06/10/20

- Outline: 1 → 2 → 3.

$$\begin{array}{lll} & & ③\ \text{Perf Spc. [Bha17]} \\ \text{Sch.} & & \downarrow \\ \text{formal sch.} & & \\ \text{Berk} & \hookrightarrow & ②\ \text{Adic spc. [Wei17]} \\ ①\ \text{Rigid Spc. [Ked04, 18.727] [BGR84]} & & \downarrow \\ & & \text{Ring Spc} \end{array}$$

I. Rigid Analytic Spc.

- Tate: Elliptic cures.
- class. rig. geo. = thy of analytic f'ns over fields that cple. + narc.
  - often interested in loc. field. = cplt. + disc. narc + finite residue field.
  - classically we have $\mathbb{R}, \mathbb{C}$. ⟷ $\mathbb{Q}_p, \mathbb{C}_p$. wrt prime p.

Ex: Consider p-adic val'n on $\mathbb{Z}$, and extend it on to $\mathbb{Q}$. $\forall x \in \mathbb{Q}$.

$$|x|_p := \begin{cases} 0 & \text{if } x=0 \\ p^{n} & \text{if } x = p^n \frac{a}{b},\ a,b,n \in \mathbb{Z},\ p \nmid ab. \end{cases}$$

- There's a way to construct a minimal alg. clos. + cpl. ext'n of $(\mathbb{Q}, |\cdot|_p)$.
  [3.4.1, BGR] (He does it for general K, nac. val'n)
  1. Take cpltion of $\mathbb{Q}$ wrt $|\cdot|_p$. $\mathbb{Q}_p$
  2. $(\mathbb{Q}_p)^{alg}$. is not necessarily complete. There's a canonical extension of $|\cdot|_p$ to alg closure.
  3. $((\mathbb{Q}_p)^{alg})' := \mathbb{C}_p$. is cplt. but is also alg. closed.

[3.1-3, BGR] extension is the specified norm

[3.4.3, BGR] That for if K is alg ⇒ $\hat{K}$ is also alg. closed.

• k cpt. narc. field.

Def'n L/k an alg ext. $\forall y \in L$.

$|y|_{sp} :=$ spec val $r(q)$, $q \in k[x]$ is min poly of y over k.

$r(q) = \sup_{1 \le i \le m} |a_i|^{1/i}$ where a_i are coeff'ns of $q = x^m + a_1 x^{m-1} + \dots + a_m$, [1.5.4, BGR].

• Tate generalizes this phenomenon.

$$\{\text{elliptic curves} / \mathbb{C}\} \leftrightarrow \{ T \xhookrightarrow{\text{emb}} \mathbb{P}^2_{\mathbb{C}} \}.$$

Def'n: $\wp(z, \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\substack{m\omega_1 + n\omega_2 = \lambda \neq 0 \\ (m,n) \in \mathbb{Z}^2}} \frac{1}{(z+\lambda)^2} - \frac{1}{z^2}$ $\omega_1, \omega_2 \in \mathbb{C}$, Wei. ell. fn.

$\Lambda := \{ m\omega_1 + n\omega_2 : (m,n) \in \mathbb{Z}^2 \}$, period lattice.

$\Rightarrow \wp(z, \omega_1, \omega_2)$ is in fact a fn on $\mathbb{C}/\Lambda \simeq T$. a torus.

• T is a Riemann surface of genus 1.

Prop'n: $\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$, where g_2, g_3 are consts.

"vested cs formula for elliptic curve".

$T \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$, $z \mapsto [1 : \wp(z) : \wp(z)'_{/2}]$

Prop'n: This is an iso Riemann surface $T \xhookrightarrow{\sim} E_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}}$

• punit is there is alg. def'n $E_{\mathbb{C}}$: non singular $\underbrace{\text{curve}}_{\text{proj. var. of dim 1.}}$ of genius one, with a Special pt $0 \in E_{\mathbb{C}}$. [Sil09, III.3].

• Goal of Tate is to make a similar formulation in terms of $\mathbb{Q}_p, \mathbb{C}_p$. what you need is the notion of rigid analytic spaces. [1, 17.8.27, Ked04].

I. Tate alg.

<u>Conv'n</u>: k cple. narc. field.

<u>Def'n</u>: [5.1.1, BGR] For $n \geq 1$, Tate alg over k is

$$T_n := T_n(k) = k\langle X_1, \dots, X_n\rangle = \{ \sum a_y X^y \mid |a_y| \to 0,\ |y| \to \infty \}. \quad a_y \in k.$$

- $y = (y_1, \dots, y_n)$, $y_i \geq 0$, $|y| = \sum y_i$. $X^y = X_1^{y_1} X_2^{y_2} \dots X_n^{y_n}$.

<u>Def'n</u>: The Gauss norm on T_n is given by

$$\| \sum a_y X_y \| = \max_y |a_y|$$

- We can develop quite a few general properties via [1.4, BGR]
- Now we go back to basics. , G ab. gp.
-

<u>Def'n</u>: $|\cdot| : G \to \mathbb{R}_{\geq 0} \cup \{0\}$. is a narc. semi norm if
- $|0| = 0$.
- $|x-y| \leq \max |x|, |y|$.

Define this cat $(G, |\cdot|)$ as SnGp morphisms.

bdd morphisms. $|\varphi x| \leq M|x|$ for sme $M > 0$.

Contractive morphisms. $|\varphi x| \leq |x|$.

<u>Def'n</u> $(A, |\,|) \in$ SnGp. is a semi-nmed ring if
1. Submultiplicative. $|xy| \leq |x|\,|y|$. $\forall x, y \in A$. (*)
2. $|1| \leq 1$.

<u>Rmk</u>:
- It is sufficient that $|xy| \leq K|x||y|$ $\forall x,y$, $K > 0$. (*)'
- But you can show, there $\exists$ an equivalent norm to (*)', satisfying (*).

Def'n: semi norm is norm if $|x| \neq 0$, $x \neq 0$

Def'n: if G has a semi norm, $|\cdot|$. It has an associated topology. This is defined by metric.

$$d(x,y) := |x-y|$$

Def'n: $A \in \text{SnRng}$, ($A \in \text{TopRng}$). $T \subseteq A$ (subset), [Wed(9)] $T(n) := \{t_1 \cdots t_n : t_i \in T\}$.

- T is top. nil if $\forall$ nbhd U of 0, $\exists N$ s.t. $T(n) \subset U$ $\forall n \geq N$.
  $a \in A$ is top nil iff $T = \{a\}$ is. $\check{A}$
- T is bdd. If $\forall$ nbhd U of 0, $\exists$ nbhd V s.t. $VT \subset U$.
  T is pow bdd if $\bigcup_{n \geq 1} T(n)$ is bdd. $\mathring{A}$
  $a \in A$ is pow bdd if $T = \{a\}$ is pow bdd.

Note: Def'n here is formulated for Top Rngs. If we apply to SnRngs

Def'n: $a \in A$ is pow bdd iff $\{|a^n| : n \in \mathbb{N}\}$ is a bdd set in $\mathbb{R}_{\geq 0}$.
top nil iff $\lim_n a_n = 0$. / $|a_n| \to 0$.

- There is another def'n: if $(A, |\cdot|) \in \text{SnRng}$

$$A^{\vee} := \{a \in A : |a| < 1\}.$$
$$A^{\circ} := \{a \in A : |a| \leq 1\}.$$

} ring of def'n for local rngs.

- Can define the residue ring. There are two versions.
  1. $A^{\sim} := A^{\circ}/A^{\vee}$
  2. $\tilde{A} := \mathring{A}/\check{A}$.

you have $A^{\vee} \subseteq \check{A}$, $A^{\circ} \subseteq \mathring{A}$.

$|a| < 1 \Rightarrow \lim_n |a^n| \leq \lim_n |a|^n \to 0$

Q: when are these obj. equal?

$(A, |\cdot|)$ $\in$ SnRing

Def'n: • We say an element $a \in A$ power-mult. (pm) if $|a^n| = |a|^n$ $\forall n \geq 1$.

• $|\cdot|$ is pm.-norm on A, if every element in A is pm.

Prop: If $|\cdot|$ is pm.-norm. then $\mathring{A} = A^0$, $A^{\vee} = \check{A}$.

Pf. Def'n chase

• Residues ring are "usually" how we deduce info. about Tate alg / $(A, |\cdot|)$ SnRing.
[BGR, I-V]

• Construct Tate alg inductively.

Def'n: $(A, |\cdot|)$ $\in$ SnRing. strictly convergent power series.

• A FPS $\sum_0^{\infty} a_v X^v \in A[[X]]$ scp if $\lim |a_v| = 0$. $A\langle X\rangle$. Set of scp.

• $f \in A\langle X\rangle$ we define the Gauss semi-norm

$|f|' := \max_v |a_v|$.

Def'n: $A\langle X_1, \ldots, X_n\rangle = A\langle X_1, \ldots, X_{n-1}\rangle\langle X_n\rangle$. , ($A = k$. we get $T_n(k)$)

• This inductive def'n allows us to prove properties $T_n(k)$ by induction.

Prop: Properties of $(A\langle X\rangle, |\cdot|')$ (can replace $A\langle X\rangle$ with $A\langle X_1, \ldots, X_n\rangle$)

1) $A \subset A[X] \subset A\langle X\rangle \subset A[[X]]$

2) $|\ |'$ is norm on $A\langle X\rangle$ iff is norm on A.

3) If $(A, |\ |)$ is complete then $(A\langle X\rangle, |\ |')$ is complete.

4) $A[X]$ is dense in $A\langle X\rangle$.

Rmk $A\langle X\rangle$ is a subring, nec. condition.

Pf. 4) $f = \sum_0^\infty a_\nu X^\nu$, $\quad |f - \sum_0^n a_\nu X^\nu| \leq \max_{j \geq n} |a_j| \to 0$ as $n \to \infty$.

3). • let $(f_i) = \left(\sum_0^\infty a_{i\nu} X^\nu\right)_{i \in \mathbb{N}}$ is a Cauchy seq in $A\langle X\rangle$.

• $f := \sum_0^\infty a_\nu X^\nu$

To define a_ν, fix a ν.

$$|a_{i+1,\nu} - a_{i,\nu}| \leq |f_{i+1} - f_i|'.$$

This implies seque $(a_{i\nu})$ is Cauchy. let $a_\nu := \lim a_{i\nu} \in A$.

$\square$

Prop: [5.1.2, BGR]

1. The Gauss norm is val'n on $T_n(k)$
2. $\tilde{T}_n = \tilde{k}[X_1, \dots, X_n]$ ← polynomial ring,

$\tilde{k} := k^\circ / k^{\circ\circ}$.

Def'n: $(A, |\cdot|)$ k-alg. Say $|\cdot|$ is a val'n on A, if $|\cdot|$ is multiplicative.

Rmk: By above analysis $T_n(k)$ is in fact a k-Banach alg.
This follows from 3. $\because$ we assume k is cplt. nac field.

Def'n: A k-Banach Algebra, is k-affinoid if $\exists$ where $n \geq 0$, continuous epi.
$$\alpha : T_n \twoheadrightarrow A.$$

Thm: (Banach open mapping) let V, W be k-Banach vec spc, $\Phi : V \to W$ be bdd + surj k-linear map, then

1) Φ is open
2) W carries the quotient nrf to Φ. $\quad$ (2) follows from 1)).

- There is no canonical R-algebra structure, as in choice of "norm" on A.

<u>Goal next time</u>:

Affinoid alg $\leftrightarrow$ rings

affinoid varieties $\leftrightarrow$ affine schemes

?.?. ?. $\leftrightarrow$ Alex. Formal Sch. ?.?

"live in".

Adic spaces

~ [18.727, Led04]