UNITS OF RING SPECTRA AND THOM SPECTRA

MATTHEW ANDO, ANDREW J. BLUMBERG, DAVID GEPNER, MICHAEL J. HOPKINS, AND CHARLES REZK

ABSTRACT. We review and extend the theory of Thom spectra and the associated obstruction theory for orientations. Specifically, we show that for an E_∞ ring spectrum A, the classical construction of $gl_1 A$, the spectrum of units, is the right adjoint of the functor

$$\Sigma_+^\infty \Omega^\infty : \text{ho(connective spectra)} \to \text{ho}(E_\infty \text{ ring spectra}).$$

To a map of spectra

$$f : b \to bgl_1 A,$$

we associate an E_∞ A-algebra Thom spectrum Mf, which admits an E_∞ A-algebra map to R if and only if the composition

$$b \to bgl_1 A \to bgl_1 R$$

is null; the classical case developed by [MQRT77] arises when A is the sphere spectrum. We develop the analogous theory for A_∞ ring spectra. If A is an A_∞ ring spectrum, then to a map of spaces

$$f : B \to BGL_1 A$$

we associate an A-module Thom spectrum Mf, which admits an R-orientation if and only if

$$B \to BGL_1 A \to BGL_1 R$$

is null. We note that $BGL_1 A$ classifies the twists of A-theory. We take two different approaches to the A_∞ theory which are of independent interest. The first involves a rigidified model of A_∞ (and E_∞) spaces, as developed in [Blum05, BCS08]. The second uses the theory of ∞-categories as described in [HTT] and involves an ∞-categorical account of parametrized spectra. In order to compare these approaches to one another and to the classical theory, we characterize the Thom spectrum functor from the perspective of Morita theory.

CONTENTS

1. Introduction

The study of orientations of vector bundles and spherical fibrations is a classical topic in algebraic topology, deeply intertwined with the modern development of the subject. In his 1970 MIT notes [Sul05], Sullivan observed that to describe the obstructions to orientability and to classify orientations, one should take seriously the idea of the "units" of a multiplicative cohomology theory. In order to make such algebraic notions about multiplicative cohomology theories precise, May, Quinn, Ray, and Tornehave [MQRT77] developed the notion of an E_∞ ring spectrum and used it to describe the obstruction theory for orientations of spherical fibrations. Indeed the Thom spectra of infinite Grassmannians such as BO and BU provided their starting examples of E_∞ ring spectra. It is interesting to recall that the theory of structured ring spectra and "brave new algebras" in part had roots in this extremely concrete geometric theory.

In a forthcoming paper, three of us (Ando, Hopkins, Rezk) construct an E_∞ orientation of tmf, the spectrum of topological modular forms [Hop02, AHR]: more precisely, we construct a map of E_∞ ring spectra from the Thom spectrum $MO\langle 8\rangle$, also known as $MString$, to the spectrum tmf. That paper requires a formulation of the obstruction theory of [MQRT77] in terms of the adjoint relationship between units and $\Sigma_+^\infty \Omega^\infty$ described in Theorem 2.1 below. It also requires a formulation of the obstruction theory in terms of modern topological or simplicial model categories of E_∞ ring spectra.

This paper began as an effort to connect the results in [MQRT77] to the results required by [AHR]. In the course of doing this, we realized that modern technology makes it possible to develop an analogous theory of orientations for A_∞ ring spectra. We also discovered that recent work on ∞-categories provides an excellent framework for the study of Thom spectra and orientations, allowing constructions which are very close to the original sketch in Sullivan's notes.

In this paper we review and extend the theory of orientations of [MQRT77], taking advantage of intervening technical developments, particularly in the theory of multiplicative spectra and of ∞-categories. We recover the obstruction theory of [MQRT77], expressed in terms of the adjunction of Theorem 2.1, and extend it to A_∞ ring spectra. We also extend the theory to more general kinds of fibrations, such as those which appear in the study of twisted generalized cohomology.

We begin by reviewing the analogy between the theory of orientations and the theory of locally free sheaves of rank one. This analogy, which appears in Sullivan's notes, is an excellent guide to the basic results of the subject. Modern technology enables us to hew more closely to this intuitive picture than was possible thirty years ago, and so it is remarkable how much of the theory was worked out in the early seventies.

1.1. **The Thom isomorphism and invertible sheaves.** Let A be a commutative ring spectrum; at this point, we remain vague about exactly what we mean by the term. An A-module M is free of rank one if there is a weak equivalence of A-modules

$$A \to M.$$

If X is a space, and V is a virtual vector bundle of rank zero over X, then let X^V be the associated Thom spectrum. A Thom isomorphism for V is a weak equivalence of A^{X_+}-modules

$$A^{X_+} \simeq A^{X^V}.$$

The local triviality of V and the suspension isomorphism together imply that, locally on X, A^{X^V} is free of rank one, and suggest that there is a cohomological obstruction to the existence of a Thom isomorphism.

Thus let

$$\underline{A} : (\text{spaces})^{\text{op}} \to (\text{commutative ring spectra})$$

be the presheaf of ring spectra defined by

$$\underline{A}(Y) = A^{Y_+},$$

and let

$$A_X = \underline{A}|_{\text{Open}(X)^{\text{op}}}$$

be the associated presheaf of ring spectra on X. Similarly, for U open in in X, let U^V be the Thom space of V restricted to U, and let A_V be the presheaf of A_X-modules whose value on the open set U is given by the formula

$$A_V(U) = A^{U^V}.$$

An open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ of X and trivializations

$$h_\alpha : U_\alpha^V \cong (U_\alpha)_+$$

of V over $\mathfrak{U}$ give equivalences

$$A_X(U_\alpha) = A^{(U_\alpha)_+} \xrightarrow{A^{h_\alpha}} A^{(U_\alpha^V)} = A_V(U_\alpha).$$

Over an intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have an equivalence of $A_X(U_{\alpha\beta})$-modules

$$g_{\alpha\beta} : A_X(U_{\alpha\beta}) \xrightarrow{h_\alpha} A_V(U_{\alpha\beta}) \xrightarrow{h_\beta^{-1}} A_X(U_{\alpha\beta}).$$

This analysis suggests that we let $\underline{A}^\times$ be the presheaf on spaces which on Y consists of $\underline{A}(Y)$-module equivalences

$$\underline{A}(Y) \to \underline{A}(Y)$$

Then the $g_{\alpha\beta}$ are a Cech 1-cocycle for the cover $\mathfrak{U}$, with values in the presheaf $\underline{A}^\times$, whose class

$$[V] \in H^1(X; \underline{A}^\times)$$

depends only on (the isomorphism class of) the vector bundle V. However $[V]$ is defined, it should be both the Cech cohomology class classifying the invertible A_X-module A_V and the obstruction to giving a Thom isomorphism for V in A-theory.

The difficulty in making this program precise is at once subtle and familiar, and arises from the suppleness of the presheaf $\underline{A}$. For example, in trivializing the cocycle $g_{\alpha\beta}$, one expects to encounter a family of sections

$$f_\alpha^0 \in \underline{A}^\times(U_\alpha)$$

and homotopies

$$f_{\alpha\beta}^1 : f_\beta^0 \simeq f_\alpha^0 g_{\alpha\beta} \tag{1.1}$$

in $\underline{A}^\times(U_{\alpha\beta} \times I)$. The $f_{\alpha\beta}^1$ will satisfy a coherence condition of their own, and so forth. Thus, we are brought quickly to the need for homotopical coherence machinery.

1.2. **Units.** The classical approach to addressing these issues goes as follows. Let A be an associative ring spectrum, that is, a monoid in the homotopy category of spectra. Following [MQRT77], let GL_1A be the pull-back in the diagram of (unpointed) spaces[1]

$$\begin{array}{ccc} GL_1A & \longrightarrow & \Omega^\infty A \\ \downarrow & & \downarrow \\ (\pi_0 A)^\times & \longrightarrow & \pi_0 A. \end{array} \tag{1.2}$$

If X is a space, then

$$[X, GL_1A] = \{f \in A^0(X_+) | \pi_0 f(X) \subset (\pi_0 A)^\times\} = A^0(X_+)^\times,$$

so GL_1A is called the *space of units* of A, and so a more refined definition of $\underline{A}^\times$ is

$$\underline{A}^\times(X) = \mathrm{map}(X, GL_1A).$$

---

[1]$GL_1A \to \Omega^\infty A$ is the inclusion of a set of path components, so this is a homotopy pull-back. To make the functor $A \mapsto GL_1A$ homotopically well-behaved, we should require A to be a fibrant object in a model category of algebras. For the model category of A_∞ ring spectra in Lewis-May-Steinberger spectra, all objects are fibrant, and the simple definition (1.2) suffices. See §6 for further discussion on this point.

If A is A_∞ ring spectrum in the sense of Lewis-May-Steinberger, then $GL_1 A$ is a group-like A_∞ space, so it has a delooping $BGL_1 A$, and the appropriate cohomology group is

$$H^1(X, \underline{A}^\times) = [X, BGL_1 A].$$

Moreover if A is an E_∞ ring spectrum, then there is a spectrum $gl_1 A$ such that

$$GL_1 A \simeq \Omega^\infty gl_1 A.$$

One could also work in a modern symmetric monoidal category of spectra, such as the S-modules of [EKMM96] or one of the categories of diagram spectra [MMSS01]. As we discuss in §6, in that case we should require A to be a cofibrant-fibrant algebra, and then (Proposition 6.2) the homotopy type of $GL_1 A$ is that of the subspace of the *derived* mapping space of A-module endomorphisms $\hom_A(A, A)$ consisting of weak equivalences. This approach is related to the approach studied in [MS06].

For now we continue by imitating how one would proceed if $GL_1 A$ were a topological group, acting on the spectrum A. Associated to the group $GL_1 A$ we would have the principal fibration

$$GL_1 A \to EGL_1 A \to BGL_1 A,$$

and given a map $f : X \to BGL_1 A$, we would form the pull-back

$$
\begin{array}{ccc}
P & \longrightarrow & EGL_1 A \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & BGL_1 A.
\end{array}
$$

One expects that the Thom spectrum associated to f is a sort of Borel construction

$$Mf = P \times_{GL_1 A} A, \tag{1.3}$$

and that the space of A-orientations of Mf is the space of sections of P/X. To motivate this picture, consider the case that $G = GL_1 R$ is the group of units of a discrete ring R, so that BG, the moduli space of G-torsors, is equivalently the moduli space of free R-modules of rank 1. To a principal G-bundle

$$G \to P \to X$$

we can attach the family of free rank-one R-modules

$$\pi : \xi = P \times_G R \to X$$

parametrized by X; conversely to such a family ξ we can attach its G-torsor of trivializations

$$P = \xi^\times = \{z \in \xi | z \text{ is an } R\text{-module generator of } \xi_{\pi(z)}\}$$

An obvious R-module associated to this situation, at least if X is discrete, is

$$\bigoplus_{x \in X} \xi_x \cong \operatorname*{colim}_{x \in X} \xi_x \tag{1.4}$$

If X is discrete then P is the G-space

$$P = \coprod_{x \in X} P_x,$$

and we can also form the R-module

$$\mathbb{Z}[P] \otimes_{\mathbb{Z}[G]} R. \tag{1.5}$$

Here $\mathbb{Z}[P]$ is the free abelian group on the points of P, $\mathbb{Z}[G]$ is the group ring of G, and the natural map

$$\mathbb{Z}[G] \to R$$

is the counit of the adjunction

$$\mathbb{Z} : (\text{groups}) \rightleftarrows (\text{rings}) : GL_1. \tag{1.6}$$

In fact the two R-modules (1.4) and (1.5) are isomorphic, as they represent the same functor:

$$(R\text{–modules})\Big(\bigoplus_{x\in X}\xi_x, T\Big) \cong \prod_{x\in X}(R\text{–modules})(\xi_x, T) \tag{1.7}$$
$$\cong \prod_{x\in X}(G\text{–sets})(\xi_x^{\times}, T)$$
$$\cong (G\text{–sets})(P, T)$$
$$\cong (\mathbb{Z}[G]\text{–modules})(\mathbb{Z}[P], T)$$
$$\cong (R\text{–modules})(\mathbb{Z}[P]\otimes_{\mathbb{Z}[G]}R, T).$$

These constructions generalize to give equivalent notions of A-module Thom spectrum when A is an A_∞ ring spectrum. The idea that this should be so is due to the senior author (Hopkins), and was the starting point for this paper. For now we set

$$Mf = \mathbb{Z}[P]\otimes_{\mathbb{Z}[G]}R, \tag{1.8}$$

and observe that with $T = R$ above we have

$$(R\text{–modules})(Mf, R) \cong (G\text{–sets})(P, R).$$

With respect to this isomorphism, the set of *orientations* of Mf is to be the subset

$$
\begin{array}{ccc}
(R\text{–modules})(Mf, R) & \xrightarrow{\cong} & (G\text{–sets})(P, R) \\
\uparrow & & \uparrow \\
(\text{orientations})(Mf, R) & \xrightarrow{\cong} & (G\text{–sets})(P, G),
\end{array}
$$

which in turn is isomorphic to the set of sections of the principal G-bundle P/X, as expected.

Returning to the provisional definition of equation (1.3), one approach to generalizing to the input of a ring spectrum A and a space X is to develop the machinery to mimic that definition, in the form (1.5), in the setting of A_∞ spaces. We carry this out in §5.

Another possible approach to this problem involves coming to grips with homotopy sheaves of spectra, and construct ξ as a homotopy sheaf of A-modules. If X is a paracompact Hausdorff space, then as in [Seg68] or [HTT, 7.1], it is equivalent to consider homotopy local systems of A-modules parametrized by X.

The parametrized homotopy theory of May and Sigurdsson provides one context for doing so, and they have discussed twisted generalized cohomology from this point of view in [MS06]. The apparatus of ∞-categories provides another setting for such questions. In §7 we show how to develop a framework for parametrized spectra in the context of the theory of quasicategories of Joyal and Lurie that makes it possible to develop the theory of Thom spectra and orientations essentially as we have described it above, with a construction of Thom spectra generalizing (1.4).

As mentioned above, it is remarkable the extent to which these ideas can and have been implemented using classical methods. To be more specific, the space BGL_1S associated to the sphere spectrum is the classifying space for stable spherical fibrations, and if A is an A_∞ ring spectrum, then the unit $S \to A$ gives rise to a map $BGL_1S \to BGL_1A$.

Given a spherical fibration classified by a map $g : X \to BGL_1S$, we can form the solid diagram

$$
\begin{array}{ccccc}
P & \longrightarrow & B(S, A) & \longrightarrow & EGL_1A \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
X & \longrightarrow & BGL_1S & \longrightarrow & BGL_1A, \\
 & g & & &
\end{array}
\tag{1.9}
$$

$$\underbrace{}_{f}$$

in which the rectangles are homotopy pull-backs. In that case,

$$Mf = (Mg) \wedge A,$$

the space $B(S, A)$ is the space of A-oriented spherical fibrations, and to give an orientation

$$Mg \to A$$

is to give a lift as indicated in the diagram.

In his 1970 MIT notes [Sul05] (in the version available at `http://www.maths.ed.ac.uk/~aar/books/gtop.pdf`, see the note on page 236), Sullivan introduced this picture, and suggested that Dold's theory of homotopy functors [Dol66] could be used to construct the space $B(S, A)$ of A-oriented spherical fibrations. He also mentioned that the technology to construct the delooping $BGL_1 A$ was on its way. Soon thereafter, May, Quinn, Ray, and Tornehave in [MQRT77] constructed the space $BGL_1 A$ in the case that A is an E_∞ ring spectrum, and described the associated obstruction theory for orientations of spherical fibrations. Thus, at least for spherical fibrations and E_∞ ring spectra, the obstruction theory for Thom spectra and orientations has been available for over thirty years.

Various aspects of the theory of units and Thom spectra have been revisited by a number of authors as the foundations of stable homotopy theory have advanced. For example, Schlichtkrull [Sch04] studied the units of a symmetric ring spectrum, and May and Sigurdsson [MS06] have studied units and orientations in light of their categories of parametrized spectra. Very recently May has prepared an authoritative paper revisiting operad (ring) spaces and operad (ring) spectra from a modern perspective, which has substantial overlap with some of the discussion in this paper [May08] (notably Section 3).

2. OVERVIEW

In this section, we give a detailed summary of the contents of the paper and state the main results.

2.1. **The spectrum of units and E_∞ orientations.** We begin in §3 by revisiting the construction, due to May et al. [MQRT77], of the spectrum of units $gl_1 R$ associated to an E_∞ ring spectrum R. Motivated by the adjunction (1.6), or more precisely its restriction to abelian groups

$$\mathbb{Z} : (\text{abelian groups}) \rightleftarrows (\text{commutative rings}) : GL_1,$$

we prove the following (see also [May08]).

Theorem 2.1 (Theorem 3.2). *The functor gl_1 participates as the right adjoint in an adjunction*

$$\Sigma_+^\infty \Omega^\infty : \text{ho}((-1)\text{-connected spectra}) \rightleftarrows \text{ho}(E_\infty \text{ ring spectra}) : gl_1 \qquad (2.2)$$

which preserves the homotopy types of derived mapping spaces.

With this in place, in §4 we recall and extend the obstruction theory of [MQRT77] for E_∞ orientations. Let R be an E_∞ ring spectrum, and suppose that b is a spectrum over $bgl_1 R = \Sigma gl_1 R$. Let p be the homotopy

pull-back in the solid diagram

$$
\begin{array}{ccc}
gl_1R & =\!=\!=\!= & gl_1R \\
\downarrow & & \downarrow \\
p & \longrightarrow & egl_1R \simeq * \\
\downarrow & & \downarrow \\
b & \xrightarrow{\ f\ } & bgl_1R.
\end{array}
\tag{2.3}
$$

Note that if $B = \Omega^\infty b$ and $P = \Omega^\infty p$, then after looping down we have a fibration sequence

$$
GL_1R \to P \to B \to BGL_1R.
\tag{2.4}
$$

Note also that an E_∞ map $\varphi : R \to A$ gives a diagram

$$
\begin{array}{ccc}
gl_1R & \longrightarrow & gl_1A \\
\downarrow & & \downarrow \\
p & \longrightarrow & egl_1A \simeq * \\
\downarrow & & \downarrow \\
b & \xrightarrow{\tilde{\varphi}\circ f} & bgl_1A,
\end{array}
\tag{2.5}
$$

where we write $\tilde{\varphi} : bgl_1R \to bgl_1A$ for the induced map.

Definition 2.6. The Thom spectrum $M = Mf$ of $f : b \to bgl_1R$ is the homotopy push-out M of the diagram of E_∞ spectra

$$
\begin{array}{ccc}
\Sigma_+^\infty \Omega^\infty gl_1R & \longrightarrow & R \\
\downarrow & & \downarrow \\
\Sigma_+^\infty \Omega^\infty p & \longrightarrow & M,
\end{array}
\tag{2.7}
$$

where the top map is the counit of the adjunction (2.2). Note that the spectrum underlying M is the derived smash product

$$
M = \Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1R}^L R,
\tag{2.8}
$$

generalizing the construction (1.8).

The adjunction between $\Sigma_+^\infty \Omega^\infty$ and gl_1 shows that, for any E_∞ map $\varphi : R \to A$, we have a homotopy pull-back diagram of derived mapping spaces

$$
\begin{array}{ccc}
(E_\infty \text{ ring spectra})(M, A) & \longrightarrow & \mathscr{S}(p, gl_1A) \\
\downarrow & & \downarrow \\
(E_\infty \text{ ring spectra})(R, A) & \longrightarrow & \mathscr{S}(gl_1R, gl_1A),
\end{array}
\tag{2.9}
$$

and comparing fibers over φ gives the following.

Theorem 2.10. *There is a homotopy pull-back diagram*

$$
\begin{array}{ccc}
(E_\infty \text{ } R\text{-algebras})(M, A) & \longrightarrow & \mathscr{S}(p, gl_1A) \\
\downarrow & & \downarrow \\
\{\varphi\} & \longrightarrow & \mathscr{S}(gl_1R, gl_1A).
\end{array}
$$

That is, the space of R-algebra maps $M \to A$ is weakly equivalent to the space of lifts in the diagram (2.5).

An important feature of our construction is that it produces a Thom spectrum from any map $b \to bgl_1 R$, not just from spherical fibrations. To make contact with the classical situation, let S be the sphere spectrum, and suppose we are given a map

$$g : b \to bgl_1 S,$$

so that

$$G = \Omega^{\infty} g : B \to BGL_1 S$$

classifies a stable spherical fibration. Using Definition 2.6, we can form the E_{∞} Thom spectrum Mg. In §8, we show that the spectrum underlying Mg is the usual Thom spectrum of the spherical fibration classified by G, as constructed for example in [LMSM86].

Now suppose that R is an E_{∞} spectrum with unit $\iota : S \to R$, and let

$$f = bgl_1 \iota \circ g : b \to bgl_1 S \to bgl_1 R.$$

Then

$$Mf \simeq Mg \wedge^L R,$$

and so we have an equivalence of derived mapping spaces

$$\mathscr{S}[E_{\infty}](Mg, R) \simeq (E_{\infty} \ R\text{-algebras})(Mf, R).$$

If we let $b(S, R)$ be the pull-back in the solid diagram

$$
\begin{array}{ccccc}
p & \longrightarrow & b(S,R) & \longrightarrow & * \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
b & \longrightarrow & bgl_1 S & \longrightarrow & bgl_1 R,
\end{array}
\tag{2.11}
$$

then Theorem 2.10 specializes to a result of May, Quinn, Ray, and Tornehave [MQRT77].

Corollary 2.12. *The derived space of E_{∞} maps $Mg \to R$ is weakly equivalent to the derived space of lifts in the diagram* (2.11).

2.2. **The space of units and orientations.** The authors of [MQRT77] describe E_{∞} orientations for E_{∞} ring spectra, and they also discuss not-necessarily E_{∞} orientations of E_{∞} spectra. As far as we know, there is no treatment in the literature of the analogous orientation theory for A_{∞} ring spectra.

In §5 and in §7, we give two independent approaches to the theory of Thom spectra and orientations for A_{∞} ring spectra. Our first approach begins by adapting the ideas of the E_{∞} construction in §3. In the associative case the analogue of the adjunction (1.6) is

$$(\text{group-like } A_{\infty} \text{ spaces}) \underset{GL_1}{\overset{}{\rightleftarrows}} (A_{\infty} \text{ spaces}) \overset{\Sigma_+^{\infty}}{\underset{\Omega^{\infty}}{\rightleftarrows}} (A_{\infty} \text{ ring spectra}) : GL_1,$$

where the right-hand adjunction is a special case of [LMSM86, p. 366]. If R is an A_{∞} spectrum, then we have the related adjunction

$$\Sigma_+^{\infty} : (\text{right } \Omega^{\infty} R\text{-modules}) \rightleftarrows (\text{right } R\text{-modules}) : \Omega^{\infty}.$$

The main difficulty in using this classical operadic approach is that $GL_1 R$ is a not a topological group but rather only a group-like A_{∞} space, and so it is not immediately apparent how to form the (quasi)fibration

$$GL_1 R \to EGL_1 R \to BGL_1 R,$$

and then make sense of the constructions suggested in §1.2.

In §5 we present technology to realize the picture sketched in §1.2. The essential strategy is to adapt the operadic smash product of [KM95, EKMM96] to the category $\mathscr{T}$ of spaces. Specifically, we produce a symmetric monoidal product on a subcategory of $\mathscr{T}$ such that monoids for this product are precisely A_{∞}-spaces; this allows us to work with models of $GL_1 R$ which are strict monoids for the new product. The observation that one could carry out the program of [EKMM96] in the setting of spaces is due to Mike

Mandell, and was worked out in the thesis of the second author [Blum05] (see also the forthcoming paper [BCS08]).

The relevant category of spaces for this product is $*$-modules, the space-level analogue of EKMM's S-modules. Precisely, let $\mathbb{L}$-spaces be the category of spaces with an action of the 1-space $\mathcal{L}(1)$ of the linear isometries operad. On this category we can define an operadic product $\times_{\mathcal{L}}$ which is associative and commutative but not unital. The category of $*$-modules is the subcategory of $\mathbb{L}$-spaces for which the unit $* \times_{\mathcal{L}} X \to X$ is an isomorphism. The category of $*$-modules is Quillen equivalent to the category of spaces and admits an operadic symmetric monoidal product.

In this setting, we can form a model of GL_1R which is a group-like monoid, and then model $EGL_1R \to BGL_1R$ as a quasi-fibration with an action of GL_1R. Given a fibration of $*$-modules

$$f : B \to BGL_1R,$$

GL_1R acts on the pull-back P in the diagram

$$
\begin{array}{ccc}
P & \longrightarrow & EGL_1R \\
\downarrow & & \downarrow \\
B & \xrightarrow{\ f\ } & BGL_1R,
\end{array}
$$

and the S-module $\Sigma_+^\infty P$ is a right $\Sigma_+^\infty GL_1R$-module. We can then imitate (1.8) to form an R-module Thom spectrum.

Definition 2.13. Given a map of spaces $f \colon B \to BGL_1R$, form P as the pullback in the diagram above associated to a fibrant replacement in $*$-modules of the free map $* \times_{\mathcal{L}} \mathbb{L}B \to BGL_1R$. The Thom spectrum of f is defined to be the derived smash product

$$Mf = \Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1R}^L R. \tag{2.14}$$

With this definition, we have

$$(\text{right } R\text{–modules})(M, R) \simeq (\text{right } GL_1R\text{–spaces})(P, \Omega^\infty R), \tag{2.15}$$

where here (and in the remainder of this subsection) we are referring to derived mapping spaces.

Definition 2.16. The space of *orientations* of M is the subspace of R-module maps $M \to R$ which correspond to

$$(\text{right } GL_1R\text{–modules})(P, GL_1R) \subset (\text{right } GL_1R\text{–modules})(P, \Omega^\infty R).$$

under the weak equivalence (2.15). That is, we have a homotopy pull-back diagram

$$
\begin{array}{ccc}
(\text{orientations})(M, R) & \xrightarrow{\ \simeq\ } & (\text{right } GL_1R\text{–spaces})(P, GL_1R) \\
\downarrow & & \downarrow \\
(\text{right } R\text{–modules})(M, R) & \xrightarrow{\ \simeq\ } & (\text{right } GL_1R\text{–modules})(P, \Omega^\infty R).
\end{array}
$$

To make contact with classical notions of orientation, for $x \in B$ let M_x be the Thom spectrum associated to the map

$$\{x\} \to B \xrightarrow{\ f\ } BGL_1R.$$

The map $x \to B$ on passage to Thom spectra gives rise to a map of R-modules $M_x \to M$. Then we have the following (Theorem 5.38 and Proposition 5.42).

Theorem 2.17.

(1) *A map of right R-modules $u : M \to R$ is an orientation if and only if for each $x \in B$, the map of R-modules*

$$M_x \to M \to R$$

is a weak equivalence.

(2) *If $f : B \to BGL_1R$ is a fibration, then the space of orientations $M \to R$ is weakly equivalent to the derived space of lifts*

$$
\begin{array}{ccc}
P & \longrightarrow & EGL_1R \\
\downarrow & \nearrow & \downarrow \\
B & \xrightarrow{\;f\;} & BGL_1R.
\end{array}
$$

As for the Thom isomorphism, just as in the classical situation we have an R-module Thom diagonal

$$M \to \Sigma_+^\infty B \wedge M.$$

and given a map of right R-modules $u : M \to R$, we obtain the composite map of right R-modules

$$\rho(u) : M \to \Sigma_+^\infty B \wedge M \xrightarrow{1 \wedge f} \Sigma_+^\infty B \wedge R,$$

about which we have the following (See [MR81] and [LMSM86, §IX]).

Proposition 2.18 (Proposition 5.45; see also Corollary 7.34)**.** *If $u : M \to R$ is an orientation, then $\rho(u)$ is a weak equivalence.*

As in the E_∞ case, we emphasize that our construction associates an R-module Thom spectrum to a map $B \to BGL_1R$, which need not arise from a spherical fibration. To compare to the classical situation, we suppose that F arises from a stable spherical fibration via

$$F : B \xrightarrow{G} BGL_1S \xrightarrow{BGL_1\iota} BGL_1R.$$

Then we can form the Thom spectrum MG using Definition 2.13 (in §8 we show that this coincides with the Thom spectrum associated to G as in for example [LMSM86]), and it follows directly from the definition that

$$MF \simeq MG \wedge^L R. \tag{2.19}$$

We define an R-orientation of MG to be a map of spectra

$$MG \to R$$

such that the induced map of R-modules

$$MF \to R$$

is an orientation as above. We then recover the result of Sullivan and of May, Quinn, Ray, and Tornehave [Sul05, MQRT77].

Corollary 2.20. *Let $B(S, R)$ to be the pull-back in the solid diagram*

$$
\begin{array}{ccccc}
P & \longrightarrow & B(S,R) & \longrightarrow & EGL_1R \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
B & \longrightarrow & BGL_1S & \longrightarrow & BGL_1R.
\end{array}
$$

Then the space of R-orientations of MG is the space of indicated lifts.

2.3. **Quasicategories and units.** As we observed in §1, the useful notion of GL_1R-bundle is that of a homotopy sheaf. Joyal's theory of quasicategories, as developed in Lurie's book [HTT], allows us to be precise about this. Specifically, we use this theory in §7 and §B to give an account of parametrized spectra (homotopy sheaves), Thom spectra and orientations which is very close to the intuitive picture discussed in §1.

In the new theory, the analogue of BGL_1R is R-line, the subcategory of the ∞-category R-mod of R-modules consisting of equivalences of free rank-one cofibrant and fibrant R-modules. To see the virtues of R-line, we note suggestively that it contains all two-simplices of the form

$$
\begin{array}{ccc}
L & \xrightarrow{\ f\ } & M \\
 & \searrow^{\sigma}_{h} & \downarrow g \\
 & & N,
\end{array}
$$

where f, g, and h are R-module weak equivalences, and σ is a homotopy of weak equivalences from gf to h. This is the sort of data mentioned at (1.1), and it exhibits R-line as the classifying space for "homotopy local systems of free R-modules of rank 1" (we call them bundles of R-lines).

Specifically, for a fibrant simplicial set X, there is an equivalence of ∞-categories between maps $X \to R$-mod and bundles of R-modules over X. The identity map R-mod $\to R$-mod classifies the "universal bundle of R-modules", and pulling back along the inclusion R-line $\to R$-mod gives the the universal bundle of R-lines. We discuss this general approach to parametrized spectra in detail in Section 7.4.

Now suppose that R is a cofibrant and fibrant algebra in spectra. Let R° be an object of R-line, and let

$$
\mathrm{Aut}(R^\circ) \stackrel{\mathrm{def}}{=} R\text{-line}(R^\circ, R^\circ) \subset R\text{-mod}(R^\circ, R^\circ)
$$

be the subspace of weak equivalences. It is a group in the ∞-categorical sense: as the automorphisms of R° in the ∞-category R-line, it is a group-like monoidal ∞-groupoid, or equivalently a group-like A_∞ monoid. Note that R-mod(R°, R°) is the derived space of endomorphisms of R, and so $\mathrm{Aut}(R^\circ)$ is the derived space of self weak equivalences of R. We show (Proposition 6.2 and §7.7) that

$$
GL_1(R^\circ) \simeq \mathrm{Aut}(R^\circ).
$$

Now the full ∞-subcategory of R-line on the single object R° is just $B\,\mathrm{Aut}(R^\circ)$. By definition, R-line is a connected ∞-groupoid (connected Kan complex), and so the inclusion

$$
B\,\mathrm{Aut}(R^\circ) \to R\text{-line}
$$

is an equivalence, and it follows that

$$
BGL_1(R^\circ) \simeq R\text{-line}.
$$

The analogue of EGL_1R is R-triv, the ∞-category of trivialized R-lines: R-modules equipped with a specific equivalence to R. It is a contractible Kan complex, and the natural map

$$
R\text{-triv} \to R\text{-line}
$$

is a Kan fibration, our model for the fibration $EGL_1R \to BGL_1R$.

In this setting, a map $X \to BGL_1R$ corresponds to a map of simplicial sets

$$
f : \Pi_\infty X \to R\text{-line},
$$

where $\Pi_\infty X$ is the ∞-groupoid (the singular complex) of the space X. By construction, R-line comes equipped with a universal bundle $\mathscr{L}$ of R-lines, and the map f classifies the bundle of R-lines $f^*\mathscr{L}$ over X. A lift in the diagram

$$
\begin{array}{ccc}
 & & R\text{-triv} \\
 & \nearrow & \downarrow \\
\Pi_\infty X & \xrightarrow{\ f\ } & R\text{-line;}
\end{array}
\tag{2.21}
$$

corresponds to an equivalence of bundles of R-lines

$$
f^*\mathscr{L} \xrightarrow{\ \simeq\ } R_X,
$$

where R_X denotes the trivial bundle of R-lines over X.

In analogy to (1.4), we have the following.

Definition 2.22. The Thom spectrum Mf of f is the colimit of the map of ∞-categories

$$\Pi_\infty X \xrightarrow{f} R\text{-line} \to R\text{-mod}.$$

Note that the colimit is the same as the left Kan extension along the map to a point, so this definition is an analogue of the May-Sigurdsson description of the Thom spectrum as the composite of the pullback of a universal parametrized spectrum followed by the base change along the map to a point [MS06, 23.7.1,23.7.4]. Using this definition, it is straightforward to prove the following.

Theorem 2.23. *The space of orientations of Mf is weakly equivalent to the space of lifts in the diagram* (2.21), *equivalently, to the space of trivializations*

$$f^*\mathscr{L} \to R_X.$$

This definition also implies the following characterization, which plays a role in §8 when we compare our approaches to Thom spectra. Recall that $\mathrm{Aut}(R^\circ) = R\text{-line}(R^\circ, R^\circ)$ is a group in the ∞-categorical sense: it is a group-like monoidal ∞-groupoid, or equivalently a group-like A_∞ monoid. In general if G is a group-like monoidal ∞-groupoid, then it has a classifying ∞-groupoid BG.

Proposition 2.24. *Let G be a group-like monoidal ∞-groupoid. Then the Thom spectrum of a map $BG \to R\text{-mod}$ is equivalent to the homotopy quotient R°/G.*

The Proposition follows immediately from the construction of the Thom spectrum, since by definition the quotient in the statement is the colimit of the map of ∞-categories

$$BG \to R\text{-line} \to R\text{-mod}.$$

In these introductory remarks, we have taken some care to relate the ∞-categorical treatment to other approaches. In fact, as explained in [HTT, DAGI, DAGII], it is possible to develop the ∞-categories of of spectra, algebras in spectra, and R-modules entirely in the setting of quasicategories, without appeal to external models of spectra or even of spaces. We take this approach to Thom spectra and orientations in §7. It leads to a clean discussion, free from distractions of comparison to other models for spectra. We also hope that it will serve as a useful introduction to the results of [DAGI].

For the reader who prefers to begin with a classical model for spectra, in §B we recapitulate some of the discussion in §7, building the ∞-category R-mod from a monoidal simplicial model category of spectra. We make use of some basic results from [HTT], but do not require anything from [DAGI, DAGII]. We hope that this section offers a useful introductory example of doing homotopy theory with quasicategories.

2.4. **Comparison of Thom spectra.** So far we have given three constructions of Thom spectra, Definitions 2.6, 2.13, and 2.22. In section §8, we compare these notions to each other and to the Thom spectra of Lewis-May-Steinberger and May-Sigurdsson [LMSM86, MS06].

An inspection of Definitions 2.6 and 2.13 shows that the following result is a consequence of the familiar fact that, if Mf is the homotopy pushout in a diagram of E_∞ ring spectra

$$
\begin{array}{ccc}
\Sigma_+^\infty \Omega^\infty gl_1 R & \longrightarrow & R \\
\downarrow & & \downarrow \\
\Sigma_+^\infty \Omega^\infty p & \longrightarrow & Mf,
\end{array}
$$

then the spectrum underlying Mf is the derived smash product

$$Mf \simeq \Sigma_+^\infty \Omega^\infty p \wedge^L_{\Sigma_+^\infty \Omega^\infty gl_1 R} R.$$

Proposition 2.25. *Let R be an E_∞ ring spectrum, and let $f : b \to bgl_1 R$ be a map. The spectrum underlying the E_∞ Thom spectrum Mf of Definition 2.6 is equivalent to the spectrum $M\Omega^\infty f$ of Definition 2.13.*

Given this simple observation, our main focus in §8 is on comparing the A_∞ construction of Definition 2.13 and §5, the quasicategorical construction of Definition 2.22 and §7, and the constructions of [LMSM86, MS06].

In §7.7 (see Proposition 7.38) we show that $BGL_1R \simeq R$-line, so in §8 we focus on more conceptual matters. Essentially, we are confronted with topological analogues of the two definitions (1.4) and (1.5). Definition 2.13 associates to a fibration of $*$-modules $f : B \to BGL_1R$ the pull-back in the diagram

$$
\begin{array}{ccc}
P & \longrightarrow & EGL_1R \\
\downarrow & & \downarrow \\
B & \xrightarrow{\ f\ } & BGL_1R,
\end{array}
$$

The Thom spectrum is then defined to be

$$\Sigma_+^\infty P \wedge^L_{\Sigma_+^\infty GL_1R} R.$$

Definition 2.22 associates to f the (∞-category) colimit of the map

$$\Pi_\infty B \xrightarrow{f} \Pi_\infty BGL_1R \simeq B\operatorname{Aut}(R^\circ) \simeq R\text{-line} \to R\text{-mod}. \tag{2.26}$$

The ∞-categorical setting makes it possible for the comparison of these two constructions to proceed much as in the discrete case, discussed in §1.2 (see (1.7)). Just as a set X is the colimit of the constant map

$$X \cong \operatorname{colim}(X \to (\text{sets})),$$

so if X is a space then it is weakly equivalent to the ∞-categorical colimit of the constant map

$$X \simeq \operatorname{colim}(\Pi_\infty X \to (\text{spaces})),$$

where (spaces) denotes the ∞-category of spaces. More generally, if $P \to B$ is principal G-bundle and (free G-spaces) is the ∞-category of free G-spaces, then

$$P \simeq \operatorname{colim}(\Pi_\infty B \to (\text{free } G\text{-spaces}) \simeq BG);$$

in our setting this becomes

$$P \simeq \operatorname{colim}(\Pi_\infty B \to (GL_1R\text{-spaces}) \simeq BGL_1R),$$

and so $\Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1R} R$ is the colimit of

$$\Pi_\infty B \to \Pi_\infty BGL_1R \simeq (GL_1R\text{-modules}) \xrightarrow{\Sigma_+^\infty(-)} (\Sigma_+^\infty GL_1R\text{-spaces}) \xrightarrow{(-)\wedge_{\Sigma_+^\infty GL_1R} R} R\text{-mod}. \tag{2.27}$$

From this point of view our job is to show that the two functors (2.26) and (2.27)

$$\Pi_\infty BGL_1R \simeq R\text{-line} \to R\text{-mod}$$

are equivalent, which amounts to showing that in each case the functor is equivalent to the standard inclusion of R-line in R-mod.

We develop an efficient proof along these lines in §8.5. However, much of §8 is devoted to a more general characterization of the Thom spectrum functor from the point of view of Morita theory. This viewpoint is implicit in the definition of the Thom spectrum in Definition 2.13 as the derived smash product with R regarded as an $\Sigma_+^\infty GL_1R$-R bimodule specified by the canonical action of $\Sigma_+^\infty GL_1R$ on R. Recalling that the target category of R-modules is stable, we can regard this Thom spectrum as essentially given by a functor from (right) $\Sigma_+^\infty GL_1R$-modules to R-modules.

Now, roughly speaking, Morita theory (more precisely, the Eilenberg-Watts theorem) implies that any continuous functor from (right) $\Sigma_+^\infty GL_1R$-modules to R-modules which preserves homotopy colimits and takes GL_1R to R can be realized as tensoring with an appropriate $(\Sigma_+^\infty GL_1R)$-R bimodule. In particular, this tells us that the Thom spectrum functor is characterized amongst such functors by the additional data of the action of GL_1R on R.

In §8 we develop these ideas in the setting of ∞-categories. Given a functor

$$F : \mathscr{T}_{/B\operatorname{Aut}(R^\circ)} \to R\text{-mod}$$

which takes $*/B\operatorname{Aut}(R^\circ)$ to R°, we can restrict along the Yoneda embedding $(8.3)^2$

$$B\operatorname{Aut}(R^\circ) \to \mathscr{T}_{/B\operatorname{Aut}(R^\circ)} \xrightarrow{F} R\text{-mod};$$

since it takes the object of $B\operatorname{Aut}(R^\circ)$ to R°, we may view this as a functor (map of simplicial sets)

$$k : B\operatorname{Aut}(R^\circ) \to B\operatorname{Aut}(R^\circ).$$

Conversely, given a map $k : B\operatorname{Aut}(R^\circ) \to B\operatorname{Aut}(R^\circ)$, we get a colimit-preserving functor

$$F : \mathscr{T}_{/B\operatorname{Aut}(R^\circ)} \to R\text{-mod}$$

whose value on $B \to B\operatorname{Aut}(R^\circ)$ is

$$F(B/B\operatorname{Aut}(R^\circ)) = \operatorname{colim}(B \to B\operatorname{Aut}(R^\circ) \xrightarrow{k} B\operatorname{Aut}(R^\circ) \hookrightarrow R\text{-mod}).$$

About this correspondence we prove the following.

Proposition 2.28 (Corollary 8.13)**.** *A functor F from the ∞-category $\mathscr{T}_{/B\operatorname{Aut}(R)}$ to the ∞-category of R-modules is equivalent to the Thom spectrum functor if and only if it preserves colimits and its restriction along the Yoneda embedding*

$$B\operatorname{Aut}(R^\circ) \to \mathscr{T}_{/B\operatorname{Aut}(R^\circ)} \xrightarrow{F} R\text{-mod}$$

is equivalent to the canonical inclusion

$$B\operatorname{Aut}(R^\circ) \xrightarrow{\simeq} R\text{-line} \to R\text{-mod}.$$

It follows easily (Corollary 8.18) that the Thom spectrum functors of Definitions 2.13 and 2.22 are equivalent. It also follows that, as in Proposition 2.24, the Thom spectrum of a group-like A_∞ map

$$\phi : G \to GL_1 S$$

is the homotopy quotient

$$\operatorname{colim}(BG \to R\text{-mod}) \simeq R_{hG}.$$

This observation is the basis for our comparison with the Thom spectrum of Lewis and May. In §8.6 we show that the Lewis-May Thom spectrum associated to the map

$$B\phi : BG \to BGL_1 S$$

is a model for the homotopy quotient S_{hG}, and it follows easily that we have the following.

Proposition 2.29 (Corollary 8.27)**.** *The Lewis-May Thom spectrum associated to a map*

$$f : B \to BGL_1 S$$

is equivalent to the Thom spectrum associated by Definition 2.22 to the map of ∞-categories

$$\Pi_\infty B \xrightarrow{\Pi_\infty f} \Pi_\infty BGL_1 S \simeq S\text{-line}.$$

---

[2]If X is a space, viewed as an ∞-groupoid, then the equivalence of ∞-categories $\mathscr{T}/X \simeq \operatorname{Fun}(X^{op}, \mathscr{T})$ allows us to regard the Yoneda embedding as a functor $X \to \mathscr{T}/X$ which, roughly, sends the point p of X to the "path fibration" $X_{/p} \to X$.

2.5. **Twisted generalized cohomology.** Our construction of Thom spectra begins with an A_∞ or E_∞ ring spectrum R, and attaches to a map

$$f : X \to BGL_1R$$

an R-module Thom spectrum Mf. As we have explained, BGL_1R can be thought of as the classifying space for bundles of free R-modules of rank 1. As such, it is the classifying space for "twists" of R-theory. Let F_R denote the R-module function spectrum. Given the map f, the f-*twisted R-homology* of X is by definition

$$R_k^f X \overset{\text{def}}{=} \pi_0 R\text{-mod}(\Sigma^k R, Mf) \cong \pi_k Mf,$$

while the f-*twisted R-cohomology* of X is

$$R_f^k X \overset{\text{def}}{=} \pi_0 R\text{-mod}(Mf, \Sigma^k R).$$

If f factors as

$$f : X \xrightarrow{g} BGL_1S \xrightarrow{i} BGL_1R, \tag{2.30}$$

then as in (2.19) we have

$$Mf \simeq (Mg) \wedge^L R,$$

and so

$$R_k^f(X) = \pi_k Mf \cong \pi_k Mg \wedge^L R = R_k Mg$$

$$R_f^k(X) = \pi_0 R\text{-mod}(Mf, \Sigma^k R) \cong \pi_0 S\text{-mod}(Mg, \Sigma^k R) \cong R^k Mg,$$

so the f-twisted homology and cohomology coincide with untwisted R-homology and cohomology of the usual Thom spectrum of the spherical fibration classified by g. Thus the constructions in this paper exhibit twisted generalized cohomology as the cohomology of a generalized Thom spectrum. In general the twists correspond to maps

$$X \to BGL_1R;$$

the ones which arise from Thom spectra of spherical fibrations are the ones which factor as in (2.30). We shall discuss the relationship to other approaches to generalized twisted cohomology in another paper in preparation.

3. Units after May-Quinn-Ray

Let A be an E_∞ ring spectrum: then there is a spectrum gl_1A such that

$$\Omega^\infty gl_1A \simeq GL_1A. \tag{3.1}$$

We recall the construction of gl_1A, which is due to [MQRT77]. Since A is an E_∞ spectrum, GL_1A is a group-like E_∞ space, and group-like E_∞ spaces model connective spectra. More precisely, we prove the following result.

Theorem 3.2. *The functors $\Sigma_+^\infty \Omega^\infty$ and gl_1 induce adjunctions*

$$\Sigma_+^\infty \Omega^\infty : \text{ho}((-1)\text{-}connected\ spectra) \rightleftarrows \text{ho}\,\mathscr{S}[E_\infty] : gl_1 \tag{3.3}$$

of categories enriched over the homotopy category of spaces.

In more detail, in §3.1–3.3 we recall that if C is an operad over the linear isometries operad, then there are Quillen model categories $\mathscr{S}[C]$ and $\mathscr{T}[C]$ of C-algebras in spectra and spaces, and that the adjunction

$$\Sigma_+^\infty : \mathscr{T} \rightleftarrows \mathscr{S} : \Omega^\infty$$

induces by restriction a continuous Quillen adjunction

$$\Sigma_+^\infty : \mathscr{T}_*[C_*] \cong \mathscr{T}[C] \rightleftarrows \mathscr{S}[C] : \Omega^\infty$$

We also recall that if C and D are E_∞ operads, then there is a zig-zag of continuous Quillen equivalences

$$\mathscr{S}[C] \simeq \mathscr{S}[D],$$

so we have a robust notion of the homotopy category of E_∞ ring spectra, which we denote $\mathrm{ho}\,\mathscr{S}[E_\infty]$.

If X is a C-space then $\pi_0 X$ is a monoid, and X is said to be *group-like* if $\pi_0 X$ is a group. We write $\mathscr{T}[C]^\times$ for the full-subcategory of group-like C-spaces. If X is a C-space, then in §3.4 we define $GL_1 X$ to be the pull-back in the diagram

$$
\begin{array}{ccc}
GL_1 X & \longrightarrow & X \\
\downarrow & & \downarrow \\
\pi_0(X)^\times & \longrightarrow & \pi_0 X.
\end{array}
$$

The functor $X \mapsto GL_1 X$ is the *right* adjoint of the inclusion

$$\mathscr{T}[C]^\times \to \mathscr{T}[C].$$

One of the main results of [May72, May74] is that, for suitable E_∞ operads C, $\mathscr{T}[C]^\times$ is a model for connective spectra. In §3.5, we express this result in the language of model categories. Let C be a unital operad (i.e. the zero space of the operad is a point), equipped with a map of monads on pointed spaces

$$f : C_* \to Q.$$

Then $\Omega^\infty : \mathscr{S} \to \mathscr{T}$ factors through $\mathscr{T}[C]$, and we show that it has a left adjoint

$$\Sigma^f : \mathscr{T}[C] \to \mathscr{S}.$$

If the basepoint $* \to C(1)$ is non-degenerate, and if for each n the space $C(n)$ has the homotopy type of of a Σ_n-CW complex, then the adjoint pair $(\Sigma^f, \Omega^\infty)$ induces an equivalence of enriched homotopy categories

$$\Sigma^f : \mathrm{ho}\,\mathscr{T}[C]^\times \xrightleftharpoons{\qquad} \mathrm{ho}(\text{connective spectra}) : \Omega^\infty .$$

In §3.6 we put all this together. For a suitable E_∞ operad C, we have a sequence of adjunctions (the left adjoints are listed on top, and equivalence of homotopy categories is indicated by $\approx$)

$$\Sigma_+^\infty \Omega^\infty : ((-1)\text{-connected spectra}) \underset{\Omega^f}{\overset{\Sigma^f, \approx}{\rightleftharpoons}} \mathscr{T}[C]^\times \underset{GL_1}{\rightleftharpoons} \mathscr{T}[C] \underset{\Omega^\infty}{\overset{\Sigma_+^\infty}{\rightleftharpoons}} \mathscr{S}[C] : gl_1 .$$

This is our model for the adjunction of Theorem 3.2.

The rest of this paper depends on this section only through the relationship (3.1) between gl_1 and GL_1 and Theorem 3.2. The reader will notice that our construction of gl_1 and the proof of Theorem 3.2 mostly amount to assembling results from the literature, particularly [May72, May74, MQRT77, LMSM86, EKMM96]. We wrote this section in the hope that it can serve as a useful guide to the literature. In the meantime May has prepared a review of the relevant multiplicative infinite loop space theory [May08], which also includes the results we need.

Remark 3.4. Theorem 3.2 can be formulated as an adjunction of ∞-categories

$$\Sigma_+^\infty \Omega^\infty : ((-1)\text{-connected spectra}) \xrightleftharpoons{\qquad} \mathscr{S}[E_\infty] : gl_1 .$$

3.1. E_∞ **spectra.** In this section we review the notion of a C-spectrum, where C is an operad (in spaces) over the linear isometries operad. We also recall the fact that the homotopy category of E_∞ spectra is well defined, in the sense that if C and D are two E_∞ operads over the linear isometries operad, then the categories of C-spectra and D-spectra are connected by a zig-zag of continuous Quillen equivalences.

If C is an operad, then for $k \geq 0$ we write $C(k)$ for the kth space of the operad. We also write C for the associated monad. Let $\mathscr{S} = \mathscr{S}_\mathcal{U}$ denote the category of spectra based on a universe $\mathcal{U}$, in the sense of [LMSM86]. Let $\mathcal{L}$ denote the linear isometries operad of $\mathcal{U}$, and let $C \to \mathcal{L}$ be an operad over $\mathcal{L}$. Then

$$CV = \bigvee_{k \geq 0} C(k) \ltimes_{\Sigma_k} V^{\wedge k}.$$

is the free C-algebra on V. We write $\mathscr{S}[C]$ for the category of C-algebras in $\mathscr{S}$, and we call its objects C-spectra.

In general $C(*) \cong \Sigma_+^\infty C(0)$ is the initial object of the category of C-spectra. We shall say that C is *unital* if $C(0) = *$, so that $C(0) \cong S$ is the sphere spectrum.

Lewis-May-Steinberger work with unital operads and the free C-spectrum with prescribed unit. If $S \to V$ is a spectrum under the sphere, then we write $C_* V$ for the free C spectrum on V with unit $\iota : S \to V \to C_* V$. This is the pushout in the category of C-spectra in the diagram

$$
\begin{array}{ccc}
CS & \longrightarrow & C(*) = S \\
{\scriptstyle C\iota}\big\downarrow & & \big\downarrow \\
CV & \longrightarrow & C_* V.
\end{array}
\tag{3.5}
$$

By construction, C_* participates in a monad on the category $\mathscr{S}_{S/}$ of spectra under the sphere spectrum.

As explained in [EKMM96, II, Remark 4.9],

$$\mathbb{S}(V) = S \vee V$$

defines a monad on $\mathscr{S}$, and we have an equivalence of categories

$$\mathscr{S}_{S/} \cong \mathscr{S}[\mathbb{S}].$$

It follows that there is a natural isomorphism

$$C(V) \cong C_* \mathbb{S}(V). \tag{3.6}$$

and ([EKMM96, II, Lemma 6.1]) an equivalence of categories

$$\mathscr{S}[C] \cong \mathscr{S}_{S/}[C_*].$$

We recall the following, which can be proved easily using the argument of [EKMM96, MMSS01], in particular an adaptation of the "Cofibration Hypothesis" of §VII of [EKMM96].

Proposition 3.7. *The category of C-spectra has the structure of a cofibrantly generated topological closed model category, in which the forgetful functor to $\mathscr{S}$ creates fibrations and weak equivalences. If $\{A \to B\}$ is a set of generating (trivial) cofibrations of $\mathscr{S}$, then $\{CA \to CB\}$ is a set of generating (trivial) cofibrations of $\mathscr{S}[C]$.*

In particular, the category of C-spectra is cocomplete (this is explained on pp. 46—49 of [EKMM96]), a fact we use in the following construction. Let $f : C \to D$ be a map of operads over $\mathcal{L}$, so there is a forgetful functor

$$f^* : \mathscr{S}[D] \to \mathscr{S}[C].$$

We construct the left adjoint $f_!$ of f^* as a certain coequalizer in C-algebras; see [EKMM96, §II.6] for further discussion of this construction.

Denote by $m : DD \to D$ the multiplication for D, and let A be a C-algebra with structure map $\mu : CA \to A$. Define $f_! A$ to be the coequalizer in the diagram of D-algebras

$$
\begin{array}{ccc}
DCA & \underset{}{\overset{D\mu}{\rightrightarrows}} & DA \longrightarrow f_! A. \\
{\scriptstyle Df}\searrow & & \nearrow {\scriptstyle m} \\
& DDA &
\end{array}
\tag{3.8}
$$

In fact, it's enough to construct $f_! A$ as the coequalizer in spectra. Then D, applied to the unit $A \to CA$, makes the diagram a reflexive coequalizer of spectra, and so $f_! A$ has the structure of a D-algebra, and as such is the D-algebra coequalizer [EKMM96, Lemma 6.6].

Proposition 3.9. *The functor $f_!$ is a continuous left adjoint to f^*; moreover, for any spectrum V, the natural map*

$$f_! CV \to DV \tag{3.10}$$

is an isomorphism.

Proof. It is straightforward if time-consuming to check that $f_!$ is continuous and the left adjoint of f^*; in fact, the statement and argument work for any map of monads $f : C \to D$ on any category $\mathcal{C}$, provided that the coequalizer (3.8) exists (naturally in A).

For the second part, note that if T is any D-algebra, then we have

$$\mathscr{S}[D](f_! CV, T) \cong \mathscr{S}[C](CV, f^* T) \cong \mathscr{S}(V, T) \cong \mathscr{S}[D](DV, T).$$

In the last two terms in this sequence of isomorphisms we have omitted the notation for the forgetful functors, and we have used the fact that the diagram

$$\begin{array}{ccc} \mathscr{S}[D] & \xrightarrow{\ f^* \ } & \mathscr{S}[C] \\ & \searrow & \downarrow \\ & & \mathscr{S} \end{array} \tag{3.11}$$

commutes. □

Remark 3.12. The reader may prefer to write $C \otimes V$ for the free C-algebra CV, and then $D \otimes_C A$ for $f_! A$. With this notation, the coequalizer diagram defining $D \otimes_C A$ takes the form

$$D \otimes C \otimes A \rightrightarrows D \otimes A \longrightarrow D \otimes_C A,$$

and the isomorphism of Proposition 3.9 becomes

$$D \otimes_C C \otimes V \cong D \otimes V.$$

About this adjoint pair there is the following well-known result.

Proposition 3.13. *Let $f : C \to D$ be a map of operads over $\mathcal{L}$. The pair $(f_!, f^*)$ is a continuous Quillen pair.*

Proof. Since the diagram (3.11) commutes and the forgetful functor to spectra creates fibrations and weak equivalences in C-algebras, f^* preserves fibrations and weak equivalences. By Proposition 3.7, C carries a generating set of cofibrations of $\mathscr{S}$ to a generating set of cofibrations of $\mathscr{S}[C]$. The isomorphism $f_! C \cong D$ (3.10) shows that $f_!$ carries this set to a generating set of cofibrations of $\mathscr{S}[D]$. □

It is folklore that various E_∞ operads over $\mathcal{L}$ give rise to the same homotopy theory. Over the years, various arguments have been given to show this, starting with May's use of the bar construction to model $f_!$ (see [EKMM96, §II.4.3] for the most recent entry in this line). We present a model-theoretic formulation of this result in the remainder of the subsection.

Proposition 3.14. *If f is a map of E_∞ operads, then $(f_!, f^*)$ is a Quillen equivalence. More generally, if each map*

$$f : C(n) \to D(n)$$

is a weak equivalence of spaces, then $(f_!, f^*)$ is a Quillen equivalence.

Before giving the proof, we make a few remarks. Assume f is a weak equivalence of operads. Since the pullback $f^* : \mathscr{S}[D] \to \mathscr{S}[C]$ preserves fibrations and weak equivalences, to show that $(f_!, f^*)$ is a Quillen equivalence it suffices to show that for a cofibrant C-algebra X the unit of the adjunction $X \to f^* f_! X$ is a weak equivalence.

If $X = CZ$ is a *free* C-algebra, then $f_! X = f_! CZ \cong DZ$ by (3.10), and so the map in question is the natural map

$$CZ \to DZ.$$

It follows from Propositions X.4.7, X.4.9, and A.7.4 of [EKMM96] that if the operad spaces $C(n)$ and $D(n)$ are CW-complexes, and if Z is a wedge of spheres or disks, then $CZ \to DZ$ is a homotopy equivalence. In fact, this argument applies to the wider class of *tame* spectra, whose definition we now recall.

Definition 3.15 ([EKMM96], Definition I.2.4)**.** A prespectrum D is Σ-*cofibrant* if each of the structure maps $\Sigma^W D(V) \to D(V \oplus W)$ is a (Hurewicz) cofibration. A spectrum Z is Σ-*cofibrant* if it is isomorphic to one of the form LD, where D is a Σ-cofibrant prespectrum. A spectrum Z is *tame* if it is homotopy equivalent to a Σ-cofibrant spectrum. In particular, a spectrum Z of the homotopy type of a CW-spectrum is tame.

For a general cofibrant X, the argument proceeds by reducing to the free case $X = CZ$. In this paper, we present an inductive argument due to Mandell [Mand97]. A different induction of this sort appeared in [Mand03] in the algebraic setting; that argument can be adapted to the topological context with minimal modifications.

Our induction will involves the geometric realization of simplicial spectra. As usual, we would like to ensure that a map of simplicial spectra

$$f_\bullet : K_\bullet \to K'_\bullet$$

in which each $f_n : K_n \to K'_n$ is a weak equivalence yields a weak equivalence upon geometric realization. The required condition is that the spectra K_n and K'_n are tame: Theorem X.2.4 of [EKMM96] says that the realization of weak equivalences of tame spectra is a weak equivalence if $K_\bullet$ and $K'_\bullet$ are "proper" [EKMM96, §X.2.1]. Recall that a simplicial spectrum $K_\bullet$ is proper if the natural map of coends

$$\int^{D_{q-1}} K_p \wedge D(q,p)_+ \to \int^{D_q} K_p \wedge D(q,p)_+ \cong K_q$$

is a Hurewicz cofibration, where D is the subcategory of Δ consisting of the monotonic surjections (i.e. the degeneracies), and D_q is the full subcategory of D on the objects $0 \le i \le q$. This is a precise formulation of the intuitive notion that the inclusion of the union of the degenerate spectra $s_j K_{q-1}$ in K_q should be a Hurewicz cofibration.

Thus, to ensure that the spectra that arise in our argument are tame and the simplicial objects proper, we make the following simplifying assumptions on our operads.

  (1) We assume that the spaces $C(n)$ and $D(n)$ have the homotopy type of Σ_n-CW-complexes.
  (2) We assume that $C(1)$ and $D(1)$ are equipped with nondegenerate basepoints.

We believe these assumptions are reasonable, insofar as they are satisfied by many natural examples; for instance, the linear isometries operad and the little n-cubes operad both satisfy the hypotheses above (see [EKMM96, XI.1.4, XI.1.7] and [May72, 4.8] respectively). More generally, if $\mathcal{O}$ is an arbitrary operad over the linear isometries operad, then taking the geometric realization of the singular complex of the spaces $\mathcal{O}$ produces an operad $|S(\mathcal{O})|$ with the properties we require.

Goerss and Hopkins have proved two versions of Proposition 3.14 using resolution model structures to resolve an arbitrary cofibrant C-space by a simplicial C-space with free k-simplices for every k. A first version [GH] proves the Proposition for LMS spectra, avoiding our simplifying assumptions on the operads via a detailed study of "flatness" for spectra (as an alternative to the theory of "tameness"). A more modern treatment [GH03] works with operads of simplicial sets and symmetric spectra in topological spaces. In that case, as they explain, a key point is that if X is a cofibrant spectrum, then $X^{(n)}$ is a *free* Σ_n-spectrum (see Lemma 15.5 of [MMSS01]). This observation helps explain why the general form of the Proposition is reasonable, even though the analogous statement for spaces is much too strong.

Proof. A cofibrant C-spectrum is a retract of a cell C-spectrum, and so we can assume without loss of generality that X is a cell C-spectrum. That is, $X = \operatorname{colim}_n X_n$, where $X_0 = C(*)$ and X_{n+1} is obtained from X_n via a pushout (in C-algebras) of the form

$$
\begin{array}{ccc}
CA & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
CB & \longrightarrow & X_{n+1}
\end{array}
$$

where $A \to B$ is a wedge of generating cofibrations of spectra. By the proof of Proposition 3.7 (specifically, the Cofibration Hypothesis), the map $X_n \to X_{n+1}$ is a Hurewicz cofibration of spectra. The hypotheses on C and the fact that A and B are CW-spectra imply that that CA and CB have the homotopy type of CW-spectra, and thus inductively so does X_n. Therefore, since $f_!$ is a left adjoint, it suffices to show that $X_n \to f^* f_! X_n$ is a weak equivalence for each X_n — under these circumstances, a sequential colimit of weak equivalences is a weak equivalence.

We proceed by induction on the number of stages required to build the C-spectrum. The base case follows from the remarks preceding the proof. For the induction hypothesis, assume that $f_!$ is a weak equivalence for all cell C-algebras that can be built in n or fewer stages. The spectrum X_{n+1} is a pushout $CB \coprod_{CA} X_n$ in C-algebras, and this pushout is homeomorphic to a bar construction $B(CB, CA, X_n)$. Since $f_!$ is a continuous left adjoint, it commutes with geometric realization and coproducts in C-algebras, and so $f_!(B(CB, CA, X_n))$ is homeomorphic to $B(DB, DA, f_! X_n)$.

The bar constructions we are working with are proper simplicial spectra by the hypothesis that $C(1)$ and $D(1)$ have nondegenerate basepoints, and thus it suffices to show that at each level in the bar construction

$$B_q(CB, CA, X_n) \to B_q(DB, DA, f_! X_n)$$

we have a weak equivalence of tame spectra. This follows from the inductive hypothesis: we have already shown that the spectra are tame, and $CB \coprod^q CA \coprod X_n$ can be built in n stages, since X_n can be built in n stages and the free algebras can be built and added in a single stage. $\square$

The idea of the following corollary goes all the way back to [May72].

Corollary 3.16. *If C and D are any two E_∞ operads over the linear isometries operad, then the categories of C-algebras and D-algebras are connected by a zig-zag of continuous Quillen equivalences.*

Proof. Apply Proposition 3.13 to the maps of E_∞ operads over $\mathcal{L}$

$$C \leftarrow C \times D \to D.$$

$\square$

Backed by this result, we adopt the following convention.

Definition 3.17. We write $\operatorname{ho} \mathscr{S}[E_\infty]$ for the homotopy category of E_∞ ring spectra. By this we mean the homotopy category $\operatorname{ho} \mathscr{S}[C]$ for any E_∞ operad C over the linear isometries operad.

3.2. E_∞ **spaces.** We adopt notation for operad actions on spaces analogous to our notation for spectra in §3.1. Let C be an operad in topological spaces. The free C-algebra on a space X is

$$CX = \coprod_{k \geq 0} C(k) \times_{\Sigma_k} X^k. \tag{3.18}$$

We set $C(\emptyset) = C(0)$. The category of C-algebras in spaces, or C-spaces, will be denoted $\mathscr{T}[C]$.

Note that the sequence of spaces given by

$$P(0) = * = P(1)$$
$$P(k) = \emptyset \text{ for } k > 1$$

has a unique structure of operad, whose associated monad is

$$PX = X_+,$$

so

$$\mathscr{T}[P] \cong \mathscr{T}_*.$$

If C is a unital operad and if Y is a pointed space, let C_*Y be the pushout in the category of C-algebras

$$
\begin{array}{ccc}
C* & \longrightarrow & C(\emptyset) = * \\
\downarrow & & \downarrow \\
CY & \longrightarrow & C_*Y.
\end{array}
\tag{3.19}
$$

Then C_* participates in a monad on the category of pointed spaces. Indeed C_* is isomorphic to the monad C_{May} introduced in [May72], since for a test C-space T,

$$\mathscr{T}[C](C_*Y, T) \cong \mathscr{T}_*(Y, T) \cong \mathscr{T}[C](C_{May}Y, T).$$

There is a natural isomorphism

$$CX \cong C_*(X_+),$$

and an equivalence of categories

$$\mathscr{T}[C] \cong \mathscr{T}_*[C_*].
\tag{3.20}$$

Part of this equivalence is the observation that, if X is a C-algebra, then it is a C_* algebra via

$$C_*X \to C_*(X_+) \cong CX \to X.$$

We have the following analogue of Proposition 3.7.

Proposition 3.21.

  (1) *The category $\mathscr{T}[C]$ has the structure of a cofibrantly generated topological closed model category, in which the forgetful functor to $\mathscr{T}$ creates fibrations and weak equivalences. If $\{A \to B\}$ is a set of generating (trivial) cofibrations of $\mathscr{T}$, then $\{CA \to CB\}$ is a set of generating (trivial) cofibrations of $\mathscr{T}[C]$.*
  (2) *The analogous statements hold for C_* and $\mathscr{T}_*[C_*]$.*
  (3) *Taking $C = P$, the resulting model category structure on the category $\mathscr{T}[P] \cong \mathscr{T}_*$ is the usual one.*
  (4) *The equivalence $\mathscr{T}[C] \cong \mathscr{T}_*[C_*]$ (3.20) carries the model structure arising from part (1) to the model structure arising from part (2).*

Proof. The statements about the model structure on $\mathscr{T}[C]$ or on $\mathscr{T}_*[C_*]$ can be proved for example by adapting the argument in [EKMM96, MMSS01]. The third part is standard, and together the first three parts imply the last. $\square$

We conclude this subsection with two results which will be useful in §3.5. For the first, note that a point of $C(0)$ determines a map of operads

$$P \to C,$$

and so we have a forgetful functor

$$\mathscr{T}[C] \to \mathscr{T}[P] \cong \mathscr{T}_*.$$

We say that a point of Y is *non-degenerate* if $(Y, *)$ is an NDR pair, i.e. that $* \to Y$ is a Hurewicz cofibration.

Proposition 3.22. *Suppose that C is a unital operad in topological spaces (or more generally, an operad in which the base point of $C(0)$ is nondegenerate). If X is a cofibrant object $\mathscr{T}_*[C_*]$, then its base point is nondegenerate.*

Note that Rezk [Rez] and Berger and Moerdijk [BM03] have proved a similar result, for algebras in a general model category over an *cofibrant* operad. In our case, we need only assume that the zero space $C(0)$ of our operad has a non-degenerate base point.

Proof. In the model structure described in Proposition 3.21, a cofibrant object is a retract of a cell object, and so we can assume without loss of generality that X is a cell C-space. That is,

$$X = \operatorname*{colim}_{n} X_n \tag{3.23}$$

where $X_0 = C(\emptyset)$ and X_{n+1} is obtained from X_n as a pushout in C-spaces

$$
\begin{array}{ccc}
CA & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
CB & \longrightarrow & X_{n+1},
\end{array}
\tag{3.24}
$$

where $A \to B$ is a disjoint union of generating cofibrations of $\mathscr{T}$.

Our argument relies on a form of the Cofibration Hypothesis of §VII of [EKMM96]. The key points are the following.

(1) By assumption $X_0 = C(\emptyset) = C(0)$ is non-degenerately based.
(2) The space underlying the C-algebra colimit X in (3.23) is just the space-level colimit.
(3) In the pushout above,

$$X_n \to X_{n+1}$$

is a based map and an unbased Hurewicz cofibration.

The second point is easily checked (and is the space-level analog of Lemma 3.10 of [EKMM96]). For the last part, the argument in Proposition 3.9 of §VII of [EKMM96] (see also Lemma 15.9 of [MMSS01]) shows that the pushout (3.24) is isomorphic to a two-sided bar construction $B(CB, CA, X_n)$: this is the geometric realization of a simplicial space where the k-simplices are given as

$$CB \coprod_C (CA)^{\coprod k} \coprod_C X_n,$$

and the simplicial structure maps are induced by the folding map and the maps $CA \to CB$ and $CA \to X_n$. Note that by $\coprod_C$ we mean the coproduct in the category of C-spaces. Recall that coproducts (and more generally all colimits) in C-spaces admit a description as certain coequalizers in $\mathscr{T}$. Specifically, for C-spaces X and Y the coproduct $X \coprod_C Y$ can be described as the coequalizer in $\mathscr{T}$

$$C(CX \coprod CY) \rightrightarrows C(X \coprod Y) \longrightarrow X \coprod_C Y,$$

where the unmarked coproducts are taken in $\mathscr{T}$ and the maps are induced from the action maps and the monadic structure map, respectively. Following an argument along the lines of [EKMM96, §VII.6] we obtain the following lemma.

Lemma 3.25. *Let C be an operad in spaces. Let A be a C-space and B a space. The map $A \to A \coprod_C CB$ is an inclusion of a component in a disjoint union.*

This implies that the simplicial degeneracy maps in the bar construction are unbased Hurewicz cofibrations and hence that the simplicial space is proper, that is, Reedy cofibrant in the Hurewicz/Strøm model structure. Thus the inclusion of the zero simplices $CB \coprod_C X_n$ in the realization is an unbased Hurewicz cofibration, and hence the map $X_n \to X_{n+1}$ is itself a unbased Hurewicz cofibration. As a map of C-algebras, it's also a based map. □

The second result we need is the following.

Proposition 3.26. *Let C be an operad and suppose that each $C(n)$ has the homotopy type of a Σ_n-CW complex. Let X be a C-space with the homotopy type of a cofibrant C-space. Then CX has the homotopy type of a cofibrant C-space and the underlying space of X has the homotopy type of a CW-complex.*

Proof. The first statement is an easy consequence of the fact that C preserves homotopies and cofibrant objects. To see the second, observe that the forgetful functor preserves homotopies, so it suffices to suppose that X is a cofibrant C-space. Under our hypotheses on C, if A has the homotopy type of a CW-complex then so does the underlying space of CA (see for instance page 372 of [LMSM86] for a proof). The result now follows from an inductive argument along the lines of the preceding proposition. $\square$

3.3. **E_∞ spaces and E_∞ spectra.** Suppose that $C \to \mathcal{L}$ is an operad over $\mathcal{L}$. In this section we recall the proof of the following result:

Proposition 3.27 ([MQRT77], [LMSM86] p. 366)**.** *The continuous Quillen pair*

$$\Sigma_+^\infty : \mathscr{T} \leftrightarrows \mathscr{S} : \Omega^\infty \tag{3.28}$$

induces by restriction a continuous Quillen adjunction

$$\Sigma_+^\infty : \mathscr{T}_*[C_*] \cong \mathscr{T}[C] \leftrightarrows \mathscr{S}[C] : \Omega^\infty \tag{3.29}$$

between topological model categories.

The first thing to observe is that C and Σ_+^∞ satisfy a strong compatibility condition.

Lemma 3.30. *There is a natural isomorphism*

$$C\Sigma_+^\infty X \cong \Sigma_+^\infty CX. \tag{3.31}$$

Proof. It follows from §VI, Proposition 1.5 of [LMSM86] that, if X is a space, then

$$C(k) \ltimes_{\Sigma_k} (\Sigma_+^\infty X)^{\wedge k} \cong \Sigma_+^\infty (C(k) \times_{\Sigma_k} X^k),$$

and so

$$C\Sigma_+^\infty X = \bigvee_{k \geq 0} C(k) \ltimes_{\Sigma_k} (\Sigma_+^\infty X)^{\wedge k} \cong \bigvee_{k \geq 0} \Sigma_+^\infty (C(k) \times X^k) \cong \Sigma_+^\infty \left(\coprod_{k \geq 0} C(k) \times X^k \right) = \Sigma_+^\infty CX.$$

$\square$

Next we have the following, from [LMSM86, p. 366].

Lemma 3.32. *The adjoint pair*

$$\Sigma_+^\infty : \mathscr{T} \leftrightarrows \mathscr{S} : \Omega^\infty \tag{3.33}$$

induces an adjunction

$$\Sigma_+^\infty : \mathscr{T}[C] \leftrightarrows \mathscr{S}[C] : \Omega^\infty \tag{3.34}$$

and so also

$$\Sigma_+^\infty : \mathscr{T}_*[C_*] \cong \mathscr{T}[C] \leftrightarrows \mathscr{S}[C] : \Omega^\infty$$

Proof. We show that the adjunction (3.33) restricts to the adjunction (3.34). If X is a C-space with structure map $\mu : CX \to X$, then, using the isomorphism (3.31), $\Sigma_+^\infty X$ is a C-algebra via

$$C\Sigma_+^\infty X \cong \Sigma_+^\infty CX \xrightarrow{\Sigma_+^\infty \mu} \Sigma_+^\infty X.$$

If A is a C-spectrum, then $\Omega^\infty A$ is a C-space via

$$C\Omega^\infty A \to \Omega^\infty CA \to \Omega^\infty A.$$

The second map is just Ω^∞ applied to the C-structure on A; the first map is the adjoint of the map

$$\Sigma_+^\infty C\Omega^\infty A \cong C\Sigma_+^\infty \Omega^\infty A \to CA$$

obtained using the counit of the adjunction. $\square$

This adjunction allows us to prove the pointed analogue of Lemma 3.30.

Lemma 3.35 ([LMSM86], §VII, Prop. 3.5)**.** *If C is a unital operad over $\mathcal{L}$, then there is a natural isomorphism*

$$\Sigma_+^\infty C_* Y \cong C_* \Sigma_+^\infty Y \cong C \Sigma^\infty Y. \tag{3.36}$$

Proof. Let Y be a pointed space. By Lemma 3.32 and the isomorphism (3.31), applying the left adjoint Σ_+^∞ to the pushout diagram (3.19) defining $C_* Y$ identifies $\Sigma_+^\infty C_* Y$ with the pushout of the diagram (3.5) defining $C_* \Sigma_+^\infty Y$. The second isomorphism is just the isomorphism (3.6) together with the isomorphism (for pointed spaces) Y

$$\Sigma_+^\infty Y \cong \Sigma^\infty (S \vee Y).$$

$\square$

Proof of Proposition 3.27. It remains to show that the adjoint pair $(\Sigma_+^\infty, \Omega^\infty)$ induces a Quillen adjunction. For this it suffices to show that the right adjoint Ω^∞ preserves fibrations and weak equivalences (see, for example, [Hov99, Lemma 1.3.4]). Now recall that the forgetful functor $\mathscr{S}[C] \to \mathscr{S}$ creates fibrations and weak equivalences, and similarly for $\mathscr{T}$ [EKMM96, MMSS01]. It follows that the functor

$$\Omega^\infty : \mathscr{S}[C] \to \mathscr{T}[C]$$

preserves fibrations and weak equivalences, since

$$\Omega^\infty : \mathscr{S} \to \mathscr{T}$$

does. $\square$

Remark 3.37. Note that if A is an E_∞ ring spectrum, then $\Omega^\infty A$ is an E_∞ space in two ways: one is described above, and arises from the multiplication on A. The other arises from the additive structure of A, i.e. the fact that $\Omega^\infty A$ is an infinite loop space. Together these two E_∞ structures give an E_∞ ring space in the sense of [MQRT77] (see also [May08]).

3.4. **E_∞ spaces and group-like E_∞ spaces.** Suppose that C is a unital operad (pointed would be enough), and let X be a C-algebra in spaces. The structure maps

$$* \to C(0) \to X$$
$$C(2) \times X \times X \to X$$

correspond to a family of H-space structures on X and give to $\pi_0 X$ the structure of a monoid.

Definition 3.38. X is said to be *group-like* if $\pi_0 X$ is a group. We write $\mathscr{T}[C]^\times$ for the full subcategory of $\mathscr{T}[C]$ consisting of group-like C-spaces.

Note that if $f : X \to Y$ is a weak equivalence of C-spaces, then X is group-like if and only Y is.

Definition 3.39. We write $\text{ho}\,\mathscr{T}[C]^\times$ for the image of $\mathscr{T}[C]^\times$ in $\text{ho}\,\mathscr{T}[C]$. It is the full subcategory of homotopy types represented by group-like spaces.

If X is a C-space, let $GL_1 X$ be the (homotopy) pull-back in the diagram

$$\begin{array}{ccc} GL_1 X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \pi_0(X)^\times & \longrightarrow & \pi_0 X. \end{array} \tag{3.40}$$

Then $GL_1 X$ is a group-like C-space.

Proposition 3.41. *The functor GL_1 is the right adjoint of the inclusion*

$$\mathscr{T}[C]^\times \to \mathscr{T}[C]$$

Proof. If X is a group-like C-space, and Y is a C-space, then

$$\mathcal{T}[C](X,Y) \cong \mathcal{T}[C]^\times(X, GL_1 Y);$$

just as, if G is a group and M is a monoid, then

$$(\text{monoids})(G, M) = (\text{groups})(G, GL_1 M).$$

$\square$

3.5. **Group-like E_∞ spaces and connective spectra.** A guiding result of infinite loop space theory is that group-like E_∞ spaces provide a model for connective spectra. We take a few pages to show how the primary sources (in particular [BV73, May72, May74]) may be used to prove a formulation of this result in the language of model categories.

To begin, suppose that C is a *unital* E_∞ operad, and f is a map of monads (on pointed spaces)

$$f : C_* \to Q \overset{\text{def}}{=} \Omega^\infty \Sigma^\infty.$$

For example, we can take C to be a unital E_∞ operad over the infinite little cubes operad, but it is interesting to note that any map of monads will do. If V is a spectrum, then $\Omega^\infty V$ is a group-like C-algebra, via the map

$$C_* \Omega^\infty V \overset{f}{\to} \Omega^\infty \Sigma^\infty \Omega^\infty V \to \Omega^\infty V.$$

Thus we have a factorization

$$\mathcal{S} \xrightarrow{\Omega^f} \mathcal{T}[C]^\times \qquad (3.42)$$

$$\Omega^\infty \searrow \qquad \downarrow$$

$$\mathcal{T}_*$$

We next show that the functor Ω^f has a left adjoint Σ^f. By regarding a C-space X as a pointed space via $* \to C(0) \to X$, we may form the spectrum $\Sigma^\infty X$. Let $\Sigma^f X$ be the coequalizer in the diagram of spectra

$$\Sigma^\infty C_* X \underset{\Sigma^\infty f}{\overset{\Sigma^\infty \mu}{\rightrightarrows}} \Sigma^\infty X \longrightarrow \Sigma^f X.$$

$$\Sigma^\infty \Omega^\infty \Sigma^\infty X$$

Then we have the following.

Lemma 3.43. *The pair*

$$\Sigma^f : \mathcal{T}[C] \leftrightarrows \mathcal{S} : \Omega^f \qquad (3.44)$$

are a Quillen pair. Moreover, the natural transformation

$$\Sigma^f C_* \to \Sigma^\infty$$

is an isomorphism.

Proof. As mentioned in the proof of Proposition 3.9, it is essentially a formal consequence of the construction that Σ^f is the left adjoint of Ω^f. Given the adjunction, we find that $\Sigma^f C_* \cong \Sigma^\infty$, since, for any pointed space X and any spectrum V, we have

$$\mathcal{S}(\Sigma^f C_* X, V) \cong \mathcal{T}[C](C_* X, \Omega^f V)$$
$$\cong \mathcal{T}_*(X, \Omega^\infty V)$$
$$\cong \mathcal{S}(\Sigma^\infty X, V).$$

To show that we have a Quillen pair, it suffices ([Hov99, Lemma 1.3.4]) to show that Ω^f preserves weak equivalences and fibrations. This follows from the commutativity of the diagram (3.42), the fact that Ω^∞ preserves weak equivalences and fibrations, and the fact that the forgetful functor

$$\mathscr{T}[C] \to \mathscr{T}$$

creates fibrations and weak equivalences.   □

Lemma 3.43 implies that the pair (Σ^f, Ω^f) induce a continuous Quillen adjunction

$$\Sigma^f : \mathscr{T}[C] \leftrightarrows \mathscr{S} : \Omega^f.$$

It is easy to see that this cannot be a Quillen equivalence. Instead, one expects that it induces an equivalence between the homotopy categories of *group-like C-spaces* and *connective* spectra. In [MMSS01], this situation is called a "connective Quillen equivalence." The rest of this subsection is devoted to the proof of the following result along these lines:

Theorem 3.45. *Suppose that C is a unital operad, equipped with a map of monads*

$$f : C \to \Omega^\infty \Sigma^\infty.$$

Suppose moreover that

   (1) *the base point $* \to C(1)$ is non-degenerate, and*
   (2) *for each n, the n-space $C(n)$ has the homotopy type of a Σ_n-CW-complex.*

Then the adjunction (Σ^f, Ω^f) induces an equivalence of categories

$$\Sigma^f : \mathrm{ho}\,\mathscr{T}[C]^\times \rightleftarrows \mathrm{ho}(\textit{connective spectra}) : \Omega^f$$

enriched over $\mathrm{ho}\,\mathscr{T}$.

Remark 3.46. As observed in [May72], adding a whisker to a degenerate basepoint produces a new operad C' from C. Also if C is a unital E_∞ operad equipped with a map of monads $f : C \to \Omega^\infty \Sigma^\infty$, then taking the geometric realization of the singular complex of the spaces $C(n)$ produces an operad $|S(C)|$ with the properties we require.

The following Lemma, easily checked, is implicit in [MMSS01]. Let

$$F : \mathcal{M} \leftrightarrows \mathcal{M}' : G$$

be a Quillen adjunction between topological closed model categories. Let $\mathcal{C} \subseteq \mathcal{M}$ and $\mathcal{C}' \subseteq \mathcal{M}'$ be full subcategories, stable under weak equivalence, so we have sensible subcategories $\mathrm{ho}\,\mathcal{C} \subseteq \mathrm{ho}\,\mathcal{M}$ and $\mathrm{ho}\,\mathcal{C}' \subseteq \mathrm{ho}\,\mathcal{M}'$. Suppose that F takes $\mathcal{C}$ to $\mathcal{C}'$, and G takes $\mathcal{C}'$ to $\mathcal{C}$.

Lemma 3.47. *If, for every cofibrant $X \in \mathcal{C}$ and every fibrant $Y \in \mathcal{C}'$, a map*

$$\phi : FX \to Y$$

is a weak equivalence if and only if its adjoint

$$\psi : X \to GY$$

is, then F and G induce equivalences

$$F : \mathrm{ho}\,\mathcal{C} \leftrightarrows \mathrm{ho}\,\mathcal{C}' : G$$

of categories enriched over $\mathrm{ho}\,\mathscr{T}$.

The key result in our setting is the following classical proposition; we recall the argument from [May72, May74].

Proposition 3.48. *Let C be a unital E_∞ operad, equipped with a map of monads*

$$f : C \to \Omega^\infty \Sigma^\infty.$$

Suppose that the basepoint $* \to C(1)$ is non-degenerate, and that each $C(n)$ has the homotopy type of a Σ_n-CW-complex. If X is a cofibrant C-space, then the unit of the adjunction

$$X \to \Omega^f \Sigma^f X$$

is group completion, and so a weak equivalence if X is group-like.

The proof of the proposition follows from analysis of the following commutative diagram of simplicial C-spaces:

$$
\begin{array}{ccc}
B_\bullet(C_*, C_*, X) & \longrightarrow & \Omega^f \Sigma^f B_\bullet(C_*, C_*, X) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Omega^f \Sigma^f X.
\end{array}
\tag{3.49}
$$

Specifically, we will show that under the hypotheses, on passage to realization the vertical maps are weak equivalences and the top horizontal map is group completion.

We begin by studying the left-hand vertical map; the usual simplicial contraction argument shows the underlying map of spaces is a homotopy equivalence, and so on passage to realizations we have a weak equivalence of C-spaces.

Lemma 3.50. *For any operad C and any C-space X, the left vertical arrow is a map of simplicial C-spaces and a homotopy equivalence of simplicial spaces, and so induces a weak equivalence of C-spaces*

$$B(C_*, C_*, X) \to X$$

upon geometric realization.

The right vertical map is more difficult to analyze, because we do not know that Σ^f preserves homotopy equivalences of spaces. May [May72] shows that, for suitable simplicial pointed spaces $Y_\bullet$, the natural map

$$|\Omega Y_\bullet| \to \Omega |Y_\bullet|
\tag{3.51}$$

is a weak equivalence, and he explains in [May72, May08] how this weak equivalence gives rise to a weak equivalence of C-spaces

$$|\Omega^f \Sigma^f B_\bullet(C_*, C_*, X)| \to \Omega^f |\Sigma^f B_\bullet(C_*, C_*, X)| \cong \Omega^f \Sigma^f B(C_*, C_*, X)$$

by passage to colimits. We note that in [May08], May describes proving that (3.51) is a weak equivalence as the hardest thing in [May72]. Therefore, to show that the map

$$|\Omega^f \Sigma^f B_\bullet(C_*, C_*, X)| \to \Omega^f \Sigma^f X$$

is a weak equivalence, it suffices to show that for cofibrant X, the map $\Sigma^f B(C_*, C_*, X) \to X$ is a weak equivalence. As it is straightforward to check from the definition that Σ^f does preserve weak equivalences between C-spaces with the homotopy type of cofibrant C-spaces, the desired result will follow once we show that $B(C_*, C_*, X)$ has the homotopy type of a cofibrant C-space if X is cofibrant.

Lemma 3.52. *Suppose that C is a unital operad, such that the base point $* \to C(1)$ is non-degenerate and each $C(n)$ has the homotopy type of a Σ_n-CW-complex. Let X be a cofibrant C-space. Then $B(C_*, C_*, X)$ has the homotopy type of a cofibrant C-space.*

Proof. With our hypotheses, it follows from Proposition 3.26 that the spaces $C_*^n X$ have the homotopy type of cofibrant C-spaces. By Proposition 3.22, the simplicial space $B_\bullet(C_*, C_*, X)$ is proper. Finally, we apply an argument analogous to that of Theorem X.2.7 of [EKMM96] to show that if $Y_\bullet$ is a proper C-space in

which each level has the homotopy type of a cofibrant C-space, then $|Y_\bullet|$ has the the homotopy type of a cofibrant C-space. □

Finally, we consider the top horizontal map in (3.49). We have *isomorphisms* of simplicial C-spaces

$$\Omega^f \Sigma^f B_\bullet(C_*, C_*, X) \cong B_\bullet(\Omega^f \Sigma^f C_*, C_*, X) \cong B_\bullet(\Omega^f \Sigma^\infty, C_*, X) \cong B_\bullet(Q, C_*, X)$$

(we used the isomorphism $\Sigma^f C_* \cong \Sigma^\infty$ of Lemma 3.43), and so an isomorphism of C-spaces

$$B(Q, C_*, X) \cong |\Omega^f \Sigma^f B_\bullet(C_*, C_*, X)|$$

We then apply the following result from [May74].

Lemma 3.53. *Let C be a unital E_∞ operad, equipped with a map of monads*

$$f : C_* \to \Omega^\infty \Sigma^\infty.$$

Let X be a C-space (and so pointed via $C(0) \to X$). Suppose that the base point of $C(1)$ and the base point of X are non-degenerate. Then the map

$$B(C_*, C_*, X) \to B(Q, C_*, X),$$

and so

$$B(C_*, C_*, X) \to |\Omega^f \Sigma^f B_\bullet(C_*, C_*, X)|,$$

is group-completion.

Proof. The point is that in general

$$C_* Y \to \Omega^\infty \Sigma^\infty Y$$

is group-completion [Coh73, CLM76, MS76], and so we have the level-wise group completion

$$C_*(C_*)^n X \to \Omega^\infty \Sigma^\infty (C_*)^n X$$

(see [May74]).

The argument requires the simplicial spaces involved to be "proper," that is, Reedy cofibrant with respect to the Hurewicz/Strøm model structure on topological spaces, so that the homology spectral sequences have the expected E_2-term. May proves that they are, provided that $(C(1), *)$ and $(X, *)$ are NDR-pairs. □

We can now finish the proof of Theorem 3.45.

Proof. It remains to show that if X is a group-like cofibrant C-algebra and V is a (fibrant) (-1)-connected spectrum, then a map

$$\phi : \Sigma^f X \to V$$

is a weak equivalence if and only if its adjoint

$$\psi : X \to \Omega^f V$$

is. These two maps are related by the factorization

$$\psi : X \to \Omega^f \Sigma^f X \xrightarrow{\Omega^f \phi} \Omega^f V.$$

The unit of adjunction is a weak equivalence by Proposition 3.48. It follows that ψ is a weak equivalence if and only if $\Omega^f \phi$ is. Certainly if ϕ is a weak equivalence, then so is $\Omega^f \phi$. Since both $\Sigma^f X$ and V are (-1)-connected, if $\Omega^f \phi$ is a weak equivalence, then so is ϕ. □

Remark 3.54. There is another perspective on Theorem 3.45 which elucidates the role of the "group-like" condition on C-spaces. Let's define a map

$$\alpha : X \to Y$$

of C-spaces to be a *stable* equivalence if the induced map

$$\Sigma^f \alpha' : \Sigma^f X' \to \Sigma^f Y'$$

is a weak equivalence (X' and Y' are cofibrant replacements of X and Y). The "stable" model structure on C-spaces is the localization of the model structure we have been considering in which the weak equivalences are the stable equivalences, and the cofibrations are as before.

In this stable model structure a C-space is fibrant if and only if it is group-like; compare the model structure on Γ-spaces discussed in [Sch99, MMSS01]. The homotopy category associated with the stable model structure is exactly $\mathrm{ho}\,\mathscr{T}[C]^\times$, and so this is a better encoding of the homotopy theory of $\mathcal{C}$-spaces. We have avoided discussing this approach in detail in order to minimize technical complications, as we do not need it for the applications.

Remark 3.55. In [MMSS01] it is shown that

$$\mathrm{ho}(\text{group-like } \Gamma\text{-spaces}) \cong \mathrm{ho}(\text{connective spectra}).$$

Rekha Santhanam [San08] has shown that the work of May and Thomason [MT78] can be used to prove that the category of C-spaces is Quillen equivalent to the category of Γ-spaces. These two results give another proof of the equivalence

$$\mathrm{ho}\,\mathscr{T}[C]^\times \simeq \mathrm{ho}(\text{connective spectra}).$$

3.6. **Units: proof of Theorem 3.2.** Let C be unital E_∞ operad, equipped with a map of operads

$$C \to \mathcal{L},$$

a map of monads on pointed spaces

$$f : C_* \to \Omega^\infty \Sigma^\infty,$$

and satisfying the hypotheses of Theorem 3.45. For example, we can take C to be

$$C = |\operatorname{Sing}(\mathcal{C} \times \mathcal{L})|,$$

the geometric realization of the singular complex on the product operad $\mathcal{C} \times \mathcal{L}$, where $\mathcal{C}$ is infinite little cubes operad of Boardman and Vogt [BV73].

Then we have a sequence of continuous adjunctions (the left adjoints are listed on top, and connective Quillen equivalence is indicated by $\approx$).

$$\Sigma_+^\infty \Omega^\infty : ((-1)\text{-connected spectra}) \underset{\Omega^f}{\overset{\Sigma^f, \approx}{\rightleftarrows}} \mathscr{T}[C]^\times \underset{GL_1}{\overset{}{\rightleftarrows}} \mathscr{T}[C] \underset{\Omega^\infty}{\overset{\Sigma_+^\infty}{\rightleftarrows}} \mathscr{S}[C] : gl_1$$

By Proposition 3.13, $\mathscr{S}[C]$ is a model for the category of E_∞ spectra. This completes the proof of Theorem 3.2.

4. E_∞ THOM SPECTRA AND ORIENTATIONS

With the adjunction of Theorem 3.2 in hand, one can construct and orient E_∞ Thom spectra as described in §2.1, where we emphasize the more novel case of the Thom spectrum associated to a map of spectra

$$b \to bgl_1 R.$$

We add a few additional remarks here, emphasizing the classical E_∞ Thom spectra associated to maps of spectra $b \to bgl_1 S$.
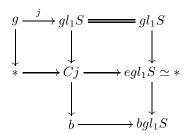
4.1. **Commutative S-algebra Thom spectra.** We write S for the sphere spectrum, bgl_1S for Σgl_1S, and BGL_1S for $\Omega^\infty bgl_1S$. BGL_1S is the classifying space for stable spherical fibrations. Theorem 3.2 gives a map (in ho $\mathscr{S}[E_\infty]$)

$$\epsilon : \Sigma_+^\infty \Omega^\infty gl_1S \to S.$$

Given a map

$$\zeta : b \to bgl_1S,$$

let $j = \Sigma^{-1}\zeta : g = \Sigma^{-1}b \to gl_1S$, and form the diagram

$$
\begin{array}{ccc}
g & \xrightarrow{\ j\ } gl_1S === gl_1S \\
\downarrow & \downarrow \quad\quad \downarrow \\
* \longrightarrow Cj \longrightarrow egl_1S \simeq * \\
\downarrow \quad\quad \downarrow \\
b \longrightarrow bgl_1S
\end{array}
$$

by requiring that the upper left and bottom right squares are homotopy Cartesian. Note that we may also view b as an infinite loop map

$$f : B \to BGL_1S.$$

As in Definition 2.6, the *Thom spectrum* of f, or of ζ, or of j, is the homotopy pushout $M = M\zeta$ in the diagram of E_∞ spectra

$$
\begin{array}{ccccc}
\Sigma_+^\infty \Omega^\infty g & \xrightarrow{\Sigma_+^\infty \Omega^\infty j} & \Sigma_+^\infty \Omega^\infty gl_1S & \xrightarrow{\ \epsilon\ } & S \\
{\scriptstyle \Sigma_+^\infty \Omega^\infty *}\downarrow & & \downarrow & & \downarrow \\
S = \Sigma_+^\infty \Omega^\infty * & \longrightarrow & \Sigma_+^\infty \Omega^\infty Cj & \longrightarrow & M,
\end{array}
\tag{4.1}
$$

which is to say that

$$M \cong \Sigma_+^\infty \Omega^\infty Cj \wedge_{\Sigma_+^\infty \Omega^\infty gl_1S}^L S \cong S \wedge_{\Sigma_+^\infty \Omega^\infty g}^L S.$$

(In §8 we compare this notion to classical definitions). Note that when writing this homotopy pushout, we are suppressing the choice of a point-set representative of the homotopy class ϵ. Since all objects are fibrant in the model structure of Proposition 3.7, it suffices to choose a cofibrant model for $\Omega^\infty gl_1S$ (and subsequently of $\Omega^\infty g$).

Now suppose that R is an E_∞ spectrum with unit $\iota : S \to R$; let $i = gl_1\iota$, and let $k = ij : g \to gl_1R$, so that we have the solid arrows of the diagram

$$
\begin{array}{ccc}
g & \xrightarrow{\ j\ } gl_1S \longrightarrow Cj \\
& {\scriptstyle k}\searrow \quad {\scriptstyle i}\downarrow \quad \nearrow {\scriptstyle u} \\
& \quad gl_1R,
\end{array}
\tag{4.2}
$$

in which the row is a cofiber sequence. The homotopy pushout diagram (4.1) and the adjunction of Theorem 3.2 gives the following.

Theorem 4.3. *Each of the squares in the commutative diagram of derived mapping spaces*

$$
\begin{array}{ccccc}
\mathscr{S}[E_\infty](M,R) & \longrightarrow & \mathscr{S}(Cj, gl_1R) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\{i\} & \longrightarrow & \mathscr{S}(gl_1S, gl_1R) & \longrightarrow & \mathscr{S}(g, gl_1R).
\end{array}
\tag{4.4}
$$

are homotopy cartesian. That is, the map k is the obstruction to the existence of an E_∞ map $M \to R$, and $\mathscr{S}[E_\infty](M,R)$ is weakly equivalent to the space of lifts in the diagram (4.2).

If $\mathscr{S}[E_\infty](M, R)$ is non-empty (i.e. if i is homotopic to the trivial map $g \to gl_1 R$) then we have equivalences of derived mapping spaces
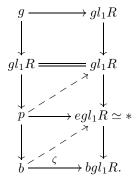
$$\mathscr{S}[E_\infty](M, R) \simeq \Omega\mathscr{S}(g, gl_1 R) \simeq \mathscr{S}(b, gl_1 R) \simeq \mathscr{S}[E_\infty](\Sigma_+^\infty B, R);$$

this is an E_∞ analogue of the usual Thom isomorphism.

4.2. **R-algebra Thom spectra.** More generally, suppose that R is an E_∞ ring spectrum. Given a map

$$\zeta : b \to bgl_1 R,$$

we obtain a map of cofiber sequences

in which $g = \Sigma^{-1} b$ and p is the fiber of $b \to bgl_1 R$.

Definition 4.5. The *R-algebra Thom spectrum* of ζ is the E_∞ R-algebra M which is the pushout in the diagram of E_∞ spectra

$$
\begin{array}{ccccc}
\Sigma_+^\infty \Omega^\infty g & \longrightarrow & \Sigma_+^\infty \Omega^\infty gl_1 R & \longrightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_+^\infty \Omega^\infty * & \longrightarrow & \Sigma_+^\infty \Omega^\infty p & \longrightarrow & M.
\end{array}
$$

If ζ factors as

$$b \xrightarrow{\zeta'} bgl_1 S \xrightarrow{bgl_1 \iota} bgl_1 R,$$

then $M\zeta$ is the derived smash product

$$M\zeta' \wedge_S^L R,$$

and so the following result is a generalization of Theorem 4.3.

Theorem 4.6. *Let A be a commutative R-algebra, and write*

$$i : gl_1 R \to gl_1 A$$

for the induced map on unit spectra. Then each of the squares in the commutative diagram

$$
\begin{array}{ccccc}
(E_\infty \ R\text{-algebras})(M, A) & \longrightarrow & \mathscr{S}(p, gl_1 A) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\{i\} & \longrightarrow & \mathscr{S}(gl_1 R, gl_1 A) & \longrightarrow & \mathscr{S}(g, gl_1 A)
\end{array}
\qquad (4.7)
$$

is homotopy cartesian.

Taking $A = R$, we see that the space of R-algebra orientations of $M\zeta$ is the space of lifts

In this form the obstruction theory generalizes to the associative case. We discuss this generalization operadically in §5 and again using quasicategories in §7.

5. A_∞ THOM SPECTRA AND ORIENTATIONS

5.1. **Sketch of the construction.** In §2.2, we outlined how one might generalize the study of orientations of E_∞ ring spectra in §3 and §4 to the associative case. We briefly review what was proposed there. We have adjunctions

$$(A_\infty \text{ spaces})^\times \underset{GL_1}{\overset{}{\rightleftarrows}} (A_\infty \text{ spaces}) \underset{\Omega^\infty}{\overset{\Sigma_+^\infty}{\rightleftarrows}} \mathscr{S}^{A_\infty} : GL_1. \tag{5.1}$$

Moreover, if R is an A_∞ spectrum, then Σ_+^∞ and Ω^∞ induce continuous Quillen adjunctions

$$\Sigma_+^\infty : (\text{right } \Omega^\infty R\text{-modules}) \rightleftarrows (\text{right } R\text{-modules}) : \Omega^\infty.$$

Using the fact that GL_1R is a group-like A_∞ space, one would like to form a "principal GL_1R-bundle"

$$GL_1R \to EGL_1R \to BGL_1R, \tag{5.2}$$

so that EGL_1R is a right A_∞ GL_1R-module. Given a map of spaces

$$f : X \to BGL_1R, \tag{5.3}$$

we would pull back this bundle as in the diagram

$$
\begin{array}{ccc}
GL_1R & =\!=\!=\!= & GL_1R \\
\downarrow & & \downarrow \\
P & \longrightarrow & EGL_1R \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & BGL_1R.
\end{array}
\tag{5.4}
$$

Then P should be a right A_∞ GL_1R-module, and so $\Sigma_+^\infty P$ should be a right $\Sigma_+^\infty GL_1R$-module. The Thom spectrum of f is to be the derived smash product

$$Mf = \Sigma_+^\infty P \wedge^L_{\Sigma_+^\infty GL_1R} R.$$

We would then have a weak equivalence

$$(\text{right } R\text{-modules})(Mf, R) \simeq (GL_1R\text{–modules})(P, \Omega^\infty R).$$

With respect to this isomorphism, the space of *orientations* of Mf should be the homotopy pull-back in the diagram

$$
\begin{array}{ccc}
(\text{orientations})(M, R) & \xrightarrow{\ \simeq\ } & (\text{right } GL_1R\text{–spaces})(P, GL_1R) \\
\downarrow & & \downarrow \\
(\text{right } R\text{–modules})(M, R) & \xrightarrow{\ \simeq\ } & (\text{right } GL_1R\text{–spaces})(P, \Omega^\infty R),
\end{array}
$$

and we should have weak equivalences

$$(\text{orientations})(M, R) \simeq (\text{right } GL_1R\text{–spaces})(P, GL_1R) \tag{5.5}$$

$$\simeq \mathscr{T}_{/BGL_1R}(B, EGL_1R). \tag{5.6}$$

The difficulties in making this sketch precise arise from the fact that GL_1R is not a topological group but rather only a group-like A_∞ space. This means for example it is more delicate to form the space P on which GL_1R will act, in such a way that we can prove the homotopy equivalence (5.6).

In this section we present an approach to carrying out the program described above. The essential strategy is to adapt the operadic smash product of [KM95, EKMM96] to the category of spaces. Specifically,

we produce a symmetric monoidal product on a subcategory of $\mathscr{T}$ such that monoids for this product are precisely A_∞-spaces; this allows us to work with models of $GL_1 R$ which are strict monoids for the new product. The observation that one could carry out the program of [EKMM96] in the setting of spaces is due to Mike Mandell, and was worked out in the thesis of the second author [Blum05]. In this paper we present a streamlined exposition covering the part of the theory we need for our applications. The interested reader should consult the forthcoming paper [BCS08] for further discussion and in particular proofs of the foundational theorems stated below.

5.2. $\mathbb{L}$-spectra and $\mathbb{L}$-spaces. We mimic the definitions of [EKMM96, §I]. Fix a universe $\mathcal{U}$ (a countably infinite-dimensional real vector space), and let $\mathcal{L}(1)$ denote the space of linear isometries $\mathcal{U} \to \mathcal{U}$. As the notation suggests, this is the first piece of the linear isometries operad.

There is a monad $\mathbb{L}$ on spaces with
$$\mathbb{L}X = \mathcal{L}(1) \times X,$$
where the product map comes from the composition on $\mathcal{L}(1)$ and the unit from the inclusion of the identity map. This is the space-level analogue of the monad $\mathbb{L}$ on spectra defined as $\mathbb{L}Y = \mathcal{L}(1) \ltimes Y$. An $\mathbb{L}$-space is precisely a space with an action of $\mathcal{L}(1)$. In direct analogy with the commutative and associative product on the category of $\mathbb{L}$-spectra [EKMM96, §I.5.1] we can define an operadic product on $\mathbb{L}$-spaces:

Definition 5.7. Let X, Y be $\mathbb{L}$-spaces. Define the operadic product $X \times_\mathcal{L} Y = \mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} X \times Y$ to be the coequalizer in the diagram

$$\mathcal{L}(2) \times (\mathcal{L}(1) \times \mathcal{L}(1)) \times (X \times Y) \underset{1 \times \xi}{\overset{\gamma \times 1}{\rightrightarrows}} \mathcal{L}(2) \times X \times Y \longrightarrow X \times_\mathcal{L} Y.$$

Here ξ denotes the map using the $\mathbb{L}$-algebra structure of X and Y, and γ denotes the operad structure map $\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \to \mathcal{L}(2)$. The left action of $\mathcal{L}(1)$ on $\mathcal{L}(2)$ induces an action of $\mathcal{L}(1)$ on $X \times_\mathcal{L} Y$.

With this definition, many of the results and arguments of [EKMM96, §I] carry over directly to the case of $\mathscr{T}[\mathbb{L}]$. For instance, a result of the senior author (see [EKMM96, §I.5.4]) implies that $\times_\mathcal{L}$ is associative and commutative:

Proposition 5.8.

(1) *The operation $\times_\mathcal{L}$ is associative. Precisely, for any $\mathbb{L}$-spaces $X_1, \ldots, X_k$ and any way of associating the product on the left, there is a canonical and natural isomorphism of $\mathbb{L}$-spaces*
$$X_1 \times_\mathcal{L} \cdots \times_\mathcal{L} X_k \cong \mathcal{L}(k) \times_{\mathcal{L}(1)^k} X_1 \times \cdots \times X_k.$$

(2) *The operation $\times_\mathcal{L}$ is commutative in the sense that there is a natural isomorphism of $\mathbb{L}$-spaces*
$$\tau : X \times_\mathcal{L} Y \cong Y \times_\mathcal{L} X$$
with the property that $\tau^2 = 1$.

There is a corresponding mapping space $F_{\times_\mathcal{L}}(X, Y)$ which satisfies the usual adjunction; in fact, the definition is forced by the adjunctions.

Definition 5.9. The mapping space $F_{\times_\mathcal{L}}(X, Y)$ is the equalizer of the diagram
$$\mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(\mathcal{L}(2) \times X, Y) \rightrightarrows \mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times X, Y).$$

Here one map is given by the action of $\mathcal{L}(1) \times \mathcal{L}(1)$ on $\mathcal{L}(2)$ and the other via the adjunction
$$\mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times X, Y) \cong \mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(\mathcal{L}(2) \times \mathcal{L}(1) \times X, \mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(\mathcal{L}(1), Y))$$
along with the action $\mathcal{L}(1) \times X \to X$ and coaction
$$Y \to \mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(\mathcal{L}(1), Y).$$

A diagram chase verifies the following proposition.

Proposition 5.10. *Let X, Y, and Z be $\mathcal{L}(1)$-spaces. Then there is an adjunction homeomorphism*

$$\mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(X \times_{\mathcal{L}} Y, Z) \cong \mathrm{Map}_{\mathscr{T}[\mathbb{L}]}(X, F_{\times_{\mathcal{L}}}(Y, Z)).$$

The unital properties of $\times_{\mathcal{L}}$ are again precisely analogous to those spelled out in [EKMM96, §I.8]. For a general $\mathbb{L}$-space, there is always a unit map

$$\lambda : * \times_{\mathcal{L}} X \to X.$$

The unit map is compatible with $\times_{\mathcal{L}}$. Specifically, one can adapt the arguments of [EKMM96, §I.8.5] to prove the following proposition.

Proposition 5.11.

(1) $* \times_{\mathcal{L}} * \to *$ *is an isomorphism.*
(2) *For any $\mathbb{L}$-space X, the unit map $* \times_{\mathcal{L}} X \to X$ is a weak equivalence.*
(3) *The unit map and $\times_{\mathcal{L}}$ specify the structure of a weak symmetric monoidal category on $\mathscr{T}[\mathbb{L}]$.*

Recall that a weak symmetric monoidal category is a category with a product that satisfies all of the axioms of a symmetric monoidal category with the exception that the unit map is not required to be an isomorphism [EKMM96, §II.7.1].

The product $\times_{\mathcal{L}}$ on $\mathscr{T}[\mathbb{L}]$ is a version of the cartesian product; in order to make a precise statement of the relationship between $\times_{\mathcal{L}}$ and $\times$, we need to discuss model structures.

Proposition 5.12. *There is a compactly generated topological model structure on $\mathscr{T}[\mathbb{L}]$ in which*

(1) *The weak equivalences are the maps which are weak equivalences of spaces,*
(2) *The fibrations are the maps which are fibrations of spaces,*
(3) *and the cofibrations are determined by the left-lifting property.*

The generating cofibrations and generating acyclic cofibrations are the sets $\{\mathbb{L}A \to \mathbb{L}B\}$ for $A \to B$ a generating cofibration in $\mathscr{T}$ and $\{\mathbb{L}C \to \mathbb{L}D\}$ for $C \to D$ a generating acyclic cofibration in $\mathscr{T}$, respectively.

The resulting model category is Quillen equivalent to spaces, since $\mathcal{L}(1)$ is contractible:

Proposition 5.13. *The free-forgetful adjunction induces a Quillen equivalence between the usual model structure on $\mathscr{T}$ and the model structure on $\mathscr{T}[\mathbb{L}]$ given in the preceding proposition.*

Furthermore, we have the following key comparison result, which says that the derived functor of $\times_{\mathcal{L}}$ is $\times$.

Proposition 5.14. *Let X and Y be cofibrant $\mathbb{L}$-spaces. Then the natural map*

$$X \times_{\mathcal{L}} Y \to X \times Y$$

is a weak equivalence.

The force of the construction of $\times_{\mathcal{L}}$ is the fact that it gives us control of A_∞-spaces and E_∞-spaces on the "point-set" level; just as in the setting of spectra, this makes it simple to define monoids and modules, and more generally to carry out definitions from homological algebra. Specifically, define monads $\mathbb{T}$ and $\mathbb{P}$ on $\mathscr{T}[\mathbb{L}]$ and $\mathbb{A}$ and $\mathbb{E}$ on $\mathscr{T}$ by the formulae

$$\mathbb{T}X = \coprod_{n \geq 0} X^{\times_{\mathcal{L}} n} \qquad\qquad \mathbb{A}X = \coprod_{n \geq 0} \mathcal{L}(n) \times X^n$$

$$\mathbb{P}X = \coprod_{n \geq 0} X^{\times_{\mathcal{L}} n}/\Sigma_n \qquad\qquad \mathbb{E}X = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} X^n.$$

Recall that $\mathbb{T}$-algebras in $\mathscr{T}[\mathbb{L}]$ are monoids (i.e. $\mathbb{L}$-spaces X equipped with multiplication maps $X \times_{\mathcal{L}} X \to X$ which are coherently associative and unital) and similarly $\mathbb{P}$-algebras are commutative monoids. As

discussed in Section 3.2, $\mathbb{A}$-algebras in $\mathscr{T}$ are A_∞-spaces structured by the (non-Σ) linear isometries operad and $\mathbb{E}$-algebras in $\mathscr{T}$ are E_∞-spaces structured by the linear isometries operad. Just as in [EKMM96, §I.4.6], these monads are closely related.

Proposition 5.15. *There are canonical isomorphisms*

$$\mathbb{A} \cong \mathbb{TL} \qquad\qquad \mathbb{E} \cong \mathbb{PL}$$

of monads on $\mathscr{T}$.

Via [EKMM96, §I.6.1], this has the following consequence.

Corollary 5.16.

(1) *The categories of $\mathbb{A}$-algebras in $\mathscr{T}$ (A_∞-spaces) and of $\mathbb{T}$-algebras in $\mathscr{T}[\mathbb{L}]$ are equivalent.*
(2) *The categories of $\mathbb{E}$-algebras in $\mathscr{T}$ (E_∞-spaces) and $\mathbb{P}$-algebras in $\mathscr{T}[\mathbb{L}]$ are equivalent.*

Finally, the work of [LMSM86, p. 366] reviewed in §3.3 and [EKMM96] implies that the category of $\mathbb{L}$-spaces has the expected relationship to the category of $\mathbb{L}$-spectra. There is a subtle point here, however: for an $\mathbb{L}$-space X, the Lewis-May suspension spectrum $\Sigma^\infty_+ X$ admits two structures as an $\mathbb{L}$-space. There is a trivial structure described in [EKMM96, §I.4.5], and the structure induced by the isomorphism $\mathcal{L}(1) \ltimes \Sigma^\infty(-) \cong \Sigma^\infty(\mathcal{L}(1) \times X)$. In the following discussion, we always use the latter.

Proposition 5.17.

(1) *If X and Y are $\mathbb{L}$-spaces, there is a natural isomorphism of $\mathbb{L}$-spectra*

$$\Sigma^\infty_+(X \times_{\mathcal{L}} Y) \cong \Sigma^\infty_+ X \wedge_{\mathcal{L}} \Sigma^\infty_+ Y$$

which is compatible with the commutativity isomorphism τ.
(2) *The Quillen pair*

$$\Sigma^\infty_+ : \mathscr{T} \leftrightarrows \mathscr{S} : \Omega^\infty \tag{5.18}$$

induces by restriction a continuous Quillen adjunction

$$\Sigma^\infty_+ : \mathscr{T}[\mathbb{L}] \leftrightarrows \mathscr{S}[\mathbb{L}] : \Omega^\infty \tag{5.19}$$

between topological model categories.
(3) *If X is a space then*

$$\Sigma^\infty_+ \mathbb{A}X \cong \bigvee_n \mathcal{L}(n) \ltimes (\Sigma^\infty_+ X)^{\wedge n} \qquad\qquad \Sigma^\infty_+ \mathbb{E}X \cong \bigvee_n \mathcal{L}(n) \ltimes_{\Sigma_n} (\Sigma^\infty_+ X)^{\wedge n},$$

and if X is an $\mathbb{L}$-space then

$$\Sigma^\infty_+ \mathbb{T}X \cong \bigvee_n (\Sigma^\infty_+ X)^{\wedge_{\mathcal{L}} n} \qquad\qquad \Sigma^\infty_+ \mathbb{P}X \cong \bigvee_n (\Sigma^\infty_+ X)^{\wedge_{\mathcal{L}} n}/\Sigma_n.$$

(4) *If R is an A_∞ spectrum, then $\Omega^\infty R$ is a monoid in $\mathscr{T}[\mathbb{L}]$, and $GL_1 R$ is a group-like monoid in $\mathscr{T}[\mathbb{L}]$. Similarly, if R is an E_∞ spectrum, then $\Omega^\infty R$ is a commutative monoid in $\mathscr{T}[\mathbb{L}]$, and $GL_1 R$ is a group-like commutative monoid in $\mathscr{T}[\mathbb{L}]$.*

5.3. S-modules and $*$-modules.
In order to work with modules over an A_∞ space, it is convenient to work with a symmetric monoidal category. In this section, we discuss the analogue of S-modules in the context of $\mathbb{L}$-spaces. Just as in [EKMM96], one can restrict to the subcategory of $\mathbb{L}$-spaces which are unital: $\mathbb{L}$-spaces X such that the unit map $* \times_{\mathcal{L}} X \to X$ is an isomorphism.

Definition 5.20. The category $\mathscr{M}_*$ of $*$-modules is the subcategory of $\mathcal{L}(1)$-spaces such that the unit map $\lambda \colon * \times_{\mathcal{L}} X \to X$ is an isomorphism. For $*$-modules X and Y, define $X \boxtimes Y$ as $X \times_{\mathcal{L}} Y$ and $F_{\boxtimes}(X, Y)$ as $* \times_{\mathcal{L}} F_{\times_{\mathcal{L}}}(X, Y)$.

The category $\mathscr{M}_*$ is a closed symmetric monoidal category with unit $*$ and product $\boxtimes$. The inclusion functor $\mathscr{M}_* \to \mathscr{T}[\mathbb{L}]$ has both a left and a right adjoint: the right adjoint from $\mathbb{L}$-spaces to $*$-modules is given by $* \times_{\mathcal{L}} (-)$, and the left adjoint by $F_{\times_{\mathcal{L}}}(*, -)$, just as in the stable setting the inclusion functor from S-modules to $\mathbb{L}$-spectra has both a left and a right adjoint [EKMM96, §II.2]. Proposition 2.7 of [EKMM96] shows that these adjunctions are respectively monadic and comonadic, and therefore as discussed in the proofs of [EKMM96, §II.1.4] and [EKMM96, §VII.4.6], standard arguments establish the following theorem.

Theorem 5.21. *The category $\mathscr{M}_*$ admits a cofibrantly generated topological model structure in which the weak equivalences are detected by the forgetful functor to $\mathbb{L}$-spaces. A map $f \colon X \to Y$ of $*$-modules is a fibration if the induced map $F_{\boxtimes}(*, X) \to F_{\boxtimes}(*, Y)$ is a fibration of spaces. Colimits are created in the category of $\mathbb{L}$-spaces, and limits are created by applying $* \times_{\mathcal{L}} (-)$ to the limit in the category of $\mathbb{L}$-spaces.*

The functor $* \times_{\mathcal{L}} (-)$ is part of a Quillen equivalence between $\mathscr{T}[\mathbb{L}]$ and $\mathscr{M}_*$. This implies in particular that there is a composite Quillen equivalence between $\mathscr{T}$ and $\mathscr{M}_*$. In addition, just as for $\mathcal{L}$-spaces, for cofibrant $*$-modules X and Y there is a weak equivalence $X \boxtimes Y \to X \times Y$.

Next, observe that the monads $\mathbb{T}$ and $\mathbb{P}$ on $\mathbb{L}$-spaces restrict to define monads on $\mathscr{M}_*$. The algebras over these monads are monoids and commutative monoids for $\boxtimes$, respectively. Thus, a $\boxtimes$-monoid in $\mathscr{M}_*$ is a $\times_{\mathcal{L}}$-monoid in $\mathbb{L}$ which is also a $*$-module. The functor $* \times_{\mathcal{L}} (-)$ gives us a means to functorially replace A_∞ and E_∞ spaces with $\boxtimes$-monoids and commutative $\boxtimes$-monoids which are weakly equivalent as A_∞ and E_∞ spaces respectively. Standard lifting techniques provide model structures on $\mathscr{M}_*[\mathbb{T}]$ and $\mathscr{M}_*[\mathbb{P}]$ in which the weak equivalences and fibrations are determined by the forgetful functor to $\mathscr{M}_*$.

We are now in a position to define categories of modules.

Definition 5.22. If G is a monoid in $\mathscr{M}_*$, then a G-module is a $*$-module P together with a map

$$G \boxtimes P \to P$$

satisfying the usual associativity and unit conditions. We write $\mathscr{M}_G$ for the category of G-modules.

Once again, there is a model structure on the category of G-modules in which the weak equivalences and fibrations are determined by the forgetful functor to $*$-modules.

Let $\Omega_{\mathcal{L}}^\infty$ denote the composite functor $* \times_{\mathcal{L}} \Omega^\infty F_{\mathcal{L}}(S, -)$ from S-modules to $\mathscr{M}_*$. Again, recall that the inclusion functor $\mathscr{M}_* \to \mathscr{T}[\mathbb{L}]$ has both a left and a right adjoint: the right adjoint from $\mathbb{L}$-spaces to $*$-modules is given by $* \times_{\mathcal{L}} (-)$, and the left adjoint by $F_{\times_{\mathcal{L}}}(*, -)$. As a consequence, the right adjoint of the functor Σ_+^∞ from $\mathscr{M}_*$ to $\mathscr{M}_S$ turns out to be $\Omega_{\mathcal{L}}^\infty$. In addition, recall that this implies that to lift right adjoints from $\mathscr{T}[\mathbb{L}]$ to $\mathscr{M}_*$ we forget from $\mathscr{M}_*$ to $\mathscr{T}[\mathbb{L}]$, apply the functor, and then compose with $* \times_{\mathcal{L}} (-)$.

In particular, for an object of $\mathscr{M}_*[\mathbb{T}]$ or $\mathscr{M}_*[\mathbb{P}]$ we compute GL_1 as the composite $* \times_{\mathcal{L}} GL_1(-)$ and therefore for an object of $\mathscr{M}_S[\mathbb{T}]$ or $\mathscr{M}_S[\mathbb{P}]$ we find that GL_1 is computed as

$$GL_1 R = * \times_{\mathcal{L}} GL_1 \Omega_{\mathcal{L}}^\infty R. \tag{5.23}$$

We have the following elaboration of Proposition 5.17, connecting the category $\mathscr{M}_*$ to the category of EKMM S-modules.

Proposition 5.24.

(1) *If X and Y are $*$-modules, then is a natural isomorphism of S-modules*

$$\Sigma_+^\infty(X \boxtimes Y) \cong \Sigma_+^\infty X \wedge_S \Sigma_+^\infty Y$$

which is compatible with the commutativity isomorphism τ. That is, Σ_+^∞ is a strong symmetric monoidal functor from $*$-modules to S-modules.

(2) *There is a continuous Quillen adjunction*

$$\Sigma_+^\infty : \mathscr{M}_* \leftrightarrows \mathscr{M}_S : \Omega_{\mathcal{L}}^\infty \tag{5.25}$$

between topological model categories.

(3) If R is a associative S-algebra, then $\Omega_{\mathcal{L}}^{\infty}$ is a monoid in $\mathscr{M}_*$ and $GL_1 R$ is a group-like monoid in $\mathscr{M}_*$. Similarly, if R is a commutative S-algebra, then $\Omega_{\mathcal{L}}^{\infty}$ is a commutative monoid in $\mathscr{M}_*$ and $GL_1 R$ is a group-like commutative monoid in $\mathscr{M}_*$.

(4) Let G be a monoid in $*$-modules, and let P be a G-module. Then $\Sigma_+^{\infty} P$ is an $\Sigma_+^{\infty} G$-module, and Σ_+^{∞} and $\Omega_{\mathcal{L}}^{\infty}$ restrict to give a continuous adjunction

$$\Sigma_+^{\infty} : (G\text{-modules}) \leftrightarrows (\Sigma_+^{\infty} G\text{-modules}) : \Omega_{\mathcal{L}}^{\infty}.$$

5.4. **Principal bundles.** Suppose that G is a monoid in $*$-modules. We will be interested in studying "classifying space" constructions on G, and for this we must say something about geometric realization of simplicial $\mathbb{L}$-spaces and $*$-modules.

Proposition 5.26.

(1) If $X_\bullet$ is a simplicial $\mathbb{L}$-space, then its geometric realization (as a simplicial space) $|X_\bullet|$ is a $\mathbb{L}$-space.

(2) If $X_\bullet$ and $Y_\bullet$ are simplicial $\mathbb{L}$-spaces, then there is a natural isomorphism of $\mathbb{L}$-spaces

$$|X_\bullet \times_{\mathcal{L}} Y_\bullet| \cong |X_\bullet| \times_{\mathcal{L}} |Y_\bullet|.$$

(3) If $X_\bullet$ is a simplicial $*$-module, then its geometric realization (as a simplicial space) $|X_\bullet|$ is a $*$-module.

(4) If $X_\bullet$ and $Y_\bullet$ are simplicial $*$-modules, then there is a natural isomorphism of $*$-modules

$$|X_\bullet \boxtimes Y_\bullet| \cong |X_\bullet| \boxtimes |Y_\bullet|.$$

Thus, we can form the $*$-modules

$$E_{\mathcal{L}} G \stackrel{\text{def}}{=} |B_\bullet(*, G, G)| \qquad\qquad B_{\mathcal{L}} G \stackrel{\text{def}}{=} B_\bullet(*, G, *).$$

That is, $B_{\mathcal{L}} G$ is the geometric realization of the simplicial $*$-module which has simplices

$$[n] \mapsto G^{\boxtimes(n-1)}. \tag{5.27}$$

The face maps are induced by the multiplication $G \boxtimes G \to G$, and the degeneracies from the unit $* \to G$.

We now will establish that $B_{\mathcal{L}} G$ is in fact a model of the usual classifying space. Associated to the map $\mathcal{L}(1) \to *$, there is the change of monoids functor Q from $\mathscr{T}[\mathbb{L}]$ to $\mathscr{T}$ given by $QX = * \times_{\mathcal{L}(1)} X$. The behavior of Q is described by the following proposition:

Proposition 5.28.

(1) Q is a strong symmetric monoidal functor from $\mathbb{L}$-spaces to spaces.

(2) Let U be the forgetful functor from $\mathbb{L}$-spaces to spaces. There is a natural transformation $U \to Q$ which is a weak equivalence for cofibrant objects of $\mathscr{T}[\mathbb{L}]$, $\mathscr{M}_*$, and $\mathscr{M}_*[\mathbb{T}]$.

Proof. Let X and Y be $\mathcal{L}(1)$-spaces. To show that Q is strong symmetric monoidal, we need to compare $* \times_{\mathcal{L}(1)} (X \times_{\mathcal{L}} Y)$ and $(* \times_{\mathcal{L}(1)} X) \times (* \times_{\mathcal{L}(1)} Y)$. Observe that $\mathcal{L}(2)$ is homeomorphic to $\mathcal{L}(1)$ as a left $\mathcal{L}(1)$-space, by composing with an isomorphism $U^2 \to U$. Therefore we have isomorphisms

$$
\begin{aligned}
* \times_{\mathcal{L}(1)} (X \boxtimes_{\mathcal{L}} Y) &= * \times_{\mathcal{L}(1)} \mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} (X \times Y) \\
&\cong (* \times_{\mathcal{L}(1)} X) \times (* \times_{\mathcal{L}(1)} Y)
\end{aligned}
$$

One checks that the required coherence diagrams commute. The result now follows, as $* \times_{\mathcal{L}(1)} * \cong *$.

The second part of the proposition follows in each case by observing that it suffices to work with cell objects, and then inducting over the cellular decomposition. The interested reader should consult [BCS08] for details. $\square$

As a consequence of the first property, Q takes monoids in $\mathcal{M}_*$ to topological monoids and G-modules in $\mathcal{M}_*$ to QG-spaces. The second property allows us to retain homotopical control. Let G be a cofibrant $\boxtimes$-monoid; equivalently, G is an A_∞-space. Denote by G' a weakly equivalent topological monoid produced by any of the standard rectification techniques. Then the Proposition and the fact that Q evidently commutes with geometric realization implies that

$$Q(B_{\mathcal{L}}G) \cong B(QG) \simeq BG' \qquad \text{and} \qquad Q(E_{\mathcal{L}}G) \cong E(QG) \simeq EG'.$$

Remark 5.29. Note however that as a consequence of the first property, nothing like the second property can be true in the setting of commutative monoids in $\mathcal{M}_*$. That is, Q takes commutative $\boxtimes$-monoids to commutative topological monoids. Since we know that commutative topological monoids have the homotopy type of a product of Eilenberg-Mac Lane spaces, in general this rectification process cannot be a weak equivalence.

Now, $E_{\mathcal{L}}G$ is a right G-space, and the projection

$$\pi : E_{\mathcal{L}}G \to B_{\mathcal{L}}G$$

is a map of G-spaces if $B_{\mathcal{L}}G$ is given the "trivial" action

$$B_{\mathcal{L}}G \boxtimes G \to B_{\mathcal{L}}G \boxtimes * \to B_{\mathcal{L}}G.$$

Furthermore, the comparison afforded by Q allows us to deduce that the projection map π is a model for the universal quasifibration.

Proposition 5.30. *Let G be a group-like cofibrant monoid in $\mathcal{M}_*[\mathbb{T}]$ with a nondegenerate basepoint. Then the map $\pi\colon E_{\mathcal{L}}G \to B_{\mathcal{L}}G$ is a quasifibration of underlying spaces.*

Proof. By the remarks above, $QE_{\mathcal{L}}G \cong E(QG)$ and $QB_{\mathcal{L}}G \cong B(QG)$. By naturality, there is a commutative diagram

$$
\begin{array}{ccc}
UE_{\mathcal{L}}G & \longrightarrow & E(QG) \\
\downarrow{\scriptstyle U\pi} & & \downarrow{\scriptstyle Q\pi} \\
UB_{\mathcal{L}}G & \xrightarrow{f} & B(QG).
\end{array}
$$

For any $p \in UB_{\mathcal{L}}G$, $(U\pi)^{-1}(p) = UG$, $(Q\pi)^{-1}(fp) = QG$, and the map between them is induced from the natural transformation $U \to Q$. Writing $F(U\pi)_p$ for the homotopy fiber of $U\pi$ at p and $F(Q\pi)_{fp}$ for the homotopy fiber of $Q\pi$ at fp, we have a commutative diagram

$$
\begin{array}{ccc}
UG \cong (U\pi)^{-1}(p) & \longrightarrow & F(U\pi)_p \\
\downarrow & & \downarrow \\
QG \cong (Q\pi)^{-1}(fp) & \longrightarrow & F(Q\pi)_{fp},
\end{array}
$$

where the horizontal maps are the natural inclusions of the actual fiber in the homotopy fiber. The hypotheses on G ensure that the vertical maps are weak equivalences: on the left, this follows directly from Proposition 5.28, and on the right, we use the fact that $UE_{\mathcal{L}}G \to QE_{\mathcal{L}}G$ and $UB_{\mathcal{L}}G \to QB_{\mathcal{L}}G$ are weak equivalences since U and Q commute with geometric realization and all the simplicial spaces involved are proper. Furthermore, since QG is a group-like topological monoid with a nondegenerate basepoint, $Q\pi$ is a quasifibration [May75, 7.6], and so the inclusion of the actual fiber of $U\pi$ in the homotopy fiber of $U\pi$ is an equivalence. That is, the bottom horizontal map is an equivalence. Thus, we deduce that the top horizontal map is an equivalence and so that $U\pi$ is a quasifibration. $\qquad\square$

Given a map of $*$-modules $f : X \to B_{\mathcal{L}} G$, let P be the pull-back in the category of G-modules

$$\begin{array}{ccc} P & \longrightarrow & E_{\mathcal{L}} G \\ \downarrow & \nearrow & \downarrow \pi \\ X & \xrightarrow{\ f\ } & B_{\mathcal{L}} G. \end{array} \qquad (5.31)$$

About this situation we have the following.

Theorem 5.32. *Suppose that G is a cofibrant group-like monoid in $\mathscr{M}_*$, and f is a fibration. Then there is a natural zigzag of weak equivalences between the derived mapping space $\mathrm{map}_{\mathscr{M}_*/B_{\mathcal{L}}G}(f, \pi)$ of lifts in the diagram (5.31) and the derived mapping space $\mathrm{map}_{\mathscr{M}_G}(P, G)$.*

Proof. We will deduce this result from the corresponding result for group-like monoids, using the functorial rectification process provided by the functor Q. Although this theorem in the classical setting is folklore, only recently have modern proofs appeared in the literature [Shu08]. We discuss the situation in Appendix A.2, where the result appears as Corollary A.3.

It is straightforward to verify that for cofibrant G, Q induces a Quillen equivalence between $\mathscr{M}_*/B_{\mathcal{L}}G$ and $\mathscr{T}/B(QG)$, and so there is an equivalence of derived mapping spaces

$$\mathrm{map}_{\mathscr{M}_*/B_{\mathcal{L}}G}(X, E_{\mathcal{L}}G) \simeq \mathrm{map}_{\mathscr{T}/B(QG)}(QX, E(QG)).$$

If G is group-like, then QG is a group-like topological monoid which has the homotopy type of a CW-complex and a nondegenerate basepoint, and so Corollary A.3 gives us a weak equivalence of derived mapping spaces

$$\mathrm{map}_{\mathscr{T}/B(QG)}(QX, E(QG)) \simeq \mathrm{map}_{(QG)\mathscr{T}}(P', QG),$$

where P' is the homotopy pullback in the diagram

$$\begin{array}{ccc} P' & \longrightarrow & B(QG) \\ \downarrow & & \downarrow \\ QX & \longrightarrow & E(QG). \end{array}$$

It is similarly straightforward to show that for cofibrant G, Q induces a Quillen equivalence between $G\mathscr{M}_*$ and $QG\mathscr{T}$, and so there is an equivalence of derived mapping spaces

$$\mathrm{map}_{G\mathscr{M}_*}(P, G) \to \mathrm{map}_{(QG)\mathscr{T}}(QP, QG).$$

The proof of the theorem will be complete once we have shown that QP is naturally weakly equivalent to P' as a QG-space. But this follows because Q preserves homotopy limits up to a zigzag of natural weak equivalences. $\square$

5.5. Thom spectra. Now suppose that R is a cofibrant EKMM S-algebra; by forgetting the unit homomorphism, an A_∞ ring spectrum. For the work of this section, we need a model of $GL_1 R$ such that $\Sigma_+^\infty GL_1 R$ is cofibrant as an S-algebra and there is an action of $\Sigma_+^\infty GL_1 R$ on R which makes R into a left $\Sigma_+^\infty GL_1 R$-module.

Based on Proposition 5.17, we observe that the adjunction described in equation 5.1 passes through $\mathscr{T}[\mathbb{L}]$ and takes values in A_∞ ring spectra, regarded as monoids under $\wedge_{\mathcal{L}}$ which are not necessarily unital. Therefore, we can use the adjunction in which GL_1 participates to obtain the desired action and promote to S-algebras.

Even better, Proposition 5.24 implies that we have a version $GL_1 R$ which produces a group-like monoid in $\mathscr{M}_*$ such that R obtains the structure of a $\Sigma_+^\infty GL_1 R$-module directly from the defining adjunction. That

is, we have the following structured version of the adjunction 5.1:

$$(\mathscr{M}_*[\mathbb{T}])^\times \xLeftarrow[GL_1]{} \mathscr{M}_*[\mathbb{T}] \xrightleftharpoons[\Omega_{\mathcal{L}}^\infty]{\Sigma_+^\infty} \mathscr{M}[\mathbb{T}] : GL_1. \tag{5.33}$$

where here recall that $GL_1 \colon \mathscr{M}_*[\mathbb{T}] \to (\mathscr{M}_*[\mathbb{T}])^\times$ is given by equation 5.23. The adjoint of the identity $GL_1 R \to GL_1 R$ under this adjunction is the map $\Sigma_+^\infty GL_1 R \to R$. Taking a cofibrant replacement $(GL_1 R)^c$ in the category of monoids in $\mathscr{M}_*$, we have a map of S-algebras

$$\Sigma_+^\infty (GL_1 R)^c \to \Sigma_+^\infty GL_1 R \to R.$$

For the remainder of the section, we will abusively refer to such a model as $GL_1 R$.

Definition 5.34. The *Thom spectrum* of f is the derived smash product in the homotopy category of R-modules

$$Mf \overset{\mathrm{def}}{=} \Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1 R}^L R.$$

It is often useful to have a specific point-set model for this derived smash product which involves only operations on f. For this purpose, we sometimes prefer to work with a model of Mf given by replacing a given map f with a cofibrant-fibrant replacement in the model structure on $GL_1 R$-modules over $B_{\mathcal{L}} GL_1 R$. While the resulting spectrum $\Sigma_+^\infty P$ will not necessarily be cofibrant as a $\Sigma_+^\infty GL_1 R$-module, it can be shown to be an extended cell module (in the sense of [EKMM96]); this is a flatness condition that ensures that smashing with $\Sigma_+^\infty P$ computes the correct homotopy type. Alternatively, we can replace R as a $\Sigma_+^\infty GL_1 R$-R bimodule by a cofibrant bimodule object R'; it then suffices to work with a fibration to compute the derived functor of the Thom spectrum.

To describe orientations in this setting, we first observe that, by construction, Mf is a right R-module, and if T is a right R-module then there is a natural weak equivalence of derived mapping spaces

$$\mathscr{M}_R(Mf, T) \simeq \mathscr{M}_{\Sigma_+^\infty GL_1 R}(\Sigma_+^\infty P, T).$$

By Proposition 5.24 there is a further adjoint weak equivalence of derived mapping spaces

$$\mathscr{M}_R(Mf, T) \simeq \mathscr{M}_{GL_1 R}(P, \Omega_{\mathcal{L}}^\infty T).$$

For example taking $T = R$ we have

$$\mathscr{M}_R(Mf, R) \simeq \mathscr{M}_{GL_1 R}(P, \Omega_{\mathcal{L}}^\infty R). \tag{5.35}$$

Definition 5.36. The space of *orientations* of Mf is the subspace of components of $\mathscr{M}_R(Mf, R)$ which correspond to

$$\mathscr{M}_{GL_1 R}(P, GL_1 R) \subseteq \mathscr{M}_{GL_1 R}(P, \Omega_{\mathcal{L}}^\infty R)$$

under the adjunction (5.35). That is, we have a pull-back diagram

$$\tag{5.37}$$

$$\begin{array}{ccc}
(\text{orientations})(Mf, R) & \xrightarrow{\simeq} & \mathscr{M}_{GL_1 R}(P, GL_1 R) \\
\downarrow & & \downarrow \\
\mathscr{M}_R(Mf, R) & \xrightarrow{\simeq} & \mathscr{M}_{GL_1 R}(P, \Omega_{\mathcal{L}}^\infty R).
\end{array}$$

With this definition, Theorem 5.32 implies the following.

Theorem 5.38. *The space of orientations of Mf is weakly equivalent to the space of lifts in the diagram (5.31). In particular, the spectrum Mf is orientable if and only if $f : X \to B_{\mathcal{L}} GL_1 R$ is null homotopic.*

To make contact with familiar notions of orientation, we'll be more explicit about the adjunctions in Definition 5.36. For this it it helpful to observe that the Thom spectrum of a point is equivalent to R.

Lemma 5.39. *The Thom spectrum of*

$$\{q\} \to X \to B_{\mathcal{L}}GL_1R$$

is weakly equivalent to R.

Proof. Let $\{q\} \to B_{\mathcal{L}}GL_1R$ be the inclusion of a point. The instructions in Definition 5.34 tell us that the Thom spectrum is $\Sigma_+^\infty P \wedge^L_{\Sigma_+^\infty GL_1R} R$, where P is the homotopy pull-back in

$$
\begin{array}{ccc}
P & \longrightarrow & E_{\mathcal{L}}GL_1R \\
\downarrow & & \downarrow \\
\{q\} \longrightarrow Z & \longrightarrow & B_{\mathcal{L}}GL_1R,
\end{array}
$$

and $\{q\} \to Z \to B_{\mathcal{L}}GL_1R$ is a fibrant replacement of $\{q\}$ in $*$-modules over $B_{\mathcal{L}}GL_1R$. Since by Proposition 5.30, $E_{\mathcal{L}}GL_1R \to B_{\mathcal{L}}GL_1R$ is a quasifibration (with fiber GL_1R,) it follows that $GL_1R \simeq P$ as GL_1R-modules. The result follows easily from this. $\qquad\square$

Corollary 5.40. *Since $E_{\mathcal{L}}GL_1R \simeq *$, we have*

$$M(E_{\mathcal{L}}GL_1R \to B_{\mathcal{L}}GL_1R) \simeq R.$$

Now suppose that $f : X \to B_{\mathcal{L}}GL_1R$ is a fibration of $*$-modules, and let P be the pull-back in the diagram

$$
\begin{array}{ccc}
P & \longrightarrow & E_{\mathcal{L}}G \\
\downarrow & \overset{\tilde{a}}{\nearrow} & \downarrow{\scriptstyle \pi} \\
X & \xrightarrow{\ f\ } & B_{\mathcal{L}}G,
\end{array}
\qquad (5.41)
$$

and let $M = Mf$. If $\tilde{a}$ is a lift as indicated, then by passing to Thom spectra along $\tilde{a}$ we get a map of R-modules

$$a : M \to R$$

which is the orientation associated to the lift $\tilde{a}$.

Conversely, suppose that $a : M \to R$ is a map of R-modules. More precisely, fix a cofibrant $\Sigma_+^\infty GL_1R$-R-bimodule R°, so that the R-module $\Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1R} R^\circ$ models M. Each point $p \in P$ determines a GL_1R-map

$$GL_1R \to P$$

and so a map of R-modules

$$j_p : R^\circ \simeq \Sigma_+^\infty GL_1R \wedge_{\Sigma_+^\infty GL_1R} R^\circ \to M \to R.$$

As p varies the j_p assemble into a map

$$P \to \mathscr{M}_R(R^\circ, R).$$

Put another way, we're studying the adjoint of the composite

$$\Sigma_+^\infty P \longrightarrow F_{\Sigma_+^\infty GL_1R}(\Sigma_+^\infty GL_1R, \Sigma_+^\infty P)$$

$$\longrightarrow F_R(\Sigma_+^\infty GL_1R \wedge_{\Sigma_+^\infty GL_1R} R^\circ, \Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1R} R^\circ) = F_R(R^\circ, M) \xrightarrow{a} F_R(R^\circ, R).$$

In Proposition 6.2, we show that

$$\mathscr{M}_R(R^\circ, R) \simeq \Omega_{\mathcal{L}}^\infty R,$$

and the resulting map

$$j : P \to \Omega_{\mathcal{L}}^\infty R$$
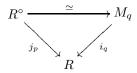
corresponds to a under the equivalence of derived mapping spaces

$$\mathscr{M}_R(Mf, R) \simeq \mathscr{M}_{GL_1R}(P, \Omega_{\mathcal{L}}^\infty R).$$

Put another way, for each $q \in X$, Lemma 5.39 implies that the Thom spectrum M_q of $\{q\} \to X \to B_{\mathcal{L}}GL_1R$ is equivalent to R. Passing to Thom spectra gives a map

$$i_q : M_q \to M \xrightarrow{a} R.$$

By Lemma 5.39, M_q is non-canonically equivalent to R: indeed, a choice of point $p \in P$ lying over q fixes an equivalence $R^\circ \simeq M_q$ making the diagram

$$
\begin{array}{ccc}
R^\circ & \xrightarrow{\ \simeq\ } & M_q \\
 & {}_{j_p}\searrow \quad \swarrow_{i_q} & \\
 & R &
\end{array}
$$

commute. Thus we have the following analogue of the standard description of Thom classes as in for example [GH81].

Proposition 5.42. *Suppose that $a : M \to R$ is a map of R-modules. Then the following are equivalent.*

(1) *a is an orientation.*
(2) *For each $q \in X$, the map of R-modules*

$$i_q : M_q \to M \xrightarrow{a} R$$

is a weak equivalence.
(3) *For each $p \in P$, the map of R-modules*

$$j_p : R^\circ \to M \xrightarrow{a} R$$

is a weak equivalence.

We now move on to discuss the Thom isomorphism in this setting. In this part of the section we tacitly assume we are working a point-set model of the Thom spectrum functor throughout, although this is suppressed from the notation and discussion. Suppose we are given an orientation in the form of a GL_1R-map $s : P \to GL_1R$, corresponding to an R-module map

$$a : M \to R,$$

where M is the Thom spectrum associated to a map of $*$-modules $f : X \to B_{\mathcal{L}}GL_1R$. The diagonal map $\Delta : X \to X \times X$ induces a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \Delta\ } & X \times X \\
{\scriptstyle f}\downarrow & \swarrow_{f\pi_2} & \\
B_{\mathcal{L}}GL_1R & &
\end{array}
\tag{5.43}
$$

in the category of $*$-modules over $B_{\mathcal{L}}GL_1R$, where here π_1 is the projection onto the first factor. Passing to Thom spectra, we obtain the R-module Thom diagonal map

$$M \xrightarrow{\Delta} \Sigma_+^\infty X \wedge M.$$

Now, we can form the composite

$$M \xrightarrow{\Delta} \Sigma_+^\infty X \wedge M \xrightarrow{1 \wedge a} \Sigma_+^\infty X \wedge R \tag{5.44}$$

as in [MR81]. To see directly that this map is a weak equivalence, we proceed by analyzing the diagonal on the level of $*$-modules. Specifically, passing to pullbacks the diagram (5.43) gives rise to a map $P \to X \times P$ of GL_1R-modules, where the action on $X \times P$ is induced from the actions on X and P via the composite

$$(X \times P) \boxtimes Z \longrightarrow ((X \times *) \boxtimes Z) \times ((* \times P) \boxtimes Z) \longrightarrow (X \boxtimes Z) \times (P \boxtimes Z).$$

Applying the map s, we obtain the composite map of GL_1R-modules

$$P \to X \times P \to X \times GL_1R.$$

Since s corresponds to a section of the map $P \to X$ induced by the universal property of the pullback, this composite is a weak equivalence, and so there is an induced weak equivalence of $\Sigma_+^\infty GL_1 R$-modules

$$\Sigma_+^\infty P \simeq X_+ \wedge \Sigma_+^\infty GL_1 R.$$

Now, passing to Thom spectra we have an induced weak equivalence of R-modules

$$M = \Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1 R} R \simeq (X_+ \wedge \Sigma_+^\infty GL_1 R) \wedge_{\Sigma_+^\infty GL_1 R} \wedge R \simeq \Sigma_+^\infty X \wedge R.$$

It is straightforward to check that this weak equivalence is the same as the composite (5.44). Summarizing, we have the following proposition.

Proposition 5.45. *If $a : M \to R$ is an orientation, then the map of right R-modules*

$$M \xrightarrow{\Delta} \Sigma_+^\infty X \wedge M \xrightarrow{1 \wedge a} \Sigma_+^\infty X \wedge R$$

is a weak equivalence.

6. The space of units is the derived space of homotopy automorphisms

In the preceding sections, we associated to an A_∞ or E_∞ ring spectrum R the group-like A_∞ or E_∞ space $GL_1 R$ which is the pull-back in the diagram

$$
\begin{array}{ccc}
GL_1 R & \longrightarrow & \Omega^\infty R \\
\downarrow & & \downarrow \\
(\pi_0 R)^\times & \longrightarrow & \pi_0 R.
\end{array}
$$

In this section, we connect this definition to a more conceptual definition of $GL_1 R$ as a derived space of automorphisms of R. Specifically, we show that this pullback description of $GL_1 R$ is weakly equivalent to the derived space of homotopy automorphism of R, considered as a module over itself. One of the appealing facets of the ∞-categorical approach to these matters is that, as we shall see in §7, it is this conceptual definition of $GL_1 R$ which naturally presents itself.

Our analysis is closely related to the evident question of how to define $GL_1 R$ in other modern categories of spectra (e.g. diagram spectra). We do not make any particular claim to novelty in this section; in particular, May and Sigurdsson provide an excellent discussion of the situation in [MS06, §22.2] (although note that our use of End and Aut is slightly different than theirs), and the conceptual description we describe is of course implicit in the original definition in [MQRT77]. Nonetheless, there are subtleties associated to the interaction between cofibrant and fibrant replacements in categories of commutative algebras and the underlying module categories that are worth exposing.

To begin, let us be clear about what we mean by the space of homotopy automorphisms of R as an R-module. Suppose that $\mathcal{M}$ is a symmetric monoidal simplicial or topological category of spectra, and let R be a monoid in $\mathcal{M}$, that is, an S-algebra. By the space of homotopy automorphisms of R, we mean the subspace of R-mod(R, R) consisting of weak equivalences. In order to make this notion homotopically well-defined, we need to "derive" it in two ways. First, we should replace R with a weakly equivalent cofibrant-fibrant *algebra* R'. Then, we should find a cofibrant-fibrant replacement R° of R' as a module over itself.

Definition 6.1. If R' is a cofibrant-fibrant algebra, and M is a cofibrant-fibrant R'-module, then the *space of endomorphisms* of M is

$$\mathrm{End}(M) \overset{\mathrm{def}}{=} \mathcal{M}_{R'}(M, M).$$

This is a monoid, and by definition the *space of homotopy automorphisms* of M is the subspace of group-like components: that is, $\mathrm{Aut}(M) = GL_1 \mathrm{End}(M)$ is the pull-back in the diagram

$$
\begin{array}{ccc}
\mathrm{Aut}(M) & \longrightarrow & \mathrm{End}(M) \\
\downarrow & & \downarrow \\
(\pi_0(\mathrm{End}(M))^\times) & \longrightarrow & \pi_0 \mathrm{End}(M).
\end{array}
$$

Since M is cofibrant and fibrant, we can equivalently define $\mathrm{Aut}(M)$ to be the subspace of $\mathrm{End}(M)$ consisting of weak equivalences. If R is an arbitrary algebra, then the *derived space of endomorphisms* of R is the homotopy type

$$
\mathrm{End}(R) = \mathrm{End}(R^\circ) \stackrel{\mathrm{def}}{=} \mathcal{M}_{R'}(R^\circ, R^\circ),
$$

where R' is a cofibrant-fibrant replacement of R as an algebra, and R° is a cofibrant-fibrant replacement of R' as a module over itself. The *derived space of homotopy automorphisms* of R is the homotopy type of the subspace

$$
\mathrm{Aut}(R) = \mathrm{Aut}(R^\circ) \subset \mathrm{End}(R^\circ) = \mathcal{M}_{R'}(R^\circ, R^\circ).
$$

We have elected to use the notation $\mathrm{Aut}(R)$ for the space of homotopy automorphisms of R°, even though it is not a group, just as we have written $GL_1 R$ for the space of units, even though it is not a group. This is because both are groups in the ∞-categorical sense, which is to say that they arise as ∞-groupoids of automorphisms of objects in ∞-categories; equivalently, but from the homotopical point of view, they are loop spaces. This notation is nearly inevitable in the setting of ∞-categories: as we shall see in §7, in the ∞-category of (cofibrant-fibrant) R-modules, the maximal ∞-groupoid on the single object R° is the space $B\,\mathrm{Aut}(R^\circ)$ which is a delooping of $\mathrm{Aut}(R^\circ)$.

As written, we have presented $\mathrm{Aut}(R)$ as a group-like topological or simplicial monoid. In practice, it is easier to access this homotopy type if we let R^c be a cofibrant replacement of R', and R^f a fibrant replacement. Then we have a homotopy equivalence of spaces

$$
\mathrm{End}(R) \simeq \mathcal{M}_{R'}(R^c, R^f),
$$

with $\mathrm{Aut}(R)$ equivalent to the subspace of weak equivalences.

We shall compare this notion of $GL_1 R$ to the classical one, in the setting of the S-modules of [EKMM96]. Let $\mathscr{S}$ be the Lewis-May-Steinberger category of spectra, and let $\mathscr{M}$ be the associated category of S-modules.

Proposition 6.2. *Let R be a cofibrant and fibrant S-algebra or commutative S-algebra in $\mathscr{M}$. Then the inclusion of derived mapping spaces*

$$
\mathrm{Aut}(R) \to \mathrm{End}(R)
$$

is a model for the inclusion

$$
GL_1 R \to \Omega^\infty R
$$

considered elsewhere in this paper.

Proof. If R is an associative or commutative S-algebra, then the underlying R-module associated to R will be fibrant. Thus, we can use R for R^f. In the notation of [EKMM96], $S \wedge_\mathcal{L} \mathbb{L}\Sigma^\infty S$ is a cofibrant replacement for S as an S-module, and $R \wedge_S \mathbb{L}\Sigma^\infty S$ is a cofibrant replacement for R as an R-module. So the derived mapping space $\mathscr{M}_R(R^c, R^f)$ is given by

$$
\begin{aligned}
\mathscr{M}_R(R \wedge_S \mathbb{L}\Sigma^\infty S^0, R) &\cong \mathscr{M}(S \wedge_S \mathbb{L}\Sigma^\infty S^0, R) \\
&\cong \mathscr{S}[\mathbb{L}](\mathbb{L}\Sigma^\infty S^0, F_\mathcal{L}(S, R)) \\
&\cong \mathscr{S}(\Sigma^\infty S^0, F_\mathcal{L}(S, R)) \\
&\cong \Omega^\infty F_\mathcal{L}(S, R).
\end{aligned}
$$

By [EKMM96, §I, Cor 8.7], the natural map of $\mathbb{L}$-spectra

$$R \to F_{\mathcal{L}}(S, R)$$

is a weak equivalence of $\mathbb{L}$-spectra, and so of spectra. The weak equivalence

$$\mathscr{M}_R(R \wedge_S \mathbb{L}\Sigma^{\infty} S^0, R) \simeq \Omega^{\infty} R$$

follows since Ω^{∞} preserves weak equivalences. It is then easy to see that the subspace of R-module weak equivalences corresponds to $GL_1 R$. □

The preceding proposition illustrates how useful it is that in the Lewis-May-Steinberger and EKMM categories of spectra, an algebra or commutative algebra R is automatically fibrant as a module over itself. In particular, since $GL_1 R$ is identified as a subspace of $\Omega^{\infty} R$, it is straightforward to see how to identify the multiplicative structure on $GL_1 R$.

In the setting of a category of diagram spectra $\mathscr{C}$, (e.g. orthogonal spectra), the situation is somewhat more complicated. For an associative S-algebra R, one can carry out a similar analysis after passing to a cofibrant-fibrant replacement of R as an S-algebra, and the pullback description of $GL_1 R$ in fact itself yields a genuine topological monoid [MS06, 22.2.3]. But the situation for commutative S-algebras in the diagrammatic setting is different. The model structure on commutative S-algebras is lifted from the positive model structure on (orthogonal) spectra, and in this model structure the underlying S-module of a cofibrant-fibrant commutative S-algebra will not be fibrant; indeed its zero space will be S^0, and so

$$\mathscr{C}(S, R) = S^0 \neq \mathscr{C}(S^0, R^f) \simeq h \operatorname{End}(R).$$

Of course, one can instead replace the given commutative S-algebra by an associative S-algebra instead, but in this case it is impossible to recover the E_{∞} structure on $GL_1 R$. To describe $GL_1 R$ in this setting requires a different construction; see [Sch04] or [L09] for a description.

The problem that arises above is a manifestation of Lewis's theorem [Lew91] about the nature of symmetric monoidal categories of spectra. If $S = \Sigma^{\infty} S^0$ is cofibrant (as it is in diagram categories of spectra), then the zero space of a cofibrant-fibrant commutative S-algebra must not be homotopically meaningful, as otherwise we could make a cofibrant-fibrant replacement S' of S, and

$$\mathscr{C}(S, S') \simeq QS^0$$

would realize QS^0 as a commutative topological monoid. On the other hand, if the zero space of a cofibrant-fibrant commutative S-algebra is homotopically meaningful, then S cannot be cofibrant, and the $(\Sigma^{\infty}, \Omega^{\infty})$ adjunction must take a modified form (as it does in the setting of EKMM spectra).

7. Parametrized spectra, units and Thom spectra via ∞-categories

7.1. Introduction.
In this section, we show that the theory of ∞-categories developed by Joyal and Lurie provides a powerful technical and conceptual framework for the study of Thom spectra and orientations.

In this setting, an A_{∞} ring spectrum R has an associated ∞-category R-mod of (right) R-modules. We define an R-*line* to be a R-module L which admits an equivalence

$$L \simeq R.$$

We define R-line to be the sub-∞-category of R-mod in which the objects are R-lines and in which the morphism space

$$R\text{-line}(L, M) \subset R\text{-mod}(L, M)$$

is the subspace of equivalences. As such R-line is an ∞-groupoid, i.e. a Kan complex. A *trivialization* of an R-line L is an equivalence

$$L \xrightarrow{\simeq} R, \tag{7.1}$$

and we define R-triv to be the ∞-groupoid

$$R\text{-triv} = R\text{-line}_{/R} \tag{7.2}$$

of trivialized R-lines. The forgetful map

$$R\text{-triv} \to R\text{-line}$$

is a Kan fibration, and is our model for the fibration

$$EGL_1R \to BGL_1R.$$

If X is a Kan complex, then a map (equivalently of simplicial sets or ∞-categories)

$$f : X \to R\text{-line} \tag{7.3}$$

is a family of R-lines parametrized by X. The Thom spectrum of f is just the (∞-categorical) colimit

$$Mf \stackrel{\text{def}}{=} \operatorname{colim}(X \xrightarrow{f} R\text{-line} \to R\text{-mod}).$$

Using this definition and the description (7.2) of R-triv, one sees that the space of orientations $Mf \to R$ is equivalent to the space of lifts

$$\begin{array}{ccc}
 & & R\text{-triv} \\
 & \nearrow & \downarrow{\scriptstyle \iota} \\
X & \xrightarrow[f]{} & R\text{-line.}
\end{array} \tag{7.4}$$

It is possible to develop the theory of Thom spectra and orientations using only these observations, together with some basic facts about ∞-categories from [HTT] and about symmetric monoidal model categories of spectra; in fact, this is our approach in §B. We recommend that the reader who is unfamiliar with ∞-categories begin with that treatment before reading this section. Nonetheless, this geodesic approach to the construction obscures some of the essential clarification provided by the ∞-categorical point of view.

For example, in this setting, we can make precise the slogan that R-line is the classifying space for bundles of R-lines. Specifically, there is a universal bundle of R-lines $\mathscr{L}$ over R-line, and the map of simplicial sets (7.3) classifies the bundle of R-lines $f^*\mathscr{L}$ over X. Moreover a lift in (7.4) corresponds to an equivalence

$$f^*\mathscr{L} \to R_X,$$

where R_X denotes the trivial bundle of R-lines over X.

In this section, we sketch the theory of bundles of R-modules, and use it to discuss Thom spectra and orientations. The story we tell reflects the close connection between abstract homotopy theory and ∞-category theory arising from the fact that ∞-groupoids are a model for spaces. However, given an existing theory of ∞-categories, it is possible to construct the ∞-category of ∞-groupoids without any mention of spaces whatsoever. We adopt this approach here: beginning with the notion of ∞-category, we summarize and expand on ideas from [HTT, DAGI, DAGII, DAGIII] to review the construction of ∞-categories of ∞-groupoids (spaces), stable ∞-groupoids (spectra), and bundles of stable ∞-groupoids (bundles of spectra). With these foundations in place, we discuss Thom spectra and orientations.

This approach has the advantage of being both self-contained and concise, but it has the disadvantage of being less concrete and potentially hiding the relationship to the established theory. In particular, as discussed in Remark 7.15, in the interests of expositional manageability we have chosen not to discuss the comparison between the approach to parametrized spectra discussed in this section and the parametrized spectra of May-Sigurdsson; this comparison will appear in a future paper.

However, at the level of Thom spectra and orientations, we do take pains to relate the abstract theory of this section to more concrete approaches: as mentioned above, in §B we take in some sense the opposite approach and show how to develop the ∞-categorical approach to Thom spectra and orientations, starting with an existing symmetric monoidal category of spectra such as S-modules or symmetric spectra. Furthermore, in §8, we provide a comparison between our various approaches to Thom spectra.

Our story in this section is independent of the material in §3—§5.

7.2. **∞-Categories and ∞-Groupoids.** For the purposes of this paper, an ∞-category will always mean a quasicategory in the sense of Joyal [Joy02]. This is the same as a weak Kan complex in the sense of Boardman and Vogt [BV73]; the different terminology reflects the fact that these objects simultaneously generalize the notions of category and of topological space. Specifically, recall that a quasicategory is a simplicial set which satisfies certain lifting properties.

Given two ∞-categories $\mathscr{C}$ and $\mathscr{D}$, the ∞-category of functors from $\mathscr{C}$ to $\mathscr{D}$ is simply the simplicial set of maps from $\mathscr{C}$ to $\mathscr{D}$ as simplicial sets. More generally, for any simplicial set X there is an ∞-category of functors from X to $\mathscr{C}$, $\mathrm{Fun}(X, \mathscr{C})$: By [HTT, Proposition 1.2.7.2, 1.2.7.3], the simplicial set $\mathrm{Fun}(X, \mathscr{C})$ is a quasicategory whenever $\mathscr{C}$ is, even for an arbitrary simplicial set X.

An ∞-groupoid is an ∞-category with the property that its homotopy category is a groupoid (cf. [HTT, §1.2.5]); equivalently [Joy02], an ∞-category is a Kan complex, and so the homotopy category is the fundamental groupoid of the underlying simplicial set. Just as ordinary categories are categories enriched over 0-groupoids, or sets, an ∞-category is essentially a category enriched over ∞-groupoids, or spaces. Indeed, if $\mathscr{C}$ is an ∞-category, and a and b are a pair of objects (vertices) of $\mathscr{C}$, then the ∞-groupoid $\mathscr{C}(a, b)$ of maps from a to b in $\mathscr{C}$ may be modeled as the fiber

$$\mathscr{C}(a, b) \longrightarrow \mathrm{Fun}(\Delta^1, \mathscr{C}) \longrightarrow \mathrm{Fun}(\partial \Delta^1, \mathscr{C})$$

over the object (a, b) of $\mathscr{C} \times \mathscr{C} \cong \mathrm{Fun}(\partial \Delta^1, \mathscr{C})$ (see [HTT, §1.2.2] for more details on this and various other homotopy equivalent models for $\mathscr{C}(a, b)$). Since ∞-categories are to be thought of as enriched over ∞-groupoids in a higher categorical sense (which we will not make precise), we should regard $\mathscr{C}(a, b)$ as only being defined up to equivalence, and so we shouldn't expect a composition map $\mathscr{C}(a, b) \times \mathscr{C}(b, c) \to \mathscr{C}(a, c)$, but rather a contractible space of possible composites. Nonetheless, we still write $gf : a \to c$ for a composite of $f : a \to b$ and $g : b \to c$.

The description of $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ described above gives rise to *categories* of ∞-categories and ∞-groupoids, but the real power of this approach comes from having ∞-*categories* Cat_∞ and Gpd_∞ of ∞-categories and ∞-groupoids, respectively. We construct these ∞-categories by a general technique for converting a simplicial category to an ∞-category: there is a simplicial nerve functor N from simplicial categories to ∞-categories which is the right Quillen functor of a Quillen equivalence [HTT, §1.1.5.5, 1.1.5.12, 1.1.5.13]

$$\mathfrak{C} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : \mathrm{N}.$$

Note that this process also gives rise to a standard passage from a simplicial model category to an ∞-category which retains the homotopical information encoded by the simplicial model structure. Specifically, given a simplicial model category $\mathcal{M}$, one restricts to the simplicial category on the cofibrant-fibrant objects, $\mathcal{M}^\circ$. Then applying the simplicial nerve yields an ∞-category $\mathrm{N}\mathcal{M}^\circ$.

Given ∞-categories $\mathscr{C}$ and $\mathscr{D}$, we have the ∞-category of functors $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ from $\mathscr{D}$ to $\mathscr{C}$; from this we obtain an ∞-groupoid $\mathrm{map}(\mathscr{D}, \mathscr{C})$ by forgetting the non-invertible natural transformations. Then Cat_∞ is the simplicial nerve of the simplicial category of ∞-categories, in which the mapping spaces are made fibrant by restricting to maximal Kan subcomplexes, and Gpd_∞ is the full ∞-subcategory of Cat_∞ on the ∞-groupoids, or equivalently the simplicial nerve of the simplicial category of ∞-groupoids (here the mapping spaces are automatically fibrant since natural transformations of ∞-groupoids are always invertible).

7.3. **Bundles of ∞-groupoids.** Because ∞-categories are enriched over ∞-groupoids, ∞-groupoids play a role in ∞-category theory analogous to that of sets in ordinary category theory. In particular, the ∞-category of ∞-groupoids inherits one of the most important exactness properties enjoyed by the category of sets: colimits commute with base-change. That is, if X_α is a diagram (possibly even indexed by an ∞-category $\mathscr{C}$) of ∞-groupoids, with colimit

$$\operatorname*{colim}_{\mathscr{C}} X_\alpha \simeq X,$$

then, for any map $f : X' \to X$,

$$X' \simeq f^* X \simeq \operatorname*{colim}_{\mathscr{C}} f^* X_\alpha,$$

where X_α is regarded as sitting over X via the inclusion into the colimit

$$X_\alpha \to \operatorname*{colim}_{\mathscr{C}} X_\alpha \simeq X.$$

To see this, it will be convenient to show a slightly stronger statement.

Proposition 7.5. *The base-change functor*

$$f^* : \operatorname{Gpd}_{\infty/X} \longrightarrow \operatorname{Gpd}_{\infty/X'}$$

admits a right adjoint (in the ∞-categorical sense, as in [HTT, §5.2.2]). In particular, f^* commutes with colimits.

Proof. This can be verified directly on the level of simplicial sets. To do so, first note that, replacing X' by an equivalent ∞-groupoid if necessary, we may assume without loss of generality that $f : X' \to X$ is a fibration of (fibrant) simplicial sets. Since $\operatorname{Set}_\Delta$ is an ordinary topos,

$$f^* : \operatorname{Set}_{\Delta/X} \longrightarrow \operatorname{Set}_{\Delta/X'}$$

admits a right adjoint

$$f_* : \operatorname{Set}_{\Delta/X'} \longrightarrow \operatorname{Set}_{\Delta/X}$$

which together comprise a Quillen pair, as f^* preserves cofibrations and weak equivalences (here we are using right properness). Moreover, this Quillen pair is compatible with simplicial model structures, so it extends to a simplicial adjunction of simplicial categories. It therefore follows from [HTT, Proposition 5.2.2.12] that restricting to (co)fibrant objects and forming simplicial nerves yields the desired adjunction on the level of ∞-categories. $\qquad\square$

Given a Kan complex X, let $\operatorname{Fib}(X)$ denote the (large) simplicial category of fibrations $Y \to X$ with target X. In other words, $\operatorname{Fib}(X)$ is the full simplicial subcategory of simplicial sets over X consisting of the fibrations. This is a contravariant simplicial functor in Kan complexes X, so applying the simplicial nerve functor

$$\mathrm{N} : \operatorname{Cat}_\Delta \longrightarrow \operatorname{Set}_\Delta$$

yields a presheaf $\mathscr{F}$ of (large) ∞-categories on $\operatorname{Gpd}_\infty$.

Proposition 7.6. *The fibrant simplicial category $\operatorname{Fib}(X)$ is a model for the slice ∞-category $\operatorname{Gpd}_{\infty/X}$; that is, for each Kan complex X, there is an equivalence of ∞-categories*

$$\mathscr{F}(X) = \operatorname{NFib}(X) \simeq \operatorname{Gpd}_{\infty/X}.$$

Proof. The projection $\mathscr{F}(X) \to \operatorname{Gpd}_\infty$ induces a map $\mathscr{F}(X)_{/1_X} \to \operatorname{Gpd}_{\infty/X}$, where 1_X is the identity fibration $X \to X$. The desired map is the composite

$$\mathscr{F}(X) \longrightarrow \mathscr{F}(X)_{/1_X} \longrightarrow \operatorname{Gpd}_{\infty/X},$$

where the first map is a homotopy inverse of the projection $\mathscr{F}(X)_{/1_X} \to \mathscr{F}(X)$, which by [HTT, Proposition 1.2.12.4] is a trivial fibration as 1_X is a final object of $\mathscr{F}(X)$. It is essentially surjective because any $Z \to X$ admits a factorization $Z \to Y \to X$ with $Z \to Y$ a homotopy equivalence and $Y \to X$ a fibration, so it only remains to check that it is fully faithful. Indeed, if $Y \to X$ and $Z \to X$ are fibrations, then $\operatorname{map}_{/X}(Z/X, Y/X)$ is the homotopy fiber of the map

$$\operatorname{map}(Z, Y) \longrightarrow \operatorname{map}(Z, X)$$

given by composing with $Y \to X$ over the vertex defined by $Z \to X$; since $Y \to X$ is a fibration, this is equivalent to the ordinary fiber, which is the mapping space in $\operatorname{Fib}(X)$ and hence also in $\mathscr{F}(X)$. $\qquad\square$

Proposition 7.7. *Let X be an ∞-groupoid. Then X is a colimit (in the ∞-category of ∞-groupoids) of the constant functor $1 : X \to \operatorname{Gpd}_\infty$ with value the terminal ∞-groupoid 1.*

Proof. Let $\mathscr{C}$ denote the full ∞-subcategory of Gpd_∞ spanned by the ∞-groupoids X with the property that $\mathrm{colim}_X 1 \simeq X$; note that $\mathscr{C}$ is nonempty, as it contains the terminal ∞-groupoid 1. Now clearly $\mathscr{C}$ is closed under (possibly infinite) coproducts and, according to [HTT, Proposition 4.4.2.2], $\mathscr{C}$ is closed under pushouts. Hence, by [HTT, Proposition 4.4.2.6], $\mathscr{C}$ is closed under arbitrary colimits, and any ∞-groupoid X may be built out of colimits from the terminal ∞-groupoid: indeed, $S^0 = 1 \coprod 1$, so inductively S^n arises as the pushout of $1 \leftarrow S^{n-1} \rightarrow 1$, and an arbitrary X is a colimit $X = \mathrm{colim}_n \mathrm{sk}_n X$, where $\mathrm{sk}_0 X$ is discrete and $\mathrm{sk}_{n+1} X$ is obtained from $\mathrm{sk}_n X$ as a pushout $\coprod 1 \leftarrow \coprod S^n \rightarrow \mathrm{sk}_n X$. Hence $\mathscr{C} = \mathrm{Gpd}_\infty$. $\square$

Proposition 7.8. *Let X be an ∞-groupoid. Then the colimit functor*

$$\mathrm{Fun}(X, \mathrm{Gpd}_\infty) \longrightarrow \mathrm{Gpd}_\infty$$

factors through $\mathrm{Gpd}_{\infty/X}$*, and the induced map*

$$\mathrm{Fun}(X, \mathrm{Gpd}_\infty) \longrightarrow \mathrm{Gpd}_{\infty/X}$$

is an equivalence of ∞-categories. In particular if $X = BG$ is an ∞-groupoid with a single object $*$ and G is the group-like monoidal ∞-groupoid $G = \mathrm{map}_X(*, *)$, then there we have an equivalence

$$\mathrm{Fun}(BG, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/BG}$$

between the ∞-category of ∞-groupoids with an action of G and the ∞-category of ∞-groupoids over BG.

Remark 7.9. Since X is an ∞-groupoid, we have $X \simeq X^{op}$, and so

$$\mathrm{Fun}(X^{op}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/X}$$

as well. This is a natural model for $\mathrm{Gpd}_{\infty/X}$ when we think of an ∞-groupoid over X as a presheaf of ∞-groupoids on X, as we do for example in Proposition 8.11.

Proof. First note that, while colim is not ordinarily a functor, we may model it as the derived functor of a (functorial choice of) colimit

$$\mathrm{colim} : \mathrm{Fun}(\mathfrak{C}[X], \mathrm{Set}_\Delta) \longrightarrow \mathrm{Set}_\Delta$$

on the level of model categories, where the model category on the left is equipped with the projective model structure. Also note that the first claim is a special case of the following fact: given ∞-categories $\mathscr{C}$ and $\mathscr{D}$ such that $\mathscr{C}$ has a terminal object 1 and given a functor $f : \mathscr{C} \rightarrow \mathscr{D}$, then f factors through the projection $\mathscr{D}_{/f(1)} \rightarrow \mathscr{D}$, since f determines a functor $\mathscr{C} \simeq \mathscr{C}_{/1} \rightarrow \mathscr{D}_{/f(1)}$. By Proposition 7.7, X is a colimit for the terminal functor $1 : X \rightarrow \mathrm{Gpd}_\infty$, giving an equivalence $X \simeq f(1)$ and thus a factorization of the colimit through $\mathrm{Gpd}_{\infty/X}$.

It remains to show that this resulting map is an equivalence of ∞-categories, which is to say that it is fully faithful and essentially surjective. For this we may assume that X is connected, since otherwise $X \simeq \coprod_{\pi_0 X} X_\alpha$ with each X_α connected and, given the result for connected ∞-groupoids, we deduce that

$$\mathrm{Fun}(X, \mathrm{Gpd}_\infty) \simeq \prod_\alpha \mathrm{Fun}(X_\alpha, \mathrm{Gpd}_\infty) \simeq \prod_\alpha \mathrm{Gpd}_{\infty/X_\alpha} \simeq \mathrm{Gpd}_{\infty/X} \,.$$

But then $X \simeq BG$, where G is the group-like simplicial monoid of endomorphisms of any object of $\mathfrak{C}[X]$, and the colimit (in this case, the quotient by the action of G) determines a Quillen equivalence of simplicial model categories between G-simplicial sets over EG, equipped with the projective model structure, and simplicial sets over BG (see Appendix A.2 for further details on this). Passing to ∞-categories yields an equivalence

$$\mathrm{N}(\mathrm{Set}_\Delta^G)^\circ \simeq \mathrm{Fun}(BG, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/BG}$$

and hence also the desired equivalence

$$\mathrm{Fun}(X, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/X} \,.$$

$\square$

Said differently, $\mathscr{F}$ is represented (in the ∞-category of large ∞-categories) by the ∞-category of ∞-groupoids. Moreover, we may work with $\mathscr{F}(X)$ in place of the equivalent but much larger ∞-groupoid $\mathrm{Gpd}_{\infty/X}$ by replacing the colimit with a map

$$\mathrm{Fun}(X, \mathrm{Gpd}_\infty) \longrightarrow \mathscr{F}(X) \tag{7.10}$$

described as follows.

There is a universal bundle of ∞-groupoids $\mathscr{U} \to \mathrm{Gpd}_\infty$ over Gpd_∞, characterized by the fact that the fiber over the ∞-groupoid T (a vertex of the quasicategory Gpd_∞) is T itself (see [HTT, §3.3.2] for a treatment of universal fibrations in more general contexts). Given a functor $f : X \to \mathrm{Gpd}_\infty$, the restriction $f^*\mathscr{U} \to X$ of $\mathscr{U} \to \mathrm{Gpd}_\infty$ along f is an ∞-groupoid over X such that the fiber of $f^*\mathscr{U} \to X$ over an object x of X is the ∞-groupoid $f(x)$. We think of $Y = f^*\mathscr{U} \to X$ as the bundle of ∞-groupoids classified by the map $f : X \to \mathrm{Gpd}_\infty$. There is also an inverse procedure which associates to an arbitrary bundle of ∞-groupoids $Y \to X$ over X a functor $f : X \to \mathrm{Gpd}_\infty$, whose value on the object x is equivalent to the fiber Y_x of Y over x, but this process is less explicit (cf. [HTT, §2.2.1]).

We may also want to consider more specific types of fibrations over X; in particular, we will be interested in those fibrations $Y \to X$ such that, for each object x of X, the fiber Y_x is equivalent to a fixed ∞-groupoid F. These are precisely those fibrations equivalent to ones of the form $f^*\mathscr{U}$ for some $f : X \to \mathrm{Gpd}_\infty$ which factors through the inclusion $B\,\mathrm{End}(F) \to \mathrm{Gpd}_\infty$, where $B\,\mathrm{End}(F)$ denotes the full ∞-subcategory of Gpd_∞ on the object F (an ∞-category with one object and endomorphisms $\mathrm{End}(F)$). In other words, the functor of fibrations with fiber F is represented by the ∞-category $B\,\mathrm{End}(F)$; generally, we're more interested in its maximal ∞-subgroupoid, which is represented by the ∞-groupoid $B\,\mathrm{Aut}(F)$. This description shows that it is enough to consider principal $\mathrm{Aut}(F)$-bundles instead, since the two functors are equivalent.

An ∞-category $\mathscr{C}$ equipped with a mapping space *functor*

$$\mathscr{C}^{op} \times \mathscr{C} \longrightarrow \mathrm{Gpd}_\infty$$

induces a Yoneda embedding

$$\mathscr{C} \longrightarrow \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Gpd}_\infty).$$

Of course, in ∞-category theory, mapping spaces are not uniquely defined, much less functorial. In particular, there is not really a canonical choice of a Yoneda embedding; instead, as with other ∞-categorical constructions, it's only defined up to a contractible space of choices.

Remark 7.11. This "problem" goes away if we work instead with the more rigid model of simplicial categories. Here we have a simplicial mapping space functor

$$\mathfrak{C}[\mathscr{C}^{op} \times \mathscr{C}] \simeq \mathfrak{C}[\mathscr{C}]^{op} \times \mathfrak{C}[\mathscr{C}] \longrightarrow \mathrm{Set}_\Delta \xrightarrow{\mathrm{fib}} \mathrm{Kan}$$

which we may suppose (after taking a fibrant replacement) lands in the simplicial category of Kan complexes. Passing back via the simplicial nerve yields a functorial assignment of mapping spaces

$$\mathscr{C}^{op} \times \mathscr{C} \simeq \mathrm{N}\mathfrak{C}[\mathscr{C}^{op} \times \mathscr{C}] \longrightarrow \mathrm{NKan} = \mathrm{Gpd}_\infty$$

and hence a particular choice of Yoneda embedding $\mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Gpd}_\infty) = \mathrm{Pre}(\mathscr{C})$. In general, we define the Yoneda embedding $\mathscr{C} \to \mathrm{Pre}(\mathscr{C})$ to be any functor equivalent to the one constructed in this way.

Proposition 7.12. *Let $X \to Y$ be a map of ∞-groupoids and let $Y \to \mathrm{Gpd}_{\infty/Y}$ be the Yoneda embedding. Then, as an ∞-groupoid over Y, X is a colimit of the composite $X \to Y \to \mathrm{Gpd}_{\infty/Y}$.*

Proof. Since $Y \simeq \mathrm{colim}_Y 1$ and colimits commute with base-change, we obtain a decomposition $X \simeq \mathrm{colim}_Y X_y$, where X_y denotes the fiber (calculated in the ∞-category of ∞-groupoids) of $X \to Y$ over the object y; that is, $X_y : Y \to \mathrm{Gpd}_\infty$ is the functor obtained from $Y \to X$ via the equivalence $\mathrm{Gpd}_{\infty/Y} \simeq \mathrm{Fun}(Y, \mathrm{Gpd}_\infty)$. As in the proof of Proposition 7.7, we easily reduce to the case in which Y is the terminal ∞-groupoid 1, in which case X is evidently a colimit of the functor $1 \to \mathrm{Gpd}_\infty$ with value X. $\qquad\square$

We'll be most interested in the case where $Y = BG$ is the one-object ∞-groupoid associated to a group-like monoidal ∞-groupoid G; i.e. $G \simeq \mathrm{map}_{BG}(*, *)$. Then

$$\mathrm{Gpd}_{\infty/BG} \simeq \mathrm{Fun}(BG, \mathrm{Gpd}_\infty)$$

is the ∞-category of ∞-groupoids equipped with an action of G, and the Yoneda embedding

$$BG \to \mathrm{Gpd}_{\infty/BG} \simeq \mathrm{Fun}(BG^{op}, \mathrm{Gpd}_\infty)$$

sends $X \to BG$ (viewed as a generalized element of the ∞-groupoid BG) to $X \to BG$ (viewed as an ∞-groupoid over BG), or, equivalently, to the fiber of $X \to BG$ over $* \to BG$ equipped with its natural $G \simeq \mathrm{map}_{/BG}(*, *)$-action (viewed as a functor $BG \to \mathrm{Gpd}_\infty$).

Corollary 7.13. *Let $X \to BG$ be an ∞-groupoid over BG, and write $P(X) = X \times_{BG} *$ for the associated G-∞-groupoid. Then $P(X)$ is a colimit of the composite $X \to BG$ followed by the Yoneda embedding $BG \to \mathrm{Fun}(BG^{op}, \mathrm{Gpd}_\infty)$.* □

7.4. **Bundles of stable ∞-groupoids.** A pointed ∞-category $\mathscr{C}$ with finite limits is said to be *stable* if the endofunctor

$$\Omega : \mathscr{C} \longrightarrow \mathscr{C},$$

defined by sending X to the limit of the diagram $* \to X \leftarrow *$, is an equivalence [DAGI, Corollary 10.12]. Strictly speaking, Ω, as defined, is not actually a functor on the level of ∞-categories (though it can be strictified to form one), but it's enough to check that Ω induces an endo-equivalence of the homotopy category of $\mathscr{C}$. A morphism of stable ∞-categories is an exact functor, meaning a functor which preserves finite limits and colimits [DAGI, §5].

More generally, given any ∞-category $\mathscr{C}$ with finite limits, the *stabilization* of $\mathscr{C}$ is the limit (in the ∞-category of ∞-categories) of the tower

$$\cdots \xrightarrow{\ \Omega\ } \mathscr{C}_* \xrightarrow{\ \Omega\ } \mathscr{C}_* \ ,$$

where $\mathscr{C}_*$ denotes the pointed ∞-category associated to $\mathscr{C}$ (the full ∞-subcategory of $\mathrm{Fun}(\Delta^1, \mathscr{C})$ on those arrows whose source is a final object $*$ of $\mathscr{C}$). See [DAGI, §10] for more on stabilization. Provided $\mathscr{C}$ is presentable, $\mathrm{Stab}(\mathscr{C})$ comes equipped with a stabilization functor $\Sigma_+^\infty : \mathscr{C} \to \mathrm{Stab}(\mathscr{C})$ functor from $\mathscr{C}$ [DAGI, Proposition 17.4], formally analogous to the suspension spectrum functor, and left adjoint to the zero-space functor $\Omega_-^\infty : \mathrm{Stab}(\mathscr{C}) \to \mathscr{C}$ (the subscript indicates that we forget the basepoint). Heuristically, Stab is left adjoint to the inclusion into the ∞-category of presentable ∞-categories of the full ∞-subcategory of presentable *stable* ∞-categories (note that morphisms of presentable ∞-categories are colimit-preserving functors). In other words, a morphism of presentable ∞-categories $\mathscr{C} \to \mathscr{D}$ such that $\mathscr{D}$ is stable factors (uniquely up to a contractible space of choices) through the stabilization $\Sigma_+^\infty : \mathscr{C} \to \mathrm{Stab}(\mathscr{C})$ of $\mathscr{C}$ (cf. [DAGI, Corollary 17.5]).

According to [DAGIII, Corollary 6.24], $\mathrm{Stab}(\mathrm{Gpd}_\infty)$ is a symmetric monoidal ∞-category under the smash product, in such a way that

$$\Sigma_+^\infty : \mathrm{Gpd}_\infty \longrightarrow \mathrm{Stab}(\mathrm{Gpd}_\infty)$$

is a strong symmetric monoidal functor (with respect to the cartesian monoidal structure on Gpd_∞). In particular, $S = \Sigma_+^\infty 1$ is a unit for a symmetric monoidal structure $\wedge$ on $\mathrm{Stab}(\mathrm{Gpd}_\infty)$. Here 1 denotes the terminal ∞-groupoid, the unit for the cartesian monoidal structure on Gpd_∞.

Since S is the unit of the symmetric monoidal structure on $\mathrm{Stab}(\mathrm{Gpd}_\infty)$, an S-algebra is just a monoid for the $\wedge$-product. Given an S-algebra R, we write R-mod for the ∞-category of right R-modules. If R is commutative then R-mod inherits a symmetric monoidal structure $\wedge_R$ for which R is the unit.

Now, for any ∞-groupoid X, the equivalence

$$\mathrm{Gpd}_{\infty/X} \simeq \mathrm{Fun}(X, \mathrm{Gpd}_\infty)$$

induces an equivalence of stabilizations

$$\mathrm{Stab}(\mathrm{Gpd}_{\infty/X}) \simeq \mathrm{Stab}(\mathrm{Fun}(X, \mathrm{Gpd}_\infty)).$$

Since limits in functor categories are computed pointwise, one easily checks that

$$\mathrm{Stab}(\mathrm{Gpd}_{\infty/X}) \simeq \mathrm{Stab}(\mathrm{Fun}(X, \mathrm{Gpd}_\infty)) \simeq \mathrm{Fun}(X, \mathrm{Stab}(\mathrm{Gpd}_\infty));$$

that is, the ∞-category of bundles of stable ∞-groupoids over X — the stabilization of the ∞-category of bundles of ∞-groupoids over X — is equivalent to the ∞-category of functors from X to stable ∞-groupoids.

The resulting ∞-category $\mathrm{Stab}(\mathrm{Gpd}_{\infty/X})$ is closed symmetric monoidal under the *fiberwise* smash product $\wedge_X$ [MS06], as well as enriched, tensored and cotensored over $\mathrm{Gpd}_{\infty/X}$. We write S_X for the sphere over X, the unit of the symmetric monoidal product $\wedge_X$. Note that $\mathrm{Stab}(\mathrm{Gpd}_{\infty/X})$ is naturally equivalent to the ∞-category S_X-mod of S_X-modules.

All of this works equally well for any S-algebra R. Writing $p : X \to 1$ for the projection to the terminal Kan complex, then

$$R_X \overset{\mathrm{def}}{=} p^* R$$

is an associative S_X-algebra, and we may form the ∞-category R_X-mod of right R_X-modules. We also have an equivalence

$$\mathrm{Fun}(X, R\text{-mod}) \overset{\simeq}{\longrightarrow} R_X\text{-mod}, \tag{7.14}$$

the R-module analogue of the equivalence (7.10), which we may think of as arising from pulling back a universal bundle of R-modules $\mathscr{E}$ over R-mod.

Remark 7.15. Since they are not necessary to obtain the main results of this paper, we have chosen to omit detailed discussion of certain foundational results necessary for a full theory of parametrized spectra in the ∞-categorical context. For one thing, for consistency one should compare the ∞-category $\mathrm{Fun}(X, \mathscr{S})$ to the May-Sigurdsson model category of spectra parametrized by X. Although this comparison is conceptually straightforward, the technical details are not insubstantial and so we leave the comparison for another paper. Note however that we do implicitly perform a part of such a comparison in our discussion of the equivalence of various definitions of the Thom spectrum functor.

Furthermore, we also have chosen to defer the explicit construction of the universal bundle of R-modules over R-mod, as that requires the significantly more complicated theory of left fibrations of (stable) ∞-groupoids over ∞-categories; see [HTT, §2] for details of the unstable theory. Fortunately, this construction is again not actually necessary for our work in this paper, as any bundle of R-modules over an ∞-category $\mathscr{C}$ is classified (up to equivalence) by a functor $\mathscr{C} \to R$-mod. In other words, the reader is free to take $\mathrm{Fun}(\mathscr{C}, R\text{-mod})$ as the *definition* of $R_\mathscr{C}$-mod, in which case the equivalence $\mathrm{Fun}(\mathscr{C}, R\text{-mod}) \simeq R_\mathscr{C}$-mod is actually equality. From this point of view, the universal bundle of R-modules $\mathscr{E}$ over R-mod corresponds to the identity map R-mod $\to R$-mod, and pulling back the universal bundle along a functor $f : \mathscr{C} \to R$-mod corresponds to precomposing the identity with f.

A map of spaces $f : X \to Y$ gives rise to a restriction functor

$$f^* : R_Y\text{-mod} \to R_X\text{-mod}$$

which admits a right adjoint f_* as well as a left adjoint $f_!$. This means that, given an R_X-module L and an R_Y-module M, there are natural equivalences of ∞-groupoids

$$R_Y\text{-mod}(f_! L, M) \simeq R_X\text{-mod}(L, f^* M) \qquad \text{and} \qquad R_X\text{-mod}(f^* M, L) \simeq R_Y\text{-mod}(M, f_* L).$$

If $M \simeq \psi^* \mathscr{E}$ is the bundle of R-modules classified by the functor $\psi : Y \to R$-mod, then $f^* M \simeq f^* \psi^* \mathscr{E}$; similarly, if $L \simeq \varphi^* \mathscr{E}$ for some $\varphi : X \to R$-mod, then $f_! M$ and $f_* M$ are classified by the left and right Kan extensions, respectively, of φ along f.

The *projection formula* asserts that the adjoint

$$f_!(L \wedge_X f^* M) \longrightarrow f_! L \wedge_Y M$$

of the composite
$$L \wedge_X f^* M \longrightarrow f^* f_! L \wedge_X f^* M \simeq f^*(f_! L \wedge_Y M)$$
is an equivalence (here we are using the fact that f^* is strong monoidal). To see this, we merely need to examine the fiber over an object y of Y:
$$f_!(L \wedge_X f^* M)_y \simeq \operatorname*{colim}_{x \in X_y}(L_x \wedge f^* M_x) \simeq \operatorname*{colim}_{x \in X_y}(L_x \wedge M_y)$$
$$\simeq (\operatorname*{colim}_{x \in X_y} L_x) \wedge M_y \simeq f_! L_y \wedge M_y \simeq (f_! L \wedge_Y M)_y.$$

This has some notable consequences. Given a second R_Y-module N, the equivalences
$$\operatorname{map}(L, f^* \operatorname{Map}_X(M, N)) \simeq \operatorname{map}(f_! L \wedge_X M, N) \simeq \operatorname{map}(L \wedge_Y f^* M, f^* N) \simeq \operatorname{map}(L, \operatorname{Map}_Y(f^* M, f^* N))$$
and
$$\operatorname{map}(M, \operatorname{Map}_Y(f_! L, N)) \simeq \operatorname{map}(f_! L \wedge_Y M, N) \simeq \operatorname{map}(L \wedge_X f^* M, f^* N) \simeq (M, f_* \operatorname{Map}_X(L, f^* N))$$
imply that the natural maps
$$f^* \operatorname{Map}_Y(M, N) \longrightarrow \operatorname{Map}_X(f^* M, f^* N) \qquad \text{and} \qquad \operatorname{Map}_Y(f_! L, N) \longrightarrow f_* \operatorname{Map}_X(L, f^* N)$$
are equivalences of R_X-modules and R_Y-modules, respectively. It also follows that the tensor of the space $f : X \to Y$ over Y with the R_Y-module M is given by $f_! f^* M$: indeed, $f \simeq f_! \operatorname{id}_X$, so
$$\operatorname{map}(f_! f^* M, N) \simeq \operatorname{map}(f^* M, f^* N) \simeq \operatorname{map}(\operatorname{id}_X, f^* \operatorname{map}_Y(M, N)) \simeq \operatorname{map}(f, \operatorname{map}_Y(M, N)). \tag{7.16}$$
In other words,
$$\Sigma_Y^\infty f_+ \wedge_{S_Y} M \simeq f_! f^* M, \tag{7.17}$$
and in particular $\Sigma_Y^\infty f_+ \wedge_{S_Y} R_Y \simeq f_! f^* R_Y$ is the free R_Y-module on f, where $f_+ : X + Y \to Y$ is $X \to Y$ plus a disjoint basepoint $Y \to Y$.

7.5. Bundles of R-lines.

Definition 7.18. An R-line is an R-module M which admits an R-module equivalence $M \simeq R$.

Let R-line denote the full ∞-subgroupoid of R-mod spanned by the R-lines. This is not the same as the full ∞-subcategory of R-mod on the R-lines, as a map of R-lines is by definition an equivalence. We regard R-line as a pointed ∞-groupoid via the distinguished object R.

Remark 7.19. R-line is a subcategory of the ∞-category R-mod in the sense of [HTT, 1.2.11]. Indeed, our definition really specifies ho R-line as a subgroupoid of the homotopy category ho R-mod of R-mod (the full subgroupoid consisting of those R-modules which are *isomorphic* to R in ho R-mod). Then R-line is obtained as the pullback of simplicial sets

$$
\begin{array}{ccc}
R\text{-line} & \longrightarrow & R\text{-mod} \\
\downarrow & & \downarrow \\
\text{N ho } R\text{-line} & \longrightarrow & \text{N ho } R\text{-mod.}
\end{array}
$$

Since limits of weak Kan complexes are again weak Kan complexes, R-line is an ∞-category; moreover, its homotopy category is the groupoid ho R-line, basically by construction, so in fact R-line is an ∞-groupoid.

Fix an ∞-groupoid X and write $p : X \to 1$ for the projection to the terminal ∞-groupoid. Recall that $R_X = p^* R$ is the unit of the symmetric monoidal ∞-category R_X-mod of bundles of R-modules over X.

Given an ∞-category $\mathscr{C}$, we write $\operatorname{Iso}(\mathscr{C})$ for the maximal ∞-subgroupoid of $\mathscr{C}$, obtained by throwing away all noninvertible arrows of $\mathscr{C}$. Thus, if a and b are objects of $\mathscr{C}$, $\operatorname{Iso}(\mathscr{C})(a, b)$ is the subcomplex of $(\mathscr{C})(a, b)$ consisting of the *equivalences*.

Definition 7.20. The ∞-groupoid R_X-line is the full ∞-subgroupoid of $\operatorname{Iso}(R_X$-mod) on those R_X-modules M such that, for all objects x of X, the fiber M_x of M over x is equivalent to R.

Proposition 7.21. *There is an equivalence*

$$R_X\text{-line}(L, M) \simeq \text{Iso}(R_X\text{-mod})(L, M),$$

natural in R_X-lines L and M.

Proof. R_X-line a full ∞-subgroupoid of $\text{Iso}(R_X\text{-mod})$. $\square$

The restriction of the universal R-module bundle $\mathscr{E}$ over R-mod along the inclusion $j : R\text{-line} \to R\text{-mod}$ is the universal R-line bundle $\mathscr{L} = j^*\mathscr{E}$ over R-line. This is the analogue of the tautological line bundle in geometry: the fiber over the line $l \in \mathbb{P}^\infty$ is the line itself.

Proposition 7.22. *There is a commutative diagram of ∞-categories*

$$
\begin{array}{ccc}
\text{Fun}(X, R\text{-line}) & \longrightarrow & R_X\text{-line} \\
\downarrow & & \downarrow \\
\text{Fun}(X, R\text{-mod}) & \longrightarrow & R_X\text{-mod}
\end{array}
$$

in which the vertical maps are inclusions of full ∞-subgroupoids and the horizontal maps are equivalences.

Proof. The fibers of an R_X-module L are equivalent to R if and only if $L \simeq \varphi^*\mathscr{E}$ for some $\varphi : X \to R\text{-mod}$ such that φ factors through the inclusion $R\text{-line} \to R\text{-mod}$. Hence R_X-line is the full ∞-subgroupoid of R_X-mod on those objects in the essential image of $\text{Fun}(X, R\text{-line})$ via the map $\text{Fun}(X, R\text{-mod}) \to R_X\text{-mod}$ of 7.14; this map is an equivalence, so $\text{Fun}(X, R\text{-line}) \simeq R_X\text{-line}$. $\square$

Corollary 7.23. *The fiber over the ∞-groupoid X of the projection $\text{Gpd}_{\infty/R\text{-line}} \to \text{Gpd}_\infty$ is equivalent to the ∞-groupoid R_X-line.*

Proof. $R_X\text{-line} \simeq \text{Fun}(X, R\text{-line}) \simeq \text{Gpd}_\infty(X, R\text{-line})$, and, in general, the ∞-groupoid $\mathscr{C}(a, b)$ of maps from a to b in the ∞-category $\mathscr{C}$ may be calculated as the fiber over a of the projection $\mathscr{C}_{/b} \to \mathscr{C}$. $\square$

Definition 7.24. A *trivialization* of an R_X-module L is an R_X-module equivalence $L \to R_X$. The ∞-category R_X-triv of trivialized R-lines is the slice category
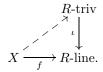
$$R_X\text{-triv} \overset{\text{def}}{=} R_X\text{-line}_{/R_X}.$$

The objects of R_X-triv are *trivialized R_X-lines*, which is to say R_X-lines L with a trivialization $L \to R_X$; more generally, an n-simplex $\Delta^n \to R_X$-triv of R_X-triv is a map $\Delta^n \star \Delta^0 \to R_X$-line of R_X-line which sends Δ^0 to R_X. There is a canonical projection

$$\iota_X : R_X\text{-triv} \longrightarrow R_X\text{-line}$$

which sends the n-simplex $\Delta^n \star \Delta^0 \to R_X$-line to the n-simplex $\Delta^n \to \Delta^n \star \Delta^0 \to R_X$-line; according to (the dual of) [HTT, Corollary 2.1.2.4], this is a right fibration, and hence a fibration as R_X-line is an ∞-groupoid [HTT, Lemma 2.1.3.2].

When X is the terminal Kan complex, we write R-triv in place of R_X-triv and $\iota : R\text{-triv} \longrightarrow R\text{-line}$ in place of ι_X. Given a map $f : X \to R$-line, we will refer to a factorization

$$
\begin{array}{ccc}
 & & R\text{-triv} \\
 & \nearrow & \downarrow \iota \\
X & \xrightarrow{\quad f \quad} & R\text{-line.}
\end{array}
$$

of f through ι as a *trivialization* of f; i.e. $\text{Triv}(f)$ is the fiber over f of $\iota_X : \text{Fun}(X, R\text{-triv}) \to \text{Fun}(X, R\text{-line})$.

Proposition 7.25. *Let X be an ∞-groupoid. Then R_X-triv is a contractible ∞-groupoid, and the fiber of the fibration*

$$\iota_X : R_X\text{-triv} \longrightarrow R_X\text{-line}$$

over an R_X-line L is the (possibly empty) ∞-groupoid $\mathrm{Triv}(L)$ of trivializations of L.

Proof. Once again, use the description of $\mathscr{C}(a, b)$ as the fiber over a of the projection $\mathscr{C}_{/b} \to \mathscr{C}$, together with the fact that if $\mathscr{C}$ is an ∞-groupoid then $\mathscr{C}_{/b}$, an ∞-groupoid with a final object, is contractible. □

Corollary 7.26. *For any $f : X \to R$-line there is a commutative diagram of ∞-groupoids*

$$
\begin{array}{ccccc}
\mathrm{Triv}(f) & \longrightarrow & \mathrm{Fun}(X, R\text{-triv}) & \longrightarrow & \mathrm{Fun}(X, R\text{-line}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Triv}(f^*\mathscr{L}) & \longrightarrow & R_X\text{-triv} & \longrightarrow & R_X\text{-line}
\end{array}
$$

in which each of the squares is cartesian and each of the vertical maps is an equivalence.

Proof. Indeed,

$$\mathrm{Fun}(X, R\text{-line}_{/R}) \simeq \mathrm{Fun}(X, R\text{-line})_{/p^*R} \simeq R_X\text{-line}_{/R_X},$$

so the right-hand square is cartesian. The left-hand square is obtained by taking the fiber over f. □

7.6. **Thom R-modules and orientations.**

Definition 7.27. The Thom R-module is the functor

$$M : \mathrm{Gpd}_{\infty/R\text{-line}} \longrightarrow R\text{-mod}$$

which sends $f : X \to R$-line to

$$Mf \overset{\mathrm{def}}{=} p_! f^* \mathscr{L},$$

the push-forward of the restriction along f of the universal R-line bundle $\mathscr{L}$.

Remark 7.28. In terms of the equivalences

$$R_X\text{-mod} \simeq \mathrm{Fun}(X, R\text{-mod})$$

and

$$R_X\text{-line} \simeq \mathrm{Fun}(X, R\text{-line}),$$

the R_X-line $f^*\mathscr{L}$ corresponds to the map

$$X \xrightarrow{f} R\text{-line} \to R\text{-mod},$$

and then Mf is the colimit

$$Mf = \mathrm{colim}(X \xrightarrow{f} R\text{-line} \to R\text{-mod}).$$

As we shall see, this is a very useful formula for the Thom spectrum. It is the only formula available in the development in §B.

We also obtain another useful characterization in the following proposition.

Proposition 7.29. *Let G be a group-like monoidal ∞-groupoid with delooping BG — that is, an ∞-groupoid with one object $*$ and an equivalence $G \simeq \Omega BG \simeq \mathrm{Aut}_{BG}(*)$ — and suppose that $X \simeq BG$. Then*

$$Mf \simeq R/G,$$

where G acts on R by R-module maps via $\Omega f : G \simeq \Omega BG \to \Omega R$-line $\simeq \mathrm{Aut}_R(R)$.

Proof. Evidently, R/G is the colimit of the composite functor

$$BG \to B\operatorname{Aut}_R(R) \simeq R\text{-line} \to R\text{-mod}$$

classifying the R_{BG}-module $f^*\mathscr{L}$. But a colimit is a special case of a left Kan extension; namely, the left Kan extension along the projection $p : BG \to *$ to the point. $\qquad\square$

With these in place, one can analyze the space of orientations in a straightforward manner, as follows. First of all observe that, by definition, we have a weak equivalence

$$R\text{-mod}(Mf, R) \simeq R_X\text{-mod}(f^*\mathscr{L}, p^*R).$$

Definition 7.30. The ∞-groupoid of *orientations* of Mf is the "pull-back"

$$
\begin{array}{ccc}
\operatorname{or}_R(Mf, R) & \longrightarrow & R\text{-mod}(Mf, R) \\
\simeq \downarrow & & \downarrow \simeq \\
R_X\text{-line}(f^*\mathscr{L}, p^*R) & \longrightarrow & R_X\text{-mod}(f^*\mathscr{L}, p^*R).
\end{array}
\tag{7.31}
$$

The ∞-groupoid $\operatorname{or}_R(Mf, R)$ enjoys an obstruction theory analogous to that of the space of orientations described in Definition 5.36. The following theorem is the analogue in this context of Theorem 5.38.

Theorem 7.32. *Let $f : X \to R$-line be a map, with associated Thom R-module Mf. Then the ∞-groupoid of orientations $Mf \to R$ is equivalent to the ∞-groupoid of lifts in the diagram*

$$
\begin{array}{ccc}
 & R\text{-triv} & \\
 & \nearrow \quad \downarrow \iota & \\
X & \xrightarrow{\quad f \quad} & R\text{-line.}
\end{array}
\tag{7.33}
$$

Proof. According to Corollary 7.26, the ∞-groupoid $\operatorname{Triv}(f)$ of factorizations of f through ι is equivalent to the ∞-groupoid $\operatorname{Triv}(f^*\mathscr{L})$ of R_X-module equivalences from $f^*\mathscr{L}$ to p^*R. $\qquad\square$

Corollary 7.34. *An orientation of Mf determines an equivalence of R-modules*

$$Mf \simeq \Sigma_+^\infty X \wedge R.$$

Proof. The desired map is the composite

$$\operatorname{or}_R(Mf, R) \to \operatorname{Iso}(R_X\text{-mod})(f^*\mathscr{L}, p^*R) \to \operatorname{Iso}(R\text{-mod})(p_! f^*\mathscr{L}, p_! p^*R) \to \operatorname{Iso}(R\text{-mod})(p_! f^*\mathscr{L}, \Sigma_+^\infty X \wedge R),$$

where the first map is the equivalence of 7.31, the second map applies $p_!$, and the last map composes with the canonical equivalence $p_! p^* R \to \Sigma_+^\infty X \wedge R$ of (7.16). $\qquad\square$

7.7. ∞-category of R-modules from model categories of spectra. Our ∞-categorical treatment of Thom spectra has been based on the the symmetric monoidal ∞-category of spectra developed in [DAGI, DAGII, DAGIII]. We associate to an algebra R in $\operatorname{Stab}(\operatorname{Gpd}_\infty)$ the ∞-category R-line, and then a Thom spectrum functor

$$(\operatorname{Gpd}_\infty)_{/R\text{-line}} \to R\text{-mod}.$$

In §5 we associate to an EKMM S-algebra R the space BGL_1R, and our Thom spectrum construction may be viewed as a functor

$$\mathscr{T}_{/BGL_1R} \to \mathscr{M}_R.$$

In §8, we will compare these two approaches to Thom spectra to each other and to Thom spectra in the literature. To prepare for this comparison, we show here how to set up our ∞-categorical analysis, starting from a (symmetric) monoidal simplicial model category of spectra, such as the S-modules of [EKMM96] or the symmetric spectra of [HSS00].

We first associate to an algebra R in a simplicial monoidal ∞-category of spectra an ∞-category R-line of R-lines, as above, and we show that R-line has the homotopy type of BGL_1R. It is straightforward then to construct a Thom spectrum functor

$$(\mathrm{Gpd}_\infty)_{/R\text{-line}} \to R\text{-mod},$$

as above.

We also show that there is an algebra R' in $\mathrm{Stab}(\mathrm{Gpd}_\infty)$ such that

$$R\text{-mod} \simeq R'\text{-mod},$$

and so

$$(\mathrm{Gpd}_\infty)_{/R\text{-line}} \simeq (\mathrm{Gpd}_\infty)_{/R'\text{-line}},$$

in such a way that the two ∞-categorical Thom spectrum functors

$$(\mathrm{Gpd}_\infty)_{/R\text{-line}} \to R\text{-mod}$$

and

$$(\mathrm{Gpd}_\infty)_{/R'\text{-line}} \to R'\text{-mod}$$

are obviously equivalent.

Let $\mathcal{M}$ be a symmetric monoidal simplicial model category of spectra such as the S-modules of [EKMM96], or the symmetric spectra of [HSS00], and let R be a cofibrant and fibrant associative algebra in $\mathcal{M}$. We require R to be cofibrant and fibrant so that the derived spaces of self-equivalences of R has the homotopy type of GL_1R; see Proposition 6.2.

Let $R\text{-mod}_\mathcal{M}$ be the simplicial model category of R-modules. From $R\text{-mod}_\mathcal{M}$ we form the ∞-category R-mod in the usual way, as we now recall. If C is a simplicial model category, let C° denote the full subcategory of C consisting of cofibrant-fibrant objects. Now take the simplicial nerve [HTT, 1.1.5.5] to obtain the ∞-category

$$R\text{-mod} \overset{\mathrm{def}}{=} \mathrm{N}R\text{-mod}_\mathcal{M}^\circ.$$

For cofibrant-fibrant L and M, the mapping spaces $R\text{-mod}_\mathcal{M}(L, M)$ are Kan complexes, and it follows [HTT, 1.1.5.9] that R-mod is an ∞-category.

Once we have R-mod, we can as in Definition 7.18 define an R-line to be an R-module L which admits a weak equivalence

$$L \to R,$$

and let R-line be the maximal ∞-subgroupoid of R-mod whose objects are the cofibrant-fibrant R-lines. Equivalently, we can define $R\text{-line}_\mathcal{M}$ to be the subcategory of $R\text{-mod}_\mathcal{M}$ in which the objects are R-lines M, and in which the space of morphisms from L to M is the subspace of $R\text{-mod}_\mathcal{M}(L, M)$ consisting of weak equivalences

$$L \overset{\simeq}{\to} M.$$

We can then set

$$R\text{-line}_T^\circ \overset{\mathrm{def}}{=} R\text{-mod}_T^\circ \cap R\text{-line}_T,$$

and

$$R\text{-line} \cong \mathrm{N}R\text{-line}_T^\circ.$$

Explicitly, the set of n-simplices of $\mathrm{N}R\text{-line}_T^\circ$ is the set of simplicial functors

$$\mathfrak{C}[\Delta^n] \to R\text{-line}_T^\circ.$$

In particular,

(1) $R\text{-line}_0$ consists of fibrant-cofibrant R-modules which are R-lines;
(2) $R\text{-line}_1$ consists of the R-module weak equivalences

$$L \to M;$$

(3) R-line$_2$ consists of diagrams (not necessarily commutative)

$$
\begin{array}{ccc}
L & \xrightarrow{\ f\ } & M \\
& {}_{h}\searrow & \downarrow{}^{g} \\
& & N,
\end{array}
$$

together with a path in R-line$_{\mathcal{M}}(L, N)$ from gf to h.

This is precisely the sort of data we proposed in (1.1) as the transitions functions for a bundle of free rank-one R-modules.

Recall from [HTT, 1.2.2] and §7.2 that if Y is any ∞-category and $a, b \in Y$, then the space of maps $Y(a, b) = \mathrm{map}_Y(a, b)$ from a to b is a homotopy type; it can be modeled as the ∞-groupoid $\hom_Y(a, b)$ which is the fiber in the diagram of simplicial sets

$$
\begin{array}{ccc}
\hom_Y(a, b) & \longrightarrow & Y^{\Delta^1} \\
\downarrow & & \downarrow \\
(a, b) & \longrightarrow & Y^{\partial\Delta^1}.
\end{array}
\tag{7.35}
$$

If $Y = \mathrm{N}C$ is the nerve of a simplicial category C, then [HTT, Theorem 1.1.5.12] implies that

$$
\hom_Y(a, b) \simeq \hom_C(a, b).
\tag{7.36}
$$

Taking $Y = R$-mod $= \mathrm{N}R\text{-mod}^\circ_{\mathcal{M}}$ and recalling that objects of R-mod are cofibrant-fibrant R-modules, we see that R-mod(L, M) has the homotopy type of the *derived* space of R-module maps from M to N. In view of the equivalence (7.36), we shall adopt the convention that for arbitrary R-modules M and N (not necessarily cofibrant-fibrant), R-mod(M, N) will mean the associated derived mapping space.

We note that R itself may not be a cofibrant-fibrant R-module, and we fix a cofibrant-fibrant replacement R°. By construction, R-line is a connected ∞-groupoid, and so equivalent to the full ∞-subgroupoid $B\,\mathrm{Aut}(R^\circ)$ with one object R° and morphisms

$$
R\text{-line}(R^\circ, R^\circ) \simeq \ (\text{by convention})\ R\text{-line}(R, R),
$$

the ∞-groupoid of derived self-equivalences of R°. This is the mapping space we studied in §6, and in this setting Proposition 6.2 becomes the following.

Lemma 7.37. *Suppose that R is a cofibrant and fibrant algebra in $\mathcal{M}$. Let L and M be objects of R-line. The inclusion of derived mapping spaces*

$$
R\text{-line}(R, R) = R\text{-line}(R^\circ, R^\circ) \to R\text{-mod}(R^\circ, R^\circ) = R\text{-mod}(R, R)
$$

is a model for the inclusion

$$
GL_1 R \to \Omega^\infty R.
$$

With the ∞-category R-line in place, we can develop the theory of Thom spectra as in §7.5 and 7.6; we do a little of this in §B. For now, we define a *trivialization* of an R-line L to be an equivalence of R-modules

$$
L \to R^\circ,
$$

and we define the ∞-category R-triv of trivialized R-lines to be the slice category

$$
R\text{-triv} = R\text{-line}_{/R^\circ}.
$$

Proposition 7.38. *The forgetful map*

$$
R\text{-triv} \to R\text{-line}
$$

is a Kan fibration. Indeed this fibration is a model for

$$
* \simeq EGL_1 R \to BGL_1 R.
$$

Proof. R-triv is the simplicial set of paths ending at R° in the ∞-groupoid (Kan complex) R-line. In particular R-triv is a contractible Kan complex, and the map

$$R\text{-triv} \to R\text{-line}$$

is a Kan fibration; by the discussion of (7.35) and Lemma 7.37, the fiber is $\mathrm{Aut}(R^\circ) \simeq R\text{-line}(R, R) \simeq GL_1 R$. $\qquad\square$

To see that the resulting theory of Thom spectra agrees with the theory developed in sections 7.5 and 7.6 amounts to comparing notions of R-mod. More precisely, we'd like to know that the ∞-category of algebras in $\mathcal{M}$ is equivalent to $\mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty))$, and that if R is an algebra in $\mathcal{M}$ corresponding to an algebra R' in $\mathrm{Stab}(\mathrm{Gpd}_\infty)$, then the ∞-categories of R-modules and R'-modules are equivalent.

The work of [MMSS01, MM02, Sch01, EKMM96] establishes the necessary Quillen equivalences for the model categories of spectra of Lewis-May-Steinberger, S-modules of [EKMM96], and symmetric and orthogonal spectra [MMSS01, HSS00]. In [DAGI, DAGII], Lurie relates the ∞-category $\mathrm{Stab}(\mathrm{Gpd}_\infty)$ to symmetric spectra.

Let $\mathcal{S}_\Sigma$ denote the category of symmetric spectra. Example 8.21 of [DAGIII] establishes an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Stab}(\mathrm{Gpd}_\infty) \simeq \mathrm{N}\mathcal{S}_\Sigma^\circ, \tag{7.39}$$

which by [DAGII, Example 1.6.14] induces an equivalence

$$\theta : \mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty)) \simeq \mathrm{N}\,\mathrm{Alg}(\mathcal{S}_\Sigma)^\circ$$

of ∞-categories of algebras.

Let R be an algebra in $\mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty))$, so $\theta(R)$ is a cofibrant-fibrant algebra in $\mathcal{S}_\Sigma$. Then we have R-mod as in §7.4, and may construct

$$\theta(R)\text{-mod} \stackrel{\mathrm{def}}{=} \mathrm{N}(\theta(R)\text{–modules in } \mathcal{S}_\Sigma)^\circ,$$

as above. According to Theorem 2.5.3 of [DAGII] we have an equivalence of ∞-categories

$$R\text{-mod} \simeq \theta(R)\text{-mod}, \tag{7.40}$$

as desired.

We summarize the discussion as we shall use it in §8. Let $\mathcal{M}$ be any simplicial symmetric monoidal model category of spectra, connected by a zig-zag of Quillen equivalences to $\mathcal{S}_\Sigma$. Suppose that $\mathcal{M}$ has a theory of algebras and modules similarly equivalent to that of $\mathcal{S}_\Sigma$. If follows that there are equivalences of ∞-categories

$$\mathrm{N}\mathcal{M}^\circ \xrightarrow{\simeq} \mathrm{Stab}(\mathrm{Gpd}_\infty)$$

$$\mathrm{N}\,\mathrm{Alg}(\mathcal{M})^\circ \xrightarrow{\simeq} \mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty)).$$

Moreover, let R' be a cofibrant and fibrant algebra in $\mathcal{M}$, and let R a corresponding algebra in $\mathrm{Stab}(\mathrm{Gpd}_\infty)$. If X is a space, let $\Pi_\infty X$ denote its singular complex (an ∞-groupoid). Then according to Proposition 7.38 and (7.40), we have equivalences

$$\Pi_\infty(BGL_1 R') \simeq R'\text{-line} \simeq R\text{-line},$$

and so we may regard a map of spaces $f : X \to BGL_1 R'$ equivalently as a map of ∞-groupoids

$$\Pi_\infty f : \Pi_\infty X \to \Pi_\infty(BGL_1 R') \simeq R\text{-line}.$$

8. Morita theory and Thom spectra

In this section we interpret the construction of the Thom spectrum from the perspective of Morita theory. This viewpoint is implicit in the "algebraic" definition of the Thom spectrum in Section 5 as the derived smash product

$$Mf \stackrel{\text{def}}{=} \Sigma_+^\infty P \wedge_{\Sigma_+^\infty GL_1 R}^L R,$$

where R is the $\Sigma_+^\infty GL_1 R$-R bimodule specified by the canonical action of $\Sigma_+^\infty GL_1 R$ on R. Recalling that the target category of R-modules is stable, we can regard the Thom spectrum as essentially given by a functor from (right) $\Sigma_+^\infty GL_1 R$-modules to R-modules.

Now, roughly speaking, Morita theory (more precisely, the Eilenberg-Watts theorem) implies that any continuous functor from (right) $\Sigma_+^\infty GL_1 R$-modules to (right) R-modules which preserves homotopy colimits and takes $GL_1 R$ to R can be realized as tensoring with an appropriate $(\Sigma_+^\infty GL_1 R)$-R bimodule. In particular, this tells us that the Thom spectrum functor is characterized amongst such functors by the additional data of the action of $GL_1 R$ on R.

Put another way, a continuous homotopy-colimit preserving functor

$$T \colon \Sigma_+^\infty GL_1 R\text{-mod} \to R\text{-mod}$$

can be determined to be the Thom spectrum functor by inspection of the induced bimodule structure on $R \simeq T(\Sigma_+^\infty GL_1 R)$ obtained from the map

$$\Sigma_+^\infty GL_1 R \longrightarrow F_{\Sigma_+^\infty GL_1 R}(\Sigma_+^\infty GL_1 R, \Sigma_+^\infty GL_1 R) \stackrel{T}{\longrightarrow} F_R(T(\Sigma_+^\infty GL_1 R), T(\Sigma_+^\infty GL_1 R)).$$

By the adjunction characterizing $GL_1 R$, this data is equivalent to specifying a multiplicative map $GL_1 R \to GL_1 R$ (or equivalently a map $BGL_1 R \to BGL_1 R$).

Beyond its conceptual appeal, this viewpoint on the Thom spectrum functor provides the basic framework for comparing the different constructions which we have discussed in this paper (both with one another, and with the "neo-classical" construction of Lewis and May and the parametrized construction of May and Sigurdsson). On the face of it, the definition of the Thom spectrum in Section 7 as the functor which sends $f : X \to R$-line to

$$Mf \stackrel{\text{def}}{=} p_! f^* \mathscr{L},$$

where $\mathscr{L}$ is the universal parametrized spectrum over R-line, has a rather different ("geometric") character. In fact, however, these definitions are essentially the same, under the equivalence between parametrized spectra and modules.

After discussing the analogue of the classical Eilenberg-Watts theorem in the context of ring spectra in Subsection 8.1, we work out a version of this classification in the setting of parametrized spectra in Subsection 8.2, using the techniques made available by the ∞-categorical perspective. Here it turns out to be possible to use the connections between modules, parametrized spectra, and presheaves to obtain a classification of colimit-preserving functors between ∞-categories that specializes to imply that a colimit-preserving functor from the infinity category of spaces over $BGL_1 R$ to R-modules is determined by a map $BGL_1 R \to BGL_1 R$.

This ∞-category machinery then provides the foundation of the comparison of the different definitions of Thom spectra in the paper. As a consistency check, we verify that the classification obtained in Subsection 8.2 agrees with that of the previous subsection; the data of a map $BGL_1 R \to BGL_1 R$ determines the same functor in each framework. We then employ this axiomatic perspective to show that the "algebraic" Thom spectrum functor induces a functor on ∞-categories which agrees with the "geometric" definition. We also give a direct argument for the comparison between the two definitions of the Thom spectrum functor; although this argument appears to skirt the underlying Morita theory, we believe it provides a useful concrete depiction of the situation.

Finally, the close relationship between our ∞-categorical construction of the Thom spectrum and the definition of May and Sigurdsson [MS06, 23.7.1,23.7.4] allows us to compare that construction (and by extension the "neo-classical" Lewis-May construction) to the ones discussed herein.

8.1. **The Eilenberg-Watts theorem for categories of module spectra.** The key underpinning of classical Morita theory is the Eilenberg-Watts theorem, which for rings A and B establishes an equivalence between the category of colimit-preserving functors A-mod $\to B$-mod and the category of (A, B)-bimodules. The proof of the theorem proceeds by observing that any functor $T\colon A$-mod $\to B$-mod specifies a bimodule structure on TA with the A-action given by the composite

$$A \to F_A(A, A) \to F_B(TA, TA).$$

It is then straightforward to check that the functor $- \otimes_A TA$ is isomorphic to the functor T, using the fact that both of these functors preserve colimits.

In this section, we discuss the generalization of this result to the setting of categories of module spectra. The situation here is more complicated than in the setting of rings; for instance, it is well-known that there are equivalences between categories of module spectra which are not given by tensoring with bimodules, and there are similar difficulties with the most general possible formulation of the Eilenberg-Watts theorem. However, much of the subtlety here comes from the fact that unlike in the classical situation, compatibility with the enrichment in spectra is not automatic (see for example the excellent recent paper of Johnson [Jo08] for a comprehensive discussion of the situation). By assuming our functors are enriched, we can recover a close analogue of the classical result.

Let A and B be (cofibrant) S-algebras, and let T be an enriched functor $T\colon A$-mod $\to B$-mod. Specifically, we assume that T induces a map of function spectra $F_A(X, Y) \to F_B(TX, TY)$. Furthermore, assume that T preserves tensors (and in particular is homotopy-preserving) and preserves homotopy colimits. For instance, these conditions are satisfied if T is a Quillen left-adjoint. The assumption that T is homotopy-preserving in particular means that T preserves weak equivalences between cofibrant objects and so admits a total left-derived functor $T^L\colon \text{ho } A$-mod $\to \text{ho } B$-mod. Furthermore, $T(A)$ is an A-B bimodule with the bimodule structure induced just as above.

Using the work of [Jo08] and an elaboration of the arguments of [SS04, 4.1.2] (see also [Sch04, 4.20]) we now can prove the following Eilenberg-Watts theorem in this setting. We will work in the EKMM categories of modules, so we can assume that all objects are fibrant.

Proposition 8.1. *Given the hypotheses of the preceding discussion, there is a natural isomorphism in* ho B-mod *between the total left-derived functor* $T^L(-)$ *and the derived smash product* $(-) \wedge^L T(A)$, *regarding* $T(A)$ *as a bimodule as above.*

Proof. By continuity, there is a natural map of B-modules

$$(-) \wedge_A T(A) \to T(-).$$

Let T' denote a cofibrant replacement of $T(A)$ as an A-B bimodule. Since the functor $(-) \wedge_A T'$ preserves weak equivalences between cofibrant A-modules, there is a total left-derived functor $(-) \wedge_A^L T'$ which models $(-) \wedge_A^L T(A)$. Thus, the composite

$$(-) \wedge_A T' \to (-) \wedge_A T(A) \to T(-).$$

descends to the homotopy category to produce a natural map

$$(-) \wedge_A^L T(A) \to T^L(-).$$

The map is clearly an equivalence for the free A-module of rank one; i.e. A. Since both sides commute with homotopy colimits, we can inductively deduce that the first map is an equivalence for all cofibrant A-modules, and this implies that the map of derived functors is an isomorphism. $\square$

We now specialize to the case of Thom spectra. Recall that for an S-algebra R, GL_1R is an A_∞ space and hence a monoid in the category of $*$-modules. As such, there is a well-defined notion of a GL_1R-module (Definition 5.22) and moreover we defined the bar constructions $B_\mathcal{L}GL_1R$ and $E_\mathcal{L}GL_1R$ such that the map $E_\mathcal{L}GL_1R \to B_\mathcal{L}GL_1R$ models the universal quasifibration (Section 5.4).

Given a fibration of $*$-modules $f\colon X \to B_\mathcal{L}GL_1R$, we first took the pullback of the diagram,

$$X \longrightarrow B_\mathcal{L}GL_1R \longleftarrow E_\mathcal{L}GL_1R$$

to obtain a GL_1R-module P. This procedure defines a functor from $*$-modules over $B_\mathcal{L}GL_1R$ to GL_1R-modules; since we are assuming f is a fibration, we are computing the derived functor. Next, since the category R-mod is stable, a continuous functor from GL_1R-modules to R-mod factors through the stabilization $\Sigma_+^\infty GL_1R$-mod. That is, up to equivalence a continuous functor from GL_1R-modules to R-modules is determined by a continuous functor from $\Sigma_+^\infty GL_1R$-modules to R-modules, which is to say a $(\Sigma_+^\infty GL_1R, R)$-bimodule.

One might like to deduce a characterization of the Thom spectrum functor as a functor from $\mathscr{T}/BGL_1R$ to R-modules from Proposition 8.1. However, it turns out to be technically involved to state such a theorem precisely, because the derived Thom spectrum functor as we have constructed it "algebraically" is presented as a composition of a right derived functor (which is an equivalence) and a left derived functor. We remark that much of the technical difficulty in the neo-classical theory of the Thom spectrum functor arises from the difficulties involved in dealing with point-set models of such composites.

Fortunately, this is the kind of formal situation that the ∞-category framework handles well. The functor $f \mapsto P$ induces an equivalence of homotopy categories; by Theorem 5.32, it is an equivalence of enriched categories. In particular, it induces an equivalence of ∞-categories

$$\mathrm{N}\mathscr{M}^\circ_{*/B_\mathcal{L}GL_1R} \simeq \mathrm{N}\mathscr{M}^\circ_{GL_1R};$$

in the ∞-category setting, this is an instance of Proposition 7.8. In light of the above discussion, the abstract characterization of the Thom spectrum functor is immediate.

Proposition 8.2. *Let $T : GL_1R\text{-mod} \to R\text{-mod}$ be a continuous, colimit-preserving functor which sends GL_1R to an R-module R' homotopy equivalent to R in such a way that*

$$GL_1R \simeq \mathrm{hEnd}_{GL_1R\text{-mod}}(GL_1R) \longrightarrow \mathrm{hEnd}_{R\text{-mod}}(R') \simeq \mathrm{hEnd}_{R\text{-mod}}(R)$$

is homotopy equivalent to the inclusion $GL_1R \simeq \mathrm{hAut}(R) \to \mathrm{hEnd}(R)$. Then T^L, the left-derived functor of T, is homotopy equivalent to $\Sigma_+^\infty(-) \wedge^L_{\Sigma_+^\infty GL_1R} R : GL_1R\text{-mod} \to R\text{-mod}$.

Proof. The stability of R-mod and Proposition 8.1 together imply that T^L is homotopy equivalent to $\Sigma_+^\infty(-) \wedge^L_{\Sigma_+^\infty GL_1R} B$ for some $(\Sigma_+^\infty GL_1R, R)$-bimodule B. Since $T(\Sigma_+^\infty GL_1R) \simeq R$, we must have $B \simeq R$; since the left action of GL_1R on itself induces (via the equivalence $R' \simeq R$) the canonical action of GL_1R on R, we conclude that $B \simeq R$ as $(\Sigma_+^\infty GL_1R, R)$-bimodules. $\qquad\square$

8.2. **Colimit-preserving functors from the ∞-categorical perspective.** We now switch to the context of ∞-categories. In this section we develop a general description of functors between ∞-categories which preserve colimits. Specializing to module categories, we will recover the version of the Eilenberg-Watts theorem which forms the basis of the comparison of the various Thom spectrum functors discussed in this paper. The setting of ∞-categories turns out to be technically felicitous for performing the general comparisons we need; our descriptions will arise from elementary considerations of cocomplete ∞-categories, the relationship between modules, parametrized objects, and presheaves, and stabilization.

We begin by considering cocomplete ∞-categories. Let $\mathscr{C}$ be a small ∞-category, and consider the ∞-topos

$$\mathrm{Pre}(\mathscr{C}) := \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty)$$

of presheaves of ∞-groupoids on $\mathscr{C}$. It comes equipped with a fully faithful functor

$$\mathscr{C} \longrightarrow \mathrm{Pre}(\mathscr{C}), \tag{8.3}$$

the Yoneda embedding, which (as we discussed in Remark 7.11) is only really defined up to a contractible space of choices.

The fact that $\mathrm{Pre}(\mathscr{C})$ is equivalent to the nerve of the simplicial model category of simplicial presheaves on $\mathfrak{C}[\mathscr{C}]$ (equipped, say, with the projective model structure) implies that $\mathrm{Pre}(\mathscr{C})$ is cocomplete as an ∞-category. Furthermore, just like in ordinary category theory, the Yoneda embedding $\mathscr{C} \to \mathrm{Pre}(\mathscr{C})$ is in some sense initial among cocomplete ∞-categories under $\mathscr{C}$.

Lemma 8.4 ([HTT, 5.1.5.6])**.** *For any cocomplete ∞-category $\mathscr{D}$, precomposition with the Yoneda embedding induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Pre}(\mathscr{C}), \mathscr{D}) \longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{D}), \tag{8.5}$$

where $\mathrm{Fun}^{\mathrm{colim}}(-, -)$ denotes the full ∞-subcategory of $\mathrm{Fun}(-, -)$ on the colimit-preserving functors.

We shall be particularly interested in the case that $\mathscr{C}$ is an ∞-groupoid, so that

$$\mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Gpd}_{\infty}) \simeq \mathrm{Gpd}_{\infty/\mathscr{C}},$$

as in Remark 7.9. Suppose given a functor $g : \mathscr{C} \to \mathscr{D}$. To an ∞-groupoid over $\mathscr{C}$

$$f : X \to \mathscr{C},$$

we can associate the colimit

$$\mathrm{colim}\, g \circ f : X \to \mathscr{C} \to \mathscr{D}.$$

Insofar as we have a functorial model for this colimit (by making use, say, of functorial homotopy colimits in simplicial categories), then this procedure determines a colimit-preserving functor

$$\mathrm{Gpd}_{\infty/\mathscr{C}} \simeq \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Gpd}_{\infty}) \simeq \mathrm{Pre}(\mathscr{C}) \to \mathscr{D},$$

with the property that its restriction along the Yoneda embedding

$$\mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Gpd}_{\infty})$$

is equivalent to g. This is just a restatement of Proposition 7.12.

Corollary 8.6. *Any inverse $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Pre}(\mathscr{C}), \mathscr{D})$ of the equivalence (8.5), given by restriction along the Yoneda embedding $\mathscr{C} \to \mathrm{Gpd}_{\infty/\mathscr{C}}$, sends g to a colimit-preserving functor whose value on $f : X \to \mathscr{C}$ is equivalent to $\mathrm{colim}\, g \circ f$.* $\square$

Lastly, we specialize to the context of stable categories.

Lemma 8.7 ([DAGI], Corollary 17.5)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be presentable ∞-categories such that $\mathscr{D}$ is stable. Then*

$$\Omega^{\infty}_{-} : \mathrm{Stab}(\mathscr{C}) \longrightarrow \mathscr{C}$$

admits a left adjoint

$$\Sigma^{\infty}_{+} : \mathscr{C} \longrightarrow \mathrm{Stab}(\mathscr{C}),$$

and precomposition with the Σ^{∞}_{+} induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Stab}(\mathscr{C}), \mathscr{D}) \longrightarrow \mathrm{Fun}^{\mathrm{colim}}(\mathscr{C}, \mathscr{D}).$$

Combining the universal properties of stabilization and the Yoneda embedding, we obtain the following equivalence of ∞-categories.

Corollary 8.8. *Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories such that $\mathscr{D}$ is stable and cocomplete. Then there are equivalences of ∞-categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Stab}(\mathrm{Pre}(\mathscr{C})), \mathscr{D}) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Pre}(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$

Proof. This follows from the last two lemmas. $\square$

Now suppose that $\mathscr{C}$ and $\mathscr{D}$ have distinguished objects, given by maps $* \to \mathscr{C}$ and $* \to \mathscr{D}$ from the trivial ∞-category $*$. Then $\mathrm{Pre}(\mathscr{C})$ and $\mathrm{Stab}(\mathrm{Pre}(\mathscr{C}))$ inherit distinguished objects via the maps

$$* \longrightarrow \mathscr{C} \xrightarrow{\ i\ } \mathrm{Pre}(\mathscr{C}) \xrightarrow{\Sigma^\infty} \mathrm{Stab}(\mathrm{Pre}(\mathscr{C})),$$

where i denotes the Yoneda embedding. Note that the fiber sequence

$$\mathrm{Fun}_{*/}(\mathscr{C}, \mathscr{D}) \longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \longrightarrow \mathrm{Fun}(*, \mathscr{D}) \simeq \mathscr{D}$$

shows that the ∞-category of pointed functors is equivalent to the fiber of the evaluation map $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ over the distinguished object of $\mathscr{D}$.

Proposition 8.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories with distinguished objects such that $\mathscr{D}$ is stable and cocomplete. Then there are equivalences of ∞-categories*

$$\mathrm{Fun}_{*/}^{\mathrm{colim}}(\mathrm{Stab}(\mathrm{Pre}(\mathscr{C})), \mathscr{D}) \simeq \mathrm{Fun}_{*/}^{\mathrm{colim}}(\mathrm{Pre}(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}_{*/}(\mathscr{C}, \mathscr{D}).$$

Proof. Take the fiber of $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Stab}(\mathrm{Pre}(\mathscr{C})), \mathscr{D}) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Pre}(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ over $* \to \mathscr{D}$. □

Corollary 8.10. *Let G be a group-like monoidal ∞-groupoid G, let BG be a one-object ∞-groupoid with $G \simeq \mathrm{Aut}_{BG}(*)$, and let $\mathscr{D}$ be a stable and cocomplete ∞-category with a distinguished object $*$. Then*

$$\mathrm{Fun}_{*/}^{\mathrm{colim}}(\mathrm{Stab}(\mathrm{Pre}(BG)), \mathscr{D}) \simeq \mathrm{Fun}_{*/}^{\mathrm{colim}}(\mathrm{Pre}(BG), \mathscr{D}) \simeq \mathrm{Fun}_{*/}(BG, \mathscr{D}) \simeq \mathrm{Fun}(BG, B\,\mathrm{Aut}_{\mathscr{D}}(*));$$

that is, specifying an action of G on the distinguished object $*$ of $\mathscr{D}$ is equivalent to specifying a pointed colimit-preserving functor from $\mathrm{Pre}(BG)$ (or its stabilization) to $\mathscr{D}$.

Proof. A base-point preserving functor $BG \to \mathscr{D}$ necessarily factors through $B\,\mathrm{End}_{\mathscr{D}}(*)$, and hence also through $B\,\mathrm{Aut}_{\mathscr{D}}(*)$ since BG is an ∞-groupoid. □

Note that the ∞-category $\mathrm{Fun}(BG, B\,\mathrm{Aut}_{\mathscr{D}}(*))$ is actually an ∞-groupoid, as $B\,\mathrm{Aut}_{\mathscr{D}}(*)$ is an ∞-groupoid.

Putting this all together, consider the case in which the target ∞-category $\mathscr{D}$ is the ∞-category of right R-modules for an associative S-algebra R, pointed by the free rank one R-module R. Then $\mathrm{Aut}_{\mathscr{D}}(*) \simeq GL_1R$, so the space of pointed colimit-preserving maps from the ∞-category of spaces over BG to the ∞-category of R-modules is equivalent to the space of monoidal maps from G to GL_1R.

8.3. **∞-categorical Thom spectra, revisited.** In this section, we return to the definition of Thom spectra from §7 and interpret that construction in light of the work of the previous subsections. Let R be an algebra in $\mathrm{Stab}(\mathrm{Gpd}_\infty)$, and form the ∞-categories R-mod and R-line. Given a map of ∞-groupoids

$$f : X \to R\text{-line},$$

the Thom spectrum is the push-forward of the restriction to X of the tautological R-line bundle $\mathscr{L}$; that is,

$$Mf = p_! f^* \mathscr{L}.$$

Let q denote the projection to the point from R-line, so that $p = q \circ f$ and we may therefore rewrite Mf as

$$Mf \simeq q_! f_! f^* \mathscr{L} \simeq q_!(\Sigma_{R\text{-line}}^\infty f_+ \wedge_{R\text{-line}} \mathscr{L}),$$

where the second equivalence follows from formula (7.17). This exhibits M as a composite of left adjoints: $\Sigma_{R\text{-line}}^\infty(-)_+$ followed by $(-) \wedge_{R\text{-line}} \mathscr{L}$ followed by $q_!$. In particular, (the ∞-categorical) M itself is a left adjoint, so it preserves (∞-categorical) colimits.

Proposition 8.11. *The restriction of $M : \mathrm{Gpd}_{\infty/R\text{-line}} \to R$-mod along the Yoneda embedding*

$$R\text{-line} \longrightarrow \mathrm{Fun}(R\text{-line}^{op}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/R\text{-line}}$$

is equivalent to the tautological inclusion

$$R\text{-line} \longrightarrow R\text{-mod}. \tag{8.12}$$

Proof. Consider the colimit-preserving functor $\mathrm{Gpd}_{\infty/R\text{-line}} \to R\text{-mod}$ induced by the canonical inclusion R-line $\to R$-mod. As we explain in Corollary 8.6, it sends $X \to R$-line to the colimit of the composite $X \to R$-line $\to R$-mod. As we explain in Remark 7.28, this is equivalent to the Thom spectrum functor M. $\qquad\square$

The following corollary is now an immediate consequence of the analysis of the previous subsection.

Corollary 8.13. *A functor* $\mathrm{Gpd}_{\infty/R\text{-line}} \to R\text{-mod}$ *is equivalent to* M *if and only if it preserves colimits and its restriction along the Yoneda embedding* R-line $\to \mathrm{Fun}(R\text{-line}^{op}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/R\text{-line}}$ *is equivalent to the tautological inclusion* (8.12) *of* R-line *into* R-mod.

8.4. **Comparing notions of Thom spectrum.** In this section, we show that, on underlying ∞-categories, the "algebraic" Thom R-module functor is equivalent to the "geometric" Thom spectrum functor via the characterization of Corollary 8.13. The proof of this result also implies that the two versions of the Eilenberg-Watts theorem we have discussed in this section agree up to natural weak equivalence.

Let $\mathscr{M}_S$ be the category of EKMM S-modules. According to the discussion in §7.7, there is an equivalence of ∞-categories

$$\mathrm{N}\mathscr{M}_S^\circ \simeq \mathrm{Stab}(\mathrm{Gpd}_\infty) \tag{8.14}$$

which induces an equivalence of ∞-categories of algebras

$$\mathrm{N}\,\mathrm{Alg}(\mathscr{M}_S)^\circ \simeq \mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty)). \tag{8.15}$$

Let R be a cofibrant-fibrant EKMM S-algebra, and let R' be the corresponding algebra in $\mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty))$. The equivalence (8.14) induces an equivalence of ∞-groupoids

$$\mathrm{N}R\text{-mod}^\circ \simeq R'\text{-mod}. \tag{8.16}$$

Proposition 7.38 gives an equivalence of ∞-groupoids

$$\Pi_\infty BGL_1 R \simeq \mathrm{N}((R\text{-line})^\circ) \tag{8.17}$$

and so putting (8.16) and (8.17) together we have an equivalence of ∞-categories

$$\mathrm{N}((\mathscr{T}/BGL_1 R)^\circ) \simeq \mathrm{Gpd}_{\infty/R'\text{-line}}\,.$$

Corollary 8.18. *The functor*

$$\mathrm{Gpd}_{\infty/R'\text{-mod}} \simeq \mathrm{N}(\mathscr{T}/BGL_1 R)^\circ \xrightarrow{\mathrm{N}M} \mathrm{N}(R\text{-mod})^\circ \simeq R'\text{-mod},$$

obtained by passing the Thom R-module functor of §5 though the indicated equivalences, is equivalent to the Thom R'-module functor of §7.

Proof. Let $\mathscr{C}$ denote the topological category with a single object $*$ and $\mathrm{map}_\mathscr{C}(*, *) = GL_1 R = \mathrm{Aut}_R(R^\circ) \simeq \mathrm{Aut}_{R'}(R')$. Note that $\mathscr{C}$ is naturally a topological subcategory of $GL_1 R$-mod (the full topological subcategory of $GL_1 R$) and by definition a topological subcategory of R-mod. Note also that

$$\mathrm{N}\mathscr{C} \simeq B\,\mathrm{Aut}(R') \simeq R'\text{-line}.$$

According to Proposition 8.2, the continuous functor

$$T^L : GL_1 R\text{-mod} \longrightarrow R\text{-mod}$$

has the property that its restriction to $\mathscr{C}$ is equivalent to the inclusion of the topological subcategory $\mathscr{C} \to R$-mod. Taking simplicial nerves, and recalling that

$$\mathrm{N}GL_1 R\text{-mod}^\circ \simeq \mathrm{N}(\mathscr{T}/BGL_1 R)^\circ \simeq \mathrm{Fun}(\mathrm{N}\mathscr{C}^{op}, \mathrm{Gpd}_\infty)GL_1 R\text{-mod},$$

we see that

$$\mathrm{N}T^L : \mathrm{Fun}(\mathrm{N}\mathscr{C}^{op}, \mathrm{Gpd}_\infty) \simeq \mathrm{N}GL_1 R\text{-mod}^\circ \longrightarrow \mathrm{N}R\text{-mod}^\circ \simeq R'\text{-mod}$$

is a colimit-preserving functor whose restriction along the Yoneda embedding $\mathrm{N}\mathscr{C} \to \mathrm{Fun}(\mathrm{N}\mathscr{C}^{op}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/R'\text{-line}}$ is equivalent to the inclusion of the ∞-subcategory $\mathrm{N}\mathscr{C} \simeq R'$-line $\to R'$-mod. It therefore follows from Proposition 8.13 that $\mathrm{N}T^L$ is equivalent to the "geometric" Thom spectrum functor of §7. $\qquad\square$

Remark 8.19. The argument also implies the following apparently more general result. Recall from 8.2 that *any* map $k : BGL_1R \to BGL_1R$ defines a functor from the ∞-category of spaces over BGL_1R to the ∞-category of R-modules, defined by sending $f : X \to BGL_1R$ to the colimit of the composite

$$X \xrightarrow{f} BGL_1R \xrightarrow{k} BGL_1R \to R\text{-mod.} \tag{8.20}$$

On the other hand, according to Proposition 8.24 below, we can describe the derived smash product from section 8.1 associated to k as the colimit of the composite

$$X \xrightarrow{f} BGL_1R \xrightarrow{k} BGL_1R \xrightarrow{\Sigma^\infty_+} \Sigma^\infty_+ GL_1R\text{-mod} \xrightarrow{(-)\wedge_{\Sigma^\infty_+ GL_1R}R} R\text{-mod.}$$

Since both functors are given by the formula $M(k \circ f)$, the Thom R-module of f composed with k, we conclude that these two procedures are equivalent for any k, not just the identity.

8.5. **The algebraic Thom spectrum functor as a colimit.** We sketch another approach to the comparison of the "geometric" and "algebraic" Thom spectrum functors. This approach has the advantage of giving a direct comparison of the two functors. It has the disadvantage that it does not characterize the Thom spectrum functor among functors

$$\mathscr{T}/BGL_1R \to R\text{-mod,}$$

and it does not exhibit the conceptual role played by Morita theory. Instead, it identifies both functors as colimits. In this sense it is a direct generalization of the argument we gave in (1.7), for the case of a discrete ring R and a discrete space X.

Suppose that R is a monoid in S-modules. Let R-mod be the associated ∞-category of R-modules, let R-line be the the sub-∞-groupoid of R-lines, and let $j : R\text{-line} \to R\text{-mod}$ denote the inclusion.

Let X be a space. The "geometric" Thom spectrum functor sends a map $f : \Pi_\infty X \to R$-line to

$$\text{colim}(\Pi_\infty X \xrightarrow{\Pi_\infty f} R\text{-line} \xrightarrow{j} R\text{-mod}).$$

As in §5.5, let $G = (GL_1R)^c$. In §7.7 we showed that $\Pi_\infty B_\mathcal{L} G \simeq R$-line. But observe that we also have a natural equivalence

$$\Pi_\infty B_\mathcal{L} G \simeq G\text{-line.}$$

That is, let G-mod $= N\mathscr{M}^\circ_G$ be the ∞-category of G-modules, and let G-line be the maximal ∞-groupoid generated by the G-lines, that is, cofibrant and fibrant G-modules which admit a weak equivalence to G. By construction, G-line is connected, and so equivalent to $B\text{Aut}(G) \simeq B_\mathcal{L} G$.

The construction of the "algebraic Thom spectrum" begins by associating to a fibration of $*$-modules $f : X \to B_\mathcal{L} G$ the G-module P which is the pull-back in

$$\begin{array}{ccc} P & \longrightarrow & E_\mathcal{L} G \\ \downarrow & & \downarrow \\ X & \longrightarrow & B_\mathcal{L} G. \end{array} \tag{8.21}$$

The association $f \mapsto P$ defines a functor

$$\mathscr{M}_{*/B_\mathcal{L} G} \to \mathscr{M}_G$$

which induces an equivalence of homotopy categories; by Theorem 5.32 it is an equivalence of enriched homotopy categories. As discussed in Proposition 7.8, this corresponds to an equivalence of ∞-categories

$$\text{Gpd}_{\infty/G\text{-line}} \simeq G\text{-mod.} \tag{8.22}$$

The key observation is the following. Let $k : G\text{-line} \to G\text{-mod}$ denote the tautological inclusion. To a map of ∞-groupoids

$$f : X \to G\text{-line,}$$

we can associate the G-module

$$P_f = \text{colim}(X \xrightarrow{f} G\text{-line} \xrightarrow{k} G\text{-mod}).$$

Lemma 8.23. *The functor $f \mapsto P_f$ gives the equivalence (8.22).*

In other words, if $f : X \to B_{\mathcal{L}}G$ is a fibration of $*$-modules, then we can form P as in (8.21). Alternatively, we can form

$$\Pi_\infty f : \Pi_\infty X \to \Pi_\infty B_{\mathcal{L}}G \simeq G\text{-mod},$$

and then form $P_{\Pi_\infty f} = \mathrm{colim}(k\Pi_\infty f)$, and then we have an equivalence of G-modules

$$P_{\Pi_\infty f} \simeq P.$$

The proof of the Lemma follows the same lines as Corollary 7.13, which treats the case that G is a group-like monoid in ∞-groupoids.

Proposition 8.24. *Let $f : X \to B_{\mathcal{L}}G$ be a fibration of $*$-modules. The "algebraic" Thom spectrum functor sends f to*

$$\mathrm{colim}(\Pi_\infty X \xrightarrow{\Pi_\infty f} \Pi_\infty B_{\mathcal{L}}G \simeq G\text{-line} \xrightarrow{k} G\text{-mod} \xrightarrow{\Sigma^\infty_+} \Sigma^\infty_+ G\text{-mod} \xrightarrow{\wedge_{\Sigma^\infty_+ G} R} R\text{-mod}).$$

Proof. By the Lemma, we have

$$P \simeq \mathrm{colim}(\Pi_\infty X \xrightarrow{\Pi_\infty f} \Pi_\infty B_{\mathcal{L}}G \simeq G\text{-line} \xrightarrow{k} G\text{-mod}), \tag{8.25}$$

and so

$$\begin{aligned} Mf = \Sigma^\infty_+ P \wedge_{\Sigma^\infty_+ G} R &\simeq \Sigma^\infty_+ \mathrm{colim}(k\Pi_\infty f) \wedge_{\Sigma^\infty_+ G} R \\ &\simeq \mathrm{colim}(\Sigma^\infty_+ k\Pi_\infty f) \wedge_{\Sigma^\infty_+ G} R \\ &\simeq \mathrm{colim}(\Sigma^\infty_+ k\Pi_\infty f \wedge_{\Sigma^\infty_+ G} R). \end{aligned}$$

This last is the colimit in the statement of the Lemma. $\qquad\square$

From this point of view, the coincidence of the two Thom spectrum functors amounts to the fact that diagram

$$\begin{array}{ccccc} B_{\mathcal{L}}G & \longrightarrow & G\text{-line} & \xrightarrow{\ \ k\ \ } & G\text{-mod} \\ & \searrow{\scriptstyle \simeq} & {\scriptstyle \Sigma^\infty_+(-)\wedge_{\Sigma^\infty_+ G}R}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma^\infty_+(-)\wedge_{\Sigma^\infty_+ G}R} \\ & & R\text{-line} & \xrightarrow{\ \ j\ \ } & R\text{-mod} \end{array}$$

evidently commutes.

8.6. **The "neo-classical" Thom spectrum functor.** In this final section we compare the Lewis-May operadic Thom spectrum functor to the Thom spectrum functors discussed in this paper. Since the May-Sigurdsson construction of the Thom spectrum in terms of a parametrized universal spectrum over BGL_1S [MS06][23.7.4] is easily seen to be equivalent to the space-level Lewis-May description, this will imply that all of the known descriptions of the Thom spectrum functor agree up to homotopy. Our comparison proceeds by relating the Lewis-May model to the quotient description of Proposition 7.29.

We begin by briefly reviewing the Lewis-May construction of the Thom spectrum functor; the interested reader is referred to Lewis' thesis, published as Chapter IX of [LMSM86], and the excellent discussion in Chapter 22 of [MS06] for more details and proofs of the foundational results below. Nonetheless, we have tried to make our discussion relatively self-contained.

The starting point for the Lewis-May construction is an explicit construction of GL_1S in terms of a diagrammatic model of infinite loop spaces. Let $\mathscr{I}_c$ be the symmetric monoidal category of finite or countably infinite dimensional real inner product spaces and linear isometries. Define an $\mathscr{I}_c$-space to be a continuous functor from $\mathscr{I}_c$ to spaces. The usual left Kan extension construction gives the diagram category of $\mathscr{I}_c$-spaces a symmetric monoidal structure. It turns out that monoids and commutative monoids for this category model

A_∞ and E_∞ spaces; for technical felicity, we focus attention on the commutative monoids which satisfy two additional properties:

(1) The map $T(V) \to T(W)$ associated to a linear isometry $V \to W$ is a homeomorphism onto a closed subspace.
(2) Each $T(W)$ is the colimit of the $T(V)$, where V runs over the finite dimensional subspaces of W and the maps in the colimit system are restricted to the inclusions.

Denote such a functor as an $\mathscr{I}_c$-FCP (functor with cartesian product) [MS06, 23.6.1]; the requirement that T be a diagrammatic commutative monoid implies the existence of a "Whitney sum" natural transformation $T(U) \times T(V) \to T(U \oplus V)$. This terminology is of course deliberately evocative of the notion of FSP (functor with smash product), which is essentially an orthogonal ring spectrum [MMSS01].

An $\mathscr{I}_c$-FCP gives rise to an E_∞ space structured by the linear isometries operad; specifically, $T(\mathbb{R}_\infty) = \mathrm{colim}_V T(V)$ is an $\mathcal{L}$-space with the operad maps induced by the Whitney sum [MQRT77, 1.9], [MS06, 23.6.3]. In fact, as alluded to above one can set up a Quillen equivalence between the category of $\mathscr{I}_c$-FCP's and the category of E_∞ spaces, although we do not discuss this matter herein (see [L09] for a nice treatment of this comparison).

Moving on, we now focus attention on the $\mathscr{I}_c$-FCP specified by taking $V \subset \mathbb{R}^\infty$ to the space of based homotopy self-equivalences of S^V; this is classically denoted by $F(V)$. Passing to the colimit over inclusions, $F(\mathbb{R}^\infty) = \mathrm{colim}_V F(V)$ becomes a $\mathbb{L}$-space which models $GL_1 S$ — this is essentially one of the original descriptions from [MQRT77]. Furthermore, since each $F(V)$ is a monoid, applying the two-side bar construction levelwise yields an FCP specified by $V \mapsto BF(V)$; here $BF(V)$ denotes the bar construction $B(*, F(V), *)$, and the Whitney sum transformation is defined using the homeomorphism $B(*, F(V), *) \times B(*, F(W), *) \cong B(*, F(V) \times F(W), *)$. The colimit $BF(\mathbb{R}^\infty)$ provides a model for $BGL_1 S$.

Now, since $F(V)$ acts on S^V, we can also form the two-sided bar construction $B(*, F(V), S^V)$, abbreviated $EF(V)$, and there is a universal quasifibration

$$\pi_V \colon EF(V) = B(*, F(V), S^V) \longrightarrow B(*, F(V), *) = BF(V)$$

which classifies spherical fibrations with fiber S^V. Given a map $X \to BF(\mathbb{R}^\infty)$, by pulling back subspaces $BF(V) \subset BF(\mathbb{R}^\infty)$ we get an induced filtration on X; denote the space corresponding to pulling back along the inclusion of $V \in \mathbb{R}^\infty$ by $X(V)$ [LMSM86, IX.3.1].

Denote by $Z(V)$ the pullback

$$X(V) \longrightarrow BF(V) \longleftarrow EF(V).$$

The Vth space of the Thom prespectrum is then obtained by taking the Thom space of $Z(V) \to X(V)$, that is by collapsing out the section induced from the base point inclusion $* \to S^V$; denote the resulting prespectrum by TF [LMSM86, IX.3.2] (note that some work is involved in checking that these spaces in fact assemble into a prespectrum).

Next, we will verify that the prespectrum TF associated to the identity map on $BF(\mathbb{R}^\infty)$ is stably equivalent to the homotopy quotient $S/GL_1 S \simeq S/F(\mathbb{R}^\infty)$. For a point-set description of this homotopy quotient, notice that it follows from Proposition 5.24 that the category of EKMM (commutative) S-algebras is tensored over (commutative) monoids in $*$-modules: the tensor of a monoid in $*$-modules M and an S-algebra A is $\Sigma^\infty_+ M \wedge A$, with multiplication

$$(\Sigma^\infty_+ M \wedge A) \wedge (\Sigma^\infty_+ M \wedge A) \cong (\Sigma^\infty_+ M \wedge \Sigma^\infty_+ M) \wedge (A \wedge A) \cong (\Sigma^\infty_+ (M \boxtimes M)) \wedge (A \wedge A) \to (\Sigma^\infty_+ M) \wedge A.$$

Thus, we can model the homotopy quotient as a bar construction in the category of (commutative) S-algebras. However, we can also describe the homotopy quotient as $\mathrm{colim}_V S/F(V)$, where here we use the structure of $F(V)$ as a monoid acting on S^V. It is this "space-level" description we will employ in the comparison below.

We find it most convenient to reinterpret the Lewis-May construction in this situation, as follows: The Thom space in this case is by definition the cofiber $(EF(V), BF(V))$ of the inclusion $BF(V) \to EF(V)$ induced from the base point inclusion $* \to S^V$. Now,

$$BF(V) \simeq */F(V)$$

and similarly

$$EF(V) \simeq S^V/F(V).$$

Hence the Thom space is likewise the cofiber $(S^V, *)/F(V)$ of the inclusion $* \to S^V$, viewed as a *pointed* space.

More generally, we can regard the prespectrum $\{MF(V)\}$ as equivalently described as

$$MF(V) \overset{\text{def}}{=} S^V/F(V),$$

the homotopy quotient of the *pointed* space S^V by $F(V)$ via the canonical action, with structure maps induced from the quotient maps $S^V \to S^V/F(V)$ together with the pairings

$$MF(V) \wedge MF(W) \simeq S^V/F(V) \wedge S^W/F(W) \longrightarrow S^{V \oplus W}/F(V) \times F(W) \longrightarrow S^{V \oplus W}/F(V \oplus W),$$

where $F(V) \times F(W) \to F(V \oplus W)$ is the Whitney sum map of F. It is straightforward to check that the structure maps in terms of the bar construction described in [LMSM86, IX.3.2] realize these structure maps.

The associated spectrum MF can then be identified as $\operatorname{colim}_V S/F(V) \simeq S/F(\mathbb{R}^\infty)$. A key point is that the Thom spectrum functor can be described as the colimit over shifts of the Thom spaces [LMSM86, IX.3.7,IX.4.4]:

$$MF = \operatorname*{colim}_V \Sigma^{-V} \Sigma^\infty MF(V).$$

Furthermore, using the bar construction we can see that the spectrum quotient $(\Sigma^V S)/F(V)$ is equivalent to $\Sigma^\infty S^V/F(V)$. Putting these facts together, we have the following chain of equivalences:

$$
\begin{aligned}
MF &= \operatorname*{colim}_V \Sigma^{-V} \Sigma^\infty MF(V) = \operatorname*{colim}_V \Sigma^{-V} \Sigma^\infty S^V/F(V) \\
&\simeq \operatorname*{colim}_V \Sigma^{-V}(\Sigma^V S)/F(V) \simeq \operatorname*{colim}_V (\Sigma^{-V} \Sigma^V S)/F(V) \simeq S/F(\mathbb{R}^\infty).
\end{aligned}
$$

More generally, a slight elaboration of this argument implies the following proposition.

Proposition 8.26. *The Lewis-May Thom spectrum MG associated to a group-like A_∞ map $\varphi : G \to GL_1 S$ modeled by the map of $\mathscr{I}_c$-FCPs $G \to F$ is equivalent to the spectrum S/G, the homotopy quotient of the sphere by the action of φ.*

Note that any A_∞ map $X \to F(\mathbb{R}^\infty)$ can be rectified to a map of $\mathscr{I}_c$-FCPs $X' \to F$ [L09].

Corollary 8.27. *Given a map of spaces $f : X \to BGL_1 S$, write $\Pi_\infty Mf$ for the stable ∞-groupoid associated to the Lewis-May Thom spectrum Mf. Then $\Pi_\infty Mf \simeq M\Pi_\infty f$, where*

$$\Pi_\infty f : \Pi_\infty X \to \Pi_\infty BGL_1 S \simeq S\text{-line}$$

is the associated map of ∞-groupoids.

Proof. A basic property of this (and any) Thom spectrum functor is that it preserves colimits [LMSM86, IX.4.3]. Thus, we can assume that X is connected. In this case, $X \simeq BG$ for some group-like A_∞ space G, and $f : BG \to BGL_1 S$ is the delooping of an A_∞ map $G \to GL_1 S$, so $Mf \simeq S/G$ by Proposition 8.26, so $\Pi_\infty Mf \simeq M\Pi_\infty f$ by Proposition 7.29. $\qquad\square$

APPENDIX A. SPACES OVER BG AND G-SPACES

A.1. **The simplicial case.** In this subsection we give a quick proof of the Quillen equivalence between simplicial sets over BG and G-simplicial sets (with respect to the projective model structure) for a simplicial monoid G, where BG denotes the diagonal of the bisimplicial set obtained by regarding G as a simplicial category with one object. This result is folklore, and in the case of simplicial groups is an old result of Dwyer and Kan [DK85]. Nonetheless, since no proof of the specific result we need appears in the literature, we include one here. Note that, throughout this subsection, all model categories are simplicial model categories, and all morphisms of model categories are simplicial Quillen adjunctions; furthermore, we will sometimes refer to simplicial sets as spaces.

Write $\operatorname{Set}_{\Delta/BG}$ and $\operatorname{Set}_\Delta^G$ for the simplicial model categories of spaces over BG, equipped with the over-category model structure, and G-spaces, equipped with the projective model structure, respectively. Interpolating between these is the simplicial model category $\operatorname{Set}_{\Delta/EG}^G$ of G-spaces over the G-space EG. The projection $p : EG \to *$ to the terminal object induces an adjunction of simplicial model categories

$$p_! : \operatorname{Set}_{\Delta/EG}^G \rightleftarrows \operatorname{Set}_\Delta^G : p^*$$

(we always write the left adjoint on the left) which is in fact a Quillen equivalence, as p is a weak equivalence of G-spaces. Note that that pullback functor p^* is also the left adjoint of a Quillen equivalence

$$p^* : \operatorname{Set}_\Delta^G \rightleftarrows \operatorname{Set}_{\Delta/EG}^G : p_*,$$

where the right adjoint p_* sends $P \to EG$ to the G-space of sections $\operatorname{map}_{/EG}(EG, P)$.

Now, if we regard BG as a trivial G-space and $q : EG \to BG$ as a G-map, then we obtain a similar base-change Quillen adjunction

$$q_! : \operatorname{Set}_{\Delta/EG}^G \rightleftarrows \operatorname{Set}_{\Delta/BG}^G : q^*$$

which clearly is not in general a Quillen equivalence; nevertheless, we claim that restriction along the unique simplicial monoid morphism $r : G \to *$ induces yet another Quillen adjunction

$$r_! : \operatorname{Set}_{\Delta/BG}^G :\rightleftarrows \operatorname{Set}_{\Delta/BG} : r^*$$

such that the composite Quillen adjunction

$$r_! q_! : \operatorname{Set}_{\Delta/EG}^G \rightleftarrows \operatorname{Set}_{\Delta/BG}^G \rightleftarrows \operatorname{Set}_{\Delta/BG} : q^* r^*$$

is a Quillen equivalence. To see this, it's enough to show that the derived unit and counit transformations are equivalences. But actually, more is true: $r_! q_!$ and $q^* r^*$ are inverse isomorphisms. Indeed,

$$r_! q_!(P \to EG) = (P/G \to EG/G = BG)$$

and

$$q^* r^*(X \to BG) = (X \times_{BG} EG \to BG \times_{BG} EG = BG),$$

so the fact that colimits are compatible with base-change implies that

$$q^* r^* r_! q_!(P \to EG) = (P/G \times_{BG} EG \to BG \times_{BG} EG) \cong (P \to EG)$$

and

$$r_! q_! q^* r^*(X \to BG) = (X \times_{BG} EG/G \to BG \times_{BG} EG/G) \cong (X \to BG).$$

Combining this with the Quillen pair (p^*, p_*) above we obtain the following result.

Proposition A.1. *Let G be a simplicial monoid. Then the composite adjunction*

$$r_! q_! p^* : \operatorname{Set}_\Delta^G \rightleftarrows \operatorname{Set}_{\Delta/EG}^G \rightleftarrows \operatorname{Set}_{\Delta/BG}^G \rightleftarrows \operatorname{Set}_{\Delta/BG} : p_* q^* r^*$$

is a Quillen equivalence.

A.2. **Principal bundles and lifting for group-like topological monoids.** In this section we give a proof of the analogue of Theorem 5.32 in the context of topological monoids. Our proof depends on a comparison between the homotopy theory of free G-spaces and spaces over BG; it is an immediate consequence of the recent work of Shulman [Shu08], who uses the technical foundations laid in the very careful development of May-Sigurdsson [MS06].

We can assume without loss of generality that G is a group-like topological monoid with the homotopy type of a CW-complex and a nondegenerate base point (i.e. the inclusion map $* \to G$ is a Hurewicz cofibration); we refer to such a G as satisfying the standard hypotheses. Associated to G is the free right G-space EG defined as the two-sided bar construction $B(*, G, G)$. There is a projection map $\pi \colon B(*, G, G) \to B(*, G, *) = BG$ which is a quasifibration (and a fibration if G is in fact a topological group) [May75, 7.6].

Mimicking the notation of the previous subsection, let $G\mathcal{U}$ denote the category of G-spaces, let $\mathcal{U}/BG$ denote the category of spaces over BG, and let $G\mathcal{U}/EG$ denote the category of G-spaces over EG. Again we have an adjoint pair $(p_!, p^*)$ of functors

$$p_! \colon G\mathcal{U}/EG \rightleftarrows G\mathcal{U} \colon p^*,$$

where p^* is the pullback and $p_!$ is simply given by composition. There is also an adjoint pair (p^*, p_*), where p_* takes $X \to EG$ to the G-space of sections $\mathrm{map}_{\mathcal{U}/EG}(EG, X)$.

Regarding BG as a trivial G-space, the projection $\pi \colon EG \to BG$ is a G-map and so there is a similar pair of adjoint functors $(q_!, q_*)$

$$q_! \colon G\mathcal{U}/EG \rightleftarrows G\mathcal{U}/BG \colon q_*.$$

Finally, associated to the map of monoids $r \colon G \to *$ there is an adjoint pair $(r_!, r^*)$ of "change of monoids" functors

$$r_! \colon G\mathcal{U}/BG \rightleftarrows \mathcal{U}/BG \colon r^*,$$

where r^* is the pullback (which assigns the trivial action) and $r_!$ takes X to $X \times_G *$.

Putting this together, we obtain a composite adjunction

$$r_! q_! \colon G\mathcal{U}/EG \rightleftarrows G\mathcal{U}/BG \rightleftarrows \mathcal{U}/BG \colon q^* r^*,$$

and combining this with the adjunction (p^*, p_*) above we obtain the composite adjunction

$$r_! q_! p^* \colon G\mathcal{U} \rightleftarrows G\mathcal{U}/EG \rightleftarrows G\mathcal{U}/BG \rightleftarrows \mathcal{U}/BG \colon p_* q^* r^*.$$

The homotopy theory of free G-spaces can be encoded in the model structure on $G\mathcal{U}$ in which the weak equivalences and fibrations are detected via the forgetful functor to $\mathcal{U}$. For $\mathcal{U}/BG$, we use the m-model structure considered in [MS06] based on work of Cole on mixed model structures [Cole06]. Here the weak equivalences are the maps which induce a weak equivalence after forgetting to $\mathcal{U}$ and the fibrations are the Hurewicz fibration. Cofibrant objects have the homotopy type of CW-complexes.

In this setting, and under the standard hypotheses on G, Shulman proves that $(r_! q_! p^*, p_* q^* r^*)$ is a Quillen equivalence [Shu08, 8.5]. An immediate consequence of this is the desired comparison of mapping spaces.

Corollary A.2. *There are equivalences of derived mapping spaces*

$$\mathrm{map}_{\mathcal{U}/BG}(X, Y) \simeq \mathrm{map}_{G\mathcal{U}}(p_* q^* r^* X, p_* q^* r^* Y)$$

and

$$\mathrm{map}_{G\mathcal{U}}(P, Q) \simeq \mathrm{map}_{\mathcal{U}/BG}(r_! q_! p^* P, r_! q_! p^* Q)$$

which are natural in spaces X and Y over BG and G-spaces P and Q.

Corollary A.3. *There are equivalences of derived mapping spaces*

$$\mathrm{map}_{\mathcal{U}/BG}(X, EG) \simeq \mathrm{map}_{G\mathcal{U}}(X \times_{BG} EG, EG \times_{BG} EG) \simeq \mathrm{map}_{G\mathcal{U}}(X \times_{BG} EG, G),$$

natural in spaces X over BG.

Proof. We simply need to know that $X \times_{BG} EG$ and G are weakly equivalent to $p_* q^* r^* X$ and $p_* q^* r^* EG$, respectively. But

$$X \times_{BG} EG \simeq \mathrm{map}_{/EG}(EG, X \times_{BG} EG) = p_* q^* r^* X,$$

and the equivalence for G follows from the equivalence $G \simeq EG \times G \cong EG \times_{BG} EG$. $\square$

APPENDIX B. ∞-CATEGORIES AND SYMMETRIC MONOIDAL MODEL CATEGORIES OF SPECTRA

In order to give a self-contained treatment of Thom spectra and orientations in the setting of ∞-categories, in §7 we used the symmetric monoidal ∞-category of spectra developed in [DAGI, DAGII, DAGIII]. In this section, we show that one can give an account of the obstruction theory for orientations using only the techniques developed in [HTT] and general facts about associative ring spectra and their categories of modules. We hope that our treatment of Thom spectra and orientations from this point of view will serve as a useful invitation to ∞-categories, for those familiar with symmetric monoidal model categories of spectra. With this in mind, we have made this section somewhat more self-contained than strictly necessary, at the price of some redundancy with §7.

B.1. **Symmetric monoidal categories of R-modules.** Let $\mathcal{M}$ be a symmetric monoidal simplicial model category of spectra such as the S-modules of [EKMM96], or the symmetric spectra of [HSS00], and let R be a cofibrant and fibrant associative algebra in $\mathcal{M}$. We build the ∞-category R-line as in §7.7.

Namely, let R-mod$_{\mathcal{M}}$ be the simplicial model category of R-modules, and let R-mod$_{\mathcal{M}}^{\circ}$ be the full subcategory of R-mod$_{\mathcal{M}}$ consisting of cofibrant-fibrant R-modules. Now take the simplicial nerve [HTT, 1.1.5.5] to obtain the ∞-category

$$R\text{-mod} \overset{\text{def}}{=} \mathrm{N}R\text{-mod}_{\mathcal{M}}^{\circ}.$$

For cofibrant-fibrant L and M, the mapping spaces R-mod$_{\mathcal{M}}(L, M)$ are Kan complexes, and it follows [HTT, 1.1.5.9] that R-mod is an ∞-category.

As in Definition 7.18, define an R-line to be an R-module L which admits a weak equivalence

$$L \to R,$$

and let R-line be the full ∞-subgroupoid of R-mod whose objects are the R-lines. As discussed in §7.7,

(1) R-line$_0$ consists of fibrant-cofibrant R-modules which are R-lines;
(2) R-line$_1$ consists of the R-module weak equivalences

$$L \to M;$$

(3) R-line$_2$ consists of diagrams (not necessarily commutative)

$$\begin{array}{ccc} L & \overset{f}{\longrightarrow} & M \\ & {}_{h}\searrow & \downarrow{g} \\ & & N, \end{array}$$

together with a path in R-mod$_{\mathcal{M}}(L, N)$ from gf to h,

and so forth. This is precisely the sort of data we proposed in (1.1) as the transitions functions for a bundle of free rank-one R-modules.

Fix a cofibrant-fibrant R-line R°. Let

$$\mathrm{Aut}(R^{\circ}) = R\text{-line}(R^{\circ}, R^{\circ}) \subset R\text{-mod}(R^{\circ}, R^{\circ}),$$

and let $B\,\mathrm{Aut}(R^{\circ})$ be the full sub-∞-category of R-line on the single object R°. Since by construction R-line is a connected ∞-groupoid, the inclusion

$$B\,\mathrm{Aut}(R^{\circ}) \to R\text{-line}$$

is an equivalence. Define a *trivialization* of an R-line L to be an equivalence of R-modules

$$L \to R^\circ,$$

and let the ∞-category R-triv of trivialized R-lines be the slice category

$$R\text{-triv} = R\text{-line}_{/R^\circ}.$$

As discussed in Proposition 7.38, the forgetful map

$$R\text{-triv} \to R\text{-line}$$

is a Kan fibration, which is a model in this setting for $EGL_1R \to BGL_1R$.

B.2. Parametrized R-lines and Thom spectra.

Now let X be a Kan complex. Suppose that we are given a map of ∞-categories (i.e. of simplicial sets)

$$f : X \to R\text{-line}.$$

Thus f assigns

(1) to each 0-simplex $p \in X$ an R-line $f(p)$;
(2) to each path γ from p to q a weak equivalence of R-modules

$$f(\gamma) : f(p) \simeq f(q);$$

(3) to each 2-simplex $\sigma : \Delta^2 \to X$, say

a path $f(\sigma)$ in R-line$(f(p), f(r))$ from $f(\sigma_{12})f(\sigma_{01})$ to $f(\sigma_{02})$;

and so forth.

This illustrates nicely the idea, developed in §7, that the ∞-category R-line^X = Fun$(X, R$-line$)$ is a model for the ∞-category of R-lines parametrized by X, and that f corresponds to a bundle

$$\mathscr{L} \to X$$

of R-lines over X.

For now, consider the diagram

Lemma B.1. *The map of ∞ categories jf admits a colimit; equivalently, there is a left Kan extension of jf along p. (See [HTT, 1.2.13,4.3]).*

Remark B.2. We can't take the colimit in R-line, since R-line does not have colimits (it doesn't even have sums).

Proof. Again we take R-mod = NR-mod$_T^\circ$ as our ∞-category of R-modules. According to [HTT, 4.2.3.14], there is an ordinary category I and a cofinal map

$$k : \mathrm{N}(I) \to X.$$

Consider

$$\mathrm{N}(I) \xrightarrow{k} X \xrightarrow{f} R\text{-line} \xrightarrow{j} R\text{-mod}.$$

According to [HTT, 4.1.1.8], a colimit of jfk is the same thing as a colimit of jf. According to [HTT, 4.2.4.1], a colimit of jfk is the same thing as a homotopy colimit of

$$I \to R\text{-mod}_{\mathcal{M}}.$$

This homotopy colimit exists because $R\text{-mod}_{\mathcal{M}}$ is a simplicial model category. □

Definition B.3. The *Thom spectrum* associated to f is the R-module spectrum

$$Mf = \operatorname{colim} jf = \operatorname{ho} L_p jf.$$

For example, let MR be the Thom spectrum associated to identity map of R-line, or equivalently and more suggestively, to the inclusion

$$B\operatorname{Aut}(R^\circ) \xrightarrow{\simeq} R\text{-line}.$$

Proposition B.4.

$$MR \simeq R^\circ / \operatorname{Aut}(R^\circ).$$

Proof. By definition, $R^\circ / \operatorname{Aut}(R^\circ)$ is the colimit of the composite functor

$$B\operatorname{Aut}(R^\circ) \simeq R\text{-line} \to R\text{-mod}.$$

But this is just the standard construction of the (homotopy) quotient. □

Before discussing orientations, we describe explicitly the mapping property of this colimit. Note that a functor $* \to R$-line is just a choice of R-line, and we write ι for the composition

$$\iota : X \xrightarrow{p} * \xrightarrow{R^\circ} R\text{-line}.$$

Lemma B.5. *There is an adjunction*

$$R\text{-mod}(Mf, R^\circ) \simeq R\text{-mod}^X(jf, j\iota) \tag{B.6}$$

Remark B.7. The "adjunction" (B.6) is the ∞-category analog of the usual property of the colimit. It is a homotopy equivalence between the indicated mapping spaces.

Proof. According to [HTT, 4.3.3.7], the left Kan extension

$$p_! : R\text{-mod}^X \to R\text{-mod}$$

is a left adjoint of the constant functor $p^* : R\text{-mod} \to R\text{-mod}^X$. By [HTT, 5.2.2.7], this means in particular that for any object g of $R\text{-mod}^X$ and Y of $R\text{-mod}$, there is a natural homotopy equivalence

$$R\text{-mod}(p_! g, Y) \simeq R\text{-mod}^X(g, p^* Y).$$

Taking $g = jf$ and $Y = R^\circ$ so $p_! g = Mf$ and $p^* Y = j\iota$ gives the result. □

B.3. **The space of orientations.** With these in place, one can analyze the space of orientations in a straightforward manner, just as in §7. Notice that we have a natural map

$$R\text{-line}^X(f, \iota) \to R\text{-mod}^X(jf, j\iota);$$

by definition, this is just the inclusion of a set of path components. The following Definition is equivalent to Definition 7.30, except that we need the material in §7 to justify the notation

$$R\text{-line}^X(f, \iota) \simeq R_X\text{-line}(f^* \mathscr{L}, p^* R^\circ),$$

and so forth.

Definition B.8. The space of *orientations* of Mf is the pull-back

$$
\begin{array}{ccc}
\mathrm{or}_R(Mf,R) & \longrightarrow & R\text{-mod}(Mf,R) \\
{\scriptstyle\simeq}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} \\
R\text{-line}^X(f,\iota) & \longrightarrow & R\text{-mod}^X(jf,j\iota),
\end{array}
\tag{B.9}
$$

where the right vertical equivalence is the adjunction of Lemma B.5.

To describe the obstruction theory associated to ∞-groupoid $\mathrm{or}_R(Mf,R)$, let $\mathrm{map}_f(X,R\text{-triv})$ be the simplicial set which is the pull-back in the diagram

$$
\begin{array}{ccc}
\mathrm{map}_f(X,R\text{-triv}) & \longrightarrow & \mathrm{map}(X,R\text{-triv}) \\
\big\downarrow & & \big\downarrow \\
\{f\} & \longrightarrow & \mathrm{map}(X,R\text{-line}).
\end{array}
$$

That is, $\mathrm{map}_f(X,R\text{-triv})$ is the mapping simplicial set of lifts in the diagram

$$
\begin{array}{ccc}
 & & R\text{-triv} \\
 & \nearrow & \big\downarrow \\
X & \xrightarrow{\ f\ } & R\text{-line}.
\end{array}
\tag{B.10}
$$

We recapitulate in the current setting the statement and proof of Theorem 7.32.

Theorem B.11. *Let $f : X \to R\text{-line}$ be a map, and let Mf be the associated R-module Thom spectrum. Then there is an equivalence*

$$
\mathrm{map}_f(X,R\text{-triv}) \simeq R\text{-line}^X(f,\iota).
$$

Proof. Let $Z = R\text{-line}^X$. We have a fibration

$$
\begin{array}{ccc}
Z(f,\iota) & \longrightarrow & Z^{\Delta^1} \\
\big\downarrow & & \big\downarrow \\
\{(f,\iota)\} & \longrightarrow & Z^{\partial\Delta^1}.
\end{array}
$$

Pulling back along the inclusion of $Z \times \iota$ gives the fibration

$$
\begin{array}{ccc}
Z(f,\iota) & \longrightarrow & (Z,\iota)^{(\Delta^1,1)} \\
\big\downarrow & & \big\downarrow \\
\{f\} & \longrightarrow & Z.
\end{array}
$$

The adjunction (of mapping simplicial sets)

$$
\mathrm{map}(\Delta^1, \mathrm{map}(X,R\text{-line})) \cong \mathrm{map}(X, \mathrm{map}(\Delta^1, R\text{-line}))
$$

identifies

$$
(Z,\iota)^{(\Delta^1,1)} \cong \mathrm{map}(X, R\text{-triv}),
$$

and so this fibration becomes

$$
\begin{array}{ccc}
Z(f,\iota) & \longrightarrow & \mathrm{map}(X,R\text{-triv}) \\
\big\downarrow & & \big\downarrow \\
\{f\} & \longrightarrow & Z,
\end{array}
$$

as required. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL 61801, USA

E-mail address: `mando@math.uiuc.edu`

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD CA 94305

E-mail address: `blumberg@math.Stanford.edu`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO IL 60607, USA

E-mail address: `gepner@math.uic.edu`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE MA 02138

E-mail address: `mjh@math.harvard.edu`

THE UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL 61801, USA

E-mail address: `rezk@math.uiuc.edu`