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REPRESENTATION THEORY OF ALGEBRAS II: AUSLANDER-REITEN THEORY

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Abstract. This is the second part of a planned book "Introduction to Representation Theory of Algebras". This part gives an introduction to Homological Algebra and to Auslander-Reiten Theory.

Preliminary version

Contents

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1. **Introduction**

This is the second part of notes for a lecture course "Introduction to Representation Theory".

Around 1970 Peter Gabriel proved that a connected quiver is representation-finite if and only if the underlying graph is a Dynkin graph of type $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$ or $\mathbb{E}_n (n = 6, 7, 8)$. He also showed that the dimension vectors of the indecomposable representations correspond to the positive roots of the corresponding Lie algebra. This celebrated result can be seen as a starting point of modern representation theory of finite-dimensional algebras. Equally important was the discovery of almost split sequences (now called Auslander-Reiten sequences) by Maurice Auslander and Idun Reiten. We will prove both results. Furthermore, we will explain the knitting algorithm for preprojective components.

1.1. **Acknowledgements.** The second author thanks his student Tim Eickmann for typo hunting.

Part 1. Homological Algebra I: Resolutions and extension groups

2. Homological Algebra

2.1. The Snake Lemma.

Theorem 2.1 (Snake Lemma). *Given the following commutative diagram of homomorphisms*

$$\begin{array}{ccccccccc}
 & & U_1 & \xrightarrow{f_1} & V_1 & \xrightarrow{g_1} & W_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & U_2 & \xrightarrow{f_2} & V_2 & \xrightarrow{g_2} & W_2 & &
\end{array}$$

such that the two rows are exact. Taken kernels and cokernels of the homomorphisms a, b, c we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & U_0 & \xrightarrow{f_0} & V_0 & \xrightarrow{g_0} & W_0 & & \\
 & & \downarrow{\scriptstyle a_0} & & \downarrow{\scriptstyle b_0} & & \downarrow{\scriptstyle c_0} & & \\
 & & U_1 & \xrightarrow{f_1} & V_1 & \xrightarrow{g_1} & W_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & U_2 & \xrightarrow{f_2} & V_2 & \xrightarrow{g_2} & W_2 & & \\
 & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle c_2} & & \\
 & & U_3 & \xrightarrow{f_3} & V_3 & \xrightarrow{g_3} & W_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Then

$$\delta(x) := (a_2 \circ f_2^{-1} \circ b \circ g_1^{-1} \circ c_0)(x)$$

defines a homomorphism (the "connecting homomorphism")

$$\delta\colon \operatorname{Ker}(c) \to \operatorname{Cok}(a)$$

such that the sequence

$$\operatorname{Ker}(a) \xrightarrow{f_0} \operatorname{Ker}(b) \xrightarrow{g_0} \operatorname{Ker}(c) \xrightarrow{\delta} \operatorname{Cok}(a) \xrightarrow{f_3} \operatorname{Cok}(b) \xrightarrow{g_3} \operatorname{Cok}(c)$$

is exact.

Proof. The proof is divided into two steps: First, we define the map δ, second we verify the exactness.

<u>Relations</u>

We need some preliminary remarks on relations: Let V and W be modules. A submodule $\rho \subseteq V \times W$ is called a **relation**. If $f\colon V \to W$ is a homomorphism, then the graph

$$\Gamma(f) = \{(v, f(v)) \mid v \in V\}$$

of f is a relation. Vice versa, a relation $\rho \subseteq V \times W$ is the graph of a homomorphism, if for every $v \in V$ there exists exactly one $w \in W$ such that $(v, w) \in \rho$.

If $\rho \subseteq V \times W$ is a relation, then the **opposite relation** is defined as $\rho^{-1} = \{(w, v) \mid (v, w) \in \rho\}$. Obviously this is a submodule again, namely of $W \times V$.

If V_1, V_2, V_3 are modules and $\rho \subseteq V_1 \times V_2$ and $\sigma \subseteq V_2 \times V_3$ are relations, then

$$\sigma \circ \rho := \{(v_1, v_3) \in V_1 \times V_3 \mid \text{there exists some } v_2 \in V_2 \text{ with } (v_1, v_2) \in \rho, (v_2, v_3) \in \sigma\}$$

is the **composition** of ρ and σ. It is easy to check that $\sigma \circ \rho$ is a submodule of $V_1 \times V_3$.

For homomorphisms $f\colon V_1 \to V_2$ and $g\colon V_2 \to V_3$ we have $\Gamma(g) \circ \Gamma(f) = \Gamma(gf)$.

The composition of relations is associative: If $\rho \subseteq V_1 \times V_2$, $\sigma \subseteq V_2 \times V_3$ and $\tau \subseteq V_3 \times V_4$ are relations, then $(\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho)$.

Let $\rho \subseteq V \times W$ be a relation. For a subset X of V define $\rho(X) = \{w \in W \mid (x, w) \in \rho \text{ for some } x \in X\}$. If $x \in V$, then set $\rho(x) = \rho(\{x\})$.

For example, if $f\colon V \to W$ is a homomorphism and X a subset of V, then

$$(\Gamma(f))(X) = f(X).$$

Similarly, $(\Gamma(f)^{-1})(Y) = f^{-1}(Y)$ for any subset Y of W.

Thus in our situation, $a_2 f_2^{-1} b g_1^{-1} c_0$ stands for

$$\Gamma(a_2) \circ \Gamma(f_2)^{-1} \circ \Gamma(b) \circ \Gamma(g_1)^{-1} \circ \Gamma(c_0).$$

First, we claim that this is indeed the graph of some homomorphism δ.

δ is a homomorphism

We show that $a_2 f_2^{-1} b g_1^{-1} c_0$ is a homomorphism: Let S be the set of tuples

$$(w_0, w_1, v_1, v_2, u_2, u_3) \in W_0 \times W_1 \times V_1 \times V_2 \times U_2 \times U_3$$

such that

$$\begin{aligned} w_1 &= c_0(w_0) = g_1(v_1), \\ v_2 &= b(v_1) = f_2(u_2), \\ u_3 &= a_2(u_2). \end{aligned}$$

$$\begin{tikzcd} & & w_0 \arrow[d, mapsto, "c_0"] \\ & v_1 \arrow[r, mapsto, "g_1"] \arrow[d, mapsto, "b"] & w_1 \\ u_2 \arrow[r, mapsto, "f_2"] \arrow[d, mapsto, "a_2"] & v_2 & \\ u_3 & & \end{tikzcd}$$

We have to show that for every $w_0 \in W_0$ there exists a tuple

$$(w_0, w_1, v_1, v_2, u_2, u_3)$$

in S, and that for two tuples $(w_0, w_1, v_1, v_2, u_2, u_3)$ and $(w_0', w_1', v_1', v_2', u_2', u_3')$ with $w_0 = w_0'$ we always have $u_3 = u_3'$.

Thus, let $w \in W_0$. Since g_1 is surjective, there exists some $v \in V_1$ with $g_1(v) = c_0(w)$. We have

$$g_2 b(v) = c g_1(v) = c c_0(w) = 0.$$

Therefore $b(v)$ belongs to the kernel of g_2 and also to the image of f_2. Thus there exists some $u \in U_2$ with $f_2(u) = b(v)$. So we see that

$$(w, c_0(w), v, b(v), u, a_2(u)) \in S.$$

Now let $(w, c_0(w), v', b(v'), u', x)$ also be in S. We get

$$g_1(v - v') = c_0(w) - c_0(w) = 0.$$

Thus $v - v'$ belongs to the kernel of g_1, and therefore to the image of f_1. So there exists some $y \in U_1$ with $f_1(y) = v - v'$. This implies

$$f_2(u - u') = b(v - v') = b f_1(y) = f_2 a(y).$$

Since f_2 is injective, we get $u - u' = a(y)$. But this yields

$$a_2(u) - x = a_2(u - u') = a_2 a(y) = 0.$$

Thus we see that $a_2(u) = x$, and this implies that δ is a homomorphism.

Exactness

Next, we want to show that $\mathrm{Ker}(\delta) = \mathrm{Im}(g_0)$: Let $x \in V_0$. To compute $\delta g_0(x)$ we need a tuple $(g_0(x), w_1, v_1, v_2, u_2, u_3) \in S$. Since $g_1 b_0 = c_0 g_0$ and $b b_0 = 0$ we can choose $(g_0(x), c_0 g_0(x), b_0(x), 0, 0, 0)$. This implies $\delta g_0(x) = 0$. Vice versa, let $w \in \mathrm{Ker}(\delta)$. So there exists some $(w, w_1, v_1, v_2, u_2, 0) \in S$. Since u_2 belongs to the kernel of a_2 and therefore to the image of a, there exists some $y \in U_1$ with $a(y) = u_2$. We have

$$b f_1(y) = f_2 a(y) = f_2(u_2) = b(v_1).$$

Thus $v_1 - f_1(y)$ is contained in $\mathrm{Ker}(b)$. This implies that there exists some $x \in V_0$ with $b_0(x) = v_1 - f_1(y)$. We get

$$c_0 g_0(x) = g_1 b_0(x) = g_1(v_1 - f_1(y)) = g_1(v_1) = c_0(w).$$

Since c_0 is injective, we have $g_0(x) = w$. So we see that w belongs to the image of g_0.

Finally, we want to show that $\mathrm{Ker}(f_3) = \mathrm{Im}(\delta)$: Let $(w_0, w_1, v_1, v_2, u_2, u_3) \in S$, in other words $\delta(w_0) = u_3$. We have

$$f_3(u_3) = f_3 a_2(u_2) = b_2 f_2(u_2).$$

Since $f_2(u_2) = v_2 = b(v_1)$, we get $b_2 f_2(u_2) = b_2 b(v_1) = 0$. This shows that the image of δ is contained in the kernel of f_3. Vice versa, let u_3 be an element in U_3, which belongs to the kernel of f_3. Since a_2 is surjective, there exists some $u_2 \in U_2$ with $a_2(u_2) = u_3$. We have $b_2 f_2(u_2) = f_3 a_2(u_2) = f_3(u_3) = 0$, and therefore $f_2(u_2)$ belongs to the kernel of b_2 and also to the image of b. Let $f_2(u_2) = b(v_1) =: v_2$. This implies $c g_1(v_1) = g_2 b(v_1) = g_2 f_2(u_2) = 0$. We see that $g_1(v_1)$ is in the kernel of c and therefore in the image of c_0. So there exists some $w_0 \in W_0$ with $c_0(w_0) = g_1(v_1)$. Altogether, we constructed a tuple $(w_0, w_1, v_1, v_2, u_2, u_3)$ in S. This implies $u_3 = \delta(w_0)$. This finishes the proof of the Snake Lemma. $\square$

Next, we want to show that the connecting homomorphism is "natural": Assume we have two commutative diagrams with exact rows:

$$\begin{tikzcd}
 & U_1 \arrow[r, "f_1"] \arrow[d, "a"] & V_1 \arrow[r, "g_1"] \arrow[d, "b"] & W_1 \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & U_2 \arrow[r, "f_2"] & V_2 \arrow[r, "g_2"] & W_2, &
\end{tikzcd}$$

$$\begin{tikzcd}
 & U_1' \arrow[r, "f_1'"] \arrow[d, "a'"] & V_1' \arrow[r, "g_1'"] \arrow[d, "b'"] & W_1' \arrow[r] \arrow[d, "c'"] & 0 \\
0 \arrow[r] & U_2' \arrow[r, "f_2'"] & V_2' \arrow[r, "g_2'"] & W_2'. &
\end{tikzcd}$$

Let $\delta\colon \mathrm{Ker}(c) \to \mathrm{Cok}(a)$ and $\delta'\colon \mathrm{Ker}(c') \to \mathrm{Cok}(a')$ be the corresponding connecting homomorphisms.

Additionally, for $i = 1, 2$ let $p_i\colon U_i \to U_i'$, $q_i\colon V_i \to V_i'$ and $r_i\colon W_i \to W_i'$ be homomorphisms such that the following diagram is commutative:

$$\begin{tikzcd}
 & U_1 \arrow[rr] \arrow[dr, "p_1"] \arrow[dd] & & V_1 \arrow[rr] \arrow[dr, "q_1"] \arrow[dd] & & W_1 \arrow[rr] \arrow[dr, "r_1"] \arrow[dd] & & 0 & \\
 & & U_1' \arrow[rr] \arrow[dd] & & V_1' \arrow[rr] \arrow[dd] & & W_1' \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[r] & U_2 \arrow[rr] \arrow[dr, "p_2"] & & V_2 \arrow[rr] \arrow[dr, "q_2"] & & W_2 \arrow[dr, "r_2"] & & & \\
 & 0 \arrow[r] & U_2' \arrow[rr] & & V_2' \arrow[rr] & & W_2' & &
\end{tikzcd}$$

The homomorphisms $p_i\colon U_i \to U_i'$ induce a homomorphism $p_3\colon \mathrm{Cok}(a) \to \mathrm{Cok}(a')$, and the homomorphisms $r_i\colon W_i \to W_i'$ induce a homomorphism $r_0\colon \mathrm{Ker}(c) \to \mathrm{Ker}(c')$.

Lemma 2.2. *The diagram*

$$\begin{tikzcd} \mathrm{Ker}(c) \arrow[r, "\delta"] \arrow[d, "r_0"] & \mathrm{Cok}(a) \arrow[d, "p_3"] \\ \mathrm{Ker}(c') \arrow[r, "\delta'"] & \mathrm{Cok}(a') \end{tikzcd}$$

is commutative.

Proof. Again, let S be the set of tuples

$$(w_0, w_1, v_1, v_2, u_2, u_3) \in W_0 \times W_1 \times V_1 \times V_2 \times U_2 \times U_3$$

such that

$$\begin{aligned} w_1 &= c_0(w_0) = g_1(v_1), \\ v_2 &= b(v_1) = f_2(u_2), \\ u_3 &= a_2(u_2), \end{aligned}$$

and let S' be the correspondingly defined subset of $W_0' \times W_1' \times V_1' \times V_2' \times U_2' \times U_3'$. Now one easily checks that for a tuple $(w_0, w_1, v_1, v_2, u_2, u_3)$ in S the tuple

$$(r_0(w_0), r_1(w_1), q_1(v_1), q_2(v_2), p_2(u_2), p_3(u_3))$$

belongs to S'. The claim follows. □

2.2. **Complexes.** A **complex** of A-modules is a tuple $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ (we often just write $(C_n, d_n)_n$ or (C_n, d_n)) where the C_n are A-modules and the $d_n : C_n \to C_{n-1}$ are homomorphisms such that

$$\mathrm{Im}(d_n) \subseteq \mathrm{Ker}(d_{n-1})$$

for all n, or equivalently, such that $d_{n-1}d_n = 0$ for all n.

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

A **cocomplex** is a tuple $C^\bullet = (C^n, d^n)_{n\in\mathbb{Z}}$ where the C^n are A-modules and the $d^n : C^n \to C^{n+1}$ are homomorphisms such that $d^{n+1}d^n = 0$ for all n.

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

Remark: We will mainly formulate results and definitions by using complexes, but there are always corresponding results and definitions for cocomplexes. We leave it to the reader to perform the necessary reformulations.

In this lecture course we will deal only with (co)complexes of modules over a K-algebra A and with (co)complexes of vector spaces over the field K.

A complex $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ is an **exact sequence** of A-modules if

$$\mathrm{Im}(d_n) = \mathrm{Ker}(d_{n-1})$$

for all n. In this case, for $a > b$ we also call

$$C_a \xrightarrow{d_a} C_{a-1} \xrightarrow{d_{a-1}} \cdots \xrightarrow{d_{b+1}} C_b,$$

$$\cdots \xrightarrow{d_{b+2}} C_{b+1} \xrightarrow{d_{b+1}} C_b,$$

$$C_a \xrightarrow{d_a} C_{a-1} \xrightarrow{d_{a-1}} \cdots$$

exact sequences. An exact sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is a **short exact sequence**. We denote such a sequence by (f, g). Note that this implies that f is a monomorphism and g is an epimorphism.

Example: Let M be an A-module, and let $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ be a complex of A-modules. Then

$$\operatorname{Hom}_A(M, C_\bullet) = (\operatorname{Hom}_A(M, C_n), \operatorname{Hom}_A(M, d_n))_{n\in\mathbb{Z}}$$

is a complex of K-vector spaces and

$$\operatorname{Hom}_A(C_\bullet, M) = (\operatorname{Hom}_A(C_n, M), \operatorname{Hom}_A(d_{n+1}, M))_{n\in\mathbb{Z}}$$

is a cocomplex of K-vector spaces. (Of course, K is a K-algebra, and the K-modules are just the K-vector spaces.)

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Given two complexes $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ and $C'_\bullet = (C'_n, d'_n)_{n\in\mathbb{Z}}$, a **homomorphism of complexes** (or just **map of complexes**) is given by $f_\bullet = (f_n)_{n\in\mathbb{Z}} \colon C_\bullet \to C'_\bullet$ where the $f_n \colon C_n \to C'_n$ are homomorphisms with $d'_n f_n = f_{n-1} d_n$ for all n.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The maps $C_\bullet \to C'_\bullet$ of complexes form a vector space: Let $f_\bullet, g_\bullet \colon C_\bullet \to C'_\bullet$ be such maps, and let $\lambda \in K$. Define $f_\bullet + g_\bullet := (f_n + g_n)_{n\in\mathbb{Z}}$, and let $\lambda f_\bullet := (\lambda f_n)_{n\in\mathbb{Z}}$.

If $f_\bullet = (f_n)_n \colon C_\bullet \to C'_\bullet$ and $g_\bullet = (g_n)_n \colon C'_\bullet \to C''_\bullet$ are maps of complexes, then the **composition**

$$g_\bullet f_\bullet = g_\bullet \circ f_\bullet \colon C_\bullet \to C''_\bullet$$

is defined by $g_\bullet f_\bullet := (g_n f_n)_n$.

Let $C_\bullet = (C_n, d_n)_n$ be a complex. A **subcomplex** $C'_\bullet = (C'_n, d'_n)_n$ of $C_\bullet$ is given by submodules $C'_n \subseteq C_n$ such that d'_n is obtain via the restriction of d_n to C'_n. (Thus we require that $d_n(C'_n) \subseteq C'_{n-1}$ for all n.) The corresponding **factor complex** $C_\bullet/C'_\bullet$ is of the form $(C_n/C'_n, d''_n)_n$ where d''_n is the homomorphism $C_n/C'_n \to C_{n-1}/C'_{n-1}$ induced by d_n.

Let $f_\bullet = (f_n)_n\colon C'_\bullet \to C_\bullet$ and $g_\bullet = (g_n)_n\colon C_\bullet \to C''_\bullet$ be homomorphisms of complexes. Then

$$0 \to C'_\bullet \xrightarrow{f_\bullet} C_\bullet \xrightarrow{g_\bullet} C''_\bullet \to 0$$

is a **short exact sequence of complexes** provided

$$0 \to C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \to 0$$

is a short exact sequence for all n.

2.3. **From complexes to modules.** We can interpret complexes of J-modules (here we use our terminology from the first part of the lecture course) as J'-modules where

$$J' := J \cup \mathbb{Z} \cup \{d\}.$$

(We assume that J, $\mathbb{Z}$ and $\{d\}$ are pairwise disjoint sets.)

If $C_\bullet = (C_n, d_n)_n$ is a complex of J-modules, then we consider the J-module

$$C := \bigoplus_{n \in \mathbb{Z}} C_n.$$

We add some further endomorphisms of the vector space C, namely for $n \in \mathbb{Z}$ take the projection $\phi_n\colon C \to C$ onto C_n and additionally take $\phi_d\colon C \to C$ whose restriction to C_n is just d_n. This converts C into a J'-module.

Now if $f_\bullet = (f_n)_n\colon C_\bullet \to C'_\bullet$ is a homomorphism of complexes, then

$$\bigoplus_{n \in \mathbb{Z}} f_n\colon \bigoplus_{n \in \mathbb{Z}} C_n \to \bigoplus_{n \in \mathbb{Z}} C'_n$$

defines a homomorphism of J'-modules, and one obtains all homomorphisms of J'-modules in such a way.

We can use this identification of complexes of J-modules with J'-modules for transferring the terminology we developed for modules to complexes: For example subcomplexes or factor complexes can be defined as J'-submodules or J'-factor modules.

2.4. **Homology of complexes.** Given a complex $C_\bullet = (C_n, d_n)_n$ define

$$H_n(C_\bullet) = \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1}),$$

the nth **homology module** (or **homology group**) of $C_\bullet$. Set $H_\bullet(C_\bullet) = (H_n(C_\bullet))_n$.

Similarly, for a cocomplex $C^\bullet = (C^n, d^n)$ let

$$H^n(C^\bullet) = \operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1})$$

be the nth **cohomology group** of $C^\bullet$.

Each homomorphism $f_\bullet\colon C_\bullet \to C'_\bullet$ of complexes induces homomorphisms

$$H_n(f_\bullet)\colon H_n(C_\bullet) \to H_n(C'_\bullet).$$

(One has to check that $f_n(\operatorname{Im}(d_{n+1})) \subseteq \operatorname{Im}(d'_{n+1})$ and $f_n(\operatorname{Ker}(d_n)) \subseteq \operatorname{Ker}(d'_n)$.)

$$\begin{tikzcd}
C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d_{n+1}"] & C'_{n+1} \arrow[d, "d'_{n+1}"] \\
C_n \arrow[r, "f_n"] \arrow[d, "d_n"] & C'_n \arrow[d, "d'_n"] \\
C_{n-1} \arrow[r, "f_{n-1}"] & C'_{n-1}
\end{tikzcd}$$

It follows that H_n defines a functor from the category of complexes of A-modules to the category of A-modules.

Let $C_\bullet = (C_n, d_n)$ be a complex. We consider the homomorphisms

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2}.$$

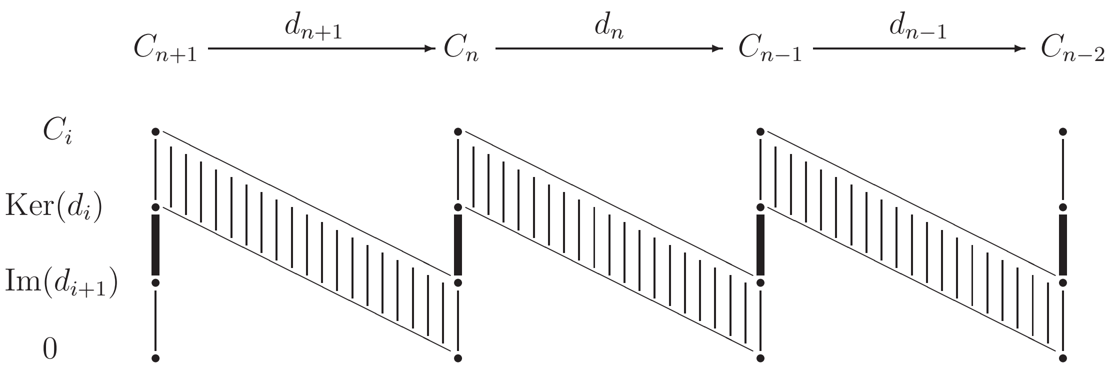
By assumption we have $\operatorname{Im}(d_{i+1}) \subseteq \operatorname{Ker}(d_i)$ for all i.

The following picture illustrates the situation. Observe that the homology groups

$$H_i(C_\bullet) = \operatorname{Ker}(d_i)/\operatorname{Im}(d_{i+1})$$

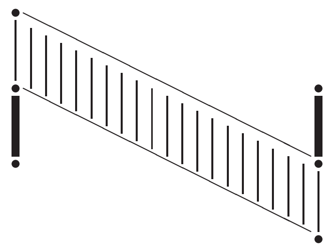
are highlighted by the thick vertical lines. The marked regions indicate which parts of C_i and C_{i-1} get identified by the map d_i. Namely d_i induces an isomorphism

$$C_i/\operatorname{Ker}(d_i) \to \operatorname{Im}(d_i).$$



The map d_n factors through $\operatorname{Ker}(d_{n-1})$ and the map $C_n \to \operatorname{Ker}(d_{n-1})$ factors through $\operatorname{Cok}(d_{n+1})$. Thus we get an induced homomorphism $\overline{d_n}\colon \operatorname{Cok}(d_{n+1}) \to \operatorname{Ker}(d_{n-1})$. The following picture describes the situation:

$$\operatorname{Cok}(d_{n+1}) \xrightarrow{\overline{d_n}} \operatorname{Ker}(d_{n-1})$$



So we obtain a commutative diagram

$$\begin{tikzcd}
C_n \arrow[r, "d_n"] \arrow[d] & C_{n-1} \\
\operatorname{Cok}(d_{n+1}) \arrow[r, "\overline{d_n}"] & \operatorname{Ker}(d_{n-1}) \arrow[u]
\end{tikzcd}$$

The kernel of $\overline{d_n}$ is just $H_n(C_\bullet)$ and its cokernel is $H_{n-1}(C_\bullet)$. Thus we obtain an exact sequence

$$0 \to H_n(C_\bullet) \xrightarrow{i_n^C} \operatorname{Cok}(d_{n+1}) \xrightarrow{\overline{d_n}} \operatorname{Ker}(d_{n-1}) \xrightarrow{p_{n-1}^C} H_{n-1}(C_\bullet) \to 0$$

where i_n^C and p_{n-1}^C denote the inclusion and the projection, respectively. The inclusion $\operatorname{Ker}(d_n^C) \to C_n$ is denoted by u_n^C.

2.5. **Homotopy of morphisms of complexes.** Let $C_\bullet = (C_n, d_n)$ and $C'_\bullet = (C'_n, d'_n)$ be complexes, and let $f_\bullet, g_\bullet \colon C_\bullet \to C'_\bullet$ be homomorphisms of complexes. Then $f_\bullet$ and $g_\bullet$ are called **homotopic** if for all $n \in \mathbb{Z}$ there exist homomorphisms $s_n \colon C_n \to C'_{n+1}$ such that

$$h_n := f_n - g_n = d'_{n+1} s_n + s_{n-1} d_n.$$

In this case we write $f_\bullet \sim g_\bullet$. (This defines an equivalence relation.) The sequence $s = (s_n)_n$ is a **homotopy** from $f_\bullet$ to $g_\bullet$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & C_n \arrow[r, "d_n"] \arrow[d, "h_n"'] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The morphism $f_\bullet \colon C_\bullet \to C'_\bullet$ is **zero homotopic** if $f_\bullet$ and the zero homomorphism $0 \colon C_\bullet \to C'_\bullet$ are homotopic. The class of zero homotopic homomorphisms forms an ideal in the category of complexes of A-modules.

Proposition 2.3. *If $f_\bullet, g_\bullet \colon C_\bullet \to C'_\bullet$ are homomorphisms of complexes such that $f_\bullet$ and $g_\bullet$ are homotopic, then $H_n(f_\bullet) = H_n(g_\bullet)$ for all $n \in \mathbb{Z}$.*

Proof. Let $C_\bullet = (C_n, d_n)$ and $C'_\bullet = (C'_n, d'_n)$, and let $x \in \mathrm{Ker}(d_n)$. We get

$$\begin{aligned} f_n(x) - g_n(x) &= (f_n - g_n)(x) \\ &= (d'_{n+1}s_n + s_{n-1}d_n)(x) \\ &= d'_{n+1}s_n(x) \end{aligned}$$

since $d_n(x) = 0$. This shows that $f_n(x)$ and $g_n(x)$ only differ by an element in $\mathrm{Im}(d'_{n+1})$. Thus they belong to the same residue class modulo $\mathrm{Im}(d'_{n+1})$. □

Corollary 2.4. *Let $f_\bullet \colon C_\bullet \to C'_\bullet$ be a homomorphism of complexes. Then the following hold:*

(i) *If $f_\bullet$ is zero homotopic, then $H_n(f_\bullet) = 0$ for all n;*
(ii) *If there exists a homomorphism $g_\bullet \colon C'_\bullet \to C_\bullet$ such that $g_\bullet f_\bullet \sim 1_{C_\bullet}$ and $f_\bullet g_\bullet \sim 1_{C'_\bullet}$, then $H_n(f_\bullet)$ is an isomorphism for all n.*

Proof. As in the proof of Proposition 2.3 we show that $f_n(x) \in \mathrm{Im}(d'_{n+1})$. This implies (i). We have $H_n(g_\bullet)H_n(f_\bullet) = H_n(g_\bullet f_\bullet) = H_n(1_{C_\bullet})$ and $H_n(f_\bullet)H_n(g_\bullet) = H_n(f_\bullet g_\bullet) = H_n(1'_{C_\bullet})$. Thus $H_n(f_\bullet)$ is an isomorphism. □

2.6. **The long exact homology sequence.** Let

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

be a short exact sequence of complexes. We would like to construct a homomorphism

$$\delta_n \colon H_n(C_\bullet) \to H_{n-1}(A_\bullet).$$

Recall that the elements in $H_n(C_\bullet)$ are residue classes of the form $x + \mathrm{Im}(d^C_{n+1})$ with $x \in \mathrm{Ker}(d^C_n)$. Here we write $A_\bullet = (A_n, d^A_n)$, $B_\bullet = (B_n, d^B_n)$ and $C_\bullet = (C_n, d^C_n)$.

For $x \in \mathrm{Ker}(d^C_n)$ set

$$\delta_n(x + \mathrm{Im}(d^C_{n+1})) := z + \mathrm{Im}(d^A_n)$$

where $z \in (f^{-1}_{n-1} d^B_n g^{-1}_n)(x)$.

Theorem 2.5 (Long Exact Homology Sequence)**.** *With the notation above, we obtain a well defined homomorphism*

$$\delta_n \colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$$

and the sequence

$$\cdots \xrightarrow{\delta_{n+1}} H_n(A_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(B_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(C_\bullet) \xrightarrow{\delta_n} H_{n-1}(A_\bullet) \xrightarrow{H_{n-1}(f_\bullet)} \cdots$$

is exact.

Proof. Taking kernels and cokernels of the maps d_n^A, d_n^B and d_n^B we obtain the following commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker}(d_n^A) \arrow[r, "f_n'"] \arrow[d] & \mathrm{Ker}(d_n^B) \arrow[r, "g_n'"] \arrow[d] & \mathrm{Ker}(d_n^C) \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "d_n^A"] & B_n \arrow[r, "g_n"] \arrow[d, "d_n^B"] & C_n \arrow[r] \arrow[d, "d_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Cok}(d_n^A) \arrow[r, "f_{n-1}''"] \arrow[d] & \mathrm{Cok}(d_n^B) \arrow[r, "g_{n-1}''"] \arrow[d] & \mathrm{Cok}(d_n^C) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

(The arrows without label are just the canonical inclusions and projections, respectively. By f_n', g_n' and f_{n-1}'', g_{n-1}'' we denote the induced homomorphisms on the kernels and cokernels of the maps d_n^A, d_n^B and d_n^C, respectively.)

The map f_n' is a restriction of the monomorphism f_n, thus f_n' is also a monomorphism. Since g_{n-1} is an epimorphism and $g_{n-1}(\mathrm{Im}(d_n^B)) \subseteq \mathrm{Im}(d_n^C)$, we know that g_{n-1}'' is an epimorphism as well.

We have seen above that the homomorphism $d_n^A\colon A_n \to A_{n-1}$ induces a homomorphism

$$a = \overline{d_n^A}\colon\ \mathrm{Cok}(d_{n+1}^A) \to \mathrm{Ker}(d_{n-1}^A).$$

Similarly, we obtain $b = \overline{d_n^B}$ and $c = \overline{d_n^C}$. The kernels and cokernels of these homomorphisms are homology groups. We obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n(A_\bullet) \arrow[r, "H_n(f_\bullet)"] \arrow[d, "i_n^A"] & H_n(B_\bullet) \arrow[r, "H_n(g_\bullet)"] \arrow[d, "i_n^B"] & H_n(C_\bullet) \arrow[d, "i_n^C"] & \\
 & \mathrm{Cok}(d_{n+1}^A) \arrow[r, "f_n''"] \arrow[d, "a"] & \mathrm{Cok}(d_{n+1}^B) \arrow[r, "g_n''"] \arrow[d, "b"] & \mathrm{Cok}(d_{n+1}^C) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d_{n-1}^A) \arrow[r, "f_{n-1}'"] \arrow[d, "p_{n-1}^A"] & \mathrm{Ker}(d_{n-1}^B) \arrow[r, "g_{n-1}'"] \arrow[d, "p_{n-1}^B"] & \mathrm{Ker}(d_{n-1}^C) \arrow[d, "p_{n-1}^C"] & \\
 & H_{n-1}(A_\bullet) \arrow[r, "H_{n-1}(f_\bullet)"] \arrow[d] & H_{n-1}(B_\bullet) \arrow[r, "H_{n-1}(g_\bullet)"] \arrow[d] & H_{n-1}(C_\bullet) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now we can apply the Snake Lemma: For our n we obtain a connecting homomorphism

$$\delta\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$$

which yields the required exact sequence. It remains to show that $\delta = \delta_n$.

Let T be the set of all triples (x, y, z) with $x \in \mathrm{Ker}(d_n^C)$, $y \in B_n$, $z \in A_{n-1}$ such that $g_n(y) = x$ and $f_{n-1}(z) = d_n^B(y)$.

(1) For every $x \in \mathrm{Ker}(d_n^C)$ there exists a triple $(x, y, z) \in T$:

Let $x \in \mathrm{Ker}(d_n^C)$. Since g_n is surjective, there exists some $y \in B_n$ with $g_n(y) = x$. We have

$$g_{n-1}d_n^B(y) = d_n^C g_n(y) = d_n^C(x) = 0.$$

Thus $d_n^B(y)$ belongs to the kernel of g_{n-1} and therefore to the image of f_{n-1}. Thus there exists some $z \in A_{n-1}$ with $f_{n-1}(z) = d_n^B(y)$.

(2) If $(x, y_1, z_1,), (x, y_2, z_2) \in T$, then $z_1 - z_2 \in \mathrm{Im}(d_n^A)$:

We have $g_n(y_1 - y_2) = x - x = 0$. Since $\mathrm{Ker}(g_n) = \mathrm{Im}(f_n)$ there exists some $a_n \in A_n$ such that $f_n(a_n) = y_1 - y_2$. It follows that

$$f_{n-1}d_n^A(a_n) = d_n^B f_n(a_n) = d_n^B(y_1 - y_2) = f_{n-1}(z_1 - z_2).$$

Since f_{n-1} is a monomorphism, we get $d_n^A(a_n) = z_1 - z_2$. Thus $z_1 - z_2 \in \mathrm{Im}(d_n^A)$.

(3) If $(x, y, z) \in T$ and $x \in \mathrm{Im}(d_{n+1}^C)$, then $z \in \mathrm{Im}(d_n^A)$:

Let $x = d_{n+1}^C(r)$ for some $r \in C_{n+1}$. Since g_{n+1} is surjective there exists some $s \in B_{n+1}$ with $g_{n+1}(s) = r$. We have

$$g_n(y) = x = d_{n+1}^C(r) = d_{n+1}^C g_{n+1}(s) = g_n d_{n+1}^B(s).$$

Therefore $y - d_{n+1}^B(s)$ is an element in $\mathrm{Ker}(g_n)$ and thus also in the image of f_n. Let $y - d_{n+1}^B(s) = f_n(t)$ for some $t \in A_n$. We get

$$f_{n-1}d_n^A(t) = d_n^B f_n(t) = d_n^B(y) - d_n^B d_{n+1}^B(s) = d_n^B(y) = f_{n-1}(z).$$

Since f_{n-1} is injective, this implies $d_n^A(t) = z$. Thus z is an element in $\mathrm{Im}(d_n^A)$.

(4) If $(x, y, z) \in T$, then $z \in \mathrm{Ker}(d_{n-1}^A)$:

We have

$$f_{n-2}d_{n-1}^A(z) = d_{n-1}^B f_{n-1}(z) = d_{n-1}^B d_n^B(y) = 0.$$

Since f_{n-2} is injective, we get $d_{n-1}^A(z) = 0$.

Combining (1),(2),(3) and (4) yields a homomorphism $\delta_n\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ defined by

$$\delta_n(x + \mathrm{Im}(d_{n+1}^C)) := z + \mathrm{Im}(d_n^A)$$

for each $(x, y, z) \in T$.

The set of all pairs $(p_n^C(x), p_{n-1}^A(z))$ such that there exists a triple $(x, y, z) \in T$ is given by the relation

$$\Gamma(p_{n-1}^A) \circ \Gamma(u_{n-1}^A)^{-1} \circ \Gamma(f_{n-1})^{-1} \circ \Gamma(d_n^B) \circ \Gamma(g_n)^{-1} \circ \Gamma(u_n^C) \circ \Gamma(p_n^C)^{-1}.$$

This is the graph of our homomorphism δ_n.

$$\begin{tikzcd}
 & & & & \mathrm{Ker}(d_n^C) \arrow[r, "p_n^C"] \arrow[d, "u_n^C"] & H_n(C_\bullet) \\
 & & & B_n \arrow[r, "g_n"] \arrow[d, "d_n^B"] & C_n & \\
H_{n-1}(A_\bullet) & \mathrm{Ker}(d_{n-1}^A) \arrow[l, "p_{n-1}^A"'] \arrow[r, "u_{n-1}^A"] & A_{n-1} \arrow[r, "f_{n-1}"] & B_{n-1} & &
\end{tikzcd}$$

Now it is not difficult to show that this relation coincides with the relation

$$\Gamma(p_{n-1}^A) \circ \Gamma(f'_{n-1})^{-1} \circ \Gamma(b) \circ \Gamma(g''_n)^{-1} \circ \Gamma(i_n^C)$$

which is the graph of δ.

$$\begin{tikzcd}
 & & H_n(C_\bullet) \arrow[d, "i_n^C"] \\
 & \mathrm{Cok}(d_{n+1}^B) \arrow[r, "g''_n"] \arrow[d, "b"] & \mathrm{Cok}(d_{n+1}^C) \\
\mathrm{Ker}(d_{n-1}^A) \arrow[r, "f'_{n-1}"] \arrow[d, "p_{n-1}^A"] & \mathrm{Ker}(d_{n-1}^B) & \\
H_{n-1}(A_\bullet) & &
\end{tikzcd}$$

This implies $\delta = \delta_n$. $\square$

The exact sequence in the above theorem is called the **long exact homology sequence** associated to the given short exact sequence of complexes. The homomorphisms δ_n are called **connecting homomorphisms**.

The connecting homomorphisms are "natural": Let

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "p_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] \arrow[d, "q_\bullet"] & C_\bullet \arrow[r] \arrow[d, "r_\bullet"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r, "f'_\bullet"] & B'_\bullet \arrow[r, "g'_\bullet"] & C'_\bullet \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows. Then the diagram

$$\begin{tikzcd} H_n(C_\bullet) \arrow[r, "\delta_n"] \arrow[d, "H_n(r_\bullet)"'] & H_{n-1}(A_\bullet) \arrow[d, "H_{n-1}(p_\bullet)"] \\ H_n(C'_\bullet) \arrow[r, "\delta'_n"] & H_{n-1}(A'_\bullet) \end{tikzcd}$$

commutes, where δ_n and δ'_n are the connecting homomorphisms coming from the two exact rows.

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3. Projective resolutions and extension groups

3.1. **Projective resolutions.** Let P_i, $i \geq 0$ be projective modules, and let M be an arbitrary module. Let $p_i\colon P_i \to P_{i-1}$, $i \geq 1$ and $\varepsilon\colon P_0 \to M$ be homomorphisms such that

$$\cdots \to P_{i+1} \xrightarrow{p_{i+1}} P_i \xrightarrow{p_i} \cdots \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

is an exact sequence. Then we call

$$P_\bullet := (\cdots \to P_{i+1} \xrightarrow{p_{i+1}} P_i \xrightarrow{p_i} \cdots \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0)$$

a **projective resolution** of M. We think of $P_\bullet$ as a complex of A-modules: Just set $P_i = 0$ and $p_{i+1} = 0$ for all $i < 0$.

Define

$$\Omega_{P_\bullet}(M) := \Omega^1_{P_\bullet}(M) := \operatorname{Ker}(\varepsilon),$$

and let $\Omega^i_{P_\bullet}(M) = \operatorname{Ker}(p_{i-1})$, $i \geq 2$. These are called the **syzygy modules** of M with respect to $P_\bullet$. Note that they depend on the chosen projective resolution.

End of Lecture 33

If all P_i are free modules, we call $P_\bullet$ a **free resolution** of M.

The resolution $P_\bullet$ is a **minimal projective resolution** of M if the homomorphisms $P_i \to \Omega^i_{P_\bullet}(M)$, $i \geq 1$ and also $\varepsilon\colon P_0 \to M$ are projective covers. In this case, we call

$$\Omega^n(M) := \Omega^n_{P_\bullet}(M)$$

the nth **syzygy module** of M. This does not depend on the chosen minimal projective resolution.

Lemma 3.1. *If*

$$0 \to U \to P \to M \to 0$$

is a short exact sequence of A-modules with P projective, then $U \cong \Omega(M) \oplus P'$ for some projective module P'.

Proof. **Exercise.** $\square$

Sometimes we are a bit sloppy when we deal with syzygy modules: If there exists a short exact sequence $0 \to U \to P \to M \to 0$ with P projective, we just write $\Omega(M) = U$, knowing that this is not at all well defined and depends on the choice of P.

Lemma 3.2. *For every module M there is a projective resolution.*

Proof. Define the modules P_i inductively. Let $\varepsilon = \varepsilon_0\colon P_0 \to M$ be an epimorphism with P_0 a projective module. Such an epimorphism exists, since every module is isomorphic to a factor module of a free module. Let $\mu_1\colon \operatorname{Ker}(\varepsilon_0) \to P_0$ be the inclusion. Let $\varepsilon_1\colon P_1 \to \operatorname{Ker}(\varepsilon_0)$ be an epimorphism with P_1 projective, and define $p_1 = \mu_1 \circ \varepsilon_1\colon P_1 \to P_0$. Now let $\varepsilon_2\colon P_2 \to \operatorname{Ker}(\varepsilon_1)$ be an epimorphism with P_2 projective, etc.

The first row of the resulting diagram

$$\begin{tikzcd}
\cdots \arrow[rr, "p_3"] & & P_2 \arrow[rr, "p_2"] \arrow[dr, "\varepsilon_2"'] & & P_1 \arrow[rr, "p_1"] \arrow[dr, "\varepsilon_1"'] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
& \cdots \arrow[ur, "\mu_3"] & & \operatorname{Ker}(\varepsilon_1) \arrow[ur, "\mu_2"] & & \operatorname{Ker}(\varepsilon_0) \arrow[ur, "\mu_1"] & & &
\end{tikzcd}$$

is exact, and we get a projective resolution

$$\cdots \xrightarrow{p_3} P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0$$

of $\operatorname{Cok}(p_1) = M$. $\square$

Theorem 3.3. *Given a diagram of homomorphisms with exact rows*

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "p_3'"] & P_2' \arrow[r, "p_2'"] & P_1' \arrow[r, "p_1'"] & P_0' \arrow[r, "\varepsilon'"] & N \arrow[r] & 0
\end{tikzcd}$$

where the P_i and P_i' are projective. Then the following hold:

(i) *There exists a "lifting" of f, i.e. there are homomorphisms $f_i\colon P_i \to P_i'$ such that*

$$p_i' f_i = f_{i-1} p_i \text{ and } \varepsilon' f_0 = f\varepsilon$$

for all i;

(ii) *Any two liftings $f_\bullet = (f_i)_{i\geq 0}$ and $f_\bullet' = (f_i')_{i\geq 0}$ are homotopic.*

Proof. (i): The map $\varepsilon'\colon P_0' \to N$ is an epimorphism, and the composition $f\varepsilon\colon P_0 \to N$ is a homomorphism starting in a projective module. Thus there exists a homomorphism $f_0\colon P_0 \to P_0'$ such that $\varepsilon' f_0 = f\varepsilon$.

We have $\mathrm{Im}(p_1) = \mathrm{Ker}(\varepsilon)$ and $\mathrm{Im}(p_1') = \mathrm{Ker}(\varepsilon')$. So we obtain a diagram with exact rows of the following form:

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & \mathrm{Im}(p_1) \arrow[r] \arrow[d, "\widetilde{f_0}"] & 0 \\
\cdots \arrow[r, "p_3'"] & P_2' \arrow[r, "p_2'"] & P_1' \arrow[r, "p_1'"] & \mathrm{Im}(p_1') \arrow[r] & 0
\end{tikzcd}$$

The homomorphism $\widetilde{f_0}$ is obtained from f_0 by restriction to $\mathrm{Im}(p_1)$. Since P_1 is projective, and since p_1' is an epimorphism there exists a homomorphism $f_1\colon P_1 \to P_1'$ such that $p_1' f_1 = \widetilde{f_0} p_1$, and this implies $p_1' f_1 = f_0 p_1$. Now we continue inductively to obtain the required lifting $(f_i)_{i\geq 0}$.

(ii): Assume we have two liftings, say $f_\bullet = (f_i)_{i\geq 0}$ and $f_\bullet' = (f_i')_{i\geq 0}$. This implies

$$f\varepsilon = \varepsilon' f_0 = \varepsilon' f_0'$$

and therefore $\varepsilon'(f_0 - f_0') = 0$.

Let $\iota_i\colon \mathrm{Im}(p_i') \to P_{i-1}'$ be the inclusion and let $\pi_i\colon P_i' \to \mathrm{Im}(p_i')$ be the obvious projection. Thus $p_i' = \iota_i \circ \pi_i$.

The image of $f_0 - f_0'$ clearly is contained in $\mathrm{Ker}(\varepsilon') = \mathrm{Im}(p_1')$. Now let $s_0'\colon P_0 \to \mathrm{Im}(p_1')$ be the map defined by $s_0'(m) = (f_0 - f_0')(m)$. The map π_1 is an epimorphism, and s_0' is a map from a projective module to $\mathrm{Im}(p_1')$. Thus by the projectivity of P_0 there exists a homomorphism $s_0\colon P_0 \to P_1'$ such that $\pi_1 \circ s_0 = s_0'$.

We obtain the following commutative diagram:

$$\begin{tikzcd}
 & & & P_0 \arrow[r, "\varepsilon"] \arrow[lll, "s_0"'] \arrow[lld, "s_0'"] \arrow[d, "f_0 - f_0'"] & M \arrow[d, "0"] \\
P_1' \arrow[r, "\pi_1"'] & \mathrm{Im}(p_1') \arrow[rr, "\iota_1"'] & & P_0' \arrow[r, "\varepsilon'"'] & N
\end{tikzcd}$$

Now assume $s_{i-1}\colon P_{i-1} \to P_i'$ is already defined such that

$$f_{i-1} - f_{i-1}' = p_i' s_{i-1} + s_{i-2} p_{i-1}.$$

We claim that $p'_i(f_i - f'_i - s_{i-1}p_i) = 0$: We have

$$\begin{aligned} p'_i(f_i - f'_i - s_{i-1}p_i) &= p'_i f_i - p'_i f'_i - p'_i s_{i-1} p_i \\ &= f_{i-1}p_i - f'_{i-1}p_i - p'_i s_{i-1}p_i \\ &= (f_{i-1} - f'_{i-1})p_i - p'_i s_{i-1} p_i \\ &= (p'_i s_{i-1} + s_{i-2}p_{i-1})p_i - p'_i s_{i-1} p_i \\ &= s_{i-2}p_{i-1}p_i \\ &= 0 \end{aligned}$$

(since $p_{i-1}p_i = 0$).

$$\begin{tikzcd} P_i \arrow[r, "p_i"] \arrow[d, "f_i"'] & P_{i-1} \arrow[r, "p_{i-1}"] \arrow[d, "f_{i-1}"] \arrow[dl, "s_{i-1}"] & P_{i-2} \arrow[d, "f_{i-2}"] \arrow[dl, "s_{i-2}"] \\ P'_i \arrow[r, "p'_i"'] & P'_{i-1} \arrow[r, "p'_{i-1}"'] & P'_{i-1} \end{tikzcd}$$

Therefore

$$\operatorname{Im}(f_i - f'_i - s_{i-1}p_i) \subseteq \operatorname{Ker}(p'_i) = \operatorname{Im}(p'_{i+1}).$$

Let $s'_i \colon P_i \to \operatorname{Im}(p'_{i+1})$ be defined by $s'_i(m) = (f_i - f'_i - s_{i-1}p_i)(m)$.

Since P_i is projective there exists a homomorphism $s_i \colon P_i \to P'_{i+1}$ such that $\pi_{i+1} \circ s_i = s'_i$. Thus we get a commutative diagram

$$\begin{tikzcd} & & P_i \arrow[dll, "s_i"'] \arrow[dl, "s'_i"] \arrow[d, "f_i - f'_i - s_{i-1}p_i"] \\ P'_{i+1} \arrow[r, "\pi_{i+1}"'] & \operatorname{Im}(p'_{i+1}) \arrow[r, "\iota_{i+1}"'] & P'_i \end{tikzcd}$$

Thus $p'_{i+1}s_i = f_i - f'_i - s_{i-1}p_i$ and therefore $f_i - f'_i = p'_{i+1}s_i + s_{i-1}p_i$, as required. This shows that $f_\bullet - f'_\bullet$ is zero homotopic. Therefore $f_\bullet = (f_i)_i$ and $f'_\bullet = (f'_i)_i$ are homotopic. $\square$

3.2. **Ext.** Let

$$P_\bullet = (\cdots \xrightarrow{p_{n+1}} P_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0)$$

be a projective resolution of $M = \operatorname{Cok}(p_1)$, and let N be any A-module. Define

$$\operatorname{Ext}^n_A(M, N) := H^n(\operatorname{Hom}_A(P_\bullet, N)),$$

the nth **cohomology group of extensions** of M and N. This definition does not depend on the projective resolution we started with:

Lemma 3.4. *If $P_\bullet$ and $P'_\bullet$ are projective resolutions of M, then for all modules N we have*

$$H^n(\operatorname{Hom}_A(P_\bullet, N)) \cong H^n(\operatorname{Hom}_A(P'_\bullet, N)).$$

Proof. Let $f_\bullet = (f_i)_{i \geq 0}$ and $g_\bullet = (g_i)_{i \geq 0}$ be liftings associated to

$$\begin{tikzcd} \cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\ \cdots \arrow[r, "p_3'"] & P_2' \arrow[r, "p_2'"] & P_1' \arrow[r, "p_1'"] & P_0' \arrow[r, "\varepsilon'"] & M \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} \cdots \arrow[r, "p_3'"] & P_2' \arrow[r, "p_2'"] & P_1' \arrow[r, "p_1'"] & P_0' \arrow[r, "\varepsilon'"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\ \cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0. \end{tikzcd}$$

By Theorem 3.3 these liftings exist and we have $g_\bullet f_\bullet \sim 1_{P_\bullet}$ and $f_\bullet g_\bullet \sim 1_{P'_\bullet}$. Thus, we get a diagram

$$\begin{tikzcd} \cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] \arrow[d, "g_2 f_2 - 1_{P_2}"] \arrow[dl, "s_2"'] & P_1 \arrow[r, "p_1"] \arrow[d, "g_1 f_1 - 1_{P_1}"] \arrow[dl, "s_1"'] & P_0 \arrow[d, "g_0 f_0 - 1_{P_0}"] \arrow[dl, "s_0"'] \\ \cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \end{tikzcd}$$

such that $g_i f_i - 1_{P_i} = p_{i+1} s_i + s_{i-1} p_i$ for all i. (Again we think of $P_\bullet$ as a complex with $P_i = 0$ for all $i < 0$.)

Next we apply $\mathrm{Hom}_A(-, N)$ to all maps in the previous diagram and get

$$\mathrm{Hom}_A(g_\bullet f_\bullet, N) \sim \mathrm{Hom}_A(1_{P_\bullet}, N).$$

Similarly, one can show that $\mathrm{Hom}_A(f_\bullet g_\bullet, N) \sim \mathrm{Hom}_A(1_{P'_\bullet}, N)$. Now Corollary 2.4 tells us that $H^n(\mathrm{Hom}_A(g_\bullet f_\bullet, N)) = H^n(\mathrm{Hom}_A(1_{P_\bullet}, N))$ and $H^n(\mathrm{Hom}_A(f_\bullet g_\bullet, N)) = H^n(\mathrm{Hom}_A(1_{P'_\bullet}, N))$. Thus

$$H^n(\mathrm{Hom}_A(f_\bullet, N))\colon H^n(\mathrm{Hom}_A(P'_\bullet, N)) \to H^n(\mathrm{Hom}_A(P_\bullet, N))$$

is an isomorphism. □

End of Lecture 34

3.3. **Induced maps between extension groups.** Let $P_\bullet$ be a projective resolution of a module M, and let $g\colon N \to N'$ be a homomorphism. Then we obtain an induced map

$$\mathrm{Ext}^n_A(M, g)\colon H^n(\mathrm{Hom}_A(P_\bullet, N)) \to H^n(\mathrm{Hom}_A(P_\bullet, N'))$$

defined by $[\alpha] \mapsto [g \circ \alpha]$. Here $\alpha\colon P_n \to N$ is a homomorphism with $\alpha \circ p_{n+1} = 0$.

There is also a contravariant version of this: Let $f\colon M \to M'$ be a homomorphism, and let $P_\bullet$ and $P'_\bullet$ be projective resolutions of M and M', respectively. Then for any module N we obtain an induced map

$$\mathrm{Ext}^n_A(f, N)\colon H^n(\mathrm{Hom}_A(P'_\bullet, N)) \to H^n(\mathrm{Hom}_A(P_\bullet, N))$$

defined by $[\beta] \mapsto [\beta \circ f_n]$. Here $\beta\colon P'_n \to N$ is a homomorphism with $\beta \circ p'_{n+1} = 0$ and $f_n\colon P_n \to P'_n$ is part of a lifting of f.

3.4. **Some properties of extension groups.** Obviously, we have $\mathrm{Ext}^n_A(M,N) = 0$ for all $n < 0$.

Lemma 3.5. $\mathrm{Ext}^0_A(M,N) = \mathrm{Hom}_A(M,N)$.

Proof. The sequence $P_1 \to P_0 \to M \to 0$ is exact. Applying $\mathrm{Hom}_A(-,N)$ yields an exact sequence

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(P_0,N) \xrightarrow{\mathrm{Hom}_A(p_1,N)} \mathrm{Hom}_A(P_1,N).$$

By definition $\mathrm{Ext}^0_A(M,N) = \mathrm{Ker}(\mathrm{Hom}_A(p_1,N)) = \mathrm{Hom}_A(M,N)$. □

Let M be a module and

$$0 \to \Omega(M) \xrightarrow{\mu_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

a short exact sequences with P_0 projective.

Lemma 3.6. $\mathrm{Ext}^1_A(M,N) \cong \mathrm{Hom}_A(\Omega(M),N)/\{s \circ \mu_1 \mid s\colon P_0 \to N\}$.

Proof. It is easy to check that $\mathrm{Hom}_A(\Omega(M),N) \cong \mathrm{Ker}(\mathrm{Hom}_A(p_2,N))$ and $\{s \circ \mu_1 \mid s\colon P_0 \to N\} \cong \mathrm{Im}(\mathrm{Hom}_A(p_1,N))$. □

Lemma 3.7. *For all $n \geq 1$ we have* $\mathrm{Ext}^{n+1}_A(M,N) \cong \mathrm{Ext}^n_A(\Omega M,N)$.

Proof. If $P_\bullet = (P_i,p_i)_{i\geq 0}$ is a projective resolution of M, then $\cdots P_3 \xrightarrow{p_3} P_2 \xrightarrow{p_2} P_1$ is a projective resolution of $\Omega(M)$. □

3.5. **Long exact Ext-sequences.** Let

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence of A-modules, and let M be any module and $P_\bullet$ a projective resolution of M. Then there exists an exact sequence of cocomplexes

$$0 \to \mathrm{Hom}_A(P_\bullet,X) \to \mathrm{Hom}_A(P_\bullet,Y) \to \mathrm{Hom}_A(P_\bullet,Z) \to 0.$$

This induces an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r] & \mathrm{Hom}_A(M,Y) \arrow[r] & \mathrm{Hom}_A(M,Z) \arrow[dll] \\
& \mathrm{Ext}^1_A(M,X) \arrow[r] & \mathrm{Ext}^1_A(M,Y) \arrow[r] & \mathrm{Ext}^1_A(M,Z) \arrow[dll] \\
& \mathrm{Ext}^2_A(M,X) \arrow[r] & \mathrm{Ext}^2_A(M,Y) \arrow[r] & \mathrm{Ext}^2_A(M,Z) \arrow[dll] \\
& \mathrm{Ext}^3_A(M,X) \arrow[r] & \cdots &
\end{tikzcd}$$

which is called a **long exact Ext-sequence**.

To obtain a "contravariant long exact Ext-sequence", we need the following result:

Lemma 3.8 (Horseshoe Lemma). *Let*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be a short exact sequence of A-module. Then there exists a short exact sequence of complexes

$$\eta\colon 0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

where $P'_\bullet$, $P_\bullet$ and $P''_\bullet$ are projective resolutions of X, Y and Z, respectively. We also have $P_\bullet \cong P'_\bullet \oplus P'_\bullet$.

Proof. ... □

Let N be any A-module. In the situation of the above lemma, we can apply $\mathrm{Hom}_A(-, N)$ to the exact sequence η. Since η splits, we obtain an exact sequence of cocomplexes

$$0 \to \mathrm{Hom}_A(P''_\bullet, N) \to \mathrm{Hom}_A(P_\bullet, N) \to \mathrm{Hom}_A(P'_\bullet, N) \to 0.$$

Thus we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(Z,N) \arrow[r] & \mathrm{Hom}_A(Y,N) \arrow[r] & \mathrm{Hom}_A(X,N) \arrow[dll] \\
& \mathrm{Ext}^1_A(Z,N) \arrow[r] & \mathrm{Ext}^1_A(Y,N) \arrow[r] & \mathrm{Ext}^1_A(X,N) \arrow[dll] \\
& \mathrm{Ext}^2_A(Z,N) \arrow[r] & \mathrm{Ext}^2_A(Y,N) \arrow[r] & \mathrm{Ext}^2_A(X,N) \arrow[dll] \\
& \mathrm{Ext}^3_A(Z,N) \arrow[r] & \cdots &
\end{tikzcd}$$

which is again called a **(contravariant) long exact Ext-sequence**.

3.6. **Short exact sequences and the first extension group.** Let M and N be modules, and let

$$P_\bullet = (\cdots \xrightarrow{p_{n+1}} P_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0)$$

be a projective resolution of $M = \mathrm{Cok}(p_1)$. Let $P_0 \xrightarrow{\varepsilon} M$ be the cokernel map of p_1, i.e.

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is an exact sequence.

We have

$$H^n(\mathrm{Hom}_A(P_\bullet, N)) := \mathrm{Ker}(\mathrm{Hom}_A(p_{n+1}, N))/\,\mathrm{Im}(\mathrm{Hom}_A(p_n, N)).$$

Let $[\alpha] := \alpha + \mathrm{Im}(\mathrm{Hom}_A(p_n, N))$ be the residue class of some homomorphism $\alpha\colon P_n \to N$ with $\alpha \circ p_{n+1} = 0$.

Clearly, we have

$$\mathrm{Im}(\mathrm{Hom}_A(p_n, N)) = \{s \circ p_n \mid s\colon P_{n-1} \to N\} \subseteq \mathrm{Hom}_A(P_n, N).$$

For an exact sequence

$$0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$$

let

$$\psi(f, g)$$

be the set of homomorphisms $\alpha\colon P_1 \to N$ such that there exists some $\beta\colon P_0 \to E$ with $f \circ \alpha = \beta \circ p_1$ and $g \circ \beta = \varepsilon$.

$$\begin{tikzcd}
P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, "\alpha"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\beta"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "f"] & E \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

Observe that $\psi(f, g) \subseteq \mathrm{Hom}_A(P_1, N)$.

Lemma 3.9. *The set $\psi(f, g)$ is a cohomology class, i.e. it is the residue class of some element $\alpha \in \mathrm{Ker}(\mathrm{Hom}_A(p_2, N))$ modulo $\mathrm{Im}(\mathrm{Hom}_A(p_1, N))$.*

Proof. **(a)**: If $\alpha \in \psi(f, g)$, then $\alpha \in \mathrm{Ker}(\mathrm{Hom}_A(p_2, N))$:

We have

$$f \circ \alpha \circ p_2 = \beta \circ p_1 \circ p_2 = 0.$$

Since f is a monomorphism, this implies $\alpha \circ p_2 = 0$.

(b): Next, let $\alpha, \alpha' \in \psi(f, g)$. We have to show that $\alpha - \alpha' \in \mathrm{Im}(\mathrm{Hom}_A(p_1, N))$:

There exist β and β' with $g \circ \beta = \varepsilon = g \circ \beta'$, $f \circ \alpha = \beta \circ p_1$ and $f \circ \alpha' = \beta' \circ p_1$. This implies $g(\beta - \beta') = 0$. Since P_0 is projective and $\mathrm{Im}(f) = \mathrm{Ker}(g)$, there exists some $s\colon P_0 \to N$ with $f \circ s = \beta - \beta'$. We get

$$f(\alpha - \alpha') = (\beta - \beta')p_1 = f \circ s \circ p_1.$$

Since f is a monomorphism, this implies $\alpha - \alpha' = s \circ p_1$. In other words, $\alpha - \alpha' \in \mathrm{Im}(\mathrm{Hom}_A(p_1, N))$.

(c): Again, let $\alpha \in \psi(f, g)$, and let $\gamma \in \mathrm{Im}(\mathrm{Hom}_A(p_1, N))$. We claim that $\alpha + \gamma \in \psi(f, g)$:

Clearly, $\gamma = s \circ p_1$ for some homomorphism $s\colon P_0 \to N$. There exists some β with $g \circ \beta = \varepsilon$ and $f \circ \alpha = \beta \circ p_1$. This implies

$$f(\alpha + \gamma) = \beta p_1 + f s p_1 = (\beta + f s)p_1.$$

Set $\beta' := \beta + fs$. We get

$$g\beta' = g(\beta + fs) = g\beta + gfs = g\beta = \varepsilon.$$

Here we used that $g \circ f = 0$. Thus $\alpha + \gamma \in \psi(f, g)$. $\square$

End of Lecture 35

Theorem 3.10. *The map*

$$\psi\colon \{0 \to N \to \star \to M \to 0\}/\!\sim \;\longrightarrow\; \mathrm{Ext}^1_A(M,N)$$
$$(f,g) \mapsto \psi(f,g)$$

defines a bijection between the set of equivalence classes of short exact sequences

$$0 \to N \xrightarrow{f} \star \xrightarrow{g} M \to 0$$

and $\mathrm{Ext}^1_A(M,N)$.

Proof. First we show that ψ is surjective: Let $\alpha\colon P_1 \to N$ be a homomorphism with $\alpha \circ p_2 = 0$. Let

$$P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

be a projective presentation of M. Set $\Omega(M) := \mathrm{Ker}(\varepsilon)$.

Thus $p_1 = \mu_1 \circ \varepsilon_1$ where $\varepsilon_1\colon P_1 \to \Omega(M)$ is the projection, and $\mu_1\colon \Omega(M) \to P_0$ is the inclusion. Since $\alpha \circ p_2 = 0$, there exists some $\alpha'\colon \Omega(M) \to N$ with $\alpha = \alpha' \circ \varepsilon_1$. Let $(f,g) := \alpha'_*(\mu_1, \varepsilon)$ be the short exact sequence induced by α'. Thus we have a commutative diagram

$$\begin{tikzcd}
P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, "\varepsilon_1"] \arrow[dd, bend right, "\alpha"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega(M) \arrow[r, "\mu_1"] \arrow[d, "\alpha'"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "f"] & E \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

This implies $\alpha \in \psi(f,g)$.

Next, we will show that ψ is injective: Assume that $\psi(f_1,g_1) = \psi(f_2,g_2)$ for two short exact sequence (f_1,g_1) and (f_2,g_2), and let $\alpha \in \psi(f_1,g_1)$. Let $\alpha'\colon \Omega(M) \to N$ and $\mu_1\colon \Omega(M) \to P_0$ be as before. the restriction of α to $\Omega(M)$ and let $p_1''\colon \Omega(M) \to P_0$ be the obvious inclusion.

We obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega(M) \arrow[r, "\mu_1"] \arrow[d, "\alpha'"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\beta_1"'] \arrow[dd, bend left, "\beta_2"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "f_1"] \arrow[d, equal] & E_1 \arrow[r, "g_1"] \arrow[d, dashed, "\gamma"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "f_2"] & E_2 \arrow[r, "g_2"] & M \arrow[r] & 0
\end{tikzcd}$$

with exact rows and where all squares made from solid arrows commute.

By the universal property of the pushout there is a homomorphism $\gamma\colon E_1 \to E_2$ with $\gamma \circ f_1 = f_2$ and $\gamma \circ \beta_1 = \beta_2$. Now as in the proof of **Skript 1, Lemma 10.10** we also get $g_2 \circ \gamma = g_1$. Thus the sequences (f_1, g_1) and (f_2, g_2) are equivalent. This finishes the proof. $\square$

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence, and let M and N be modules. Then the connecting homomorphism

$$\mathrm{Hom}_A(M, Z) \to \mathrm{Ext}^1_A(M, X)$$

is given by $h \mapsto [\eta]$ where η is the short exact sequence $h^*(f, g)$ induced by h via a pullback.

$$\begin{tikzcd}
\eta : & 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & \star \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, "h"] & 0 \\
& 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

Similarly, the connecting homomorphism

$$\mathrm{Hom}_A(X, N) \to \mathrm{Ext}^1_A(Z, N)$$

is given by $h \mapsto [\eta]$ and where η is the short exact sequence $h_*(f, g)$ induced by h via a pushout.

$$\begin{tikzcd}
& 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[r, "g"] \arrow[d, dashed] & Z \arrow[r] \arrow[d, equal] & 0 \\
\eta : & 0 \arrow[r] & N \arrow[r] & \star \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

If (f, g) is a split short exact sequence, then $\psi(f, g) = 0 + \mathrm{Im}(\mathrm{Hom}_A(p_1, N))$ is the zero element in $\mathrm{Ext}^1_A(M, N)$: Obviously, the diagram

$$\begin{tikzcd}
& P_1 \arrow[r, "p_1"] \arrow[d, dashed, "0"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\left[\begin{smallmatrix} 0 \\ \varepsilon \end{smallmatrix}\right]"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]"] & N \oplus M \arrow[r, "\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]"] & M \arrow[r] & 0
\end{tikzcd}$$

is commutative. This implies

$$\psi(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]) = 0 + \mathrm{Im}(\mathrm{Hom}_A(p_1, N)).$$

In fact, $\mathrm{Ext}^1_A(M, N)$ is a K-vector space and ψ is an isomorphism of K-vector spaces. So we obtain the following fact:

Lemma 3.11. *For an A-module M we have $\mathrm{Ext}^1_A(M, M) = 0$ if and only if each short exact sequence*

$$0 \to M \to E \to M \to 0$$

splits. In other words, $E \cong M \oplus M$.

3.7. **The vector space structure on the first extension group.** Let

$$\eta_M \colon 0 \to \Omega(M) \to P_0 \to M \to 0$$

be a short exact sequence with P_0 projective. For $i = 1, 2$ let

$$\eta_i \colon 0 \to N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \to 0$$

be short exact sequences.

Take the direct sum $\eta_1 \oplus \eta_2$ and construct the pullback along the diagonal embedding $M \to M \oplus M$. This yields a short exact sequence η'.

We know that every short exact sequence $0 \to X \to \star \to M \to 0$ is induced by η_M. Thus we get a homomorphism $\left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right] : \Omega(M) \to N \oplus N$ such that the diagram

$$\begin{tikzcd}
\eta_M : & 0 \arrow[r] & \Omega(M) \arrow[r, "u"] \arrow[d, "\left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right]"] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta' : & 0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]"] & 0 \\
\eta_1 \oplus \eta_2 : & 0 \arrow[r] & N \oplus N \arrow[r, "\left[\begin{smallmatrix}f_1 & 0\\ 0 & f_2\end{smallmatrix}\right]"'] & E_1 \oplus E_2 \arrow[r, "\left[\begin{smallmatrix}g_1 & 0\\ 0 & g_2\end{smallmatrix}\right]"'] & M \oplus M \arrow[r] & 0
\end{tikzcd}$$

commutes. Taking the pushout of η' along $[1, 1] \colon N \oplus N \to N$ we get the following commutative diagram:

$$\begin{tikzcd}
\eta_M : & 0 \arrow[r] & \Omega(M) \arrow[r, "u"] \arrow[d, "\left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right]"] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta' : & 0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "{[1,1]}"] & E' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta'' : & 0 \arrow[r] & N \arrow[r] & E'' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

In other words,

$$\eta' = \left[\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right]_*(\eta_M),$$
$$\eta'' = [1, 1]_*(\eta').$$

This implies $\eta'' = (\alpha_1 + \alpha_2)_*(\eta_M)$. Define

$$\eta_1 + \eta_2 := \eta''.$$

Note that there exists some β_i, $i = 1, 2$ such that the diagram

$$\begin{tikzcd}
\eta_M : & 0 \arrow[r] & \Omega(M) \arrow[r, "u"] \arrow[d, "\alpha_i"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\beta_i"] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta_i : & 0 \arrow[r] & N \arrow[r, "f_i"] & E_i \arrow[r, "g_i"] & M \arrow[r] & 0
\end{tikzcd}$$

commutes. Thus $\eta_i = (\alpha_i)_*(\eta_M)$.

Similarly, let $\eta\colon 0 \to N \to E \to M \to 0$ be a short exact sequence. For $\lambda \in K$ let Let $\eta' := (\lambda\cdot)_*(\eta)$ be the short exact sequence induced by the multiplication map with λ. We also know that there exists some $\alpha\colon \Omega(M) \to N$ which induces η. Thus we obtain a commutative diagram

$$\begin{tikzcd}
\eta_M : & 0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d, "\alpha"] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta : & 0 \arrow[r] & N \arrow[r] \arrow[d, "\lambda\cdot"] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta' : & 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Define $\lambda\eta := \eta'$.

Thus, we defined an addition and a scalar multiplication on the set of equivalence classes of short exact sequences. We leave it as an (easy) exercice to show that this really defines a K-vector space structure on $\mathrm{Ext}^1_A(M, N)$.

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4. **Injective modules**

A module I is called **injective** if the following is satisfied: For any monomorphism $f\colon X \to Y$, and any homomorphism $h\colon X \to I$ there exists a homomorphism $g\colon Y \to I$ such that $gf = h$.

$$\begin{tikzcd}
 & Y \arrow[dl, dashed, "g"'] \\
I & X \arrow[l, "h"] \arrow[u, "f"']
\end{tikzcd}$$

Lemma 4.1. *The following are equivalent:*

(i) *I is injective;*
(ii) *The functor $\mathrm{Hom}_A(-, I)$ is exact;*
(iii) *Every monomorphism $I \to N$ splits;*
(iv) *For all A-modules M we have $\mathrm{Ext}^1_A(M, I) = 0$.*

Proof. (i) $\iff$ (ii): By (i) we know that for all monomorphisms $f\colon X \to Y$ the map $\mathrm{Hom}_A(f, I)\colon \mathrm{Hom}_A(Y, I) \to \mathrm{Hom}_A(X, I)$ is surjective. This implies that $\mathrm{Hom}_A(-, I)$ is an exact contravariant functor. The converse is also true.

(i) $\Longrightarrow$ (iii): Let $f\colon I \to N$ be a monomorphism. Thus there exists some $g\colon N \to I$ such that the diagram

$$\begin{tikzcd} & N \arrow[dl, dashed, "g"'] \\ I & I \arrow[l, "1_I"] \arrow[u, "f"'] \end{tikzcd}$$

commutes. Thus f is a split monomorphism.

(iii) $\Longrightarrow$ (i): Let $f\colon X \to Y$ be a monomorphism, and let $h\colon X \to I$ be an arbitrary homomorphism. Taking the pushout along h we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[r] \arrow[d, "h'"] & \mathrm{Cok}(f) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r, "f'"] & E \arrow[r] & \mathrm{Cok}(f) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. By (iii) we know that f' is a split monomorphism. Thus there exists some $f''\colon E \to I$ with $f'' \circ f' = 1_I$. Observe that $\mathrm{Im}(h' \circ f) \subseteq \mathrm{Im}(f')$. Set $g := f'' \circ h'$. This implies $g \circ f = h$. In other words, I is injective.

(iii) $\Longleftrightarrow$ (iv): We have $\mathrm{Ext}_A^1(X, I) = 0$ if and only if each short exact sequence $0 \to I \to E \to X \to 0$ splits. This is obviously equivalent to (iii). $\square$

Lemma 4.2. *For an algebra A the following are equivalent:*

- (i) *A is semisimple;*
- (ii) *Every A-module is projective;*
- (iii) *Every A-module is injective.*

Proof. Recall that A is semisimple if and only if all A-modules are semisimple. A module M is semisimple if and only if every submodule of M is a direct summand. Thus A is semisimple if and only if each short exact sequence

$$0 \to X \to Y \to Z \to 0$$

of A-modules splits. Now the lemma follows from the basic properties of projective and injective modules. $\square$

For any left A-module ${}_AM$ let $D({}_AM) = \mathrm{Hom}_K({}_AM, K)$ be the **dual module** of ${}_AM$. This is a right A-module, or equivalently, a left A^{op}-module: For $\alpha \in D({}_AM)$, $a \in A^{\mathrm{op}}$ and $x \in {}_AM$ define $(a\alpha)(x) := \alpha(ax)$. It follows that $((ab)\alpha)(x)\alpha(abx) = (a\alpha)(bx) = (b(a\alpha))(x)$. Thus $(b \star a)\alpha = (ab)\alpha = b(a\alpha)$ for all $x \in M$ and $a, b \in A$.

Similarly, let M_A now be a right A-module. Then $D(M_A)$ becomes an A-module as follows: For $\alpha \in D(M_A)$ and $a \in A$ set $(a\alpha)(x) := \alpha(xa)$. Thus we have $((ab)\alpha)(x) = \alpha(xab) = (b\alpha)(xa) = (a(b\alpha))(x)$ for all $x \in M$ and $a, b \in A$.

Lemma 4.3. *The A-module $\mathrm{D}(A_A) = \mathrm{D}({}_{A^{\mathrm{op}}}A)$ is injective.*

Proof. Let $f\colon X \to Y$ be a monomorphism of A-modules, and let

$$e\colon \operatorname{Hom}_K(A_A, K) \to K$$

be the map defined by $\alpha \mapsto \alpha(1)$. Clearly, e is K-linear, but in general it will not be A-linear. Let $h\colon X \to \operatorname{Hom}_K(A_A, K)$ be a homomorphism of A-modules.

Let us now just think of K-linear maps: There exists a K-linear map $e'\colon Y \to K$ such that $e' \circ f = e \circ h$. Define a map $h'\colon Y \to \operatorname{Hom}_K(A_A, K)$ by $h'(y)(a) := e'(ay)$ for all $y \in Y$ and $a \in A$.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"] & \operatorname{Hom}_A(A_A, K) \arrow[r, "e"] & K \\
Y \arrow[ur, dashed, "h'"] \arrow[urr, "e'"'] & &
\end{tikzcd}$$

It is easy to see that h' is K-linear. We want to show that h' is A-linear. (In other words, h' is a homomorphism of A-modules.)

For $y \in Y$ and $a, b \in A$ we have $h'(by)(a) = e'(aby)$. Furthermore, $(bh'(y))(a) = h'(y)(ab) = e'(aby)$. This finishes the proof. □

Lemma 4.4. *There are natural isomorphisms*

$$\operatorname{Hom}_A\left(-, \prod_{i\in I} M_i\right) \cong \prod_{i\in I} \operatorname{Hom}_A(-, M_i)$$

and

$$\operatorname{Hom}_A\left(\bigoplus_{i\in I} M_i, -\right) \cong \prod_{i\in I} \operatorname{Hom}_A(M_i, -).$$

Proof. **Exercise.** □

Lemma 4.5. *The following hold:*

(i) *Direct summands of injective modules are injective;*
(ii) *Direct products of injective modules are injective;*
(iii) *Finite direct sums of injective modules are injective.*

Proof. Let $I = I_1 \oplus I_2$ be a direct sum decomposition of an injective A-module I, and let $f\colon X \to Y$ be a monomorphism. If $h\colon X \to I_1$ is a homomorphism, then $\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right] : X \to I_1 \oplus I_2$ is a homomorphism, and since I is injective, we get some $g = \left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right] : Y \to I_1 \oplus I_2$ such that

$$g \circ f = \left[\begin{smallmatrix} g_1 f \\ g_2 f \end{smallmatrix}\right] \circ f = \left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right].$$

Thus $g_1 \circ f = h$ and therefore I_1 is injective. This proves (i).

Let I_i, $i \in I$ be injective A-modules, let $f\colon X \to Y$ be a monomorphism, and suppose that $h\colon X \to \prod_{i\in I} I_i$ is any homomorphism. Clearly, $h = (h_i)_{i\in I}$ where h_i is obtained by composing h with the obvious projection $\prod_{i\in I} I_i \to I_i$. Since

I_i is injective, there exists a homomorphism $g_i \colon Y \to I_i$ with $g_i \circ f = h_i$. Set $g := (g_i)_{i \in I} \colon Y \to \prod_{i \in I} I_i$. It follows that $g \circ f = h$. This proves (ii).

The statement (iii) follows obviously form (ii). □

Warning: Infinite direct sums of injective modules are often not injective. The reason is that in general we have

$$\bigoplus_{i \in I} \mathrm{Hom}_A(-, M_i) \ncong \bigoplus_{i \in I} \mathrm{Hom}_A(M_i, -) \ncong \mathrm{Hom}_A\left(\bigoplus_{i \in I} M_i, -\right).$$

Lemma 4.6. *If P_A is a projective A^{op}-module, then $D(P_A)$ is an injective A-module.*

Proof. First assume that $P_A = \bigoplus_{i \in I} A_A$ is a free A^{op}-module. We know already by Lemma 4.3 that $D(A_A)$ is an injective A-module. By Lemma 4.4 we have

$$D(P_A) = \mathrm{Hom}_K\left(\bigoplus_{i \in I} A_A, K\right) \cong \prod_{i \in I} \mathrm{Hom}_K(A_A, K) = \prod_{i \in I} D(A_A).$$

Now Lemma 4.5 (ii) implies that $D(P_A)$ is projective. Any projective module is a direct summand of a free module. Thus Lemma 4.5 (i) yields that $D(P_A)$ is an injective A-module for all projective A^{op}-module P_A. □

Lemma 4.7. *Every A-module can be embedded into an injective A-module.*

Proof. Let ${}_AM$ be an A-module. There exists a projective A^{op}-module P_A and an epimorphism $P_A \to D({}_AM)$. Applying the duality $D = \mathrm{Hom}_K(-, K)$ gives a monomorphism $DD({}_AM) \to D(P_A)$. Lemma 4.6 says that $D(P_A)$ is an injective A-module. It is also clear that there exists a monomorphism ${}_AM \to DD({}_AM)$. This finishes the proof. □

One can now define injective resolutions, and develop Homological Algebra with injective instead of projective modules.

Recall that a submodule U of a module M is called **large** if for any non-zero submodule V of M the intersection $U \cap V$ is non-zero.

A homomorphism $f \colon M \to I$ is called an **injective envelope** if the following hold:

(i) I is injective;
(ii) f is a monomorphism;
(iii) $f(M)$ is a large submodule of I.

End of Lecture 36

Lemma 4.8. *Let U_1 and U_2 be large submodules of M_1 and M_2, respectively. Then $U_1 \oplus U_2$ is large in $M_1 \oplus M_2$.*

Proof. Let W be a non-zero submodule of $M_1 \oplus M_2$. For $i = 1, 2$ let $\pi_i \colon M_1 \oplus M_2 \to M_i$ be the obvious projection. Without loss of generality assume $\pi_1(W) \neq 0$. Since U_1 is large in M_1 and $\pi_1(W)$ is a non-zero submodule of M_1, there exists some $w = (w_1, w_2) \in W$ with $w_1 \neq 0$ and $w_1 \in U_1$. If $w_2 = 0$, then $w \in (U_1 \oplus U_2) \cap W$. If $w_2 \neq 0$, then we look at the submodule Aw_2 of U_2. Again there has to be some $a \in A$ with $0 \neq aw_2 \in U_2$. This implies $0 \neq (aw_1, aw_2) \in (U_1 \oplus U_2) \cap W$. □

Lemma 4.9. *Let I be an injective module, and let U and V be submodules of I such that $U \cap V = 0$. Assume that U and V are maximal with this property (i.e. if $U \subseteq U'$ with $U' \cap V = 0$, then $U = U'$, and if $V \subseteq V'$ with $U \cap V' = 0$, then $V = V'$). Then $I = U \oplus V$.*

Proof. It is easy to check that the map
$$f \colon I \to I/U \oplus I/V$$
defined by $m \mapsto (m + U, m + V)$ is a monomorphism: Namely, $m \in \mathrm{Ker}(f)$ implies $m \in U \cap V = 0$.

There is an embedding $(U + V)/U \to I/U$. We claim that $(U + V)/U$ is large in I/U: Let U'/U be a submodule of I/U (thus $U \subseteq U' \subseteq I$) with
$$(U + V)/U \cap (U'/U) = 0 = U/U.$$
In other words, $(U + V) \cap U' = U + (V \cap U') = U$. This implies $(V \cap U') \subseteq U$ and (obviously) $(V \cap U') \subseteq V$. Thus $V \cap U' = 0$. By the maximality of U we get $U = U'$ and therefore $U'/U = 0$.

Similarly one shows that $(U + V)/V$ is a large submodule of I/V.

We get
$$(U + V)/U \oplus (U + V)/V \cong V \oplus U \subseteq M \subseteq M/U \oplus M/V.$$
By Lemma 4.8 we know that M is large in $M/U \oplus M/V$. But M is injective and therefore a direct summand of $M/U \oplus M/V$. Thus $M \oplus C = M/U \oplus M/V$ for some C. Since M is large, we get $C = 0$. So $M = M/U \oplus M/V$. By the maximality of U and V we get $V = M/U$ and $U = M/V$ and therefore $U \oplus V = M$. □

The dual statement for projective modules is also true:

Lemma 4.10. *Let P be a projective module, and let U and V be submodules of P such that $U + V = P$. Assume that U and V are minimal with this property (i.e. if $U' \subseteq U$ with $U' + V = P$, then $U = U'$, and if $V' \subseteq V$ with $U + V' = P$, then $V = V'$). Then $P = U \oplus V$.*

Lemma 4.11. *Let U be a submodule of a module M. Then there exists a submodule V of M which is maximal with the property $U \cap V = 0$.*

Proof. Let
$$\mathcal{V} := \{W \subseteq M \mid U \cap W = 0\}.$$

Take a chain $(V_i)_{i\in J}$ in $\mathcal{V}$. (Thus for all V_i and V_j we have $V_i \subseteq V_j$ or $V_j \subseteq V_i$.) Set $V = \bigcup_i V_i$. We get

$$U \cap V = U \cap \left(\bigcup_i V_i \right) = \bigcup_i (U \cap V_i) = 0.$$

Now the claim follows from Zorn's Lemma. □

Warning: For a submodule U of a module M there does not necessarily exist a minimal V such that $U + V = M$.

Example: Let $M = K[T]$ and $U = (T)$. Then for each $n \geq 1$ we have $(T) + (T + 1)^n = M$.

Theorem 4.12. *Every A-module has an injective envelope.*

Proof. Let X be an A-module, and let $X \to I$ be a monomorphism with I injective. Let V be a submodule of I with $X \cap V = 0$ and we assume that V is maximal with this property. Such a V exists by the previous lemma.

Next, let

$$\mathcal{U} := \{U \subseteq I \mid U \cap V = 0 \text{ and } X \subseteq U\}.$$

Again, by Zorn's Lemma we obtain a submodule U of I which is maximal with $U \cap V = 0$ and $X \subseteq U$.

Thus, U and V are as in the assumptions of the previous lemma, and we obtain $I = U \oplus V$ and $X \subseteq U$. We know that U is injective, and we have our embedding $X \to U$.

We claim that X is a large submodule of U:

Let U' be a submodule of U with $X \cap U' = 0$. We have to show that $U' = 0$. We have $X \cap (U' \oplus V) = 0$: If $x = u' + v$, then $x - u' = v$ and therefore $v = 0$. Thus $x = u' \in X \cap U' = 0$. By the maximality of V we have $U' \oplus V = V$. Thus $U' = 0$. □

Warning: Projective covers do not exist in general.

If X is an A-module, we denote its injective hull by $I(X)$.

Lemma 4.13. *Injective envelopes are uniquely determined up to isomorphism.*

Proof. **Exercise.** □

Recall that a module M is **uniform**, if for all non-zero submodules U and V of M we have $U \cap V \neq 0$.

Lemma 4.14. *Let I be an indecomposable injective A-module. Then the following hold:*

(i) *I is uniform (i.e. if U and V are non-zero submodules of I, then $U \cap V \neq 0$);*

(ii) *Each injective endomorphism of I is an automorphism;*
(iii) *If $f, g \in \mathrm{End}_A(I)$ are both not invertible, then $f + g$ is not invertible;*
(iv) $\mathrm{End}_A(I)$ *is a local ring.*

Proof. (i): Let U and V be non-zero submodules of I. Assume $U \cap V = 0$. Let U' and V' be submodules which are maximal with the properties $U \subseteq U'$, $V \subseteq V'$ and $U' \cap V' = 0$. Lemma 4.9 implies that $I = U' \oplus V'$. But I is indecomposable, a contradiction.

(ii): Let $f \colon I \to I$ be an injective homomorphism. Since I is injective, f is a split monomorphism. Thus $I = f(I) \oplus \mathrm{Cok}(f)$. Since I is indecomposable and $f(I) \neq 0$, we get $\mathrm{Cok}(f) = 0$. Thus f is also surjective and therefore an automorphism.

(iii): Let f and g be non-invertible elements in $\mathrm{End}_A(I)$. by (ii) we know that f and g are not injective. Thus $Ker(f) \neq 0 \neq \mathrm{Ker}(g)$. By (i) we get $\mathrm{Ker}(f) \cap \mathrm{Ker}(g) \neq 0$. This implies $Ker(f + g) \neq 0$.

We know already from the theory of local rings that (iii) and (iv) are equivalent statements. □

injective resolution

...

minimal injective resolution

...

Theorem 4.15. *Let $I^\bullet$ be an injective resolution of an A-module N. Then for any A-module M we have an isomorphism*

$$\mathrm{Ext}^1_A(M, N) \cong H^n(\mathrm{Hom}_A(M, I^\bullet)).$$

which is "natural in M and N".

Proof. **Exercise.** □

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5. **Digression: Homological dimensions**

5.1. **Projective, injective and global dimension.** Let A be a K-algebra. For an A-module M let $\mathrm{proj.\,dim}(M)$ be the minimal $j \geq 0$ such that there exists a projective resolution $(P_i, d_i)_i$ of M with $P_j = 0$, if such a minimum exists, and define $\mathrm{proj.\,dim}(M) = \infty$, otherwise.

We call $\operatorname{proj.dim}(M)$ the **projective dimension** of M. The **global dimension** of A is by definition

$$\operatorname{gl.dim}(A) = \sup\{\operatorname{proj.dim}(M) \mid M \in \operatorname{mod}(A)\}.$$

Here sup denote the supremum of a set.

It often happens that the global dimension of an algebra A is infinite, for example if we take $A = K[X]/(X^2)$. One proves this by constructing the minimal projective resolution of the simple A-module S. Inductively one shows that $\Omega^i(S) \cong S$ for all $i \geq 1$.

Proposition 5.1. *Assume that A is finite-dimensional. Then we have*

$$\operatorname{gl.dim}(A) = \max\{\operatorname{proj.dim}(S) \mid S \text{ a simple } A\text{-module}\}.$$

Proof. Use the Horseshoe Lemma. □

Similarly, let $\operatorname{inj.dim}(M)$ be the minimal $j \geq 0$ such that there exists an injective resolution $(I_i, d_i)_i$ of M with $I_j = 0$, if such a minimum exists, and define $\operatorname{inj.dim}(M) = \infty$, otherwise.

We call $\operatorname{inj.dim}(M)$ the **injective dimension** of M.

Theorem 5.2 (No loop conjecture)**.** *Let A be a finite-dimensional K-algebra. If* $\operatorname{Ext}^1_A(S,S) \neq 0$ *for some simple A-module S, then* $\operatorname{gl.dim}(A) = \infty$*.*

Conjecture 5.3 (Strong no loop conjecture)**.** *Let A be a finite-dimensional K-algebra. If* $\operatorname{Ext}^1_A(S,S) \neq 0$ *for some simple A-module S, then* $\operatorname{proj.dim}(S) = \infty$*.*

5.2. **Hereditary algebras.** A K-algebra A is **hereditary** if $\operatorname{gl.dim}(A) \leq 1$.

5.3. **Selfinjective algebras.**

5.4. **Finitistic dimension.** For an algebra A let

$$\operatorname{fin.dim}(A) := \sup\{\operatorname{proj.dim}(M) \mid M \in \operatorname{mod}(A), \operatorname{proj.dim}(M) < \infty\}$$

be the **finitistic dimension** of A. The following conjecture is unsolved for several decades and remains wide open:

Conjecture 5.4 (Finitistic dimension conjecture)**.** *If A is finite-dimensional, then* $\operatorname{fin.dim}(A) < \infty$*.*

5.5. **Representation dimension.** The **representation dimension** of a finite-dimensional K-algebra A is the infimum over all $\operatorname{gl.dim}(C)$ where C is a generator-cogenerator of A, i.e. each indecomposable projective module and each indecomposable injective module occurs (up to isomorphism) as a direct summand of C.

Theorem 5.5 (Auslander)**.** *For a finite-dimensional K-algebra A the following hold:*

(i) $\operatorname{rep.dim}(A) = 0$ *if and only if A is semisimple;*

(ii) $\mathrm{rep.dim}(A) \neq 1$;
(iii) $\mathrm{rep.dim}(A) = 2$ *if and only if A is representation-finite, but not semisimple.*

Theorem 5.6 (Iyama). *If A is a finite-dimensional algebra, then* $\mathrm{rep.dim}(A) < \infty$.

Theorem 5.7 (Rouquier). *For each $n \geq 3$ there exists a finite-dimensional algebra A with* $\mathrm{rep.dim}(A) = n$.

5.6. **Dominant dimension. dominant dimension** of A

5.7. **Auslander algebras.** Let A be a finite-dimensional representation-finite K-algebra. The **Auslander algebra** of A is defined as $\mathrm{End}_A(M)$ where M is the direct sum of a complete set of representatives of isomorphism classes of the indecomposable A-modules.

Theorem 5.8 (Auslander). *...*

5.8. **Gorenstein algebras.**

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6. Tensor products, adjunction formulas and Tor-functors

6.1. **Tensor products of modules.** Let A be a K-algebra. Let X be an A^{op}-module, and let Y be an A-module. Recall that X can be seen as a right A-module as well. For $x \in X$ and $a \in A$ we denote the action of A^{op} and A on X by $a \star x = x \cdot a$.

By $V(X, Y)$ we denote a K-vector space with basis
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Let $R(X, Y)$ be the subspace of $V(X, Y)$ which is generated by all vectors of the form

(1) $((x + x'), y) - (x, y) - (x', y)$,
(2) $(x, (y + y')) - (x, y) - (x, y')$,
(3) $(xa, y) - (x, ay)$,
(4) $\lambda(x, y) - (\lambda x, y)$.

where $x \in X$, $y \in Y$, $a \in A$ and $\lambda \in K$. The vector space
$$X \otimes_A Y := V(X, Y)/R(X, Y)$$
is the **tensor product** of X_A and ${}_AY$. The elements z in $X \otimes_A Y$ are of the form
$$\sum_{i=1}^{m} x_i \otimes y_i,$$
where $x \otimes y := (x, y) + R(X, Y)$. But note that this expression of z is in general not unique.

End of Lecture 37

Warning

From here on there are only fragments, incomplete proofs or no proofs at all, typos, wrong statements and other horrible things...

A map $\beta\colon X \times Y \to V$ where V is a vector space is called **balanced** if for all $x, x' \in X$, $y, y' \in Y$, $a \in A$ and $\lambda \in K$ the following hold:

(1) $\beta(x + x', y) = \beta(x, y) + \beta(x', y)$,
(2) $\beta(x, y + y') = \beta(x, y) + \beta(x, y')$,
(3) $\beta(xa, y) = \beta(x, ay)$,
(4) $\beta(\lambda x, y) = \lambda\beta(x, y)$.

In particular, a balanced map is K-bilinear.

For example, the map
$$\omega\colon X \times Y \to X \otimes_A Y$$
defined by $(x, y) \mapsto x \otimes y$ is balanced. This map has the following universal property:

Lemma 6.1. *For each balanced map $\beta\colon X \times Y \to V$ there exists a unique K-linear map $\gamma\colon X \otimes_A Y \to V$ with $\beta = \gamma \circ \omega$.*

$$\begin{tikzcd} X \times Y \arrow[r, "\omega"] \arrow[d, "\beta"'] & X \otimes_A Y \arrow[dl, dashed, "\gamma"] \\ V & \end{tikzcd}$$

Furthermore, this property characterizes $X \otimes_A Y$ up to isomorphism.

Proof. We can extend β and ω (uniquely) to K-linear maps $\beta'\colon V(X, Y) \to V$ and $\omega'\colon V(X, Y) \to X \otimes_A Y$, respectively. We have $R(X, Y) \subseteq \operatorname{Ker}(\beta')$, since β is balanced. Let $\iota\colon R(X, Y) \to \operatorname{Ker}(\beta')$ be the inclusion map. Now it follows easily that there is a unique K-linear map $\gamma\colon X \otimes_A Y \to V$ with $\beta = \gamma \circ \omega$ and $\beta' = \gamma \circ \omega'$.

$$\begin{tikzcd} 0 \arrow[r] & R(X,Y) \arrow[r] \arrow[d, "\iota"] & V(X,Y) \arrow[r, "\omega'"] \arrow[d, equal] & X \otimes_A Y \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\ 0 \arrow[r] & \operatorname{Ker}(\beta') \arrow[r] & V(X,Y) \arrow[r, "\beta'"] & V \arrow[r] & 0 \end{tikzcd}$$

□

Let A, B, C be K-algebras, and let ${}_AX_B$ be an A-B^{op}-bimodule and ${}_BY_C$ a B-C^{op}-bimodule. We claim that $X \otimes_B Y$ is an A-C^{op}-bimodule with the bimodule structure defined by

$$a(x \otimes y) = (ax) \otimes y,$$
$$(x \otimes y)c = x \otimes (yc)$$

where $a \in A$, $c \in C$ and $x \otimes y \in X \otimes_B Y$: One has to check that everything is well defined. It is clear that we obtain an A-module structure and a C^{op}-module structure. Furthermore, we have

$$(a(x \otimes y))c = ((ax) \otimes y)c = (ax) \otimes (yc) = a((x \otimes y)c).$$

Thus we get a bimodule structure on $X \otimes_B Y$.

Lemma 6.2. *For any A-module M, we have*

$$_AA_A \otimes_A M \cong M$$

as A-modules.

Proof. The A-module homomorphisms $\eta \colon A \otimes_A M \to M$, $a \otimes m \mapsto am$ and $\phi \colon M \to A \otimes_A M$, $m \mapsto 1 \otimes m$ are mutual inverses. □

Let $f \colon X_A \to X'_A$ and $g \colon {}_AY \to {}_AY'$ be homomorphisms. Then the map $\beta \colon X \times Y \to X' \otimes_A Y'$ defined by $(x, y) \mapsto f(x) \otimes g(y)$ is balanced. Thus there exists a unique K-linear map

$$f \otimes g \colon X \otimes_A Y \to X' \otimes_A Y'$$

with $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$.

$$\begin{tikzcd} X \times Y \arrow[r, "\omega"] \arrow[d, "\beta"] & X \otimes_A Y \arrow[dl, dashed, "f \otimes g"] \\ X' \otimes_A Y' & \end{tikzcd}$$

Now let $f = 1_X$, and let g be as above. We obtain a K-linear map

$$X \otimes g := 1_X \otimes g \colon X \otimes_A Y \to X \otimes_A Y'$$

Lemma 6.3. (i) *For any right A-module X_A we get an additive right exact functor*

$$X \otimes_A - \colon \mathrm{Mod}(A) \to \mathrm{Mod}(K)$$

defined by $Y \mapsto X \otimes_A Y$ and $g \mapsto X \otimes g$.

(ii) *For any A-module $_AY$ we get an additive right exact functor*

$$- \otimes_A Y \colon \mathrm{Mod}(A) \to \mathrm{Mod}(K)$$

defined by $X \mapsto X \otimes_A Y$ and $f \mapsto f \otimes Y$.

Proof. We just prove (i) and leave (ii) as an exercise. Clearly, $X \otimes_A -$ is a functor: We have $X \otimes_A (g \circ f) = (X \otimes_A g) \circ (X \otimes_A f)$. In particular, $X \otimes_A 1_Y = 1_{X \otimes_A Y}$.

Additivity:

$$\begin{aligned} (X \otimes_A (f + g))(x \otimes y) &= x \otimes (f + g)(y) \\ &= x \otimes (f(y) + g(y)) \\ &= (x \otimes f(y)) + (x \otimes g(y)) \\ &= (X \otimes f)(x \otimes y) + (X \otimes g)(x \otimes y). \end{aligned}$$

Right exactness:

...

□

Lemma 6.4. (i) *Let X_A be a right A-module. If $(Y_i)_i$ is a family of A-modules, then*
$$X \otimes_A \left(\bigoplus_i Y_i \right) \cong \bigoplus_i (X \otimes_A Y_i)$$
where an isomorphism is defined by $x \otimes (y_i)_i \mapsto (x \otimes y_i)_i$.

(ii) *Let ${}_AY$ be an A-module. If $(X_i)_i$ is a family of right A-modules, then*
$$\left(\bigoplus_i X_i \right) \otimes_A Y \cong \bigoplus_i (X_i \otimes_A Y)$$
where an isomorphism is defined by $(x_i)_i \otimes y \mapsto (x_i \otimes y)_i$.

Proof. Again, we just prove (i).

...

□

Corollary 6.5. *If P_A is a projective right A-module and ${}_AQ$ a projective left A-module, then*
$$P \otimes_A -\colon \mathrm{Mod}(A) \to \mathrm{Mod}(K)$$
and
$$- \otimes_A Q\colon \mathrm{Mod}(A^{\mathrm{op}}) \to \mathrm{Mod}(K)$$
are exact functor.

Proof. We know that $A \otimes_A -$ is exact. It follows that $\bigoplus_i A \otimes_A -$ is exact. Since $P_A \oplus Q_A = \bigoplus_i A$ for some Q_A, we use the additivity of $\otimes$ and get that $P_A \otimes -$ is exact as well. The exactness of $- \otimes_A Q$ is proved in the same way. □

Lemma 6.6. *Let A be a finite-dimensional algebra, and let X_A be a right A-module. If $X \otimes_A -$ is exact, then X_A is projective.*

Proof. **Exercise.** □

End of Lecture 38

6.2. **Adjoint functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be functors. If
$$\mathrm{Hom}_{\mathcal{B}}(F(X), Y)) \cong \mathrm{Hom}_{\mathcal{A}}(X, G(Y))$$
for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ and if this isomorphism is "natural", then F and G are **adjoint functors**. One calls F the **left adjoint of** G, and G is the **right adjoint of** F.

Theorem 6.7 (Adjunction formula)**.** *Let A and B be K-algebras, let ${}_AX_B$ be an A-B^{op}-bimodule, ${}_BY$ a B-module and ${}_AZ$ an A-module. Then there is an isomorphism*
$$\mathrm{Adj} := \eta\colon \mathrm{Hom}_A(X \otimes_B Y, Z) \to \mathrm{Hom}_B(Y, \mathrm{Hom}_A(X, Z))$$
where η is defined by $\eta(f)(y)(x) := f(x \otimes y)$. Furthermore, η is "natural in X, Y, Z".

Proof. ...

$\square$

6.3. **Tor.** We will not need any Tor-functors, but at least we will define them and acknowledge their existence.

Let $P_\bullet$ be a projective resolution of ${}_AY$, and let X_A be a right A-module. This yields a complex
$$\cdots \to X \otimes_A P_1 \to X \otimes_A P_0 \to X \otimes_A 0 \to \cdots$$
For $n \in \mathbb{Z}$ define
$$\mathrm{Tor}_n^A(X, Y) := H_n(X \otimes_A P_\bullet).$$

Let $P_\bullet$ be a projective resolution of a right A-module X_A. Then one can show that for all A-modules ${}_AY$ we have
$$\mathrm{Tor}_n^A(X, Y) \cong H_n(P_\bullet \otimes_A Y).$$

Similarly as for $\mathrm{Ext}_A^1(-,-)$ one can prove that $\mathrm{Tor}_n^A(X, Y)$ does not depend on the choice of the projective resolution of Y.

The following hold:

(i) $\mathrm{Tor}_n^A(X, Y) = 0$ for all $n < 0$;
(ii) $\mathrm{Tor}_0^A(X, Y) = X \otimes_A Y$;
(iii) If ${}_AP$ is projective, then $\mathrm{Tor}_n^A(X, P) = 0$ for all $n \geq 1$.
(iv)

Again, similarly as for $\mathrm{Ext}_A^1(-,-)$ we get long exact Tor-sequences:

(i) Let
$$\eta\colon 0 \to X'_A \to X_A \to X''_A \to 0$$
be a short exact sequence of right A-modules. For every A-module ${}_AY$ this induces an exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_2^A(X'', Y) \arrow[dll] \\
\mathrm{Tor}_1^A(X', Y) \arrow[r] & \mathrm{Tor}_1^A(X, Y) \arrow[r] & \mathrm{Tor}_1^A(X'', Y) \arrow[dll] \\
X' \otimes_A Y \arrow[r] & X \otimes_A Y \arrow[r] & X'' \otimes_A Y \arrow[r] & 0
\end{tikzcd}$$

(ii) Let

$$\eta\colon 0 \to {}_AY' \to {}_AY \to {}_AY'' \to 0$$

be a short exact sequence of A-modules. For every right A-module X_A this induces an exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{Tor}_2^A(X,Y'') \arrow[dlll] \\
\mathrm{Tor}_1^A(X,Y') \arrow[r] & \mathrm{Tor}_1^A(X,Y) \arrow[rr] & & \mathrm{Tor}_1^A(X,Y'') \arrow[dlll] \\
X \otimes_A Y' \arrow[r] & X \otimes_A Y \arrow[rr] & & X \otimes_A Y'' \arrow[r] & 0
\end{tikzcd}$$

Note that the bifunctor $\mathrm{Tor}_n^A(-,-)$ is covariant in both arguments. This is not true for $\mathrm{Ext}_A^n(-,-)$.

Theorem 6.8 (General adjunction formula)**.** *Let A and B be K-algebras, let ${}_AX_B$ be an A-B^{op}-bimodule, ${}_BY$ a B-module and ${}_AZ$ an A-module. If ${}_AZ$ is injective, then there is an isomorphism*

$$\mathrm{Hom}_A(\mathrm{Tor}_n^B(X,Y),Z) \cong \mathrm{Ext}_B^n(Y,\mathrm{Hom}_A(X,Z))$$

for all $n \geq 1$.

****

Part 2. Homological Algebra II: Auslander-Reiten Theory

7. Auslander-Reiten Theory

7.1. The transpose of a module. ...

7.2. The Auslander-Reiten formula. An A-module M is **finitely presented** if there exists an exact sequence

$$P_1 \xrightarrow{p} P_0 \xrightarrow{q} M \to 0$$

with P_0 and P_1 are finitely generated projective A-modules. Our aim is to prove the following result:

Theorem 7.1 (Auslander-Reiten formula)**.** *For a finitely presented A-module M we have*

$$\mathrm{Ext}^1_A(N, \tau(M)) \cong \mathrm{D}\underline{\mathrm{Hom}}_A(M, N).$$

Before we can prove Theorem 7.1 we need some preparatory results:

Lemma 7.2. *Let $X \to Y \xrightarrow{p} Z \to 0$ be exact, and let*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\xi_x"] & Y \arrow[r, "p"] \arrow[d, "\xi_y"] & Z \arrow[r] \arrow[d, "\xi"] & 0 \\ X' \arrow[r, "f"] & Y' \arrow[r, "g"] & Z' & \end{tikzcd}$$

be a commutative diagram where ξ_x and ξ_y are isomorphisms and $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$*. Then*

$$\mathrm{Ker}(g)/\operatorname{Im}(f) \cong \mathrm{Ker}(\xi).$$

Proof. ...

□

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Lemma 7.3. *Let $f\colon X \to Y$ be a homomorphism, and let $u\colon Y \to Z$ be a monomorphism. Then*

$$\mathrm{Ker}(\mathrm{Hom}_A(N, f)) = \mathrm{Ker}(\mathrm{Hom}_A(N, u \circ f)).$$

Proof. Let $h\colon N \to X$ be a homomorphism. Then $h \in \mathrm{Ker}(\mathrm{Hom}_A(N, f))$ if and only if $f \circ h = 0$. This is equivalent to $u \circ f \circ h = 0$, since u is injective. Furthermore $u \circ f \circ h = 0$ if and only if $h \in \mathrm{Ker}(\mathrm{Hom}_A(N, u \circ f))$. □

Let A be a K-algebra, and let X be an A-module. Set

$$X^* := \mathrm{Hom}_A(X, {}_AA).$$

Observe that X^* is a right A-module.

For an A-module Y define

$$\eta_{XY} \colon X^* \otimes_A Y \to \mathrm{Hom}_A(X,Y)$$

by $(\alpha \otimes y)(x) := \alpha(x) \cdot y$. In other words

$$\eta_{XY}(\alpha \otimes y) := \rho_y \circ \alpha$$

where ρ_y is the right multiplication with y.

$$X \xrightarrow{\alpha} {}_A A \xrightarrow{\rho_y} Y$$

Clearly, X^* is a right A-module: For $\alpha \in X^*$ and $a \in A$ set $(\alpha \cdot a)(x) := \alpha(x) \cdot a$.

The map $X^* \times Y \to \mathrm{Hom}_A(X,Y)$, $(\alpha, y) \mapsto \rho_y \circ \alpha$ is bilinear, and we have

$$\begin{aligned}(\alpha a, y) &\mapsto \rho_y \circ (\alpha a)\\ (\alpha, ay) &\mapsto \rho_{ay} \circ \alpha.\end{aligned}$$

We also know that

$$(\rho_y \circ (\alpha a))(x) = \rho_y(\alpha(x) \cdot a) = \alpha(x) \cdot ay = (\rho_{ay} \circ \alpha)(x).$$

In other words, the map $(\alpha, y) \mapsto \rho_y \circ \alpha$ is balanced.

$$\begin{tikzcd} X^* \times Y \arrow[r, "\omega"] \arrow[d] & X^* \otimes_A Y \arrow[dl, "\eta_{XY}"] \\ \mathrm{Hom}_A(X,Y) & \end{tikzcd}$$

Lemma 7.4. *The image of η_{XY} consists of the homomorphisms $X \to Y$ which factor through finitely generated projective modules.*

Proof. We have

$$\begin{aligned}\eta_{XY}\left(\sum_{i=1}^n \alpha_i \otimes y_i\right) &= \sum_{i=1}^n \eta_{XY}(\alpha_i \otimes y_i)\\ &= \sum_{i=1}^n \rho_{y_i} \circ \alpha_i.\end{aligned}$$

$$X \xrightarrow{\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}} \bigoplus_{i=1}^n {}_A A \xrightarrow{[\rho_{y_1},\ldots,\rho_{y_n}]} Y$$

To prove the other direction, let P be a finitely generated projective module, and assume $h = g \circ f$ for some homomorphisms $h \colon X \to Y$, $f \colon X \to P$ and $g \colon P \to Y$. There exists a module C such that $P \oplus C$ is a free module of finite rank. Thus without loss of generality we can assume that P is free of finite rank. Let $e_1, \ldots, e_n$

be a free generating set of P. Then $f(x) = \sum_i \alpha_i(x)e_i$ for some $\alpha_i(x) \in A$. This defines some homomorphisms $\alpha_i \colon X \to {}_AA$. Set $y_i := g(e_i)$. It follows that

$$\begin{aligned} \eta_{XY}\left(\sum_i \alpha_i \otimes y_i\right)(x) &= \sum_i \alpha_i(x)y_i \\ &= \sum_i \alpha_i(x)g(e_i) \\ &= g\left(\sum_i \alpha_i(x)e_i\right) \\ &= (g \circ f)(x) = h(x). \end{aligned}$$

This finishes the proof. □

Lemma 7.5. *Assume that X is finitely generated, and let $f \colon X \to Y$ be a homomorphism. Then the following are equivalent:*

(i) *f factors through a projective module;*
(ii) *f factors through a finitely generated projective module;*
(iii) *f factors through a free module of finite rank.*

Proof. **Exercise.** □

Let $\mathrm{Hom}_A(X,Y)_{\mathcal{P}} := \mathcal{P}_A(X,Y)$ be the set of homomorphisms $X \to Y$ which factor through a projective module. Clearly, this is a subspace of $\mathrm{Hom}_A(X,Y)$. As before, define

$$\underline{\mathrm{Hom}}_A(X,Y) := \mathrm{Hom}_A(X,Y)/\mathcal{P}_A(X,Y).$$

Lemma 7.6. *If X is a finitely generated projective A-module, then η_{XY} is bijective.*

Proof. It is enough to show that

$$\eta_{{}_AA,Y} \colon ({}_AA)^* \otimes_A Y \to \mathrm{Hom}_A({}_AA, Y)$$

is bijective. (Note that $\eta_{X \oplus X',Y}$ is bijective if and only if η_{XY} and $\eta_{X'Y}$ are bijective.)

Recall that $({}_AA)^* = \mathrm{Hom}_A({}_AA, {}_AA) \cong A_A$, $A_A \otimes_A Y \cong {}_AY$ and $\mathrm{Hom}_A({}_AA, {}_AY) \cong {}_AY$.

Thus we have isomorphisms $A_A \otimes_A Y \to Y$, $\alpha \otimes y \mapsto \alpha(1)y$ and $Y \to \mathrm{Hom}_A({}_AA, Y)$, $y \mapsto \rho_y$. Composing these yields a map $\alpha \otimes y \mapsto \rho_{\alpha(1)y} = \rho_y \circ \alpha$. We have

$$\rho_{\alpha(1)y}(a) = a\alpha(1)y = \alpha(a)y = (\rho_y \circ \alpha)(a).$$

□

7.3. **The Nakayama functor.** Let

$$\nu\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

be the **Nakayama functor** defined by

$$\nu(X) := \mathrm{D}(X^*) = \mathrm{Hom}_K(X^*, K) = \mathrm{Hom}_K(\mathrm{Hom}_A(X, {}_AA), K).$$

Since X^* is a right A-module, we know that $\nu(X)$ is an A-module.

Lemma 7.7. *The functor ν is right exact, and it maps finitely generated projective modules to injective modules.*

Proof. We know that for all modules N the functor $\mathrm{Hom}_A(-, N)$ is left exact. It is also clear that D is contravariant and exact. Thus ν is right exact.

Now let P be finitely generated projective. It follows that $\mathrm{D}(P^*)$ is injective: Without loss of generality assume $P = {}_AA$. Then $P^* = A_A$ and $\mathrm{Hom}_K(A_A, K)$ is injective. $\square$

Set $\nu^{-1} := \mathrm{Hom}_A(\mathrm{D}(A_A), -)$.

7.4. **Proof of the Auslander-Reiten formula.** Now we can prove Theorem 7.1: Let M be a finitely presented module. Thus there exists an exact sequence

$$P_1 \xrightarrow{p} P_0 \xrightarrow{q} M \to 0$$

where P_0 and P_1 are finitely generated projective modules. Applying ν yields an exact sequence

$$\nu(P_1) \xrightarrow{\nu(p)} \nu(P_0) \xrightarrow{\nu(q)} \nu(M) \to 0$$

where $\nu(P_0)$ and $\nu(P_1)$ are now injective modules. Define

$$\tau(M) := \mathrm{Ker}(\nu(p)).$$

We obtain an exact sequence

$$0 \to \tau(M) \to \nu(P_1) \xrightarrow{\nu(p)} \nu(P_0) \xrightarrow{\nu(q)} \nu(M) \to 0.$$

Warning: $\tau(M)$ is not uniquely determined by M, since it depends on the chosen projective presentation of M. But if $\mathrm{Mod}(A)$ has projective covers, then we take a minimal projective presentation of M. In this case, $\tau(M)$ is uniquely determined up to isomorphism.

Notation: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are homomorphisms with $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$, then set

$$H(X \xrightarrow{f} Y \xrightarrow{g} Z) := \mathrm{Ker}(g)/\mathrm{Im}(f).$$

We know that $\mathrm{Ext}^1_A(N, \tau(M))$ is equal to

$$H\left(\mathrm{Hom}_A(N, \nu(P_1)) \xrightarrow{\mathrm{Hom}_A(N,\nu(p))} \mathrm{Hom}_A(N, \nu(P_0)) \xrightarrow{\mathrm{Hom}_A(N,\nu(q))} \mathrm{Hom}_A(N, \nu(M))\right).$$

Let $u\colon \nu(M) \to I$ be a monomorphism where I is injective. We get

$$\begin{aligned}\operatorname{Ext}^1_A(N,\tau(M)) &= \operatorname{Ker}(\operatorname{Hom}_A(N,\nu(q)))/\operatorname{Im}(\operatorname{Hom}_A(N,\nu(p)))\\ &= \operatorname{Ker}(\operatorname{Hom}_A(N,u)\circ \operatorname{Hom}_A(N,\nu(q)))/\operatorname{Im}(\operatorname{Hom}_A(N,\nu(p))).\end{aligned}$$

For the last equality we used Lemma 7.3.

Define a map

$$\xi_{XY} := i \circ \mathrm{D}(\eta_{XY}\colon \mathrm{D}\operatorname{Hom}_A(X,Y) \to \operatorname{Hom}_A(Y,\nu(X))$$

by

$$\begin{tikzcd}
\mathrm{D}\operatorname{Hom}_A(X,Y) \arrow[r, "\mathrm{D}(\eta_{XY})"] \arrow[ddrr, "\xi_{XY}"'] & \mathrm{D}(X^* \otimes_A Y) \arrow[r, equal] & \operatorname{Hom}_K(X^* \otimes_A Y, K) \arrow[d, "i"] \\
 & & \operatorname{Hom}_A(Y, \operatorname{Hom}_K(X^*, K)) \arrow[d, equal] \\
 & & \operatorname{Hom}_A(Y,\nu(X))
\end{tikzcd}$$

where $i := \mathrm{Adj}$ is the isomorphism given by the adjunction formula Theorem 6.7. We know by Lemma 7.6 that ξ_{XY} is bijective, provided X is finitely generated projective.

Using this, we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathrm{D}\operatorname{Hom}_A(P_1,N) \arrow[r] \arrow[d, "\xi_{P_1N}"] & \mathrm{D}\operatorname{Hom}_A(P_0,N) \arrow[r] \arrow[d, "\xi_{P_0N}"] & \mathrm{D}\operatorname{Hom}_A(M,N) \arrow[r] \arrow[d, "\xi_{MN}"] & 0 \\
\mu\colon & \operatorname{Hom}_A(N,\nu(P_1)) \arrow[r] & \operatorname{Hom}_A(N,\nu(P_0)) \arrow[r] & \operatorname{Hom}_A(N,\nu(M)) &
\end{tikzcd}$$

whose first row is exact and whose second row is a complex. This is based on the facts that the functor D is exact, and the functor $\operatorname{Hom}_A(-,N)$ is left exact.

Thus we can apply Lemma 7.2 to this situation and obtain

$$\begin{aligned}H(\mu) &= \operatorname{Ker}(\xi_{MN})\\ &= \operatorname{Ker}(\mathrm{D}(\eta_{MN}))\\ &= \{\alpha \in \mathrm{D}\operatorname{Hom}_A(M,N) \mid \alpha(\operatorname{Im}(\eta_{MN})) = 0\}.\end{aligned}$$

(If $f\colon V \to W$ is a K-linear map, then the kernel of $f^*\colon \mathrm{D}W \to \mathrm{D}V$ consists of all $g\colon W \to K$ such that $g \circ f = 0$. This is equivalent to $g(\operatorname{Im}(f)) = 0$.)

Recall that

$$\xi_{MN} = \mathrm{Adj} \circ \mathrm{D}(\eta_{MN}).$$

If M is finitely generated, then Lemma 7.4 and Lemma 7.5 yield that

$$\operatorname{Im}(\eta_{MN}) = \operatorname{Hom}_A(M,N)_{\mathcal{P}}.$$

This implies

$$\{\alpha \in \mathrm{D}\operatorname{Hom}_A(M,N) \mid \alpha(\operatorname{Im}(\eta_{MN})) = 0\} = \mathrm{D}\underline{\operatorname{Hom}}_A(M,N).$$

This finishes the proof of Theorem 7.1.

The isomorphism

$$\mathrm{D}\underline{\mathrm{Hom}}_A(M,N) \to \mathrm{Ext}^1_A(N,\tau(M))$$

is "natural in M and N":

Let M be a finitely presented A-module, and let $f\colon M \to M'$ be a homomorphism. This yields a map

$$\mathrm{D}\,\mathrm{Hom}_A(f,N)\colon \mathrm{D}\,\mathrm{Hom}_A(M,N) \to \mathrm{D}\,\mathrm{Hom}_A(M',N)$$

and a homomorphism $\tau(f)\colon \tau(M) \to \tau(M')$. Now one easily checks that the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_A(N,\tau(M)) \arrow[d, "\mathrm{Ext}^1_A(N,\tau(f))"] & \mathrm{D}\underline{\mathrm{Hom}}_A(M,N) \arrow[l] \arrow[d, "\mathrm{D}\underline{\mathrm{Hom}}_A(f,N)"] \\
\mathrm{Ext}^1_A(N,\tau(M')) & \mathrm{D}\underline{\mathrm{Hom}}_A(M',N) \arrow[l]
\end{tikzcd}$$

commutes, and that $\mathrm{Ext}^1_A(N,\tau(f))$ is uniquely determined by f.

Similarly, if $g\colon N \to N'$ is a homomorphism, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_A(N,\tau(M)) & \mathrm{D}\underline{\mathrm{Hom}}_A(M,N) \arrow[l] \\
\mathrm{Ext}^1_A(N',\tau(M)) \arrow[u, "\mathrm{Ext}^1_A(g,\tau(M))"] & \mathrm{D}\underline{\mathrm{Hom}}_A(M,N') \arrow[l] \arrow[u, "\mathrm{D}\underline{\mathrm{Hom}}_A(M,g)"]
\end{tikzcd}$$

Explicit construction of the isomorphism

$$\phi_{MN}\colon \mathrm{D}\underline{\mathrm{Hom}}_A(M,N) \to \mathrm{Ext}^1_A(N,\tau(M)).$$

...

7.5. **Existence of Auslander-Reiten sequences.** Now we use the Auslander-Reiten formula to prove the existence of Auslander-Reiten sequences:

Let $M = N$ be a finitely presented A-module, and assume that $\mathrm{End}_A(M)$ is a local ring. We have $\underline{\mathrm{End}}_A(M) := \underline{\mathrm{Hom}}_A(M,M) = \mathrm{End}_A(M)/I$ where

$$I := \mathrm{End}_A(M)_{\mathcal{P}} := \{f \in \mathrm{End}_A(M) \mid f \text{ factors through a projective module}\}.$$

If M is projective, then $\underline{\mathrm{Hom}}_A(M,M) = 0$. Thus, assume M is not projective. The identity 1_M does not factor through a projective module: If $1_M = g \circ f$ for some homomorphisms $f\colon M \to P$ and $g\colon P \to M$ with P projective, then f is a split monomorphism. Since M is indecomposable, it follows that M is projective, a contradiction.

Note that $\mathrm{End}_A(M)_{\mathcal{P}}$ is an ideal in $\mathrm{End}_A(M)$. It follows that

$$\mathrm{End}_A(M)_{\mathcal{P}} \subseteq \mathrm{rad}(\mathrm{End}_A(M)).$$

Thus we get a surjective homomorphism of rings

$$\underline{\mathrm{Hom}}_A(M,M) \to \mathrm{End}_A(M)/\mathrm{rad}(\mathrm{End}_A(M)).$$

Recall that $\mathrm{End}_A(M)/\operatorname{rad}(\mathrm{End}_A(M))$ is a skew field.

Set
$$U := \{\alpha \in \mathrm{D}\underline{\mathrm{End}}_A(M) \mid \alpha(\operatorname{rad}(\underline{\mathrm{End}}_A(M))) = 0\},$$
and let ε be a non-zero element in U.

Now our isomorphism
$$\phi_{MM} \colon \mathrm{D}\underline{\mathrm{Hom}}_A(M,M) \to \mathrm{Ext}^1_A(M,\tau(M))$$
sends ε to a non-split short exact sequence
$$\eta \colon 0 \to \tau(M) \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

Let
$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$
be a short exact sequence of A-modules. Then g is a **right almost split homomorphism** if for every homomorphism $h \colon N \to Z$ which is not a split epimorphism there exists some $h' \colon N \to Y$ with $g \circ h' = h$.

$$\begin{tikzcd} & & & N \arrow[dl, dashed, "h'"'] \arrow[d, "h"] & \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

Dually, f is a **left almost split homomorphism** if for every homomorphism $h \colon X \to M$ which is not a split monomorphism there exists some $h' \colon Y \to M$ with $h' \circ f = h$.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[r, "g"] \arrow[dl, dashed, "h'"] & Z \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

Now let
$$\eta \colon 0 \to \tau(M) \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$
be the short exact sequence we constructed above.

Lemma 7.8. *g is a right almost split homomorphism.*

Proof. Let $h \colon N \to M$ be a homomorphism, which is not a split epimorphism. We have to show that there exists some $h' \colon N \to Y$ such that $gh' = h$, or equivalently that the induced short exact sequence $h^*(f,g)$ splits.

Since h is not a split epimorphism, the map
$$\mathrm{Hom}_A(M,h) \colon \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,M)$$
defined by $f \mapsto hf$ is not surjective: If $hf = 1_M$, then h is a split epimorphism, a contradiction.

The induced map

$$\underline{\mathrm{Hom}}_A(M,h)\colon \underline{\mathrm{Hom}}_A(M,N) \to \underline{\mathrm{Hom}}_A(M,M)$$

is also not surjective, since its image is contained in $\mathrm{rad}(\underline{\mathrm{End}}_A(M))$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{D}\underline{\mathrm{Hom}}_A(M,M) \arrow[r, "\phi_{MM}"] \arrow[d, "\mathrm{D}\underline{\mathrm{Hom}}_A(M,h)"] & \mathrm{Ext}^1_A(M,\tau(M)) \arrow[d, "\mathrm{Ext}^1_A(h,\tau(M))"] \\
\mathrm{D}\underline{\mathrm{Hom}}_A(M,N) \arrow[r, "\phi_{MN}"] & \mathrm{Ext}^1_A(N,\tau(M))
\end{tikzcd}$$

where $\phi_{MM}(\varepsilon) = \eta$ and $\mathrm{D}\underline{\mathrm{Hom}}_A(M,h)(\varepsilon) = 0$. This implies $\mathrm{Ext}^1_A(h,\tau(M))(\eta) = 0$.

Note that the map $\mathrm{Ext}^1_A(h,\tau(M))$ sends a short exact sequence ψ to the short exact sequence $h^*(\psi)$ induced by h via a pullback.

So we get $h^*(\eta) = 0$ for all $h\colon N \to M$ which are not split epimorphisms. In other words, g is a right almost split morphism. □

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Lemma 7.9. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a non-split short exact sequence. Assume that g is right almost split and that $\mathrm{End}_A(X)$ is a local ring. Then f is left almost split.*

Proof. Let $h\colon X \to X'$ be a homomorphism which is not a split monomorphism. Taking the pushout we obtain a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[r, "g"] \arrow[d, "h'"] & Z \arrow[r] \arrow[d, equal] & 0 \\
\psi: & 0 \arrow[r] & X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. Assume ψ does not split. Thus g' is not a split epimorphism.

Since g is right almost split, there exists some $g''\colon Y' \to Y$ with $g \circ g'' = g'$. It follows that $g(g''f') = g'f' = 0$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\
& & & Y' \arrow[u, "g'"] \arrow[ul, dashed, "g''"] &
\end{tikzcd}$$

Since $\mathrm{Im}(f) = \mathrm{Ker}(g)$ this implies $g''f' = ff''$ for some homomorphism $f''\colon X' \to X$. Thus

$$g(g''h') = (gg'')h' = g'h' = g.$$

In other words, $g(g''h' - 1_Y) = 0$. Again, since $\mathrm{Im}(f) = \mathrm{Ker}(g)$, there exists some $p\colon Y \to X$ with $g''h' - 1_Y = fp$. This implies

$$\begin{aligned} ff''h &= g''f'h \\ &= g''h'f \\ &= (fp + 1_Y)f \\ &= fpf + f \end{aligned}$$

and therefore $f(f''h - pf - 1_X) = 0$. Since f is injective, $f''h - pf - 1_X = 0$. In other words, $1_X = f''h - pf$. By assumption, $\mathrm{End}_A(X)$ is a local ring. So $f''h$ or pf is invertible in $\mathrm{End}_A(X)$. Thus f is a split monomorphism or h is a split monomorphism. In both cases, we have a contradiction. $\square$

Recall the following result:

Lemma 7.10 (Fitting Lemma). *Let M be a module of length m, and let $h \in \mathrm{End}_A(M)$. Then $M = \mathrm{Im}(h^m) \oplus \mathrm{Ker}(h^m)$.*

A homomorphism $g\colon M \to N$ is **right minimal** if all $h \in \mathrm{End}_A(M)$ with $gh = g$ are automorphisms. Dually, a homomorphism $f\colon M \to N$ is **left minimal** if all $h \in \mathrm{End}_A(N)$ with $hf = f$ are automorphisms.

Lemma 7.11. *Let $g\colon M \to N$ be a homomorphism, and assume that M has length m. Then there exists a decomposition $M = M_1 \oplus M_2$ with $g(M_2) = 0$, and the restriction $g\colon M_1 \to N$ is right minimal.*

Proof. Let $M = M_1 \oplus M_2$ with $M_2 \subseteq \mathrm{Ker}(g)$ and M_2 is of maximal length with this property. If now $M_1 = M_1' \oplus M_1''$ with $M_1'' \subseteq \mathrm{Ker}(g)$, then $M_1'' \oplus M_2 \subseteq \mathrm{Ker}(g)$. Thus $M_1'' = 0$.

So without loss of generality assume that $g(M') \neq 0$ for each non-zero direct summand M' of M. Assume that $gh = g$ for some $h \in \mathrm{End}_A(M)$.

By the Fitting Lemma we have $M = \mathrm{Im}(h^m) \oplus \mathrm{Ker}(h^m)$ for some m. If $\mathrm{Ker}(h^m) \neq 0$, then $g(\mathrm{Ker}(h^m)) \neq 0$, and therefore there exists some $0 \neq x \in \mathrm{Ker}(h^m)$ with $g(x) \neq 0$. We get $g(x) = gh^m(x) = 0$, a contradiction. Thus $\mathrm{Ker}(h^m) = 0$. This implies $M = \mathrm{Im}(h^m)$, which implies that h is surjective. It follows that h is an isomorphism. $\square$

Lemma 7.12. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a non-split short exact sequence. If X is indecomposable, then g is right minimal.*

Proof. Without loss of generality we assume that f is an inclusion map. By Lemma 7.11 We have a decomposition $Y = Y_1 \oplus Y_2$ such that $Y_2 \subseteq \mathrm{Ker}(g)$ and the restriction $g\colon Y_1 \to Z$ is right minimal. It follows that $X = \mathrm{Ker}(g) = (\mathrm{Ker}(g) \cap Y_1) \oplus Y_2$.

Case 1: $\mathrm{Ker}(g) \cap Y_1 = 0$. This implies $X = Y_2$, thus f is a split monomorphism, a contradiction since our sequence does not split.

Case 2: $Y_2 = 0$. Then $Y = Y_1$ and the restriction $g\colon Y_1 \to Z$ coincides with g. □

We leave it as an exercise to formulate and prove the dual statements of Lemma 7.11 and 7.12.

Theorem 7.13. *Let*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be a short exact sequence of A-modules. Then the following are equivalent:

(i) *g is right almost split, and X is indecomposable;*
(ii) *f is left almost split, and Z is indecomposable;*
(iii) *f and g are irreducible.*

Proof. Use **Skript 1, Cor. 11.5 and the dual statement Cor. 11.10** and **Skript 1, Lemma 11.6 (Converse Bottleneck Lemma) and the dual statement Lemma 11.11**. Furthermore, we need **Skript 1, Cor. 11.3 and Cor. 11.8**. □

7.6. **Properties of τ, Tr and ν.**

Lemma 7.14. *For any indecomposable A-module M we have*

$$\nu^{-1}(\tau(M)) \cong \Omega_2(M).$$

Proof. Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M. Thus we get ab exact sequence

$$0 \to \Omega_2(M) \to P_1 \xrightarrow{p} P_0 \to M \to 0.$$

Applying ν yields an exact sequence

$$0 \to \tau(M) \to \nu(P_1) \xrightarrow{\nu(p)} \nu(P_0).$$

Now we apply ν^{-1} and obtain an exact sequence

$$0 \to \nu^{-1}(\tau(M)) \to P_1 \xrightarrow{p} P_0.$$

Here we use that $\nu^{-1}(\nu(P)) \cong P$, which comes from the fact that ν induces an equivalence between the category of projective A-modules and the category of injective A-modules. This implies $\nu^{-1}(\tau(M)) \cong \Omega_2(M)$. □

Here is the dual statement:

Lemma 7.15. *For any indecomposable A-module M we have*

$$\nu(\tau^{-1}(M)) \cong \Sigma_2(M).$$

Lemma 7.16. *Let A be a finite-dimensional K-algebra. For an A-module M the following are equivalent:*

(i) $\operatorname{proj.dim}(M) \leq 1$;
(ii) *For each injective A-module I we have* $\operatorname{Hom}_A(I, \tau(M)) = 0$.

Proof. Clearly, $\operatorname{proj.dim}(M) \leq 1$ if and only if $\Omega_2(M) = 0$. By the Lemma above this is equivalent to $\operatorname{Hom}_A(\mathrm{D}(A_A), \tau(M)) = 0$. But we know that each indecomposable injective A-module is isomorphic to a direct summand of $\mathrm{D}(A_A)$. (Let I be an indecomposable injective A-module. Then $\mathrm{D}(I)$ is an indecomposable projective right A-module. It follows that $\mathrm{D}(I)$ is isomorphic to a direct summand of A_A. Thus $I \cong \mathrm{DD}(I)$ is a direct summand of $\mathrm{D}(A_A)$.) This finishes the proof. $\square$

Here is the dual statement, which can be proved accordingly:

Lemma 7.17. *Let A be a finite-dimensional K-algebra. For an A-module M the following are equivalent:*

(i) $\operatorname{inj.dim}(M) \leq 1$;
(ii) *For each projective A-module P we have* $\operatorname{Hom}_A(\tau^{-1}(M), P) = 0$.

7.7. **Properties of Auslander-Reiten sequences.** Let A be a finite-dimensional K-algebra. In this section, by a “module” we mean a finite-dimensional module. A homomorphism $f\colon X \to Y$ is a **source map** for X if the following hold:

(i) f is not a split monomorphism;
(ii) For each homomorphism $f'\colon X \to Y'$ which is not a split monomorphism there exists a homomorphism $f''\colon Y \to Y'$ with $f' = f'' \circ f$;

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & Y \arrow[dl, dashed, "f''"] \\ Y' & \end{tikzcd}$$

(iii) If $h\colon Y \to Y$ is a homomorphism with $f = h \circ f$, then h is an isomorphism.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[loop right, "h"] \end{tikzcd}$$

Dually, a homomorphism $g\colon Y \to Z$ is a **sink map** for Z if the following hold:

(i)* g is not a split epimorphism;
(ii)* For each homomorphism $g'\colon Y' \to Z$ which is not a split epimorphism there exists a homomorphism $g''\colon Y' \to Y$ with $g' = g \circ g''$;

$$\begin{tikzcd} & Y' \arrow[dl, dashed, "g''"'] \arrow[d, "g'"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

(iii)* If $h\colon Y \to Y$ is a homomorphism with $g = g \circ h$, then h is an isomorphism.

$$\begin{tikzcd} Y \arrow[loop left, "h"] \arrow[r, "g"] & Z \end{tikzcd}$$

We know already the following facts:

- If
$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$
is an Auslander-Reiten sequence, then f is a source map for X, and g is a sink map for Z.
- If X is an indecomposable module which is not injective, then there exists a source map for X.
- If Z is an indecomposable module which is not projective, then there exists a sink map for Z.

Lemma 7.18. (i) *If $f\colon X \to Y$ is a source map, then X is indecomposable;*
(ii) *If $g\colon Y \to Z$ is a sink map, then Z is indecomposable.*

Proof. We just prove (i): Let $X = X_1 \oplus X_2$ with $X_1 \neq 0 \neq X_2$, and let $\pi\colon X \to X_i$, $i = 1, 2$ be the projection. Clearly, π_i is not a split monomorphism, thus there exists some $g_i Y \to X_i$ with $g_i \circ f = \pi_i$. This implies $1_X = [\pi_1, \pi_2]^t = [g_1, g_2^t] \circ f$. Thus f is a split monomorphism, a contradiction. □

Lemma 7.19. *Let P be an indecomposable projective module. Then the embedding*
$$\mathrm{rad}(P) \to P$$
is a sink map.

Proof. Denote the embedding $\mathrm{rad}(P) \to P$ by g. Clearly, g is not a split epimorphism. This proves (i)*. Let $g'\colon Y' \to P$ be a homomorphism which is not a split epimorphism. Since P is projective, we can conclude that g' is not an epimorphism. Thus $\mathrm{Im}(g') \subset P$ which implies $\mathrm{Im}(g') \subseteq \mathrm{rad}(P)$. Here we use that P is a local module. So we proved (ii)*. Finally, assume $g = gh$ for some $h \in \mathrm{End}_A(\mathrm{rad}(P))$. Since g is injective, this implies that h is the identity $1_{\mathrm{rad}(P)}$. This proves (iii)*. □

Lemma 7.20. *Let I be an indecomposable injective module. Then the projection*
$$Q \to Q/\operatorname{soc}(Q)$$
is a source map.

Proof. Dualize the proof of Lemma 7.19. □

Corollary 7.21. *There a source map and a sink map for every indecomposable module.*

Lemma 7.22. *Let $f\colon X \to Y$ be a source map, and let $f'\colon X \to Y'$ be an arbitrary homomorphism. Then the following are equivalent:*

(i) *There exists a homomorphism $f''\colon X \to Y''$ and an isomorphism $h\colon Y \to Y' \oplus Y''$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\begin{bmatrix} f' \\ f'' \end{bmatrix}"'] & Y \arrow[dl, "h"] \\ Y' \oplus Y'' & \end{tikzcd}$$

commutes.

(ii) *f' is irreducible or $Y' = 0$.*

Proof. (ii) $\Longrightarrow$ (i): If $Y' = 0$, then choose $f'' = f$. Thus, let f' be irreducible. It follows that f' is not a split monomorphism. Thus there exists some $h' \colon Y \to Y'$ with $f' = h'f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"] & Y \arrow[dl, dashed, "h'"] \\ Y' & \end{tikzcd}$$

Now f' is irreducible and f is not a split monomorphism. Thus h' is a split epimorphism. Let $Y'' = \mathrm{Ker}(h')$. This is a direct summand of of Y. Let $p \colon Y \to Y''$ be the corresponding projection. We obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, swap, "\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right]"] & Y \arrow[d, "\left[\begin{smallmatrix} h' \\ p \end{smallmatrix}\right]"] \\ & Y' \oplus Y'' \end{tikzcd}$$

Clearly, $\left[\begin{smallmatrix} h' \\ p \end{smallmatrix}\right]$ is an isomorphism. Now set $f'' := pf$.

(i) $\Longrightarrow$ (ii): Without loss of generality we assume $h = 1$. Thus $f = \left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right] : X \to Y = Y' \oplus Y''$. We have to show: If $Y' \neq 0$, then f' is irreducible.

(a): f' is not a split monomorphism: Otherwise f would be a split monomorphism, a contradiction.

(b): f' is not a split epimorphism: We know that $Y' \neq 0$ and X is indecomposable. If f' is a split epimorphism, we get that f' is an isomorphism and therefore a split monomorphism, a contradiction.

(c): Let $f' = hg$.

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g"] & Y' \\ C \arrow[ur, swap, "h"] & \end{tikzcd}$$

There is a source map $\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right] : X \to Y' \oplus Y''$. Assume g is not a split monomorphism. Then there exists some $[g', g''] \colon Y' \oplus Y''$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right]"] \arrow[d, "g"] & Y' \oplus Y'' \arrow[dl, "{[g',g'']}"] \\ C & \end{tikzcd}$$

commutes. Thus $g = g'f' + g''f''$. It follows that the diagram

$$\begin{tikzcd} X \arrow[r, "\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right]"'] & Y' \oplus Y'' \arrow[dl, "\left[\begin{smallmatrix} hg' & hg'' \\ 0 & 1 \end{smallmatrix}\right]"] \\ Y' \oplus Y'' & \end{tikzcd}$$

commutes. Since $\left[\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right]$ is left minimal, the map $\left[\begin{smallmatrix} hg' & hg'' \\ 0 & 1 \end{smallmatrix}\right]$ is an automorphism. Thus hg' is an automorphism. This implies that h is a split epimorphism. So we have shown that f' is irreducible. $\square$

Corollary 7.23. *Let $f: X \to Y$ be a source map, and let $h: Y \to M$ be a split epimorphism. Then $h \circ f: X \to M$ is irreducible.*

Here is the dual statement which is proved accordingly:

Lemma 7.24. *Let $g: Y \to Z$ be a sink map, and let $g': Y' \to Z$ be an arbitrary homomorphism. Then the following are equivalent:*

(i) *There exists a homomorphism $g'': Y'' \to Z$ and an isomorphism $h: Y' \oplus Y'' \to Y$ such that the diagram*

$$\begin{tikzcd} & Y' \oplus Y'' \arrow[dl, "h"'] \arrow[d, "\left[\begin{smallmatrix} g' \\ g'' \end{smallmatrix}\right]"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutes.

(ii) *g' is irreducible or $Y' = 0$.*

Corollary 7.25. *Let $g: Y \to Z$ be a sink map, and let $h: M \to Y$ be a split monomorphism. Then $g \circ h: M \to Z$ is irreducible.*

Here is again the (preliminary) definition of the **Auslander-Reiten quiver** Γ_A of A: The vertices are the isomorphism classes of indecomposable A-modules, and there is an arrow $[X] \to [Y]$ if and only if there exists an irreducible map $X \to Y$. Furthermore, we draw a dotted arrow $[\tau(X)] \dashleftarrow [X]$ for each non-projective indecomposable A-module X.

A **(connected) component** of Γ_A is a full subquiver $\Gamma = (\Gamma_0, \Gamma_1)$ of Γ_A such that the following hold:

(i) For each arrow $[X] \to [Y]$ in Γ_A with $\{[X], [Y]\} \cap \Gamma_0 \neq \emptyset$ we have $\{[X], [Y]\} \subseteq \Gamma_0$;

(ii) If $[X]$ and $[Y]$ are vertices in Γ, then there exists a sequence

$$([X_1], [X_2], \dots, [X_t])$$

of vertices in Γ with $[X] = [X_1]$, $[Y] = [X_t]$, and for each $1 \leq i \leq t-1$ there is an arrow $[X_i] \to [X_{i+1}]$ or an arrow $[X_{i+1}] \to [X_i]$.

Corollary 7.26. *Let $X \to Y$ be a source map, and let $Y = \bigoplus_{i=1}^t Y_i^{n_i}$ where Y_i is indecomposable, $n_i \geq 1$ and $Y_i \ncong Y_j$ for all $i \neq j$. Then there are precisely t arrows in Γ_A starting at $[X]$, namely $[X] \to [Y_i]$, $1 \leq i \leq t$.*

Lemma 7.27. *A vertex $[X]$ is a source in Γ_A if and only if X is simple projective.*

Proof. Assume P is a simple projective module. Then any non-zero homomorphism $X \to P$ is a split epimorphism. So $[P]$ has to be a source in Γ_A. Now assume P is projective, but not simple. Then the embedding $\mathrm{rad}(P) \to P$ is a non-zero sink map. It follows that $[P]$ cannot be a source in Γ_A. Finally, if Z is an indecomposable non-projective A-module, then again there exists a non-zero sink map $Y \to Z$. So $[Z]$ cannot be a source. This finishes the proof. □

Lemma 7.28. *A source map $X \to Y$ is not a monomorphism if and only if X is injective.*

We leave it to the reader to formulate the dual statements.

Corollary 7.29. *Γ_A is a locally finite quiver.*

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an Auslander-Reiten sequence in $\mathrm{mod}(A)$. Thus, by definition f and g are irreducible. We proved already that X and Z have to be indecomposable (Skript 1). It follows that we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f'"] \arrow[d, equal] & E \arrow[r, "g'"] \arrow[d, "h"] & \tau^{-1}(X) \arrow[r] \arrow[d, "h'"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

where h and h' are isomorphisms.

Here $\tau^{-1}(X) := \mathrm{Tr}D(X)$.

Source maps are unique in the following sense: Let X be an indecomposable A-module which is not injective, and let $f\colon X \to Y$ and $f\colon X \to Y'$ be source maps. By $g\colon Y \to Z$ and $g'\colon Y' \to Z'$ we denote the projections onto the cokernel of f and f', respectively. Then we get a cimmutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f'"] \arrow[d, equal] & Y' \arrow[r, "g'"] \arrow[d, "h"] & Z' \arrow[r] \arrow[d, "h'"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

where h and h' are isomorphisms.

Dually, sink maps are unique as well.

End of Lecture 41

7.8. **Digression: The Brauer-Thrall Conjectures.** Assume that A is a finite-dimensional K-algebra, and let $S_1, \dots, S_n$ be a set of representatives of isomorphism classes of simple A-modules. Then the **quiver of** A has vertices $1, \dots, n$ and there are exactly $\dim \mathrm{Ext}^1_A(S_i, S_j)$ arrows from i to j.

The algebra A is **connected** if the quiver of A is connected.

Lemma 7.30. *For a finite-dimensional algebra A the following are equivalent:*

(i) *A is connected;*
(ii) *For any indecomposable projective A-modules $P \not\cong P'$ there exists a tuple $(P_1, P_2, \dots, P_m)$ of indecomposable projective modules such that $P_1 = P$, $P_m = P'$ and for each $1 \leq i \leq m-1$ we have $\mathrm{Hom}_A(P_i, P_{i+1}) \oplus \mathrm{Hom}_A(P_{i+1}, P_i) \neq 0$;*
(iii) *For any simple A-modules S and S' there exists a tuple $(S_1, S_2, \dots, S_m)$ of simple modules such that $S_1 = S$, $S_m = S'$ and for each $1 \leq i \leq m-1$ we have $\mathrm{Ext}^1_A(S_i, S_{i+1}) \oplus \mathrm{Ext}^1_A(S_{i+1}, S_i) \neq 0$;*
(iv) *If $A = A_1 \times A_2$ then $A_1 = 0$ or $A_2 = 0$;*
(v) *0 and 1 are the only central idempotents in A.*

Proof. **Exercise**. Hint: If $\mathrm{Ext}_A(S_i, S_j) \neq 0$, then there exists a non-split short exact sequence

$$0 \to S_j \xrightarrow{f} E \xrightarrow{g} S_i \to 0.$$

Then there exists an epimorphism $p_i \colon P_i \to S_i$. This yields a homomorphism $p_i' \colon P_i \to E$ such that $gp_i' = p_i$. Clearly, h' has to be an epimorphism. (Why?) Let $p_j \colon P_j \to S_j$ be the obvious epimorphism. Then there exists an epimorphism $p_j' \colon P_j \to E$ such that $fp_j = p_j'$. Next, there exists a non-zero homomorphism $q \colon P_j \to P_i$ such that $p_i q = fp_j$. $\square$

Theorem 7.31 (Auslander)**.** *Let A be a finite-dimensional connected K-algebra, and let $\mathcal{C}$ be a component of the Auslander-Reiten quiver of A. Assume that there exists some b such that all indecomposable modules in $\mathcal{C}$ have length at most b. Then $\mathcal{C}$ is a finite component and it contains all indecomposable A-modules. In particular, A is representation-finite.*

Proof. (a): Let X be an indecomposable A-module such that there exists a non-zero homomorphism $h \colon X \to Y$ for some $[Y] \in \mathcal{C}$. We claim that $[X] \in \mathcal{C}$: Let

$$g^{(1)} = [g_1^{(1)}, \dots, g_{t_1}^{(1)}] \colon \bigoplus_{i=1}^{t_1} Y_i^{(1)} \to Y$$

be the sink map ending in Y, where $Y_i^{(1)}$ is indecomposable for all $1 \leq i \leq t_1$. If h is a split epimorphism, then h is an isomorphism and we are done. Thus, assume $h_0 := h$ is not a split epimorphism. It follows that there exists a homomorphism

$$f^{(1)} = \begin{bmatrix} f_1^{(1)} \\ \vdots \\ f_{t_1}^{(1)} \end{bmatrix} \colon X \to \bigoplus_{i=1}^{t_1} Y_i^{(1)}$$

such that

$$h_0 = g^{(1)} f^{(1)} = \sum_{i=1}^{t_1} g_i^{(1)} f_i^{(1)} \colon X \to Y.$$

Since $h_0 \neq 0$, there exists some $1 \leq i_1 \leq t_1$ such that $g_{i_1}^{(1)} \circ f_{i_1}^{(1)} \neq 0$. Set $h_1 := f_{i_1}^{(1)}$ and $h_1' := g_{i_1}^{(1)}$. Next, assume that for each $1 \leq k \leq n-1$ we already constructed a non-invertible homomorphism

$$h_k' \colon Y_{i_k}^{(k)} \to Y_{i_{k-1}}^{(k-1)},$$

where $[Y_{i_k}^{(k)}] \in \mathcal{C}$ and $Y_{i_0}^{(0)} := Y$, and a homomorphism

$$h_k \colon X \to Y_{i_k}^{(k)}$$

such that $h_1' \circ \cdots \circ h_k' \circ h_k \neq 0$. So we get the following diagram:

$$\begin{tikzcd}
X \arrow[d, "h_{n-1}"] & X \arrow[d, "h_{n-2}"] & \cdots & X \arrow[d, "h_1"] & X \arrow[d, "h_0"] \\
Y_{i_{n-1}}^{(n-1)} \arrow[r, "h_{n-1}'"'] & Y_{i_{n-2}}^{(n-2)} \arrow[r, "h_{n-2}'"'] & \cdots \arrow[r, "h_2'"'] & Y_{i_1}^{(1)} \arrow[r, "h_1'"'] & Y
\end{tikzcd}$$

with $h_1' \circ h_2' \circ \cdots \circ h_{n-1}' \circ h_{n-1} \neq 0$.

If h_{n-1} is an isomorphism, then $X \cong Y_{i_{n-1}}^{(n-1)}$ and therefore $[X] \in \mathcal{C}$.

Thus assume that $h_{n-1} \colon X \to Y_{i_{n-1}}^{(n-1)}$ is non-invertible. Let

$$g^{(n)} = [g_1^{(n)}, \ldots, g_{t_n}^{(n)}] \colon \bigoplus_{i=1}^{t_n} Y_i^{(n)} \to Y_{i_{n-1}}^{(n-1)}$$

be the sink map ending in $Y_{i_{n-1}}^{(n-1)}$, where $Y_i^{(n)}$ is indecomposable for all $1 \leq i \leq t_n$. Since h_{n-1} is not a split epimorphism, there exists a homomorphism

$$f^{(n)} = \begin{bmatrix} f_1^{(n)} \\ \vdots \\ f_{t_n}^{(n)} \end{bmatrix} : X \to \bigoplus_{i=1}^{t_n} Y_i^{(n)}$$

such that

$$h_{n-1} = g^{(n)} f^{(n)} = \sum_{i=1}^{t_n} g_i^{(n)} f_i^{(n)} \colon X \to Y_{i_{n-1}}^{(n-1)}.$$

Since $h_1' \circ h_2' \circ \cdots \circ h_{n-1}' \circ h_{n-1} \neq 0$, there exists some $1 \leq i_n \leq t_n$ such that

$$h_1' \circ h_2' \circ \cdots \circ h_{n-1}' \circ g_{i_n}^{(n)} \circ f_{i_n}^{(n)} \neq 0.$$

Set $h_n := f_{i_n}^{(n)}$ and $h_n' := g_{i_n}^{(n)}$. Thus

$$h_1' \circ h_2' \circ \cdots \circ h_{n-1}' \circ h_n' \circ h_n \neq 0.$$

Clearly, h_n' is non-invertible, since h_n' is irreducible.

If $n \geq 2^b - 2$ we know by the Harada-Sai Lemma that h_n has to be an isomorphism. This finishes the proof of (a).

(b): Dually, if Z is an indecomposable A-module such that there exists a non-zero homomorphism $Y \to Z$ for some $[Y] \in \mathcal{C}$, then $[Z] \in \mathcal{C}$.

(c): Let Y be an indecomposable A-module with $[Y] \in \mathcal{C}$, and let S be a composition factor of Y. Then there exists a non-zero homomorphism $P_S \to Y$ where P_S is the indecomposable projective module with top S. By (a) we know that $[P_S] \in \mathcal{C}$. Now we use Lemma 7.30, (iii) in combination with (a) and (b) to show that all indecomposable projective A-modules lie in $\mathcal{C}$. Finally, if Z is an arbitrary indecomposable A-module, then again there exists an indecomposable projective module P and a non-zero homomorphism $P \to Z$. Now (b) implies that $[Z] \in \mathcal{C}$. It follows that $\mathcal{C} = (\Gamma_A, d_A)$. By the proof of (a) and (b) we know that there is a path of length at most $2^b - 2$ in $\mathcal{C}$ which starts in $[P]$ and ends in $[Z]$. It is also clear that $\mathcal{C}$ has only finitely many vertices: Since Γ_A is a locally finite quiver, for each projective vertex $[P]$ there are only finitely many paths of length at most $2^b - 2$ starting in $[P]$. □

Corollary 7.32 (1st Brauer-Thrall Conjecture)**.** *Let A be a finite-dimensional K-algebra. Assume there exists some b such that all indecomposable A-modules have length at most b. Then A is representation-finite.*

Thus the 1st Brauer-Thrall Conjecture says that *bounded representation type* implies *finite representation type*. There exists a completely different proof of the 1st Brauer-Thrall conjecture due to Roiter, using the Gabriel-Roiter measure.

Conjecture 7.33 (2nd Brauer-Thrall Conjeture)**.** *Let A be a finite-dimensional algebra over an infinite field K. If A is representation-infinite, then there exists some $d \in \mathbb{N}$ such that the following hold: For each $n \geq 1$ there are infinitely many isomorphism classes of indecomposable A-modules of dimension nd.*

Theorem 7.34 (Smalø)**.** *Let A be a finite-dimensional algebra over an infinite field K. Assume there exists some $d \in \mathbb{N}$ such that there are infinitely many isomorphism classes of indecomposable A-modules of dimension d. Then for each $n \geq 1$ there are infinitely many isomorphism classes of indecomposable A-modules of dimension nd.*

Thus to prove Conjecture 7.33, the induction step is already known by Theorem 7.34. Just the beginning of the induction is missing...

Conjecture 7.33 is true if K is algebraically closed. This was proved by Bautista using the well developed theory of representation-finite algebras over algebraically closed fields.

7.9. **The bimodule of irreducible morphisms.** Let A be a finite-dimensional K-algebra, and as before let $\mathrm{mod}(A)$ be the category of finitely generated A-modules. All modules are assumed to be finitely generated.

For indecomposable A-modules X and Y let

$$\mathrm{rad}_A(X, Y) := \{f \in \mathrm{Hom}_A(X, Y) \mid f \text{ is not invertible}\}.$$

In particular, if $X \ncong Y$, then $\mathrm{rad}_A(X,Y) = \mathrm{Hom}_A(X,Y)$. If $X = Y$, then
$$\mathrm{rad}_A(X,X) = \mathrm{rad}(\mathrm{End}_A(X)) := J(\mathrm{End}_A(X)).$$

Now let $X = \bigoplus_{i=1}^s X_i$ and $Y = \bigoplus_{j=1}^t Y_j$ be A-modules with X_i and Y_j indecomposable for all i and j. Recall that we can think of an endomorphism $f\colon X \to Y$ as a matrix
$$f = \begin{pmatrix} f_{11} & \cdots & f_{s1} \\ \vdots & & \vdots \\ f_{1t} & \cdots & f_{st} \end{pmatrix}$$
where $f_{ij}\colon X_i \to Y_j$ is an homomorphism for all i and j. Set
$$\mathrm{rad}_A(X,Y) := \begin{pmatrix} \mathrm{rad}_A(X_1,Y_1) & \cdots & \mathrm{rad}_A(X_s,Y_1) \\ \vdots & & \vdots \\ \mathrm{rad}_A(X_1,Y_t) & \cdots & \mathrm{rad}_A(X_s,Y_t) \end{pmatrix}.$$
Thus $\mathrm{rad}_A(X,Y) \subseteq \mathrm{Hom}_A(X,Y)$.

Lemma 7.35. *For A-modules X and Y we have $f \notin \mathrm{rad}_A(X,Y)$ if and only if there exists a split monomorphism $u\colon X' \to X$ and a split epimorphism $p\colon Y \to Y'$ such that $p \circ f \circ u\colon X' \to Y'$ is an isomorphism and $X' \neq 0$.*

Proof. **Exercise**. □

For A-modules X and Y let $\mathrm{rad}_A^2(X,Y)$ be the set of homomorphisms $f\colon X \to Y$ with $f = h \circ g$ for some $g \in \mathrm{rad}_A(X,M)$, $h \in \mathrm{rad}_A(M,Y)$ and M.

Lemma 7.36. *Let X and Y be indecomposable A-modules. For a homomorphism $f\colon X \to Y$ the following are equivalent:*

(i) *f is irreducible;*
(ii) *$f \in \mathrm{rad}_A(X,Y) \setminus \mathrm{rad}_A^2(X,Y)$.*

Proof. Assume $f\colon X \to Y$ is irreducible. Since X and Y are indecomposable we know that f is an isomorphism if and only if f is a split monomorphism if and only if f is a split epimorphism. Thus $f \in \mathrm{rad}_A(X,Y)$. Assume $f \in \mathrm{rad}_A^2(X,Y)$.

...

□

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For indecomposable A-modules X and Y define
$$\mathrm{Irr}_A(X,Y) := \mathrm{rad}_A(X,Y)/\mathrm{rad}_A^2(X,Y).$$
We call $\mathrm{Irr}_A(X,Y)$ the **bimodule of irreducible maps** from X to Y.

Set $F(X) := \mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X))$ and $F(Y) := \mathrm{End}_A(Y)/\mathrm{rad}(\mathrm{End}_A(X))$. Since X and Y are indecomposable, we know that $F(X)$ and $F(Y)$ are skew fields.

Lemma 7.37. $\mathrm{Irr}_A(X,Y)$ *is an* $F(X)^{\mathrm{op}}$-$F(Y)$*-bimodule.*

Proof. Let $\overline{f} \in \mathrm{Irr}_A(X,Y)$, $\overline{g} \in F(X)$ and $\overline{h} \in F(Y)$, where $f \in \mathrm{rad}_A(X,Y)$, $g \in \mathrm{End}_A(X)$ and $h \in \mathrm{End}_A(Y)$. Define

$$\overline{g} \star \overline{f} := \overline{fg},$$
$$\overline{h} \cdot \overline{f} := \overline{hf}.$$

We have to check that this is well defined: We have a map

$$\mathrm{End}_A(Y) \times \mathrm{Hom}_A(X,Y) \times \mathrm{End}_A(X) \to \mathrm{Hom}_A(X,Y)$$

defined by $(h,f,g) \mapsto hfg$. Clearly, if $f \in \mathrm{rad}_A(X,Y)$, then hf and fg are in $\mathrm{rad}_A(X,Y)$. It follows that $\mathrm{rad}_A(X,Y)$ is an $\mathrm{End}_A(X)^{\mathrm{op}}$-$\mathrm{End}_A(Y)$-bimodule. It is also clear that $\mathrm{rad}_A^2(X,Y)$ is a subbimodule: Let $f = f_2f_1 \in \mathrm{rad}_A^2(X,Y)$ where $f_1 \in \mathrm{rad}_A(X,C)$ and $f_2 \in \mathrm{rad}_A(C,Y)$ for some C. Then $hf = (hf_2)f_1$ and $fg = f_2(f_1g)$, so they are both in $\mathrm{rad}_A^2(X,Y)$. Furthermore, the images of the maps $\mathrm{rad}_A(X,Y) \times \mathrm{rad}(\mathrm{End}_A(X)) \to \mathrm{rad}_A(X,Y)$, $(f,g) \mapsto fg$ and $\mathrm{rad}_A(X,Y) \times \mathrm{rad}(\mathrm{End}_A(Y)) \to \mathrm{rad}_A(X,Y)$, $(h,f) \mapsto hf$ are both contained in $\mathrm{rad}_A^2(X,Y)$. Thus $\mathrm{Irr}_A(X,Y)$ is annihilated by $\mathrm{rad}(\mathrm{End}_A(X)^{\mathrm{op}})$ and $\mathrm{rad}(\mathrm{End}_A(Y))$. This implies that $\mathrm{Irr}_A(X,Y)$ is an $F(X)^{\mathrm{op}}$-$F(Y)$-bimodule. □

Lemma 7.38. *Let Z be indecomposable and non-projective. Then* $F(Z) \cong F(\tau(Z))$.

Proof. **Exercise**. □

Lemma 7.39. *Assume K is algebraically closed. If X is an indecomposable A-module, then* $F(X) \cong K$.

Proof. **Exercise**. □

Theorem 7.40. *Let M and N be indecomposable A-modules. Let $g\colon Y \to N$ be a sink map for N. Write*

$$Y = M^t \oplus Y'$$

with t maximal. Thus $g = [g_1, \ldots, g_t, g']$ where $g_i\colon M \to N$, $1 \le i \le t$ and $g'\colon Y' \to N$ are homomorphisms. Then the following hold:

(i) *The residue classes of $g_1, \ldots, g_t$ in $\mathrm{Irr}_A(M,N)$ form a basis of the $F(M)^{\mathrm{op}}$-vector space $\mathrm{Irr}_A(M,N)$;*
(ii) *We have*

$$t = \dim_{F(M)^{\mathrm{op}}}(\mathrm{Irr}_A(M,N)) = \frac{\dim_K(\mathrm{Irr}_A(M,N))}{\dim_K(F(M))}.$$

Dually, let $f\colon M \to X$ be a source map for M. Write

$$X = N^s \oplus X'$$

with s maximal. Thus $f = {}^t[f_1, \ldots, f_s, f']$ where $f_i\colon M \to N$, $1 \le i \le s$ and $f'\colon M \to X'$ are homomorphisms. Then the following hold:

(iii) *The residue classes of* $f_1, \dots, f_s$ *in* $\mathrm{Irr}_A(M,N)$ *form a basis of the* $F(N)$-*vector space* $\mathrm{Irr}_A(M,N)$;

(iv) *We have*

$$s = \dim_{F(N)}(\mathrm{Irr}_A(M,N)) = \frac{\dim_K(\mathrm{Irr}_A(M,N))}{\dim_K(F(N))}.$$

We have $s = t$ *if and only if* $\dim_K(F(M)) = \dim_K(F(N))$ *or* $s = t = 0$.

Proof. (a): First we show that the set $\{\overline{g_1}, \dots, \overline{g_t}\}$ is linearly independent in the $F(M)^{\mathrm{op}}$-vector space $\mathrm{Irr}_A(M,N)$:

Assume

$$\sum_{i=1}^{t} \overline{\lambda_i} \star \overline{g_i} = \overline{0} \tag{1}$$

where $\lambda_i \in \mathrm{End}_A(M)$, $g_i \in \mathrm{rad}_A(M,N)$, $\overline{\lambda_i} = \lambda_i + \mathrm{rad}(\mathrm{End}_A(M))$, $\overline{g_i} = g_i + \mathrm{rad}_A^2(M,N)$ and $\overline{0} = 0 + \mathrm{rad}_A^2(M,N)$. By definition $\overline{\lambda_i} \star \overline{g_i} = \overline{g_i \lambda_i}$. We have to show that $\overline{\lambda_i} = 0$, i.e. $\lambda_i \in \mathrm{rad}(\mathrm{End}_A(M))$ for all i.

Assume $\lambda_1 \notin \mathrm{rad}(\mathrm{End}_A(M))$. In other words, λ_1 is invertible. We get

$$\sum_{i=1}^{t} g_i \lambda_i = [g_1, \dots, g_t, g'] \circ \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_t \\ 0 \end{bmatrix} = g \circ \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_t \\ 0 \end{bmatrix} : M \to N.$$

By Equation (1) we know that this map is contained in $\mathrm{rad}_A^2(M,N)$.

Clearly, $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_t \\ 0 \end{bmatrix}$ is a split monomorphism, since

$$[\lambda_1^{-1}, 0, \dots, 0] \circ \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_t \\ 0 \end{bmatrix} = 1_M.$$

Using Lemma 7.24 this implies that $\sum_{i=1}^{t} g_i \lambda_i$ is irreducible and can therefore not be contained in $\mathrm{rad}_A^2(M,N)$, a contradiction.

(b): Next, we show that $\{\overline{g_1}, \dots, \overline{g_t}\}$ generates the $F(M)^{\mathrm{op}}$-vector space $\mathrm{Irr}_A(M,N)$:

Let $u \colon M \to N$ be a homomorphism with $u \in \mathrm{rad}_A(M,N)$. We have to show that $\overline{u} := u + \mathrm{rad}_A^2(M,N)$ is a linear combination of $\overline{g_1}, \dots, \overline{g_t}$.

Since g is a sink map and u is not a split epimorphism, we get a commutative diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, swap, "{\begin{bmatrix} u_1 \\ \vdots \\ u_t \\ u' \end{bmatrix}}"] \arrow[d, "u"] \\ M^t \oplus Y' \arrow[r, swap, "{[g_1,\dots,g_t,g']}"] & N \end{tikzcd}$$

such that $u = \sum_{i=1}^t g_i u_i + g'u'$.

We know that $g' \in \mathrm{rad}_A(Y', N)$, since g' is just the restriction of the sink map g to a direct summand Y' of Y. Thus g' is irreducible or $g' = 0$. Furthermore, M is indecomposable and Y' does not contain any direct summand isomorphic to M. So $u' \in \mathrm{rad}_A(M, Y')$. Thus implies $g'u' \in \mathrm{rad}_A^2$ and therefore $\overline{g'u'} = \overline{0}$. It follows that

$$\overline{u} = \sum_{i=1}^t \overline{u_i} \star \overline{g_i} + \overline{g'u'} = \sum_{i=1}^t \overline{u_i} \star \overline{g_i}.$$

This finishes the proof.

The second part of the theorem is proved dually. □

Corollary 7.41. *Let*

$$0 \to \tau(Z) \to Y \to Z \to 0$$

be an Auslander-Reiten sequence, and let M be indecomposable. Then

$$\dim_K \mathrm{Irr}_A(M, Z) = \dim_K \mathrm{Irr}_A(\tau(Z), M).$$

Proof. Let t be maximal such that $Y = M^t \oplus Y'$ for some module Y'. Then we get

$$t = \frac{\dim_K \mathrm{Irr}_A(M, Z)}{\dim_K F(M)} = \frac{\dim_K \mathrm{Irr}_A(\tau(Z), M)}{\dim_K F(M)}.$$

□

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It is often quite difficult to construct Auslander-Reiten sequences. But if there exists a projective-injective module, one gets one such sequence for free:

Lemma 7.42. *Let I be an indecomposable projective-injective A-module, and assume that I is not simple. Then there is an Auslander-Reiten sequence of the form*

$$0 \to \mathrm{rad}(I) \to \mathrm{rad}(I)/\mathrm{soc}(I) \oplus I \to I/\mathrm{soc}(I) \to 0.$$

Proof. ...

□

7.10. **Translation quivers and mesh categories.** Let $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ be a quiver (now we allow Γ_0 and Γ_1 to be infinite sets).

We call Γ **locally finite** if for each vertex y there are at most finitely many arrows ending at y and there are are most finitely many arrows starting at y.

If there is an arrow $x \to y$ then x is called a **direct predecessor** of y, and if there is an arrow $y \to z$ then z is a **direct successor** of y.

Let y^- be the set of direct predecessors of y, and let y^+ be the set of direct successors of y. Note that we do not assume that y^- and y^+ are disjoint.

A **path** of length $n \geq 1$ in Γ is of the form $w = (\alpha_1, \dots, \alpha_n)$ where the α_i are arrows such that $s(\alpha_i) = t(\alpha_{i+1})$ for $1 \leq i \leq n-1$. We say that w **starts** in $s(w) := s(\alpha_n)$, and w **ends** in $t(w) := t(\alpha_1)$. In this case, $s(w)$ is a **predecessor** of $t(w)$, and $t(w)$ is a **successor** of $s(w)$.

Additionally, for each vertex x of Γ there is a path 1_x of length 0 with $s(1_x) = t(1_x) = x$. For vertices x and y let $W(x, y)$ be the set of paths from x to y. If a path w in Γ starts in x and ends in y, we say that x is a predecessor of y, and y is a successor of x. If $w = (\alpha_1, \dots, \alpha_n)$ has length $n \geq 1$, and if $s(w) = t(w)$, then w is called a **cycle** in Γ. In this case, we say that $s(\alpha_1), \dots, s(\alpha_n)$ **lie on the cycle** w.

A vertex x in a quiver Γ is **reachable** if there are just finitely many paths in Γ which end in x.

It follows immediately that a vertex x is reachable if and only if x has only finitely many predecessors and none of these lies on a cycle. Of course, every predecessor of a reachable vertex is again reachable. We define a chain

$$\emptyset = {}_{-1}\Gamma \subseteq {}_0\Gamma \subseteq \cdots \subseteq {}_{n-1}\Gamma \subseteq {}_n\Gamma \subseteq \cdots$$

of subsets of Γ_0.

By definition ${}_{-1}\Gamma = \emptyset$. For $n \geq 0$, if ${}_{n-1}\Gamma$ is already defined, then let ${}_n\Gamma$ be the set of all vertices z of Γ such that $z^- \subseteq {}_{n-1}\Gamma$.

By ${}_n\underline{\Gamma}$ we denote the full subquiver of Γ with vertices ${}_n\Gamma$. Set

$${}_\infty\underline{\Gamma} := \bigcup_{n \geq 0} {}_n\underline{\Gamma} \quad \text{and} \quad {}_\infty\Gamma := \bigcup_{n \geq 0} {}_n\Gamma.$$

Clearly, ${}_\infty\Gamma$ is the set of all reachable vertices of Γ.

Now let K be a field. We define the **path category** $K\Gamma$ as follows:

The objects in $K\Gamma$ are the vertices of Γ. For vertices $x, y \in \Gamma_0$, we take as morphism set $\operatorname{Hom}_{K\Gamma}(x, y)$, the K-vector space with basis $W(x, y)$.

The composition of morphisms is by definition K-bilinear, so it is enough to define the composition of two basis elements: First, the path 1_x of length 0 is the unit element for the object x. Next, if $w = (\alpha_1, \dots, \alpha_n) \in W(x, y)$ and $v = (\beta_1, \dots, \beta_m) \in W(y, z)$, then define

$$vw := v \cdot w := (\beta_1, \dots, \beta_m, \alpha_1, \dots, \alpha_n) \in W(x, z).$$

This is again a path since $s(\beta_m) = t(\alpha_1)$.

We call $\Gamma = (\Gamma_0, \Gamma_1, s, t, \tau, \sigma)$ a **translation quiver** if the following hold:

- (T1) $(\Gamma_0, \Gamma_1, s, t)$ is a locally finite quiver without loops;
- (T2) $\tau\colon \Gamma_0' \to \Gamma_0$ is an injective map where Γ_0' is a subset of Γ_0, and for all $z \in \Gamma_0'$ and every $y \in \Gamma_0$ the number of arrows $y \to z$ equals the number of arrows $\tau(z) \to y$;

(T3) $\sigma\colon \Gamma'_1 \to \Gamma_1$ is an injective map with $\sigma(\alpha)\colon \tau(z) \to y$ for each $\alpha\colon y \to z$, where Γ'_1 is the set of all arrows $\alpha\colon y \to z$ with $z \in \Gamma'_0$.

Note that a translation quiver can have multiple arrows between two vertices.

If $\Gamma = (\Gamma_0, \Gamma_1, s, t, \tau, \sigma)$ is a translation quiver, then τ is called the **translation** of Γ. The vertices in $\Gamma_0 \setminus \Gamma'_0$ are the **projective vertices**, and $\Gamma_0 \setminus \tau(\Gamma'_0)$ are the **injective vertices**. If Γ does not have any projective or injective vertices, then Γ is **stable**.

A translation quiver Γ is **preprojective** if the following hold:

(P1) There are no oriented cycles in Γ;
(P2) If z is non-projective vertex, then $z^- \neq \emptyset$;
(P3) For each vertex z there exists some $n \geq 0$ such that $\tau^n(z)$ is a projective vertex.

A translation quiver Γ is **preinjective** if the following hold:

(I1) There are no oriented cycles in Γ;
(I2) If z is non-injective vertex, then $z^+ \neq \emptyset$;
(I3) For each vertex z there exists some $n \geq 0$ such that $\tau^{-n}(z)$ is an injective vertex.

Again, let Γ be a translation quiver. A function $f\colon \Gamma_0 \to \mathbb{Z}$ is **additive** if for all non-projective vertices z we have

$$f(\tau(z)) + f(z) = \sum_{y \in z^-} f(y).$$

For example, if $\mathcal{C}$ is a component of the Auslander-Reiten quiver of an algebra A with $\dim_K \operatorname{Irr}_A(X, Y) \leq 1$ for all $X, Y \in \mathcal{C}$, then $f([X]) := l(X)$ is an additive function on the translation quiver $\mathcal{C}$.

We will often investigate translation quivers without multiple arrows. In this case, we do not mention the map σ, since it is uniquely determined by the other data.

By condition (T2) we know that each non-projective vertex z of Γ yields a subquiver of the form

$$\begin{tikzcd}
& y_1 \arrow[dr, "\alpha_1"] & \\
\tau(z) \arrow[ur, "\sigma(\alpha_1)"] \arrow[dr, "\sigma(\alpha_n)"'] & \vdots & z \\
& y_n \arrow[ur, "\alpha_n"'] &
\end{tikzcd}$$

Such a subquiver is called a **mesh** in Γ. (Recall that there could be more than one arrow from $\tau(z)$ to y_i and therefore also from y_i to z. In this case, the map σ yields a bijection between the set of arrows $y_i \to z$ and the set of arrows $\tau(z) \to y_i$.)

Now let K be a field, and let $\Gamma = (\Gamma_0, \Gamma_1, s, t, \tau, \sigma)$ be a translation quiver. We look at the path category $K\Gamma := K(\Gamma_0, \Gamma_1, s, t)$ of the quiver $(\Gamma_0, \Gamma_1, s, t)$. For each non-projective vertex z we call the linear combination

$$\rho_z := \sum_{\alpha\colon y \to z} \alpha \cdot \sigma(\alpha)$$

the **mesh relation** associated to z, where the sum runs over all arrows ending in z. This is an element in the path category $K\Gamma$.

The **mesh category** $K\langle\Gamma\rangle$ of the translation quiver Γ is by definition the factor category of $K\Gamma$ modulo the ideal generated by all mesh relations ρ_z where z runs through the set Γ_0' of all non-projective vertices of Γ.

Example: Let Γ be the following translation quiver:

$$\begin{tikzcd}
& & w \arrow[dr, "\delta"] & & \\
& v \arrow[ur, "\beta"] \arrow[dr, "\gamma"] & & y \arrow[ll, dashed] \arrow[dr, "\xi"] & \\
u \arrow[ur, "\alpha"] & & x \arrow[ll, dashed] \arrow[ur, "\epsilon"] & & z \arrow[ll, dashed]
\end{tikzcd}$$

This is a translation quiver without multiple arrows. The dashed arrows describe τ, they start in some z and end in $\tau(z)$. Thus we have three projective vertices u, v, w and three injective vertices w, y, z. The mesh relations are

$$\begin{aligned} \gamma\alpha &= 0, \\ \delta\beta + \epsilon\gamma &= 0, \\ \xi\epsilon &= 0. \end{aligned}$$

For example, in the path category $K\Gamma$ we have $\dim \mathrm{Hom}_{K\Gamma}(u, y) = 2$. But in the mesh category $K\langle\Gamma\rangle$, we obtain $\mathrm{Hom}_{K\langle\Gamma\rangle}(u, y) = 0$.

Assume that $\Gamma = (\Gamma_0, \Gamma_1, s, t, \tau, \sigma)$ is a translation quiver without multiple arrows. A function

$$d\colon \Gamma_0 \cup \Gamma_1 \to \mathbb{N}_1$$

is a **valuation** for Γ if the following hold:

(V1) If $\alpha\colon x \to y$ is an arrow, then $d(x)$ and $d(y)$ divide $d(\alpha)$;
(V2) We have $d(\tau(z)) = d(z)$ and $d(\tau(z) \to y) = d(y \to z)$ for every non-projective vertex z and every arrow $y \to z$.

If d is a valuation for Γ, then we call (Γ, d) a **valued translation quiver**. If d is a valuation for Γ with $d(x) = 1$ for all vertices x of Γ, then d is a **split valuation**.

Our main and most important examples of valued translation quivers are the following:

Let A be a finite-dimensional K-algebra. For an A-module X denote its isomorphism class by $[X]$. If X and Y are indecomposable A-modules, then as before define

$$F(X) := \mathrm{End}_A(X)/\,\mathrm{rad}(\mathrm{End}_A(X))$$

and

$$\mathrm{Irr}_A(X,Y) := \mathrm{rad}_A(X,Y)/\,\mathrm{rad}_A^2(X,Y).$$

Let τ_A be the Auslander-Reiten translation of A.

The **Auslander-Reiten quiver** Γ_A of A has as vertices the isomorphism classes of indecomposable A-modules. If X and Y are indecomposable A-modules, there is an arrow $[X] \longrightarrow [Y]$ if and only if $\mathrm{Irr}_A(X,Y) \neq 0$. Define $\tau([Z]) := [\tau_A(Z)]$ if Z is indecomposable and non-projective. In this case, we draw a dotted arrow $[\tau_A(Z)] \dashleftarrow [Z]$.

For each vertex $[X]$ of Γ_A define

$$d_X := d_A([X]) := \dim_K F(X),$$

and for each arrow $[X] \to [Y]$ let

$$d_{XY} := d_A([X] \to [Y]) := \dim_K \mathrm{Irr}_A(X,Y).$$

When we display arrows in Γ_A we often write $[X] \xrightarrow{d_{XY}} [Y]$.

For an indecomposable projective module P and an indecomposable module X let r_{XP} be the multiplicity of X in a direct sum decompositions of $\mathrm{rad}(P)$ into indecomposables, i.e.

$$\mathrm{rad}(P) = X^{r_{XP}} \oplus C$$

for some module C and r_{XP} is maximal with this property.

Lemma 7.43. *For a finite-dimensional K-algebra the following hold:*

(i) $\Gamma(A) := (\Gamma_A, d_A)$ *is a translation quiver;*
(ii) *The valuation d_A is split if and only if for each indecomposable A-module X we have* $\mathrm{End}_A(X)/\,\mathrm{rad}(\mathrm{End}_A(X)) \cong K$ *(For example, if K is algebraically closed, then d_A is a split valuation.);*
(iii) *A vertex $[X]$ of (Γ, d_A) is projective (resp. injective) if and only if X is projective (resp. injective).*

Proof. We have $\mathrm{Irr}_A(X,X) = 0$ for every indecomposable A-module X. (Recall that every irreducible map between indecomposable modules is either a monomorphism or an epimorphism.) Thus the quiver Γ_A does not have any loops. If Z is an indecomposable non-projective module, then the skew fields $F(\tau_A(Z))$ and $F(Z)$ are isomorphic, and $\dim_K \mathrm{Irr}_A(\tau_A(Z), Y) = \dim_K \mathrm{Irr}_A(Y, Z)$ for each indecomposable module Y. This shows that Γ_A is locally finite, and that the conditions (T1), (T2), (T3) and (V2) are satisfied. Since $\mathrm{Irr}_A(X,Y)$ is an $F(X)^{\mathrm{op}}$-$F(Y)$-bimodule, also (V1) holds.

...

□

If $\mathcal{C}$ is a connected component of (Γ_A, d_A) such that $\mathcal{C}$ is a preprojective (resp. preinjective) translation quiver, then $\mathcal{C}$ is called a **preprojective (resp. preinjective) component** of Γ_A.

An indecomposable A-module X is **preprojective** (resp. **preinjective**) if $[X]$ lies in a preprojective (resp. preinjective) component of Γ_A.

Let Γ be a translation quiver with a split valuation d. Then we define the **expansion** $(\Gamma, d)^e$ of Γ as follows:

The quiver $(\Gamma, d)^e$ has the same vertices as (Γ, d), and also the same translation τ. For every arrow $\alpha\colon x \to y$ in Γ, we get a sequence of $d(x \to y)$ arrows $\alpha^i\colon x \to y$ where $1 \leq i \leq d(\alpha)$. (Thus the arrows in $(\Gamma, d)^e$ starting in x and ending in y are enumerated, there is a first arrow, a second arrow, etc.) Now σ sends the ith arrow $y \to z$ to the ith arrow $\tau(z) \to y$ provided z is a non-projective vertex.

7.11. **Examples of Auslander-Reiten quivers. (a)**: Let $K = \mathbb{R}$ and set

$$A = \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} \subset M_2(\mathbb{C}).$$

Clearly, A is a 5-dimensional K-algebra. Let $e_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Set

$$M = Ae_{11} = \left[\begin{smallmatrix} \mathbb{R} \\ 0 \end{smallmatrix}\right] \quad \text{and} \quad N = Ae_{22} = \left[\begin{smallmatrix} \mathbb{C} \\ \mathbb{C} \end{smallmatrix}\right].$$

These are the indecomposable projective A-modules, and we have ${}_AA = M \oplus N$.

We can identify $\operatorname{Hom}_A(M, N)$ with $\mathbb{C}$ since

$$\operatorname{Hom}_A(M, N) = \operatorname{rad}_A(M, N) \cong e_{11}Ae_{22} \cong \mathbb{C}.$$

Next, we observe that $\operatorname{rad}(M) = 0$ and $\operatorname{rad}(N) = \left[\begin{smallmatrix} \mathbb{C} \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{R} \\ 0 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} \mathbb{R} \\ 0 \end{smallmatrix}\right]$. It follows that the obvious map $M \oplus M \to N$ is a sink map. Furthermore, it is easy to check that $\operatorname{End}_A(M) \cong \mathbb{R}$, $F(M) \cong \mathbb{R}$, $\operatorname{End}_A(N) \cong \mathbb{C}$ and $F(N) \cong \mathbb{C}$.

We have

$$2 = r_{MN} = \frac{\dim_K \operatorname{Irr}_A(M, N)}{\dim_K F(M)} = \frac{\dim_K \operatorname{Irr}_A(M, N)}{1}$$

This implies $\dim_K \operatorname{Irr}_A(M, N) = 2$. Thus $M \to N$ is a source map. We get an Auslander-Reiten sequence $0 \to M \to N \to Q \to 0$ where $Q = \left[\begin{smallmatrix} \mathbb{C}/\mathbb{R} \\ \mathbb{C} \end{smallmatrix}\right]$.

Next, we look for the source map starting in N: We have $\dim_K \operatorname{Irr}_A(N, Q) = \dim_K \operatorname{Irr}_A(M, N) = 2$ and $\dim_K F(Q) = 1$. Thus $N \to Q \oplus Q$ is a source map. We obtain an Auslander-Reiten sequence $0 \to N \to Q \oplus Q \to R \to 0$ where $R = \left[\begin{smallmatrix} 0 \\ \mathbb{C} \end{smallmatrix}\right]$.

The modules $\tau^{-1}(M)$ and $\tau^{-1}(N)$ are injective, thus the following is the Auslander-Reiten quiver of A:

$$
\begin{tikzcd}
d_N = 2 & & \left[\begin{smallmatrix}\mathbb{C}\\ \mathbb{C}\end{smallmatrix}\right] \arrow[dr, "2"] & & \left[\begin{smallmatrix}0\\ \mathbb{C}\end{smallmatrix}\right] \arrow[ll, dashed] \\
d_M = 1 & \left[\begin{smallmatrix}\mathbb{R}\\ 0\end{smallmatrix}\right] \arrow[ur, "2"] & & \left[\begin{smallmatrix}\mathbb{C}/\mathbb{R}\\ \mathbb{C}\end{smallmatrix}\right] \arrow[ur, "2"] \arrow[ll, dashed] &
\end{tikzcd}
$$

So there are just four indecomposable A-modules up to isomorphism. Using dimension vectors it looks as follows:

$$
\begin{tikzcd}
2 & & \begin{smallmatrix}2\\ 1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\ 1\end{smallmatrix} \arrow[ll, dashed] \\
1 & \begin{smallmatrix}1\\ 0\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1\\ 1\end{smallmatrix} \arrow[ur] \arrow[ll, dashed] &
\end{tikzcd}
$$

Note that the valuation of the vertices remains constant on τ-orbits (and τ^{-1}-orbits), so it is enough to display them only once per orbit.

(b): Next, let

$$A = \begin{pmatrix} k & K \\ 0 & K \end{pmatrix} \subset M_2(K)$$

where $k \subset K$ is a field extension of dimension three, e.g. $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt[3]{2})$. The indecomposable projective A-modules are

$$M = Ae_{11} = \left[\begin{smallmatrix}k\\ 0\end{smallmatrix}\right] \quad \text{and} \quad N = Ae_{22} = \left[\begin{smallmatrix}K\\ K\end{smallmatrix}\right].$$

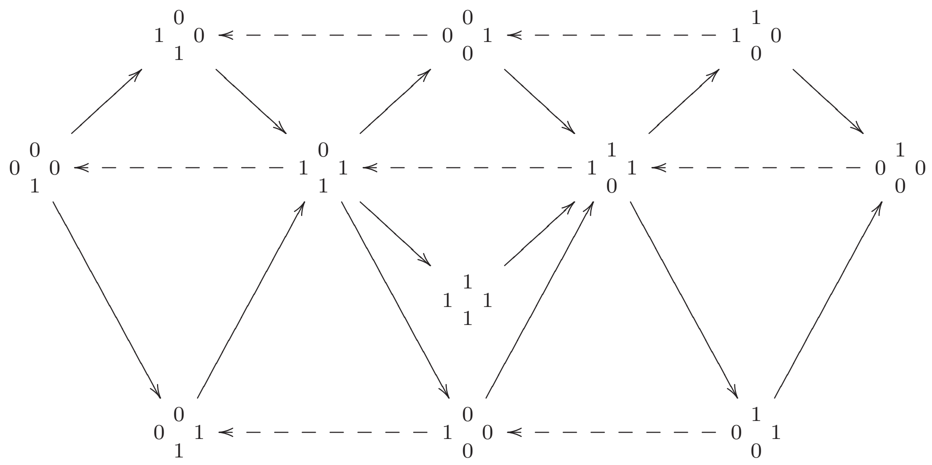
In this case there are 6 indecomposable A-modules, and the Auslander-Reiten quiver Γ_A looks like this:

$$
\begin{tikzcd}
d_N = 3 & & \begin{smallmatrix}3\\ 1\end{smallmatrix} \arrow[dr, "3"] & & \begin{smallmatrix}3\\ 2\end{smallmatrix} \arrow[dr, "3"] \arrow[ll, dashed] & & \begin{smallmatrix}0\\ 1\end{smallmatrix} \arrow[ll, dashed] \\
d_M = 1 & \begin{smallmatrix}1\\ 0\end{smallmatrix} \arrow[ur, "3"] & & \begin{smallmatrix}2\\ 1\end{smallmatrix} \arrow[ur, "3"] \arrow[ll, dashed] & & \begin{smallmatrix}1\\ 1\end{smallmatrix} \arrow[ur, "3"] \arrow[ll, dashed] &
\end{tikzcd}
$$

(c): Here is the Auslander-Reiten quiver of the algebra $A = KQ/I$ where Q is the quiver

$$
\begin{tikzcd}
& 1 \arrow[dl, "a"'] \arrow[dr, "c"] & \\
2 \arrow[dr, "b"'] & & 3 \arrow[dl, "d"] \\
& 4 &
\end{tikzcd}
$$

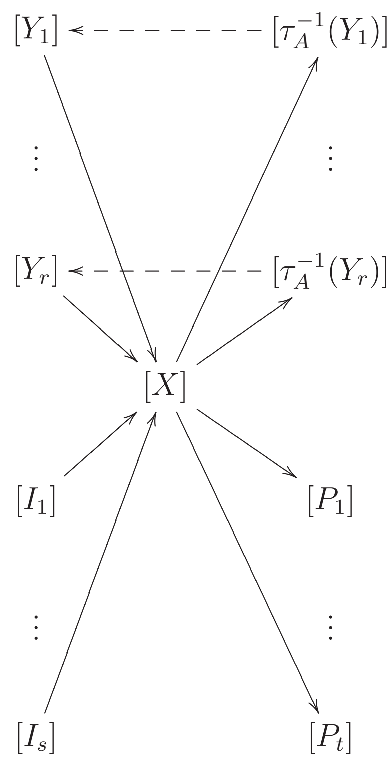
and I is the ideal generated by $ba - dc$:



End of Lecture 44

7.12. **Knitting preprojective components.** Let A be a finite-dimensional K-algebra.

Basic idea: Let X be an indecomposable A-module. Whenever the sink map ending in X is known, we can construct the source map starting in X. In $\Gamma(A) = (\Gamma_A, d_A)$ the situation around the vertex $[X]$ looks like this:



Here the Y_i are non-injective modules, the I_i are injective, and the P_i are projective. The sink map ending in X is of the form $Y \to X$ where

$$Y = \bigoplus_{i=1}^{r} Y_i^{d_{Y_i X}/d_{Y_i}} \oplus \bigoplus_{i=1}^{s} I_i^{d_{I_i X}/d_{I_i}}.$$

To get the source map $X \to Z$, we have to translate the non-injective modules Y_i using τ_A^{-1}. Note that

$$d_{X\tau_A^{-1}(Y_i)} = d_{Y_i X} \quad \text{and} \quad d_{\tau_A^{-1}(Y_i)} = d_{Y_i}$$

for all i. Furthermore, we have to check if X occurs as a direct summand of $\mathrm{rad}(P)$ where P runs through the set of indecomposable projective modules. In this case, there is an arrow $[X] \to [P]$ with valuation

$$d_{XP} = \dim_K \mathrm{Irr}_A(X, P) = r_{XP} \cdot \dim_K F(X).$$

We get

$$Z = \bigoplus_{i=1}^{r} \tau_A^{-1}(Y_i)^{d_{X\tau_A^{-1}(Y_i)}/d_{\tau_A^{-1}(Y_i)}} \oplus \bigoplus_{i=1}^{t} P_i^{d_{XP_i}/d_{P_i}}.$$

If X is non-injective, we get a mesh

$$\begin{tikzcd}
 & {[\tau_A^{-1}(Y_1)]} \arrow[dddr] & \\
 & \vdots & \\
 & {[\tau_A^{-1}(Y_r)]} \arrow[dr] & \\
{[X]} \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & {[\tau_A^{-1}(X)]} \arrow[ll, dashed] \\
 & {[P_1]} \arrow[ur] & \\
 & \vdots & \\
 & {[P_t]} \arrow[uuur] &
\end{tikzcd}$$

in the Auslander-Reiten quiver $\Gamma(A)$ of A. We have

$$d_{\tau_A^{-1}(Y_i)\tau_A^{-1}(X)} = d_{X\tau_A^{-1}(Y_i)} \quad \text{and} \quad d_{\tau_A^{-1}(X)} = d_X.$$

Knitting preparations

(i) Determine all indecomposable projectives $P_1, \ldots, P_n$ and all indecomposable injectives $I_1, \ldots, I_n$.

(ii) For each $1 \leq i \leq n$ determine $\operatorname{rad}(P_i)$ and decompose it into indecomposable modules, say
$$\operatorname{rad}(P_i) = \bigoplus_{j=1}^{r_i} R_{ij}^{r_{ij}}$$
where $r_{ij} \geq 1$, and the R_{ij} are indecomposable such that $R_{ik} \cong R_{il}$ if and only if $k = l$.

(iii) For each $1 \leq i \leq n$ determine $d_{P_i} = \dim_K F(P_i)$.

Note that
$$d_{R_{ij}P_i} = \dim_K \operatorname{Irr}_A(R_{ij}, P_i) = r_{ij} \cdot d_{R_{ij}}$$
where $r_{ij} = r_{R_{ij}P_i}$. Furthermore, we know that
$$F(P_i) = \operatorname{End}_A(P_i)/\operatorname{rad}(\operatorname{End}_A(P_i)) \cong \operatorname{End}_A(P_i/\operatorname{rad}(P_i)) \cong \operatorname{End}_A(S_i)$$
where S_i is the simple A-module with $S_i \cong P_i/\operatorname{rad}(P_i)$.

The knitting algorithm

Let ${}_{-1}\underline{\Delta}$ be the empty quiver.

We define inductively quivers ${}_n\underline{\Delta}$, ${}_n\underline{\Delta}'$, ${}_n\underline{\Delta}''$, $n \geq 0$ which are subquivers of (Γ_A, d_A).

For all $n \geq 1$ these quivers will satisfy
$${}_{n-1}\underline{\Delta} \subseteq {}_n\underline{\Delta} \subseteq {}_{n-1}\underline{\Delta}'' \subseteq {}_n\underline{\Delta}' \subseteq {}_n\underline{\Delta}''.$$

By ${}_n\Delta$, ${}_n\Delta'$, ${}_n\Delta''$, we denote the set of vertices of ${}_n\underline{\Delta}$, ${}_n\underline{\Delta}'$, ${}_n\underline{\Delta}''$, respectively.

(a_0) **Define** ${}_0\underline{\Delta}$: Let ${}_0\underline{\Delta}$ be the quiver (without arrows) with vertices $[S]$ where S is simple projective.

(b_0) **Add projectives**: For each $[S] \in {}_0\Delta$, if $S \cong R_{ij}$ for some i, j, then (if it wasn't added already) add the vertex $[P_i]$ with valuation d_{P_i}, and add an arrow $[S] \to [P_i]$ with valuation $d_{SP_i} = r_{SP_i} \cdot d_S$. We denote the resulting quiver by ${}_0\underline{\Delta}'$.

(c_0) **Translate the non-injectives in** ${}_0\Delta$: For each $[S] \in {}_0\Delta$ with S non-injective, add the vertex $[\tau_A^{-1}(S)]$ to ${}_0\underline{\Delta}'$ with valuation $d_{\tau_A^{-1}(S)} = d_S$, and for each arrow $[S] \to [Y]$ constructed so far add an arrow $[Y] \to [\tau_A^{-1}(S)]$ to ${}_0\underline{\Delta}'$ with valuation $d_{Y\tau_A^{-1}(S)} = d_{SY}$. We denote the resulting quiver by ${}_0\underline{\Delta}''$.

Note that any source map starting in a simple projective module S is of the form $S \to P$ where P is projective. (Proof: Assume there is an indecomposable non-projective module X and an arrow $[S] \to [X]$. Then there has to be an arrow $[\tau_A(X)] \to [S]$, a contradiction since $[S]$ is a source in (Γ_A, d_A).) Thus we get P from the data collected in (i), (ii) and (iii). More precisely, we have
$$P = \bigoplus_{i=1}^{n} P_i^{d_{SP_i}/d_{P_i}},$$
and we know that $d_{SP_i} = r_{SP_i} \cdot d_S$.

Now assume that for $n \geq 1$ the quivers ${}_{n-1}\underline{\Delta}$, ${}_{n-1}\underline{\Delta}'$ and ${}_{n-1}\underline{\Delta}''$ are already defined. We also assume that for each vertex $[X] \in {}_{n-1}\Delta''$ and each arrow $[X] \to [Y]$ in ${}_{n-1}\underline{\Delta}''$ we defined valuations d_X and d_{XY}, respectively.

- (a_n) **Define** ${}_n\underline{\Delta}$: Let ${}_n\underline{\Delta}$ be the full subquiver of ${}_{n-1}\underline{\Delta}''$ with vertices $[X]$ such that all direct predecessors of $[X]$ in ${}_{n-1}\underline{\Delta}''$ are contained in ${}_{n-1}\underline{\Delta}$, and if $[X]$ is a vertex with $X \cong P_i$ projective, then we require additionally that $[R_{ij}] \in {}_{n-1}\underline{\Delta}$ for all j.
- (b_n) **Add projectives**: For each $[X] \in {}_n\Delta$, if $X \cong R_{ij}$ for some i, j, then (if it wasn't added already) add the vertex $[P_i]$ to ${}_{n-1}\underline{\Delta}''$ with valuation d_{P_i}, and add an arrow $[X] \to [P_i]$ to ${}_{n-1}\underline{\Delta}''$ with valuation $d_{XP_i} = r_{XP} \cdot d_X$. We denote the resulting quiver by ${}_n\underline{\Delta}'$.
- (c_n) **Translate the non-injectives in** ${}_n\Delta \setminus {}_{n-1}\Delta$: For each $[X] \in {}_n\Delta \setminus {}_{n-1}\Delta$ with X non-injective, add the vertex $[\tau_A^{-1}(X)]$ to ${}_n\underline{\Delta}'$ with valuation $d_{\tau_A^{-1}(X)} = d_X$, and for each arrow $[X] \to [Y]$ constructed to far add an arrow $[Y] \to [\tau_A^{-1}(X)]$ to ${}_n\underline{\Delta}'$ with valuation $d_{Y\tau_A^{-1}(X)} = d_{XY}$. We denote the resulting quiver by ${}_n\underline{\Delta}''$.

The algorithm stops if ${}_n\Delta \setminus {}_{n-1}\Delta$ is empty for some n. It can happen that the algorithm never stops.

Define

$$
{}_\infty\underline{\Delta} = \bigcup_{n \geq 0} {}_n\underline{\Delta} \quad \text{and} \quad {}_\infty\Delta = \bigcup_{n \geq 0} {}_n\Delta.
$$

Let $[X] \in {}_n\Delta$, and let $[X] \to [Z_i]$, $1 \leq i \leq t$ be the arrows in ${}_n\underline{\Delta}'$ starting in $[X]$. Then the corresponding homomorphism

$$
X \to \bigoplus_{i=1}^{t} Z_i^{d_{XZ_i}/d_{Z_i}}
$$

is a source map. Similarly, let $[Y_i] \to [X]$, $1 \leq i \leq s$ be the arrows in ${}_n\underline{\Delta}$ ending in $[X]$. Then the corresponding homomorphism

$$
\bigoplus_{i=1}^{s} Y_i^{d_{Y_iX}/d_{Y_i}} \to X
$$

is a sink map. The following lemma is now easy to prove:

Lemma 7.44. *For all $n \geq -1$ we have*

$$
{}_n\underline{\Delta} = {}_n(\underline{\Gamma}_A).
$$

In particular, ${}_\infty\underline{\Delta} = {}_\infty(\underline{\Gamma}_A)$.

Clearly, ${}_\infty\underline{\Delta}$ is a full subquiver of (Γ_A, d_A). One easily checks that ${}_\infty\underline{\Delta}$ is a translation subquiver of (Γ_A, d_A) in the obvious sense.

The number of connected components of ${}_\infty\underline{\Delta}$ is bounded by the number of simple projective A-modules.

If we know the dimension vectors $\underline{\dim}(P_i)$ and $\underline{\dim}(R_{ij})$ for all i, j, then our knitting algorithm yields an algorithm to determine $\underline{\dim}(X)$ for any vertex $[X] \in {}_\infty\underline{\Delta}$:

Let $[X]$ be a vertex in ${}_n\Delta \setminus {}_{n-1}\Delta$, and let $[X] \to [Z_i]$, $1 \leq i \leq t$ be the arrows in ${}_n\underline{\Delta}'$ starting in $[X]$. Then X is non-injective if and only if

$$l(X) < \sum_{i=1}^{t} d_{XZ_i} \cdot l(Z_i).$$

In this case, we have

$$\underline{\dim}(\tau^{-1}(X)) = -\underline{\dim}(X) + \sum_{i=1}^{t} d_{XZ_i} \cdot \underline{\dim}(Z_i).$$

These considerations provide a knitting algorithm which is only based on dimension vectors. We will prove the following result:

Theorem 7.45. *Let $[X], [Y] \in {}_\infty\Delta$. Then $[X] = [Y]$ if and only if $\underline{\dim}(X) = \underline{\dim}(Y)$.*

Lemma 7.46. *Let $\mathcal{C}$ be a connected component of (Γ_A, d_A). If*

$$\mathcal{C} \subseteq {}_\infty\underline{\Delta},$$

then $\mathcal{C}$ is a preprojective component of (Γ_A, d_A).

Proof. (a): By construction, for each $[X] \in {}_n\Delta''$ we have $\tau_A^n(X)$ is projective for some $n \geq 0$.

(b): The quiver ${}_n\underline{\Delta}$ has no oriented cycles: One shows by induction on n that if $[X] \to [Y]$ is an arrow in ${}_n\underline{\Delta}$, then there exists a unique $t \leq n$ such that $[Y] \in {}_t\Delta \setminus {}_{t-1}\Delta$ and $[X] \in {}_{t-1}\Delta$. The result follows.

(c): Let $[X] \in {}_n\Delta$. Then $[X]$ has a direct predecessor in ${}_n\Delta$ if and only if X is not in ${}_0\Delta$. $\square$

Often knitting does not work. For example, we cannot even start with the knitting procedure, if there is no simple projective module. Furthermore, if an indecomposable projective module P_i is inserted such that an indecomposable direct summand of $\mathrm{rad}(P_i)$ does not show up in some step of the knitting prodedure, then we are doomed and cannot continue.

But the good news is that in many interesting situations knitting does work. Here are the two most important situations: Path algebras and directed algebras. In fact, using covering theory, one can use knitting to construct the Auslander-Reiten quiver of any representation-finite algebra (provided the characteristic of the ground field is not two).

The dual situation: Obviously, there is also a "dual knitting algorithm" by starting with the simple injective A-modules. As a knitting preparation one needs to decompose $I_i/\operatorname{soc}(I_i)$ into a direct sum of indecomposables, and one needs the values

$d_{I_i} = \dim_K F(I_i)$. If $\mathcal{C}$ is a component of $\Gamma(A)$ which is obtained by the dual knitting algorithm, then $\mathcal{C}$ is a preinjective component.

Lemma 7.47. *Let Q be a finite connected quiver without oriented cycles. Then the following hold:*

(i) *$\Gamma(KQ)$ has a unique preprojective component $\mathcal{P}$ and a unique preinjective component $\mathcal{I}$;*
(ii) *$\mathcal{P} = \mathcal{I}$ if and only if KQ is representation-finite.*

Proof. **Exercise**. □

7.13. **More examples of Auslander-Reiten quivers.** **(a)**: Let Q be the quiver

$$\begin{tikzcd} 2 \arrow[dr] & 3 \arrow[d] & 4 \arrow[dl] \\ & 1 & \end{tikzcd}$$

and let $A = KQ$. Using the dimension vector notation, Γ_A looks as follows:

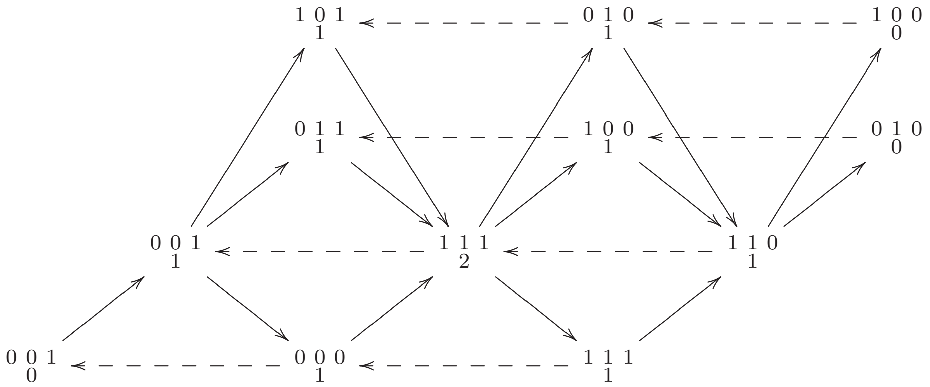
$$\begin{tikzcd}
& \substack{1\,0\,0\\1} \arrow[drr] & & \substack{0\,1\,1\\1} \arrow[drr] \arrow[ll, dashed] & & \substack{1\,0\,0\\0} \arrow[ll, dashed] \\
\substack{0\,0\,0\\1} \arrow[ur] \arrow[dr] \arrow[ddr] & & \substack{1\,1\,1\\2} \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[ll, dashed] & & \substack{1\,1\,1\\1} \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[ll, dashed] & \\
& \substack{0\,1\,0\\1} \arrow[ur] & & \substack{1\,0\,1\\1} \arrow[ur] \arrow[ll, dashed] & & \substack{0\,1\,0\\0} \arrow[ll, dashed] \\
& \substack{0\,0\,1\\1} \arrow[uur] & & \substack{1\,1\,0\\1} \arrow[uur] \arrow[ll, dashed] & & \substack{0\,0\,1\\0} \arrow[ll, dashed]
\end{tikzcd}$$

Here is an interesting question: What happens with the Auslander-Reiten quiver of KQ if we change the orientation of an arrow in Q?

For example, the path algebra of the quiver

$$\begin{tikzcd} 2 \arrow[dr] & 3 \arrow[d] & 4 \\ & 1 \arrow[ur] & \end{tikzcd}$$

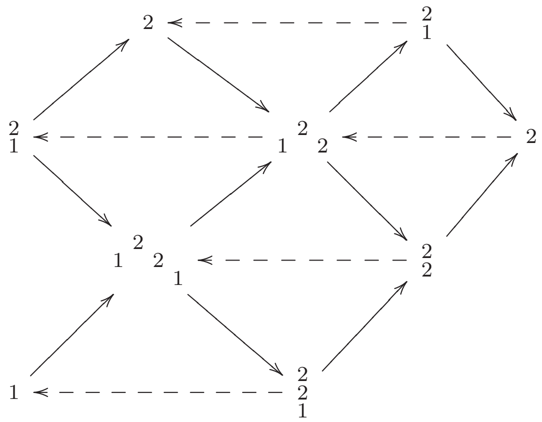
has the following Auslander-Reiten quiver:



(b): Let Q be the quiver

$$1 \xleftarrow{b} 2 \circlearrowleft a$$

and let $A = KQ/I$ where I is generated by the path aa. Clearly, A is finite-dimensional, and has two simple modules, whch we denote by 1 and 2. The Auslander-Reiten quiver of A looks like this:



Note that this time, we did not display the dimension vectors of each indecomposable module. Instead we used the composition factors 1 and 2 to indicate how the modules look like. For example, the 4-dimensional A-module

$$\begin{smallmatrix} & 2 & & \\ 1 & & 2 & \\ & & & 1 \end{smallmatrix}$$

has a simple top 2, its socle is isomorphic to $1 \oplus 1$. Note also that one has to identify the two vertices on the upper left with the two vertices on the upper right. Thus Γ_A has in fact just 7 vertices. Sometimes one displays certain vertices more than once, in order to obtain a nicer and easier to understand picture.

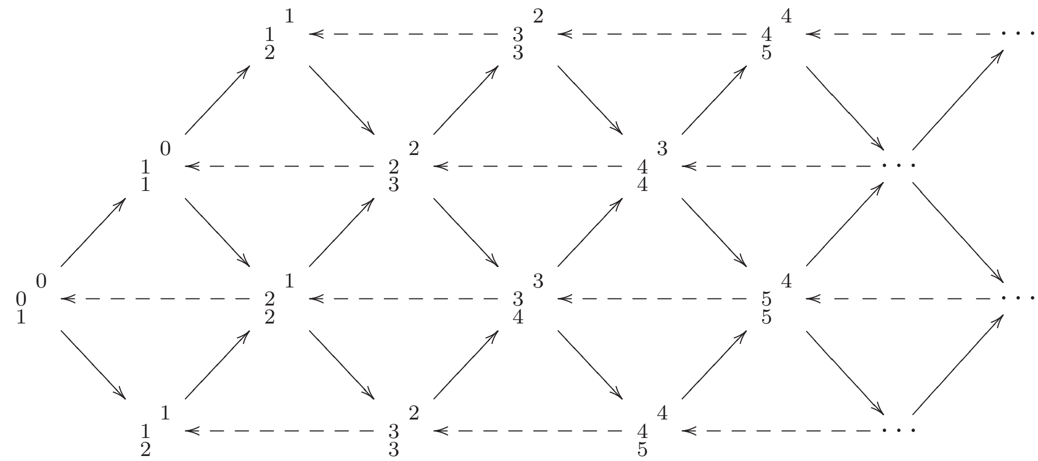
Clearly, Γ_A does not contain a preprojective component. We have a simple projective module, namely 1. So ${}_0\Delta = \{1\}$. But then we see that ${}_1\Delta \setminus {}_0\Delta = \emptyset$. So there is just one reachable vertex in Γ_A.

We constructed Γ_A "by hand". In other words, our methods are not yet developed enough to prove that this is really the Auslander-Reiten quiver of A.

(c): Let A be the path algebra of the quiver

$$\begin{tikzcd} & 3 \arrow[dl] \arrow[ddl] \\ 2 \arrow[d] & \\ 1 & \end{tikzcd}$$

Then there is an infinite preprojective component in (Γ_A, d_A), which can be obtained from the following picture by identifying the vertices in the first with the corresponding vertices in the fourth row:

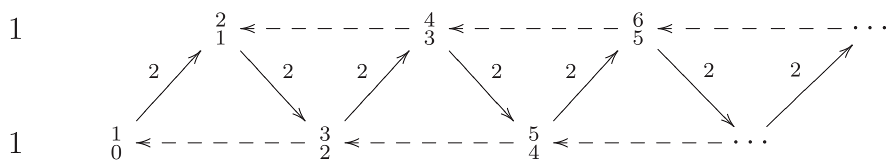


Exercise: Determine ${}_n\Delta$ for all $n \geq 0$.

(d): Let

$$A = \begin{bmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix} \subset M_2(\mathbb{C}).$$

Using the dimension vector notation, we obtain an infinite preprojective component of (Γ_A, d_A):



(e): Let

$$A = \begin{bmatrix} \mathbb{R} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{R} \end{bmatrix} \subset M_2(\mathbb{C}).$$

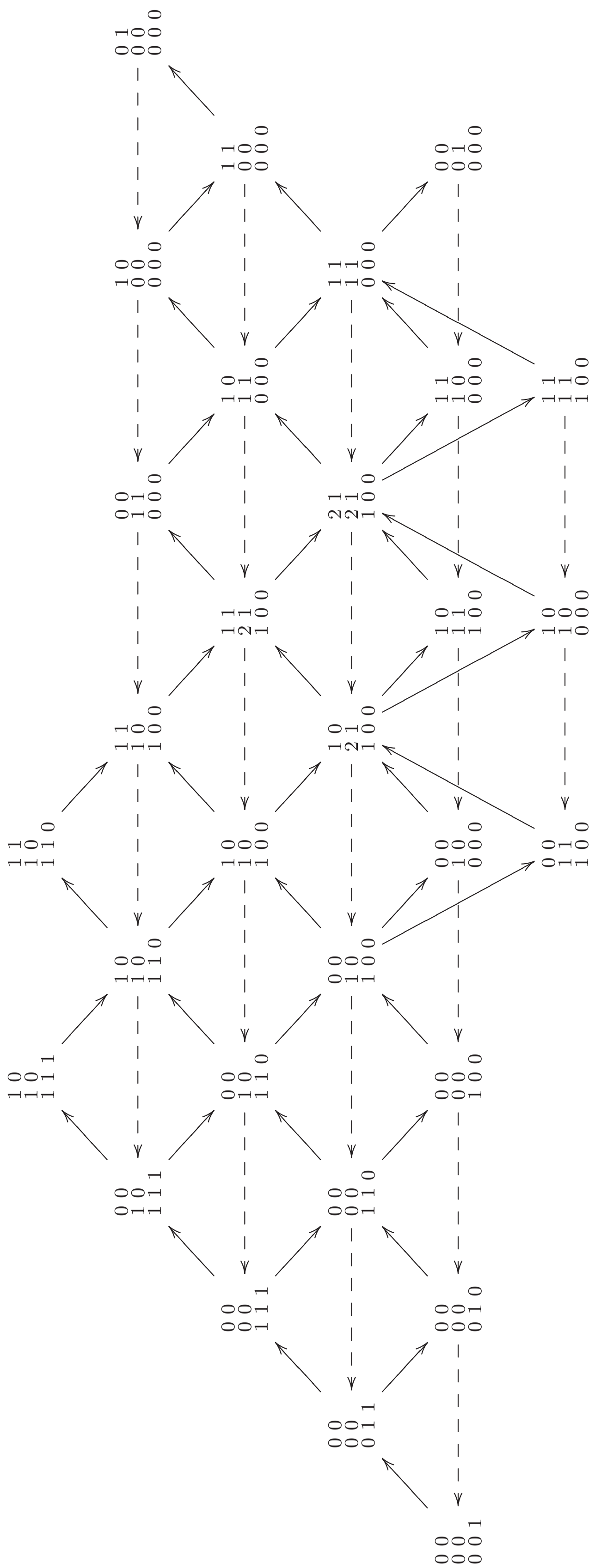
Again using the dimension vector notation we get an infinite preprojective component:

$$
\begin{tikzcd}[column sep=small]
1 & & & \substack{2\\1\\1} \arrow[dr, "2"] & & \substack{2\\2\\1} \arrow[ll, dashed] \arrow[dr, "2"] & & \substack{4\\3\\3} \arrow[ll, dashed] \arrow[dr, "2"] & & \cdots \arrow[ll, dashed] \\
2 & & \substack{2\\1\\0} \arrow[ur, "2"] \arrow[dr, "2"] & & \substack{4\\3\\2} \arrow[ll, dashed] \arrow[ur, "2"] \arrow[dr, "2"] & & \substack{6\\5\\4} \arrow[ll, dashed] \arrow[ur, "2"] \arrow[dr, "2"] & & \substack{8\\7\\6} \arrow[ll, dashed] \arrow[ur, "2"] \arrow[dr, "2"] & \\
1 & \substack{1\\0\\0} \arrow[ur, "2"] & & \substack{1\\1\\0} \arrow[ll, dashed] \arrow[ur, "2"] & & \substack{3\\2\\2} \arrow[ll, dashed] \arrow[ur, "2"] & & \substack{3\\3\\2} \arrow[ll, dashed] \arrow[ur, "2"] & & \cdots \arrow[ll, dashed]
\end{tikzcd}
$$

(f): Let $A = KQ/I$ where Q is the quiver

$$
\begin{tikzcd}
6 \arrow[d, "e"'] & 7 \arrow[l, "f"'] & & \\
5 \arrow[d, "d"'] & 8 \arrow[l, "g"'] & & \\
4 \arrow[r, "c"] & 3 \arrow[r, "b"] & 2 \arrow[r, "a"] & 1
\end{tikzcd}
$$

and the ideal I is generated by $abcdef$ and cdg. It turns out that (Γ_A, D_A) consists of a single preprojective component:



(g): Let $A = KQ/I$ where Q is the quiver

$$\begin{tikzcd} 3 \arrow[d, "a"'] \\ 2 \arrow[d, shift right, "b"'] \arrow[d, shift left, "c"] \\ 1 \end{tikzcd}$$

and I is the ideal generated by ba. The indecomposable projective A-modules are of the form $P_1 = 1$, $P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}$, $P_3 = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$. Then ${}_\infty\underline{\Delta}$ consists of a preprojective component

$$\begin{tikzcd}
& \begin{smallmatrix}0\\1\\2\end{smallmatrix} \arrow[dr, "2"] & & \begin{smallmatrix}0\\3\\4\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \begin{smallmatrix}0\\5\\6\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \cdots \arrow[ll, dashed] \\
\begin{smallmatrix}0\\0\\1\end{smallmatrix} \arrow[ur, "2"] & & \begin{smallmatrix}0\\2\\3\end{smallmatrix} \arrow[ll, dashed] \arrow[ur, "2"] & & \begin{smallmatrix}0\\4\\5\end{smallmatrix} \arrow[ll, dashed] \arrow[ur, "2"] & & \cdots \arrow[ll, dashed] \arrow[ur] &
\end{tikzcd}$$

which does not contain P_3.

(h): Let $A = KQ/I$ where Q is the quiver

$$\begin{tikzcd} 3 \arrow[d, "a"'] \arrow[dd, bend left, "d"] \\ 2 \arrow[d, shift right, "b"'] \arrow[d, shift left, "c"] \\ 1 \end{tikzcd}$$

and I is the ideal generated by ba. The indecomposable projective A-modules are of the form $P_1 = 1$, $P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}$, $P_3 = \begin{smallmatrix} & & 3 & \\ & 2 & & 1 \\ 1 & & & \end{smallmatrix}$. Then ${}_\infty\underline{\Delta}$ consists of two points, namely P_1 and P_2:

$$\begin{tikzcd}
& \begin{smallmatrix}0\\1\\2\end{smallmatrix} \arrow[dr, "2"] & & \begin{smallmatrix}4\\7\\12\end{smallmatrix} \arrow[ll, dashed] \\
\begin{smallmatrix}0\\0\\1\end{smallmatrix} \arrow[ur, "2"] \arrow[dr, "2"] & & \begin{smallmatrix}2\\4\\7\end{smallmatrix} \arrow[ur, "2"] & \\
& \begin{smallmatrix}1\\1\\2\end{smallmatrix} \arrow[ur, "2"] & &
\end{tikzcd}$$

Note that one of the direct summands of the radical of P_3 does not show up in the course of the knitting algorithm. So we get ${}_2\Delta \setminus {}_1\Delta = \emptyset$.

(i): Let $A = KQ/I$ where Q is the quiver

$$\begin{tikzcd}
3 \arrow[d, "a"'] \arrow[dr, "d"] & \\
2 \arrow[d, shift right, "b"'] \arrow[d, shift left, "c"] & 4 \\
1 &
\end{tikzcd}$$

and I is the ideal generated by ba. The indecomposable projective A-modules are of the form $P_1 = {\scriptstyle 1}$, $P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}$, $P_3 = \begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & & \end{smallmatrix}$, $P_4 = {\scriptstyle 4}$. Then ${}_\infty\underline{\Delta}$ has two connected components, one is an (infinite) preprojective component, and the other one consists just of the vertex P_4:

$$\begin{tikzcd}
 & \begin{smallmatrix}0\\1\;0\\2\end{smallmatrix} \arrow[dr, "2"] & & \begin{smallmatrix}0\\3\;0\\4\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \begin{smallmatrix}0\\5\;0\\6\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \cdots \arrow[ll, dashed] \\
\begin{smallmatrix}0\\0\;0\\1\end{smallmatrix} \arrow[ur, "2"] & & \begin{smallmatrix}0\\2\;0\\3\end{smallmatrix} \arrow[ll, dashed] \arrow[ur, "2"] & & \begin{smallmatrix}0\\4\;0\\5\end{smallmatrix} \arrow[ll, dashed] \arrow[ur, "2"] & & \cdots \arrow[ll, dashed] \arrow[ur, "2"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{smallmatrix}0\\0\;1\\0\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\1\;0\\1\end{smallmatrix} \arrow[ll, dashed] \\
 & \begin{smallmatrix}1\\1\;1\\1\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

(j): Let $A = KQ/I$ where Q is the quiver

$$\begin{tikzcd}
3 \arrow[d, shift right, "b"'] \arrow[d, shift left, "a"] \\
2 \arrow[d, "c"'] \\
1
\end{tikzcd}$$

and I is the ideal generated by ca and cb. The indecomposable projective A-modules are of the form $P_1 = {\scriptstyle 1}$, $P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}$, $P_3 = \begin{smallmatrix} & 3 & \\ 2 & & 2 \end{smallmatrix}$. Then ${}_\infty\underline{\Delta}$ consists of an infinite preprojective component containing an injective module:

$$\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}0\\1\\1\end{smallmatrix} \arrow[dr] & & & & & & & & \\
\begin{smallmatrix}0\\0\\1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\1\\0\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \begin{smallmatrix}2\\3\\0\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \begin{smallmatrix}4\\5\\0\end{smallmatrix} \arrow[ll, dashed] \arrow[dr, "2"] & & \cdots \arrow[ll, dashed] \arrow[dr, "2"] & \\
 & & & \begin{smallmatrix}1\\2\\0\end{smallmatrix} \arrow[ur, "2"] & & \begin{smallmatrix}3\\4\\0\end{smallmatrix} \arrow[ll, dashed] \arrow[ur, "2"] & & \begin{smallmatrix}5\\6\\0\end{smallmatrix} \arrow[ll, dashed] \arrow[ur, "2"] & & \cdots \arrow[ll, dashed]
\end{tikzcd}$$

(l): Let $A = K[T]/(T^4)$. There is just one simple A-modules S, and all indecomposable A-modules are uniserial. The Auslander-Reiten quiver looks like this:

$$\begin{tikzcd}
\begin{smallmatrix}S\\S\\S\\S\end{smallmatrix} \arrow[d, shift left] \\
\begin{smallmatrix}S\\S\\S\end{smallmatrix} \arrow[u, shift left] \arrow[d, shift left] \arrow[loop right, dashed] \\
\begin{smallmatrix}S\\S\end{smallmatrix} \arrow[u, shift left] \arrow[d, shift left] \arrow[loop right, dashed] \\
S \arrow[u, shift left] \arrow[loop right, dashed]
\end{tikzcd}$$

The only indecomposable projective A-module has length 4. For the other three indecomposables we have $\tau_A(X) \cong X$. For example, the obvious sequence of the form

$$0 \to \begin{smallmatrix}S\\S\end{smallmatrix} \to S \oplus \begin{smallmatrix}S\\S\\S\end{smallmatrix} \to \begin{smallmatrix}S\\S\end{smallmatrix} \to 0$$

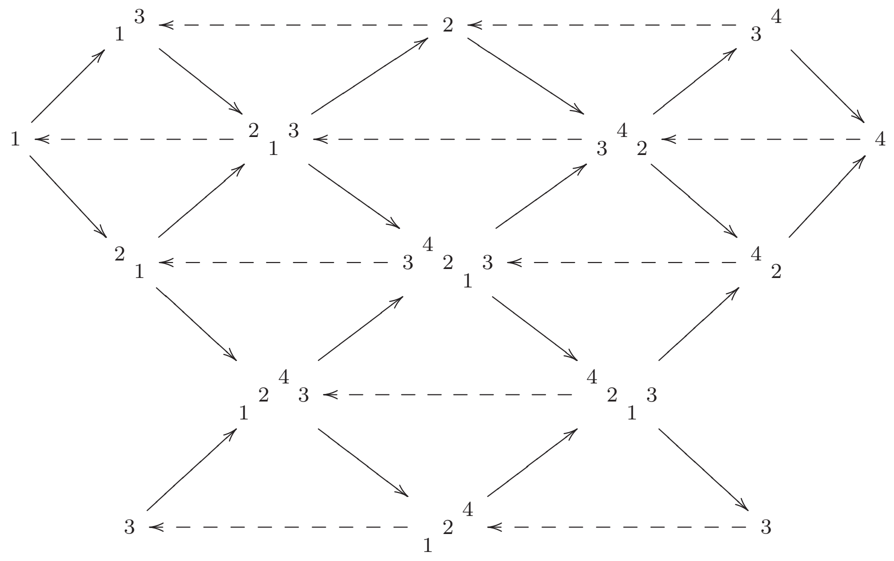
is an Auslander-Reiten sequence.

(m): Let Q be the quiver

$$\begin{tikzcd}
 & 4 \arrow[dl] \arrow[dr, "a"] & \\
2 \arrow[dr] & & 3 \arrow[dl, "b"] \\
 & 1 &
\end{tikzcd}$$

and set $A = KQ/(ba)$.

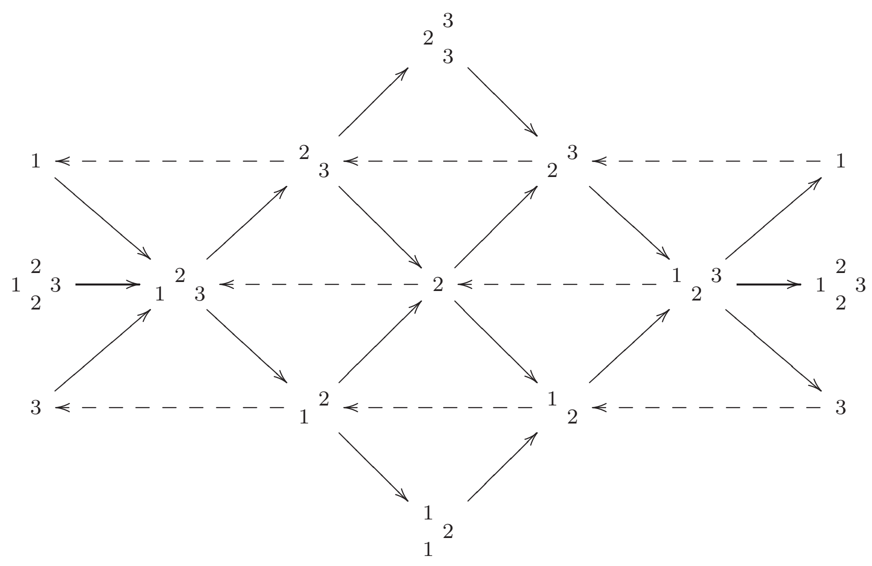
Using the socle series notation the Auslander-Reiten quiver of A looks as follows:



(n): Let Q be the quiver

$$1 \underset{c}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{b}{\rightleftarrows}} 3$$

and let $A = KQ/I$ where I is generated by $ba, cd, ac - db$. The (Γ_A, d_A) looks as follows (one has to identify the three modules on the left with the three modules on the right):

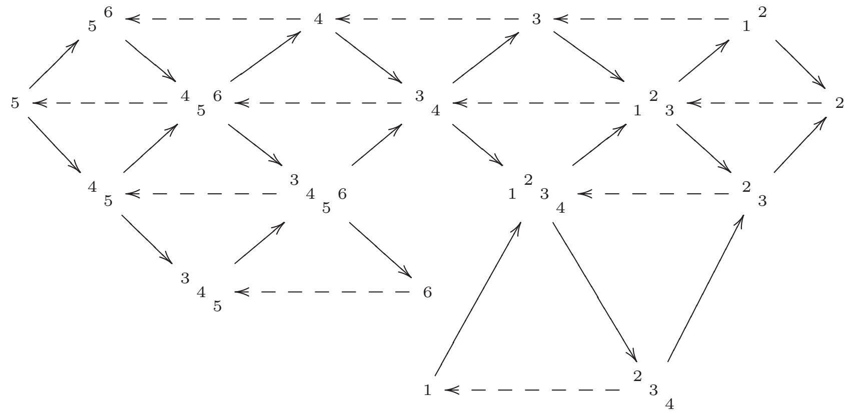


Note that A is a selfinjective algebra, i.e. an A-module is projective if and only if it is injective.

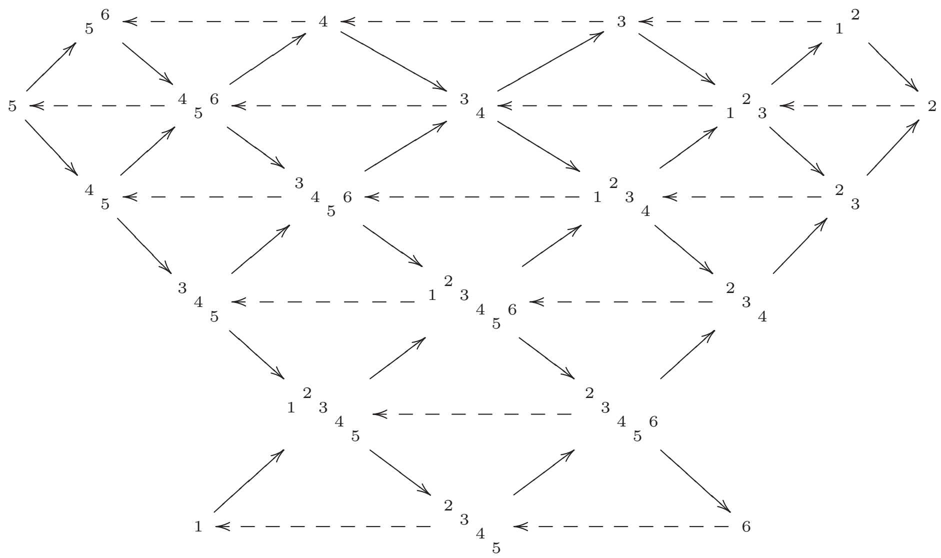
(o): Let Q be the quiver

$$1 \longleftarrow 2 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{c} 5 \longleftarrow 6$$

and let $A = KQ/I$ where I is generated by cba. Then (Γ_A, d_A) looks as follows:



Next, we display the Auslander-Reiten quiver of KQ:



8. **Grothendieck group and Ringel form**

8.1. **Grothendieck group.** As before, let A be a finite-dimensional K-algebra, and let $S_1, \ldots, S_n$ be a complete set of representatives of isomorphism classes of the simple A-modules. For a finite-dimensional module M let

$$\underline{\dim}(M) := ([M : S_1], \ldots, [M : S_n])$$

be its dimension vector. Here $[M : S_i]$ is the Jordan-Hölder multiplicity of S_i in M. Note that $\underline{\dim}(M) \in \mathbb{N}_0^n \subset \mathbb{Z}^n$. Set $e_i := \underline{\dim}(S_i)$. Then

$$G(A) := K_0(A) := \mathbb{Z}^n$$

is the **Grothendieck group** of mod(A), and $e_1, \ldots, e_n$ is a free generating set of the abelian group $G(A)$.

We can see $\underline{\dim}$ as a map

$$\underline{\dim}\colon \{A\text{-modules}\}/\cong \longrightarrow G(A)$$

which associates to each modules M, or more precisely to each isomorphism class $[M]$, the dimension vector $\underline{\dim}(M)$.

Note that

$$\sum_{i=1}^{n} [M : S_i] = l(X).$$

Furthermore, $\underline{\dim}$ is **additive on short exact sequences** , i.e. if $0 \to X \to Y \to Z \to 0$ is a short exact sequence, then $\underline{\dim}(Y) = \underline{\dim}(X) + \underline{\dim}(Z)$.

Lemma 8.1. *If*

$$f\colon \{A\text{-modules}\}/\cong \longrightarrow H$$

is a map which is additive on short exact sequences and H is an abelian group, then there exists a unique group homomorphism $f'\colon G(A) \to H$ such that the diagram

$$\begin{tikzcd}
\{A\text{-modules}\}/\cong \arrow[r, "\underline{\dim}"] \arrow[d, "f"'] & G(A) \arrow[dl, dashed, "f'"] \\
H &
\end{tikzcd}$$

commutes.

Proof. Define a group homomorphism $f'\colon G(A) \to H$ by $f'(e_i) := f(S_i)$ for $1 \leq i \leq n$. We have to show that $f'(\underline{\dim}(M)) = f(M)$ for all finite-dimensional A-modules M. We proof this by induction on the length $l(M)$ of M. If $l(M) = 1$, then $M \cong S_i$ and we are done, since $f'(\underline{\dim}(M)) = f'(e_i) = f(S_i)$.

Next, assume $l(M) > 1$. Then there exists a submodule U of M such that $U \neq 0 \neq M/U$. We obtain a short exact sequence

$$0 \to U \to M \to M/U \to 0.$$

Clearly, $l(U) < l(M)$ and $l(M/U) < l(M)$. Thus by induction $f'(\underline{\dim}(U)) = f(U)$ and $f'(\underline{\dim}(M/U)) = f(M/U)$. Since f is additive on short exact sequences, we get

$$f(M) = f(U) + f(M/U) = f'(\underline{\dim}(U)) + f'(\underline{\dim}(M/U)) = f'(\underline{\dim}(M)).$$

It is obvious that f' is unique. This finishes the proof. $\square$

Here is an alternative construction of $G(A)$: Let $F(A)$ be the free abelian group with generators the isomorphism classes of finite-dimensional A-modules. Let $U(A)$ be the subgroup of $F(A)$ which is generated by the elements of the form

$$[X] - [Y] + [Z]$$

if there is a short exact sequence $0 \to X \to Y \to Z \to 0$. Define

$$G(A) := F(A)/U(A).$$

For an A-module M set $\overline{[M]} := [M] + U(A)$. It follows that $G(A)$ is isomorphic to $\mathbb{Z}^n$ with generators $\overline{[S_i]}$, $1 \leq i \leq n$. By induction on $l(M)$ one shows that

$$\overline{[M]} = \sum_{i=1}^{n} [M : S_i] \cdot \overline{[S_i]}.$$

8.2. **The Ringel form.** We assume now that A is a finite-dimensional K-algebra with $\mathrm{gl.dim}(A) = d < \infty$. In other words, we assume $\mathrm{Ext}_A^{d+1}(X, Y) = 0$ for all A-modules X and Y and d is minimal with this property.

Define

$$\langle X, Y\rangle_A := \sum_{t=0}^{d} (-1)^t \dim \mathrm{Ext}_A^t(X, Y).$$

(If $\mathrm{gl.dim}(A) = \infty$, but $\mathrm{proj.dim}(X) < \infty$ or $\mathrm{inj.dim}(Y) < \infty$, then we can still define $\langle X, Y\rangle_A := \sum_{t\geq 0}(-1)^t \dim \mathrm{Ext}_A^t(X, Y)$.)

Recall that $\mathrm{Ext}_A^0(X, Y) = \mathrm{Hom}_A(X, Y)$. We know that for each short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

and an A-module Y we get a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}_A^0(X'', Y) \arrow[r] & \mathrm{Ext}_A^0(X, Y) \arrow[r] & \mathrm{Ext}_A^0(X', Y) \arrow[dll] \\
& \mathrm{Ext}_A^1(X'', Y) \arrow[r] & \mathrm{Ext}_A^1(X, Y) \arrow[r] & \mathrm{Ext}_A^1(X', Y) \arrow[dll] \\
& \mathrm{Ext}_A^2(X'', Y) \arrow[r] & \mathrm{Ext}_A^2(X, Y) \arrow[r] & \mathrm{Ext}_A^2(X', Y) \arrow[dll] \\
& \mathrm{Ext}_A^3(X'', Y) \arrow[r] & \cdots &
\end{tikzcd}$$

Now one easily checks that this implies

$$\sum_{t=0}^{d} (-1)^t \dim \mathrm{Ext}_A^t(X'', Y) - \sum_{t=0}^{d} (-1)^t \dim \mathrm{Ext}_A^t(X, Y) + \sum_{t=0}^{d} (-1)^t \dim \mathrm{Ext}_A^t(X', Y) = 0.$$

In other words,

$$\langle X'', Y\rangle_A - \langle X, Y\rangle_A + \langle X, Y\rangle_A = 0.$$

It follows that

$$\langle -, Y\rangle_A \colon \{A\text{-modules}\}/\cong \to \mathbb{Z}$$

is a map which is additive (on short exact sequences). Thus $\langle \underline{\dim}(X), Y\rangle_A := \langle X, Y\rangle_A$ is well defined.

Similarly, we get that

$$\langle X, Y'\rangle_A - \langle X, Y\rangle_A + \langle X, Y''\rangle_A = 0.$$

if $0 \to Y' \to Y \to Y'' \to 0$ is a short exact sequence.

Thus $\langle \underline{\dim}(M), \underline{\dim}(N)\rangle_A := \langle M, N\rangle_A$ is well defined, and we obtain a bilinear map

$$\langle -, -\rangle_A \colon G(A) \times G(A) \to \mathbb{Z}.$$

This map is determined by the values

$$\langle e_i, e_j\rangle_A = \sum_{t=0}^{d} (-1)^t \dim \operatorname{Ext}_A^t(S_i, S_j)$$

since $\underline{\dim}(M) = \sum_{i=1}^n [M : S_i] e_i$.

---

9. **Reachable and directing modules**

Let K be a field, and let A be a finite-dimensional K-algebra. By $\mathcal{M} = \mathcal{M}(A)$ we denote the category $\mathrm{mod}(A)$ of all finite-dimensional A-modules.

9.1. **Reachable modules.** A **path** of length $n \geq 0$ in $\mathcal{M}$ is a finite sequence $([X_0], [X_1], \dots, [X_n])$ of isomorphism classes of indecomposable A-modules X_i such that for all $1 \leq i \leq n$ there exists a homomorphism $X_{i-1} \to X_i$ which is non-zero and not an isomorphism, in other words we assume $\mathrm{rad}_A(X_{i-1}, X_i) \neq 0$. We say that such a path $([X_0], [X_1], \dots, [X_n])$ starts in X_0 and ends in X_n. If $n \geq 1$ and $[X_0] = [X_n]$, then $([X_0], [X_1], \dots, [X_n])$ is a **cycle** in $\mathcal{M}$. In this case, we say that the modules $X_0, \dots, X_{n-1}$ **lie on a cycle**.

If X and Y are indecomposable A-modules, we write $X \preceq Y$ if there exists a path from X to Y, and we write $X \prec Y$ if there is such a path of length $n \geq 1$.

An indecomposable module X in $\mathcal{M}$ is **reachable** if there are only finitely many paths in $\mathcal{M}$ which end in X. Let

$$\mathcal{E}(A)$$

be the subcategory of reachable modules in $\mathcal{M}$.

Furthermore, we call X **directing** if X does not lie on a cycle, or equivalently, if $X \not\prec X$.

The following two statements are obvious:

Lemma 9.1. *Every reachable module is directing.*

Lemma 9.2. *If X is a directing module, then* $\mathrm{rad}(\mathrm{End}_A(X)) = 0$.

Examples: (a): Let $A = K[T]/(T^m)$ for some $m \geq 2$. Then none of the indecomposable A-modules is directing.

(b): If A is the path algebra of a quiver of type $\mathbb{A}_2$, then each indecomposable A-module is directing.

Let $\Gamma(A) = (\Gamma_A, d_A)$ be the Auslander-Reiten quiver of A. If Y is a reachable A-module, and $[X]$ is a predecessor of $[Y]$ in $\Gamma(A)$, then by definition there exists a path from $[X]$ to $[Y]$ in Γ_A. Thus, we also get a path from X to Y in $\mathcal{M}$. This implies that X is a reachable module as well. In particular, if Z is a reachable non-projective module, then $\tau_A(Z)$ is reachable. So the Auslander-Reiten translation maps the set of isomorphism classes of reachable modules into itself.

We define classes

$$\emptyset = {}_{-1}\mathcal{M} \subseteq {}_{0}\mathcal{M} \subseteq \cdots \subseteq {}_{n-1}\mathcal{M} \subseteq {}_{n}\mathcal{M} \subseteq \cdots$$

of indecomposable modules as follows: Set ${}_{-1}\mathcal{M} = \emptyset$. Let $n \geq 0$ and assume that ${}_{n-1}\mathcal{M}$ is already defined. Then let ${}_{n}\mathcal{M}$ be the subcategory of all indecomposable modules M in $\mathcal{M}$ with the following property: If N is indecomposable with $\mathrm{rad}_A(N, M) \neq 0$, then $N \in {}_{n-1}\mathcal{M}$.

Let

$${}_{\infty}\mathcal{M} = \bigcup_{n \geq 0} {}_{n}\mathcal{M}$$

be the full subcategory of $\mathcal{M}$ containing all $M \in {}_{n}\mathcal{M}$, $n \geq 0$.

Then the following hold:

(a) ${}_{n-1}\mathcal{M} \subseteq {}_{n}\mathcal{M}$ (Proof by induction on $n \geq 0$);
(b) ${}_{0}\mathcal{M}$ is the class of simple projective modules;
(c) ${}_{1}\mathcal{M}$ contains additionally all indecomposable projective modules P such that $\mathrm{rad}(P)$ is semisimple and projective;
(d) ${}_{2}\mathcal{M}$ can contain non-projective modules (e.g. if A is the path algebra of a quiver of type $\mathbb{A}_2$);
(e) ${}_{n}\mathcal{M}$ is closed under indecomposable submodules;
(f) If $g\colon Y \to Z$ is a sink map, and

$$Y = \bigoplus_{i=1}^{t} Y_i$$

a direct sum decomposition with Y_i indecomposable and $Y_i \in {}_{n-1}\mathcal{M}$ for all i, then $Z \in {}_{n}\mathcal{M}$; (Proof: Let N be indecomposable, and let $0 \neq h \in \mathrm{rad}_A(N, Z)$. Then there exists some $h'\colon N \to Y$ with $h = g \circ h'$.

$$\begin{tikzcd} & N \arrow[dl, dashed, "h'"'] \arrow[d, "h"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Thus we can find some $0 \neq h_i'\colon N \to Y_i$. If h_i' is an isomorphism, then $N \cong Y_i \in {}_{n-1}\mathcal{M}$. If h_i' is not an isomorphism, then $N \in {}_{n-2}\mathcal{M} \subseteq {}_{n-1}\mathcal{M}$.)

(g) If $Z \in {}_n\mathcal{M}$ is non-projective, then $\tau_A(Z) \in {}_{n-2}\mathcal{M}$;
(h) We have
$$\mathcal{E}(A) = {}_\infty\mathcal{M}.$$

Lemma 9.3. *Let A be a finite-dimensional K-algebra. If Z is an indecomposable A-module, then $Z \in {}_n\mathcal{M}$ if and only if $[Z] \in {}_n(\Gamma_A)$.*

Proof. The staatement is correct for $n = -1$. Thus assume $n \geq 0$. If $Z \in {}_n\mathcal{M}$ and
$$\bigoplus_{i=1}^{t} Y_i \to Z$$
is a sink map with Y_i indecomposable for all i, then $Y_i \in {}_{n-1}\mathcal{M}$ for all i. Thus by induction assumption $[Y_i] \in {}_{n-1}(\Gamma_A)$, and therefore $[Z] \in {}_n(\Gamma_A)$. Vice versa, if $[Z] \in {}_n(\Gamma_A)$, then $[Y_i] \in {}_{n-1}(\Gamma_A)$. Thus $Y_i \in {}_{n-1}\mathcal{M}$. Using (f) we get $Z \in {}_n\mathcal{M}$. □

Let
$$E(A)$$
be the full subquiver of all vertices $[X]$ of Γ_A such that X is a reachable module. One easily checks that $E(A)$ is again a valued translation quiver.

Summarizing our results and notation, we obtain
$$E(A) = {}_\infty(\underline{\Gamma}_A) = {}_\infty\underline{\Delta}, \quad \text{and} \quad \mathcal{E}(A) = {}_\infty\mathcal{M}.$$
Furthermore, $\mathcal{E}(A)$ is the full subcategory of all A-modules X such that $[X] \in E(A)$.

We say that K is a **splitting field** for A if $\mathrm{End}_A(S) \cong K$ for all simple A-modules S.

Examples: If K is algebraically closed, then K is a splitting field for K. Also, if $A = KQ$ is a finite-dimensional path algebra, then K is a splitting field for A.

Roughly speaking, if K is a splitting field for A, then there are more combinatorial tools available, which help to understand (parts of) $\mathrm{mod}(A)$. The most common tools are mesh categories and integral quadratic forms.

Theorem 9.4. *Let A be a finite-dimensional K-algebra, and assume that K is a splitting field for A. Then the valuation for $E(A)$ splits, and there is an equivalence of categories*
$$\eta\colon K\langle E(A)^e\rangle \to \mathcal{E}(A).$$

Proof. Let $\mathcal{I}$ be a complete set of indecomposable A-modules (thus we take exactly one module from each isomorphism class). Set
$${}_n\mathcal{I} = \mathcal{I} \cap {}_n\mathcal{M} \quad \text{and} \quad {}_\infty\mathcal{I} = \mathcal{I} \cap \mathcal{E}(A).$$

For $X, Y \in {}_\infty\mathcal{I}$ we want to construct homomorphisms
$$a^i_{XY} \in \mathrm{Hom}_A(X, Y)$$
with $1 \leq i \leq d_{XY} := \dim_K \mathrm{Irr}_A(X, Y)$.

If $Y = P$ is projective, we choose a direct decomposition

$$\operatorname{rad}(P) = \bigoplus_{X \in \mathcal{I}} X^{d_{XP}}.$$

We know that $d_{XP} = \dim_K \operatorname{Irr}_A(X, P)$. Let

$$a_{XP}^i \colon X \to P$$

with $1 \leq i \leq d_{XP}$ be the inclusion maps.

By induction we assume that for all $X, Y \in {}_n\mathcal{I}$ we have chosen homomorphisms $a_{XY}^i \colon X \to Y$ where $1 \leq i \leq d_{XY}$.

Let $Z \in {}_{n+1}\mathcal{I}$ be non-projective, and let

$$0 \to X \xrightarrow{f} \bigoplus_{Y \in {}_n\mathcal{I}} Y^{d_{XY}} \xrightarrow{g} Z \to 0$$

be the Auslander-Reiten sequence ending in Z, where the d_{XY} component maps $X \to Y$ of f are given by a_{XY}^i, $1 \leq i \leq d_{XY}$. Now g together with the direct sum decomposition

$$\bigoplus_{Y \in {}_n\mathcal{I}} Y^{d_{XY}}$$

yields homomorphisms $a_{YZ}^i \colon Y \to Z$, $1 \leq i \leq d_{XY} = d_{YZ}$. These homomorphisms obviously satisfy the equation

$$\sum_{Y \in {}_n\mathcal{I}} \sum_{i=1}^{d_{XY}} a_{YZ}^i a_{XY}^i = 0.$$

Denote the corresponding arrows from $[X]$ to $[Y]$ in

$$\Gamma := E(A)^e$$

by α_{XY}^i where $1 \leq i \leq d_{XY}$.

We obtain a functor

$$\eta \colon K\langle\Gamma\rangle \to \mathcal{E}(A)$$

as follows: For $X \in {}_\infty\mathcal{I}$ define

$$\eta([X]) := X \quad \text{and} \quad \eta\left(\alpha_{XY}^i\right) := a_{XY}^i.$$

This yields a functor $K\langle\Gamma\rangle \to \mathcal{E}(A)$, since by the equation above the mesh relations are mapped to 0.

Now we will show that η is bijective on the homomorphism spaces.

Before we start, note that $\operatorname{End}_A(X) \cong K$ for all $X \in \mathcal{E}(A)$. (Proof: A reachable module X does not lie on a cycle in $\mathcal{M}(A)$, thus $\operatorname{rad}(\operatorname{End}_A(X)) = 0$. This shows that $F(X) \cong \operatorname{End}_A(X)$. Let $X \in {}_\infty\mathcal{M} = \mathcal{E}(A)$. If $X = P$ is projective, then

$$F(X) \cong \operatorname{End}_A(P/\operatorname{rad}(P)) \cong \operatorname{End}_A(S) \cong K$$

where S is the simple A-module isomorphic to $P/\operatorname{rad}(P)$. Here we used that K is a splitting field for A. If X is non-projective, then $F(X) \cong F(\tau_A(X))$. Furthermore we know that $\tau_A^n(X)$ is projective for some $n \geq 1$. Thus by induction we get $F(X) \cong \mathrm{End}_A(X) \cong K$.)

Surjectivity of η: Let $h\colon M \to Z$ be a homomorphism in ${}_\infty\mathcal{I}$, and let $Z \in {}_n\mathcal{I}$. We use induction on n. If $M = Z$, then $h = c \cdot 1_M$ for some $c \in K$. Thus $h = \eta(c \cdot 1_{[M]})$. Assume now that $M \neq Z$. This implies that h is not an isomorphism. The sink map ending in Z is

$$g = (a_{YZ}^i)_{Y,i}\colon \bigoplus_{Y \in {}_{n-1}\mathcal{I}} Y^{d_{YZ}} \to Z.$$

We get

$$h = \sum_{Y,i} a_{YZ}^i h_{Y,i}.$$

By induction the homomorphisms $h_{Y,i}\colon M \to Y$ are in the image of η, and by the construction of η also the homomorphisms a_{YZ}^i are contained in the image of η. Thus h lies in the image of η

Injectivity of η: Let $\mathcal{R}$ be the mesh ideal in the path category $K\Gamma$. We investigate the kernel $\mathcal{K}$ of

$$\eta\colon K\Gamma \to {}_\infty\mathcal{I}.$$

Clearly, $\mathcal{R} \subseteq \mathcal{K}$. Next, let $\omega \in \mathcal{K}$. Thus $\omega \in \mathrm{Hom}_{K\Gamma}([M],[Z])$ for some $[M]$ and $[Z]$. We have to show that $\omega \in \mathcal{R}$. Assume $[Z] \in {}_n\mathcal{I}$. We use induction on n. Additionally, we can assume that $\omega \neq 0$. Thus there exists a path from $[M]$ to $[Z]$.

If $[M] = [Z]$, then $\omega = c \cdot 1_{[M]}$ and $\eta(\omega) = c \cdot 1_M = 0$. This implies $c = 0$ and therefore $\omega = 0$.

Thus we assume that $[M] \neq [Z]$. Now ω is a linear combination of paths from $[M]$ to $[Z]$, i.e. ω is of the form

$$\omega = \sum_{Y,i} \alpha_{YZ}^i \omega_{Y,i}$$

where the $\omega_{Y,i}$ are elements in $\mathrm{Hom}_{K\Gamma}([M],[Y])$. Note that $[Y] \in {}_{n-1}\mathcal{I}$. Applying η we obtain

$$0 = \eta(\omega) = \sum_{Y,i} a_{YZ}^i \eta(\omega_{Y,i}).$$

If Z is projective, then each $a_{YZ}^i\colon Y \to Z$ is an inclusion map, and we have

$$\mathrm{Im}(a_{Y_1Z}^{i_1}) \cap \mathrm{Im}(a_{Y_2,Z}^{i_2}) \neq 0$$

if and only if $Y_1 = Y_2$ and $i_1 = i_2$. This implies $a_{YZ}^i \eta(\omega_{Y,i}) = 0$ for all Y, i. Since a_{YZ}^i is injective, we get $\eta(\omega_{Y,i}) = 0$. Thus by induction $\omega_{Y,i} \in \mathcal{R}$ and therefore $\omega \in \mathcal{R}$.

Thus assume Z is not projective. Then we know the kernel of the map

$$g = (a_{YZ}^i)_{Y,i}\colon \bigoplus_{Y \in {}_{n-1}\mathcal{I}} Y^{d_{YZ}} \to Z$$

namely

$$f = (a_{XY}^i)_{Y,i}\colon X \to \bigoplus_{Y \in {}_{n-1}\mathcal{I}} Y^{d_{YZ}}.$$

Thus the map

$$h := (\eta(\omega_{Y,i}))_{Y,i}\colon M \to \bigoplus_{Y \in {}_{n-1}\mathcal{I}} Y^{d_{YZ}}$$

factorizes through f, since $g \circ h = 0$. So we obtain a homomorphism $h'\colon M \to X$ such that

$$(a^i_{XY})_{Y,i} \circ h' = (\eta(\omega_{Y,i}))_{Y,i}$$

and therefore $a^i_{XY} \circ h' = \eta(\omega_{Y,i})$.

By the surjectivity of η there exists some $\omega'\colon [M] \to [X]$ such that $\eta(\omega') = h'$. Thus we see that

$$\eta\left(\alpha^i_{XY}\omega'\right) = a^i_{XY} \circ h' = \eta(\omega_{Y,i}).$$

In other words, $\eta\left(\omega_{Y,i} - \alpha^i_{XY}\omega'\right) = 0$. By induction $\omega_{Y,i} - \alpha^i_{XY}\omega'$ belongs to the mesh ideal. Thus also

$$\begin{aligned}\omega &= \sum_{Y,i} \alpha^i_{YZ}\omega_{Y,i}\\ &= \sum_{Y,i} \alpha^i_{YZ}\left(\omega_{Y,i} - \alpha^i_{XY}\omega'\right) + \sum_{Y,i}\left(\alpha^i_{YZ}\alpha^i_{XY}\right)\omega'\end{aligned}$$

is contained in the mesh ideal. This finishes the proof. □

9.2. **Computations in the mesh category.** Let M and X be non-isomorphic indecomposable A-modules such that X is non-projective. Let $0 \to \tau_A(X) \to E \to X \to 0$ be the Auslander-Reiten sequence ending in X. Then

$$0 \to \operatorname{Hom}_A(M, \tau_A(X)) \to \operatorname{Hom}_A(M, E) \to \operatorname{Hom}_A(M, X) \to 0$$

is exact.

Let $\Gamma = (\Gamma_A, d_A)$. If $[X]$ and $[Z]$ are vertices in $E(A)$ such that none of the paths in Γ starting in $[X]$ and ending in $[Z]$ contains a subpath of the form $[Y] \to [E] \to [\tau_A^{-1}(Y)]$, then we have

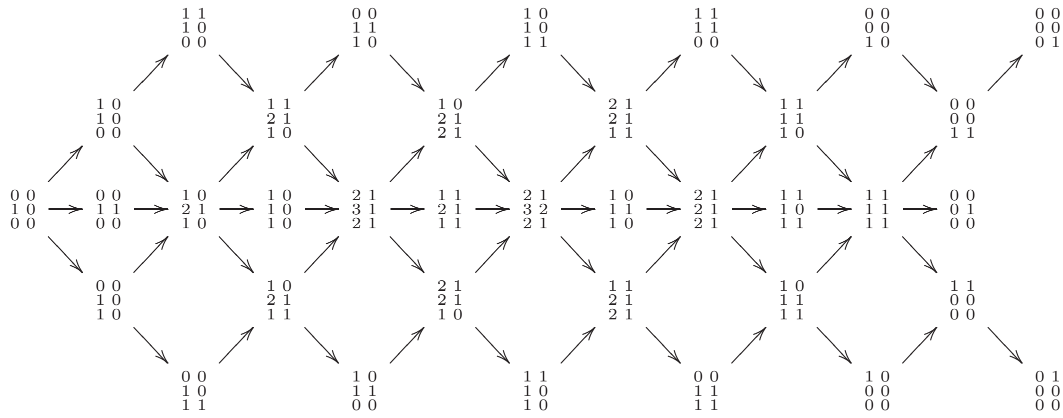
$$\operatorname{Hom}_{K\langle E(A)^e\rangle}([X],[Z]) = \operatorname{Hom}_{K\Gamma}([X],[Z]).$$

Using this and the considerations above, we can now calculate dimensions of homomorphism spaces using in the mesh category $K\langle E(A)^e\rangle$.

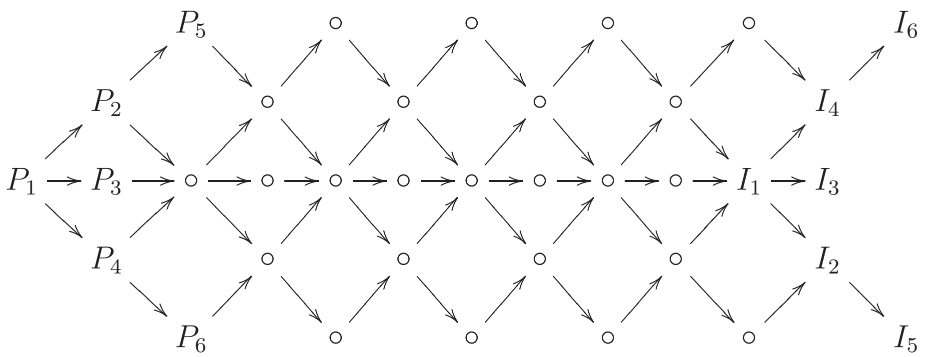
Let Q be the quiver

$$\begin{tikzcd} 2 \arrow[d] & 5 \arrow[l] \\ 1 & 3 \arrow[l] \\ 4 \arrow[u] & 6 \arrow[l] \end{tikzcd}$$

and let $A = KQ$. Here is the Auslander-Reiten quiver of A, using the dimension vector notation:

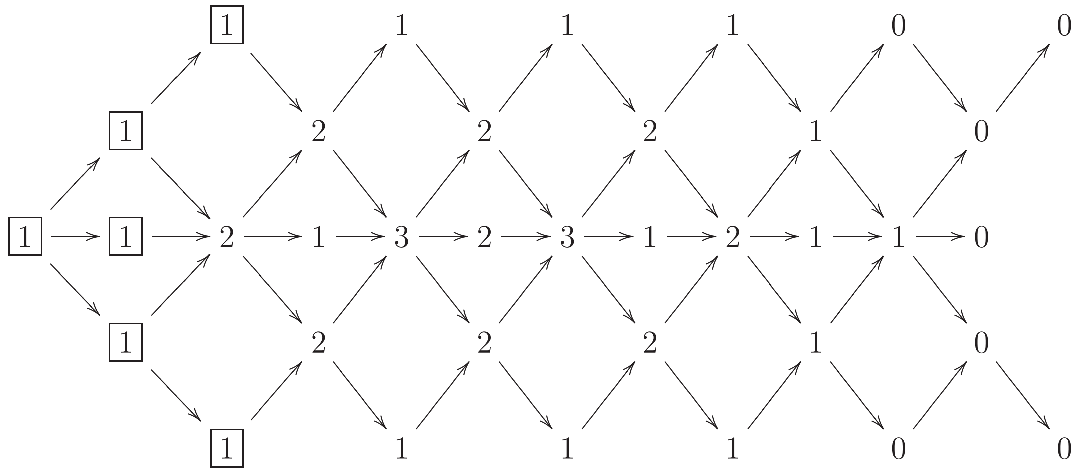


Here we display the locations of the indecomposable projective and the indecomposable injective A-modules:

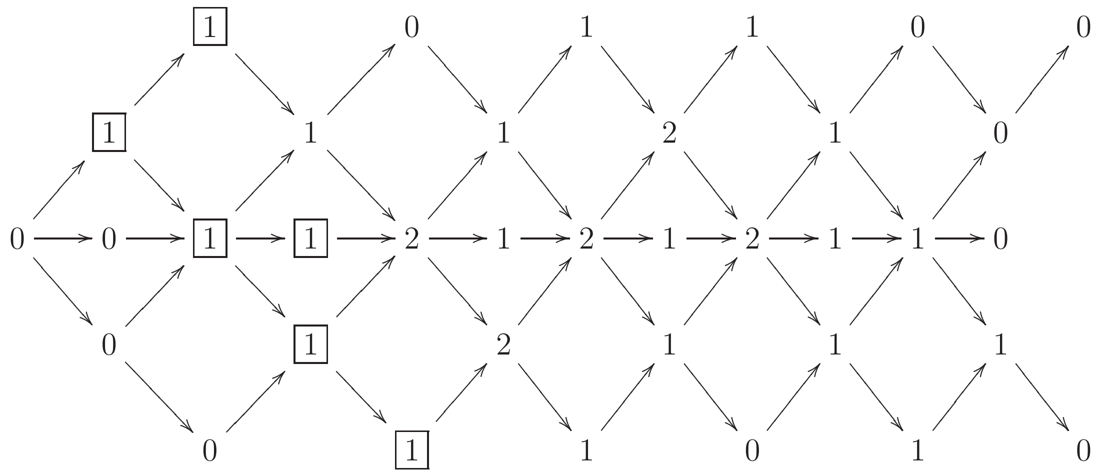


The following pictures show how to compute $\dim \operatorname{Hom}_A(P_i, -)$ for all indecomposable projective A-modules P_i. Note that the cases P_2 and P_4, and also P_5 and P_6 are dual to each other. We marked the vertices $[Z]$ by $\boxed{\text{a}}$ where $a = \dim \operatorname{Hom}_A(P_i, Z)$, provided none of the paths in $E(A)$ starting in $[P_i]$ and ending in $[Z]$ contains a subpath of the form $[Y] \to [E] \to [\tau_A^{-1}(Y)]$. Of course, we can compute $\dim \operatorname{Hom}_A(X, -)$ for any indecomposable A-module.

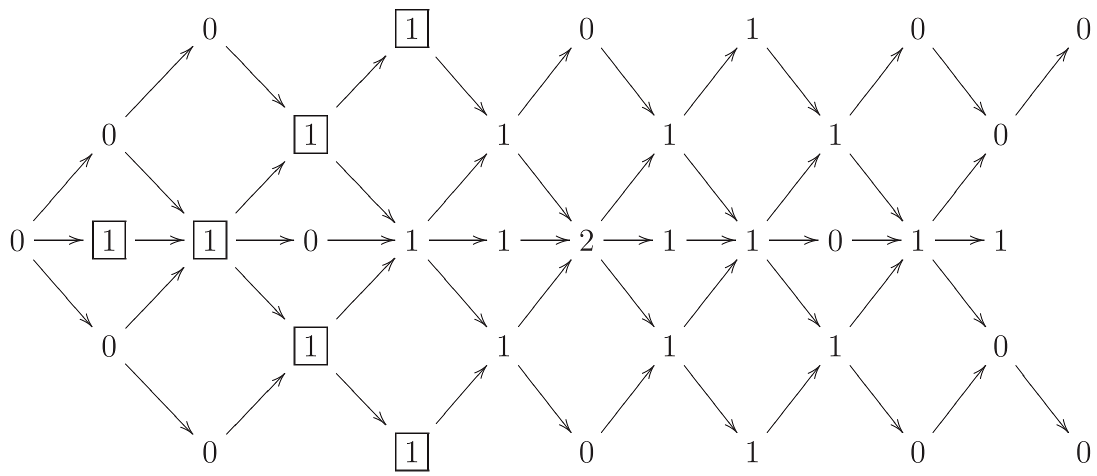
dim $\mathrm{Hom}_A(P_1, -)$:



dim $\mathrm{Hom}_A(P_2, -)$:



dim $\mathrm{Hom}_A(P_3, -)$:



dim $\mathrm{Hom}_A(P_5, -)$:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \boxed{1} \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & 1 \arrow[dr] & & 0 \arrow[dr] & & 0 \\
 & 0 \arrow[ur] \arrow[dr] & & \boxed{1} \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & 1 \arrow[ur] \arrow[dr] & & 1 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
0 \arrow[r] \arrow[ur] \arrow[dr] & 0 \arrow[r] & 0 \arrow[r] \arrow[ur] \arrow[dr] & 0 \arrow[r] & \boxed{1} \arrow[r] \arrow[ur] \arrow[dr] & \boxed{1} \arrow[r] & 1 \arrow[r] \arrow[ur] \arrow[dr] & 0 \arrow[r] & 1 \arrow[r] \arrow[ur] \arrow[dr] & 1 \arrow[r] & 1 \arrow[r] \arrow[ur] \arrow[dr] & 0 & \\
 & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & \boxed{1} \arrow[ur] \arrow[dr] & & 1 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & 1 \arrow[dr] & \\
 & & 0 \arrow[ur] & & 0 \arrow[ur] & & \boxed{1} \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 1
\end{tikzcd}
$$

9.3. **Directing modules.**

Lemma 9.5. *Let X be a directing A-module, then $\mathrm{End}_A(X)$ is a skew-field, and we have $\mathrm{Ext}^i_A(X,X) = 0$ for all $i \geq 1$.*

Proof. Since $\mathrm{rad}(\mathrm{End}_A(X)) = 0$, we know that $\mathrm{End}_A(X)$ is a skew-field. It is also clear that $\mathrm{Ext}^1_A(X,X) = 0$: If $0 \to X \to M \to X \to 0$ is a short exact sequence which does not split, then we immediately get a cycle (X, M_i, X) where M_i is an indecomposable direct summand of M.

Let $\mathcal{C}$ be the class of indecomposable A-modules M with $M \preceq X$. We will show by induction that $\mathrm{Ext}^j_A(M,X) = 0$ for all $M \in \mathcal{C}$ and all $j \geq 1$:

The statement is clear for $j = 1$. Namely, if $\mathrm{Ext}^1_A(M,X) \neq 0$, then any non-split short exact sequence

$$0 \to X \to \bigoplus_i Y_i \to M \to 0$$

yields $X \prec M \preceq X$, a contradiction.

Next, assume $j > 1$. Without loss of generality assume M is not projective. Let $0 \to \Omega(M) \to P_0 \xrightarrow{\varepsilon} M \to 0$ be a short exact sequence where $\varepsilon\colon P_0 \to M$ is a projective cover of M. We get

$$\mathrm{Ext}^j_A(M,X) \cong \mathrm{Ext}^{j-1}_A(\Omega(M), X).$$

If $\mathrm{Ext}^j_A(M,X) \neq 0$, then there exists an indecomposable direct summand M' of $\Omega(M)$ such that $\mathrm{Ext}^{j-1}_A(M',X) \neq 0$. But for some indecomposable direct summand P of P_0 we have $M' \preceq P \prec M \preceq X$, and therefore $M' \in \mathcal{C}$. This is a contradiction to our induction assumption. □

Corollary 9.6. *Assume* $\mathrm{gl.dim}(A) < \infty$, *and let X be a directing A-module. Then the following hold:*

(i) $\chi_A(X) = \langle X, X \rangle_A = \dim_K \mathrm{End}_A(X)$;

(ii) *If K is algebraically closed, then $\chi_A(X) = 1$;*
(iii) *If K is a splitting field for A, and if X is preprojective or preinjective, then $\chi_A(X) = 1$.*

As before, let A be a finite-dimensional K-algebra. An A-module M is **sincere** if each simple A-module occurs as a composition factor of M.

We call the algebra A **sincere** if there exists an indecomposable sincere A-module.

Lemma 9.7. *For an A-module M the following are equivalent:*

(i) *M is sincere;*
(ii) *For each simple A-module S we have $[M : S] \neq 0$;*
(iii) *If e is a non-zero idempotent in A, then $eM \neq 0$;*
(iv) *For each indecomposable projective A-module P we have $\mathrm{Hom}_A(P, M) \neq 0$;*
(v) *For each indecomposable injective A-module I we have $\mathrm{Hom}_A(M, I) \neq 0$*

Proof. **Exercise**. □

Theorem 9.8. *Let M be a sincere directing A-module. Then the following hold:*

(i) $\mathrm{proj.dim}(M) \leq 1$;
(ii) $\mathrm{inj.dim}(M) \leq 1$;
(iii) $\mathrm{gl.dim}(A) \leq 2$.

Proof. (i): We can assume that M is not projective. Assume there exists an indecomposable injective A-module I with $\mathrm{Hom}_A(I, \tau(M)) \neq 0$. Since M is sincere, we have $\mathrm{Hom}_A(M, I) \neq 0$. This yields $M \preceq I \prec \tau(M) \prec M$, a contradiction. Thus $\mathrm{proj.dim}(M) \leq 1$.

(ii): This is similar to (i).

(iii): Assume $\mathrm{gl.dim}(A) > 2$. Thus there are indecomposable A-modules with $\mathrm{Ext}^3_A(U, V) \neq 0$. Let $0 \to \Omega(U) \to P_0 \xrightarrow{\varepsilon} U \to 0$ be a short exact sequence with $\varepsilon\colon P_0 \to U$ a projective cover. It follows that $\mathrm{Ext}^2_A(\Omega(U), V) \cong \mathrm{Ext}^3_A(U, V) \neq 0$. Thus $\mathrm{proj.dim}(\Omega(U)) \geq 2$. Let U' be an indecomposable direct summand of $\Omega(U)$ with $\mathrm{proj.dim}(U') \geq 2$. This implies $\mathrm{Hom}_A(I, \tau_A(U')) \neq 0$ for some indecomposable injective A-module I. It follows that

$$M \preceq I \prec \tau_A(U') \prec U' \prec P \preceq M$$

where P is an indecomposable direct summand of P_0, a contradiction. The first and the last inequality follows from our assumption that M is sincere. This finishes the proof. □

Theorem 9.9. *Let X and Y be indecomposable finite-dimensional A-modules with $\underline{\dim}(X) = \underline{\dim}(Y)$. If X is a directing module, then $X \cong Y$.*

Proof. (a): Without loss of generality we can assume that X and Y are sincere:

Assume X is not sincere. Then let R be the two-sided ideal in A which is generated by all primitive idempotents $e \in A$ such that $eX = 0$. It follows that $R \subseteq \mathrm{Ann}_A(X) := \{a \in A \mid aX = 0\}$ and $R \subseteq \mathrm{Ann}_A(Y) := \{a \in A \mid aY = 0\}$. Clearly, $eX = 0$ if and only $eY = 0$, since $\underline{\dim}(X) = \underline{\dim}(Y)$. We also know that $\mathrm{Ann}_A(X)$ is a two-sided ideal: If $a_1X = 0$ and $a_2X = 0$, then $(a_1 + a_2)X = 0$. Furthermore, if $aX = 0$, then $a'aX = 0$ and also $aa''X \subseteq aX = 0$ for all $a', a'' \in A$. It follows that X and Y are indecomposable sincere A/R-modules. Furthermore, X is also directing as an A/R-module, since a path in $\mathrm{mod}(A/R)$ can also be seen as a path in $\mathrm{mod}(A)$. Thus from now on assume that X and Y are sincere.

(b): Since X is directing, we get $\mathrm{proj.dim}(X) \leq 1$, $\mathrm{inj.dim}(X) \leq 1$ and $\mathrm{gl.dim}(A) \leq 2$. Furthermore, we know that $\langle \underline{\dim}(X), \underline{\dim}(X) \rangle_A = \dim_K \mathrm{End}_A(X) > 0$, and therefore

$$\begin{aligned}\langle \underline{\dim}(X), \underline{\dim}(X) \rangle_A &= \langle \underline{\dim}(X), \underline{\dim}(Y) \rangle_A \\ &= \dim \mathrm{Hom}_A(X, Y) - \dim \mathrm{Ext}^1_A(X, Y) + \dim \mathrm{Ext}^2_A(X, Y).\end{aligned}$$

We have $\mathrm{Ext}^2_A(X, Y) = 0$ since $\mathrm{proj.dim}(X) \leq 1$. It follows that $\mathrm{Hom}_A(X, Y) \neq 0$. Similarly,

$$\langle \underline{\dim}(X), \underline{\dim}(X) \rangle_A = \langle \underline{\dim}(Y), \underline{\dim}(X) \rangle_A = \dim \mathrm{Hom}_A(Y, X) - \mathrm{Ext}^1_A(Y, X)$$

since $\mathrm{inj.dim}(X) \leq 1$. This implies $\mathrm{Hom}_A(Y, X) \neq 0$. Thus, if $X \not\cong Y$, we get $X \prec Y \prec X$, a contradiction. □

Motivated by the previous theorem, we say that an indecomposable A-module X is **determined by composition factors** if $X \cong Y$ for all indecomposable A-modules Y with $\underline{\dim}(X) = \underline{\dim}(Y)$.

Summary

Let A be a finite-dimensional K-algebra. By $\mathrm{mod}(A)$ we denote the category of finite-dimensional left A-modules. Let $\mathrm{ind}(A)$ be the subcategory of $\mathrm{mod}(A)$ containing all indecomposable A-modules.

The two general problems are these:

Problem 9.10. *Classify all modules in* $\mathrm{ind}(A)$.

Problem 9.11. *Describe* $\mathrm{Hom}_A(X, Y)$ *for all modules* $X, Y \in \mathrm{ind}(A)$.

Note that we do not specify what "classify" and "describe" should exactly mean.

(a) Let $\mathcal{E}(A)$ be the subcategory of $\mathrm{ind}(A)$ containing all reachable A-modules. For all $X \in \mathcal{E}(A)$ and all $Y \in \mathrm{ind}(A)$ we have $\underline{\dim}(X) = \underline{\dim}(Y)$ if and only if $X \cong Y$.
(b) The knitting algorithm gives ${}_\infty\underline{\Delta} = {}_\infty(\underline{\Gamma}_A) = E(A)$, and for each $[X] \in E(A)$ we can compute $\underline{\dim}(X)$.
(c) For $X \in \mathrm{ind}(A)$ we have $[X] \in E(A)$ if and only if $X \in \mathcal{E}(A)$.

(d) If K is a splitting field for A (for example, if K is algebraically closed), then the mesh category $K\langle E(A)^e\rangle$ is equivalent to $\mathcal{E}(A)$.
(e) We can use the mesh category of compute dim $\mathrm{Hom}_A(X,Y)$ for all $X, Y \in \mathcal{E}(A)$.

We cannot hope to solve Problems 9.10 and 9.11 in general, but for the subcategory $\mathcal{E}(A) \subseteq \mathrm{ind}(A)$ of reachable A-modules, we get a complete classification of reachable A-modules (the isomorphism classes of reachable modules are in bijection with the dimension vectors obtained by the knitting algorithm), and we know a lot of things about the morphism spaces between them.

Keep in mind that there is also a dual theory, using "coreachable modules" etc.

Furthermore, for some classes of algebras we have $\mathcal{E}(A) = \mathrm{ind}(A)$, for example if A is a representation-finite path algebra, or more generally if Γ_A is a union of preprojective components.

9.4. **The quiver of an algebra.** Let A be a finite-dimensional K-algebra. The **valued quiver** Q_A of A has vertices $1, \ldots, n$, and there is an arrow $i \to j$ if and only if $\dim_K \mathrm{Ext}^1_A(S_i, S_j) \neq 0$. In this case, the arrow has valuation

$$d_{ij} := \dim_K \mathrm{Ext}^1_A(S_i, S_j).$$

Each vertex i of Q_A has valuation $d_i := \dim_K \mathrm{End}_A(S_i)$.

Let Q_A^{op} be the opposite quiver of A, which is obtained from Q_A by reversing all arrows. The valuation of arrows and vertices stays the same.

Note that Q_A and Q_A^{op} can be seen as valued translation quivers, where all vertices are projective and injective.

Special case: Assume that A is hereditary. Then we have

$$d_{P_jP_i} = d_{ij} \quad \text{and} \quad d_{P_i} = d_{S_i} = d_i.$$

Thus, the subquiver $\mathcal{P}_A$ of preprojective components of (Γ_A, d_A) is (as a valued translation quiver) isomorphic to $\mathbb{N}Q_A^{\mathrm{op}}$.

We define the **valued graph** $\overline{Q}_A$ of A as follows: The vertices are again $1, \ldots, n$. There is a (non-oriented) edge between i and j if and only if

$$\mathrm{Ext}^1_A(S_i, S_j) \oplus \mathrm{Ext}^1_A(S_j, S_i) \neq 0.$$

Such an edge has as a valuation the pair

$$(\dim_{\mathrm{End}_A(S_j)} \mathrm{Ext}^1_A(S_i, S_j), \dim_{\mathrm{End}_A(S_i)^{\mathrm{op}}} \mathrm{Ext}^1_A(S_i, S_j)) = (d_{ij}/d_j, d_{ij}/d_i).$$

Example of a valued graph:

$$\cdot \;\text{———}\; \cdot \;\overset{(2,1)}{\text{———}}\; \cdot \;\text{———}\; \cdot$$

The representation-finite hereditary algebras can be characterized as follows:

Theorem 9.12. *A hereditary algebra A is representation-finite if and only if $\overline{Q}_A$ is a Dynkin graph.*

The list of Dynkin graphs can be found in **Skript 3**. Note that non-isomorphic hereditary algebras can have the same valued graph.

9.5. **Exercises.** **1**: Let A be an algebra with $\mathrm{gl.dim}(A) \geq d$. Show that there exist indecomposable A-modules X and Y with $\mathrm{Ext}^d_A(X, Y) \neq 0$.

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10. **Cartan and Coxeter matrix**

Let A be a finite-dimensional K-algebra. We use the usual notation:

- $P_1, \dots, P_n$ are the indecomposable projective A-modules;
- $I_1, \dots, I_n$ are the indecomposable injective A-modules;
- $S_1, \dots, S_n$ are the simple A-modules;
- $S_i \cong \mathrm{top}(P_i) \cong \mathrm{soc}(I_i)$.

(Of course, the modules P_i, I_i and S_i are just sets of representatives of isomorphism classes of projective, injective and simple A-modules, respectively.)

Let X and Y be A-modules.

If $\mathrm{proj.dim}(X) < \infty$ or $\mathrm{inj.dim}(Y) < \infty$, then

$$\langle X, Y \rangle_A := \langle \underline{\dim}(X), \underline{\dim}(Y) \rangle_A := \sum_{t \geq 0} (-1)^t \dim_K \mathrm{Ext}^i_A(X, Y)$$

is the Ringel form of A. This defines a (not necessarily symmetric) bilinear form $\langle -, - \rangle_A \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$.

If $\mathrm{proj.dim}(X) < \infty$ or $\mathrm{inj.dim}(X) < \infty$, then set

$$\chi_A(X) := \chi_A(\underline{\dim}(X)) := \langle X, X \rangle_A = \sum_{t \geq 0} (-1)^t \dim_K \mathrm{Ext}^i_A(X, X).$$

This defines a quadratic form $\chi_A(-) \colon \mathbb{Z}^n \to \mathbb{Z}$.

10.1. **Coxeter matrix.**

We did all the missing proofs in this section in the lectures. But you also find them in Ringel's book.

If $\underline{\dim}(P_1), \dots, \underline{\dim}(P_n)$ are linearly independent, then define the **Coxeter matrix** Φ_A of A by

$$\underline{\dim}(P_i)\Phi_A = -\underline{\dim}(I_i)$$

for $1 \leq i \leq n$. It follows that $\Phi_A \in M_n(\mathbb{Q})$.

Lemma 10.1. *If* $\operatorname{gl.dim}(A) < \infty$, *then* $\underline{\dim}(P_1), \dots, \underline{\dim}(P_n)$ *are linearly independent.*

Proof. We know that $\operatorname{gl.dim}(A) < \infty$ if and only if $\operatorname{proj.dim}(S) < \infty$ for all simple A-modules S. Furthermore $\{\underline{\dim}(S_i) \mid 1 \leq i \leq n\}$ are a free generating set of the Grothendieck group $G(A)$. Let

$$0 \to P^{(d)} \to \cdots \to P^{(1)} \to P^{(0)} \to S \to 0$$

be a minimal projective resolution of a simple A-module S. This implies

$$\sum_{i=0}^{d} (-1)^i \underline{\dim}(P^{(i)}) = \underline{\dim}(S).$$

Thus the vectors $\underline{\dim}(P_i)$ generate $\mathbb{Z}^n$. The result follows. □

Dually, if $\operatorname{gl.dim}(A) < \infty$, then $\underline{\dim}(I_1), \dots, \underline{\dim}(I_n)$ are also linearly independent. So Φ_A is invertible in this case.

By the definition of Φ_A, for each $P \in \operatorname{proj}(A)$ we have

$$\underline{\dim}(P)\Phi_A = -\underline{\dim}(\nu(P)). \tag{2}$$

Let M be an A-module, and let $P^{(1)} \xrightarrow{p} P^{(0)} \to M \to 0$ be a minimal projective presentation of M. Thus we obtain an exact sequence

$$0 \to M'' \to P^{(1)} \to P^{(0)} \to M \to 0 \tag{3}$$

where $M'' = \operatorname{Ker}(p) = \Omega_2(M)$. We also get an exact sequence

$$0 \to \tau_A(M) \to \nu_A(P^{(1)}) \xrightarrow{\nu_A(p)} \nu_A(P^{(0)}) \to \nu_A(M) \to 0 \tag{4}$$

since the Nakajama functor ν_A is right exact.

There is the dual construction of τ_A^{-1}: For an A-module N let

$$0 \to N \to I^{(0)} \xrightarrow{q} I^{(1)} \to N'' \to 0 \tag{5}$$

be an exact sequence where $0 \to N \to I^{(0)} \xrightarrow{q} I^{(1)}$ is a minimal injective presentation of N.

Applying ν_A^{-1} yields an exact sequence

$$0 \to \nu_A^{-1}/N) \to \nu_A^{-1}(I^{(0)}) \xrightarrow{\nu_A^{-1}(q)} \nu_A^{-1}(I^{(1)}) \to \tau_A^{-1}(N) \to 0 \tag{6}$$

Lemma 10.2. *We have*

$$\underline{\dim}(\tau_A(M)) = \underline{\dim}(M)\Phi_A - \underline{\dim}(M'')\Phi_A + \underline{\dim}(\nu_A(M)). \tag{7}$$

Proof. From Equation (3) we get

$$-\underline{\dim}(P^{(1)}) + \underline{\dim}(P^{(0)}) = \underline{\dim}(M) - \underline{\dim}(M'').$$

Applying Φ_A to this sequence, and using $\underline{\dim}(P)\Phi_A = -\underline{\dim}(\nu_A(P))$ for all projective modules P, we get

$$\underline{\dim}(\nu_A(P^{(1)})) - \underline{\dim}(\nu_A(P^{(0)})) = \underline{\dim}(M)\Phi_A - \underline{\dim}(M'')\Phi_A.$$

From the injective presentation of $\tau_A(M)$ (see in Equation (4)) we get

$$\begin{aligned}\underline{\dim}(\tau_A(M)) &= \underline{\dim}(\nu_A(P^{(1)})) - \underline{\dim}(\nu_A(P^{(0)})) + \underline{\dim}(\nu_A(M)) \\ &= \underline{\dim}(M)\Phi_A - \underline{\dim}(M'')\Phi_A + \underline{\dim}(\nu_A(M)).\end{aligned}$$

□

Lemma 10.3. *If* $\operatorname{proj.dim}(M) \leq 2$*, then*

$$\underline{\dim}(\tau_A(M)) \geq \underline{\dim}(M)\Phi_A. \tag{8}$$

If $\operatorname{proj.dim}(M) \leq 2$ *and* $\operatorname{inj.dim}(\tau_A(M)) \leq 2$*, then*

$$\underline{\dim}(\tau_A(M)) - \underline{\dim}(M)\Phi_A = \underline{\dim}(I) \tag{9}$$

for some injective module I.

Proof. If $\operatorname{proj.dim}(M) \leq 2$, then M'' is projective, which implies $\underline{\dim}(M'')\Phi_A = -\underline{\dim}(\nu_A(M''))$. Therefore

$$\underline{\dim}(\tau_A(M)) - \underline{\dim}(M)\Phi_A = \underline{\dim}(\nu_A(M'') \oplus \nu_A(M)),$$

and therefore this vector is non-negative. Note that $\nu_A(M'')$ is injective. If we assume additionally that $\operatorname{inj.dim}(\tau_A(M)) \leq 2$, then $\nu_A(M)$ is also injective, since it is the cokernel of the homomorphism

$$\nu_A(p)\colon \nu_A(P^{(1)}) \to \nu_A(P^{(0)})$$

with $\nu_A(P^{(1)})$ and $\nu_A(P^{(0)})$ being injective. □

Lemma 10.4. *If* $\operatorname{proj.dim}(M) \leq 1$ *and* $\operatorname{Hom}_A(M, {}_AA) = 0$*, then*

$$\underline{\dim}(\tau_A(M)) = \underline{\dim}(M)\Phi_A. \tag{10}$$

Proof. If $\operatorname{proj.dim}(M) \leq 1$, then $M'' = 0$, since Equation (3) gives a minimal projective presentation of M. By assumption $\nu_A(M) = \mathrm{D}\operatorname{Hom}_A(M, {}_AA) = 0$. Thus the result follows directly from Equation (7). □

Note that Equation (10) has many consequences and applications. For example, if A is a hereditary algebra, then each A-module M satisfies $\operatorname{proj.dim}(M) \leq 1$, and if M is non-projective, then $\operatorname{Hom}_A(M, {}_AA) = 0$.

Lemma 10.5. *Assume* $\operatorname{proj.dim}(M) \leq 2$*. If* $\underline{\dim}(\tau_A(M)) = \underline{\dim}(M)\Phi_A$*, then* $\operatorname{proj.dim}(M) \leq 1$ *and* $\operatorname{Hom}_A(M, {}_AA) = 0$.

Proof. Clearly, $\underline{\dim}(\tau_A(M)) = \underline{\dim}(M)\Phi_A$ implies $\nu_A(M'') oplus \nu_A(M) = 0$. Since M'' is projective, we have $\nu_A(M'') = 0$ if and only if $M'' = 0$. □

Using the notations from Equation (5) and (6) we obtain the following dual statements:

(i) We have
$$\underline{\dim}(\tau_A^{-1}(N)) = \underline{\dim}(N)\phi_A^{-1} - \underline{\dim}(N'')\Phi_A^{-1} + \underline{\dim}(\nu_A^{-1}(N)).$$

(ii) If $\operatorname{inj.dim}(N) \leq 2$, then
$$\underline{\dim}(\tau_A^{-1}(N)) \geq \underline{\dim}(N)\Phi_A^{-1}.$$
If $\operatorname{inj.dim}(N) \leq 2$ and $\operatorname{proj.dim}(\tau_A^{-1}(N)) \leq 2$, then
$$\underline{\dim}(\tau_A^{-1}(N)) - \underline{\dim}(N)\Phi_A^{-1} = \underline{\dim}(P)$$
for some projective module P.

(iii) If $\operatorname{inj.dim}(N) \leq 1$ and $\operatorname{Hom}_A(\mathrm{D}(A_A), N) = 0$, then
$$\underline{\dim}(\tau_A^{-1}(N)) = \underline{\dim}(N)\Phi_A^{-1}.$$

Lemma 10.6. *If $0 \to U \to X \to V \to 0$ is a non-split short exact sequence of A-modules, then*
$$\dim \operatorname{End}_A(X) < \dim \operatorname{End}_A(U \oplus V).$$

Proof. Applying $\operatorname{Hom}_A(-,U)$, $\operatorname{Hom}_A(-,X)$ and $\operatorname{Hom}_A(-,V)$ we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_A(V,U) \arrow[r] \arrow[d] & \operatorname{Hom}_A(X,U) \arrow[r] \arrow[d] & \operatorname{Hom}_A(U,U) \arrow[r, "\delta"] \arrow[d] & \operatorname{Ext}^1_A(V,U) \\
0 \arrow[r] & \operatorname{Hom}_A(V,X) \arrow[r] \arrow[d] & \operatorname{Hom}_A(X,X) \arrow[r] \arrow[d] & \operatorname{Hom}_A(U,X) \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_A(V,V) \arrow[r] & \operatorname{Hom}_A(X,V) \arrow[r] & \operatorname{Hom}_A(U,V) &
\end{tikzcd}$$

with exact rows and columns. Since η does not split, we know that the connecting homomorphism δ is non-zero. This implies
$$\dim \operatorname{Hom}_A(X,U) \leq \dim \operatorname{Hom}_A(V,U) + \dim \operatorname{Hom}_A(U,U) - 1.$$
Thus we get
$$\begin{aligned}
\dim \operatorname{Hom}_A(X,X) &\leq \dim \operatorname{Hom}_A(X,U) + \dim \operatorname{Hom}_A(X,V) \\
&\leq \dim \operatorname{Hom}_A(V,U) + \dim \operatorname{Hom}_A(U,U) - 1 \\
&\quad + \dim \operatorname{Hom}_A(V,V) + \dim \operatorname{Hom}_A(U,V) \\
&= \dim \operatorname{End}_A(U \oplus V) - 1.
\end{aligned}$$
This finishes the proof. □

Recall that for an indecomposable A-module X we defined

$$F(X) = \mathrm{End}_A(X)/\operatorname{rad}(\mathrm{End}_A(X)),$$

which is a K-skew field. If K is algebraically closed, then $F(X) \cong K$ for all indecomposables X. If K is a splitting field for K, then $F(\tau^{-n}(P_i)) \cong K$ and $F(\tau^n(I_i)) \cong K$ for all $n \geq 0$.

An algebra A is **directed** if every indecomposable A-module is directing.

Let A be of finite-global dimension. Then we call the quadratic form χ_A **weakly positive** if $\chi_A(x) > 0$ for all $x > 0$ in $\mathbb{Z}^n$. If $x \in \mathbb{Z}^n$ with $\chi_A(x) = 1$, then x is called a **root** of χ_A.

Theorem 10.7. *Let A be a finite-dimensional directed algebra. If* $\operatorname{gl.dim}(A) \leq 2$, *then the following hold:*

(i) *χ_A is weakly positive;*
(ii) *If K is algebraically closed, then $\underline{\dim}$ yields a bijection between the set of isomorphism classes of indecomposable A-modules and the set of positive roots of χ_A.*

Proof. (i): Let $x > 0$ in $G(A) = \mathbb{Z}^n$. Thus $x = \underline{\dim}(X)$ for some non-zero A-module X. We choose X such that $\dim \mathrm{End}_A(X)$ is minimal. In other words, if Y is another module with $\underline{\dim}(Y) = x$, then $\dim \mathrm{End}_A(X) \leq \dim \mathrm{End}_A(Y)$.

Let $X = X_1 \oplus \cdots \oplus X_t$ with X_i indecomposable for all i. It follows from Lemma 10.6 that $\mathrm{Ext}^1_A(X_i, X_j) = 0$ for all $i \neq j$. (Without loss of generality assume $\mathrm{Ext}^1_A(X_2, X_1) \neq 0$. Then there exists a non-split short exact sequence

$$0 \to X_1 \to Y \to \bigoplus_{i=2}^{t} X_i \to 0$$

and Lemma 10.6 implies that $\dim \mathrm{End}_A(Y) < \dim \mathrm{End}_A(X)$, a contradiction.) Furthermore, since X_i is directing, we have $\mathrm{Ext}^1_A(X_i, X_i) = 0$ for all i. Thus we get $\mathrm{Ext}^1_A(X, X) = 0$. Since $\operatorname{gl.dim}(A) \leq 2$, we have

$$\chi_A(x) = \chi_A(\underline{\dim}(X)) = \dim \mathrm{End}_A(X) + \dim \mathrm{Ext}^2_A(X, X) > 0.$$

Thus χ_A is weakly positive.

(ii): If Y is an indecomposable A-module, then we know that

$$\chi_A(Y) = \dim \mathrm{End}_A(Y),$$

since Y is directing. We also know that $\mathrm{End}_A(Y)$ is a skew field, which implies $F(Y) \cong \mathrm{End}_A(Y)$. Thus, $\chi_A(Y) = 1$ in case $F(Y) \cong K$.

Furthermore, we know that any two non-isomorphic indecomposable A-modules Y and Z satisfy $\underline{\dim}(Y) \neq \underline{\dim}(Z)$. So the map $\underline{\dim}$ is injective.

Assume additionally that x is a root of χ_A. Now

$$1 = \chi_A(x) = \dim \mathrm{End}_A(X) + \dim \mathrm{Ext}^2_A(X, X)$$

shows that $\mathrm{End}_A(X) \cong K$. This implies that X is indecomposable.

It follows that the map $\underline{\dim}$ from the set of isomorphism classes of indecomposable A-modules to the set of positive roots is surjective. $\square$

Note that a sincere directed algebra A always satisfies $\mathrm{gl.\,dim}(A) \leq 2$.

Corollary 10.8. *If Q is a representation-finite quiver, then χ_{KQ} is weakly positive.*

Proof. If KQ is representation-finite, then Γ_{KQ} consists of a union of preprojective components. Therefore all KQ-modules are directed. Furthermore, $\mathrm{gl.\,dim}(KQ) \leq 1$. Now one can apply the above theorem. $\square$

Proposition 10.9 (Drozd)**.** *A weakly positive integral quadratic form χ has only finitely many positive roots.*

Proof. Use partial derivations of χ and some standard results from Analysis. For details we refer to [Ri1]. $\square$

From now on we assume that K is a splitting field for A.

10.2. **Cartan matrix.** As before, we denote the transpose of a matrix M by M^T. For a ring or field R we denote the elements in R^n as row vectors.

The **Cartan matrix** $C_A = (c_{ij})_{ij}$ of A is the $n \times n$-matrix with ijth entry equal to

$$c_{ij} := [P_j : S_i] = \underline{\dim}(P_j)_i.$$

Thus the jth column of C_A is given by $\underline{\dim}(P_j)^T$.

Recall that the Nakayama functor $\nu = \nu_A = \mathrm{D}\,\mathrm{Hom}_A(-, {}_AA)$ induces an equivalence

$$\nu\colon \mathrm{proj}(A) \to \mathrm{inj}(A)$$

where $\nu(P_i) = I_i$. It follows that

$$\underline{\dim}(I_i)_j = \dim\, \mathrm{Hom}_A(I_i, I_j) = \dim\, \mathrm{Hom}_A(P_i, P_j) = c_{ij}.$$

(Here we used our assumption that K is a splitting field for A.)

Thus the ith row of C_A is equal to $\underline{\dim}(I_i)$. So we get

$$\underline{\dim}(P_i) = e_i C_A^T \quad \text{and} \quad \underline{\dim}(I_i) = e_i C_A. \tag{11}$$

Lemma 10.10. *If* $\mathrm{gl.\,dim}(A) < \infty$*, then C_A is invertible over $\mathbb{Z}$.*

Proof. Copy the proof of Lemma 10.1. $\square$

But note that there are algebras A where C_A is invertible over $\mathbb{Q}$, but not over $\mathbb{Z}$, for example if A is a local algebra with non-zero radical.

Assume now that the Cartan matrix C_A of A is invertible. We get a (not necesssarily symmetric) bilinear form

$$\langle -,-\rangle'_A \colon \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}$$

defined by

$$\langle x,y\rangle'_A := xC_A^{-T}y^T.$$

Here C_A^{-T} denote the inverse of the transpose C_A^T of C. Furthermore, we define a symmetric bilinear form

$$(-,-)'_A \colon \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}$$

by

$$(x,y)'_A := \langle x,y\rangle'_A + \langle y,x\rangle'_A = x(C_A^{-1} + C_A^{-T})y^T.$$

Set $\chi'_A(x) := \langle x,x\rangle'_A$. This defines a quadratic form

$$\chi'_A \colon \mathbb{Q}^n \to \mathbb{Q}.$$

It follows that

$$(x,y)'_A = \chi'_A(x+y) - \chi'_A(x) - \chi'_A(y).$$

The **radical of** χ'_A is defined by

$$\operatorname{rad}(\chi'_A) = \{w \in \mathbb{Q}^n \mid (w,-)'_A = 0\}.$$

The following lemma shows that the form $\langle -,-\rangle'_A$ we just defined using the Cartan matrix, coincides with the Ringel form we defined earlier:

Lemma 10.11. *Assume that C_A is invertible. If X and Y are A-modules with* $\operatorname{proj.dim}(X) < \infty$ *or* $\operatorname{inj.dim}(Y) < \infty$, *then*

$$\langle \underline{\dim}(X), \underline{\dim}(Y)\rangle'_A = \langle X,Y\rangle_A = \sum_{t\geq 0}(-1)^t \dim \operatorname{Ext}_A^t(X,Y).$$

In particular, $\chi'_A(\underline{\dim}(X)) = \chi_A(X)$.

Proof. Assume $\operatorname{proj.dim}(X) = d < \infty$. (The case $\operatorname{inj.dim}(Y) < \infty$ is done dually.) We use induction on d.

If $d = 0$, then X is projective. Without loss of generality we assume that X is indecomposable. Thus $X = P_i$ for some i. Let $y = \underline{\dim}(Y)$. We get

$$\langle \underline{\dim}(X), \underline{\dim}(Y)\rangle'_A = \langle \underline{\dim}(P_i), y\rangle'_A = \underline{\dim}(P_i)C_A^{-T}y^T = e_iy^T = \dim \operatorname{Hom}_A(P_i,Y).$$

Furthermore, we have $\operatorname{Ext}_A^t(P_i,Y) = 0$ for all $t > 0$.

Next, let $d > 0$. Let $P \to X$ be a projective cover of X and let X' be its kernel. It follows that $\operatorname{proj.dim}(X') = d-1$. We apply $\operatorname{Hom}_A(-,Y)$ to the exact sequence

$$0 \to X' \to P \to X \to 0.$$

Using the long exact homology sequence we obtain

$$\begin{aligned}\sum_{t\geq 0}(-1)^i\dim \operatorname{Ext}_A^t(X,Y) &= \sum_{t\geq 0}(-1)^i\dim \operatorname{Ext}_A^t(P,Y) - \sum_{t\geq 0}(-1)^i\dim \operatorname{Ext}_A^t(X',Y)\\ &= \langle \underline{\dim}(P), Y\rangle_A' - \langle \underline{\dim}(X'), \underline{\dim}(Y)\rangle_A'\\ &= \langle \underline{\dim}(X), \underline{\dim}(Y)\rangle_A'.\end{aligned}$$

Here the second equality is obtained by induction. This finishes the proof. □

Let δ_{ij} be the Kronecker function.

Corollary 10.12. *If A is hereditary, then*

$$\langle e_i, e_j\rangle_A = \begin{cases} 1 & \text{if } i=j,\\ -\dim \operatorname{Ext}_A^1(S_i,S_j) & \text{otherwise.}\end{cases}$$

Proof. This holds since $\operatorname{gl.dim}(A)\leq 1$ and since K is a splitting field for A. □

Lemma 10.13. *Let $A = KQ$ be a finite-dimensional path algebra. Then for any simple A-module S_i and S_j we have $\dim \operatorname{Ext}_A^1(S_i,S_j)$ is equal to the number of arrows $i\to j$ in Q.*

Proof. Let a_{ij} be the number of arrows $i\to j$. Since A is finite-dimensional we have $a_{ii}=0$ for all i. The minimal projective resolution of the simple A-module S_i is of the form

$$0\to \bigoplus_{j=1}^n P_j^{a_{ij}} \to P_i \to S_i \to 0$$

Applying $\operatorname{Hom}_A(-,S_j)$ yields an exact sequence

$$0\to \operatorname{Hom}_A(S_i,S_j)\to \operatorname{Hom}_A(P_i,S_j)\to \operatorname{Hom}_A(P_j^{a_{ij}},S_j)\to \operatorname{Ext}_A^1(S_i,S_j)\to 0.$$

Thus $\dim \operatorname{Ext}_A^1(S_i,S_j) = a_{ij}$. □

Corollary 10.14. *Let $A = KQ$ be a finite-dimensional path algebra, and let X and Y be A-modules with $\underline{\dim}(X)=\alpha$ and $\underline{\dim}(Y)=\beta$. Then*

$$\langle X,Y\rangle_{KQ} = \langle \alpha,\beta\rangle_{KQ} = \sum_{i\in Q_0}\alpha_i\beta_i - \sum_{a\in Q_1}\alpha_{s(a)}\beta_{t(a)}$$

and

$$\chi_{KQ}(X) = \langle \alpha,\alpha\rangle_{KQ} = \sum_{i=1}^n \alpha_i^2 - \sum_{i<j} q_{ij}\alpha_i\alpha_j$$

where q_{ij} is the number of arrows $a\in Q_1$ with $\{s(a),t(a)\}=\{i,j\}$.

Lemma 10.15. *Assume that C_A is invertible. Then*

$$\Phi_A = -C_A^{-T}C_A.$$

Proof. For each $1 \leq i \leq n$ we have to show that

$$\underline{\dim}(P_i)\Phi_A = -\underline{\dim}(I_i). \tag{12}$$

We have

$$\underline{\dim}(P_i)(-C_A^{-T}C_A) = -\underline{\dim}(I_i) \quad \text{if and only if} \quad -\underline{\dim}(I_i)^T = -C_A^T C_A^{-1}\underline{\dim}(P_i)^T.$$

Clearly, $C_A^{-1}\underline{\dim}(P_i)^T = e_i^T$, and $-C_A^T e_i^T = -\underline{\dim}(I_i)^T$. $\square$

Example: Let Q be the quiver

$$\begin{tikzcd} 2 \arrow[rrd] & 3 \arrow[rd] & & 4 \arrow[ld] & 5 \arrow[lld] \\ & & 1 & & \end{tikzcd}$$

and let $A = KQ$. Then

$$C_A = \begin{pmatrix} 1&1&1&1&1 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}$$

and

$$\Phi_A = \begin{pmatrix} -1&-1&-1&-1&-1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{pmatrix}.$$

Here are some calculations:

- $(3,1,1,1,1)\Phi_A = (1,0,0,0,0)$ and $(3,1,1,1,1)\Phi_A^2 = -(1,1,1,1,1)$,
- $(1,1,1,0,0)\Phi_A = (1,0,0,1,1)$ and $(1,1,1,0,0)\Phi_A^2 = (1,1,1,0,0)$,
- $(2,1,1,1,1)\Phi_A = (2,1,1,1,1)$.

Lemma 10.16. *For all $x, y \in \mathbb{Q}^n$ we have*

$$\langle x, y\rangle_A' = -\langle y, x\Phi_A\rangle_A' = \langle x\Phi_A, y\Phi_A\rangle_A'.$$

Proof. We have

$$\begin{aligned} \langle x, y\rangle_A' &= xC_A^{-T}y^T = (xC_A^{-T}y^T)^T = yC_A^{-1}x^T \\ &= yC_A^{-T}C_A^TC_A^{-1}x^T = -yC_A^{-T}\Phi_A^Tx^T = -\langle y, x\Phi_A\rangle_A'. \end{aligned}$$

This proves the first equality. Repeating this calculation we obtain the second equality. $\square$

Lemma 10.17. *If there exists some $x > 0$ such that $x\Phi_A = x$, then χ_A is not weakly positive.*

Proof. We have $(x, y)_A' = 0$ for all y if and only if $x(C_A^{-1} + C_A^{-T}) = 0$ if and only if $xC_A^{-1} = -xC_A^{-T}$ if and only if $x\Phi_A = x$. $\square$

Corollary 10.18. *If there exists some $x > 0$ such that $x\Phi_A = x$, then χ_A' is not weakly positive.*

Proof. If $x \in \mathrm{rad}(\chi_A')$, then $\chi_A'(x) = 0$. $\square$

Assume there exists an indecomposable KQ-module X with $\tau_{KQ}^m(X) \cong X$ and assume $m \geq 1$ is minimal with this property. Set

$$Y = \bigoplus_{i=1}^{m} \tau_{KQ}^i(X).$$

Then $\tau_{KQ}(Y) \cong Y$ which implies

$$\underline{\dim}(Y) = \underline{\dim}(Y)\Phi_{KQ}.$$

We get

$$\begin{aligned}(Y,Z)_{KQ} &= \langle Y,Z\rangle_{KQ} + \langle Z,Y\rangle_{KQ} \\ &= -\langle \underline{\dim}(Z), \underline{\dim}(Y)\Phi_{KQ}\rangle - \langle \underline{\dim}(Y)\Phi_{KQ}^{-1}, \underline{\dim}(Z)\rangle \\ &= -(\langle Y,Z\rangle_{KQ} + \langle Z,Y\rangle_{KQ}).\end{aligned}$$

This implies $\underline{\dim}(Y) \in \mathrm{rad}(\chi_{KQ})$.

Lemma 10.19. *For an A-module M the following hold:*

(i) *If* $\mathrm{proj.dim}(M) \leq 1$, *then*
$$\tau_A(M) \cong \mathrm{D}\,\mathrm{Ext}_A^1(M, {}_AA).$$

(ii) *If* $\mathrm{inj.dim}(M) \leq 1$, *then*
$$\tau_A^{-1}(M) \cong \mathrm{Ext}_{A^{\mathrm{op}}}^1(\mathrm{D}(M), A_A).$$

Proof. Assume $\mathrm{proj.dim}(M) \leq 1$. Then in Equation (3) we have $M'' = 0$. Applying $\mathrm{Hom}_A(-, {}_AA)$ yields an exact sequence

$$0\,\mathrm{Hom}_A(M, {}_AA) \to \mathrm{Hom}_A(P^{(0)}, {}_AA) \to \mathrm{Hom}_A(P^{(1)}, {}_AA) \to \mathrm{Ext}_A^1(M, {}_AA) \to 0$$

of right A-modules. Keeping in mind that $\nu_A = \mathrm{D}\,\mathrm{Hom}_A(-, {}_AA)$ we dualize the above sequence get an exact sequence

$$0\mathrm{D}\,\mathrm{Ext}_A^1(M, {}_AA) \to \nu_A(P^{(1)}) \to \nu_A(P^{(0)}) \to \nu_A(M) \to 0.$$

This implies (i). Part (ii) is proved dually. $\square$

10.3. **Exercises. 1**: Show the following: If the Cartan matrix C_A is an upper triangular matrix, then C_A is invertible over $\mathbb{Q}$. In this case, C_A is invertible over $\mathbb{Z}$ if and only if $\mathrm{End}_A(P_i) \cong K$ for all i.

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11. **Representation theory of quivers**

Parts of this section are copied from Crawley-Boevey's lecture notes "Lectures on representations of quivers", which you can find on his homepage.

11.1. **Bilinear and quadratic forms.** Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver with vertices $Q_0 = \{1, \ldots, n\}$, and let $A = KQ$ be the path algebra of Q.

For vertices $i, j \in Q_0$ let $q_{ij} = q_{ji}$ be the number of arrows $a \in Q_1$ with $\{s(a), t(a)\} = \{i, j\}$. Note that the numbers q_{ij} do not depend on the orientation of Q.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ define

$$q_Q(\alpha) := \sum_{i=1}^{n} \alpha_i^2 - \sum_{i \leq j} q_{ij}\alpha_i\alpha_j.$$

We call the quadratic form $q_Q \colon \mathbb{Z}^n \to \mathbb{Z}$ the **Tits form** of Q.

The **symmetric bilinear form** $(-,-)_Q \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ of Q is defined by

$$(e_i, e_j)_Q := \begin{cases} -q_{ij} & \text{if } i \neq j, \\ 2 - 2q_{ii} & \text{otherwise.} \end{cases}$$

As before, e_i denotes the canonical basis vector of $\mathbb{Z}^n$ with ith entry 1 and all other entries 0.

We have

$$\begin{aligned} (\alpha, \alpha)_Q &= 2q_Q(\alpha), \\ (\alpha, \beta)_Q &= q_Q(\alpha + \beta) - q_Q(\alpha) - q_Q(\beta). \end{aligned}$$

Note that q_Q and $(-,-)_Q$ do not depend on the orientation of the quiver Q.

For $\alpha, \beta \in \mathbb{Z}^n$ define

$$\langle \alpha, \beta, \rangle_Q := \sum_{i \in Q_0} \alpha_i\beta_i - \sum_{a \in Q_1} \alpha_{s(a)}\beta_{t(a)}.$$

This defines a (not necessarily symmetric) bilinear form

$$\langle -,- \rangle_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

which is called the **Euler form** of Q. Clearly, we have

$$\begin{aligned} q_Q(\alpha) &= \langle \alpha, \alpha \rangle_Q, \\ (\alpha, \beta)_Q &= \langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q. \end{aligned}$$

The bilinear form $\langle -,- \rangle_Q$ does depend on the orientation of Q.

The Tits form q_Q is **positive definite** if $q_Q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{Z}^n$, and q_Q is **positive semi-definite** if $q_Q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^n$.

The **radical** of q is defined by

$$\operatorname{rad}(q_Q) = \{\alpha \in \mathbb{Z}^n \mid (\alpha, -)_Q = 0\}.$$

For $\alpha, \beta \in \mathbb{Z}^n$ set $\beta \geq \alpha$ if $\beta - \alpha \in \mathbb{N}^n$. This defines a partial ordering on $\mathbb{Z}^n$.

An element $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ is **sincere** if $\alpha_i \neq 0$ for all i. We write $\alpha \geq 0$ if $\alpha_i \geq 0$ for all i, and $\alpha > 0$ if $\alpha \geq 0$ and $\alpha_i > 0$ for some i.

Let $S_1, \dots, S_n$ be the simple KQ-modules corresponding to the vertices of Q. (These are the only simple KQ-modules if and only if Q has no oriented cycles.) It is easy to check that $\dim \mathrm{Ext}^1_{KQ}(S_i, S_j)$ equals the number of arrows $i \to j$ in Q. (Just construct the minimal projective resolution

$$0 \to \bigoplus_{j \in Q_0} P_j^{a_{ij}} \to P_i \to S_i \to 0$$

of S_i, where a_{ij} is the number of arrows $i \to j$ in Q. Then apply the functor $\mathrm{Hom}_{KQ}(-, S_j)$.)

Lemma 11.1. *Let Q be a connected quiver, and let $\beta \geq 0$ be a non-zero element in* $\mathrm{rad}(q_Q)$. *Then the following hold:*

(i) *β is sincere;*
(ii) *q_Q is positive semi-definite;*
(iii) *For $\alpha \in \mathbb{Z}^n$ the following are equivalent:*
 (a) *$q_Q(\alpha) = 0$;*
 (b) *$\alpha \in \mathbb{Q}\beta$;*
 (c) *$\alpha \in \mathrm{rad}(q_Q)$.*

Proof. (a): By assumption we have

$$(\beta, e_i)_Q = (2 - 2q_{ii})\beta_i - \sum_{j \neq i} q_{ij}\beta_j = 0.$$

If $\beta_i = 0$, then

$$\sum_{j \neq i} q_{ij}\beta_j = 0,$$

and since $q_{ij} \geq 0$ for all i, j and $\beta \geq 0$, we get $\beta_j = 0$ whenever $q_{ij} > 0$. Since Q is connected, we get $\beta = 0$, a contradiction. Thus we proved that β is sincere.

(b): The following calculation shows that q_Q is positive semi-definite:

$$\begin{aligned}
\sum_{i<j} q_{ij} \frac{\beta_i \beta_j}{2} \left(\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} \right)^2 &= \sum_{i<j} q_{ij} \frac{\beta_j}{2\beta_i} \alpha_i^2 - \sum_{i<j} q_{ij}\alpha_i\alpha_j + \sum_{i<j} q_{ij} \frac{\beta_i}{2\beta_j} \alpha_j^2 \\
&= \sum_{i \neq j} q_{ij} \frac{\beta_j}{2\beta_i} \alpha_i^2 - \sum_{i<j} q_{ij}\alpha_i\alpha_j \\
&= \sum_{i} (2 - 2q_{ii})\beta_i \frac{1}{2\beta_i} \alpha_i^2 - \sum_{i<j} q_{ij}\alpha_i\alpha_j = q_Q(\alpha).
\end{aligned}$$

For the last equality we used n times the equation

$$(2 - 2q_{ii})\beta_i = \sum_{j \neq i} q_{ij}\beta_j.$$

(c): If $q_Q(\alpha) = 0$, then the calculation above shows that $\alpha_i/\beta_i = \alpha_j/\beta_j$ whenever $q_{ij} > 0$. Since Q is connected it follows that $\alpha \in \mathbb{Q}\beta$.

(d): If $\alpha \in \mathbb{Q}\beta$, then $\alpha \in \mathrm{rad}(q_Q)$, since $\beta \in \mathrm{rad}(q_Q)$.

(e): Clearly, if $\alpha \in \mathrm{rad}(q_Q)$, then $q_Q(\alpha) = 0$. $\square$

Theorem 11.2. *Suppose that Q is connected.*

(i) *If Q is a Dynkin quiver, then q_Q is positive definite;*
(ii) *If Q is an Euclidean quiver, then q_Q is positive semi-definite and* $\mathrm{rad}(q_Q) = \mathbb{Z}\delta$*, where δ is the dimension vector for Q listed in Figure 2;*
(iii) *If Q is not a Dynkin and not an Euclidean quiver, then there exists some $\alpha \geq 0$ in $\mathbb{Z}^n$ with $q_Q(\alpha) < 0$ and $(\alpha, e_i)_Q \leq 0$ for all i.*

Proof. (ii): It is easy to check that $\delta \in \mathrm{rad}(q_Q)$: If there are no loops or multiple edges we have to check that for all vertices i we have

$$2\delta_i = \sum_j \delta_j$$

where j runs over the set of neighbours of i in Q. By Lemma 11.1 this implies that q_Q is positive semi-definite.

In each case there exists some vertex i such that $\delta_i = 1$. Thus $\mathrm{rad}(q_Q) = \mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta$.

(i): Any Dynkin quiver Q with n vertices can be seen as a full subquiver of some Euclidean quiver $\widetilde{Q}$ with $n+1$ vertices. We have $q_{\widetilde{Q}}(x) > 0$ for all non-sincere elements in $\mathbb{Z}^{n+1}$, since the x with $q_{\widetilde{Q}}(x) = 0$ are all multiples of the sincere element δ. So q_Q is positive definite. (The form q_Q is obtained from $q_{\widetilde{Q}}$ via restriction to the subquiver Q of $\widetilde{Q}$.)

(iii): Let Q be a quiver which is not Dynkin and not Euclidean. Then Q contains a (not necessarily full) subquiver Q' such that Q' is a Euclidean quiver. Note that any dimension vector of Q' can be seen as a dimension vector of Q by just adding some zeros in case Q has more vertices than Q'.

Let δ be the radical vector associated to Q'. If the vertex sets of Q' and Q coincide, then $\alpha := \delta$ satisfies $q_Q(\alpha) < 0$.

Otherwise, if i is a vertex of Q which is not a vertex of Q' but which is connected to a vertex in Q' by an edge, then $\alpha := 2\delta + e_i$ satisfies $q_Q(\alpha) < 0$. $\square$

Let Q be a Euclidean quiver. If i is a vertex of Q with $\delta_i = 1$, then i is called an **extending vertex**. Observe that there always exists such an extending vertex. Furthermore, if we delete an extending vertex (and the arrows attached to it), then we will obtain a corresponding Dynkin diagram.

For Q a Dynkin or an Euclidean quiver, let

$$\Delta_Q := \{\alpha \in \mathbb{Z}^n \mid \alpha \neq 0, q_Q(\alpha) \leq 1\}$$

be the set of **roots** of Q.

A root α of Q is **real** if $q_Q(\alpha) = 1$. Otherwise, if $q_Q(\alpha) = 0$, it is called an **imaginary root**. Let Δ_Q^{re} and Δ_Q^{im} be the set of real and imaginary roots, respectively.

Proposition 11.3. *Let Q be a Dynkin or a Euclidean quiver. Then the following hold:*

(i) *Each e_i is a root;*
(ii) *If $\alpha \in \Delta_Q \cup \{0\}$, then $-\alpha$ and $\alpha + \beta$ are in $\Delta_Q \cup \{0\}$ where $\beta \in \mathrm{rad}(q_Q)$;*
(iii) *We have*

$$\Delta_Q^{\mathrm{im}} = \begin{cases} \emptyset & \text{if } Q \text{ is Dynkin,} \\ \{r\delta \mid 0 \neq r \in \mathbb{Z}\} & \text{if } Q \text{ is Euclidean;} \end{cases}$$

(iv) *Every root $\alpha \in \Delta_Q$ is either positive or negative;*
(v) *If Q is Euclidean, then the set $(\Delta_Q \cup \{0\})/\mathbb{Z}\delta$ of residue classes modulo $\mathbb{Z}\delta$ is finite;*
(vi) *If Q is Dynkin, then Δ_Q is finite.*

Proof. (i): Clearly, we have $q_Q(e_i) = 1$, so e_i is a root.

(ii): Let $\alpha \in \Delta_Q \cup \{0\}$ and $\beta \in \mathrm{rad}(q_Q)$. Since $(\beta, \alpha)_Q = 0 = q_Q(\beta)$, we have

$$\begin{aligned} q_Q(\alpha) &= q_Q(\beta + \alpha) = q_Q(\beta) + q_Q(\alpha) + (\beta, \alpha)_Q \\ &= q_Q(\beta - \alpha) = q_Q(\beta) + q_Q(\alpha) - (\beta, \alpha)_Q \end{aligned}$$

Thus $-\alpha$ and $\alpha + \beta$ are in $\Delta_Q \cup \{0\}$. (The case $\beta = 0$ yields $q_Q(-\alpha) = q_Q(\alpha)$.)

(iii): This follows directly from Lemma 11.1.

(iv): Let α be a root. So we can write $\alpha = \alpha^+ - \alpha^-$ where $\alpha^+, \alpha^- \geq 0$ and have disjoint supports. Assume that both α^+ and α^- are non-zero. It follows immediately that $(\alpha^+, \alpha^-)_Q \leq 0$. This implies

$$1 \geq q_Q(\alpha) = q_Q(\alpha^+) + q_Q(\alpha^-) - (\alpha^+, \alpha^-)_Q \geq q_Q(\alpha^+) + q_Q(\alpha^-).$$

Thus one of α^+ and α^- is an imaginary root and is therefore sincere. So the other one is zero, a contradiction.

(v): Let Q be an Euclidean quiver, and let e be an extending vertex of Q. If α is a root with $\alpha_e = 0$, then $\delta - \alpha$ and $\delta + \alpha$ are roots which are positive at the vertex e. Thus both are positive roots. This implies

$$\{\alpha \in \Delta \cup \{0\} \mid \alpha_e = 0\} \subseteq \{\alpha \in \mathbb{Z}^n \mid -\delta \leq \alpha \leq \delta\},$$

and obviously this is a finite set.

If $\beta \in \Delta \cup \{0\}$, then $\beta - \beta_e\delta$ belongs to the finite set

$$\{\alpha \in \Delta \cup \{0\} \mid \alpha_e = 0\}.$$

(vi): If Q is a Dynkin quiver, we can consider Q as a full subquiver of the corresponding Euclidean quiver $\widetilde{Q}$ with extending vertex e. (Thus, we obtain Q by deleting e from $\widetilde{Q}$.) We can now view a root α of Q as a root of $\widetilde{Q}$ with $\alpha_e = 0$. Thus by the proof of (v) we get that Δ is a finite set. □

11.2. **Gabriel's Theorem.** Combining our results obtained so far, we obtain the following famous theorem:

Theorem 11.4 (Gabriel)**.** *Let Q be a connected quiver. Then KQ is representation-finite if and only if Q is a Dynkin quiver. In this case $\underline{\dim}$ yields a bijection between the set of isomorphism classes of indecomposable KQ-modules and the set of positive roots of q_Q.*

Proof. (a): We know that there is a unique preprojective component $\mathcal{P}_{KQ}$ of the Auslander-Reiten quiver Γ_{KQ}.

(b): We have $\chi_{KQ}(X) = q_Q(\underline{\dim}(X))$ for all KQ-modules X.

(c): Assume KQ is representation-finite. This is the case if and only if $\mathcal{P}_{KQ} = \Gamma_{KQ}$. Since all indecomposable preprojective modules are directed, we know that KQ is a directed algebra. Furthermore, we have $\operatorname{gl.dim}(KQ) \leq 1 \leq 2$. So we can apply Theorem **xx** and obtain a bijection between the isomorphism classes of indecomposable KQ-modules and the set of positive roots of χ_{KQ}. Furthermore, an element $\alpha \in \mathbb{N}^n$ is a positive root of χ_{KQ} if and only if $\alpha \in \Delta_Q$. We also know that $\chi_{KQ} = q_Q$ is weakly positive. But this implies that Q has to be a Dynkin quiver. (For all quivers Q which are not Dynkin we found some $\alpha > 0$ with $q_Q(\alpha) \leq 0$.)

(d): If KQ is representation-infinite, the component $\mathcal{P}_{KQ}$ is infinite. Each indecomposable module X in $\mathcal{P}_{KQ}$ is directed, and K is a splitting field for KQ. Thus

$$\chi_{KQ}(X) = q_Q(\underline{\dim}(X)) = 1.$$

Furthermore, we know that there is no other indecomposable KQ-module Y with $\underline{\dim}(X) = \underline{\dim}(Y)$. So we found infinitely many $\alpha \in \mathbb{Z}^n$ with $q_Q(\alpha) = 1$.

Suppose that Q is a Dynkin quiver. Then

$$\Delta_Q = \{\alpha \in \mathbb{Z}^n \mid q_Q(\alpha) = 1\}$$

is a finite set, a contradiction. □

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12. **Cartan matrices and (sub)additive functions**

In Figure 1 we define a set of valued graphs called **Dynkin graphs**. By definition each of the graphs A_n, B_n, C_n and D_n has n vertices. The graphs A_n, D_n, E_6, E_7 and E_8 are the **simply laced Dynkin graphs**.

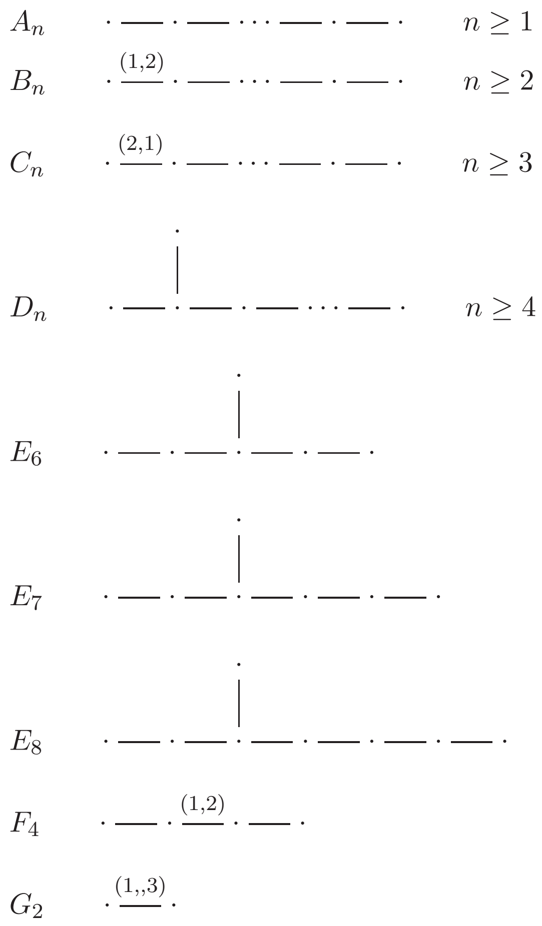

FIGURE 1. **Dynkin graphs**

In Figure 2 we define a set of valued graphs called **Euclidean graphs**. By definition each of the graphs $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$, $\widetilde{BC}_n$, $\widetilde{BD}_n$ and $\widetilde{CD}_n$ has $n+1$ vertices. The graphs $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ are the **simply laced Euclidean graphs**. By definition the graph $\widetilde{A}_0$ has one vertex and one loop, and $\widetilde{A}_1$ has two vertices joined by two edges. Our table of Euclidean graphs does not only contain the graphs themselves, but for each graph we also display a dimension vector which we will denote by δ.

A quiver Q is a **Dynkin quiver** or an **Euclidean quiver** of the underlying graph of Q (replace each arrow of Q by a non-oriented edge) is a simply laced Dynkin graph or a simply laced Euclidean graph, respectively.

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$$\widetilde{A}_n \qquad \begin{tikzcd} 1 \arrow[r, dash] \arrow[rrrrr, dash, bend left] & 1 \arrow[r, dash] & 1 \cdots 1 \arrow[r, dash] & 1 \arrow[r, dash] & 1 \end{tikzcd}$$

$$\widetilde{B}_n \qquad 1 \overset{(1,2)}{\text{——}} 1 \text{ — } 1 \cdots 1 \text{ — } 1 \overset{(2,1)}{\text{——}} 1$$

$$\widetilde{C}_n \qquad 1 \overset{(2,1)}{\text{——}} 2 \text{ — } 2 \cdots 2 \text{ — } 2 \overset{(1,2)}{\text{——}} 1$$

$$\widetilde{D}_n \qquad \begin{tikzcd} & 1 \arrow[d, dash] & & & 1 \arrow[d, dash] & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 2 \cdots 2 \arrow[rr, dash] & & 2 \arrow[r, dash] & 1 \end{tikzcd}$$

$$\widetilde{E}_6 \qquad \begin{tikzcd} & & 1 \arrow[d, dash] & & \\ & & 2 \arrow[d, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1 \end{tikzcd}$$

$$\widetilde{E}_7 \qquad \begin{tikzcd} & & & 2 \arrow[d, dash] & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1 \end{tikzcd}$$

$$\widetilde{E}_8 \qquad \begin{tikzcd} & & 3 \arrow[d, dash] & & & & & \\ 2 \arrow[r, dash] & 4 \arrow[r, dash] & 6 \arrow[r, dash] & 5 \arrow[r, dash] & 4 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1 \end{tikzcd}$$

$$\widetilde{A}_{11} \qquad 2 \overset{(1,4)}{\text{——}} 1 \qquad\qquad \widetilde{A}_{12} \qquad 1 \overset{(2,2)}{\text{——}} 1$$

$$\widetilde{BC}_n \qquad 2 \overset{(1,2)}{\text{——}} 2 \text{ — } 2 \cdots 2 \text{ — } 2 \overset{(1,2)}{\text{——}} 1$$

$$\widetilde{BD}_n \qquad \begin{tikzcd} & 1 \arrow[d, dash] & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 2 \cdots 2 \arrow[r, dash] & 2 \arrow[r, dash, "{(2,1)}"] & 2 \end{tikzcd}$$

$$\widetilde{CD}_n \qquad \begin{tikzcd} & 1 \arrow[d, dash] & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 2 \cdots 2 \arrow[r, dash] & 2 \arrow[r, dash, "{(1,2)}"] & 1 \end{tikzcd}$$

$$\widetilde{F}_{41} \qquad 1 \text{ — } 2 \text{ — } 3 \overset{(1,2)}{\text{——}} 2 \text{ — } 1 \qquad\qquad \widetilde{F}_{42} \qquad 1 \text{ — } 2 \text{ — } 3 \overset{(2,1)}{\text{——}} 4 \text{ — } 2$$

$$\widetilde{G}_{21} \qquad 1 \text{ — } 2 \overset{(1,3)}{\text{——}} 1 \qquad\qquad \widetilde{G}_{22} \qquad 1 \text{ — } 2 \overset{(3,1)}{\text{——}} 3$$

FIGURE 2. **Euclidean graphs and additive functions δ**

Part 3. Extras

13. Classes of modules

simple modules

serial modules

uniserial modules

cyclic modules

cocyclic modules

indecomposable modules

projective modules

injective modules

preprojective modules (which should really be called postprojective modules)

preinjective modules

regular modules

bricks

stones

exceptional modules

Schur modules

tree modules (2 different definitions)

string modules

band modules

(generalized) tilting modules

(generalized) partial tilting modules

torsion modules

torsion free modules

In the world of infinite dimensional modules we find names like the following:

Prüfer modules

p-adic modules

generic modules

pure-injective modules

algebraically compact module

Classifications of modules

For some algebras of infinite representation type, a complete classification of indecomposable modules is known. We list some of these classes of algebras:

Solved:

tame hereditary algebras

tubular algebras

Gelfand-Ponomarev algebras

dihedral 2-group algebras

quaternion algebra

special biserial algebras

clannish algebras

multicoil algebras

Open:

biserial algebras

However, one still has to be careful what it means to have a classification of all indecomposable modules over an algebra. For example for tubular algebras, one can parametrize all indecomposable modules by roots of a quadratic form. But given a root, it is still very difficult to write down explicitely the corresponding indecomposable module(s). In fact, for tubular algebras this remains an open problem.

14. **Classes of algebras**

We list some names of classes of mostly finite-dimensional algebras which were studied in the literature:

Basic algebras

Biserial algebras

Canonical algebras

Clannish algebras

Cluster-tilted algebra

Directed algebras

Dynkin algebras

Euclidean algebras

Gentle algebras

Group algebras

Hereditary algebras

Multicoil algebras

Nakayama algebras

Path algebras

Poset algebras

Preprojective algebras

Quasi-hereditary algebras

Quasi-tilted algebras

Representation-finite algebras

Selfinjective algebras

Semisimple algebras

Simply connected algebras

Special biserial algebras

String algebras

Strongly simply connected algebras

Symmetric algebras

Tame algebras

Tilted algebras

Tree algebras

Triangular algebras

Trivial extension algebras

Tubular algebras

Wild algebras

Here are some classes of algebras, which are not finite-dimensional, but linked to the finite-dimensional world:

Repetitive algebras

Enveloping algebras of Lie algebras

Quantized enveloping algebras

Ringel-Hall algebras

Cluster algebras

Hecke algebras

15. **Dimensions**

The concept of "dimension" occurs frequently and with different meanings in the representation theory of algebras. Here just some of the most common dimensions:

dimension of a module as a vector space

projective dimension of a module

injective dimension of a modules

global dimension of an algebra

finitistic dimension of an algebra

dominant dimension of an algebra

representation dimension of an algebra

Krull-Gabriel dimension of an algebra

Krull-dimension of a commutative ring

dimension of a variety

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