Notes on Perfectoid Spaces

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1 Introduction

Program

A: Rigid Analytic Spaces

1.1. Baiscs of normed and valued rings, [BGR84, I]. Disctiono between the operations: $^\circ$ $\check{}$ $\tilde{}$ which makes sense both wrt to a topology and a (semi) norm, [BGR84, I.2-4]. Tate algebras T_n, their residue $\widetilde{T}_n$, and Weierstrass Preparation theorem. [BGR84, V.1-2].

1.2. Affinoid algebras, [BGR84, VI.1].

B: Adic Spaces

1.3. Huber rings. Tate rings.

1.4. Theory of valutations, [Wed19, 1]. Explain bijection 5.24. Height of tot. ord. ab. grp. Rank of valuation. Review of dvr, [Wed19, 2.2].

C: Perfectoid spaces

1.5. Perfectoid fields.

D: Applications

1.6. Formulating Fontaine-Winterberger theorem, [Ked15, 1].

2 Semi-norms (narc)

2.1. All rings are assumed commutative with identity 1.

2.1 Semi-normed and normed rings

2.2. We begin with the general theory of narc. (semi) norms. On

- grps.
- rngs.
- strictly convergent power series (scps.)

2.3. A tool to analyze such objects is to consider *residue rings*.

2.4. It is important to note that much of the theory of adic spaces is done in the ctx. of *completion*.[1]

2.1.1 Semi-norm

2.5. In this section G will always denote an abelian group written additively.

Definition 2.6. A function $|\ \ |: G \to \mathbb{R}_{>0} \cup \{0\}$ is *an ultrametric function/a narc. seminorm/a valuation*[2] if

1. $|0| = 0$.
2. $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in G$.

Definition 2.7. A pair $(G, |\ \ |)$ is a *semi-normed grp.* It is *normed* if $|x| = 0$ implies $x = 0$. We denote the cat. of snGrp with semi-normed grps and bdd. homomorphisms by snGrp.

Definition 2.8. Pseudometric topology. For generalization, see defn. 6.23. We can give G a topology via

$$d(x, y) := |x - y|$$

2.9. [BGR84, I.1] Some features of this topology:

1. Every sphere $S(a, r) := \{x \in G\,;\, |x - a| = r\}$ is open.
2.

Definition 2.10. $(A, |\ \ |) \in \mathrm{snGrp}\,(\mathrm{nGrp})$ is called *semi-normed (normed) ring* if

1. Submultiplicative. $|xy| \leq |x||y|$ for all $x, y \in A$.
2. $|1| \leq 1$.

Remark 2.11. It seems natural to make the following definition.

$$\begin{tikzcd} \text{"snRng"} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{TopRng} \arrow[d] \\ \mathrm{snGrp} \arrow[r] & \mathrm{TopGrp} \end{tikzcd}$$

or that "snRng" should be a rng object in snGrp. Neither seems to be true.

[1] Make precise.

[2] The first two are used in [BGR84] whilst last in [Sch11].

2.1.2 Examples of semi-norms

2.12. Let us now explore three standard examples, [BGR84, 1.3.3].

2.13. All these constructions follow from giving a simple *filtration* on a ring, which also called an *additive valuation*. The idea is a filtration $f : A \to \mathbb{R} \cup \{\infty\}$ should induce a norm, where

$$|\quad| := \varepsilon^f$$

where $0 < \varepsilon < 1$.

2.14. What is really happening: we are simply changing the valued grp from $(\mathbb{R}, +) \cup \{\infty\}$ to $(\mathbb{R}_{>0}, \times) \cup \{0\}$.

Example 2.15. The $\mathfrak{a}$-adic semi-norm. Let $A \in \mathrm{Rng}$, with an ideal $\mathfrak{a} \neq A$. Define

$$v_{\mathfrak{a}}(x) := \begin{cases} \infty & \text{if } x \in \mathfrak{a}^i \, \forall i \in \mathbb{N} \\ \max\{i : x \in \mathfrak{a}^i\} & \text{otherwise} \end{cases} \tag{1}$$

It is not hard to check that the following three axioms are satisfied:

1. $v(1) = 0, \quad v(0) = \infty$.
2. $v(xy) \geq v(x) + v(y)$.
3. $v(x - y) \geq \min\{v(x), v(y)\}$

These are the axioms of an *additive valuation* $v : A \to \Gamma \cup \{\infty\}$ written additively. See,

Example 2.16. The (X)-*adic norm* on $A[\![X]\!]$. Let $f = \sum_{v \geq 0}^{\infty} a_v X^v$

- Condition 1 translates to
$$v(f) := \min\{v : a_v \neq 0\}$$
- $v(f) = \infty$ iff $f \in \bigcap_{i \geq 1}(X)^i = 0$. Hence, the induced semi-norm is in fact a *norm*.

Example 2.17. The (p)-*adic norm* on $\mathbb{Z}$.

- Condition 1, translates to $v_p(m) = \sup\{i : p^i | m\}$
- Then the associated *normalized norm*, $|m|_p := (1/p)^{v_p(f)}$

The (p)-*adic norm* on $\mathbb{Z}[x]$. Let us denote the (p)-adic filtration on $\mathbb{Z}$ by v_p.

- Condition 1 translates to $v(f) = \min\{v_p(a_k)\}$
- The associated norm is then by given by $|f| = \sup_k\{|a_k|_p\}$

2.1.3 Residue ring

2.18. From now on, unless stated otherwise, $A \in \mathrm{snRng}$. If the statement is true more generally for $A \in \mathrm{TopRng}$, I will state this.

Definition 2.19. Let $A \in \mathrm{TopRng}$. For any subset $T \subset A$, we denote

$$T(n) := \{t_1 \cdots t_n : t_i \in T\}$$

- T is *top. nil.* if for all nhood U of 0, exists N, $T(n) \subset U$, for all $n \geq N$.

  $a \in A$ is *top. nil.* if $\{a\}$ is, i.e. $\lim a^n = 0$. We let $\check{A}$ denote the set of such elements.
- T is bdd if for all nhood U of 0 in A exists nhood V, $VT \subset U$.
- T is *power bdd* if $\bigcup_{n\geq 1} T(n)$ bdd.[3] $a \in A$ is *power bdd* if $\{a\}$ is power bdd. We let $\mathring{A}$ denote set of such elements.

2.20. It is useful to think about the above defn. when $A \in$ snRng. i.e. a is power bdd iff

$$\{|a^n| : n \in \mathbb{N}\}$$

is a bdd set in $\mathbb{R}_{>0} \cup \{0\}$.

Example 2.21. Let $A \in$ TopRng. Finite subset are bdd, $\{b_i\}_{i\in I}$. Let $V \subset A$ be any open set. This follows as $\bigcap b_i^{-1}(V)$ is open, where we regard $b_i : A \to A$ as multiplication map.

Proposition 2.22. [BGR84, I.2.4.2] The set $\check{A}$ is a mult. closed, open and closed subgrp of A.

Proof. Step 0: Let $a, b \in \check{A}$, we show $a - b \in \check{A}$.
This is some standard boundedness arguments.
Step 1: $\check{A}$ is open and closed. This follows from the observations

- $A\check{}\subset \check{A}$, where the former object is open in A.
- Any subgrp. containing an open nhood of 0 is open.
- Any open subgrp is closed.

□

Proposition 2.23. We have a similar result for the power bdd. elements:

- The set $\mathring{A}$ is an open and closed subring of A.
- $\check{A}$ is an ideal of $\mathring{A}$.

2.24. Now we discuss some invertibility consequences from compltness.

Proposition 2.25. [BGR84, I.2.4]. Suppose A is cplt.

1. $\check{A}$ is cplt.
2. Each element of the form $e = 1 - y$, $y \in \check{A}$, is a unit in A.
3. An element $a \in \mathring{A}$ is a unit iff its residue class $\tilde{a}$ is a unit in $\widetilde{A}$.

Proof. 1. By 2.22, $\check{A}$ is closed subgroup. Standard (say, Banach algebra arguments) show $\check{A}$ is complete.

2. Inverse is given by $e^{-1} := \sum_0^\infty y^i =: 1 + z$ where $z \in \check{A}$ by 2.22.

3. One direction is clear. For the converse, if $ab = 1 - x$, where $x \in \check{A}$, then by 2. a is a unit. □

Example 2.26. Take $A = \mathbb{Q}_p$.Then $\check{A} = \mathbb{Z}_p$ is cplt. wrt to the topology.

[3] This is not to be confused with Tate algebras

2.2 Multiplicative norm

2.27. There are adjectives which describe the extent to which a semi-norm on $A \in \mathrm{snRng}$ is multiplicative. This is a distinctive part of the foundational theory of rigid analytic space that we shall focus on - compared to valuations, see 5, which are generalized *mult. norms*.

Definition 2.28. Let $(A, |\quad|) \in \mathrm{snRng}$.

1. An element $a \in A$ is *pm* if $|a^n| = |a|^n$ for all n.

2. $|\quad|$ is *pm* if all elements of A are pm.

Corollary 2.29. If $|\quad|$ is a pm-semi norm then

1. $\mathring{A} = A^\circ, \check{A} = A^{\check{}}, \widetilde{A} = A^{\tilde{}}$.

2.30. One may think that most examples so far are multiplicative - but we would see in the case of affinoid algebra, this is *not* the case.

2.2.1 Valued ring

2.31. A special full subcat of nRng are the *valued rings*.

Definition 2.32. [BGR84, I.5.1] $A \in \mathrm{nRng}$ is a *valued ring* if all nonzero elements are mult. [4] We will generalize this definition when the valuation takes values in $\Gamma \cup \{0\}$, see

Lemma 2.33. [BGR84, I.5.3.1] Let $A \in \mathrm{nRng}$ such that

1. Every element is pm by a mult. elem. For each $a \in A, a \neq 0$, there exists a multiplicative element $m \in A$, and an exponent $s \in \mathbb{N}$,
$$|ma^s| = |m||a|^s = 1$$

2. $\tilde{A} = A^\circ / A^{\check{}}$ is an integral dom.

Then $|\quad|$ is a valuation on A.

3 Affinoid algebras

3.1 Tate Algebras

3.1. Our goal now is to apply the general theory to our main algebra of interest.

Definition 3.2. The *free Tate algebra in n indet. over k*

$$T_n(k) := k\langle X_1, \ldots, X_n \rangle := \left\{ \sum a_v X^v \,:\, a_v \in k, |a_v| \to 0 \text{ for } |v| \to \infty \right\}$$

where the summation is taken over all $(v_1, \ldots, v_n)$ where each $v_i \geq 0$.

[4] Do not confuse this with *valuation ring*.

3.1.1 An inductive construction

3.3. We can give an inductive definition of this construction: begining with theory of formal power series over semi-normed ring, $(A, |\ \ |)$.

Definition 3.4. Let $(A, |\ \ |) \in \mathrm{snRng}$.

1. A fps. $\sum_{v=0}^{\infty} a_v X^v$ is *strictly convergent* (scps.) if $\lim_v |a_v| = 0$. We denote $A\langle X\rangle$ the set of scps.
2. We expand 1, by defining for subset $M \subset A$,

$$M\langle X\rangle := \{\sum a_v X^v \in A\langle X\rangle : a_v \in M \text{ for all } v \geq 0\}$$

3. For each $f \in A$, we define the *Gauss semi-norm.*

$$|f|' := \max |a_v|$$

Proposition 3.5. Properties of $(A\langle X\rangle, |\ \ |')$.

1. We have $A \subset A[X] \subset A\langle X\rangle \subset A[\![X]\!]$
2. $|\ \ |'$ is a norm on $A\langle X\rangle$ iff $|\ \ |$ is a norm on A.
3. If $(A, |\ \ |)$ is complete, then $(A\langle X\rangle, |\ \ |')$ is complete.

Definition 3.6. We define $A\langle X_1, \ldots, X_n\rangle := A\langle X_1, \ldots, X_{n-1}\rangle\langle X_n\rangle$.

Corollary 3.7. 1. $T_n(k)$ is a k subalg. of $k[\![X_1, \ldots, X_n]\!]$

2.

Proposition 3.8. [BGR84, I.4.2] $\mathring{A}\langle X\rangle = \overbrace{A\langle X\rangle}^{\circ}$ and $\check{A}\langle X\rangle$ [5]

Proposition 3.9. [BGR84, I.4.3].

- An element $f = \sum a_v X^v \in \mathring{A}\langle X\rangle$.

Remark 3.10. T_n is a field iff $n = 0$.

3.2 Affinoid algebras

3.11. We will need some theory of k-Banach algebras, [BGR84, III.7].

3.3 Banach algebras

3.12. We begin by defining normed A-algebra for $A \in \mathrm{nRng}$.

Definition 3.13. A pair $(M, |\ \ |)$ is a *normed A-module* if tfh:

1. Norm on M. $(M, |\ \ |) \in \mathrm{nGrp}$.
2. Continuity of A-action. $|ax| \leq |a||x|$ for all $a \in A, x \in M$.

[5] The overbracket is to indicate we take the operation wrt whole ring.

If in 2, rather than $\leq$ we have $=$, we have a *faithful* normed A-module.

3.14. Most cases of interests sre faithful. In particular, if A is a valued field, then each A-module norm is faithful. Indeed, for all $a \in A^\times$,

$$|x| = |a^{-1}ax| \leq |a^{-1}||ax| = |a|^{-1}|ax|, \quad |ax| \geq |a||x|$$

3.15. We denote cat. of normed A-modules with bdd. A-linear map by nMod_A.

Definition 3.16. We define a *ring norm* $|\quad|$ *on an A-algebra B* to be an A-algebra norm if $|\quad|$ is an A-module norm on B.

Definition 3.17. 1. A cplt. normed k-vs V is a *Banach space.*

2. A cplt. normed k-alg. B is a *Banach alg.*

Theorem 3.18. Open mapping theorem. Let V, W be Banach spaces, $\Phi : V \to W$ be bounded and surj K-linear map. Then Φ is open and W carries the quot. top. with respect to Φ.

3.19. Let us now give a characterization of Noetherian A-modules and the ideals of A. For this we will require a version of *Nakayama lemma* in the ctx. of semi-normed rings.

Lemma 3.20. [BGR84, I.2.4] Let A be cplt. M an A-module. N a submodule of M such that exists $\{x_i\}_1^n \in M$, satisfying

$$M \subset N + \sum_1^n \check{A} x_i$$

Then $N = M$.

Proof. We prove the converse inclusion. *Step 0. Imitate classical arg.* For $x \in M$, write

$$x_i = y_i + \sum c_{ij} x_j$$

$$y = (I - C)x$$

where $C = (c_{ij})$, and $y = (y_i)$ and $x = (x_i)$ are column vectors.

Step 1. $I - C$ is invertible. This is true iff $\det(I - C) = 1 - c$ for some $c \in \check{A}$ is invertible. This is 2.25. ☐

Lemma 3.21. Let A be a k-Banach alg. M a normed A-module such that $\hat{M}$ of M is a finite A-module. Then M is complete.

Proposition 3.22. [BGR84, III.7.2] L

3.3.1 Affinoid algebras

[BGR84, VI.1].

Definition 3.23. A k-Banach algebra A is called *affinoid*, if there exist an integer $n \geq 0$ and a cont. epi

$$\mathfrak{a} \to T_n \xrightarrow{\mathfrak{a}} A \tag{2}$$

where for convenience we denote the epi. by $\mathfrak{a}$.

3.24. In 2, we can endow A with the *residue norm.* That is,

$$|\bar{f}|_{\mathfrak{a}} := \inf\{|h| : h \in T_n, \bar{h} = \bar{f}\}$$

3.25. This norm is *not* pm in general.

Theorem 3.26. [BGR84, VI.1.2] Noether normalization.

1. Let A be a non-zero k-affinoid Algebra.

Remark 3.27. An affinoid algebra can have an affinoid subalg. of greater Krull dimension.

Corollary 3.28. For any max ideal $\mathfrak{m}$ of T_n, the field $T_n/\mathfrak{m}$ is finite over k.

3.4 Maximal spectrum and spectral seminorm

3.29. In this section we introduce the superemum semi-norm [BGR84, III.8].

Definition 3.30. Let L/k be an alg. ext. For al $y \in L$, we set

$$|y|_{sp} := \text{ spec val. } \sigma(q) \text{ of the min poly } q \in k[x] \text{ of } y \text{ over } k$$

We call

$$|\quad|_{sp} : L \to \mathbb{R}_{>0}$$

the *spectral norm on L.*

Theorem 3.31. 1. $|\quad|_{sp}$ is a pm k-alg. norm on L, extending the norm on k.

Definition 3.32. The spectrum of k-algebraic

3.33. Let $A \in \mathrm{Afnd}_k$.

4 G-topologies

5 Valuations

[Sch11, 2].

5.1. We may generalize the various definitions in Subsec 2 when the norms are valued in a totally ordered group, Γ.

Definition 5.2. A tot. ord. ab. grp. is an abelian group $(\Gamma, \cdot)$ with order $\leq$ that respects $\cdot$: for all $a, b, c \in \Gamma$,

$$a \leq b \Rightarrow a \cdot c \leq b \cdot c$$

We let TotAbGrp be the cat. of tot. ord. ab. grps, with morphisms a ordering preserivng grp homo.

Definition 5.3. Maximal/minimal element. Our convention in this note is that we will usually write $(\Gamma, \cdot) \in \mathrm{TotAbGrp}$ *multiplicatively*, $(\Gamma, \times)$.

In which case we denote the unit of Γ by 1, and define $\Gamma \cup \{0\} \in \mathrm{TotAbGrp}$ satisfying, for all $a \in \Gamma$

- $a \times 0 = 0 \times a = 0$
- $0 \leq a$.

In the case when we consider $(\Gamma, +)$ *additively*, we write its unit 0 and add a maximal element $\Gamma \cup \{\infty\}$, satisfying, for all $a \in \Gamma$

- $a + \infty = \infty + a = \infty$
- $a \leq \infty$.

Example 5.4. There are two common ways to write, for a prime l, the l-adic valuation

$$v_l : \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$$

$$|\quad|_l : \mathbb{Q} \to \mathbb{R}_{>0} \cup \{0\}$$

additively and multiplicatively resp.

Example 5.5. $(\mathbb{R}, +)$, $(\mathbb{R}_{>0}, \times)$ with their usual order relation.

Definition 5.6. Let $A \in \mathrm{Rng}$, a *valuation* $|\quad| : A \to \Gamma \cup \{0\}$, written *multiplicatively*, satisfies

- Normed. $|x| = 0$ iff $x = 0$.
- Narc. $|a + b| \leq \max(|a|, |b|)$
- Multiplicative. $|ab| = |a||b|$.
- Identity. $|1| = 1$.

Definition 5.7. Let $|\quad| : A \to \Gamma \cup \{0\}$ be a val.

- The sugrp of Γ generated[6] by $\mathrm{im}(|\quad|)$ is the *value group of* $|\quad|$ denoted $\Gamma_{|\quad|}$.
- The *support* of $|\quad|$ is denoted $\mathrm{supp}(|\quad|) := |\quad|^{-1}(\{0\})$.

[6] When $|\quad|$ restricted to $(A \smallsetminus \{0\}, \times) \to \Gamma$, we have a morphism of *monoids*.

5.8. Wrt to such a valuation, and the distinguished multiplicative unit $1 \in \Gamma$, we can define the notion of fraction field, ring of integers, and residue rings.

Example 5.9. If $\Gamma = \mathbb{R}^\times_{>0}$. We retrieve the definition of multiplicative norm.

Example 5.10. The *trivial valuation.*

Example 5.11. Let A be an integral domain

Definition 5.12. Two valuations $|\ \ |, |\ \ |'$ are equivalent if tfec are satisfied.

1. There is an iso. of tot. ordered monoids
$$\alpha : \Gamma_{|\ \ |} \cup \{0\} \simeq \Gamma_{|\ \ |'} \cup \{0\}$$
such that $|\ \ |' = \alpha \circ |\ \ |$.
2. $\operatorname{supp}(|\ \ |_1) = \operatorname{supp}(|\ \ |_2)$ and $A^{\circ_1} = A^{\circ_2}$, where
$$A^{\circ_i} := \{a \in A : |a|_i \leq 1\}$$
3. For all $a, b \in R$, $|a| \geq |b|$ iff $|a|' \geq |b|'$.

Proof. Step 0. Trivial implicials of 1-3. We observe that

- All 3 conditions imply $\operatorname{supp}(|\ \ |_1) = \operatorname{supp}(|\ \ |_2)$.

Step 1. $1 \Rightarrow 3 \Rightarrow 2$ is clear. □

Definition 5.13. Let $A \in \mathrm{Rng}$. The *val. spec.*, $\operatorname{Spv}(A)$ is the set, whose

- elements are set of equiv. class of valuations on A under equiv relations 5.12. We will often abuse notation and write an element $x \in \operatorname{Spv}(A)$ by its representative.
- topology is given by open basis:
$$U\left(\frac{f}{s}\right) := \{|\ \ | \in \operatorname{Spv}(A) : |f| \leq |s| \neq 0\}$$

5.14. The first step to understanding val. spec is by the correspondence 5.24.

5.1 Rank of valuation

5.15. Our first goal is to define height.

Definition 5.16. Let $\Gamma \in \mathrm{TotAbGrp}$. A *convex subgroup* of Γ is a subgroup Δ such that for all $a, b, c \in \Gamma$, $a \le b \le c$,

$$a, c \in \Delta \Rightarrow b \in \Delta$$

Proposition 5.17. Let $\Gamma \neq 0 \in \mathrm{TotAbGrp}$. Tfae

1. $\mathrm{hgt}(\Gamma) = 1$.
2. for all $a, b \in \Gamma$, $a > 0$ and $b \ge 0$, exists $n \in \mathbb{N}$ such that $b \le na$.
3. there is an injective ordered grp. morphism $\Gamma \to (\mathbb{R}, +)$.

Proof. $1 \Rightarrow 2$. Let

$$\Delta := \{y \in \Gamma :, \text{exists } n, -y, y \le na\}$$

It is to check by definition that this is a nontrivial $(a \in \Delta)$ convex subgroup. By hypothesis $\Delta = \Gamma$. □

Definition 5.18. The *rank* of a val. is the height of its val. grp.

5.2 Valuation ring

[Wed19, 2], [SP, 10.49].

Definition 5.19. Let A, B be local rings, $A \subset B$, B *dominates* A if $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. This induces a *partial order* on the collection of local subrings of a field K.

Definition 5.20. Let A be an integral domain. It is a *valuation ring* (of $\mathrm{Frac}(A) := K$) if it satisfies tfec:

1. for every $x \in K^\times$, either x or x^{-1} lies in A.
2. $\mathrm{Frac}\, A = K$ and the set of ideals of A is tot. ord. by inclusion.
3. A is local and a max. element in the set of local subrings of K wrt the domination order.
4. it is the ring of integers of some valuation $|\ \ |$ on K.

Proof. *1* ⇒ *4*. Clear. *4* ⇒ *1*. The key observation is that a valuation ring comes naturally with a valuation

$$|\ \ | : K \to K^\times / A^\times \cup \{0\} \tag{3}$$

where we declare[7] for $a, b \in K^\times$,

$$\bar{a} \le \bar{b} \Leftrightarrow a/b \in A$$

and $|\ \ |$ is given by the quotient map on $K^\times$, with $|0| = 0$. □

5.21. In the literature, one often denotes denote the value grp. in 3, by $\Gamma_A := K^\times / A^\times$.

5.22. The argument 1 to 4. implies the following bijection. Let K be a field.

$$\mathrm{ValSub}(K) \leftrightarrow \mathrm{Spv}(K)$$

$$A \mapsto (|\ \ | : K \to \Gamma_A \cup \{0\}), \quad (|\ \ | : K \to \Gamma \cup \{0\}) \mapsto |\ \ |^{-1}(\Gamma_{\le 1} \cup \{0\})$$

Example 5.23. Consider $|\ \ |_l : \mathbb{Q} \to \mathbb{R}_{>0} \cup \{0\}$. Then the correspondence yields $\mathbb{Z}_{(l)}$ as valn. subrng.

Proposition 5.24. We have a bijection

$$\mathrm{Spv}\, A \leftrightarrow \{(\mathfrak{p}, |\ \ |) : \mathfrak{p} \in \mathrm{Spec}\, A, |\ \ | \in \mathrm{Spv}(\kappa(\mathfrak{p}))\}$$

$$|\ \ | \mapsto (\mathrm{supp}\, |\ \ |, |\ \ |')$$

Or equivalenltly, a pb. in Set.

$$\begin{tikzcd} \mathrm{Spv}(\kappa(\mathfrak{p})) \arrow[r] \arrow[d] & \mathrm{Spv}\, A \arrow[d, "\mathrm{Supp}\, |\ \ |"] \\ * \arrow[r, "\mathfrak{p}"'] & \mathrm{Spec}\, A \end{tikzcd}$$

This will be shown to be a pb. in Top.

[7] This is simply realizing the necessary condition $|a| \le |b| \Leftrightarrow |a/b| \le 1 \Leftrightarrow a/b \in R$ for $a, b \in K^\times$.

Proof. The first correspondence is direct check. The inverse map is given by composition

$$A \to \kappa(\mathfrak{p}) \xrightarrow{|\ \ |} \Gamma \cup \{0\}$$

□

5.25. We have another bijection

$$\{(\mathfrak{p}, |\ \ |) : \mathfrak{p} \in \operatorname{Spec} A, |\ \ | \in \operatorname{Spv}(\kappa(\mathfrak{p}))\} \leftrightarrow \{(\mathfrak{p}, B) : B \text{ is } a \text{ val. ring of } \kappa(\mathfrak{p})\}$$

$$(\mathfrak{p}, |\ \ |) \mapsto \left(\mathfrak{p}, \overbrace{\kappa(\mathfrak{p})}^{\circ}\right)$$

Example 5.26. Under this correspondence,

- $$U\left(\frac{f_1, \dots, f_n}{g}\right) \leftrightarrow \left\{(\mathfrak{p}, R) \in X : g \notin \mathfrak{p}, \quad \bar{f}_i/\bar{g} \in R\right\}$$

Proof. Indeed, by definition $|g|_x \neq 0$, so $g \notin \mathfrak{p} := \operatorname{supp} |\ \ |_x$. Further, $|\bar{f}_i|_x \leq |\bar{g}|_x \Leftrightarrow |\bar{f}_i/\bar{g}|_x \leq 1$. □

5.2.1 Valuation subrings and convex subgroups

5.27. Our goal now is to prove the following bijection, [Wed19, 2.14]

$$\operatorname{Spec} A \leftrightarrow \text{ValSub(A,K)} \leftrightarrow \text{ConSub}\Gamma$$

5.2.2 Discrete valuations

[Wed19, 2.2]

5.28. One may safely omit this section, returning when we discuss perfectoid fields, 8.

Proposition 5.29. Let A be an integral domain which is not a field. Tfae

1. A is a noetheiran val. ring.
2. A is a local principal domain.
3. A is a vl. ring and Γ_A is iso to the totally ordered group $\mathbb{Z}$.

If any of the above is satisfied, A is a *dvr*. It is *normed* if its value group is $\mathbb{Z}$.

Example 5.30. An alg. closed field admits no disc. val.

Proof. Step 0. Properties of div. grp. An ab. A grp is *divisible* if $n : A \to A$ is surjective. We deduce: a quot. of div. grp is div.

Step 1. $(k^\times, \times)$ *is divisible.* This follows as the polynomial $x^n - a$ has a root for $a \in k^\times$.

Now as any valution on k restricts to a group homo

$$(k^\times, \times) \to \Gamma$$

where the map is surj by defn, Γ cannot be isomoprhic to $\mathbb{Z}$ by step 0. ☐

5.2.3 Microbial valuations

Theorem 5.31. Let K be a field. $|\ \ | \in \operatorname{Spv} K$. We give K the val. top. Tfae.

1. The topology on K coincides with the val. top. defined by a rank 1 val.
2. There exists a nonzero topologically nilpotent element in K.

5.3 A study of Riemann-Zariski spaces

[Con14, 2].

5.32. The case of $\operatorname{Spv}(A)$ when $A = K$ a field has been studied classically.

Definition 5.33. Ctx. Let A be an integral dom. K a field containing A.

- A vsl. subring $R \subset K$ has *center in* A if $A \subset R$.
- The set of valution rings of K centered in A is the *Riemann-Zariski space of* K *wrt* A.

Example 5.34. $\mathrm{RZ}(K, K)$ is the one point set of trivial valuation.

Example 5.35. $\mathrm{RZ}(K) := \mathrm{RZ}(K, \mathbb{Z})$ is the set of all valutions K since $|\ \ |$ is narc.

5.36. We define topology on $\mathrm{RZ}(K, A)$ as follows.

$$U(x_1, \dots, x_n) := \mathrm{RZ}(K, A[x_1, \dots, x_n]), \quad \text{where } x_1, \dots, x_n \in K$$

5.4 Topology of Spv

[Mor19, I.2].

Definition 5.37. Let $A \in \mathrm{Rng}$.

5.38. As we are following [Mor19, I.2.3] presentation, we will give an equivalent definition of constructible topology, for the case on a qcqs top. space. The general definition is in [SP, 5.15].

Definition 5.39. Let $X \in \mathrm{Top}_{\mathrm{qcqs}}$.

Definition 5.40. X be qcqs.

- The *constructible topology* is the top. with base the collection of constructible subsets of X.

5.5 Specialization relation Spv

5.41. Recall that for any spectral space, X, the specialization relation is an *order rel.*

5.42. Specializations in $\mathrm{Spv}(A)$ breaks in to two cases.

5.5.1 Vert. specialization

5.5.2 Hor. specialization

[Wed19, 4.2].

Proposition 5.43. Let $x \in \mathrm{Spv}(A)$.

Definition 5.44. Let $x \in \mathrm{Spv}(A)$, with value grp. Γ_x.

6 Adic Spaces

6.1. We will fix a base field k now, which would be complete and narc, following conventions in [Bha17].

6.1 Huber Rings

Definition 6.2. Let $A \in$ TopRng.

- It is *I-adic* ifor some ideal I of A if $\{I^n : n \geq 0\}$ forms a fsn. of 0 in A.
- More generally, A is *narc* if it admits a neighborhood of 0 consisting of subgrps of $(A, +)$.

Example 6.3. Narc fields and (more generally) k-affinoid algebras are narc rings.

Definition 6.4. A *f-adic/Huber ring*[8] is an $A \in$ TopRng satisfying the equivalent definition.

1. A contains an *open* subring A_0 such that subspc top on A_0 is I-adic for *a fg. ideal I of A_0*.

Example 6.5. Discrete rings. $A_0 = A$, $I = 0$.

Example 6.6. Formal schemes. An adic ring A which has a fg. ideal of definition.

Example 6.7. K-Banach algebras, where K is a narch field.

- Let $A_0 \subset A$ be the unit ball.
- We let $g \in K^\times$ be such that $|g| < 1$. Then
$$\{g^n A_0\}$$
forms a fsn. of 0.

Proposition 6.8. $A \in$ HubRng. Then $A_0 \subset A$ is a rng of def iff it is open and bdd.

Proof. $\Rightarrow$ It is open by def. We now check A_0 is bdd. Let $U \subset A$ be open. Wlog, we may assume $U = I^n$. Hence, $I^n A = U$, implies A is bdd.

$\Leftarrow$ As $A \in$ HubRng, let T be a finite set of generators for U, where (B, U) is rng of def of A. What we really use are the following

- $\{U^n : n \geq 1\}$ is a fsn. of 0 in A.
- $T \subset U$.
- $T \cdot U = U^2 \subset U$, where $T \cdot U$ is the ab. grp gen, wrt to +, by $\{tu : t \in T, u \in U\}$.

Step 0. Construct an ideal of defn. Since A_0 is open, exists some k, such that $T(k) \subset A_0$. Let I be the ideal generated by $T(k)$. There are two things to check.

1. For all n, I^n is an open nhood of 0. Chose l such that $U^l \subset A_0$. Consider
$$I^n = T(nk)A_0 \supset T(nk)U^l = U^{l+nk}$$
2. I^n forms a basis. Let V be any nhood. Exists m, such that
$$U^m A_0 \subset V$$
Then we also have
$$I^m = T(mk)A_0 \subset U^{mk} A_0 \subset (U^m A_0)^k \subset V$$

□

[8] We take the latter as concention for this notes. The former confuses one when f is also used elsewhere.

6.1.1 Continuous morphisms

6.9. A morphism in TopRng does not necessarily preserve bdd sets. Hence we need a notion of morphism that preserves the structure.

Definition 6.10. Let $A, B \in \text{HubRng}$. A morphism is *adic* if there exists a couple of definition (A_0, I)

6.2 Tate rings

Definition 6.11. Let $A \in \text{HubRng}$, it is *Tate* if it has a topologically nilpotent unit.

6.12. We denote the cat. of Tate rings as $\text{TateRng} \subset \text{TopRng}$, the full subcat spanned by Tate rings.

6.3 Huber Pairs

Definition 6.13. Let $A \in \text{HubRng}$.

- A subring $A^+ \subset \mathring{A} \subset A$ is a *ring of integral elements* if it is open and integrally closed in A.
- An *affinoid ring/Huber pair* is the datum (A, A^+). We use the latter name.
- A Huber pair is *complete* (resp. *adic*, $\cdots$) if A has this property.
-

6.3.1 Ring of polynomials

[Mor19, III.3.3], [Wed19, 5.6]

6.14. Our goal now is to define topologies on rings of polynomials over A to make them Huber rings.

6.15. Ctx.

1. A is a narc top. ring.
2. $X = ((X_i)_{i \in I}$ is a family of indeterminates.
3. $T = (T_i)_{i \in I}$ is a family of subsets of A satisfying that for all $i \in I$, $m \in \mathbb{N}$, nhood U of 0 in A, the subgrp $T_i^m \cdot U$ is an nhood of 0.

6.3.2 Analytic points

[Mor19, II.2.4]

6.16. We assume $A \in \text{HubRng}$.

Definition 6.17. A point $x \in \text{Cont}(A)$ is *analytic* if $\mathfrak{p}_x$ is *not* open.

Proposition 6.18. Let $x \in \text{Cont}(A)$. Tfae

1. x is analytic.
2. $|\check{A}|_x \neq \{0\}$.

Proof. $1 \Rightarrow 2$. If $\mathfrak{p}_x$ is not open, it cannot contain the *open* additive subgroup $\check{A}$ of A. ☐

6.4 The adic spectrum

Definition 6.19. Let $(A, A^+) \in \mathrm{HubPair}$. The *adic spectrum* $\mathrm{Spa}(A, A^+)$ is a set whose

- elements are set of equiv. class of cont. val. $| \quad |$ on A such that $|A^+| \leq 1$.
- topology is generated by
$$\{x : |f(x)| \leq |g(x)| \neq 0\}$$
$f, g \in A$.

Proposition 6.20.

6.4.1 Quotients

6.4.2 Completion

Proposition 6.21. Let $(A, A^+) \in \mathrm{HubRng}$

6.5 Example: Spa($\mathbb{Z}, \mathbb{Z}$)

6.22. In general for any ring R, $\mathrm{Spa}(R, R)$ consists of valuations bounded by 1.

6.6 Continuous valuation

[Wed19, 7.2]

Definition 6.23. Topology associated from valuation. If $A \in \mathrm{TopRng}$.

- The *valuation topology* is topology by by basis

$$B(a, \gamma) := \{x \in R : |x - a| < \gamma\}, \quad a \in R, \gamma \in \Gamma$$

- We let $\mathrm{Cont}(A) \subset \mathrm{Spv}(A)$.

Definition 6.24. [Wed19, 5.39] Let A be a ring, v a valuation on A, for $\gamma \in \Gamma_v$, set

$$A_\gamma := \{a \in A : v(a) < \gamma\}$$

7 Motivations

7.1. The goal of this section is to describe several of the many applications of perfectoid spaces.

7.1 Fontaine-Winterberger

7.2. The goal of this section is to make sense of the following classical result.

Theorem 7.3. (Fontaine-Winterberger). For μ_{p^∞} the grp. of all p-power roots of unit in an algebraic closure of $\mathbb{Q}_p$, the absolute galois group of the fields $\mathbb{F}_p((\bar{\pi}))$ and $\mathbb{Q}_p(t_{p^\infty})$ are iso. (and homeomoprhic as profintie top. grps).

8 Perfectoid fields

8.1. We will follow the convention as proposed in [Bha17]. Where we fix a base narc complete field k.

8.1 Perfections and tilting

[Bha17, 2], [Mat18, 2].

8.2. Our goal this section is to study the category of perfect $\mathbb{F}_p$-algebras.

Definition 8.3. A ring R of char. p, i.e. an $\mathbb{F}_p$-algebra, is (semi) *perfect* if Frob homo. [9]

$$x \mapsto x^p$$

is a (surjection) bijection. We let $\mathrm{Perf}_{\mathbb{F}_p}$ denote cat. of perfect rings.

Example 8.4. Perfect polynomial rings.

1. Let $R := \mathbb{F}_p[x^{1/p^\infty}] := \varinjlim_{n \geq 1} \mathbb{F}_p[x^{1/p^n}]$ is a perfect rng. The transition maps are frob.
2. For any perfect ring S,
$$\mathrm{Hom}_{\mathrm{Perf}_{\mathbb{F}_p}}(R, S) \simeq S$$

Proof. 2. For any $y \in S$, we can choose a compatible sequence of elements

$$(y, y^{1/p}, \cdots)$$

via the Frob. iso, inducing a unique map $\psi_y : R \to S$, $x \mapsto y$. The bijection is given by

$$\varphi \mapsto \mathrm{ev}_x\, \varphi, \quad y \mapsto \psi_y$$

□

Definition 8.5. Perfection. Let $A \in \mathrm{Perf}_{\mathbb{F}_p}$. We construct in two ways a perfect ring.

1. The direct limit perfection.

Proposition 8.6. There is an adjunction

$$\begin{tikzcd} \mathrm{Perf}_{\mathbb{F}_p} \arrow[r] & \mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p} \arrow[l, bend right, "(\ \ )_{\mathrm{perf}}"'] \arrow[l, bend left, "(\ \ )^{\mathrm{perf}}"] \end{tikzcd}$$

8.1.1 Perfectoid fields

8.7. Most sources begin with the case of fields, [Lur18, 1,2]. We briefly state the case here, but dedicate a subsection on the general case following [Mor16, 2], this nicely leads to the results in [BMS19, 3.2].

8.8. We fix a prime p throughout this discussion.

Definition 8.9. A *perf. field*, is a field K equipped with narc. $|\ \ |_K : K \to \mathbb{R}_{>0} \cup \{0\}$ satisfying tfc.

[9] For this to be a homo we require char p.

A3 Semi-perfect.

A4a The max. ideal $\mathfrak{m}_K$ is not gen. by p.

8.10. If K has char p, then A1, A3 are trivially satisfied.

8.11. Let us give brief remarks on each axioms.

8.12. A4 is satisfied when

- K is alg.
- more generally, if p has a pth root.

A4 can equally be phrased in the following way, which is useful in many arguments.

A4b There exists some $x \in K$ such that $|p|_K < |x|_K < 1$.

Proof. $p = 0$ is clear. If $p \neq 0$,

$$|t| \leq |p| \Leftrightarrow |t/p| \leq 1 \Leftrightarrow t \in (p)$$

□

8.1.2 Strict p-rings

Definition 8.13. A *strict p-ring* is a p-torsion-free, p-adically complete ring R for which $R/(p)$ is perfect $\mathbb{F}_p$ algebra.

Example 8.14. $\mathbb{Z}_p$ is a strict p-ring. Completeness.

1. $\mathbb{Z}_p$ is an open, hence closed, subgrp of semi-normed top. grp. $\mathbb{Q}_p$.
2. A closed subgroup of a cplt. semi-normed grp is cplt.

p-torsion free.

p-adically complete. This follows as $\mathbb{Z}_p$ is open, [BGR84, 1.2.5]

8.2 Integral perfectoid rings

[Mor16, 1].

Definition 8.15. Let $A \in \mathrm{TopRng}$. A is *integral perfectoid* iff exists a nonzero divisor $\pi \in A$, st.

B1 The topology on A is the π-adic top and A is complete for this topology. [10]

B2 $p \in \pi^p A$.

B3 $\Phi : A/\pi A \to A/\pi^p A$, $a \mapsto a^p$ is an iso.

In the lang. of [Mor16, 1], we call any such π a *perfectoid pseudo uniformizer* (ppu.)

[10] In the sense [SP, 07E7].

9 Appendix: Infinite Galois Theory

9.1. This is for personal use - as I am not familiar with this theory. Helpful references are [Conb], [Sut19, 26].

Example 9.2. Three examples to keep in mind.

- Finite extensions of quad. fields.
- p-power cyclotomic extensions.

Theorem 9.3. For an alg. ext. L/K tfae

1. L/K is both separable and normal.

9.4. Via Zorn's lemma, we have the following: for every L/K Galois extension, $K \subset E \subset L$, and K-homo. $E \to L$. There is a K-homo extension $L \to L$.

$$\begin{tikzcd} L \arrow[r, dashed, "\exists"] \arrow[d, no head] & L \\ E \arrow[ur] \arrow[d, no head] & \\ K & \end{tikzcd}$$

9.1 Recollection of finite theory

9.5. We will L/K denote finite ext of fields.

Theorem 9.6. $\mathrm{Aut}(L/K)$ is finite.

9.7. Collecting the above results we shall have tfec

1. $|\mathrm{Aut}(L/K)| = |L : K|$.

Example 9.8. Below we list some failures of Galois extension.

- $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.
- $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$.

9.1.1 Cyclotomic extensions

[Cona, 1].

Definition 9.9. Let $A \in \mathrm{Ab}$ written multiplicatively. An *nth roots of unity*, is $a \in A$ such that

$$a^n = 1$$

In practice, $A = K^\times$, for some field K.

9.10. For any field K, we denote $K(\zeta_n)$, where ζ_n is a root of unit of order n, a *cyclotomic ext.*

Proposition 9.11. The group of nth roots of unity in a field is cyclic. More generally any finite subgroup of nonzero elements of a field form a cyclic group.

Example 9.12. Fix a prime p, $r \geq 1$ positive int.

- For any finite field $\mathbb{F}_q$, $q = p^r$, $\mathbb{F}_q^\times$ are precisely the $q-1$th roots of unity, hence cyclic.
- $(\mathbb{Z}/p^r)^\times$ is *not necess.* cyclic if $r > 1$.

9.13. A separability reduction. Given a cyc. ext. $K(\zeta)/K$. Let n be order of ζ. Then

$$x^n - 1$$

is separable. Thus, we make the following assumption in pursuing discussion

$$x^n - 1 \text{ is } \textit{seaprable over } k.$$

This is equivalent to $n \neq 0$ and

1. If char $k = p > 0$. $(p, n) = 1$.
2. char $k = 0$.

Definition 9.14. When there are n diff. nth roots of unit, we denote the group by μ_n.

Theorem 9.15. There is an injective group homo

$$\mathrm{Gal}(K(\mu_n)/K) \to (\mathbb{Z}/n)^\times$$

Proof. Step 1. *$\sigma \in \mathrm{Gal}(K(\mu_n)/K$ is determined by its an integer a_σ*

$$\sigma(\zeta) = \zeta^a \quad \textit{for all } \zeta \in \mu_n$$

Recall μ_n is a cyclic group. Fix a choice of gen[11] ζ_n by mult. and inj. of σ, we must have

$$\sigma(\zeta_n) = \zeta_n^a$$

where $(a, n) = 1$. By multiplicative we obtain the claimed equality.

This choice of a is well defined modulo n as ζ_n is a gen. This induces a map $\sigma \mapsto a_\sigma \pmod n \in (\mathbb{Z}/n\mathbb{Z})^\times$.

Step 2. Injectivity. If σ in kernel, then $a_\sigma = 1 \pmod n$. All other elements are fixed - and K. □

9.16. We have thus shown, that *cyc. ext. are always abelian.*

Example 9.17. When $K \hookrightarrow \mathbb{C}$.

- Realizing complex conj. Consider $\mathbb{Q}(\mu_n)/\mathbb{Q}$. Cplx. conj. corresponds to $a_\sigma = -1 \in (\mathbb{Z}/n\mathbb{Z})^\times$.
- Counter ex. to surjectivity. Let $K = \mathbb{R}$. $n \geq 3$, then

$$K(\mu_n)/K = \mathbb{C}/\mathbb{R}$$

  is a quad. ext, and this can never surject.

9.18. Standard example. $K = \mathbb{Q}$. $L = \mathbb{Q}(\zeta_m)$ where ζ_m is a root of unit of order m.

1. All other roots of unit are given ζ_m^a where $(a, m) = 1$.
2. L is split field over $\mathbb{Q}$ of $x^m - 1$, which has dist. roots.
3. So $L/\mathbb{Q}$ is galois. This implies $|L : \mathbb{Q}| = |\mathrm{Gal}(L/\mathbb{Q})| = \varphi(m)$.

9.19. As a warmup of the theory, let us apply results form 9.1.

Example 9.20. $\zeta_8 := e^{2\pi i/8}$. $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ is Galois.

[11] a *prim. nth root of unit*

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