2020 / 11 / 3 perfectoid seminar

⓪ Review up to last time

K : algebraically closed completely valued (non-Archimedean) field of residue characteristic p ($p > 0$)

$\rightsquigarrow$ $K^\flat := \lim (\cdots \xrightarrow{\varphi} K \xrightarrow{\varphi} K)$: a field of char p

$\mathcal{O}_K^\flat := \lim (\cdots \longrightarrow \mathcal{O}_K \longrightarrow \mathcal{O}_K)$: a valuation ring of $K^\flat$

φ : Frobenius map

$$\begin{array}{c} \{\text{AC CV field of res. } p\} \\ (-)^\flat \text{ tilting} \downarrow \quad \uparrow \text{ untilting} \\ \{\text{AC CV field of char } p\} \end{array}$$

$\rightsquigarrow$ generalized version

$$\begin{array}{c} \{\text{perfectoid field of res. } p\} \\ \text{tilting} \downarrow \quad \uparrow \text{ untilting} \\ \{\text{perfectoid field of char } p\} \end{array}$$

$\lrcorner$

◎ Today

Fix $C^\flat$: a perfectoid field of char p.

Q. How can we classify the untilts of $C^\flat$?

Perfectoid geometry	Complex analysis
$\{\text{untilts } (K, \iota) \text{ of } C^\flat\}/\sim$	open disk $\{z \in \mathbb{C} \mid \lvert z \rvert < 1\}$
$\lvert p \rvert_K$	$\lvert z \rvert$
prime number p	coordinate z
the period ring $A_{\inf}$	$\{\text{Power series } \sum_{n \geq 0} c_n z^n \mid \lvert c_n \rvert \leq 1\}$

$B_{[a,b]} \rightsquigarrow \mathcal{O}^{an}(\{z \in \mathbb{C} \mid a < \lvert z \rvert < b\})$

@ Fact from lecture 2

<u>Prop</u> (Prop 15 & Thm 17)

K : perfectoid field of res. char p

$\rightsquigarrow$ (1) $K^\flat$: perfectoid field of char p

(2) $\mathcal{O}^\flat_K$: valuation ring of $K^\flat$

$\forall \; \pi \in \mathcal{O}^\flat_K$

$0 < |\pi|_K < 1 \Rightarrow \mathcal{O}_K^\flat [\pi^{-1}] \simeq K^\flat$

<u>Rem</u> $\# : K^\flat \longrightarrow K$: multiplicative map

$x = \{x_i\}_{i \geq 0} \mapsto x^\# := x_0$

$|x|_{K^\flat} := |x^\#|_K$

<u>Prop</u> (Lem 18) $\pi \in K^\flat$

$|p|_K \leq |\pi|_{K^\flat} < 1$

$\Longrightarrow \mathcal{O}_K^\flat / (\pi) \xrightarrow{\simeq} \mathcal{O}_K / (\pi^\#)$

⓪ Classification of the untilts of $C^\flat$

$C^\flat$: perfectoid field of char p
(Lecture 3)

$\boxed{\underline{\text{Goal}}}$: classification of the untilts (K, ι) of $C^\flat$

For an untilt (K, ι) of $C^\flat$,

$$\# : C^\flat \to K$$
$$:= \quad C^\flat \underset{\simeq}{\xrightarrow{\iota}} K^\flat \xrightarrow{\#} K.$$

$\# : \mathcal{O}_K^\flat \to \mathcal{O}_K$ is surjective mod p
(Prop 13, Lecture 2)

$\rightsquigarrow \forall x \in \mathcal{O}_K \ \exists \ c_0 \in \mathcal{O}_C^\flat \ \& \ x' \in \mathcal{O}_K$

s.t.
$$x = c_0^\# + p x'$$

$$\rightsquigarrow \quad x = c_0^\# + c_1^\# p + c_2^\# p^2 + \cdots$$

(This makes sence because $\mathcal{O}_K$ is p-adically complete.)

Remark. This expansion is not unique.

Notation 1: R : perfect ring of char p

(i.e. ∘ $p = 0$ in R

∘ $\forall x \in R$ $\exists ! y$ s.t. $y^p = x$.)

$\exists$ $W(R)$: ring of Witt vectors

s.t. (1) $W(R)/(p) \simeq R$

(2) p is not a zero-divisor in $W(R)$

(3) $W(R)$: p-adically complete

Notation 2: $\exists$ $R \to W(R)$: multiplicative map

$x \mapsto [x]$

s.t.

(1) $R \to W(R) \longrightarrow W(R)/(p) \simeq R$

$x \mapsto [x] \longmapsto x$

(2) $[x] \in W(R)$ admits a p^n-th root for $n \geq 0$

$[x]$: Teichmuller representative

$\forall\ x \in W(R)\quad \exists\ c_0 \in R$ s.t. & $x' \in W(R)$

$$x = [c_0] + x'p$$

$$\rightsquigarrow \quad x = [c_0] + [c_1]p + [c_2]p^2 + \cdots$$

: Teichmüller expansion

(⚠ this expansion is unique.)

<u>Prop.</u> $\forall\ A$: p-adically complete ring

$$\mathrm{Hom}(W(R), A) \xrightarrow{\simeq} \mathrm{Hom}(R, A/pA)$$

<u>Def.</u> $C^\flat$: perfectoid field of char p

$$A_{\inf} := W(\mathcal{O}_C^\flat)$$

where $\mathcal{O}_C^\flat$: the valuation ring of $C^\flat$
$(= \mathcal{O}_{C^\flat})$

☺ $\mathcal{O}_C^\flat$ is perfect ring since

- $\mathcal{O}_C^\flat \xrightarrow{\varphi} \mathcal{O}_C^\flat$ is surjective by the def'n of perfectiod fields
- $\mathcal{O}_C^\flat$ is integral $\rightsquigarrow$ φ is injective.

(K, ι) : an untilt of $C^\flat$

$$\# : \mathcal{O}_C^\flat \xrightarrow{\iota} \underset{\text{s.t.}}{\mathcal{O}_K^\flat} \xrightarrow[\text{ring hom}]{\#} \mathcal{O}_K/p\mathcal{O}_K$$

$$\varprojlim_{\varphi} \left(\xrightarrow{\varphi} \mathcal{O}_K/p\mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_K/p\mathcal{O}_K \right) \to \mathcal{O}_K/p\mathcal{O}_K$$

φ : ring hom

$\rightsquigarrow$ We get

$$\theta : A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat) \longrightarrow \mathcal{O}_K$$

$$\theta\left(\sum_{n\geq 0} [c_n] p^n\right) = \sum_{n \geq 0} c_n^\# p^n$$

<u>Remark.</u> θ is surjective.

<u>Remark.</u> θ is local : i.e.

$\sum [c_n] p^n$ is invertible

iff $\sum c_n^\# p^n$ is invertible

$\because$ $\sum [c_n] p^n$ is invertible

$\Leftrightarrow$ c_0 is invertible

$\Leftrightarrow$ $|c_0|_{C^\flat} = 1$

$\Leftrightarrow$ $|c_0^\#|_K = 1$

$\Leftrightarrow$ $\sum c_n^\# p^n$ is invertible.

$\exists\ \pi \in \mathcal{O}_C^\flat$ s.t.

$$|\pi|_{C^\flat} = |p|_K \ \& \ \mathcal{O}_C^\flat/(\pi) \simeq \mathcal{O}_K/(p)$$

$$\rightsquigarrow \begin{tikzcd} A_{\inf} \arrow[r, two heads, "\theta"] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_K \arrow[d, two heads] \\ \mathcal{O}_C^\flat \arrow[r, two heads, "\#"] & \mathcal{O}_K/p\mathcal{O}_K \end{tikzcd}$$

$$|\pi|_{C^\flat} = |p|_K$$

$$\rightsquigarrow |\pi^\#|_K = |p|_K$$

u is invertible
$\Leftrightarrow$ $\theta(u)$: inv.
$\Leftrightarrow$ $|\theta(u)|_K = 1$
$\Leftrightarrow$ u : invertible.
(by Milnor)

$\rightsquigarrow \exists\ u \in A_{\inf}$ s.t.

$$\pi^\# = \theta(u)\cdot p \quad \text{in } \mathcal{O}_K$$

$$\rightsquigarrow \theta\Big([\pi] - up\Big) = \pi^\# - \theta(u)\cdot p \qquad [\pi] - up \in \operatorname{Ker}\theta$$

$\theta([\pi]) = \pi^\#$ $\quad \underline{\theta(p)} = \underline{p}$

Def. $\xi \in A_{\inf}$ is distinguished

if $\exists\ \pi \in \mathcal{O}_C^\flat\ \&\ u \in A_{\inf}$ s.t.

- $\xi = [\pi] - up$
- $|\pi|_{C^\flat} < 1$
- u is invertible in $A_{\inf}$

$\Longleftrightarrow \quad \xi = [c_0] + [c_1]p + \cdots$

s.t. $\quad |c_0| < 1 \quad \& \quad |c_1| = 1$

$[c_0] = [\pi] \qquad |c_0| = |\pi|$

$\xi = [\pi] - \sum_{n \geq 0} [c_n] p^{n+1}$

Prop. 3 $\{$ distinguished elements of $A_{inf} \}$ / $\sim$ unit-multiplication

(Cor. 18, Lec 3)

$\simeq \{$ untilts of $C^\flat \} / \sim$ isom

To see this, we use the following proposition.

Prop. 4 (Prop. 16, Lec 3)

For any $\xi \in A_{inf}$: distinguished element,

$\exists\, K$: perfectoid field s.t.

(1) $A_{inf}/(\xi) \simeq \mathcal{O}_K$

(2) $\mathcal{O}_C^\flat = A_{inf}/(p)$

$\longrightarrow \quad A_{inf}/(\xi, p) \simeq \mathcal{O}_K/(p)$

exhibits K as an untilt of $C^\flat$

Remark K : perfectoid field

$$\mathcal{O}_C^\flat/(\pi) \simeq \mathcal{O}_K/(p)$$

$$\Rightarrow \quad \mathcal{O}_C^\flat \simeq \mathcal{O}_K^\flat$$

$$\mathcal{O}_C^\flat \simeq \varprojlim\left(\cdots \xrightarrow{\varphi} \mathcal{O}_C^\flat/(\pi) \xrightarrow{\varphi} \mathcal{O}_C^\flat/(\pi)\right)$$

$$\mathcal{O}_K^\flat \simeq \varprojlim\left(\cdots \xrightarrow{\varphi} \mathcal{O}_K/(p) \xrightarrow{\varphi} \mathcal{O}_K/(p)\right)$$

claim $\forall \pi \in \mathcal{O}_C^\flat \quad 0 < |\pi| < 1$

$$\mathcal{O}_C^\flat \cong \varprojlim\left(\cdots \xrightarrow{\varphi} \mathcal{O}_C^\flat/(\pi)\right)$$

proof.

$$\begin{tikzcd}
\mathcal{O}_C^\flat/(\pi^{p^{n+1}}) \arrow[r] \arrow[d] & \mathcal{O}_C^\flat/(\pi^{p^{n+1}}) \arrow[d] \\
\mathcal{O}_C^\flat/(\pi^{p^n}) \arrow[r, "\varphi"] \arrow[ur, dashed] & \mathcal{O}_C^\flat/(\pi^{p^n})
\end{tikzcd}$$

$$x \equiv y \mod \pi^{p^n}$$

$\rightsquigarrow$

$$x^p \equiv y^p \mod \pi^{p^{n+1}}$$

$\rightsquigarrow \quad Z(n) := \varprojlim\left(\cdots \xrightarrow{\varphi} \mathcal{O}_C^\flat/(\pi^{p^n})\right)$

$$\mathcal{O}_C^\flat \simeq \varprojlim\left(\to \mathcal{O}_C^\flat/(\pi^{p^{n+1}}) \longrightarrow \mathcal{O}_C^\flat/(\pi^{p^n}) \to \cdots\right)$$

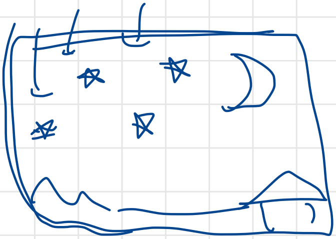
$\rightsquigarrow \quad \mathcal{O}_C^\flat \simeq \varprojlim Z(1)$

$\square$

proof of Prop 3 (using Prop 4)

$$\{\text{distinguished elements}\}/\sim \;\longrightarrow\; \{\text{untilt}\}/\sim$$

$$\xi \longmapsto \mathrm{Frac}(A_{\inf}/(\xi))$$



$[\pi]$ - up $\qquad \xleftarrow[(*)]{} \quad (K, \iota)$

- Well-definedness of $(*)$

Fix an untilt (K, ι).

Lem $\theta : A_{\inf} \to \mathcal{O}_K$

$\rightsquigarrow$ $\forall\, \xi \in \ker\theta$: distinguished element

$\ker\theta = (\xi)$

proof: $\bar\theta : A_{\inf}/(\xi) \longrightarrow \mathcal{O}_K$

$$\begin{array}{c} A_{\inf}/(\xi) \\ \wr\| \\ \mathcal{O}_{K'} \end{array}$$

$\mathcal{O}_K$: integral

$\rightsquigarrow$ $\ker\bar\theta$: prime ideal of $\mathcal{O}_{K'}$

$\rightsquigarrow$ $\ker\bar\theta = 0$ or $\mathfrak{m}_{K'}$

$\mathcal{O}_K$ is not a field. $\square$

$\square$