Geometric Representation Theory in positive characteristic

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Présentation

Ce mémoire présente les travaux que j'ai effectués depuis ma thèse. J'ai choisi de ne présenter en détail que certains travaux qui s'appliquent (ou tout du moins qu'on espère appliquer) à la théorie des représentations des groupes algébriques réductifs (et des objets associés) définis sur un corps de caractéristique positive. Mes autres travaux sont brièvement résumés dans un appendice.

Plus précisément, je présente trois ensembles d'articles :

(1) Le premier, écrit en collaboration avec Pramod Achar, Anthony Henderson et Daniel Juteau, construit une *correspondance de Springer généralisée modulaire*, c'est-à-dire un analogue en caractéristique positive de la *correspondance de Springer généralisée* de Lusztig. Cet énoncé est d'intérêt essentiellement géométrique, mais on espère qu'il aura des applications en théorie des représentations modulaires des groupes finis de type Lie.

(2) Le second, écrit en collaboration avec Pramod Achar, concerne la *dualité de Koszul géométrique*. On y construit un analogue en caractéristique positive d'une équivalence de catégories due à Bezrukavnikov–Yun reliant les catégories dérivées constructibles d'une variété de drapeaux et de la variété de drapeaux duale. (Cette construction a elle-même ses racines dans des travaux antérieurs de Beĭlinson–Ginzburg–Soergel.) Ces travaux s'inscrivent dans la continuité de travaux effectués précédemment en collaboration avec Wolfgang Soergel et Geordie Williamson. Encore une fois, il s'agit d'un énoncé géométrique, mais nous espérons l'utiliser (ou plus précisément en utiliser une généralisation) pour étudier la théorie des représentations des groupes algébriques réductifs en caractéristique positive.

(3) Enfin, le troisième ensemble d'articles, qui comprend des collaborations avec Carl Mautner, avec Geordie Williamson, et avec Pramod Achar, concerne la géométrie qui sous-tend la théorie des représentations des groupes algébriques réductifs en caractéristique positive. On y construit certaines équivalences de catégories qui devraient permettre de donner des formules de caractères pour les représentations simples et les représentations basculantes. (Ces formules de caractères ont déjà été démontrées d'une manière très différente et plus directe, dans les travaux avec Geordie Williamson, dans le cas particulier du groupe linéaire $\mathrm{GL}_n(\Bbbk)$.)

Ce mémoire se termine par trois appendices. Le premier rassemble des résultats “bien connus” (mais non disponibles dans la littérature dans ce cadre, à ma connaissance) sur les catégories de plus haut poids, en suivant un point de vue dû à Beĭlinson–Ginzburg–Soergel. Le second donne une liste de mes publications. Enfin, le troisième propose un bref résumé de chacun de mes articles, et quelques commentaires sur les liens entre ces différents articles.

1. Introduction

1.1. Presentation. This report gives an exposition of my main contributions since my PhD thesis. I have chosen to concentrate on three results that I believe are the most significant:

(1) the construction of the modular generalized Springer correspondence (obtained in [AHJR2, AHJR3, AHJR4, AHJR5, AHJR6]);
(2) a geometric Koszul duality for constructible sheaves on flag varieties (obtained in [AR3, AR4, AR5], building on earlier work in [AR1, RSW]);
(3) a new (partly conjectural) geometric approach to the modular representation theory of connected reductive groups (developed in [MaR2, RW, AR6] building on earlier work in [R1, BR, R3, MaR1]).

My other articles are summarized more briefly in Section 9.

1.2. Geometric Representation Theory: motivating examples. The guiding principle of Geometric Representation Theory is that in order to solve some problems in Representation Theory, one should first translate them into geometric problems, and then try to solve this new problem using some tools from Geometry. Two of the most notable applications of this idea are:

(1) the proof, by Beĭlinson–Bernstein [BB] and Brylinsky–Kashiwara [BK], of the Kazhdan–Lusztig conjecture [KL1] on multiplicities of simple modules in Verma modules in a regular block of category $\mathcal{O}$ of a complex semisimple Lie algebra;
(2) Lusztig's theory of character sheaves [Lu3], which provides a geometric way to compute characters of complex representations of finite groups of Lie type.

In both of these examples, the representations under consideration are over a field $\Bbbk$ of characteristic 0, and the geometry used to obtain the representation-theoretic information takes the form of perverse sheaves with coefficients in $\Bbbk$. In this report we present works which provide first steps towards analogues of the results in (1) and (2) above in the setting of positive-characteristic representation theory, involving perverse (or more generally constructible) sheaves with coefficients in a field $\Bbbk$ of positive characteristic (but also, as an intermediate step, coherent sheaves on some algebraic varieties over $\Bbbk$).

As a "modular analogue" of (1), we have obtained (in a joint work with Geordie Williamson [RW]) a conjectural character formula for indecomposable tilting representations in regular blocks of the category of finite-dimensional algebraic representations of a connected reductive algebraic group G defined over an algebraically closed field $\Bbbk$ of positive characteristic $p > h$, where h is the Coxeter number of G. (As is well known, from such a character formula one can deduce also a character formula for simple modules if $p \geq 2h - 2$). We have also proved this formula in the case of the group $\mathrm{GL}_n(\Bbbk)$ (see again [RW]), and made a first important step in the direction of a general proof (in a joint work with Pramod Achar [AR6]).

Concerning (2), our results are admittedly less satisfactory so far. We have only obtained (in joint works with Pramod Achar, Anthony Henderson and Daniel Juteau, see [AHJR2, AHJR3, AHJR4]) a modular analogue of a theory which is a preliminary step to the theory of character sheaves, namely Lusztig's theory of the generalized Springer correspondence [Lu2]. We hope this will lead to a theory of "modular character sheaves," but certain important and subtle questions remain to be understood; see [AHJR6] for more comments on this question.

1.3. Geometric Representation Theory in the modular setting. As explained above, the most spectacular results obtained in Geometric Representation Theory so far concern problems over fields of characteristic 0, and use some categories of sheaves with coefficients in characteristic 0. More recently, some authors have started to use geometric methods to study representation theory over fields of positive characteristic. Important examples include:

(1) the Geometric Satake Equivalence of Mirković–Vilonen [MV];

(2) Juteau's modular Springer correspondence [Ju2];

(3) the "localization theory in positive characteristic" of Bezrukavnikov–Mirković–Rumynin [BMR, BMR2, BM].

In (1) and (2), the geometry takes the form of perverse sheaves on a complex algebraic variety, with coefficients in a field of positive characteristic. These results are of course very important from the theoretical point of view, but it is difficult to extract from them concrete information of representation-theoretic interest; in particular, the Geometric Satake Equivalence cannot be used to say anything about the characters of modules of interest over a connected reductive algebraic group.

On the other hand, in (3) the geometry involved concerns coherent sheaves on an algebraic variety over a base field of positive characteristic. This theory can be used to provide new combinatorial information on the representation theory of reductive Lie algebras in positive characteristic, see [BM]; however for the most interesting information this uses comparison with a characteristic-0 setting, hence can only be applied in "sufficiently large" characteristic (with no explicit bound).

Some new tools introduced recently by Juteau–Mautner–Williamson [JMW1] and Elias–Williamson [EW] make it now conceivable to extract combinatorial information from geometry without leaving the positive-characteristic setting; in this way one can hope to obtain results valid under more reasonable assumptions on the characteristic. This idea can be considered a guiding principle of my recent works. Some concrete evidence for this philosophy is provided by the results presented in §1.8.

1.4. Modular generalized Springer correspondence. Now we start presenting our results more concretely, starting with those concerned with the *modular generalized Springer correspondence.*

Let G be a *complex* connected reductive algebraic group, with nilpotent cone $\mathscr{N}_G$ and Weyl group W_f. The *Springer correspondence* is an injective map

$$\mathsf{Irr}(\mathsf{Rep}(W_\mathrm{f}, \mathbb{C})) \hookrightarrow \mathsf{Irr}(\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{C})) \tag{1.1}$$

from the set of isomorphism classes of simple objects in the category $\mathsf{Rep}(W_\mathrm{f}, \mathbb{C})$ of finite-dimensional representations of W_f over $\mathbb{C}$, to the set of isomorphism classes of simple objects in the category $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{C})$ of G-equivariant perverse sheaves on $\mathscr{N}_G$, with coefficients in $\mathbb{C}$. This construction, which generalizes the well-known bijection between isomorphism classes of simple complex representations of the symmetric group S_n and the nilpotent orbits for $\mathrm{GL}_n(\Bbbk)$ (through the parametrization of both of these sets by partitions of n), is intially due to Springer [Sp1], and was reformulated in many different ways by several authors.

Lusztig's *generalized Springer correspondence* [Lu2] is a way to "complete" the left-hand side in (1.1) in order to obtain a bijection rather than an injection. For this one has to add to W_{f} a family of "relative Weyl groups" associated with Levi subgroups which support a "cuspidal pair." This result was the starting point for the theory of character sheaves [Lu3].

The Springer correspondence (1.1) was generalized in a different direction in Juteau's thesis, replacing the field of coefficients $\mathbb{C}$ by a field $\mathbb{F}$ of possibly positive characteristic, to obtain the *modular Springer correspondence*, which takes the form of an injection

$$\mathsf{Irr}(\mathsf{Rep}(W_{\mathrm{f}}, \mathbb{F})) \hookrightarrow \mathsf{Irr}(\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F}))$$

(where the meaning of the notation should be clear).

In a series of joint works with Pramod Achar, Anthony Henderson and Daniel Juteau, we show that Lusztig's generalized Springer correspondence can also be formulated to work in this setting of positive-characteristic coefficients. Here also we have to consider some relative Weyl groups associated with Levi subgroups supporting a cuspidal pair; see §§2.4–2.5 for precise statements. The theory needed for this generalization was developed in three steps. First, in [AHJR2] we proved what was necessary to construct the bijection in the case $G = \mathrm{GL}_n(\mathbb{C})$. (Several simplifications appear in this setting, but some important differences with Lusztig's setting are already visible in this "easier" case.) Then, in [AHJR3], we developed the theory further to be able to treat all classical groups. Finally, in [AHJR4] we found some general proof, which in particular applies to exceptional groups.

In Section 2 we have tried to present this general proof in logical (rather than historical) order. We have also tried to emphasize the "canonicity" of this construction.

1.5. "Constructible" Koszul duality. Fundamental work of Beĭlinson–Ginzburg–Soergel allows to construct a "Koszul duality" equivalence relating Bruhat-constructible $\mathbb{Q}_\ell$-sheaves on the flag variety of a complex connected reductive group G and Bruhat-constructible $\mathbb{Q}_\ell$-sheaves on the flag variety of the Langlands dual group $G^\vee$. This construction was generalized in [BY] to all Kac–Moody groups. An important ingredient of this generalization is the idea (suggested in [BG]) that, in order to obtain a more favorable duality, one should compose the original duality from [BGS] with a Ringel duality, so as to obtain a "Ringel–Koszul" duality exchanging simple perverse sheaves and indecomposable tilting perverse sheaves.

In a series of joint works with Pramod Achar [AR3, AR4, AR5], we have started to generalize the constructions from [BY] to the case of positive characteristic coefficients. In this setting, one should *not* consider simple perverse sheaves, but rather the *parity sheaves* of [JMW1]. (This idea was suggested by [So4], and already used in [RSW].) The other diffculty one has to overcome is to understand the concept of "mixed perverse sheaves" in this setting. Indeed Deligne's notion of mixed perverse sheaves, which was used in [BGS], does not make sense for positive characteristic coefficients. In [AR4] we propose a new point of view on the construction of mixed perverse sheaves, which does not rely of eigenvalues of the Frobenius, but uses parity sheaves instead. This allows to construct an appropriate Koszul duality equivalence in the case of reductive groups, as explained in Section 3. We expect similar constructions to be possible for general Kac–Moody groups, see §3.4 for details.

1.6. Towards character formulas in the modular representation theory of reductive algebraic groups. Finally, we consider our results concerned with the modular representation theory of reductive algebraic groups.

Let G be now a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic p (assumed to be bigger than the Coxeter number h of G). Classical works of Jantzen and Andersen (among others) show that most of the combinatorial information on the category $\mathsf{Rep}(G)$ of finite-dimensional algebraic representations of G (in particular, characters of simple objects and indecomposable tilting objects) can be derived from the similar information in the "principal block" $\mathsf{Rep}_0(G)$. (See Part 2 for more details on the definitions and notation.) Until recently, the main conjectures describing this combinatorial information were:

(1) Lusztig's conjecture [**Lu1**] giving multiplicities of simple modules in induced modules in a certain region, from which one can derive character formulas for all simple modules;

(2) Andersen's conjecture [**An**] giving multiplicities of induced modules in indecomposable tilting modules in a certain region, from which one can derive character formulas for many (but not all) indecomposable tilting modules.

Lusztig's conjecture was proved in 1995/96, under the assumption that p is "big enough" (with no explicit bound), by a combination of works by Kashiwara–Tanisaki [**KT**], Kazhdan–Lusztig [**KL2**], Lusztig [**Lu5**], and Andersen–Jantzen–Soergel [**AJS**]. Later, Fiebig [**Fi**] obtained a bound for the validity of this conjecture. (This bound is difficult to compute explicitly, and in any case several orders of magnitude larger than h.) It is known that Andersen's conjecture implies Lusztig's conjecture (if $p \geq 2h-2$), but as far as we know no proof of this conjecture (under any assumptions) is available unless G has semisimple rank 1.

On the other hand, in [**Wi**], G. Williamson has shown that Lusztig's conjecture does *not* hold for all $p > h$, and even that there cannot exist any general polynomial bound in h which guarantees the validity of this conjecture. In view of this, Lusztig's conjecture (and also Andersen's conjecture) should rather be considered as "asymptotic" character formulas (when p is very large), and a finer point of view should be adopted in order to obtain formulas valid under reasonable bounds on p. Our main contributions in this direction so far are:

- a conjectural description of this "finer" point of view (obtained in joint work with Geordie Williamson);
- a proof of these character formulas in the case $G = \mathrm{GL}_n(\Bbbk)$ (also joint with Geordie Williamson);
- and some steps towards a general proof of the character formula (obtained in joint works with Carl Mautner and Pramod Achar).

More precisely, our conjecture takes the form of a correction to Andersen's conjecture, which we expect to hold for *all* indecomposable tilting modules in $\mathsf{Rep}_0(G)$. Our formula involves the *p-canonical basis* of the affine Hecke algebra, introduced in two different forms

$$\begin{tikzcd}
 & \mathsf{Rep}_0(\mathsf{U}_\zeta) \arrow[dr, dotted, leftrightarrow, "\text{quantum FM conjecture}"] & \\
\mathsf{Coh}^{G_{\mathbb{C}}}(\widetilde{\mathcal{N}_{\mathbb{C}}}) \arrow[ur, leftrightarrow, "(1)"] \arrow[rr, leftrightarrow, "(2)"] \arrow[dr, leftrightarrow, "(3)"'] & & \mathsf{Perv}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \mathbb{C}). \\
 & \mathsf{Perv}_{\mathcal{IW}}(\mathcal{F}\mathrm{l}, \mathbb{C}) \arrow[ur, leftrightarrow, "(4)"'] &
\end{tikzcd}$$

FIGURE 1. Bezrukavnikov's geometric framework for representations of quantum groups at roots of unity.

by Juteau–Mautner–Williamson [JMW1] and Williamson [JW]. (The fact that these two definitions are equivalent is proved in [RW].)

1.7. The case of quantum groups (after Bezrukavnikov et al.) Our point of view on the geometry underlying the representation theory of reductive algebraic groups over fields of positive characteristic has been suggested by the results of Bezrukavnikov and his collaborators in the early 2000's on the geometry underlying the representation theory of Lusztig's quantum groups at a root of unity. These results can be roughly summarized in the diagram of Figure 1.

Here U_ζ is Lusztig's quantum group at a root of unity attached to a semisimple complex algebraic group $G_{\mathbb{C}}$ of adjoint type, $\mathsf{Rep}_0(\mathsf{U}_\zeta)$ is the principal block of the category of finite-dimensional U_ζ-modules, $\widetilde{\mathcal{N}_{\mathbb{C}}}$ is the Springer resolution of $G_{\mathbb{C}}$, $\mathcal{F}\mathrm{l}$ and $\mathcal{G}\mathrm{r}$ are the affine flag variety and the affine Grassmannian of the Langlands dual group $G^\vee_{\mathbb{C}}$, and the symbols "(Iw)" and $\mathcal{IW}$ mean "Iwahori constructible" and "Iwahori–Whittaker" conditions on perverse sheaves (whose coefficients are in the field $\mathbb{C}$ of complex numbers). The arrows labelled (1) and (2) are the main results of [ABG]; taken together they allow to prove an equivalence of abelian (highest weight) categories corresponding to the dotted arrow, which provides the natural quantum group analogue of a conjecture of Finkelberg–Mirković [FM]. The arrow labelled (3) is the main result of [AB]; finally, the arrow labelled (4) follows from the results of [BY].

At the time when these results were obtained, most of the combinatorial information on the category $\mathsf{Rep}_0(\mathsf{U}_\zeta)$ was already understood. In particular, the characters of simple modules were obtained from the works of Kashiwara–Tanisaki [KT], Kazhdan–Lusztig [KL2] and Lusztig [Lu5] (following a conjecture of Lusztig), and the characters of tilting modules were obtained (in most cases) by Soergel [So2]. But these results can be used to obtain more direct proofs of the simple character formula and, in combination with later work of Yun [Yu], of the tilting character formula, which bypass the comparison with affine Kac–Moody algebras.

1.8. Counterpart for reductive groups. Our "modular counterpart" of the diagram of Figure 1 is depicted in Figure 2. Here G is a connected reductive algebraic group with simply-connected derived subgroup over an algebraically closed field $\Bbbk$ of positive

$$\begin{tikzcd}
 & \mathsf{Rep}_0(G) \arrow[dr, leftrightarrow, dotted, "\text{FM conjecture}"] & \\
\mathsf{Coh}^G(\widetilde{\mathscr{N}}) \arrow[ur, leftrightarrow, "(1)"] \arrow[rr, leftrightarrow, "(2)"] & & \mathsf{Perv}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk). \\
 & \mathsf{Perv}_{\mathcal{IW}}(\mathcal{F}\mathrm{l}, \Bbbk) \arrow[uu, leftrightarrow, "(3)"] \arrow[ur, leftrightarrow, dashed, "(4)"] &
\end{tikzcd}$$

FIGURE 2. Geometric framework for modular representations of reductive groups.

characteristic p (assumed to be bigger than the Coxeter number h), $\mathsf{Rep}_0(G)$ is the principal block of the category of finite-dimensional algebraic G-modules, $\widetilde{\mathscr{N}}$ is the Springer resolution of G, $\mathcal{F}\mathrm{l}$ and $\mathcal{G}\mathrm{r}$ are as above the affine flag variety and affine Grassmannian of the complex Langlands dual group, but now we consider perverse sheaves with coefficients in $\Bbbk$.

The motivation for building this diagram comes from a conjectural formula for the multiplicities of costandard G-modules in indecomposable tilting modules in $\mathsf{Rep}_0(G)$ in terms of parity complexes on $\mathcal{F}\mathrm{l}$ formulated in [**RW**] and corresponding to the arrow labelled (3); see §4.4 below for a precise statement. Following work of Jantzen, Donkin and Andersen, this formula would imply character formulas for simple modules in $\mathsf{Rep}_0(G)$, hence can be considered as a replacement for Lusztig's conjecture [**Lu1**] considered in §1.6. The combinatorial data concerning parity complexes in this conjectural formula is encoded in the "p-canonical basis" of the affine Hecke algebra, which can be computed algorithmically using the "Soergel calculus" of [**EW**]; see [**JW**] for details. In [**RW**] we prove this conjectural formula in the case $G = \mathrm{GL}_n(\Bbbk)$, but there is no hope to generalize our methods beyond classical groups.

The arrows labelled (1), (2) and (4) provide a plan of proof of this conjecture for a general reductive group G as above. First, the arrow labelled (1) is the main result of [**AR6**]. It provides a modular analogue of the first part of [**ABG**]; see §5.5 for details. The arrow labelled (2) is the main result of [**MaR2**] (and is also proved independently in [**ARd2**]). It provides a modular analogue of the second part of [**ABG**]; see §5.4 for details. Taken together, these results allow us to prove a "graded version" of the Finkelberg–Mirković conjecture mentioned in §1.7; see §5.6.

Finally, the dashed arrow labelled (4) would follow from a conjectural modular generalization of the "Koszul duality" of [**BY**], discussed in §3.4. (As explained in §1.5, we have already obtained a modular version of this duality in the case of finite flag varieties. But the crucial case here is that of *affine* Kac–Moody algebras.) This generalization is the subject of a work in progress with Pramod Achar, Shotaro Makisumi and Geordie Williamson; a more detailed discussion of the expected application to the conjecture from [**RW**] is given in Section 6.

1.9. Comments on Koszul duality. A unifying theme of my research so far is *Koszul duality*. The importance of this construction in Representation Theory was discovered in [**BGS**]. There, Koszul duality was defined as an equivalence of triangulated categories between some derived categories of modules over two rings which are "Koszul dual," this construction generalizing the celebrated equivalence between the bounded derived categories of graded finitely-generated modules over the symmetric algebra $\mathcal{S}(V)$ of a finite-dimensional vector space V and the exterior algebra $\bigwedge V^*$ of the dual vector space. A central property of this equivalence is that it exchanges simple modules over a ring with projective (or injective, depending on the conventions) modules over the dual ring. As explained in §1.5, in Lie-theoretic contexts it is sometimes more convenient to compose such a duality with a Ringel duality, to obtain a Ringel–Koszul duality exchanging simple modules and *tilting* modules.

Since then, this idea has been generalized in many directions. In this report, what we mean by a "Koszul duality" is an equivalence which is either based on the same kind of constructions as for the duality between $\mathcal{S}(V)$ and $\bigwedge V^*$, or which exchanges some simple objects with some tilting objects. As explained in §1.5, in a modular context, one sometimes wants to replace "simple objects" by "parity objects." From this point of view, Koszul duality equivalences are ubiquitous in my work, see in particular:

(1) the "linear Koszul duality" (from some coherent sheaves to some coherent sheaves) of [**MR1, MR2**], which has found applications to the modular representation theory of reductive Lie algebras [**R2**] and to a categorification of the Iwahori–Matsumoto involution [**MR3, MR4**];
(2) a "constructible" Ringel–Koszul duality for sheaves on flag varieties, see §1.5;
(3) the equivalence constructed in [**MaR2**], which is also a Ringel–Koszul duality, see Remark 5.7(2);
(4) the equivalence constructed in [**AR6**], which we can once again think of as a Ringel–Koszul duality.

If one considers Koszul duality to be of the same nature as Fourier transforms (as suggested in particular in [**MR4**, §0.1], and as illustrated by the main result of [**MR4**]), then one can also add to this list the results of [**AHJR2, AHJR3, AHJR4, AHJR5**], which rely in a crucial way on the use of the Fourier–Sato transform.

1.10. Contents of the report. In Part 1 we present two of our contributions which are of more geometric interest. First, in Section 2 we give a detailed account of our present understanding of the modular generalized Springer correspondence (see §1.4). Then, in Section 3 we consider a generalization of the Bezrukavnikov–Yun geometric Koszul duality [**BY**] (which itself stems from the Beĭlinson–Ginzburg–Soergel Koszul duality for category $\mathcal{O}$ of a complex semisimple Lie algebra [**BGS**]) to the case of coefficients in positive characteristic. This generalization uses the concept of the "mixed derived category" of sheaves on a (nice) algebraic variety, which we believe is of independent interest, and plays a key role also in the results of Part 2.

In Part 2 we give a detailed account on our results evoked in §1.8. First, in Section 4 we state the conjecture on tilting characters from [**RW**]. Then, in Section 5 we present the main results of [**MaR2**] and [**AR6**], which provide steps towards a general proof of this character formula. (These constructions rely in an important way on the results previously obtained in [**R1, BR, R3, MaR1**].) Finally, in Section 6 we explain how we expect to complete this proof using a modular version of the results from [**BY**].

The report finishes with three appendices. In Section 7 we provide definitions and proofs or references for some "well known" results on highest weight categories, which are sometimes not available in the literature in the form we want to use. Section 8 contains a list of my publications so far. Finally, Section 9 contains summaries of all my articles, together with some comments on the relations between certain of these articles.

Part 1

Geometry

In this part, G denotes a complex connected reductive algebraic group, and $\mathbb{F}$ is a field of characteristic p.

2. Modular generalized Springer correspondence

In this section we explain our joint work with Pramod Achar, Anthony Henderson and Daniel Juteau on the modular generalized Springer correspondence: see [AHJR2, AHJR3, AHJR4], and some complements in [AHJR5]. See also [AHJR6] for a different overview of these results (with an emphasis on the possible application to modular character sheaves).

2.1. Notations. We will denote by $\mathfrak{g}$ the Lie algebra of G, and by $\mathscr{N}_G \subset \mathfrak{g}$ the nilpotent cone (i.e. the cone consisting of elements $x \in \mathfrak{g}$ such that the endomorphism $y \mapsto [x,y]$ of $\mathfrak{g}$ is nilpotent). We will consider $\mathscr{N}_G$ as a complex algebraic variety, endowed with an action of the group G induced by the adjoint action. We fix a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$; this allows to identify the G-modules $\mathfrak{g}$ and $\mathfrak{g}^*$. For any Levi subgroup [1] L of G, with Lie algebra $\mathfrak{l}$, this form restricts to a non-degenerate L-invariant symmetric bilinear form on $\mathfrak{l}$, which also allows to identify the L-modules $\mathfrak{l}$ and $\mathfrak{l}^*$.

Our main object of study in this section is the category $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F})$ of G-equivariant $\mathbb{F}$-perverse sheaves on $\mathscr{N}_G$. This category is the heart of the perverse t-structure on the equivariant derived category $D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F})$. More precisely, the main goal of the series [AHJR2, AHJR3, AHJR4] is to give a "representation-theoretic" parametrization of simple objects in $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F})$, adapting known results in the case of ℓ-adic sheaves (on the analogue of $\mathscr{N}_G$ over an algebraically closed field of positive characteristic different from ℓ) due to Lusztig [Lu2]. In [AHJR5] we use this information to deduce some structural properties of $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F})$ and $D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F})$.

We will denote by $\mathfrak{N}_{G,\mathbb{F}}$ the quotient of the set of pairs $(\mathscr{O}, \mathcal{E})$ where $\mathscr{O} \subset \mathscr{N}_G$ is a G-orbit and $\mathcal{E}$ is an irreductible G-equivariant $\mathbb{F}$-local system on $\mathscr{O}$, by the relation

$$(\mathscr{O}, \mathcal{E}) \sim (\mathscr{O}', \mathcal{E}') \quad \Leftrightarrow \quad \mathscr{O} = \mathscr{O}' \text{ and } \mathcal{E} \cong \mathcal{E}'.$$

(By abuse, we will often write $(\mathscr{O}, \mathcal{E}) \in \mathfrak{N}_{G,\mathbb{F}}$ to mean that $\mathscr{O}$ is a G-orbit in $\mathscr{N}_G$ and that $\mathcal{E}$ is an irreducible G-equivariant local system on $\mathscr{O}$.)

The general theory of perverse sheaves [BBD] ensures that the map $(\mathscr{O}, \mathcal{E}) \mapsto \mathcal{IC}(\mathscr{O}, \mathcal{E})$ induces a bijection between $\mathfrak{N}_{G,\mathbb{F}}$ and the set of isomorphism classes of simple objects in the abelian category $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F})$. On the other hand, the theory of local systems ensures that, if we fix for any G-orbit $\mathscr{O} \subset \mathscr{N}_G$ a point $x_{\mathscr{O}} \in \mathscr{O}$ and set $A_G(\mathscr{O}) = G_{x_{\mathscr{O}}}/(G_{x_{\mathscr{O}}})^\circ$ (where $G_{x_{\mathscr{O}}}$ is the centralizer of $x_{\mathscr{O}}$ in G, and $(-)^\circ$ means the connected component of the identity), then the set $\mathfrak{N}_{G,\mathbb{F}}$ is in natural bijection with the set of equivalence classes of pairs $(\mathscr{O}, V)$ where $\mathscr{O} \subset \mathscr{N}_G$ is a G-orbit and V is a simple $\mathbb{F}$-representation of $A_G(x_{\mathscr{O}})$.

For any Levi subgroup L of G we can likewise consider the set $\mathfrak{N}_{L,\mathbb{F}}$. In the whole section we will make the following assumption on our field $\mathbb{F}$: for any Levi subgroup $L \subset G$ and any pair $(\mathcal{O}_L, \mathcal{E}_L)$ in $\mathfrak{N}_{L,\mathbb{F}}$, the local system $\mathcal{E}_L$ is absolutely irreducible. It is clear that this condition is satisfied if $\mathbb{F}$ is algebraically closed. In fact it is satisfied under

1. By *Levi subgroup* we mean a Levi factor of a parabolic subgroup.

much weaker assumptions: see [AHJR4, Proposition 3.2] for an explicit characterization of when this condition holds in many important cases.

2.2. Cuspidal pairs and triples. For any parabolic subgroup $P \subset G$ and Levi factor $L \subset P$, as above we can consider the nilpotent cones $\mathscr{N}_P$ and $\mathscr{N}_L$ in the Lie algebras of P and L respectively, and the associated categories $D^{\mathrm{b}}_P(\mathscr{N}_P, \mathbb{F})$, $\mathsf{Perv}_P(\mathscr{N}_P, \mathbb{F})$, $D^{\mathrm{b}}_L(\mathscr{N}_L, \mathbb{F})$, $\mathsf{Perv}_L(\mathscr{N}_L, \mathbb{F})$. We have natural maps

$$\mathscr{N}_G \xleftarrow{i_{L\subset P}} \mathscr{N}_P \xrightarrow{p_{L\subset P}} \mathscr{N}_L$$

where $i_{L\subset P}$ is induced by the embedding $P \hookrightarrow G$, and $p_{L\subset P}$ is induced by the projection $P \twoheadrightarrow L$, where we identify L with the quotient of P by its unipotent radical. In this way we can define two "restriction" functors

$$\mathbf{R}^G_{L\subset P} := (p_{L\subset P})_* \circ (i_{L\subset P})^! : D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F}) \to D^{\mathrm{b}}_L(\mathscr{N}_L, \mathbb{F})$$
$$'\mathbf{R}^G_{L\subset P} := (p_{L\subset P})_! \circ (i_{L\subset P})^* : D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F}) \to D^{\mathrm{b}}_L(\mathscr{N}_L, \mathbb{F})$$

and an "induction" functor

$$\mathbf{I}^G_{L\subset P} := \gamma^G_P \circ (i_{L\subset P})_! \circ (p_{L\subset P})^* : D^{\mathrm{b}}_L(\mathscr{N}_L, \mathbb{F}) \to D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F}),$$

where γ^G_P is the left adjoint to the forgetful functor $D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F}) \to D^{\mathrm{b}}_P(\mathscr{N}_G, \mathbb{F})$ (see [BL, Theorem 3.7.1]). It follows from the usual theory of derived functors for sheaves that these functors form adjoint pairs $('\mathbf{R}^G_{L\subset P}, \mathbf{I}^G_{L\subset P})$ and $(\mathbf{I}^G_{L\subset P}, \mathbf{R}^G_{L\subset P})$. Moreover, it is known that they are exact for the perverse t-structures, hence restrict to exact functors

$$\mathbf{R}^G_{L\subset P} : \mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F}) \to \mathsf{Perv}_L(\mathscr{N}_L, \mathbb{F}),$$
$$'\mathbf{R}^G_{L\subset P} : \mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F}) \to \mathsf{Perv}_L(\mathscr{N}_L, \mathbb{F}),$$
$$\mathbf{I}^G_{L\subset P} : \mathsf{Perv}_L(\mathscr{N}_L, \mathbb{F}) \to \mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F});$$

see [AHJR2, §2.1] for references.

The following lemma is an application of Braden's hyperbolic localization theorem; see [AHJR2, Proposition 2.1] for details.

Lemma 2.1. *Let $\mathcal{F}$ be a simple object in $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F})$. Then the following conditions are equivalent:*

(1) for any pair (L, P) as above with $P \subsetneq G$, we have $\mathbf{R}^G_{L\subset P}(\mathcal{F}) = 0$;

(2) for any pair (L, P) as above with $P \subsetneq G$, we have $'\mathbf{R}^G_{L\subset P}(\mathcal{F}) = 0$;

(3) for any pair (L, P) as above with $P \subsetneq G$, and any $\mathcal{G}$ in $\mathsf{Perv}_L(\mathscr{N}_L, \mathbb{F})$, $\mathcal{F}$ does not appear in the head of $\mathbf{I}^G_{L\subset P}(\mathcal{G})$;

(4) for any pair (L, P) as above with $P \subsetneq G$, and any $\mathcal{G}$ in $\mathsf{Perv}_L(\mathscr{N}_L, \mathbb{F})$, $\mathcal{F}$ does not appear in the socle of $\mathbf{I}^G_{L\subset P}(\mathcal{G})$.

The simple perverse sheaves which satisfy the conditions of Lemma 2.1 are called *cuspidal*. A pair $(\mathscr{O}, \mathcal{E})$ is called *cuspidal* if $\mathcal{IC}(\mathscr{O}, \mathcal{E})$ is cuspidal; the subset of $\mathfrak{N}_{G,\mathbb{F}}$ consisting of classes of cuspidal pairs will be denoted $\mathfrak{N}^{\mathrm{cusp}}_{G,\mathbb{F}}$. Similarly, for any Levi subgroup $L \subset G$ we can consider the subset $\mathfrak{N}^{\mathrm{cusp}}_{L,\mathbb{F}}$ of $\mathfrak{N}_{L,\mathbb{F}}$. A triple $(L, \mathscr{O}_L, \mathcal{E}_L)$ where L is a Levi subgroup of G and $(\mathscr{O}_L, \mathcal{E}_L) \in \mathfrak{N}^{\mathrm{cusp}}_{L,\mathbb{F}}$ will be called a *cuspidal triple* for G. The set of cuspidal triples for G admits a natural action of G by conjugation, and we denote by $\mathfrak{M}_{G,\mathbb{F}}$ the set of orbits for this action. If $(L, \mathscr{O}_L, \mathcal{E}_L)$ is a cuspidal triple, we will write $[L, \mathscr{O}_L, \mathcal{E}_L] \in \mathfrak{M}_{G,\mathbb{F}}$ for the corresponding G-orbit.

For any Levi subgroup $L \subset G$, we denote by $N_G(L)$ the normalizer of L in G. For any L-orbit $\mathscr{O}_L \subset \mathscr{N}_L$, we denote by $N_G(L, \mathscr{O}_L)$ the subgroup of $N_G(L)$ consisting of elements g such that $g\mathscr{O}_L g^{-1} = \mathscr{O}_L$. This group acts naturally on the set of isomorphism classes of L-equivariant local systems on $\mathscr{O}_L$. The following result is based on the observation that cuspidal pairs are supported on distinguished orbits (see [AHJR3, Proposition 2.6]) and then on an explicit verification in each type; see [AHJR3, Lemma 2.9] and [AHJR4, Proposition 3.1].

Lemma 2.2. *If $L \subset G$ is a Levi subgroup and if $(\mathscr{O}_L, \mathcal{E}_L) \in \mathfrak{N}^{\mathrm{cusp}}_{L,\mathbb{F}}$, then we have $N_G(L, \mathscr{O}_L) = N_G(L)$. Moreover, the action of this group on the set of isomorphism classes of L-equivariant local systems on $\mathscr{O}_L$ is trivial.*

If $\mathfrak{L}$ is a set of pairwise non-conjugate representatives of conjugacy classes of Levi subgroups of G, then Lemma 2.2 implies that the natural map

$$\bigsqcup_{L \in \mathfrak{L}} \mathfrak{N}^{\mathrm{cusp}}_{L,\mathbb{F}} \to \mathfrak{M}_{G,\mathbb{F}} \tag{2.1}$$

is a bijection.

2.3. Induction series. If $(L, \mathscr{O}_L, \mathcal{E}_L)$ is a cuspidal triple, and if $P \subset G$ is a parabolic subgroup having L as a Levi factor, then we can consider the perverse sheaf

$$\mathbf{I}^G_{L \subset P}(\mathcal{IC}(\mathscr{O}_L, \mathcal{E}_L)).$$

Since we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$, we can consider the Fourier–Sato transform $\mathbb{T}_{\mathfrak{g}}$ as an auto-equivalence of the abelian category of conical G-equivariant $\mathbb{F}$-perverse sheaves on $\mathfrak{g}$. (Note that any G-equivariant perverse sheaf on $\mathfrak{g}$ which is supported on $\mathscr{N}_G$ is conical, as follows e.g. from [Ja2, Lemma 2.10].) Using a geometric analysis due to Lusztig (but explained in more detail by Letellier [Le]), we check in [AHJR2, §2.6] that there exists a canonical isomorphism

$$\mathbb{T}_{\mathfrak{g}}\Big(\mathbf{I}^G_{L \subset P}(\mathcal{IC}(\mathscr{O}_L, \mathcal{E}_L))\Big) \cong \mathcal{IC}\left(Y_{(L,\mathscr{O}'_L)}, (\varpi_{(L,\mathscr{O}'_L)})_* \widetilde{\mathcal{E}'_L}\right) \tag{2.2}$$

for some pair $(\mathscr{O}'_L, \mathcal{E}'_L) \in \mathfrak{N}^{\mathrm{cusp}}_{L,\mathbb{F}}$ which might potentially be different from $(\mathscr{O}_L, \mathcal{E}_L)$ (although we do not know any example where this actually occurs), and which is characterized by the fact that

$$\mathbb{T}_{\mathfrak{l}}(\mathcal{IC}(\mathscr{O}_L, \mathcal{E}_L)) \cong \mathcal{IC}(\mathscr{O}'_L + \mathfrak{z}_L, \mathcal{E}'_L \boxtimes \underline{\mathbb{F}}_{\mathfrak{z}_L}),$$

where $\mathfrak{z}_L$ is the center of the Lie algebra $\mathfrak{l}$ of L and $\mathbb{T}_{\mathfrak{l}}$ is the Fourier–Sato transform on $\mathfrak{l}$. In (2.2) we have used the notation

$$Y_{(L,\mathscr{O}'_L)} := G \cdot (\mathscr{O}'_L + \mathfrak{z}_L^\circ) \subset \mathfrak{g}, \quad \text{where} \quad \mathfrak{z}_L^\circ = \{z \in \mathfrak{z}_L \mid G_z^\circ = L\},$$

and

$$\varpi_{(L,\mathscr{O}'_L)} : G \times^L (\mathscr{O}'_L + \mathfrak{z}_L^\circ) \to Y_{(L,\mathscr{O}'_L)}$$

is the natural morphism, which is known to be a Galois covering with group $N_G(L, \mathscr{O}_L)/L = N_G(L)/L$ (see Lemma 2.2). Finally, $\widetilde{\mathcal{E}'_L}$ is the unique local system on $G \times^L (\mathscr{O}'_L + \mathfrak{z}_L^\circ)$ whose pullback to $G \times (\mathscr{O}'_L + \mathfrak{z}_L^\circ)$ is $\underline{\mathbb{F}}_G \boxtimes (\mathcal{E}'_L \boxtimes \underline{\mathbb{F}}_{\mathfrak{z}_L^\circ})$.

From (2.2) we see that the perverse sheaf $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$ is independent of P up to isomorphism, and in fact that it only depends on the G-conjugacy class of $(L,\mathscr{O}_L,\mathcal{E}_L)$. Hence we can define $\mathfrak{N}_{G,\Bbbk}^{[L,\mathscr{O}_L,\mathcal{E}_L]}\subset\mathfrak{N}_{G,\Bbbk}$ as the subset consisting of classes of pairs $(\mathscr{O},\mathcal{E})$ such that $\mathcal{IC}(\mathscr{O},\mathcal{E})$ is isomorphic to a quotient of $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$. This subset is called the *induction series* attached to the class $[L,\mathscr{O}_L,\mathcal{E}_L]$.

Remark 2.3. (1) We insist that $\mathfrak{N}_{G,\Bbbk}^{[L,\mathscr{O}_L,\mathcal{E}_L]}$ is the set of pairs associated with *quotients* of $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$, and not with all *sub*quotients of this perverse sheaf. If $p=0$, then it follows from the Decomposition Theorem that $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$ is semisimple; so the two sets coincide. However, if $p>0$, in general there exist much more simple subquotients of this object than quotients. For instance when $G=\mathrm{GL}_n(\mathbb{C})$, L is a maximal torus, and $(\mathscr{O}_L,\mathcal{E}_L)$ is the unique pair in $\mathfrak{N}_{L,\mathbb{F}}$, then the simple subquotients of $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$ correspond to all the pairs $(\mathscr{O},\underline{\mathbb{F}}_{\mathscr{O}})$ with $\mathscr{O}\subset\mathscr{N}_G$ a G-orbit, see [AHJR2, Remark 3.2]. In particular, they are parametrized by all partitions of n. On the other hand, the simple quotients of $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$ are in bijection with simple $\mathbb{F}$-representations of the Weyl group $\mathfrak{S}_n$ of G (via Juteau's modular Springer correspondence [Ju2]), hence with the set of partitions of n which are p-restricted (in other words whose transpose is p-regular, in the sense that no part appears at least p times in this transposed partition).

(2) The pairs $(\mathscr{O},\mathcal{E})$ such that $\mathcal{IC}(\mathscr{O},\mathcal{E})$ does not occur as a *subquotient* of a nontrivially induced perverse sheaf are called *supercuspidal*. They are closely related to the cuspidal pairs appearing in characteristic 0, see [AHJR5, Theorem 1.6].

(3) There seems to be a choice in our conventions since we could have considered simple *subobjects* of $\mathbf{I}_{L\subset P}^G\big(\mathcal{IC}(\mathscr{O}_L,\mathcal{E}_L)\big)$ rather than simple quotients. However the set of isomorphism classes of simple subobjects of this perverse sheaf coincides with the set of isomorphism classes of its simple quotients; see [AHJR3, Lemma 2.3].

2.4. Generalized Springer correspondence – Part 1. We can now state the first part of the (modular) *generalized Springer correspondence.*

Theorem 2.4. *We have*

$$\mathfrak{N}_{G,\mathbb{F}}=\bigsqcup_{[L,\mathscr{O}_L,\mathcal{E}_L]\in\mathfrak{M}_{G,\mathbb{F}}}\mathfrak{N}_{G,\mathbb{F}}^{[L,\mathscr{O}_L,\mathcal{E}_L]}.$$

The fact that $\mathfrak{N}_{G,\mathbb{F}}$ is the union of the induction series follows from the definitions and an easy induction on the semisimple rank of G; see [AHJR2, Corollary 2.7]. The real content of Theorem 2.4 is the fact that the distinct induction series are *disjoint*. The general proof of this property uses a "Mackey formula" for our geometric induction and restriction functors; see [AHJR4, Theorem 2.2] for the statement of this Mackey formula, and [AHJR4, Theorem 2.5] for the application to the disjointness of induction series.

In [AHJR2, AHJR3] we use a different approach to prove disjointness in some special cases. For this we remark that from (2.2) we can deduce that the series $\mathfrak{N}_{G,\mathbb{F}}^{[L,\mathscr{O}_L,\mathcal{E}_L]}$ and $\mathfrak{N}_{G,\mathbb{F}}^{[M,\mathscr{O}_M,\mathcal{E}_M]}$ are disjoint unless there exists $g\in G$ such that $g\cdot(L,\mathscr{O}_L)=(M,\mathscr{O}_M)$ and such that the L-equivariant local systems $\mathcal{E}_L$ and $(\mathrm{Ad}_g)^*\mathcal{E}_M$ on $\mathscr{O}_L$ have the same central character.[2] Hence, a sufficient condition for disjointness of all induction series attached to non-conjugate cuspidal triples is that for any Levi subgroup $L \subsetneq G$, there does not exist two distinct cuspidal pairs in $\mathfrak{N}_{L,\mathbb{F}}$ which are supported on the same L-orbit and which have the same central character. In the case of classical groups, we are able to check this condition by explicitly classifying cuspidal pairs (by induction on the rank). In the case of exceptional groups, in general we do not know (nor have a conjecture for) the precise description of cuspidal pairs; hence we cannot even try to apply such ideas.

REMARK 2.5. Using the bijection (2.1) one can write the disjoint union in Theorem 2.4 also as a disjoint union over a fixed set of pairwise non-conjugate representatives of conjugacy classes of Levi subgroups, and then over cuspidal pairs for these Levi subgroups. This is the way this result was stated in [AHJR2, AHJR3].

2.5. Generalized Springer correspondence – Part 2. For any Levi subgroup $L \subset G$, we consider the set $\mathsf{Irr}(\mathbb{F}[N_G(L)/L])$ of isomorphism classes of simple $\mathbb{F}$-representations of the finite group $N_G(L)/L$. If L and M are two Levi subgroups of G which are conjugate, then there exists a canonical bijection

$$\imath_{L,M} : \mathsf{Irr}(\mathbb{F}[N_G(L)/L]) \xrightarrow{\sim} \mathsf{Irr}(\mathbb{F}[N_G(M)/M]).$$

Indeed, for any $g \in G$ such that $gLg^{-1} = M$ we have a group isomorphism

$$N_G(L)/L \xrightarrow{\sim} N_G(M)/M \tag{2.3}$$

induced by the morphism $h \mapsto ghg^{-1}$. Using this isomorphism one can identify the set $\mathsf{Irr}(\mathbb{F}[N_G(L)/L])$ with $\mathsf{Irr}(\mathbb{F}[N_G(M)/M])$. This identification does not depend on the choice of g since changing g only replaces (2.3) by a composition with an inner automorphism of $N_G(L)/L$, and inner automorphisms fix isomorphism classes of simple representations.

Now we can state the second part of the generalized Springer correspondence.

THEOREM 2.6. *For any cuspidal triple $(L, \mathscr{O}_L, \mathcal{E}_L)$, there exists a canonical bijection*

$$\varphi_{(L,\mathscr{O}_L,\mathcal{E}_L)} : \mathfrak{N}_{G,\mathbb{F}}^{[L,\mathscr{O}_L,\mathcal{E}_L]} \longleftrightarrow \mathsf{Irr}(\mathbb{F}[N_G(L)/L]).$$

This bijection is invariant under conjugation, in the sense that for any $g \in G$ the diagram

$$\begin{tikzcd}
 & \mathsf{Irr}(\mathbb{F}[N_G(L)/L]) \arrow[dd, "\imath_{L,gLg^{-1}}"] \\
\mathfrak{N}_{G,\mathbb{F}}^{[L,\mathscr{O}_L,\mathcal{E}_L]} \arrow[ur, leftrightarrow, "\varphi_{(L,\mathscr{O}_L,\mathcal{E}_L)}"] \arrow[dr, leftrightarrow, "\varphi_{g\cdot(L,\mathscr{O}_L,\mathcal{E}_L)}"'] & \\
 & \mathsf{Irr}(\mathbb{F}[N_G(gLg^{-1})/gLg^{-1}])
\end{tikzcd}$$

commutes.

To prove Theorem 2.6 we use some constructions due to Bonnafé [Bo]. Let us fix some cuspidal triple $(L, \mathscr{O}_L, \mathcal{E}_L)$, and recall the pair $(\mathscr{O}'_L, \mathcal{E}'_L)$ associated with $(\mathscr{O}_L, \mathcal{E}_L)$ as in §2.3. Then, using geometric constructions copied from [Bo], in [AHJR3, Theorem 3.1] we prove that

2. Here, the central character of a simple L-equivariant local system $\mathcal{E}_L$ on an L-orbit $\mathscr{O}_L = L \cdot x \subset \mathscr{N}_L$ is the character of $Z(L)$ on the stalk $(\mathcal{E}_L)_x$, where $Z(L)$ acts via the natural morphism $Z(L) \to G_x/(G_x)^\circ$. (This character does not depend on the choice of x.)

(1) there exists (up to isomorphism) a unique direct summand of the local system $(\varpi_{(L,\mathscr{O}'_L)})_* \widetilde{\mathcal{E}'_L}$ whose $\mathcal{IC}$-extension has a nonzero restriction to the induced orbit $\mathrm{Ind}_L^G(\mathscr{O}_L)$;
(2) the head $\overline{\mathcal{E}'_L}$ of this direct summand is absolutely irreducible;
(3) there exists (up to scalar) a unique nonzero morphism $(\varpi_{(L,\mathscr{O}'_L)})_* \widetilde{\mathcal{E}'_L} \to \overline{\mathcal{E}'_L}$;
(4) the morphism $\widetilde{\mathcal{E}'_L} \to (\varpi_{(L,\mathscr{O}'_L)})^* \overline{\mathcal{E}'_L}$ obtained by adjunction from any such morphism is an isomorphism.

By the projection formula we deduce a canonical isomorphism

$$(\varpi_{(L,\mathscr{O}'_L)})_* \widetilde{\mathcal{E}'_L} \xrightarrow{\sim} \overline{\mathcal{E}'_L} \otimes_{\mathbb{F}} (\varpi_{(L,\mathscr{O}'_L)})_* \underline{\mathbb{F}}_{G \times^L (\mathscr{O}'_L + \mathfrak{z}_L^\circ)},$$

and finally that, under this isomorphism, the simple quotients of $(\varpi_{(L,\mathscr{O}'_L)})_* \widetilde{\mathcal{E}'_L}$ correspond to the local systems of the form $\overline{\mathcal{E}'_L} \otimes \mathcal{L}$, where $\mathcal{L}$ is a simple quotient of the local system $(\varpi_{(L,\mathscr{O}'_L)})_* \underline{\mathbb{F}}_{G \times^L (\mathscr{O}'_L + \mathfrak{z}_L^\circ)}$. Since $\varpi_{(L,\mathscr{O}'_L)}$ is a Galois covering of Galois group $N_G(L)/L$ (see §2.3), the latter simple quotients are in bijection with $\mathsf{Irr}(\mathbb{F}[N_G(L)/L])$, and we finally obtain the bijection $\varphi_{(L,\mathscr{O}_L,\mathcal{E}_L)}$.

The invariance by conjugation of $\varphi_{(L,\mathscr{O}_L,\mathcal{E}_L)}$ is clear from the characterization of the local system $\overline{\mathcal{E}'_L}$ and the construction of the bijection.

Remark 2.7. In Lusztig's characteristic-0 setting, the groups $N_G(L)/L$ which appear in Theorem 2.6 are Coxeter groups; see [**Lu2**, Theorem 9.2]. In our setting this is not always the case; see [**AHJR3**, Remark 3.5] for an explicit example.

2.6. Summary. Combining Theorems 2.4 and 2.6, we obtain a canonical decomposition of the set $\mathfrak{N}_{G,\mathbb{F}}$ as a disjoint union of subsets, each of which is in a canonical bijection with the set of isomorphism classes of simple representations of a certain finite group. In this decomposition, the subset corresponding to the case where the Levi subgroup $L = T$ is a maximal torus (and $(\mathscr{O}_L, \mathcal{E}_L)$ is the unique pair in $\mathfrak{N}_{L,\Bbbk}$), and its bijection with $\mathsf{Irr}(\mathbb{F}[W_{\mathrm{f}}])$ where $W_{\mathrm{f}} = N_G(T)/T$ is the Weyl group, is the celebrated *Springer correspondence*. In the case $p = 0$, one recovers Lusztig's generalized Springer correspondence from [**Lu2**] (with the difference that we work with usual complexes of sheaves on the complex version of $\mathscr{N}_G$, while Lusztig works in the étale setting, over the version of $\mathscr{N}_G$ over a field of positive characteristic; but essential differences occur only when this positive characteristic is bad for G).

Although the construction of this correspondence is based on geometry, the set $\mathfrak{N}_{G,\mathbb{F}}$ has a combinatorial description in all cases (e.g. in terms of various types of partitions in the case of classical groups). The groups $N_G(L)/L$ for Levi subgroups $L \subset G$, together with their simple representations, can also be described combinatorially. In this way, the generalized Springer correspondence becomes a bijection between two finite sets, which one can try to describe explicitly.

A first step towards this is the classification of the conjugacy classes of cuspidal triples $(L, \mathscr{O}_L, \mathcal{E}_L)$, or in other words of the cuspidal pairs of each Levi subgroup of G. This step was accomplished for classical groups in [**AHJR3**], and in some cases for exceptional groups in [**AHJR4**, Appendix A]. But even when this step is understood, the actual description of the correspondence is still a difficult problem. The cases where this description is known are the following:

(1) G arbitrary, with p not dividing $\#W_{\mathrm{f}}$, see [AHJR4, §7.1] (in this case the correspondence is the same as in characteristic 0, in which case it was computed by Lusztig and Spaltenstein);
(2) $G = \mathrm{SL}_n(\mathbb{C})$, with p arbitrary, see [AHJR3, Theorem 9.1];
(3) $G = \mathrm{Sp}_N(\mathbb{C})$ or $\mathrm{SO}_N(\mathbb{C})$ with $p = 2$, see [AHJR3, Theorem 9.5];
(4) G quasi-simple of type $\mathbf{G}_2$ with p arbitrary, see [AHJR4, §7.3].

2.7. Remarks on cleanness. In [AHJR5] we study some aspects of the generalized Springer correspondence in the case where p is rather good for G, i.e. good for G and not dividing $\#(Z(G)/Z(G)^\circ)$. In this case there exists a canonical bijection $\mathfrak{N}_{G,\mathbb{F}} \leftrightarrow \mathfrak{N}_{G,\mathbb{C}}$. It is known that under this bijection the set $\mathfrak{N}^{\mathrm{cusp}}_{G,\mathbb{C}}$ is a subset of $\mathfrak{N}^{\mathrm{cusp}}_{G,\mathbb{F}}$, see [AHJR2, Proposition 2.22]. Carl Mautner conjectured (in unpublished work) that for all pairs $(\mathscr{O}, \mathcal{E}) \in \mathfrak{N}_{G,\mathbb{F}}$ whose image in $\mathfrak{N}_{G,\mathbb{C}}$ belongs to $\mathfrak{N}^{\mathrm{cusp}}_{G,\mathbb{C}}$, the local system $\mathcal{E}$ is clean; in other words, for any such pair $(\mathscr{O}, \mathcal{E})$, we have

$$\mathcal{IC}(\mathscr{O}, \mathcal{E}) = (j_{\mathscr{O}})_! \mathcal{E}[\dim(\mathscr{O})] = (j_{\mathscr{O}})_* \mathcal{E}[\dim(\mathscr{O})],$$

where $j_{\mathscr{O}} : \mathscr{O} \hookrightarrow \mathscr{N}_G$ is the embedding. In the case $p = 0$, this fact is a well-known observation of Lusztig. In [AHJR5, Theorem 1.3] we proved this conjecture in a number of cases, including the case when p does not divide $\#W_{\mathrm{f}}$, and the case when G has only simple factors of exceptional type. We also observed that this conjecture has a number of interesting consequences, including an orthogonal decomposition of the category $D^{\mathrm{b}}_G(\mathscr{N}_G, \Bbbk)$ which generalizes a result of Lusztig in the characteristic-0 setting; see [AHJR5, Theorem 1.6].

3. Mixed derived categories and Koszul duality

This section is concerned with joint work with Pramod Achar [AR3, AR4, AR5] on the construction of a geometric "Koszul duality" equivalence for constructible sheaves on flag varieties. These constructions were inspired by earlier work with Wolfgang Soergel and Geordie Williamson [RSW] (based on more "traditional" techniques) and subsequent discussions with Geordie Williamson, and rely on the use of the new notion of *mixed derived category*. (We believe this notion is of independent interest; it has also found other applications in [R3, MaR2, ARd2, AR6].)

3.1. Reminder on parity complexes. Let us consider some complex algebraic variety X endowed with an algebraic stratification

$$X = \bigsqcup_{s \in \mathscr{S}} X_s \tag{3.1}$$

(in the sense of [CG, Definition 3.2.23]) such that each X_s is isomorphic to an affine space. We denote by $D^{\mathrm{b}}_{\mathscr{S}}(X, \mathbb{F})$ the derived category of $\mathbb{F}$-sheaves which are constructible with respect to the stratification (3.1). We will denote by $\mathbb{D}_X : D^{\mathrm{b}}_{\mathscr{S}}(X, \mathbb{F})^{\mathrm{op}} \xrightarrow{\sim} D^{\mathrm{b}}_{\mathscr{S}}(X, \mathbb{F})$ the Verdier duality functor.

The following definition is due to Juteau–Mautner–Williamson [JMW1].

Definition 3.1. An object $\mathcal{F}$ in $D^{\mathrm{b}}_{\mathscr{S}}(X,\mathbb{F})$ is called *even* if it satisfies

$$\mathcal{H}^i(\mathcal{F}) = \mathcal{H}^i(\mathbb{D}_X(\mathcal{F})) = 0 \qquad \text{unless } i \text{ is even.}$$

An object $\mathcal{F}$ is called a *parity complex* if $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{G}'[1]$ for some even objects $\mathcal{G}$ and $\mathcal{G}'$.

We will denote by $\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$ the full additive subcategory of $D^{\mathrm{b}}_{\mathscr{S}}(X,\mathbb{F})$ consisting of parity complexes. This subcategory is stable under direct summands and under the cohomological shift [1]. For some reason that will appear in §3.2 below, we will denote by $\{1\}$ the restriction of [1] to the subcategory $\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$. It is known that the category $D^{\mathrm{b}}_{\mathscr{S}}(X,\mathbb{F})$, as well as its subcategory $\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$, are Krull–Schmidt categories; see [**JMW1**, §2.1].

For the following result, see [**JMW1**, Theorem 2.12 and Corollary 2.28].

Theorem 3.2. *For any $s \in \mathscr{S}$, there exists a unique indecomposable parity complex $\mathcal{E}_s$ in $D^{\mathrm{b}}_{\mathscr{S}}(X,\mathbb{F})$ which is supported on $\overline{X_s}$ and whose restriction to X_s is $\underline{\mathbb{F}}_{X_s}[\dim(X_s)]$. Moreover, any indecomposable object in $\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$ is isomorphic to $\mathcal{E}_s[i]$ for some unique $s \in \mathscr{S}$ and $i \in \mathbb{Z}$.*

Remark 3.3. Standard arguments allow to generalize the theory of parity complexes from the case of an algebraic variety with a finite stratification to the case of an ind-variety X which can be written as an increasing union of closed subvarieties endowed with compatible finite stratifications satisfying the conditions above, see e.g. [**JMW1**, §2.7]. This comment also applies to the various constructions considered in the remainder of this section.

3.2. Mixed derived category and mixed perverse sheaves. In [**AR4**] we defined the *mixed derived category* of X as the triangulated category

$$D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F}) := K^{\mathrm{b}}\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F}).$$

Any object of $\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$ can be considered as a complex concentrated in degree 0, hence as an object of $D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$. In particular, for $s \in \mathscr{S}$, we will denote by $\mathcal{E}^{\mathrm{mix}}_s$ the image of $\mathcal{E}_s$ in $D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$.

We consider this construction to be a replacement for the notion of mixed sheaves (in the sense of [**BBD**, §5.1]) for coefficient fields of positive characteristic, which is suitable at least in some situations of interest in Representation Theory (in particular for flag varieties and their generalizations). To support this idea, we next explain the analogues in this setting of some basic constructions for mixed sheaves.

First we explain the definition of the "Tate twist" autoequivalence

$$\langle 1 \rangle : D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F}) \xrightarrow{\sim} D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F}).$$

As explained in §3.1, the category $\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$ admits an autoequivalence $\{1\}$. We will denote similarly the induced autoequivalence of $K^{\mathrm{b}}\mathsf{Parity}_{\mathscr{S}}(X,\mathbb{F})$. On the other hand, since $D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ is a triangulated category, it also has a "cohomological shift" autoequivalence [1]. We define the Tate twist as

$$\langle 1 \rangle := \{-1\}[1].$$

In [**AR4**, §2.4–2.5], we explain that for any locally closed inclusion of strata $h : Y \to X$ one can define functors

$$h^*, h^! : D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F}) \to D^{\mathrm{mix}}_{\mathscr{S}}(Y,\mathbb{F}), \quad h_*, h_! : D^{\mathrm{mix}}_{\mathscr{S}}(Y,\mathbb{F}) \to D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$$

such that

(1) there exist canonical adjunctions (h^*, h_*) and $(h_!, h^!)$; moreover the adjunction morphisms $h^* h_* \to \mathrm{id}$ and $\mathrm{id} \to h^! h_!$ are isomorphisms;
(2) if h is a closed embedding then $h_* = h_!$ and this functors is induced by the natural functor $h_* : \mathsf{Parity}_{\mathscr{S}}(Y, \mathbb{F}) \to \mathsf{Parity}_{\mathscr{S}}(X, \mathbb{F})$;
(3) if h is an open embedding then $h^* = h^!$ and this functor is induced by the natural functor $h^* : \mathsf{Parity}_{\mathscr{S}}(X, \mathbb{F}) \to \mathsf{Parity}_{\mathscr{S}}(Y, \mathbb{F})$;
(4) if h is an open embedding and k is the embedding of the (closed) complement, then for any $\mathcal{F}$ in $D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})$ there exist functorial distinguished triangles
$$h_! h^* \mathcal{F} \to \mathcal{F} \to k_* k^* \mathcal{F} \xrightarrow{[1]} \quad \text{and} \quad k_* k^! \mathcal{F} \to \mathcal{F} \to h_* h^* \mathcal{F} \xrightarrow{[1]}$$
where the first and second morphisms are induced by adjunction.

In particular, consider some $s \in \mathscr{S}$. Then the constant sheaf $\underline{\mathbb{F}}_{X_s}$ is obviously a parity complex on X_s, hence it defines an object in $D^{\mathrm{mix}}_{\{s\}}(X_s, \mathbb{F})$ which we also denote $\underline{\mathbb{F}}_{X_s}$. Then, if we denote by $i_s : X_s \to X$ the embedding, we can consider the objects
$$\Delta^{\mathrm{mix}}_s := (i_s)_! \underline{\mathbb{F}}_{X_s}\{\dim(X_s)\}, \qquad \nabla^{\mathrm{mix}}_s := (i_s)_* \underline{\mathbb{F}}_{X_s}\{\dim(X_s)\}$$
in $D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})$. These objects satisfy
$$\mathrm{Hom}_{D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})}(\Delta^{\mathrm{mix}}_s, \nabla^{\mathrm{mix}}_t\langle i\rangle[j]) = \begin{cases} \mathbb{F} & \text{if } s = t \text{ and } i = j = 0; \\ 0 & \text{otherwise;} \end{cases}$$
see [AR4, Lemma 3.2].

Using the gluing formalism from [BBD, §1.4], we can then prove the following result; see [AR4, §3.1]. (Here, given a triangulated category $\mathcal{D}$ and a collection $\mathcal{X}$ of objects, we denote by $\langle\!\langle \mathcal{X} \rangle\!\rangle$ the strictly full subcategory of $\mathcal{D}$ generated under extensions by the objects in $\mathcal{X}$.)

THEOREM 3.4. *Set*
$$\begin{aligned} {}^{\mathrm{p}}D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})^{\leq 0} &:= \langle\!\langle \Delta^{\mathrm{mix}}_s\langle i\rangle[j], \ s \in \mathscr{S}, \, i \in \mathbb{Z}, \, j \in \mathbb{Z}_{\geq 0} \rangle\!\rangle; \\ {}^{\mathrm{p}}D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})^{\geq 0} &:= \langle\!\langle \nabla^{\mathrm{mix}}_s\langle i\rangle[j], \ s \in \mathscr{S}, \, i \in \mathbb{Z}, \, j \in \mathbb{Z}_{\leq 0} \rangle\!\rangle. \end{aligned}$$
Then $({}^{\mathrm{p}}D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})^{\leq 0}, {}^{\mathrm{p}}D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})^{\geq 0})$ *is a t-structure on on* $D^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})$.

This t-structure is called the *perverse t-structure*. Its heart is called the category of *mixed perverse sheaves* on X, and will be denoted $\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F})$. It follows from the general gluing formalism that this category is a finite-length abelian category, and that its simple objects are the perverse sheaves $\mathcal{IC}^{\mathrm{mix}}_s\langle j\rangle$, where $\mathcal{IC}^{\mathrm{mix}}_s$ is the image of the unique (up to scalar) nonzero morphism ${}^{\mathrm{p}}\mathcal{H}^0(\Delta^{\mathrm{mix}}_s) \to {}^{\mathrm{p}}\mathcal{H}^0(\nabla^{\mathrm{mix}}_s)$.

To proceed further, we need to make the following assumption:
$$\text{for any } s \in \mathscr{S}, \text{ the objects } \Delta^{\mathrm{mix}}_s \text{ and } \nabla^{\mathrm{mix}}_s \text{ belong to } \mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X, \mathbb{F}).$$
In the case of ordinary perverse sheaves, the analogous property follows from the fact that the morphism i_s is affine. We were not able to show that it holds in general in our setting; however in [AR4, Theorem 4.7] we proved that it is satisfied if X is a partial flag

variety of a Kac–Moody group, stratified by orbits of a Borel subgroup (i.e. by the Bruhat decomposition).

Under these assumptions, we check in [**AR4**, §3.2] that $\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ is a graded highest weight category with weight poset $(\mathscr{S},\preceq)$ (where $\preceq$ is the partial order on $\mathscr{S}$ induced by inclusions of closures of strata), standard objects Δ^{mix}_s, and costandard objects ∇^{mix}_s. (See Definition 7.1 and Remark 7.2(3) for these notions.) In particular we can consider the *tilting objects* in this category; see §7.5. We will denote by $\mathsf{Tilt}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ the full subcategory of $\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ whose objects are the tilting objects, and for $\mathcal{F}$ in $\mathsf{Tilt}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ we will denote by

$$(\mathcal{F}:\nabla^{\mathrm{mix}}_s\langle i\rangle),\quad\text{resp.}\quad(\mathcal{F}:\Delta^{\mathrm{mix}}\langle i\rangle),$$

the multiplicity of $\nabla^{\mathrm{mix}}_s\langle i\rangle$, resp. $\Delta^{\mathrm{mix}}_s\langle i\rangle$, in a costandard, resp. standard, filtration of $\mathcal{F}$ (see §7.1). The general theory of graded highest weight categories ensures that the category $\mathsf{Tilt}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ is Krull–Schmidt, and that its indecomposable objects are parametrized by $\mathscr{S}\times\mathbb{Z}$. More precisely, for any $s\in\mathscr{S}$ there exists a unique indecomposable tilting object $\mathsf{T}^{\mathrm{mix}}_s$ such that

$$(\mathsf{T}^{\mathrm{mix}}_s:\nabla^{\mathrm{mix}}_s)=1\quad\text{and}\quad\big((\mathsf{T}^{\mathrm{mix}}_s:\nabla^{\mathrm{mix}}_t\langle i\rangle)\neq 0\ \Rightarrow\ t\preceq s\big).$$

Then the isomorphism classes of indecomposable objects in $\mathsf{Tilt}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ are the isomorphism classes of the objects $\mathsf{T}^{\mathrm{mix}}_s\langle i\rangle$ for $s\in\mathscr{S}$ and $i\in\mathbb{Z}$.

REMARK 3.5. (1) If X is a partial flag variety and $\mathbb{F}=\overline{\mathbb{Q}}_\ell$, then one can deduce from the results of [**AR1**] that the category $\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ is equivalent to the category considered in [**BGS**, §4.4]. (This category is a certain subcategory of the category of Deligne's mixed perverse sheaves on the version of X over a finite field.) More generally, this comment applies to varieties endowed with an affine even stratification in the sense of [**AR1**, Definition 7.1] and which are defined over a localization of $\mathbb{Z}$ (so that we can compare the constructible derived categories in the étale sense for the scheme $X_{\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$ with the ordinary derived category $D^{\mathrm{b}}_{\mathscr{S}}(X,\overline{\mathbb{Q}}_\ell)$ considered here using the principles of [**BBD**, §6.1]). In fact, in this setting there exists an equivalence of categories between the category denoted $\mathrm{Pure}(X_{\mathbb{F}_q})$ in [**AR1**] and the category $\mathsf{Parity}_{\mathscr{S}}(X,\overline{\mathbb{Q}}_\ell)$.

(2) The definition of the mixed derived category makes sense as soon as the category of parity complexes makes reasonable sense; see [**JMW1**, §2.1] for the required conditions. In particular, we can consider an equivariant setting, see [**AR4**, §3.5], or more general stratifications, see [**ARd2**, Appendix A]. In both of these cases one also has a perverse t-structure; however its heart will not be graded highest weight in general.

(3) One should keep in mind the fact that from the general theory it is not clear that there exists any "forgetful functor" from $D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ to $D^{\mathrm{b}}_{\mathscr{S}}(X,\mathbb{F})$, contrary to the situation for Deligne's mixed perverse sheaves. The existence of such a functor is an important question; see Remark 3.7(5) below.

(4) One can consider on $D^{\mathrm{mix}}_{\mathscr{S}}(X,\mathbb{F})$ several notions which play a role similar to Deligne's theory of weights for ordinary mixed sheaves, see [**AR5**]. However, in general these notions do not have a behavior as favorable as in the $\overline{\mathbb{Q}}_\ell$-setting.

3.3. Application to Koszul duality. Our first application of the theory of §3.2 is an adaptation to the modular setting of the Bezrukavnikov–Yun geometric Koszul duality from [BY]. More precisely, we consider a complex connected reductive algebraic group G, we choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset G$, and denote by $G^\vee$ the Langlands dual complex connected reductive group, with maximal torus $T^\vee$ such that $X^*(T^\vee) = X_*(T)$, and denote by $B^\vee \subset G^\vee$ the Borel subgroup containing $T^\vee$ whose roots are the coroots of B. The Weyl group W_{f} of (G,T) identifies canonically with the Weyl group of $(G^\vee, T^\vee)$, so that if we set

$$\mathscr{B} := G/B, \qquad \mathscr{B}^\vee := B^\vee \backslash G^\vee,$$

we have the Bruhat decompositions

$$\mathscr{B} = \bigsqcup_{w \in W_{\mathrm{f}}} \mathscr{B}_w, \qquad \mathscr{B}^\vee := \bigsqcup_{w \in W_{\mathrm{f}}} \mathscr{B}^\vee_w$$

where, for $w \in W_{\mathrm{f}}$, we have set

$$\mathscr{B}_w = BwB/B, \qquad \mathscr{B}^\vee_w = B^\vee \backslash B^\vee w B^\vee.$$

Each of these strata is isomorphic to an affine space, so that we can consider the categories defined in §§3.1–3.2, which we denote

$$D^{\mathrm{b}}_{(B)}(\mathscr{B}, \mathbb{F}), \quad \mathsf{Parity}_{(B)}(\mathscr{B}, \mathbb{F}), \quad D^{\mathrm{mix}}_{(B)}(\mathscr{B}, \mathbb{F}), \quad \mathsf{Tilt}^{\mathrm{mix}}_{(B)}(\mathscr{B}, \mathbb{F})$$

and

$$D^{\mathrm{b}}_{(B^\vee)}(\mathscr{B}^\vee, \mathbb{F}), \quad \mathsf{Parity}_{(B^\vee)}(\mathscr{B}^\vee, \mathbb{F}), \quad D^{\mathrm{mix}}_{(B^\vee)}(\mathscr{B}^\vee, \mathbb{F}), \quad \mathsf{Tilt}^{\mathrm{mix}}_{(B^\vee)}(\mathscr{B}^\vee, \mathbb{F})$$

respectively. For $w \in W$, we will denote by Δ^{mix}_w, ∇^{mix}_w, $\mathsf{T}^{\mathrm{mix}}_w$, $\mathcal{E}^{\mathrm{mix}}_w$ the objects of the category $D^{\mathrm{mix}}_{(B)}(\mathscr{B}, \mathbb{F})$ attached to w, and by $\Delta^{\vee,\mathrm{mix}}_w$, $\nabla^{\vee,\mathrm{mix}}_w$, $\mathsf{T}^{\vee,\mathrm{mix}}_w$, $\mathcal{E}^{\vee,\mathrm{mix}}_w$ the objects of the category $D^{\mathrm{mix}}_{(B^\vee)}(\mathscr{B}^\vee, \mathbb{F})$ attached to w.

The following result is equivalent to [AR4, Theorem 5.4].

THEOREM 3.6. *Assume that $\mathbb{F}$ is finite,* $\mathrm{char}(\mathbb{F})$ *is good for G, and that $\mathbb{F} \neq \mathbb{F}_2$. There exists an equivalence of triangulated categories*

$$\kappa : D^{\mathrm{mix}}_{(B)}(\mathscr{B}, \mathbb{F}) \xrightarrow{\sim} D^{\mathrm{mix}}_{(B^\vee)}(\mathscr{B}^\vee, \mathbb{F})$$

such that $\kappa \circ \langle 1 \rangle \cong \langle -1 \rangle [1] \circ \kappa$ and

$$\kappa(\Delta^{\mathrm{mix}}_w) \cong \Delta^{\vee,\mathrm{mix}}_w, \quad \kappa(\nabla^{\mathrm{mix}}_w) \cong \nabla^{\vee,\mathrm{mix}}_w, \quad \kappa(\mathsf{T}^{\mathrm{mix}}_w) \cong \mathcal{E}^{\vee,\mathrm{mix}}_w, \quad \kappa(\mathcal{E}^{\mathrm{mix}}_w) \cong \mathsf{T}^{\vee,\mathrm{mix}}_w$$

for all $w \in W$.

REMARK 3.7. (1) In [AR4], the theorem is stated for the variety $G^\vee/B^\vee$ rather than $B^\vee \backslash G^\vee$. Each version is deduced from the other via the equivalence induced by the isomorphism $G^\vee/B^\vee \xrightarrow{\sim} B^\vee \backslash G^\vee$ sending $gB^\vee$ to $B^\vee g^{-1}$, which exchanges the orbits of w and w^{-1}.

(2) We believe this statement should hold without any assumption on $\mathbb{F}$. (A more general statement, covering all fields except those of characteristic 2 in some cases, should follow from the methods considered in §3.4 below.) Under the assumptions of the theorem, it follows from its proof that the categories $\mathsf{Parity}_{(B)}(\mathscr{B}, \mathbb{F})$ and $\mathsf{Parity}_{(B^\vee)}(G^\vee/B^\vee, \mathbb{F})$ are equivalent, hence the same holds for the mixed derived categories. This fact is known to be false in general.

(3) In the characteristic-0 setting, an equivalence as in Theorem 3.6 was obtained previously by Bezrukavnikov–Yun in [BY]. These authors work with ordinary étale derived categories, and not with our mixed derived categories; however their constructions would also apply in our setting. Their construction has its roots in the work of Beĭlinson–Ginzburg–Soergel [BGS] on category $\mathcal{O}$ for a complex semisimple Lie algebra. A fundamental idea used in [BY] (and suggested earlier in [BG]) is that, in order to obtain a more favorable Koszul duality equivalence (in particular, valid for all Kac–Moody groups), one should compose the original Koszul duality of [BGS] with Ringel duality, hence work with tilting objects instead of projective objects.

(4) The main difference between our Theorem 3.6 and the characteristic-0 setting considered in (3) is that the duals of tilting objects are *parity complexes* and not semisimple complexes. (In the characteristic-0 setting the two classes of objects coincide, but not in the modular setting in general.) This idea was already used crucially in [RSW].

(5) At the same time as proving Theorem 3.6, we also construct a "forgetful" t-exact functor $D^{\mathrm{mix}}_{(B)}(\mathscr{B},\mathbb{F}) \to D^{\mathrm{b}}_{(B)}(\mathscr{B},\mathbb{F})$ sending standard, costandard, simple, indecomposable tilting mixed perverse sheaves to standard, costandard, simple, indecomposable tilting ordinary perverse sheaves.

Our proof of Theorem 3.6 is analogous to the proof of Bezrukavnikov–Yun in the characteristic-0 setting. The main step is accomplished in [AR3], where we relate the additive category $\mathsf{Tilt}_{(B)}(\mathscr{B},\mathbb{F})$ of (ordinary) tilting perverse sheaves on $\mathscr{B}$ to the additive category $\mathsf{Parity}_{(B^\vee)}(\mathscr{B}^\vee,\mathbb{F})$, by describing both sides in terms of some "Soergel modules" over the coinvariant algebra of the Lie algebra of T. The functor from parity complexes to Soergel modules is provided by total cohomology (as usual), while the functor from tilting perverse sheaves to Soergel modules is constructed using the "logarithm of the monodromy." (In fact, the main new ideas required for our proof are those used to make sense of this notion of logarithm of the monodromy in a positive-characteristic setting.)

Once this step is established, we can consider the functor

$$\nu : D^{\mathrm{mix}}_{(B^\vee)}(\mathscr{B}^\vee,\mathbb{F}) = K^{\mathrm{b}}\mathsf{Parity}_{(B^\vee)}(\mathscr{B}^\vee,\mathbb{F}) \to K^{\mathrm{b}}\mathsf{Tilt}_{(B)}(\mathscr{B},\mathbb{F}) \cong D^{\mathrm{b}}_{(B)}(\mathscr{B},\mathbb{F}).$$

(See Proposition 7.17 for the equivalence on the right-hand side.) We observe that this functor restricts to an equivalence between the additive subcategories $\mathsf{Tilt}^{\mathrm{mix}}_{(B^\vee)}(\mathscr{B}^\vee,\mathbb{F})$ and $\mathsf{Parity}_{(B)}(\mathscr{B},\mathbb{F})$, and use this induced equivalence to construct the functor κ.

Remark 3.8. In [AR3] we use the construction of the equivalence κ to prove that if p is bigger than the Coxeter number of G, then Soergel's modular category $\mathcal{O}$ associated with the split simply-connected semisimple $\mathbb{F}$-algebraic group with the same root system as G, as defined in [So4], is equivalent, as a highest weight category, to $\mathsf{Perv}_{(B)}(\mathscr{B},\mathbb{F})$. This result can be considered as a "finite analogue" of the Finkelberg–Mirković conjecture considered in §5.6 below.

3.4. A conjectural generalization to Kac–Moody groups. We expect Theorem 3.6 to admit the following generalization. Let $A=(a_{i,j})_{i,j\in I}$ be a generalized Cartan matrix, with rows and columns parametrized by a finite set I, and let $(\Lambda, \{\alpha_i : i\in I\}, \{\alpha_i^\vee :$

$i \in I\}$), be an associated Kac–Moody root datum. In other words, Λ is a finitely generated free $\mathbb{Z}$-module, $\{\alpha_i : i \in I\}$ is a collection of elements of Λ, $\{\alpha_i^\vee : i \in I\}$ is a collection of elements of $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$, and we assume that $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$. To such a datum one can associate following Mathieu [Ma][3] a $\mathbb{Z}$-group scheme. We denote by $\mathcal{G}$ the set of $\mathbb{C}$-points of this group scheme. We also denote by $\mathcal{B} \subset \mathcal{G}$ the ($\mathbb{C}$-points of the) standard Borel subgroup, and by $\mathcal{T} \subset \mathcal{B}$ the ($\mathbb{C}$-points of the) standard maximal torus. Then the group of (algebraic) characters of $\mathcal{T}$ is Λ. We let

$$\mathscr{B} := \mathcal{G}/\mathcal{B}$$

be the associated flag (ind-)variety. As in the case of reductive groups we have a Weyl group $\mathcal{W}$ (a Coxeter group which is not finite in general), and a Bruhat decomposition

$$\mathscr{B} = \bigsqcup_{w \in \mathcal{W}} \mathscr{B}_w$$

(see [Ro, §3.16]), and we can consider the associated categories

$$D^{\mathrm{b}}_{(\mathcal{B})}(\mathscr{B}, \mathbb{F}), \quad \mathsf{Parity}_{(\mathcal{B})}(\mathscr{B}, \mathbb{F}), \quad D^{\mathrm{mix}}_{(\mathcal{B})}(\mathscr{B}, \mathbb{F}), \quad \mathsf{Tilt}^{\mathrm{mix}}_{(\mathcal{B})}(\mathscr{B}, \mathbb{F}).$$

Similarly, the transposed matrix ${}^{\mathrm{t}}A$ is a generalized Cartan matrix, and the triple $(\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}), \{\alpha_i^\vee : i \in I\}, \{\alpha_i : i \in I\})$ is an associated Kac–Moody root datum. Hence we can consider the "Langlands dual" Kac–Moody group $\mathcal{G}^\vee$ with its Borel subgroup $\mathcal{B}^\vee$, and the flag variety

$$\mathscr{B}^\vee := \mathcal{B}^\vee \backslash \mathcal{G}^\vee.$$

The Weyl group of $\mathcal{G}^\vee$ identifies naturally with $\mathcal{W}$, and we have the Bruhat decomposition

$$\mathscr{B}^\vee = \bigsqcup_{w \in \mathcal{W}} \mathscr{B}^\vee_w,$$

so that we can consider the associated categories

$$D^{\mathrm{b}}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \mathbb{F}), \quad \mathsf{Parity}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \mathbb{F}), \quad D^{\mathrm{mix}}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \mathbb{F}), \quad \mathsf{Tilt}^{\mathrm{mix}}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \mathbb{F}).$$

As for reductive groups, for $w \in \mathcal{W}$ we will denote by Δ^{mix}_w, ∇^{mix}_w, $\mathsf{T}^{\mathrm{mix}}_w$, $\mathcal{E}^{\mathrm{mix}}_w$ the objects of $D^{\mathrm{mix}}_{(\mathcal{B})}(\mathscr{B}, \mathbb{F})$ attached to w, and by $\Delta^{\vee,\mathrm{mix}}_w$, $\nabla^{\vee,\mathrm{mix}}_w$, $\mathsf{T}^{\vee,\mathrm{mix}}_w$, $\mathcal{E}^{\vee,\mathrm{mix}}_w$ the objects of $D^{\mathrm{mix}}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \mathbb{F})$ attached to w.

We expect the following generalization of Theorem 3.6 to hold.

Conjecture 3.9. *Assume that the "Demazure surjectivity" condition of* [EW] *holds over* $\mathbb{F}$*, i.e. that the morphisms* $\alpha_i^\vee : \mathbb{F} \otimes_{\mathbb{Z}} \Lambda \to \mathbb{F}$ *and* $\alpha_i : \mathbb{F} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \to \mathbb{F}$ *are surjective for any* $i \in I$.[4] *There exists an equivalence of triangulated categories*

$$\kappa : D^{\mathrm{mix}}_{(\mathcal{B})}(\mathscr{B}, \mathbb{F}) \xrightarrow{\sim} D^{\mathrm{mix}}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \mathbb{F})$$

such that $\kappa \circ \langle 1 \rangle \cong \langle -1 \rangle [1] \circ \kappa$ *and*

$$\kappa(\Delta^{\mathrm{mix}}_w) \cong \Delta^{\vee,\mathrm{mix}}_w, \quad \kappa(\nabla^{\mathrm{mix}}_w) \cong \nabla^{\vee,\mathrm{mix}}_w, \quad \kappa(\mathsf{T}^{\mathrm{mix}}_w) \cong \mathcal{E}^{\vee,\mathrm{mix}}_w, \quad \kappa(\mathcal{E}^{\mathrm{mix}}_w) \cong \mathsf{T}^{\vee,\mathrm{mix}}_w$$

for all $w \in \mathcal{W}$.

3. To be more precise, Mathieu works under some technical conditions on the Kac–Moody root datum; see [Ti, §6.5] or [Ro, Remarque 3.5]. See [Ti, §6.8] for a sketch of an argument explaining how to generalize this construction to general Kac–Moody root data, and [Ro, §3.19] for more details.

4. Note that this condition is always satisfied if $p \neq 2$.

Remark 3.10. In a joint project involving Pramod Achar, Shotaro Makisumi and Geordie Williamson, we expect to prove Conjecture 3.9 using a strategy similar to the one used to prove Theorem 3.6, but this time using a "diagrammatic" version of Soergel modules instead of the "actual" Soergel modules considered in [**AR3**]. In fact, already the first half of this proof (relating parity complexes to diagrammatic Soergel modules) can be deduced from [**RW**, Theorems 10.5–6]. It remains to obtain the second half, i.e. to describe the category of mixed tilting perverse sheaves in terms of these diagrammatic Soergel modules.

Part 2

Representation Theory

In this part we let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and G be a connected reductive algebraic group over $\Bbbk$ with simply-connected derived subgroup.

4. A new approach to character formulas for reductive algebraic groups in positive characteristic

This section is devoted to my joint work [RW] with Geordie Williamson, where we propose a new point on view on the modular representation theory of reductive algebraic groups based on the use of the p-canonical basis of the affine Hecke algebra. Here we have chosen to concentrate on the combinatorial aspects of our conjectures and results. See [RW] for more precise categorical considerations related to diagrammatic Soergel bimodules.

4.1. Notations. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset G$. We will denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ the respective Lie algebras of G, B, T, and by h the Coxeter number of G.

We let $\mathbf{X}$ be the lattice of characters of T, $\Phi \subset \mathbf{X}$ be the set of roots of G relative to T, and W_{f} be the corresponding Weyl group. For $\gamma \in \Phi$, we will denote the corresponding reflection by s_γ. We denote by $\Phi^+ \subset \Phi$ the subset of positive roots consisting of the T-weights in $\mathfrak{g}/\mathfrak{b}$, by $\Phi^s \subset \Phi^+$ the corresponding set of simple roots, by $S_{\mathrm{f}} \subset W_{\mathrm{f}}$ the corresponding set of simple reflections, and by $\mathbf{X}^+ \subset \mathbf{X}$ the corresponding set of dominant weights. As usual we set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \quad \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{X}.$$

We also denote by $\Phi^\vee$ the system of coroots of (G, T). For $\alpha \in \Phi$ we will denote by $\alpha^\vee$ the associated coroot. Then we set $\Phi^\vee_+ = \{\alpha^\vee : \alpha \in \Phi^+\}$.

We consider the affine Weyl group

$$W := W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi.$$

For $\mu \in \mathbb{Z}\Phi$, we will denote by t_μ the image of μ in W. This group acts on $\mathbf{X}$ via the "dot-action" defined by

$$(t_\lambda v) \cdot_p \mu = v(\mu + \rho) - \rho + p \cdot \lambda$$

for $\lambda \in \mathbb{Z}\Phi$, $v \in W_{\mathrm{f}}$ and $\mu \in \mathbf{X}$. The group W has a natural Coxeter group structure with simple reflections S consisting of S_{f} together with the elements $t_\gamma s_\gamma$ where γ runs over maximal short roots in Φ. We will denote by ${}^{\mathrm{f}}W$ the subset of W consisting of elements w which are minimal in their coset $W_{\mathrm{f}} w$.

For any $\lambda \in \mathbf{X}$, we denote by $\mathsf{dom}(\lambda)$ the unique dominant weight in the W_{f}-orbit of λ, and by v_λ the unique element in W_{f} of minimal length such that $v_\lambda(\lambda) = \mathsf{dom}(\lambda)$. Then it is well known that if $p \geq h$ (so that 0 is a regular weight) we have bijections

$$\mathbb{Z}\Phi \xrightarrow{\sim} {}^{\mathrm{f}}W \xrightarrow{\sim} \mathbf{X}^+ \cap \{w \cdot_p 0 : w \in W\} \tag{4.1}$$

defined by

$$\mu \mapsto t_{\mathsf{dom}(\mu)} \cdot v_\mu \quad \text{and} \quad w \mapsto w \cdot_p 0$$

for $\mu \in \mathbb{Z}\Phi$ and $w \in W$. (For the first bijection, see e.g. [MaR1, Lemma 2.4]; of course the condition $p \geq h$ is not necessary for this part.)

We will denote by $\mathsf{Rep}(G)$ the abelian category of finite-dimensional algebraic G-modules. For any $\lambda \in \mathbf{X}^+$ we set

$$\mathsf{N}(\lambda) := \mathrm{Ind}_B^G(\lambda), \quad \mathsf{M}(\lambda) := \big(\mathrm{Ind}_B^G(-w_0(\lambda))\big)^*$$

(where $w_0 \in W_{\mathrm{f}}$ is the longest element). We denote by $\mathsf{L}(\lambda)$ the image of the only (up to scalar) nonzero morphism from $\mathsf{M}(\lambda)$ to $\mathsf{N}(\lambda)$. Then $\mathsf{L}(\lambda)$ is a simple G-module of highest weight λ. It is well known that the category $\mathsf{Rep}(G)$ is a highest weight category in the sense of Section 7, with standard objects $\mathsf{M}(\lambda)$ and costandard objects $\mathsf{N}(\lambda)$ for $\lambda \in \mathbf{X}^+$ (see Remark 7.7). Hence for any $\lambda \in \mathbf{X}^+$ we can consider the indecomposable tilting G-module $\mathsf{T}(\lambda)$ with highest weight λ, see §7.5.

If $p \geq h$, we will also denote by $\mathsf{Rep}_0(G)$ the "principal block" of $\mathsf{Rep}(G)$, i.e. the Serre subcategory generated by the simple objects $\mathsf{L}(\mu)$ with $\mu \in \mathbf{X}^+ \cap \{w \cdot_p 0 : w \in W\}$. (The bijections (4.1) show that these dominant weights are in a natural bijection with $\mathbb{Z}\Phi$ and with ${}^{\mathrm{f}}W$.) The "linkage principle" implies that $\mathsf{Rep}_0(G)$ is a direct factor in $\mathsf{Rep}(G)$, and using translation functors it is well known that most of the combinatorial information on the category $\mathsf{Rep}(G)$ (in particular, characters of simple and indecomposable tilting modules) can be derived from the corresponding combinatorial information in $\mathsf{Rep}_0(G)$. Therefore, understanding the category $\mathsf{Rep}_0(G)$ is the major problem in the representation theory of G.

4.2. The affine Hecke algebra and the p-canonical basis. We will denote by $\mathcal{H}$ the Hecke algebra of (W, S), i.e. the $\mathbb{Z}[v, v^{-1}]$-algebra generated by elements H_w for $w \in W$, with relations

$$(H_s + v)(H_s - v^{-1}) = 0 \quad \text{for } s \in S,$$
$$H_v \cdot H_w = H_{vw} \quad \text{for } v, w \in W \text{ such that } \ell(vw) = \ell(v) + \ell(w).$$

It is well known that the elements $\{H_w : w \in W\}$ form a $\mathbb{Z}[v, v^{-1}]$-basis of this algebra.

One can define a "geometric" basis of this algebra, called the *p-canonical basis*, as follows. Consider the complex connected reductive group $G^\wedge$ with maximal torus $T^\wedge$ whose root datum is $(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Phi, \mathbb{Z}), \Phi^\vee, \mathbb{Z}\Phi, \Phi)$. Then $G^\wedge$ is semisimple and simply-connected, and we have an identification $X_*(T^\wedge) = \mathbb{Z}\Phi$. Moreover, the Weyl group of $(G^\wedge, T^\wedge)$ identifies canonically with W_{f}. Let also $B^\wedge \subset G^\wedge$ be the Borel subgroup of $G^\wedge$ containing $T^\wedge$ whose set of roots is $-\Phi^\vee_+$. We set

$$\mathscr{K} := \mathbb{C}((z)), \qquad \mathscr{O} := \mathbb{C}[[z]],$$

denote by $I^\wedge \subset G^\wedge(\mathscr{O})$ the inverse image of $B^\wedge$ under the morphism $G^\wedge(\mathscr{O}) \to G^\wedge$ induced by $z \mapsto 0$, and consider the affine flag variety

$$\mathcal{F}\mathrm{l}^\wedge := G^\wedge(\mathscr{K})/I^\wedge.$$

Any $\lambda \in \mathbb{Z}\Phi = X_*(T^\wedge)$ defines a point $z^\lambda \in T^\wedge(\mathscr{K})$. If $v \in W_{\mathrm{f}}$, and $w = vt_\lambda$, we set

$$\mathcal{F}\mathrm{l}^\wedge_w := I^\wedge \cdot \tilde{v} z^\lambda \cdot I^\wedge / I^\wedge,$$

where $\tilde{v}$ is any lift of v in $N_{G^\wedge}(T^\wedge)$. Then we have a "Bruhat decomposition"

$$\mathcal{F}\mathrm{l}^\wedge_w = \bigsqcup_{w \in W} \mathcal{F}\mathrm{l}^\wedge_w, \tag{4.2}$$

and each $\mathcal{F}l^{\wedge}_w$ is isomorphic to an affine space of dimension $\ell(w)$. For $w \in W$, we denote by $i_w : \mathcal{F}l^{\wedge}_w \to \mathcal{F}l^{\wedge}$ the embedding.

Using the stratification (4.2) one can consider the associated constructible derived category $D^{\mathrm{b}}_{(I^{\wedge})}(\mathcal{F}l^{\wedge}, \Bbbk)$ of sheaves of $\Bbbk$-vector spaces, and the full additive subcategory $\mathsf{Parity}_{I^{\wedge}}(\mathcal{F}l^{\wedge}, \Bbbk)$ of parity complexes (see §3.1). As usual, we will denote by $\mathcal{E}_w$ the parity complex associated with the stratum $\mathcal{F}l^{\wedge}_w$.

Following Springer (see [Sp2, §2.5]), given an object $\mathcal{F}$ in $D^{\mathrm{b}}_{(I^{\wedge})}(\mathcal{F}l^{\wedge}, \Bbbk)$, we define an element $\mathrm{ch}(\mathcal{F})$ in $\mathcal{H}$ by the following formula:

$$\mathrm{ch}(\mathcal{F}) = \sum_{\substack{w \in W \\ i \in \mathbb{Z}}} \dim_{\Bbbk}\big(\mathsf{H}^{-\ell(w)-i}(\mathcal{F}l^{\wedge}_w, i_w^* \mathcal{F})\big) \cdot v^i H_w.$$

For $w \in W$, we also set

$$\mathrm{ch}_w(\mathcal{F}) = \sum_{i \in \mathbb{Z}} \dim_{\Bbbk}\big(\mathsf{H}^{-\ell(w)-i}(\mathcal{F}l^{\wedge}_w, i_w^* \mathcal{F})\big) \cdot v^i \in \mathbb{Z}[v, v^{-1}].$$

In this way we have $\mathrm{ch}(\mathcal{F}) = \sum_{w \in W} \mathrm{ch}_w(\mathcal{F}) \cdot H_w$.

Definition 4.1. (1) The *p-canonical basis* of $\mathcal{H}$ is the $\mathbb{Z}[v, v^{-1}]$-basis $\{{}^p\underline{H}_w : w \in W\}$ defined by ${}^p\underline{H}_w := \mathrm{ch}(\mathcal{E}_w)$.

(2) The associated p-Kazhdan–Lusztig polynomials are defined by ${}^p h_{y,w} := \mathrm{ch}_y(\mathcal{E}_w)$, so that ${}^p\underline{H}_w = \sum_{y \in W} {}^p h_{y,w} \cdot H_y$.

Remark 4.2. (1) Standard arguments show that the elements ${}^p\underline{H}_w$ only depend on the characteristic of $\Bbbk$, and not on the choice of field with this characteristic. This justifies the name of the basis.

(2) In [JW], the authors give a different definition for the p-canonical basis, in terms of an Elias–Williamson category of diagrammatic Soergel bimodules. The fact that the two definitions coincide can be easily deduced from the results of [RW, §11]. In particular, this means that the algorithm for computing this basis presented in [JW] applies.

(3) The same construction also makes sense for a field of characteristic 0, and it is known that the basis of $\mathcal{H}$ obtained in this way coincides with the usual Kazhdan–Lusztig (or canonical) basis $\{\underline{H}_w : w \in W\}$, see [Sp2, Theorem 2.8]. In general, it is known that the coefficients of ${}^p\underline{H}_w$ in the Kazhdan–Lusztig basis are polynomials in v with non-negative coefficients, which are invariant under $v \leftrightarrow v^{-1}$. It is also known that for every $w \in W$ there exists $N(w) \in \mathbb{Z}_{\geq 0}$ such that ${}^p\underline{H}_w = \underline{H}_w$ as soon as $p \geq N(w)$.

4.3. The antispherical module and its p-canonical basis. We will denote by $\mathcal{H}_{\mathrm{f}}$ the Hecke algebra of the Coxeter group $(W_{\mathrm{f}}, S_{\mathrm{f}})$, which coincides with the subalgebra of $\mathcal{H}$ generated by the elements H_w with $w \in W_{\mathrm{f}}$. We let sgn be the "sign module" of $\mathcal{H}_{\mathrm{f}}$, i.e. the right module given by $\mathbb{Z}[v, v^{-1}]$, with H_w acting by multiplication by $(-v)^{\ell(w)}$. The *anti-spherical module* of $\mathcal{H}$ is the right $\mathcal{H}$-module

$$\mathcal{M}^{\mathrm{asph}} := \mathrm{sgn} \otimes_{\mathcal{H}_{\mathrm{f}}} \mathcal{H}.$$

For $w \in {}^{\mathrm{f}}W$ we set $N_w := 1 \otimes H_w$. These elements form a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{M}^{\mathrm{asph}}$. Then we define the p-canonical basis of $\mathcal{M}^{\mathrm{asph}}$ by the formula

$$ {}^p\underline{N}_w := 1 \otimes {}^p\underline{H}_w $$

for $w \in {}^{\mathrm{f}}W$. The associated p-Kazhdan–Lusztig polynomials ${}^p n_{y,w} \in \mathbb{Z}[v, v^{-1}]$ are defined by the formula

$$ {}^p\underline{N}_w = \sum_{y \in {}^{\mathrm{f}}W} {}^p n_{y,w} \cdot N_y. $$

Remark 4.3. (1) In the analogous situation in characteristic 0, the basis defined by $\underline{N}_w := 1 \otimes \underline{H}_w$ is the Kazhdan–Lusztig basis of $\mathcal{M}^{\mathrm{asph}}$ considered in [So1, §3]; see [So1, Proof of Proposition 3.4(2)].

(2) The same arguments as in [So1, Proof of Proposition 3.4(2)] show that the polynomials ${}^p n_{y,w}$ are related to the polynomials ${}^p h_{z,w}$ by the formula

$$ {}^p n_{y,w} = \sum_{x \in W_{\mathrm{f}}} (-v)^{\ell(x)} \cdot {}^p h_{xy,w} \tag{4.3} $$

for $y, w \in {}^{\mathrm{f}}W$.

4.4. Relation with representation theory. From now on we assume that $p > h$.[5] The relation of these constructions with the principal block is as follows. The Grothendieck group $[\mathsf{Rep}_0(G)]$ has a basis (as a $\mathbb{Z}$-module) given by the classes $[\mathsf{N}(\lambda)]$ for $\lambda \in \mathbf{X}^+ \cap \{w \cdot_p 0 : w \in W\}$. In view of (4.1), we therefore have an isomorphism of $\mathbb{Z}$-modules

$$ \begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{asph}} & \cong & [\mathsf{Rep}_0(G)] \\ 1 \otimes N_w & \leftrightarrow & [\mathsf{N}(w \cdot_p 0)] \end{array} \tag{4.4} $$

(where here $w \in {}^{\mathrm{f}}W$, and the morphism $\mathbb{Z}[v, v^{-1}] \to \mathbb{Z}$ is induced by $v \mapsto 1$). It follows from the well-known combinatorics of translations functors (see [Ja1, §§II.7.11–12]) that this isomorphism is actually an isomorphism of W-modules, where $s \in S$ acts on the left-hand side via the action of $H_s \in \mathcal{H}$, and on the right-hand side via $[\Xi_s] - 1$, where Ξ_s is the composition of a translation functor from 0 to a weight on the s-wall of the fundamental alcove, and a translation functor back to 0.

In [RW] we formulate the following conjecture.

Conjecture 4.4. *Under the isomorphism (4.4), the class $[\mathsf{T}(w \cdot_p 0)]$ corresponds to $1 \otimes {}^p\underline{N}_w$. In other words, for any $y, w \in W$ we have*

$$ (\mathsf{T}(w \cdot_p 0) : \mathsf{N}(y \cdot_p 0)) = {}^p n_{y,w}(1). $$

Remark 4.5. (1) This conjecture is inspired by the earlier conjecture of Andersen [An] considered in §1.6 (which itself was inspired by earlier work of Soergel on tilting modules for Lusztig's quantum groups at roots of unity, see [So1, So2, So3]). The main differences between his conjecture and ours are that our formula is expected to be valid for *all* indecomposable tilting modules, and that it is formulated in terms of the p-canonical basis rather than the ordinary Kazhdan–Lusztig basis.

5. This condition is equivalent to requiring that $p \geq h$ and p is very good for G.

(2) In [RW] we show that this conjecture follows from a "categorical" conjecture on some properties of translation functors, and we prove this stronger conjecture in the case of the group $G = \mathrm{GL}_n(\Bbbk)$ (if $n \geq 3$) using the Khovanov–Lauda–Rouquier theory of categorical actions of Lie algebras. In particular, this means that Conjecture 4.4 holds in these cases.

(3) It follows from works of Jantzen, Donkin and Andersen (see [RW, §1.8]) that, if $p \geq 2h-2$, from the knowledge of the multiplicities $(\mathsf{T}(w \cdot_p 0) : \mathsf{N}(y \cdot_p 0))$ one can obtain the multiplicities $[\mathsf{M}(x \cdot_p 0) : \mathsf{L}(y \cdot_p 0)]$ for any $x \in W$ such that $\langle x \cdot_p 0, \alpha^\vee \rangle < p(h-1)$ for all $\alpha \in \Phi^+$. From this one can deduce character formulas for the simple modules $\mathsf{L}(x \cdot_p 0)$ (for x satisfying the same condition). Since these weights include all the restricted weights in the $(W, \cdot_p)$-orbit of 0, this is sufficient (using Steinberg's tensor product theorem and Jantzen's translation functors) to deduce character formulas for all simple modules.

(4) If $p < h$, then our approach does not make sense since regular weights do not exist. However from the characters of tilting modules with regular highest weight one can deduce characters of tilting modules with singular highest weight, see [RW, Conjecture 1.6] for an explicit formula. This formula also makes sense for $p \leq h$, and we conjecture that it holds also in this setting.

5. A geometric framework for the modular representation theory of reductive algebraic groups

This section is devoted to my joint works with Carl Mautner [MaR2] and with Pramod Achar [AR6], where we adapt some constructions due to Arkhipov–Bezrukavnikov–Ginzburg [ABG] to obtain a geometric description of the principal block of a connected reductive algebraic group defined over an algebraically closed field of positive characteristic. These works build on the earlier works [R1, BR, R3, MaR1]. We expect to use these results to obtain a proof of Conjecture 4.4 valid in full generality; see Section 6 for details.

5.1. Notations. We continue with the notation of Section 4 (but not assuming that $p > h$ at this point.) We also denote by U the unipotent radical of B, and by $\mathfrak{u}$ its Lie algebra. The natural bijections between Φ^s and S_{f} will be denoted

$$\alpha \mapsto s_\alpha \qquad \text{and} \qquad s \mapsto \alpha_s$$

respectively. For any $s \in S$ we choose $\varsigma_s \in \mathbf{X}$ such that for $t \in S$ we have

$$\langle \varsigma_s, \alpha_t^\vee \rangle = \begin{cases} 1 & \text{if } s = t; \\ 0 & \text{otherwise.} \end{cases}$$

(Such weights exist thanks to our assumption that the derived subgroup of G is simply-connected.)

For any subset $I \subset S$, we denote by P_I the standard parabolic subgroup of G associated with I, by U_I the unipotent radical of P_I, by $\mathfrak{p}_I$ the Lie algebra of P_I, and by $\mathfrak{u}_I$ the Lie algebra of U_I.

We will also denote by $\dot{G}$, $\dot{B}$, $\dot{T}$, $\dot{U}$, $\dot{P}_I$, $\dot{U}_I$ the Frobenius twists of G, B, T, U, P_I, U_I, and by $\dot{\mathfrak{g}}$, $\dot{\mathfrak{b}}$, $\dot{\mathfrak{t}}$, $\dot{\mathfrak{u}}$, $\dot{\mathfrak{p}}_I$, $\dot{\mathfrak{u}}_I$ their respective Lie algebras.

For any $I \subset S$, we will consider the partial flag variety $\mathscr{P}_I$, the Grothendieck resolution $\widetilde{\mathfrak{g}}_I$ and the Springer resolution $\widetilde{\mathscr{N}_I}$ defined as

$$\mathscr{P}_I := \dot{G}/\dot{P}_I, \qquad \widetilde{\mathfrak{g}}_I := \dot{G} \times^{\dot{P}_I} (\dot{\mathfrak{g}}/\dot{\mathfrak{u}}_I)^*, \qquad \widetilde{\mathscr{N}_I} := \dot{G} \times^{\dot{P}_I} (\dot{\mathfrak{g}}/\dot{\mathfrak{p}}_I)^*.$$

If $J \subset I$, then there exists a natural morphism $\pi_{J,I} : \widetilde{\mathfrak{g}}_J \to \widetilde{\mathfrak{g}}_I$. These varieties are naturally endowed with actions of $\dot{G} \times \mathbb{G}_{\mathrm{m}}$, where $t \in \mathbb{G}_{\mathrm{m}}$ acts by multiplication by t^{-2} along the fibers of the projection to $\mathscr{P}_I$.

We will identify the lattice of weights of $\dot{T}$ with $\mathbf{X}$, in such a way that the inverse image under the Frobenius morphism $T \to \dot{T}$ induces the morphism $\mathbf{X} \to \mathbf{X}$ given by multiplication by p. Then for any $\lambda \in \mathbf{X}$ which satisfies $\langle \lambda, \alpha_s^\vee \rangle = 0$ for all $s \in I$, we have an associated line bundle $\mathcal{O}_{\mathscr{P}_I}(\lambda)$ on $\mathscr{P}_I$. We denote by $\mathcal{O}_{\widetilde{\mathscr{N}_I}}(\lambda)$ the pullback of this line bundle under the projection $\widetilde{\mathscr{N}_I} \to \mathscr{P}_I$.

5.2. Braid group action. To the Coxeter group $(W_{\mathrm{f}}, S_{\mathrm{f}})$ one can associate the braid group B_{f}, which can be defined as the group generated by elements T_w° for $w \in W_{\mathrm{f}}$, with relations

$$T_v^\circ \cdot T_w^\circ = T_{vw}^\circ \quad \text{if } \ell(vw) = \ell(v) + \ell(w).$$

In this subsection we recall the construction of an action [6] of this group on the category $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_\varnothing})$.

Let $s \in S$, and consider the fiber product of schemes $\widetilde{\mathfrak{g}}_\varnothing \times_{\widetilde{\mathfrak{g}}_{\{s\}}} \widetilde{\mathfrak{g}}_\varnothing$. One can check by explicit computation in coordinates (see [R1, §1.4]) that this scheme is reduced, and has two connected components: one is the diagonal copy of $\widetilde{\mathfrak{g}}_\varnothing$, and the other one will be denoted Z_s. Then we define Z_s' as the (scheme-theoretic) intersection of Z_s with $\widetilde{\mathscr{N}_\varnothing} \times \widetilde{\mathscr{N}_\varnothing} \subset \widetilde{\mathfrak{g}}_\varnothing \times \widetilde{\mathfrak{g}}_\varnothing$. Once again, this scheme is reduced; see [R1, Remark 4.2]. By definition Z_s' is a closed subscheme of $\widetilde{\mathscr{N}_\varnothing} \times \widetilde{\mathscr{N}_\varnothing}$, hence it is endowed with natural (proper) projections

$$\begin{tikzcd} & Z_s' \arrow[dl, "p_1^s"'] \arrow[dr, "p_2^s"] & \\ \widetilde{\mathscr{N}_\varnothing} & & \widetilde{\mathscr{N}_\varnothing}. \end{tikzcd}$$

We consider the functors

$$\mathsf{F}_s, \mathsf{G}_s : D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_\varnothing}) \to D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_\varnothing})$$

defined by

$$\begin{aligned} \mathsf{F}_s(\mathcal{F}) &:= R(p_2^s)_* L(p_1^s)^*(\mathcal{F})\langle -1 \rangle, \\ \mathsf{G}_s(\mathcal{G}) &:= R(p_2^s)_* \big(L(p_1^s)^*(\mathcal{G}) \otimes_{\mathcal{O}_{Z_s'}} \mathcal{O}_{Z_s'}(-\varsigma_s, \varsigma_s - \alpha_s) \big) \langle -1 \rangle. \end{aligned}$$

Here, $\mathcal{O}_{Z_s'}(-\varsigma_s, \varsigma_s - \alpha_s)$ is the line bundle on Z_s' obtained by pullback from the line bundle $\mathcal{O}_{\mathscr{P}_\varnothing}(-\varsigma_s) \boxtimes \mathcal{O}_{\mathscr{P}_\varnothing}(\varsigma_s - \alpha_s)$ on $\mathscr{P}_\varnothing \times \mathscr{P}_\varnothing$. (In this formula, ς_s can be replaced by any weight whose pairing with $\alpha_s^\vee$ is 1 without changing the functor G_s up to isomorphism, see [R1, Lemma 1.5.1]; in particular G_s does not depend on the choice of ς_s.) The following statement can also be checked by explicit computation; see [R1, Proof of Corollary 4.4].

Lemma 5.1. *The functors F_s and G_s are quasi-inverse equivalences of triangulated categories.*

6. Here by an action of a group on a category we mean a *weak* action, i.e. a group morphism from the given group to the group of isomorphism classes of autoequivalences of the given category.

Using these functors one can describe the desired action of the group B_{f} on the category $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$, whose existence was proved in full generality in [**BR**].

THEOREM 5.2. *There exists a unique right action of B_{f} on $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$ such that for any $s \in S$, the element T_s° acts via the functor F_s.*

REMARK 5.3. (1) In [**R1, BR**] we consider instead a *left* action of B_{f}. When writing [**MaR1**] we realized that it is more convenient to work with a right version of this action. In any case the two constructions are equivalent, see the comments at the beginning of [**MaR1**, §3.3].

(2) The statement also holds in characteristic 0. In fact, in [**BR**] we construct several variations of this action: for $\widetilde{\mathfrak{g}}_\varnothing$ instead of $\widetilde{\mathcal{N}_\varnothing}$, with different equivariance conditions, over rings instead of fields, etc.

(3) In [**BR**] we also show that this action can be extended to an action of the *affine* braid group. For simplicity, we do not consider this extension here.

(4) Theorem 5.2 can be considered as a "categorical upgrade" of the Kazhdan–Lusztig–Ginzburg description of the affine Hecke algebra $\widetilde{\mathcal{H}}$ (defined as in §4.2, but using the lattice $\mathbf{X}$ instead of $\mathbb{Z}\Phi$) in the following sense. The functor F_s is the Fourier–Mukai transform with kernel $\mathcal{O}_{Z'_s}\langle -1\rangle$. Theorem 5.2 (and its proof) imply in particular that the assignment

$$T_s \mapsto [\mathcal{O}_{Z'_s}\langle -1\rangle] \in \mathsf{K}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing} \times_{\dot{\mathfrak{g}}^*} \widetilde{\mathcal{N}_\varnothing})$$

extends to an algebra morphism from the Hecke algebra $\mathcal{H}_{\mathrm{f}}$ of W_{f} (see §4.3) to $\mathsf{K}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing} \times_{\dot{\mathfrak{g}}^*} \widetilde{\mathcal{N}_\varnothing})$, where the latter is endowed with the convolution product. Now $\widetilde{\mathcal{N}_\varnothing} \times_{\dot{\mathfrak{g}}^*} \widetilde{\mathcal{N}_\varnothing}$ is the *Steinberg variety of* $\dot{G}$, and the morphism so constructed is (at least in the characteristic-0 setting, see (2)) the restriction to $\mathcal{H}_{\mathrm{f}}$ of the algebra isomorphism $\widetilde{\mathcal{H}} \xrightarrow{\sim} \mathsf{K}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing} \times_{\dot{\mathfrak{g}}^*} \widetilde{\mathcal{N}_\varnothing})$ due to Kazhdan–Lusztig and Ginzburg, in the version of Lusztig in [**Lu6**, §8]; see [**R1**, §6] for details. [7]

(5) The construction of the action considered in Theorem 5.2 stems from the Bezrukavnikov–Mirković–Rumynin localization theory. Namely, this theory provides (if $p > h$) equivalences of categories between certain full subcategories of $D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathcal{N}_\varnothing})$ and certain derived categories of modules over $\mathcal{U}(\mathfrak{g})$. Under these equivalences (suitably normalized), the non-equivariant version of the action of Theorem 5.2 corresponds to an action on the representation-theoretic side constructed in terms of some wall-crossing functors; see [**R1**, §5] for details. This was my original motivation for studying this action, and this construction played an important role in [**R2**] and in [**BM**].

(6) Using the representation-theoretic description of the action evoked in (5), in [**BR**, §2] we prove that if either $p > h$ or $p = 0$, [8] the action of the elements T_w° with $w \in W_{\mathrm{f}}$ can be described as follows. Consider the open subvariety $\dot{\mathfrak{g}}^*_{\mathrm{reg}} \subset \dot{\mathfrak{g}}^*$ consisting of regular elements (see [**BR**, §1.8]), and the inverse image $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ of $\dot{\mathfrak{g}}^*_{\mathrm{reg}}$

7. In [**R1**, §6] I incorrectly suggest that the constructions in [**CG**] and in [**Lu6**] coincide; in fact they use different normalizations.

8. We expect these restrictions to be unnecessary.

under the natural morphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$. Then there exists a well-known action of W_{f} on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$, see [BR, §1.9]. If Z_w is the closure (in $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$) of the graph of the action of w, then T_w° acts via the Fourier–Mukai transform with kernel

$$\mathcal{O}_{Z_{w^{-1}} \cap (\widetilde{\mathcal{N}_\varnothing} \times \widetilde{\mathcal{N}_\varnothing})} \langle -\ell(w) \rangle,$$

where we consider the scheme-theoretic intersection.

To prove the theorem it is more convenient to first consider the variant of the action for $\widetilde{\mathfrak{g}}_\varnothing$ instead of $\widetilde{\mathcal{N}_\varnothing}$ (see Remark 5.3(2)), and then deduce the case of $\widetilde{\mathcal{N}_\varnothing}$; see [R1, §4].

A first proof of Theorem 5.2 is given (under the assumptions that G is semisimple, that $p \neq 2$ if G is not simply laced, and that G has no component of type $\mathbf{G}_2$) in [R1]. This proof is based on explicit computations in coordinates. In fact what one has to prove is that the kernels $\mathcal{O}_{Z_s'} \langle -1 \rangle$ satisfy some braid relations, which only involve two different simple reflections. Hence they can be proved by some computations which are essentially "rank 2" computations. We are able to make this computation in types $\mathbf{A}_1 \times \mathbf{A}_1$ and $\mathbf{A}_2$ (in full generality) and in type $\mathbf{B}_2$ (if $p \neq 2$), which is sufficient to imply the theorem in the stated generality.

In [BR] we give a second proof of Theorem 5.2. This proof is based on the analysis of the image of line bundles under the functors F_s and G_s. It applies for any connected reductive group with simply-connected derived subgroup (although the theorem is stated only for semisimple groups in [BR]).

For $w \in W_{\mathrm{f}}$, we will denote by F_w the action of T_w° on $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$, and by G_w the action of $(T_w^\circ)^{-1}$. (These functors are well defined up to isomorphism.)

5.3. The exotic t-structure. Recall the notation introduced in §4.1. For $\lambda \in \mathbf{X}$ we set

$$\begin{aligned}
\mathcal{N}_\varnothing(\lambda) &:= \mathsf{G}_{(v_\lambda)^{-1}}\big(\mathcal{O}_{\widetilde{\mathcal{N}_\varnothing}}(\mathsf{dom}(\lambda))\big), \\
\mathcal{M}_\varnothing(\lambda) &:= \mathsf{G}_{(v_{-\lambda})^{-1}}\big(\mathcal{O}_{\widetilde{\mathcal{N}_\varnothing}}(-\mathsf{dom}(-\lambda))\big)\langle -\ell(v_{-\lambda}) - \ell(v_\lambda) \rangle.
\end{aligned}$$

In the following statement we use the notation $\langle\langle \mathcal{X} \rangle\rangle$ introduced in §3.2. We also consider the notion of highest weight category reviewed in Section 7 (see in particular Remarks 7.2(3) and 7.7).

Theorem 5.4. *(1) There exists a unique t-structure $({}^{\mathrm{e}}D^{\leq 0}, {}^{\mathrm{e}}D^{\geq 0})$ on the category $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$ such that*

$$\begin{aligned}
{}^{\mathrm{e}}D^{\leq 0} &= \langle\langle\, \mathcal{M}_\varnothing(\lambda)\langle n \rangle[m],\ \lambda \in \mathbf{X}, n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0} \,\rangle\rangle, \\
{}^{\mathrm{e}}D^{\geq 0} &= \langle\langle\, \mathcal{N}_\varnothing(\lambda)\langle n \rangle[m],\ \lambda \in \mathbf{X}, n \in \mathbb{Z}, m \in \mathbb{Z}_{\leq 0} \,\rangle\rangle.
\end{aligned}$$

(2) The heart $\mathcal{E}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$ of this t-structure is a graded highest weight category, with standard objects $\mathcal{M}_\varnothing(\lambda)$ and costandard objects $\mathcal{N}_\varnothing(\lambda)$.

(3) If we denote by $\mathsf{Tilt}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$ the category of tilting objects in $\mathcal{E}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$, then the natural functors

$$K^{\mathrm{b}}\mathsf{Tilt}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing}) \to D^{\mathrm{b}}\mathcal{E}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing}) \to D^{\mathrm{b}}\mathsf{Coh}^{\dot{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$$

are equivalences of triangulated categories (where the second functor is the realization functor, see [BBD, §3.1]*).*

This theorem is proved in [MaR1]; see in particular [MaR1, Proposition 3.8 and §3.5]. [9] The proof uses a very different description of the objects $\mathcal{M}_\varnothing(\lambda)$ and $\mathcal{N}_\varnothing(\lambda)$, in terms of mutation of exceptional sequences. The obvious analogue of part (1) in the characteristic-0 setting is due to Bezrukavnikov [Be], and the generalization to arbitrary characteristic is not difficult. The relation with the braid group action is made explicit in [BM]. Part (2) is suggested in Bezrukavnikov's work, but not explicitly stated. (The crucial fact one has to prove is that the objects $\mathcal{M}_\varnothing(\lambda)$ and $\mathcal{N}_\varnothing(\lambda)$ belong to $\mathcal{E}^{\dot{G}\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}_\varnothing})$; then the result follows from basic properties of exceptional sequences.) Finally, part (3) is an easy consequence of (2) and general properties of exceptional sequences.

The general theory of (graded) highest weight categories shows that the indecomposable objects in the category $\mathsf{Tilt}^{\dot{G}\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}_\varnothing})$ are parametrized by $\mathbf{X}\times\mathbb{Z}$ (see §7.5 for the analogous ungraded setting). We will denote by $\mathcal{T}(\lambda)$ the indecomposable object attached to $(\lambda, 0)$; then the object attached to (λ, n) is $\mathcal{T}(\lambda)\langle n\rangle$. It is an interesting question to try to describe the objects $\mathcal{T}(\lambda)$ explicitly. In [MaR1, §4.2] (following earlier ideas of Bezrukavnikov–Mirković [BM] and Dodd [Do]) we give a "Bott–Samelson type" way of generating these objects. We also prove the following fact in [MaR1, Corollary 4.8 and Corollary 4.16].

PROPOSITION 5.5. *Assume that p is a good prime for $\dot{G}$, and that $\dot{\mathfrak{g}}$ admits a non degenerate $\dot{G}$-invariant symmetric bilinear form.*

(1) For any $\lambda \in \mathbf{X}$, the complex $\mathcal{T}(\lambda)$ is concentrated in degree 0.

(2) If $\lambda \in \mathbf{X}^+$, then we have $\mathcal{T}(\lambda) \cong \dot{\mathsf{T}}(\lambda) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}$, where $\dot{\mathsf{T}}(\lambda)$ is the indecomposable tilting $\dot{G}$-module of highest weight λ.

In [AR6, §9] we begin the study of some analogues of these constructions for "partial" Springer resolutions $\widetilde{\mathcal{N}_I}$, which we will consider below.

For any $I \subset S$ we set
$$\varsigma_I = \sum_{s\in I} \varsigma_s.$$
We define
$$\widetilde{\mathcal{N}}_{\varnothing,I} := \dot{G} \times^{\dot{B}} (\dot{\mathfrak{g}}/\dot{\mathfrak{p}}_I)^*.$$
Then we have natural morphisms
$$\begin{tikzcd} & \widetilde{\mathcal{N}}_{\varnothing,I} \arrow[dl, "\mathsf{e}_I"'] \arrow[dr, "\mu_I"] & \\ \widetilde{\mathcal{N}_\varnothing} & & \widetilde{\mathcal{N}_I} \end{tikzcd}$$
where μ_I is proper and e_I is a closed embedding. Hence we can consider the functor
$$\Pi_I : D^\mathrm{b}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}_\varnothing}) \to D^\mathrm{b}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}_I})$$
defined by
$$\Pi_I(\mathcal{F}) = R(\mu_I)_* L(\mathsf{e}_I)^*(\mathcal{F} \otimes \mathcal{O}_{\widetilde{\mathcal{N}_\varnothing}}(-\varsigma_I)).$$

9. Here we follow the notation of [AR6]. In [MaR1], the object $\mathcal{M}_\varnothing(\lambda)$ is denoted $\Delta^\lambda_{\widetilde{\mathcal{N}}}$, and the object $\mathcal{N}_\varnothing(\lambda)$ is denoted $\nabla^\lambda_{\widetilde{\mathcal{N}}}$.

We set

$$\mathbf{X}_I^{+,\mathrm{reg}} := \{\lambda \in \mathbf{X} \mid \forall s \in I,\ \langle \lambda, \alpha_s^\vee \rangle \geq 1\}.$$

Then for $\lambda \in \mathbf{X}_I^{+,\mathrm{reg}}$ we set

$$\mathcal{N}_I(\lambda) = \Pi_I(\mathcal{N}_\varnothing(\lambda))\langle |\Phi_I^+| \rangle [|\Phi_I^+|], \quad \mathcal{M}_I(\lambda) = \Pi_I(\mathcal{M}_\varnothing(\lambda))\langle -|\Phi_I^+| \rangle [-|\Phi_I^+|], \tag{5.1}$$

where $\Phi_I^+ = \Phi \cap (\sum_{s \in I} \mathbb{Z} \cdot \alpha_s)$. Then it follows from [**AR6**, Proposition 9.24] that these objects satisfy

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_I})}(\mathcal{M}_I(\lambda), \mathcal{N}_I(\mu)\langle n \rangle [m]) = \begin{cases} \Bbbk & \text{if } \lambda = \mu \text{ and } n = m = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

As for the objects $\mathcal{M}_\varnothing(\lambda)$ and $\mathcal{N}_\varnothing(\lambda)$, these objects admit a description in terms of mutations of exceptional sequences, and the Hom-vanishing statement is clear from this construction. What is not so clear (and is proved in [**AR6**, §9]) is that these objects constructed from an exceptional sequence satisfy (5.1).

5.4. Relating exotic sheaves to parity complexes on affine Grassmannians. In this subsection we assume that G is a product of simply connected quasi-simple groups and of general linear groups, and that p is very good for each quasi-simple factor of G. (We do not have to impose any condition related to the factors which are general linear groups.)

We denote by $\dot{G}^\vee$ the complex connected reductive algebraic group with maximal torus $\dot{T}^\vee$ whose root datum is dual to that of $\dot{G}$. (In particular $\mathbf{X}$, which we identified with the lattice of characters of $\dot{T}$, now gets identified further with the lattice of coweights of $\dot{T}^\vee$.) We also denote by $\dot{B}_+^\vee$ the Borel subgroup of $\dot{G}^\vee$ containing $\dot{T}^\vee$ whose roots are the positive coroots of $(\dot{G}, \dot{T})$. As in §4.2 we set

$$\mathscr{K} := \mathbb{C}((z)), \qquad \mathscr{O} := \mathbb{C}[[z]],$$

and we consider the affine Grassmannian

$$\mathcal{G}\mathrm{r} := \dot{G}^\vee(\mathscr{K})/\dot{G}^\vee(\mathscr{O}).$$

Any $\lambda \in \mathbf{X} = X_*(\dot{T}^\vee)$ defines an element $z^\lambda \in \dot{T}^\vee(\mathscr{K})$, hence a point $L_\lambda := z^\lambda \dot{G}^\vee(\mathscr{O})$ in $\mathcal{G}\mathrm{r}$.

We define the Iwahori subgroup $\mathrm{Iw} \subset \dot{G}^\vee(\mathscr{O})$ as the inverse image of $\dot{B}_+^\vee$ under the morphism $\dot{G}^\vee(\mathscr{O}) \to \dot{G}^\vee$ induced by the assignment $z \mapsto 0$. For $\lambda \in \mathbf{X}$ we denote by $\mathcal{G}\mathrm{r}_\lambda$ the Iw-orbit of L_λ. Then we have a "Bruhat decomposition"

$$\mathcal{G}\mathrm{r} = \bigsqcup_{\lambda \in \mathbf{X}} \mathcal{G}\mathrm{r}_\lambda, \tag{5.3}$$

and each $\mathcal{G}\mathrm{r}_\lambda$ is isomorphic to an affine space of dimension

$$\dim(\mathcal{G}\mathrm{r}_\lambda) = \langle \mathsf{dom}(\lambda), 2\rho^\vee \rangle - \ell(v_\lambda),$$

where $2\rho^\vee$ is the sum of positive coroots of $(\dot{G}, \dot{T})$. (To obtain this formula, compare [**AR6**, Remark 11.3(2)] and [**MaR1**, Lemma 2.4].) In particular, it makes sense to consider the derived category $D^{\mathrm{b}}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk)$ of $\Bbbk$-sheaves on $\mathcal{G}\mathrm{r}$ which are constructible with respect to the stratification (5.3), its subcategory $\mathsf{Parity}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk)$ of parity complexes, and the mixed derived category

$$D^{\mathrm{mix}}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk) := K^{\mathrm{b}}\mathsf{Parity}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk).$$

As in §3.2, we denote by $\Delta^{\mathrm{mix}}_\lambda$, $\nabla^{\mathrm{mix}}_\lambda$ the standard and costandard mixed perverse sheaves attached to the stratum $\mathcal{G}\mathrm{r}_\lambda$, and by $\mathcal{E}_\lambda$ the indecomposable object in $\mathsf{Parity}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk)$ labelled by λ (for $\lambda \in \mathbf{X}$).

The following theorem is proved in [MaR2] (see also [AR6, Remark 11.3(2)] and §6.3 below for the difference of conventions between this statement and that of [MaR2]).

THEOREM 5.6. *There exists an equivalence of additive categories*

$$P : \mathsf{Parity}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk) \xrightarrow{\sim} \mathsf{Tilt}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$$

which intertwines the "shift functors" $\{1\}$ and $\langle -1\rangle$, and satisfies

$$P(\mathcal{E}_\lambda) \cong \mathcal{T}(\lambda) \tag{5.4}$$

for all $\lambda \in \mathbf{X}$. The induced equivalence

$$\mathsf{P} : D^{\mathrm{mix}}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk) = K^{\mathrm{b}}\mathsf{Parity}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk) \xrightarrow[\sim]{K^{\mathrm{b}}(P)} K^{\mathrm{b}}\mathsf{Tilt}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing}) \xrightarrow[\sim]{\textit{Theorem 5.4}} D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$$

satisfies $\mathsf{P} \circ \langle 1\rangle \cong \langle 1\rangle[1] \circ \mathsf{P}$, and

$$\mathsf{P}(\Delta^{\mathrm{mix}}_\lambda) \cong \mathcal{M}_\varnothing(\lambda), \qquad \mathsf{P}(\nabla^{\mathrm{mix}}_\lambda) \cong \mathcal{N}_\varnothing(\lambda)$$

for all $\lambda \in \mathbf{X}$.

REMARK 5.7. (1) The theorem also holds in the characteristic-0 setting. In this case, a different construction of the equivalence P is due to Arkhipov–Bezrukavnikov–Ginzburg [ABG]. The exotic t-structure does not play any role in their approach.

(2) The property (5.4) shows that the functor P is a Koszul–Ringel duality in the sense of §1.9: it relates parity objects to tilting objects.

(3) These results were also independently obtained by Achar–Rider in [ARd2]. Their approach is closer to the one in [ABG], although they also notice the relation with the exotic t-structure. One bonus of their construction is that it is compatible with the Geometric Satake Equivalence [MV] in the suitable sense (which is not clear from our approach). The drawback, however, is that it requires to know from the beginning that the parity complexes $\mathcal{E}_\lambda$ with $\lambda \in \mathbf{X}^+$ are perverse. This fact was proved in [JMW2] under certain conditions on p. The most general proof of this property, however, is obtained as a consequence of Theorem 5.6; see part (4) of this remark.

(4) Apart from the application to the modular representation theory of reductive algebraic groups presented below, another motivation for Theorem 5.6 is the application to the Mirković–Vilonen conjecture, see [MV, Conjecture 6.3]. This conjecture asserts that the cohomology objects of the stalks of the perverse sheaves ${}^p(j_\lambda)_!\underline{\mathbb{Z}}_{\mathcal{G}\mathrm{r}^\lambda}[\dim(\mathcal{G}\mathrm{r}^\lambda)]$ (for $\lambda \in \mathbf{X}^+$) are free over $\mathbb{Z}$. Here $\mathcal{G}\mathrm{r}^\lambda := \dot{G}^\vee(\mathscr{O}) \cdot L_\lambda$, and

$j_\lambda : \mathcal{G}\mathrm{r}^\lambda \hookrightarrow \mathcal{G}\mathrm{r}$ is the embedding. It was noticed by Juteau [**Ju1**] that this conjecture is false, and more precisely that these stalks can have p-torsion if p is a bad prime. On the other hand, it was proved recently by Achar–Rider [**ARd1**] that if, for some given prime p, the parity sheaves $\mathcal{E}_\lambda$ as above for a field of coefficients of characteristic p are perverse for all $\lambda \in \mathbf{X}^+$, then the stalks in question have no p-torsion. As explained in part (3) of this remark, this fact was known under certain assumptions on p thanks to [**JMW2**]. As a consequence of Theorem 5.6 and the description of the "dominant" tilting exotic sheaves in Proposition 5.5(2), we proved this fact for any good p, see [**MaR2**, Corollary 1.6]. The combination of these results settles the question raised by the Mirković–Vilonen conjecture completely.

The approach to Theorem 5.6 developed in [**MaR2**] uses some kind of "Soergel theory." Namely, we describe both sides of the desired equivalence P in "combinatorial terms," and more precisely in terms of "Bott–Samelson type" modules over the algebra $\mathcal{O}(\dot{\mathfrak{t}}^* \times_{\dot{\mathfrak{t}}^*/W} \mathsf{T}(\dot{\mathfrak{t}}^*/W))$ (where $\mathsf{T}(-)$ denotes the tangent bundle), and identify the two descriptions.

The description of the left-hand side is based on standard techniques using the global cohomology functor, and a description of the equivariant cohomology $\mathsf{H}^\bullet_{\mathrm{Iw}}(\mathcal{G}\mathrm{r};\mathbb{Q})$ in terms of the characteristic-0 analogue of the algebra $\mathcal{O}(\dot{\mathfrak{t}}^* \times_{\dot{\mathfrak{t}}^*/W} \mathsf{T}(\dot{\mathfrak{t}}^*/W))$ which can be deduced from results of Bezrukavnikov–Finkelberg, see [**BF**, Theorem 1].

The description of the right-hand side adapts some ideas of Dodd [**Do**] (which were introduced in the characteristic-0 setting): we "deform" the coherent sheaves on $\widetilde{\mathcal{N}_\varnothing}$ to coherent sheaves on $\widetilde{\mathfrak{g}}_\varnothing$, and then use a "Kostant–Whittaker reduction functor" to obtain modules over the desired algebra. For this we use a result of [**R3**] (which adapts another result of [**BF**] to the positive characteristic setting) identifying the Lie algebra of the universal centralizer of $\dot{G}$ (the group-scheme over the regular part $\dot{\mathfrak{g}}_{\mathrm{reg}} \subset \dot{\mathfrak{g}}$ whose fiber over $x \in \dot{\mathfrak{g}}_{\mathrm{reg}}$ is the centralizer $\dot{G}_x$) with

$$\dot{\mathfrak{g}}_{\mathrm{reg}} \times_{\dot{\mathfrak{t}}/W} \mathsf{T}^*(\dot{\mathfrak{t}}/W).$$

(To compare the two descriptions, we identify $\dot{\mathfrak{g}}$ with $\dot{\mathfrak{g}}^*$ by choosing a non-degenerate $\dot{G}$-invariant symmetric bilinear form on $\dot{\mathfrak{g}}$.)

To be more precise, we in fact run this strategy over a localization of $\mathbb{Z}$ rather than over $\mathbb{k}$ (in order to be able to ignore the difference between the algebras $\mathsf{H}^\bullet_{\mathrm{Iw}}(\mathcal{G}\mathrm{r};\mathbb{k})$ and $\mathcal{O}(\dot{\mathfrak{t}}^* \times_{\dot{\mathfrak{t}}^*/W} \mathsf{T}(\dot{\mathfrak{t}}^*/W))$, which are not isomorphic in the positive characteristic setting). Our technical assumptions are due to a lack of reference for certain technical results on split reductive groups over $\mathbb{Z}$ which are used in [**R3**].

Remark 5.8. The equivalence of Theorem 5.6 has some "equivariant variants" as follows. First, in [**MaR2**, Theorem 1.4] we use similar methods to construct (under the same assumptions) an equivalence of categories

$$D^{\mathrm{mix}}_{\mathrm{Iw}}(\mathcal{G}\mathrm{r},\mathbb{k}) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathfrak{g}}_\emptyset)$$

with properties similar to those of P, where the left-hand side is defined as the bounded homotopy category of the category of Iw-equivariant parity complexes on $\mathcal{G}\mathrm{r}$. (In this setting however, no analogue of the exotic t-structure is known.) Secondly, in [**R3**, §5] we use a similar strategy (which, this time, does not require to work over the integers) to construct an equivalence of triangulated categories

$$D^{\mathrm{mix}}_{\dot{G}^\vee(\mathscr{O})}(\mathcal{G}r, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\dot{\mathfrak{g}})$$

(under the assumption that G is quasi-simple of adjoint type and that p is very good for G) which is a "mixed modular" counterpart of the main result of [**BF**]. Here the left-hand side is defined as the bounded homotopy category of the category of $\dot{G}^\vee(\mathscr{O})$-equivariant $\Bbbk$-parity complexes on $\mathcal{G}r$.

5.5. Relation to the representation theory of G. In this section we come back to the assumptions of §5.1 on G, assuming in addition that $p > h$ (where h is the Coxeter number of G).

Recall the G-modules $\mathsf{M}(\lambda)$, $\mathsf{L}(\lambda)$, $\mathsf{N}(\lambda)$, $\mathsf{T}(\lambda)$ defined in §4.1. Recall also that these objects satisfy

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(G)}(\mathsf{M}(\lambda), \mathsf{N}(\mu)[n]) = \begin{cases} \Bbbk & \text{if } \lambda = \mu \text{ and } n = 0; \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

(see [**Ja1**, Proposition II.4.13]).

Let $I \subset S$ be a subset, and recall the weight ς_I considered in §5.3. We denote by $\mathsf{Rep}_I(G)$ the Serre subcategory of the category $\mathsf{Rep}(G)$ of finite-dimensional algebraic G-modules generated by the simple modules $\mathsf{L}(\lambda)$ with $\lambda \in \mathbf{X}^+$ of the form $v(-\varsigma_I + \rho) - \rho + p\mu$ with $v \in W_{\mathrm{f}}$ and $\mu \in \mathbf{X}$. The *linkage principle* ensures that $\mathsf{Rep}_I(G)$ is a direct summand in the category $\mathsf{Rep}(G)$.

The following theorem is the main result of [**AR6**].

THEOREM 5.9. *There exists a functor*

$$\Upsilon_I : D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_I}) \to D^{\mathrm{b}}\mathsf{Rep}_I(G)$$

and an isomorphism $\varepsilon : \Upsilon_I \circ \langle 1 \rangle [1] \xrightarrow{\sim} \Upsilon_I$ *such that:*

(1) *for any* $\mathcal{F}, \mathcal{G}$ *in* $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_I})$, *the functor* Υ_I *and the isomorphism* ε *induce an isomorphism*

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_I})}(\mathcal{F}, \mathcal{G}\langle n \rangle [n]) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}_I(G)}(\Upsilon_I(\mathcal{F}), \Upsilon_I(\mathcal{G}));$$

(2) *for any* $\lambda \in \mathbf{X}_I^{+,\mathrm{reg}}$ *we have*

$$\Upsilon_I(\mathcal{M}_I(\lambda)) \cong \mathsf{M}(v_\lambda(-\varsigma_I + \rho) - \rho + p \cdot \mathsf{dom}(\lambda)),$$
$$\Upsilon_I(\mathcal{N}_I(\lambda)) \cong \mathsf{N}(v_\lambda(-\varsigma_I + \rho) - \rho + p \cdot \mathsf{dom}(\lambda));$$

(3) *for any* V *in* $\mathsf{Rep}(\dot{G})$ *and* $\mathcal{F}$ *in* $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_I})$, *we have a bifunctorial isomorphism*

$$\Upsilon_I(V \otimes \mathcal{F}) \cong \Upsilon_I(\mathcal{F}) \otimes \mathrm{Fr}^*(V),$$

where $\mathrm{Fr} : G \to \dot{G}$ *is the Frobenius morphism.*

Remark 5.10. (1) In the case $I = \varnothing$, an analogue of Theorem 5.9 in the setting of Lusztig's quantum groups at a root of unity was obtained in 2004 by Arkhipov–Bezrukavnikov–Ginzburg [ABG]. In their case, $\mathsf{Rep}_{\varnothing}(G)$ is replaced by the principal block of the category of finite dimensional representations of the given quantum group, and the Springer resolution is considered over a field of characteristic 0.

(2) The property of Υ_I stated in (1) can be formulated as saying that this functor is a *degrading functor* with respect to the autoequivalence $\langle 1 \rangle[1]$. Instead of a degrading functor, our methods can be used to obtain an *equivalence of categories* between $D^{\mathrm{b}}\mathsf{Rep}_I(G)$ and a "dg-version" of the category $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}}(\widetilde{\mathscr{N}_I})$; namely, a certain derived category of equivariant dg-modules over the symmetric algebra of the tangent field of $\mathscr{P}_I$ placed in degree 2, with trivial differential. However, this version is less interesting because it cannot be combined with the constructions of §5.4.

As explained in Remark 5.10(1), Theorem 5.9 has a characteristic-0 predecessor due to Arkhipov–Bezrukavnikov–Ginzburg, and our proof follows the same general strategy as theirs. Namely, we use as an "intermediate step" between $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_I})$ and $D^{\mathrm{b}}\mathsf{Rep}_I(G)$ the triangulated category $D^{\mathrm{b}}_{\mathrm{Stein}}(P_I)$, defined as the full triangulated subcategory of the category $D^{\mathrm{b}}\mathsf{Rep}(P_I)$ generated by the objects of the form $\mathrm{St}_I \otimes \mathrm{Fr}^*(V)$ for V in $\mathsf{Rep}(\dot{P}_I)$, where $\mathrm{St}_I = \mathrm{Ind}_B^{P_I}((p-1)\varsigma_I)$, and $\mathrm{Fr} : P_I \to \dot{P}_I$ is the Frobenius morphism. We first construct a degrading functor

$$D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}_I}) \to D^{\mathrm{b}}_{\mathrm{Stein}}(P_I), \tag{5.6}$$

and then show that the functor

$$R\,\mathrm{Ind}_{P_I}^G : D^{\mathrm{b}}_{\mathrm{Stein}}(P_I) \to D^{\mathrm{b}}\mathsf{Rep}_I(G) \tag{5.7}$$

is an equivalence of categories.

In the case $I = \varnothing$, the functor (5.6) can be considered as a "categorification" of a well-known isomorphism of $\dot{B}$-equivariant algebras

$$\mathrm{Ext}^{\bullet}_{B_1}(\Bbbk, \Bbbk) \cong \mathrm{Sym}(\dot{\mathfrak{u}}^*_{\varnothing})$$

due to Friedlander–Parshall [FP], where B_1 is the (first) Frobenius kernel of B. The main new idea of our construction compared to what is done in [ABG] is to use the following diagram of dg-algebras and dg-algebra morphisms:

$$\textstyle\bigwedge^{\bullet}(\dot{\mathfrak{u}}_I) \longleftarrow \bigwedge^{\bullet}(\dot{\mathfrak{u}}_I) \otimes \mathcal{U}(\mathfrak{u}_I) \longrightarrow \mathrm{Dist}(U_{I,1}). \tag{5.8}$$

Here the middle term is endowed with a Koszul-type differential, and the left-hand morphism is induced by the augmentation morphism $\mathcal{U}(\mathfrak{u}_I) \to \Bbbk$. On the other hand, $U_{I,1}$ is the Frobenius kernel of U_I, $\mathrm{Dist}(U_{I,1})$ is identified with the restricted enveloping algebra of $\mathfrak{u}_I$, and the right-hand morphism is the natural quasi-isomorphism. The diagram (5.8) allows to relate dg-modules over $\bigwedge^{\bullet}(\dot{\mathfrak{u}}_I)$ (which, via Koszul duality, are derived-equivalent to $\mathrm{Sym}(\dot{\mathfrak{u}}_I^*)$-dg-modules) to $U_{I,1}$-modules. Taking further equivariance conditions into account, from this we construct the degrading functor (5.6).

Our proof that (5.7) is an equivalence is very different from the corresponding proof in [ABG]. In fact the crucial step is to show that the composition of (5.6) and (5.7) satisfies statement (2) of Theorem 5.9. Then, comparing (5.2) and (5.5), it is not difficult to prove that the functor (5.7) is fully faithful, and finally to deduce Theorem 5.9.

REMARK 5.11. (1) By construction, the functor (5.6) sends the structure sheaf to St_I. Now, consider the case $I = S$; in this case we have $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_S}) = D^{\mathrm{b}}\mathsf{Rep}(\dot{G}\times\mathbb{G}_{\mathrm{m}})$, and for any $\lambda \in \mathbf{X}_S^{+,\mathrm{reg}}$ we have

$$\mathcal{M}_S(\lambda) = \mathsf{M}(\lambda - \varsigma_S)\langle -|\Phi^+|\rangle[-|\Phi^+|], \quad \mathcal{N}_S(\lambda) = \mathsf{N}(\lambda - \varsigma_S)\langle -|\Phi^+|\rangle[-|\Phi^+|].$$

Hence parts (2) and (3) of Theorem 5.9 say in this case that for $\lambda \in \mathbf{X}_S^{+,\mathrm{reg}}$

$$\mathsf{N}\big((p-1)\cdot\varsigma_S\big) \otimes \mathrm{Fr}^*\big(\mathsf{N}(\lambda - \varsigma_S)\big) \cong \mathsf{N}(-\varsigma_S + p\cdot\lambda),$$

and similarly for Weyl modules. This isomorphism is well known, see [Ja1, §II.3.19].

(2) In [HKS], the authors provide an alternative proof of the fact that the functor (5.7) is an equivalence of categories in the case $I = \varnothing$. (Their proof is closer to the corresponding proof in [ABG].)

5.6. A graded analogue of a conjecture of Finkelberg–Mirković. In this subsection we restrict our attention to the case $I = \varnothing$. We assume that the conditions of §5.4 are satisfied, and moreover that $p > h$.

Theorem 5.9 gives a "geometric model" for the category $\mathsf{Rep}_\varnothing(G)$. However, this model might not be very easy to work with because it is not clear how to describe the inverse image under this equivalence of the tautological t-structure on $D^{\mathrm{b}}\mathsf{Rep}_\varnothing(G)$. In particular, this t-structure is neither the tautological t-structure nor the exotic t-structure. (With respect to the exotic t-structure on $D^{\mathrm{b}}\mathsf{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}_\varnothing})$ and the tautological t-structure on $D^{\mathrm{b}}\mathsf{Rep}_\varnothing(G)$, the functor $\Upsilon_\varnothing$ should be thought of as some kind of Koszul–Ringel duality, exchanging tilting objects and parity objects.) But this construction can be combined with those of §5.4 to provide a relation between constructible sheaves on $\mathcal{G}r$ and the category $\mathsf{Rep}_\varnothing(G)$. Since both of these constructions are Koszul–Ringel dualities, combining them should provide a t-exact equivalence. This is indeed the case, and we deduce the following result (see [AR6, Theorem 11.7]), where for $\lambda \in \mathbf{X}$ we denote by $\mathsf{T}_\lambda^{\mathrm{mix}}$ the indecomposable tilting mixed perverse sheaf on $\mathcal{G}r$ associated with the stratum $\mathcal{G}r_\lambda$.

THEOREM 5.12. *There exists an exact functor*

$$\mathbf{Q} : \mathsf{Perv}^{\mathrm{mix}}_{(\mathrm{Iw})}(\mathcal{G}r, \Bbbk) \to \mathsf{Rep}_\varnothing(G)$$

and an isomorphism $\varepsilon : \mathbf{Q} \to \mathbf{Q} \circ \langle 1 \rangle$ *such that:*

(1) for any $\mathcal{F}, \mathcal{G}$ *in* $\mathsf{Perv}^{\mathrm{mix}}_{(\mathrm{Iw})}(\mathcal{G}r, \Bbbk)$ *and any* $n \in \mathbb{Z}$, $\mathbf{Q}$ *and* ε *induce an isomorphism*

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Ext}^n_{\mathsf{Perv}^{\mathrm{mix}}_{(\mathrm{Iw})}(\mathcal{G}r,\Bbbk)}(\mathcal{F}, \mathcal{G}\langle m\rangle) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathsf{Rep}_\varnothing(G)}(\mathbf{Q}(\mathcal{F}), \mathbf{Q}(\mathcal{G}));$$

(2) for any $\lambda \in \mathbf{X}$ *we have*

$$\mathbf{Q}(\Delta_\lambda^{\mathrm{mix}}) \cong \mathsf{M}\big(v_\lambda(\rho) - \rho + p\cdot\mathsf{dom}(\lambda)\big), \quad \mathbf{Q}(\nabla_\lambda^{\mathrm{mix}}) \cong \mathsf{N}\big(v_\lambda(\rho) - \rho + p\cdot\mathsf{dom}(\lambda)\big),$$
$$\mathbf{Q}(\mathcal{IC}_\lambda^{\mathrm{mix}}) \cong \mathsf{L}\big(v_\lambda(\rho) - \rho + p\cdot\mathsf{dom}(\lambda)\big), \quad \mathbf{Q}(\mathsf{T}_\lambda^{\mathrm{mix}}) \cong \mathsf{T}\big(v_\lambda(\rho) - \rho + p\cdot\mathsf{dom}(\lambda)\big).$$

Remark 5.13. (1) Theorem 5.12 proves a "graded version" of a conjecture of Finkelberg–Mirković, see [FM, §1.5]. (This conjecture predicts an equivalence of highest weight categories between $\mathsf{Perv}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk)$ and $\mathsf{Rep}_{\varnothing}(G)$.) Our version might be more suited to the computation of combinatorial data in $\mathsf{Rep}_{\varnothing}(G)$ than the original conjecture, since it would be possible to combine it with the conjectural statement in §3.4; see Section 6 for details.

(2) Theorem 5.12 should have analogues for singular blocks $\mathsf{Rep}_I(G)$ (involving Iwahori–Whittaker sheaves on $\mathcal{G}\mathrm{r}$), which would follow from singular analogues of the constructions of §5.4. We plan to consider this in a future publication.

(3) If one uses the version of Theorem 5.9 from [ARd2] instead of that from [MaR2], one gets a functor $\mathbf{Q}$ which, in addition, is compatible with the Geometric Satake Equivalence in the suitable sense; see [AR6, Theorem 11.7(2)].

6. Comparison between Sections 4 and 5

We consider Theorem 5.12 as a first step towards a proof of Conjecture 4.4 for a general reductive group. In fact, recall the bijections (4.1). For $\mu \in \mathbb{Z}\Phi$, we denote by $w_\mu = t_{\mathsf{dom}(\mu)} v_\mu$ the image of μ in ${}^{\mathrm{f}}W$. Then from Theorem 5.12 we deduce that we have

$$(\mathsf{T}(w_\lambda \cdot_p 0), \mathsf{N}(w_\mu \cdot_p 0)) = \sum_{i \in \mathbb{Z}} (\mathsf{T}^{\mathrm{mix}}_\lambda, \nabla^{\mathrm{mix}}_\mu \langle i \rangle).$$

Hence what remains in order to prove Conjecture 4.4 is the equality

$$\sum_{i \in \mathbb{Z}} (\mathsf{T}^{\mathrm{mix}}_\lambda, \nabla^{\mathrm{mix}}_\mu \langle i \rangle) = {}^p n_{w_\mu, w_\lambda}(1).$$

In view of (4.3), an equivalent formulation of this equality is given by

$$\sum_{i \in \mathbb{Z}} (\mathsf{T}^{\mathrm{mix}}_\lambda, \nabla^{\mathrm{mix}}_\mu \langle i \rangle) = \sum_{x \in W_{\mathrm{f}}} (-1)^{\ell(x)} \cdot {}^p h_{x \cdot w_\mu, w_\lambda}(1). \tag{6.1}$$

In this section we explain how this equality would follow from a special case of the conjectural statements in §3.4.

6.1. Koszul duality for affine Kac–Moody groups. Recall that in §4.2 we defined the connected reductive group $G^\wedge$ with maximal torus $T^\wedge$ such that the root datum of $(G^\wedge, T^\wedge)$ is $(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Phi, \mathbb{Z}), \Phi^\vee, \mathbb{Z}\Phi, \Phi)$. On the other hand, the root datum of $(\dot{G}^\vee, \dot{T}^\vee)$ is $(\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}, \mathbb{Z}), \Phi^\vee, \mathbf{X}, \Phi)$. The morphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Phi, \mathbb{Z})$ induced by restriction is a homomorphism from the root datum $(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Phi, \mathbb{Z}), \Phi^\vee, \mathbb{Z}\Phi, \Phi)$ to the root datum $(\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}, \mathbb{Z}), \Phi^\vee, \mathbf{X}, \Phi)$. Hence there exists an algebraic group morphism $G^\wedge \to \dot{G}^\vee$ sending $T^\wedge$ into $\dot{T}^\vee$ and inducing this morphism of root data (see [Ja1, §II.1.13–15]). In fact, this morphism identifies $G^\wedge$ with the simply-connected cover of the derived subgroup of $\dot{G}^\vee$. Consider the ind-varieties

$$'\mathcal{F}\mathrm{l}^\wedge = I^\wedge \backslash G^\wedge(\mathscr{K}), \quad '\mathcal{G}\mathrm{r}^\wedge = G^\wedge(\mathscr{O}) \backslash G^\wedge(\mathscr{K}), \quad \mathcal{G}\mathrm{r}^\wedge = G^\wedge(\mathscr{K}) / G^\wedge(\mathscr{O}).$$

Then there exists a natural morphism $\pi^\wedge : {}'\mathcal{F}\mathrm{l}^\wedge \to {}'\mathcal{G}\mathrm{r}^\wedge$, and $\mathcal{G}\mathrm{r}^\wedge$ identifies with the connected component of $\mathcal{G}\mathrm{r}$ containing the base point L_0. As for $\mathcal{F}\mathrm{l}$ (and as in §3.4) we have a Bruhat decomposition

$$'\mathcal{F}\mathrm{l}^{\wedge} = \bigsqcup_{w \in W} {'\mathcal{F}\mathrm{l}^{\wedge}_w}$$

into orbits of $I^{\wedge}$ (for the action induced by multiplication on the right on $G^{\wedge}(\mathscr{K})$), and we can consider the associated categories $D^{\mathrm{b}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$, $\mathsf{Parity}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$, $D^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$, $\mathsf{Perv}^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$, and finally $\mathsf{Tilt}^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$. We will denote by $'\nabla^{\mathrm{mix}}_w$, resp. by $'\mathsf{T}^{\mathrm{mix}}_w$, the costandard, resp. indecomposable tilting, mixed perverse sheaf associated with $w \in W$. Similarly we have a Bruhat decomposition

$$'\mathcal{G}\mathrm{r}^{\wedge} = \bigsqcup_{\lambda \in \mathbb{Z}\Phi} {'\mathcal{G}\mathrm{r}^{\wedge}_\lambda}$$

into orbits of $I^{\wedge}$, and we can consider the category $\mathsf{Tilt}^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{G}\mathrm{r}^{\wedge}, \Bbbk)$. We will denote by $'\mathsf{T}^{\mathrm{mix}}_\lambda$ the indecomposable object associated with λ.

The functor $(\pi^{\wedge})_* : D^{\mathrm{b}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk) \to D^{\mathrm{b}}_{(I^{\wedge})}('\mathcal{G}\mathrm{r}^{\wedge}, \Bbbk)$ sends parity complexes to parity complexes, hence induces a functor from $D^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$ to $D^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{G}\mathrm{r}^{\wedge}, \Bbbk)$, see [AR4, §2.6]. This functor is not t-exact, but the same arguments as in [Yu, Proposition 2.4.1] show that it sends tilting mixed perverse sheaves to tilting mixed perverse sheaves.

We expect the following statement to hold.

CONJECTURE 6.1. *Assume that $G^{\wedge}$ is quasi-simple, and that p is very good for G.*

(1) *There exists an equivalence of additive categories*

$$\kappa : \mathsf{Tilt}^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk) \xrightarrow{\sim} \mathsf{Parity}_{(I^{\wedge})}(\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$$

which satisfies

$$\kappa \circ \langle 1 \rangle \cong \{1\} \circ \kappa, \quad \kappa('\mathsf{T}^{\mathrm{mix}}_w) \cong \mathcal{E}_w \quad \textit{for all } w \in W,$$

and moreover

$$\mathrm{ch}_y(\kappa(\mathcal{F})) = \sum_{i \in \mathbb{Z}} (\mathcal{F}, {'\nabla^{\mathrm{mix}}_y}\langle i \rangle) \cdot v^i$$

for any $y \in W$ and $\mathcal{F}$ in $\mathsf{Tilt}^{\mathrm{mix}}_{(I^{\wedge})}('\mathcal{F}\mathrm{l}^{\wedge}, \Bbbk)$.

(2) *For any $\lambda \in \mathbb{Z}\Phi$, we have*

$$(\pi^{\wedge})_*('\mathsf{T}^{\mathrm{mix}}_{w_\lambda}) \cong {'\mathsf{T}^{\mathrm{mix}}_\lambda}.$$

In §6.2 we discuss the relation between this conjecture and Conjecture 3.9, and in §6.3 we explain how to deduce (6.1) from this conjecture.

6.2. Relation between Conjecture 6.1 and Conjecture 3.9. First, we claim that part (1) of Conjecture 6.1 would follow from Conjecture 3.9. Indeed, this conjecture easily reduces to the case where $G^{\wedge}$ is quasi-simple, which we will assume from now on. In this case, let A be the affine Cartan matrix associated with $G^{\wedge}$, and let $(\Lambda, \{\tilde{\alpha}_i : i = 0, \cdots, r\}, \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}), \{\tilde{\alpha}_i : i = 0, \cdots, r\})$ be the realization of A as considered in [RW, Remark 11.2]. (Note that the roots $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_r$ are the simple roots of $G^{\wedge}$, which are also the simple coroots of G.) Let $\mathcal{G}$ be the associated Kac–Moody group. Then $\mathcal{F}\mathrm{l}^{\wedge}$ is the flag variety associated with $\mathcal{G}$ (denoted $\mathscr{B}$ in §3.4).[10] If $\mathscr{B}^\vee$ is the Langlands dual flag variety as defined in §3.4, to prove that indeed part (1) follows from Conjecture 3.9, it remains to construct an equivalence of categories

$$\mathsf{Parity}_{(I^\wedge)}({}'\mathcal{F}l^\wedge, \Bbbk) \cong \mathsf{Parity}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \Bbbk) \tag{6.2}$$

compatible with the labeling of indecomposable objects by W, and with characters.

The equivalence (6.2) can be deduced from the results of [**RW**, Part 3]. Indeed, from the results of [**RW**, §11.1] one obtains a description[11] of the category $\mathsf{Parity}_{(\mathcal{B}^\vee)}(\mathscr{B}^\vee, \Bbbk)$ in terms of the Elias–Williamson diagrammatic category $\mathcal{D}_{\mathrm{BS}}$ associated with the realization of W with underlying vector space $\Bbbk \otimes_{\mathbb{Z}} \Lambda$ (see [**RW**, §10.1 & §11.1] for details). Similarly, the same results provide a description of the category $\mathsf{Parity}_{(I^\wedge)}({}'\mathcal{F}l^\wedge, \Bbbk)$ in terms of the diagrammatic category $\mathcal{D}'_{\mathrm{BS}}$ associated with the realization with underlying vector space $\Bbbk \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Now, following [**Ku**, Proposition 1.5.2] we define a symmetric W-invariant bilinear form on $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ by setting

$$\langle d, d\rangle = 0, \quad \langle h, \tilde{\alpha}_i^\vee\rangle = \langle \tilde{\alpha}_i^\vee, h\rangle = \tilde{\alpha}_i(h)\epsilon_i$$

for any $h \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ and $i \in \{0, \cdots, r\}$. (Here, $D = \mathrm{diag}(\varepsilon_0, \cdots, \varepsilon_r)$ is a minimal matrix such that $D^{-1}A$ is symmetric, as in [**Ku**, §1.5].) Writing the matrix of this form in the basis $(d, \tilde{\alpha}_0^\vee, \cdots, \tilde{\alpha}_r^\vee)$, we see that under the assumption that p is very good for G this form induces a non-degenerate form on $\Bbbk \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$, hence a W-equivariant isomorphism

$$\varphi : \Bbbk \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\Bbbk}(\Bbbk \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}), \Bbbk) = \Bbbk \otimes_{\mathbb{Z}} \Lambda,$$

which satisfies, for any $i \in \{0, \cdots, r\}$,

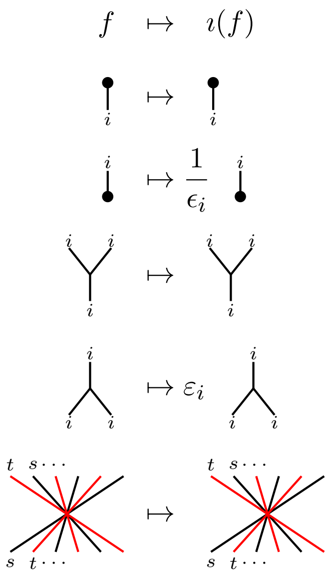
$$\varphi(\tilde{\alpha}_i^\vee) = \epsilon_i \cdot \tilde{\alpha}_i.$$

If we denote by $\imath : \mathcal{O}(\Bbbk \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}) \xrightarrow{\sim} \mathcal{O}(\Bbbk \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^\vee)$ the isomorphism induced by φ, we can then define an equivalence of categories $\mathcal{D}_{\mathrm{BS}} \xrightarrow{\sim} \mathcal{D}'_{\mathrm{BS}}$ which is the identity on objects, and

10. This fact is well known using e.g. the construction of Kac–Moody groups as in [**Ku**]. In our conventions (i.e. those of [**Ma**]), we were not able to find a reference for this fact. However we do not really need this property: one can observe (as in [**RW**, Section 11]) that the same constructions as for Kac–Moody groups work without modification for the group scheme $G^\wedge(\mathscr{K})$.

11. More precisely, the results of [**RW**] provide a description of the category of *equivariant* parity complexes. To pass from equivariant parity complexes to constructible parity complexes one simply quotients by the augmentation ideal of $\mathsf{H}^\bullet_{\mathcal{B}^\vee}(\mathrm{pt}, \Bbbk)$; see e.g. [**MaR2**, Lemma 2.2].

which is induced on morphisms by the assignment



(In fact, the only thing that has to be checked is that this assignment indeed defines a functor, i.e. that it is compatible with the relations defining $\mathcal{D}_{\mathrm{BS}}$ and $\mathcal{D}'_{\mathrm{BS}}$. This can be done either directly, or by remarking that the functor is defined over a localization of $\mathbb{Z}$, so that it is enough to check the relations over $\mathbb{Q}$; in this setting one can invoke the equivalence between the diagrammatic categories and the corresponding categories of Soergel bimodules and simply observe that our functor corresponds to the equivalence on Soergel bimodules induced by $\imath$.)

In the characteristic-0 setting, part (2) of Conjecture 6.1 follows from [**Yu**, Proposition 3.4.1] (see also [**BY**, Corollary 5.5.2(2)]). In the modular setting, we expect to prove this property as follows. As explained above, we know that $(\pi^\wedge)_*({}'\mathsf{T}^{\mathrm{mix}}_{w_\lambda})$ is a tilting mixed perverse sheaf, and it is not difficult to show that it is supported on $\overline{{}'\mathcal{G}\mathrm{r}^\wedge_\lambda}$ with appropriate restriction to ${}'\mathcal{G}\mathrm{r}^\wedge_\lambda$. Hence all that remains to be proved is indecomposability. This property should follow from the "Koszul dual" statement proved in [**RW**, Proposition 11.12]. (In fact, a similar result should hold in the general setting of §3.4, for any projection on a partial flag variety.)

6.3. Application to (6.1). Consider now the Grothendieck groups $[D^{\mathrm{mix}}_{(I^\wedge)}({}'\mathcal{F}\mathrm{l}^\wedge, \Bbbk)]$ and $[D^{\mathrm{mix}}_{(I^\wedge)}({}'\mathcal{G}\mathrm{r}^\wedge, \Bbbk)]$ (in the sense of triangulated categories), and the morphism

$$[(\pi^\wedge)_*] : [D^{\mathrm{mix}}_{(I^\wedge)}({}'\mathcal{F}\mathrm{l}^\wedge, \Bbbk)] \to [D^{\mathrm{mix}}_{(I^\wedge)}({}'\mathcal{G}\mathrm{r}^\wedge, \Bbbk)]$$

induced by the functor $(\pi^\wedge)_*$. Then it follows from [**AR4**, Lemma 3.7] that for any $x \in W_{\mathrm{f}}$ and $\lambda \in \mathbb{Z}\Phi$ we have

$$[(\pi^\wedge)_*]\big([{}'\nabla^{\mathrm{mix}}_{xw_\lambda}\langle i\rangle]\big) = (-1)^{\ell(x)} \cdot [{}'\nabla^{\mathrm{mix}}_{\lambda}\langle i - \ell(x)\rangle]. \tag{6.3}$$

(Here we denote by ${}'\nabla^{\mathrm{mix}}_\lambda$ the costandard mixed perverse sheaves on ${}'\mathcal{G}\mathrm{r}^\wedge$.) Now by definition, for $\lambda \in \mathbb{Z}\Phi$ we have

$$[{}'\mathsf{T}^{\mathrm{mix}}_{w_\lambda}] = \sum_{\substack{\mu\in\mathbb{Z}\Phi \\ x\in W_{\mathrm{f}}}} \sum_{i\in\mathbb{Z}} ({}'\mathsf{T}^{\mathrm{mix}}_{w_\lambda}, {}'\nabla^{\mathrm{mix}}_{xw_\mu}\langle i\rangle) \cdot [{}'\nabla^{\mathrm{mix}}_{xw_\mu}\langle i\rangle].$$

Hence from Conjecture 6.1(2) and (6.3) we deduce that

$$[{}'\mathsf{T}^{\mathrm{mix}}_\lambda] = \sum_{\substack{\mu\in\mathbb{Z}\Phi \\ x\in W_{\mathrm{f}}}} \sum_{i\in\mathbb{Z}} (-1)^{\ell(x)} \cdot ({}'\mathsf{T}^{\mathrm{mix}}_{w_\lambda}, {}'\nabla^{\mathrm{mix}}_{xw_\mu}\langle i\rangle) \cdot [{}'\nabla^{\mathrm{mix}}_{\mu}\langle i-\ell(x)\rangle],$$

hence in particular that

$$\sum_{i\in\mathbb{Z}} ({}'\mathsf{T}^{\mathrm{mix}}_\lambda, {}'\nabla^{\mathrm{mix}}_\mu\langle i\rangle) = \sum_{x\in W_{\mathrm{f}}} (-1)^{\ell(x)} \cdot \left(\sum_{j\in\mathbb{Z}} ({}'\mathsf{T}^{\mathrm{mix}}_{w_\lambda}, {}'\nabla^{\mathrm{mix}}_{xw_\mu}\langle j\rangle) \right).$$

On the other hand, from Conjecture 6.1(1) and the definitions we deduce that for $\lambda, \mu \in \mathbb{Z}\Phi$ and $x \in W_{\mathrm{f}}$ we have

$$\sum_{j\in\mathbb{Z}} ({}'\mathsf{T}^{\mathrm{mix}}_{w_\lambda}, {}'\nabla^{\mathrm{mix}}_{xw_\mu}\langle j\rangle) = \mathrm{ch}_{xw_\mu}(\mathcal{E}_{w_\lambda})(1) = {}^p h_{xw_\mu, w_\lambda}(1).$$

Hence we finally deduce that

$$\sum_{i\in\mathbb{Z}} ({}'\mathsf{T}^{\mathrm{mix}}_\lambda, {}'\nabla^{\mathrm{mix}}_\mu\langle i\rangle) = \sum_{x\in W_{\mathrm{f}}} (-1)^{\ell(x)} \cdot {}^p h_{xw_\mu, w_\lambda}(1). \tag{6.4}$$

To compare the tilting mixed perverse sheaves ${}'\mathsf{T}^{\mathrm{mix}}_\lambda$ on ${}'\mathcal{G}\mathrm{r}^\wedge$ (which are defined using the Iwahori subgroup $I^\wedge$ associated with the negative Borel subgroup of $G^\wedge$) with the corresponding objects $\mathsf{T}^{\mathrm{mix}}_\lambda$ on $\mathcal{G}\mathrm{r}$ (which are defined using the Iwahori subgroup Iw of $\dot{G}^\vee(\mathscr{K})$ associated with the positive Borel subgroup of $\dot{G}^\vee$), one considers an antiautomorphism τ of $G^\wedge$ such that $\tau_{|T^\wedge} = \mathrm{id}_{T^\wedge}$ and which sends each root subgroup $U^\wedge_\alpha \subset B^\wedge$ associated with a negative root α to the root subgroup $U^\wedge_{-\alpha}$ (see [Ja1, Corollary II.1.16]). Then $\tau(B^\wedge)$ is the Borel subgroup whose roots are $\Phi^\vee_+$. This antiautomorphism induces an isomorphism ${}'\mathcal{G}\mathrm{r}^\wedge \xrightarrow{\sim} \mathcal{G}\mathrm{r}^\wedge$. Identifying $\mathcal{G}\mathrm{r}^\wedge$ with the connected component of the base point in $\mathcal{G}\mathrm{r}$, we obtain an injective morphism

$$\vartheta : {}'\mathcal{G}\mathrm{r}^\wedge \to \mathcal{G}\mathrm{r}$$

which satisfies $\vartheta({}'\mathcal{G}\mathrm{r}^\wedge_\lambda) = \mathcal{G}\mathrm{r}_\lambda$ for any $\lambda \in \mathbb{Z}\Phi$. This morphism induces a fully-faithful functor

$$\vartheta_* : D^{\mathrm{mix}}_{(I^\wedge)}({}'\mathcal{G}\mathrm{r}^\wedge, \Bbbk) \to D^{\mathrm{mix}}_{(\mathrm{Iw})}(\mathcal{G}\mathrm{r}, \Bbbk)$$

which satisfies

$$\vartheta_*({}'\mathsf{T}^{\mathrm{mix}}_\lambda) \cong \mathsf{T}^{\mathrm{mix}}_\lambda, \qquad \vartheta_*({}'\nabla^{\mathrm{mix}}_\mu) \cong \nabla^{\mathrm{mix}}_\mu$$

for any $\lambda, \mu \in \mathbb{Z}\Phi$. In particular, we deduce that

$$\sum_{i\in\mathbb{Z}} ({}'\mathsf{T}^{\mathrm{mix}}_\lambda, {}'\nabla^{\mathrm{mix}}_\mu\langle i\rangle) = \sum_{i\in\mathbb{Z}} (\mathsf{T}^{\mathrm{mix}}_\lambda, \nabla^{\mathrm{mix}}_\mu\langle i\rangle).$$

Comparing with (6.4), we finally obtain the desired equality (6.1).

Part 3

Appendices

7. Complements on highest weight categories

The theory of highest weight categories was initially studied by Cline–Parshall–Scott in connection with the theory of quasi-hereditary algebras, see [CPS]. However we prefer to use a different, more "categorical," point of view introduced in [BGS, §3.2]. In this appendix we gather references or proofs for some standard results on these categories using this point of view. (These results are sometimes available in the literature only in the Cline–Parshall–Scott setting, which seems to justify a complete treatment from the Beĭlinson–Ginzburg–Soergel perspective.)

7.1. Definitions. Throughout the appendix, $\Bbbk$ will be a field, and $\mathcal{A}$ will be a finite-length $\Bbbk$-linear abelian category such that $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is finite-dimensional for any M, N in $\mathcal{A}$. Note that such a category is Krull–Schmidt, see [CYZ, Remark A.2].

Let $\mathscr{S}$ be the set of isomorphism classes of irreducible objects of $\mathcal{A}$. Assume that $\mathscr{S}$ is equipped with a partial order $\leq$, and that for each $s \in \mathscr{S}$ we have a fixed representative simple object L_s. Assume also we are given, for any $s \in \mathscr{S}$, objects Δ_s and ∇_s, and morphisms $\Delta_s \to \mathsf{L}_s$ and $\mathsf{L}_s \to \nabla_s$. For $\mathscr{T} \subset \mathscr{S}$, we denote by $\mathcal{A}_{\mathscr{T}}$ the Serre subcategory of $\mathcal{A}$ generated by the objects L_t for $t \in \mathscr{T}$. We write $\mathcal{A}_{\leq s}$ for $\mathcal{A}_{\{t \in \mathscr{S} \mid t \leq s\}}$, and similarly for $\mathcal{A}_{<s}$. Finally, recall that an *ideal* of $\mathscr{S}$ is a subset $\mathscr{T} \subset \mathscr{S}$ such that if $t \in \mathscr{T}$ and $s \in \mathscr{S}$ are such that $s \leq t$, then $s \in \mathscr{T}$.

DEFINITION 7.1. The category $\mathcal{A}$ (together with the above data) is said to be a *highest weight category* if the following conditions hold:

(1) for any $s \in \mathscr{S}$, the set $\{t \in \mathscr{S} \mid t \leq s\}$ is finite;
(2) for each $s \in \mathscr{S}$, we have $\mathrm{Hom}_{\mathcal{A}}(\mathsf{L}_s, \mathsf{L}_s) = \Bbbk$;
(3) for any $s \in \mathscr{S}$ and any ideal $\mathscr{T} \subset \mathscr{S}$ such that $s \in \mathscr{T}$ is maximal, $\Delta_s \to \mathsf{L}_s$ is a projective cover in $\mathcal{A}_{\mathscr{T}}$ and $\mathsf{L}_s \to \nabla_s$ is an injective envelope in $\mathcal{A}_{\mathscr{T}}$;
(4) the kernel of $\Delta_s \to \mathsf{L}_s$ and the cokernel of $\mathsf{L}_s \to \nabla_s$ belong to $\mathcal{A}_{<s}$;
(5) we have $\mathrm{Ext}^2_{\mathcal{A}}(\Delta_s, \nabla_t) = 0$ for all $s, t \in \mathscr{S}$.

In this case, the poset $(\mathscr{S}, \leq)$ is called the *weight poset* of $\mathcal{A}$.

If $\mathcal{A}$ satisfies Definition 7.1, the objects Δ_s are called *standard objects*, and the objects ∇_s are called *costandard objects*. We say that an object M *admits a Δ-filtration*, resp. *admits a ∇-filtration*, if there exists a finite filtration of M whose subquotients are standard objects, resp. costandard objects.

From the axioms (3) and (4) we see in particular that

$$\Delta_s \text{ and } \nabla_s \text{ belong to } \mathcal{A}_{\leq s} \text{ and satisfy } [\Delta_s : \mathsf{L}_s] = [\nabla_s : \mathsf{L}_s] = 1. \tag{7.1}$$

REMARK 7.2. (1) The axioms in Definition 7.1 are exactly those in [BGS, §3.2], except that we replace the condition that $\mathscr{S}$ is finite by the weaker condition (1).
(2) In [AR4] we used the term *quasihereditary category* instead of *highest weight category*. We now believe that the latter term is more appropriate than the former, and we changed our terminology in [MaR1, AR6].
(3) The axioms in Definition 7.1 can be easily modified to define a *graded highest weight category*, where we consider in addition a "shift" autoequivalence $\langle 1 \rangle$ of $\mathcal{A}$; see [AR4,

Appendix A] for details. All the statements below have analogues in this context, but for simplicity we will not state them explicitly.

We start with the following observations.

LEMMA 7.3. *Let $\mathcal{A}$ be a highest weight category, with weight poset $(\mathscr{S}, \leq)$, standard objects $\{\Delta_s : s \in \mathscr{S}\}$ and costandard objects $\{\nabla_s : s \in \mathscr{S}\}$.*

(1) The category $\mathcal{A}^{\mathrm{op}}$ is a highest weight category, with weight poset $(\mathscr{S}, \leq)$, standard objects $\{\nabla_s : s \in \mathscr{S}\}$, and costandard objects $\{\Delta_s : s \in \mathscr{S}\}$.

(2) If $\mathscr{T} \subset \mathscr{S}$ is an ideal, then $\mathcal{A}_{\mathscr{T}}$ is a highest weight category with weight poset $(\mathscr{T}, \leq)$, standard objects $\{\Delta_t : t \in \mathscr{T}\}$ and costandard objects $\{\nabla_t : t \in \mathscr{T}\}$.

PROOF. Part (1) is clear. In part (2), the only axiom which might not be clear is (5). However, this axiom for $\mathcal{A}_{\mathscr{T}}$ follows from the similar axiom for $\mathcal{A}$ using [BGS, Lemma 3.2.3]. □

LEMMA 7.4. *For any $s, t \in \mathscr{S}$, we have*

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \nabla_t) = \begin{cases} \Bbbk & \text{if } s = t; \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, \nabla_t) = \{0\}.$$

PROOF. If $s \not\leq t$, then s is maximal in the ideal $\mathscr{T} = \{u \in \mathscr{S} \mid u \leq s \text{ or } u \leq t\}$, and both Δ_s and ∇_t belong to $\mathcal{A}_{\mathscr{T}}$ by (7.1). Then we have $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \nabla_t) = \mathrm{Hom}_{\mathcal{A}_{\mathscr{T}}}(\Delta_s, \nabla_t)$ and $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, \nabla_t) = \mathrm{Ext}^1_{\mathcal{A}_{\mathscr{T}}}(\Delta_s, \nabla_t)$, and the claim follows from axiom (3) and (7.1).

If $s < t$, then t is maximal in the ideal $\mathscr{T} = \{u \in \mathscr{S} \mid u \leq t\}$, and both Δ_s and ∇_t belong to $\mathcal{A}_{\mathscr{T}}$ by (7.1); then the claim follows again from axiom (3) and (7.1). □

From Lemma 7.4 we see that if M is an object of $\mathcal{A}$ which admits a Δ-filtration, then the number of times Δ_s appears as a subquotient in such a filtration is equal to $\dim_{\Bbbk}(\mathrm{Hom}_{\mathcal{A}}(M, \nabla_s))$. In particular this number does not depend on the filtration, and will be denoted $(M : \Delta_s)$. Similarly, if M admits a ∇-filtration, then the number of times ∇_s appears as a subquotient in such a filtration is well defined, and will be denoted $(M : \nabla_s)$.

7.2. Existence of projectives and some consequences. The following result is proved in [BGS, Theorem 3.2.1 & Remarks following the theorem].

THEOREM 7.5. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$ and assume that $\mathscr{S}$ is finite. Then $\mathcal{A}$ has enough projective objects, and any projective object admits a Δ-fitration. Moreover, if P_s is the projective cover of L_s, we have*

$$(\mathsf{P}_s : \Delta_t) = [\nabla_t : \mathsf{L}_s]. \tag{7.2}$$

Applying Theorem 7.5 to the category $\mathcal{A}^{\mathrm{op}}$ (see Lemma 7.3(1)), we see that if $\mathscr{S}$ is finite, then $\mathcal{A}$ also has enough injective objects, and any injective object admits a ∇-filtration.

COROLLARY 7.6. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$. Then for any $s, t \in \mathscr{S}$ we have*

$$\mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, \nabla_t) = \begin{cases} \Bbbk & \text{if } s = t \text{ and } i = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

PROOF. The case when $i \in \{0, 1\}$ is proved in Lemma 7.4, and we only have to prove the vanishing when $i \geq 2$.

First, we assume that $\mathscr{S}$ is finite, and prove the claim by descending induction on s. If s is maximal in $\mathscr{S}$, then Δ_s is a projective cover of L_s in $\mathcal{A}$ by axiom (3), and the claim follows. In general, consider the projective cover P_s of L_s. By Theorem 7.5, this object admits a Δ-filtration. Moreover, the last term in such a filtration must be Δ_s, since the top of P_s is L_s. In particular, we have an exact sequence

$$\ker \hookrightarrow \mathsf{P}_s \twoheadrightarrow \Delta_s$$

where ker admits a Δ-filtration. Moreover, (7.2) and (7.1) imply that if $(\ker : \Delta_t) \neq 0$, then $t > s$. Then the desired vanishing follows from induction and a long exact sequence consideration.

Now we prove the general case. Let $i \geq 2$, and consider a morphism $f : \Delta_s \to \nabla_t[i]$ in $D^{\mathrm{b}}(\mathcal{A})$. This morphism is represented by a fraction $\frac{g}{h}$, where M is a bounded complex of objects of $\mathcal{A}$, $h : M \xrightarrow{\text{qis}} \Delta_s$ is a quasi-isomorphism of complexes, and $g : M \to \nabla_t[i]$ is a morphism of complexes. Choose a finite ideal $\mathscr{S}' \subset \mathscr{S}$ which contains s, t, and the isomorphism classes of all composition factors of nonzero terms of M. (Such an ideal exists thanks to axiom (1).) Then $\frac{g}{h}$ defines a morphism in $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{S}'})$, which must be the 0 morphism by Lemma 7.3(2) and the case of finite weight posets. We deduce that f is also 0 in $D^{\mathrm{b}}(\mathcal{A})$, which concludes the proof. □

REMARK 7.7. Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$. Let $\preceq$ be the preorder generated by the relation

$$s \preceq t \quad \text{if} \quad [\Delta_t : \mathsf{L}_s] \neq 0 \text{ or } [\nabla_t : \mathsf{L}_s] \neq 0.$$

Then (7.1) implies that $\preceq$ is an order such that $\leq$ refines $\preceq$. We claim that $\mathcal{A}$ is also a highest weight category for the poset $(\mathscr{S}, \preceq)$. Indeed, the only axiom which might not be clear is (3). However, as in the proof of Corollary 7.6, to check this axiom we can assume that $\mathscr{S}$ is finite. Then $\mathcal{A}$ has enough projective objects by Theorem 7.5, and the reciprocity formula (7.2) ensures that, if P_t is the projective cover of L_t in $\mathcal{A}$, then we have an exact sequence

$$\ker \hookrightarrow \mathsf{P}_t \twoheadrightarrow \Delta_t \tag{7.3}$$

where ker admits a Δ-filtration such that if $(\ker : \Delta_s) \neq 0$, then $s \succ t$. Now if $u \in \mathscr{S}$, considering the long exact sequence associated with (7.3) we obtain a surjection

$$\mathrm{Hom}_{\mathcal{A}}(\ker, \mathsf{L}_u) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, \mathsf{L}_u).$$

Hence if $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, \mathsf{L}_u) \neq \{0\}$ then $\mathrm{Hom}_{\mathcal{A}}(\ker, \mathsf{L}_u) \neq \{0\}$, so that there exists $s \in \mathscr{S}$ such that $(\ker : \Delta_s) \neq 0$ and $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \mathsf{L}_u) \neq \{0\}$. Then $u = s$, so that $u \succ t$. From this it is easy to see that if $\mathscr{T}$ is an ideal in $(\mathscr{S}, \preceq)$ in which t is maximal, then Δ_t is projective in $\mathcal{A}_{\mathscr{T}}$, hence the projective cover of L_t.

This remark shows that it makes sense to say that a category is highest weight without specifying the order $\leq$ (if one specifies the standard and costandard objects).

7.3. Ideals and associated subcategories and quotients. The following results show that highest weight categories satisfy some "gluing" formalism which turns out to be very useful to run inductive arguments.

Lemma 7.8. *Let $\mathcal{A}$ be a highest weight category, with weight poset $(\mathscr{S}, \leq)$, standard objects $\{\Delta_s : s \in \mathscr{S}\}$ and costandard objects $\{\nabla_s : s \in \mathscr{S}\}$. If $\mathscr{T} \subset \mathscr{S}$ is an ideal, then the Serre quotient $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ is a highest weight category with weight poset $(\mathscr{S} \smallsetminus \mathscr{T}, \leq)$, standard objects $\{\pi_{\mathscr{T}}(\Delta_s) : s \in \mathscr{S} \smallsetminus \mathscr{T}\}$, and costandard objects $\{\pi_{\mathscr{T}}(\nabla_s) : s \in \mathscr{S} \smallsetminus \mathscr{T}\}$, where $\pi_{\mathscr{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_{\mathscr{T}}$ is the quotient functor.*

Proof. It is clear that the category $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ satisfies axioms (1), (2) and (4).

Now we check axiom (3) in the case of Δ_s; the case of ∇_s is similar. First, we claim that for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ and N in $\mathcal{A}$, the morphism

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, N) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N)) \tag{7.4}$$

induced by $\pi_{\mathscr{T}}$ is an isomorphism. Indeed, consider a morphism $f : \pi_{\mathscr{T}}(\Delta_s) \to \pi_{\mathscr{T}}(N)$. By definition, this morphism is represented by a morphism $f' : M' \to N/N'$ in $\mathcal{A}$, where $M' \subset \Delta_s$ and $N' \subset N$ are subobjects such that Δ_s/M' and N' belong to $\mathcal{A}_{\mathscr{T}}$. Since the head of Δ_s is L_s and $s \notin \mathscr{T}$, we have necessarily $M' = \Delta_s$. And since $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, N') = \{0\}$, the morphism f' factors through a morphism $f'' : \Delta_s \to N$. These arguments show that (7.4) is surjective. Since the image of any nonzero morphism from Δ_s to N contains L_s as a composition factor, its image under $\pi_{\mathscr{T}}$ is nonzero, hence the image of the morphism itself is nonzero. This shows that (7.4) is also injective, hence an isomorphism.

Now, let $\mathscr{U} \subset \mathscr{S} \smallsetminus \mathscr{T}$ be an ideal, and let $s \in \mathscr{U}$ be maximal. The isomorphisms (7.4) show that the top of $\pi_{\mathscr{T}}(\Delta_s)$ is $\pi_{\mathscr{T}}(\mathsf{L}_s)$. It remains to prove that this object is projective. If $f : \pi_{\mathscr{T}}(M) \to \pi_{\mathscr{T}}(N)$ is a surjection with $\pi_{\mathscr{T}}(M)$ and $\pi_{\mathscr{T}}(N)$ in $(\mathcal{A}/\mathcal{A}_{\mathscr{T}})_{\mathscr{U}}$, then M and N belong to $\mathcal{A}_{\mathscr{U} \sqcup \mathscr{T}}$, and f is represented by a morphism $f' : M' \to N/N'$ in $\mathcal{A}$ whose cokernel C belongs to $\mathcal{A}_{\mathscr{T}}$, where $M' \subset M$ and $N' \subset N$ are subobjects such that M/M' and N' belong to $\mathcal{A}_{\mathscr{T}}$. Then using isomorphisms (7.4) we see that we have

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M)) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M')) \cong \mathrm{Hom}_{\mathcal{A}}(\Delta_s, M')$$

and

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N)) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N/N')) \cong \mathrm{Hom}_{\mathcal{A}}(\Delta_s, N/N'),$$

and that the morphism

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M)) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N))$$

induced by f coincides with the morphism

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, M') \to \mathrm{Hom}_{\mathcal{A}}(\Delta_s, N/N')$$

induced by f'. Hence the desired surjectivity follows from the facts that Δ_s is projective in $\mathcal{A}_{\mathscr{U} \sqcup \mathscr{T}}$ and that $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, C) = \{0\}$.

Finally, we need to check axiom (5). For this we first assume that $\mathscr{S}$ is finite. Then $\mathcal{A}$ has enough projective objects by Theorem 7.5. Moreover, the proof of Corollary 7.6 shows that to prove the desired vanishing it suffices to prove that for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ there exists a projective object P in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ and a surjection $P \twoheadrightarrow \pi_{\mathscr{T}}(\Delta_s)$ whose kernel admits a filtration with subquotients $\pi_{\mathscr{T}}(\Delta_t)$ with $t > s$. We claim that $P = \pi_{\mathscr{T}}(\mathsf{P}_s)$ satisfies these properties. In fact, the only property which is not clear is that P is projective. If this were not the case, there would exist a non-split and non-trivial surjection $f : \pi_{\mathscr{T}}(M) \to \pi_{\mathscr{T}}(\mathsf{P}_s)$ for some M in $\mathcal{A}$. This morphism is represented by a morphism $f' : M' \to \mathsf{P}_s/N'$ whose cokernel D belongs to $\mathcal{A}_{\mathscr{T}}$, where $M' \subset M$ and $N' \subset \mathsf{P}_s$ are subobjects such that M/M' and N' belong to $\mathcal{A}_{\mathscr{T}}$. Now D is a quotient of P_s; hence if it belongs to $\mathcal{A}_{\mathscr{T}}$ it must be 0, so that f' is surjective. Since P_s is projective, there exists a morphism $g' : \mathsf{P}_s \to M'$ such that $f' \circ g'$ is the quotient morphism $\mathsf{P}_s \twoheadrightarrow \mathsf{P}_s/N'$. Then $\pi_{\mathscr{T}}(f') \circ \pi_{\mathscr{T}}(g')$ is an isomorphism in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$, so that $\pi_{\mathscr{T}}(f')$ is split. This is absurd, and finishes the proof of axiom (5) in the case $\mathscr{S}$ is finite.

Property (5) in the general case follows from the same property for finite weight posets using the same arguments as in the proof of Corollary 7.6. □

Proposition 7.9. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$ and let $\mathscr{T} \subset \mathscr{S}$ be an ideal.*

(1) The functor $\imath_{\mathscr{T}} : D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}}) \to D^{\mathrm{b}}(\mathcal{A})$ induced by the embedding $\mathcal{A}_{\mathscr{T}} \to \mathcal{A}$ is fully faithful.

(2) The quotient functor $\pi_{\mathscr{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_{\mathscr{T}}$ induces an equivalence of categories

$$D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}}) \xrightarrow{\sim} D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}}),$$

where $D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ is the Verdier quotient.

(3) The functor $\imath_{\mathscr{T}}$ and the quotient functor $\Pi_{\mathscr{T}} : D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ admit (triangulated) left and right adjoints $\imath^{\mathrm{L}}_{\mathscr{T}}$, $\imath^{\mathrm{R}}_{\mathscr{T}}$ and $\Pi^{\mathrm{L}}_{\mathscr{T}}$, $\Pi^{\mathrm{R}}_{\mathscr{T}}$ respectively. Moreover, we have isomorphisms

$$\imath^{\mathrm{R}}_{\mathscr{T}} \circ \imath_{\mathscr{T}} \cong \mathrm{id}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})} \cong \imath^{\mathrm{L}}_{\mathscr{T}} \circ \imath_{\mathscr{T}}$$
$$\text{and} \quad \Pi_{\mathscr{T}} \circ \Pi^{\mathrm{R}}_{\mathscr{T}} \cong \mathrm{id}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})} \cong \Pi_{\mathscr{T}} \circ \Pi^{\mathrm{L}}_{\mathscr{T}},$$

for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ we have

$$\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_s) \cong \Delta_s, \quad \Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\nabla_s) \cong \nabla_s,$$

and for any M in $D^{\mathrm{b}}(\mathcal{A})$ there exist functorial distinguished triangles

$$\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M) \to M \to \imath_{\mathscr{T}} \circ \imath^{\mathrm{L}}_{\mathscr{T}}(M) \xrightarrow{[1]}$$
$$\text{and} \quad \imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M) \to M \to \Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M) \xrightarrow{[1]}$$

where the first and second morphisms are induced by adjunction.

Proof. This result is proved in [AR5, Lemma 2.2]. Here we explain the construction in more detail.

For part (1), we remark that the category $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ is generated (as a triangulated category) by the objects $\{\Delta_t : t \in \mathscr{T}\}$ as well as by the objects $\{\nabla_t : t \in \mathscr{T}\}$. Hence to prove the claim if suffices to prove that for $s, t \in \mathscr{T}$ the morphism

$$\mathrm{Ext}^i_{\mathcal{A}_{\mathscr{T}}}(\Delta_s, \nabla_t) \to \mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, \nabla_t)$$

induced by $\imath_{\mathscr{T}}$ is an isomorphism. This follows from Corollary 7.6 (applied to $\mathcal{A}$ and $\mathcal{A}_{\mathscr{T}}$).

Then we prove part (3). Consider the full triangulated subcategory $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$ of $D^{\mathrm{b}}(\mathcal{A})$ generated by the objects ∇_s with $s\in\mathscr{S}\smallsetminus\mathscr{T}$. Then for M in $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and N in $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$, by Corollary 7.6 we have $\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(M,N)=0$. From this one can deduce that for any M in $D^{\mathrm{b}}(\mathcal{A})$ and N in $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$, the morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(M,N)\to\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(M),\Pi_{\mathscr{T}}(N))$$

induced by $\Pi_{\mathscr{T}}$ is an isomorphism.

Now the category $D^{\mathrm{b}}(\mathcal{A})$ is generated, as a triangulated category, by (the essential image of) $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and by $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$. Using the octahedral axiom, we deduce that for any M in $D^{\mathrm{b}}(\mathcal{A})$ there exists a distinguished triangle

$$M'\to M\to M''\xrightarrow{[1]} \tag{7.5}$$

where M' belongs to $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and M'' belongs to $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$. Moreover, [BBD, Proposition 1.1.9] implies that this triangle is unique and functorial.

These facts show that the restriction of $\Pi_{\mathscr{T}}$ to $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$ is an equivalence, and that if we define $\Pi^{\mathrm{R}}_{\mathscr{T}}:D^{\mathrm{b}}(\mathcal{A})\to D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ as the composition of the inverse equivalence with the embedding $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}\to D^{\mathrm{b}}(\mathcal{A})$, then $\Pi^{\mathrm{R}}_{\mathscr{T}}$ is right adjoint to $\Pi_{\mathscr{T}}$. (In more concrete terms, $\Pi^{\mathrm{R}}_{\mathscr{T}}$ sends an object M to the object M'' in (7.5).)

Finally we define the functor $\imath^{\mathrm{R}}_{\mathscr{T}}$ as the functor sending an object M to the object M' in (7.5). Again, it is easily checked that this functor is right adjoint to $\imath_{\mathscr{T}}$. The isomorphisms $\imath^{\mathrm{R}}_{\mathscr{T}}\circ\imath_{\mathscr{T}}\cong\mathrm{id}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}$, $\Pi_{\mathscr{T}}\circ\Pi^{\mathrm{R}}_{\mathscr{T}}\cong\mathrm{id}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}$, and $\Pi^{\mathrm{R}}_{\mathscr{T}}\circ\Pi_{\mathscr{T}}(\nabla_s)\cong\nabla_s$, and the existence of the functorial triangles $\imath_{\mathscr{T}}\circ\imath^{\mathrm{R}}_{\mathscr{T}}(M)\to M\to\Pi^{\mathrm{R}}_{\mathscr{T}}\circ\Pi_{\mathscr{T}}(M)\xrightarrow{[1]}$, are clear from the construction of $\Pi^{\mathrm{R}}_{\mathscr{T}}$ and $\imath^{\mathrm{R}}_{\mathscr{T}}$.

The construction of the functors $\Pi^{\mathrm{L}}_{\mathscr{T}}$ and $\imath^{\mathrm{L}}_{\mathscr{T}}$ is completely similar, using the full triangulated subcategory $\mathcal{D}^{\Delta}_{\mathscr{S}\smallsetminus\mathscr{T}}$ generated by the objects Δ_s with $s\in\mathscr{S}\smallsetminus\mathscr{T}$ instead of $\mathcal{D}^{\nabla}_{\mathscr{S}\smallsetminus\mathscr{T}}$.

Finally we prove part (2). The universal property of the Verdier quotient guarantees the existence of a natural functor $D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})\to D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}})$, and what we have to prove is that this functor is an equivalence of categories. Both $D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and $D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}})$ are generated, as triangulated categories, by the images of the objects Δ_s with $s\in\mathscr{S}\smallsetminus\mathscr{T}$, as well as by the images of the objects ∇_s with $s\in\mathscr{S}\smallsetminus\mathscr{T}$. Hence what we have to prove is that for any $s,t\in\mathscr{S}\smallsetminus\mathscr{T}$ the induced morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_s),\Pi_{\mathscr{T}}(\nabla_t)[i])\to\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}})}(\pi_{\mathscr{T}}(\Delta_s),\pi_{\mathscr{T}}(\nabla_t)[i])$$

is an isomorphism. However we have

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_s),\Pi_{\mathscr{T}}(\nabla_t)[i])&\cong\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_s,\Pi^{\mathrm{R}}_{\mathscr{T}}\circ\Pi_{\mathscr{T}}(\nabla_t)[i])\\&\cong\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_s,\nabla_t[i]),\end{aligned}$$

and then the claim follows from Corollary 7.6 applied to the highest weight categories $\mathcal{A}$ and $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$, see Lemma 7.8. $\square$

7.4. Criterion for the existence of ∇-filtrations.

PROPOSITION 7.10. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$, and let M be in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) M admits a ∇-filtration;

(2) for any $s \in \mathscr{S}$ and $i \in \mathbb{Z}_{>0}$, we have $\mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, M) = \{0\}$;

(3) for any $s \in \mathscr{S}$, we have $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, M) = \{0\}$.

REMARK 7.11. It follows in particular from Proposition 7.10 that a direct summand of an object which admits a ∇-filtration also admits a ∇-filtration.

PROOF. The fact that (1) $\Rightarrow$ (2) follows from Corollary 7.6, and the implication (2) $\Rightarrow$ (3) is clear. It remains to prove that (3) $\Rightarrow$ (1). For this we can assume that $\mathscr{S}$ is finite, and argue by induction on $\#\mathscr{S}$, the case $\#\mathscr{S} = 1$ being obvious.

Assume that $\#\mathscr{S} > 1$, let $t \in \mathscr{S}$ be a minimal element, and let $\mathscr{T} = \{t\}$. Let M be an object in $\mathcal{A}$ such that $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, M) = 0$ for all $s \in \mathscr{S}$. Then for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$, using Proposition 7.9 we see that

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M)) &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_s), \Pi_{\mathscr{T}}(M)[1]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_s), M[1]) \cong \mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, M) = \{0\}.\end{aligned}$$

Hence, by induction, $\pi_{\mathscr{T}}(M)$ admits a ∇-filtration in the highest weight category $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$. Using again Proposition 7.9, it follows that $\Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M)$ belongs to $\mathcal{A}$, and admits a ∇-filtration.

Consider now the distinguished triangle

$$\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M) \to M \to \Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M) \xrightarrow{[1]} \tag{7.6}$$

provided once again by Proposition 7.9. Since the second and third terms belong to $\mathcal{A}$, the first term can have nonzero cohomology objects only in degrees 0 and 1. Moreover, we have

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_t, \imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M)[1]) &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\imath^{\mathrm{L}}_{\mathscr{T}} \circ \imath_{\mathscr{T}}(\Delta_t), \imath^{\mathrm{R}}_{\mathscr{T}}(M)[1]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Delta_t, \imath^{\mathrm{R}}_{\mathscr{T}}(M)[1]) \cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_t, M[1]),\end{aligned}$$

hence

$$\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_t, \imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M)[1]) = \{0\}. \tag{7.7}$$

We claim that $\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M)$ belongs to $\mathcal{A}$. Indeed, consider the truncation distinguished triangle

$$H^0(\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M)) \to \imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M) \to H^1(\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M))[-1] \xrightarrow{[1]} .$$

Since the category $\mathcal{A}_{\mathscr{T}}$ is semisimple, this triangle is split. Hence if $H^1(\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M))$ were nonzero there would exist a nonzero morphism $\Delta_t[-1] \to \imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M)$, which would contradict (7.7).

Finally, since the functor $\imath_{\mathscr{T}}$ is exact and does not kill any object (since it is fully-faithful), we deduce that $\imath^{\mathrm{R}}_{\mathscr{T}}(M)$ belongs to $\mathcal{A}_{\mathscr{T}}$, hence that $\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M)$ is a direct sum of copies of ∇_t. Then the distinguished triangle (7.6) is an exact sequence in $\mathcal{A}$, and shows that M admits a ∇-filtration. $\square$

Applying Proposition 7.10 to the opposite category $\mathcal{A}^{\mathrm{op}}$, we obtain the following "dual" statement.

PROPOSITION 7.12. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$, and let M be in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) M admits a Δ-filtration;

(2) for any $s \in \mathscr{S}$ and $i \in \mathbb{Z}_{>0}$, we have $\mathrm{Ext}^i_{\mathcal{A}}(M, \nabla_s) = \{0\}$;

(3) for any $s \in \mathscr{S}$, we have $\mathrm{Ext}^1_{\mathcal{A}}(M, \nabla_s) = \{0\}$.

7.5. Tilting objects. In this subsection we fix a highest weight category $\mathcal{A}$ with weight poset $(\mathscr{S}, \leq)$.

DEFINITION 7.13. An object M in $\mathcal{A}$ is said to be *tilting* if admits both a Δ-filtration and a ∇-filtration.

In this subsection we prove the following theorem.

THEOREM 7.14. *For any $s \in \mathscr{S}$, there exists (up to isomorphism) a unique indecomposable tilting object T_s such that*

$$[\mathsf{T}_s : \mathsf{L}_s] = 1 \quad \text{and} \quad [\mathsf{T}_s : \mathsf{L}_t] \neq 0 \ \Rightarrow \ t \leq s. \tag{7.8}$$

Moreover there exists an embedding $\Delta_s \hookrightarrow \mathsf{T}_s$ whose cokernel admits a Δ-filtration, and a surjection $\mathsf{T}_s \twoheadrightarrow \nabla_s$ whose kernel admits a ∇-filtration. Finally, any indecomposable tilting object is isomorphic to T_s for a unique $s \in \mathscr{S}$.

Our proof is inspired by the proof of [So3, Proposition 3.1] (where the author considers a much more general setting). We begin with the following preliminary result.

LEMMA 7.15. *For any $s \in \mathscr{S}$, there exists a tilting object T endowed with an embedding $\Delta_s \hookrightarrow T$ whose cokernel admits a Δ-filtration with subquotients Δ_t with $t < s$.*

PROOF. We proceed by induction on $\#\{t \in \mathscr{S} \mid t \leq s\}$. If s is minimal then we can take $T = \Delta_s = \nabla_s$. Otherwise, consider some minimal $t \in \mathscr{S}$ with $t < s$. We set $\mathscr{T} = \{t\}$. By induction, we have an object M in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ with the desired properties, and we consider $M' := \Pi^{\mathrm{L}}_{\mathscr{T}}(M)$. Using Proposition 7.9, we see that there exists an embedding from $\Delta_s = \Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_s)$ to M', whose cokernel admits a Δ-filtration with subquotients Δ_u with $u < s$. Moreover, for any $u \neq t$ we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, M') &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_u), M'[1]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_u), \Pi_{\mathscr{T}}(M')[1]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_u), M[1]) = \{0\}.\end{aligned}$$

Now, let $E := \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, M')$. Consider the image of id_E in

$$\mathrm{Hom}_{\Bbbk}(E, E) \cong E^* \otimes_{\Bbbk} E \cong \mathrm{Ext}^1_{\mathcal{A}}(E \otimes_{\Bbbk} \Delta_t, M').$$

This element corresponds to a short exact sequence

$$M' \hookrightarrow T \twoheadrightarrow E \otimes_{\Bbbk} \Delta_t. \tag{7.9}$$

Clearly, there exists an embedding $\Delta_s \hookrightarrow T$ whose cokernel admits a Δ-filtration with subquotients Δ_u with $u < s$. Hence to conclude our construction we only have to prove that T also admits a ∇-filtration. By Proposition 7.10, for this it suffices to prove that

$$\mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, T) = \{0\}$$

for any $u \in \mathscr{S}$. If $u \neq t$, this property follows from the similar vanishing for M' proved above and the fact that $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, \Delta_t) = \mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, \nabla_t) = \{0\}$. And to prove it for $u = t$ we consider the following part of the long exact sequence obtained by applying $\mathrm{Hom}_{\mathcal{A}}(\Delta_t, -)$ to (7.9):

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_t, E \otimes_{\Bbbk} \Delta_t) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, M') \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, T) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, E \otimes_{\Bbbk} \Delta_t).$$

Here by construction the first morphism is the identity of E, and the fourth term vanishes; hence the third term vanishes also, as desired. $\square$

Now we prove Theorem 7.14.

Proof of Theorem 7.14. For any $s \in \mathscr{S}$ there exists an indecomposable tilting object T_s endowed with an embedding $\Delta_s \hookrightarrow \mathsf{T}_s$ whose cokernel admits a Δ-filtration with subquotients Δ_t with $t < s$. Indeed, Lemma 7.15 provides an object T with such properties, which is not necessarily indecomposable. But then T admits an indecomposable direct summand T_s with $(\mathsf{T}_s : \Delta_s) = 1$. The composition $\Delta_s \hookrightarrow T \twoheadrightarrow \mathsf{T}_s$ is still injective, and its cokernel still admits the required filtration, since there exists no nonzero morphism from Δ_s to any other direct summand of T.

We fix such objects (and the corresponding embeddings), and now prove that any indecomposable tilting object is isomorphic to T_s for some $s \in \mathscr{S}$. Indeed, let T be an indecomposable tilting object, and choose $t \in \mathscr{S}$ and an embedding $\Delta_t \hookrightarrow T$ whose cokernel admits a Δ-filtration. Consider the diagram

$$\begin{tikzcd} \Delta_t \arrow[r, hook] \arrow[d, equal] & T \arrow[r, two heads] \arrow[d, dashed] & \mathrm{coker} \\ \Delta_t \arrow[r, hook] & \mathsf{T}_t \arrow[r, two heads] & \mathrm{coker}'. \end{tikzcd}$$

Since coker admits a Δ-filtration and T_t is tilting, we have $\mathrm{Ext}^1_{\mathcal{A}}(\mathrm{coker}, \mathsf{T}_t) = 0$. Hence there exists a morphism $\varphi : \mathsf{T}_t \to T$ which restricts to the identity on Δ_t. Similarly, there exists $\psi : T \to \mathsf{T}_t$ which restricts to the identity on Δ_t. Then $\varphi \circ \psi$ is an element of the artinian local ring $\mathrm{End}_{\mathcal{A}}(T)$ which is not nilpotent, hence invertible by Fitting's lemma. Similarly $\psi \circ \varphi$ is invertible, hence φ and ψ are isomorphisms.

We have proved that the objects $\{\mathsf{T}_s : s \in \mathscr{S}\}$ constructed above provide representatives for all isomorphism classes of indecomposable tilting objects in $\mathcal{A}$. Among these objects, it is clear that T_s is characterized by (7.8). Hence to conclude it suffices to prove that there exists a surjection $\mathsf{T}_s \twoheadrightarrow \nabla_s$ whose kernel admits a ∇-filtration. However, Lemma 7.15 applied to $\mathcal{A}^{\mathrm{op}}$ guarantees the existence, for any $s \in \mathscr{S}$, of a tilting object T'_s with a surjection $\mathsf{T}'_s \twoheadrightarrow \nabla_s$ whose kernel admits a ∇-filtration with subquotients of the form ∇_t with $t < s$. Moreover, as above this object can be assumed to be indecomposable. This object satisfies the conditions (7.8); hence it must be isomorphic to T_s. $\square$

Remark 7.16. The proof of Theorem 7.14 shows also that if T is an indecomposable tilting object in $\mathcal{A}$, then the first term in any Δ-filtration of T is Δ_s, where s is the (unique) maximal element of $\mathscr{S}$ such that $[T : \mathsf{L}_s] \neq 0$. In particular this first term does not depend on the chosen Δ-filtration, and characterizes T up to isomorphism.

We denote by $\mathsf{Tilt}(\mathcal{A})$ the additive full subcategory of $\mathcal{A}$ whose objects are the tilting objects. The following is an easy but very useful observation.

Proposition 7.17. *The natural functor*

$$K^{\mathrm{b}}\mathsf{Tilt}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{A})$$

is an equivalence of triangulated categories.

Proof. The category $D^{\mathrm{b}}(\mathcal{A})$ is generated as a triangulated category by the objects Δ_s for $s \in \mathscr{S}$, hence also (using Theorem 7.14) by the tilting objects. So, to prove the proposition it suffices to prove that our functor is fully-faithful. However, this follows directly from the observation that

$$\mathrm{Ext}^i_{\mathcal{A}}(T, T') = 0 \quad \text{for all } i > 0$$

if T and T' are tilting objects, as follows from Corollary 7.6. $\square$

8. List of publications

8.1. Articles published or accepted.

(1) (with Roman Bezrukavnikov) *Computations for $\mathfrak{sl}(3)$*, Annals of Mathematics **167** (2008), 988–991. Appendix to *Localization of modules for a semisimple Lie algebra in prime characteristic* (R. Bezrukavnikov, I. Mirković and D. Rumynin), 945–987.

(2) *Geometric braid group action on derived categories of coherent sheaves* (with an appendix joint with Roman Bezrukavnikov), Representation Theory **12** (2008), 131–169. (HAL: 00476253.)

(3) *Koszul duality and modular representations of semi-simple Lie algebras*, Duke Mathematical Journal **154** (2010), 31–134. (arXiv: 0803.2076; HAL: 00476255.)

(4) (with Ivan Mirković) *Linear Koszul duality*, Compositio Mathematica **146** (2010), 233–258. (arXiv: 0804.0923; HAL: 00476254.)

(5) (with Roman Bezrukavnikov) *Affine braid group actions on Springer resolutions*, Annales Scientifiques de l'École Normale Supérieure **45** (2012), 535–599. (arXiv: 1101.3702; HAL: 00557665.)

(6) (with Pramod Achar) *Koszul duality and semisimplicity of Frobenius*, Annales de l'Institut Fourier **63** (2013), 1511–1612. (arXiv: 1102.2820; HAL: 00565745.)

(7) (with Wolfgang Soergel and Geordie Williamson) *Modular Koszul duality*, Compositio Mathematica **150** (2014), 273–332. (arXiv: 1209.3760; HAL: 00733191.)

(8) (with Pramod Achar) *Constructible sheaves on affine Grassmannians and coherent sheaves on the dual nilpotent cone*, Israel Journal of Mathematics **205** (2015), 247–315. (arXiv: 1102.2821; HAL: 00565748.)

(9) (with Ivan Mirković) *Iwahori–Matsumoto involution and linear Koszul duality*, International Mathematics Research Notices **2015** (2015), 150–196. (arXiv: 1301.-4098; HAL: 00777769.)

(10) (with Pramod Achar, Anthony Henderson and Daniel Juteau) *Weyl group actions on the Springer sheaf*, Proceedings of the London Mathematical Society **108** (2014), 1501–1528. (arXiv: 1304.2642; HAL: 00813156.)

(11) (with Julien Bichon) *Hopf algebras having a dense big cell*, Transactions of the American Mathematical Society **368** (2016), 515–538. (arXiv: 1307.3567; HAL: 00843969.)

(12) (with Victor Ginzburg) *Differential operators on G/U and the affine Grassmannian*, Journal de l'Institut de Mathématiques de Jussieu **14** (2015), 493–575. (arXiv: 1306.6754; HAL: 00839864.)

(13) (with Pramod Achar, Anthony Henderson and Daniel Juteau) *Modular generalized Springer correspondence I: the general linear group*, Journal of the European Mathematical Society **18** (2016), 1405–1436. (arXiv: 1307.2702; HAL: 00843303.)

(14) (with Pramod Achar) *Modular perverse sheaves on flag varieties I: tilting and parity sheaves* (with an appendix joint with Geordie Williamson), Annales Scientifiques de l'École Normale Supérieure **49** (2016), 325–370. (arXiv: 1401.7245; HAL: 00937989.)

(15) (with Pramod Achar, Anthony Henderson and Daniel Juteau) *Modular generalized Springer correspondence II: classical groups*, to appear in Journal of the European Mathematical Society. (arXiv: 1404.1096; HAL: 00974454.)

(16) (with Pramod Achar) *Modular perverse sheaves on flag varieties II: Koszul duality and formality*, Duke Mathematical Journal **165** (2016), 161–215. (arXiv: 1401.7256; HAL: 00938001.)

(17) (with Pramod Achar and Anthony Henderson) *Geometric Satake, Springer correspondence, and small representations II*, Representation Theory **19** (2015), 94–166. (arXiv: 1205.5089; HAL: 00700454.)

(18) (with Ivan Mirković) *Linear Koszul duality and Fourier transform for convolution algebras*, Documenta Mathematica **20** (2015), 989–1038. (arXiv: 1401.7186; HAL: 00937414.)

(19) (with Ivan Mirković) *Linear Koszul duality II – Coherent sheaves on perfect sheaves*, Journal of the London Mathematical Society **93** (2016), 1–24. (arXiv: 1301.3924; HAL: 00777196.)

(20) (with Carl Mautner) *On the exotic t-structure in positive characteristic*, to appear in International Mathematics Research Notices. (arXiv: 1412.6818; HAL: 01098152.)

(21) (with Pramod Achar, Anthony Henderson and Daniel Juteau) *Constructible sheaves on nilpotent cones in rather good characteristic*, to appear in Selecta Mathematica. (arXiv: 1507.06581; HAL: 01223124.)

(22) (with Pramod Achar and Laura Rider) *Complements on mixed modular derived categories*. Appendix to *The affine Grassmannian and the Springer resolution in positive characteristic* (P. Achar and L. Rider), to appear in Compositio Mathematica. (arXiv: 1408.7050.)

(23) (with Pramod Achar) *Modular perverse sheaves on flag varieties III: positivity conditions*, to appear in Transactions of the American Mathematical Society. (arXiv: 1408.4189; HAL: 01059854.)

(24) (with Carl Mautner) *Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture*, to appear in Journal of the European Mathematical Society. (arXiv: 1501.07369; HAL: 01110852.)

(25) *Kostant section, universal centralizer, and a modular derived Satake equivalence*, to appear in Mathematische Zeitschrift. (arXiv: 1411.3112; HAL: 01081988.)

(26) (with Pramod Achar, Anthony Henderson and Daniel Juteau) *Modular generalized Springer correspondence III: exceptional groups*, to appear in Mathematische Annalen. (arXiv: 1507.00401; HAL: 01171168.)

8.2. Submitted preprints.

(1) (with Pramod Achar, Anthony Henderson and Daniel Juteau) *Modular generalized Springer correspondence: an overview.* (arXiv: 1510.08962. HAL: 01223125.)

(2) (with Geordie Williamson) *Tilting modules and the p-canonical basis.* (arXiv: 1512.-08296; HAL: 01249796.)

(3) (with Pramod Achar) *Reductive groups, the loop Grassmannian, and the Springer resolution.* (arXiv: 1602.04412; HAL: 01273980.)

8.3. Preprints not submitted.

(1) *Koszul duality and semi-simple Lie algebras in positive characteristic.* PhD thesis (Université Paris 6, under the supervision of Roman Bezrukavnikov and Patrick Polo), available on TEL (00416471).

(2) *Koszul duality and Frobenius structure for restricted enveloping algebras.* (arXiv: 1010.0495; HAL: 00522983.)

(3) *Borel–Moore homology and the Springer correspondence by restriction*, notes available on `http://math.univ-bpclermont.fr/~riche/`.

8.4. Lecture notes. All the notes referred to below are available on my web page (`http://math.univ-bpclermont.fr/~riche/`).

(1) *D-modules, faisceaux pervers et conjecture de Kazhdan–Lusztig*, after Beĭlinson–Bernstein, notes from a talk given in Caen on Jan. 15th, 2010.

(2) *Perverse sheaves on flag manifolds and Kazhdan–Lusztig polynomials*, after Kazhdan–Lusztig, Springer, MacPherson, ..., notes from a talk given in Strasbourg on Apr. 16th, 2010.

(3) *Modèles minimaux pour les dg-algèbres pseudo-compactes complètes*, after Van den Bergh, notes from a talk given in Clermont-Ferrand on Oct. 14th and Oct. 20th, 2011.

(4) *Tilting exotic sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture*, slides from a talk given in Irako (Japan) on Feb. 17th, 2015.

(5) *Algèbres de Lie.* Notes from a "Master 2" course (45 hours) given in Clermont-Ferrand in 2015/2016.

9. Summaries of the articles

In this section I give a summary of each of my articles, gathered by themes.

9.1. Linear Koszul duality. The following articles are all based on the "linear Koszul duality" formalism, and are either part of my PhD thesis or continuations of work started during the preparation of my thesis.

9.1.1. *Computations for* $\mathfrak{sl}(3)$ *(with R. Bezrukavnikov),* [**BMR**]. This article is part of my PhD thesis. We describe explicitly the complexes of coherent sheaves corresponding, under some equivalences proved in the main body of the paper [**BMR**], to some simple and projective modules over the enveloping algebra of the Lie algebra $\mathfrak{sl}_3(\Bbbk)$, for $\Bbbk$ an algebraically closed field of positive characteristic. These computations provided some evidence for the theory developed in [**R2**].

9.1.2. *Koszul duality and modular representations of semi-simple Lie algebras,* [**R2**]. This article is part of my PhD thesis. If G is a simply connected semisimple algebraic group over an algebraically closed field $\Bbbk$, with Lie algebra $\mathfrak{g}$, we use the Bezrukavnikov–Mirković–Rumynin localization theory to construct (in a geometric way) a "Koszul duality" relating two different derived categories of $\mathcal{U}(\mathfrak{g})$-modules. We prove that, if p is large enough so that Lusztig's conjecture [**Lu1**] holds, then this duality exchanges projective and simple restricted $\mathcal{U}(\mathfrak{g})$-modules, and deduce that the restricted enveloping algebra $\mathcal{U}_0(\mathfrak{g})$ admits a Koszul grading (under the same assumption). The latter statement generalizes a result of Andersen–Jantzen–Soergel [**AJS**], who treated the case of regular blocks of $\mathcal{U}_0(\mathfrak{g})$.

9.1.3. *Linear Koszul duality (with I. Mirković),* [**MR1**]. This article is part of my PhD thesis. Given a noetherian, integral, separated, regular scheme X, a vector bundle E over X, and two subbundles $F_1, F_2 \subset E$, we construct an equivalence of triangulated categories between the derived categories of $\mathbb{G}_{\mathrm{m}}$-equivariant coherent dg-sheaves on the dg-schemes $F_1 \overset{R}{\cap}_E F_2$ and $F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}$ (where $\mathbb{G}_{\mathrm{m}}$ acts by dilation along the fibers of the projection $E \to X$). This construction generalizes the standard Koszul duality between graded modules over the symmetric algebra of a vector space V and over the exterior algebra of the dual vector space V^*.

This construction was later generalized in [**MR2**].

9.1.4. *Linear Koszul duality II – Coherent sheaves on perfect sheaves (with I. Mirković),* [**MR2**]. We generalize the construction of the "linear Koszul duality" of [**MR1**] by weakening the assumptions on the base scheme X, now only required to be separated, noetherian, of finite Krull dimension, and admitting a dualizing complex. For this we use ideas due to Positselski [**Po**] which allow to replace some considerations in [**MR1**] needed to ensure the convergence of some spectral sequences by much simpler arguments. We also show that this construction is compatible (in the natural sense) with base change and morphisms of vector bundles.

9.1.5. *Iwahori–Matsumoto involution and linear Koszul duality (with I. Mirković),* [**MR3**]. We generalize the constructions of [**MR1, MR2**] to the setting of *equivariant* coherent (dg-)sheaves, and use it to construct a "categorification" of the Iwahori–Matsumoto involution of the affine Hecke algebra $\mathcal{H}$ of a reductive group, i.e. an equivalence between certain triangulated categories whose Grothendieck group is naturally isomorphic to $\mathcal{H}$, such that the induced automorphism of $\mathcal{H}$ is the Iwahori–Matsumoto involution (up to a correction factor). This construction uses the Kazhdan–Lusztig–Ginzburg description of $\mathcal{H}$ in terms of the equivariant K-theory of the Steinberg variety of G.

9.1.6. *Linear Koszul duality and Fourier transform for convolution algebras (with I. Mirković),* [MR4]. We show that, under certain technical assumptions, a certain isomorphism in K-theory induced by the linear Koszul duality of [MR1, MR2] and a similar isomorphism in Borel–Moore homology induced by a Fourier–Sato transform are related via the Chern character. In a specific geometric situation, this statement explains the relation between the main result of [MR3] and a construction for graded affine Hecke algebras in [EM].

9.2. Braid group action. The following articles are devoted to the construction of the "categorical" braid group action considered in §5.2.

9.2.1. *Geometric braid group action on derived categories of coherent sheaves (with an appendix joint with R. Bezrukavnikov),* [R1]. This article is part of my PhD thesis. If G is a simply connected semisimple algebraic group over an algebraically closed field $\Bbbk$ with no factor of type $\mathbf{G}_2$ (and assuming that $\mathrm{char}(\Bbbk) \neq 2$ if G is not simply-laced), and if $\widetilde{\mathscr{N}}$ and $\widetilde{\mathfrak{g}}$ are the Springer and Grothendieck resolutions of G, we construct an action of the (extended affine) braid group associated with G on the categories $D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathscr{N}_\varnothing})$ and $D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathfrak{g}}_\varnothing)$ (and the equivariant versions), as considered in §5.2.

This action originates in the Bezrukavnikov–Mirković–Rumynin localization theory (see in particular [BMR2]), and can also be considered as a categorical upgrade of the Kazhdan–Lusztig–Ginzburg description of the affine Hecke algebra in terms of the K-theory of the Steinberg variety. It plays an important technical role in particular in [BM, R2, MaR1]. We later generalized this construction in [BR].

9.2.2. *Affine braid group actions on Springer resolutions (with R. Bezrukavnikov),* [BR]. In this paper we generalize the results of [R1], removing the assumptions on G (still assumed to be semisimple and simply-connected) and p. We also develop the theory of dg-sheaves on dg-schemes (originally due to Ciocan-Fontanine–Kapranov [CK]), and use some base change constructions to generalize the action also to the derived categories of more general schemes.

9.3. Koszul duality for constructible sheaves on flag varieties. The following articles are concerned with variations on and extensions of the Beĭlinson–Ginzburg–Soergel Koszul duality for constructible sheaves on flag varieties.

9.3.1. *Koszul duality and semisimplicity of Frobenius (with P. Achar),* [AR1]. The starting point of this paper is the fact, due to Beĭlinson–Ginzburg–Soergel [BGS], that if X is an algebraic variety over a finite field, endowed with a stratification $\mathscr{S}$ and satisfying certain assumptions (verified in particular if X is a partial flag variety of a Kac–Moody group stratified by the Bruhat decomposition), then a certain full subcategory $\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X, \overline{\mathbb{Q}}_\ell)$ of the category of Deligne's mixed $\overline{\mathbb{Q}}_\ell$-perverse sheaves on X is a Koszul category. We develop various tools to adapt the construction of the usual functors for constructible sheaves (in particular, !- and $*$-extension and restriction for a locally closed inclusion of a union of strata) to the setting of the derived category $D^{\mathrm{b}}\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X, \overline{\mathbb{Q}}_\ell)$.

An observation contained in this paper, and which is crucial for the later work [AR4], is that the triangulated category $D^{\mathrm{b}}\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X, \overline{\mathbb{Q}}_\ell)$ is equivalent to the bounded homotopy category of the additive category of pure (semisimple) objects of weight 0.

9.3.2. *Modular Koszul duality (with W. Soergel and G. Williamson),* [RSW]. Given a split connected reductive algebraic group G over a finite field $\mathbb{F}$, we construct a "Koszul duality" equivalence relating the derived category of Soergel's modular category $\mathcal{O}$ attached to G (defined as a certain subquotient of the regular block $\mathsf{Rep}_0(G)$ "around the Steinberg weight") and the bounded derived category of Bruhat-constructible $\mathbb{F}$-sheaves on the flag variety of the Langlands dual complex reductive group, under the assumption that the characteristic of $\mathbb{F}$ is at least the number of roots of G plus 2.

The key technical statement is a "formality" result for this constructible derived category, describing it in terms of dg-modules over the dg-algebra of extensions between certain parity complexes. This formality result is obtained using a classical trick (usually attributed to Deligne) involving the study of eigenvalues of a certain Frobenius action. The classical setting for this trick is that of $\mathbb{Q}_\ell$-sheaves; but here we adapt it to the setting of $\mathbb{Z}_\ell$- and $\mathbb{F}_\ell$-sheaves. In [AR4] we generalize this formality result to the case when the characteristic of $\mathbb{F}$ is good for G, using a completely different approach.

This article is the first concrete example of the idea (which is suggested by [So4]) that, in a "Koszul duality" equivalence in a modular context, the role usually played by simple modules (or perverse sheaves) should be played by some parity objects (or parity complexes) instead. This idea is crucial for the constructions in [AR4, MaR2] in particular.

9.3.3. *Modular perverse sheaves on flag varieties I: tilting and parity sheaves (with P. Achar),* [AR3]. We construct a "degrading functor" from the category of Bruhat-constructible $\mathbb{F}$-parity complexes on the flag variety of a complex connected reductive algebraic group to the category of tilting $\mathbb{F}$-perverse sheaves on the flag variety of the Langlands dual group, where $\mathbb{F}$ is as in Theorem 3.6. The bridge between these two categories is provided by usual Soergel modules (with coefficients in $\mathbb{F}$). As applications, we show that the multiplicities of simple perverse sheaves in standard perverse sheaves can be computed in terms of the Langlands dual p-canonical basis (generalizing the Kazhdan–Lusztig inversion formula for Kazhdan–Lusztig polynomials), and we show that Soergel's modular category $\mathcal{O}$ is equivalent to a category of Bruhat-constructible perverse sheaves on a flag variety. This result can be considered as a "finite analogue" of the Finkelberg–Mirković conjecture considered in §5.6.

9.3.4. *Modular perverse sheaves on flag varieties II: Koszul duality and formality (with P. Achar),* [AR4]. We develop the theory of the mixed derived category as presented in §3.2. (The starting point of this approach is that the description of the category $D^{\mathrm{b}}\mathsf{Perv}^{\mathrm{mix}}_{\mathscr{S}}(X, \overline{\mathbb{Q}}_\ell)$ in terms of pure objects as explained in §9.3.1 makes sense for arbitrary coefficients, if one understands "pure objects of weight 0" as "parity complexes.") Then we build on the results of [AR3] to construct a Koszul duality equivalence as presented in §3.3. We also use this construction to generalize the "formality" results of [RSW] (see §9.3.2) to coefficients of good characteristic.

9.3.5. *Modular perverse sheaves on flag varieties III: positivity conditions (with P. Achar),* [AR5]. We continue the study of the mixed derived category of sheaves on the flag variety of a complex connected reductive algebraic group, begun in [**AR5**]. In particular, we study some analogues of Deligne's notion of weights for usual mixed perverse sheaves. We use these tools to try to determine when the category $\mathsf{Perv}^{\mathrm{mix}}_{(B)}(\mathscr{B}, \mathbb{F})$ satisfies some forms of Koszul properties. In particular we prove that this category is Koszul iff the parity complexes $\mathcal{E}_w$ are the simple perverse sheaves, and that this category is positively graded iff the parity complexes $\mathcal{E}^\vee_w$ (on $\mathscr{B}^\vee$) are perverse. (However, we are not able to give any condition on p which ensures that these properties are satisfied.)

9.3.6. *Complements on mixed modular derived categories (with P. Achar and L. Rider),* [**ARd2**]. We prove some complements on the theory of mixed derived categories of §3.2, considering in particular more general stratifications. These results are used by Achar–Rider in the main body of [**ARd2**].

9.4. Variations on the Geometric Satake Equivalence. The following articles study some aspects of the Geometric Satake Equivalence. The articles [**AR2**] and [**GR**] are concerned with characteristic-0 coefficients, and can be considered as preparatory for [**MaR2**] (in the sense that I learnt important tools for the constructions in [**MaR2**] when working on these articles), and [**AHR**] can be considered as preparatory for the work on the generalized Springer correspondence.

9.4.1. *Constructible sheaves on affine Grassmannians and coherent sheaves on the dual nilpotent cone (with P. Achar),* [**AR2**]. Following ideas of Ginzburg and Arkhipov–Bezrukavnikov–Ginzburg, we construct an equivalence of categories between the $G(\mathscr{O})$-constructible derived category of sheaves on $\mathcal{G}\mathrm{r}_G$ (where G is a connected complex reductive group, and we consider sheaves with coefficients in a field $\Bbbk$ of characteristic 0) and some derived category of equivariant coherent sheaves on the nilpotent cone of the Langlands dual reductive group over $\Bbbk$. We also study the compatibility of this construction with restriction to a Levi subgroup.

9.4.2. *Differential operators on G/U and the affine Grassmannian (with V. Ginzburg),* [**GR**]. We describe, in terms of the geometry of the Langlands dual group, the corestriction to torus-fixed points of the affine Grassmannian $\mathcal{G}\mathrm{r}_G$ of a complex connected reductive group G of the $G(\mathscr{O})$-equivariant $\mathbb{C}$-perverse sheaves on $\mathcal{G}\mathrm{r}_G$. This allows us in particular to obtain a cleaner proof of the description (due to Ginzburg) of the Brylinski–Kostant filtration in terms of the Geometric Satake Equivalence.

9.4.3. *Geometric Satake, Springer correspondence, and small representations II (with P. Achar and A. Henderson),* [**AHR**]. We study the relation between the Geometric Satake Equivalence and the Springer correspondence. More precisely, we show that the functor sending a "small" representation of a reductive group to its 0-weight space (considered as a representation of the Weyl group) can be realized geometrically in terms of perverse sheaves on the affine Grassmannian and the nilpotent cone of the Langlands dual group. This result generalizes to arbitrary coefficients a previous result of Achar–Henderson [**AH**] for characteristic-0 coefficients.

9.5. Springer correspondence and generalizations. The following articles are concerned with the Springer correspondence and the generalized Springer correspondence.

9.5.1. *Weyl group actions on the Springer sheaf (with P. Achar, A. Henderson and D. Juteau),* [**AHJR1**]. We show that two Weyl group actions on the Springer sheaf with arbitrary coefficients, one defined "by Fourier transform" and one defined "by restriction," agree up to a twist by the sign character. These actions arise in two possible definitions of the Springer correspondence. This result generalizes a familiar fact from the characteristic-0 setting. We also define a Springer correspondence for coefficients in any noetherian commutative ring of finite global dimension, and using results of [**AHR**] and [**Ju2**] we identify the 0-weight spaces of "small" representations in terms of this Springer correspondence.

9.5.2. *Modular generalized Springer correspondence I: the general linear group (with P. Achar, A. Henderson and D. Juteau),* [**AHJR2**]. We begin the study of the modular generalized Springer correspondence, see Section 2. In this article we construct and describe explicitly the correspondence in the special case of the group $G = \mathrm{GL}_n(\mathbb{C})$. We also use this study to prove that, in this case, the category $\mathsf{Perv}_G(\mathscr{N}_G, \mathbb{F})$ can be obtained by 'gluing' from the categories of $\mathbb{F}$-representations of the relative Weyl groups arising in this correspondence.

9.5.3. *Modular generalized Springer correspondence II: classical groups (with P. Achar, A. Henderson and D. Juteau),* [**AHJR3**]. We continue the study of the modular generalized Springer correspondence, see Section 2. In this article we push the theory further to treat all classical groups. In this case we construct the correspondence, describe the cuspidal pairs in all cases (the case $\mathrm{char}(\mathbb{F}) = 2$ being radically different from the case $\mathrm{char}(\mathbb{F}) \neq 2$), and describe the correspondence explicitly in some cases.

9.5.4. *Modular generalized Springer correspondence III: exceptional groups (with P. Achar, A. Henderson and D. Juteau),* [**AHJR4**]. We continue the study of the modular generalized Springer correspondence, see Section 2. In this article we obtain a general proof of the correspondence (valid for any group G), and study the case of exceptional groups in more detail.

9.5.5. *Constructible sheaves on nilpotent cones in rather good characteristic (with P. Achar, A. Henderson and D. Juteau),* [**AHJR5**]. We study some aspects of the modular generalized Springer theory of Section 2 in the case when p is rather good for G, i.e. is good and does not divide the order of the component group of the centre of G. Under this assumption the set $\mathfrak{N}_{G,\mathbb{F}}$ of §2.1 is in a natural bijection with the corresponding set $\mathfrak{N}_{G,\mathbb{C}}$, and we prove that the partition of this set given by the mod-p generalized Springer correspondence is a refinement of the partition given by the characteristic-0 correspondence. We also consider Mautner's "cleanness conjecture" (see §2.7). We prove this conjecture in some cases and deduce some consequences, including a classification of supercuspidal pairs (see Remark 2.3) and an orthogonal decomposition of the category $D^{\mathrm{b}}_G(\mathscr{N}_G, \mathbb{F})$, see §2.7.

9.5.6. *Modular generalized Springer correspondence: an overview (with P. Achar, A. Henderson and D. Juteau),* [**AHJR6**]. We give an overview of our results on the modular generalized Springer correspondence, see Section 2. In particular, we discuss the motivating idea of modular character sheaves.

9.6. Towards a geometric framework for modular representation theory of reductive groups. These articles allow to construct a geometric framework for the modular representation theory of reductive algebraic groups, as presented in Part 2.

9.6.1. *On the exotic t-structure in positive characteristic (with C. Mautner),* [**MaR1**]. We generalize to arbitrary characteristic a construction (originally due to Bezrukavnikov [**Be**]), of an "exotic" t-structure on the category $D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathscr{N}})$, where G is a connected reductive group with simply-connected derived subgroup, and $\widetilde{\mathscr{N}}$ is its Springer resolution. We emphasize in particular the role of the braid group action from [**R1, BR**] in this construction, which was implicit in [**Be**] (and mentioned more explicitly in [**BM**] and [**Do**]). We show that the heart of this t-structure is a graded highest weight category, and study the corresponding tilting objects; see §5.3 for more details. These results are used in a crucial way in [**MaR2**], and also in [**AR6**].

9.6.2. *Kostant section, universal centralizer, and a modular derived Satake equivalence,* [**R3**]. We generalize to the positive characteristic setting (and also to some rings of integers) some fundamental results of Kostant on regular elements in the Lie algebra of a connected reductive algebraic group and their centralizers. These results play an important technical role in [**MaR2**], which was our main motivation. As a more direct application, we also give a "mixed modular" analogue of a "derived Satake equivalence" due to Bezrukavnikov–Finkelberg [**BF**], describing the equivariant mixed derived category $D^{\mathrm{mix}}_{G^\vee(\mathscr{O})}(\mathscr{G}\mathrm{r}_{G^\vee}, \mathbb{F})$ (defined as the bounded homotopy category of the category of equivariant parity complexes) in terms of coherent sheaves on the Lie algebra of G. (Here, G is a connected reductive algebraic group over a field of positive characteristic, satisfying some technical assumptions, and $G^\vee$ is the Langlands dual group.)

9.6.3. *Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture (with C. Mautner),* [**MaR2**]. We construct the equivalence of categories stated in §5.4, and use it to complete the proof of the Mirković–Vilonen conjecture, as explained in Remark 5.7(4). A surprising feature of our approach is that it does *not* rely on the Geometric Satake Equivalence; instead we use a "Soergel approach" involving some kinds of Soergel bimodules.

9.6.4. *Tilting modules and the p-canonical basis (with G. Williamson),* [**RW**]. We conjecture that a certain diagrammatic category of Soergel bimodules acts on the principal block $\mathsf{Rep}_0(G)$ of a connected reductive group G as in Section 4. We observe that this conjecture has as a consequence a character formula for tilting and simple modules (as stated here in §4.4), and we prove our conjecture in the case of the group $\mathrm{GL}_n(\Bbbk)$ using a categorical action of the Lie algebra $\widehat{\mathfrak{gl}}_p$ on the category of representations of $\mathrm{GL}_n(\Bbbk)$ constructed by Chuang–Rouquier, and an adaptation of results of Mackaay (and some collaborators) relating the 2-Kac–Moody algebra associated with $\widehat{\mathfrak{gl}}_p$ to an Elias–Williamson diagrammatic category. Finally, we describe the category of parity complexes on the flag variety of a Kac–Moody groups in terms of diagrammatic Soergel bimodules.

9.6.5. *Reductive groups, the loop Grassmannian, and the Springer resolution (with P. Achar),* [**AR6**]. We adapt the main results of Part 1 of [**ABG**] to the setting of modular representations of reductive groups, as presented in §§5.5–5.6.

9.7. Other. The following article is not directly connected with the rest of my work.

9.7.1. *Hopf algebras having a dense big cell (with J. Bichon),* [**BiR**]. We study some axioms on a general (possibly noncommutative) Hopf algebra which ensure that its simple comodules admit a Borel–Weil classification in terms of highest weights. We illustrate this theory on the example of the universal cosovereign Hopf algebras, for which the weight group is the free group on two generators.

Bibliography

[AH] P. Achar and A. Henderson, *Geometric Satake, Springer correspondence, and small representations*, Selecta Math. (N.S.) **19** (2013), 949–986.

[AHR] P. Achar, A. Henderson, and S. Riche, *Geometric Satake, Springer correspondence, and small representations II*, Represent. Theory **19** (2015), 94–166.

[AHJR1] P. Achar, A. Henderson, D. Juteau, and S. Riche, *Weyl group actions on the Springer sheaf*, Proc. Lond. Math. Soc. **108** (2014), no. 6, 1501–1528.

[AHJR2] P. Achar, A. Henderson, D. Juteau, and S. Riche, *Modular generalized Springer correspondence I: the general linear group*, J. Eur. Math. Soc. **18** (2016), 1405–1436.

[AHJR3] P. Achar, A. Henderson, D. Juteau, and S. Riche, *Modular generalized Springer correspondence II: classical groups*, preprint arXiv:1404.1096, to appear in J. Eur. Math. Soc.

[AHJR4] P. Achar, A. Henderson, D. Juteau, and S. Riche, *Modular generalized Springer correspondence III: exceptional groups*, preprint arXiv:1507.00401, to appear in Math. Ann.

[AHJR5] P. Achar, A. Henderson, D. Juteau, and S. Riche, *Constructible sheaves on nilpotent cones in rather good characteristic*, preprint arXiv:1507.06581, to appear in Selecta Math. (N.S.).

[AHJR6] P. Achar, A. Henderson, D. Juteau, and S. Riche, *Modular generalized Springer correspondence: an overview*, preprint arXiv:1510.08962.

[AR1] P. Achar and S. Riche, *Koszul duality and semisimplicity of Frobenius*, Ann. Inst. Fourier **63** (2013), 1511–1612.

[AR2] P. Achar and S. Riche, *Constructible sheaves on affine Grassmannians and coherent sheaves on the dual nilpotent cone*, Israel J. Math. **205** (2015), 247–315.

[AR3] P. Achar and S. Riche, *Modular perverse sheaves on flag varieties I: tilting and parity sheaves* (with a joint appendix with G. Williamson), Ann. Sci. Éc. Norm. Supér. **49** (2016), 325–370.

[AR4] P. Achar and S. Riche, *Modular perverse sheaves on flag varieties II: Koszul duality and formality*, Duke Math. J. **165** (2016), 161–215.

[AR5] P. Achar and S. Riche, *Modular perverse sheaves on flag varieties III: positivity conditions*, preprint arXiv:1408.4189, to appear in Trans. Amer. Math. Soc.

[AR6] P. Achar and S. Riche, *Reductive groups, the loop Grassmannian, and the Springer resolution*, preprint arXiv:1602.04412.

[ARd1] P. Achar and L. Rider, *Parity sheaves on the affine Grassmannian and the Mirković–Vilonen conjecture*, Acta Math. **215** (2015), 183–216.

[ARd2] P. Achar and L. Rider, *The affine Grassmannian and the Springer resolution in positive characteristic* (with an appendix joint with S. Riche), preprint arXiv:1408.7050, to appear in Compos. Math.

[An] H. H. Andersen, *Tilting modules for algebraic groups*, in *Algebraic groups and their representations (Cambridge, 1997)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 517, Kluwer Acad. Publ., 1998, 25–42.

[AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque **220** (1994), 1–321.

[AB] S. Arkhipov and R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group* (with an appendix by R. Bezrukavnikov and I. Mirković), Israel J. Math. **170** (2009), 135–183.

[ABG] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17** (2004), 595–678.

[BB] A. Beĭlinson and J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 15–18.

[BBD] A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, Astérisque **100** (1982), 5–171.

[BG] A. Beĭlinson and V. Ginzburg, *Wall-crossing functors and $\mathcal{D}$-modules*, Represent. Theory **3** (1999), 1–31.

[BGS] A. Beĭlinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.

[BL] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994.

[Be] R. Bezrukavnikov, *Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves*, Invent. Math. **166** (2006), 327–357.

[BF] R. Bezrukavnikov and M. Finkelberg, *Equivariant Satake category and Kostant–Whittaker reduction*, Mosc. Math. J. **8** (2008), 39–72.

[BM] R. Bezrukavnikov and I. Mirković, *Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution*, Ann. of Math. **178** (2013), 835–919.

[BMR] R. Bezrukavnikov, I. Mirković, and D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic* (with an appendix by R. Bezrukavnikov and S. Riche), Ann. of Math. **167** (2008), 945–991.

[BMR2] R. Bezrukavnikov, I. Mirković, and D. Rumynin, *Singular localization and intertwining functors for reductive Lie algebras in prime characteristic*, Nagoya Math. J. **184** (2006), 1–55.

[BR] R. Bezrukavnikov and S. Riche, *Affine braid group actions on Springer resolutions*, Ann. Sci. Éc. Norm. Supér. **45** (2012), 535–599.

[BY] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac–Moody groups*, Represent. Theory **17** (2013), 1–98.

[BiR] J. Bichon and S. Riche, *Hopf algebras having a dense big cell*, Trans. Amer. Math. Soc. **368** (2016), 515–538.

[Bo] C. Bonnafé, *Actions of relative Weyl groups. I*, J. Group Theory **7** (2004), 1–37.

[Br] T. Braden, *Hyperbolic localization of intersection cohomology*, Transform. Groups **8** (2003), 209–216.

[BK] J.-L. Brylinski and M. Kashiwara, *Kazhdan–Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.

[CYZ] X.-W. Chen, Y. Ye, P. Zhang, *Algebras of derived dimension zero*, Comm. Algebra **36** (2008), 1–10.

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, 1997.

[CK] I. Ciocan-Fontanine, M. Kapranov, *Derived quot schemes*, Ann. Scient. Éc. Norm. Sup. **34** (2001), 403–440.

[CPS] E. Cline, B. Parshall, and L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[Do] C. Dodd, *Equivariant coherent sheaves, Soergel bimodules, and categorification of affine Hecke algebras*, preprint arXiv:1108.4028.

[EW] B. Elias and G. Williamson, *Soergel calculus*, preprint arXiv:1309.0865.

[EM] S. Evens and I. Mirković, *Fourier transform and the Iwahori–Matsumoto involution*, Duke Math. J. **86** (1997), 435–464.

[Fi] P. Fiebig, *An upper bound on the exceptional characteristics for Lusztig's character formula*, J. Reine Angew. Math. **673** (2012), 1–31.

[FM] M. Finkelberg and I. Mirković, *Semi-infinite flags I. Case of global curve $\mathbb{P}^1$*, in *Differential topology, infinite-dimensional Lie algebras, and applications*, Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., 1999, 81–112.

[FP] E. Friedlander and B. Parshall, *Cohomology of Lie algebras and algebraic groups*, Amer. J. Math. **108** (1986), 235–253.

[GR] V. Ginzburg and S. Riche, *Differential operators on G/U and the affine Grassmannian*, J. Inst. Math. Jussieu **14** (2015), 493–575.

[HKS] T. Hodge, P. Karuppuchamy, and L. Scott, *Remarks on the ABG induction theorem*, preprint arXiv:1603.05699.

[Ja1] J. C. Jantzen, *Representations of Algebraic Groups, second edition*, Mathematical Surveys and Monographs 107, Amer. Math. Soc., 2003.

[Ja2] J. C. Jantzen, *Nilpotent orbits in representation theory*, in *Lie theory*, 1–211, Progr. Math. 228, Birkhäuser, 2004.

[JW] L. T. Jensen and G. Williamson, *The p-canonical basis for Hecke algebras*, preprint arXiv:1510.01556.

[Ju1] D. Juteau, *Modular representations of reductive groups and geometry of affine Grassmannians*, preprint arXiv:0804.2041.

[Ju2] D. Juteau, *Modular Springer correspondence, decomposition matrices and basic sets*, preprint arXiv:1410.1471, to appear in Bull. Soc. Math. France.

[JMW1] D. Juteau, C. Mautner, and G. Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), 1169–1212.

[JMW2] D. Juteau, C. Mautner, and G. Williamson, *Parity sheaves and tilting modules*, Ann. Sci. Éc. Norm. Supér. **49** (2016), 257–275.

[KT] M. Kashiwara and T. Tanisaki, *Kazhdan–Lusztig conjecture for affine Lie algebras with negative level I*, Duke Math. J. **77** (1995), 21–62; *II, non-integral case*, Duke Math. J. **84** (1996), 771–813.

[KL1] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[KL2] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras I*, J. Amer. Math. Soc. **6** (1993), 905–947; *II*, J. Amer. Math. Soc. **6** (1993), 949–1011; *III*, J. Amer. Math. Soc. **7** (1994), 335–381; *IV*, J. Amer. Math. Soc. **7** (1994), 383–453.

[Ku] S. Kumar, *Kac–Moody groups, their flag varieties and representation theory*, Progress in Mathematics 204, Birkhäuser, 2002.

[Le] E. Letellier, *Fourier transforms of invariant functions on finite reductive Lie algebras*, Lecture Notes in Math., vol. 1859, Springer-Verlag, 2005.

[Lu1] G. Lusztig, *Some problems in the representation theory of finite Chevalley groups*, in *The Santa Cruz Conference on Finite Groups*, Proc. Sympos. Pure Math. 37, Amer. Math. Soc., 1980, 313–317.

[Lu2] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), 205–272.

[Lu3] G. Lusztig, *Character sheaves. I*, Adv. in Math. **56** (1985), 193–237; *II*, Adv. in Math. **57** (1985), 226–265; *III*, Adv. in Math. **57** (1985), 266–315; *IV*, Adv. in Math. **59** (1986), 1–63; *V*, Adv. in Math. **61** (1986), 103–155.

[Lu4] G. Lusztig, *Fourier transforms on a semisimple Lie algebra over $\mathbb{F}_q$*, in *Algebraic groups (Utrecht 1986)*, Lecture Notes in Math., vol. 1271, Springer, 1987, 177–188.

[Lu5] G. Lusztig, *Monodromic systems on affine flag manifolds*, Proc. Roy. Soc. London **445** (1994), 231–246. Errata in **450** (1995), 731–732.

[Lu6] G. Lusztig, *Bases in equivariant K-theory*, Represent. Theory **2** (1998), 298–369.

[Ma] O. Mathieu, *Construction d'un groupe de Kac–Moody et applications*, Compositio Math. **69** (1989), 37–60.

[MaR1] C. Mautner and S. Riche, *On the exotic t-structure in positive characteristic*, preprint arXiv:1412.6818, to appear in Int. Math. Res. Not.

[MaR2] C. Mautner and S. Riche, *Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture*, preprint arXiv:1501.07369, to appear in J. Eur. Math. Soc.

[MR1] I. Mirković and S. Riche, *Linear Koszul duality*, Compos. Math. **146** (2010), 233–258.

[MR2] I. Mirković and S. Riche, *Linear Koszul duality II: coherent sheaves on perfect sheaves*, J. London Math. Soc. **93** (2016), 1–24.

[MR3] I. Mirković and S. Riche, *Iwahori-Matsumoto involution and linear Koszul duality*, Int. Math. Res. Not. **2015** (2015), 150–196.

[MR4] I. Mirković and S. Riche, *Linear Koszul duality and Fourier transform for convolution algebras*, Doc. Math. **20** (2015), 989–1038.

[MV] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166** (2007), 95–143.

[Po] L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, Memoirs of the A.M.S., Vol. 212, N. 996, Amer. Math. Soc., 2011.

[R1] S. Riche, *Geometric braid group action on derived categories of coherent sheaves* (with an appendix joint with R. Bezrukavnikov), Represent. Theory **12** (2008), 131–169.

[R2] S. Riche, *Koszul duality and modular representations of semisimple Lie algebras*, Duke Math. J. **154** (2010), 31–134.

[R3] S. Riche, *Kostant section, universal centralizer, and a modular derived Satake equivalence*, preprint arXiv:1411.3112, to appear in Math. Z.

[RSW] S. Riche, W. Soergel, and G. Williamson, *Modular Koszul duality*, Compos. Math. **150** (2014), 273–332.

[RW] S. Riche and G. Williamson, *Tilting modules and the p-canonical basis*, preprint arXiv:1512.08296.

[Ro] G. Rousseau, *Groupes de Kac–Moody déployés sur un corps local, II. Masures ordonnées*, preprint arXiv:1009.0138.

[So1] W. Soergel, *Kazhdan–Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114.

[So2] W. Soergel, *Character formulas for tilting modules over Kac–Moody algebras*, Represent. Theory **2** (1998), 432–448.

[So3] W. Soergel, *Character formulas for tilting modules over quantum groups at roots of one*, in *Current developments in mathematics, 1997 (Cambridge, MA)*, Int. Press, 1999, 161–172.

[So4] W. Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*, in *Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998)*, J. Pure Appl. Algebra **152** (2000), 311–335.

[Sp1] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.

[Sp2] T. A. Springer, *Quelques applications de la cohomologie d'intersection*, in *Bourbaki Seminar, Vol. 1981/1982*, Astérisque, 92–93, Soc. Math. France, 1982, 249–273.

[Ti] J. Tits, *Groupes associés aux algèbres de Kac–Moody*, Séminaire Bourbaki, Vol. 1988/89. Astérisque **177–178** (1989), Exp. 700, 7Ð31.

[Wi] G. Williamson, *Schubert calculus and torsion explosion* (with an appendix joint with A. Kontorovich and P. J. McNamara), preprint arXiv:1309.5055.

[Yu] Z. Yun, *Weights of mixed tilting sheaves and geometric Ringel duality*, Selecta Math. (N.S.) **14** (2009), 299–320.