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EQUIVARIANT MAPS WHICH ARE SELF HOMOTOPY EQUIVALENCES[1]

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ABSTRACT. The aim of this note is (i) to give (in §2) a precise statement and proof of the (to some extent well-known) fact that *the most elementary homotopy theory of "simplicial sets on which a fixed simplicial group H acts" is equivalent to the homotopy theory of "simplicial sets over the classifying complex $\overline{W}H$"*, and (ii) to use this (in §1) to prove *a classification theorem for simplicial sets with an H-action, which provides classifying complexes for their equivariant maps which are self homotopy equivalences.*

1. The classification theorem. In order to formulate our classification theorem (1.2) we need some definitions involving

1.1. SIMPLICIAL SETS WITH A SIMPLICIAL GROUP ACTION. (i) Let V and V' be simplicial sets on which a simplicial group H acts (from the right). A map $V \to V'$ then will be called an *equivariant weak homotopy equivalence* if it is compatible with the action of H and is a weak homotopy equivalence. This should not be confused with the stronger notion of *weak equivariant-homotopy equivalence*, which is a map compatible with the action of H, which induces weak homotopy equivalences on the fixed point sets of all simplicial subgroups of H (and not merely the identity subgroup).

(ii) Let V be a simplicial set on which a simplicial group H acts. Then we denote by $\operatorname{haut}_H V$ the simplicial monoid of *equivariant weak self homotopy equivalences* of V, i.e., the simplicial monoid which has as n-simplices the commutative diagrams

$$\begin{array}{ccc} \Delta[n] \times V & \rightarrow & \Delta[n] \times V \\ \text{proj.}\searrow & & \swarrow \text{proj.} \\ & \Delta[n] & \end{array}$$

in which the top map is an equivariant weak homotopy equivalence.

1.2. CLASSIFICATION THEOREM. *Let H be a simplicial group and $\overline{W}H$ its classifying complex* [4, *p.* 87] *and let M be a minimal simplicial set* [4, *p.* 35] *and* $\operatorname{aut} M$ *its simplicial group of automorphisms* [1, 1.3], [4, *p.* 74]. *Then the function complex* $(\overline{W}\operatorname{aut} M)^{\overline{W}H}$ *has the following properties*:

(i) *The components of* $(\overline{W}\operatorname{aut} M)^{\overline{W}H}$ *are in natural* 1-1 *correspondence with the equivariant weak homotopy equivalence classes of simplicial sets with an H-action, which have the weak homotopy type of M. Moreover each such class contains*

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simplicial sets which satisfy the extension condition [**4**, *p*. 2] *and on which the H-action is free* (*i.e. principal* [**4**, *p*. 70]).

(ii) *If V is a simplicial set which satisfies the extension condition and has the homotopy type of M and on which H acts freely, then* $\overline{W}\,\mathrm{haut}_H V$ *has the weak homotopy type of the component of* $(\overline{W}\,\mathrm{aut}\,M)^{\overline{W}H}$ *which* (*see* (i)) *corresponds to V.*

PROOF. This is an immediate consequence of the classification theorem [**1**, 1.4] for fibrations over $\overline{W}H$ and the results of §3.

1.3. COROLLARY. *Let H be a simplicial group and let G be a group which has no center and no outer automorphisms* (*such as, for instance, the symmetric group on n letters, where $n \neq 2$ or* 6). *Then*

(i) *there is only one equivariant weak homotopy equivalence class of simplicial sets with an H-action, which have the weak homotopy type of* $K(G, 1)$, *and*

(ii) *if V is a simplicial set which satisfies the extension condition and has the homotopy type of* $K(G, 1)$ *and on which H acts freely, then* $\mathrm{haut}_H V$ *is contractible* (*i.e. has the weak homotopy type of a point*).

PROOF. This follows immediately from the fact that aut $K(G, 1)$ is contractible.

1.4. COROLLARY. *If V is a contractible simplicial set which satisfies the extension condition and on which a simplicial group H acts freely, then* $\mathrm{haut}_H V$ *is contractible.*

2. The equivalence of homotopy theories. Let H be a simplicial group, let $\mathbf{S}_H$ denote the category of *simplicial sets with a* (*right*) *H-action* and equivariant maps between them, and let $\mathbf{S}/\overline{W}H$ be the category of *simplicial sets over the classifying complex* [**4**, p. 87] $\overline{W}H$. Then the homotopy theories of $\mathbf{S}_H$ and $\mathbf{S}/\overline{W}H$ are equivalent in the sense that Theorem 2.1 and Corollary 2.5 below hold.

2.1. THEOREM. *The simplicial localizations* [**3**, §3] *of* $\mathbf{S}_H$ *with respect to the equivariant weak homotopy equivalences* (1.1) *and of* $\mathbf{S}/\overline{W}H$ *with respect to the weak homotopy equivalences over* $\overline{W}H$ *are simplicial categories in the sense of* [**3**, §2], *which are homotopically equivalent* [**3**, 2.5].

This follows immediately from [**2**, 3.5], [**3**, 4.1] and the following propositions.

2.2. PROPOSITION. *The categories* $\mathbf{S}/\overline{W}H$ *and* $\mathbf{S}_H$, *with fibrations, cofibrations and weak equivalences as defined below, are closed model categories in the sense of Quillen* [**5**]:

(i) *The model category structure on* $\mathbf{S}/\overline{W}H$ *is the one induced by the usual one on the category of simplicial sets* $\mathbf{S}$ [**5**, II, 2.8]; *in particular the weak equivalences are the weak homotopy equivalences over* $\overline{W}H$.

(ii) *A map in* $\mathbf{S}_H$ *is a fibration if the underlying map in* $\mathbf{S}$ *is so, is a cofibration if it is* 1-1 *and its H-action is relatively free* (*i.e. no nonidentity simplex of H fixes a simplex of the range which is not in the image of the domain*), *and is a weak equivalence if it is an equivariant weak homotopy equivalence* (1.1).

The proof is straightforward.

2.3. PROPOSITION. *There is a pair of adjoint functors* A: $\mathbf{S}/\overline{W}H \to \mathbf{S}_H$ (*the left adjoint*) *and* B: $\mathbf{S}_H \to \mathbf{S}/\overline{W}H$ (*the right adjoint*) *such that, in the terminology of* 2.2

(i) *both functors send weak equivalences into weak equivalences, and*

(ii) *for every object* $U \in \mathbf{S}/\overline{W}H$ *and every object* $V \in \mathbf{S}_H$, *the adjunction maps* $U \to BAU$ *and* $ABV \to V$ *are weak equivalences.*

PROOF. Given an object U: $Y \to \overline{W}H \in \mathbf{S}/\overline{W}H$, one defines AU by [4, §21] $AU = Y \times_{\overline{W}H} WH$, with its H-action induced by the one on WH and, given an object $V \in \mathbf{S}_H$, one defines BV as the map $(V \times WH)/H \to \overline{W}H \in \mathbf{S}$ induced by the projection $V \times WH \to WH \in \mathbf{S}_H$. The adjoint of a map f: $AU \to V \in \mathbf{S}_H$ is the map $Y \approx AU/H \to (V \times WH)/H \in \mathbf{S}$ over $\overline{W}H$ induced by the map $AU \to V \times WH \in \mathbf{S}_H$ which is the product of f and the obvious map $AU = Y \times_{\overline{W}H} WH \to WH \in \mathbf{S}_H$. The rest of the proof is a straightforward verification.

Also not hard to verify is

2.4. PROPOSITION. *Let* $(\mathbf{S}/\overline{W}H)_*$ *and* $(\mathbf{S}_H)_*$ *denote the simplicial categories* [3, §2] *obtained from the model categories* $\mathbf{S}/\overline{W}H$ *and* $\mathbf{S}_H$ (2.2) *by giving them the obvious simplicial structure* (*i.e. function complexes*). *Then* $(\mathbf{S}/\overline{W}H)_*$ *and* $(\mathbf{S}_H)_*$ *are closed simplicial model categories in the sense of Quillen* [5]. *Moreover, the functors A and B of* 2.3 *induce functors* A_*: $(\mathbf{S}/\overline{W}H)_* \to (\mathbf{S}_H)_*$ *and* B_*: $(\mathbf{S}_H)_* \to (\mathbf{S}/\overline{W}H)_*$.

Let $(\mathbf{S}/\overline{W}H)_*^{\mathrm{cf}} \subset (\mathbf{S}/\overline{W}H)_*$ and $(\mathbf{S}_H)_*^{\mathrm{cf}} \subset (\mathbf{S}_H)_*$ denote the subcategories generated by the objects which are cofibrant as well as fibrant (i.e. the *fibrations with* $\overline{W}H$ *as base* and the *fibrant simplicial sets with free H-actions*). Then the functors A_* and B_* (2.4) send these simplicial subcategories into each other and one has, in view of [3, 4.8]:

2.5. COROLLARY. *The simplicial categories* $(\mathbf{S}/\overline{W}H)_*^{\mathrm{cf}}$ *and* $(\mathbf{S}_H)_*^{\mathrm{cf}}$ *are homotopically equivalent* [3, 2.5]. *Moreover, they are weakly equivalent* [3, 2.4] *to the simplicial categories of Theorem* 2.1.

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