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Ref. BO78. Notes on Crystalline Cohomology.
Gro68 Crystals and the De Rham Cohomology of Schemes.

Aim: To define Crystalline cohomology.

- Initially we have $\ell$-adic coho:
  $X$ is smth + proj + geo conected $/ k = \mathbb{F}_q$ $q = p^r$, $p$ prime.

$\rightsquigarrow$ means "this gives".

$\rightsquigarrow$ nice invariant: $\zeta_X(t)$

$\rightsquigarrow$ $\zeta_X(t)$: rational f'n + satisfies f'nl eq'n + "zero"
i.e. Weil conjectures.

Weil: Proof of above properties follows from $\exists$ of Weil coho. theories.

An example: is when $\ell \neq p$
$$H^i(X; \mathbb{Q}_\ell) := \varprojlim H^i(X_{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \mathbb{Q}_\ell .$$
$X_{\bar{k}} := X \times_k \bar{k}$. We take the étale cohomology on rhs.

- If $\ell = p$, the above formula does not give a Weil coho. theory.

N: Can't lift it globally.

- The idea of Groth. of defining a Weil coho. for $X/k$ $k$ char $p>0$,
  1. lift to $\tilde{X} / W(k)$. $W(k)$ ring of witt vectors of $k$.
  2. Take de Rham coho of $\tilde{X} / W(k)$.

De Rham: algebraic deRham coho.
$H^i_{dR}(X/S) := R\Gamma^i(X, \Omega^{\bullet}_{X/S})$.
$\Omega^{\bullet}_{X/S}$: cplx in $D^+(\mathcal{O}_X)$.
Note: Mod $\mathcal{O}_X$ has enough injectives.

- Gro68.4. Introduced two sites. Infinitesmal site + Stratified site.

4.1. Gro68. let $X \in \mathrm{Sch}_S$. Define $\mathrm{Inf}(X/S)$.

Objects: nilpotent $S$-immersions. $\mathcal{U} \overset{\mathrm{cl}}{\hookrightarrow} T$.
where $\mathcal{U} \overset{\mathrm{op}}{\hookrightarrow} X$. (zariski. open).

$$\begin{tikzcd} T \arrow[d] & \mathcal{U} \arrow[l, hook', "\mathrm{cl}"'] \arrow[d, hook, "\mathrm{op}"] \\ S & X \arrow[l] \end{tikzcd}$$
denoted by $(\mathcal{U}, T)$.

Defn: $\mathcal{U} \overset{\mathrm{cl}}{\hookrightarrow} T$ is nilpotent imm. if $\mathcal{U} \overset{\mathrm{cl}}{\hookrightarrow} T$ is closed immersion + (recall closed means $\exists$ $\mathcal{Y}$ qc. of ideals on $T$) the corresponding ideal shf $\mathcal{Y}$ is nilpotent.

morphisms:
$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d, hook, "\mathrm{incl.}"'] & T \arrow[d, "u"] \\ \mathcal{U}' \arrow[r] & T' \end{tikzcd}$$

- This implies $|\mathcal{U}| = |T|$ as Topological spaces.

Covering: $(\mathcal{U}_i, T_i) \to (\mathcal{U}, T)$. st. $T_i \to T$ Zariski open.
$\bigcup T_i = T$.

Ex: $\mathrm{Spec}\, \frac{k[x]}{x} \hookrightarrow \mathrm{Spec}\, \frac{k[x]}{x^2}$.

- Inf Topos = Shv ( Inf $(X/S)$ )

• obj: $\{ F_{(U,T)} \in Sh_{zar}(T) \}_{(U,T) \in Inf(X/S)}$

$+ \{ \rho_u : u^* F_{(U',T')} \to F_{(U,T)} \}_{u : (U',T') \to (U,T)}$.

+ some compatibility,

An important example:

$\mathcal{O}_{X/S} : (U,T) \mapsto \mathcal{O}_T$.

"Structure sheaf" of inf. site. of $X/S$.

• $S$ is char 0 ⇝ $S \to \operatorname{Spec} \mathbb{Q}$.

Thm 4.1. Gro 6.8. If $S$ is sch of char 0. $X/S$ smooth. Then.

$$H^i((X/S)_{inf}, \mathcal{O}_{X/S}) \simeq H^i_{DR}(X/S) \simeq R^i\Gamma(X, \Omega^*_{X/S}).$$

N: nilpotent thicking ⇝ differentials.

• if $\mathcal{C}$ is a site. ($Inf(X/S)$.) $\mathcal{F} \in PSh(\mathcal{C})$.

$\Gamma(\mathcal{C}, \mathcal{F}) := Hom_{PSh(\mathcal{C})}(e, \mathcal{F})$.

where $e$ is a terminal obj in $PSh(\mathcal{C})$.

$H^i(\mathcal{C}, \mathcal{F}) := R^i\Gamma(\mathcal{C}, \mathcal{F})$.

• Apply defn to $\mathcal{C} = Inf(X/S)$

$\mathcal{F} = \mathcal{O}_{X/S}$.

5.3 Gro 6.8. If $X/S$ $S_0 \hookrightarrow S$ is nilpotent thickening. (37.2.SP)

$X_0 := X \times_S S_0$.

$$H^i((X_0/S)_{inf}, \mathcal{O}_{X_0/S}) \simeq H^i((X/S)_{inf}, \mathcal{O}_{X/S}) \quad \text{(p340. Gro68).}$$

∴ 4.1+5.3 ⇒ $H^i_{DR}(X/S) \simeq H^i((X_0/S)_{inf}, \mathcal{O}_{X_0/S})$. if $S$ is of char 0.

01.2.SP.

• Let us suppose we have a lift of $X_0$ to $X/W(k)$.

1.12 BO78.

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] \arrow[rr, bend left] & X_n \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec}(W_n(k)) = S_n \arrow[r] & \operatorname{Spec}(W(k)) = S
\end{tikzcd}$$

• $X/S$ is smooth + proper.

Lem: $R^i\Gamma(X, \Omega^*_{X/S}) \simeq \varprojlim R^i\Gamma(X_n, \Omega^*_{X_n/S_n})$.

• Don't know why.

Also if: A). $H^i_{inf}(X_n/W_n, \mathcal{O}_{X_n}) \simeq H^i_{inf}(X_0/W_n, \mathcal{O}_{X_0})$

B). $H^i_{inf}(X_n/W_n, \mathcal{O}_{X_n}) \simeq R^i\Gamma(X_n, \Omega^*_{X_n/W_n})$.

holds then,

$$H^*_{dR}(X/S) := R^i\Gamma(X, \Omega^*_{X/S}) \simeq \varprojlim H^i_{inf}(X_0/W_n, \mathcal{O}_{X_0}).$$

A. holds by constr'n.

B. does not hold. This holds ⟸ Poincaré lemma which requires $\frac{t^i}{i!}$. Ex: Consider $Inf(X/S)$ $(U \hookrightarrow T)$

which are equipped with a PD str.

Divided Powers 3. B078.

- $A \in CAlg^{\heartsuit}$, $\mathcal{J} \triangleleft A$.
- A PD str. on $\mathcal{J}$ is a family of maps.

  $\gamma_n : \mathcal{J} \to \mathcal{J}$ $\forall n \geq 1$, st.

0. $\gamma_0(x) = 1$.
1. $\gamma_1(x) = x$. $\forall x \in \mathcal{J}$.
2. $x, y \in \mathcal{J}$, $\gamma_n(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(x)$.
3. For $\lambda \in A$, $x \in \mathcal{J}$, $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$.
4. For $x \in \mathcal{J}$, $\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$
5. $\gamma_p(\gamma_q(x)) = C_{p,q}(x)\,\gamma_{pq}(x)$. $C_{p,q} := \frac{(pq)!}{p!(q!)^p}$.

Morally $\gamma_n(x) = \frac{x^n}{n!}$

Ex: If $A$ is $\mathbb{Q}$-alg. Every ideal $I \triangleleft A$ has a uique PD str.

Pf: 1+4 of defn $\Rightarrow$ $n!\,\gamma_n(x) = x^n$.

$\therefore$ $\gamma_n(x) = \frac{x^n}{n!}$. check axioms are satisfied. $\square$.

3.2.3 B078.
- $V$ is dvr of mix chr $p > 0$
- uniformizer $\pi$. Supose $p = u\pi^e$.
- $(\pi)$ has a PD str iff $e \leq p-1$.

$W(k)$ chr 0. $k$ perfect chr $p$.

$\leadsto$ $pW(k) \triangleleft W(k)$ has a PD str.

Defn A PD-mph.

$(A, I, \gamma) \to (A', I', \gamma')$.
- $\varphi: A \to A'$ ring homo.
- $$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[d] & I' \arrow[d] \\ A \arrow[r, "\varphi"] & A' \end{tikzcd}$$
- $$\begin{tikzcd} I \arrow[r] \arrow[d, "\gamma_n"'] & I' \arrow[d, "\gamma_n"] \\ I \arrow[r] & I' \end{tikzcd}$$ $\forall n \geq 1$.

3.5 B078. If $(A, I, \gamma)$ is a PD-ring. $\mathcal{J} \triangleleft A$.
There is a PD str. on $\bar{I} = I(A/\mathcal{J})$ s.t.
$(A, I, \gamma) \to (A/\mathcal{J}, \bar{I}, \bar{\gamma})$ is a PD-morphism.
iff $\mathcal{J} \cap I \subseteq I$ is a sub PD ideal.

Defn A $(A, I, \gamma)$

$\mathcal{J} \subseteq I$ is a sub PD ideal

if $\gamma_i(x) \in \mathcal{J}$ $\forall x \in \mathcal{J}$ $i \in \mathbb{N}$

3.14 B078 If $(A, I, \gamma)$ is a PD ring.
$B$ is an $A$-alg. We say $\gamma$-extends
$(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a PD-morphism.
where $(B, IB, \bar{\gamma})$ is a PD-ring.

Rmk: The extension is unique if $\exists$.

Pf: Use Condition 2+3. $\bar{\gamma}_k(\sum x_i b_i) = \sum \prod b_i^{*} \gamma_{*}(x_i)$.
from requirement of PD-morphism. $\square$.

Various conditions that allows extensions: eg if $I$ is principal.
3.15. B078.

## Globalizing the notions. 59.7.1.

- If $(\mathcal{C},\tau)$ is a site,
  $\mathcal{O} \in \mathrm{Shv}_\tau(\mathcal{C})$ is ring obj.
- $\mathcal{I} \subset \mathcal{O}$ is an ideal shf.
- A PD str. on $\mathcal{I}$ is given by $\gamma_n : \mathcal{I} \to \mathcal{I}$ st. $\forall u \in \mathcal{C}$.
  $(\mathcal{O}(u), \mathcal{I}(u), \gamma)$ is a PD-ring.

- A morphism betwn such PD topos.
  $(f, f^*) : (\mathrm{Sh}(\mathcal{C}), \mathcal{I}, \mathcal{O}) \to (\mathrm{Sh}(\mathcal{C}'), \mathcal{I}', \mathcal{O}')$.
  as similarly as a morphism of PD-rigs.

59.7.2. A PD-sch. is $(S, \mathcal{I}, \gamma)$.
- $S$ zar the small zariski site.
- Require $\mathcal{I}$ to be a qc. shf.

5. B078. The crystalline topos.
- let $(S, \mathcal{I}, \gamma)$ be a PD-sch.
- $X$ is a $S$-sch. extends $\gamma$
  (ring defn of PD extendy)
- We will require that all schs are locally nilpotent.
  i.e. $p \cdot \mathcal{O}$ is locally nilpotent.

Cris$(X/S)$ := obj $(\mathcal{U} \xhookrightarrow{cl} T, \delta)$ $\quad \mathcal{U} \underset{\text{zaski open}}{\hookrightarrow} X$
- $\mathcal{U} \leftrightarrow \mathcal{J}$ qc. of ideal.
- Require a PD-structure on $\mathcal{J}$.
  i.e. data. of PD-sch. $(T, \mathcal{J}, \delta)$

summary.

Obj:
- $\mathcal{U} \underset{\text{zariski open}}{\hookrightarrow} X$
- $(\mathcal{U} \xhookrightarrow{cl} T, \delta)$ closed immersion over $S$.
- $(T, \mathcal{J}, \delta)$ PD str on $\mathcal{J}$, $\mathcal{J}$ is qcoh shf corresponding to $\mathcal{U} \hookrightarrow T$.

- Errata: replace $T$ by $\mathcal{U}$ below.

Important: ~~$T$~~ $\mathcal{U}$ has two PD-str.

1. $\mathcal{U}$ ~~$T$~~ $\to S$. ~~this indcs a PD-str on $T$.~~ has induced PD-str fr $\mathcal{U} \hookrightarrow X$, $\searrow S$.

2. has PD str. $(T, \mathcal{J}, \delta)$ from closed imersion.
   require these two to be compatible.

- obj:

$$\begin{tikzcd} T \arrow[d] & \mathcal{U} \arrow[l, hook', "cl"'] \arrow[d, hook, "open"] \\ S & X \arrow[l] \end{tikzcd}$$

- As a consequence of locally nilpotent can show $\mathcal{U} \hookrightarrow T$ is nil imersion, 23.2.6 SP.
  thickenings, 37.2 SP.

7.16. B078.

$(A, I, \gamma) \in$ PD ring. $B$ is a $A$-alg.
$(J, \delta)$ is a PD-ideal of $B$.
If $\gamma$ extends to $B$
ie $(B, IB, \bar{\gamma})$ on $B$.
Then we say PD-str are compatible
if $\bar{\gamma} = \delta$ on $J \cap IB$.

nil imersion: the $\mathcal{J}$ qc shf
defining $\mathcal{U}$ is nil-ideal.

$\hookrightarrow$ $\mathcal{J}$ nil-ideal $\Rightarrow$ $\mathcal{J}(u)$ nil ideal.
$J \triangleleft A$ is nil. iff $\forall x \in J$ $\exists n$ st $x^n = 0$.

$X \underset{\text{cl. im.}}{\hookrightarrow} X'$ $(\mathcal{J})$
s.t. $|X| \simeq |X'|$
$\Leftrightarrow$ $\mathcal{J}$ is nil-ideal.

- Covering family: $\{u_i : T_i \to T\}$, s.t.
  - $T_i \hookrightarrow T$ Zariski open.
  - $T = \bigcup T_i$.

If $\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec} A$. has same underly space, $\Leftrightarrow$ $I \subseteq \bigcap_{\text{prime}} \mathfrak{p}$.
$\Leftrightarrow$ $I \subseteq$ nilpotent of $A$.

$\mathrm{Shv}(\mathrm{Cris}(X/S))$.
$\|$
- Similar to inf-site. we can describe sheaves on $F \in (X/S)_{cris}$
  - $\{F_T \in \mathrm{Shv}_{Zar}(T)\}_{(U,T,\delta) \in \mathrm{Cris}(X/S)}$
  - $+$ morphisms $+$ capatibility.

Pf: Zariski shf: Given any $(U, T, \delta)$
Each Zariski open $T' \hookrightarrow T$.
- let $U' := U \cap T$, $\delta' := \delta|_{T'}$.
  PD-sch. $(U', T', \delta') \hookrightarrow (U, T, \delta)$. $\rightsquigarrow$ $F(T) \to F(T')$.
  $\therefore$ Assignment. $T' \mapsto F(T')$ is a shf on Zariski top of $T$. $\square$.

In partical we have the str. shf $\mathcal{O}_{X/S}$ on $\mathrm{Cris}(X/S)$
$\mathcal{O}_{X/S} : (U, T, \delta) \mapsto \mathcal{O}_T$.

- define coho: $H^i((X/S)_{cris}, \mathcal{O}_{X/S})$
  (as $R^i\Gamma(e, \mathcal{O}_{X/S})$).

$W_n :=$ witt vectors of length $n$.

- $H^i(X/W(k)) := \varprojlim H^i((X/W_n)_{crys}, \mathcal{O}_{X/W_n})$.
  in our case of interest. people refer this as Crystalline coho.