal{H}^i(A)$ is ULA and $\mathbb{D}_{X/S}(A)$ is ULA.*

Proposition 18.7. *A is ULA + perverse over S implies all subquotients of A are ULA and $\mathbb{D}_{X/S}(A)$ is ULA and perverse over S.*

All the proofs are based on two key ingredients:

(1) v-descent: this allows us to reduce all question to case when $S = \operatorname{Spec} V$, where V is a valuation ring with fraction field K, algebraically closed. [19]

(2) theory of nearby cycles

[19] These are pretty big rings with very nice properties.

19. Fargues Fontaine curve, I

References: [L2, S], [Zhang1]
Set up:

- Fix some narc local field E, res. field $\mathbb{F}_q$ char $\varpi \in \mathcal{O}_E$.

Example 19.1. Let us first recall the perspective of Zariski site. $\operatorname{Spec} E_{\text{ét}} \simeq \text{pt}$. The étale site

$$(\operatorname{Spec} E)_{\text{ét}} = \{\text{fintie sep. } E\text{-alg}\}^{\text{op}} = B\Gamma_E = \{\text{finite set with cts } \Gamma_E \text{ act}\}$$

Note that we have

$$0 \to I_E \to \Gamma_E \twoheadrightarrow \Gamma_{\mathbb{F}_q} \to 1$$

One can say a little more about the inertia:, which we recall in

$$0 \to P_E \to I_E \to \prod_{l \neq p} \mathbb{Z}_l =: \widehat{\mathbb{Z}}^p \to 0$$

More or less using this input, Tate proved Local Tate duality : For all torsion $M \in \operatorname{Rep}^{\text{tors}}(\Gamma_E)$ There is a cup product pairing[20]

$$H^i_{\text{ét}}(\operatorname{Spec} E, M) \otimes H^{2-i}_{\text{ét}}(\operatorname{Spec}, M^*(1)) \xrightarrow{\cup} H^2_{\text{ét}}(\operatorname{Spec} E, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z}$$

- $M^* \simeq \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z})$.
- $\mathbb{Q}/\mathbb{Z}(1) :=$ Tate Twiste $= \bigcup_n \mu_n$.

This looks like Poincaré duality on compact Riemann surface. Except, that there is a twist here, accounted for orientation. Motivated from above: we want to turn E closer to a compact Riemann surface.

It would be useful to consider the case in equal characteristic. In this case $\breve{E} = W(\bar{\mathbb{F}}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p((t))$, where we define the Definition 20.1.

Definition 19.2. $\mathbb{D} := \varprojlim_{K/E, z \mapsto z^{|K:E|}} \mathbb{G}_m$ over E. $X^*(\mathbb{D}) \simeq \varinjlim_{K/E} \frac{1}{|K:E|}\mathbb{Z} \xrightarrow{\simeq} \mathbb{Q}$.

where K is taken over all finite exntesion. We have a map

$$\hat{\mathbb{Z}} \simeq H^2_{\text{fppf}}\left(E, \varprojlim_{K/E} \mu_{[K:E]}\right)$$

Proof. Let us explain some parts of the map: the left hand side isomorphism. We have exact sequence for each finite exntesion $\mu \to \mathbb{G}_m \to \mathbb{G}_m \to \mathbb{Q}/\mathbb{Z} \xrightarrow{|K:E|} \mathbb{Q}/\mathbb{Z}$. □

[20] Again, the last isomorphism is true away from p if $E/\mathbb{F}_q$. is of char p.

20. Equal characteristic Fargues Fontaine Curve

References: [Far16], [L2,S]

We fix $C/\bar{\mathbb{F}}_q$ a complete algebraically closed narc field. [21] There is well defined notion of base change,

$$\operatorname{Spa}(C, \mathcal{O}_C) \times_{\operatorname{Spa}\mathbb{F}_q} \operatorname{Spa}\mathbb{F}_q((t)) \simeq \mathbb{D}_C^{\times} \to \operatorname{Spa}(C, \mathcal{O}_C) \times_{\operatorname{Spa}\mathbb{F}_q} \operatorname{Spa}\mathbb{F}_q[[t]] \simeq \operatorname{Spa} C[[t]] \simeq \mathbb{D}_C$$

here Spa is the adic spectrum. $\mathbb{D}_C^{\times}$ is the puncture open unit disc. over C. This has many points in contrary to the previous spaces we looked at. The same construction applies if we take C to be a perfectoid field. In this case, we would want to add in all pth power root of unity.

Definition 20.1. The equal characteristic Fargues Fontine curve for local field E, Then we can define a family of curves

$$\begin{gathered} \mathrm{Pftd}_{\mathbb{F}_p} \to \mathrm{AdicStack}_E \\ C \mapsto X_{C,E} := \mathbb{D}_C^{\times}/(\mathrm{id} \times \phi_C)^{\mathbb{Z}} \to \operatorname{Spa} E \end{gathered}$$

$\mathbb{D}_C^{\times}$ is a classical object. However the quotient is not. We can think of $\mathbb{D}_C^{\times} = \operatorname{Spa}\mathcal{O}_C[[t]]\backslash V(\varpi, t)$, where ϖ is uniformizer of $\mathcal{O}_C$. So the punctured union of annulis,

$$\bigcup_{m,n} \mathbb{D}_{m,n,C}, \quad \mathbb{D}_{m,n,C} := \operatorname{Spa}\left(\widehat{\mathcal{O}_C[[t]]\left[\frac{\varpi^m}{t}, \frac{t^n}{\varpi}\right]}^{\varpi}[\frac{1}{\varpi}], \overline{R^+}_{m,n}\right)$$

where $R_{m,n}$ is the ring of integral elements (before inverting ϖ. Here we also have that t is invertible. We have the frobenius given by

$$\sum_{i\in\mathbb{Z}} a_i t^i \mapsto \sum_i a_i^p t^i$$

where $a_i \in \mathcal{O}_C$.

Generally, fiber products do not exists in the category of adic spaces.

In what sense is this a geometrization?

Proposition 20.2. *Setting as above,*

- $H^0(X_{C,E}, \mathcal{O}_{X_{C,E}}) = E$.
- *fét*$(X_{C,E}) = \operatorname{Spec} E_{\acute{e}t}$
- $H^i_{\acute{e}t}(X_{C,E}, M) \simeq H^i_{\acute{e}t}(\operatorname{Spec} E, M)$

Note however there is always a *choice of* C.

21. First Informal discussion session

21.1. Random stuff. Begins [7:37] So what the FF. curve as X is scheme, Dedekind,[22] and is over $\operatorname{Spec}\mathbb{Q}_p$. This is not locally of finite type, because.

$$H^0(X, \mathcal{O}) = \mathbb{Q}_p$$

but residue fields at closed points fininte extnesions of C, i.e. $\mathbb{C}_q = \widehat{\overline{\mathbb{Q}_p}}$.

[21]E.g. $C := \widehat{\overline{\mathbb{F}_q((t))}} = \mathbb{F}_q(\widehat{(t^{1/p^{\infty}}}))$. This is some how the minimal algebraically closed choice.

[22]This justifies the name that it is a curve.

One can naively: look at

$$\Omega^*_X/\mathbb{Q}$$

which is horrible. There is no duality

$$H^i(X,\mathcal{F}). \times H^{n-i}(X, F^\vee \otimes \mathbb{Q}) \to \mathbb{Q}$$

how does one see that there is no cohomology. Later: usually these cohomology spaces $\dim_{\mathbb{Q}_p} H^i(X,F) = \infty$.

Fact $H^1(X,\mathcal{O}) = 0$, shows that X "behaves" like $\mathbb{P}^1$.

Pick $x \in X$ a closed point. We have

$$0 \to \mathcal{O} \to \mathcal{O}(x) \to i_*\kappa(x) \to 0$$

By taking H^1, we have ses

$$0 \to \mathbb{Q}_p \to H^1 \to \mathbb{C}_p \to 0$$

by this picture this shows that we cannot ...[15:00]

21.1. **[15:00] Explaination of.**

21.2. **Recap of reductive groups [52:45].**

21.2. Let k be a field, k alg. closed.

22. The Fargues Fontaine Curve, II

On Monday: I talked about the adic space a little. Last time we focused on equal char. $E = \mathbb{F}_q((t))$. We consider

$$\operatorname{Spa} E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C := \mathbb{D}_C^*$$

punctured open unit disk. There are two frobenius: ϕ_C. We defined the *Fargues Fontaine curve* as $X_{C,E} = \mathbb{D}_C^\times / \phi_C^{\mathbb{Z}}$. Today:

(1) In what sense is this a curve?

(2) What are classical points?

22.1. **Classical points.** There is an equivalence of categories

$$\{\text{Rigid analytic varieties over } C\} \simeq \{\text{adic spaces localy of finite type over } \operatorname{Spa} C\}$$

We have the following:

$$X(C) \hookrightarrow |X| \hookleftarrow X$$

The LHS is a category defined by Tate., are the "classical poitns". Locally $X = \mathrm{Sp}A$, $A = C\langle T_1, \ldots, T_n \rangle / I$, then

$$X \mapsto \mathrm{Sp}A$$

$$|\mathrm{Sp}A| = X(C) = \mathrm{Spm}A = \{(x_1, \ldots, x_n) \in C^n \,:\, |x_i| \leq 1 \text{ for all } f \in I f(x) = 0\}$$

Theorem 22.1. *$(SpA)_{rig}$ defined a Grothendieck topology on SpA. This is iso. to* $\operatorname{Spa} A$.

- *q c adm. open = qc open of* $\operatorname{Spa} A$.
- *adm covers = covers.*

Tate: found a round about way - there is a reconnstruction theorem for spectral topological spaces. In particular, this tells you that qc cpt open subsets of adic specturm is completely determiend by theri clasasical points.

22.2. For $\mathbb{D}_C^*$ the classical points are

$$\{x \in C \,:\, 0 < |x| < 1\}$$

We have the Frobenius acting on this:

$$x \mapsto x^{1/q}$$

22.3. There is something hapenning

- It

For any cn. affiinoid $\operatorname{Spa} A \hookrightarrow \mathbb{D}_C^\times$, A is principal ideal domain. [23] By descent can also define classical points of $X_{C,E} \supset \{0 < |x| < 1\} / \phi = X_{C,E}^{\mathrm{cl}}$. Again, any cn. affonid subset of $X_{C,E}$ is the Spa of principal ideal domain.

22.4. Now let us addpat this to the mixed char. case. *Question: what is* "$\operatorname{Spa} E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C$*?* Problem is there is just no map.

[23] Just by hand one can enumrate alll the points. ar egen. by $t - x$

Idea:: what we did in char p. We deformed. any $\mathbb{F}_q$ algebra R to $\mathbb{F}_q[[t]]$ by taking $R[[t]]$. Note: if R is a *perfect*. $\mathbb{F}_q$-algebra, there is a unique (upto unique iso) lift $\tilde{R}/\mathcal{O}_E$. that is flat π-adically complete with red. mod π

$$\tilde{R}/\pi = R$$

There are several ways to prove this. One chosie, using p-typical Witt vectors, is $\tilde{R} := W(R). \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$., this is the "ramified Witt vectors" $W_{O_E}(R)$.

The idea behind this construct: ... [19:40]

- the choice becomes more and more irrelevant...

There is a Tecihmüller map.

$$[\cdot] : R \to \tilde{R}$$

$$x \mapsto \lim_{n\to\infty} \tilde{x}_n^{p^n}$$

Let $\tilde{x}_n \in \tilde{R}$ any lift of x^{1/p^n}. This is ultiplicative, but not additive. [24]

Any element of $\tilde{R}$ admits a unique expression as $\sum_{n\geq 0} \pi^n[r_n]$. , $r_n \in R$. It's a subtle to compute in this ring, however the recipe would tell you how to do it.

22.5. Q: What is the topology on R? [23:07]...

22.6. The analogue of

$$\operatorname{Spa} \mathbb{F}_q[[t]] \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} O_C[[t]]$$

in MC is

$$\text{"}\operatorname{Spa} \mathcal{O}_E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathcal{O}_C\text{"} = \operatorname{Spa} W_{O_E}(O_C)$$

the analogue of

$$\operatorname{Spa} \mathbb{F}_q((t) \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C = \mathbb{D}_C^{\times}.$$

is.

$$\text{"}\operatorname{Spa} E \times_{\operatorname{Spa} .\mathbb{F}_q} \operatorname{Spa} C\text{"} := Y_{C,E} = \{\pi \neq 0[\pi] \neq 0\} \hookrightarrow \operatorname{Spa} W_{O_E}(O_C)$$

The lhs doesn't carry an action .... $Y_{C,E}$ still carries an action ϕ_C.

Definition 22.7. The <u>FF curve is</u> $X_{C,E} = Y_{C,E}/\phi^{\mathbb{Z}}$. ...[28:45]

Many of the structural features above

- Classical points.
- principal domainn.

actually extends.

Theorem 22.8. *FF, Kedlaya. There is a notion of classical points.* $Y_{C,E}^{cl} \hookrightarrow Y_{C,E}$ *st. for any cn. affinnoid subset* $\operatorname{Spa} A \hookrightarrow Y_{C,E}$ A *is a prinncipal ideal domain, and (the maximal spectrum)*

$$SpmA \xrightarrow{\simeq} \operatorname{Spa} A \cap Y_{C,E}^{cl} \hookrightarrow Y_{C,E}$$

and

[24]And it cannot because the former is char p, and the latter is char 0.

(1) for any classical point $y \in Y^{cl}_{C,E}$*, there exists one* $x \in C$*,* $0 < |x| < 1$ *st.*

$$y = V(\pi - [x])$$

Beware: this element x *is not unique.*

- *In equal characteristic...*
- *This says that in mixed characteristic this is almost the same way. The unfortunate thing is that this is not always unique.*

(2) For any classical point $y \in Y^{cl}_{C,E}$*, the cplt. residue field at* y *is a cplt. alg. closed narc field, w/ a distinguished iso.*

$$C(y)^\flat \xrightarrow{\simeq} C$$

This gives a bijection

$$Y^{cl}_{C,E} = \textit{Untilts } C^{\#}/E \textit{ of } C$$

22.2. **Tilting.**

22.9. For a cplt. alg. closed narc field.[25] field K, st. $|p|_K < 1$ onne can define a cplt. alg closed field of char p.

$$K^\flat := \varprojlim_{x \mapsto x^p} K$$

as a *topological multiplicative monoids.* The issue:

- p-power mpa is not a ringn map, so the limit is a priori not clear to be a rnig.

Resolutoin:

$$O_{K^\flat} = \varprojlim O_K \xrightarrow{\simeq} \varprojlim O_K/p$$

Now rhs: are *actuallly ring maps.* Now you actually get a norm.

So tilting: takes char 0 stuff to char p.

22.3. **Sketch of theorem.**

22.10. We will go the other way: define the set of classical points as the set of untilts.

22.11. Do we need K to be *algebriaaclaly clsoed* for definition of $K^\flat$? A: no, not exactly, we just want the transition maps to be surjective. ... [45:35]

[25]Normally called C.

23. 2nd informal disccussion

11-11-20 Let C be an algebriacallaly closed narc field. Then we have a valuation $| \quad | : C \to \mathbb{R}_{\geq 0}, \mathcal{O}_C := \{x \in C \,:\, |x| \leq 1\}$. Then

$$\mathbb{B}_C := \mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C(T)) = \left(\sum x_i T^I \,:\, x_i \in C, |x_i| \to 0, i \to \infty\right)$$

Recall that $\mathbb{B}_C$ is cts valuations : [26]

$$\{C\langle T\rangle \to \Gamma \cup \{0\} \,:\, v(\mathcal{O}_C\langle T\rangle) \leq 1\} / \sim$$

We will assume the rank 1 case. $\Gamma = \mathbb{R}_{\geq 0}$. We will always assume that the valuation extends that of $| \quad |_C$. In the lecture there was the Gauss point

$$v_{0,1} : C\langle T\rangle \to \mathbb{R}_{\geq 0} \quad f = \sum x_i T^i \mapsto \max\{|x_i|\}$$

More generally, let $c \in \mathcal{O}_C$, and for each $r \in [0,1]$, the function

$$v_{c,r} : C\langle t\rangle \to \mathbb{R}_{\geq 0}, \quad f = \sum x_i (T-c)^i \mapsto \max\left\{|x_i|, r^i\right\}$$

In otherwords,

$$v_{c,r}(f) := \sup(f(x) \,:\, x \in B(c,r)) \quad B(c,r) = \{y \in \mathcal{O}_C \,:\, |y-x| \leq r\}$$

Discuss a picture:

- We need to check when $v_{c,r} = v_{c',r'}$. This happens if and only if $r = r'$ nad that $|c - c'| \leq r$.

[26]The valuation is taking up to some equivalence relation. Γ is an ordered abelian group.

24. The Fargues Fontaine III: Isocrystals

Date(dmy): 09-11-22 Referenced in ??.

- Classification of isocrystals.

Let E be narc local field.

- $\pi \in O_E$ uniformizer. residue field $\mathbb{F}_q$.
- $C/\mathbb{F}_q$ algebraically closed nonarchimedian field[27]Associate to this we defined the Fargues-Fontaine curve.
$$X_{C,E} = Y^{\text{cl}}_{C,E}/\phi^{\mathbb{Z}}$$
with abbreviation X_C, X. [28]
- $Y_{C,E} \hookrightarrow_{\text{open}} \operatorname{Spa} W_{O_E}(O_C)$. [29] It is defined as the space where
$$\pi \neq 0, [\varpi] \neq 0$$
where $\varpi \in C$ is psu.
- $X^{\text{cl}}_{C,E} := \left\{\text{untilts } C^{\#}/E \text{ of } C\right\}/\phi^{\mathbb{Z}} \hookrightarrow |X_{C,E}|$.
- Any cn. affonid open subset $\operatorname{Spa} A \hookrightarrow X_{C,E}$ has A PID. and
$$\text{Spm}A = X^{\text{cl}}_{C,E} \cap |\operatorname{Spa} A| \hookrightarrow |X_{C,E}|$$
- For any classical point $y \in Y^{\text{cl}}_{C,E}$ exsits $t \in C$, with $0 < |t|, 1$ with $y = V(\pi - [t])$.

The most important theorem is the classification of vector bundles. There are various refinent of the local Langlands correspondence. Here we wish to introduce the *Kottwitz set*

Definition 24.1. Let $\breve{E}$ be completion of maximal unramified extension. define an equivalence relation on $G(\breve{E})$ as follows $b \sim b'$ iff $b' = gb\sigma(g)^{-1}$ for some $g \in G(\breve{E})$. We define
$$B(G) := G(\breve{E})/\sim$$

To better understand the the Kottwitz set, it is understood by the Kottwitz and Newton map [MS22]. In that we have the following diagram
$$B(G) \xrightarrow{\nu} \text{Hom}_{\breve{E}}(\mathbb{D}, G)/G(\breve{E})^{\sigma}$$

Example 24.2. vjays notes. There is a natural map
$$B(-) \to X^*(\widehat{T})^{\Gamma}$$
The natural map would be
$$b \mapsto (\lambda \mapsto v(\lambda(b)))$$
for Tori. Note that $\breve{G}$ naturally has both action of Weyl group, and $\Gamma_{\bar{E}/E}$.

[27] e.g. $\widehat{\overline{\mathbb{F}_q}}$.
[28] Often we will fix E, but we will allow C to be varying.
[29] The latter is flat deformation of O_C to O_E.

Example 24.3. Consider if $G = \mathrm{GL}_1$, $E = \breve{\mathbb{Q}}_p$. Take uniformizer is it the case that

$$\breve{E}^\times = \langle \pi \rangle^{\mathbb{Z}} \times \mathcal{O}_{\breve{E}}^\times$$

what induces the decomposition? Note that the fixed points of $\varphi \circlearrowright W(k)$ is precisely $W(\mathbb{F}_p)$ for any finite exntesion $k/\mathbb{F}_p$. Indeed we have the following diagram

$$\begin{tikzcd} k^{\varphi} \arrow[r] \arrow[d] & k \arrow[d, "\mathrm{id}"] \\ k \arrow[r, "\varphi : x \mapsto x^p"] & k \end{tikzcd}$$

and that Witt vector is a right adjoint functor.

Definition 24.4. An F *isocrystal* is a pair (V, ϕ_V) where V is finite dimensional $\breve{E}$ vector space. Let $\breve{E}/E$[30] be completion of maximal unrmaified extension. Note that $W_{\mathcal{O}_E}(\bar{\mathbb{F}})$ naturally has a a Frobenius $\phi_{\breve{E}}$ induced from the Frobenius of $\bar{\mathbb{F}}$.

- Structure of $\phi_V : V \xrightarrow{\simeq} V$ is a $\phi_{\breve{E}}$-lienar automorphism.

This becomes a E-linear $\otimes$ category, we denote as Isoc_E. [31]

Example 24.5. (1) Unit $(\breve{E}, \phi_{\breve{E}})$.

(2) 1 dimensional object, $(\breve{E}, b\phi_{\breve{E}})$ for some $b \in \breve{E}^\times$. any such is isomorphic to $\breve{E}_n := (\breve{E}, \pi^n \phi_{\breve{E}})$ for a unique integer $n \in \mathbb{Z}$. Change of basis replaces b by an inverse $a^{-1} b\phi(a) = a^{-1}\phi(a) b$. where $a \in \breve{E}^\times$. The first term lies in $\mathcal{O}_{\breve{E}}^\times$, can be any possible element. [32] Also, note that

$$\mathrm{Map}_{\mathrm{FIso}_k}((\breve{E}, \pi^n \phi_{\breve{E}}), (\breve{E}, \pi^m \phi_{\breve{E}})) = \breve{E}^{\phi = \pi^{m-n}} = \begin{cases} E & m = n \\ 0 & m \neq n \end{cases}$$

Thus the m would be the slopes. In this this says that $\mathrm{End}_{\mathrm{FIso}_k}(\breve{E}_n) \simeq E$.

Example 24.6. If $\lambda = s/r \in \mathbb{Q}$ where we use the unique representation $(s, r) = 1$, $r > 0$. Then

$$(V_\lambda, \phi_{V_\lambda}) = \left(\breve{E}^r, A_{s,r}\phi_{\breve{E}}\right)$$

where A is the matrix of the following form

$$\begin{pmatrix} 0 & 1 & \cdots & & & \\ & 0 & 1 & \cdots & & \\ & 0 & 0 & 1 & & \\ \vdots & & 0 & 0 & \ddots & \\ 0 & \cdots & & & & 1 \\ \pi^s & & & & 0 & 0 \end{pmatrix}$$

where we have an off-diagonal of 1s. Explicitly, this also means that

$$\phi_{V_\lambda}(x_1, \ldots, x_r) = \left(\phi_{\breve{E}}(x_2), \ldots, \phi_{\breve{E}}(x_r), \varpi^s \phi_{\breve{E}}(x_1)\right)$$

[30] E.g. $W_{\mathcal{O}_E}(\bar{\mathbb{F}}_q)[1/\pi]$

[31] Since we take the Frobenius fix points.

[32] This follows from Lang's lemma, from solving polynomial equations.

This also has a universal property.

$$\mathrm{Hom}_{\mathrm{FIso}_k}(V_\lambda, W)$$

where k is residue of $\breve{E}$ we have chosen V_λ to have basis $\{e_1, \ldots, e_r\}$ so that the the matrix for ϕ_{V_λ} is given as $A_{r,s}$. Indeed we know that

$$\phi^r(e_r) = \phi^{r-1}(\phi(e_r)) = \phi^{r-1}(e_{r-1}) = \phi(e_1) = \pi^s e_r$$

Thus a morphism $f : V_\lambda \to W$, is determined by $f(e_r)$ and must be an element in $W^{\phi^r = \varpi^s}$.

Theorem 24.7 (Dieudonné Manin)**.**

$$Isoc_E \simeq \bigoplus_{\lambda \in \mathbb{Q}} Isoc_E^\lambda$$

where $Isoc_E^\lambda$ where is the isoclinic of slope λ. Where

$$Isoc_E^\lambda := \text{"}\, Vect_E^{\textit{fin. dim}} \otimes V_\lambda \text{"}$$

25. Informal Discusions

Date(d.m.y): 09-12-20. Today:

(1) Divisor.

(2) Ex. for non-representability.

Let us fix some geometric point $S = \mathrm{Spa}(C, \mathcal{O}_C)$. Then on $\mathcal{Y}_{S,E}$, one can define:

$$\mathrm{Div}^1_{\mathcal{Y}} := \left\{ \sum a_y y \text{locally finite sums } a_y \in \mathbb{Z},\, y \in \mathcal{Y}^{\mathrm{cl}}_{S,E} \right\}$$

why does locally finite mean? If $U \hookrightarrow_{\mathrm{qcpt}} Y$, then $\{y \in U^{\mathrm{cl}} \,:\, a_y \neq 0\}$ is a finite set. This works on $X_{S,E} = \mathcal{Y}_{S,E}/\phi^{\mathbb{Z}}$. Then

$$\mathrm{Div}^1_X := \left\{ \sum a_x x \text{finite sum } a_x \in \mathbb{Z} \quad x \in X^{\mathrm{cl}} \right\}$$

As divisors are meant to capture things of codimension 1, we restrict to the classical point[33] The classical points of Fargues Fontaine curve also relly works when we fix a base. One can define

$$\deg : \mathrm{Div}^1_X - c \to \mathbb{Z}$$
$$\deg(x) = 1$$

A crucial statement, is that we want

$$\mathrm{Div}^1_{X,E} \to \mathrm{Pic}(X_{S,E})$$

this is the business

[33] other points are "open".

26. Untilts $\mathcal{O}(1)$, Lubin Tate Theory

Date(d.m.y) 20-11-20 Let E be narc field. Let us recall where we were, choose $\pi \in \mathcal{O}_E$, $\mathbb{F}_q \subset \overline{\mathbb{F}_q}$. Let $\breve{E} := W_{\mathcal{O}_E}(\bar{\mathbb{F}}_q)[1/\pi]$ completion of maximal unramified extension. Let $S \in \mathrm{Pftd}_{\mathbb{F}_q}$ a perfectoid space. Then we define

$$\begin{array}{c} \mathcal{Y}_{S,E} \\ \big\downarrow \\ X_{S,E} := Y_{S,E}/\phi_S^{\mathbb{Z}} \end{array}$$

Then we had $\mathcal{Y}_{S,E}^{\diamondsuit} = S \times (\mathrm{Spa}\, E)^{\diamondsuit}$. Then we have

$$\text{"unitls} = \text{degree 1 Cartier divisor on } Y_{S,E}\text{"}$$

Given an until $S^{\#}/E$ of S, locally

$$S = \mathrm{Spa}(R, R^+), S^{\#} = \mathrm{Spa}(R^{\#}, R^{\#+})$$

and there is a canonical surjection

$$\theta : W_{\mathcal{O}_E}(R^+) \to R^{\#+}$$

By rigidity, this map is determined by a map $R^+ \to R^{\#+}/\pi$. This will then be the induced map $R^+ = \varprojlim R^{\#+}/\pi \to R^{\#+}/\pi$. We have

$$\ker \theta = (\xi)$$

where ξ is a nonzero divisor in $W_{\mathcal{O}_E}(R^+)$. In the modern language, this is saying: integral perfectoid ring is the same as the category of perfect prisms. As last time, this gives one a closed immersion

$$S^{\#} = V(\xi) \hookrightarrow \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{\pi = 0, [\varpi] = 0\}$$

Thus, this presents $S^{\#}$ as a Cartier divisor in $\mathcal{Y}_{S,E}$.

Definition 26.1. Let X be a uniform [34] an analytic[35] adic space. A *closed Cartier divisor* on X is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ that locally free of rank 1, such that for all affinoid $U \hookrightarrow X$, the map

$$\mathcal{I}(U) \to \mathcal{O}_X(U)$$

has a closed image. [36]

This really means that $Z = (V(\mathcal{I}), \mathcal{O}_X/I, \text{valuations})$ defines an adic space.[37]

Proposition 26.2. *[SW20].*

(1) $V(\xi) = S^{\#} \hookrightarrow Y_{S,E}$ is a closed Cartier divisor.

(2) $S^{\#} \hookrightarrow \mathcal{Y}_{S,E} \to X_{S,E}$ defines a closed Cartier divisor[38]

Definition 26.3. Let $\mathrm{Div}^1_Y, \mathrm{Div}^1_X : \mathrm{Pftd}_{\bar{\mathbb{F}}_q} \to \mathrm{Set}$ [39] be the functors taking [40]

$$S \mapsto \Big\{ \text{ closed carrier divisor on } \mathcal{Y}_{S,E} \text{ that locally arise as } S^{\#} \hookrightarrow \mathcal{Y}_{S,E} \Big\}$$

---

[34]The spectral norm is a norm i.e. i.e. $R^{\circ} \subset R$ is a bounded subset. Perfectoid things are always uniform.

[35]locally $\mathrm{Spa}(R, R^+)$ where R is Tate.

[36]It is really necessary here to check on all affinoids.

[37]which is another way to define a closed Cartier divisor.

[38]You have to show that they don't have identification after passing to quotient.

[39]In principle it is not really necessary to work over $\bar{\mathbb{F}}_q$.

[40]It is usually more practical to ask for *closed* Cartier divisor. In this case there is a cleaner correspondence.

by locally means locally on S. This is the moduli space of degree 1 Cartier divisors.

For many purposes it is convenient to work over the geometric closure so one doesn't see the arithmetic.

Proposition 26.4. *(1)* $Div^1_{\mathcal{Y}} = \left(\operatorname{Spa}\breve{E}\right)^{\Diamond}$.

(2) $Div^1_X = Div^1_{\mathcal{Y}}/\phi_E^{\mathbb{Z}} = \left(\operatorname{Spa}\breve{E}\right)^{\Diamond}/\phi_E^{\mathbb{Z}}$.

In 2 we observe that any functor defined on $\mathrm{Pftd}_{\mathbb{F}_q}$ acquires a frobenius. So $\operatorname{Spa} E^{\Diamond}$ admits a Frobenius action ϕ_E and

$$\operatorname{Spa}(\breve{E})^{\Diamond} = (\operatorname{Spa} E)^{\Diamond} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$$

This is in contrast to when previously when one take base change to an algebric closure.

Proof. By definition $\operatorname{Spa}\breve{E}^{\Diamond} \twoheadrightarrow \mathrm{Div}^1_{\mathcal{Y}}$ as any closed carrier divisor paratmerized by Div^1_Y locally comes from an until $S^{\#}/\breve{E}$. But conversely, a closed Cartier divisor on $Y_{S,E}$ determines $Z \subset Y_{S,E}$, locally on S, this gives an until $S^{\#}$ of S. □

[25:25] ...

Question A. What is $\operatorname{Spa}\breve{E}^{\Diamond}$? Maybe ponder about classical local CFT here.

Remark 26.5. there is something quite funny here: to construct the Fargues fontaine curve, we take quotient on the first factor.

$$X^{\Diamond}_{S,E} = S/\phi_S^{\mathbb{Z}} \times (\operatorname{Spa} E)^{\Diamond}$$

To construct we take quotient on the second factor.

$$\mathrm{Div}^1_X = (\operatorname{Spa}\breve{E})^{\Diamond}/\phi_E^{\mathbb{Z}}$$

Faruges referred this as this as the *mirror curve*. Note that topologically this is a point, but as this a quotient, this is not quasi-seperated, and not locally spatial. So this is something quite weird: your moduli space of points on your curve is not your curve!

26.1. **Lubin Tate Theory.** Recall that $\mathcal{O}_{X_{S,E}}(1)$ is the line bundle on $X_{S,E}$ corresponding to isocrystals $(\breve{E}, \pi^{-1})$. Then

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})^{\phi_S = \pi}$$

Our gaol is to see that if $S^{\#}$ any until $/E$ of S, $\mathcal{I}_{S^{\#}} \subset \mathcal{O}_{X_{S,E}}$ is (at least after proétale locally on S) isomorphic to $\mathcal{O}_{X_{S,E}}(-1)$. In this sense, $S^{\#} \hookrightarrow X_{S,E}$ is of degree 1. To do this I need to construct $\mathcal{O}(-1) \simeq I_{S^{\#}} \hookrightarrow \mathcal{O}$. This is equivalent to maps, this corresponds to maps

$$\mathcal{O} \to \mathcal{O}(1)$$

or a global section $H^0(X_{S,E}, \mathcal{O}(1))$. We will give a formula for $H^0(X_{S,E}, \mathcal{O}(1))$ in terms of Lubin Tate formal group.

Definition 26.6. A Lubin-Tate group is a 1-dimensonal formal group $G/\mathcal{O}_{\breve{E}}$ with an action of $\mathcal{O}_E$ such that the two induced action on Lie$G$ agree and of height 1.

Definition 26.7. Height 1: there's a multiplication endomorphism

$$[\pi]_G[x] = \pi X + a_2 X^2$$

If one looks at this $\pmod{\pi}$, then the first nonzero coordinate is necessarily

$$a_{q^h} x^{q^h}$$

where $h = 1, 2, \ldots, \infty$, with h the height of G. So height 1 is the most nondegenerate case.

Theorem 26.8. *Such a G is unique up to isomorphism.*

Example 26.9. $E = \mathbb{Q}_p$, G is the formal multiplicative group Spf$\breve{\mathbb{Z}}_p[[x]]$. In this case

$$X +_G Y = (1 + X)(1 + X) - 1$$

Proposition 26.10. $G \times_{\mathcal{O}_E} \breve{E} \simeq_{\log} \mathbb{G}_a$.

One can choose $G \simeq \mathrm{Spf}\mathcal{O}_{\breve{E}}[[X]]$ so that $\log(x) = x + \frac{1}{\pi}X^q + \frac{1}{\pi}X^{q^2}$.

27. Recollection on adic spaces

References: Bhatt. Roughly adic spaces are variants of schemes but associated with certain topological rings. (e.g. Banach algebras).

- In common with schemes: they have specializations of points.
- They are quite general. There are *no finiteness assumptoins.*

These are the first step to define perfectoid fields. These are inparticular perfectoid Tate ring R. A simple motivation can be observed as follows: that $(\mathbb{Z}_p, \mathbb{Q}_p)$ is somewhat similar to $(\mathbb{F}_p[[t]], \mathbb{F}_p((t)))$, in that both have residue field $\mathbb{F}_p$. This can be made precise if we add in sufficiently many pth power roots of unity.

Theorem 27.1 (Fontaine-Wintenberger)**.** *The absolute Galois group of* $L := \mathbb{Q}_p[p^{1/p^\infty}]$ *and* $L^\flat = \bigcup_n \mathbb{F}_p((t^{1/p^n}))$ *are canonically isomorphic.*

Note that $\Gamma_{\hat{L}} \simeq \Gamma_{\hat{L^\flat}}$, so that it does not hurt to consider, the p-adic (t-adic) completions of the fields.

Example 27.2. The following examples are typical

- $\mathbb{Q}_p^{\text{cycl}} := \widehat{\mathbb{Q}_p(\mu_{p^\infty}})$
- $\overline{\mathbb{F}_p[[t]]}$, [Ked99].

First of all, I have to specify the topological rings.

Definition 27.3. $A \in \text{TopRng}$.

(1) A is *adic* if there is some ideal $I \hookrightarrow A$ st. $\{I^n : n \geq 0\}$ is an nhd basis of 0. I is called an <u>ideal of definition</u> [41]

(2) A is <u>Huber</u> (f-adic in Huber's paper[42]) if exists open subring $A_0 \hookrightarrow A$ that si adic wrt fg. ideal of defn.

Remark 27.4. Any such A admits completion $\widehat{A}$ containing $\widehat{A}_0 \hookrightarrow \widehat{A}$ as open subring, where $\widehat{A}_0$ is jst I-adic completion of A_0. [43] As well see later

$$\text{Spa}(-,-) : \text{HuberPairs}^{\text{op}} \to \text{SpectralSpaces}$$

induces an equivalence

$$\text{Spa}(\widehat{A}, \widehat{A}) \xrightarrow{\simeq} \text{Spa}(A, A^+)$$

In practice, one thus takes completion.

Most important case:

Example 27.5. The final object $\text{Spa}\,\mathbb{Z} := \text{Spa}(\mathbb{Z}, \mathbb{Z})$.

[41] Not unique, but any two I, J exists n st, $I^n \hookrightarrow J, J^n \hookrightarrow I$. Thus, if you have one ideal of defn: aall other are precisely those which satisfies this property.

[42] But I find this unfortunate, as often you would have an element f

[43] Basically, all possible types of completion coincide.

Example 27.6. The open *unit disk*, $\mathbb{D} := \operatorname{Spa}\mathbb{Z}[[t]]$. Indeed, mapping into $\operatorname{Spa}(R, R^\circ)$, where R is I adic, implies $f : \mathbb{Z}[[t]] \to R$. need to complete this argument.

Definition 27.7. A is Tate if it contains a topologically nilpotent unit[44] $\varpi \in A$.

Example 27.8. •

- Any nonarchimedean field, such as the following: $t \in \mathbb{F}_p((t))$, $p \in \mathbb{Q}_p$, $\pi \in E$, all are pseudouniformizer.
- Any Huber ring over a narc field.

Remark: If K is any narc field. $\varpi \in K$ is psu [45] A has natural structure as Banach over K, with

$$\{f \in A \,:\, |f_0| = 1\} = A_0$$

the "unit" ball. Any $A_0 \hookrightarrow A$ ring of the definition has ϖ-adic topology.

- For a Tate ring: one always has A_0 being generated *one* uniformizer.
- One can rescale an element: [46]

$$|| \quad || : A \to \mathbb{R}_{\geq 0}$$
$$a \mapsto \inf_{n\,:\,\varpi^n a \in A_0} 2^{-n}$$

- We have equi. of categories

$$\{\text{Banach alg}/K \text{ ts maps}\} \simeq \{\text{Tate-Huber rings}/K\}$$

  There are two advantaanges to our way:
  - In Banach algebra one endows with a norm, and a base field. I think there is too much structure here.

27.0.1. *Valuation specrum.*

27.9. Just like a scheme, we need to look at the "prmeis". This is something that happens before the school of Grothendieck[47] [42:30]

$$\operatorname{Cont} A := \{| \quad | \,:\, A \to \Gamma \cup \{0\}\} \,/\sim$$

with topology generated by opens

$$\{|f| \leq |g| \neq 0\}$$

- We want to define two things: to defin $f \neq 0$.and closed bals. [44:20]

Here: Γ is a totally ordered monoid.

- $\mathbb{R}_{>0}$
- $\mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$. where $r > \gamma > 1$ for all r...

[44] also called psu.

[45] In local field, the uniformizer ϖ is one generator. This has something to do with a very ramified field. [35:30]

[46] To think about this: if $\varpi^{n_0} a \in A_0$ for some n, then it holds for all $N \geq n_0$, thus we take the infimum.

[47] This kind of passed aside untli they resurfaced in narc geometry.

Where

$$|\quad| : A \to \Gamma \cup \{0\}$$

staisfies

- $|ab| = |a||b|$.
- $|a+b| \leq \max(|a|, |b|)$
- $|0| = 0$, $|1| = 1$.

for all $\gamma \in \Gamma$, $\{a : |a| < \gamma\} \hookrightarrow A$ open. [48] Two cts vaaluation $|\quad|_1, |\quad|_2$ are eqi. if $|a|_1 \geq |b|_1$ iff $|a|_2 \geq |b|_2$.

If Γ is chosen minimal, then exists equiv. $\Gamma_1 \simeq \Gamma_2$ st.

$$\begin{tikzcd} & \Gamma_1 \cup 0 \\ A \arrow[ur] \arrow[dr] & \\ & \Gamma_2 \cup 0 \end{tikzcd}$$

[51:50]

Remark 27.10. Why in definition of continuity of inequality is strict by that of Spa is non strict?

Turns out it is better to keep track of certain additional daatum

Definition 27.11. A <u>Huber pair</u> is a pair. is a pair (A, A^+) where A is a Huber ring, $A^+ \hookrightarrow A$ is open integraally closed subgrin of power bdd elemetns [49]. ($\bigcup A_0 = A^\circ \hookrightarrow A$ subgring of power bdd elements).

- ...

Definition 27.12. (1) $\mathrm{Spa}(A, A^+) := \{|\quad| : |A^+| \leq 1\} \hookrightarrow \mathrm{Cont} A$

(2) $\mathrm{Spa}\, A = \mathrm{Spa}(A, A^\circ)$. [50]

Example 27.13. What can we say about basic properties of Huber Paris?

(1) $\mathrm{Spa}(A, A^+)$ is empty if and only if $A = 0$. How does one prove $\Rightarrow$?

Thus we can endow Spec A with presheaf $\mathcal{O}_{\mathrm{Spec}\, A}$ of Huber rings $\mathcal{O}^+_{\mathrm{Spec}\, A}$. on basis of <u>rational subsets.</u> These are basis of $\mathrm{Spa}(A, A^+)$ given by

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) = \{|f_i| \leq g \neq 0\}, \quad (f_1, \ldots, f_n, g) \text{ generate an open ideal}$$

[48] Q: I don't think you get the right topology. [49:18]

[49] I recommend to ignroe

[50] In practice, almost either A^+ has no role, or we take A°.

Thus

$$\mathcal{O}_{\operatorname{Spec} A}(U(f_1,\dots,f_n/g)) = A\left\langle \frac{f_1}{g},\dots,\frac{f_n}{g}\right\rangle$$

this "allows convergent seires in $\frac{f_i}{g}$s. "

Theorem 27.14 (Huber,...)**.** *In "all practical cases" $\mathcal{O}_{\operatorname{Spa}(A,A^+)}$ is a sheaf. But nont always!.* [51]

It has been an ongoing in issue on what to do with this. Recently, we realized we can correct this by allowing the structure sheaf to be derived. It's subtle about how to do this, I did this with Dustin Clausen[52], and we get a sheaf of condensed animated algebra.

In all what we can say: the structure sheaf is concentrated in deg. 0. One has to go in nonnoetherian cases. Thisi s however not relevant for thsi course

Definition 27.15. An adic space is a triple $(X,\mathcal{O}_X,\mathcal{O}_X^+)$, where X is a topological spc,

- $\mathcal{O}_X$ is a sheaf of cplt topological rings.
- $\mathcal{O}_X^+$ subsehaf of $\mathcal{O}_X$.
- that is locall of the form $\operatorname{Spa}(A,A^+),\mathcal{O},\mathcal{O}^+$.

One has to formullate correct analogue of "locally ringed". ... comments on valuations... [1:07:16]

27.16. Q: whaat is the advantanget of completing everything? A: Only by completing you can formulate the universal property. But also in the examples - on rigid spaces/ [1:09:14]

Q: You have defined $\mathcal{O}_{\operatorname{Spa}}^+$? A: it is the minimal choice that containns A_i^+ for all f_i/g.

Further remarks: on *open int. closed condition.*

27.1. **An example.** This object's points seemed to be classified in Scholze's paper.

C can be any narc field.

$$\mathbb{D}_C^* = \text{"}\{x\,:\,0<|x|<1\}\text{"}$$

We have the Tate algebra:

$$C\langle T\rangle := \left\{\sum a_nT^n\,:\,a_n\in C, a_n\to 0\right\}$$

These are the convergent power series in 1 var. for all $x\in C, |x|<1$. We have a map

$$C\langle T\rangle \to C$$
$$\sum a_n t^n \mapsto \sum a_n x^n$$

Example 27.17. A classification of few more points.

(1) Type 2 or 3.

---

[51]

[52] Bambozzi-Kreminizer too

So

$$\operatorname{Spa} C\langle T\rangle =: \mathbb{B}_C = " \{x. : 0 \le |x| \le 1\}$$

is the "closed unit disk". First of all. we have

$$\mathbb{B}_C^* = \mathbb{B}_C \backslash \{0\} = \bigcup_{\varepsilon > 0} \mathbb{A}(\varepsilon, 1)$$

where $\varepsilon \in |C|$.

$$\mathbb{A}(\varepsilon, 1) = \operatorname{Spa} A_\varepsilon = \left\{ \sum a_n t^n \ : , a_n \in C, \epsilon^n |a_n| \to 0, |a_n| \to 0 \text{ as } n \to \infty \right\}$$

This is similar condition on convergence condition for fps in narc series. Observe: $\mathbb{B}_C$ is qc. But $\mathbb{B}_C^*$ is not qc. [1:17:50] Similarly

$$\mathbb{D}_C^* = \bigcup_{\varepsilon > 0, r < 1} \mathbb{A}(\varepsilon, r) \quad \mathbb{A}(\varepsilon, r) = \{\varepsilon < |t| < r\}$$

again you can write the the corresponding algebra of functions.

Beware: $\mathbb{D}_C^* \hookrightarrow \mathbb{B}_C$ is open but is not equal to the lcous $\{0 < |t| < 1\}$. Problem, there was one point $x \in \mathbb{B}_C = \operatorname{Spa}\langle T\rangle$ st. $r < |T(x)|| < 1$ for all $r \in |C|$, $r < 1$. SO there is an point that is infinitely close to 1. [53]

In fact there is. a natural map

$$\mathrm{rad} : \mathbb{B}_C \longrightarrow [0, 1]$$

$$\mathbb{D}_C^\times \longrightarrow (0, 1)$$

$$x \mapsto T(\tilde{x})$$

We have a frobenius map ϕ_C on $\mathbb{D}_C^\times$, that satisfies

$$\mathrm{rad}\phi_C = \mathrm{rad}^{1/q}$$

This ensures that the action of ϕ_C is properly discontinuous. Thus

$$X_{C,E} := \mathbb{D}_C^* / \phi_C^{\mathbb{Z}}$$

is well-defined. [54] So to understand this thing: take $\mathbb{A}(r, r^{1/q})$. and identify annuli boundary:

- In this annulus, the frobenius sends $\phi : \mathbb{A}(r, r) \simeq \mathbb{A}(r^{1/q}, r^{1/q})$.
- One ends up precisely with a donut situation again.

We have roughly acheived what we wanat:

- Being able to geometrically takae the frobenius.
- We get an object that somehow looks llike a cpt Riemann surface. End. [1:27:57]

---

[53] Thus if you take the rising union, this point x does not lie in it. This is a idfference of a higher rank point. Such issue would not play too much of a role.

[54] One can understand local properties of this object form that of $\mathbb{D}_C^*$.

27.2. Remarks on curves. Let us be clear the definition of a curve

Definition 27.18.
- *regular*, separated Noetherian scheme.
- It is *complete* if

$$\sum_{x \in |X|} v_x f = 0 \quad f \in \mathcal{O}_{X,\eta}^{\times}$$

Example 27.19. (1) $k(t) = \mathrm{Frac} k[t]$, is the spectrum of $\mathbb{A} = \operatorname{Spec} k[t]$.

As each $\mathcal{O}_{X,x}$ is a *valuation ring*, with valuation v_x. In ortherwords, it uniquely extends to

$$v_x : \mathrm{Frac}\mathcal{O}_{X,x} = k(X)^{\times} \to \mathbb{Z}$$

27.3. References.

- **[Bru18]** has one of the nices notes on curves.

$$\begin{tikzcd}
 & \mathrm{PftdSpc} \arrow[d] \\
\text{Sch, Formal, Rig, Berk} \arrow[r, hook] & \mathrm{AdicSpc} \arrow[d] \\
\text{Manifold, cplx analy} \arrow[r, hook] & \text{ringed spaces}\,(X, \mathcal{O}_X)
\end{tikzcd}$$

Theorem 27.20. *[**Hub94**], [**Wei17**]*

$$\begin{array}{cc}
 & \mathit{Fml}\,\mathrm{Sch} \\
 & \\
\mathrm{Sch} & \mathit{AdicSpc}
\end{array}$$

Where

$$\mathit{Spf}A \mapsto \mathrm{Spa}(A, A)$$

Example 27.21. The open disk.

- We may consider

$$U(1/T) = \{| \quad | : 1 \leq |T|\}$$

Example 27.22. Key examples:

(1) $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ is the final object.

(2) $\mathrm{Hom}(\mathrm{Spa}(R, R^+), \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T]) \simeq \mathrm{Hom}(\mathbb{Z}[T], R^+) = R^+$.

(3) $\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z})$ represents $X \mapsto \mathcal{O}_X(X)$. Note this allows one sees that the difference in ring of integral elements.

28. Perfectoid Spaces

It would be useful to consider the case of rings.

Theorem 28.1. *There is an adjunction*

$$\begin{tikzcd} Perf_{\mathbb{F}_p} \arrow[r, shift left] & Alg^{p-cpl} \arrow[l, shift left, "R\mapsto R^\flat"] \end{tikzcd}$$

Proposition 28.2. *$PftdField_{\mathbb{F}_p} \simeq$ complete perfect field.*

Example 28.3. One can begin with a field of characteristic p, snd construct its perfection.

29. Perfect geometry, char p geometry

Let k be perfect field of char $p > 0$. We have an adjunction

$$\begin{tikzcd} \mathrm{CAlg}_k^{\mathrm{pf}} \arrow[r, shift right] & \mathrm{CAlg}_k \arrow[l, shift right, "1/p^{-\infty}"'] \end{tikzcd}$$

30. The diamond functor

Let $\mathrm{Pftd}_{\mathbb{F}_p}$ be the category of perfectoid spaces of char p.

Definition 30.1 (Topologies on Pftd)**.** Ordering by fine-ness of topology, we have

$$v \quad \subset \quad \text{Étale} \quad \subset \quad \text{Analytic} \tag{7}$$

One can access any v-sheaf via reversing the following properties

(1) A v-sheaf diamond is quotient of perfectoid space under pro-étale equivalence [**Sch22**]

The diamond functor generalizes tilting.

$$\begin{tikzcd} \mathrm{AdicSpc}_{\mathrm{Spa}_{\mathbb{Z}_p}} \arrow[r] \arrow[rrdd, dashed, bend left] & \mathrm{AdicSpc}_{\mathbb{F}_p} \arrow[rdd] & \\ \mathrm{Pftd}_{\mathbb{Z}_p} \arrow[u, hook] \arrow[r, "(\ \ )^\flat"] & \mathrm{Pftd}_{\mathbb{F}_p} \arrow[u, hook] & \\ \mathrm{AffPftdAlg}_{\mathbb{Z}_p} \arrow[u, hook] \arrow[r] & \mathrm{AffPftd}_{\mathbb{F}_p} \arrow[u, hook] \arrow[r, hook] & \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_p}, v) \end{tikzcd} \tag{8}$$

Diamonds are the algebraic spaces under the pro-'etale equivalence relation in the characteristic p world. This comes from the phenomena that

Proposition 30.2. *If $X \in AdicSpc_{\mathrm{Spa}\,\mathbb{Z}_p}$ then X is a pro-étale quotient of perfectoid space.*

This implies an intuitive construction of $\Diamond$: given $X \in \mathrm{AdicSpc}_{\mathrm{Spa}_{\mathbb{Z}_p}}$, choose a proétale surjection $\tilde{X} \to X$, such that $\tilde{X}/R \simeq X$ where $R \subset \tilde{X} \times \tilde{X}$ is an equivalence relation. Then

$$X^{\Diamond} \simeq \tilde{X}^\flat / R^\flat$$

Definition 30.3. A *diamond* is $X \in \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_q}, \text{pro-ét})$ such that there eixsts a perfectoid space $\tilde{X}$

Note that one define a proétale sheaf $\mathrm{Spd}\,\mathbb{Z}_p$ is *is not* a diamond.

30.0.1. *Embedding schemes as v-sheaves.* We birefly recall [AGLR22]*2.2. For schemes locally of finite type over $\mathcal{O}$ one has two ways of embedding as v-sheaves,

$$\begin{tikzcd} & \mathrm{Sch}_{\mathcal{O}}^{\mathrm{lft}} \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, "(-)^{\diamond}"] \arrow[drr, shift right, swap] & & \\ \mathrm{DiscFml\,Sch}_{\mathcal{O}} \arrow[r] & \mathrm{DiscAdic}_{\mathcal{O}_{\mathrm{disc}}} \arrow[u] \arrow[rr, "(-)^{\lozenge}"] & & v\,\mathrm{Shv}_{\mathrm{Spd}\,\mathcal{O}} \end{tikzcd} \tag{9}$$

There is a 1-morphism

$$(-)^{\diamond} \to (-)^{\lozenge}$$

which is equivalent on proper schemes.

We describe the points:

$$\mathrm{Spec}\, A^{\diamond} : \mathrm{Spa}(R, R^+) \mapsto \left\{ \left(R^{\#}, i, f^+ \right) \right\} / \sim$$
$$\mathrm{Spec}\, A^{\lozenge} : \mathrm{Spa}(R, R^+) \mapsto \left\{ \left(R^{\#}, i, f \right) \right\} / \sim$$

where $(R^{\#}, i)$ is untilt and $f^+ : A \to R^{\#,+}$, and $f : A \to R^{\#}$ are ring homomorphisms.

Proposition 30.4. *$X^{\lozenge} \simeq (X_{\flat})^{\lozenge}$, [**Han16**].*

Equivalently (relative to a finite extension $E/\mathbb{Q}_p$)

$$\mathbb{G}_a^{\lozenge} : \mathrm{Pftd}_{\mathbb{F}_p, \mathrm{Spd}\, E} \to \mathrm{Grp}$$
$$((R, R^+) \to \mathrm{Spd}\, E) \mapsto \{\mathrm{Spa}(R^{\sharp}, R^{\sharp,+}) \to \mathbb{G}_a\} = \mathbb{G}_a(R^{\sharp}) = R^{\sharp}$$

31. Adic Spaces

Definition 31.1. A top. ring A *continuous valuation* is

$$| \quad | : A \to \Gamma \cup \{0\}$$

Γ is totally ordered abelian group. St.

- $|0| = 0, |1| = 1$
- $|ab| = |a||b|$
- $|a + b| \leq \max |a|, |b|$

For all $\gamma \in \Gamma$,

$$\{a \in \Gamma \,:\, |a| < \gamma\}$$

- It wouldn't make a difference if we take $\leq$.
- it only makes a difference if Γ is trivial.

Two such valuations, $| \quad |, | \quad |'$ are equivalent: if we have a commuting diagram

$$\begin{tikzcd} & A \arrow[dl, "| \quad |"'] \arrow[dr, "| \quad |'"] & \\ \Gamma_{| \quad |} \arrow[rr] & & \Gamma_{| \quad |'} \end{tikzcd}$$

where lhs rhs are the minimal choices associated to the valuation - i.e. taking the image.

Example 31.2. Reinterpreting $\operatorname{Spec} A$ as continuous valuations. Let $A \in \mathrm{CAlg}^{\heartsuit}$, We have

$$\operatorname{Spec} A \simeq \{x : A \to K\} / \sim$$

Example 31.3. Let K be a field with a valuation.

$$R_v := \{x \in K \,:\, |v(x)| \leq 1\}$$

Definition 31.4.

Let

$$\operatorname{Spa}(A, A^+) := \left\{| \quad | \in \operatorname{cts}(A) \,:\, x \in A^+, |x| \leq 1\right\} / \sim$$

Definition 31.5. Topology is generated by

$$\{x \,:\, |f(x)| \leq |g(x)| \neq 0, f, g \in A\} := X(f, g)$$

Note: the notation on rhs here means that if $x : A \to \Gamma \cup \{0\}$, $x(f) = |f(x)|$. In contrast to AG, we allow open sets that tell us when a function is smaller than the other.

Remark 31.6. This shows that both $\{x \,:\, |f(x)| \neq 0\}$ and $\{x \,:\, |f(x)| \leq 1\}$ are open.

Definition 31.7. $\varpi \in A$ is a psu if it is a topologically nilpotent unit.

- A priori we can ignore A^+, which is a perfectly sensible object.
-

31.0.1. *The points of* $\operatorname{Spa}\mathbb{Z}_p[[T]]$. Let us describe the points of $\operatorname{Spa}(\mathbb{Z}_p[[t]], \mathbb{Z}_p[[t]])$.

(1) There is a unique non analytic, def. 31.9, adic point

$$x_{\mathbb{F}_p} : \mathbb{Z}_p[[t]] \to \mathbb{F}_p \to \{0, 1\}$$

(2) There is a point given by

$$x_{\mathbb{F}_p((t))} : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p/p\mathbb{Z}_p[[t]] \hookrightarrow \mathbb{F}_p((t)) \simeq \mathbb{F}_p[[t]][1/t]$$

(3) There is a point given by

$$x_{\mathbb{Q}_p} : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \simeq \mathbb{Z}_p[1/p]$$

Once we have the *analytic points*, we can define a map

$$\kappa : \mathcal{Y} := X \backslash \{x_{\mathbb{F}_p}\} \to [0, \infty]$$

given by

$$\kappa(x) := -\frac{\log p(\tilde{x})}{\log T(\tilde{x})}$$

where $\tilde{x}$ is the unique maximal generalization. A similar story would be repeated in **??**.

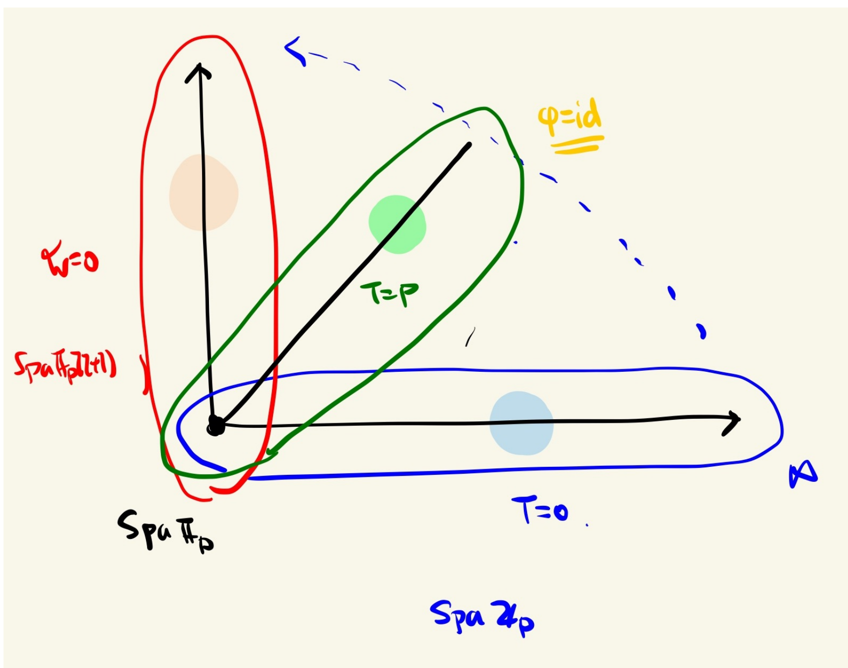

FIGURE 1. Geometry of $\operatorname{Spa}\mathbb{Z}_p[[t]]$. Note that there is no Frobenius.

Example 31.8. The closed unit disk.

$$\mathbb{D} := \operatorname{Spa}(\mathbb{Z}[T], \mathbb{Z}[T]) = \operatorname{Spa}\mathbb{Z}[T]$$

This represent the functor [55]

$$X \mapsto \mathcal{O}^+(X)$$

[55]which is thought of as the closed disk of the ring of functions.

If K is cplt. narc field, then

$$\mathbb{D} \times_{\operatorname{Spa}\mathbb{Z}} \operatorname{Spa}(K, K^{\times}) = \operatorname{Spa}\left(K\langle T\rangle, K^{\circ}\langle T\rangle\right) = \mathbb{D}_K$$

31.1. **Analytic points.**

Definition 31.9. A point $x \in \operatorname{Spa}(A, A^+)$ is *non analytic* if the kernel is $|\quad|_x$ is open.

A more algebraic characterization can be given as follows:

Proposition 31.10. *Let $(A, A^+) \in HubRng$. The following are equivalent:*

(1) *The ring of ideal A is analytic.*
(2) *Every open ideal of A is trivial.*

Definition 31.11. Let A be a Huber ring. A subring is a *ring of integral elements* if

- Integrally closed, open.
- $A^+ \subseteq A^{\circ}$, where A° is ring of power boundeded elements.

Example 31.12. This is closed unit disk.

$$\operatorname{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \operatorname{Spa} K = [\operatorname{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \operatorname{Spa}(\mathcal{O}_K, \mathcal{O}_K)]$$

How does one prove this? This is by universal property.

Example 31.13. Tate algebra. Let K be nonarchimedean complete field. Then we have

$$K\langle t_1, \ldots, t_n\rangle = K\widehat{[T_1, \ldots, T_n]} =$$

Plans for next 2 weeks.

- Today.
- Th. Tu. Perfectoid spaces.
- Th. Diamons. Preview: C = Perf spaces o char. p. C-étale = pro-étale. This gives rise to C-spaces.=: diamonds.
- Setp 29. Oct 3 Jared.
- Oct 7. Shtuka's finally.

Let me correct to last time: Rational subsets are <u>not</u> stable under pullback! e.g.

$$\operatorname{Spa}(\mathbb{Z}[T), \mathbb{Z}_p[[t]]) \to \operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) = \{s, \eta\} \supset \{\eta\}$$

but the preimage is equal to $\mathcal{Y}_{[0,\infty)}$. This is not evne qc. cf. lecture notes. For the lectures we only need honest adic spaces! so we need not worry about this. The reason I believe rational subsets are stable udner pb. is that this is often true.

Definition 31.14. A moprhism $f : A \to B$ of Huber rings is <u>adic</u> if for (any, hence any) choice of rings of defn. $A_0 \to B_0$, $I \subset A_0$ ideal of definition $IB_0 \subset B_0$ is an ideal of defn.

<u>Remark</u> One condition that implies this behavior: if A is Tate, then any $f : A \to B$ is adic. This is because B would also be tate as it contains a topologically nilpotent unit form A.

Proposition 31.15. *[SW20].*

(1) If $(A, A^+) \to (B, B^+)$ is adic. (i.e. $A \to B$) is. the pullback along

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$$

preserves rational subset.

(2) pushout exists.

$$\begin{tikzcd} & (B, B^+) \\ (A, A^+) \arrow[ur] \arrow[dr] & \\ & (C, C^+) \end{tikzcd}$$

then $D := B \otimes_A C$ with the topology making the image D_0 of $B_0 \otimes_{A_0} C_0$ in D the ideal of definition $I \cdot D_0$, where $I \hookrightarrow A_0$ is an ideal of definition.

I noticed my definition of rational subset is unnecessarily complicate.

Proposition 31.16. *Let (A, A^+) be Huber pair. Let $T \subset A$ be a finite subset, such that $T \cdot A \subseteq A$ is open*[56]*. Then any rational subset $U \hookrightarrow \mathrm{Spa}(A, A^+)$ is of the form*

$$U(T/g) := \{x \,:\, |f_i(x)| \leq |g(x)| \neq 0, f_i \in T\}$$

Now we check that the intersection of finitely many rational subsets is again rational. Let us take

$$U_1 = \{x \,:\, |t(x)| \leq s(x)| \neq 0, t \in T\}$$

$$U_2 = \left\{x \,:\, |t'(x)| \leq |s'(x)| \neq 0, t' \in T'\right\}$$

31.2. **Meaning of the ring A^+.** [17:24]

For any huber ring A, let

$$\mathrm{Cont}(A) = \{\text{cts valuation of } A\} / \simeq$$

Then there is bijection

$$\{\text{Intersections of subsets of } \mathrm{Cont}(A)\}$$

In otherwords the A^+ keeps track of which inequalities have been enforced....

In interesting remarks in [Berkvid, 4,2:30]

[56] This is the ideal generated by these elements

[SW20]

Geometry of diamonds. Idea: there should be a functor

$$\mathrm{AdicSpc}_{\mathbb{Q}_p} \to \mathrm{Dia}$$

Thought of as "forgetting structure morphism to $\mathbb{Q}_p$". On Pftd spaces $/\mathbb{Q}_p$, we already have a map

$$X \mapsto X^\flat$$

In general for X an adic space $\mathbb{Q}_p$, X is pro-étale locally perfectoid. why? i.e.

$$X = \mathrm{coeq}(Y \to \tilde{X})$$

with

$$Y \overset{\simeq}{\rightrightarrows} \tilde{X} \times_X \tilde{X}$$

is a pro-étale perfectoid cover.

Thus it is clear what the functor is

$$X \mapsto \mathrm{coeq}(Y^\flat \to \tilde{X}^\flat)$$

The question is then which category?

32.1. Ex.

$$\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \mapsto \mathrm{Spa}((\mathbb{Q}_p^{\mathrm{cyc}})^\flat, (\mathbb{Z}_p^{\mathrm{cyc}})^\flat)/\mathbb{Z}_p^\times$$

- An étale is locally composition of rational embeddings and finite étale morphisms. [19:13]

From now on, we are gonna look at all adic spaces by regarding them via perfectoid spaces mapping into them.

Definition 32.2. A moprhism $f : X \to Y$ in Pftd, is pro-étale, if locally it is of the form

$$\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$$

where A_∞, A are perfectoid [57] where

$$(A_\infty A_\infty^+) = \widehat{\varinjlim (A_i, A_i^+)}$$

32.3. Let us analyze the in the affinoid case.

$$\begin{tikzcd} \mathrm{Spa}(A_\infty, A_\infty^+) \arrow[dr] & & \mathrm{Spa}(B_\infty, B_\infty^+) \arrow[dl] \\ & \mathrm{Spa}(A, A^+) & \end{tikzcd}$$

Situation as above: all algebra objects areperfectoid. And the objects are filtered colimits. Then

[57]It is an unknown question whether $\mathrm{Spa}(R, R^+)$ perfectoid implies R is perfectoid.

$$\begin{aligned}&\mathrm{Map}_{\mathrm{Spa}(A,A^+)}\big(\mathrm{Spa}(A_\infty, A_\infty^+), \mathrm{Spa}(B_\infty, B_\infty^+)\big)\\&= \varprojlim_j \varinjlim_i \mathrm{Map}(\mathrm{Spa}(A_i, A_i^+), \mathrm{Spa}(B_j, B_j^+))\end{aligned}$$

Proof. Wlog . J is just a singleton. So $(B, B^+) = (B_\infty, B_\infty^+)$. Now this can be checked lcoally on (B, B^+). ∎

33. Diamonds associated with adic spaces

Refernces: [**SW20**] We will study the functor

$$\mathrm{AnaAdicSpc}_{\mathbb{Z}_p} \to \mathrm{Dia}$$

Thought of roughly as the functor

$$\{\text{complex analytic spaces}\} \to \{\text{topological spaces}\}$$

Definition 33.1. Let $X \in \mathrm{AnaAdicSpc}_{\mathbb{Z}_p}$, define

$$X^{\diamond} : \mathrm{Perf} \to \mathrm{Set}$$
$$Y \mapsto \left\{(Y^\sharp, i), Y^\sharp \to X\right\} / \sim$$

This is roughly mimicking what happened to $\mathbb{Q}_p$. It turns out that this already shift. i is part of the data.

- The fucntoriality is slightly not clear.

Theorem 33.2. *$X^{\diamond}$ is a diamond.*

Proof. Same as $\mathbb{Q}_p$ last time.

- Assume $X = \mathrm{Spa}(R, R^+)$ affinoid. R a Tate ring. Because we are working over $\mathrm{Spa}\,\mathbb{Z}_p$,. this shows $p \in R$ is topologically nilpotent - but not nnecessarily unit.

Proposition 33.3 (Faltings)**.** *Let R be a Tate ring, $p \in R$ top. nilpotennt. Let $\varinjlim R_i$ be filtered direct limit R_i finite étale over R.*

$$\tilde{R} := \varinjlim R_i, R_i \xrightarrow{\text{fét}} R$$

such that has no nonsplit finite étale cover. Endow $\tilde{R}$ with topology making $\varinjlim R_i^\circ$, open and bounded. Then $\widehat{\tilde{R}}$ its ocmpletionn. Then $\widehat{\tilde{R}}$ is perfectoid.

Proof. (1) Find $\varpi \in \tilde{R}$ a psu st. $\varpi^p | p$ [58]in $\tilde{R}^\circ$. To do this , let $\varpi_0 \in R$ be psu. Let N be large enough so that [59]

$$\varpi_0 | p^N \quad N \gg 0$$

do not understand argument. [10:00]

∎

∎

---

[58]So that $p = \varpi^P a$

[59]This is possible as p is top. nilpotent. Not that $x|y$ iff $(y) \hookrightarrow (x)$, iff $|y| \leq |x|$.

34. Examples of diamonds

Ref: [Lec 16]

$\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p$. By which we analytic adic spaces $X/\operatorname{Spa}\mathbb{Z}_p$ + an action by a profinite group G such that $X^{\Diamond}/G \simeq \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p$. One incarnation of this was $\widetilde{\mathbb{D}^{\times}_{\mathbb{Q}_p}}/\mathbb{Z}_p^{\times}$. We have the usual perfectoid open unit disk

$$\widetilde{\mathbb{D}_{\mathbb{Q}_p}} = \varprojlim_{x \mapsto (1+x)^p - 1} \mathbb{D}_{\mathbb{Q}_p}$$

Then we have Drinfeld's lemma:

$$\pi_1(\widetilde{\mathbb{D}}^{\times}_{\mathbb{Q}_p}/\mathbb{Z}_p^{\times}) = G_{\mathbb{Q}_p} \times G_{\mathbb{Q}_p}$$

- It's clear how one copy of Galois group comes from.
- For any $C/\mathbb{Q}_p$ alge braically closed compete, we have $\pi_1(\tilde{\mathbb{D}}^{\times}_C/\mathbb{Q}_p^{\times}) \simeq \Gamma_{\mathbb{Q}_p}$ In this one we see that $\Gamma_{\mathbb{Q}_p}$ as a geometric fundamental group. This is also explained in the paper of J. Weinstein.

35. the different incaranations

Ref: [Lec 18]

Theorem 35.1 (Fargue). *The following categories are equivalent:*

(1) Shutkas/$\operatorname{Spa}(C^{\flat}, \mathcal{O}_{C^{\flat}})$ *with one part "at* C*".*

(2) (T, Ξ) *with* T *finite free* $\mathbb{Z}_p$ *module.* Ξ

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