10 / 11 / 2020 perfectoid seminar

Let $C^\flat$ be a perfectoid field of char p.

perfectoid geometry	complex analysis
$\{\text{untilts of } C^\flat\}/\sim$	unit disk in $\mathbb{C}$
$\mathbb{A}_{\inf}$	$\{f = \sum_{n\geq 0} c_n z^n \mid \lvert c_n\rvert \leq 1\}$
$B_{[a,b]}$	the ring of holo. func. on the annulus $\{z \in \mathbb{C} \mid a \leq \lvert z\rvert \leq b\}$
$\Big\}$ lim	
B	the ring of holo. functions on the punctured disk $\{z \in \mathbb{C} \mid 0 < \lvert z\rvert < 1\}$

Def. The Fargues–Fontaine curve of $C^\flat$ is the scheme $\mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}} B^{\varphi = p^n}\right)$

where
- φ is an automorphism (of B) induced by the Frobenius map of $C^\flat$
- $B^{\varphi = p^n} := \{ x \in B \mid \varphi(x) = p^n x \}$

◎ Today's Aim

- Construct the ring $B = \varprojlim B_{[a,b]}$ (Lecture 4)

- Characterize $B_{[a,b]}$ as the completion w.r.t. Gauss norms $|\cdot|_a, |\cdot|_b$

⊙ Holomorphic functions of the variable p

$C^\flat$: perfectoid field of char p

<u>Def.</u> (K, ι) : untilt of $C^\flat$

$r(K, \iota) := |p|_K$

(We fix an absolute values on K which satisfies that $\forall\ c \in C^\flat \quad |c^\#|_K = |c|_{C^\flat}$)

$C^\flat \xrightarrow{\simeq} K^\flat \xrightarrow{\#} K$

<u>Rem</u> $[\cdot] : \mathcal{O}_C^\flat \longrightarrow A_{\inf}$ Teichmüller representatives

$\rightsquigarrow \quad 0 < |\pi| < 1$ ($\pi \in \mathcal{O}_C^\flat$)

$$\underline{[\cdot] : C^\flat \simeq \mathcal{O}_C^\flat[\tfrac{1}{\pi}]} \longrightarrow \underline{A_{\inf}[\tfrac{1}{[\pi]}]}$$

multiplicative map

Let's consider various enlargements of $A_{\inf}$!

① $\mathbb{A}_{\inf}[\frac{1}{[\pi]}]$

Fix a quasi-uniformizer $\pi \in C^\flat$

i.e. $0 < |\pi|_{C^\flat} < 1$.

$\mathbb{A}_{\inf}[\frac{1}{[\pi]}]$: localization

$$\mathbb{A}_{\inf}\left[\frac{1}{[\pi]}\right] \ni f = \sum_{n\geq 0} [c_n]\, p^n$$

$\{|c_n|\}$: bounded $\qquad |c_n| \leq \frac{1}{|\pi|^m}$

$\Downarrow \quad \Uparrow$

☺ $f = \dfrac{\sum [c_n \pi^m]\, p^n}{[\pi^m]}$

any element of $\mathbb{A}_{\inf}[\frac{1}{[\pi]}]$ can represented by

~~where $|c_n \pi^m| \leq 1$~~ ☹

$\hookrightarrow \quad f = \sum \left[\dfrac{c_n'}{\pi^m}\right] p^n \qquad \forall n \ |c_n'| \leq 1$

<u>Rem</u> This Teichmüller expansion is unique.

<u>Rem</u> $\mathbb{A}_{\inf}[\frac{1}{[\pi]}]$ <u>does not</u> depend on the choice of the quasi-uniformizer π.

∵ $0 < |\pi| \leq |\pi'| < 1$

$\rightsquigarrow$ $\pi = a \cdot \pi'$ $\quad a \in \mathcal{O}_C^\flat$

$\rightsquigarrow$ We get a map

$$A_{\inf}\left[\frac{1}{[\pi']}\right] \longrightarrow A_{\inf}\left[\frac{1}{[\pi]}\right]$$

$$\frac{f}{[\pi'^m]} \longmapsto \frac{[a^m]\cdot f}{[\pi^m]}$$

where $f \in A_{\inf}$.

This map is bijective.

∵ $\exists\, n \in \mathbb{Z}_{\geq 0}$, $b \in \mathcal{O}_C^\flat$

$$\pi'^n = b \cdot \pi$$

$\rightsquigarrow$ we get the inverse map

$$A_{\inf}\left[\frac{1}{\pi}\right] \to A_{\inf}\left[\frac{1}{\pi'}\right]$$

$$\frac{f}{[\pi^m]} \longmapsto \frac{[b^m] f}{[\pi'^{nm}]}$$

∴

∴

$$\theta : A_{\inf}\left[\frac{1}{[\pi]}\right] \longrightarrow O_K\left[\frac{1}{\pi^\#}\right] \cong K$$

(2) $A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right]$

(K, ι) : untilt of char 0

$$\rightsquigarrow \quad A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right] \xrightarrow{\theta} K$$

$$\sum_{n \geq -k} [c_n] p^n \qquad \{|c_n|\} : \text{bounded}$$

(3) $B_{[a,b]}$ "holomorphic functions on the annulus"

Fix $0 < a \leq b < 1$ s.t.

$\exists \; \pi_a, \pi_b \in C^b$ s.t.

$$|\pi_a| = a, \quad |\pi_b| = b.$$

$$A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right] \subseteq A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right]$$

This does not depend on the choice of π_a, π_b.

$$\frac{[\pi_a]}{p} = \frac{1}{p} \cdot [\pi_a] \in$$

$$\left(\pi^n = a \cdot \pi_b \quad a \in O_C^b\right) \quad \frac{p}{[\pi_b]} = \frac{[a]\, p}{[\pi^n]} \in$$

$\rightsquigarrow$ p-adic completion

$$B_{[a,b]} := \widehat{A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]}\left[\frac{1}{p}\right]$$

For an untilt (K, ι) of char 0 s.t.

$$a \leq |p|_K \leq b,$$

we have

$$\begin{array}{ccc} A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right] & \xrightarrow{\theta} & \mathcal{O}_K \\ \downarrow & & \downarrow \\ A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right] & \xrightarrow{\theta} & K. \end{array}$$

$$\left(\because \quad \frac{\pi_a^\#}{p}, \frac{p}{\pi_b^\#} \in \mathcal{O}_K \right)$$

Since $\mathcal{O}_K$: p-adically complete

&

K : characteristic 0,

we get

$$B_{[a,b]} = \widehat{A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]}\left[\frac{1}{p}\right]$$

$$\longrightarrow \mathcal{O}_K\left[\frac{1}{p}\right] = K$$

$$0 < a \leq a' \leq b' \leq b < 1$$

$$\rightsquigarrow \quad \pi_a = c \cdot \pi_{a'}$$

$$\frac{1}{\pi_b} = c' \cdot \frac{1}{\pi_{b'}} \qquad (c, c' \in \mathcal{O}_C^b)$$

$$\rightsquigarrow \quad A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right] \subseteq A_{\inf}\left[\frac{[\pi_{a'}]}{p}, \frac{p}{[\pi_{b'}]}\right]$$

$$\rightsquigarrow \quad B_{[a,b]} \longrightarrow B_{[a',b']}$$

④ B

"holomorphic function on the punctured disk"

<u>Def.</u> $$B := \varprojlim_{\substack{[a,b] \subseteq (0,1) \\ a,b \in \text{value grp of } C^b}} B_{[a,b]}$$

(i.e. $\exists\ \pi_a, \pi_b \in C^b$
$|\pi_a| = a, \ |\pi_b| = b$)

$$\mathrm{Im}\,|\cdot| \subseteq \mathbb{R}_{\geq 0}$$

② Gauss norm

<u>Def.</u>: (Gauss norm) $0 < \rho < 1 \in \mathbb{R}$

$$f = \sum_{n \geq -k} [c_n] p^n \in A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right]$$

$(c_n \in C^\flat)$

$$|f|_\rho := \sup \{ |c_n| \cdot \rho^n \}$$

(this is well-defined since $\{ |c_n| \}$ is bounded)

" punctured disk "

<u>Notation</u>:

(1) $Y = \{ \text{untilt of } C^\flat \text{ of char } 0 \} / \text{isom}$

(2) For every $y = (K, \iota)$, we have

$$\theta_y : A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right] \longrightarrow K.$$

$$f(y) := \theta_y(f) \in K$$

$\left(f \in A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right] \right)$

(3) $Y_{[a,b]} \subseteq Y$: annulus

$= \{ y = (K, \iota) \mid a \leq |p|_K \leq b \}$

Rem $(k, \iota) \in Y.$ $\qquad \rho = |p|_k$

$$\rightsquigarrow \quad |f(y)|_k = \left|\sum c_n^\# p^n\right| \le \sup\{|c_n^\#| \cdot \rho^n\}$$

$$= |f|_\rho$$

$\forall y \in Y$ s.t. $|p|_y = \rho$

Rem $\forall f \in A_{inf}$ $\{\rho \in (0,1) \mid |f|_\rho \ne |f(y)|_k\}$ is a discrete subset of $(0,1)$.

Prop $\pi_a, \pi_b \in C^b$ $\qquad 0 < a = |\pi_a| \le b = |\pi_b| < 1$

$$V_0 := \{f \in A_{inf}[\tfrac{1}{p}, \tfrac{1}{[\pi]}] \mid |f|_a \le 1, \; |f|_b \le 1\}$$

$$\rightsquigarrow \quad V_0 = A_{inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]$$

proof. (1) $A_{inf} \subseteq V_0$

$\left(\because \; f \in A_{inf} \rightsquigarrow |f|_\rho = \sup\{c_n \cdot \rho^n\} \le 1 \right.$
$f = \sum c_n p^n$, $|c_n| \le 1$ $\left.\right)$

(2) $\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]} \in V_0$

$\left(\because \; \left|\frac{[\pi_a]}{p}\right|_a = 1, \quad \left|\frac{[\pi_a]}{p}\right|_b = \frac{a}{b} < 1 \right.$

$\left|\frac{p}{[\pi_b]}\right|_a = \frac{a}{b} < 1, \quad \left|\frac{p}{[\pi_b]}\right|_b = 1$

(3) V_0 is closed under addition & multiplication

∵ $f, g \in V_0 \rightsquigarrow |f+g|_\rho \le \sup\{|f|_\rho, |g|_\rho\}$ by definition

$|fg|_\rho = |f|_\rho \cdot |g|_\rho$ omit the proof of this. (Prop. 17, Lecture 5)

$$\rightsquigarrow \quad V_0 \supseteq A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]$$

(4) $f \in V_0 \rightsquigarrow f \in A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]$

∵ $$f = \sum c_n p^n = \left(\sum_{n<m} c_n p^n\right) + \underbrace{\left(\sum_{n\ge 0} [c_{n+m}\pi_b^m] p^n\right)\cdot\left(\frac{p}{[\pi_b]}\right)^m}_{\in A_{\inf}\left[\frac{p}{[\pi_b]}\right]}$$

where $m \gg 0$ satisfies that $\forall_n \quad c_n \pi_b^m \in \mathcal{O}_C^b$

$\rightsquigarrow$ We reduce to the case where the expansion of f is finite.

$|f|_a, |f|_b \leq 1$ means

$\forall n$

$$|c_n| \cdot a^n \leq 1, \quad |c_n| \cdot b^n \leq 1.$$

$$\rightsquigarrow \quad c_n \pi_a^n, \quad c_n \pi_b^n \in \mathcal{O}_C^\flat$$

For $n \leq 0$,

$$[c_n] p^n = [c_n \pi_a^n] \left(\frac{[\pi_a]}{p} \right)^{-n}$$

$$\in A_{\inf}\left[\frac{[\pi_a]}{p}\right]$$

For $n \geq 1$, (≥ 0)

$$[c_n] p^n = [c_n \pi_b^n] \left(\frac{p}{[\pi_b]} \right)^n \in A_{\inf}\left(\frac{p}{[\pi_b]}\right)$$

$$\rightsquigarrow \quad f \in A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right]$$

$\square$

Fact V : $\mathbb{Q}_p$ - vector space non-arc.

equipped w/ a norm $|\cdot|$

$V_0 :=$ unit ball of V $\leftarrow$ submodule of V

p-adic completion

completion wrt $|\cdot|$

$$\rightsquigarrow \quad \widehat{V_0}\left[\frac{1}{p}\right] \xrightarrow{\simeq} \overline{V}$$

: an isomorphism

$$
\begin{tikzcd}
V_0 \arrow[r, hook] \arrow[dr] & V \arrow[r] & \overline{V} \\
 & \hat{V}_0 \arrow[ur] &
\end{tikzcd}
$$

$\rightsquigarrow$

$$\hat{V}_0\left[\frac{1}{p}\right] \longrightarrow \overline{V}$$

$\rightsquigarrow$

<u>claim</u> (1) V_0 : closed unit ball $\subseteq V$

(2) $\hat{V}_0$: closed unit ball $\subseteq \overline{V}$

<u>proof</u>. (2)

$\{x_i\} \subseteq \hat{V}_0 \qquad x_i \longrightarrow x \in \overline{V}$

$x_i = \{y_{ij}\} \qquad y_{ij} \in V_0$

$\rightsquigarrow \quad y_{ii} \longrightarrow x \quad$ in $\overline{V}$

$$V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_0$$

$$\hat{V} = \widehat{\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_0} = \underbrace{\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{V}_0}$$

$\cong$

$\overline{V}$

$$\overline{V} \cong \varprojlim V/p^n V_0$$

$$\hat{V}_0 = \varprojlim V_0/p^n V_0$$

$$\hat{V}_0 \hookrightarrow \overline{V}$$

$$\begin{tikzcd} V_0/p^n V_0 \arrow[r] \arrow[d, dotted] & V/p^n V_0 \arrow[r] \arrow[d, dotted] & \operatorname{coker} \arrow[d, dotted] \\ & & \varprojlim \operatorname{coker} \end{tikzcd}$$

$$\operatorname{coker} = V/V_0, \qquad \varprojlim \operatorname{coker} = V/V_0$$

$$[\tfrac{1}{p}]$$

$$\hat{V}_0[\tfrac{1}{p}] \longrightarrow \overline{V}[\tfrac{1}{p}] \cong \overline{V} \ : \text{surjective}$$

$\square$

Cor $\quad B_{[a,b]}$ is the completion of

$A_{\inf}\left[\frac{1}{p}, \frac{1}{[\pi]}\right]$ w.r.t. $|\cdot|_a + |\cdot|_b$

$$\left(\begin{aligned} & V_0 = A_{\inf}\left[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}\right] \\ & B_{[a,b]} = \hat{V}_0\left[\frac{1}{p}\right] \simeq \overline{V} \end{aligned} \right)$$