A MINI COURSE ON DISTINGUISHED REPRESENTATIONS

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Disclaimer: These notes are based on a mini-course I gave in the research school "Introduction to Relative Aspects in Representation Theory, Langlands Functoriality and Automorphic Forms", May 2016 in Luminy. It is meant to introduce the reader new to the subject, with some aspects of the theory of period integrals of automorphic forms. As there is much interplay between the local and global aspects of the theory, both are addressed here. However, these notes are by no means comprehensive. In particular, they do not touch upon the archimedean part of the theory.

Some parts are written in an informal way. While the author takes full responsibility for possible inaccuracies, readers are encouraged to use these notes at their own risk.

1. Local distinction-definitions and a simple example

1.1. **The local setting.** Let F be a non-archimedean local field, $\underline{G}$ a connected reductive group defined over F and $G = \underline{G}(F)$.

Let $\mathrm{Rep}(G)$ be the category of complex valued smooth representations of G. That is, a representation (π, V) of G on a complex vector space V is in $\mathrm{Rep}(G)$ if $\{g \in G : \pi(g)v = v\}$ is an open subgroup of G for every $v \in V$.

Consider a closed subgroup H of G and a character χ of H. For $(\pi, V) \in \mathrm{Rep}(G)$ let $\mathrm{Hom}_H(\pi, \chi)$ be the space of linear forms $\ell : V \to \mathbb{C}$ such that $\ell(\pi(h)v) = \chi(h)v$, $v \in V$, $h \in H$.

Definition 1.1. *A representation $\pi \in \mathrm{Rep}(G)$ is (H, χ)-distinguished if*

$$\mathrm{Hom}_H(\pi, \chi) \neq 0.$$

If $\chi = \mathbf{1}_H$ is the trivial character of H, we simply say that π is H-distinguished.

Example 1.2. *Let $G = \mathrm{GL}_n(F)$, N the group of unipotent upper triangular matrices in G, ψ_0 a non-trivial character of F and*

$$\psi(x) = \psi_0(x_{1,2} + \cdots + x_{n-1,n}), \quad x \in N.$$

The (N, ψ)-distinguished representations of G are also called generic *and play an important role in the theory of automorphic forms.*

1.2. **A few words of motivation.** Distinguished representations in $\hat{G}$, the unitary dual of G, are the building blocks for harmonic analysis on the homogeneous space $H\backslash G$. Indeed, the unitary representation $L^2(H\backslash G)$ of G decomposes as a direct integral

$$L^2(H\backslash G) = \int_{\hat{G}}^{\oplus} \pi \quad \mu\pi$$

and the support of the Plancherel measure μ is contained in the class of H-distinguished representations (see e.g. [Ber88]).

Further motivation, comes from the theory of period integrals of automorphic forms. The relation between local and global distinction is discussed in §3.2.

In order to become more familiar with some of the techniques to study distinguished representation, we begin with a fairly simple example.

1.3. **Distinction by the split torus in GL_2.** Let $G = \mathrm{GL}_2(F)$ and

$$T = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) : a,\, d \in F^*\} \simeq F^* \times F^*$$

the diagonal torus in G. The first question we address is: which principal series representations of G are T-distinguished?

In order to define the principal series we introduce some further notation. Let

$$N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in F\} \simeq F \quad \text{and} \quad B = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a,\, d \in F^*, b \in F\}.$$

Then $B = T \ltimes N$ and every character of T will also be considered as a character of $B/N \simeq T$. Let $\delta_B\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = |a/d|$ be the modulus function of B. Here and henceforth $|\cdot|$ is the standard absolute value on F.

For a character χ of T let $I(\chi) = \mathrm{Ind}_B^G(\chi)$ be the associated principal series representation. Namely,

$$\begin{aligned} I(\chi) = \{f : G \to \mathbb{C} : f(bg) = (\delta_B^{1/2}\chi)(b)f(g),\ b \in B,\, g \in G, \\ f \text{ is right invariant by some open subgroup of } G\} \end{aligned}$$

with the action

$$(R(g)f)(x) = f(xg), f \in I(\chi),\, g,\, x \in G.$$

The first problem we address is therefore

Question 1.3. *For which χ is $I(\chi)$ a T-distinguished representation?*

Note that the centre Z of G is contained in T. Indeed,

$$Z = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in F^*\} \simeq F^*.$$

Furthermore, a character χ of T is associated to a pair of characters (χ_1, χ_2) of F^* via

$$\chi\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \chi_1(a)\chi_2(d)$$

and $\chi|_Z = \mathbf{1}_Z$ if and only if $\chi_2 = \chi_1^{-1}$.

Lemma 1.4 (A simple necessary condition)**.** *If $I(\chi)$ is T-distinguished then $\chi|_Z = \mathbf{1}_Z$.*

Proof. The representation $I(\chi)$ has central character $\chi|_Z$, that is,

$$R(z)f = \chi(z)f,\ f \in I(\chi),\, z \in Z.$$

For $\ell \in \mathrm{Hom}_T(I(\chi), \mathbf{1})$, we have

$$\chi(z)\ell(f) = \ell(\chi(z)f) = \ell(R(z)f) = \ell(f),\ f \in I(\chi),\, z \in Z.$$

Therefore, if some such $\ell \neq 0$ then $\chi|_Z = \mathbf{1}_Z$. $\square$

In fact, the above necessary condition is also a sufficient condition for distinction by T. In order to observe this, we first obtain the necessary condition in a more complicated way (one that in a more general setting will provide a necessary condition finer then the one given by the central character argument).

The property '$I(\chi)$ is T-distinguished' depends only on the restriction $I(\chi)|_T$. Since $I(\chi)$ consists of functions on G that are B equivariant, it makes sense to look at the double coset structure $B\backslash G/T$. Since $T \subseteq B$, this is a refinement of the Bruhat decomposition

$$G = B \sqcup BwN$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that,

$$B = \{g \in G : g_{21} = 0\}$$

is closed and

$$BwN = \{g \in G : g_{21} \neq 0\}$$

is open in G. Clearly, $B = BT$ is a (B, T)-double coset and since w normalizes T also $Bw = BwT$. Furthermore, for $x \in F^*$ we have

$$w\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) w \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in B\eta T \quad \text{where} \quad \eta = w\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right).$$

Therefore

$$G = B \sqcup Bw \sqcup B\eta T$$

is the double coset decomposition of $B\backslash G/T$. Note that

$$B = \{g \in G : g_{21} = 0\} \quad \text{and} \quad Bw = \{g \in G : g_{22} = 0\}$$

are closed and

$$B\eta T = \{g \in G : g_{21}g_{22} \neq 0\}$$

is open in G. It therefore makes sense to define the following subspace of $I(\chi)$

$$V = \{f \in I(\chi) : \mathrm{Supp}(f) \subseteq B\eta T\}.$$

It is a T-invariant subspace of $I(\chi)$ or, otherwise put,

$$0 \subseteq V \subseteq I(\chi)|_T$$

is a filtration of T-representations.

Remark 1.5. *By the smoothness condition, the support of a function in $I(\chi)$ is open. Therefore, for example, $\{f \in I(\chi) : \mathrm{Supp}(f) \subseteq Bw\} = 0$ while $V \neq 0$.*

If $0 \neq \ell \in \mathrm{Hom}_T(I(\chi), \mathbf{1}_T)$ then either $\ell|_V \neq 0$ (in which case $\mathrm{Hom}_T(V, \mathbf{1}_T) \neq 0$) or ℓ defines a non-zero element of $\mathrm{Hom}_T(I(\chi)/V, \mathbf{1}_T)$.

The representations V and $I(\chi)/V$ of T are analyzed as follows.

For $f \in V$ let $\phi_f : T \to \mathbb{C}$ be defined by

$$\phi_f(t) = f(\eta t).$$

Note that if $t \in T \cap \eta^{-1}B\eta$ then

$$\phi_f(t) = f((\eta t \eta^{-1})\eta) = (\delta_B^{1/2}\chi)(\eta t \eta^{-1}) f(\eta).$$

It is easy to see that $T \cap \eta^{-1}B\eta = Z$ and therefore:

- $\phi_f(zt) = \chi(z)\phi_f(t)$;
- ϕ_f is right invariant by an open subgroup of T and of compact support mod Z.

In fact, any function of T satisfying these two properties arises as a ϕ_f for some $f \in V$. In other words, as a representation of T, V is isomorphic to the compactly induced representation $\mathrm{ind}_Z^T(\chi|_Z)$.

Note that by Frobenius reciprocity [BZ76, Proposition 2.29],

$$\mathrm{Hom}_T(V, \mathbf{1}_T) = \mathrm{Hom}_T(\mathrm{ind}_Z^T(\chi|_Z), \mathbf{1}_T) = \mathrm{Hom}_Z(\chi|_Z, \mathbf{1}_Z) = \begin{cases} \mathbb{C} & \chi|_Z = \mathbf{1}_Z \\ 0 & \text{otherwise.} \end{cases}$$

Conclusion 1.6. $\mathrm{Hom}_T(V, \mathbf{1}_T) \neq 0$ *if and only if* $\chi|_Z = \mathbf{1}_Z$.

Note further that

$$I(\chi)/V \simeq \delta_B^{1/2}\chi \oplus (\delta_B^{1/2}\chi)^w$$

is two dimensional. The isomorphism is induced by the map $f \mapsto (f(e), f(w)) : I(\chi) \to \mathbb{C}^2$ (which clearly vanishes on V). Here $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore,

$$\mathrm{Hom}_T(I(\chi)/V, \mathbf{1}_T) = \begin{cases} \mathbb{C}^2 & \chi = \delta_B^{-1/2} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Conclusion 1.7. $\mathrm{Hom}_T(I(\chi)/V, \mathbf{1}_T) \neq 0$ *if and only if* $\chi = \delta_B^{-1/2}$.

Since $\delta_B|_Z = \mathbf{1}_Z$, combining Conclusions 1.6 and 1.7 we recover Lemma 1.4.

Next, we consider sufficient conditions for distinction.

For the case $\chi = \delta_B^{-1/2}$, note that $\ell_e(f) = f(e)$ and $\ell_w(f) = f(w)$ satisfy

$$\mathbb{C}\ell_e \oplus \mathbb{C}\ell_w \subseteq \mathrm{Hom}_T(I(\delta_B^{-1/2}), \mathbf{1}_T)$$

and in particular $I(\delta_B^{-1/2})$ is T-distinguished.

Remark 1.8. *Note that ℓ_e and ℓ_w emerge from the two closed (B,T)-double cosets. In fact, we have* $\mathrm{Hom}_T(I(\delta_B^{-1/2}), \mathbf{1}_T) = \mathbb{C}\ell_e \oplus \mathbb{C}\ell_w$.

Assume now that $\chi|_Z = \mathbf{1}_Z$. We can explicitly construct $j_\chi \in \mathrm{Hom}_T(V, \mathbf{1}_T)$ as follows:

$$j_\chi(f) = \int_{Z\backslash T} f(\eta t)\, dt = \int_{F^*} f(\eta\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right))\, da, \ f \in V.$$

It is easy to see that j_χ is not identically zero and therefore $\mathrm{Hom}_T(V, \mathbf{1}_T) \neq 0$. But can j_χ be extended to $I(\chi)$? Not in general, but it can in the sense of meromorphic continuation.

Let $j_\chi(f) = \int_{Z\backslash T} f(\eta t)\, dt$ for any $f \in I(\chi)$ for which the integral converges and let

$$\chi_s = \delta_B^s \chi.$$

Applying the Iwasawa decomposition $G = BK$, where $K = \mathrm{GL}_2(\mathcal{O}_F)$ is the standard maximal compact subgroup of G and $\mathcal{O}_F$ is the ring of integers of F, we can extend δ_B^s to the function

$$\xi_s(g) = \delta_B^s(b), \quad g = bk,\, b \in B,\, k \in K.$$

Then $f_s = \xi_s f \in I(\chi_s)$ whenever $f \in I(\chi)$. References for the general principle of meromorphic continuation that we apply are given in §2.7.1. For now we state the relevant

Fact 1.9. *The integral j_{χ_s} converges absolutely on $I(\chi_s)$ for $\mathrm{Re}(s) \gg 1$. There is a rational function $P \in \mathbb{C}(q^s)$ (where q is the size of the residual field of F) such that $P(q^s)j_{\chi_s}(f_s)$ (a-priorily defined on a right half plane) is a polynomial in $q^{\pm s}$ for all $f \in I(\chi)$.*

In order to give some flavours for this fact, we explain the absolute convergence. The computation also indicates why j_{χ_s} is, in fact, a rational function of q^s. Without loss of generality assume that χ is unitary and s is real. Note that if

$$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right)k,\ \ k \in K$$

then

$$|\beta| = \max(|c|\,, |d|) \quad \text{and} \quad |\alpha\beta| = |\det g|\,.$$

Therefore

$$\left|f_s\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right| \ll \left|\frac{\det g}{\max(|c|\,, |d|)}\right|^{s+1/2} \max(|c|\,, |d|)^{-s-1/2}.$$

Since $\eta\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ a & 1 \end{smallmatrix}\right)$ we get that

$$\int_{F^*} |f_s|\left(\left(\begin{smallmatrix} 0 & 1 \\ a & 1 \end{smallmatrix}\right)\right) da \ll \int_{F^*} |a|^{s+1/2} \max(|a|\,, 1)^{-2s-1}\, da = \int_{|a|\le 1} |a|^{s+1/2}\ da + \int_{|a|>1} |a|^{-s-1/2}\ da.$$

This is a sum of two convergent geometric series precisely when $s + 1/2 > 0$.

Lemma 1.10. *Let χ be a character of T such that $\chi|_Z = \mathbf{1}_Z$. Then $I(\chi)$ is T-distinguished.*

Proof. It follows from Fact 1.9 that there exits $n \ge 0$ such that $s^n j_{\chi_s}$ is holomorphic and not identically zero at $s = 0$. Therefore,

$$J(f) = \lim_{s\mapsto 0} s^n j_{\chi_s}(f_s) \tag{2}$$

is well defined and $0 \ne J \in \mathrm{Hom}_T(I(\chi), \mathbf{1}_T)$. In other words, the leading term of j_{χ_s} at $s = 0$ shows that $I(\chi)$ is T-distinguished. □

Lemmas 1.4 and 1.10 give the characterization of T-distinguished principal series of G. We formulate the complete solution to Question 1.3 as

Conclusion 1.11. *The representation $I(\chi)$ is T-distinguished if and only if $\chi|_Z = \mathbf{1}_Z$.*

The above considerations can also help study distinction of other representations. To illustrate this, we show that the Steinberg representation St_2 of G is T-distinguished. Recall the short exact sequence

$$0 \to \mathbf{1}_G \to I(\delta_B^{-1/2}) \to \mathrm{St}_2 \to 0.$$

By (1) we have $\dim \mathrm{Hom}_T(I(\delta_B^{-1/2}), \mathbf{1}_T) \ge 2$ (in fact, equality holds) while clearly, $\mathrm{Hom}_T(\mathbf{1}_G, \mathbf{1}_T)$ is one dimensional. It follows that St_2 is T-distinguished.

Similar consideration can be used to characterize all the T-distinguished, essentially square integrable representations of G. Any essentially square integrable representation of G is of the form $\mathrm{St}_2(\xi) = \mathrm{St}_2 \otimes (\xi \circ \det)$ for a character ξ of F^*. We already considered above the case where ξ is trivial and we assume now that $\xi \neq \mathbf{1}_{F^*}$. The relevant exact sequence is

$$0 \to \xi \circ \det \to I((\xi \circ \det)\delta_B^{-1/2}) \to \mathrm{St}_2(\xi) \to 0.$$

Since, by assumption, $\xi \circ \det|_T \neq \mathbf{1}_T$ we see that $\mathrm{St}_2(\xi)$ is T-distinguished if and only if $I((\xi \circ \det)\delta_B^{-1/2})$ is. By Conclusion 1.11 we have

Conclusion 1.12. *The essentially square integrable representation* $\mathrm{St}_2(\xi)$ *of* G *is* T*-distinguished if and only if* ξ *is a quadratic character.*

We do not address distinguished supercuspidal representations.

2. Local distinction and the geometric lemma for symmetric spaces

We keep the notation set in §1.1. For a subgroup Q of G let δ_Q be the modulus function of Q.

2.1. **The geometric lemma.** Consider a parabolic subgroup $P = M \ltimes U$ of G with its Levi decomposition (M a Levi subgroup and U the unipotent radical) and a representation $\sigma \in \mathrm{Rep}(M)$. Consider σ also as a representation of P that is trivial on U via the isomorphism $M \simeq P/U$.

Let $\mathrm{Ind}_P^G(\sigma)$ be the representation of G by right translations $(R(g)f)(x) = f(xg)$, g, $x \in G$ on the space of functions $f : G \to \sigma$ satisfying

$$f(pg) = (\delta_P^{1/2}\sigma)(p)f(g), \quad p \in P,\ g \in G$$

and $R(g)f = f$ for all g in some open subgroup of G.

Our goal is to analyze

$$\mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma), \mathbf{1}_H).$$

This space depends only on $\mathrm{Ind}_P^G(\sigma)|_H$ and we study it using a version of Mackey theory formulated by Bernstein and Zelevinsky in[BZ77] (see also [BD08]).

From now on we make the following

Assumption 2.1. *The double coset space* $P\backslash G/H$ *is finite.*

Under this assumption it follows from [BZ76, §1.5] (see also [BD08, Lemma 3.1]) that the (P, H)-double cosets can be ordered as follows.

Lemma 2.2. *There is an ordering* $\{P\eta_i H\}_{i=1}^N$ *on the double coset space* $P\backslash G/H$ *so that*

$$Y_i := \cup_{j=1}^i P\eta_j H$$

is open in G *for all* $i = 1, \dots, N$.

Based on this ordering let

$$V_i = \{f \in \mathrm{Ind}_P^G(\sigma) : \mathrm{Supp}(f) \subseteq Y_i\}.$$

Then

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_N = \mathrm{Ind}_P^G(\sigma)$$

is a filtration of $\mathrm{Ind}_P^G(\sigma)|_H$.

Observation 2.3. *If* $\mathrm{Ind}_P^G(\sigma)$ *is* H*-distinguished then there exits* i *such that*

$$\mathrm{Hom}_H(V_i/V_{i-1}, \mathbf{1}_H) \neq 0.$$

Indeed, if $0 \neq \ell \in \mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma), \mathbf{1}_H)$ then there exists $i > 0$ minimal such that $\ell|_{V_i} \neq 0$ and therefore ℓ defines a non-zero element of $\mathrm{Hom}_H(V_i/V_{i-1}, \mathbf{1}_H)$.

Lemma 2.4. *We have*

$$\mathrm{Hom}_H(V_i/V_{i-1}, \mathbf{1}_H) \simeq \mathrm{Hom}_{P_i}(\sigma, \delta_{P_i}\delta_P^{-1/2})$$

where $P_i := \eta_i H \eta_i^{-1} \cap P$.

Proof. For $f \in V_i$ let

$$\phi_f(h) = f(\eta_i h), \ h \in H.$$

Clearly, $\ker(f \mapsto \phi_f) = V_{i-1}$. Furthermore, [BZ77, Theorem 5.2] (see also [BD08, Proposition 1.17]), this defines an isomorphism

$$V_i/V_{i-1} \simeq \mathrm{ind}_{H_i}^H((\delta_P^{1/2}\sigma)^{\eta_i}|_{H_i})$$

where $H_i = H \cap \eta_i^{-1} P \eta_i = \eta_i^{-1} P_i \eta_i$ and $(\delta_P^{1/2}\sigma)^{\eta_i}$ is the representation of $\eta_i^{-1} P \eta_i$ defined from $\delta_P^{1/2}\sigma$ by η_i conjugation. Here ind is non-normalized induction of compact support.

Now by Frobenious reciprocity [BZ76, Proposition 2.29] we have

$$\mathrm{Hom}_H(V_i/V_{i-1}, \mathbf{1}_H) \simeq \mathrm{Hom}_{H_i}(\delta_{H_i}^{-1}(\delta_P^{1/2}\sigma)^{\eta_i}, \mathbf{1}_{H_i}).$$

Conjugating by η_i gives an isomorphism of the right hand side with

$$\mathrm{Hom}_{P_i}(\delta_{P_i}^{-1}\delta_P^{1/2}\sigma, \mathbf{1}_{P_i}) \simeq \mathrm{Hom}_{P_i}(\sigma, \delta_{P_i}\delta_P^{-1/2}).$$

□

As an immediate consequence of Observation 2.3 and Lemma 2.4 we conclude

Corollary 2.5 (A necessary condition for distinction)**.** *If the representation* $\mathrm{Ind}_P^G(\sigma)$ *is* H*-distinguished then there exits* i *such that* σ *is* $(P_i, \delta_{P_i}\delta_P^{-1/2})$*-distinguished.*

In order to apply this to particular cases and obtain explicit necessary conditions for distinction we need to better understand the structure of the groups P_i, and the characters $\delta_{P_i}\delta_P^{-1/2}$ on them. We explain how this can be done when (G, H) is a symmetric pair.

2.2. **The setting for p-adic symmetric spaces.** For the rest of this section let θ be an involution on $\underline{G}$ defined over F and $\underline{H} = \underline{G}^\theta$. Set $G = \underline{G}(F)$ and $H = \underline{H}(F)$.

It is convenient to introduce the algebraic set

$$X = \{g \in G : g\theta(g) = e\}$$

with the G-action $g \cdot x = gx\theta(g)^{-1}$ (twisted conjugation). For a subgroup Q of G and $x \in X$ let Q_x be the stabilizer of x in Q. In particular, $H = G_e$. Note that $g \mapsto g \cdot e$ defines an imbedding $G/H \hookrightarrow X$.

For any parabolic subgroup P of G this induces an imbedding $P\backslash G/H \hookrightarrow P\backslash X$. It is well known ([HW93, Proposition 6.15]) that there are finitely many P-orbits in X and in particular Assumption 2.1 holds.

2.3. **Orbits for a minimal parabolic.** Fix a minimal parabolic subgroup P_0 of G. Helminck and Wang showed in [HW93, Lemma 2.4]) that there exists a maximal split torus T of G contained in P_0 and θ-stable. Write $P_0 = M_0 \ltimes U_0$ where $M_0 = C_G(T)$ and U_0 is the unipotent radical of P_0.

We provide a description of the P_0 orbits in X based on a generalization (as in [HW93] and [LR03]) of Springer's study of involutions in Weyl groups [Spr86].

Let $W = N_G(T)/M_0$ be the Weyl group of G with respect to T. By the Bruhat decomposition

$$W \simeq P_0\backslash G/P_0.$$

Since $\theta(T) = T$, θ defines an involution on W that we will also denote by θ.

In order to avoid some technicalities, for the sake of these notes we make the following simplification.

Assumption 2.6. *P_0 can be chosen so that $\theta(P_0) = P_0$.*

Remark 2.7. *In general, $\theta(P_0)$ is another minimal parabolic subgroup of G containing T (semi-standard) and there exists $\tau \in W$ such that $\theta(P_0) = \tau P_0 \tau^{-1}$. We can then replace the involution θ on W with $\theta'(w) = \tau^{-1}\theta(w)\tau$. It is an involution on W satisfying $\theta'(P_0) = P_0$ (but it does not arise from an involution on G). It is explained in* [Off17] *why this is enough in order to remove Assumption 2.6.*

Note that for $x \in X$ we have $P_0 \cdot x \subseteq P_0 x P_0$ and there is a unique w such that $P_0 \cdot x \subseteq P_0 w P_0$. Note further that since $\theta(x)^{-1} = x$ we must have $P_0\theta(w)^{-1}P_0 = P_0 w P_0$ and therefore $w\theta(w) = e$. Let

$$\mathfrak{J}_0(\theta) = \{w \in W : w\theta(w) = e\}$$

be the set of θ-twisted involutions on W. Let

$$\imath_0 : P\backslash X \to \mathfrak{J}_0(\theta)$$

be defined by $\imath_0(P_0 \cdot x) = w$ whenever $P_0 \cdot x \subseteq P_0 w P_0$.

Lemma 2.8. *The map $P_0 \cdot x \mapsto P_0 \cdot x \cap N_G(T)$ defines a bijection*

$$P_0\backslash X \simeq M_0\backslash(X \cap N_G(T)).$$

Proof. First, we observe that $P_0 \cdot x \cap N_G(T)$ is not empty. Let $\imath_0(P_0 \cdot x) = w$. We can decompose $x = u_1 n u_2$ with $u_1, u_2 \in U_0$ and $n \in w$ a representative of w in G. Then, n is uniquely determined and since $x = \theta(x)^{-1} = \theta(u_2)^{-1}\theta(n)^{-1}\theta(u_1)^{-1}$ we get that $n = \theta(n)^{-1}$ (that is, that $n \in X \cap w$). Furthermore, we must have $u_2 = y\theta(u_1)^{-1}$ with $y \in U' := U_0 \cap w^{-1}U_0 w$. Therefore, $x = u_1 \cdot (ny)$. In particular, $ny \in X$ satisfies $ny\theta(n)\theta(y) = e$. Note that $\theta_n(u) := n^{-1}\theta(u)n$ is an involution on U' satisfying the cocycle condition $y^{-1}\theta_n(y^{-1}) = e$. Since unipotent groups have trivial cohomology ($H^1(\langle\theta_n\rangle, U') = 1$ by [HW93, Lemma 0.1]), y^{-1} must be a coboundary, that is, there exists $u \in U'$ such that $y^{-1} = u\theta_n(u)^{-1} = un^{-1}\theta(u)^{-1}n$. Then $ny = \theta(u) \cdot n$ and therefore $n \in P_0 \cdot x \cap N_G(T)$.

It is left to show that $P_0 \cdot x \cap N_G(T)$ is a unique M_0-orbit. Let $n, n' \in X \cap N_G(T)$ and $p \in P_0$ be such that $p \cdot n = n'$. Then, in particular, $P_0 n P_0 = P_0 n' P_0$ and therefore $n'n^{-1} \in M_0$. Therefore $p = n'\theta(p)n^{-1} \in P_0 \cap nP_0n^{-1} = M_0 \ltimes (U_0 \cap wU_0w^{-1})$. If $p = mu$

with $m \in M_0$ and $u \in U_0 \cap wU_0w^{-1}$ then $mu = n'\theta(m)n^{-1}n\theta(u)n^{-1}$ and projecting to M_0 we get that $m = n'\theta(m)n^{-1}$ or in other words that $m \cdot n = n'$. $\square$

Example 2.9. *We explicate the bijection of the lemma in two examples.*

- *Let E/F be a quadratic extension, $\underline{G} = \mathrm{Res}_{E/F}((\mathrm{GL}_n)_E)$ and $\theta(g) = \bar{g}$ the Galois action, so that $G = \mathrm{GL}_n(E)$, $H = \mathrm{GL}_n(F)$ and $X = \{g \in G : g\bar{g} = e\}$. Then, for the diagonal torus $T = M_0$ of G we have*
$$M_0 \backslash (X \cap N_G(T)) \simeq W_2 := \{w \in W : w^2 = e\}.$$
Thus, for the standard Borel subgroup P_0 of upper triangular matrices in G, $P_0\backslash X$ is in bijection with W_2, the set of involutions in W.
- *Let $G = \mathrm{GL}_{2n}(F)$ and $\theta(g) = J\,{}^tg^{-1}J^{-1}$ where $J = J_n = \left(\begin{smallmatrix} 0 & w_n \\ -w_n & 0 \end{smallmatrix}\right)$ and $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n(F)$. Then $H = \mathrm{Sp}_{2n}(F)$ and $XJ = \{g \in G : {}^tg = -g\}$. For the diagonal torus $T = M_0$ we have*
$$T\backslash (X \cap N_G(T)) \simeq [w_{2n}]w_{2n}$$
where $[w]$ is the conjugacy class of w in W. In this case,
$$\mathfrak{J}_0(\theta) = \{w \in W : (ww_{2n})^2 = e\}$$
and unlike in the first example, $[w_{2n}]w_{2n}$ is strictly contained in $\mathfrak{J}_0(\theta)$ so that not every twisted involution lies in the image of $\imath_0$.

Remark 2.10. *If θ is a Galois involution then $\imath_0 : P_0\backslash(G \cdot e) \to \mathfrak{J}_0(\theta)$ is surjective* [LR03]*. In our second example $\imath_0$ is not surjective. In general it also needs not be injective.*

2.4. **The stabilizer $(P_0)_x$.** Let $x \in X \cap N_G(T)$ and suppose that $x \in w$ (i.e., $w = \imath_0(P_0 \cdot x)$). Note that $\theta_x(g) = x\theta(g)x^{-1}$ defines an involution on G and $G_x = G^{\theta_x}$. Therefore
$$(P_0)_x \subseteq P_0 \cap \theta_x(P_0) = P_0 \cap wP_0w^{-1} = M_0 \ltimes (U_0 \cap wU_0w^{-1}).$$
Also, $\theta_x(M_0) = M_0$ and $\theta_x(U_0 \cap wU_0w^{-1}) = U_0 \cap wU_0w^{-1}$. Therefore,
$$(P_0)_x = (M_0)_x \ltimes (U_0 \cap wU_0w^{-1})_x.$$

2.5. **Parabolic orbits.** Let $P = M \ltimes U \supseteq P_0$ be a standard parabolic subgroup of G with its standard Levi decomposition. Let $W_M = N_M(T)/M_0$ be the Weyl group of M. The Bruhat decomposition gives a bijection
$$P\backslash G/\theta(P) \simeq W_M\backslash W/W_{\theta(M)}.$$
Furthermore, every double coset in $W_M\backslash W/W_{\theta(M)}$ has a unique representative of minimal length. We denote by ${}_MW_{\theta(M)}$ the set of such representatives. It is the set of all left W_M-reduced and right $W_{\theta(M)}$-reduced elements of W. As before, we can define a map
$$\imath_M : P\backslash X \to \mathfrak{J}_0(\theta) \cap {}_MW_{\theta(M)}$$
characterized by $w = \imath_M(P \cdot x)$ if $Pw\theta(P) = Px\theta(P)$.

Recall that if $w \in {}_MW_{\theta(M)}$ then
$$P \cap w\theta(P)w^{-1} = (M \cap w\theta(P)w^{-1})(U \cap w\theta(P)w^{-1})$$

where $P(w) := M \cap w\theta(P)w^{-1}$ is a standard parabolic subgroup of M with Levi decomposition

$$P(w) = M(w) \ltimes U(w) \quad \text{where} \quad M(w) = M \cap w\theta(M)w^{-1} \quad \text{and} \quad U(w) = M \cap w\theta(U)w^{-1}.$$

In fact, we can decompose $P \cap w\theta(P)w^{-1} = L \ltimes Z$ where $L = M(w)$ is a standard Levi subgroup and the unipotent radical

$$Z = U(w)(U \cap w\theta(M)w^{-1})(U \cap w\theta(U)w^{-1}).$$

Note that $L = w\theta(L)w^{-1}$ and $Z = w\theta(Z)w^{-1}$. For the next result see e.g. [Off17, Lemmas 3.2, 3.3 and 6.8].

Lemma 2.11. *For a P-orbit $\mathcal{O} \in P\backslash X$ let $w = \imath_M(\mathcal{O})$, $L = M(w)$ and $Q \subseteq P$ the standard parabolic subgroup of G with Levi subgroup L. Then*

1. *$\mathcal{O} \cap Lw$ is not empty, in fact, it is a unique L-orbit.*
2. *For $x \in \mathcal{O} \cap Lw$ we have $P_x = L_x \ltimes Z_x = Q_x$.*
3. *For $x \in \mathcal{O} \cap Lw$ we have $\mathrm{pr}_M(Z_x) = U(w)$.*
4. *$\delta_{P_x}\delta_P^{-1/2}|_{L_x} = \delta_{Q_x}\delta_Q^{-1/2}|_{L_x}$.*

2.6. **A necessary condition for distinction-the case of a symmetric space.** Recall that $H = G_e$ and therefore, for $\eta \in G$ we have $\eta H \eta^{-1} = G_x$ where $x = \eta \cdot e \in X$. We then have

$$\eta H \eta^{-1} \cap P = P_x.$$

For a representation $\sigma \in \mathrm{Rep}(M)$, Corollary 2.5 says that if $\mathrm{Ind}_P^G(\sigma)$ is H-distinguished then there exists $x \in G \cdot e$ so that

$$\mathrm{Hom}_{P_x}(\delta_{P_x}^{-1}\delta_P^{1/2}\sigma, \mathbf{1}_{P_x}) \simeq \mathrm{Hom}_{P_x}(\sigma, \delta_{P_x}\delta_P^{-1/2}) \neq 0.$$

Taking §2.5 into considerations, we can choose $x \in Lw$ and then factoring through Z_x we obtain that

$$\mathrm{Hom}_{P_x}(\sigma, \delta_{P_x}\delta_P^{-1/2}) \simeq \mathrm{Hom}_{L_x}(r_{L,M}(\sigma), \delta_{Q_x}\delta_Q^{-1/2}) \tag{3}$$

where $r_{L,M}(\sigma)$ is the normalized Jacquet module of σ. As a consequence we therefore have

Theorem 2.12. *Let σ be a representation of M. If $\mathrm{Ind}_P^G(\sigma)$ is H-distinguished then there exists $x \in G \cdot e$ such that $L = M \cap x\theta(M)x^{-1}$ is the standard Levi of a standard parabolic $Q \subseteq P$ and $r_{L,M}(\sigma)$ is $(L_x, \delta_{Q_x}\delta_Q^{-1/2})$-distinguished.*

We call $x \in X$, M-admissible if $x\theta(M)x^{-1} = M$.

Remark 2.13. *In Theorem 2.12, x is always L-admissible. So the computation of the modulus function $\delta_{Q_x}\delta_Q^{-1/2}$ is already reduced to the case of admissible orbits. In fact, it can further be reduced to orbits that are minimal in a sense introduced by Springer (see* [Off17, §6]*).*

Remark 2.14. *If θ is a Galois involution then $\delta_{Q_x}\delta_Q^{-1/2}$ is trivial on L_x (*[LR03, Proposition 4.3.2]*). The necessary condition in Theorem 2.12 then becomes: $r_{L,M}(\sigma)$ is L_x-distinguished.*

2.7. **Sufficient conditions for distinction-the case of a symmetric space.** We formulate two sufficient conditions for distinction of $\mathrm{Ind}_P^G(\sigma)$. For more details see [Off17, §7].

2.7.1. *Open orbit contribution.*

Proposition 2.15. *Let $x \in G \cdot e$ be M-admissible and such that $\theta_x(P) = x\theta(P)x^{-1}$ is opposite to P (that is, $P \cap \theta_x(P) = M$). If σ is M_x-distinguished then $\mathrm{Ind}_P^G(\sigma)$ is H-distinguished.*

We explain how this is deduced from the theory of Blanc and Delorme [BD08]. Recall that if $x = \eta \cdot e$ then by the assumption on x, $P\eta H$ is open in G and

$$V = \{f \in \mathrm{Ind}_P^G(\sigma) : \mathrm{Supp}(f) \subseteq P\eta H\} \simeq \mathrm{ind}_{M_x}^{G_x}(\sigma)^\eta$$

(note that $P_x = M_x$). We can construct a non-zero H-invariant linear form $\ell \in \mathrm{Hom}_H(V, \mathbf{1})$ by

$$\ell(f) = \int_{M_x \backslash G_x} L(f(h\eta))\, dh$$

where $0 \neq L \in \mathrm{Hom}_{M_x}(\sigma, \mathbf{1}_{M_x})$. In general, ℓ does not extend to $\mathrm{Ind}_P^G(\sigma)$ but Blanc and Delorme showed that it can in a meromorphic family.

There is a non-zero meromorphic family of linear forms $\ell_\lambda \in \mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma \otimes \chi_\lambda), \mathbf{1})$. Here λ lies in a certain complex vector space (the -1 eigenspace of θ_x on $X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}$ where $X^*(M)$ is the lattice of F-rational characters of M) and χ_λ is the character of M associated with λ. We can define a non-zero element of $\mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma), \mathbf{1}_H)$ by taking a leading term in a generic line at $\lambda = 0$. This is illustrated in our first example in (2) for a rank one case.

2.7.2. *Closed orbit contribution.*

Proposition 2.16. *Let $x \in G \cdot e$ be M-admissible and such that $\theta_x(P) = P$. If σ is $(M_x, \delta_{P_x}\delta_P^{-1/2})$-distinguished then $\mathrm{Ind}_P^G(\sigma)$ is H-distinguished.*

Proof. If $x = \eta \cdot e$ then the assumption implies that $P\eta H$ is closed in G and can therefore be ordered last in the order assigned to $P\backslash G/H$ by Lemma 2.2. In the H-filtration of $\mathrm{Ind}_P^G(\sigma)$ that the geometric lemma provides, the associated component V is in fact a quotient of $\mathrm{Ind}_P^G(\sigma)$. Composition with the projection to V gives and imbedding

$$\mathrm{Hom}_H(V, \mathbf{1}_H) \hookrightarrow \mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma), \mathbf{1}_H).$$

By (3),

$$\mathrm{Hom}_{P_x}(\sigma, \delta_{P_x}\delta_P^{-1/2})) \simeq \mathrm{Hom}_{M_x}(\sigma, \delta_{P_x}\delta_P^{-1/2})$$

and by Lemma 2.4

$$\mathrm{Hom}_H(V, \mathbf{1}_H) \simeq \mathrm{Hom}_{P_x}(\sigma, \delta_{P_x}\delta_P^{-1/2}).$$

The proposition follows. □

2.8. **Standard modules and further notation.** Recall the Langlands classification. A standard module is a representation of the form $\mathrm{Ind}_P^G(\tau\otimes\lambda)$ where τ is a tempered representation of a Levi M of G and λ is a positive exponent with respect to P. The representation $\mathrm{Ind}_P^G(\tau\otimes\lambda)$ admits a unique irreducible quotient π. The relation $\mathrm{Ind}_P^G(\tau\otimes\lambda)\leftrightarrow\pi$ is a bijection between standard modules and the set of irreducible representations $\mathrm{Irr}(G)\subseteq\mathrm{Rep}(G)$.

Clearly the projection $\mathrm{Ind}_P^G(\tau\otimes\lambda)\to\pi$ defines an imbedding

$$\mathrm{Hom}_H(\pi,\mathbf{1}_H)\hookrightarrow\mathrm{Hom}_H(\mathrm{Ind}_P^G(\tau\otimes\lambda),\mathbf{1}_H).$$

We may therefore obtain necessary conditions for H-distinction of irreducible representations in terms of their standard modules by applying the geometric lemma.

This works particularly well when $G=\mathrm{GL}_n(F)$ since much more is known about the classification of irreducible representations.

Assume now that $G=\mathrm{GL}_n(F)$. Every tempered representation in $\mathrm{Irr}(G)$ is parabolically induced from a square integrable representation. We use the product symbol for parabolic induction:

$$\pi_1\times\pi_2=\mathrm{Ind}(\pi_1\otimes\pi_2).$$

Every standard module I for G is of the form

$$I=\delta_1\times\cdots\times\delta_k$$

where each δ_i is essentially square integrable (a twist by a character of square integrable). We remark that the order on the δ_i's is not unique. For example, any ordering for which the central exponents of the δ_i's are weakly decreasing works. This plays a role in applications of the gemetric lemma.

Forthermore, let ρ be a supercuspidal representation of $\mathrm{GL}_d(F)$ and let $\nu=|\det|$ on $\mathrm{GL}_d(F)$ for any d. For integers $a\le b$ let $\Delta(\rho,a,b)$ be the unique irreducible quotient of

$$\nu^a\rho\times\nu^{a+1}\rho\times\cdots\times\nu^b\rho.$$

Then $\Delta(\rho,a,b)$ is essentially square integrable and every essentially square integrable representation is obtained in this way. The Jacquet modules of $\delta=\Delta(\rho,a,b)$ are easy to describe. Suppose that δ is a representation of $\mathrm{GL}_n(F)$ and that $n=s+t$ and $M\simeq\mathrm{GL}_s(F)\times\mathrm{GL}_t(F)$. Then

$$r_{M,G}(\delta)=\begin{cases}\Delta(\rho,c,b)\otimes\Delta(\rho,a,c-1) & s=d(b-c+1)\\ 0 & d\not| s.\end{cases}$$

This allows us to explicate Theorem 2.12 further, for standard modules of G.

2.9. **Example: GL_n over a quadratic extension.** Let E/F be a quadratic extension, with Galois involution θ, $G=\mathrm{GL}_n(E)$ and $H=\mathrm{GL}_n(F)=G^\theta$. Fix a decomposition $n=n_1+\cdots+n_k$ and let P be the standard parabolic subgroup of G of type $(n_1,\dots,n_k)$.

For every admissible M-orbit $\mathcal{O}\in P\backslash X$, there is a representative $x\in\mathcal{O}$ so that there is an involution $\tau\in S_k$ such that $n_{\tau(i)}=n_i$ and

$$x\,\mathrm{diag}(g_1,\dots,g_k)x^{-1}=\mathrm{diag}(g_{\tau(1)},\dots,g_{\tau(k)}),\quad g_i\in\mathrm{GL}_{n_i}(F)\tag{4}$$

for all i. Then

$$M_x = \{\mathrm{diag}(g_1,\dots,g_k) : g_i = g_{\tau(i)}^\theta, i = 1,\dots,k\} \simeq \left(\times_{\tau(i)=i} \mathrm{GL}_{n_i}(F)\right) \times \left(\times_{i<\tau(i)} \mathrm{GL}_{n_i}(E)\right).$$

Note that for representations $\sigma_i \in \mathrm{Rep}(\mathrm{GL}_{n_i}(E))$, $i = 1,\dots,k$ the representation $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ of M is M_x-distinguished if and only if $\sigma_{\tau_i}^\theta = \tilde{\sigma}_i$, $i \neq \tau(i)$ and σ_i is $\mathrm{GL}_{n_i}(F)$-distinguished if $\tau(i) = i$.

We simply call distinguished, an H-distinguished representation of G. A complete classification of the distinguished standard modules was obtained by Max Gurevich in [Gur15].

Proposition 2.17. *A standard module of* $\mathrm{GL}_n(E)$ *is* $\mathrm{GL}_n(F)$*-distinguished if and only if it is of the form*

$$\delta_1^\theta \times \cdots \times \delta_s^\theta \times \sigma_1 \times \cdots \times \sigma_t \times \tilde{\delta}_s \times \cdots \times \tilde{\delta}_1$$

for essentially square integrable representations δ_i *and* σ_j *such that each* σ_j *is distinguished.*

For the 'if' part, $\sigma_1 \times \cdots \times \sigma_t$ is distinguished by a closed orbit argument (Proposition 2.16) and therefore the standard module at hand is distinguished by an open orbit argument (Proposition 2.15).

The key to the proof of the 'only if' part is a careful application of Theorem 2.12. For any standard module I, there exists an ordering of the essentially square integrable data $I = \delta_1 \times \cdots \times \delta_k$ so that only M-admissible orbits can contribute to $\mathrm{Hom}_H(I, \mathbf{1}_H)$. The proposition is immediate from this fact.

2.10. **Example: Symplectic periods.** Let $G = \mathrm{GL}_{2n}(F)$ and $H = \mathrm{Sp}_{2n}(F)$. Fix a decomposition $2n = n_1 + \cdots + n_k$ and let P be the standard parabolic subgroup of G of type $(n_1,\dots,n_k)$.

The admissible M-orbits in X can be described as follows. They are in bijection with the set

$$S_k(n_1,\dots,n_k) = \{\tau \in S_k : \tau^2 = 1, n_{\tau(i)} = n_i \text{ and if } \tau(i) = i \text{ then } n_i \text{ is even}\}.$$

Let $g^* = w_n {}^t g^{-1} w_n^{-1}$ define an involution on $\mathrm{GL}_n(F)$. For $\tau \in S_k(n_1,\dots,n_k)$ an admissible $x = x_\tau \in X$ can be chosen so that

$$M_x = \{\mathrm{diag}(g_1,\dots,g_k) : g_{\tau(i)} = g_i^* \text{ if } \tau(i) \neq i \text{ and } g_i \in \mathrm{Sp}_{n_i}(F) \text{ if } \tau(i) = i\}.$$

We then have

$$\delta_{P_x}\delta_P^{-1/2}(\mathrm{diag}(g_1,\dots,g_k)) = \prod_{i<\tau(i)} |\det(g_i)|. \tag{5}$$

By results of Gelfand-Kazdhan for an irreducible representation π of $\mathrm{GL}_n(F)$ we have $\pi^* \simeq \tilde{\pi}$. Thus, an irreducible representation $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ of M is $(M_x, \delta_{P_x}\delta_P^{-1/2})$-distinguished if and only if:

$$\sigma_i = \nu\sigma_{\tau(i)} \text{ if } i < \tau(i) \text{ and } \sigma_i \text{ is } \mathrm{Sp}_{n_i}(F)\text{-distinguished if } \tau(i) = i.$$

We remark further that if σ is a generic representation then for the above distinction condition to be satisfied the involution τ cannot have any fixed points. This is based on the following result ([HR90]).

Proposition 2.18. *An irreducible generic representation of G is not H-distinguished.*

For this case, we cannot currently classify all distinguished standard modules. The following partial result is proved in [MOS].

Proposition 2.19. *Let $I = \delta_1 \times \cdots \times \delta_k$ be a standard module of G so that $\delta_i \neq \delta_j$ for all $i \neq j$. If I is H-distinguished, then there exists an involution $\tau \in S_k$ with no fixed points so that $\delta_i = \nu\delta_{\tau(i)}$ whenever $i < \tau(i)$. (In particular, k must be even.)*

Again the key is to observe that for some ordering of the δ_i's only admissible orbits contribute.

Distinguished standard modules are all reducible. (By the Zelevinsky classification irreducible standard modules are generic, hence this follows from Proposition 2.18.) In order to deduce results about distinguished irreducible representations, we first explicate the consequences of the sufficient conditions for distinction.

Suppose that P is of type $(2m_1, \ldots, 2m_k)$. Then to the trivial involution $\tau \in S_k$ we associated $x = x_\tau$ such that

$$M_x = \mathrm{Sp}_{2m_1}(F) \times \cdot \times \mathrm{Sp}_{2m_k}(F)$$

and $\theta_x(P)$ is opposite to P. In fact, explicitly x is defined by $xJ_n = \mathrm{diag}(J_{m_1}, \ldots, J_{m_k})$.

Lemma 2.20 (Open orbit). *Let σ_i be an $\mathrm{Sp}_{2m_i}(F)$-distinguished representation of $\mathrm{GL}_{2m_i}(F)$, $i = 1, \ldots, k$. Then $\sigma_1 \times \cdots \times \sigma_k$ is $\mathrm{Sp}_{2n}(F)$-distinguished.*

Also, if P is of type $(n_1, \ldots, n_t, 2m, n_t, \ldots, n_1)$ then PH is closed in G and we have

Lemma 2.21 (Closed orbit). *If σ is an $\mathrm{Sp}_{2m}(F)$-distinguished representation of $\mathrm{GL}_{2m}(F)$ and π_i an irreducible representation of $\mathrm{GL}_{n_i}(F)$, $i = 1, \ldots, t$ then*

$$\nu\pi_1 \times \cdots \times \nu\pi_t \times \sigma \times \pi_t \times \cdots \times \pi_1$$

is $\mathrm{Sp}_n(F)$-distinguished.

In what follows, we say that a representation π of $\mathrm{GL}_d(F)$ is Sp-distinguished if d is even and π is $\mathrm{Sp}_d(F)$-distinguished.

2.10.1. *Distinction of Speh representations.* For an essentially square integrable representation δ of $\mathrm{GL}_d(F)$ let $U(\delta, k)$ be the unique irreducible quotient of

$$I = \nu^{(k-1)/2}\delta \times \cdots \times \nu^{(1-k)/2}\delta.$$

These are the Speh representations. If δ has a unitary central character then $U(\delta, k)$ is unitary and in general, Speh representations are the building blocks for the unitary dual of the general linear groups (see [Tad86]).

Question 2.22. *Which Speh representations are $\mathrm{Sp}(F)$-distinguished?.*

If k is odd then I is not distinguished (by Proposition 2.19) and therefore $U(\delta, k)$ is not distinguished. We need finer information for k even.

There is an exact sequence

$$0 \to \mathcal{K} \to I \to U(\delta, k) \to 0.$$

Applying a determinental formula due to Tadic one can obtain a better description of $\mathcal{K}$. Recall that δ has the form $\delta = \Delta(\rho, a, b)$ and then $\nu\delta = \Delta(\rho, a+1, b+1)$. If $\nu\delta \times \delta = \delta_1 \times \delta_2$ then write $\delta_1' \times \delta_2' = \Delta(\rho, a, b+1) \times \Delta(\rho, a+1, b)$. Write $I = \delta_1 \times \cdots \times \delta_k$ so that $\delta_i = \nu^{(k+1)/2-i}\delta$ and let

$$I_i = \delta_1 \times \cdots \times \delta_{i-1} \times \delta_i' \times \delta_{i+1}' \times \delta_{i+2} \times \cdots \times \delta_k.$$

Then $\mathcal{K} = \sum_{i=1}^{k-1} I_i$ ([LM14, Theorem 1]). The representations I_i are themselves standard modules that are not Sp-distinguished (this can be deduced from Proposition 2.19). Therefore $\mathcal{K}$ is not Sp-distinguished.

Assume $k = 2m$ is even. Then I is Sp-distinguished (Indeed, by Lemma 2.21 $\delta_{2i-1} \times \delta_{2i}$ is Sp-distinguished, $i = 1, \ldots, m$ and by Lemma 2.20 the product over all i is also Sp-distinguished). We therefore provided a purely local proof to the following result that was first obtained using global methods in [OS08a, OS08b, OS07].

Proposition 2.23. *A Speh representation $U(\delta, k)$ is* Sp*-distinguished if and only if k is even.*

By Lemma 2.20 we now also recover [OS07, Proposition 2].

Corollary 2.24. *Any representation of the form $U(\delta_1, 2m_1) \times \cdots \times U(\delta_t, 2m_t)$, where all δ_i's are essentially square integrable, is* Sp*-distinguished.*

Remark 2.25. *In fact, based on Tadic' classification of the unitary dual, every irreducible, unitary representation of a general linear group is a product of Speh representations. A natural question would then be, how to classify the* Sp*-distinguished unitary dual, or more generally, can we classify all* Sp*-distinguished, irreducible products of Speh representations? It turns out, that Corollary 2.24 provides the complete list. But in order to show this one needs to introduce more tools. We explain them next.*

2.11. **Klyachko models.** Let $G = \mathrm{GL}_n(F)$ and $n = 2k + r$. We define the Klyachko group

$$H_r = \{\left(\begin{smallmatrix} h & x \\ 0 & u \end{smallmatrix}\right) : h \in \mathrm{Sp}_{2k}(F),\, x \in M_{2k\times r}(F),\, u \in N_r\}$$

where N_r is the group of unipotent upper-triangular matrices in $\mathrm{GL}_r(F)$. Let ψ_r be the character of H_r defined by

$$\psi_r \begin{pmatrix} h & x \\ 0 & u \end{pmatrix} = \psi(u_{1,2} + \cdots + u_{r-1,r})$$

where ψ is a non-trivial character of F. Then we have the following result.

Theorem 2.26. (1) *Let π be an irreducible representation of G. Then (*[OS08b]*)*

$$\sum_r \dim \mathrm{Hom}_{H_r}(\pi, \psi_r) \leq 1.$$

(2) *Let π_i be a representation of $\mathrm{GL}_{n_i}(F)$ that is (H_{r_i}, ψ_{r_i})-distinguished, $i = 1, \ldots, k$. Then $\pi_1 \times \cdots \times \pi_k$ is (H_r, ψ_r)-distinguished for $r = r_1 + \cdots + r_k$. (*[MOS]*)*

(3) *Let δ be an essentially square integrable representation of $\mathrm{GL}_d(F)$. Then for k-odd the representation $U(\delta, k)$ is (H_d, ψ_d)-distinguished. (*[OS08a]*)*

From here we easily get the classification of the Sp-distinguished unitary dual.

Corollary 2.27. *Let $\delta_1, \dots, \delta_t$ be essentially square integrable irreducible representations and $k_1, \dots, k_t$ be such that $\pi = U(\delta_1, k_1) \times \cdots \times U(\delta_t, k_t)$ is irreducible. Then π is* Sp-*distinguished if and only if k_i is even for all i.*

The uniqueness and disjointness of Klyachko models (Theorem 2.26(1)) are based on another important tool in the study of distinguished representations, the theory of invariant distributions. We briefly touch upon this rich subject.

2.12. **Multiplicity one results.** We say that (G, H) is a Gelfand pair if $\dim_H(\pi, \mathbf{1}_H) \le 1$ for every irreducible representation π of G. It appears to be difficult to directly prove (G, H) is a Gelfand pair without proving a stronger result.

A central tool in the direction of multiplicity one results is the Gelfand trick. Let $D(G) = C_c^\infty(G)^*$ be the space of distributions on G.

Proposition 2.28. *Suppose that τ is an anti-involution on G such that for every $D \in D(G)^{H \times H^\tau}$ we also have $D^\tau = D$. Then*

$$\dim \operatorname{Hom}_H(\pi, \mathbf{1}_H) \cdot \dim \operatorname{Hom}_{H^\tau}(\tilde{\pi}, \mathbf{1}_{H^\tau}) \le 1.$$

Proof. Let $\ell \in \operatorname{Hom}_H(\pi, \mathbf{1}_H)$ and $\ell' \in \operatorname{Hom}_{H^\tau}(\tilde{\pi}, \mathbf{1}_{H^\tau})$. Consider the spherical distribution $D(f) = \ell'(\pi^*(f)\ell)$ (note that $\pi^*(f)\ell = \ell \circ \pi(f) \in \tilde{\pi}$). Then clearly $D \in D(G)^{H \times H^\tau}$. Therefore, $D = D^\tau$. Now define the linear form $B(f_1, f_2) = D(f_1 * f_2)$. Then $B(f_1, f_2) = B(f_2^\tau, f_1^\tau)$ (since $(f_1 * f_2)^\tau = f_2^\tau * f_1^\tau$)) and we get that

$$L\ker(B) = R\ker(B)^\tau$$

($L\ker$ (resp. $R\ker$) is the left (resp. right) kernel). Recall that

$$B(f_1, f_2) = \langle \ell \circ \pi(f_1), \ell' \circ \tilde{\pi}(f_2) \rangle.$$

Assume that the product of dimensions is not zero and take $\ell \ne 0$ and $\ell' \ne 0$. Then $L\ker(B) = \{f : \ell \circ \pi(f) = 0\}$ and $R\ker(B)^\tau = \{f^\tau : \ell' \circ \tilde{\pi}(f) = 0\}$. But the relation between the left and right kernels of B then implies that the kernel of the operator $(f \mapsto \ell \circ \pi(f)) : C_c^\infty(G) \to \tilde{\pi}$ is uniquely determined by ℓ'. Thus, for $0 \ne \ell_1, \ell_2 \in \operatorname{Hom}_H(\pi, \mathbf{1}_H)$ there exists $c \in \mathbb{C}$ such that $(\ell_1 - c\ell_2) \circ \pi(f) = 0$ for all f. This clearly implies $\ell_1 = c\ell_2$ and we get that $\dim \operatorname{Hom}_H(\pi, \mathbf{1}_H) = 1$ and by symmetry also $\operatorname{Hom}_{H^\tau}(\tilde{\pi}, \mathbf{1}_{H^\tau}) = 1$. □

An even simpler statement of a similar nature is

Lemma 2.29. *Let H_i be a closed subgroup of G and χ_i a character of H_i, $i = 1, 2$. If $D(G)^{H_1 \times H_2, \chi_1 \otimes \chi_2} = 0$ then*

$$\dim \operatorname{Hom}_{H_1}(\pi, \chi_1) \cdot \dim \operatorname{Hom}_{H_2}(\tilde{\pi}, \chi_2) = 0, \quad \pi \in \operatorname{Irr}(G).$$

This means, no irreducible π imbeds in both $\operatorname{Ind}_{H_i}^G(\chi_i)$, $i = 1, 2$.

Next we recall how this was applied in our two main examples of symmetric pairs. We recall that in those cases one can show that if an irreducible representation π of G is H-distinguished then so is $\tilde{\pi}$.

For the case $G = \mathrm{GL}_n(E)$ and $H = \mathrm{GL}_n(F)$ Flicker showed that for the anti-involution $g^\tau = \theta(g)^{-1}$ (inverse of Galois conjugate) we have: if $D \in D(G)^{H\times H}$ then $D^\tau = D$ (note that $H = H^\tau$) and deduced:

Theorem 2.30 ([Fli91])**.** *Let π be an irreducible representaion of $\mathrm{GL}_n(E)$. Then*

- $\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \mathbf{1}_{\mathrm{GL}_n(F)}) \le 1$
- *If* $\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \mathbf{1}_{\mathrm{GL}_n(F)}) = 1$ *then* $\pi^\theta \simeq \tilde{\pi}$.

For $G = \mathrm{GL}_{2n}(F)$ and $H = \mathrm{Sp}_{2n}(F)$ Heummos and Rallis showed that $D \in D(G)^{H\times H}$ also satisfies $D^\tau = D$ for g^τ the transpose of g and deduced

Theorem 2.31 ([HR90])**.** *For an irreducible representation π of $\mathrm{GL}_{2n}(F)$ we have*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(F)}(\pi, \mathbf{1}_{\mathrm{Sp}_{2n}(F)}) \le 1.$$

For $G = \mathrm{GL}_n(F)$, $n = 2k + r$ and H_r the associated Klyachko subgroup, with τ being the transpose again, we showed in [OS08b] that

- $D(G)^{H_r \times H^\tau_{r'}, \psi_r \otimes \psi^\tau_{r'}} = 0$ for $r \neq r'$;
- Every $D \in D(G)^{H_r \times H^\tau_r, \psi_r \otimes \psi^\tau_r}$ also satisfies $D = D^\tau$.

From this we deduced Theorem 2.26(1).

We remark that a key tool in order to prove the vanishing of a space of invariant distributions is the following.

Proposition 2.32. *Let Q be a group acting on G and χ a character of Q. If $\chi|_{\mathrm{Stab}_Q(g)} \neq \mathbf{1}$ for all $g \in G$ then $D(G)^{Q,\chi} = 0$.*

3. Period integrals of automorphic forms

Let k be a number field and $\mathbb{A} = \mathbb{A}_k$ the ring of adeles of k. Let G be a reductive group defined over k and let

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : |\chi(G)|_{\mathbb{A}} = 1,\ \chi \in X^*(G)\}$$

where $|\cdot|_{\mathbb{A}}$ is the standard absolute value on $\mathbb{A}$ and $X^*(G)$ is the lattice of k-rational characters of G.

Fix a k-subgroup H of G and let

$$H(\mathbb{A})^{(1)} = H(\mathbb{A}) \cap G(\mathbb{A})^1.$$

For an automorphic form $\phi : G(k)\backslash G(\mathbb{A}) \to \mathbb{C}$, whenever defined by a convergent integral, we denote its H-period integral by

$$\mathcal{P}_H(\phi) = \int_{H(k)\backslash H(\mathbb{A})^{(1)}} \phi(h)\ dh.$$

3.1. **Motivating example.** As a classical example, let

$$E(z,s)=\sum_{(m,n)\in\mathbb{Z}^2_{\geq 0}\setminus\{(0,0)\}}\frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}}$$

be the normalized real analytic Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. It is straightforward that evaluation at $z=i:=\sqrt{-1}$ gives the Dedekind zeta function

$$E(i,s)=\zeta_{\mathbb{Q}(i)}(s).$$

Since

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\simeq\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/SO(2)\simeq\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})/K$$

where $K=SO(2)\prod_p\mathrm{SL}_2(\mathbb{Z}_p)$, we can lift $E(z,s)$ to an Eisenstein series $E(g,s)$ on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$ and observe that the special value of the classical Eisenstein series is, in fact, the compact period integral of the adelic version

$$E(i,s)=\int_{SO_2(\mathbb{Q})\backslash SO_2(\mathbb{A}_{\mathbb{Q}})}E(h,s)\,dh=\mathcal{P}_{SO_2}(E(s)).$$

3.2. **Definitions and generalities.**

Definition 3.1. *Let π be an automorphic representation of $G(\mathbb{A})$. We say that π is H-distinguished if $\mathcal{P}_H|_\pi$ is defined by a convergent integral and is not identically zero.*

Remark 3.2. *When H is reductive $\mathcal{P}_H$ is convergent on cusp forms (*[AGR93]*).*

Recall that an irreducible, cuspidal automorphic representation π of $G(\mathbb{A})$ decomposes (representation theoretically) as a restricted tensor product over all places v of k, $\pi\simeq\otimes\pi_v$ ([Fla79]).

Observation 3.3. *If π is an H-distinguished, irreducible cuspidal automorphic representation then π_v is $H(k_v)$-distinguished for all places v of k.*

Indeed, there exists a pure tensor $\phi\simeq\otimes_v\varphi_v$ in π such that $\mathcal{P}_H(\phi)\neq 0$. Fix a place v_0 of k and let $\phi^0=\otimes_{v\neq v_0}\varphi_v$ and ℓ the linear form on π_{v_0} defined by

$$\ell(\xi)=\mathcal{P}_H(\phi^0\otimes\xi).$$

Then $0\neq\ell\in\mathrm{Hom}_{H(k_{v_0})}(\pi_{v_0},\mathbf{1}_{H(k_{v_0})})$.

Therefore, the local components of irreducible, H-distinguished cuspidal automorphic representations are locally $H(k_v)$-distinguished. This remains true for any irreducible automorhic representation in the discrete spectrum as long as $\mathcal{P}_H$ converges on it.

In many cases we expect the period integral to be an Euler product.

Definition 3.4. *The period integral $\mathcal{P}_H$ is called factorizable on π, if there exist $l_v\in\mathrm{Hom}_{H(k_v)}(\pi_v,\mathbf{1}_{H(k_v)})$ for all places v of k such that*

$$\mathcal{P}_H(\phi)=\prod_v\ell_v(\varphi_v),\quad \phi\simeq\otimes_v\varphi_v.$$

This is the case, for example, if local multiplicity one is satisfied for G and H, that is, if $\dim \mathrm{Hom}_{H(k_v)}(\pi_v, \mathbf{1}_{H(k_v)}) \leq 1$ for all irreducible representations π_v of $G(k_v)$ and all places v of k.

For the rest of this section, unless otherwise specified let $G = \mathrm{GL}_{2n}$ and $H = \mathrm{Sp}_{2n}$.

3.3. **Symplectic periods-results of Jacquet and Rallis.** Jacquet and Rallis observed the following.

Lemma 3.5 ([JR92b]). *Let π be an irreducible, cuspidal automorphic representation of $G(\mathbb{A})$. Then π is not H-distinguished.*

The reason is that every cuspidal π is generic, so for any place v of k, π_v is generic. The Lemma is therefore immediate from Observation 3.3 and Proposition 2.18.

Question 3.6. *Can we find Sp_{2n}-distinguished automorphic representations of GL_{2n}?*

Yes, the first example was provided by Jacquet and Rallis.

Let ρ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_k)$. Denote by

$$I(\rho, s) = \mathrm{Ind}_{P(\mathbb{A}_k)}^{G(\mathbb{A}_k)}(|\det|^{s/2} \rho \otimes |\det|^{-s/2} \rho)$$

the induced representation where P is the standard parabolic of type (n, n).

Let K be the standard maximal compact of $G(\mathbb{A}_k)$ so that $G(\mathbb{A}_k) = P(\mathbb{A}_k)K$. Let ξ be the function on $G(\mathbb{A}_k)$ defined by $\xi(g) = \left|\frac{\det a}{\det b}\right|$ where $g = pk$, $p = \left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \in P$ and $k \in K$. For $s \in \mathbb{C}$ and $\varphi \in I(\rho) := I(\rho, 0)$ let $\varphi_s = \xi^s \varphi \in I(\rho, s)$.

The Eisenstein series, is an intertwining operation

$$E(\cdot, s) : I(\rho, s) \to Aut(\mathrm{GL}_{2n}(k) \backslash \mathrm{GL}_{2n}(\mathbb{A}_k))$$

defined by the meromorphic continuation of the series

$$E(g, \varphi, s) = \sum_{P(k) \backslash \mathrm{GL}_{2n}(k)} \varphi_s(\gamma g)$$

that converges for $\mathrm{Re}(s) \gg 1$. The Eisenstein series has a simple pole at $s = 1$ and we can define the residue

$$\mathcal{E}_{-1}(\varphi) = \lim_{s \to 1} (s-1) E(\varphi, s).$$

The space spanned by these residues

$$U(\rho, 2) = \{\mathcal{E}_{-1}(\varphi) : \varphi \in I(\rho)\}$$

is an irreducible representation in the discrete automorphic spectrum of $\mathrm{GL}_{2n}(\mathbb{A}_k)$. From a representation theoretic point of view

$$\varphi_1 \mapsto \mathcal{E}_{-1}(\varphi) : I(\rho, 1) \to U(\rho, 2)$$

is a surjective intertwining operator and $U(\rho, 2)$ is the unique irreducible quotient of $I(\rho, 1)$. It can be viewed as a global analogue of the Speh representations introduced in the local setting. By a closed orbit argument, it is easy to define a non-zero $H(\mathbb{A}_k)$-invariant linear form on the induced representation $I(\rho,1)$ as follows

$$\ell_H(\varphi)=\int_{(H\cap P)(\mathbb{A}_k)\backslash H(\mathbb{A}_k)}\int_{\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k)^1}\varphi(\left(\begin{smallmatrix} g & 0\\ 0 & g^*\end{smallmatrix}\right)k)\ dg\ dk.$$

Jacquet and Rallis showed that this linear form factors through its unique irreducible quotient via the symplectic period $\mathcal{P}_H$, that is:

Theorem 3.7 ([JR92b]). *We have*

$$\mathcal{P}_H(\mathcal{E}_{-1}(\varphi))=\ell_H(\varphi_1).$$

In particular, the representation $U(\rho,2)$ is H-distinguished.

In fact, with some extra work this can be generalized to the entire discrete spectrum. Mœglin and Waldspurger provided an explicit decomposition of the discrete automorphic spectrum of a general linear group. Based on their decomposition we can classify the entire H-distinguished discrete automorphic spectrum.

3.4. **The discrete automorphic spectrum of** GL_n**.** The discrete spectrum of $G=\mathrm{GL}_n(\mathbb{A}_k)$ admits the following decomposition ([MW89])

$$L^2_{disc}(G(k)\backslash G(\mathbb{A}_k)^1)=\hat{\oplus}U(\rho,m)$$

where the (Hilbert) direct sum is over all divisors m of n and all cuspidal irreducible automorphic representations of $\mathrm{GL}_d(\mathbb{A}_k)$ where $n=md$. The representation $U(\rho,m)$ is the unique irreducible quotient of

$$\mathrm{Ind}_{P(\mathbb{A}_k)}^{G(\mathbb{A}_k)}(|\det|^{(m-1)/2}\rho\otimes\cdots\otimes|\det|^{(1-m)/2}\rho).$$

Again, it is spanned by residues of Eisenstein series

$$\mathcal{E}_{-1}(g,\varphi)=\lim_{s\mapsto((m-1)/2,\dots,(1-m)/2)}\prod_{i=1}^{m-1}(s_{i+1}-s_i-1)E(g,\varphi,s).$$

3.5. **Symplectic periods on the discrete spectrum of** G**.** Let $2n=dt$ so that $U(\rho,t)$ is a discrete automorphic representation of $G(\mathbb{A}_k)$. The symplectic periods on $U(\rho,t)$ are studied in [Off06a, Off06b]. In particular, it is proved that $\mathcal{P}_H$ converges on discrete automorphic forms. The classification of the distinguished discrete spectrum is as follows.

Theorem 3.8. *The representation $U(\rho,t)$ is Sp_{2n}-distinguished if and only if t is even.*

We explain the key ideas behind this generalization of the results of Jacquet and Rallis.

Let $P=MU$ be the standard parabolic subgroup of G of type $(d,\dots,d)$. Recall that

$$M_H=M\cap H=\{m=\mathrm{diag}(g_1,\dots,g_{[t/2]},h,g^*_{[t/2]},\dots,g_1^*):g_i\in\mathrm{GL}_d,h\in\mathrm{Sp}_d\ \text{ if } t \text{ is odd}\}.$$

For $s\in\mathbb{C}^t$ let

$$I(\rho,s):=\mathrm{Ind}_{P(\mathbb{A}_k)}^{G(\mathbb{A}_k)}(|\det|^{s_1}\rho\otimes\cdots\otimes|\det|^{s_t}\rho)$$

and consider the linear form

$$\ell_H(\varphi) = \int_{K_H} \int_{M_H(k)\backslash M_H(\mathbb{A}_k)^1} \varphi(mk)\ dm\ dk$$

on $I(\rho, s)$. Here $K_H = K \cap H(\mathbb{A}_k)$.

If t is odd then the inner integral vanishes by Lemma 3.5 and it is easier to generalize the results of Jacquet-Rallis and show that $U(\rho, t)$ is not H-distinguished:

$$\mathcal{P}_{\mathrm{Sp}_{2n}}(\mathcal{E}_{-1}(\varphi)) = 0.$$

Assume now that $t = 2k$ is even. As in the local setting (see (5)) we have

$$\delta_{P_H(\mathbb{A}_k)}\delta_{P(\mathbb{A}_k)}^{-1/2}(m) = \prod_{i=1}^{k} |\det g_i| \,.$$

As in the $t = 2$ case it is therefore easy to see that ℓ_H is $H(\mathbb{A}_k)$-invariant on $I(\rho, s)$ whenever $s_i - s_{t+1-i} = 1$ for all $i = 1, \dots, [t/2]$ and it is not identically zero for such s.

Let

$$\Lambda = (\frac{t-1}{2}, \dots, \frac{1-t}{2}).$$

In order to define an H-invariant linear form on $I(\rho, \Lambda)$ we apply an intertwining operator. Let $w(2i-1) = i$ and $w(2i) = t+1-i$, $i = 1, \dots, k$. Then ℓ_H is H-invariant on $I(\rho, w\Lambda)$. For the residue at Λ of the standard intertwining operator $M_{-1}(w) : I(\rho, \Lambda) \to I(\rho, w\Lambda)$ the linear form

$$\ell_H \circ M_{-1}(w)$$

on $I(\rho, \Lambda)$ is H-invariant and we show in [Off06a] that

$$\mathcal{P}_H(\mathcal{E}_{-1}(\varphi)) = \ell_H(M_{-1}(w)(\varphi)).$$

We therefore obtain that $\mathcal{P}_H|_{U(\rho,2k)} \neq 0$ if and only if $\ell_H|_{\mathrm{Im}\, M_{-1}(w)} \neq 0$. The linear form $\ell_H = \otimes_v \ell_v$ and the intertwining operator $M_{-1}(w) = \otimes_v M_v(w)$ are both factorizable. Therefore, this becomes a local problem:

For a generic unitary representation σ of $\mathrm{GL}_d(k_v)$ we can define the linear form $\ell_v \circ M_v(w)$ on $I(\sigma, \Lambda) = \nu^{k-1/2}\sigma \times \cdots \times \nu^{1/2-k}\sigma$. The linear form ℓ_v is known to be non-zero on $I(\sigma, w\Lambda)$. We show in [Off06b] that

$$\mathrm{Hom}_{H_v}(I(\sigma, w\Lambda)/\,\mathrm{Im}\, M_v(w), \mathbf{1}_{H_v}) = 0$$

and therefore, that ℓ_v cannot vanish on $\mathrm{Im}\, M_v(w)$. In the finite places, this is shown using the geometric lemma. For the archimedean places, other techniques are required. We show, that $\ell_v(\varphi) \neq 0$ for φ the minimal K-type in $I(\sigma, w\Lambda)$ in the sense of Vogan. This is using the Cartan-Helgason theorem. The minimal K-type lies in the image of the intertwining operators and we are therefore done.

4. The relative trace formula

The relative trace formula was developed by Jacquet as a tool to study period integrals of automorphic forms. It is a distribution that has both a geometric and a spectral expansion where contribution to the spectral expansion arises solely by distinguished representations.

4.1. **Setting and definition of the relative trace formula.** Let G be a connected reductive group defined over a number field k. A relative trace formula (RTF) is attached to a quintuple $\mathcal{D} = (G, H_1, H_2, \chi_1, \chi_2)$ where H_i is a k-subgroup of G and χ_i a character of $H_i(k)\backslash H_i(\mathbb{A}_k)$, $i = 1, 2$.

In order for everything to be well defined without further convergence issues the reader may assume that G is anisotropic. In general, some regularization is required and to date it is only available for some special cases (see e.g. [Lap06]). Without the anisotropic assumption the reader should take the notes bellow only as heuristics.

For $f \in C_c^\infty(G(\mathbb{A}_k))$ let

$$\mathrm{RTF}_{\mathcal{D}}(f) = \int_{H_1(k)\backslash H_1(\mathbb{A}_k)} \int_{H_2(k)\backslash H_2(\mathbb{A}_k)} K_f(h_1, h_2)\chi_1(h_1)\chi_2(h_2)\ dh_1\ dh_2$$

where

$$K_f(x, y) = \sum_{\gamma \in G(k)} f(x^{-1}\gamma y),\ x,\ y \in G(\mathbb{A}_k)$$

is the standard kernel function associated to the test function f.

4.2. **Geometric expansion.** The geometric expansion of the RTF is based on the fact that

$$K_f(x, y) = \sum_{\eta} \sum_{(\alpha_1, \alpha_2)} f(x^{-1}\alpha_1^{-1}\eta\alpha_2 y)$$

where η is summed over a set of representatives for $H_1(k)\backslash G(k)/H_2(k)$ and (α_1, α_2) over a set of representatives for $\mathcal{H}_\eta(F)\backslash\mathcal{H}(F)$ where $\mathcal{H} = H_1 \times H_2$ and $\mathcal{H}_\eta = \mathrm{Stab}_{\mathcal{H}}(\eta)$.

Expanding and unfolding we see that

$$\mathrm{RTF}_{\mathcal{D}}(f) = \sum_{\eta} \int_{\mathcal{H}_\eta(k)\backslash\mathcal{H}(\mathbb{A}_k)} f(h_1^{-1}\eta h_2)\chi_1(h_1)\chi_2(h_2)\ d(h_1, h_2) = \tag{6}$$

$$\sum_{\eta} \int_{\mathcal{H}_\eta(A_k)\backslash\mathcal{H}(\mathbb{A}_k)} \left[\int_{\mathcal{H}_\eta(k)\backslash\mathcal{H}_\eta(A_k)} \chi_1(t_1)\chi_2(t_2) d(t_1, t_2)\right] f(h_1^{-1}\eta h_2)\chi_1(h_1)\chi_2(h_2)\ d(h_1, h_2).$$

Definition 4.1. *A double coset $H_1(k)\eta H_2(k)$ is $\mathcal{D}$-relevant if $\chi_1 \times \chi_2|_{\mathcal{H}_\eta(\mathbb{A}_k)} = \mathbf{1}$. We denote by $[H_1\backslash G/H_2]_{\mathcal{D}-\mathrm{rel}}$ a set of representatives for the $\mathcal{D}$-relevant double cosets in $H_1(k)\backslash G(k)/H_2(k)$.*

As a consequence of the above unfolding we obtain the geometric expansion of the relative trace formula

$$\mathrm{RTF}_{\mathcal{D}}(f) = \sum_{\eta \in [H_1\backslash G/H_2]_{\mathcal{D}-\mathrm{rel}}} v(\eta)\ \mathcal{O}_{\mathcal{D}}(\eta, f)$$

where

$$v(\eta) = \mathrm{vol}(\mathcal{H}_\eta(k)\backslash\mathcal{H}_\eta(\mathbb{A}_k))$$

and $\mathcal{O}_\mathcal{D}(\eta, f)$ is the orbital integral

$$\mathcal{O}_\mathcal{D}(\eta, f) = \int_{\mathcal{H}_\eta(\mathbb{A}_k)\backslash\mathcal{H}(\mathbb{A}_k)} f(h_1^{-1}\eta h_2)\chi_1(h_1)\chi_2(h_2)\ d(h_1, h_2).$$

Note that the orbital integral is factorizable. If $f = \otimes_v f_v$ then $\mathcal{O}_\mathcal{D}(\eta, f) = \prod_v \mathcal{O}_{\mathcal{D}_v}(\eta, f_v)$ where the local orbital integrals are defined similarly.

4.3. **Global periods and distinction with respect to a character.** In order to explain the relation with distinction, as in the local setting, we extend the notion of period integrals and distinction to allow twists by characters.

For a subgroup H of G, a character χ of $H(k)\backslash H(\mathbb{A}_k)$ and an automorphic form ϕ on $G(\mathbb{A}_k)$, whenever well-defined by a convergent integral let

$$\mathcal{P}_{H,\chi}(\phi) = \int_{H(k)\backslash H(\mathbb{A}_k)^{(1)}} \phi(h)\chi(h)\ dh.$$

An automorphic representation of $G(\mathbb{A}_k)$ is called (H, χ)-distinguished if $\mathcal{P}_{H,\chi}|_\pi$ is absolutely convergent and defines a non-zero linear form on π.

4.4. **On the spectral expansion.** For general G, Arthur expanded spectrally

$$K_f(x, y) = \sum_{\mathfrak{X}} K_{f,\mathfrak{X}}(x, y)$$

where $\mathfrak{X}$ ranges over equivalence classes of cuspidal data for G. Under our simplifying anisotropic assumption this gives an expansion of $\mathrm{RTF}_\mathcal{D}$ as a sum over the irreducible cuspidal automorphic representations of $G(\mathbb{A}_k)$. We look more explicitly into the contribution of an irreducible, cuspidal automorphic representation.

Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_k)$ and

$$K_{f,\pi}(x, y) = \sum_\phi (\pi(f)\phi)(x)\overline{\phi(y)}$$

where ϕ ranges over an orthonormal basis of the isotypic component $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k)^1)_\pi$ of π in the cuspidal automorphic spectrum of G. The contribution of π to the relative trace formula is expressed by the spherical distribution

$$B_\pi^\mathcal{D}(f) = \int_{\mathcal{H}(k)\backslash\mathcal{H}(\mathbb{A}_k)} K_{f,\pi}(h_1, h_2)\chi_1(h_1)\chi_2(h_2)\ d(h_1, h_2) = \sum_\phi \mathcal{P}_{H_1,\chi_1}(\pi(f)\phi)\overline{\mathcal{P}_{H_2,\chi_2}(\phi)}.$$

The following simple statement holds for a general G as long as $\mathcal{P}_{H_i,\chi_i}$ converge on cuspidal representations $i = 1, 2$. (If H_i is either reductive or unipotent for $i = 1, 2$ then convergence is granted.)

Fact 4.2. *Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_k)$. If $B_\pi^\mathcal{D} \neq 0$ then π is both (H_i, χ_i)-distinguished $i = 1, 2$.*

In many applications of trace formulas in general and the relative trace formula in particular one compares between formulas for two sets of data $\mathcal{D}$ and $\mathcal{D}'$ in order to deduce results on one from the other. In what follows we give one of the earlier examples due to Jacquet and Rallis ([JR92a]) of such a comparison of relative trace formulas in general rank.

4.5. **The Kuznetsov trace formula.** Let $\mathcal{D}' = (G', {}^tN', N', {}^t\psi', \psi')$ where $G' = \mathrm{GL}_n$ considered as an algebraic group over the number field k, N' is its subgroup of upper triangular unipotent matrices and $\psi'(u) = \psi_0(u_{1,2} + \cdots + u_{n-1,n})$, $u \in N'(\mathbb{A}_k)$ where ψ_0 is a fixed non-trivial character of $k\backslash\mathbb{A}_k$. The group ${}^tN'$ (resp. the character ${}^t\psi'$) is obtained from N' (resp. ψ') by applying transpose. The Kuznetsov trace formula is the RTF defined by

$$\mathrm{KTF} = \mathrm{RTF}_{\mathcal{D}'}.$$

Following Jacquet and Rallis we explicitly describe the $\mathcal{D}'$-relevant orbits $[{}^tN'\backslash G'/N']_{\mathcal{D}'-\mathrm{rel}}$. Let A' be the maximal torus of diagonal matrices in G'. By the Bruhat decomposition we have

$${}^tN'(k)\backslash G'(k)/N'(k) \simeq N_{G'(k)}(A'(k)).$$

That is, the scaled permutation matrices provide a complete set of representatives for the double coset space. Let $\mathcal{N}' = {}^tN' \times N'$. We begin by a computation of relevant orbits for the $n = 2$ case.

4.5.1. *Relevant orbits for* GL_2. For $n = 2$ we have $N_{G'(k)}(A'(k)) = A'(k) \sqcup wA'(k)$ where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The double coset of $a = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) \in A'(k)$ lies in the open Bruhat cell and is in bijection with $\mathcal{N}'(k)$. It follows that the stabilizer $\mathcal{N}'_a$ of a is trivial and in particular a is $\mathcal{D}'$-relevant. Indeed,

$$\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & a_1 y \\ xa_1 & a_2 + xa_1 y \end{smallmatrix}\right)$$

and the right hand side equals $\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)$ if and only if $x = y = 0$.

For wa we note that

$$\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & a_2 \\ a_1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & a_2 \\ a_1 & xa_2 + ya_1 \end{smallmatrix}\right)$$

and therefore

$$\mathcal{N}'_{wa} = \{(\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)) : y = -a_1^{-1}a_2 x\}.$$

It follows that wa is relevant if and only if $a_1 = a_2$. Thus,

$$[{}^tN'\backslash G'/N']_{\mathcal{D}'-\mathrm{rel}} = \{\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) : a_1,\, a_2 \in k^*\} \sqcup \{\left(\begin{smallmatrix} 0 & a \\ a & 0 \end{smallmatrix}\right) : a \in k^*\}.$$

4.5.2. *Relevant orbits.* Let $w_n = (\delta_{i,n+1-j}) \in G'(k)$ be the longest permutation matrix. For any standard Levi subgroup $M \simeq \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t}$ of G' let $w_M = \mathrm{diag}(w_{n_1}, \ldots, w_{n_t})$ and Z_M the centre of M so that

$$Z_M(k) = \mathrm{diag}(a_1 I_{n_1}, \ldots, a_t I_{n_t}) : a_1, \ldots, a_t \in k^*\}.$$

The $\mathcal{D}'$-relevant orbits are described as follows

$$[{}^tN'\backslash G'/N']_{\mathcal{D}'-\mathrm{rel}} = \sqcup_M \{w_M a : a \in Z_M(k)\}$$

where the disjoint union is over all standard Levi subgroups M of G'.

Let $P = M \ltimes U_M$ be the standard parabolic subgroup of G' with Levi subgroup M and unipotent radical U_M. Then $N' = (N' \cap M) \ltimes U_M$ and it is easy to verify that ${}^tN'w_MaN' \simeq {}^tN' \times U_M$ and

$$\mathcal{N}_{w_Ma} = \{({}^tn_1, n_2) : n_2 \in N' \cap M,\, {}^tn_1 = w_M n_2^{-1} w_M^{-1}\}$$

from which it easily follows that each w_Ma is indeed relevant.

4.5.3. *The geometric expansion.* Recall that for a nilpotent group N defined over k, the automorphic quotient $N(k)\backslash N(\mathbb{A}_k)$ is compact. We can further normalize measures so that all volumes $v(w_ma)$ equal one. The geometric expansion of the KTF therefore becomes

$$\mathrm{KTF}(f') = \sum_M \sum_{a \in Z_M(k)} \mathcal{O}'(w_Ma, f')$$

where $f' \in C_c^\infty(G'(\mathbb{A}_k)$, M is summed over standard Levi subgroups of G' and

$$\mathcal{O}'(w_Ma, f') = \int_{U_M(\mathbb{A}_k)} \int_{N'(A_k)} f({}^tn_1 w_M a n_2)\psi(n_1n_2)\; dn_1\; dn_2.$$

4.5.4. *The spectral expansion.* For an irreducible cuspidal automorphic representation $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_t$ of $M(\mathbb{A}_k)$ and $\lambda \in \mathbb{C}^t$ let $I(\sigma, \lambda)$ be the parabolically induced representation $\mathrm{Ind}_{P(\mathbb{A}_k)}^{G'(\mathbb{A}_k)}(|\det|^{\lambda_1}\sigma_1 \otimes \cdots \otimes |\det|^{\lambda_t}\sigma_t)$ realised in the representation space $I(\sigma) = I(\sigma, 0)$. For $\varphi \in I(\sigma)$ let $E(\varphi, \lambda)$ be the associated Eisenstein series. It is meromorphic in λ and $E(\cdot, \lambda)$ intertwines $I(\sigma, \lambda)$ with the space of automorphic forms on G'. Since only compact periods are involved, the spectral expansion of the KTF can be written explicitly. It takes the form

$$\mathrm{KTF}(f') = \sum_M \sum_\sigma \int_{i\mathbb{R}^t} B_{(\sigma,\lambda)}(f')\; d\lambda$$

where

$$B_{(\sigma,\lambda)}(f') = \sum_\varphi \mathcal{P}_{({}^tN', {}^t\psi')}(E(f' * \varphi, \lambda))\overline{\mathcal{P}_{N',\psi'}(E(\varphi, \lambda))}$$

and φ is summed over an orthonormal basis of $I(\sigma)$.

4.6. **A relative Kuznetsov trace formula.** Let $\mathcal{D} = (G, H, \mathbf{1}, N, \psi)$ be given by $G = \mathrm{GL}_{2n}$, $H = \mathrm{Sp}_{2n}$, $\mathbf{1}$ the trivial character on $H(\mathbb{A}_k)$, N the group of upper triangular unipotent matrices in G and ψ the degenerate character of $N(k)\backslash N(\mathbb{A}_k)$ defined by

$$\psi(\left(\begin{smallmatrix} u_1 & x \\ 0 & u_2 \end{smallmatrix}\right) = \psi'(u_1u_2), u_1,\, u_2 \in N'(\mathbb{A}_k),\, x \in M_n(\mathbb{A}_k).$$

We consider the following distribution on $G(\mathbb{A}_k)$

$$\mathrm{RTF} = \mathrm{RTF}_{\mathcal{D}}\,.$$

We realise the symplectic group as

$$\{h \in G : {}^thJh = J\}$$

where we recall that $J = \left(\begin{smallmatrix} 0 & w_n \\ -w_n & 0 \end{smallmatrix}\right)$. It is convenient to identify $H\backslash G$ with the space $X = \{g \in G : {}^t g = -g\}$ of alternating matrices in G. The identification is $Hg \mapsto {}^t gJg$. We also consider the projection $f \mapsto \Phi = \Phi_f : C_c^\infty(G(\mathbb{A}_k)) \twoheadrightarrow C_c^\infty(X(\mathbb{A}_k))$ defined by

$$\Phi({}^t gJg) = \int_{H(\mathbb{A}_k)} f(hg)\, dh.$$

Note that after unfolding the RTF becomes

$$\mathrm{RTF}(f) = \int_{N(k)\backslash N(\mathbb{A}_k)} \left[\sum_{x \in X(k)} \Phi({}^t nxn)\right] \psi(n)\, dn.$$

For $x \in X(k)$ let $N_x = \{n \in N : {}^t nxn = x\}$. We say that an element $x \in X(k)$ is relevant if $\psi|_{N_x(\mathbb{A}_k)}$ is trivial.

It is easy to see that $g \in G(k)$ is $\mathcal{D}$-relevant if and only if $x = {}^t gJg \in X(k)$ is relevant. Thus, a set of representatives $[X/N]_{\mathrm{rel}}$ for the relevant $N(k)$-orbits in $X(k)$ is in natural bijection with $[H\backslash G/N]_{\mathcal{D}-\mathrm{rel}}$.

Another simple exercise shows that $g \in G'(k)$ is $\mathcal{D}'$-relevant if and only if $x = \left(\begin{smallmatrix} 0 & g \\ -{}^t g & 0 \end{smallmatrix}\right) \in X(k)$ is relevant (note that $N_x = \{\mathrm{diag}(n_1, n_2) : (n_1, n_2) \in \mathcal{N}'_g\}$).

It can therefore be deduced that there is a natural bijection between relevant orbits for $\mathcal{D}$ and for $\mathcal{D}'$. Expicitly,

$$[X/N]_{\mathrm{rel}} = \sqcup_M \{\left(\begin{smallmatrix} 0 & w_M a \\ -w_M a & 0 \end{smallmatrix}\right) : a \in Z_M(k)\}$$

where the disjoint union is over standard Levi subgroups of G'.

Remark 4.3. *If we replaced ψ with a generic character of $N(\mathbb{A}_k)$ then no $N(k)$-orbit in $X(k)$ would be relevant. This reflects on the fact that every cuspidal automorphic representation of $G(\mathbb{A}_k)$ being automatically generic is never H-distinguished.*

The geometric expansion of RTF is therefore

$$\mathrm{RTF}(f) = \sum_M \sum_{a \in Z_M(k)} \mathcal{O}(\left(\begin{smallmatrix} 0 & w_M a \\ -w_M a & 0 \end{smallmatrix}\right), f)$$

where

$$\mathcal{O}(\left(\begin{smallmatrix} 0 & w_M a \\ -w_M a & 0 \end{smallmatrix}\right), f) = \int_{N_x(\mathbb{A}_k)\backslash N(\mathbb{A}_k)} \Phi({}^t n\left(\begin{smallmatrix} 0 & w_M a \\ -w_M a & 0 \end{smallmatrix}\right) n)\psi(n)\, dn.$$

4.7. **Transfer of orbital integrals.** We say that $f \in C_c^\infty(G(\mathbb{A}_k))$ and $f' \in C_c^\infty(G'(\mathbb{A}_k))$ have matching orbital integrals and write $f \leftrightarrow f'$ if for every standard Levi subgroup M of G' we have

$$\mathcal{O}(\left(\begin{smallmatrix} 0 & w_M a \\ -w_M a & 0 \end{smallmatrix}\right), f) = \mathcal{O}'(w_M a, f'),\ a \in Z_M(k).$$

If this is the case then $\mathrm{KTF}(f') = \mathrm{RTF}(f)$. Given that there are 'enough matching functions' this identity can be applied to the spectral expansions in order to relate between distinguished representations associated to these trace formulas.

Given the local nature of the orbital integrals, matching reduces to a local problem. From the local matching of orbital integrals obtain by Jacquet and Rallis it follows that for any $f \in C_c^\infty(G(\mathbb{A}_k))$ there exists $f' \in C_c^\infty(G'(\mathbb{A}_k))$ such that $f \leftrightarrow f'$.

4.8. **Possible consequences of the comparison.** We explain in an informal way how the comparison of RTF and KTF above is related to the results mentioned in §3.3.

Recall that no cuspidal automorphic representation of $G(\mathbb{A}_k)$ is H-distinguished and that if ρ is an irreducible cuspidal authomorphic representation of $G'(\mathbb{A}_k)$ then $U(\rho, 2)$ is H-distinguished.

This could be recovered by completing the Jacquet-Rallis RTF comparison. Given enough matching functions, it should follow from comparison of the spectral expansions and standard linear independence of characters argument that

$$B_\rho^{\mathcal{D}'}(f') = B_{U(\rho,2)}^{\mathcal{D}}(f)$$

whenever $f \leftrightarrow f'$ where

$$B_{U(\rho,2)}(f) = \sum_\phi \mathcal{P}_H(f * \phi)\overline{\mathcal{P}_{N,\psi}(\phi)}$$

is the contribution to KTF of the discrete automorphic representation $U(\rho, 2)$ of $G(\mathbb{A}_k)$ (the sum is over an orthonormal basis of $U(\rho, 2)$). Since ρ is generic, it follows that $B_\rho(f') \neq 0$ for some f'. If we can find f such that $f \leftrightarrow f'$ the H-distinction of $U(\rho, 2)$ immediately follows.

Replacing ψ with other degenerate characters, one could also attempt to recover the results described in §3.5 using a RTF comparison. It is possible that similar RTF comparisons hold when we replace k with a quaternion algebra. For this case, much less is known about distinction by the symplectic group and the RTF approach could yield new results.

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