2 Lecture 2: Spaces of valuations

2.1 Introduction

Let A be a commutative ring. Recall that

$$\mathrm{Spec}(A) = \{x : A \to K\}/\sim$$

that is, $\mathrm{Spec}(A)$ is the set of equivalence classes of homomorphisms $x : A \to K$ to fields K, where two such are considered *equivalent* as soon as they can be compatibly embedded into third field (giving the same homomorphism on A). Indeed, we can shrink K so that we can assume K is generated by the image of A under x. Thus, we have

$$0 \to \mathfrak{p} \to A \to x(A) \to 0$$

where $\mathfrak{p}$ is the kernel of x, and is a prime ideal, as $x(A)$ is a domain whose fraction field is the target of x. We therefore recover our usual notion of Spec of a ring as the set of prime ideals (with a suitable topology on this set). Note that if A is finitely generated over a field k and we consider only x making $K := \mathrm{Frac}(x(A))$ of finite degree over k then we recover the more classical MaxSpec construction in such cases.

Similarly, if we fix a non-archimedean field k, and a k-affinoid algebra A, we can view the Berkovich spectrum of A as

$$M(A) = \{x : A \to K \text{ continuous}\}/\sim;$$

that is, the set of continuous homomorphisms $A \to K$ into non-archimedean field extensions of k (with compatible absolute value), taken up to a suitable equivalence. In contrast with the classical rigid-analytic use of MaxSpec, here we most definitely do *not* require K to be k-finite.

Though it may seem ungeometric at first, with adic spaces we are going to work with *higher-rank* valuations on a field, valued in some totally ordered abelian group. As we shall discuss, this additional freedom will allow the theory of certain *spaces of valuations*, which we begin to introduce in this lecture.

This lecture is organized as follows. We give an overview of the theory of valuation rings, the main reference for this being [Mat, §10]. Then, we review the classical notion of Riemann–Zariski space for fields, and then reach the valuation spectrum of a commutative ring A as a generalization. Although our ultimate interest is in the case of certain topological rings and valuations satisfying a continuity condition, the quasi-compactness properties of such refined spaces of continuous valuations we shall introduce later will follow from those of the valuation spectrum $\mathrm{Spv}(A)$ of general commutative rings A. Hence, we first study Spv in this lecture and the next one before moving on to the case of topological rings closer to those of interest in non-archimedean geometry.

2.2 Valuation rings

Definition 2.2.1 For a field K, a *valuation* on K is a map

$$v : K \to \Gamma \cup \{0\}$$

where Γ is a totally ordered abelian group with the additional rules:

$$0 < \gamma, \quad \forall\, \gamma \in \Gamma$$

$$0 \cdot 0 = 0, \;\; \gamma \cdot 0 = 0 \;\; \forall \gamma \in \Gamma,$$

which satisfies the following properties:

(1) $v(0) = 0$, $v(1) = 1$.

(2) $v(xy) = v(x)v(y)$, for all $x, y \in K$.

(3) $v(x + y) \leq \max(v(x), v(y))$, for all $x, y \in K$.

Note that $v(x) \neq 0$ for all $x \in K^\times$ since $v(1) = 1$.

Example 2.2.2 The following assignment

$$v(x) := \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

is called the *trivial* valuation on K.

Remark 2.2.3 Often v is denoted with the symbol $|\cdot|$; our definition is actually that of an *absolute value* on K, but since we will be using higher-rank cases it is traditional not to call such things "absolute values".

Remark 2.2.4 We say a valuation v on K has *rank* 1 if it is non-trivial and $\Gamma \subset \mathbf{R}^\times_{>0}$ as ordered abelian groups, and it has *higher rank* if it is nontrivial and not of rank 1. Sometimes valuations that are either trivial or rank 1 (i.e., $\Gamma \subset \mathbf{R}^\times_{>}$ as ordered abelian groups) are called *archimedean* (a potentially confusing terminology!) because an ordered abelian group Γ is a subgroup of $\mathbf{R}^\times_{>0}$ if and only if for every $\gamma < 1$ in Γ the powers $\{\gamma^n\}_{n>0}$ are cofinal for the order structure: for any γ' there exists $n > 0$ such that $\gamma^n < \gamma'$; see [Mat, Thm. 10.6]. (This is the multiplicatively reciprocated formulation of the classical "archimedean property" in the axioms for the real line.)

Remark 2.2.5 It is harmless to shrink Γ to $v(K^\times)$. This is technically convenient to avoid getting side-tracked into the notion of "equivalence" for valuations. This will become more relevant as we move on to valuations on rings.

We recall that, given a field K and a valuation v on K, we are naturally given a subring of K:

$$R_v := \{x \in K \mid v(x) \leq 1\}$$

the *valuation ring* of v in K. The units $R_v^\times$ of R_v are easily seen to be

$$\{x \in K \mid v(x) = 1\}.$$

As a direct consequence of this, it follows that $R_v - R_v^\times$ is an ideal of R_v, which we shall denote $\mathfrak{m}_v$, and therefore R_v is a local ring with unique maximal ideal $\mathfrak{m}_v$.

We can give the notion of valuation ring of a field K without mentioning the valuation on K, as follows. From the above discussion, we notice that a property of R_v is that for all $x \in K^\times$, either $x \in R_v$ or $x^{-1} \in R_v$. It also follows that $\mathrm{Frac}(R_v) = K$.

Definition 2.2.6 A *valuation ring* is a domain R, with fraction field K, such that for all x in $K^\times$, either $x \in R$ or $x^{-1} \in R$.

We can rephrase the Definition 2.2.6 in terms of divisibility of non-zero elements of R, that is: for all $a, b \in R - \{0\}$, either a divides b or b divides a. This latter reformulation has the immediate consequence that as soon as an ideal of a valuation ring R, with fraction field K, is finitely generated, then it is principal.

To relate Definition 2.2.6 back to 2.2.1, we need to explain how a valuation ring as in Definition 2.2.6 comes implicitly with an associated valuation and a value group. Observe that if R is a valuation ring with fraction field K, then

$$\Gamma_R := K^\times/R^\times$$

is a totally ordered abelian group where, for all $a, b \in K^\times$, we say

$$a \bmod R^\times \leq b \bmod R^\times$$

if and only if $x := a/b$ is in R. We are naturally given a map

$$v : K \to \Gamma_R \cup \{0\}$$

by letting $v(a)$ be $a \bmod R^\times$ if $a \neq 0$ or 0 if $a = 0$.

Remark 2.2.7 There may well exist more than one valuation v on K with valuation ring R and value group isomorphic to Γ. For example, let $K = \mathbf{Q}$, and $v_{p,\varepsilon}$ be the p-adic valuation on $\mathbf{Q}$, that is, given an $0 < \varepsilon < 1$,

$$v_{p,\varepsilon}(a/b) := \varepsilon^{-(\mathrm{ord}_p(a) - \mathrm{ord}_p(b))}$$

for nonzero $a, b \in \mathbf{Z}$. For all such ε, the value group is always *isomorphic* to $\mathbf{Z}$. The valuations $v_{p,\varepsilon}$ are distinct $\mathbf{R}^\times_{>0}$-valued *functions* on $\mathbf{Q}$, but they yield the *same* valuation ring $\mathbf{Z}_{(p)}$. By regarding v as having target $v(K^\times)$, the data of the map v and totally ordered abelian group Γ are absorbed into the data of the valuation ring itself. Hence, it is really the valuation ring that is the fundamental structure of interest.

A recipe to produce higher rank valuations

Let K be a field equipped with a valuation, and call R its valuation ring, and $\mathfrak{m}$ its unique maximal ideal. Let us assume the residue field $k := R/\mathfrak{m}$ is equipped again with a valuation. We call

$$\overline{R}' \subset k$$

its corresponding valuation ring. We define the subset of R

$$R' := \{x \in R \mid x \bmod \mathfrak{m} \in \overline{R}'\}.$$

Clearly R' is a subring of R. We have $\mathfrak{m} \subset R' \subset R$, and $x' \in R'$ lies in $R'^\times$ if and only if its reduction in k is in $\overline{R}'^\times$.

Assume $\mathfrak{m} \neq (0)$. Let a and b be in $R - \{0\}$, and pick $c \in \mathfrak{m} - \{0\}$. Since

$$\frac{a}{b} = \frac{ca}{cb},$$

we have $\mathrm{Frac}(R') = K$. We observe that in the case $\mathfrak{m} = (0)$, this is true as well.

Proposition 2.2.8 *R' is a valuation ring of K.*

We call R' the *composite* of the valuation rings R and $\overline{R}'$.

Proof. Choose $x \in K^\times$. If $x \in R'$, then we are done, so let us assume x is not in R'. We must show $x^{-1} \in R'$. We first let x be such that $x \notin R$. Then, since R is a valuation ring, $x^{-1} \in R$, and this latter is not a unit. It follows $x^{-1} \in R'$.

Now, if $x \in R - R'$, then $\overline{x} := x \bmod \mathfrak{m}$ is not in $\overline{R}'$. Since $\overline{R}'$ is a valuation ring of k, it follows $\overline{x}^{-1} \in \overline{R}'$. Since $\mathfrak{m}$ is contained in R', so x is a unit in R, we have that $x^{-1} \in R$ and this reduces to an element of $\overline{R}'$, modulo $\mathfrak{m}$. Thus, $x^{-1} \in R'$ as desired. $\square$

In the Appendix, we introduce the notion of *convex* subgroup of a totally ordered abelian group, discuss a total order structure modulo such a subgroup, and explain the relationship between the orderings on Γ_R, $\Gamma_{\overline{R}'}$, and $\Gamma_{R'}$:

Theorem 2.2.9 *The value group $\Gamma_{\overline{R}'}$ is naturally a subgroup of $\Gamma_{R'}$, and as such is a convex subgroup for which there is a natural isomorphism of ordered abelian groups $\Gamma_{R'}/\Gamma_{\overline{R}'} \simeq \Gamma_R$.*

The procedure described in the above discussion is, in fact, exhaustive in the following sense (see [Mat, Thm. 10.1]):

Theorem 2.2.10 *Let R be a valuation ring with fraction field K. The process described above produces all valuation subrings R' of R with fraction field K. Moreover, the maximal ideal $\mathfrak{m}$ of R is a prime ideal of R' and $R = R'_{\mathfrak{m}}$.*

Let R be a valuation ring with fraction field K. Let $\Gamma := \Gamma_R = K^\times/R^\times$ be the value group of R, and v the valuation on K given by reduction modulo $R^\times$. Some properties of R can be read off from Γ. For example, since $v(R - \{0\})$ is contained in

$$\Gamma_{\leq 1} := \{\gamma \in \Gamma \mid \gamma \leq 1\}$$

we see that ideals of R give rise to submonoids of Γ contained in $\Gamma_{\leq 1}$ under the mapping

$$I \mapsto v(I \setminus \{0\}) =: M_I.$$

Note that if $I = (0)$ then M_I is empty.

We leave to the reader as an exercise to prove that this is an inclusion-preserving bijection from the set of ideals of R onto the set of submonoids $M \subset \Gamma_{\leq 1}$ such that $m \in M, \gamma \leq m \Rightarrow \gamma \in M$. (Informally, such M behave like intervals in $\mathbf{R}$ that are unbounded to the left.) The condition that I is a finitely generated nonzero ideal of R (or equivalently, principal and nonzero) is exactly the condition that M has a "right endpoint": that is, $M = \Gamma_{\leq \gamma_0}$ for some $\gamma_0 \in \Gamma_{\leq 1}$.

An immediate consequence of the inclusion-preserving correspondence $I \mapsto M_I$ between ideals of R and submonoids of Γ that are "unbounded to the left" is that the set of ideals of R is totally ordered set with respect to inclusion (as this property is readily verified to hold for the corresponding collection of monoids: if M and M' are two such monoids and $M \not\subset M'$ then for $m \in M$ not in M' we cannot have $m \leq m'$ for an $m' \in M'$, so by the total ordering necessarily $m' \leq m$ and hence $m' \in M$ for all $m' \in M'$, which is to say $M' \subset M$).

A few general facts about valuation rings

The following are proved in [Mat, §10]:

(1) Any valuation ring R is integrally closed (proof exactly as in the case of discrete valuation rings).

(2) If we have a local inclusion of valuation rings $R \subset R'$, with fraction fields K and K' respectively, then we have an induced inclusion of the corresponding value groups:

$$\Gamma_R \subset \Gamma_{R'},$$

as totally ordered abelian groups (i.e., if $\gamma, \delta \in \Gamma_R$ then $\gamma \leq \delta$ in Γ_R if and only if $\gamma \leq \delta$ in $\Gamma_{R'}$).

(3) Valuation rings with fraction field K are exactly the local domains contained in K maximal with respect to local domination.

(4) Let A be a domain with fraction field K. Then the normalization of A is the intersection of all the valuation rings of K containing A, that is:

$$\widetilde{A} = \bigcap_{R \supset A} R,$$

the intersection running over all valuation rings of K containing A.

2.3 Riemann-Zariski spaces for fields

Definition 2.3.1 Let A be an integral domain, and K field containing it (eg., $K = \mathrm{Frac}(A)$). We say a vauation ring R of K with $\mathrm{Frac}(R) = K$ is *centered* in A when R contains A. In which case, the prime ideal $\mathfrak{m}_R \cap A$ is called the *center* of R in A. The set of valuation rings of K centered in A is called the *Riemann-Zariski space* of K with respect to A, and will be denoted by $\mathrm{RZ}(K, A)$.

Throughout the next lectures we will be studying more intricate spaces of valuations, but $\mathrm{RZ}(K, A)$ will be the prototype, and indeed Riemann-Zariski spaces for fields will play a useful technical role even in later generalizations. The Riemann-Zariski space $\mathrm{RZ}(K, A)$ has a topology as follows: for $x_1, ..., x_n \in K$ we define

$$U(x_1, ..., x_n) := \mathrm{RZ}(K, A[x_1, ..., x_n]).$$

More explicitly, since $\mathrm{RZ}(K, A[x_1, ..., x_n])$ is, by definition, the set of valuation rings of K centered at $A[x_1, ..., x_n]$, $U(x_1, ..., x_n)$ is nothing but

$$\{v \text{ valuation on } K \mid A \subset R_v, v(x_i) \leq 1,\ i = 1, ..., n\}.$$

Trivially, we have

$$U(x_1, ..., x_s) \cap U(y_1, ..., y_t) = U(x_1, ..., x_s, y_1, ..., y_y),$$

which ensures that the family

$$\mathscr{F} := \{U(x_1, ..., x_n) \mid n \geq 0, x_i \in K\}$$

indeed generates a topology on $\mathrm{RZ}(K, A)$. In analogy with the case of $\mathrm{Spec}(A)$, this topology is called *Zariski topology* on $\mathrm{RZ}(K, A)$. The special case $A = \mathbf{Z}$ gives the set of all valuations on K (as any valuation on $\mathbf{Z}$ takes values ≤ 1 since $\mathbf{Z}$ is additively generated by 1), and we write $\mathrm{RZ}(K)$ to denote $\mathrm{RZ}(K, \mathbf{Z})$.

Remark 2.3.2 In the case $K = \mathrm{Frac}(A)$, the basic open sets $U(x_1, ..., x_n)$ have a simpler form. Via finite intersections, we have that the topology on $\mathrm{RZ}(K)$ is actually generated by sets of the form

$$U(x) := \{v \text{ valuation on } K \mid A \subset R_v, v(x) \leq 1\}$$

with $x \neq 0$. Since $x \in K$, there exist a and b in A with $b \neq 0$ such that $x = a/b$, and so the topology on $\mathrm{RZ}(A)$ is generated by basic open sets of the form

$$U(a, b) := \{v \text{ valuation on } K \mid A \subset R_v, v(a) \leq v(b)\}.$$

The condition $b \neq 0$ can equally well be written as the condition $v(b) \neq 0$. By writing instead

$$U(a, b) = \{v \text{ valuation on } K \mid A \subset R_v, v(a) \leq v(b) \neq 0\},$$

we see that the right side also makes sense when $b = 0$ but $U(a, 0) = \emptyset$. This may seem like a silly remark, but it will be of (slightly) more interest when we pass to valuations on rings (in which case there will be lots of non-zero non-units b).

Example 2.3.3 Let us take $A = K$. Then, according to Definition 2.3.1, $\mathrm{RZ}(K, K)$ is simply a one-point space.

Example 2.3.4 Let $K = k((y))((x)) = \mathrm{Frac}(k((y))[\![x]\!])$ for a field k, and $A = k[\![x]\!]$. Then $\mathrm{RZ}(K, A)$ is the set of those valuations v on K trivial on k such that $v(x) \leq 1$, by property (3) of Definition 2.2.1. One such example is the evident "x-adic" discrete valuation v. Its valuation ring R is $k((y))[\![x]\!]$. Let us put in practice the procedure described in connection with Proposition 2.2.8. The maximal ideal $\mathfrak{m} \subset R$ is xR, so that, if we equip $R/\mathfrak{m} \simeq k((y))$ with the evident y-adic valuation (trivial on k) then we get $\Gamma_R = x^{\mathbf{Z}} \simeq \mathbf{Z}$, $\Gamma_{\overline{R}'} = y^{\mathbf{Z}} \simeq \mathbf{Z}$, and the exact sequence of abelian groups

$$1 \to \Gamma_{\overline{R}'} \to \Gamma_{R'} \to \Gamma_R \to 1$$

naturally splits to give

$$\Gamma_{R'} \simeq \Gamma_R \times \Gamma_{\overline{R}'} \simeq \mathbf{Z}^2$$

in which (exercise!) $\mathbf{Z}^2$ is totally ordered with the lexicographical ordering that $(n, m) < (n', m')$ if and only if either $n < n'$ or $n = n'$ and $m < m'$. This value group violates the "archimedean property" as in Remark 2.2.4, so R' is a higher-rank valuation ring.

We have the following:

Theorem 2.3.5 $\mathrm{RZ}(K, A)$ *is quasi-compact with respect to the Zariski topology.*

This is achieved via a Zorn's Lemma argument to establish the "finite intersection property" criterion. See [Mat, Thm. 10.5] for the details.

Remark 2.3.6 Zariski deeply studied $\mathrm{RZ}(K, k)$ for a finitely generated extension of fields K/k, especially with k algebraically closed of characteristic 0. He gave a classification of the valuation rings of K containing k (see [H, Ch. II, Exer. 4.12(b)] for the 2-dimensional case). This enabled him to prove the resolution of singularities of algebraic varieties in dimensions 2 and 3 over such k; see [Zar1], and [Zar2]. Hironaka's general proof of resolution of singularities in characteristic 0 was obtained with other methods, with no use of valuation theory.

In general, if K/k is a finitely generated extension of fields then the collection $\{X_i\}$ of projective integral k-schemes with function field K is an inverse system and the valuation criterion for properness provides a compatible system of maps $\mathrm{RZ}(K, k) \to X_i$ under which the preimage of a quasi-compact open subspace is a quasi-compact open subspace (the preimage of an affine open $\mathrm{Spec}(B) \subset X_i$ is $\mathrm{RZ}(K, B)$). Zariski proved that the resulting continuous map $\mathrm{RZ}(K, k) \to \varprojlim X_i$ is a homeomorphism. Recent work of Temkin (see [T]) has revived the role of valuation theory and (generalizations of) Riemann-Zariski spaces in the study of resolution problems in the scheme-theoretic framework; e.g., see [T, Cor. 3.4.7] for a vast generalization of Zariski's inverse limit description of $\mathrm{RZ}(K, k)$, applicable to any relatively affine morphism between arbitrary qcqs schemes and inspired by Huber's work on adic spaces.

2.4 $\mathrm{Spv}(A)$ as a Riemann-Zariski space

Valuations on commutative rings

We are ready to generalize the notion of valuation on a field K to a commutative ring A.

Definition 2.4.1 Let A be a commutative ring. A *valuation* on A is the datum of a map $v : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group, with the additional compatibility rule between Γ and 0 as in Definition 2.2.1, satisfying the following properties:

(1) $v(0) = 0, v(1) = 1$.

(2) $v(xy) = v(x)v(y)$, for all $x, y \in A$.

(3) $v(x+y) \leq \max(v(x), v(y))$ for all $x, y \in A$.

If $A = 0$ then there is no such v, and if $A \neq 0$ then it admits a homomorphism to a field, on which the trivial valuation pulls back to one on A. Of course, if A is not a field then $A - \{0\}$ is larger than $A^\times$, so it is no longer the case that necessarily $v(x) \neq 0$ when $x \neq 0$. We define

$$\mathrm{supp}(v) := \{x \in A : v(x) = 0\} = v^{-1}(0)$$

the *support* of v, which is easily seen to be a prime ideal, due to Definition 2.4.1.

Letting $\mathfrak{p}_v$ be the support of v, naturally v induces a well-defined valuation $\widetilde{v}$ on the residue field $\kappa(\mathfrak{p}_v)$ at $\mathfrak{p}_v$. Let $\Gamma_v := \widetilde{v}(\kappa(\mathfrak{p}_v)^\times)$. Write R_v for the valuation ring of $\widetilde{v}$ in $\kappa(\mathfrak{p}_v)$, so that

$$\Gamma_v = \kappa(\mathfrak{p}_v)^\times / R_v^\times.$$

It is not at all true that, in general, $v(A) \subset \Gamma_{\leq 1} \cup \{0\}$, as shown by the following example:

Example 2.4.2 Let $A = K$ a field. Then, we recover our old notion of valuation, as in Definition 2.2.1. If v is nontrivial, then $K = A \not\subseteq R_v$. Every field algebraic over a finite field admits only the trivial valuation (as all nonzero elements have finite order yet a totally ordered abelian group is torsion-free), but any other field does admit (many!) non-trivial valuations. Indeed, any such field contains either $\mathbf{Q}$ or $\mathbf{F}_p(x)$ for some prime p, and valuations on a subfield always extend to any extension field (perhaps at the cost of a huge value group) due to the characterization of valuation rings as maximality with respect to local domination in a field.

Definition 2.4.3 Two valuations v and v' on A are *equivalent* if the following equivalent conditions are satisfied:

(1) There is an isomorphism $\Gamma_v \simeq \Gamma_{v'}$ (necessarily unique) such that the following diagram commutes

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ \Gamma_v \cup \{0\} \arrow[r, "\sim"'] & \Gamma_{v'} \cup \{0\} \end{tikzcd}$$

(2) $\mathfrak{p}_v = \mathfrak{p}_{v'}$, and $R_v = R_{v'}$ as valuation rings of the residue field at this common prime ideal of A;

(3) For all $x, y \in A$, $v(x) \geq v(y)$ if and only if $v'(x) \geq v'(y)$.

We view condition (2) as the most efficient means of encoding a valuation on a ring A without the hassle of "equivalence": we simply pick a prime ideal $\mathfrak{p}$ of A and choose a valuation subring R of $\kappa(\mathfrak{p})$ with $\mathrm{Frac}(R) = \kappa(\mathfrak{p})$.

Definition 2.4.4 Let A be a commutative ring. The *valuation spectrum* of A, denoted $\mathrm{Spv}(A)$, is the set of equivalence classes of valuations on A (more concretely: the set of pairs $(\mathfrak{p}, R)$ where $\mathfrak{p}$ is a prime ideal of A and $R \subset \kappa(\mathfrak{p})$ is a valuation subring with full fraction field). The set $\mathrm{Spv}(A)$ is made into a topological space by letting the topology being generated by the sets

$$U\left(\frac{f}{g}\right) := \{v \in \mathrm{Spv}(A) \mid v(f) \leq v(g) \neq 0\},$$

for $f, g \in A$. (The set $U(f/g)$ is empty if $g = 0$, but for functoriality purposes as we vary A it is good to allow this silly case in the definition.)

In other words a base for the topology is given by finite intersections of sets $U(f/g)$. Informally, $U(f/g)$ should be viewed as the set of valuations v for which $v(f/g) \le 1$, even though the fraction f/g doesn't generally make sense in A. This notation is borrowed from work with rational domains in rigid-analytic geometry.

Remark 2.4.5 It may initially seem hard to visualize the topology on $\mathrm{Spv}(A)$. For example, if $v, w \in \mathrm{Spv}(A)$ then what does it means that $v \in \overline{\{w\}}$ (i.e., v is a "specialization" of w)? We will see later how to probe the topology with some concrete specialization constructions (generalizing Proposition 2.2.8 and Riemann-Zariski spaces of fields). To build a structure sheaf we will need to enrich our assumptions on A and restrict the class of valuations on A that we consider.

A good way to organize our study of the topology of $\mathrm{Spv}(A)$ is to think about the natural map

$$\begin{array}{c}\mathrm{Spv}(A)\\ \downarrow{\scriptstyle\varphi}\\ \mathrm{Spec}(A)\end{array}$$

sending a valuation v on A to its support $\mathfrak{p}_v$. As a warm-up, suppose $\mathrm{Spec}(A)$ is open in a proper scheme X over a field k (thereby making A a k-algebra) and v is trivial on k, so the valuation ring $R_v \subset \kappa(\mathfrak{p}_v)$ is a k-subalgebra. The valuative criterion for properness ensures that the k-morphism $\mathrm{Spec}(\kappa(\mathfrak{p}_v)) \to X$ extends (uniquely) to a k-morphism $\mathrm{Spec}(R_v) \to X$, and this lands inside the open subscheme $\mathrm{Spec}(A)$ if and only if the image of A inside $\kappa(\mathfrak{p}_v)$ lands inside R_v; generally it doesn't! That is, the closed point of $\mathrm{Spec}(R_v)$ typically will land outside $\mathrm{Spec}(A)$ in X. Geometrically speaking, $\varphi(v)$ is the image in X of the *generic point* of $\mathrm{Spec}(R_v)$, *not* the image of the closed point of $\mathrm{Spec}(R_v)$.

Proposition 2.4.6 *The map φ is continuous with respect to the respective topologies on* $\mathrm{Spv}(A)$ *and* $\mathrm{Spec}(A)$.

Proof. Indeed, consider a basic open set $D(b)$ in $\mathrm{Spec}(A)$. Then

$$\varphi^{-1}(D(b)) = \{v \in \mathrm{Spv}(A) \mid v(b) \neq 0\} = \{v \in \mathrm{Spv}(A) \mid v(0) \le v(b) \neq 0\}$$

which is $U\left(\frac{0}{b}\right)$, and is open. □

The fiber of φ over any point $\mathfrak{p} \in \mathrm{Spec}(A)$ is topologically the set $\mathrm{RZ}(\kappa(\mathfrak{p})) = \mathrm{RZ}(\kappa(\mathfrak{p}), \mathbf{Z})$ equipped with its Zariski topology as defined earlier in this lecture (check the topological aspect!). So, roughly speaking, $\mathrm{Spv}(A)$ is an amalgamation of Riemann-Zariski spaces at all points of $\mathrm{Spec}(A)$ (but its topology is much richer than the one on each fiber separately; "horizontal specialization" relative to φ will be a topic in a couple of lectures, closely related to Theorem 2.2.9 that is proved in the Appendix).

Example 2.4.7 Let us fix ideas on the case $A = K$ a field. Then $\mathrm{Spv}(K)$, as just remarked, is nothing but $\mathrm{RZ}(K)$. Consider $v, w \in \mathrm{Spv}(A)$ with the property that v is a specialization of w; i.e., $v \in \overline{\{w\}}$. This is equivalent to saying that all open subsets of $\mathrm{Spv}(A)$ containing v also contain w, and this is in turn equivalent to the same statement, but involving just (finite intersections of) the basic open sets containing v. More explicitly, the condition is exactly that if a basic open set $U\left(\frac{f}{g}\right)$ contains v then it contains w. The containment of v says

$$v(f) \le v(g) \neq 0,$$

and this is meant to imply $w(f) \le w(g) \neq 0$. Since $A = K$, this implication holds true for all $f, g \in K$ precisely when for any $x \in K$ we have that $v(x) \le 1$ implies $w(x) \le 1$. But this means exactly that we have a local inclusion of valuation rings:

$$R_v \subseteq R_w.$$

(Recall from Theorem 2.2.10 that in such cases R_w is the localization of R_v at a prime ideal, exactly as in the picture for "generization" in algebraic geometry.) For example, the trivial valuation v_0 corresponds to the valuation ring $R_{v_0} = K$ and this contains all others, so v_0 is the unique generic point of $\mathrm{Spv}(K)$.

For $v, w \in \mathrm{Spv}(A)$, we say that v is a *vertical specialization* of w if it is a specialization and has the same support. Letting $\mathfrak{p}$ denote this common support, this is exactly the specialization relation inside the topological fiber $\varphi^{-1}(\mathfrak{p}) = \mathrm{RZ}(\kappa(\mathfrak{p}))$ for which Example 2.4.7 describes the meaning of specialization in algebraic terms.

Next time we discuss quasi-compactness for $\mathrm{Spv}(A)$, and to do so it will be convenient to introduce a special class of topological spaces, as in the following definition:

Definition 2.4.8 A topological space X is *spectral* if:

(1) X is quasi-compact.

(2) X is sober.

(3) There exists a basis of quasi-compact open subsets of X stable under finite intersections (which implies that any finite intersection of quasi-compact open subsets is quasi-compact; i.e., X is a "quasi-separated" topological space).

The above definition is motivated by well-known topological properties of the spectrum of a ring.

2.5 Appendix: value groups of composite valuation rings

We now provide a proof of Theorem 2.2.9, using notation from that result. We call $\overline{v}'$ the valuation on $k = R/\mathfrak{m}$ with valuation ring $\overline{R}'$.

Recall that the value group for v is $K^\times/R^\times$ with the usual ordering that expresses divisibility relative to R, and the value group for $\overline{v}'$ is $k^\times/(\overline{R}')^\times$ with order structure expressing divisibility relative to $\overline{R}'$. Finally, the value group for R' is $K^\times/(R')^\times$ with order structure expressing divisibility relative to R'. Note that since R' is contained in R, v' is a specialization of v in $\mathrm{Spv}(K) = \mathrm{RZ}(K)$, as explained in Example 2.4.7.

There is an evident exact sequence of abelian groups

$$1 \to R^\times/(R')^\times \to K^\times/R'^\times \to K^\times/R^\times \to 1$$

and under the quotient map $R \to k$ we have $R^\times \to k^\times$ is surjective by locality of R and also $R' \to \overline{R}'$ is a surjective local homomorphism, so also $R'^\times \to \overline{R}'^\times$ is. Thus,

$$R^\times/R'^\times \to k^\times/\overline{R}'^\times$$

is an isomorphism. Hence, we have a short exact sequence of groups

$$(*) \quad 1 \to k^\times/\overline{R}'^\times \to K^\times/R'^\times \to K^\times/R^\times \to 1$$

expressing the value group for the composite valuation v' as an extension of the value group for v by the one for $\overline{v}'$.

But what about the relationship among the *order structures* for the three groups in $(*)$? The subgroup $k^\times/\overline{R}'^\times$ of the totally ordered abelian group $K^\times/(R')^\times$ is not any old subgroup: we will show that it satisfies the following condition.

Definition 2.5.1 Let Γ be a totally ordered abelian group, and H a subgroup. We say H is *convex* if for any h, h' in H every γ in Γ satisfying $h \leq \gamma \leq h'$, then necessarily $\gamma \in H$. (That is, an element of Γ sandwiched between two elements of H is also in H).

The interesting thing with convexity is that it is exactly what we need to have a well-defined structure of totally ordered abelian group on the quotient $\overline{\Gamma} := \Gamma/H$. Namely, we want to define $\overline{\Gamma}_{\leq 1}$ to be the image of $\Gamma_{\leq 1}/H$, which is to say that for elements g and g' in $\overline{\Gamma}$ we want to say $g \leq g'$ precisely when $g/g' \in \Gamma_{\leq 1}H$.

It is clear that this is a translation-invariant relation that is transitive and reflexive, and certainly for any $g, g' \in \overline{\Gamma}/H$ we have $g \leq g'$ or $g' \leq g$ (since the same holds at the level of choices of representatives in Γ). The key point is to check that this is anti-symmetric (i.e., $g \leq g'$ and $g' \leq g$ imply $g = g'$), and that holds precisely because H is convex: the task is to show that

$$\Gamma_{\leq 1}H \cap \Gamma_{\geq 1}H = H,$$

which is to say that if $\gamma, \gamma' \in \Gamma_{\leq 1}$ and $\gamma/\gamma' = h \in H$ then $\gamma, \gamma' \in H$. We have $1 \leq \gamma = \gamma' h \leq h$, so by convexity $\gamma \in H$, so $\gamma' = \gamma/h \in H$ too.

In fact, convexity is also *necessary* for this to be a total order (really to be anti-symmetric), as follows. Suppose $h \leq \gamma \leq h'$ with $h, h' \in H$ and $\gamma \in \Gamma$, so $h/\gamma \in \Gamma_{\leq 1}H$ and $\gamma/h' \in \Gamma_{\leq 1}H$. Thus, the class of γ in the "residue order" is both ≥ 1 and ≤ 1. Hence, if anti-symmetric would hold then this class must be 1, which is to say $\gamma \in H$. That is exactly the property of convexity.

Coming back to $(*)$, we cannot expect to express the ordered value group of the composite valuation v' in terms of those of v and $\overline{v}'$, as $(*)$ doesn't generally split even as abelian groups. Instead, we claim that the left term in $(*)$ is carried order-isomorphically onto a convex subgroup of the middle term and that the order structure on the right term is thereby induced from the middle via the general definition given above for quotients by convex subgroups.

As a first step, we check that the inclusion

$$k^\times/\overline{R}'^\times = R^\times/R'^\times \hookrightarrow K^\times/R'^\times$$

is order-preserving and in fact an order-isomorphism onto its image. This amounts to checking that for elements $r_1, r_2 \in R^\times$ with reductions $\overline{r}_1, \overline{r}_2 \in k^\times$, we have $\overline{r}_1/\overline{r}_2 \in \overline{R}'$ if and only if $r_1/r_2 \in R'$. But this is obvious since $R^\times \to k^\times$ is a homomorphism and R' is defined to be the preimage of $\overline{R}'$ under $R \twoheadrightarrow k$.

In the order structure on the quotient term $K^\times/R^\times$ in $(*)$, the part ≤ 1 is exactly $(R - \{0\})/R^\times$, and for $H := k^\times/\overline{R}'^\times = R^\times/R'^\times$ we have

$$(K^\times/R'^\times)_{\leq 1}H = ((R' - \{0\})/R'^\times)H = ((R' - \{0\})R^\times)/R'^\times = (R - \{0\})/R'^\times,$$

the final equality expressing that the inclusion $(R' - 0)R^\times \subset R - \{0\}$ is an equality since $\mathfrak{m}_R - \{0\} \subset R - \{0\}$ whereas $R - \mathfrak{m}_R = R^\times$. Hence, by the necessity established above it follows that the image of $H = \Gamma_{\overline{R}'}$ in $\Gamma_{R'} = K^\times/R'^\times$ has to be convex (as one could also check by bare hands) and we've just shown the quotient total ordering induced on Γ_R is the one imposed on the value group of v from that of the composite valuation v' via the value group of $\overline{v}'$.

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