Next time

1. [7, Con(4)] ⇝ Natural domains.

2. [34, Lem(8)] ⇝ finishes the tilting equivalence.

Valuation Spectra.

- $A \in \mathrm{CRing}$. The val'n Spectrum $\mathrm{Spv}(A) := \{ |\cdot|_v : |\ | \text{ is val'n on } A \} / \sim$
  - (v) $|\cdot|_v : A \to \Gamma \cup \{0\}$, Γ is a tot ord. gp.

    (narc). $|x+y|_v \leq \max\{|x|_v, |y|_v\}$.
- $\mathrm{Spv}\, A$ has a topology given by, for $f, s \in A$.

  $\mathrm{Spv}(A)\left(\frac{f}{s}\right) := \{ |\ |_v \in \mathrm{Spv}(A) : |f| \leq |s| \neq 0 \}$

- $\mathrm{Spv}(A)\left(\frac{f_1}{s_1}\right) \cap \mathrm{Spv}(A)\left(\frac{f_2}{s_2}\right) = \{ |\cdot| : |f_1| \leq |s_1|, |f_2| \leq |s_2| \}$

  $= \{ |\cdot| : |f_1 s_2| \leq |s_1 s_2|, |f_2 s_1| \leq |s_1 s_2| \}$.

  $= \mathrm{Spv}\, A\left(\frac{\{f_1 s_2, f_2 s_1\}}{s_1 s_2}\right)$. $|f_1 s_2 + f_2 s_1|_v = \max\{|f_1 s_2|, |f_2 s_1|\} \leq |s_1 s_2|$.

Rmk: $\mathrm{Spv}\, A\left(\frac{T}{s}\right) := \{ |\ |_v : |t| \leq |s|\ \forall t \in T \}$.

T finite subset of A,

Ex: $\mathrm{Spv}(A)\left(\frac{0}{s}\right) = \{ |\ |_v : |s|_v \neq 0 \}$.

$\mathrm{Spv}(A)\left(\frac{1}{1}\right) = \mathrm{Spv}\, A$.

Cor: $\mathrm{Spv}\, A \xrightarrow{\text{Supp.}} \mathrm{Spec}\, A$. $\overset{(v)}{|\cdot|} : A \to \Gamma \cup \{0\} \mapsto \mathrm{Supp}|\cdot| := |\ |^{-1}(0)$.

Pf: $D(s) := \{ \mathfrak{p} \in \mathrm{Spec}\, A : s \notin \mathfrak{p} \}$, $s \in A$.

$\mathrm{Supp}^{-1}(D(s)) = \{ v \in \mathrm{Spv}\, A \mid |s|_v \neq 0 \} = \mathrm{Spv}\, A\left(\frac{0}{s}\right)$.   □.

Ex: $\mathrm{Spv}\, \mathbb{Q} := \{ |\cdot|_p, \underbrace{|\cdot|_{triv}}_{\leadsto\, 0\, \in \mathrm{Spec}\, \mathbb{Z}.}, p \text{ prime} \}$. (narc.) $(\mathrm{Spv}\, \mathbb{Q} \to \mathrm{Spec}\, \mathbb{Z})$

- Basic open $\mathrm{Spv}(\mathbb{Q})\left(\frac{f}{s}\right) = \{ p : |f|_p \leq |s|_p, |s|_p \neq 0. \}$.

  $= \{ p : |x|_p \leq 1, x = f/s \}$.
- We can't have inf. many primes st. $|x|_p > 1$.

$\therefore$ Open sets are complement of a set of finite primes.

$\therefore$ $\mathrm{Spv}\, \mathbb{Q} \simeq \mathrm{Spec}\, \mathbb{Z}$. (Zariski topology).

Ex: Spv K. is irreducible and the triv val'n is the generic pt.

i.e. $|\cdot|_{triv} \rightsquigarrow v \quad \forall v \in \mathrm{Spv}\, K$.

$\iff v \in \overline{\{|\cdot|_{triv}\}}$.

$|\cdot|_{triv} : K \longrightarrow \{0,1\}$. $(0 \mapsto 0, \quad x \mapsto 1 \quad \forall x \neq 0.)$

Pf: let $v \in \mathrm{Spv}K(\frac{f}{s})$. $\Rightarrow |s|_v \neq 0$. $(s \neq 0)$.

$\Rightarrow |s|_{triv} \neq 0$.

$\therefore |\cdot|_{triv} \in \mathrm{Spv}K(\frac{f}{s})$. $\square$.

Thm: • Spv A is in fact a Spectral Space. ($\simeq$ Spec R, for some ring R)

in particular, it is sober (irred has a unique generic pt).

• Spv is functorial, $A \xrightarrow{\varphi} B$

$\mathrm{Spv}\, B \longrightarrow \mathrm{Spv}\, A$.

this map is also spectral.

Rmk: most val'n we think of are usually height 1.

equivalently the image can be embedded into $\mathbb{R}_{>0} \cup \{0\}$.

Defn: • The convex subgps of Tot Ab Gp. forms a well-ordered set.

The cardinality of this set, is its height. of Γ.

Def'n: • If $|\cdot|_v : A \to \Gamma \cup \{0\}$.

• let Γ_v be the convex subgp gen by the image of $A \setminus \{0\}$.

- this is the value gp of v.

• the height of v is the height of Γ_v.

eg. the p-adic val'n. on $\mathbb{Q}$. its image $\{p^{\mathbb{Z}} : p^n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$.

$ht(|\cdot|_p) = 1$. $\because$ only convex subgp of $\mathbb{Z}$ are 0 and itself.

• $a \leq \delta \leq b$, $a, b \in H$, $H \leq \Gamma$, $\delta \in$ tot ab gp, then $\delta \in H$.

2. An Example of rank 2-val'n.

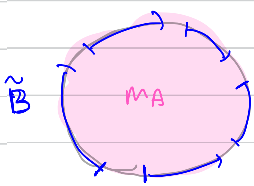
- This "recipe" is given [2.5, Wed 19], [2, Con 14] ← in here.
- Ctr:
  - let A be a val'n subrg of K [2.1, Wed 19]

(i.e. $\exists\ v : K \to \Gamma \cup \{0\}$, st. A = val'n subrg of $v = \{x \in K : |x|_v \le 1\}$.

-

$$\begin{tikzcd} m_A \arrow[r, hook] & \tilde{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\dashv"] & A \arrow[r, hook] \arrow[d] & K \\ m_B \arrow[r, hook] & B \arrow[r] & A/m_A = k & \end{tikzcd}$$

Prop: $\{B : B \text{ val'n subring of } k\} \longleftrightarrow \{\tilde{B} : \tilde{B} \text{ val'n subrg of } K \text{ contained in } A\}$.



- $|\cdot|_B : k \to \Gamma' \cup \{0\}$, whose val'n ring is B.
- $\exists\ |\cdot|_{\tilde{B}} : K \to \Gamma'' \cup \{0\}$. st. val'n ring wrt $|\cdot|_{\tilde{B}}$ is $\tilde{B}$.

Thm: In the above case, we have ses

$$1 \to \Gamma_B \to \Gamma_{\tilde{B}} \to \Gamma_A \to 1.$$

Pf: Step 0: recall [2.2, Wed 19]

- (A, K), Then there is a canonical choice of val'n gp. $\Gamma_A \simeq K^\times / A^\times$.

$K^\times \to K^\times/A^\times \cup \{0\}$ [$\bar{x} \le \bar{y} \Leftrightarrow xy^{-1} \in A$] is total order on RHS.

$(K^\times \to K^\times/A^\times,\ 0 \mapsto 0)$.

$(A, K, |\cdot|_v)$. st. A is val'n subrg of $|\cdot|_v$.)

$$\begin{tikzcd} m_A \arrow[r, hook] & \tilde{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r, hook] \arrow[d] & K \\ m_B \arrow[r, hook] & B \arrow[r] & A/m_A = k. \end{tikzcd}$$

Step 1:

$$\Gamma_{\tilde{B}} \simeq K^\times / \tilde{B}^\times.$$

$$\Gamma_A \simeq K^\times / A^\times.$$

$$\Gamma_B \simeq k^\times / B^\times \simeq (A/m_A)^\times / (\tilde{B}/m_B)^\times.$$

$$\therefore\ 1 \to \underbrace{(A/m_A)^\times / (\tilde{B}/m_B)^\times \xrightarrow{\sim} A^\times / \tilde{B}^\times}_{\Gamma_B} \hookrightarrow \underbrace{K^\times / \tilde{B}^\times}_{\Gamma_{\tilde{B}}} \to \underbrace{K^\times / A^\times}_{\Gamma_A} \to 1$$

$\square$.

Ex: • let $K = \overset{F}{\overset{\|}{k((y))}}((x))$ (Laurent series with coeff. $k((y))$).

• There is the (x)-adic val'n on K.

i) $K = F((x))$. [2.6, Wed 19]

val'n subring is $F[[x]]$.

$$\begin{tikzcd} m_A = (x) \arrow[r, dash] & \overset{\tilde{B}}{k[[y]][[x]]} \arrow[r] \arrow[d] & \overset{A}{k((y))[[x]]} \arrow[r] \arrow[d] & \overset{(K)}{k((y))((x))} \\ (y). \arrow[r, hook] & \overset{B}{k[[y]]} \arrow[r] & \overset{k}{k((y))}. \end{tikzcd}$$

Consider the y-adic val'n.

Now:

$$1 \to \Gamma_B \to \Gamma_{\tilde{B}} \to \Gamma_A \to 1.$$

as $\Gamma_B \simeq \Gamma_A \simeq \mathbb{Z}$, $\Gamma_{\tilde{B}} \simeq \mathbb{Z} \times \mathbb{Z}$., which is of height 2.

$\therefore$ $\tilde{B} \hookrightarrow k((y))[[x]]$ as a val'n subring, came from a height 2 val'n.

General thy: (I write this additively first).

Prop: A is local ring with max ideal genertd by an elemt p. $\bigcap_{n\geq 1} p^n = 0$

$v: A \to \mathbb{Z} \cup \{\infty\}$,

$x \mapsto \begin{cases} n & \text{if } x \in (p^n) \setminus (p^{n+1}) \\ \infty & \text{otherwise} \end{cases}$.

Then: if p is not nilpotent, A is a val'n ring.

Ex: $A = F[[x]]$. (x) is max ideal. is local. $\bigcap_{n\geq 1}(x^n) = 0$.

$\sum a_n t^n \longmapsto \max\{n : a_n \neq 0\}$.

$v: A \longrightarrow \mathbb{Z} \cup \{\infty\} \xrightarrow{c^{(-)}} \mathbb{R}_{\geq 0} \cup \{0\}$ where $0 < c < 1$, $c^{\infty} := 0$.

This extends to a val'n on $\mathrm{Frac} A \subset F((x))$.

Note this valutn is -ixpost a discrete val'n, i.e. $\Gamma_v \cong \mathbb{Z}$.

• Now we have two key examples.

• $\mathbb{Q}$, $\mathbb{Z}$.

• $k((y))((x))$. $\supseteq k[[y]][[x]]$

• In [4, Cor (4)] He gives an analysis of the specifcations in

3. Specializations.

- let $v, w \in \mathrm{Spv}\, A$ $\quad w \rightsquigarrow v \quad (v \in \overline{\{w\}})$

$$\begin{array}{ccc} v,w \in \mathrm{Spv}\,A & & w \rightsquigarrow v \\ \big\downarrow \text{supp.} & & \big\downarrow \\ \mathrm{Spec}\,A & & \mathrm{supp}\, w \subseteq \mathrm{supp}\, v. \end{array}$$

- $v \in \overline{\{w\}}$, $v \in \mathrm{Spv}\, A(f/s) \Rightarrow w \in \mathrm{Spv}\, A(f/s)$. Take $f=0$.
  - $|s|_v \neq 0 \Rightarrow |s|_w \neq 0$. $\quad (|s|_w = 0 \Rightarrow |s|_v = 0)$. $\Rightarrow \mathrm{supp}\, w \subseteq \mathrm{supp}\, v$.

There's two cases to analyze.

I. $\mathrm{supp}\, w = \mathrm{supp}\, v$. we say w is a vertical generization.

II. $\mathrm{supp}\, w \subsetneq \mathrm{supp}\, v$. $\rightsquigarrow$ we will consider horizontal generization.

Def'n: • Let $H \subseteq \Gamma_v$ be a convex subgp of Γ_v.

• $v : A \longrightarrow \Gamma_v \cup \{0\}$.

(want to construct $v \rightsquigarrow v_H$ i.e. v_H would move "support").

$$|\ |_H = v_H : A \longrightarrow H \cup \{0\} \hookrightarrow \Gamma_v \cup \{0\}.$$
$$f \longmapsto \begin{cases} |f|_v & \text{if } |f|_v \in H \\ 0 & \text{if } |f|_v \notin H. \end{cases}$$

- This is not a val'n in general.
- Sanity $\mathrm{supp}\, v \subseteq \mathrm{supp}\, v_H$.

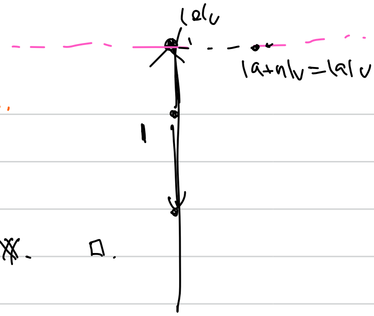
Prop: • Let H be cvx subgp of Γ_v.

• v_H is a val'n $\Longleftrightarrow$ H contains $c\Gamma_v$.

Def'n $c\Gamma_v$ = characteristic gp = convex subgp gen. by $\Gamma_{v,\geq 1} \cap \mathrm{im}\, v$.

Ex: $v(A) \leq 1 \Rightarrow c\Gamma_v = \{1\}$.

Pf: ($\Rightarrow$) Step 0. $v_H(0) = 0$, $v_H(1) = 1$.



Step 1: $\Gamma_{\geq 1} \cap vA \subseteq H$. Prove by contradiction.

- If $1 < |a|_v$, but $|a|_v \notin H$.
- $|a+1|_v = |a|_v \notin H$. ∵ nonarch.
- $a+1, a \in \mathrm{Supp}\, v_H \Rightarrow 1 \in \mathrm{Supp}\, v_H$. ※. □

Def'n: • let $v : A \to \Gamma \cup \{0\}$ be a val'n.

- The a 'horizontal spec' of v is a val'n on A of the form v_H, $H \in \Gamma$, containing $c\Gamma_v$ + convex.

[4, Con KG]

Prop: 4.3.1 If $v \overset{h}{\rightsquigarrow} w$, $w \overset{v}{\rightsquigarrow} w'$ then $\exists$

"Exchange lemma".

$$\begin{tikzcd} v \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow, "\exists v"'] & w \arrow[d, rightsquigarrow] \\ v' \arrow[r, rightsquigarrow, "h"'] & w' \end{tikzcd}$$

Thm: 4.3.2. Suppose $v \rightsquigarrow w$ (w is a specialization of v) then $\exists$ factorization.

$$\begin{tikzcd} v \arrow[d, rightsquigarrow, "\text{verti}"'] \arrow[dr, rightsquigarrow] & \\ v' \arrow[r, rightsquigarrow, "\text{horiz}"'] & w. \end{tikzcd}$$