Lecture 3: Untilting

October 5, 2018

In this lecture, we let $C^\flat$ denote a perfectoid field of characteristic p.

Warning 1. We will often use the superscript $^\flat$ to signal that an object under consideration "lives in characteristic p". In particular, declaring that $C^\flat$ is a perfectoid field of characteristic p is not meant to signal that $C^\flat$ is given as the tilt of a perfectoid field C. In fact, our emphasis is the opposite: we take the field $C^\flat$ as given, and would like to understand all possible *untilts* of $C^\flat$. Recall that an untilt of $C^\flat$ is defined to be a pair (K, ι), where K Is a perfectoid field and $\iota : C \simeq K^\flat$ is an (continuous) isomorphism.

Question 2. Let $C^\flat$ be a perfectoid field of characteristic p. How can one classify the untilts of $C^\flat$?

Remark 3. Let K be a perfectoid field of characteristic zero. Note that giving a continuous isomorphism $\iota : C^\flat \simeq K^\flat$ is equivalent to giving an isomorphism of valuation rings

$$\mathcal{O}_C^\flat \to \mathcal{O}_K^\flat = \varprojlim(\cdots \to \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_K / p\,\mathcal{O}_K).$$

We saw in the previous lecture that this induces an isomorphism of quotient rings $\iota_0 : \mathcal{O}_C^\flat / (\pi) \simeq \mathcal{O}_K / (p)$ for some element $\pi \in C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$. Conversely, any such isomorphism ι_0 can be lifted to an isomorphism of valuation rings $\mathcal{O}_C^\flat \simeq \mathcal{O}_K^\flat$, since $\mathcal{O}_C^\flat$ is isomorphic to the inverse limit

$$\cdots \to \mathcal{O}_C^\flat / (\pi) \xrightarrow{\varphi} \mathcal{O}_C^\flat / (\pi) \xrightarrow{\varphi} \mathcal{O}_C^\flat / (\pi).$$

We may therefore rephrase Question 2 as follows: how can we classify perfectoid fields K of characteristic zero equipped with an isomorphism $\mathcal{O}_K / (p) \simeq \mathcal{O}_C^\flat / (\pi)$?

For an untilt (K, ι) of C, let us abuse notation by writing $\sharp : C^\flat \to K$ for the composite map $C^\flat \xrightarrow{\iota} K^\flat \xrightarrow{\sharp} K$. This map does not need to be surjective. However, it is not too far off. We saw in the previous lecture that every element $x \in \mathcal{O}_K$ is congruent modulo p to an element in the image of the map $\sharp$: that is, we can find an element $c_0 \in \mathcal{O}_C^\flat$ satisfying $x = c_0^\sharp + x'p$, for some $x' \in \mathcal{O}_K$. Applying the same argument to x', we obtain $x = c_0^\sharp + c_1^\sharp p + x''p^2$, for some $x'' \in \mathcal{O}_K$. Iterating this argument, we obtain a description of x as an infinite sum

$$x = c_0^\sharp + c_1^\sharp p + c_2^\sharp p^2 + c_3^\sharp p^3 + \cdots,$$

for some sequence of elements $c_0, c_1, c_2, \ldots \mathcal{O}_C^\flat$; note that this infinite sum makes sense because the ring $\mathcal{O}_K$ is p-adically complete. The decomposition above is not at all unique: generally an element $x \in \mathcal{O}_K$ can be decomposed as a sum $\sum_{n\geq 0} c_n^\sharp p^n$ in many different ways. For example, if K is algebraically closed, then any element $x \in \mathcal{O}_K$ can be written in the form $c_0^\sharp$ by choosing a compatible sequence of p^nth roots of x; in characteristic zero, these pth roots are not unique.

One virtue of working with expressions like $\sum_{n\geq 0} c_n^\sharp p^n$ is that they make sense simultaneously in *every* untilt K of $C^\flat$. Moreover, it is possible to give work out formulas for adding and multiplying these expressions which are independent of the choice of K. To make this idea precise, it will be convenient to review the theory of Witt vectors.

Notation 4. Let R be a perfect ring of characteristic p: that is, a commutative ring such that $p = 0$ in R and every element $x \in R$ has a unique pth root. We let $W(R)$ denote the ring of Witt vectors of R. Then $W(R)$ is characterized up to (unique) isomorphism by the following properties:

(1) There is an isomorphism $W(R)/pW(R) \simeq R$.

(2) The element p is not a zero-divisor in $W(R)$.

(3) The ring $W(R)$ is p-adically complete.

Example 5. Let $R = \mathbf{F}_p$ be the finite field with p-elements. Then $W(R) \simeq \mathbf{Z}_p$ can be identified with the ring of p-adic integers.

Notation 6. For every element $x \in R$, we let $[x]$ denote its *Teichmüller representative* in $W(R)$. Then $[x]$ is uniquely determined by the following properties:

- The quotient map $W(R) \twoheadrightarrow W(R)/pW(R) \simeq R$ carries $[x]$ to x.
- The element $[x] \in W(R)$ admits a p^nth root, for every $n \geq 0$.

Concretely, one can construct the Teichmüller representative $[x]$ as the limit $\lim_{n\to\infty}(\overline{x^{1/p^n}})^{p^n}$, where $\overline{x^{1/p^n}}$ is any element of $W(R)$ representing the p^nth root $x^{1/p^n} \in R$. The construction of Teichmüller representatives determines a map

$$[\bullet] : R \to W(R)$$

which is multiplicative (that is, we have $[xy] = [x][y]$) but not additive.

Remark 7. Let R be a perfect ring of characteristic p and let x be an element of $W(R)$. Then x has some image $c_0 \in R$ under the quotient map $W(R) \twoheadrightarrow W(R)/pW(R) \simeq R$. The Teichmüller lift $[c_0]$ is then congruent to x modulo p, so we can write $x = [c_0] + x'p$ for some $x' \in W(R)$. Iterating this observation, we obtain an identity

$$x = [c_0] + [c_1]p + [c_2]p^2 + [c_3]p^3 + \cdots,$$

called the *Teichmüller expansion* of x. Note that, in contrast to the situation before, this expansion is unique: if

$$\sum [c_n]p^n = \sum [c'_n]p^n,$$

then an easy induction shows that $c_n = c'_n$ for each n.

Remark 8. Let R be a perfect ring of characteristic p. Then the ring of Witt vectors $W(R)$ can be characterized by a universal property:

($*$) For any p-adically complete ring A, reduction modulo p induces a bijection

$$\mathrm{Hom}(W(R), A) \to \mathrm{Hom}(R, A/pA).$$

In other words, every ring homomorphism $R \to A/pA$ can be lifted uniquely to a ring homomorphism $W(R) \to A$.

Let us now specialize to the situation of interest to us.

Construction 9. Let $C^\flat$ be a perfectoid field of characteristic p and let $\mathcal{O}_C^\flat$ be the valuation ring of $C^\flat$. Then $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p. We let $\mathbf{A}_{\mathrm{inf}}$ denote the ring of Witt vectors $W(\mathcal{O}_C^\flat)$.

Remark 10. The ring $\mathbf{A}_{\mathrm{inf}}$ is one of Fontaine's *period rings*; it will play an essential role in this course.

Remark 11. Let $C^\flat$ be a perfectoid field of characteristic p and let (K, ι) be an untilt of $C^\flat$. The map $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$ is not a ring homomorphism (unless K has characteristic p). However, it induces a ring homomorphism $\mathcal{O}_C^\flat \to \mathcal{O}_K / p\,\mathcal{O}_K$. Since $\mathcal{O}_K$ is p-adically complete, the universal property of Remark 8 to lifts this to a ring homomorphism

$$\theta : \mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat) \to \mathcal{O}_K .$$

Concretely, this map is given by the formula

$$\theta([c_0] + [c_1]p + [c_2]p^2 + \cdots) = c_0^\sharp + c_1^\sharp p + c_2^\sharp p^2 + \cdots .$$

From the discussion at the beginning of the lecture, we deduce that θ is surjective.

Remark 12. In the situation of Remark 11, the map θ is *local*: that is, an element $\sum [c_n]p^n$ of $\mathbf{A}_{\text{inf}}$ is invertible if and only if its image $\sum c_n^\sharp p^n \in \mathcal{O}_K$ is invertible (in both cases, invertibility is equivalent to the requirement that $|c_0|_{C^\flat} = 1$).

Remark 13. Let $C^\flat$ be a perfectoid field of characteristic p and let K be a characteristic zero untilt of $C^\flat$. We saw in the previous lecture that it is possible to choose an element $\pi \in \mathcal{O}_C^\flat$ with $|\pi|_{C^\flat} = |p|_K$, and that the map $\sharp : C^\flat \to K$ induces an isomorphism of commutative rings $\mathcal{O}_C^\flat / \pi\,\mathcal{O}_C^\flat \simeq \mathcal{O}_K / p\,\mathcal{O}_K$. In other words, $\mathcal{O}_C^\flat$ and $\mathcal{O}_K$ have a common quotient ring. Remark 11 shows that both $\mathcal{O}_C^\flat$ and $\mathcal{O}_K$ can be realized as a quotient of the same ring $\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat)$: one gets $\mathcal{O}_C^\flat$ from $\mathbf{A}_{\text{inf}}$ by reducing modulo p, and $\mathcal{O}_K$ by reducing modulo the kernel $\ker(\theta)$. These quotient maps fit into a commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{\text{inf}} \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_K \arrow[d] \\
\mathcal{O}_C^\flat \arrow[r, "\sharp"] & \mathcal{O}_K / p\,\mathcal{O}_K .
\end{tikzcd}$$

In the situation of Remark 11, the map θ is never injective: in other words, it is always possible to write an element $x \in \mathcal{O}_K$ as a sum $\sum_{n \geq 0} c_n^\sharp p^n$ in multiple ways.

Example 14. Let $C^\flat$ be a perfectoid field of characteristic p and let K be an untilt of $C^\flat$. We saw in the previous lecture that there exists an element $\pi \in \mathcal{O}_C^\flat$ satisfying $|\pi|_{C^\flat} = |p|_K$ (if K has characteristic p, we just take $\pi = 0$). It follows that the elements p and $\pi^\sharp$ have the same absolute value in K, and therefore differ by multiplication by an invertible element $\overline{u} \in \mathcal{O}_K$. Write $\overline{u} = \theta(u)$ for $u \in \mathbf{A}_{\text{inf}}$ (so that u is also invertible, by Remark 12). The identity $\pi^\sharp = \overline{u}p$ then implies that $[\pi] - up \in \mathbf{A}_{\text{inf}}$ belongs to the kernel of θ.

Definition 15. Let $C^\flat$ be a perfectoid field of chacteristic p. We say that an element $\xi \in \mathbf{A}_{\text{inf}}$ is *distinguished* if it has the form $[\pi] - up$, where $|\pi|_{C^\flat} < 1$ and u is an invertible element of $\mathbf{A}_{\text{inf}}$. In other words, ξ is distinguished if its Teichmüller expansion

$$\xi = [c_0] + [c_1]p + [c_2]p^2 + \cdots$$

has the property that $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$.

Example 14 shows that, for every untilt K of $C^\flat$, the kernel of the induced map $\theta : \mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat) \to \mathcal{O}_K$ always contains a distinguished element ξ. We now establish a converse:

Proposition 16. *Let $C^\flat$ be a perfectoid field of characteristic p and let ξ be a distinguished element of $\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat)$. Then the quotient $\mathbf{A}_{\text{inf}}/(\xi)$ can be identified with the valuation ring $\mathcal{O}_K$ in a perfectoid field K. Moreover, the canonical map*

$$\mathcal{O}_C^\flat = \mathbf{A}_{\text{inf}}/(p) \to \mathbf{A}_{\text{inf}}/(\xi, p) \simeq \mathcal{O}_K / (p)$$

exhibits K as an untilt of $C^\flat$ (see Remark 3).

Corollary 17. *Let $C^\flat$ be a perfectoid field of characteristic p, let K be an untilt of $C^\flat$, and let $\theta : \mathbf{A}_{\rm inf} \to \mathcal{O}_K$ be as above. Then $\ker(\theta)$ is a principal ideal, generated by any choice of distinguished element $\xi \in \ker(\theta)$.*

Proof. It follows from Example 14 that $\ker(\theta)$ contains a distinguished element ξ, so that θ induces a surjection $\overline{\theta} : \mathbf{A}_{\rm inf}/(\xi) \twoheadrightarrow \mathcal{O}_K$. Proposition 16 shows that we can identify $\mathbf{A}_{\rm inf}/(\xi)$ with $\mathcal{O}_{K'}$ for some untilt K' of $C^\flat$. Since $\mathcal{O}_K$ is an integral domain, the kernel of θ is a prime ideal of $\mathcal{O}_{K'}$, and is therefore either (0) (in which case $\overline{\theta}$ is an isomorphism) or the maximal ideal $\mathfrak{m}_{K'}$ (which is impossible, since $\mathcal{O}_K$ is not a field). $\square$

Corollary 18. *Let $C^\flat$ be a perfectoid field of characteristic p. Then the construction*

$$\xi \mapsto \textit{ Fraction field of } \mathbf{A}_{\rm inf}/(\xi)$$

induces a bijection

$$\{\textit{Distinguished elements of } \mathbf{A}_{\rm inf}\}/\textit{multiplication by units} \simeq \{\textit{Untilts of } C^\flat\}/\textit{isomorphism.}$$

To prove Proposition 16, we will need the following purely algebraic fact, which we leave to the reader.

Exercise 19. Let R be a commutative ring (not necessarily Noetherian!) containing a pair of elements x and y. Suppose that:

- The element x is not a zero-divisor in R, and R is x-adically complete.
- The image of y is not a zero-divisor in R/xR, and R/xR is y-adically complete.

Show that:

- The element y is not a zero-divisor in R, and R is y-adically complete.
- The image of x is not a zero-divisor in R/yR, and R/yR is x-adically complete.

Proof of Proposition 16. Let ξ be a distinguished element of $\mathbf{A}_{\rm inf}$, so that we can write $\xi = [\pi] - up$ for some $\pi \in \mathfrak{m}_C^\flat$ and some invertible element u in $\mathbf{A}_{\rm inf}$. If $\pi = 0$, then $\mathbf{A}_{\rm inf}/(\xi) \simeq \mathbf{A}_{\rm inf}/(p) \simeq \mathcal{O}_C^\flat$ and we have nothing to prove. Let us therefore assume that π is not zero. Let $\mathcal{O}_K$ denote the quotient ring $\mathbf{A}_{\rm inf}/\xi\mathbf{A}_{\rm inf}$. (Beware that this notation is misleading, since we do not yet know that there is a valued field K having $\mathcal{O}_K$ as its valuation ring.) We then have a canonical map $\theta : \mathbf{A}_{\rm inf} \twoheadrightarrow \mathcal{O}_K$ (with kernel generated by ξ); for each element $x \in \mathcal{O}_C^\flat$, we will denote $\theta([x])$ by $x^\sharp \in \mathcal{O}_K$.

We now apply Exercise 19 to the elements $x = p$ and $y = \xi$ of the ring $\mathbf{A}_{\rm inf}$. The construction of $\mathbf{A}_{\rm inf}$ as the ring of Witt vectors $W(\mathcal{O}_C^\flat)$ shows that $\mathbf{A}_{\rm inf}$ is p-adically complete and p-torsion free. Moreover, the image of ξ in the quotient $\mathbf{A}_{\rm inf}/p\mathbf{A}_{\rm inf} \simeq \mathcal{O}_C^\flat$ is π, satisfying $0 < |\pi|_{C^\flat} < 1$. It follows that $\mathcal{O}_C^\flat$ is ξ-adically complete and ξ-torsion free. Applying Exercise 19, we deduce the following:

- The ring $\mathbf{A}_{\rm inf}$ is ξ-adically complete and ξ-torsion free.
- The quotient ring $\mathcal{O}_K = \mathbf{A}_{\rm inf}/(\xi)$ is p-adically complete and p-torsion free.

We next prove the following:

(a) For any element $y \in \mathcal{O}_K$, there exists an element $x \in \mathcal{O}_C^\flat$ such that y is a unit multiple of $x^\sharp$.

To prove (a), we may assume without loss of generality that $y \neq 0$. Since $\mathcal{O}_K$ is p-adically complete, we can write $y = p^n y'$ for some $y' \in \mathcal{O}_K$ which is not divisible by p. Replacing y by y', we may assume that y is not divisible by p. Since θ is surjective, we can choose $x \in \mathcal{O}_C^\flat$ such that $y \equiv x^\sharp \pmod p$. Then $x^\sharp$ is not divisible by p, so x is not divisible by π. We can therefore write $\pi = xx'$ for some $x' \in \mathfrak{m}_C^\flat$. We have $y = x^\sharp + \pi^\sharp w = x^\sharp(1 + x'^\sharp w)$ for some $w \in \mathcal{O}_K$. Since some power of x' is divisible by π in the ring $\mathcal{O}_C^\flat$, some power of $x'^\sharp$ is divisible by p in the ring $\mathcal{O}_K$. It follows that $1 + x'^\sharp w$ is an invertible element of $\mathcal{O}_K$ (with inverse given by the p-adically convergent sum $1 - x'^{\sharp}w + (x'^{\sharp}w)^2 - (x'^{\sharp}w)^3 + \cdots$). This proves that y is a unit multiple of $x^{\sharp}$, as desired.

Note that the element x appearing in (a) is not uniquely determined: we are free to multiply it by any unit in $\mathcal{O}_C^{\flat}$. However, this is our only freedom:

(b) Let $x, x' \in \mathcal{O}_C^{\flat}$ be elements such that $x^{\sharp}$ is divisible by $x'^{\sharp}$ in $\mathcal{O}_K$. Then x is divisible by x' in $\mathcal{O}_C^{\flat}$: that is, we have $|x|_{C^{\flat}} \leq |x'|_{C^{\flat}}$.

Suppose otherwise. We then have $|x|_{C^{\flat}} > |x'|_{C^{\flat}}$, so we can write $x' = tx$ for some $t \in \mathfrak{m}_{C^{\flat}}$. Since x is not zero, it divides π^n for $n \gg 0$. Consequently, our assumption that $x^{\sharp}$ is a multiple of $x'^{\sharp}$ guarantees that $(\pi^n)^{\sharp}$ is a unit multiple of $(\pi^n t)^{\sharp}$ in $\mathcal{O}_K$. Since $\pi^{\sharp}$ is a unit multiple of p and $\mathcal{O}_K$ is p-torsion-free, it follows that $t^{\sharp}$ is a unit in $\mathcal{O}_K$. This is impossible, since the image of $t^{\sharp}$ is nilpotent in the ring $\mathcal{O}_K / p\,\mathcal{O}_K \simeq \mathcal{O}_C^{\flat} / \pi\,\mathcal{O}_C^{\flat}$.

We next claim:

(c) The ring $\mathcal{O}_K$ is an integral domain.

To prove (c), let y be any nonzero element of $\mathcal{O}_K$; we wish to show that y is not a zero divisor. By virtue of (a), we may assume that $y = x^{\sharp}$ for some nonzero element $x \in \mathcal{O}_C^{\flat}$. Then x divides π^n for some large n; we may therefore replace x by π^n. In this case, y is a unit multiple of p^n, which we have already seen is not a zero divisor in $\mathcal{O}_K$.

For each element $y \in \mathcal{O}_K$, let us define $|y|_K = |x|_{C^{\flat}}$, where x is any element of $\mathcal{O}_{C^{\flat}}$ satisfying $y = x^{\sharp} \cdot \text{unit}$. It follows from (a) and (b) that $|y|_K$ is well-defined. Moreover, we have obvious identities

$$|0|_K = 0 \qquad |1|_K = 1 \qquad |y \cdot z|_K = |y|_K \cdot |z|_K.$$

Moreover, it follows from (b) that for each $y, z \in \mathcal{O}_K$, we have $|y|_K \leq |z|_K$ if and only if y is divisible by z. This immediately implies that $|y + z|_K \leq \max(|y|_K, |z|_K)$. We can therefore extend $|\bullet|_K$ uniquely to a non-archimedean absolute value on the fraction field K of $\mathcal{O}_K$. Moreover, an element $\frac{y}{z}$ of K satisfies $|\frac{y}{z}|_K = \frac{|y|_K}{|z|_K} \leq 1$ if and only if $|y|_K \leq |z|_K$: that is, if and only if y is divisible by $z \in \mathcal{O}_K$. It follows that $\mathcal{O}_K$ is the valuation ring of K (with respect to the absolute value $|\bullet|_K$).

Note that $|p|_K = |\pi|_{C^{\flat}} < 1$, so that K has residue characteristic p. Moreover, since p is not a zero-divisor in $\mathcal{O}_K$, the field K has characteristic zero: that is, p is a pseudo-uniformizer of $\mathcal{O}_K$. Consequently, the assertion that $\mathcal{O}_K$ is p-adically complete guarantees that it is complete with respect to its absolute value. The maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_K$ is not generated by p: for example, it contains the element $(\pi^{1/p})^{\sharp}$, which is not divisible by p. Finally, we note that the isomorphisms

$$\mathcal{O}_K / p\,\mathcal{O}_K \simeq \mathbf{A}_{\mathrm{inf}}/(\xi, p) \simeq (\mathbf{A}_{\mathrm{inf}}/p\mathbf{A}_{\mathrm{inf}})/(\xi) = \mathcal{O}_C^{\flat} / \pi\,\mathcal{O}_C^{\flat}$$

guarantee that the Frobenius map is surjective on $\mathcal{O}_K / p\,\mathcal{O}_K$. Consequently, K is a perfectoid field of characteristic zero which is an untilt of C (Remark 3). $\square$