ALGEBRAIC THEORIES

A CATEGORICAL INTRODUCTION TO GENERAL ALGEBRA

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With a Foreword by F. W. Lawvere

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Foreword

The study that was initiated by Birkhoff in 1935 was named "general algebra" by Kurosh in his classic text; the subject is also called universal algebra, as in the text by Cohn. The purpose of general algebra is to make explicit common features of the practice of commutative algebra, group theory, linear algebra, Lie algebra, lattice theory, et cetera, in order to illuminate the path for that practice. Less than twenty years after the 1945 debut of the Eilenberg-Mac Lane method of categorical transformations, the obvious possibility of its application to general algebra began to be realized in 1963; that realization continues in the present book.

Excessive iteration of the passage

$$\mathcal{T}' = \text{ theory of } \mathcal{T}$$

would be sterile if pursued as idle speculation without attention to that fundamental motion of theory: concentrate the essence of practice, in order to guide practice. Such theory is necessary to clear the way for the advance of teaching and research. General algebra can and should be used in particular algebra (i.e. in algebraic geometry, functional analysis, homological algebra et cetera) much more than it has been. There are several important instruments for such application, including the partial structure theorem in Birkhoff's "Nullstellensatz", the "commutator" construction, and the general framework itself.

Birkhoff's theorem was inspired by theorems of Hilbert and Noether in algebraic geometry (as indeed was the more general model theory of Robinson and Tarski). His big improvement was not only in generality: beyond mere existence of generalized points, he showed they are sufficient to give a monomorphic embedding. Nevertheless, in commutative algebra his result is rarely mentioned (although it is closely related to Gorenstein algebras). The categorical formulation of Birkhoff's theorem ([65] and [91]), like the pre-categorical ones, involves sub-direct irreducibility and Zorn's lemma. Finitely-generated algebras, in particular, are partially dissected by the theorem into (often qualitatively simpler) finitely-generated pieces. For example, when verifying consequences of a system of polynomial equations over a field, it suffices to consider all possible finite-dimensional interpretations, where constructions of linear algebra such as trace are available.

Another accomplishment of general algebra is the so-called commutator theory (named for its realization in the particular category of groups); a categorical treatment of this theory can be found in [78] and [56]. In other categories this theory specializes to a construction important in algebraic geometry and number theory, namely the product of ideals [51]. In the geometrical classifying topos for the algebraic category of K-rigs, this construction yields an internal multiplicative semi-lattice of closed sub-varieties.

In the practice of group theory and ring theory the roles of presentations and of the algebras presented have long been distinguished, giving a syntactic approach to calculation in particular algebraic theories. Yet many works in general algebra (and model theory generally) continue anachronistically to confuse a presentation in terms of signatures with the theory itself that is presented, thus ignoring the application of general algebra to specific theories, like that of $\mathcal{C}^\infty$-rings, for which no presentation is feasible.

Apart from the specific accomplishments mentioned above, the most effective illumination of algebraic practice by general algebra, both classical and categorical, has come from the explicit nature of the framework itself. The closure properties of certain algebraic sub-categories, the functorality of semantics itself, the ubiquitous existence of functors adjoint to algebraic functors, the canonical method for extracting algebraic information from non-algebraic categories, have served (together with their many particular ramifications) as a partial guidance to mathematicians in dealing with the inevitably algebraic content of their subjects. The careful treatment of these basics, by Adámek, Rosický, and Vitale, will facilitate future mutual applications of algebra, general algebra, and category theory. They have achieved in this book a new resolution of several issues that should lead to further research.

What is General Algebra?

The bedrock ingredient for all its aspects is the use of finite cartesian products. Therefore, as a framework for the subject, it is appropriate to recognize the 2-category of categories that have finite categorical products and of functors preserving these products. Among such categories there are the linear ones whose products are simultaneously co-products; that is a crucial property of linear algebra in that maps between products are then uniquely represented as matrices of smaller maps between the factors (though of course there is no unique decomposition of objects into products, so it would be incorrect to say inversely that maps "are" matrices). General categories with products can be forced to become linear and this reflection 2-functor is an initial ingredient in linear representation theory. However, I want to emphasize instead a strong analogy between general algebra as a whole and any particular linear monoidal category, because that will reveal some of the features of the finite product framework that make possible the more profound results.

The 2-category of all categories with finite products has (up to equivalence) three characteristic features of a linear category such as the category of modules over a rig:

1. It is "additive" because if $A \times B$ is the product of two categories with finite products, it is also their coproduct, the evident injections from A, B having the universal property for maps into any third such category.

2. It is "symmetric closed"; indeed $\mathrm{Hom}(A, B)$ is the category of algebras in the background B according to the theory A. The unit I for this Hom is the opposite of the category of finite sets. The category J of finite sets itself satisfies $\mathrm{Hom}(J, J) = I$, and the category $Hom(J, B)$ is the category of Boolean algebras in B. As dualizer, the case B = small sets is most often considered in abstract algebra.

3. It is "tensored" because a functor of two variables that is product-preserving in each variable separately can be represented as a product-preserving functor on a suitable tensor-product category. Such functors occur in recent work of Zurab Janelidze [55]; specifically, there is a canonical evaluation $A \otimes \mathrm{Hom}(A, B) \to B$, where the domain is "a category whose maps involve both algebraic operations and their homomorphisms".

A feature not present in abstract linear algebra (though it has an analogue in the cohesive linear algebra of functional analysis) is Ross Street's bo-ff factorization of any map (an abbreviation of "bijective on objects followed by full and faithful"), see [89] and [90]. A single-sorted algebraic theory is a map $I \otimes A$ that is bijective on objects; such a map induces a single "underlying" functor $\mathrm{Hom}(A, B) \otimes B$ on the category of A-algebras in B. The factorization permits the definition of the full "algebraic structure" of any given map $u\colon X \to B$ as follows: the map $I \to \mathrm{Hom}(X, B)$ that represents u has its bo-ff factorization, with its bo part the algebraic theory $I \to A(u)$, the full X-natural structure (in its abstract general guise) of all values of u. The original u lifts across the canonical $\mathrm{Hom}(A(u), B) \to B$ by a unique $u^{\#}$. This is a natural first step in one program for "inverting" u, because if we ask whether an object of B is a value of u, we should perhaps consider the richer (than B) structure that any such object would naturally have; that is, we change the problem to one of inverting $u^{\#}$. Jon Beck called this program "descent" with respect to the "doctrine" of general algebra. (A second step is to consider whether $u^{\#}$ has an adjoint.)

Frequently, the dualizing background B is cartesian-closed, i.e., it has not only products but also finite co-products and exponentiation, where exponentiation is a map

$$B^{op} \otimes B \to B$$

in our 2-category. This permits the construction of the important family of function algebras $B^{op} \to \mathrm{Hom}(A, B)$ given any A-algebra (of "constants") in B.

On a higher level, the question whether a given C is a value of the 2-functor $U = \mathrm{Hom}(-, B)$ (for given B), leads to the discovery that such values belong to a much richer doctrine, involving as operations all limits that B has and all colimits that exist in B and preserve finite products. As in linear algebra, where dualization in a module B typically leads to modules with a richer system of operators, conversely such a richer structure assumed on C is a first step toward 2-descent back along U.

The power of the doctrine of natural 2-operations on $\mathrm{Hom}(-, B)$ is enhanced by fixing B to be the category of small sets, where smallness specifically excludes measurable cardinals (although they may be present in the categorical universe at large).

A contribution of Birkhoff's original work had been the characterization of varieties, that is, of those full subcategories of a given algebraic category $\mathrm{Hom}(A, B)$ that are equationally defined by a surjective map $A \to A'$ of theories. Later, the algebraic categories themselves were characterized. Striking refinements of those characterization results, in particular, the clarification of a question left open in the 1968 treatment of categorical general algebra [64], are among the new accomplishments explained in the present book. As Grothendieck had shown in his very successful theory of abelian categories, the exactness properties found in abstract linear algebra continue to be useful for the concretely variable linear algebras arising as sheaves in complex analysis; should something similar be true for non-linear general algebras? More specifically, what are the natural 2-operations and exactness properties shared by all the set-valued categories concretely arising in general algebra? In particular, can that class of categories be characterized by further properties, such as sufficiency of projectives, in terms of these operations? It was clear that small limits and filtered colimits were part of the answer, as with the locally-finitely-presentable categories of Gabriel and Ulmer. But the further insistence of general algebra, on algebraic operations that are total, leads to a further functorial operation, needed to isolate equationally the correct projectives and also useful in dealing with non-Mal'cev categories: that further principle is the ubiquitous preservation of Linton's reflexive co-equalizers, which are explained in this book as a crucial case of Lair's sifted colimits.

Bill Lawvere
Buffalo, October 31, 2009

Introduction

The concept of an algebraic theory, introduced in 1963 by F. W. Lawvere, was a fundamental step towards a categorical view on general algebra in which varieties of algebras are formalized without details of equational presentations. An algebraic theory, as originally introduced, is roughly speaking a category whose objects all are finite powers of a given object. An algebra is then a set-valued functor preserving finite products and a homomorphism between algebras is a natural transformation. In the almost half of a century that followed, this idea has gone through a number of generalizations, ramifications, and applications in areas such as algebraic geometry, topology, and computer science. The generalization from one-sorted algebras to many-sorted ones (of particulat interest in computer science) leads to a simplification: an algebraic theory is now simply a small category with finite products.

Abstract Algebraic Categories

In the first part of this book, consisting of Chapters 1 – 10, we develop the approach in which algebraic theories are studied without a reference to sorting. Consequently, algebraic categories are investigated as abstract categories. We study limits and colimits of algebras, paying special attention to the sifted colimits since they play a central role in the development. For example, algebraic categories are characterized as precisely the free completions under sifted colimits of small categories with finite coproducts. And algebraic functors are precisely the functors preserving limits and sifted colimits. This leads to an algebraic duality: the 2-category of algebraic categories is dually biequivalent to the 2-category of canonical algebraic theories.

Here we present the concept of equation as a parallel pair of morphisms in the algebraic theory. An algebra satisfies the equation iff it merges the parallel pair. We prove Birkhoff's Variety Theorem: subcategories which can be presented by equations are precisely those closed under products, subalgebras, regular quotients, and directed unions. (The last item can be omitted in case of one-sorted algebras.)

Concrete Algebraic Categories

Lawvere's original concept of one-sorted theory is studied in Chapters 11 – 13. Here the categories of algebras are concrete categories over *Set*, and we prove that up to concrete equivalence they are precisely the classical equational categories of Σ-algebras for one-sorted signatures Σ. More generally, given a set S of sorts, we introduce in Chapter 14 S-sorted algebraic theories and the corresponding S-sorted algebraic categories which are concrete over S-sorted sets. Thus we distinguish between "many-sorted" where sorting is not specified and "S-sorted" where a set S of sorts is given (and this distinction leads us to considering the categories of algebras as concrete or abstract ones).

This is supplemented by Appendix A in which a short introduction to monads and monadic algebras is presented. There we prove a duality between one-sorted algebraic theories and finitary monadic categories over *Set*. And again, more generally, between S-sorted algebraic theories and finitary monadic categories over Set^S.

The non-strict version of some concepts, like morphism of one-sorted theories and concrete functor, is treated in Appendix C.

Special Topics

Chapters 15 – 18 are devoted to some more specialized topics. Here we introduce Morita equivalence, characterizing pairs of algebraic theories yielding equivalent categories of algebras. We also prove that algebraic categories are free exact categories. Finally, the finitary localizations of algebraic categories are described: they are precisely the exact locally finitely presentable categories.

Abelian categories are shortly treated in Appendix B.

Other Topics

Of the two categorical approaches to general algebra, monads and algebraic theories, only the latter is treated in this book, with the exception of the short appendix on monads. Both of these approaches make it possible to study algebras in a general category; in our book we just restrict ourselves to sets and many-sorted sets. Thus examples such as topological groups are not treated here.

Another feature not treated at all in this book are algebraic categories based on more general tensor products, not only the categorical product. Such tensor theories were already studied by Mac Lane [71] and they lead to the concept of operad, which is an important development beyond the scope of this book. The development of operads was mostly stimulated by homotopy theory with the aim to define "homotopy invariant algebraic structures" (see [17]). Let us just remark that sifted colimits play a role also in this more general setting since all monads associated to an operad preserve these colimits, see [85]. Moreover, the whole machinery of algebraic theories and sifted colimits can be modified to the homotopy setting, see [70] and [86].

Chapter 0

Preliminaries

The aim of this chapter is to fix some notation and recall well-known facts concerning basic concepts of category theory used throughout the book. The reader may well skip it and return to it where needed. Only the most usual definitions and results of the theory of categories are mentioned here, more about them can be found in any of the book mentioned at the end of this chapter.

0.1 Foundations. In category theory one needs to distinguish between *small* collections (sets) and *large* ones (classes). An arbitrary set theory making such a distinction possible is sufficient for our book. The category of (small) sets and functions is denoted by

$$\mathit{Set}.$$

All categories we work with have small hom-sets.

0.2 Properties of functors. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is

1. *faithful* if for every parallel pair of morphisms $f, g\colon A \rightrightarrows A'$ in $\mathcal{A}$, one has $f = g$ whenever $Ff = Fg$,

2. *full* if for every morphism $b\colon FA \to FA'$ in $\mathcal{B}$ there exists a morphism $a\colon A \to A'$ in $\mathcal{A}$ such that $Fa = b$,

3. *essentially surjective* if for every object B in $\mathcal{B}$ there exists an object A in $\mathcal{A}$ with B isomorphic to FA,

4. an *equivalence* if there exists a functor $F'\colon \mathcal{B} \to \mathcal{A}$ such that both $F \cdot F'$ and $F' \cdot F$ are naturally isomorphic to the identity functors. Such a functor F' is called a *quasi-inverse* of F,

5. an *isomorphism* if there exists a functor $F'\colon \mathcal{B} \to \mathcal{A}$ such that both $F \cdot F'$ and $F' \cdot F$ are equal to the identity functors,

6. *conservative* if it reflects isomorphisms, that is, $a\colon A \to A'$ is an isomorphism whenever $Fa\colon FA \to FA'$ is.

0.3 Remark. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor.

1. If F is full and faithful, then it is conservative.

2. F is an equivalence iff it is full, faithful, and essentially surjective.

3. If F is an equivalence and F' a quasi-inverse of F, it is possible to choose natural isomorphisms $\eta\colon\ \mathrm{Id}_{\mathcal{B}} \to F \cdot F'$ and $\varepsilon\colon\ F' \cdot F \to \mathrm{Id}_{\mathcal{A}}$ such that

$$F\varepsilon \cdot \eta F = F \quad \text{and} \quad \varepsilon F' \cdot F'\eta = F'$$

   (compare with 0.8). (Observe that in equations like $F\varepsilon \cdot \eta F = F$ we write F for the identity natural transformation on a functor F. We adopt the same convention in diagrams having functors as vertices and natural transformations as edges.)

4. F is an isomorphism iff it is full, faithful, and bijective on objects.

0.4 Functor categories and Yoneda embedding.

1. Given a category $\mathcal{A}$ and a small category $\mathcal{C}$ we denote by $\mathcal{A}^{\mathcal{C}}$ the category of functors from $\mathcal{C}$ to $\mathcal{A}$ and natural transformations, and by $\mathcal{C}^{op}$ the dual category.

2. In case $\mathcal{A} = \mathit{Set}$ we have the *Yoneda embedding*

$$Y_{\mathcal{C}}\colon \mathcal{C}^{op} \to \mathit{Set}^{\mathcal{C}}\ , \quad Y_{\mathcal{C}}(X) = \mathcal{C}(X, -)$$

   which is full and faithful. This follows from the *Yoneda Lemma* which states that for every $X \in \mathcal{C}$ and for every functor $F\colon \mathcal{C} \to \mathit{Set}$, the map assigning to every natural transformation $\alpha\colon Y_{\mathcal{C}}(X) \to F$ the value $\alpha_X(\mathrm{id}_X)$ is a bijection natural in X and F.

0.5 Diagrams.

1. A *diagram* in a category $\mathcal{K}$ is a functor from a small category into $\mathcal{K}$.

2. A *finite diagram* is a diagram $D\colon \mathcal{D} \to \mathcal{K}$ such that $\mathcal{D}$ is a *finitely generated category.* This means that $\mathcal{D}$ has finitely many objects and a finite set of morphisms whose closure under composition gives all the morphisms of $\mathcal{D}$.

3. A category is *complete* if every diagram in it has a limit; dually: *cocomplete.*

0.6 Colimits in Set. In the category of sets:

1. Coproducts are disjoint unions.

2. Coequalizers of $i, j\colon X \rightrightarrows Z$ can be described as the canonical maps $c\colon Z \to Z/\sim$ where $\sim$ is the smallest equivalence relation with $i(z) = j(z)$ for every $z \in Z$. This equivalence relation merges elements z and z' of Z iff there exists a zig-zag of elements

$$\begin{tikzcd}[column sep=small] & y_0' \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & & y_1' \arrow[dl, "f_2"'] \arrow[dr, "f_3"] & & \cdots\cdots & & y_n' \arrow[dl, "f_{2n}"'] \arrow[dr, "f_{2n+1}"] & \\ z & & y_1 & & y_2 & \cdots & y_n & & z' \end{tikzcd}$$

where each f_k is equal to i or to j for $k = 0, \ldots, 2n+1$.

3. A category $\mathcal{D}$ is called *filtered* if every finitely generated subcategory has a cocone in $\mathcal{D}$ (for more about this concept see Chapter 2). A colimit of a diagram $D\colon \mathcal{D} \to \mathit{Set}$ is described as the quotient

$$\coprod_{x \in obj\,\mathcal{D}} Dx/\sim$$

where for elements $u_i \in Dx_i$ we have $u_1 \sim u_2$ iff there exist morphisms $f_i\colon x_i \to y$ in $\mathcal{D}$ such that $Df_1(u_1) = Df_2(u_2)$.

0.7 Construction of colimits. In a category $\mathcal{A}$ with coproducts and coequalizers all colimits exist. Given a diagram $D\colon \mathcal{D} \to \mathcal{A}$ form a parallel pair

$$\coprod_{f \in \mathrm{mor}\,\mathcal{D}} Df_d \overset{i}{\underset{j}{\rightrightarrows}} \coprod_{x \in obj\,\mathcal{D}} Dx$$

where f_d and f_c denote the domain and codomain of f. The f-component of i is the coproduct injection of Df_d, that of j is the composite of Df and the coproduct injection of Df_c.

1. If
$$c\colon \coprod_{x \in obj\,\mathcal{D}} Dx \longrightarrow C$$
is the coequalizer of i and j, then $C = \mathit{colim}\, D$ and the components of c form the colimit cocone.

2. The pair i, j above is reflexive, that is, there exists a morphism
$$\delta\colon \coprod_{x \in obj\,\mathcal{D}} Dx \longrightarrow \coprod_{f \in \mathrm{mor}\,\mathcal{D}} Df_d$$
such that $i \cdot \delta = \mathrm{id} = j \cdot \delta$. Indeed, the x-component of δ is the coproduct injection of id_X .

0.8 Adjoint functors. Given functors $U\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{A}$, then F is a left adjoint of U, notation $F \dashv U$, if there exist natural transformations $\eta\colon \mathrm{Id}_{\mathcal{B}} \to UF$ and $\varepsilon\colon FU \to \mathrm{Id}_{\mathcal{A}}$ satisfying

$$\varepsilon F \cdot F\eta = F \quad \text{and} \quad U\varepsilon \cdot \eta U = U\,.$$

This is equivalent to the existence of a bijection

$$\mathcal{A}(FB, A) \simeq \mathcal{B}(B, UA)$$

natural in $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

1. Every left adjoint preserves colimits.
2. Dually, every right adjoint preserves limits.
3. A *solution set* for a functor $U\colon \mathcal{A} \to \mathcal{B}$ and an object X of $\mathcal{B}$ is a set of morphisms $f_i\colon X \to UA_i$ ($i \in I$) with $A_i \in \mathcal{A}$ such that every other morphism $f\colon X \to UA$ has a factorization $f = Uh \cdot f_i$ for some $i \in I$ and some morphism $h\colon A_i \to A$ in $\mathcal{A}$.
4. The *Adjoint Functor Theorem* states that if $\mathcal{A}$ has limits, then a functor $U\colon \mathcal{A} \to \mathcal{B}$ has a left adjoint iff it
   (a) preserves limits, and
   (b) has a solution set for every object X of $\mathcal{B}$.

0.9 Reflective subcategories. Given a category $\mathcal{B}$, by a *reflective subcategory* of $\mathcal{B}$ is meant a full subcategory $\mathcal{A}$ such that the inclusion functor $\mathcal{A} \to \mathcal{B}$ has a left adjoint (called a reflector for $\mathcal{B}$). We denote by $R\colon \mathcal{B} \to \mathcal{A}$ the reflector and by $r_B\colon B \to RB$ the reflections.

0.10 Representable functors. A functor from a category $\mathcal{A}$ to *Set* is *representable* if it is naturally isomorphic to a hom-functor $\mathcal{A}(A, -)$.

1. If $\mathcal{A}$ has coproducts, then $\mathcal{A}(A, -)$ has a left adjoint assigning to a set X a coproduct of X copies of A.
2. The colimit of $\mathcal{A}(A, -)$ is a singleton set.
3. The Adjoint Functor Theorem can be stated in terms of representable functors as follow:
   A functor $F\colon \mathcal{B} \to \mathit{Set}$, with $\mathcal{B}$ complete, has a left adjoint iff it is representable. This is the case iff it preserves limits and satisfies the solution set condition: there exists a set $\mathcal{G}$ of objects of $\mathcal{B}$ such that for any object B of $\mathcal{B}$ and any element $b \in FB$, there are $X \in \mathcal{G}$, $x \in FX$ and $f\colon X \to B$ such that $Ff(x) = b$.

0.11 Example.

1. For every set X the functor $X \times - : \mathit{Set} \to \mathit{Set}$ is left adjoint to $\mathit{Set}(X, -)$.

2. For a category $\mathcal{A}$ the diagonal functor

$$\Delta \colon \mathcal{A} \to \mathcal{A} \times \mathcal{A}\,, \quad A \mapsto (A, A)$$

has a left adjoint iff $\mathcal{A}$ has finite products. Then

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}\,, \quad (A, B) \mapsto A \times B$$

is a left adjoint to Δ.

0.12 Remark. The contravariant hom-functors $\mathcal{B}(-, B) \colon \mathcal{B} \to \mathit{Set}^{op}$, $B \in \mathit{obj}\,\mathcal{B}$, collectively reflect colimits. That is, for every cocone C of a diagram $D \colon \mathcal{D} \to \mathcal{B}$ we have: C is a colimit of D iff the image of C under any $\mathcal{B}(-, B)$ is a colimit of the diagram $\mathcal{B}(-, B) \cdot D$ in *Set*.

0.13 Slice categories. Given functors $F \colon \mathcal{A} \to \mathcal{K}$ and $G \colon \mathcal{B} \to \mathcal{K}$, the *slice category* $(F \downarrow G)$ has as objects all triples (A, f, B) with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $f \colon FA \to GB$, and as morphisms $(A, f, B) \to (A', f', B')$ all pairs $a \colon A \to A'$, $b \colon B \to B'$ such that $Gb \cdot f = f' \cdot Ga$.

1. As special cases, we have $K \downarrow G$ and $F \downarrow K$ where an object $K \in \mathcal{K}$ is seen as a functor from the one-arrow category to $\mathcal{K}$.

2. If F is the identity functor on $\mathcal{K}$, we write $\mathcal{K} \downarrow K$ instead of $\mathrm{id}_{\mathcal{K}} \downarrow K$.

0.14 Set functors as colimits of representables. Every functor $A \colon \mathcal{T} \to \mathit{Set}$ ($\mathcal{T}$ small) is in a canonical way a colimit of representable functors. In fact, consider the Yoneda embedding $Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to \mathit{Set}^{\mathcal{T}}$ and the slice category $\mathit{El}\,A = Y_{\mathcal{T}} \downarrow A$ of "elements of A". Its objects can be represented as pairs (X, x) with $X \in \mathit{obj}\,\mathcal{T}$ and $x \in A(X)$, and its morphisms $f \colon (X, x) \to (Z, z)$ are morphisms $f \colon Z \to X$ of $\mathcal{T}$ such that $Af(z) = x$. We denote by $\Phi_A \colon \mathit{El}\,A \to \mathcal{T}^{op}$ the canonical projection which to every element of the set AX assigns the object X. Then A is a colimit of the diagram of representable functors as follows

$$\mathit{El}\,A \xrightarrow{\Phi_A} \mathcal{T}^{op} \xrightarrow{Y_{\mathcal{T}}} \mathit{Set}^{\mathcal{T}}$$

Indeed, the colimit injection $Y_{\mathcal{T}}(\Phi_A(X, x)) \to A$ is the natural transformation corresponding, by Yoneda Lemma, to the element $x \in AX$.

0.15 Kernel pair. A *kernel pair* of a morphism $f \colon A \to B$ is a parallel pair $f_1, f_2 \colon N(f) \rightrightarrows A$ forming a pullback of f and f.

0.16 Classification of quotient objects. A *quotient object* of an object A is represented by an epimorphism $e \colon A \to B$, and an epimorphism $e' \colon A \to B'$ represents the same quotient iff $e' = i \cdot e$ holds for some isomorphism $i \colon B \to B'$. We use the same adjective for quotient objects and (any of) the representing epimorphisms $e \colon A \to B$:

1. *Split* means that there exists $i \colon B \to A$ with $e \cdot i = \mathrm{id}_B$. Then B is called a *retract* of A.

2. *Regular* means that e is a coequalizer of a parallel pair with codomain A.
3. *Strong* means that in every commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dotted, "d"'] \\ X \arrow[r, "m"'] & Z \end{tikzcd}$$

   where m is a monomorphism, there is a "diagonal" morphism $d\colon B \to X$ such that $m \cdot d = v$ and $d \cdot e = u$.

4. *Extremal* means that in every commutative triangle

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "u"'] & & B \\ & X \arrow[ur, "m"'] & \end{tikzcd}$$

   where m is a monomorphism, then m is an isomorphism.

Dually, a *subobject* of A is represented by a monomorphism $m\colon B \to A$, and a monomorphism $m'\colon B' \to A$ represents the same subobject iff $m' = m \cdot i$ holds for some isomorphism $i\colon B' \to B$.

0.17 Remark. Let us recall some elementary facts on extremal, strong and regular epimorphisms.

1. Every regular epimorphism is strong and every strong epimorphism is extremal. If the category $\mathcal{A}$ has finite limits, then extremal = strong.
2. If the category $\mathcal{A}$ has binary products, then the condition of being an epimorphism in the definition of strong epimorphism is redundant. The same holds for extremal epimorphisms if the category $\mathcal{A}$ has equalizers.
3. If a composite $f \cdot g$ is a strong epimorphism, then f is a strong epimorphism. The same holds for extremal epimorphisms, but in general this fails for regular epimorphisms.
4. If f is a monomorphism and an extremal epimorphism, then it is an isomorphism.

0.18 Concrete categories. Let $\mathcal{K}$ be a category.

1. By a *concrete category over* $\mathcal{K}$ is meant a category $\mathcal{A}$ together with a faithful functor $U\colon \mathcal{A} \to \mathcal{K}$.
2. Given concrete categories $(\mathcal{A}, U)$ and $(\mathcal{A}', U')$ over $\mathcal{K}$, a *concrete functor* is a functor $F\colon \mathcal{A} \to \mathcal{A}'$ such that $U = U' \cdot F$.

Further Reading

For standard concepts of category theory the reader may consult e.g. the following monographs: [4], [27], [71].

Chapter 1

Algebraic theories and algebraic categories

In this chapter we introduce algebraic theories and algebraic categories and study basic concepts, such that limits of algebras, and representable algebras. We also introduce some of the main examples of algebraic categories.

1.1 Definition. An *algebraic theory* is a small category $\mathcal{T}$ with finite products. An *algebra* for the theory $\mathcal{T}$ is a functor $A\colon \mathcal{T} \to Set$ preserving finite products. We denote by $Alg\,\mathcal{T}$ the category of algebras of $\mathcal{T}$. Morphisms, called *homomorphisms*, are the natural transformations. That is, $Alg\,\mathcal{T}$ is a full subcategory of the functor category $Set^{\mathcal{T}}$.

1.2 Definition. A category is *algebraic* if it is equivalent to $Alg\,\mathcal{T}$ for some algebraic theory $\mathcal{T}$.

1.3 Remark. An algebraic theory is by definition a small category. However, throughout the book we do not take care of the difference between *small* and *essentially small*: a category is essentially small if it is equivalent to a small one.

We will see in 10.15, 13.11 and 14.23 that algebraic categories correspond well with varieties, i.e., equational categories of (many-sorted, finitary) algebras. Before giving some examples of algebraic categories, we need two simple facts.

1.4 Remark. Every object t of an algebraic theory $\mathcal{T}$ yields the algebra $Y_{\mathcal{T}}(t)$ representable by t :

$$Y_{\mathcal{T}}(t) = \mathcal{T}(t, -)\colon \mathcal{T} \to Set\,.$$

This, together with the Yoneda transformations, defines a full and faithful functor

$$Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to Alg\,\mathcal{T}\,.$$

1.5 Lemma. *For every algebraic theory $\mathcal{T}$, the Yoneda embedding*

$$Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to Alg\,\mathcal{T}$$

preserves finite coproducts.

Proof. If 1 is a terminal object of $\mathcal{T}$ then $\mathcal{T}(1,-)$ is an initial object of $Alg\,\mathcal{T}$: for every algebra A we know that $A1$ is a terminal object, thus there is a unique morphism $\mathcal{T}(1,-) \to A$.

Given two objects t_1, t_2 in $\mathcal{T}$ then $\mathcal{T}(t_1 \times t_2, -)$ is a coproduct of $\mathcal{T}(t_1,-)$ and $\mathcal{T}(t_2,-)$ since for every algebra A the morphisms $\mathcal{T}(t_1 \times t_2, -) \to A$ correspond to elements of $A(t_1 \times t_2) = A(t_1) \times A(t_2)$. □

The simplest examples of algebraic categories are the category of sets and the category of many-sorted sets. We will treat them in 1.9 and 1.10 as special cases of the following more general example which plays a prominent role throughout our book.

1.6 Example. Set-valued functors. If $\mathcal{C}$ is a small category, the functor category $Set^{\mathcal{C}}$ is algebraic. An algebraic theory of $Set^{\mathcal{C}}$ is a *free completion* $\mathcal{T}_{\mathcal{C}}$ *of* $\mathcal{C}$ *under finite products*. This means a functor $E_{Th} : \mathcal{C} \to \mathcal{T}_{\mathcal{C}}$ such that

1. $\mathcal{T}_{\mathcal{C}}$ is a category with finite products

and

2. for every functor $F : \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is a category with finite products, there exists an essentially unique functor (that is, unique up to natural isomorphism) $F^* : \mathcal{T}_{\mathcal{C}} \to \mathcal{B}$ preserving finite products with F naturally isomorphic to $F^* \cdot E_{Th}$.

In other words, composition with E_{Th} gives an equivalence between the category of finite product preserving functors from $\mathcal{T}_{\mathcal{C}}$ to $\mathcal{B}$ and the category of functors from $\mathcal{C}$ to $\mathcal{B}$. In particular, the categories $Set^{\mathcal{C}}$ and $Alg\,\mathcal{T}_{\mathcal{C}}$ are equivalent.

1.7 Remark. The free finite-product completion $\mathcal{T}_{\mathcal{C}}$ can be described as follows:

(i) Objects of $\mathcal{T}_{\mathcal{C}}$ are all finite families

$$(C_i)_{i \in I}\,,\ I \text{ finite}$$

of objects of $\mathcal{C}$, and morphisms from $(C_i)_{i \in I}$ to $(C'_j)_{j \in J}$ are pairs (a, α) where $a : J \to I$ is a function and $\alpha = (\alpha_j)_{j \in J}$ is a family of morphisms $\alpha_j : C_{a(j)} \to C'_j$ of $\mathcal{C}$. The composition and identity morphisms are defined as expected. A terminal object in $\mathcal{T}_{\mathcal{C}}$ is the empty family, and a product of two objects is the disjoint union of the families. Finally, the functor $E_{Th} : \mathcal{C} \to \mathcal{T}_{\mathcal{C}}$ is given by $E_{Th}(C) = (C)$. It is easy to verify the universal property: since for every object $(C_i)_{i \in I}$ in $\mathcal{T}_{\mathcal{C}}$ we have $(C_i)_{i \in I} = \Pi_I E_{Th}(C_i)$, then necessarily $F^*((C_i)_{i \in I}) = \Pi_I F C_i$.

(ii) Equivalently, $\mathcal{T}_{\mathcal{C}}$ can be described as the category of all words over $obj\,\mathcal{C}$ (the set of objects of $\mathcal{C}$). That is, objects have the form of n-tuples $C_0 \dots C_{n-1}$ where each C_i is an object of $\mathcal{C}$ (and where n is identified with the set $\{0, \dots, n-1\}$) including the case $n = 0$ (empty word). Morphisms from $C_0 \dots C_{n-1}$ to $C'_0 \dots C'_{k-1}$ are pairs (a, α) consisting of a function $a : k \to n$ and a k-tuple of $\mathcal{C}$-morphisms $\alpha = (\alpha_0, \dots, \alpha_{k-1})$ with $\alpha_i : C_{a(i)} \to C'_i$.

1.8 Remark. Since the Yoneda embedding $Y_{\mathcal{T}_\mathcal{C}}: {\mathcal{T}_\mathcal{C}}^{op} \to Alg\,\mathcal{T}_\mathcal{C} \simeq Set^{\mathcal{C}}$ preserves finite coproducts (1.5), the category ${\mathcal{T}_\mathcal{C}}^{op}$ is equivalent to the full subcategory of $Set^{\mathcal{C}}$ given by finite coproducts of representable functors.

1.9 Example. Sets. The simplest algebraic category is the category of sets itself. An algebraic theory $\mathcal{N}$ for *Set* can be described as the full subcategory of Set^{op} whose objects are the natural numbers. In fact, since $n = 1 \times \ldots \times 1$ in Set^{op}, every algebra $A\colon \mathcal{N} \to Set$ is determined, up to isomorphism, by the set $A(1)$. More precisely, we have an equivalence functor

$$E\colon Alg\mathcal{N} \to Set\,, \quad A \mapsto A(1)\,.$$

The category *Set* has other algebraic theories – we describe them in Chapter 15.

1.10 Example. Many-sorted sets. The algebraic theory $\mathcal{N}$ for *Set* described in 1.9 is nothing else than the theory $\mathcal{T}_\mathcal{C}$ of 1.6 when $\mathcal{C}$ is the one-object discrete category. More generally, if in 1.6 the category $\mathcal{C}$ is discrete, i.e. it is a set S, we get the power category Set^S of S-sorted sets and S-sorted functions. Following 1.7, $\mathcal{T}_S^{op}$ is equivalent to the full subcategory of Set^S of *finite S-sorted sets* (an S-sorted set $\langle A_s\rangle_{s\in S}$ is finite if the coproduct $\coprod_S A_s$ is a finite set). Another equivalent description of $\mathcal{T}_S$ is obtained by taking finite words over S as objects (including the empty word). Morphisms from $s_0 \ldots s_{n-1}$ to $s'_0 \ldots s'_{k-1}$ are functions $a\colon k \to n$ such that $s_{a(i)} = s'_i$ $(i = 0, \ldots, k-1)$.

1.11 Example. Abelian groups. We denote by $\mathcal{T}_{ab}$ the category having natural numbers as objects, and morphisms from n to k are matrices of integers with n columns and k rows. Composition of $P\colon m \to n$ and $Q\colon n \to k$ is given by matrix multiplication $Q \cdot P = Q \times P\colon m \to k$, and identity morphisms are the unit matrices. If $n = 0$ or $k = 0$, the only $n \times k$ matrix is the empty one $[\;]$. $\mathcal{T}_{ab}$ has finite products. For example, 2 is the product 1×1 with projections $[1,0]\colon 2 \to 1$ and $[0,1]\colon 2 \to 1$. (In fact, given one-row matrices $P, Q\colon n \to 1$, there exists a unique two-row matrix $R\colon n \to 2$ such that $[1,0] \cdot R = P$ and $[0,1] \cdot R = Q$: the matrix with rows P and Q.) Here is a direct argument proving that the category Ab of abelian groups is equivalent to $Alg\,\mathcal{T}_{ab}$ (see also 1.13). Every abelian group G defines an algebra $\widehat{G}\colon \mathcal{T}_{ab} \to Set$ whose object function is $\widehat{G}n = G^n$. For every morphism $P\colon n \to k$ we define $\widehat{G}P\colon G^n \to G^k$ by matrix multiplication

$$\widehat{G}(P)\colon \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \mapsto P \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

The function $G \mapsto \widehat{G}$ extends to a functor $\widehat{(-)}\colon Ab \to Alg\,\mathcal{T}_{ab}$ in a rather obvious way: given a group homomorphism $h\colon G_1 \to G_2$, then $\hat{h}\colon \widehat{G}_1 \to \widehat{G}_2$ is the natural transformation whose components are $h^n\colon G_1^n \to G_2^n$. It is clear that $\widehat{(-)}$ is a well defined, full and faithful functor. To prove that it is an equivalence functor, we need, for every algebra $A\colon \mathcal{T}_{ab} \to Set$, to present an abelian group G with

$A \simeq \widehat{G}$. The underlying set of G is $A1$. The binary group operation is obtained from the morphism $[1,1]\colon 2 \to 1$ in $\mathcal{T}_{ab}$ by $A[1,1]\colon G^2 \to G$, the neutral element is $A[\,]\colon 1 \to G$ for the morphism $[\,]\colon 0 \to 1$ of $\mathcal{T}_{ab}$, and the inverse is given by $A[-1]\colon G \to G$. It is not difficult to check that the axioms of abelian group are fulfilled. For example, the axiom $x+0 = x$ follows from the fact that A preserves the composition of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}\colon 1 \to 2$ with $[1,1]\colon 2 \to 1$. Clearly, $A \simeq \widehat{G}$ (consider the canonical isomorphism $An = A(1 \times \ldots \times 1) \simeq A1 \times \ldots \times A1 = G^n = \widehat{G}n$).

1.12 Example. Modules. Let R be a ring with unit. The category R-*Mod* of left modules and module homomorphisms is algebraic. A theory directly generalizing that of abelian groups above has as objects natural numbers and as morphisms matrices over R. Algebraic categories of the form R-*Mod* are treated in greater detail in Appendix B.

1.13 Remark.

1. In the example of abelian groups we have the forgetful functor $U\colon Ab \to Set$ assigning to every abelian group its underlying set. Observe that the above groups $\mathbb{Z}^n$ are free objects of Ab on n generators, and the full subcategory of all these objects is the dual of the theory $\mathcal{T}_{ab}$ above.

2. Analogously, if the category $\mathcal{C}$ of Example 1.6 has object set S, then we have a forgetful functor which forgets the action of $A\colon \mathcal{C} \to Set$ on morphisms:
$$U\colon Set^{\mathcal{C}} \to Set^{S}\,, \quad UA = \langle A(s)\rangle_{s\in S}.$$
The functor U has a left adjoint
$$F\colon Set^{S} \to Set^{\mathcal{C}}\,, \quad F(\langle A_s\rangle_{s\in S}) = \coprod_{s\in S}\left(\coprod_{A_s}\mathcal{C}(s,-)\right)$$
(this easily follows from the Yoneda Lemma because F preserves coproducts). Following 1.8, the objects of the theory $\mathcal{T}_{\mathcal{C}}$ are precisely the finitely generated free objects, that is, the image of finite S-sorted sets under F.

3. In Chapters 11 and 14 we will see that this is not a coincidence: for every S-sorted algebraic category $\mathcal{A}$ the free objects on finitely many S-sorted generators form a full subcategory whose dual is a theory for $\mathcal{A}$ (see 11.22 for one-sorted algebraic categories and 14.13 for S-sorted algebraic categories).

1.14 Remark.

1. Let Σ be a *signature*, that is, a set Σ (of operation symbols) together with an arity function
$$\mathrm{ar}\colon \Sigma \to \mathbb{N}\,.$$
A Σ-*algebra* consists of a set A and, for every n-ary symbol $\sigma \in \Sigma$, an n-ary operation $\sigma^A\colon A^n \to A$. An homomorphism of Σ-algebras is a function

preserving the given operations. The category Σ-*Alg* of Σ-algebras and homomorphisms as well as its equational subcategories are algebraic, as we demonstrate in Chapter 13. Thus e.g. Boolean algebras form an algebraic category.

2. Also the more general many-sorted signatures yield algebraic categories that we study in Chapter 14. Here is a concrete example:

1.15 Example. Graphs. We denote by *Graph* the category of *directed graphs* G with multiple edges: they are given by a set G_v of vertices, a set G_e of edges, and two functions from G_e to G_v determining the target (τ) and the source (σ) of every edge. The morphisms are called *graph homomorphisms*: given graphs G and G' a graph homomorphism is a pair of functions $h_v \colon G_v \to G'_v$ and $h_e \colon G_e \to G'_e$ such that the source and the target of every edge is preserved. A theory $\mathcal{T}_{\mathcal{G}}$ of *Graph* is a special case of 1.6 with the obvious category $\mathcal{G}$:

$$\begin{tikzcd} e \arrow[loop left, "\mathrm{id}_e"] \arrow[r, shift left, "\tau"] \arrow[r, shift right, "\sigma"'] & v \arrow[loop right, "\mathrm{id}_v"] \end{tikzcd}$$

The category of algebras of an algebraic theory is quite rich. We already know that every object t of an algebraic theory $\mathcal{T}$ yields the representable algebra $Y_{\mathcal{T}}(t) = \mathcal{T}(t, -)$. Other examples of algebras can be obtained e.g. by the formation of limits and colimits. We will now show that limits always exist and are built up at the level of sets. Also colimits always exist, but they are seldom built up at the level of sets. We will study colimits in the subsequent Chapters.

1.16 Proposition. *For every algebraic theory $\mathcal{T}$, the category $\mathit{Alg}\,\mathcal{T}$ is closed in $\mathit{Set}^{\mathcal{T}}$ under limits.*

Proof. Limits are formed objectwise in $\mathit{Set}^{\mathcal{T}}$. Since limits and finite products commute, given a diagram in $\mathit{Set}^{\mathcal{T}}$ whose objects are functors preserving finite products, then a limit of that diagram also preserves finite products. □

1.17 Corollary. *Every algebraic category is complete.*

1.18 Remark.

1. The previous proposition means that limits of algebras are formed objectwise at the level of sets. For example, a product of two graphs has both the vertex set given by the cartesian product of the vertex sets, and the set of edges given by the cartesian product of the edge sets.

2. Monomorphisms in the category $\mathit{Alg}\,\mathcal{T}$ are precisely the homomorphisms that are componentwise monomorphisms (i.e., injective functions) in *Set*. In fact, this is true in $\mathit{Set}^{\mathcal{T}}$, and $\mathit{Alg}\,\mathcal{T}$ is closed under monomorphisms (being closed under limits) in $\mathit{Set}^{\mathcal{T}}$.

3. In every algebraic category kernel pairs (0.15) exist and are formed objectwise (in *Set*).

1.19 Example. One of the most important data types in computer science is a stack, or finite list, of elements of a set (of "letters") called an alphabet. Here we consider stacks of natural numbers: we will have elements of sort n (natural number) and s (stack) and the following basic operations:

succ, the successor of a natural number,

push, which adds a new letter to the left-most position of a stack,

pop, which deletes the left-most position,

and

top, which reads the top element of the stack.

We will also have two constants: e, for the empty stack, and 0 of sort n. For simplicity we put $\mathrm{top}(e) = 0$. This leads us to the concept of algebras A of two sorts

$$s \text{ (stack)} \quad \text{and} \quad n \text{ (natural number)}$$

with operations

$\mathrm{succ} : A_n \to A_n$,

$\mathrm{push} : A_s \times A_n \to A_s$,

$\mathrm{pop} : A_s \to A_s$,

and

$\mathrm{top} : A_s \to A_n$,

and with constants $0 \in A_n$ and $e \in A_s$.

We can consider stacks as equationally specified algebras of sorts $\{s, n\}$, and the algebraic theory is then obtained from the corresponding finitely generated free algebras.

1.20 Example. Sequential automata. Recall that a deterministic sequential automaton A is given by a set A_s of states, a set A_i of input symbols, a set A_o of output symbols, and by three functions

$\delta\colon A_s \times A_i \to A_s$ (next-state function)

$\gamma\colon A_s \to A_o$ (output)

$\varphi\colon 1 \to A_s$ (initial state)

Given two sequential automata A and $A' = (A'_s, A'_i, A'_o, \delta', \gamma', \varphi')$, a morphism (simulation) is given by a triple of functions

$$h_s\colon A_s \to A'_s \ , \ \ h_i\colon A_i \to A'_i \ , \ \ h_o\colon A_o \to A'_o$$

such that the diagram

$$\begin{tikzcd}
A_s \times A_i \arrow[rr, "\delta"] \arrow[dd, "h_s \times h_i"'] & & A_s \arrow[r, "\gamma"] \arrow[dd, "h_s"] & A_o \arrow[dd, "h_o"] \\
& 1 \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & & \\
A'_s \times A'_i \arrow[rr, "\delta'"'] & & A'_s \arrow[r, "\gamma'"'] & A'_o
\end{tikzcd}$$

commutes. This is the category of algebras of three sorts s, i and o given by the signature

$$\delta\colon si \to s\,, \quad \gamma\colon s \to o\,, \quad \varphi\colon \ \to s$$

(see Chapter 14 for the general notion of many-sorted signature). Again, an algebraic theory of automata is formed by considering finitely generated free algebras.

Historical Remarks for Chapter 1

Algebraic theories were introduced by F. W. Lawvere in his dissertation [63]. He considered the one-sorted case, which we study in Chapter 11. This corresponds to (one-sorted) equational theories of G. Birkhoff [24] which we treat in Chapter 13.

Many-sorted equational theories were first considered by P. J. Higgins [53] and were later popularized by G. Birkhoff and J. D. Lipson [25]. In the review of Higgins' paper A. Heller [52] suggested to look for the connection with Lawvere's approach. This was done by J. Bénabou [21] who dealt with many-sorted algebraic theories. Our definition of an algebraic theory is given without a reference to sorting. This "sort-free" approach corresponds to the more general theory of sketches initiated by C. Ehresmann [41] (see [20] for an exposition). The S-sorted approach is presented in Chapter 14.

The interested reader can find expositions of various aspects of algebraic theories e.g. in [98], [77], [27], [7] and [80].

Chapter 2

Sifted and filtered colimits

Colimits in algebraic categories are, in general, not formed objectwise. In this chapter we study the important case of sifted colimits, which are always formed objectwise. Prominent examples of sifted colimits are filtered colimits and reflexive coequalizers (see Chapter 3).

2.1 Definition. A small category $\mathcal{D}$ is called

1. *sifted* if finite products in *Set* commute with colimits over $\mathcal{D}$.
2. *filtered* if finite limits in *Set* commute with colimits over $\mathcal{D}$.

Colimits of diagrams over sifted (or filtered) categories are called sifted (or filtered) colimits.

2.2 Remark. Explicitly, a small category $\mathcal{D}$ is sifted iff, given a diagram

$$D \colon \mathcal{D} \times \mathcal{J} \to Set$$

where $\mathcal{J}$ is a finite discrete category, then the canonical map

$$\underset{\mathcal{D}}{colim} \left(\prod_{\mathcal{J}} D(d,j) \right) \to \prod_{\mathcal{J}} (\underset{\mathcal{D}}{colim} \; D(d,j)) \qquad [2.1]$$

is an isomorphism. $\mathcal{D}$ is filtered iff it satisfies the same condition, but with respect to every finitely generated category $\mathcal{J}$ (replace $\prod_{\mathcal{J}}$ by $lim_{\mathcal{J}}$ in [2.1]).

2.3 Example.

1. Colimits of ω-chains are filtered. Here the category $\mathcal{D}$ is the poset of natural numbers, considered as a category.
2. More generally, colimits of chains (where $\mathcal{D}$ is an infinite ordinal considered as a poset) are filtered. These are the "typical" filtered colimits: a category having colimits of chains has all filtered colimits, see 1.5 and 1.7 in [AR].

3. Generalizing still further: *directed colimits* are filtered. Recall that a poset is called (upwards) directed if it is nonempty and every pair of elements has an upper bound. Directed colimits are colimits of diagrams whose schemes are directed posets.

4. An example of filtered colimits that are not directed: the colimits of idempotents. Let f be an endomorphism of an object A which is idempotent, that is, $f \cdot f = f$. This can be considered as a diagam whose domain $\mathcal{D}$ has one object and, besides the identity, precisely one idempotent morphism. This category is filtered. In fact, the colimit of the above diagram is the coequalizer of f and id_A. It is not difficult to verify directly (or using 2.18) that in *Set* these coequalizers commute with finite limits.
   Colimits of idempotents are the only filtered colimits of finite diagrams: every category with colimits of idempotents has all finite filtered colimits, and every functor preserving colimits of idempotents preserves all finite filtered colimits, see e.g. [7].

5. Filtered colimits are of course sifted.

6. Coequalizers are colimits that are not sifted (see 2.16). As we will see in Chapter 3, reflexive coequalizers are sifted (but not filtered); these are coequalizers of parallel pairs $a_1, a_2 \colon A \rightrightarrows B$ for which $d \colon B \to A$ exists with $a_1 \cdot d = \mathrm{id}_B = a_2 \cdot d$.

   In fact, in a sense made precise in Chapter 7, we can state that

   $$\text{sifted colimits} = \text{filtered colimits} + \text{reflexive coequalizers.}$$

2.4 Remark. Sifted categories have an easy characterization: they

(i) are nonempty

and

(ii) have, for every pair of objects, the category of all cospans connected.

This will be proved in 2.14. Before doing that, we need to recall the standard concepts of connected category and final functor. But we first present a result showing why sifted colimits are important.

2.5 Proposition. *For every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under sifted colimits.*

Proof. Since sifted colimits and finite products commute in *Set*, they do so in $Set^{\mathcal{T}}$ (where they are computed objectwise). It follows that a sifted colimit in $Set^{\mathcal{T}}$ of functors preserving finite products also preserves finite products. $\square$

2.6 Example. Coproducts are not sifted colimits. In fact, for almost no algebraic theory $\mathcal{T}$ is $Alg\,\mathcal{T}$ closed under coproducts in $Set^{\mathcal{T}}$. Concrete example: if $\mathcal{T}_{ab}$ is the theory of abelian groups (1.11), then binary coproducts in $Alg\,\mathcal{T}_{ab}$ are products and in $Set^{\mathcal{T}_{ab}}$ they are disjoint union.

2.7 Corollary. *In every algebraic category sifted colimits commute with finite products.*

In fact, this follows from the fact that the category $Alg\,\mathcal{T}$ is closed under limits and sifted colimits in $Set^{\mathcal{T}}$ and such limits and colimits in $Set^{\mathcal{T}}$ are formed objectwise.

2.8 Example. In the category of abelian groups

1. a directed union of abelian groups carries a canonical structure of abelian group: the directed colimit of the diagram of inclusion homomorphisms,

2. let $a_1, a_2 \colon A \rightrightarrows B$ be a pair of homomorphisms with a common splitting $d \colon B \to A$ (i.e., $a_1 \cdot d = \mathrm{id}_B = a_2 \cdot d$) and let $c \colon B \to C$ be its coequalizer in *Set*. The set C carries a canonical structure of abelian group (the unique one for which c is a homomorphism). Reflexive coequalizers will be studied in detail in Chapter 3.

2.9 Remark. Generalizing 2.8.1, a *directed union* in $Alg\,\mathcal{T}$ is a directed colimit of subalgebras. That is, an algebra A is a directed union of subalgebras $m_i \colon A_i \to A$ $(i \in I)$ provided that the poset on I given by $i \le j$ iff $m_i \subseteq m_j$ is directed and A is the colimit (with colimit cocone m_i) of the directed diagram of all A_i, $i \in I$ and all connecting morphisms m_{ij}:

$$\begin{tikzcd} & A & \\ A_i \arrow[ur, "m_i"] \arrow[rr, "m_{ij}"'] & & A_j \arrow[ul, "m_j"'] \end{tikzcd}$$

Thus $Alg\,\mathcal{T}$ is closed under directed unions in $Set^{\mathcal{T}}$.

2.10 Definition. A category $\mathcal{A}$ is called *connected* if it is nonempty and for every pair of objects X and X' in $\mathcal{A}$ there exists a zig-zag of morphisms connecting X and X':

$$X \to X_1 \leftarrow X_2 \to \ldots \to X_n \leftarrow X' \,.$$

This is equivalent to saying that $\mathcal{A}$ cannot be decomposed as a coproduct (that is, disjoint union) of two nonempty subcategories.

2.11 Remark. A small category $\mathcal{A}$ is connected iff the constant functor $\mathcal{A} \to$ *Set* of value 1 has colimit 1.

2.12 Definition. A functor $F \colon \mathcal{D}' \to \mathcal{D}$ is called *final* if, for every diagram $D \colon \mathcal{D} \to \mathcal{A}$ such that $colim\,(D \cdot F)$ exists in $\mathcal{A}$, then $colim\,D$ exists and the canonical morphism $colim\,D \cdot F \to colim\,D$ is an isomorphism.

2.13 Lemma. *The following conditions on a functor $F \colon \mathcal{D}' \to \mathcal{D}$ are equivalent:*

1. *F is final;*

2. *F satisfies the finality condition with respect to all representable functors* $D = \mathcal{D}(d, -)$;

3. *For every object* d *of* $\mathcal{D}$, *the slice category* $d \downarrow F$ *of all arrows* $d \to Fd'$, $d' \in obj\,\mathcal{D}'$, *is connected.*

Proof. $1 \Rightarrow 2$ is trivial and $2 \Rightarrow 3$ follows from the usual description of colimits in *Set* (see 0.6) and the fact that since the diagram $\mathcal{D}(d, -)$ has colimit 1 (see 0.10), so does the diagram $\mathcal{D}(d, F-) = \mathcal{D}(d, -) \cdot F$.
To prove $3 \Rightarrow 1$, let $D\colon \mathcal{D} \to \mathcal{A}$ be a diagram and let $c_{d'}\colon D(Fd') \to C$ ($d' \in obj\,\mathcal{D}'$) be a colimit of $D \cdot F$. For every object d of $\mathcal{D}$, choose a morphism $u_d\colon d \to Fd'$ for some $d' \in obj\,\mathcal{D}'$, and put $g_d = c_{d'} \cdot Du_d\colon Dd \to C$. We claim that $g_d\colon Dd \to C$ ($d \in obj\,\mathcal{D}$) is a colimit of D. In fact, since $d \downarrow F$ is connected, it is easy to verify that g_d does not depend on the choice of d' and u_d, and that these morphisms form a cocone of D. The rest of the proof is straightforward. ☐

2.14 Theorem. *A small category* $\mathcal{D}$ *is sifted if and only if it is nonempty and the diagonal functor* $\Delta\colon \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ *is final.*

In order to simplify the proof, let us observe two preliminary facts:

a) Following 2.13 the finality of Δ means that for every pair of objects A, B of $\mathcal{D}$ the category $(A, B) \downarrow \Delta$ of cospans on A, B is connected. That is:

   (i) a cospan $A \to X \leftarrow B$ exists, and

   (ii) every pair of cospans on A, B is connected by a zig-zag of cospans.

   Therefore, the statement

   $\mathcal{D}$ is nonempty and the diagonal functor Δ is final

   is equivalent to the statement

   $\mathcal{D}$ is connected and the diagonal functor Δ is final.

b) The canonical map

$$\delta\colon \operatorname*{colim}_{\mathcal{D}} \left(\prod_{\mathcal{J}} D(d, j) \right) \to \prod_{\mathcal{J}} (\operatorname*{colim}_{\mathcal{D}} D(d, j))$$

   of 2.2 is an isomorphism for every finite discrete category $\mathcal{J}$ iff δ is an isomorphism when $\mathcal{J}$ is the empty set and when $\mathcal{J}$ is the two-element set.

We are going to prove that $\mathcal{D}$ is sifted iff it is connected and the diagonal functor Δ is final. More precisely, we are going to prove that

1. $\mathcal{D}$ is connected iff δ is an isomorphism when $\mathcal{J}$ is the empty set,

and

2. Δ is final iff δ is an isomorphism when $\mathcal{J}$ is the two-element set (that is, binary products in *Set* commute with colimits over $\mathcal{D}$).

Proof. 1. When $\mathcal{J}$ is the empty set the codomain of δ is 1 whereas its domain is 1 iff $\mathcal{D}$ is connected (see 2.11).
2. Let $\mathcal{J}$ be the two-element set. Given diagrams $D, D' \colon \mathcal{D} \to Set$, consider the functor

$$D \times D' \colon \mathcal{D} \times \mathcal{D} \to Set\,, \quad (d, d') \mapsto Dd \times D'd'\,.$$

Since, for every set X, the functor $X \times - \colon Set \to Set$ preserves colimits (see 0.11), the colimit of the diagram $D \times D'$ is

$$\operatorname*{colim}_{\mathcal{D}\times\mathcal{D}'} (Dd{\times}D'd') \simeq \operatorname*{colim}_{d\in\mathcal{D}} \big(\operatorname*{colim}_{d'\in\mathcal{D}} (Dd{\times}D'd')\big) \simeq \operatorname*{colim}_{d\in\mathcal{D}} Dd \times \operatorname*{colim}_{d'\in\mathcal{D}} D'd'\,.$$

Consider now the following commutative diagram of canonical maps

$$\begin{tikzcd}
\operatorname*{colim}_{\mathcal{D}} (Dd \times D'd) \arrow[r, "\delta"] \arrow[d, "="'] & (\operatorname*{colim}_{\mathcal{D}} Dd) \times (\operatorname*{colim}_{\mathcal{D}} D'd) \\
\operatorname*{colim}_{\mathcal{D}} ((D \times D') \cdot \Delta) \arrow[r, "\beta"] & \operatorname*{colim}_{\mathcal{D}\times\mathcal{D}} (D \times D') \arrow[u, "\simeq"']
\end{tikzcd}$$

If Δ is final, then β is an isomorphism and therefore δ is also an isomorphism.

Conversely, assume that δ is an isomorphism. Given two objects d and d' in $\mathcal{D}$, the representable functor $(\mathcal{D} \times \mathcal{D})((d, d'), -)$ is nothing but $D \times D' \colon \mathcal{D} \times \mathcal{D} \to Set$, with $D = \mathcal{D}(d, -)$ and $D' = \mathcal{D}(d', -)$. If δ is an isomorphism, the previous diagram shows that Δ satisfies the finality condition with respect to all representable functors. Following 2.13, Δ is final. $\square$

2.15 Example. Every small category with finite coproducts is sifted. In fact, it contains an initial object, and the slice category $(A, B) \downarrow \Delta$ is connected because it has an initial object (the coproduct of A and B).

2.16 Example. Consider the category $\mathcal{D}$ given by the morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & B \end{tikzcd}$$

(identity morphisms are not depicted). $\mathcal{D}$ is not sifted. In fact, the slice category $(A, B) \downarrow \Delta$ is the discrete category with objects (f, id_B) and (g, id_B).

2.17 Remark. Filtered colimits are closely related to sifted ones. In fact, our definition in 2.1 stresses this fact. The more usual definition of filtered category $\mathcal{D}$ is to say that every finitely generated subcategory of $\mathcal{D}$ has a cocone in $\mathcal{D}$. (This includes the condition that $\mathcal{D}$ is nonempty.) And a well-known result states that this implies the property of Definition 2.1.2. The converse is also true:

2.18 Theorem. *For a small category $\mathcal{D}$ the following conditions are equivalent:*

1. *$\mathcal{D}$ is filtered,*
2. *every finitely generated subcategory of $\mathcal{D}$ has a cocone,*

and

3. *$\mathcal{D}$ is nonempty and fulfils*
   (a) *for every pair of object A, B there exists a cospan $A \to X \leftarrow B$, and*
   (b) *for every parallel pair of morphisms $u, v\colon A \rightrightarrows B$ there exists a morphism $f\colon B \to C$ merging u and v: $f \cdot u = f \cdot v$.*

Proof. The proof of the implications $3 \Leftrightarrow 2 \Rightarrow 1$ is standard, the reader can find it e.g. in [27], Vol. 1, Theorem 2.13.4. The proof of the implication $1 \Rightarrow 3$ is easy: for (a) argue as in 2.14, for (b) use, analogously, the equalizer of $\mathcal{D}(u, -), \mathcal{D}(v, -)\colon \mathcal{D}(B, -) \rightrightarrows \mathcal{D}(A, -)$ which is the diagram D of all morphisms merging u and v: since $\mathit{colim}\, D = 1$ the diagram is nonempty. □

With the next proposition we establish a further analogy to 2.14:

2.19 Proposition. *A small category $\mathcal{D}$ is filtered if and only if for any finitely generated category $\mathcal{J}$ the diagonal functor $\Delta\colon \mathcal{D} \to \mathcal{D}^{\mathcal{J}}$ is final.*

Proof. 1. Let all such functors Δ be final. We are to show that every finitely generated subcategory $\mathcal{J}$ of $\mathcal{D}$ has a cocone in $\mathcal{D}$. The inclusion functor $d\colon \mathcal{J} \to \mathcal{D}$ is an object of the functor category $\mathcal{D}^{\mathcal{J}}$. Following 2.13.3, the slice category $d \downarrow \Delta$ is connected, thus, nonempty. Since $d \downarrow \Delta$ is precisely the category of cocones of $\mathcal{J}$ in $\mathcal{D}$, we obtain the desired cocone.
2. Conversely, let $\mathcal{D}$ be filtered. We are to verify that for every object d of $\mathcal{D}^{\mathcal{J}}$, where $\mathcal{J}$ is finitely generated, the slice category $d \downarrow \Delta$ is connected.
2a. If $\mathcal{J}$ is a subcategory of $\mathcal{D}$ and $d\colon \mathcal{J} \to \mathcal{D}$ is the inclusion functor, then the category $d \downarrow \Delta$ of all cocones is nonempty. To prove that it is in fact connected, consider two cocones C_1 and C_2. Since $\mathcal{J}$ is finitely generated (say, by a finite set M of morphisms), it has finitely many objects, thus, C_1 and C_2 are finite sets of morphisms. Put $\overline{M} = M \cup C_1 \cup C_2$ and let $\overline{\mathcal{J}}$ be the subcategory of $\mathcal{D}$ generated by $\overline{M}$. Then $\overline{\mathcal{J}}$ has a cocone in $\mathcal{D}$, and this cocone defines an obvious cocone of $\mathcal{J}$ with cocone morphisms to C_1 and C_2. Thus, we obtain a zig-zag of length 2.
2b. Let $d\colon \mathcal{J} \to \mathcal{D}$ be arbitrary and let M be a finite set of morphisms generating $\mathcal{J}$. Then the set

$$d(M) \cup \{\mathrm{id}_{d(x)}\ ;\ x \in \mathit{obj}\,\mathcal{J}\}$$

is finite and generates a subcategory $\mathcal{J}_0$ of $\mathcal{D}$. The slice category $d \downarrow \Delta$ is clearly equivalent to the category of cocones of $\mathcal{J}_0$ in $\mathcal{D}$ which is connected by 2a. □

2.20 Remark. Every colimit can be expressed as a filtered colimit of finite colimits. That is, given a diagram $D\colon \mathcal{D} \to \mathcal{A}$ with $\mathcal{A}$ cocomplete, then $\mathit{colim}\, D$ can be constructed as the filtered colimit of the diagram of all $\mathit{colim}\, D'$ where $D'\colon \mathcal{D}' \to \mathcal{A}$ ranges over all domain restrictions of D to finitely generated subcategories $\mathcal{D}'$ of $\mathcal{D}$.

2.21 Remark. In Chapter 7 we study functors preserving filtered and sifted colimits. In case of endofunctors of *Set* these two properties coincide (see 6.30) but in general the latter one is stronger, see 2.25. We use the following terminology:

2.22 Definition. A functor is called *finitary* if it preserves filtered colimits.

2.23 Example. Here we mention some endofunctors of *Set* that are finitary.

1. The functor
$$H_n \colon Set \to Set \quad H_n X = X^n$$
is finitary for every natural number n since finite products commute in *Set* with filtered colimits.

2. A coproduct of finitary functors is finitary.

3. Let Σ be a signature (1.14). We define the corresponding *polynomial functor*
$$H_\Sigma \colon Set \to Set$$
as the coproduct of the functors $H_{\mathrm{ar}(\sigma)}$ for $\sigma \in \Sigma$. Explicitly,
$$H_\Sigma X = \coprod_{n \in \mathbb{N}} \Sigma_n \times X^n$$
where Σ_n is the set of all symbols of arity n ($n = 0, 1, 2, \ldots$). This is a finitary functor.

2.24 Example. Let $H \colon Set \to Set$ be a functor. An *H-algebra* is a pair (A, a) where A is a set and $a \colon HA \to A$ a function. A *homomorphism* from (A, a) to (B, b) is a function $f \colon A \to B$ such that the square

$$\begin{tikzcd} HA \arrow[r, "Hf"] \arrow[d, "a"'] & HB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. The resulting category is denoted *H-Alg*.

1. In 13.23 we will see that if H is finitary then the category *H-Alg* is algebraic.

2. The special case of a polynomial endofunctor H_Σ leads to Σ-algebras. Indeed, for every one-sorted signature Σ the category Σ-*Alg* is precisely the category H_Σ-*Alg* : if (A, a) is a H_Σ-algebra then the operations $\sigma^A \colon A^n \to A$ are the domain restrictions of a to the summand A^n corresponding to $\sigma \in \Sigma_n$. This case will be treated in Chapter 13.

2.25 Example. An example of a finitary functor not preserving sifted colimits is the forgetful functor $U\colon Pos \to Set$, where Pos is the category of posets and order-preserving maps. Consider the refexive pair $u, v\colon 1 + 2 \rightrightarrows 2$ from the coproduct of the terminal poset 1 and the two-element chain, where both morphisms are identity on the second summand, and they map the first one to the top and bottom of 2, respectively. Whereas the coequalizer in Pos is given by the terminal poset, the coequalizer of Uu and Uv in Set has two elements.

Historical Remarks for Chapter 2

Filtered colimits are a natural generalization of directed colimits known from algebra and topology since the beginning of the twentieth century. The general concept can already be found in Bourbaki [29] including the fact that directed colimits commute with finite products in Set. Both [48] and [13] contain the general definition of filtered colimits and the fact that they are precisely those colimits which commute with finite limits in Set. P. Gabriel and F. Ulmer even speculated about general commutation of colimits with limits in Set (see Chapter 15 in [48]) and characterized colimits commuting with finite products in Set; this is the source of 2.14. This was later re-discovered by C. Lair [61] who called these colimits “tamisantes”. The term sifted was suggested by P. T. Johnstone.

The concept of a final functor and the characterization 2.13 is a standard result of category theory which can be found in [71].

The fact that sifted colimits play an analogous role for algebraic categories that filtered colimits play for the locally finitely presentable ones was presented in [8].

Chapter 3

Reflexive coequalizers

An important case of sifted colimits are reflexive coequalizers:

3.1 Definition. *Reflexive coequalizers* are coequalizers of reflexive pairs, that is, parallel pairs of split epimorphisms having a common splitting.

3.2 Example. In other words, reflexive coequalizers are colimits of diagrams over the category $\mathcal{M}$ given by the morphisms

$$\begin{tikzcd} A \arrow[r, shift left=2, "a_1"] \arrow[r, shift right=2, "a_2"'] & B \arrow[l, "d" description] \end{tikzcd}$$

(identity morphisms are not depicted) composed freely modulo $a_1 \cdot d = \mathrm{id}_B = a_2 \cdot d$. This category is sifted: it is an easy exercise to check that the categories $(A, A) \downarrow \Delta$, $(A, B) \downarrow \Delta$ and $(B, B) \downarrow \Delta$ are connected.

Another method of verifying that $\mathcal{M}$ is a sifted category is to prove directly that reflexive coequalizers commute in *Set* with binary products. In fact, suppose that

$$\begin{tikzcd} A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & B \arrow[r, "c"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A' \arrow[r, shift left, "a'_1"] \arrow[r, shift right, "a'_2"'] & B' \arrow[r, "c'"] & C' \end{tikzcd}$$

are reflexive coequalizers in *Set*. We can assume, without loss of generality, that c is the canonical function of the quotient $C = B/\sim$ modulo the equivalence relation described as follows: two elements $x, y \in B$ are equivalent iff there exists a zig-zag

$$\begin{tikzcd}[column sep=small] A\colon & & z_1 \arrow[dl, "a_{i_1}"'] \arrow[dr, "a_{i_2}"] & & z_2 \arrow[dl, "a_{i_3}"'] \arrow[dr, "a_{i_4}"] & & \cdots\cdots & & z_k \arrow[dl, "a_{i_{2k-1}}"'] \arrow[dr, "a_{i_{2k}}"] & \\ B\colon & x & & {} & & {} & \cdots & {} & & y \end{tikzcd}$$

where $i_1, i_2, \ldots, i_{2k}$ are 1 or 2. For reflexive pairs a_1, a_2 the zig-zags can always be chosen to have the form

$$\begin{array}{lccccccccccc}
A\colon & & z_1 & & & z_2 & & \ldots\ldots & & z_{2k} & & \\
& \swarrow^{a_1} & & \searrow^{a_2} & \swarrow^{a_2} & & \searrow^{a_1} & & \swarrow^{a_2} & & \searrow^{a_1} & \\
B\colon & x & & & & & & \ldots & & & & y
\end{array} \qquad [3.1]$$

(here for the elements z_{2i} of A we use a_1, a_2 and for the elements z_{2i+1} we use a_2, a_1). In fact, let $d\colon B \to A$ be a joint splitting of a_1, a_2.) Given a zig-zag, say,

$$\begin{tikzcd} & z \arrow[dl, "a_2"'] \arrow[dr, "a_1"] & \\ x & & y \end{tikzcd}$$

we can modify it as follows:

$$\begin{tikzcd} & dx \arrow[dl, "a_1"'] \arrow[dr, "a_2"] & & z \arrow[dl, "a_2"'] \arrow[dr, "a_1"] & \\ x & & x & & y \end{tikzcd}$$

Analogously for the general case. Moreover, the length $2k$ of the zig-zag [3.1] can be prolonged to $2k+2$ or $2k+4$ etc. by using d. Analogously, we can assume $C' = B'/\sim'$ where $\sim'$ is the equivalence relation given by zig-zags of a_1' and a_2' of the above form [3.1]. Now we form the parallel pair

$$A \times A' \overset{a_1 \times a_1'}{\underset{a_2 \times a_2'}{\rightrightarrows}} B \times B'$$

and obtain its coequalizer by the zig-zag equivalence $\approx$ on $B \times B'$. Given $(x, x') \approx (y, y')$ in $B \times B'$, we obviously have zig-zags both for $x \sim y$ and for $x' \sim' y'$ (use projections of the given zig-zag). But also the other way round: whenever $x \sim y$ and $x' \sim' y'$, then we choose the two zig-zags so that they both have the above type [3.1] and have the same lengths. They create an obvious zig-zag for $(x, x') \approx (y, y')$. From this it follows that the map

$$A \times A' \overset{a_1 \times a_1'}{\underset{a_2 \times a_2'}{\rightrightarrows}} B \times B' \xrightarrow{\ c \times c'\ } (B/\sim) \times (B'/\sim')$$

is a coequalizer, as required.

3.3 Corollary. *For every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under reflexive coequalizers.*

In fact, this follows from 2.5 and 3.2.

3.4 Example. In a category with kernel pairs every regular epimorphism is a reflexive coequalizer. In fact, if r_1, r_2 is a kernel pair of a regular epimorphism $e\colon A \to B$

$$\begin{tikzcd} & A \arrow[d, "d"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\ & R \arrow[dl, "r_1"] \arrow[dr, "r_2"'] & \\ A \arrow[dr, "e"'] & & A \arrow[dl, "e"] \\ & B & \end{tikzcd}$$

then e is a coequalizer of r_1, r_2. And since $e \cdot \mathrm{id} = e \cdot \mathrm{id}$, there exists a unique d with $r_1 \cdot d = \mathrm{id} = r_2 \cdot d$.

3.5 Corollary. *For every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under regular epimorphisms. Therefore regular epimorphisms in $Alg\,\mathcal{T}$ are precisely the homomorphisms which are componentwise epimorphisms (i.e., surjective functions) in Set.*

In fact, the first part of the statement follows from 2.5 and 3.4. The second one is clear since the statement is true in $Set^{\mathcal{T}}$.

3.6 Remark. In particular, every algebraic category is *co-wellpowered* with respect to regular epimorphisms. This means that, for a fixed object A, the regular quotients of A constitute a set (not a proper class). In fact, this is true in *Set* and therefore, by 3.5, in every algebraic category.

More is true: algebraic categories are co-wellpowered with respect to all epimorphisms. We do not present a proof of this fact here because we do not need it. The interested reader can consult [7], 1.52 and 1.58.

3.7 Corollary. *Every algebraic category has* regular factorizations, *i.e., every morphism is a composite of a regular epimorphism followed by a monomorphism.*

Proof. The category $Set^{\mathcal{T}}$ has regular factorizations: given a morphism $f\colon A \to B$, form a kernel pair $r_1, r_2\colon R \rightrightarrows A$ and its coequalizer $e\colon A \to C$. The factorizing morphism m

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "f"] \arrow[d, "e"'] & B \\ & C \arrow[ur, "m"'] & \end{tikzcd}$$

is a monomorphism. This follows from the fact that kernel pairs and coequalizers are formed objectwise (in *Set*). Since $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under kernel pairs (1.18) and their coequalizers (3.3), it inherits the regular factorizations from $Set^{\mathcal{T}}$. □

3.8 Example. In *Ab* we know that:

1. Coproducts are not formed at the level of sets. In fact, $A + B = A \times B$ for all abelian groups A, B.

2. Reflexive coequalizers are formed at the level of sets, but general coequalizers are not. Consider e.g. the pair $x \mapsto 2x$ and $x \mapsto 0$ of endomorphisms of $\mathbb{Z}$ whose coequalizer in *Ab* is finite and in *Set* it is infinite.

3.9 Remark. We provided a simple characterization of monomorphisms (1.18) and regular epimorphisms (3.5) in algebraic categories. There does not seem to be a simple characterization of the dual concepts (epimorphisms and regular monomorphisms). In fact, there exist algebraic categories with non-surjective epimorphisms and with non-regular monomorphisms, as we show in the following example.

3.10 Example. Monoids. These are algebras with one associative binary operation and one constant which is a neutral element. The category *Mon* of monoids and homomorphisms is algebraic, see 13.14.

An example of an epimorphism which is not regular is the embedding

$$i \colon \mathbb{Z} \to \mathbb{Q}$$

of the multiplicative monoid of integers into that of rational numbers. In fact, consider monoid homomorphisms $h, k \colon \mathbb{Q} \to A$ such that $h \cdot i = k \cdot i$. That is, $h(n) = k(n)$ for every integer n. To prove $h = k$, it is sufficient to verify $h(1/m) = k(1/m)$ for all integers $m \neq 0$: this follows from $h(m) \cdot h(1/m) = k(m) \cdot k(1/m) = 1$ (since $h(1) = k(1) = 1$). Consequently, i is not a regular epimorphism. Observe that i is also a monomorphism, but not a regular one.

3.11 Remark. Recall that in a finitely complete category $\mathcal{A}$ *relations* on an object A are the subobjects of $A \times A$. A relation can be represented by a monomorphism $r \colon R \to A \times A$ or by a parallel pair $r_1, r_2 \colon R \rightrightarrows A$ of morphisms that are jointly monic. The following definitions generalize the corresponding concepts for relations in *Set*:

3.12 Definition. A relation $r_1, r_2 \colon R \rightrightarrows A$ in a category $\mathcal{A}$ is called

1. *reflexive* if the pair r_1, r_2 is reflexive ($r_1 \cdot d = \mathrm{id} = r_2 \cdot d$ for some $d \colon A \to R$),

2. *symmetric* if there exists $s \colon R \to R$ with $r_1 = r_2 \cdot s$ and $r_2 = r_1 \cdot s$

and

3. *transitive* provided that for a pullback $\overline{R}$ of r_1 and r_2 there exists a mor-

phism $t\colon \overline{R} \to R$ such that the diagram

$$\begin{tikzcd}
 & & R \arrow[dddll, bend right, "r_1"'] \arrow[dddrr, bend left, "r_2"] & & \\
 & & \overline{R} \arrow[u, "t"'] \arrow[dl, "\overline{r}_1"'] \arrow[dr, "\overline{r}_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
A & & A & & A
\end{tikzcd}$$

commutes.

An *equivalence relation* is a relation which is reflexive, symmetric and transitive.

3.13 Remark.

1. If $r_1, r_2\colon R \rightrightarrows A$ is an equivalence relation, then the morphisms d, s and t of 3.12 are necessarily unique.

2. An equivalence relation in *Set* is precisely an equivalence relation in the usual sense.

3. Given a relation $r_1, r_2\colon R \rightrightarrows A$ and an object X, we can define a relation $\sim_R$ on the hom-set $\mathcal{A}(X, A)$ as follows: $f \sim_R g$ if there exists a morphism $h\colon X \to R$ such that $r_1 \cdot h = f$ and $r_2 \cdot h = g$. It is easy to check that $r_1, r_2\colon R \rightrightarrows A$ is an equivalence relation in $\mathcal{A}$ iff $\sim_R$ is an equivalence relation in *Set* for all X in $\mathcal{A}$.

4. Kernel pairs are equivalence relations.

5. In the category of Σ-algebras, see 1.14, an equivalence relation on an algebra A is precisely a subobject of $A \times A$ which, as a relation on the underlying set of A, is an equivalence relation in *Set*. These relations are usually called congruences on A. We refer to them as equivalence relations in Σ-*Alg* because the concept "congruence" is reserved (with the only exceptions of 11.31 – 11.33) for congruences of algebraic theories, see Chapter 10.

3.14 Definition. A category is said to have *effective equivalence relations* provided that every equivalence relation is a kernel pair.

3.15 Example. The category of posets does not have this property: take an arbitrary poset B and an equivalence relation R on the underlying set of B equipped with the discrete ordering, then the two projections $R \rightrightarrows B$ form an equivalence relation which is seldom a kernel pair.

3.16 Definition. A category is called *exact* if it has

1. finite limits
2. coequalizers of kernel pairs

and

3. effective equivalence relations,

and if

4. regular epimorphisms are stable under pullback. That is, in every pullback

$$\begin{tikzcd} A \arrow[r, "e'"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

if e is a regular epimorphism, then so is e'.

3.17 Example. *Set* is an exact category. In fact:
1. If $r_1, r_2 \colon R \rightrightarrows A$ is an equivalence relation, its coequalizer is

$$q \colon A \to A/\sim_R$$

where $\sim_R$ is as in 3.13 (with $X = 1$) and q is the canonical morphism. Clearly, $r_1, r_2 \colon R \rightrightarrows A$ is a kernal pair of q.
2. Given a pullback as in 3.16 and an element $x \in B$, we choose $z \in C$ with $g(x) = e(z)$ using that e is an epimorphism. Then (x, z) is an element of the pullback A and $e'(x, z) = x$. This proves that e' is surjective.

3.18 Corollary. *Every algebraic category is exact.*

In fact, since *Set* is exact, so is Set^S. Since equivalence relations are reflexive pairs, the exactness of $Alg\,\mathcal{T}$ follows using 1.18 and 3.3.

3.19 Definition. We say that colimits in a category $\mathcal{A}$ *distribute over products* if given diagrams $D_i \colon \mathcal{D}_i \to \mathcal{A}$ $(i \in I)$ and forming the diagram

$$D \colon \prod_{i \in I} \mathcal{D}_i \to \mathcal{A}\,, \quad Dd_i = \prod_{i \in I} D_i d_i$$

then the canonical morphism

$$colim\, D \to \prod_{i \in I} colim\, D_i$$

is an isomorphism.

If all $\mathcal{D}_i$ are of a certain type we say that colimits of that type distribute over products.

The concept of distributing over finite products is defined as above but I is required to be finite.

3.20 Example. In the category *Set* it is easy to verify that

1. filtered colimits distribute over products

and

2. all colimits distribute over finite products.

However reflexive coequalizers do not distribute over infinite products. In fact, consider the coequalizers

$$\begin{tikzcd} n+n \arrow[r, shift left, "f_n"] \arrow[r, shift right, "g_n"'] & n+1 \arrow[r, "c_n"] & 1 \end{tikzcd}$$

where the left-hand components of f_n and g_n are the inclusion maps $n \to n+1$, and the right-hand ones are $i \mapsto i$ and $i \mapsto i+1$, respectively. Then $\prod_{n\in\mathbb{N}} f_n, \prod_{n\in\mathbb{N}} g_n$ have a coequalizer with infinite codomain, thus, distinct from $\prod_{n\in\mathbb{N}} c_n$.

3.21 Corollary. *In every algebraic category:*

1. *Regular epimorphisms are stable under products: given regular epimorphisms* $e_i\colon A_i \to B_i$ $(i \in I)$, *then*
$$\prod_{i\in I} e_i\colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$
is a regular epimorphism.

2. *Filtered colimits distribute over products.*

3. *Sifted colimits distribute over finite products.*

In fact, since each of the three statements holds in *Set*, it holds in $Set^{\mathcal{T}}$, where limits and colimits are formed objectwise. Following 1.16, 2.5 and 3.5, the statements hold in $Alg\,\mathcal{T}$ for every algebraic theory $\mathcal{T}$.

3.22 Remark.

1. In the above corollary we implicitly assume the existence of colimits in an algebraic category. This is not a restriction, because every algebraic category is cocomplete, as we prove in the next chapter.

2. Although in *Set* all colimits distribute over finite products, this is not true in algebraic categories in general: consider the empty diagram in the category of abelian groups. For $I = \{1,2\}$ in 3.19 and $\mathcal{D}_1 = \emptyset = \mathcal{D}_2$ we get $colim\, D = \mathbb{Z}$ and $colim\, D_1 \times colim\, D_2 = \mathbb{Z} \times \mathbb{Z}$.

Historical Remarks for Chapter 3

Reflexive coequalizers were probably first applied by F. E. J. Linton [67]. The fact that they commute with finite products in *Set* (in fact, in every cartesian closed category) is contained in the unpublished thesis of P. T. Johnstone written in the early 1970s. The importance of reflexive coequalizers for algebraic categories was first understood by Y. Diers [39] but his paper remained unnoticed. It was later rediscovered by M. C. Pedicchio and R. Wood [81]. A decisive step was taken in [5] and [6] where reflexive coequalizers were used for establishing the algebraic duality (see Chapter 9) and for the study of abstract "operations" present in algebraic categories except limits and filtered colimits.

Effective equivalence relations were introduced by M. Artin, A. Grothendieck and J. L. Verdier in [13] and exact categories by M. Barr in [16]. The fact that filtered colimits distribute with all products in *Set* goes back to [13].

Chapter 4

Algebraic categories as free completions

In this chapter we prove that every algebraic category has colimits. Moreover, the category $Alg\,\mathcal{T}$ is a free completion of $\mathcal{T}^{op}$ under sifted colimits.

4.1 Remark. For the existence of colimits, since we already know that $Alg\,\mathcal{T}$ has sifted colimits and, in particular, reflexive coequalizers (see 2.5 and 3.3), all we need to establish is the existence of finite coproducts. Indeed, coproducts then exist because they are filtered colimits of finite coproducts. And coproducts and reflexive coequalizers construct all colimits, see 0.7. The first step towards the existence of finite coproducts has already been done in Lemma 1.5: finite coproducts of representable algebras, including an initial object, exist in $Alg\,\mathcal{T}$.

In the next lemma we use the category of elements $El\,A$ of a functor $A\colon \mathcal{T} \to Set$ introduced in 0.14.

4.2 Lemma. *Given an algebraic theory $\mathcal{T}$, for every functor A in $Set^{\mathcal{T}}$ the following conditions are equivalent:*

1. *A is an algebra,*

2. *$El\,A$ is a sifted category*

and

3. *A is a sifted colimit of representable algebras.*

Proof. $2 \Rightarrow 3$: This follows from 0.14.
$3 \Rightarrow 1$: Representable functors are objects of $Alg\,\mathcal{T}$ (1.4), and $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under sifted colimits (2.5).
$1 \Rightarrow 2$: Following 2.15, it suffices to prove that $(El\,A)^{op}$ has finite products. This is obvious: for example, the product of (X, x) and (Z, z) is $(X \times Z, (x, z))$: recall that $(x, z) \in AX \times AZ = A(X \times Z)$. $\square$

4.3 Remark. An analogous result (with a completely analogous proof) holds for small categories $\mathcal{T}$ with finite limits: a functor $A\colon \mathcal{T} \to \mathit{Set}$ preserves finite limits iff $\mathit{El}\,A$ is a filtered category iff A is a filtered colimit of representable functors.

4.4 Lemma.

1. *If two functors $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{A}$ are final, then the product functor $F \times G\colon \mathcal{D} \times \mathcal{B} \to \mathcal{C} \times \mathcal{A}$ is final.*

2. *A product of two sifted categories is sifted.*

Proof. 1. This follows from 2.13.3 because, for any object (d, b) in $\mathcal{D} \times \mathcal{B}$, the slice category $(d, b) \downarrow F \times G$ is nothing but the product category $(d \downarrow F) \times (b \downarrow G)$, and the product of two connected categories is connected.
2. Obvious from 1 and 2.14. ☐

4.5 Theorem. *Every algebraic category is cocomplete.*

Proof. As explained at the beginning of this chapter, we only need to establish finite coproducts $A + B$ in $\mathit{Alg}\,\mathcal{T}$. Express A as a sifted colimit of representable algebras (4.2)

$$A = \mathit{colim}\,(Y_{\mathcal{T}} \cdot \Phi_A)$$

and analogously for B. The category

$$\mathcal{D} = \mathit{El}\,A \times \mathit{El}\,B$$

is sifted by 4.4 and for the projections P_1, P_2 of $\mathcal{D}$ we have two colimits in $\mathit{Alg}\,\mathcal{T}$ over $\mathcal{D}$:

$$A = \mathit{colim}\,Y_{\mathcal{T}} \cdot \Phi_A \cdot P_1 \quad \text{and} \quad B = \mathit{colim}\,Y_{\mathcal{T}} \cdot \Phi_B \cdot P_2\ .$$

The diagram $D\colon \mathcal{D} \to \mathit{Alg}\,\mathcal{T}$ assigning to every pair (X, x) and (Z, z) a coproduct of the representable algebras (see 1.5)

$$D((X, x), (Z, z)) = Y_{\mathcal{T}} \cdot \Phi_A(x) + Y_{\mathcal{T}} \cdot \Phi_B(z) \qquad (\text{in } \mathit{Alg}\,\mathcal{T})$$

is sifted, thus it has a colimit in $\mathit{Alg}\,\mathcal{T}$. Since colimits over $\mathcal{D}$ commute with finite coproducts, we get

$$\mathit{colim}\,D = \mathop{\mathit{colim}}_{(x,z)} Y_{\mathcal{T}} \cdot \Phi_A(x) + \mathop{\mathit{colim}}_{(x,z)} Y_{\mathcal{T}} \cdot \Phi_B(z) = A + B.$$

☐

4.6 Example. Coproducts.

1. In the category Ab of abelian groups finite coproducts are finite products: the abelian group $A \times B$ together with the homomorphisms

$$(-, 0)\colon A \to A \times B \ \text{ and } \ (0, -)\colon B \to A \times B$$

is a coproduct of A and B.

2. Infinite coproducts $\coprod_{i \in I} A_i$ are directed colimits of finite subcoproducts $\coprod_{j \in J} A_j = \prod_{j \in J} A_j$ (for $J \subseteq I$ finite).

3. In the category of sequential automata (1.20) the product $A \times B$ of two automata is the machine working simultaneously in A and B on the given (joint) input streams. Whereas the coproduct $A + B$ is the machine working, on a given input streams, completely in A or completely in B.

4. A coproduct of graphs in *Graph* is given by the disjoint union of vertices and the disjoint union of edges.

4.7 Example. Coequalizers.

1. In Ab a coequalizer of homomorphisms $f, g\colon A \rightrightarrows B$ is the quotient $c\colon B \to B/B_0$ modulo the subgroup $B_0 \subseteq B$ of the elements $f(a) - g(a)$ for all $a \in A$.

2. A coequalizer of a parallel pair $f, g\colon A \rightrightarrows B$ in *Graph* is given by forming the coequalizers in *Set* of (i) the two vertex functions and (ii) the two edge functions.

4.8 Remark. Before characterizing algebraic categories as free completions under sifted colimits, let us recall the general concept of a free completion of a category $\mathcal{C}$: this is, roughly speaking, a cocomplete category $\mathcal{A}$ in which $\mathcal{C}$ is a full subcategory such that every functor from $\mathcal{C}$ to a cocomplete category has an *essentially unique* extension (that is, unique up to natural isomorphism) to a colimit-preserving functor with domain $\mathcal{A}$. In the following definition we say this more precisely. Also, for a given class $\mathbb{D}$ of small categories we define a free completion under $\mathbb{D}$-colimits, meaning that all colimits considered are colimits of diagrams with domains that are elements of $\mathbb{D}$.

4.9 Definition. Let $\mathbb{D}$ be a class of small categories. By a *free completion of a category $\mathcal{C}$ under $\mathbb{D}$-colimits* is meant a functor $E_{\mathbb{D}}\colon \mathcal{C} \to \mathbb{D}(\mathcal{C})$ such that

1. $\mathbb{D}(\mathcal{C})$ is a category with $\mathbb{D}$-colimits

and

2. for every functor $F\colon \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is a category with $\mathbb{D}$-colimits, there exists an essentially unique functor $F^*\colon \mathbb{D}(\mathcal{C}) \to \mathcal{B}$ preserving $\mathbb{D}$-colimits with F naturally isomorphic to $F^* \cdot E_{\mathbb{D}}$

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "E_{\mathbb{D}}"] \arrow[dr, "F"'] & & \mathbb{D}(\mathcal{C}) \arrow[dl, "F^*"] \\
& \mathcal{B} &
\end{tikzcd}$$

If $\mathbb{D}$ consists of all small categories, then $E_{\mathbb{D}}\colon \mathcal{C} \to \mathbb{D}(\mathcal{C})$ is called a *free completion of $\mathcal{C}$ under colimits.*

4.10 Theorem. *For every small category $\mathcal{C}$, the Yoneda embedding*

$$Y_{\mathcal{C}^{op}} \colon \mathcal{C} \to \mathit{Set}^{\mathcal{C}^{op}}$$

is a free completion of $\mathcal{C}$ under colimits.

Proof. The category $\mathit{Set}^{\mathcal{C}^{op}}$ is of course cocomplete. Let $F \colon \mathcal{C} \to \mathcal{B}$ be a functor, where $\mathcal{B}$ has colimits. Since $F^* \colon \mathit{Set}^{\mathcal{C}^{op}} \to \mathcal{B}$ should extend F and preserve colimits, we are forced to define it on objects $A \colon \mathcal{C}^{op} \to \mathit{Set}$ (using the notation of 0.14 applied to $\mathcal{T} = \mathcal{C}^{op}$) by

$$F^*A = \operatorname*{\mathit{colim}}_{\mathit{El}\,A} (F \cdot \Phi_A)\,.$$

The definition on morphisms (that is, natural transformations) $h \colon A_1 \to A_2$ is also obvious: h induces a functor $\mathit{El}\,h \colon \mathit{El}\,A_1 \to \mathit{El}\,A_2$ which to every element (X, x) of A_1 assigns the corresponding element $(X, h_X(x))$ of A_2. By the universal property of colimits, $\mathit{El}\,h$ induces a morphism

$$h' \colon \mathit{colim}\,(F \cdot \Phi_{A_1}) \to \mathit{colim}\,(F \cdot \Phi_{A_2})$$

and we are forced to define $F^*h = h'$.

The above rule $A \mapsto \mathit{colim}\,(F \cdot \Phi_A)$ defines a functor $F^* \colon \mathit{Set}^{\mathcal{C}^{op}} \to \mathcal{B}$ which fulfils $F^* \cdot Y_{\mathcal{C}^{op}} \simeq F$ because, for $A = Y_{\mathcal{C}^{op}}(X) = \mathcal{C}(-, X)$, a colimit of $F \cdot \Phi_A = \mathcal{B}(-, FX)$ is FX. It remains to prove that F^* preserves colimits: for this we prove that F^* has the following right adjoint

$$R \colon \mathcal{B} \to \mathit{Set}^{\mathcal{C}^{op}}, \quad RB = \mathcal{B}(F-, B)\,.$$

We prove the adjunction $F^* \dashv R$ by verifying that there is a bijection

$$\mathcal{B}(F^*A, B) \simeq \mathit{Set}^{\mathcal{C}^{op}}(A, RB)$$

natural in $A \colon \mathcal{C}^{op} \to \mathit{Set}$ and $B \in \mathcal{B}$. In fact, the definition of F^* makes it clear that the left-hand side consists of precisely all cocones of the diagram $F \cdot \Phi_A$ with codomain B in $\mathcal{B}$:

$$\mathcal{B}(F^*A, B) = \mathcal{B}(\mathit{colim}\,(F \cdot \Phi_A), B) \simeq \mathit{lim}\,\mathcal{B}(F(\Phi_A(X, x)), B)$$

The same is true about the right-hand side: recall that $A = \mathit{colim}\,(Y_{\mathcal{C}^{op}} \cdot \Phi_A)$ (0.14), thus a natural transformation from A to RB is a cocone of the diagram $Y_{\mathcal{C}^{op}} \cdot \Phi_A$ with codomain RB:

$$\mathit{Set}^{\mathcal{C}^{op}}(A, RB) = \mathit{Set}^{\mathcal{C}^{op}}(\mathit{colim}\,(Y_{\mathcal{C}^{op}} \cdot \Phi_A), RB) \simeq \mathit{lim}\,\mathit{Set}^{\mathcal{C}^{op}}(Y_{\mathcal{C}^{op}}(\Phi_A(X, x)), RB)\,.$$

Yoneda Lemma tells us that morphisms from the objects $Y_{\mathcal{C}^{op}}(X)$ of that diagram to $RB = \mathcal{B}(F-, B)$ are precisely the members of the set $\mathcal{B}(FX, B)$:

$$\mathit{Set}^{\mathcal{C}^{op}}(Y_{\mathcal{C}^{op}}(X), RB) \simeq \mathcal{B}(FX, B)\,.$$

In this sense, morphisms from A to RB in $\mathit{Set}^{\mathcal{C}^{op}}$ encode precisely the cocones of $F \cdot \Phi_A$ with codomain B. $\square$

4.11 Remark.

1. Although the triangle in Definition 4.9 commutes up to natural isomorphism only, in 4.10 it is actually always possible to choose F^* so that the (strict) equality
$$F = F^* \cdot Y_{\mathcal{C}}$$
holds. This is easily seen from the above proof since if the algebra A has the form $A = Y_{\mathcal{C}^{op}}(X)$ a colimit of $F \cdot \Phi_A$ can be chosen to be FX.

2. Let $Colim(Set^{\mathcal{C}^{op}}, \mathcal{B})$ be the full subcategory of $\mathcal{B}^{\mathcal{C}}$ of all functors preserving colimits. Then composition with $Y_{\mathcal{C}^{op}}$ defines a functor
$$- \cdot Y_{\mathcal{C}^{op}} \colon Colim(Set^{\mathcal{C}^{op}}, \mathcal{B}) \to \mathcal{B}^{\mathcal{C}}.$$
The above universal property tells us that this functor is an equivalence. (It is, however, not an isomorphism of categories even assuming the choice in 1. above.)

4.12 Example.

1. A famous classical example is the free completion under filtered colimits denoted by
$$E_{Ind} : \mathcal{C} \to Ind\mathcal{C}\,.$$
For a small category $\mathcal{C}$, $Ind\mathcal{C}$ can be described as the category of all filtered colimits of representable functors in $Set^{\mathcal{C}^{op}}$, and the functor E_{Ind} is the codomain restriction of the Yoneda embedding.

2. One can proceed analogously with sifted colimits: we denote the free completion under sifted colimits by
$$E_{Sind} : \mathcal{C} \to Sind\mathcal{C}\,.$$
For a small category $\mathcal{C}$, $Sind\mathcal{C}$ can be described as the category of all sifted colimits of representable functors in $Set^{\mathcal{C}^{op}}$, and E_{Sind} is the codomain restriction of the Yoneda embedding. We are not going to prove these results in full generality here (the interested reader can find them in [8]). We only prove 2. under the assumption that $\mathcal{C}$ has finite coproducts:

4.13 Theorem. *For every algebraic theory $\mathcal{T}$ the Yoneda embedding*
$$Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to Alg\,\mathcal{T}$$
is a free completion of $\mathcal{T}^{op}$ under sifted colimits. In other words,
$$Alg\,\mathcal{T} = Sind(\mathcal{T}^{op})\,.$$

Analogously to 4.11 we have, for every functor $F\colon \mathcal{T}^{op} \to \mathcal{B}$, a choice of a sifted colimit preserving functor $F^*\colon Alg\,\mathcal{T} \to \mathcal{B}$ satisfying $F = F^* \cdot Y_{\mathcal{T}}$.

Proof. This is analogous to the proof of 4.10 with $\mathcal{T} = \mathcal{C}^{op}$. Given a functor $F\colon \mathcal{T}^{op} \to \mathcal{B}$ where $\mathcal{B}$ has sifted colimits, we prove that there exists an essentially unique functor

$$F^*\colon Alg\,\mathcal{T} \to \mathcal{B}$$

preserving sifted colimits and such that $F \simeq F^* \cdot Y_{\mathcal{T}}$. By 0.14 we are forced to define F^* on objects $A = colim\,(Y_{\mathcal{T}} \cdot \Phi_A)$ by

$$F^*A = \underset{El\,A}{colim}\,(F \cdot \Phi_A)\,.$$

This definition makes sense because, by 4.2, $El\,A$ is sifted. As in 4.10 all that needs to be proved is that the resulting functor F^* preserves sifted colimits. In the present situation F^* does not have a right adjoint. Nevertheless, since the inclusion $I\colon Alg\,\mathcal{T} \to Set^{\mathcal{T}}$ preserves sifted colimits (2.5), we still have, for $RB = \mathcal{B}(F-, B)$, a bijection

$$\mathcal{B}(F^*A, B) \simeq Set^{\mathcal{T}}(IA, RB)$$

natural in $A\colon \mathcal{C}^{op} \to Set$ and $B \in \mathcal{B}$. The argument is analogous to 4.10: both sides represent cocones of the (sifted) diagram $F \cdot \Phi_A$ with codomain B. From the above natural bijection one deduces that F^* preserves sifted colimits: for every fixed B the functor $A \mapsto \mathcal{B}(F^*A, B)$ preserves sifted colimits, now use 0.12. □

4.14 Corollary. *A category $\mathcal{A}$ is algebraic if and only if it is a free completion of a small category with finite coproducts under sifted colimits.*

4.15 Remark. Let $\mathcal{T}$ be an algebraic theory. If $\mathcal{B}$ is cocomplete and the functor $F\colon \mathcal{T}^{op} \to \mathcal{B}$ preserves finite coproducts, then its extension

$$F^*\colon Alg\,\mathcal{T} \to \mathcal{B}$$

preserving sifted colimits has a right adjoint. In fact, since F preserves finite coproducts, the functor $B \mapsto \mathcal{B}(F-, B)$ factorizes through $Alg\,\mathcal{T}$ and the resulting functor

$$R\colon \mathcal{B} \to Alg\,\mathcal{T}\ ,\quad B \mapsto \mathcal{B}(F-, B)$$

is a right adjoint to F^*.

4.16 Remark. Let $\mathcal{T}$ be a finitely complete small category and $Lex\,\mathcal{T}$ denote the full subcategory of $Set^{\mathcal{T}}$ of finite limits preserving functors.

1. $Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to Lex\,\mathcal{T}$ preserves finite colimits.
2. The embedding $Lex\,\mathcal{T} \to Set^{\mathcal{T}}$ preserves limits and filtered colimits.
3. $Lex\,\mathcal{T}$ is cocomplete.

The proofs of 1. and 2. are easy modifications of 1.5, 1.16 and 2.5. Using 4.3, the proof of 3. is analogous to that of 4.5.

4.17 Theorem. *For every finitely complete small category $\mathcal{T}$ the Yoneda embedding*

$$Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to Lex\,\mathcal{T}$$

is a free completion of $\mathcal{T}^{op}$ under filtered colimits. In other words,

$$Lex\,\mathcal{T} = Ind\,(\mathcal{T}^{op})\,.$$

Proof. A functor $A \colon \mathcal{T} \to Set$ preserves finite limits iff $El\,A$ is filtered (4.3). Moreover, the embedding $Lex\,\mathcal{T} \to Set^{\mathcal{T}}$ preserves filtered colimits (4.16). The rest of the proof is a trivial modification of the proof of 4.13, just replace "sifted" with "filtered" everywhere. □

4.18 Remark. Let $\mathcal{T}$ be a finitely complete small category. Analogously to 4.15, if $\mathcal{B}$ is cocomplete and the functor $F \colon \mathcal{T}^{op} \to \mathcal{B}$ preserves finite colimits, then its extension $F^* \colon Lex\,\mathcal{T} \to \mathcal{B}$ preserving filtered colimits has a right adjoint.

Historical Remarks for Chapter 4

Free colimit completions (see Theorem 4.10) were probably first described by F. Ulmer [92], see also [48]. The completion *Ind* was introduced by A.Grothendieck and J. L.Verdier in [13], but it is also contained in [48]. The completion *Sind* was introduced in [8], together with its relation to algebraic categories. But a general completion under a class of colimits is already treated in [48]. Later, these completions were studied by a number of authors, see e.g. the results and the references in [59] and [3].

Chapter 5

Properties of algebras

In this chapter properties of algebras such as finite presentability, regular projectivity, etc., are studied. These will be later used for a characterization of algebraic categories.

5.1 Definition. An object A of a category $\mathcal{A}$ is called *regular projective* if its hom-functor $\mathcal{A}(A,-)\colon \mathcal{A} \to Set$ preserves regular epimorphisms. That is, for every regular epimorphism $e\colon X \to Z$ and every morphism $f\colon A \to Z$ there exists a commutative triangle

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr, "f"] & \\ X \arrow[rr, "e"'] & & Z \end{tikzcd}$$

5.2 Example.

1. In *Set* all objects are regular projective.
2. In *Ab* all free abelian groups are regular projective. Conversely, a regular projective abelian group is free: express A as a regular quotient $e\colon X \to A$ of a free group X and apply the previous definition to $f = \mathrm{id}_A$. This proves that $X \simeq A \times \operatorname{Ker} e$, and then A is free (because every subgroup of a free abelian group is free).
3. A graph G is regular projective in *Graph* (see 1.15) iff its edges are pairwise disjoint. That is, both functions $G_e \to G_v$ are monomorphisms.

5.3 Definition. Let $\mathcal{A}$ be a category. An object A of $\mathcal{A}$ is:

1. *finitely presentable* if the hom-functor $\mathcal{A}(A,-)\colon \mathcal{A} \to Set$ preserves filtered colimits;
2. *perfectly presentable* if the hom-functor $\mathcal{A}(A,-)\colon \mathcal{A} \to Set$ preserves sifted colimits.

5.4 Remark. Any perfectly presentable object is finitely presentable (because filtered colimits are sifted) and, assuming the existence of kernel pairs, regular projective (because regular epimorphisms are coequalizers of reflexive pairs). We will show in 5.16 that in an algebraic category also the converse implication holds: perfectly presentable objects are precisely the finitely presentable regular projectives. In fact, the converse implication holds in any cocomplete exact category, see 18.3.

5.5 Example. Let $\mathcal{T}$ be a small category.

1. In $Set^{\mathcal{T}}$ the representable functors are perfectly presentable, in fact, they have a stronger property: their hom-functors preserve *all* colimits. This follows from Yoneda Lemma and the fact that in $Set^{\mathcal{T}}$ colimits are formed objectwise.

2. If $\mathcal{T}$ is an algebraic theory, representable functors are perfectly presentable objects in $Alg\,\mathcal{T}$. This follows from 1. and the fact that $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under sifted colimits, see 2.5.

3. Analogously, if $\mathcal{T}$ has finite limits, then representable functors are finitely presentable objects in $Lex\,\mathcal{T}$. This follows from 1. and the fact that $Lex\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under filtered colimits, see 4.16.

5.6 Example.

1. Every finite set is perfectly presentable in *Set*.

2. An abelian group A is finitely presentable in the above sense in the category Ab iff it is finitely presentable in the usual algebraic sense: that is, A can be presented by finitely many generators and finitely many equations. This is easily seen from the fact that every abelian group is a filtered colimit of abelian groups that are finitely presentable (in the algebraic sense).

   An abelian group is perfectly presentable iff it is free on finitely many generators.

3. In a poset considered as a category the finitely (or perfectly) presentable objects are precisely the *compact* elements x, i.e., such that for every directed join $y = \bigvee_{i \in I} y_i$, from $x \leq y$ it follows that $x \leq y_i$ for some $i \in I$.

4. A graph is finitely presentable in *Graph* iff it has finitely many vertices and finitely many edges. In fact, it is easy to see that for each such a graph G the hom-functor $Graph(G, -)$ preserves filtered colimits. Conversely, if G is finitely presentable, use the fact that G is a filtered colimit of all of its subgraphs on finitely many vertices and finitely many edges.

   A graph is perfectly presentable iff it has finitely many vertices and finitely many pairwise disjoint edges.

5.7 Remark. We will see in Chapter 11 that the situation described for Ab in the above example is a special case of the general fact that:

1. finite presentability has in algebraic categories the usual algebraic meaning (finitely presentable objects are precisely those which can be, in the classical sense, presented by finitely many generators and finitely many equations),
2. every free algebra is regular projective,
3. perfectly presentable algebras are just the retracts of the free algebras on finitely many generators.

5.8 Remark. As pointed out in 5.5, in categories $Set^{\mathcal{C}}$ the representable objects have the property that their hom-functors preserve all colimits. We call such objects *absolutely presentable*. In algebraic categories, absolutely presentable objects are typically rare. For example no abelian group A is absolutely presentable: for the initial object 1, the object $Ab(A, 1)$ is never intial in *Set*. However, the categories $Set^{\mathcal{C}}$ are an exception: every object is a colimit of absolutely presentable objects.

5.9 Lemma. *If an object is regular projective (or finitely presentable or perfectly presentable) then every retract has that property too.*

Proof. If $f\colon B \to A$ and $g\colon A \to B$ are such that $g \cdot f = \mathrm{id}_B$, then the natural transformations

$$\alpha = \mathcal{A}(g, -)\colon G = \mathcal{A}(B, -) \to F = \mathcal{A}(A, -) \quad \text{and} \quad \beta = \mathcal{A}(f, -)\colon F \to G$$

fulfil $\beta \cdot \alpha = G$. Therefore,

$$\begin{tikzcd} F \arrow[r, shift left, "\alpha\cdot\beta"] \arrow[r, shift right, "F"'] & F \arrow[r, "\beta"] & G \end{tikzcd}$$

is a coequalizer. By interchange of colimits, G preserves every colimit preserved by F. $\square$

5.10 Remark. Absolutely presentable objects in $Set^{\mathcal{C}}$ are precisely the retracts of the representable functors.

5.11 Lemma.

1. *Perfectly presentable objects are closed under finite coproducts.*
2. *Finitely presentable objects are closed under finite colimits.*

Proof. Let us prove the first statement (the proof of the second one is similar). Consider a finite family $(A_i)_{i\in I}$ of perfectly presentable objects. Since $\mathcal{A}(\coprod_I A_i, -) \simeq \prod_I \mathcal{A}(A_i, -)$, the claim follows from the obvious fact that a finite product of functors $\mathcal{A} \to Set$ preserving sifted colimits also preserves them. $\square$

5.12 Corollary. *Every object of an algebraic category is*

1. *a sifted colimit of perfectly presentable algebras*

and

2. *a filtered colimit of finitely presentable algebras.*

In fact, 1. follows from 4.2 and 5.3, and 2. follows from 4.2, 2.20 and 5.11.

5.13 Lemma. *Regular projective objects are closed under coproducts.*

Proof. Let $(A_i)_{i \in I}$ be a family of regular projective objects and let $e\colon X \to Z$ be a regular epimorphism. The claim follows from the formula $\mathcal{A}(\coprod_I A_i, e) \simeq \prod_I \mathcal{A}(A_i, e)$ and the fact that in *Set* regular epimorphisms are stable under products (3.21). ☐

5.14 Corollary. *In every category $Alg\,\mathcal{T}$:*

1. *The perfectly presentable objects are precisely the retracts of representable algebras.*

2. *The regular projective objects are precisely the retracts of coproducts of representable algebras.*

Proof. 1: Following 5.5 and 5.9, a retract of a representable algebra is perfectly presentable. Conversely, following 4.2, we can express every perfectly presentable algebra A as a sifted colimit of representable algebras. Since $Alg\,\mathcal{T}(A, -)$ preserves this colimit, it follows that id_A factorizes through some of the colimit morphism $e\colon Y_{\mathcal{T}}(t) \to A$. Thus e is a split epimorphism and A is a retract of $Y_{\mathcal{T}}(t)$.
2: Following 5.5, 5.9 and 5.13, a retract of a coproduct of representable algebras is regular projective. Conversely, following 4.2, we can express every algebra A as a colimit of representable algebras. Since $Alg\,\mathcal{T}$ is cocomplete, this implies, by 0.7, that A is a regular quotient of a coproduct of representable algebras

$$e\colon \coprod_{i \in I} \mathcal{T}(t_i, -) \to A\,. \qquad [5.1]$$

Therefore, if A is regular projective, it is a retract of $\coprod_{i \in I} \mathcal{T}(t_i, -)$. ☐

5.15 Corollary. *Every algebraic category has enough regular projective objects, i.e., every algebra is a regular quotient of a regular projective algebra.*

In fact, use [5.1] from the above proof: e is a regular epimorphism and, following 5.5 and 5.13, $\coprod_{i \in I} \mathcal{T}(t_i, -)$ is a regular projective object.

5.16 Corollary. *In an algebraic category, an algebra is perfectly presentable if and only if it is finitely presentable and regular projective.*

Proof. One implication holds in any category, see 5.4. Conversely, if P is a regular projective object in $Alg\mathcal{T}$, due to 5.14.2 P is a retract of a coproduct of representable algebras. Since every coproduct is a filtered colimit of its finite subcoproducts (2.20), if P is also finitely presentable, then it is a retract of a finite coproduct of representable algebras. Following 5.5, 5.9 and 5.11.1, P is perfectly presentable. $\square$

5.17 Proposition. *In every algebraic category the finitely presentable algebras are precisely the coequalizers of reflexive pairs of homomorphisms between representable algebras.*

Proof. One implication is obvious: representable algebras are finitely presentable (5.5), and finitely presentable objects are closed under finite colimits (5.11).
Conversely, let A be a finitely presentable algebra. Following 4.2, A is a (sifted) colimit of representable algebras. Thus, A is a filtered colimit of finite colimits of representable algebras, see 2.20. From 1.5 and 0.7 we deduce that every finite colimit of representable algebras is a reflexive coequalizer of a parallel pair between two representable algebras. Since A is finitely presentable, it is a retract of one of the above coequalizers:

$$\begin{tikzcd} Y_{\mathcal{T}}(t_1) \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & Y_{\mathcal{T}}(t_2) \arrow[r, "e"] & Q \arrow[r, shift left, "s"] & A \arrow[l, shift left, "u"] \end{tikzcd}$$

Here e is a coequalizer of h and k, and $s \cdot u = \mathrm{id}_A$. Since $Y_{\mathcal{T}}(t_2)$ is regular projective (see 5.4 and 5.5) and e is a regular epimorphism, there exists a homomorphism $g\colon Y_{\mathcal{T}}(t_2) \to Y_{\mathcal{T}}(t_2)$ such that $e \cdot g = u \cdot s \cdot e$. Let us observe that the morphism $s \cdot e$ is a joint coequalizer of h, k and $g \cdot h$:

$$\begin{tikzcd} Y_{\mathcal{T}}(t_1) \arrow[r, shift left=2, "h"] \arrow[r, "k"] \arrow[r, shift right=2, "g \cdot h"'] & Y_{\mathcal{T}}(t_2) \arrow[r, "s \cdot e"] & A \end{tikzcd}$$

In fact, $s \cdot e \cdot k = s \cdot e \cdot h = s \cdot u \cdot s \cdot e \cdot h = s \cdot e \cdot g \cdot h$. Assume that a morphism $f\colon Y_{\mathcal{T}}(t_2) \to X$ coequalizes h, k, and $g \cdot h$. Then there exists a unique homomorphism $v\colon Q \to X$ with $v \cdot e = f$. Hence $f \cdot h = f \cdot g \cdot h = v \cdot e \cdot g \cdot h = v \cdot u \cdot s \cdot e \cdot h$. Since h is an epimorphism (because the pair h, k is reflexive), we get $f = v \cdot u \cdot s \cdot e$.
Now, from 0.7 we conclude that A is a reflexive coequalizer of a pair of homomorphisms between finite coproducts of representable algebras. By 1.5, we get the claim. $\square$

5.18 Remark. Beside finite presentability, an important concept in general algebra is finite generation: an algebra A is finitely generated if it has a finite subset not contained in any proper subalgebra. (Or, equivalently, A is a regular quotient of a free algebra on finitely many generators.) This concept also has a categorical formulation. For this we need to introduce the following

5.19 Definition. A *directed union* is a filtered colimit of subobjects. That is, given a filtered diagram $D\colon \mathcal{D} \to \mathcal{A}$ where D maps every morphism to a monomorphism, then $colim\, D$ is called the directed union.

5.20 Remark. In *Set* directed unions have colimit cocones formed by monomorphisms. Thus, the same holds in $Set^{\mathcal{T}}$. Since $Alg\,\mathcal{T}$ is closed under filtered colimits in $Set^{\mathcal{T}}$, this is also true in $Alg\,\mathcal{T}$.

5.21 Definition. Let $\mathcal{A}$ be a category. An object A of $\mathcal{A}$ is *finitely generated* if the hom-functor $\mathcal{A}(A, -)\colon \mathcal{A} \to Set$ preserves directed unions.

5.22 Proposition. *In every algebraic category the finitely generated algebras are precisely the regular quotients of representable algebras.*

Proof. 1. Let A be finitely generated. Recall from 4.2 that A is the sifted colimit of

$$El\,A \xrightarrow{\Phi_A} \mathcal{T}^{op} \xrightarrow{Y_{\mathcal{T}}} Alg\,\mathcal{T}$$

Let (X, x) be an object of $El\,A$ and consider the regular factorization of the colimit morphism $\hat{x}$:

$$\begin{tikzcd} Y_{\mathcal{T}}(X) \arrow[rr, "\hat{x}"] \arrow[dr, "q_x"'] & & A \\ & I_x \arrow[ur, "m_x"'] & \end{tikzcd}$$

These objects I_x and the connecting monomorphisms between them one gets from the morphisms of $El\,A$ (via diagonal fill-in, see 0.16) form a filtered diagram of monomorphisms. Indeed, given two elements (X, x) and (Z, z) of A, for the element $(X \times Z, (x, z))$ we see that m_x and m_z both factorize through $m_{(x,z)}$. It is clear that $A = colim\, I_x$, and since A is finitely generated, $\mathcal{A}(A, -)$ preserves this colimit. Thus there exists $(X, x) \in El\,A$ and $f\colon A \to I_x$ such that $\mathrm{id}_A = m_x \cdot f$. Therefore m_x, being a monomorphism and a split epimorphism, is an isomorphism. This implies that $\hat{x}\colon Y_{\mathcal{T}}(X) \to A$ is a regular epimorphism.
2. For every regular epimorphism $e\colon Y_{\mathcal{T}}(X) \to A$ in $Alg\,\mathcal{T}$ we prove that A is finitely generated. Given a filtered diagram $D\colon \mathcal{D} \to Alg\,\mathcal{T}$ of subobjects with colimit cocone $c_d\colon Dd \to C$ $(d \in obj\,\mathcal{D})$, since c_d are monomorphisms, see 5.20, it is sufficient to prove that every morphism $f\colon A \to C$ factorizes through some c_d. In fact, since $Y_{\mathcal{T}}(X)$ is finitely presentable, the morphism $f \cdot e\colon Y_{\mathcal{T}}(X) \to C$ factorizes through some c_d – and then we just use the diagonal fill-in:

$$\begin{tikzcd} Y_{\mathcal{T}}(X) \arrow[r, "e"] \arrow[d] & A \arrow[d, "f"] \arrow[dl, dotted] \\ C_d \arrow[r, "c_d"'] & C \end{tikzcd}$$

□

5.23 Example. In the category $\mathbb{N}/Set$ of sets with countably many constants the finitely generated objects are those that have, beside the constants, only finitely many elements. Whereas the finitely presentable objects have, moreover, the property that only finitely many pairs of distinct natural numbers label the same constant. (Thus for example the terminal object is finitely generated but not finitely presentable.) Finally, the absolutely presentable objects are those finitely generated objects where the constants are pairwise distinct.

Historical Remarks for Chapter 5

The lecture notes [48] by P. Gabriel and F. Ulmer is the source of the concept of a finitely presentable object. In [8] perfectly presentable objects were introduced under the name of strongly finitely presentable. In algebraic categories, they coincide with objects "projectif-de-type-fini" of Y. Diers [39] and with the finitely presentable effective projectives of M. C. Pedicchio and R. J. Wood [81].

The term "perfectly presentable" was suggested by A. Joyal (see [57]), his motivation comes from perfect complexes as explained in 6.11 below.

Chapter 6

A characterization of algebraic categories

We have already characterized algebraic categories as free completions (see 4.14). The aim of this chapter is to characterize them as those cocomplete categories which have a strong generator formed by perfectly presentable objects.

6.1 Definition.

1. A set of objects $\mathcal{G}$ in a category $\mathcal{A}$ is called a *generator* if two morphisms $x, y\colon A \rightrightarrows B$ are equal whenever $x \cdot g = y \cdot g$ for every morphisms $g\colon G \to A$ with domain G in $\mathcal{G}$.

2. A generator $\mathcal{G}$ is called *strong* if a monomorphism $m\colon A \to B$ is an isomorphism whenever every morphism $g\colon G \to B$ with domain G in $\mathcal{G}$ factorizes through m.

6.2 Remark. Here is an equivalent way to express the notions of generator and strong generator. Consider the functor

$$\mathcal{A} \to \mathit{Set}^{\mathcal{G}}\ , \quad A \mapsto \langle \mathcal{A}(G, A) \rangle_{G \in \mathcal{G}}\ .$$

1. $\mathcal{G}$ is a generator iff the above functor is faithful.

2. $\mathcal{G}$ is a strong generator iff the above functor is faithful and conservative (see 0.2).

The following proposition suggests that "strong" generator should more properly be called "extremal", the present terminology has just historical reasons.

6.3 Proposition. *In a category $\mathcal{A}$ with coproducts a set of objects $\mathcal{G}$ is*

1. *a generator if and only if every object of $\mathcal{A}$ is a quotient of a coproduct of objects from $\mathcal{G}$,*

2. *a strong generator if and only if every object of $\mathcal{A}$ is an extremal quotient of a coproduct of objects from $\mathcal{G}$.*

Explicitly: $\mathcal{G}$ is a (strong) generator iff for every object A in $\mathcal{A}$, an (extremal) epimorphism

$$e\colon \coprod_{i\in I} G_i \to A$$

exists with all G_i in $\mathcal{G}$. We will see in the proof that this is equivalent to saying that the *canonical morphism*

$$e_A\colon \coprod_{(G,g)\in\mathcal{G}\downarrow A} G \to A$$

whose (G,g)-component is g, is an (extremal) epimorphism.

Proof. 1. If $\mathcal{G}$ is a generator, then the canonical morphism e_A is obviously an epimorphism. Conversely, if $e\colon \coprod_{i\in I} G_i \to A$ is an epimorphism, then given distinct morphisms $x, y\colon A \rightrightarrows B$, some component $e_i\colon G_i \to A$ fulfils $x{\cdot}e_i \neq y{\cdot}e_i$.
2. Let $\mathcal{G}$ be a strong generator. If e_A factorizes through a monomorphism m, then all of its components $g\colon G \to A$ factorize through m. Since $\mathcal{G}$ is a strong generator, this implies that m is an isomorphism. Conversely, let $e\colon \coprod_{i\in I} G_i \to A$ be an extremal epimorphism, and consider a monomorphism $m\colon B \to A$. If every morphism $g\colon G \to A$ (with G varying in $\mathcal{G}$) factorizes through m, then e factorizes through m, so that m is an isomorphism. $\square$

6.4 Corollary. *If $\mathcal{A}$ has colimits and every object of $\mathcal{A}$ is a colimit of objects from a set $\mathcal{G}$, then $\mathcal{G}$ is a strong generator.*

In fact, this follows from 0.7 and 6.3, because any regular epimorphism is extremal.

6.5 Example.

1. Every nonempty set forms a (singleton) strong generator in *Set*.
2. The group of integers forms a strong generator in *Ab*.
3. In the category of posets and order-preserving functions the terminal (one-element) poset forms a generator – but this generator is not strong. In contrast, the two-element chain is a strong generator.

6.6 Example.

1. Let $\mathcal{C}$ be a small category. Then $Set^{\mathcal{C}}$ has a strong generator formed by all representable functors. This follows from 0.14 and 6.4.
2. Analogously, for an algebraic theory $\mathcal{T}$ the category $Alg\,\mathcal{T}$ has a strong generator formed by all representable algebras: see 6.4 and 4.2.

6.7 Lemma. *Let $\mathcal{A}$ be a cocomplete category with a set of perfectly presentable objects such that every object of $\mathcal{A}$ is a sifted colimit of objects of that set. Then $\mathcal{A}$ has, up to isomorphism, only a set of perfectly presentable objects.*

Proof. Express an object A of $\mathcal{A}$ as a sifted colimit of objects from the given set $\mathcal{G}$. If A is perfectly presentable, then it is a retract of an object from $\mathcal{G}$. Since each object from $\mathcal{A}$ has only a set of retracts (because each retract of an object B gives rise to an idempotent morphism $e\colon B \to B$, $e \cdot e = e$), our claim is proved. □

6.8 Remark. A result analogous to 6.7 holds for finitely presentable objects and filtered colimits. The proof is the same.

6.9 Theorem. *(Characterization of algebraic categories) The following conditions on a category $\mathcal{A}$ are equivalent:*

1. *$\mathcal{A}$ is algebraic;*
2. *$\mathcal{A}$ is cocomplete and has a set $\mathcal{G}$ of perfectly presentable objects such that every object of $\mathcal{A}$ is a sifted colimit of objects of $\mathcal{G}$;*
3. *$\mathcal{A}$ is cocomplete and has a strong generator consisting of perfectly presentable objects.*

Moreover if the strong generator $\mathcal{G}$ in 3. is closed under finite coproducts, then the dual of $\mathcal{G}$ (seen as a full subcategory) is an algebraic theory of $\mathcal{A}$.

Proof. $1 \Rightarrow 2$: Let $\mathcal{T}$ be an algebraic theory. Then $Alg\,\mathcal{T}$ is cocomplete (4.5), the representable algebras form a set of perfectly presentable objects (5.5), and every algebra is a sifted colimit of representable algebras (4.2).
$2 \Rightarrow 3$: Consider the family $\mathcal{A}_{pp}$ of all perfectly presentable objects of $\mathcal{A}$. By 6.7, $\mathcal{A}_{pp}$ is essentially a set. By 6.4, it is a strong generator.
$3 \Rightarrow 1$: Let $\mathcal{G}$ be a strong generator consisting of perfectly presentable objects. Since perfectly presentable objects are closed under finite coproducts (5.11), we can assume without loss of generality that $\mathcal{G}$ is closed under finite coproducts (if this is not the case, we can replace $\mathcal{G}$ by its closure in $\mathcal{A}$ under finite coproducts, which still is a strong generator). We are going to prove that $\mathcal{A}$ is equivalent to $Alg\,(\mathcal{G}^{op})$, where $\mathcal{G}$ is seen as a full subcategory of $\mathcal{A}$.
(a) We prove first that $\mathcal{G}$ is *dense*, i.e., for every object K of $\mathcal{A}$ the canonical diagram of all arrows from $\mathcal{G}$

$$D_K\colon \mathcal{G}\downarrow K \to \mathcal{A}\,,\ \ (g\colon G \to K) \mapsto G$$

has K as colimit, with $(g\colon G \to K)$ as colimit cocone. To prove this, form a colimit cocone of D_K:

$$(c_g\colon G \to K^*) \quad \text{for all } g\colon G \to K \text{ in } \mathcal{G}\downarrow K.$$

We have to prove that the unique factorizing morphism $\lambda\colon K^* \to K$ with $\lambda \cdot c_g = g$ for all g in $\mathcal{G}\downarrow K$ is an isomorphism. Consider the coproduct

$$\coprod_{(G,g)\in\mathcal{G}\downarrow K} G$$

with coproduct injections $\rho_g \colon G \to \coprod_{(G,g)\in\mathcal{G}\downarrow K} G$. We have a commutative triangle

$$\begin{tikzcd} & \coprod_{(G,g)\in\mathcal{G}\downarrow K} G \arrow[dl, "v"'] \arrow[dr, "e_K"] & \\ K^* \arrow[rr, "\lambda"'] & & K \end{tikzcd}$$

where e_K is the canonical morphism 6.3 and v is defined by $v \cdot \rho_g = c_g$ for all $(G,g) \in \mathcal{G} \downarrow K$. Since e_K is an extremal epimorphism (see 6.3), then λ is an extremal epimorphism. It remains to prove that λ is a monomorphism. Consider two morphisms $x, x' \colon X \rightrightarrows K^*$ such that $\lambda \cdot x = \lambda \cdot x'$, and let us prove that $x = x'$. While $\mathcal{G}$ is a (strong) generator, we can assume without loss of generality that X is in $\mathcal{G}$. Since $\mathcal{G} \downarrow K$ is sifted (in fact, it has finite coproducts, because $\mathcal{G}$ has, see 2.15) and X is perfectly presentable, both x and x' factorize through some colimit morphism: that is, for some (G,g) and (G',g') in $\mathcal{G} \downarrow K$ we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "y"'] \arrow[dr, "x"] & & X' \arrow[dl, "x'"'] \arrow[d, "y'"] \\ G \arrow[r, "c_g"'] \arrow[dr, "g"'] & K^* \arrow[d, "\lambda"] & G' \arrow[l, "c_{g'}"] \arrow[dl, "g'"] \\ & K & \end{tikzcd}$$

Since $c_{\lambda \cdot x} = c_g \cdot y = x$ and analogoulsy $c_{\lambda \cdot x'} = x'$, we get $x = c_{\lambda \cdot x} = c_{\lambda \cdot x'} = x'$.
(b) It follows from (a) that the functor

$$E \colon \mathcal{A} \to Alg(\mathcal{G}^{op}) \,, \quad K \mapsto \mathcal{A}(-,K)$$

is full and faithful. Indeed, given a homomorphism $\alpha \colon \mathcal{A}(-,K) \to \mathcal{A}(-,L)$, for every $g \colon G \to K$ in $\mathcal{G} \downarrow K$ we have a morphism $\alpha_G(g) \colon G \to L$. Those morphisms form a cocone on $\mathcal{G} \downarrow K$, so that there exists a unique morphism $\hat{\alpha} \colon K \to L$ such that $\hat{\alpha} \cdot g = \alpha_G(g)$ for all g in $\mathcal{G} \downarrow K$. It is easy to check that $E\hat{\alpha} = \alpha$ and that $\widehat{Ef} = f$ for all $f \colon K \to L$ in $\mathcal{A}$. It remains to prove that E is essentially surjective on objects.
(c) Let us prove first that E preserves sifted colimits. Consider a sifted diagram $D \colon \mathcal{D} \to \mathcal{A}$ with colimit $(h_d \colon Dd \to H)$. For every object G in $\mathcal{G}$ a colimit of $\mathcal{A}(G,-) \cdot D$ in *Set* is $\mathcal{A}(G,H)$ with the colimit cocone $\mathcal{A}(G,Dd) \to \mathcal{A}(G,H)$ given by composition with h_d (because $\mathcal{G}$ is formed by perfectly presentable objects). This implies that $(Eh_d \colon E(Dd) \to EH)$ is a colimit of $E \cdot D$ in $Alg(\mathcal{G}^{op})$ (sifted colimits are computed objectwise in $Alg(\mathcal{G}^{op})$, see 2.5).
(d) It follows from (c) that E is essentially surjective on objects. In fact, we

have the following diagram, commutative up to natural isomorphism,

$$\begin{tikzcd}
Alg\,(\mathcal{G}^{op}) \arrow[r, "I^*"] & \mathcal{A} \arrow[r, "E"] & Alg\,(\mathcal{G}^{op}) \\
& \mathcal{G} \arrow[ul, "Y_{\mathcal{G}^{op}}"] \arrow[u, "I"'] \arrow[ur, "Y_{\mathcal{G}^{op}}"'] &
\end{tikzcd}$$

where I is the inclusion and I^* is its extension preserving sifted colimit (4.13). Since $E \cdot I^* \cdot Y_{\mathcal{G}^{op}} \simeq Y_{\mathcal{G}^{op}}$ and $E \cdot I^*$ preserves sifted colimits, it follows from 4.13 that $E \cdot I^*$ is naturally isomorphic to the identity functor. Thus E is essentially surjective on objects. $\square$

6.10 Example.

1. The category *Pos* of posets and order-preserving maps is not algebraic: only the discrete posets are perfectly presentable, and there exists no strong generator formed by discrete posets.

2. In the category *Bool* of Boolean algebras consider the free algebras $\mathcal{P}\mathcal{P}n$ on n generators (where $\mathcal{P}X$ is the algebra of all subsets of a set X). The dual of the category $\mathcal{P}\mathcal{P}n$ ($n \in \mathbb{N}$) is an algebraic theory for *Bool*. In fact, $\mathcal{P}\mathcal{P}n$ are perfectly presentable and form a strong generator closed under finite coproducts.

6.11 Example. Let R be a unitary ring. We denote by $Ch(R)$ the category of chain complexes of left R-modules. Its objects are collections $X = (X_n)_{n\in\mathbb{Z}}$ of left R-modules equipped with a *differential*, that is, a collection of module homomorphisms

$$d = (d_n \colon X_n \to X_{n-1})_{n\in\mathbb{Z}}$$

where $d_{n-1} \cdot d_n = 0$ for each n. Morphisms $f \colon X \to Y$ are *chain maps*, i.e., collections $(f_n \colon X_n \to Y_n)_{n\in\mathbb{Z}}$ of module homomorphisms such that $d_n \cdot f_n = f_{n-1} \cdot d_n$ for all n.

A complex X is *bounded* if there are only finitely many $n \in \mathbb{N}$ with $X_n \neq 0$. Since every complex is a filtered colimit of its truncations, each finitely presentable complex is bounded. Perfectly presentable objects in $Ch(R)$ are precisely the bounded complexes of perfectly presentable left R-modules. Since they form a strong generator of $Ch(R)$, the category $Ch(R)$ is algebraic.

6.12 Notation. We denote by $\mathcal{A}_{pp}$ a full subcategory of $\mathcal{A}$ representing all perfectly presentable objects of $\mathcal{A}$ up to isomorphism.

6.13 Corollary. *For every algebraic category $\mathcal{A}$ the dual of $\mathcal{A}_{pp}$ is an algebraic theory of $\mathcal{A}$: we have an equivalence functor*

$$E \colon \mathcal{A} \to Alg\,(\mathcal{A}_{pp}^{op}), \quad A \mapsto \mathcal{A}(-, A).$$

In fact, $\mathcal{A}_{pp}$ is a strong generator closed under finite coproducts.

6.14 Corollary. *Two algebraic categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if the categories $\mathcal{A}_{pp}$ and $\mathcal{B}_{pp}$ are equivalent.*

Proof. This follows immediately from 6.13 and the fact that equivalence functors preserve perfectly presentable objects. □

From 1.10 we know that the slice category $Set \downarrow S$, equivalent to the category Set^S of S-sorted sets, is algebraic. This is a particular case of a more general fact:

6.15 Proposition. *Every slice category $\mathcal{A} \downarrow A$ of an algebraic category $\mathcal{A}$ is algebraic.*

Proof. The category $\mathcal{A} \downarrow A$ is cocomplete: consider a small category $\mathcal{D}$ and a functor

$$F\colon \mathcal{D} \to \mathcal{A} \downarrow A\,, \quad FD = (F_D, f_D\colon F_D \to A)\,.$$

A colimit of F is given by (C, c), where $(C, \sigma_D\colon F_D \to C)$ is a colimit of $\Phi_A \cdot F$ where $\Phi_A\colon \mathcal{A} \downarrow A \to \mathcal{A}$ is the forgetful functor, and $c \cdot \sigma_D = f_D$ for every object D in $\mathcal{D}$. This immediately implies that an object (G, g) is perfectly presentable in $\mathcal{A} \downarrow A$ as soon as G is perfectly presentable in $\mathcal{A}$. Let now $\mathcal{G}$ be a strong generator of $\mathcal{A}$. Then the set of objects $\mathcal{G} \downarrow A = \{(G, g) \mid G \in \mathcal{G}\}$ is a strong generator of $\mathcal{A} \downarrow A$. This is so because a morphism $f\colon (X, x) \to (Z, z)$ in $\mathcal{A} \downarrow A$ is a strong epimorphism iff $f\colon X \to Z$ is a strong epimorphism in $\mathcal{A}$. Following 6.9, $\mathcal{A} \downarrow A$ is algebraic. □

6.16 Lemma. *Let the functor $I\colon \mathcal{A} \to \mathcal{B}$ have a left adjoint R.*

1. *If I is faithful and conservative and $\mathcal{G}$ is a strong generator of $\mathcal{B}$, then $R(\mathcal{G})$ is a strong generator of $\mathcal{A}$.*

2. *If I preserves sifted colimits and X is perfectly presentable in $\mathcal{B}$, then RX is perfectly presentable in $\mathcal{A}$.*

Proof. 1: $R(\mathcal{G})$ is a generator because I is a faithful right adjoint. Next, consider a monomorphism $a\colon A \to A'$ in $\mathcal{A}$ such that every morphism $RG \to A'$ with $G \in \mathcal{G}$ factorizes through a. This implies, by adjunction, that every morphism $G \to IA'$ factorizes through the monomorphism Ia. Since $\mathcal{G}$ is a strong generator, Ia is an isomorphism, and since I is conservative, a is an isomorphism.
2: Since $\mathcal{A}(RX, -) \simeq \mathcal{B}(X, I-) = \mathcal{B}(X, -) \cdot I$, we see that $\mathcal{A}(RX, -)$ is the composite of two functors preserving sifted colimits. □

6.17 Proposition. *Let $\mathcal{T}$ be an algebraic theory. Then $Alg\,\mathcal{T}$ is a reflective subcategory of $Set^{\mathcal{T}}$ closed under sifted colimits.*

Proof. This is a special case of the adjunction $F^* \dashv \mathcal{B}(F-, -)$ obtained in the proof of 4.10. Indeed, by the Yoneda Lemma, the right adjoint $Alg\,\mathcal{T}(Y_{\mathcal{T}}-, -)$ is naturally isomorphic to the full inclusion $Alg\,\mathcal{T} \to Set^{\mathcal{T}}$. □

6.18 Theorem. *A category is algebraic if and only if it is equivalent to a reflective subcategory of $Set^{\mathcal{C}}$ closed under sifted colimits, for some small category $\mathcal{C}$.*

Proof. Let $\mathcal{T}$ be an algebraic theory. Following 6.17 and 2.5, $Alg\mathcal{T}$ is a reflective subcategory closed under sifted colimits of $Set^{\mathcal{T}}$.
Conversely, let $\mathcal{C}$ be a small category. By 1.6 $Set^{\mathcal{C}}$ is an algebraic category, so that it fulfils the conditions of 6.9.3. Following 6.16, those conditions are inherited by any reflective subcategory closed under sifted colimits of $Set^{\mathcal{C}}$. $\square$

6.19 Corollary. *The functor category $\mathcal{A}^{\mathcal{D}}$ of an algebraic category $\mathcal{A}$ is algebraic.*

Proof. Due to 6.18, there exist a small category $\mathcal{C}$ and a reflection

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} Set^{\mathcal{C}}$$

with the right adjoint I preserving sifted colimits. This induces another reflection

$$\mathcal{A}^{\mathcal{D}} \underset{I\cdot -}{\overset{R\cdot -}{\leftrightarrows}} \left(Set^{\mathcal{C}}\right)^{\mathcal{D}}$$

with $I \cdot -$ preserving sifted colimits because they are formed objectwise. Since $\left(Set^{\mathcal{C}}\right)^{\mathcal{D}} \simeq Set^{\mathcal{C}\times\mathcal{D}}$, by 6.18 $\mathcal{A}^{\mathcal{D}}$ is algebraic. $\square$

6.20 Remark. Our Characterization Theorem 6.9 shows a strong parallel between algebraic categories and the following more general concept due to Gabriel and Ulmer [48].

6.21 Definition. A category is called *locally finitely presentable* if it is cocomplete and has a set $\mathcal{G}$ of finitely presentable objects such that every object of $\mathcal{A}$ is a filtered colimit of objects of $\mathcal{G}$.

6.22 Example.

1. Following 6.9, all algebraic categories are locally finitely presentable.
2. If $\mathcal{T}$ is a small category with finite limits, then $Lex\mathcal{T}$, see 4.16, is a locally finitely presentable category. In fact $Lex\mathcal{T}$ is cocomplete (4.16), the representable functors form a set of finitely presentable objects (5.5), and every object is a filtered colimit of representable functors (4.3).
3. The category *Pos* of posets (which is not algebraic, see 6.10) is locally finitely presentable: the two-element chain which forms a strong generator is finitely presentable. Thus we can apply the following

6.23 Theorem. *(Characterization of locally finitely presentable categories) The following conditions on a category $\mathcal{A}$ are equivalent:*

1. *$\mathcal{A}$ is locally finitely presentable;*

2. *$\mathcal{A}$ is equivalent to $Lex\mathcal{T}$ for a small category $\mathcal{T}$ with finite limits;*

3. *$\mathcal{A}$ is cocomplete and has a strong generator formed by finitely presentable objects.*

Moreover if the strong generator $\mathcal{G}$ in 3. is closed under finite colimits, then $\mathcal{A}$ is equivalent to $Lex(\mathcal{G}^{op})$.

Proof. This proof is quite analogous to that of Theorem 6.9: just change $Alg\mathcal{T}$ to $Lex\mathcal{T}$ for $\mathcal{T} = \mathcal{G}^{op}$, work with finitely presentable objects instead of perfectly presentable ones, and in the proof of $3 \Rightarrow 2$ use the closure under finite colimits. □

6.24 Corollary. *A category $\mathcal{A}$ is locally finitely presentable if and only if it is a free completion of a small, finitely cocomplete category under filtered colimits.*

In fact, this follows from 4.17 and 6.23.

6.25 Notation. We denote by $\mathcal{A}_{fp}$ a full subcategory of $\mathcal{A}$ representing all finitely presentable objects of $\mathcal{A}$ up to isomorphism.

6.26 Corollary. *Every locally finitely presentable category $\mathcal{A}$ is equivalent to $Lex\mathcal{T}$ for some small, finitely complete category $\mathcal{T}$. In fact, we have the equivalence functor*

$$E\colon \mathcal{A} \to Lex(\mathcal{A}_{fp}^{op}) , \quad A \mapsto \mathcal{A}(-, A) .$$

This follows from 6.23 applied to $\mathcal{G} = \mathcal{A}_{fp}$.

6.27 Proposition. *Let $\mathcal{T}$ be a small finitely complete category. Then $Lex\mathcal{T}$ is a reflective subcategory of $Set^{\mathcal{T}}$ closed under filtered colimits.*

Proof. This is a special case of the adjunction $F^* \dashv \mathcal{B}(F-, -)$ obtained in the proof of 4.10. Indeed, by the Yoneda Lemma, the right adjoint $Lex\mathcal{T}(Y_{\mathcal{T}}-, -)$ is naturally isomorphic to the full inclusion $Lex\mathcal{T} \to Set^{\mathcal{T}}$. □

6.28 Theorem. *A category is locally finitely presentable if and only if it is equivalent to a reflective subcategory of $Set^{\mathcal{C}}$ closed under filtered colimits, for some small category $\mathcal{C}$.*

6.29 Corollary. *Let $\mathcal{T}$ be a small finitely complete category. Then $Lex\mathcal{T}$ is a reflective subcategory of $Alg\mathcal{T}$ closed under filtered colimits.*

Proof. Consider the full inclusions

$$\begin{tikzcd} Lex\mathcal{T} \arrow[rr, "I_1"] \arrow[dr, "I_2"'] & & Set^{\mathcal{T}} \\ & Alg\mathcal{T} \arrow[ur, "I_3"'] & \end{tikzcd}$$

By 6.27, I_1 has a left adjoint, say R. Since I_3 is full and faithful, $R \cdot I_3$ is left adjoint to I_2. Finally, I_2 preserves filtered colimits because I_1 preserves them by 6.27 and I_3 reflects them. □

In 12.12 we will need the following fact.

6.30 Corollary. *Let $\mathcal{A}$ be an algebraic category and $\mathcal{B}$ a category with sifted colimits. If in $\mathcal{A}$ every finitely presentable object is regular projective, then a functor $F\colon \mathcal{A} \to \mathcal{B}$ preserving filtered colimits preserves also sifted colimits.*

Proof. Following 6.13 and 6.26

$$Alg\,(\mathcal{A}_{pp}^{op}) \simeq \mathcal{A} \simeq Lex\,(\mathcal{A}_{fp}^{op})\,.$$

If moreover finitely presentable objects in $\mathcal{A}$ are regular projective, then $\mathcal{A}_{pp} = \mathcal{A}_{fp}$ (5.16). The result now follows from the universal properties stated in 4.13 and 4.17. $\square$

6.31 Example. *Set* (and more generally Set^S) and the category of vector spaces over a field are examples of algebraic categories where every (finitely presentable) object is regular projective. More generally, in the category of left modules over a semi-simple ring every object is regular projective.

Let us finish this chapter by quoting from [30] another characterization theorem similar to 6.9. Recall that $Set^{\mathcal{C}}$ is the free completion of $\mathcal{C}^{op}$ under colimits 4.10; recall also the concept of absolutely presentable object in 5.8.

6.32 Theorem. *A category is equivalent to a functor category $Set^{\mathcal{C}}$ for some small category $\mathcal{C}$ if and only if it is cocomplete and has a strong generator consisting of absolutely presentable objects.*

Historical Remarks for Chapter 6

The concept of locally finitely presentable category, due to P. Gabriel and F. Ulmer [48], was an attempt of a categorical approach to categories of finitary structures generalizing Lawvere's algebraic theories. This is the source of 6.23. See [7] for a more recent monograph on locally presentable categories. In [8] the analogy between locally finitely presentable categories and algebraic categories was made explicit.

A characterization of categories of algebras for one-sorted algebraic theories is contained in the thesis of F. W. Lawvere [63]. The characterization 6.9 is taken from [8]. But the equivalence of 1. and 3. was already proved by Y. Diers [39]. A first characterization of categories of *Set*-valued functors is in the thesis of M. Bunge [30] (the first proof was published in [68]). There is a general result covering both 6.9, 6.23 and 6.32 (see [34]).

Chapter 7

From filtered to sifted

The aim of this chapter is to demonstrate that the "equation"

sifted colimits = filtered colimits + reflexive coequalizers

is almost valid – but not quite. What we have in mind are three facts:

1. A category $\mathcal{C}$ has sifted colimits iff it has filtered colimits and reflexive coequalizers. This holds whenever $\mathcal{C}$ has finite coproducts – and in general it is false.

2. A functor preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers. This holds whenever the domain category is finitely cocomplete – and in general it is false.

3. The free completion $Sind\mathcal{C}$ of a small category $\mathcal{C}$ under sifted colimits is obtained from the free completion $Rec\mathcal{C}$ under reflexive coequalizers by completing it under filtered colimits

   $$Sind\mathcal{C} = Ind(Rec\mathcal{C})\,.$$

   This holds whenever $\mathcal{C}$ has finite coproducts – and in general it is false.

We begin by describing $Rec\mathcal{C}$ in a manner analogous to the description of $Sind\mathcal{C}$ and $Ind\mathcal{C}$, see 4.13 and 4.17. A quite different approach to $Rec\mathcal{C}$ is treated in Chapter 17.

As a special case of Definition 4.9, we get the following

7.1 Definition. By a *free completion of a category* $\mathcal{C}$ *under reflexive coequalizers* is meant a functor $E_{Rec} : \mathcal{C} \to Rec\mathcal{C}$ such that

1. $Rec\mathcal{C}$ is a category with coequalizers of reflexive pairs

and

2. for every functor $F\colon \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is a category with reflexive coequalizers, there exists an essentially unique functor $F^*\colon Rec\,\mathcal{C} \to \mathcal{B}$ preserving reflexive coequalizers with F naturally isomorphic to $F^* \cdot E_{Rec}$.

Recall that for a category $\mathcal{A}$ we denote by $\mathcal{A}_{fp}$ the full subcategory of finitely presentable objects.

7.2 Lemma. *Let $\mathcal{T}$ be an algebraic theory. The inclusion $I\colon (Alg\,\mathcal{T})_{fp} \to Set^{\mathcal{T}}$ preserves reflexive coequalizers.*

Proof. This follows from the fact that $(Alg\,\mathcal{T})_{fp}$ is closed in $Alg\,\mathcal{T}$ under finite colimits (5.11) and that $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under sifted colimits (2.5), see also 3.2. $\square$

7.3 Theorem. *For every algebraic theory $\mathcal{T}$ the restricted Yoneda embedding*

$$Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to (Alg\,\mathcal{T})_{fp}$$

is a free completion of $\mathcal{T}^{op}$ under reflexive coequalizers. In other words,

$$(Alg\,\mathcal{T})_{fp} = Rec\,(\mathcal{T}^{op})\,.$$

Proof. Recall from 3.2 the category $\mathcal{M}$

$$\begin{tikzcd} P \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & Q \arrow[l, "d" description] \end{tikzcd} \quad \text{modulo} \quad f_1 \cdot d = \mathrm{id}_Q = f_2 \cdot d\,.$$

We will prove that given a finitely presentable algebra $A\colon \mathcal{T} \to Set$, there exists a final functor (see Definition 2.12)

$$M\colon \mathcal{M} \to El\,A\,.$$

The rest of the proof is analogous to the proof of Theorem 4.10: given a functor $F\colon \mathcal{T}^{op} \to \mathcal{B}$ where $\mathcal{B}$ has reflexive coequalizers, we prove that there exists an essentially unique functor

$$F^*\colon (Alg\,\mathcal{T})_{fp} \to \mathcal{B}$$

preserving reflexive coequalizers and such that $F \simeq F^* \cdot Y_{\mathcal{T}}$. In fact, using the notation of 0.14 for the above final functor M we see that the reflexive coequalizer of $F \cdot \Phi_A \cdot M(f_i)$ $(i = 1, 2)$ in $\mathcal{B}$ is just the colimit of $F \cdot \Phi_A$, thus, the latter colimit exists, and we are forced to define

$$F^*A = \operatorname*{colim}_{El\,A}\,(F \cdot \Phi_A)$$

on objects. This extends uniquely to morphisms (as in the proof of 4.10) and yields a functor $F^*\colon (Alg\,\mathcal{T})_{fp} \to \mathcal{B}$. Since the inclusion $I\colon (Alg\,\mathcal{T})_{fp} \to Set^{\mathcal{T}}$ preserves reflexive coequalizers (7.2), we have for $RB = \mathcal{B}(F-, B)$ a bijection

$$\mathcal{B}(F^*A, B) \simeq Set^{\mathcal{T}}(IA, RB)$$

natural in A and B, from which one deduces that F^* preserves reflexive coequalizers.

To prove the existence of the final functor M, recall from 5.17 that there exists a reflexive pair

$$\begin{tikzcd} \overline{P} \arrow[r, shift left=2, "\overline{f_1}"] \arrow[r, shift right=2, "\overline{f_2}"'] & \overline{Q} \arrow[l, "\overline{d}" description] \end{tikzcd}$$

in $\mathcal{T}$ such that A is a coequalizer of $Y_{\mathcal{T}}(f_i)$:

$$\begin{tikzcd} Y_{\mathcal{T}}(\overline{P}) \arrow[r, shift left, "Y_{\mathcal{T}}(\overline{f_1})"] \arrow[r, shift right, "Y_{\mathcal{T}}(\overline{f_2})"'] & Y_{\mathcal{T}}(\overline{Q}) \arrow[r, "c"] & A \end{tikzcd}$$

Put $\overline{c} = c \cdot Y_{\mathcal{T}}(\overline{f_i})$ and define objects of $El\,A$ as follows:

$$MP = (\overline{P}, \overline{c}_{\overline{P}}(\mathrm{id}_{\overline{P}})) \quad \text{and} \quad MQ = (\overline{Q}, c_{\overline{Q}}(\mathrm{id}_{\overline{Q}}))\,.$$

Since clearly $\overline{f_i}\colon MP \to MQ$ and $\overline{d}\colon MQ \to MP$ are morphisms of $El\,A$, we obtain a functor $M = \overline{(-)}\colon \mathcal{M} \to El\,A$. Let us prove its finality:

1. Every object (X, x) of $El\,A$ has a morphism into MQ. In fact,

$$c_X\colon \mathcal{T}(\overline{Q}, X) \to A(X)$$

   is an epimorphism, thus, for $x \in A(X)$ there exists $f\colon \overline{Q} \to X$ with $c_X(f) = x$, which implies that $f\colon (X, x) \to MQ$ is a morphism of $El\,A$.

2. Given two morphisms of $El\,A$ from (X, x) to MP or MQ, they are connected by a zig-zag in the slice category $(X, x) \downarrow M$. In fact, we can restrict ourselves to the codomain MQ: the general case is then solved by composing morphisms with codomain MP by Mf_1.

   Given morphisms
$$h, k\colon (X, x) \to MQ$$
   then we have
$$A(h) \cdot c_Q(\mathrm{id}_Q) = x = A(k) \cdot c_Q(\mathrm{id}_Q)$$
   which, due to naturality of c, yields
$$c_X(h) = c_X(k)\,.$$

   Now use the description of coequalizers in *Set* (see 0.6): since c_X is the coequalizer of $Y_{\mathcal{T}}(f_1)$and $Y_{\mathcal{T}}(f_2)$, there is a zig-zag of this pair connecting h and k. For example a zig-zag of length 2:

$$\begin{tikzcd} & v \arrow[dl, "Y_{\mathcal{T}}(f_1)"'] \arrow[dr, "Y_{\mathcal{T}}(f_2)"] & \\ h & & k \end{tikzcd}$$

for some $v\colon P \to X$. This means $h = v \cdot f_1$ and $k = v \cdot f_2$ and yields the following zig-zag in $(X,x) \downarrow M$:

$$\begin{tikzcd}
 & (X,x) \arrow[dl, "h"'] \arrow[d, "v"] \arrow[dr, "k"] & \\
MQ & MP \arrow[l, "Mf_1"] \arrow[r, "Mf_2"'] & MQ
\end{tikzcd}$$

Analogously for longer zig-zags.

□

7.4 Corollary. *For a small category $\mathcal{C}$ with finite coproducts we have*

$$\mathit{Sind}\,\mathcal{C} = \mathit{Ind}\,(\mathit{Rec}\,\mathcal{C})\,.$$

More precisely, the composition

$$\mathcal{C} \xrightarrow{E_{Rec}} \mathit{Rec}\,\mathcal{C} \xrightarrow{E_{Ind}} \mathit{Ind}\,(\mathit{Rec}\,\mathcal{C})$$

is a free completion of $\mathcal{C}$ under sifted colimits.

In fact, for $\mathcal{T} = \mathcal{C}^{op}$ the above theorem yields $\mathit{Rec}\,\mathcal{C} = (\mathit{Alg}\,\mathcal{T})_{fp}$ from which 6.26 and 4.17 prove $\mathit{Ind}\,(\mathit{Rec}\,\mathcal{C}) = \mathit{Alg}\,\mathcal{T}$. Now apply 4.13.

7.5 Remark. In the proof of Theorem 7.3 if $\mathcal{B}$ has finite colimits and F preserves finite coproducts, then the extension F^* preserves finite colimits. This follows from the fact that $\mathcal{B}(F-, B)$ lies now in $\mathit{Alg}\,\mathcal{T}$ and that $(\mathit{Alg}\,\mathcal{T})_{fp}$ is closed in $\mathit{Alg}\,\mathcal{T}$ under finite colimits (5.11), so that we have a bijection

$$\mathcal{B}(F^*A, B) \simeq \mathit{Alg}\,\mathcal{T}(A, \mathcal{B}(F-, B))$$

natural in $A \in (\mathit{Alg}\,\mathcal{T})_{fp}$ and $B \in \mathcal{B}$.

7.6 Remark. In the introduction to this chapter we claimed that a functor defined on a finitely cocomplete category preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers. The proof of this result can be found in [11], here we present a (simpler) proof based on 7.4 which requires cocompleteness of both categories (in fact, sifted colimits are enough as far as the codomain category is concerned).

7.7 Theorem. *A functor between cocomplete categories preserves sifted colimits if and only if it preserves filtered colimits and reflexive coequalizers.*

Proof. Necessity is clear. To prove sufficiency, let $F\colon \mathcal{E} \to \mathcal{A}$ preserve filtered colimits and reflexive coequalizers where $\mathcal{E}$ and $\mathcal{A}$ are cocomplete. For every sifted diagram $D\colon \mathcal{D} \to \mathcal{E}$ choose a small full subcategory $U\colon \mathcal{C} \hookrightarrow \mathcal{E}$ containing the image of D and closed in $\mathcal{E}$ under finite coproducts. Consider the following diagram

$$\begin{tikzcd}
 & Sind\,\mathcal{C} \arrow[r, "(F\cdot U)^*"] \arrow[ddr, "U^*"] & \mathcal{A} \\
 & Rec\,\mathcal{C} \arrow[u, "E_{Ind}"'] \arrow[dr, "U'"] & \\
\mathcal{C} \arrow[uur, "E_{Sind}"] \arrow[ur, "E_{Rec}"'] \arrow[rr, "U"'] & & \mathcal{E} \arrow[uu, "F"']
\end{tikzcd}$$

where U^* and $(F \cdot U)^*$ are the extensions of U and $F \cdot U$, respectively, preserving sifted colimits, and U' is the extension of U preserving reflexive coequalizers. Since by 7.4 we have $E_{Sind} = E_{Ind} \cdot E_{Rec}$, it follows that

$$(F \cdot U)^* \cdot E_{Ind} \cdot E_{Rec} \simeq (F \cdot U)^* \cdot E_{Sind} \simeq F \cdot U \simeq F \cdot U' \cdot E_{Rec}\,.$$

The functor E_{Ind} preserves reflexive coequalizers by 4.16 and 4.17, and so do the functors F, U' and $(F \cdot U)^*$. Thus, the universal property of E_{Rec} yields

$$(F \cdot U)^* \cdot E_{Ind} \simeq F \cdot U'\,.$$

Since

$$U^* \cdot E_{Ind} \cdot E_{Rec} \simeq U^* \cdot E_{Sind} \simeq U \simeq U' \cdot E_{Rec}$$

and, once again, the functors U^*, E_{Ind} and U' preserve reflexive coequalizers, we have

$$U^* \cdot E_{Ind} \simeq U'\,.$$

Finally, since

$$F \cdot U^* \cdot E_{Ind} \simeq F \cdot U' \simeq (F \cdot U)^* \cdot E_{Ind}$$

and the functors F, U^* and $(F \cdot U)^*$ preserve filtered colimits, we have

$$F \cdot U^* \simeq (F \cdot U)^*\,. \qquad [7.1]$$

We are ready to prove $F(colim\, D) \simeq colim\,(F \cdot D)$. Let

$$D' \colon \mathcal{D} \to \mathcal{C} \quad \text{with} \;\; D = U \cdot D'$$

be the codomain restriction of D. Then

$$F \cdot D = F \cdot U \cdot D' \simeq (F \cdot U)^* \cdot E_{Sind} \cdot D'$$

implies, since $(F \cdot U)^*$ preserves the sifted colimit of $E_{Sind} \cdot D'$, that

$$colim\,(F \cdot D) \simeq (F \cdot U^*)(colim\,(E_{Sind} \cdot D'))\,.$$

From [7.1] and the fact that U^* also preserves the sifted colimit of $E_{Sind} \cdot D'$ we derive

$$colim\,(F \cdot D) \simeq F(colim\,(U^* \cdot E_{Sind} \cdot D')) \simeq F(colim\,D)\,.$$

$\square$

7.8 Remark. We thus established proofs of the affirmative statements 2. and 3. of the introduction of this chapter. The statement 1. is easy: if $\mathcal{C}$ has filtered colimits, reflexive coequalizers, and finite coproducts, then it has all colimits (4.1).

With the following examples we demonstrate the negative parts of the statements 1.–3.

7.9 Example. Here we give an example of a category not having sifted colimits although it has (i) filtered colimits and (ii) reflexive coequalizers.
We start with the following category $\mathcal{D}$ given by the gluing of two reflexive pairs at their codomains. That is, $\mathcal{D}$ is given by the graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "a_1"] \arrow[r, shift right=2, "a_2"'] & B \arrow[l, "d" description] \arrow[r, "d'" description] & A' \arrow[l, shift right=2, "a_1'"'] \arrow[l, shift left=2, "a_2'"] \end{tikzcd}$$

and the equations making both parallel pairs reflexive:

$$a_i \cdot d = \mathrm{id}_B = a_i' \cdot d' \quad \text{for} \;\; i = 1, 2\,.$$

The proof that $\mathcal{D}$ is sifted is completely analogous to the proof of Example 3.2: we verify that colimits over $\mathcal{D}$ in *Set* commute with finite products. Assume that the above graph depicts sets A, B and A' and functions between them. Then a colimit can be described as the canonical function $c\colon B \to C = B/\sim$ where two elements $x, y \in B$ are equivalent iff they are connected by a zig-zag formed by a_1, a_2, a_1' and a_2'. Since the two pairs are reflexive, the length of the zig-zag can be arbitrarily prolonged. And the type can be chosen to be

$$\begin{tikzcd}[column sep=small] & z_1 \arrow[dl, "a_1"'] \arrow[dr, "a_2"] & & z_2 \arrow[dl, "a_2"'] \arrow[dr, "a_1"] & & z_3 \arrow[dl, "a_1'"'] \arrow[dr, "a_2'"] & & z_4 \arrow[dl, "a_2'"'] \arrow[dr, "a_1'"] & \cdots\cdots & z_{4k} \arrow[dr, "a_1'"] & \\ x & & {} & & {} & & {} & & {} & & y \end{tikzcd}$$

(here for the elements z_{4i+1} we use a_1, a_2, for z_{4i+2} we use a_2, a_1, for z_{4i+3} we use a_1', a_2' and for z_{4i} we use a_2', a_1'). From that it is easy to derive that $\mathcal{D}$ is sifted.
We now add to $\mathcal{D}$ the coequalizers c_1 of a_1, a_2 and c' of a_1', a_2': let $\mathcal{E}$ be the category given by the graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "a_1"] \arrow[r, shift right=2, "a_2"'] & B \arrow[l, "d" description] \arrow[r, "d'" description] \arrow[dl, "c'"] \arrow[dr, "c"'] & A' \arrow[l, shift right=2, "a_1'"'] \arrow[l, shift left=2, "a_2'"] \\ C' & & C \end{tikzcd}$$

with the previous equations plus the following ones

$$c \cdot a_1 = c \cdot a_2 \quad \text{and} \quad c' \cdot a'_1 = c' \cdot a'_2 \,.$$

The sifted diagram $\mathcal{D} \to \mathcal{E}$ which is the inclusion does not have a colimit. However, $\mathcal{E}$ has reflexive coequalizers because its only nontrivial reflexive pairs are a_1, a_2 (with coequalizer c) and a'_1, a'_2 (with coequalizer c'). Moreover, $\mathcal{E}$ has filtered colimits: since the category $\mathcal{E}$ is clearly finite, it does not have any nontrivial filtered diagram except those obtained by iterating an idempotent endomorphism e, see 2.3. Thus it is sufficient to verify that $\mathcal{E}$ has coequalizers of all pairs e, id_X where e is an idempotent endomorhism of X. In fact, the only idempotents of $\mathcal{E}$ are $d \cdot a_i$ and $d \cdot a'_i$. The coequalizer of $d \cdot a_1$ and id_A is clearly a_1 (because a morphism f with $f = f \cdot d \cdot a_1$ fulfils $f \neq a_2$ and then it uniquely factorizes through a_1), analogously for the other three idempotents. Thus, the above coequalizers demonstrate that $\mathcal{E}$ has filtered colimits.

7.10 Remark. For the category $\mathcal{D}$ of 7.9 we have

$$\mathit{Sind}\mathcal{D} \neq \mathit{Ind}(\mathit{Rec}\mathcal{D})\,.$$

Observe first that

$$\mathit{Rec}\mathcal{D} = \mathcal{E}$$

is obtained from $\mathcal{D}$ by freely adding the coequalizers of c and c'. The category $\mathit{Ind}\mathcal{E}$ does not have a terminal object. In fact, the full subcategory of all objects X in $\mathit{Ind}\mathcal{E}$ having a morphism from at most one of the objects C or C' into X is clearly closed under filtered colimits – thus every object X has that property. In contrast, $\mathit{Sind}\mathcal{D}$ has the terminal object $\mathit{colim}\,\mathcal{D}$.

7.11 Example. Here we give an example of a functor not preserving sifted colimits although it preserves (i) filtered colimits and (ii) reflexive coequalizers. Let $\mathcal{E}^T$ be the above category $\mathcal{E}$ with a terminal object T added. The sifted diagram $D\colon \mathcal{D} \to \mathcal{E}^T$ which is the inclusion has colimit

$$T = \mathit{colim}\, D$$

in $\mathcal{E}^T$. Let $\mathcal{A}$ be the category $\mathcal{E}^T$ with a new terminal objects S added. The functor

$$F\colon \mathcal{E}^T \to \mathcal{A} \quad \text{with} \quad FT = S \quad \text{and} \quad FX = X \quad \text{for all} \quad X \neq T$$

which is the identity map on morphisms of $\mathcal{E}$ does not preserve sifted colimits because

$$F(\mathit{colim}\, D) = S \quad \text{and} \quad \mathit{colim}\, F \cdot D = T\,.$$

However, F clearly preserves filtered colimits and reflexive coequalizers: the only nontrivial colimits of these types in $\mathcal{E}^T$ lie in $\mathcal{E}$ and are described in Example 7.9. The same description applies to $\mathcal{A}$.

Historical Remarks for Chapter 7

The open problem of [8] whether preservation of filtered colimits and reflexive coequalizers implies preservation of sifted colimits was answered by A. Joyal; his proof even works for quasicategories (see [57]). Another proof was given S. Lack (see [60]). The present proof is taken from [11] where also the stronger statements stated at the beginning of this chapter are proved.

The formula 7.4 stems from [8].

Chapter 8

Canonical theories

Every algebraic category has a number of algebraic theories which are often non-equivalent, we will study this more detailed in Chapter 15. In the present chapter we prove that there is always an essentially unique algebraic theory with split idempotents. We call it a canonical theory. We first discuss splitting of idempotents and idempotent completions.

8.1 Definition.

1. Given an idempotent morphism

$$f\colon X \to X\ , \quad f \cdot f = f$$

   in a category $\mathcal{C}$, by a *splitting* of f is meant a factorization $f = m \cdot e$ such that $e \cdot m$ is the identity morphism:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & X & Z \arrow[rr, "\mathrm{id}_Z"] \arrow[dr, "m"'] & & Z \\ & Z \arrow[ur, "m"'] & & & X \arrow[ur, "e"'] & \end{tikzcd}$$

2. A category $\mathcal{C}$ is called *idempotent-complete* provided that every idempotent in $\mathcal{C}$ has a splitting.

8.2 Remark.

1. A splitting of an idempotent f is unique up to isomorphism:

   (a) for every isomorphism $i\colon Z \to \bar{Z}$ the morphisms $\bar{e} = i \cdot e$ and $\bar{m} = m \cdot i^{-1}$ form a splitting of f, and

   (b) for every splitting $f = \bar{m} \cdot \bar{e}$, $\bar{e} \cdot \bar{m} = \mathrm{id}$, there exists a unique isomorphism i such that $i \cdot e = \bar{e}$ and $m \cdot i^{-1} = \bar{m}$ (just put $i = \bar{e} \cdot m$ and $i^{-1} = e \cdot \bar{m}$).

2. To be idempotent-complete is a self-dual notion: $\mathcal{C}$ is idempotent complete iff $\mathcal{C}^{op}$ is so.

8.3 Example.

1. Every category which has equalizers is idempotent-complete. In fact, form an equalizer m of the idempotent $f\colon X \to X$ and $\mathrm{id}\colon X \to X$

$$\begin{tikzcd} Z \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "\mathrm{id}"'] & X \end{tikzcd}$$

   Since $f \cdot f = \mathrm{id} \cdot f$, the morphism f factorizes as $f = m \cdot e$ for some $e\colon X \to Z$. Now, $m \cdot e \cdot m = f \cdot m = m$ and m is a monomorphism, so that $e \cdot m = \mathrm{id}$.

   Conversely, if an idempotent $f\colon X \to X$ splits as $f = m \cdot e$, then m is an equalizer of f and id_X.

2. Every category with coequalizers is also idempotent-complete. Conversely, if an idempotent $f\colon X \to X$ splits as $f = m \cdot e$, then e is a coequalizer of f and id_X. This is the dualization of 1.

3. A full subcategory of an idempotent-complete category $\mathcal{C}$ is idempotent-complete iff it is closed in $\mathcal{C}$ under retracts.

8.4 Definition. By an *idempotent completion of a category* $\mathcal{C}$ is meant a functor

$$E_{Ic} : \mathcal{C} \to Ic\,\mathcal{C}$$

such that

1. $Ic\,\mathcal{C}$ is idempotent-complete

and

2. for every functor $F\colon \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is an idempotent-complete category, there exists an essentially unique functor $F^*\colon Ic\,\mathcal{C} \to \mathcal{B}$ with F naturally isomorphic to $F^* \cdot E_{Ic}$.

8.5 Remark.

1. A category $\mathcal{C}$ is idempotent-complete iff the functor $E_{Ic} : \mathcal{C} \to Ic\,\mathcal{C}$ is an equivalence.

2. Clearly, $Ic\,(\mathcal{C}^{op}) \simeq (Ic\,\mathcal{C})^{op}$.

We give now an elementary description of $Ic\,\mathcal{C}$.

8.6 Definition. For every category $\mathcal{C}$ we denote by $Ic\,\mathcal{C}$ the category of idempotents: its objects are the idempotent morphisms of $\mathcal{C}$. Its morphisms from

$f\colon X \to X$ to $g\colon Z \to Z$ are the morphisms $a\colon X \to Z$ in $\mathcal{C}$ such that $a = g \cdot a \cdot f$ or, equivalently, such that the diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, "a"] & Z \arrow[d, "g"] \\ X \arrow[r, "a"'] & Z \end{tikzcd}$$

commutes. The identity of the object $f\colon X \to X$ is f itself, and composition is as in $\mathcal{C}$.

We have a full and faithful functor $E_{Ic}: \mathcal{C} \to Ic\mathcal{C}$ defined by $E_{Ic}(X) = \mathrm{id}_X$ and $E_{Ic}(a) = a$.

8.7 Proposition. *The functor $E_{Ic}: \mathcal{C} \to Ic\mathcal{C}$ defined in 8.6 is an idempotent completion of $\mathcal{C}$.*

Proof. 1. $Ic\mathcal{C}$ is idempotent-complete: let a be an idempotent endomorphism of $(f\colon X \to X)$ in $Ic\mathcal{C}$. A splitting of a is given by

$$(f\colon X \to X) \xrightarrow{a} (a\colon X \to X) \xrightarrow{a} (f\colon X \to X)$$

2. Consider a functor $F\colon \mathcal{C} \to \mathcal{B}$ with $\mathcal{B}$ idempotent-complete. Every object $(f\colon X \to X)$ of $Ic\mathcal{C}$ is obtained by splitting the idempotent $f\colon E_{Ic}(X) \to E_{Ic}(X)$. Thus, we are forced to define F^* on objects $f\colon X \to X$ as the (essentially unique) splitting of $Ff\colon FX \to FX$ in $\mathcal{B}$.

To define F^* on morphisms $a\colon (f\colon X \to X) \to (g\colon Z \to Z)$ in $Ic\mathcal{C}$. observe that a is the unique arrow making the following diagram

$$\begin{tikzcd} E_{Ic}(X) \arrow[rr, "E_{Ic}(f)"] \arrow[dr, "f"] \arrow[ddd, "E_{Ic}(a)"'] & & E_{Ic}(X) \arrow[ddd, "E_{Ic}(a)"] \\ & (f) \arrow[ur, "f"'] \arrow[d, "a"] & \\ & (g) \arrow[dr, "g"] & \\ E_{Ic}(Z) \arrow[ur, "g"] \arrow[rr, "E_{Ic}(g)"'] & & E_{Ic}(Z) \end{tikzcd}$$

commutative in $Ic\mathcal{C}$. Thus, we are forced to define $F^*(a)$ as the unique arrow making the following diagram

$$\begin{tikzcd}
FX \arrow[rr, "Ff"] \arrow[dr, "F^*f"] \arrow[ddd, "Fa"'] & & FX \arrow[ddd, "Fa"] \\
& F^*(f) \arrow[ur, "F^*f"] \arrow[d, "F^*a"] & \\
& F^*(g) \arrow[dr, "F^*g"] & \\
FZ \arrow[rr, "Fg"'] \arrow[ur, "F^*g"] & & FZ
\end{tikzcd}$$

commutative in $\mathcal{B}$. Explicitly:

$$F^*a : \; F^*(f) \xrightarrow{F^*f} FX \xrightarrow{Fa} FZ \xrightarrow{F^*g} F^*(g) \; .$$

It is easy to verify that this yields a well-defined functor F^* with $F^* \cdot I \simeq F$. ☐

8.8 Proposition. *For every small category $\mathcal{C}$ an idempotent completion of $\mathcal{C}^{op}$ is the codomain restriction of the Yoneda embedding $Y_{\mathcal{C}} \colon \mathcal{C}^{op} \to Set$ to the full subcategory of all absolutely presentable objects of $Set^{\mathcal{C}}$.*

Proof. Recall from 5.10 that a functor $\mathcal{C} \to Set$ is absolutely presentable iff it is a retract of a representable one. To prove that the full subcategory of all absolutely presentable objects of $Set^{\mathcal{C}}$ is equivalent to the category $Ic\,(\mathcal{C}^{op})$ of 8.6, consider two retracts

$$R \underset{m}{\overset{e}{\leftrightarrows}} Y_{\mathcal{C}}(X) \; , \; e \cdot m = \mathrm{id}_R \qquad \text{and} \qquad S \underset{n}{\overset{f}{\leftrightarrows}} Y_{\mathcal{C}}(Z) \; , \; f \cdot n = \mathrm{id}_S$$

and a morphism $g \colon R \to S$ in $Set^{\mathcal{C}}$. By the Yoneda Lemma, we get two idempotents $\hat{e} = m \cdot e(\mathrm{id}_X)$ and $\hat{f} = n \cdot f(\mathrm{id}_Z)$ in $\mathcal{C}$ and a morphism $\hat{g} = n \cdot g \cdot e(\mathrm{id}_X)$ forming a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "\hat{g}"] \arrow[d, "\hat{f}"'] \arrow[dr, "\hat{g}"] & X \arrow[d, "\hat{e}"] \\
Z \arrow[r, "\hat{g}"'] & X
\end{tikzcd}$$

This is a morphism in the category $Ic\,(\mathcal{C}^{op})$. The rest of the proof is straightforward. ☐

8.9 Corollary. *For every algebraic theory $\mathcal{T}$ the Yoneda embedding*

$$Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to (Alg\,\mathcal{T})_{pp}$$

is an idempotent completion of $\mathcal{T}^{op}$. In other words, $(Alg\,\mathcal{T})_{pp} \simeq Ic\,(\mathcal{T}^{op})$.

In fact, by 5.10 and 8.8, $Ic\,(\mathcal{T}^{op})$ is formed by retracts of representables in $Set^{\mathcal{T}}$, that is, by perfectly presentable objects in $Alg\,\mathcal{T}$ (see 5.14).

8.10 Corollary. *For two small categories $\mathcal{C}$ and $\mathcal{D}$ the corresponding functor categories $Set^{\mathcal{C}}$ and $Set^{\mathcal{D}}$ are equivalent if and only if $\mathcal{C}$ and $\mathcal{D}$ have a common idempotent completion:*

$$Set^{\mathcal{C}} \simeq Set^{\mathcal{D}} \quad \textit{iff} \quad Ic\,\mathcal{C} \simeq Ic\,\mathcal{D}\,.$$

Proof. The universal property of $E_{Ic} : \mathcal{C} \to Ic\,\mathcal{C}$ clearly implies $Set^{\mathcal{C}} \simeq Set^{Ic\,\mathcal{C}}$.
Conversely, if $Set^{\mathcal{C}} \simeq Set^{\mathcal{D}}$, then the subcategories of absolutely presentable objects are equivalent. By 5.10 and 8.8, this means that $Ic\,(\mathcal{C}^{op}) \simeq Ic\,(\mathcal{D}^{op})$. Now use duality, or 8.5.2. □

8.11 Definition. Let $\mathcal{A}$ be an algebraic category. An algebraic theory for $\mathcal{A}$ is called *canonical* if it is idempotent-complete.

8.12 Proposition. *Every algebraic category $\mathcal{A}$ has a canonical theory unique up to equivalence. The dual of $\mathcal{A}_{pp}$ (see 6.12) is a canonical theory for $\mathcal{A}$.*

Proof. Following 6.13, $\mathcal{A} \simeq Alg\,(\mathcal{A}_{pp}^{op})$. By 5.14 and 8.3, $\mathcal{A}_{pp}$ is idempotent-complete, and then $\mathcal{A}_{pp}^{op}$ is also idempotent-complete, see 8.2. Let us verify the uniqueness. If $\mathcal{A} \simeq Alg\,\mathcal{T}$ for some algebraic theory $\mathcal{T}$, then $\mathcal{A}_{pp} \simeq (Alg\,\mathcal{T})_{pp} \simeq Ic\,(\mathcal{T}^{op})$ by 6.14 and 8.9. Finally, $\mathcal{A}_{pp}^{op} \simeq Ic\,\mathcal{T}$ by 8.5. □

8.13 Example.

1. The canonical theory of the category *Set* is the theory $\mathcal{N}$ of natural numbers, see 1.9: it is clear that $\mathcal{N}$ is idempotent-complete.

2. The canonical theory of the category *Ab* is the theory $\mathcal{T}_{ab}$ described in 1.13. In fact, we saw in 5.6 that $\mathcal{T}_{ab}$ is dual to Ab_{pp}.

3. In the category *Bool* of boolean algebras we have the algebras $\mathcal{P}X$ of all subsets of a set X. The free algebras on n generators $\mathcal{P}\mathcal{P}n$ form, for $n \in \mathbb{N}$, a strong generator. As noted in 6.10, the dual of this full subcategory of *Bool* is a theory for *Bool*. However, this is not the canonical theory. In fact, the canonical theory is the dual of the full subcategory of all algebras $\mathcal{P}n$ for $n \in \mathbb{N} \setminus \{0\}$, or equivalently, the category of finite nonempty sets and functions. Since each $n > 0$ is injective in the category of finite sets, it is a retract of 2^n. Thus $\mathcal{P}n$ is a retract of $\mathcal{P}\mathcal{P}n$.

Historical Remarks for Chapter 8

The idempotent completion can already be found in B. Mitchell's monograph [75]. M. Bunge [30] presented it as well and used it for proving Corollary 8.10. She called it idempotent splitting closure; other names were used as well, e.g. Cauchy completion or Karoubi envelope. Corollary 8.10 was later proved in [45]. Proposition 8.12 is due to J. J. Dukarm [40].

Chapter 9

Algebraic functors

We have studied algebraic categories as individual categories so far. It turns out that there is a natural concept of morphism between algebraic categories, which we call an algebraic functor, so that we obtain a 2-category of all algebraic categories. We then prove a duality result: this 2-category is biequivalent to the 2-category of algebraic theories. We first need to introduce a concept of morphism between algebraic categories – this is quite obvious:

9.1 Definition. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be algebraic theories. A functor $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ is called a *morphism of algebraic theories* if it preserves finite products.

9.2 Notation. For a morphism of theories $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ we denote by

$$Alg\, M\colon Alg\, \mathcal{T}_2 \to Alg\, \mathcal{T}_1$$

the functor defined on objects $A\colon \mathcal{T}_2 \to Set$ by $A \mapsto A \cdot M$.

9.3 Proposition. *Let $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a morphism of algebraic theories.*

1. *$Alg\, M\colon Alg\, \mathcal{T}_2 \to Alg\, \mathcal{T}_1$ preserves limits and sifted colimits.*
2. *$Alg\, M$ has a left adjoint $M^*\colon Alg\, \mathcal{T}_1 \to Alg\, \mathcal{T}_2$ which is the essentially unique functor which preserves sifted colimits and makes the square*

$$\begin{tikzcd}
\mathcal{T}_1^{op} \arrow[r, "Y_{\mathcal{T}_1}"] \arrow[d, "M^{op}"'] & Alg\, \mathcal{T}_1 \arrow[d, "M^*"] \\
\mathcal{T}_2^{op} \arrow[r, "Y_{\mathcal{T}_2}"'] & Alg\, \mathcal{T}_2
\end{tikzcd}$$

commutative up to natural isomorphism.

Proof. The essentially unique functor M^* follows from Theorem 4.13. Since $Y_{\mathcal{T}_2} \cdot M^{op}$ preserves finite coproducts, see Lemma 1.5, to get the adjunction $M^* \dashv Alg\, M$ it suffices to apply 4.15: the right adjoint $Alg\, M$ is given by

$$B \mapsto Alg\, \mathcal{T}_2(Y_{\mathcal{T}_2}(M-), B).$$

By the Yoneda Lemma, this is nothing but composition with M. This immediately implies that $Alg\, M$ preserves sifted colimits, because they are calculated objectwise in $Alg\,\mathcal{T}_1$ and $Alg\,\mathcal{T}_2$ (see 2.5). $\square$

9.4 Definition. A functor between two algebraic categories is called *algebraic* provided that it preserves limits and sifted colimits.

9.5 Example.

1. Every functor $Alg\, M$, for a theory morphism M, is algebraic.
2. The forgetful functor $Ab \to Set$ is algebraic.
3. Given an algebra A in an algebraic category $\mathcal{A}$, then $\mathcal{A}(A, -)\colon \mathcal{A} \to Set$ is algebraic iff A is perfectly presentable.
4. A constant functor with value A between algebraic categories is algebraic iff A is a terminal object.
5. For every algebraic theory $\mathcal{T}$, the embedding $I\colon Alg\,\mathcal{T} \to Set^{\mathcal{T}}$ is an algebraic functor, see 1.16 and 2.5.

9.6 Remark. We know from 9.3 that every morphism of theories induces an algebraic functor between the corresponding algebraic categories. If, moreover, the algebraic theories are canonical (8.11), then the algebraic functors are essentially just those induced by morphisms of theories, see 9.15 below. This will motivate us to define "morphisms of algebraic categories" as the algebraic functors. We are now going to prove that every algebraic functor has a left adjoint. For this we will use Freyd's Adjoint Functor Theorem (see 0.8).

9.7 Theorem. *A functor between algebraic categories is algebraic if and only if it has a left adjoint and preserves sifted colimits.*

Proof. Let $G\colon \mathcal{B} \to \mathcal{A}$ be an algebraic functor. We are to prove that G has a left adjoint. That is, for every object A of $\mathcal{A}$ we are to prove that the functor

$$\mathcal{A}(A, G-)\colon \mathcal{B} \to Set$$

is representable.

(1) Assume first that A is perfectly presentable. Since G preserves limits, it remains to prove that $\mathcal{A}(A, G-)$ satisfies the Solution Set Condition of 0.10. For $\mathcal{B}_{pp}$ in 6.12 put

$$\mathcal{G} = \{GX \mid X \in \mathcal{B}_{pp}\}\,.$$

Every object B of $\mathcal{B}$ is a sifted colimit of objects from $\mathcal{B}_{pp}$ (see 6.9). Let us write $(\sigma_X\colon X \to B)$ for the colimit cocone. Since G preserves sifted colimits, $(G\sigma_X\colon GX \to GB)$ is also a colimit cocone. As $\mathcal{A}(A, -)$ preserves sifted colimits, every morphism $b\colon A \to GB$ factorizes as follows

$$\begin{tikzcd} & GX \arrow[d, "G\sigma_X"] \\ A \arrow[ur, "\varphi"] \arrow[r, "b"'] & GB \end{tikzcd}$$

Thus all $G(\sigma_X)$ form a solution set.
(2) If A is an arbitrary object of $\mathcal{A}$, we express it as a sifted colimit of perfectly presentable objects (see 6.9), say $A = \mathit{colim}\, A_i$. From (1) we know that $\mathcal{A}(A_i, G-)$ is representable, say $\mathcal{A}(A_i, G-) \simeq \mathcal{B}(B_i, -)$. Yoneda Lemma now allows us to define an (obvious) sifted diagram whose objects are the B_i. Therefore, $\mathcal{A}(A_i, G-)$ is representable by $\mathit{colim}\, B_i$ since

$$\mathcal{A}(A, G-) = \mathcal{A}(\mathit{colim}\, A_i, G-) \simeq \mathit{lim}\, \mathcal{A}(A_i, G-) \simeq \mathit{lim}\, \mathcal{B}(B_i, -) \simeq \mathcal{B}(\mathit{colim}\, B_i, -).$$

$\square$

9.8 Remark. The previous theorem can be refined, as we demonstrate in Chapter 18: a functor between algebraic categories is algebraic if and only if it preserves limits, filtered colimits and regular epimorphisms. One implication follows from the fact that: a) filtered implies sifted, and b) every regular epimorphism is a reflexive coequalizer (of its kernel pair). The converse implication is a particular case of 18.2.

9.9 Remark. We are going to prove a duality between algebraic categories and canonical algebraic theories. This does not really mean a contravariant equivalence of categories. Indeed, a more subtle formulation is needed: just look at the simplest algebraic category, *Set*, and the simplest endomorphism, the identity functor $\mathrm{Id}_{\mathit{Set}}$. It is easy to find a proper class of functors naturally isomorphic to $\mathrm{Id}_{\mathit{Set}}$ – and each of them is algebraic. However, in the category of all theories no such phenomenon occurs. We thus need to work with morphisms of algebraic categories "up to natural isomorphism". For this reason we have to move from categories to 2-categories. The reader does not need to know much about 2-categories. Here we summerize the needed facts.

9.10 A primer on 2-categories.

1. Let us recall that a 2-category $\mathbb{A}$ has a class $\mathit{obj}\,\mathbb{A}$ of objects and, instead of hom-sets $\mathbb{A}(A, B)$, it has hom-categories $\mathbb{A}(A, B)$ for every pairs A, B of objects. The objects of $\mathbb{A}(A, B)$ are called *1-cells* and the morphisms *2-cells*. A prototype of a 2-category is the 2-category *Cat* of all small categories (as objects). Given small categories A, B then 1-cells are functors from A to B and 2-cells are natural transformations. In our book, we essentially work just with this 2-category and its full sub-2-categories.

2. Let us recall the concept of a 2-functor $F\colon \mathbb{A} \to \mathbb{B}$ between 2-categories $\mathbb{A}$ and $\mathbb{B}$: it assigns objects FA of $\mathbb{B}$ to objects A of $\mathbb{A}$; for every pair A, A' of objects of $\mathbb{A}$, it defines a functor $F_{A,A'}\colon \mathbb{A}(A, A') \to \mathbb{B}(FA, FA')$ and fulfils some canonical requirements about compositions and identities. Example: a *full sub-2-category* $\mathbb{A}$ of a 2-category $\mathbb{B}$ is given by a choice of a class of objects $\mathit{obj}\,\mathbb{A} \subseteq \mathit{obj}\,\mathbb{B}$. The 1-cells and 2-cells in $\mathbb{A}$ are precisely those of $\mathbb{B}$. The inclusion $I\colon \mathbb{A} \to \mathbb{B}$ is a 2-functor.

3. Two objects A, B of a 2-category $\mathbb{A}$ are called *equivalent* if there exist 1-cells $f\colon A \to B$ and $\overline{f}\colon B \to A$ such that $\overline{f} \cdot f$ is isomorphic to id_A in $\mathbb{A}(A, A)$ and $f \cdot \overline{f}$ is isomorphic to id_B in $\mathbb{A}(B, B)$.

4. A 2-functor $F\colon \mathbb{A} \to \mathbb{B}$ is called a *biequivalence* if all the functors $F_{A,A'}$ are equivalence functors, and every object of $\mathbb{B}$ is equivalent to FA for some object A of $\mathbb{A}$.

5. For every 2-category $\mathbb{A}$ we denote by $\mathbb{A}^{op}$ the *dual 2-category*: it has the same objects and the direction of 1-cells is reversed (while the direction of 2-cells remains non-reversed): $\mathbb{A}^{op}(A, A') = \mathbb{A}(A', A)$.

9.11 Example.

1. We define the *2-category Th of theories* to have

   objects: algebraic theories,

   1-cells: morphisms of algebraic theories (i.e., functors preserving finite products),

   2-cells: natural transformations.

   This is a full sub-2-category of *Cat*, i.e., composition of 1-cells and 2-cells are defined in *Th* as the usual composition of functors and natural transformations, respectively.

2. The *2-category* Th_c *of canonical theories* is the full sub-2-category of *Th* on all theories that are canonical, that is, idempotent-complete.

3. We define the *2-category ALG of algebraic categories* to have

   objects: algebraic categories,

   1-cells: algebraic functors,

   2-cells: natural transformations.

   Once again, this is a full sub-2-category of *Cat* : composition is the usual composition of functors or natural transformations.

9.12 Remark. We need to be a little careful about foundations here: there is, as remarked above, a proper class of 1-cells in $ALG(\mathit{Set}, \mathit{Set})$, for example. However, if we consider the 1-cells up to natural isomorphism, all problems disappear: this is one consequence of the duality theorem below. Ignoring the foundational considerations, we consider ALG as a sub-2-category of the 2-category of all categories. (The duality we prove below tells us that ALG is essentially just the dual of Th_c.)

9.13 Definition. We denote by

$$\mathit{Alg}\colon \mathit{Th}^{op} \to ALG$$

the 2-functor assigning to every algebraic theory $\mathcal{T}$ the category $\mathit{Alg}\,\mathcal{T}$, to every 1-cell $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ the functor $\mathit{Alg}\,M = (-) \cdot M$ and to every 2-cell $\alpha\colon M \to N$ the natural transformation $\mathit{Alg}\,\alpha\colon \mathit{Alg}\,M \to \mathit{Alg}\,N$ whose component at a $\mathcal{T}_2$-algebra A is $A \cdot \alpha\colon A \cdot M \to A \cdot N$.

9.14 Remark. The 2-functor Alg is well-defined due to 9.3: for every morphism of theories M, the functor $Alg\,M$ is algebraic. The fact that for every natural transformation α we get a natural transformation $Alg\,\alpha$ is easy to verify.

9.15 Theorem. *(Duality of algebraic categories and theories) The 2-category ALG of algebraic categories is biequivalent to the dual of the 2-category Th_c of canonical algebraic theories.*
In fact, the domain restriction of the 2-functor Alg to canonical algebraic theories

$$Alg\colon\, Th_c^{op} \to ALG$$

is a biequivalence.

Proof. (1) Following 8.12, every algebraic category $\mathcal{A}$ is equivalent to $Alg\,\mathcal{T}$ for the canonical algebraic theory $\mathcal{T} = \mathcal{A}_{pp}^{op}$.
(2) We will prove that for two canonical algebraic theories $\mathcal{T}_1$ and $\mathcal{T}_2$ the functor

$$Alg_{\mathcal{T}_1,\mathcal{T}_2}\colon\, Th_c(\mathcal{T}_1,\mathcal{T}_2) \to ALG\,(Alg\,\mathcal{T}_2, Alg\,\mathcal{T}_1)$$

is an equivalence of categories.
(2a) $Alg_{\mathcal{T}_1,\mathcal{T}_2}$ is full and faithful: given morphisms $M, N\colon \mathcal{T}_1 \rightrightarrows \mathcal{T}_2$ and a natural transformation $\lambda\colon Alg\,M \to Alg\,N$ there exists a unique natural transformation $\alpha\colon M \to N$ such that $Alg\,\alpha = \lambda$. The proof follows the lines of the proof of Yoneda Lemma. Let us just indicate how to construct α. Consider an object X in $\mathcal{T}_1$. Since $\mathcal{T}_2(MX,-) \in Alg\,\mathcal{T}_2$, we have the component $\lambda_{\mathcal{T}_2(MX,-)}(X)\colon \mathcal{T}_2(MX,MX) \to \mathcal{T}_2(MX,NX)$ and we put

$$\alpha_X = \lambda_{\mathcal{T}_2(MX,-)}(X)(\mathrm{id}_{MX})\colon MX \to NX.$$

The family $(\alpha_X)_{X\in\mathcal{T}_1}$ is the required natural transformation $\alpha\colon M \to N$.
(2b) The functor $Alg_{\mathcal{T}_1,\mathcal{T}_2}$ is essentially surjective. In fact, consider an algebraic functor $G\colon Alg\,\mathcal{T}_2 \to Alg\,\mathcal{T}_1$ and let L be its left adjoint, see 9.7. We are going to prove that L restricts to a functor F which preserves finite coproducts and makes the square

$$\begin{tikzcd}
\mathcal{T}_1^{op} \arrow[r, "Y_{\mathcal{T}_1}"] \arrow[d, "F"'] & Alg\,\mathcal{T}_1 \arrow[d, "L"] \\
\mathcal{T}_2^{op} \arrow[r, "Y_{\mathcal{T}_2}"'] & Alg\,\mathcal{T}_2
\end{tikzcd}$$

commutative up to natural isomorphism. For every object X in $\mathcal{T}_1$ we have by adjunction a natural isomorphism

$$Alg\,\mathcal{T}_2(L(Y_{\mathcal{T}_1}(X)),-) \simeq Alg\,\mathcal{T}_1(Y_{\mathcal{T}_1}(X), G-)\,.$$

Since $Y_{\mathcal{T}_1}(X)$ is perfectly presentable (see 5.5) and G preserves sifted colimits, the above natural isomorphism implies that $L(Y_{\mathcal{T}_1}(X))$ is perfectly presentable. By 5.14, $L(Y_{\mathcal{T}_1}(X))$ is a retract of a representable algebra and, since $\mathcal{T}_2$ is idempotent-complete, $L(Y_{\mathcal{T}_1}(X))$ is itself a representable algebra (see 8.3.3). This means that there exists an essentially unique object in $\mathcal{T}_2$, say FX, such that $L(Y_{\mathcal{T}_1}(X)) \simeq Y_{\mathcal{T}_2}(FX)$. In this way we get a map on objects, $F\colon obj\,\mathcal{T}_1 \to obj\,\mathcal{T}_2$, which, by the Yoneda Lemma, extends to a functor $F\colon \mathcal{T}_1^{op} \to \mathcal{T}_2^{op}$ making the above square commutative up to natural isomorphism. F preserves finite coproducts because $Y_{\mathcal{T}_1}$ preserves them by 1.5, $Y_{\mathcal{T}_2}$ reflects finite coproducts, and L preserves them. It remains to prove that $G \simeq Alg\,M$, where $M = F^{op}\colon \mathcal{T}_1 \to \mathcal{T}_2$. Or, equivalently, that $L \simeq M^*$ in the notation of 9.3. But this follows from the essential commutativity of the above square and the last part of 9.3. This proves that $Alg_{\mathcal{T}_1,\mathcal{T}_2}$ is essentially surjective. $\square$

9.16 Corollary. *A functor between algebraic categories*

$$G\colon \mathcal{A}_2 \to \mathcal{A}_1$$

is algebraic if and only if it is induced by a morphism of theories. That is: there exists a morphism of the corresponding canonical algebraic theories $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ *and two equivalence functors* $E_1\colon Alg\,\mathcal{T}_1 \to \mathcal{A}_1$ *and* $E_2\colon Alg\,\mathcal{T}_2 \to \mathcal{A}_2$ *such that the square*

$$\begin{tikzcd} Alg\,\mathcal{T}_2 \arrow[r, "Alg\,M"] \arrow[d, "E_2"'] & Alg\,\mathcal{T}_1 \arrow[d, "E_1"] \\ \mathcal{A}_2 \arrow[r, "G"'] & \mathcal{A}_1 \end{tikzcd}$$

commutes up to natural isomorphism.

In fact, given G, the proof of 9.15 yields $G \simeq Alg\,M$ with the desired property. The converse implication is clear.

9.17 Remark. The category *Set* is a kind of *dualizing object* for the biequivalence $Alg\colon Th_c^{op} \to ALG$:
(1) Forgetting size considerations (an algebraic theory is by definition a small category) we have $Alg\,\mathcal{T} = Th(\mathcal{T}, Set)$ for every algebraic theory $\mathcal{T}$.
(2) For every algebraic category $\mathcal{A}$ there is an equivalence of categories

$$\mathcal{A}_{pp}^{op} \simeq ALG(\mathcal{A}, Set)\,.$$

Indeed, if $A \in \mathcal{A}_{pp}^{op}$, then $G = \mathcal{A}(A, -)\colon \mathcal{A} \to Set$ preserves limits and sifted colimits. Conversely, let $G\colon \mathcal{A} \to Set$ be an algebraic functor and let L be a left adjoint of G (see 9.7). Then $G \simeq \mathcal{A}(L1, -)$ (with 1 denoting a one-element set) and $L1$ is perfectly presentable because G preserves sifted colimits.

9.18 Remark. We conclude this chapter by mentioning the analogous Gabriel-Ulmer duality for locally finitely presentable categories. The proof is similar to that of 9.15. Whereas the morphisms of algebraic categories are the functors preserving limits and sifted colimits, the morphisms of locally finitely presentable categories are the functors preserving limits and filtered colimits. These are the

1-cells of the 2-category LFP, and the 2-cells are natural transformations. We also denote by LEX the 2-category of small categories with finite limits, finite limit preserving functors, and natural transformations. The 2-functor

$$Lex \colon LEX^{op} \to LFP$$

assigns to every small category $\mathcal{T}$ with finite limits the category $Lex\mathcal{T}$ (4.16) and it acts on 1-cells and 2-cells in the analogous way as $Alg \colon Th^{op} \to ALG$.

9.19 Theorem. *The 2-categories LFP and LEX are dually biequivalent.*

In fact, $Lex \colon LEX^{op} \to LFP$ is a biequivalence. The converse construction associates to a locally finitely presentable category $\mathcal{A}$ the small, finitely complete category $\mathcal{A}_{fp}^{op}$ (6.26).

Historical Remarks for Chapter 9

Morphisms of algebraic theories and the resulting algebraic functors were (in the one-sorted case) introduced by F. W. Lawvere [63] and belong to the main contribution of his work. The characterization 9.7 of algebraic functors and the duality theorem 9.15 are contained in [5]. This is analogous to the Gabriel-Ulmer duality 9.19 for locally finitely presentable categories (see [48]). A general result can be found in [35].

2-categories have been introduced by C. Ehresmann in [42].

Chapter 10

Birkhoff's Variety Theorem

So far we have not treated one of the central concepts of algebra: equations. In the present chapter we prove the famous characterization of varieties of algebras, due to G. Birkhoff: varieties are precisely the full subcategories of $Alg\,\mathcal{T}$ closed under

products,

subalgebras,

regular quotients,

and

directed unions.

The last item was not included in Birkhoff's formulation. The reason is that Birkhoff only considered one-sorted algebras, and for them directed unions follow from the other three items (see 11.35). For general algebraic categories directed unions cannot be omitted, see Example 10.23 below.

Classically, an equation is an expression $u = v$ where u and v are terms (say, in n variables). We will see in 13.9 that such terms are morphisms from n to 1 in the theory of Σ-algebras. We can also consider k-tuples of classical equations as pairs of morphisms from n to k. The following concept generalizes this idea.

10.1 Definition. If $\mathcal{T}$ is an algebraic theory, an *equation* in $\mathcal{T}$ is a parallel pair $u, v\colon s \rightrightarrows t$ of morphisms in $\mathcal{T}$. (Following algebraic tradition, we write $u = v$ in place of (u, v).) An algebra $A\colon \mathcal{T} \to Set$ *satisfies* the equation $u = v$ if $A(u) = A(v)$.

10.2 Example.

1. In the theory $\mathcal{T}_{ab}$ of abelian groups (1.11) recall that endomorphisms of 1 have the form $[n]$ and correspond to the operations on abelian groups given by $x \mapsto n \cdot x$. Thus, the equation

$$[2] = [0]$$

is satisfied by precisely the groups with

$$x + x = 0$$

for all elements x.

2. Graphs whose only edges are loops are given, considering the theory in 1.15, by the equation
$$\tau = \sigma\,.$$

10.3 Remark. Observe that if an algebra A satisfies the equation $u = v$, then it also satisfies all the equations of the form $u \cdot x = v \cdot x$ and $y \cdot u = y \cdot v$ for $x\colon s' \to t$ and $y\colon t \to t'$ in $\mathcal{T}$. Moreover, if the equations $u_i = v_i$, $i = 1, \ldots, n$, are satisfied by A for $u_i, v_i\colon s \to t_i$, then A also satisfies $\langle u_i \rangle = \langle v_i \rangle$, where

$$\langle u_i \rangle, \langle v_i \rangle\colon s \to t_1 \times \ldots \times t_n$$

are the correspnding morphisms. For this reason, we will state the definition of variety using congruences as well as equations.

10.4 Definition. Let $\mathcal{T}$ be an algebraic theory. A *congruence* on $\mathcal{T}$ is a collection $\sim$ of equivalence relations $\sim_{s,t}$ on hom-sets $\mathcal{T}(s,t)$, where (s,t) ranges over pairs of objecs, which is stable under composition and finite products in the following sense:

1. If $u \sim_{s,t} v$ and $x \sim_{r,s} y$, then $u \cdot x \sim_{r,t} v \cdot y$

$$\begin{tikzcd} r \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & s \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & t \end{tikzcd}$$

2. If $u_i \sim_{s,t_i} v_i$ for $i = 1, \ldots, n$, then $\langle u_1, \ldots, u_n \rangle \sim_{s,t} \langle v_1, \ldots, v_n \rangle$, where $t = t_1 \times \ldots \times t_n$

$$\begin{tikzcd} s \arrow[rr, shift left, "u_i"] \arrow[rr, shift right, "v_i"'] \arrow[d, shift right, "\langle v_1,\ldots,v_n \rangle"'] \arrow[d, shift left, "\langle u_1,\ldots,u_n \rangle"] & & t_i \\ t_1 \times \ldots \times t_n \arrow[urr] & & \end{tikzcd}$$

10.5 Example. Consider the theory $\mathcal{T}_{mon}$ of monoids (3.10) whose morphisms from n to k are all k-tuples of words in $n = \{0, 1, \ldots, n-1\}$. The commutative law corresponds to the congruence $\sim$ with $\sim_{n,1}$ defined for words v and w by

$w \sim_{n,1} v$ iff v can be obtained from w by a permutation of its letters.

The general equivalence $\sim_{n,k}$ is defined by coordinates of the k-tuples of words.

10.6 Example. Let $\mathcal{T}$ be an algebraic theory and let $M: \mathcal{T} \to \mathcal{B}$ be a finite product preserving functor. The *kernel congruence* of M is denoted by $\approx_M$. That is, for all $u, v \in \mathcal{T}(s,t)$ we put

$$u \approx_M v \ \text{ iff } \ Mu = Mv$$

This is obviously a congruence on $\mathcal{T}$. In particular, for every $\mathcal{T}$-algebra A we have a congruence $\approx_A$.

10.7 Remark.

1. Congruences on a given algebraic theory $\mathcal{T}$ are ordered in a canonical way: we write $\sim \subseteq \sim'$ in case that for every pair s, t of objects of $\mathcal{T}$ we have:

$$u \sim_{s,t} v \ \text{ implies } \ u \sim'_{s,t} v$$

2. It is easy to see that every (set theoretical) intersection of congruences is a congruence. Consequently, for every set E of equations there exists the smallest congruence $\sim_E$ on $\mathcal{T}$ containing E. We say that the congruence $\sim_E$ is *generated* by the equations of E.

10.8 Definition. A full subcategory $\mathcal{A}$ of $Alg\,\mathcal{T}$ is called a *variety* if there exists a set of equations such that a $\mathcal{T}$-algebra lies in $\mathcal{A}$ iff it satisfies all equations in that set.

10.9 Remark.

1. Varieties are also sometimes called equational classes of algebras, or equational categories. But we reserve this name for the special case of varieties of Σ-algebras treated in Chapters 13 (for one-sorted signatures) and 14 (for S-sorted ones).

2. We will use the name variety also in the loser sense of a category *equivalent* to a full subcategory of $Alg\,\mathcal{T}$ specified by equations. Every time we use the word variety it will be clear whether the above definition is meant or the loser version.

10.10 Example.

1. The abelian groups satisfying $x + x = 0$ form a variety in Ab.

2. All graphs whose edges are just loops form a variety in *Graph*.

3. Let us consider the category *Set* × *Set* of two-sorted sets with sorts called, say, s and t. This has an algebraic theory $\mathcal{T}_\mathcal{C}$ of all words in $\{s, t\}$, see 1.10. Consider the full subcategory $\mathcal{A}$ of *Set* × *Set* of all pairs $A = (A_s, A_t)$ with either $A_s = \emptyset$ or A_t has at most one element. This can be specified by the equation given by the parallel pair of projections

$$s\,t\,t \rightrightarrows t$$

10.11 Remark. Every variety is also specified by a congruence. More precisely, given a variety $\mathcal{A}$ of $Alg\,\mathcal{T}$, let $\sim$ be the congruence generated by the given set E of equations. A $\mathcal{T}$-algebra A lies in $\mathcal{A}$ iff it satisfies all equations in $\sim$, that is, iff it fulfils:

$$u \sim v \text{ implies } Au = Av\,.$$

In fact, if A satisfies all equations in E, then $E \subseteq \approx_A$ and therefore $\sim \subseteq \approx_A$.

10.12 Notation. For every congruence $\sim$ on an algebraic theory $\mathcal{T}$ we denote by

$$\mathcal{T}/\sim$$

the algebraic theory on the same objects and with morphisms given by the congruence classes of morphisms of $\mathcal{T}$:

$$(\mathcal{T}/\sim)(s,t) = \mathcal{T}(s,t)/\sim_{s,t} \ .$$

Composition and identity morphisms are inherited from $\mathcal{T}$; more precisely, they are determined by the fact that we have a functor

$$Q\colon \mathcal{T} \to \mathcal{T}/\sim$$

which is the identity map on objects and which assigns to every morphism its congruence class.

10.13 Remark.

1. It is easy to verify that $\mathcal{T}/\sim$ has finite products determined by those of $\mathcal{T}$, thus, $\mathcal{T}/\sim$ is an algebraic theory and Q is a theory morphism. Moreover, the functor Q is full and surjective on objects.

2. A morphism of theories $M\colon \mathcal{T} \to \mathcal{T}'$ factorizes through Q up to natural isomorphism

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "Q"] \arrow[dr, "M"'] & \arrow[d, phantom, "\simeq"] & \mathcal{T}/\sim \arrow[dl, "M'"] \\ & \mathcal{T}' & \end{tikzcd}$$

iff the congruence $\sim$ is contained in the kernel congruence $\approx_M$. When this is the case, the factorization M' is essentially unique and it is a theory morphism. In fact, it is clear that if $M' \cdot Q \simeq M$, then $\sim \subseteq \approx_M$. For the converse define M' to be equal to M on objects, and put $M'[f] = Mf$ on morphisms. Clearly, $M' \cdot Q = M$. From the fact that M preserves finite products and Q reflects them the desired properties of M' easily follow.

3. If in 2. we take $\sim$ equal to the congruence $\approx_M$, then the factorization M'

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "Q"] \arrow[dr, "M"'] & & \mathcal{T}/\approx_M \arrow[dl, "M'"] \\ & \mathcal{T}' & \end{tikzcd}$$

is faithful. Therefore, M is full and essentially surjective iff M' is an equivalence of categories.

10.14 Proposition. *Let $\sim$ be a congruence on an algebraic theory $\mathcal{T}$ and let $Q\colon \mathcal{T} \to \mathcal{T}/\sim$ be the corresponding quotient. The functor*

$$\mathit{Alg}\,Q\colon \mathit{Alg}\,(\mathcal{T}/\sim) \to \mathit{Alg}\,\mathcal{T}$$

is full and faithful.

Proof.
(1) Clearly, $\mathit{Alg}\,Q$ is faithful because Q is surjective.
(2) Consider objects $A, B \in \mathit{Alg}\,\mathcal{T}/\sim$ and a morphism $\beta\colon A \cdot Q \to B \cdot Q$, that is, a collection $\beta_t\colon At \to Bt$ of homomorphisms natural in t ranging through $\mathcal{T}$. Then the same collection is natural in t ranging through $\mathcal{T}/\sim$, thus, $\beta\colon A \to B$ is a morphism of $\mathit{Alg}\,(\mathcal{T}/\sim)$. Clearly, $\mathit{Alg}\,Q$ takes this morphism to the original one. ☐

10.15 Corollary. *Every variety is an algebraic category.*

More detailed: let $\sim$ be a congruence on an algebraic theory $\mathcal{T}$, and let $\mathcal{A}$ be the full subcategory of $\mathit{Alg}\,\mathcal{T}$ specified by $\sim$. There exists an isomorphism of categories

$$E\colon \mathit{Alg}\,(\mathcal{T}/\sim) \to \mathcal{A}$$

such that the triangle

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, hook] & \\
\mathit{Alg}\,(\mathcal{T}/\sim) \arrow[u, "E"] \arrow[r, "\mathit{Alg}\,Q"'] & \mathit{Alg}\,\mathcal{T}
\end{tikzcd}$$

commutes. In fact, for every $(\mathcal{T}/\sim)$-algebra B, given $u \sim v$, then $B(Qu) = B[u] = B[v] = B(Qv)$. This implies that $\mathit{Alg}\,Q$ factorizes through the inclusion of $\mathcal{A}$ in $\mathit{Alg}\,\mathcal{T}$. Moreover, if A lies in $\mathcal{A}$, then $A = (\mathit{Alg}\,Q)(B)$ where B is the $(\mathcal{T}/\sim)$-algebra defined by $B[u] = Au$. This shows that the factorization $E\colon \mathit{Alg}\,(\mathcal{T}/\sim) \to \mathcal{A}$ is bijective on objects. Since $\mathit{Alg}\,Q$ is full and faithful (see 10.14), E is an isomorphism.

10.16 Proposition. *Every variety $\mathcal{A}$ of $\mathcal{T}$-algebras is closed in $\mathit{Alg}\,\mathcal{T}$ under*

(a) *products: given a product $B = \prod_{i \in I} A_i$ in $\mathit{Alg}\,\mathcal{T}$ with all A_i in $\mathcal{A}$, then B also lies in $\mathcal{A}$,*

(b) *subalgebras: given a monomorphism $m\colon B \to A$ in $\mathit{Alg}\,\mathcal{T}$ with A in $\mathcal{A}$, then B also lies in $\mathcal{A}$,*

(c) *regular quotients: given a regular epimorphism $e\colon A \to B$ in $\mathit{Alg}\,\mathcal{T}$ with A in $\mathcal{A}$, then B also lies in $\mathcal{A}$,*

and

(d) *sifted colimits: given a sifted colimit* $B = colim\, A_i$ *in* $Alg\,\mathcal{T}$ *with all* A_i *in* $\mathcal{A}$, *then* B *also lies in* $\mathcal{A}$.

Proof. Following 10.15, the inclusion functor $\mathcal{A} \to Alg\,\mathcal{T}$ is naturally isomorphic to $Alg\,Q$, which preserves limits and sifted colimits by 9.3. This proves (a) and (d).
(b) Let $m\colon B \to A$ be a monomorphism with A in $\mathcal{A}$. We prove that B is in $\mathcal{A}$ by verifying that every equation $u_1, u_2\colon s \rightrightarrows t$ that A satisfies is also satisfied by B. We know from 1.18.2 that the component $m_t\colon Bt \to At$ is a monomorphism. From $A(u_1) = A(u_2)$ and the commutativity of the squares

$$\begin{tikzcd} Bs \arrow[r, "Bu_i"] \arrow[d, "m_s"'] & Bt \arrow[d, "m_t"] \\ As \arrow[r, "Au_i"'] & At \end{tikzcd}$$

we conclude $Bu_1 = Bu_2$.
(c) Let $e\colon A \to B$ be a regular epimorphism with A in $\mathcal{A}$. To prove that B lies in $\mathcal{A}$, observe that a kernel pair $k_1, k_2\colon N(e) \rightrightarrows A$ of e yields a subobject of $A \times A \in \mathcal{A}$, thus $N(e)$ lies in $\mathcal{A}$. by (a) and (b). And since the pair k_1, k_2 is reflexive and e is its coequalizer, B is a sifted colimit (3.2) of a diagram in $\mathcal{A}$, thus $B \in \mathcal{A}$. □

10.17 Corollary. *Every variety* $\mathcal{A}$ *of* $\mathcal{T}$*-algebras is closed in* $Alg\,\mathcal{T}$ *under limits and sifted colimits.*

10.18 Example. Not every full subcategory of an algebraic category $Alg\,\mathcal{T}$ closed under limits and sifted colimits is a variety. Consider the free completion $\mathcal{T}'$ of $\mathcal{T}$ under finite products (1.6) and the finite product preserving extension $M\colon \mathcal{T}' \to \mathcal{T}$ of the identity functor on $\mathcal{T}$. The induced functor $Alg\,M$ is naturally isomorphic to the full inclusion $Alg\,\mathcal{T} \to Alg\,\mathcal{T}' \simeq Set^{\mathcal{T}}$. But $Alg\,\mathcal{T}$ is usually not closed in $Set^{\mathcal{T}}$ under subalgebras (in contrast to 10.16). As a concrete example, let $\mathcal{T}$ be the category of finite sets and functions. The functor

$$A\colon \mathcal{T} \to Set\,, \quad AX = X \times X \times X$$

is clearly an algebra for $\mathcal{T}$, but its subfunctor A' given by all triples in $X \times X \times X$ in which at least two different coordinates have the same value is not an algebra for $\mathcal{T}$.

10.19 Remark. Following 10.13.3 and 10.14, the algebraic functor induced by a full and essentially surjective morphism of theories is full and faithful. The morphism $M\colon \mathcal{T}' \to \mathcal{T}$ in 10.18 also demonstrates that this fact cannot be inverted: $Alg\,M\colon Alg\,\mathcal{T} \to Set^{\mathcal{T}}$ is full and faithful, but M is not full.

10.20 Definition. A reflective subcategory $\mathcal{A}$ of a category $\mathcal{B}$ (see 0.9) is called *regular epireflective* if all reflections $r_B\colon B \to RB$ are regular epimorphisms.

10.21 Corollary. *Every variety of $\mathcal{T}$-algebras is a regular epireflective subcategory of $Alg\,\mathcal{T}$ closed under regular quotients and directed unions.*

Proof. We already know that the variety $\mathcal{A}$ is closed in $Alg\,\mathcal{T}$ under regular quotients and sifted colimits (10.16) and therefore under directed unions, which are a special case of sifted colimits (2.9). Moreover, following 10.15, the inclusion functor $\mathcal{A} \to Alg\,\mathcal{T}$ is naturally isomorphic to $Alg\,Q$, which has a left adjoint (9.3). It remains to prove that for every $\mathcal{T}$-algebra B the reflection $r_B \colon B \to RB$ is a regular epimorphism. Let $r_B = m \cdot e$

$$\begin{tikzcd} & A \arrow[dr, shift left, "m"] & \\ B \arrow[ur, "e"] \arrow[rr, "r_B"'] & & RB \arrow[ul, shift left, "\bar{e}"] \end{tikzcd}$$

be a regular factorization of r_B, see 3.7. By 10.16(b) $A \in \mathcal{A}$, thus there is a unique $\bar{e} \colon RB \to A$ such that $e = \bar{e} \cdot r_B$. Since e is an epimorphism, we see that $\bar{e} \cdot m = \mathrm{id}_A$. Also $m \cdot \bar{e} = \mathrm{id}_{RB}$ due to the universal property of r_B. Thus, m is an isomorphism and r_B a regular epimorphism. $\square$

10.22 Birkhoff's Variety Theorem. *Let $\mathcal{T}$ be an algebraic theory. A full subcategory $\mathcal{A}$ of $Alg\,\mathcal{T}$ is a variety if and only if it is closed in $Alg\,\mathcal{T}$ under*

(a) products,

(b) subalgebras,

(c) regular quotients,

and

(d) directed unions.

Proof. Every variety is closed under (a)-(d): see 10.16. Conversely, let $\mathcal{A}$ be closed under (a)-(d).
(1) We first prove that $\mathcal{A}$ is a regular epireflective subcategory. Let B be a $\mathcal{T}$-algebra. By 3.6 there exists a set of regular epimorphisms $e \colon B \to A_e$ $(e \in X)$ representing all regular quotients of B with codomain in $\mathcal{A}$. Denote by $b \colon B \to \prod_{e\in X} A_e$ the induced morphism and let

$$\begin{tikzcd} B \arrow[rr, "b"] \arrow[dr, "r_B"'] & & \prod_{e\in X} A_e \\ & RB \arrow[ur, "m_B"'] & \end{tikzcd}$$

be the regular factorization of b (see 3.7). The algebra RB lies in $\mathcal{A}$ (because it is a subalgebra of a product of algebras in $\mathcal{A}$) and r_B is the reflection of B

in $\mathcal{A}$. Indeed, for every morphism $f\colon B \to A$ with A in $\mathcal{A}$ we have a regular factorization

$$f = m \cdot e \quad \text{for some} \quad e \in X$$

and since e factorizes through b, so does f.
(2) We will prove that $\mathcal{A}$ is specified by the congruence $\sim$ which is the intersection of the kernel congruences of all algebras in $\mathcal{A}$:

$$u_1 \sim_{s,t} u_2 \ \text{ iff } \ Au_1 = Au_2 \ \text{ for all } \ A \in \mathcal{A}\,.$$

This is indeed a congruence, see 10.6 and 10.7.2. It is our task to prove that every $\mathcal{T}$-algebra B such that $\sim$ is contained in $\approx_B$, see 10.7, lies in $\mathcal{A}$.
(2a) Assume first that B is a regular quotient of a representable algebra $Y_{\mathcal{T}}(t)$. We thus have a regular epimorphism $e\colon Y_{\mathcal{T}}(t) \to B$. We know that the reflection morphism $r_t\colon Y_{\mathcal{T}}(t) \to R(Y_{\mathcal{T}}(t))$ is a regular epimorphism, thus, it is a coequalizer

$$\begin{tikzcd} N \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y_{\mathcal{T}}(t) \arrow[r, "r_t"] & R(Y_{\mathcal{T}}(t)) \end{tikzcd}$$

By 4.2 we can express N as a sifted colimit of representable algebras, denote the colimit cocone by $c_s\colon Y_{\mathcal{T}}(s) \to N$. Using Yoneda Lemma, we see that for every s there exist morphisms $u_s, v_s\colon t \rightrightarrows s$ representing $u \cdot c_s$ and $v \cdot c_s$, respectively:

$$\begin{tikzcd}
Y_{\mathcal{T}}(s) \arrow[dr, shift left, "Y_{\mathcal{T}}(u_s)"] \arrow[dr, shift right, "Y_{\mathcal{T}}(v_s)"'] \arrow[dd, "c_s"'] & & B \\
& Y_{\mathcal{T}}(t) \arrow[ur, "e"] \arrow[r, "r_t"] \arrow[dr, "h"'] & R(Y_{\mathcal{T}}(t)) \arrow[u, "f"'] \arrow[d] \\
N \arrow[ur, shift left, "u"] \arrow[ur, shift right, "v"'] & & A
\end{tikzcd}$$

Moreover, we have $u_s \sim_{t,s} v_s$ because every morphism $h\colon Y_{\mathcal{T}}(t) \to A$ with $A \in \mathcal{A}$ factorizes through r_t, so that $Y_{\mathcal{T}}(u_s) \cdot h = Y_{\mathcal{T}}(v_s) \cdot h$, and it follows that $Au_s = Av_s$. By assumption on B, this implies $Bu_s = Bv_s$, and then $e \cdot u \cdot c_s = e \cdot v \cdot c_s$. The cocone c_s is jointly epimorphic, thus we have $e \cdot u = e \cdot v$. Since r_t is the coequalizer of u and v, there exists $f\colon R(Y_{\mathcal{T}}(t)) \to B$ such that $f \cdot r_t = e$. Finally, f is a regular epimorphism because e is, so that B, being a regular quotient of $R(Y_{\mathcal{T}}(t))$, lies in $\mathcal{A}$.
(2b) Let B be arbitrary. Express B as a sifted colimit of representable algebras (4.2) and for each of the colimit morphisms $\sigma_s\colon Y_{\mathcal{T}}(s) \to B$ denote by B_s the image of σ_s which, by 3.7, is a subalgebra of B. Since $\mathcal{T}$ has finite products, the collection of these subalgebras of B is directed. Due to assumption (d) we only need to prove that every B_s lies in $\mathcal{A}$. This follows from (2a): we know that B_s is a regular quotient of a representable algebra, and B_s has the desired property: given $u_1 \sim_{s,t} u_2$, we have $B(u_1) = B(u_2)$ and this implies $B_s u_1 = B_s u_2$ since B_s is a subalgebra of B. $\square$

10.23 Example. The assumption of closure under directed unions cannot be omitted: consider the category $Set^{\mathbb{N}}$ of $\mathbb{N}$-sorted sets and let $\mathcal{A}$ be the full subcategory of all $A = (A_n)_{n\in\mathbb{N}}$ such that either some A_n is empty or all A_n's have precisely one element. This subcategory is clearly closed under products, subalgebras and regular quotients – we omit the easy verification. However, it is not a variety, not being closed under directed unions. In fact, every $\mathbb{N}$-sorted set $A = (A_n)_{n\in\mathbb{N}}$ is a directed union of objects A^k of $\mathcal{A}$: put $A^k = (A_0, \ldots, A_k, \emptyset, \emptyset, \ldots)$.

10.24 Corollary. *Let $\mathcal{A}$ be a full subcategory of $Alg\,\mathcal{T}$. Then $\mathcal{A}$ is a variety if and only if it is a regular epireflective subcategory closed under regular quotients and directed unions.*

In fact, following 10.21 and 10.22, we only need to observe that every regular epireflective subcategory $\mathcal{A}$ is closed in $Alg\,\mathcal{T}$ under products (and this is obvious) and subalgebras: consider the diagram

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "r_B"'] & A \arrow[d, "r_A"] \\ RB \arrow[r, "Rm"'] & RA \end{tikzcd}$$

where m is a monomorphism and A lies in $\mathcal{A}$. Then r_A is an isomorphism and r_B is a monomorphism. But r_B is also a regular epimorphism, so that it is an isomorphism, and B lies in $\mathcal{A}$.

10.25 Example. The category Ab_{tf} of torsion free abelian groups is a regular epireflective subcategory of Ab closed under filtered colimits. But this is not a variety in Ab because it is not closed under quotients. Indeed, Ab_{tf} is locally finitely presentable but not algebraic (it is not exact).

Historical Remarks for Chapter 10

The classical characterization of varieties of one-sorted algebras as HSP-classes is due to G. Birkhoff [24]; we present this version in 11.35.

For many-sorted algebras the concept of equation in 10.1 corresponds to the formulas

$$\forall\, x_1 \ldots \forall\, x_n \ (t = s)$$

where t and s are terms of the same sort in (sorted) variables $x_1, \ldots, x_n$. Example 10.23 demonstrates that with respect to the above equations we need to add the closure under directed unions. Another approach, used e.g. in [7], is to admit infinitely many variables in the equations – that is, to work in the logic $L_{\omega\infty}$ (admitting quantification over infinite sets) rather than in the finitary logic $L_{\omega\omega}$. As proved in [7], Theorem 3.9, in this logic directed unions can be omitted in Birkhoff's Variety Theorem. This is illustrated by Example 14.21 below.

Chapter 11

One-sorted algebraic categories

Classical algebraic categories, such as groups, modules, boolean algebras, etc. are not only abstract categories: their objects are sets with a structure, and their morphisms are functions preserving the structure. Thus, they are concrete categories over *Set* (see 0.18), which means that a "forgetful" functor into *Set* is given. Also, these classical algebraic categories have theories generated by a single object X in the sense that all objects of the theory are finite powers X^n of X. (Consider the free group or the free module on one generator.) In order to formalize this idea, we study in this chapter one-sorted algebraic categories. In Chapter 14 we will deal with the more general notion of S-sorted algebraic categories, for which the forgetful functor into a power of *Set* is considered rather than into *Set*.

11.1 Example. The theory

$$\mathcal{N}$$

of sets (see 1.9), which is the full subcategory of Set^{op} on natural numbers $n = \{0, \dots, n-1\}$, is a "prototype" one-sorted theory: every object n is the product of n copies of 1. Moreover, the n injections in *Set*

$$\pi_i^n \colon 1 \to n\,, \quad 0 \mapsto i \quad (i = 0, \dots, n-1)$$

yield a canonical choice of projections $\pi_i^n \colon n \to 1$ in $\mathcal{N}$ which present n as 1^n.

11.2 Remark.

1. If an algebraic theory $\mathcal{T}$ has all objects finite powers of an object X, we obtain a theory morphism $T \colon \mathcal{N} \to \mathcal{T}$ as follows: for every n choose an n-th power of X with projections $p_i^n \colon X^n \to X$ for $i = 0, \dots, n-1$. Then T is uniquely determined by

$$Tn = X^n \ \text{ and } \ T\pi_i^n = p_i^n \ \text{ for all } \ i < n \ \text{ in } \ \mathbb{N}\,.$$

2. Conversely, every theory morphism $T : \mathcal{N} \to \mathcal{T}$ represents an object and a choice of projections of all of its finite powers: put $X = T1$ and $p_i^n = T\pi_i^n : X^n \to X$ for all $i < n$.

This leads us to the following

11.3 Definition.

1. A *one-sorted algebraic theory* is a pair $(\mathcal{T}, T)$ where $\mathcal{T}$ is an algebraic theory whose objects are the natural numbers, and $T : \mathcal{N} \to \mathcal{T}$ is a theory morphism which is the identity map on objects.

2. A *morphism* of one-sorted algebraic theories $M : (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ is a functor $M : \mathcal{T}_1 \to \mathcal{T}_2$ such that $M \cdot T_1 = T_2$.

11.4 Remark.

1. We have not requested that morphisms of one sorted theories preserve finite products. In fact, this simply follows from the equation $M \cdot T_1 = T_2$. Observe that due to that equation M is the identity map on objects. Since, moreover, M takes the projections $T_1\pi_i^n$ to the projections $T_2\pi_i^n$, it clearly preserves finite powers, thus, finite products.

2. The reason why one-sorted theories are requested to be equipped with a theory morphism from $\mathcal{N}$ is that in this way the category of one-sorted theories and the category of finitary monads on *Set* are equivalent, as proved in A.38. (And, by the way, this is the original definition by Lawvere from 1963.)

3. There is an obvious non-strict version of morphism of one-sorted theories, where the equality above is weaken to a natural isomorphism between $M \cdot T_1$ and T_2. See Appendix C for this approach.

11.5 Example. Recall the theory $\mathcal{T}_{ab}$ of abelian groups whose objects are natural numbers and morphisms are matrices (see 1.11). It can be canonically considered as a one-sorted theory if we define $T_{ab} : \mathcal{N} \to \mathcal{T}_{ab}$ as the identity map on objects, and assign to $\pi_i^n : n \to 1$ the one-row matrix with i-th entry 1 and all other entries 0.

11.6 Remark. Given a one-sorted theory $(\mathcal{T}, T)$, the functor T does not influence the concept of algebra: the category $Alg\,\mathcal{T}$ thus consists, again, of all functors $A : \mathcal{T} \to Set$ preserving finite products. However, the presence of T makes the category of algebras concrete over *Set* (see 0.18). Assuming that we identify *Set* and $Alg\,\mathcal{N}$, the forgetful functor is simply

$$Alg\,T : Alg\,\mathcal{T} \to Set$$

(see 9.2), which is faithful by 11.8. More precisely, this forgetful functor takes an algebra $A : \mathcal{T} \to Set$ to the set $A1$, and a homomorphism $h : A \to B$ to the component $h_1 : A1 \to B1$.

11.7 Example. For the one-sorted theory $(\mathcal{T}_{ab}, T_{ab})$ of abelian groups the category $Alg\,\mathcal{T}_{ab}$ is equivalent to Ab. (But it is not isomorphic to Ab : this si caused by the fact that algebras for $\mathcal{T}_{ab}$ are not required to preserve products strictly. Consequently, there exist many algebras that are naturally isomorphic to algebras of the form $\widehat{G}$ (see 1.11) but are not equal to any of those.) The forgetful functor assigns, for every group G, to $\widehat{G}$ the underlying set of G. Observe that, unlike in Ab, there exist isomorphisms $f\colon A \to B$ in $Alg\,\mathcal{T}_{ab}$ for which $A1 = B1$ and $f_1 = \mathrm{id}$ but still $A \neq B$. In fact, given a group G we usually have many algebras $B \neq \widehat{G}$ naturally isomorphic to $\widehat{G}$ such that the component of the natural isomorphism at 1 is the identity. In other words, $Alg\,T_{ab}$ is not amnestic (see 13.16).

11.8 Proposition. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory. The forgetful functor $Alg\,T\colon Alg\,\mathcal{T} \to Set$ is algebraic, faithful and conservative.*

Proof. $Alg\,T$ is algebraic by 9.3. Let $f\colon A \to B$ be a homomorphism of $\mathcal{T}$-algebras. Because of the naturality of f, the following square commutes

$$\begin{tikzcd} AT^n \arrow[r, "f_{T^n}"] \arrow[d, "\simeq"'] & BT^n \arrow[d, "\simeq"] \\ (AT)^n \arrow[r, "f_T^n"'] & (BT)^n \end{tikzcd}$$

It is now obvious that $Alg\,T$ is faithful and conservative. ◻

11.9 Corollary. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory. The forgetful functor preserves and reflects limits, sifted colimits, monomorphisms and regular epimorphisms.*

11.10 Remark. For every $\mathcal{T}$-algebra A and every subset X of its underlying set $A1$ there exists the least subalgebra $\overline{X}$ of A such that $\overline{X}1$ contains X ($\overline{X}$ is called the *subalgebra generated* by X). In fact, consider the intersection of all subalgebras of A containing X.

11.11 Remark. The concept of algebraic category in Chapter 1 used equivalences of categories. For one-sorted algebraic theories we need more: the equivalence functor must be concrete. This is, in fact, not enough because a quasi-inverse of a concrete functor is in general not concrete. We are going to require that the equivalence functor admits a quasi-inverse which is concrete.

11.12 Definition. Given concrete categories $U\colon \mathcal{A} \to \mathcal{K}$ and $V\colon \mathcal{B} \to \mathcal{K}$ over $\mathcal{K}$, by a *concrete equivalence* between them we mean a pair of concrete functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "E"] \arrow[dr, "U"'] & \mathcal{B} \arrow[r, "E'"] \arrow[d, "V"] & \mathcal{A} \arrow[dl, "U"] \\ & \mathcal{K} & \end{tikzcd}$$

such that both $E \cdot E'$ and $E' \cdot E$ are naturally isomorphic to the identity functors. We then say that $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$ are *concretely equivalent*.

11.13 Definition. A *one-sorted algebraic category* is a concrete category over *Set* which is concretely equivalent to $Alg\, T\colon Alg\, \mathcal{T} \to Set$ for a one-sorted algebraic theory $(\mathcal{T}, T)$.

11.14 Remark. A non-strict version of one-sorted algebraic categories is treated in Appendix C.

11.15 Example. The category *Ab* with its canonical forgetful functor is a one-sorted algebraic category. In fact, it is concretely equivalent to the category of algebras for $(\mathcal{T}_{ab}, T_{ab})$: the functor

$$\begin{tikzcd} Ab \arrow[rr, "E"] \arrow[dr, "U"'] & & Alg\, \mathcal{T}_{ab} \arrow[dl, "Alg\, T_{ab}"] \\ & Set & \end{tikzcd}$$

which to every group G assigns the algebra $\widehat{G}\colon \mathcal{T}_{ab} \to Set$ of Example 1.11 is concrete. And we have the oncrete functor

$$\begin{tikzcd} Alg\, \mathcal{T}_{ab} \arrow[rr, "E'"] \arrow[dr, "Alg\, T_{ab}"'] & & Ab \arrow[dl, "U"] \\ & Set & \end{tikzcd}$$

which to every algebra $A\colon \mathcal{T}_{ab} \to Set$ assigns the group G with $A \simeq \widehat{G}$ from 1.11. It is easy to verify that both $E \cdot E'$ and $E' \cdot E$ are naturally isomorphic to the identity functors.

11.16 Remark. Given a concrete category $(\mathcal{A}, U)$, all subcategories of $\mathcal{A}$ are considered to be concrete by using the domain restriction of U.

11.17 Proposition. *Every variety of $\mathcal{T}$-algebras for a one-sorted theory $(\mathcal{T}, T)$ is a one-sorted algebraic category.*

Proof. Let $\sim$ be a congruence on $\mathcal{T}$ and let $Q\colon \mathcal{T} \to \mathcal{T}/\sim$ be the corresponding quotient functor. Since Q preserves finite products, $(\mathcal{T}/\sim, Q \cdot T)$ is a one-sorted theory. Consider the full subcategory $\mathcal{A}$ of $Alg\, \mathcal{T}$ specified by $\sim$. The equivalence

$$E\colon Alg\,(\mathcal{T}/\sim) \to \mathcal{A}$$

constructed in 10.15 is a concrete functor (because $Alg\, Q$ is concrete). Moreover, E is an isomorphism (not just an equivalence) because it is bijective on objects. Thus, the inverse functor $E^{-1}\colon \mathcal{A} \to Alg\,(\mathcal{T}/\sim)$ is concrete. ☐

11.18 Example. The category *Graph* of graphs is algebraic, but not one-sorted algebraic. In fact, a terminal object in *Graph* is the graph with one vertex and one edge, and it has a proper subobject given by the graph G with one vertex and no edge. Observe that G is neither terminal nor initial in *Graph*. Now use the following

11.19 Lemma. *In a one-sorted algebraic category a terminal object A has no nontrivial subobjects: for every subobject $m\colon B \to A$ either B is an initial object or a terminal one.*

Proof. Given a one-sorted algebraic theory $(\mathcal{T}, T)$, denote by A a terminal object of $Alg\,\mathcal{T}$ and by I the initial one. Since $Alg\,T\colon Alg\,\mathcal{T} \to Set$ preserves limits (11.9), $B(1)$ is a subobject of $A1 = 1$. If $B1 \simeq 1$, then m_1 is an isomorphism and thus m is an isomorphism (since $Alg\,T$ is conservative). If $B1 = \emptyset$, consider the unique monomorphism $a\colon I \to B$ and the induced map $a_1\colon I1 \to B1 = \emptyset$. Such a map is necessarily an isomorphism, and thus so is a. It is easy to see that concrete equivalences preserve the above property of terminal objects. $\square$

11.20 Example. Even though the category of graphs is not a one-sorted algebraic category, the category *RGraph* of reflexive graphs is. Here the objects are directed graphs

$$G_e \overset{\tau}{\underset{\sigma}{\rightrightarrows}} G_v$$

together with a map $d\colon G_v \to G_e$ such that $\tau \cdot d = \mathrm{id}_{G_v} = \sigma \cdot d$. A morphism from $(G_e, G_v, \tau, \sigma, d)$ to $(G'_e, G'_v, \tau', \sigma', d')$ is a graph homomorphism

$$(h_e\colon G_e \to G'_e, h_v\colon G_v \to G'_v)$$

such that $h_e \cdot d = d' \cdot h_v$. We consider this category as concrete over *Set* by $U(G_e, G_v, \tau, \sigma, d) = G_e$.

In fact, the category *RGraph* is concretely equivalent to the following one-sorted algebraic category $\mathcal{A}$: an object of $\mathcal{A}$ is a set G_e equipped with two maps

$$G_e \overset{t}{\underset{s}{\rightrightarrows}} G_e$$

such that $s \cdot t = t$ and $t \cdot s = s$. A morphism from (G_e, t, s) to (G'_e, t', s') is a map $h_e\colon G_e \to G'_e$ such that $h_e \cdot t = t' \cdot h_e$ and $h_e \cdot s = s' \cdot h_e$.
1. Define $E\colon RGraph \to \mathcal{A}$ by assigning to a reflexive graph $(G_e, G_v, \tau, \sigma, d)$ the object (G_e, t, s) of $\mathcal{A}$ by defining $t = d \cdot \tau$ and $s = d \cdot \sigma$. This is a concrete functor.
2. Define $E'\colon \mathcal{A} \to RGraph$ by assigning to an object (G_e, t, s) in $\mathcal{A}$ the reflexive graph $(G_e, G_v, \tau, \sigma, d)$ by taking as $d\colon G_v \to G_e$ a joint equalizer of

$$\begin{tikzcd} G_e \arrow[r, shift left=2, "t"] \arrow[r, "\mathrm{id}" description] \arrow[r, shift right=2, "s"'] & G_e \end{tikzcd}$$

This yields the canonical factorizations $\tau\colon G_e \to G_v$ of t through d, and $\sigma\colon G_e \to G_v$ of s through d (such factorizations exist because $t = s \cdot t = t \cdot s \cdot t = t \cdot t$ and

analogously for s). The rest of the definition of E' is straightforward. Again, E' is a concrete functor.
3. The verification that $E \cdot E'$ and $E' \cdot E$ are naturally isomorphic to the identity functors is easy.

11.21 Remark. Let $(\mathcal{T}, T)$ be a one-sorted theory.

1. The forgetful functor $Alg T \colon Alg \mathcal{T} \to Set$ has a left adjoint. In fact, due to 4.11 applied to $Y_{\mathcal{N}} \colon \mathcal{N}^{op} \to Set$ we can choose a left adjoint

$$F_T \colon Set \to Alg \mathcal{T}$$

in such a way that the square

$$\begin{tikzcd} \mathcal{N}^{op} \arrow[r, "T^{op}"] \arrow[d, "Y_{\mathcal{N}}"'] & \mathcal{T}^{op} \arrow[d, "Y_{\mathcal{T}}"] \\ Set \arrow[r, "F_T"'] & Alg \mathcal{T} \end{tikzcd}$$

commutes. Thus, for every natural number n,

(a) $F_T(n) = \mathcal{T}(n, -)$

and

(b) $F_T(\pi_i^n) = - \cdot T\pi_i^n$ for all $i = 0, \ldots, n-1$.

2. The naturality square for $\eta \colon \mathrm{Id} \to Alg T \cdot F_T$ applied to π_i^n yields the commutativity of

$$\begin{tikzcd} 1 \arrow[r, "\eta_1"] \arrow[d, "\pi_i^n"'] & \mathcal{T}(1,1) \arrow[d, "- \cdot T\pi_i^n"] \\ n \arrow[r, "\eta_n"'] & \mathcal{T}(n,1) \end{tikzcd}$$

that is

$$\eta_n(i) = T\pi_i^n \quad \text{for all} \quad i = 0, \ldots, n-1$$

(recall that π_i^n is the inclusion of i).

3. Since F_T preserves coproducts, $F_T X = \coprod_X Y_{\mathcal{T}}(1)$ for every set X.

4. $\mathcal{T}$-algebras of the form $F_T X$, for X a set, are called *free algebras.* If X is finite, they are called *finitely generated free algebras.*

11.22 Corollary. *Let $(\mathcal{T}, T)$ be a one-sorted theory. $\mathcal{T}^{op}$ is equivalent to the full subcategory of $Alg \mathcal{T}$ of finitely generated free algebras.*

Indeed by Yoneda Lemma $\mathcal{T}(n,k) \simeq Alg \mathcal{T}(Y_{\mathcal{T}}(k), Y_{\mathcal{T}}(n)) = Alg \mathcal{T}(F_T(k), F_T(n))$.

11.23 Remark. Every object of a category with finite coproducts defines an algebraic theory $\mathcal{T}(A)$ for which we need to fix coproduct injections

$$p_i^n \colon A \to nA = A + \ldots + A \ \ (n \text{ summands}).$$

The objects of $\mathcal{T}(A)$ are natural numbers, and morphisms from n to k are the morphisms of $\mathcal{A}$ from $kA = A + \ldots + A$ to nA. Then $T \colon \mathcal{N} \to \mathcal{T}(A)$ is defined by $T\pi_i^n = p_i^n$. Observe that $\mathcal{T}(A)$ is equivalent to the full subcategory of $\mathcal{A}^{op}$ on all finite copowers of A under the equivalence functor $n \mapsto nA$. The corresponding category of $\mathcal{T}(A)$-algebras can be equivalent to $\mathcal{A}$, as we have seen in the example $\mathcal{A} = Ab$ and $A = \mathbb{Z}$. (In fact, if $\mathcal{A} = Alg\,\mathcal{T}$ for a one-sorted algebraic theory $(\mathcal{T}, T)$ and $A = T1$, then $\mathcal{A}$ is always equivalent to $Alg\,\mathcal{T}(A)$.) These theories $\mathcal{T}(A)$ are often seen as the "natural" algebraic theories in classical algebra (e.g. for $\mathcal{A} =$ groups, lattices, monoids, etc.).

11.24 Remark. Extending 11.21, for every one-sorted algebraic category $(\mathcal{A}, U)$ with a left adjoint $F \dashv U$ a one-sorted theory can be constructed from the full subcategory of $\mathcal{A}^{op}$ on the objects $F\{x_0, \ldots, x_{n-1}\}$. Here a set of "standard variables" $x_0, x_1, x_2, \ldots$ is assumed. In fact, the n injections

$$\{x_0\} \to \{x_0, \ldots, x_{n-1}\}\ , \ \ x_0 \mapsto x_i$$

define n morphisms $p_i^n \colon F\{x_0, \ldots, x_{n-1}\} \to F\{x_0\}$ in $\mathcal{A}^{op}$. Let $\mathcal{T}$ be the category whose objects are the natural numbers and whose morphisms are

$$\mathcal{T}(n, k) = \mathcal{A}(F\{x_0, \ldots, x_{k-1}\}, F\{x_0, \ldots, x_{n-1}\})\,.$$

The composition in $\mathcal{T}$ is inherited from $\mathcal{A}^{op}$, and so are the identity morphisms. The functor $T \colon \mathcal{N} \to \mathcal{T}$ is determined by the above choice of morphisms p_i^n for all $i \leq n$.

11.25 Example. For the one-sorted theory $(\mathcal{T}_{ab}, T_{ab})$ (11.5) the induced adjunction $F_{T_{ab}} \dashv Alg\,T_{ab}$ is, up to concrete equivalence, the usual adjunction given by free abelian groups.

Using free algebras we can restate some facts from Chapter 5:

11.26 Proposition. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory.*

1. *Free algebras are precisely the coproducts of representable algebras.*
2. *Every algebra is a regular quotient of a free algebra.*
3. *Regular projectives are precisely the retracts of free algebras.*

Proof. 1: Following 11.21, every free algebra is a coproduct of representable algebras. Conversely, every representable algebra is free by 11.21. Thus, every coproduct of representable algebra is free because F_T preserves coproducts.
2: Following 4.2, every algebra is a regular quotient of a coproduct of representable algebras and then, by 1, it is a regular quotient of a free algebra.
3: This follows from 1. and 5.14.2. ☐

11.27 Remark. We have used "finitely generated" in two different situations above: for objects of a category, see 5.21, and as a denotation of $F_T X$ with X finite. The following proposition demonstrates that there is no conflict. Note, however, that a finitely generated free algebra can, in principle, coincide with a non-finitely generated one. In fact, in the variety of algebras satisfying, for a pair x, y of distinct variables, the equation $x = y$, all algebras are isomorphic.

11.28 Proposition. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory.*

1. *Finitely generated free algebras are precisely the representable algebras. These are precisely the free algebras which are finitely generated (in the sense of Definition 5.21).*
2. *Perfectly presentable algebras are precisely the retracts of finitely generated free algebras.*
3. *Finitely presentable algebras are precisely the coequalizers of (reflexive) pairs of morphisms between finitely generated free algebras.*

Proof. 1. follows from 11.21 and the observation that whenever the object $F_T X$ is finitely generated then it is isomorphic to $F_T X'$ for some finite set X'. In fact, the set X is the directed union of its nonempty finite subsets X'. Since F_T is a left adjoint and directed unions are directed colimits in *Set*, we see that $F_T X$ is a directed colimit with the colimit cocone formed by all Fi, where $i \colon X' \to X$ are the inclusion maps. Moreover, since each i is a split monomorphism in *Set*, $F_T X$ is the directed union of the finitely generated free algebras $F_T X'$. Since $F_T X$ is were finitely generated, there exists a finite nonempty subset $i \colon X' \hookrightarrow X$ and a homomorphism $f \colon F_T X \to F_T X'$ such that $F_T i \cdot f = id_{F_T X}$. Thus $F_T i \colon F_T X' \to F_T X$, being a monomorphism and a split epimorphism, is an isomorphism, a contradiction.
2. follows from 1. and 5.14.1, and 3. follows from 1. and 5.17. ☐

11.29 Remark. Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory and X a subset of the underlying set $A1$ of a $\mathcal{T}$-algebra A. The subalgebra of A generated by X (see 11.9) is a regular quotient of the free algebra $F_T X$. Indeed, consider the homomorphisms $\bar{i}_X \colon F_T X \to A$ corresponding to the inclusion $i_X \colon X \to A1$. Then $\bar{i}_X(F_T X)$ is a subalgebra of A because the forgetful functor preserves regular factorizations. This is obviously the least subalgebra of A containing X (use diagonal fill-in, see 0.16) and the codomain restriction of $\bar{i}_X$ is a regular quotient.

11.30 Proposition. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory and A a $\mathcal{T}$-algebra. The following conditions are equivalent:*

1. *A is finitely generated (see 5.21),*
2. *A is a regular quotient of a finitely generated free algebra,*

and

3. *there exists a finite subset X of $A(1)$ not contained in any proper subalgebra of A.*

Proof. The equivalence between 1. and 2. follows from 5.21 and 11.18.1.
$1 \Rightarrow 3$: Let A be a finitely generated object of $Alg\mathcal{T}$. Form a diagram in $Alg\mathcal{T}$ indexed by the poset of all finite subsets of $A1$ by assigning to every such $X \subseteq A1$ the subalgebra $\overline{X}$ of A generated by X (see 11.10). Given finite subsets X and Y with $X \subseteq Y \subseteq A1$, the connecting map $\overline{X} \to \overline{Y}$ is the inclusion map. Then the inclusion homomorphisms $i_X\colon \overline{X} \to A$ form a colimit cocone of this directed diagram. Since the functor $Alg\mathcal{T}(A, -)$ preserves this colimit, for $\mathrm{id}_A \in Alg\mathcal{T}(A, A)$ there exists a finite set X such that id_A lies in the image of i_X – but this proves $\overline{X} = A$.
$3 \Rightarrow 2$: If $A = \overline{X}$ for a finite subset X of $A1$, then by 11.19 A is a regular quotient of the finitely generated free algebra $F_T X$. □

11.31 Remark.

1. Recall the notion of an equivalence relation on an object A in a category from 3.12. If the category is $Alg\mathcal{T}$, following the terminology of general algebra in 11.32 and 11.33 we speak about *congruence on the algebra* A (instead of equivalence). This is a slight abuse of terminology since congruences were previously used for the theory $\mathcal{T}$ itself.

2. Similarly to 11.10, for every $\mathcal{T}$-algebra A and every subset X of $A1 \times A1$ there exists the least congruence on A whose underlying set contains X. Such a congruence is called the *congruence generated by* X.

3. The *finitely generated congruences* are those generated by finite subsets of $A1 \times A1$.

11.32 Lemma. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory. Given*

$$F_T X \overset{u}{\underset{v}{\rightrightarrows}} B$$

in $Alg\mathcal{T}$, the congruence on B generated by the image of $\langle u, v\rangle$ coincides with that generated by the image of $\langle u_1 \cdot \eta_X, v_1 \cdot \eta_X\rangle$, where $\eta_X\colon X \to (F_T X)(1)$ is the unit of the adjunction $F_T \dashv AlgT$.

Proof. Let R be the congruence by the image of $\langle u, v\rangle$ and S that generated by the image of $\langle u_1 \cdot \eta_X, v_1 \cdot \eta_X\rangle$. To check the inclusion $R \subseteq S$ use the universal property of η_X and the diagonal fill-in (cf. 0.16). The other inclusion is obvious. □

11.33 Corollary. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory. A $\mathcal{T}$-algebra A is finitely presentable if and only if there exists a coequalizer of the form*

$$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} F_T Y \xrightarrow{c} A$$

with Y a finite set and R a finitely generated congruence on $F_T Y$.

In fact, this follows from 11.28 and 11.32 by taking $B = F_T Y$ with X and Y finite sets.

As announced at the beginning of Chapter 10, we are going to prove that condition (d) in Birkhoff's Variety Theorem 10.22 can be avoided when the theory is one-sorted. We need a preliminary lemma.

11.34 Lemma. *Let $(\mathcal{T}, T)$ be a one-sorted theory with the forgetful functor $U = AlgT$. Consider a regular epimorphism $e\colon A \to B$ and a homomorphism $g\colon A \to C$ in $Alg\mathcal{T}$. If Ug factorizes through Ue in Set*

$$\begin{tikzcd} UA \arrow[r, "Ue"] \arrow[d, "Ug"'] & UB \arrow[dl, dotted] \\ UC & \end{tikzcd}$$

then g factorizes through e in $Alg\mathcal{T}$

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[dl, dotted] \\ C & \end{tikzcd}$$

Proof. Let $f\colon UB \to UC$ be such that $f \cdot Ue = Ug$. Define $h\colon B \to C$ to be the homomorphism whose component at $n \in \mathcal{T}$ is

$$h_n = f^n \colon UB^n \to UC^n \,.$$

It is strightforward to check the naturality of h using that each component e_{T^n} is surjective (3.5). Since $Uh = f$ and $AlgT$ is faithful, we conclude that $h \cdot e = g$. ☐

11.35 Proposition. *Let $(\mathcal{T}, T)$ be a one-sorted algebraic theory. If a full subcategory $\mathcal{A}$ of $Alg\mathcal{T}$ is closed under products, subalgebras, and regular quotients, then it is a variety.*

Proof. By 10.22 all we need is proving that $\mathcal{A}$ is closed under directed unions. Put $U = AlgT$ and $F = F_T$ for short. Following the first part of the proof of 10.22 we know that $\mathcal{A}$ is regular epireflective in $Alg\mathcal{T}$, with reflectons denoted by $r_B\colon B \to RB$. Then for every directed union $A = \cup_{i\in I} A_i$ of subobjects $m_i\colon A_i \to A$ with $A_i \in \mathcal{A}$ for each i we prove that A lies in $\mathcal{A}$. We use the counit $\varepsilon_A\colon FUA \to A$. Due to the definition of adjunction, $U\varepsilon_A)$ is a split and therefore regular epimorphism. Since U reflects regular epimorphisms (see 11.9), ε_A is a regular epimorphism. Form the reflection $r_{FUA}\colon FUA \to RFUA$ and prove that ε_A factorizes through it: this finishes the proof because given $h\colon RFUA \to A$ with $\varepsilon_A = h \cdot r_{UFA}$, then h is a regular epimorphism. Thus, A is a regular quotient of $RFUA$, which proves that A is in $\mathcal{A}$.

By Lemma 11.34 it is sufficient to prove that $U\varepsilon_A$ factorizes through Ur_{FUA} in *Set*. This means that given morphisms $x, y \colon 1 \rightrightarrows UFUA$ we must prove that

$$Ur_{FUA} \cdot x = Ur_{FUA} \cdot y \text{ implies } U\varepsilon_A \cdot x = U\varepsilon_A \cdot y\,.$$

The functor UFU preserves filtered colimits (because U does, see 11.9, and F is a left adjoint) thus

$$UFUA \simeq \operatorname*{\mathit{colim}}_{i \in I} UFUA_i\,.$$

Consequently, there exists $i \in I$ such that x, y factorize through $UFUm_i$

$$\begin{tikzcd}
UFUA_i \arrow[rr, "UFUm_i"] & & UFUA \\
& 1 \arrow[ul, shift left, "y'"'] \arrow[ul, shift right, "x'"'] \arrow[ur, shift left, "x"] \arrow[ur, shift right, "y"'] &
\end{tikzcd}$$

Now in *Set* the monomorphisms with nonempty domains split. Therefore, we can assume without loss of generality that Um_i is a split monomorphism. In fact, if $UA = \mathit{colim}\, UA_i$ is nonempty, then some UA_{i_0} for $i_0 \in I$ must be nonempty and it is sufficient to choose $i' \in I$ such that $A_{i'}$ contains A_i and A_{i_0}; if $UA = \emptyset$ then $UA_i = \emptyset$ and $Um_i = \mathrm{id}$. Consequently, $URFUm_i$ is a monomorphism. Consider the following commutative diagram

$$\begin{tikzcd}
RFUA_i \arrow[rrr, "R\varepsilon_{A_i}"] \arrow[ddd, "RFUm_i"'] & & & RA_i \arrow[ddd, "Rm_i"] \\
& FUA_i \arrow[ul, "r_{FUA_i}"] \arrow[r, "\varepsilon_{A_i}"] \arrow[d, "FUm_i"'] & A_i \arrow[ur, "r_{A_i}"', "\simeq"] \arrow[d, "m_i"] & \\
& FUA \arrow[dl, "r_{FUA}"] \arrow[r, "\varepsilon_A"'] & A \arrow[dr, "r_A"'] & \\
RFUA \arrow[rrr, "R\varepsilon_A"'] & & & RA
\end{tikzcd}$$

By applying U it, we see that the fact that Ur_{FUA} merges $x = UFUm_i \cdot x'$ and $y = UFUm_i \cdot y'$ implies that $URFUm_i \cdot Ur_{FUA_i}$ merges x' and y'. Since $URFUm_i$ is a monomorphism, also $UR\varepsilon_{A_i} \cdot Ur_{FUA_i}$ merges x and y. Consulting the above diagram again we conclude that $U\varepsilon_{A_i}$ merges x' and y'. This finishes the proof: from

$$U\varepsilon_A \cdot x = U\varepsilon_A \cdot UFUm_i \cdot x' = Um_i \cdot U\varepsilon_{A_i} \cdot x'$$

and the analogous equation for y we conclude $U\varepsilon_A \cdot x = U\varepsilon_A \cdot y$. □

Recall the algebraic duality of Chapter 9: if we restrict algebraic theories to the canonical ones, we obtain a contravariant biequivalence between the 2-category of algebraic categories and the 2-category of algebraic theories. In the one-sorted case a better result is obtained, since we do not have to restrict the theories at all.

11.36 Definition. We define

1. the *2-category Th^1 of one-sorted theories* to have

   objects: one-sorted algebraic theories,

   1-cells: morphisms of one-sorted algebraic theories,

   2-cells: natural transformations,

2. the *2-category ALG^1 of one-sorted algebraic categories* to have

   objects: one-sorted algebraic categories,

   1-cells: concrete functors,

   2-cells: natural transformations.

11.37 Remark. Every 1-cell in ALG^1 is a faithful and conservative algebraic functor.

11.38 Definition. We denote by

$$Alg^1 \colon (Th^1)^{op} \to ALG^1$$

the 2-functor assigning to every one-sorted theory $(\mathcal{T}, T)$ the concrete category $Alg^1(\mathcal{T}, T) = (Alg\, T \colon Alg\, \mathcal{T} \to Set)$, to every 1-cell $M \colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ the concrete functor $Alg^1 M = (-) \cdot M$, and to every 2-cell $\alpha \colon M \to N$ the natural transformation $Alg^1\alpha \colon Alg^1 M \to Alg^1 N$ whose component at a $\mathcal{T}_2$-algebra A is $A \cdot \alpha \colon A \cdot M \to A \cdot N$.

11.39 Theorem. *(One-sorted algebraic duality) The 2-category ALG^1 of one-sorted algebraic categories is biequivalent to the dual of the 2-category Th^1 of one-sorted algebraic theories. In fact, the 2-functor*

$$Alg^1 \colon (Th^1)^{op} \to ALG^1$$

is a biequivalence.

Proof. (1) Alg^1 is well-defined and essentially surjective (in the sense of the 2-category ALG^1, which means surjectivity up to concrete equivalence) by definition of one-sorted algebraic category.
(2) We will prove that for two one-sorted algebraic theories $(\mathcal{T}_1, T_1)$ and $(\mathcal{T}_2, T_2)$ the functor

$$Th^1((\mathcal{T}_1, T_1), (\mathcal{T}_2, T_2)) \xrightarrow{Alg^1_{(\mathcal{T}_1, T_1), (\mathcal{T}_2, T_2)}} ALG^1((Alg\, \mathcal{T}_2, Alg\, T_2), (Alg\, \mathcal{T}_1, Alg\, T_1))$$

is an equivalence of categories. The proof that $Alg^1_{(\mathcal{T}_1,T_1),(\mathcal{T}_2,T_2)}$ is full and faithful is the same as in Theorem 9.15. It remains to prove that $Alg^1_{(\mathcal{T}_1,T_1),(\mathcal{T}_2,T_2)}$ is essentially surjective: consider a concrete functor

$$\begin{tikzcd} Alg\,\mathcal{T}_1 \arrow[dr, "Alg\,T_1"'] & & Alg\,\mathcal{T}_2 \arrow[ll, "G"'] \arrow[dl, "Alg\,T_2"] \\ & Set & \end{tikzcd}$$

It is our task to find a theory morphism $M\colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ with $G \simeq Alg^1 M$. We have the left adjoint F_T of 11.21 and we denote by $F\colon Alg\,\mathcal{T}_1 \to Alg\,\mathcal{T}_2$ a left adjoint of G. The commutativity of the above triangle yields a natural isomorphism

$$\psi\colon F_{T_2} \to F \cdot F_{T_1}\,.$$

We are going to prove that $F \cdot Y_{\mathcal{T}_1}$ factorizes (up to natural isomorphism) through $Y_{\mathcal{T}_2}$:

$$\begin{tikzcd} \mathcal{T}_1^{op} \arrow[r, "Y_{\mathcal{T}_1}"] \arrow[d, "M^{op}"'] \arrow[dr, phantom, "\simeq"] & Alg\,\mathcal{T}_1 \arrow[d, "F"] \\ \mathcal{T}_2^{op} \arrow[r, "Y_{\mathcal{T}_2}"'] & Alg\,\mathcal{T}_2 \end{tikzcd} \qquad [11.1]$$

We define $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ to be the identity on objects. Consider a morphism $f\colon t \to s$ in $\mathcal{T}_1$. Since $Y_{\mathcal{T}_2}$ is full and faithful, there exists a unique morphism $Mf\colon Mt \to Ms$ such that the following diagram commutes

$$\begin{tikzcd} Y_{\mathcal{T}_2}M(s) = Y_{\mathcal{T}_2}T_2(s) = F_{T_2}(s) \arrow[r, "\psi_s"] \arrow[d, "Y_{\mathcal{T}_2}M(f)"'] & FF_{T_1}(s) = FY_{\mathcal{T}_1}T_1(s) = FY_{\mathcal{T}_1}(s) \arrow[d, "FY_{\mathcal{T}_1}(f)"] \\ Y_{\mathcal{T}_2}M(t) = Y_{\mathcal{T}_2}T_2(t) = F_{T_2}(t) \arrow[r, "\psi_t"] & FF_{T_1}(t) = FY_{\mathcal{T}_1}T_1(t) = FY_{\mathcal{T}_1}(t) \end{tikzcd} \qquad [11.2]$$

The equalities $F_{T_i} \cdot Y_{\mathcal{N}} = Y_{\mathcal{T}_i} \cdot T_i^{op}$ $(i = 1, 2)$ for the embedding $Y_{\mathcal{N}}\colon \mathcal{N}^{op} \to Set$ come from 11.21. The functoriality of

$$M\colon \mathcal{T}_1 \to \mathcal{T}_2$$

follows from the uniqueness of Mf. Moreover, since the isomorphism ψ is natural, if $f = T_1 g$ for some $g\colon t \to s$ in $\mathcal{N}$ then $Mf = T_2 g$. Diagram [11.2] gives also the natural isomorphism needed in [11.1]. This finishes the proof: $F \simeq M^*$ by 9.3, and then $G \simeq Alg\,M$. $\square$

11.40 Remark. A related duality involving the categories (not the 2-categories) of one-sorted algebraic theories and one-sorted algebraic categories can be obtained if we restrict the latter to the uniquely transportable ones. This result, established in Appendix C, gives an alternative approach to the classical duality between finitary monads and finitary monadic categories over *Set* presented in Appendix A.

Historical Remarks for Chapter 11

In his dissertation [63] F. W. Lawvere presents the name algebraic category for one equivalent to the category $Alg\,\mathcal{T}$ of algebras of a one-sorted algebraic theory. Our decision to use *concrete* equivalence is motivated by the precise analogy one gets to finitary monadic categories over *Set* (see Appendix A).

Another variant, based on pseudo-concrete functors in place of the concrete ones, is to take all categories pseudo-concretely equivalent to the categories $Alg\,\mathcal{T}$ above. This is shortly mentioned in Appendix C.

Chapter 12

Algebras for an endofunctor

Throughout this chapter a finitary endofunctor H of *Set* is assumed to be given. We discuss the category of algebras for H. We will prove in 13.23 that it is a one-sorted algebraic category.

12.1 Remark.

1. The concept of H-algebra in 2.24 can be formulated for the endofunctor H of an arbitrary category $\mathcal{K}$: it is a pair (A, a) consisting of an object A and a morphism $a\colon HA \to A$. The category

$$H\text{-}\mathit{Alg}$$

   has as objects H-algebras and as morphisms from (A, a) to (B, b) those morphisms $f\colon A \to B$ for which $f \cdot a = b \cdot Hf$.

2. We denote by

$$U_H\colon H\text{-}\mathit{Alg} \to \mathcal{K}$$

   the canonical forgetful functor $(A, a) \mapsto A$.

3. In the present chapter we restrict ourselves to finitary endofunctors H of *Set*. See also 12.17.

12.2 Remark. We want to show how colimits of H-algebras are obtained. We begin with the simplest case: the initial H-algebra. We will prove that it can be obtained by iterating the unique morphism $u\colon \emptyset \to H\emptyset$. More precisely, let us form the ω-chain

$$\emptyset \xrightarrow{u} H\emptyset \xrightarrow{Hu} H^2\emptyset \xrightarrow{H^2u} H^3\emptyset \longrightarrow \cdots$$

We call it the *initial chain* of H. Its colimit

$$I = \mathit{colim}_{n\in\mathbb{N}} H^n\emptyset$$

carries the structure of an H-algebra. Indeed, since H preserve colimits of ω-chains,

$$HI \simeq colim_{n\in\mathbb{N}} H(H^n\emptyset) \simeq colim_{n\in\mathbb{N}} H^n\emptyset = I\,.$$

We denote by $i\colon HI \to I$ the canonical isomorphism. More detailed, denote by

$$v_n\colon H^n\emptyset \to I \quad (n \in \mathbb{N})$$

a colimit cocone for I. Then $i\colon HI \to I$ is defined by

$$i \cdot Hv_n = v_{n-1} \quad \text{for all } n \geq 1\,.$$

12.3 Lemma. *The H-algebra $i\colon HI \to I$ is initial.*

Proof. For every algebra $a\colon HA \to A$ define a cocone $f_n\colon H^n\emptyset \to A$ of the initial chain as follows: $f_0\colon \emptyset \to A$ is unique and

$$f_{n+1} = a \cdot Hf_n\colon HH^n\emptyset \to A\,.$$

The unique morphism $f\colon I \to A$ with $f \cdot v_n = f_n$ $(n \in \mathbb{N})$ is a homomorphism: since the cocone (Hv_n) is a colimit cocone, thus collectively epimorphic, this follows from the commutative diagram

$$\begin{tikzcd}
HI \arrow[rr, "i"] \arrow[dd, "Hf"'] & & I \arrow[dd, "f"] \\
& H^n\emptyset \arrow[ul, "Hv_{n-1}"'] \arrow[ur, "v_n"] \arrow[dl, "Hf_n"] & \\
HA \arrow[rr, "a"'] & & A
\end{tikzcd}$$

Conversely, if $f\colon I \to A$ is a homomorphism, the above diagram proves that for every $n \geq 1$ the above morphism f_n is equal to $f \cdot v_n$. This shows the uniqueness. $\square$

12.4 Example. We describe the initial H_Σ-algebra for the polynomial functor

$$H_\Sigma X = \coprod_{k\in\mathbb{N}} \Sigma_k \times X^k$$

by applying labelled trees.

Recall that a *tree* is a directed graph with a distinguished node (root) such that for every node there exists a unique path from the root into it. We work with *ordered trees*, that is, for every node the set of children nodes is linearly ordered ("from left to right"). Trees which are isomorphic for an isomorphism respecting the ordering and the labels are identified.

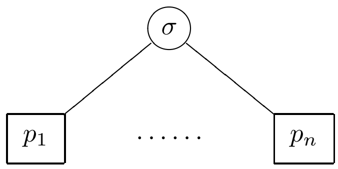
For every signature Σ, by a Σ-*tree* is meant a tree labelled in Σ so that every node has the number of children equal to the arity of its label. More generally, given a set X of variables, by a Σ-*tree on* X is meant a tree labelled in $\Sigma + X$

in such a way that every node with $n > 0$ children has a label in Σ_n and every leaf has a label in $\Sigma_0 + X$.

Initial H_Σ-algebra: We can represent $H_\Sigma\emptyset = \Sigma_0$ by the set of all singleton trees labelled by elements of Σ_0. Given a tree representation of $H_\Sigma^k\emptyset$, we represent

$$H_\Sigma^{k+1}\emptyset = \coprod_{n\in\mathbb{N}} \Sigma_n \times (H_\Sigma^k\emptyset)^n$$

by the set of all trees



[12.1]

with $\sigma \in \Sigma_n$ and $p_1, \ldots, p_n \in H_\Sigma^k\emptyset$. In this way we see that for every $k \in \mathbb{N}$

$$H_\Sigma^k\emptyset = \text{ all } \Sigma\text{-trees of depths less than } k\,.$$

The above ω-chain is the chain of inclusion maps $\emptyset \subseteq H_\Sigma\emptyset \subseteq H_\Sigma^2\emptyset \subseteq \ldots$ and its colimit

$$I = \bigcup_{k\in\mathbb{N}} H_\Sigma^k(\emptyset)$$

is the set of all finite Σ-trees. The algebraic structure $i\colon H_\Sigma I \to I$ is given by tree-tupling: to every n-tuple of trees corresponding to the summand of $\sigma \in \Sigma_n$ it assigns the tree [12.1] above.

12.5 Remark. Free H-algebras. Let H be a finitary endofunctor of *Set*. We now describe free H-algebras, that is, a left adjoint of the forgetful functor U_H. For every set X, the endofunctor

$$H(-) + X$$

is also finitary. Therefore, following 12.3, it has an initial algebra.

12.6 Proposition. *The free H-algebra on X is the initial algebra for the endofunctor $H(-) + X$.*

Explicitly, if H^*X is the initial algebra for $H(-) + X$ with structure

$$i_X\colon HH^*X + X \to H^*X,$$

then the components

$$\varphi_X\colon HH^*X \to H^*X \quad\text{and}\quad \eta_X\colon X \to H^*X$$

of i_X form the algebra structure and the universal arrow, respectively.

Proof. This follows easily from the observation that to specify an algebra for $H(-) + X$ on an set A means to specify an algebra $HA \to A$ for H and a function $X \to A$. $\square$

12.7 Corollary. *The free H-algebra on X is the colimit of the ω-chain*

$$\emptyset \to H\emptyset + X \to H(H\emptyset + X) + X \to \dots$$

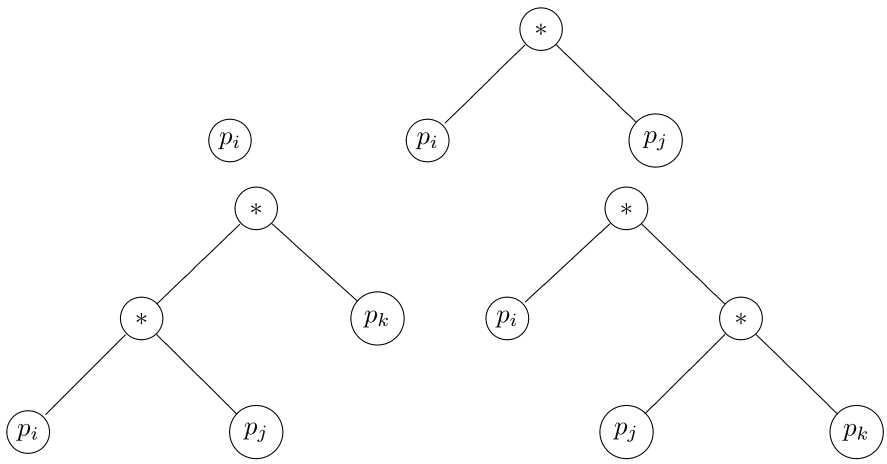
12.8 Notation. $F_H\colon \mathit{Set} \to H\text{-}\mathit{Alg}$ denotes the left adjoint of U_H. In case $H = H_\Sigma$ we use F_Σ instead of F_{H_Σ}.

12.9 Example. We describe the free H_Σ-algebra on an set X. Observe that

$$H_\Sigma(-) + X = H_{\overline{\Sigma}}$$

for the signature $\overline{\Sigma}$ obtained from Σ by adding nullary operation symbols from X. Thus, the description of initial algebra in 12.4 immediately yields a description of the free H_Σ-algebra $F_\Sigma X$ on X as the algebra of all finite Σ-trees on X. That is, finite labelled trees with leaves labelled in $X + \Sigma_0$, and nodes with $n > 0$ children labelled in Σ_n. The operations of $F_\Sigma X$ are given by tree-tupling. The universal arrow assigns to a variable $x \in X$ the singleton tree labelled x.

12.10 Example. For the signature Σ of a single binary operation $*$, we have a description of $F_\Sigma X$ for $X = \{p_1, p_2, p_3\}$ as all binary trees with leaves labelled by p_1, p_2, p_3. Examples:



12.11 Example. A commutative binary operation: this can be expressed by the functor H assigning to every set X the set HX of all unordered pairs in X, and to every function f the function Hf acting as f componentwise.

An H-algebra is a set with a commutative binary operation. The free algebra on X is the colimit of the chain

$$\emptyset \to H\emptyset + X = X \to HX + X \to H(HX + X) + X \to \dots$$

We can represent the elements of HZ as binary non-ordered trees with both subtrees elements of Z, then we see that the n-th set in the above chain consists of precisely all binary non-ordered trees of depth less than n with leaves labelled in X. Consequently the initial algebra is

$$IX = \text{ all unordered binary trees over } X\,.$$

12.12 Proposition. *For every finitary endofunctor H on Set the category H-Alg has*

(a) limits and

(b) sifted colimits

preserved by the forgetful functor.

We know from 6.30 that H preserves sifted colimits. We can generalize (b) to say: for every type of colimits preserved by H the category H-*Alg* has colimits of that type preserved by U_H.

Proof. We prove the more general formulation of (b), the proof of (a) is analogous. Let $D\colon \mathcal{D} \to H\text{-}Alg$ be a diagram with objects $Dd = (A_d, a_d)$ and let $A = colim\, A_d$ be the colimit of $U_H \cdot D$ in *Set* with the colimit cocone $c_d\colon A_d \to A$. If H preserves this colimit, there exists a unique H-algebra structure $a\colon HA \to A$ turning each c_d into a homomorphism. In fact, the commutative squares

$$\begin{tikzcd} HA_d \arrow[r, "Hc_d"] \arrow[d, "a_d"'] & HA \arrow[d, "a"] \\ A_d \arrow[r, "c_d"'] & A \end{tikzcd}$$

define a unique a since (i) $c_d \cdot a_d$ is a cocone of $U_H \cdot D$ and (ii) Mc_d is the colimit cocone of $H \cdot U_H \cdot D$. It is easy to see that the algebra (A, a) is a colimit of D in H-*Alg* with the cocone c_d. $\square$

12.13 Theorem. *For every finitary endofunctor H on Set the category H-Alg is cocomplete. It also has regular factorizations of morphisms preserved by the forgetful functor.*

Proof. (1) We start with the latter statement. Observe that regular epimorphisms split in *Set* and therefore H preserves them. Given a homomorphism $h\colon (A, a) \to (B, b)$ in H-*Alg* and a factorization $h = m \cdot e$ with

$$e\colon A \to C \ \text{ a regular epimorphism } , \quad m\colon C \to B \ \text{ a monomorphism}$$

in *Set*, use the diagonal fill-in to obtain an H-algebra structure $c\colon HC \to C$ turning e and m into homomorphisms:

$$\begin{tikzcd} HA \arrow[r, "He"] \arrow[d, "a"'] & HC \arrow[d, "Hm"] \arrow[ddl, dotted, "c"'] \\ A \arrow[d, "e"'] & HB \arrow[d, "b"] \\ C \arrow[r, "m"'] & B \end{tikzcd}$$

Since U_H is faithful, m in a monomorphism in H-*Alg*. And e is a regular epimorphism in H-*Alg* because given a pair $u, v\colon X \rightrightarrows A$ with coequalizer e in *Set*, the corresponding homomorphisms $\overline{u}, \overline{v}\colon F_H X \to A$ from the free H-algebra have the coequalizer e in H-*Alg*.
(2) Arguing as in 4.1, to prove that H-*Alg* is cocomplete it is sufficient to prove that it has finite coproducts. Since by 12.3 H-*Alg* has initial object, it remains to consider binary coproducts. Thus, we are to prove that the diagonal functor $\Delta\colon H\text{-}Alg \to H\text{-}Alg \times H\text{-}Alg$ has a left adjoint, see 0.11. Since by 12.12 the category H-*Alg* is complete, it is sufficient (using the Adjoint Functor Theorem 0.8) to find a solution set for every pair $(A_1, a_1), (A_2, a_2)$ of H-algebras. That is, we need a set of cospans

$$(A_1, a_1) \xrightarrow{f_1} (C, c) \xleftarrow{f_2} (A_2, a_2)$$

in H-*Alg* through which all cospans factorize. Consider the coproduct

$$A_1 \xrightarrow{c_1} A_1 + A_2 \xleftarrow{c_2} A_2$$

in *Set*, denote by $f\colon A_1 + A_2 \to C$ the morphism induced by the cospan (f_1, f_2), and let $\overline{f}\colon F_H(A_1 + A_2) \to (C, c)$ be the homomorphism corresponding to f by adjunction. We claim that a solution set is provided by those cospans (f_1, f_2) such that $\overline{f}$ is a regular quotient of $F_H(A_1 + A_2)$. This is indeed a set because $\overline{f}$ is a regular epimorphism also in *Set*. For any cospan

$$(A_1, a_1) \xrightarrow{g_1} (D, d) \xleftarrow{g_2} (A_2, a_2)$$

consider the regular factorization in H-*Alg*

$$\begin{tikzcd}
F_H(A_1 + A_2) \arrow[rr, "\overline{g}"] \arrow[dr, "e"'] & & (D, d) \\
& (C, c) \arrow[ur, "m"'] &
\end{tikzcd}$$

We get a new cospan in *Set* by defining

$$f_i\colon\ A_i \xrightarrow{c_i} A_1 + A_2 \xrightarrow{\eta_{A_1 + A_2}} F_H(A_1 + A_2) \xrightarrow{e} C \quad (i = 1, 2)$$

(where η is the unit of the adjunction $F_H \dashv U_H$). The cospan (g_1, g_2) factorizes through (f_1, f_2) because $g_i = m \cdot f_i$. Moreover, (f_1, f_2) is a cospan in H-*Alg* : this follows easily from the fact that g_i and m are homomorphisms of H-algebras and m is a monomorphism in *Set*. Finally, (f_1, f_2) has the desired property because $\overline{f} = e$. □

12.14 Remark. So far we have mentioned, besides the polynomial functors H_Σ, only one finitary functor that is not polynomial, see 12.11. And that functor is an obvious quotient of the polynomial functor $H_\Sigma X = X \times X$.

In general a *quotient* of a functor H is represented by a natural transformation $\alpha\colon H \to \overline{H}$ with epimorphic components. We now prove that finitary endofunctors of *Set* are indeed precisely the quotients of the polynomial ones:

12.15 Theorem. *For an endofunctor H on Set the following conditions are equivalent:*

1. *H is finitary,*
2. *H is a quotient of a polynomial functor*

and

3. *every element of HX lies in the image of Hi for the inclusion $i\colon Y \to X$ of a finite subset Y.*

Proof. $3 \Rightarrow 1$: Let

$$D\colon \mathcal{D} \to \mathit{Set}$$

be a filtered diagram with a colimit cocone

$$c_d\colon Dd \to C \quad (d \in \mathit{obj}\mathcal{D})\,.$$

We prove that the diagram $D \cdot H$ has the colimit

$$Hc_d\colon HDd \to HC$$

in *Set*. For that it is by 0.6 sufficient to verify that

(a) every element x of HC lies in the image of Hc_d for some d

and

(b) given elements $y_1, y_2 \in HDd$ merged by Hc_d, there exists a connecting morphism $f\colon d \to d'$ of $\mathcal{D}$ with Hf also merging y_1 and y_2.

For (a) choose a finite subset $i\colon Y \to X$ with x lying in the image of Hi. Since $C = \mathit{colim}\, Dd$ is a filtered colimit in *Set*, there exists a factorization

$$i = c_d \cdot j \quad \text{for some } d \in \mathit{obj}\mathcal{D} \text{ and } j\colon Y \to Dd\,.$$

Thus x lies in the image of Hc_d.

For (b) choose a finite subset $i\colon Z \to Dd$ such that y_1, y_2 lie in the image of Hi. Since $C = \mathit{colim}\, Dd$ is a filtered colimit, there exists a connecting morphism $f\colon d \to d'$ such that the domain restriction of $c_{d'}$ to the image $i'\colon Z' \to Dd'$ of $f \cdot i$ is a monomorphism. We obtain a commutative diagram

$$\begin{tikzcd}
Z \arrow[rr, "e"] \arrow[d, "i"'] & & Z' \arrow[d, "i'"] \\
Dd \arrow[rr, "Df"] \arrow[dr, "c_d"'] & & Dd' \arrow[dl, "c_{d'}"] \\
& C &
\end{tikzcd}$$

for some morphism e such that $c_{d'} \cdot i'$ is a monomorphism. Without loss of generality $Z' \neq \emptyset$ provided $C \neq \emptyset$. Then $c_{d'} \cdot i'$ is a split monomorphism. Then $H(c_{d'} \cdot i')$ also is a split monomorphism, and we conclude that HDf merges y_1, y_2.

$1 \Rightarrow 3$: Obvious from the description of fitered colimits in *Set* (see 0.6).

$1 \Rightarrow 2$: This follows from Yoneda Lemma. Define the signature Σ by using, for every $n \in \mathbb{N}$, the elements of Hn as the operation symbols σ of arity $n \in \mathbb{N}$. Shortly: $Hn = \Sigma_n$. Then we have a natural transformation $\alpha\colon H_\Sigma \to H$ which, given an operation symbol $\sigma \in \Sigma_n$ (that is, $\sigma \in Hn$), assigns to the corresponding n-tuple $f\colon n \to Z$ the value

$$\alpha_Z(\sigma(f)) = Hf(\sigma)\,.$$

In other words, the component of α_Z at the functor *Set*$(n, -)$ corresponding to $\sigma \in \Sigma_n$ is the Yoneda transformation of σ. Condition 3. tells us precisely that α_Z is a surjective map for all set Z.

$2 \Rightarrow 3$: Every polynomial functor satisfies condition 3. Indeed, to choose an element $x \in H_\Sigma X$ means to fix a symbol $\sigma \in \Sigma_n$, so that $x = (x_1, \ldots, x_n) \in X \times \ldots \times X$. Therefore, as Y we can take $\{x_1, \ldots, x_n\}$.

Let now $\alpha\colon H_\Sigma \to H$ be a quotient, and fix an element $x \in HX$. Since $\alpha_X\colon H_\Sigma X \to HX$ is surjective, there exists $y \in H_\Sigma$ such that $\alpha_X(y) = x$. Find a finite subset $i\colon Y \to X$ such that $y = (H_\Sigma i)(\overline{y})$ for some $\overline{y} \in H_\Sigma$. By naturality of α we have

$$x = \alpha_X(H_\Sigma i)(\overline{y}) = (Hi)\alpha_Y(\overline{y})\,.$$

$\square$

12.16 Remark. In 13.23 we will see that for every presentation of a functor H as a quotient functor of H_Σ, the category of H-algebras can be viewed as an equational category of Σ-algebras.

12.17 Remark. Most of the result in this chapter has an obvious generalization to endofunctors H of cocomplete categories $\mathcal{K}$ which preserve sifted colimits.

1. The initial chain of 12.2 is defined by denoting by $\emptyset$ an initial object of $\mathcal{K}$ and using the unique morphism $u\colon \emptyset \to H\emptyset$. The corresponding H-algebra is initial.
2. The free H-algebra on an object X of $\mathcal{K}$ is the intial algebra for the endofunctor $H(-) + X$.
3. The category *H-Alg* is complete and cocomplete, and the forgetful functor into $\mathcal{K}$ preserves limits and sifted colimits.

Historical Remarks for Chapter 12

Algebras for an endofunctor were introduced by J. Lambek in [62]. The initial algebra construction 12.2 and its free-algebra variation 12.7 stem from [1]. Factorizations and colimits in categories *H-Alg* were studied in [2].

The fact that finitary endofunctors on *Set* yield one-sorted algebraic categories follows from the work of M. Barr [15] on free monads, see Appendix A.

Chapter 13

Equational categories of Σ-algebras

In the present chapter we prove that one-sorted algebraic categories are precisely the equational categories of Σ-algebras for (one-sorted) signatures Σ. The case of S-sorted signatures is treated in Chapter 14.

13.1 Remark. We described a left adjoint

$$F_\Sigma \colon \mathit{Set} \to \Sigma\text{-}\mathit{Alg}$$

of the forgetful functor $U_\Sigma \colon \Sigma\text{-}\mathit{Alg} \to \mathit{Set}$ in 12.9. The more standard description is that $F_\Sigma X$ is the following Σ-term-algebra: the underlying set is the smallest set such that

- every element $x \in X$ is a Σ-term

and

- for every $\sigma \in \Sigma$ of arity n and for every n-tuple of Σ-terms $p_1, \ldots, p_n$ we have a Σ-term $\sigma(p_1, \ldots, p_n)$.

The Σ-algebra structure on $F_\Sigma X$ is given by the formation of terms $\sigma(p_1, \ldots, p_n)$. This defines a functor $F_\Sigma \colon \mathit{Set} \to \Sigma\text{-}\mathit{Alg}$ on objects. To define it on morphisms $f \colon X \to Z$, let $F_\Sigma f$ be the function which in every term p of $F_\Sigma X$ substitutes for every variable $x \in X$ the variable $f(x)$. More explicitly:

- if $x \in X$, then $F_\Sigma f(x) = f(x)$,
- if $p_1, \ldots, p_n \in F_\Sigma X$ and $\sigma \in \Sigma_n$, then

$$F_\Sigma f(\sigma(p_1, \ldots, p_n)) = \sigma(F_\Sigma f(p_1), \ldots, F_\Sigma f(p_n)).$$

It is easy to verify that F_Σ is a well-defined functor which is naturally isomorphic to the Σ-tree functor of 12.9. Thus, we have $F_\Sigma \dashv U_\Sigma$. The unit of the adjunction is the inclusion of variables into the set of Σ-terms: $\eta_X \colon X \to F_\Sigma X$.

13.2 Notation. Suppose that a set of "standard variables" $x_0, x_1, x_2, \dots$ is given. Then the free Σ-algebras

$$F_\Sigma\{x_0, \dots, x_{n-1}\}$$

yield, by 11.24, a one-sorted theory for Σ-*Alg*. We denote this theory by

$$(\mathcal{T}_\Sigma, T_\Sigma)\,.$$

Thus, morphisms from n to 1 in $\mathcal{T}_\Sigma$ are the Σ-terms in variables $x_0, \dots, x_{n-1}$. General hom-sets are given by k-tuples of these terms:

$$\mathcal{T}_\Sigma(n, k) = (F_\Sigma n)^k\,.$$

And $T_\Sigma\colon \mathcal{N} \to \mathcal{T}_\Sigma$ assigns to every function $g\colon k \to n$, that is, $g \in \mathcal{N}(n, k)$, the k-tuple of terms

$$x_{g(0)}, \dots, x_{g(k-1)}\,.$$

13.3 Lemma. *The category Σ-Alg is concretely equivalent to the category of algebras of $(\mathcal{T}_\Sigma, T_\Sigma)$.*

Proof. Define $E\colon \Sigma\text{-}Alg \to Alg\,\mathcal{T}_\Sigma$ on objects as follows. For a Σ-algebra (A, a), the corresponding functor from $\mathcal{T}_\Sigma$ to *Set* is given on objects by $n \mapsto A^n$ and on morphisms $t\colon n \to 1$ by the function $A^n \to A$ of evaluation of the term t in the given algebra. This function takes a map $f\colon n \to A$ to $\overline{f}(t) \in A$, where $\overline{f}\colon F_\Sigma\{x_0, \dots, x_{n-1}\} \to (A, a)$ is the unique homomorphism exteding f.
Conversely, if B is a $\mathcal{T}_\Sigma$-algebra we get a Σ-algebra structure on the set $B1$ as follows: if $\sigma \in \Sigma_n$, then $\sigma(x_0, \dots, x_{n-1}) \in F_\Sigma n = \mathcal{T}_\Sigma(n, 1)$ and this yields an n-ary operation on $B1$ by applying B to that morphism (recall that Bn is isomorphic to the n-th power of $B1$).
This gives a concrete equivalence $E\colon \Sigma\text{-}Alg \to Alg\,\mathcal{T}_\Sigma$. $\square$

13.4 Definition. Given signatures Σ and Σ', a *morphism of signatures* is a function $f\colon \Sigma \to \Sigma'$ preserving the arities. This leads to the category of signatures *Sign* – this is just *Set*/$\mathbb{N}$.

13.5 Definition. For every one-sorted algebraic theory $(\mathcal{T}, T)$ we define the signature

$$\mathcal{C}(\mathcal{T}, T)$$

as the signature whose n-ary symbols are precisely the morphisms from n to 1 in $\mathcal{T}$:

$$(\mathcal{C}(\mathcal{T}, T))_n = \mathcal{T}(n, 1)\,.$$

This construction can be easily extended to morphisms of one-sorted theories.

13.6 Example. The signature $\mathcal{C}(\mathcal{T}_\Sigma, T_\Sigma)$ has all Σ-terms in variables $x_0, \dots, x_{n-1}$ as n-ary operation symbols. Therefore, there is a canonical morphism of signatures

$$\eta_\Sigma\colon \Sigma \longrightarrow \mathcal{C}(\mathcal{T}_\Sigma, T_\Sigma)$$

given by $\eta_\Sigma(\sigma) = \sigma(x_0, \dots, x_{n-1}) \in F_\Sigma n$ for any σ of arity n.

13.7 Proposition. *(A free one-sorted theory on a signature) For every signature* Σ *the theory* $(\mathcal{T}_\Sigma, T_\Sigma)$ *is free on* Σ. *That is, given a one-sorted theory* $(\mathcal{T}, T)$ *for every morphism* $G\colon \Sigma \to \mathcal{C}(\mathcal{T}, T)$ *of signatures there exists a unique morphism* $\overline{G}\colon (\mathcal{T}_\Sigma, T_\Sigma) \to (\mathcal{T}, T)$ *of one-sorted theories such that* $\mathcal{C}(\overline{G}) \cdot \eta_\Sigma = G$

$$\begin{tikzcd} \Sigma \arrow[rr, "\eta_\Sigma"] \arrow[dr, "G"'] & & \mathcal{C}(\mathcal{T}_\Sigma, T_\Sigma) \arrow[dl, "\mathcal{C}(\overline{G})"] \\ & \mathcal{C}(\mathcal{T}, T) & \end{tikzcd}$$

Proof. (1) We define a functor $\overline{G}\colon \mathcal{T}_\Sigma \to \mathcal{T}$ on objects by $n \mapsto n$ and on morphisms $p \in \mathcal{T}_\Sigma(k, 1)$, that is Σ-terms on $\{x_0, \ldots, x_{k-1}\}$, by structural induction:

(i) For variables $x_i \in \mathcal{T}_\Sigma(k, 1)$ put $\overline{G}x_i = Tp_i^k$, the chosen projection in $\mathcal{T}(k, 1)$.

(ii) Given $p = \sigma(p_1, \ldots, p_n)$ where $\sigma \in \Sigma_n$, $p_i \in \mathcal{T}(k, 1)$ $(i = 1, \ldots, n)$ and $\overline{G}p_i$ are defined already, put

$$\overline{G}p\colon\ k \xrightarrow{<\overline{G}p_1, \ldots, \overline{G}p_n>} n \xrightarrow{G\sigma} 1$$

It is clear that $\overline{G} \cdot T_\Sigma = T$ and $\mathcal{C}(\overline{G}) \cdot \eta_\Sigma = G$.
(2) Uniqueness: Let $M\colon (\mathcal{T}_\Sigma, T_\Sigma) \to (\mathcal{T}, T)$ be a morphism of one-sorted theories with $\mathcal{C}(M) \cdot \eta_\Sigma = G$. Since M preserves finite products, all we have to prove is that it is determined when precomposed by η_Σ. This is the case, indeed:
(i) For variables $x_i \in \mathcal{T}_\Sigma(k, 1)$ use $M \cdot T_\Sigma = T$, so that $M(\eta_\Sigma(x_i))$ is the i-th projection p_i^k.
(ii) Consider $p = \sigma(p_1, \ldots, p_n)$ with $\sigma \in \Sigma_n$ and $p_i \in \mathcal{T}(k, 1)$ $(i = 1, \ldots, n)$. Since $\sigma(p_1, \ldots, p_n) = \sigma(x_1, \ldots, x_n)\cdot < p_1, \ldots, p_n >$ in $\mathcal{T}_\Sigma$, we have that $Mp = M(\eta_\Sigma(\sigma)) \cdot \langle Mp_1, \ldots, Mp_m \rangle$. ☐

13.8 Remark. Let $(\mathcal{T}, T)$ be a one-sorted theory. If we apply the construction described in the first part of the proof of 13.7 to the identity morphism on $\mathcal{C}(\mathcal{T}, T)$, we get a morphism

$$\varepsilon_{(\mathcal{T},T)}\colon \mathcal{T}_{\mathcal{C}(\mathcal{T},T)} \longrightarrow (\mathcal{T}, T)$$

of one-sorted theories. It is clearly full and then, by 10.13, the unique functor ε' making commutative the following diagram of morphisms of one-sorted theories is an isomorphism

$$\begin{tikzcd} \mathcal{T}_{\mathcal{C}(\mathcal{T},T)} \arrow[rr, "Q"] \arrow[dr, "\varepsilon_{(\mathcal{T},T)}"'] & & \mathcal{T}_{\mathcal{C}(\mathcal{T},T)}/\sim \arrow[dl, "\varepsilon'"] \\ & \mathcal{T} & \end{tikzcd}$$

Therefore:

1. $(\mathcal{T}, T)$ is a quotient of the free one-sorted theory $\mathcal{T}_{\mathcal{C}(\mathcal{T},T)}$.

2. $Alg\,\mathcal{T}$ is a variety of $\mathcal{C}(\mathcal{T}, T)$-algebras.

In order to improve the previous result we need the notion of equational category of Σ-algebras. We start comparing the classical notion of equation with the one introduced in Chapter 10.

13.9 Remark.

1. Classically equations are expressions

$$t = t'$$

where t and t' are terms in variables $x_0, \ldots, x_{n-1}$ for some n. This is a special case of 10.1: here we have a parallel pair $t, t' \colon n \rightrightarrows 1$ in the theory $\mathcal{T}_\Sigma$. Also, a Σ-algebra (A, a) satisfies this equation in the classical sense (that is, for every interpretation $f \colon \{x_0, \ldots, x_{n-1}\} \to A$ we have $\overline{f}(t) = \overline{f}(t')$) iff the corresponding $\mathcal{T}_\Sigma$-algebra satisfies this equation in the sense of 10.1.

2. In fact, for the theory $\mathcal{T}_\Sigma$ equations in the sense of 10.1 are equivalent to the classical equations: given a parallel pair

$$t, t' \colon n \rightrightarrows k \ \text{ in } \ \mathcal{T}_\Sigma$$

and given the k projections $p_i^k \colon k \to 1$ $(i = 0, \ldots, k-1)$ specified by the functor $T_\Sigma \colon \mathcal{N} \to \mathcal{T}_\Sigma$, we get a k-tuple of equations

$$p_i^k \cdot t = p_i^k \cdot t'$$

in the classical sense. It is clear that a Σ-algebra satisfies each of these k equations iff the corresponding $\mathcal{T}_\Sigma$-algebra satisfies $t = t'$ in the sense of 10.1.

13.10 Definition.

1. By an *equational category of Σ-algebras* is meant a full subcategory of Σ-*Alg* formed by all algebras satisfying a set E of equations. We denote such a caegory by

$$(\Sigma, E)\text{-}Alg\ .$$

The pair (Σ, E) is called an *equational theory*.

2. *Equational categories* are concrete categories over *Set* which are, for some signature Σ, equational categories of Σ-algebras.

13.11 Theorem. *One-sorted algebraic categories are precisely the equational categories.*

More detailed: a concrete category over *Set* is one-sorted algebraic iff it is concretely equivalent to an equational category of Σ-algebras for some signature Σ.

Proof. (1) Every equational category of Σ-algebras is a one-sorted algebraic category. In fact, following 13.9, the concrete equivalence $\Sigma\text{-}Alg \simeq Alg\,\mathcal{T}_\Sigma$ of 13.3 restricts to a concrete equivalence between $(\Sigma, E)\text{-}Alg$ and $Alg\,(\mathcal{T}_\Sigma/\sim_E)$, where $\sim_E$ is the congruence on $\mathcal{T}_\Sigma$ generated by E (see 10.7).
(2) Conversely, every one-sorted algebraic category is concretely equivalent to an equational category of Σ-algebras. In fact, $Alg\,\mathcal{T}$ is equivalent to $Alg\,(\mathcal{T}_\Sigma/\sim)$ for some congruence $\sim$ on $\mathcal{T}_\Sigma$ (13.8) and, therefore, to $(\Sigma, E)\text{-}Alg$ where E is the set of all equations $u = v$ where u and v are congruent terms. □

13.12 Example. Recall that a *semigroup* is an algebra on one associative binary operation. This means that we consider the Σ-algebras with $\Sigma = \{*\}$ which satisfy the equation

$$(x * y) * z \quad = \quad x * (y * z) \quad .$$

Thus, the theory of semigroups is the quotient theory $\mathcal{T}_\Sigma/\sim$ where $\sim$ is the congruence generated by the equation above.

13.13 Example. Beside the algebraic theory $\mathcal{T}_{ab}$ of abelian groups of 1.11 we now have a different one, based on the usual equational presentation: let $\Sigma = \{+, -, 0\}$ with $+$ binary, $-$ unary and 0 nullary. Then a theory of abelian groups is the quotient $\mathcal{T}_\Sigma/\sim$ modulo the congruence on $\mathcal{T}_\Sigma$ generated by the four equations

$$\begin{aligned} (x+y)+z &= x+(y+z) \\ x+y &= y+x \\ x+0 &= 0 \\ x+(-x) &= 0 \end{aligned}$$

13.14 Example. Recall that a *monoid* is a semigroup $(M, *)$ with a unit. We can consider the category of all monoids as the category $(\Sigma, E)\text{-}Alg$ where Σ has a binary symbol $*$ and a nullary symbol e, and E contains the associativity of $*$ and the equations

$$\begin{aligned} x &= x * e \\ x &= e * x \end{aligned}$$

(equivalently, as the category $Alg\,(\mathcal{T}_\Sigma/\sim)$ where $\sim$ is the congruence generated by the associativity of $*$ and the equations above).

13.15 Example. For every monoid M, an *M-set* is a pair (X, α) consisting of a set X and a monoid action $\alpha\colon M \times X \to X$ (the usual notation is mx in place of $\alpha(m, x)$) such that every element $x \in X$ satisfies $m(m'x) = (m * m')x$ for all $m, m' \in M$, and $ex = x$. The homomorphisms $f\colon (X, \alpha) \to (Z, \beta)$ of M-sets are the functions $f\colon X \to Z$ with $f(mx) = mf(x)$ for all $m \in M$ and $x \in X$. We can describe this category as $(\Sigma, E)\text{-}Alg$ where $\Sigma = M$ with all arities equal to

1, and E contains the equations

$$\begin{aligned} x &= e\,x \\ (m * m')\,x &= m\,(m'\,x) \end{aligned}$$

for all $m, m' \in M$.

13.16 Definition. A concrete category $U\colon \mathcal{A} \to \mathcal{K}$ is called:

1. *Amnestic* provided that given an isomorphism $i\colon A \to A'$ in $\mathcal{A}$ with $Ui = \mathrm{id}_{UA}$, then $A = A'$. (This implies $i = \mathrm{id}_A$ because U is faithful.)
2. *Transportable* provided that for every object A in $\mathcal{A}$ and every isomorphism $i\colon UA \to X$ in $\mathcal{K}$ there exists an isomorphism $j\colon A \to B$ in $\mathcal{A}$ with $UB = X$ and $Uj = i$.
3. *Uniquely transportable* if in 2. the isomorphism j is unique.

13.17 Example.

1. For every one-sorted algebraic theory $(\mathcal{T}, T)$ the concrete category

$$Alg\,T\colon Alg\,\mathcal{T} \to Set$$

is transportable – but almost never uniquely transportable (see 11.7).
In fact, given a $\mathcal{T}$-algebra A and a bijection $i\colon A1 \to X$, let $B\colon \mathcal{T} \to Set$ be defined on objects by $Bn = X^n$ and on morphisms $f\colon n \to k$ in the unique way making the powers of i natural:

$$\begin{tikzcd} (A1)^n \simeq An \arrow[r, "Af"] \arrow[d, "i^n"'] & Ak = (A1)^k \arrow[d, "i^k"] \\ X^n = Bn \arrow[r, "Bf"'] & Bk = X^k \end{tikzcd}$$

Then these powers form a natural isomorphism $j\colon A \to B$ with $j_1 = i$.

2. For every signature Σ the concrete category

$$U_\Sigma\colon \Sigma\text{-}Alg \to Set\,, \quad U_\Sigma(A, \sigma^A) = A$$

is uniquely transportable. In fact, given a bijection $i\colon A \to X$ there is a unique way of defining, for an n-ary symbol $\sigma \in \Sigma$, the operation σ^X so that the square

$$\begin{tikzcd} A^n \arrow[r, "\sigma^A"] \arrow[d, "i^n"'] & A \arrow[d, "i"] \\ X^n \arrow[r, "\sigma^X"'] & X \end{tikzcd}$$

commutes. The same is true for the equational categories of Σ-algebras. For example, the category of abelian groups is uniquely transportable.

3. For every endofunctor H of a category $\mathcal{K}$ the concrete category

$$U_H \colon H\text{-}Alg \to \mathcal{K}\ , \quad (A, a) \mapsto A$$

is uniquely transportable. In fact, given an algebra $a \colon HA \to A$ and an isomorphism $i \colon A \to X$ in $\mathcal{K}$, the unique algebra $x \colon HX \to X$ for which i becomes a homomorphism is $x = i \cdot a \cdot Hi^{-1}$.

13.18 Remark.

1. For every concrete category we have

$$\text{transportable} + \text{amnestic} \ \Leftrightarrow\ \text{uniquely transportable.}$$

In fact, if $(\mathcal{A}, U)$ is transportable and amnestic, and if in 13.16.2 we have another isomorphism $j' \colon A \to B'$ with $Uj' = i$, use 13.16.1 on the isomorphism $j' \cdot j^{-1} \colon B \to B'$ to conclude $B = B'$. Then $j = j'$ since U is faithful.
Conversely, if $(\mathcal{A}, U)$ is uniquely transportable, then by applying 13.16.3 to $i = \mathrm{id}_{UA}$ we deduce that it is amnestic.

2. Transportability is not invariant under concrete equivalence, thus, not all one-sorted algebraic categories are transportable. For example, let $E \colon Set' \to Set$ be the full subcategory of Set consisting of all cardinal numbers. Then (Set', E) is one-sorted algebraic because it is concretely isomorphic to (Set, Id). But it obviously fails to be transportable.

3. We will see in 13.21 a converse of 13.17.2: every uniquely transportable one-sorted algebraic category is (up to concrete isomorphism) an equational category.

4. In case of groupoids, the concept of (uniquely) transportable is precisely the concept of (discrete) fibration.

13.19 Definition. Given concrete categories $U \colon \mathcal{A} \to \mathcal{K}$ and $V \colon \mathcal{B} \to \mathcal{K}$ by a *concrete isomorphism* between them we mean a concrete functor

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "E"] \arrow[dr, "U"'] & & \mathcal{B} \arrow[dl, "V"] \\ & \mathcal{K} & \end{tikzcd}$$

and a functor $E' \colon \mathcal{B} \to \mathcal{A}$ such that both $E \cdot E'$ and $E' \cdot E$ are equal to the identity functors. (Note that such a functor E' is necessarily concrete.) We then say that $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$ are *concretely isomorphic.*

13.20 Lemma. *A concrete equivalence between uniquely transportable concrete categories is a concrete isomorphism.*

Proof. Given a concrete equivalence

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "E"] \arrow[dr, "U"'] & & \mathcal{B} \arrow[dl, "V"] \\ & \mathcal{K} & \end{tikzcd}$$

between uniquely transportable categories, we prove that E is bijective on objects – thus it is a (concrete) isomorphism.
(a) If A and A' are objects of $\mathcal{A}$ with $EA = EA'$, then for the identity morphism of EA there exists, since E is full, a morphism $f\colon A \to A'$ with $Ef = \mathrm{id}$. And f is of course an isomorphism in $\mathcal{A}$. Since $Uf = V(Ef) = \mathrm{id}$, amnesticity of U implies $A = A'$.
(b) For every object B of $\mathcal{B}$ there exists an isomorphism $i\colon EA \to B$ in $\mathcal{B}$ yielding an isomorphism $Vi\colon UA \to VB$ in $\mathcal{K}$. Let $j\colon A \to A'$ be the unique isomorphism in $\mathcal{A}$ with $Uj = Vi$. The isomorphism $Ej \cdot i^{-1}\colon B \to EA'$ fulfils $V(Ej \cdot i^{-1}) = Uj \cdot Vi^{-1} = \mathrm{id}$, thus by amnesticity of V we have $B = EA'$. ☐

13.21 Corollary. *Uniquely transportable one-sorted algebraic categories are, up to concrete isomorphism, precisely the equational categories.*

In fact, this follows from 13.11, 13.17 and 13.20.

Birkhoff's Variety Theorem (10.22) can be restated in the context of Σ-algebras:

13.22 Theorem. *Let Σ be a signature. A full subcategory $\mathcal{A}$ of Σ-Alg is equational if and only if it is closed in Σ-Alg under*

(a) products,

(b) subalgebras,

and

(c) regular quotients.

In fact, this follows from 10.22 and 13.11.

13.23 Proposition. *Let Σ be a one-sorted signature and $H\colon Set \to Set$ be a quotient of the polynomial functor H_Σ. The concrete category H-Alg is concretely isomorphic to an equational category of Σ-algebras.*

Proof. Let $\alpha\colon H_\Sigma \to H$ be a natural transformation with epimorphic components. We get a full and faithful functor

$$I\colon H\text{-}Alg \to H_\Sigma\text{-}Alg \quad I(A, a) = (A, a \cdot \alpha_A)\,.$$

Moreover, I is concrete since the diagram

$$\begin{tikzcd} H\text{-}Alg \arrow[rr, "I"] \arrow[dr, "U_H"'] & & H_\Sigma\text{-}Alg \arrow[dl, "U_\Sigma"] \; = \Sigma\text{-}Alg \\ & Set & \end{tikzcd}$$

clearly commutes.

Since I is injective on objects, H-*Alg* is concretely isomorphic to the full subcategory $I(H\text{-}Alg)$ of Σ-*Alg*. We are to prove that $I(H\text{-}Alg)$ satisfies the conditions of Proposition 13.22.

(a) Consider the above commutative diagram. Since U_H preserves and U_Σ reflects limits and sifted colimits, I preserves them. In particular, $I(H\text{-}Alg)$ is closed in Σ-*Alg* under products.

(b) Let $f\colon (A,x) \to I(B,b)$ be a monomorphism in Σ-*Alg* (and then in *Set*). Since α_A is a strong epimorphism in *Set*, we get an H-algebra structure on A by diagonal fill-in:

$$\begin{tikzcd} H_\Sigma A \arrow[rr, "H_\Sigma f"] \arrow[dr, "\alpha_A"] \arrow[dd, "x"'] & & H_\Sigma B \arrow[d, "\alpha_B"] \\ & HA \arrow[r, "Mf"] \arrow[dl, dotted, "a"] & HB \arrow[d, "b"] \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

This shows that $(A,x) = I(A,a)$.

(c) First observe that if $f\colon I(A,a) \to (B,y)$ is an isomorphism in Σ-*Alg*, then $(B,y) \in I(H\text{-}Alg)$. Indeed $f\colon (A,a) \to (B,b)$ is an isomorphism in H-*Alg*, where $b = f \cdot a \cdot Hf^{-1}$, and $I(B,b) = (B,y)$. (In other words, $(H\text{-}Alg, I)$ is transportable, see 13.16.)

Now consider a regular epimorphism $e\colon I(A,a) \to (B,y)$ in Σ-*Alg*. Its kernel pair, being a subobject of $I(A,a) \times I(A,a)$, lies in the image of I and we can take its coequalizer (Q,q) in H-*Alg*. This coequalizer is preserved by I (because it is reflexive, see 3.4). Thus $I(Q,q) \simeq (B,y)$. □

13.24 Remark. In general algebra the concepts of finitely generated and finitely presentable Σ-algebra are defined as follows: a Σ-algebra (A,a) is called

(a) finitely generated if it is generated by a finite set (see 11.10), that is, (A,a) is isomorphic to

$$F_\Sigma\{x_1, \ldots, x_n\}/\sim$$

for some congruence $\sim$ on $F_\Sigma\{x_1, \ldots, x_n\}$

and

(b) finitely presentable if it is such a quotient modulo a finitely generated congruence, that is, (A, a) is isomorphic to $F_\Sigma\{x_1, \ldots, x_n\}/\sim$ for some congruence $\sim$ generated by finitely many equations.

It turns out that these concepts coincide with the categorical concepts of 5.21 and 5.3, respectively. Let us first observe that:

(1) Every subalgebra generated by a set X (see 11.10) is a regular quotient of the free algebra $F_\Sigma X$. In fact, let B be the subalgebra of A generated by X and let $f\colon F_\Sigma X \to A$ be the unique homomorphism extending the inclusion map. Then the image $f(F_\Sigma X)$ is a subalgebra of A because the forgetful functor preserves regular factorizations, see 11.9. This is, obviously, the least subalgebra containing X, and the codomain restriction of f is a regular epimorphism.

(2) Conversely, every regular quotient $q\colon F_\Sigma X \to A$ of a free algebra generated by X is "generated by X" – more precisely, the image of the map

$$X \xrightarrow{\eta_\Sigma} U_\Sigma(F_\Sigma X) \xrightarrow{U_\Sigma q} U_\Sigma A$$

generates A.

13.25 Proposition. *A Σ-algebra is a finitely generated object of Σ-Alg if and only if it is a regular quotient of a finitely generated free algebra.*

Proof. (1) Let (A, a) be a finitely generated object of Σ-*Alg*. Form a diagram in Σ-*Alg* indexed by the poset of all finite subsets of A by assigning to every such $X \subseteq A$ the subalgebra $\overline{X}$ of A generated by X (see 11.10). Given finite subsets X and Z with $X \subseteq Z \subseteq A$, the connecting map $\overline{X} \to \overline{Z}$ is the inclusion map. Then the inclusion homomorphisms $i_X\colon \overline{X} \to A$ form a colimit cocone of this directed diagram. Since the functor Σ-*Alg*$(A, -)$ preserves this colimit, for $\mathrm{id}_A \in \Sigma$-*Alg*(A, A) there exists a finite set X such that id_A lies in the image of i_X – but this proves $\overline{X} = A$.
(2) Let us prove that every quotient $A = F\{x_1, \ldots, x_n\}/\sim$ is finitely generated. Given a directed diagram of subobjects B_i $(i \in I)$ with a colimit $B = \mathit{colim}\, B_i$, it is our task to prove that Σ-*Alg*$(A, -)$ preserves this colimit, that is, every homomorphism $h\colon A \to B$ factorizes through one of the colimit homomorphisms $b_i\colon B_i \to B$. For the finite set $\{h(\eta_X(x_k))\}_{k=1}^n$ there exists $i \in I$ such that this set lies in the image of b_i. From that it easily follows that the image of h is contained into the image of b_i. Since b_i is a monomorphism, it follows that there exists a homomorphism $g\colon A \to B_i$ with $h = b_i \cdot g$, as requested. $\square$

13.26 Proposition. *A Σ-algebra is a finitely presentable object of Σ-Alg if and only if it is a regular quotient of a finitely generated free algebra modulo a finitely generated congruence.*

Proof. (1) Let (A, a) be a finitely presentable object. By 11.28 there exists a coequalizer

$$F_\Sigma X \overset{u}{\underset{v}{\rightrightarrows}} F_\Sigma Z \xrightarrow{c} (A, a)$$

with X and Z finite. Let $\sim$ be the congruence generated by the finitely many equations $u(\eta_\Sigma(x)) = v(\eta_\Sigma(x))$ where $x \in X$, then (A, a) is clearly isomorphic to $F_\Sigma Z/\sim$: the canonical morphism $q\colon F_\Sigma Z \to F_\Sigma Z/\sim$ is namely also a coequalizer of u and v.
(2) Conversely, let Z be a finite set and $\sim$ a congruence on $F_\Sigma Z$ generated by equations $t_1 = s_1, \dots, t_k = s_k$. For $X = \{1, \dots, k\}$ define homomorphisms $u, v\colon F_\Sigma X \rightrightarrows F_\Sigma Z$ by

$$u(i) = t_i \quad \text{and} \quad v(i) = s_i \quad \text{for} \quad i = 1, \dots, k\,.$$

Then the canonical map $q\colon F_\Sigma Z \to F_\Sigma Z/\sim$ is a coequalizer of u and v. Therefore, $F_\Sigma Z/\sim$ is finitely presentable by 11.28. □

Historical Remarks for Chapter 13

This chapter contains classical results one can find in every introduction to general algebra, e.g. [37].

The concepts of amnestic and transportable concrete categories are taken from [4].

Chapter 14

S-sorted algebraic categories

In the previous chapters we have considered one-sorted algebraic categories. They are categories equipped with a forgetful functor into *Set*, like groups, abelian groups, lattices, etc. In computer science one often considers S-sorted algebras, where S is a given nonempty set (of sorts), and algebras are not sets with operations, but rather S-indexed families of sets with operations of given sort. This means that the forgetful functor is into Set^S rather than into *Set*. In this chapter we revisit one-sorted algebraic categories generalizing definitions and several results to the S-sorted case, see 14.3.

Analogously to the one-sorted case, where the theory has objects X^n (which we represented by n alone) and projections $\pi_i^n \colon X^n \to X$ are specified, in case of S-sorted theories we have objects X_s for $s \in S$ that generated the whole theory in the sense that every object of $\mathcal{T}$ is a product

$$X_{s_0} \times \ldots \times X_{s_{n-1}}$$

for some word $w = s_0 \ldots s_{n-1}$ over S. We, again, suppose that projections

$$\pi_i^w \colon X_{s_0} \times X_{s_{n-1}} \to X_{s_i} \quad (i = 0, \ldots, n-1)$$

are chosen. And, again, instead with the above product we work with the word $s_0 \ldots s_{n-1}$ alone. In other words, the theory $\mathcal{N}$ which plays a central role for one-sorted theories is generalized to the following:

14.1 Notation. For every nonempty set S (of sorts) we denote by

$$S^*$$

the category whose objects are the finite words on S, and whose morphisms from $s_0 \ldots s_{n-1}$ to $t_0 \ldots, t_{k-1}$ are all functions $f \colon k \to n$ with $s_{f(i)} = t_i$ for all

$i = 0, \dots, k-1$.
In particular, for every word $w = s_0 \dots s_{n-1}$ we have the projections

$$\pi_i^w \colon s_0 \dots s_{n-1} \to s_i \quad (i = 0, \dots, n-1)$$

given by the i-th injection $1 \mapsto n$ in *Set*.

14.2 Example. $\mathcal{N} = \{s\}^*$ provided that we identify every natural number n with the word $ss \dots s$ of length n.

14.3 Remark. We know from 1.10 that S^* is an algebraic theory for Set^S, and every word w is a product of one-letter words with the projections $\pi_0^w, \dots, \pi_{n-1}^w$ above. We are going to identify Set^S with $Alg\, S^*$. The full embedding

$$Y_{S^*} \colon (S^*)^{op} \to Set^S$$

assigns to a word $w = s_0 \dots s_{n-1}$ the S-sorted set

$$Y_{S^*}(w)_s = \{i = 0, \dots, n-1\, ;\ s_i = s\}\, .$$

14.4 Definition. Let S be a nonempty set.

1. A *S-sorted algebraic theory* is a pair $(\mathcal{T}, T)$ where $\mathcal{T}$ is an algebraic theory whose objects are the words over S, and $T \colon S^* \to \mathcal{T}$ is a theory morphism which is the identity map on objects.

2. A *morphism* of S-sorted algebraic theories $M \colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ is a functor $M \colon \mathcal{T}_1 \to \mathcal{T}_2$ such that $M \cdot T_1 = T_2$.

14.5 Remark. Analogously to 11.3 we have not requested that morphisms of one sorted theories preserve finite products since this simply follows from the equation $M \cdot T_1 = T_2$. Observe that due to that equation M is the identity map on objects.

14.6 Example.

1. For the category *Graph* of graphs we have an S-sorted theory with $S = \{v, e\}$ and $T \colon S^* \to \mathcal{T}_{graph}$ is determined by $Tv = v$ and $Te = e$ (recall that $\mathcal{T}_{graph}$ is the free completion under finite products of the category $\mathcal{C}_{graph}$ described in 1.15).

2. Let $\mathcal{C}$ be a small category, put $S = obj\,\mathcal{C}$, and let $E_{Th} : \mathcal{C} \to \mathcal{T}_{\mathcal{C}}$ denote the free completion of $\mathcal{C}$ under finite products (1.6); recall from 1.7 that the objects of $\mathcal{T}_{\mathcal{C}}$ can be viewed as words over S. Moreover, if $\mathcal{C}$ is discrete, then $\mathcal{T}_{\mathcal{C}} = S^*$. Therefore, we have a unique theory morphism $T_{\mathcal{C}} \colon S^* \to \mathcal{T}_{\mathcal{C}}$ which is the identity map on objects. We obtain an S-sorted theory

$$(\mathcal{T}_{\mathcal{C}}, T_{\mathcal{C}})\, .$$

14.7 Remark. Precisely as in the one-sorted case, the functor T does not influence the concept of algebra: the category $Alg\,\mathcal{T}$ thus consists, again, of all functors $A\colon \mathcal{T} \to Set$ preserving finite products. However, the presence of T makes the category of algebras concrete over Set^S: the forgetful functor is simply

$$Alg\,T\colon Alg\,\mathcal{T} \to Set^S$$

(see 14.4). More precisely, this forgetful functor takes an algebra $A\colon \mathcal{T} \to Set$ to the S-sorted set $\langle As\rangle_{s\in S}$, and a homomorphism $h\colon A \to B$ to the S-sorted function with components $h_s\colon As \to Bs$.

14.8 Proposition. *Let $(\mathcal{T}, T)$ be an S-sorted algebraic theory. The forgetful functor*

$$Alg\,T\colon Alg\,\mathcal{T} \to Set^S$$

is faithful, algebraic and conservative. It thus preserves and reflects limits, sifted colimits, monomorphisms and regular epimorphisms.

The proof is analogous to that of 11.8.

14.9 Remark. The concept of one-sorted algebraic category in Chapter 11 used concrete equivalences of categories over *Set*. For S-sorted algebraic theories we need, analogously, concrete equivalences over Set^S (see 11.12).

14.10 Definition. A *S-sorted algebraic category* is a concrete category over Set^S which is concretely equivalent to $Alg\,T\colon Alg\,\mathcal{T} \to Set^S$ for an S-sorted algebraic theory $(\mathcal{T}, T)$.

14.11 Proposition. *Every variety of $\mathcal{T}$-algebras for an S-sorted theory $(\mathcal{T}, T)$ is an S-sorted algebraic category.*

The proof is analogous to that of 11.17.

14.12 Remark. Let $(\mathcal{T}, T)$ be an S-sorted theory.

1. The forgetful functor $Alg\,T\colon Alg\,\mathcal{T} \to Set^S$ has a left adjoint. In fact, due to 4.11 applied to $Y_{S^*}\colon (S^*)^{op} \to Set^S$ we can choose a left adjoint

   $$F_T\colon Set^S \to Alg\,\mathcal{T}$$

   in such a way that the square

   $$\begin{tikzcd}
   (S^*)^{op} \arrow[r, "T^{op}"] \arrow[d, "Y_{S^*}"'] & \mathcal{T}^{op} \arrow[d, "Y_{\mathcal{T}}"] \\
   Set^S \arrow[r, "F_T"'] & Alg\,\mathcal{T}
   \end{tikzcd}$$

   commutes.

2. $\mathcal{T}$-algebras of the form $F_T(X)$, for X an S-sorted set, are called *free algebras.* If X is finite (1.10), they are called *finitely generated free algebras.*

14.13 Corollary. *Let $(\mathcal{T}, T)$ be an S-sorted theory. $\mathcal{T}^{op}$ is equivalent to the full subcategory of $Alg\mathcal{T}$ of finitely generated free algebras.*

14.14 Remark. Results about finitely presentable and perfectly presentable algebras and regular projectives generalize easily from the one-sorted case, see 11.26 – 11.33, to the S-sorted one, we leave this for the reader. Let us just stress that when working with variables, in the S-sorted case to every variable a sort is assigned. That is, the corresponding object of variables also lives in Set^S.

14.15 Theorem. *(One-sorted algebraic duality) The 2-category $ALG^{\,S}$ of S-sorted algebraic categories, concrete functors and natural transformations is biequivalent to the dual of the 2-category $Th^{\,S}$ of S-sorted algebraic theories, morphisms of S-sorted theories and natural transformations.*

The proof is completely analogous to that of 11.39.

The concept of signature also generalizes from one-sorted to S-sorted easily. Note however that we cannot speak about n-ary operations. Instead we need to specify $n+1$ sorts: one sort for each of the variables, and the sort of the result.

14.16 Definition.

1. An *S-sorted signature* is a set Σ (of operation symbols) together with an arity function assigning to every symbol σ in Σ an element of $S^* \times S$; we write
$$\sigma\colon s_0 \dots s_{n-1} \to s$$
if the pair is $(s_0 \dots s_{n-1}, s)$, including the case
$$\sigma\colon s$$
for $n = 0$.

2. A *Σ-algebra* is a pair (A, a) consisting of an S-sorted set $A = \langle A_s \rangle_{s \in S}$ and a function assigning to every element $\sigma\colon s_0 \dots s_{n-1} \to s$ of Σ a mapping
$$\sigma^A\colon A_{s_0} \times \dots \times A_{s_{n-1}} \to A_s\,.$$
(In case $n = 0$ we have a constant $\sigma^A \in A_s$.)

3. Σ-homomorphisms from (A, a) to (B, b) are S-sorted functions
$$f = \langle f_s \rangle \ \text{ with } \ f_s\colon A_s \to B_s \ \ (s \in S)$$
such that for every element $\sigma\colon s_0 \dots s_{n-1} \to s$ of Σ the square
$$\begin{tikzcd} A_{s_0} \times \dots \times A_{s_{n-1}} \arrow[r, "\sigma^A"] \arrow[d, "f_{s_0} \times \dots \times f_{s_{n-1}}"'] & A_s \arrow[d, "f_s"] \\ B_{s_0} \times \dots \times B_{s_{n-1}} \arrow[r, "\sigma^B"'] & B_s \end{tikzcd}$$

commutes. This yields a concrete category

$$\Sigma\text{-}Alg$$

of Σ-algebras with the forgetful functor

$$U_\Sigma \colon \Sigma\text{-}Alg \to Set^S\ ,\ \ U_\Sigma(A,a) = A\,.$$

14.17 Example.

1. The category of graphs has the form Σ-*Alg* for $S = \{v, e\}$ and Σ consisting of two operations of arity $e \to v$ (called τ and σ in 1.15).
2. For $\Sigma = \emptyset$ we have $\Sigma\text{-}Alg = Set^S$.
3. For sequential automata (see 1.20) put $S = \{s, i, o\}$ and $\Sigma = \{\delta, \gamma, \varphi\}$ with arities $\delta \colon si \to s$, $\gamma \colon s \to o$ and $\varphi \colon s$.
4. For the example of stacks 1.19 put $S = \{s, n\}$ and $\Sigma = \{$succ, push, pop, top, $0, e\}$ with the arities given in 1.19.

14.18 Remark.

1. The description of a left adjoint

$$F_\Sigma \colon Set^S \to \Sigma\text{-}Alg$$

   of U_Σ is completely analogous to 13.1. Given an S-sorted set X of variables, we form the smallest S-sorted set $F_\Sigma X$ (of terms) such that every element $x \in X_s$ is a term of sort s, and for every $\sigma \in \Sigma$ of arity $s_0 \dots s_{n-1} \to s$ and for every n-tuple of terms $p_0, \dots, p_{n-1}$ of sorts $s_0, \dots, s_{n-1}$ respectively, we have a term $\sigma(p_0, \dots, p_{n-1})$ of sort s. The Σ-algebra structure on $F_\Sigma X$ is given by the formation of terms $\sigma(p_0, \dots, p_{n-1})$.
   This defines a functor $F_\Sigma \colon Set^S \to \Sigma\text{-}Alg$ on object. To define it on morphisms, proceed as in 13.1.

2. We obtain, assuming a countable set of "standard variables x_i^s of sort s" for every $s \in S$, an S-sorted theory

$$(\mathcal{T}_\Sigma, T_\Sigma)$$

   analogous to the one described in 13.2: the words $s_0 \dots s_{n-1}$ represent the free Σ-algebra $F_\Sigma\{x_0^{s_0}, \dots, x_{n-1}^{s_{n-1}}\}$. The categories Σ-*Alg* and $Alg\,\mathcal{T}_\Sigma$ are concretely equivalent over Set^S, this is analogous to 13.3.

3. Equations in the sense of 10.1 can be substituted by expressions

$$t = t'$$

   where t and t' are two elements of $F_\Sigma X$ of the same sort (for some finite S-sorted set X of standard variables). This is analogous to 13.9, except that in the S-sorted case the quantification of variables must be made explicit. If Σ is a one-sorted signature and t, t' are terms in $F_\Sigma X$, then in place of X we can take the set $Z \subseteq X$ of all variables that appear in t or t'. An algebra satisfies $t = t'$ independently of whether we work with $F_\Sigma Z$ or $F_\Sigma X$. This is not so in S-sorted signatures, as we demonstrate below. We therefore need the following

14.19 Definition. Given an S-sorted signature Σ, by an *equation* is meant an expression

$$\forall x_0 \, \forall x_1 \ldots \forall x_{n-1} \, (t = t')$$

where x_i is a variable of sort s_i ($i = 0, \ldots, n-1$) and t, t' are elements of $F_\Sigma\{x_0, \ldots, x_{n-1}\}$ of the same sort s. A Σ-algebra (A, a) *satisfies* the equation provided that for every S-sorted function $f \colon \{x_0, \ldots, x_{n-1}\} \to A$ the unique homomorphism $\overline{f} \colon F_\Sigma\{x_0, \ldots, x_{n-1}\} \to (A, a)$ extending f fulfils $\overline{f}_s(t) = \overline{f}_s(t')$. In case $n = 0$ we write $\forall \emptyset \, (t = t')$.

14.20 Example.

1. In the signature of graphs, see 14.17, consider variables x, x' of sort v and a variable y of sort e. the equation

$$\forall x \, \forall x' \, (x = x')$$

describes graphs on at most one vertex. Whereas

$$\forall x \, \forall x' \, \forall y \, (x = x')$$

describes all graphs that either have no edge or have just one vertex.

2. In the theory of stacks there are several equations one expects to be required. For example, if a natural number x is inserted into a stack y and then deleted, the stack does not change:

$$\forall x \, \forall y \, (\mathrm{pop}(\mathrm{push}(x, y)) = y \, .$$

Other such equations are

$$\forall x \, \forall y \, (\mathrm{top}(\mathrm{push}(x, y)) = x$$

and (due to our definition of top)

$$\mathrm{top}(e) = 0 \, .$$

14.21 Example. Let us return to Example 10.23 explaining that Birkhoff's Variety Theorem requires, in general, the use of directed unions. The example worked with $\mathit{Set}^{\mathbb{N}}$ which is Σ-Alg for the empty $\mathbb{N}$-sorted signature. Let x_n and y_n be variables of sort $n \in \mathbb{N}$, and consider the equation quantifying y_n and all $x_0, x_1, x_2, \ldots$:

$$\forall y_n \, \forall x_0 \, \forall x_1 \, \forall x_2 \ldots (x_n = y_n) \, .$$

Then algebras, i.e., $\mathbb{N}$-sorted sets, satisfy these equations iff they lie in the category $\mathcal{A}$. However, infinite quantification brings us out of the finitary logic (and out of the realm of Definition 10.1).

14.22 Definition.

1. Let Σ be an S-sorted signature. By an *S-sorted equational category of Σ-algebras* is meant a full subcategory of Σ-*Alg* formed by all algebras satisfying a set E of equations (in the sense of 10.1 or, equivalently, 14.18).

2. *S-sorted equational categories* are concrete categories over Set^S which are, for some signature Σ, S-sorted equational categories of Σ-algebras.

14.23 Proposition.

1. *S-sorted algebraic categories are precisely the S-sorted equational categories. More detailed: a concrete category over Set^S is S-sorted algebraic iff it is concretely equivalent to an S-sorted equational category of Σ-algebras for some signature Σ.*

2. *Uniquely transportable S-sorted algebraic categories are, up to concrete isomorphism, precisely the S-sorted equational categories.*

The proof is completely analogous to that of 13.11 and 13.21.

14.24 Remark. Every S-sorted equational category is, of course, complete and cocomplete. In particular initial algebras exist in all S-sorted equational categories. In theoretical computer science these algebras are used as a formalization of "abstract data types": these are given by operations and equations and consist of elements generated by the given operations (no extra variables are used) and satisfy only the equations that are consequences of the given ones. An abstract data type is thus, precisely as initial objects should be, determined only up to isomorphism. We illustrate this on a couple of examples:

14.25 Example.

1. Natural numbers is a one-sorted abstract data type given by a constant 0 and a unary operation s (successor). This correspond to the initial algebra of the one-sorted signature $\Sigma = \{s, 0\}$ with arity 1 and 0, respectively. In fact, every initial Σ-algebra is a representation of natural numbers.

2. Stacks of natural numbers. Here we need the two-sorted signature Σ of 14.17.4. Its initial algebra does not resemble stacks because we will have formal terms such as top(e), top(top(e)), etc. However, the equational category given by the three equations of 14.20.2 has an initial algebra $I = \langle I_n, I_s \rangle$ where I_n is the abstract data type of natural numbers (no equation involves the operation succ) and I_s consists of stacks

$$e = [\ ],\ [x],\ [x, y],\ [x, y, z], \ldots$$

of elements $x, y, z, \ldots$ of I_n.

14.26 Remark. For one-sorted signatures we have Σ-*Alg* concretely equivalent to H_Σ-*Alg*, where $H_\Sigma A = \coprod_{\sigma\in\Sigma} A^n$ (for n = arity of σ), see 2.24.
Analogously for S-sorted signatures Σ: define

$$H_\Sigma \colon \mathit{Set}^S \to \mathit{Set}^S$$

on objects $A = \langle A_s \rangle_{s\in S}$ by setting the sort s of $H_\Sigma A$ as follows; we denote by $\Sigma_s \subseteq \Sigma$ the set of all symbols of output sort s and put

$$(H_\Sigma A)_s = \coprod_{\sigma\in\Sigma_s} A_{s_0} \times \ldots \times A_{s_{n-1}}$$

for the arity $s_0 \ldots s_{n-1} \to s$ of σ. Then there is a concrete equivalence

$$\begin{tikzcd}
\Sigma\text{-}\mathit{Alg} \arrow[rr, "E"] \arrow[dr, "U_\Sigma"'] & & H_\Sigma\text{-}\mathit{Alg} \arrow[dl, "U_{H_\Sigma}"] \\
& \mathit{Set}^S &
\end{tikzcd}$$

assigning to every Σ-algebra (A, a) the H_Σ-algebra $(A, \overline{a})$ where the coproduct components of

$$\overline{a}_s \colon \coprod_{\sigma\in\Sigma_s} A_{s_0} \times \ldots \times A_{s_{n-1}} \to A_s$$

are the given operations σ^A.

14.27 Proposition. *Every finitary endofunctor H of Set^S is a quotient of a polynomial functor H_Σ for some S-sorted signature Σ. Moreover, the concrete category H-Alg is concretely isomorphic to an equational category of Σ-algebras.*

Proof. For the first statement, the argument in 12.15 using Yoneda Lemma generalizes without a problem: define an S-sorted signature Σ whose operations σ of (an arbitrary) arity $s_0 \ldots s_{n-1} \to s$ are precisely the elements of sort s in HX, where the S-sorted set X is given by

$$X_t = \{i = 0, \ldots, n-1; s_i = t\} \quad \text{for all} \quad t \in S\,.$$

Then define $\alpha \colon H_\Sigma \to H$ by taking such an operation symbol σ and putting

$$\alpha_Z(\sigma(f)) = (Hf)_s(\sigma)$$

for all S-sorted function f from the above set X into Z.
For the second statement, the only difference to the proof of 13.23 is that in the S-sorted case we must also check that $I(H\text{-}\mathit{Alg})$ is closed in Σ-*Alg* under directed unions. This follows from the commutativity of the diagram

$$\begin{tikzcd}
H\text{-}\mathit{Alg} \arrow[rr, "I"] \arrow[dr, "U_H"'] & & H_\Sigma\text{-}\mathit{Alg} = \Sigma\text{-}\mathit{Alg} \arrow[dl, "U_\Sigma"] \\
& \mathit{Set}^S &
\end{tikzcd}$$

since U_H preserves sifted colimits and U_Σ reflects them (see 12.17.3). $\square$

14.28 Remark.

1. The converse implication of Theorem 12.15 does not generalize to the S-sorted case: a quotient of a polynomial functor on Set^S need not be finitary.
   A simple example can be presented in $Set^{\mathbb{N}}$: start with the constant functor of value $2 = 1 + 1$ (the S-sorted set having two elements in every sort). This functor is clearly polynomial. Let H be the quotient with $HX = 1$ whenever all sorts of X are nonempty, else, $HX = 2$. This functor does not preserve the filtered colimit of all finitely presentable subobjects of 1.

2. For finite sets S of sorts Theorem 12.15 fully generalizes: finitary endofuctors of Set^S are precisely the quotients of polynomial functors. In fact, the proof of 12.15 easily modifies: in part (b) of the implication $3 \Rightarrow 1$ choose the S-sorted set Z' in such a way that for every sort s we have $Z'_s \neq \emptyset$ iff $C_s \neq \emptyset$; since $\mathcal{D}$ is filtered and S is finite, this choice is clearly possible. Then, again, $c'_d \cdot i'$ is a split monomorphism.

Historical Remarks for Chapter 14

Historical comments on S-sorted algebras have been mentioned already at the end of Chapter 1. For a short introduction to applications of S-sorted algebras see [97].

Chapter 15

Morita equivalence

In this chapter we study the problem of the presentation of an algebraic category by different algebraic theories. This is inspired by the classical work of Kiiti Morita who in 1950's studied this problem for the categories *R-Mod* of left modules over a ring R. He completely characterized pairs of rings R and S such that *R-Mod* and *S-Mod* are equivalent categories; such rings are nowadays called *Morita equivalent*. We will recall the results of Morita below, and we will show in which way they generalize from *R-Mod* to $Alg\,\mathcal{T}$ where $\mathcal{T}$ is an algebraic theory.

We begin with a particularly simple example.

15.1 Example. In 1.9, we described a one-sorted algebraic theory $\mathcal{N}$ of Set : $\mathcal{N}$ is the full subcategory of Set^{op} whose objects are the natural numbers. Here is another one-sorted theory of Set : $\mathcal{T}_2$ is the full subcategory of Set^{op} whose objects are the even natural numbers $0, 2, 4, 6, \ldots$. $\mathcal{T}_2$ obviously has finite products. Observe that $\mathcal{T}_2$ is not idempotent-complete (consider the constant functions $2 \to 2$) and that $\mathcal{N}$ is an idempotent completion of $\mathcal{T}_2$: for every natural number n we can find an idempotent function $f\colon 2n \to 2n$ with precisely n fixed points. Then n is obtained by splitting f. Following 6.14 and 8.12, $Alg\,\mathcal{T}_2 \simeq Alg\,\mathcal{N} \simeq Set$.
In fact, we can repeat the previous argument for every natural number $k > 0$. In this way we get a family $\mathcal{T}_k$, $k = 1, 2, \ldots$ of one-sorted algebraic theory of Set (with $\mathcal{T}_1 = \mathcal{N}$). We will prove later that, up to equivalence, there is no other one-sorted algebraic theory of Set.

Clearly, if $\mathcal{T}$ and $\mathcal{T}'$ are algebraic theories and if there is an equivalence $\mathcal{T} \simeq \mathcal{T}'$, then $Alg\,\mathcal{T}$ and $Alg\,\mathcal{T}'$ are equivalent categories. The previous example shows that the converse is not true.

15.2 Definition. Two algebraic theories $\mathcal{T}$ and $\mathcal{T}'$ are called *Morita equivalent* if the corresponding categories $Alg\,\mathcal{T}$ and $Alg\,\mathcal{T}'$ are equivalent.

From 6.14 and 8.12. we already know a simple characterization of Morita equivalent algebraic theories: two theories are Morita equivalent iff they have equivalent idempotent completions. In case of S-sorted algebraic categories a much sharper result can be proved. Before doing so, let us recall the classical result of Morita.

15.3 Example. Let R be a unitary ring (not necessarily commutative) and denote by R-*Mod* for the category of left R-modules. There are two basic constructions:

1. Matrix ring $R^{[k]}$. This is the ring of all $k \times k$ matrices over R with the ususal addition, multiplication, and unit matrix. This ring $R^{[k]}$ is Morita equivalent to R for every $k > 0$, i.e., the category $R^{[k]}$-*Mod* is equivalent to R-*Mod*.

2. Idempotent modification uRu. Let u be an idempotent element of R, $uu = u$, and let uRu be the ring of all elements $x \in R$ with $ux = x = xu$ with the binary operation inherited from R and the neutral element u. This ring is Morita equivalent to R whenever u is pseudoinvertible, i.e., $eum = 1$ for some elements e and m of R.

Morita's original result is that the two operations above are sufficient: if a ring S is Morita equivalent to R, i.e., R-*Mod* and S-*Mod* are equivalent categories, then S is isomorphic to the ring $uR^{[k]}u$ for some pseudoinvertible idempotent $k \times k$ matrix u.

We generalize now Morita constructions to one-sorted algebraic theories. For these theories we take $\mathbb{N}$ as the set of objects, see 1.9, and write $\mathcal{T}$ instead of $(\mathcal{T}, 1)$.

15.4 Definition. Let $\mathcal{T}$ be a one-sorted algebraic theory.

1. The *matrix theory* $\mathcal{T}^{[k]}$, for $k = 1, 2, 3, \ldots$ is the one-sorted algebraic theory whose morphisms $f\colon p \to q$ are precisely the morphisms $f\colon kp \to kq$ of $\mathcal{T}$; composition and identity morphisms are defined as in $\mathcal{T}$.

2. Let $u\colon 1 \to 1$ be an idempotent morphism ($u \cdot u = u$). We call u *pseudoinvertible* provided that there exist morphisms $m\colon 1 \to n$ and $e\colon n \to 1$ such that $e \cdot u^n \cdot m = \mathrm{id}_1$. The idempotent modification of $\mathcal{T}$ is the theory $u\mathcal{T}u$ whose morphisms $f\colon p \to q$ are precisely the morphisms of $\mathcal{T}$ satisfying $f \cdot u^p = f = u^q \cdot f$. The composition is defined as in $\mathcal{T}$, the identity morphism on p is u^p.

15.5 Remark.

1. Both $\mathcal{T}^{[k]}$ and $u\mathcal{T}u$ are well defined. In fact, $\mathcal{T}^{[k]}$ has finite products with $p = 1 \times \ldots \times 1$: the i-th projection is obtained from the i-th projection in $\mathcal{T}$ of $kp = k \times \ldots \times k$. Also $u\mathcal{T}u$ has finite products with $p = 1 \times \ldots \times 1$: the i-projection $\pi_i\colon p \to 1$ of $\mathcal{T}$ yields a morphism $u \cdot \pi_i\colon p \to 1$ of $u\mathcal{T}u$ $(i = 1, \ldots, k)$ and these morphisms form a product $p = 1 \times \ldots \times 1$ in $u\mathcal{T}u$.

2. Idempotent modifications are much more "concrete" than (the equality of) idempotent completions. This will be seen on examples illustrating Morita equivalence below.

15.6 Theorem. *Let $\mathcal{T}$ be a one-sorted algebraic theory.*

1. the matrix theories $\mathcal{T}^{[k]}$ are Morita equivalent to $\mathcal{T}$ for all $k > 0$,

and

2. the idempotent modifications $u\mathcal{T}u$ are Morita equivalent to $\mathcal{T}$ for all pseudoinvertible idempotents u.

Proof. 1: Matrix theory $\mathcal{T}^{[k]}$. We have a full and faithful functor $\mathcal{T}^{[k]} \to \mathcal{T}$ defined on objects by $n \mapsto nk$ and on morphisms as the identity mapping. Every objects of $\mathcal{T}$ is a retract of an object coming from $\mathcal{T}^{[k]}$: in fact, for every n consider the diagonal morphism $\Delta\colon n \to nk = n \times \ldots \times n$. Consequently, $\mathcal{T}$ and $\mathcal{T}^{[k]}$ have the same idempotent completion. Thus, by 8.12, they are Morita equivalent.
2: Idempotent modification $u\mathcal{T}u$. Here we consider $\mathcal{T}$ as a full subcategory of $(Alg\,\mathcal{T})^{op}$ via the Yoneda embedding (1.4)

$$Y_{\mathcal{T}}\colon \mathcal{T} \to (Alg\,\mathcal{T})^{op}\,.$$

Following 8.3, the idempotent $Y_{\mathcal{T}}(u)\colon Y_{\mathcal{T}}(1) \to Y_{\mathcal{T}}(1)$ has a splitting in $(Alg\,\mathcal{T})^{op}$, say

$$\begin{tikzcd} Y_{\mathcal{T}}(1) \arrow[rr, "Y_{\mathcal{T}}(u)"] \arrow[dr, "\epsilon"'] & & Y_{\mathcal{T}}(1) \\ & A \arrow[ur, "\eta"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\eta"'] & & A \\ & Y_{\mathcal{T}}(1) \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

Consider also the subcategory $\mathcal{T}_A$ of $(Alg\,\mathcal{T})^{op}$ of all powers A^n, $n \in \mathbb{N}$. $\mathcal{T}_A$ is a one-sorted algebraic theory, and it is Morita equivalent to $\mathcal{T}$. In fact, every object of $\mathcal{T}$ is a retract of one in $\mathcal{T}_A$ and vice-versa – this clearly implies that $\mathcal{T}$ and $\mathcal{T}_A$ have a joint idempotent completion (obtained by splitting their idempotents in $(Alg\,\mathcal{T})^{op}$). Indeed, since A is a retract of $Y_{\mathcal{T}}(1)$, A^p is a retract of $Y_{\mathcal{T}}(p)$. Conversely, consider $m\colon 1 \to n$ and $e\colon n \to 1$ in $\mathcal{T}$ such that $e \cdot u^n \cdot m = \mathrm{id}_1$ as in 15.4.2. Then $Y_{\mathcal{T}}(1)$ is a retract of A^n via $\epsilon^n \cdot Y_{\mathcal{T}}(m)\colon Y_{\mathcal{T}}(1) \to A^n$ and $Y_{\mathcal{T}}(e) \cdot \eta^n\colon A^n \to Y_{\mathcal{T}}(1)$, and then $Y_{\mathcal{T}}(p)$ is a retract of A^{np}.
To complete the proof, we construct an equivalence functor $\bar{Y}\colon u\mathcal{T}u \to \mathcal{T}_1$. On objects it is defined by $\bar{Y}(p) = A^p$, and on morphisms $f\colon p \to q$ by

$$\begin{tikzcd} A^p \arrow[r, "\bar{Y}f"] \arrow[d, "\eta^p"'] & A^q \\ Y_{\mathcal{T}}(p) \arrow[r, "Y_{\mathcal{T}}(f)"'] & Y_{\mathcal{T}}(q) \arrow[u, "\epsilon^q"'] \end{tikzcd}$$

in $(Alg\,\mathcal{T})^{op}$. Observe that $\bar{Y}(\mathrm{id}_p) = \mathrm{id}_{A^p}$ because $\epsilon \cdot \eta = \mathrm{id}_A$. Now, we check the equation

$$Y_{\mathcal{T}}(f) = \eta^q \cdot \bar{Y}(f) \cdot \epsilon^p \qquad [15.1]$$

Indeed:

$$Y_{\mathcal{T}}(f) = Y_{\mathcal{T}}(u)^q \cdot Y_{\mathcal{T}}(f) \cdot Y_{\mathcal{T}}(u)^p = \eta^q \cdot \epsilon^q \cdot Y_{\mathcal{T}}(f) \cdot \eta^p \cdot \epsilon^p = \eta^q \cdot \bar{Y}(f) \cdot \epsilon^p\,.$$

From equation [15.1] since ϵ^p is a (split) epimorphism and η^q is is a (split) monomorphism, we deduce that $\bar{Y}$ preserves composition (because $Y_{\mathcal{T}}$ does), and that $\bar{Y}$ is faithful (because $Y_{\mathcal{T}}$ is). Since $\bar{Y}$ is surjective on objects, it remains to show that it is full: consider $h\colon A^p \to A^q$ in $(Alg\,\mathcal{T})^{op}$, we define $k = \eta^q \cdot h \cdot \epsilon^p \colon Y_{\mathcal{T}}(p) \to Y_{\mathcal{T}}(q)$. Since $Y_{\mathcal{T}}$ is full, there is a $f\colon p \to q$ in $\mathcal{T}$ such that $Y_{\mathcal{T}}(f) = k$. Now:

$$\bar{Y}(f) = \epsilon^q \cdot Y_{\mathcal{T}}(f) \cdot \eta^p = \epsilon^q \cdot \eta^q \cdot h \cdot \epsilon^p \cdot \eta^p = h\,.$$

It remains to check that f is in $u\mathcal{T}u$:

$$Y_{\mathcal{T}}(f) \cdot Y_{\mathcal{T}}(u^p) = \eta^q \cdot h \cdot \epsilon^p \cdot \eta^p \cdot \epsilon^p = \eta^q \cdot h \cdot \epsilon^p = k = Y_{\mathcal{T}}(f)$$

and then $f \cdot u^p = f$ because $Y_{\mathcal{T}}$ is faithful; analogously, $u^q \cdot f = f$. $\square$

15.7 Theorem. *For two one-sorted algebraic theories $\mathcal{T}$ and $\mathcal{S}$ the following conditions are equivalent:*

1. *$\mathcal{S}$ is Morita equivalent to $\mathcal{T}$;*
2. *$\mathcal{S}$ is, as a category, equivalent to an idempotent modification $u\mathcal{T}^{[k]}u$ of a matrix theory of $\mathcal{T}$ for some pseudoinvertible idempotent u of $\mathcal{T}^{[k]}$.*

Proof. Consider an equivalence functor

$$E\colon Alg\,\mathcal{S} \to Alg\,\mathcal{T}$$

and the Yoneda embeddings $Y_{\mathcal{S}}\colon \mathcal{S}^{op} \to Alg\,\mathcal{S}$, $Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to Alg\,\mathcal{T}$ (recall, from 1.5, that $Y_{\mathcal{S}}$ and $Y_{\mathcal{T}}$ preserve finite coproducts). Since $Y_{\mathcal{S}}(1)$ is perfectly presentable in $Alg\,\mathcal{S}$ (5.5), then $A = E(Y_{\mathcal{S}}(1))$ is perfectly presentable in $Alg\,\mathcal{T}$, therefore, due to (5.14) it is a retract of $Y_{\mathcal{T}}(n)$ for some n in $\mathbb{N}$:

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\eta"'] & & A \\ & Y_{\mathcal{T}}(n) \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

There exists a unique $u\colon n \to n$ in $\mathcal{T}$ such that $Y_{\mathcal{T}}(u) = \eta \cdot \epsilon$, and such a u is an idempotent. We consider u as an idempotent on 1 in $\mathcal{T}^{[n]}$ and prove that u is pseudoinvertible there. For this, choose an $\mathcal{S}$-algebra $\bar{A}$ and an isomorphism

$i\colon Y_{\mathcal{T}}(n) \to E\bar{A}$. Since E is an equivalence functor, $\bar{A}$ is perfectly presentable, thus it is a retract of $Y_{\mathcal{S}}(k)$ for some $k \in \mathbb{N}$:

$$\begin{tikzcd} \bar{A} \arrow[rr, "\mathrm{id}_{\bar{A}}"] \arrow[dr, "\bar{\eta}"'] & & \bar{A} \\ & Y_{\mathcal{S}}(k) \arrow[ur, "\bar{\epsilon}"'] & \end{tikzcd}$$

Consider now the composites

$$Y_{\mathcal{T}}(n) \xrightarrow{i} E\bar{A} \xrightarrow{E\bar{\eta}} EY_{\mathcal{S}}(k) \simeq kA \xrightarrow{k\eta} kY_{\mathcal{T}}(n) \simeq Y_{\mathcal{T}}(nk)$$

and

$$Y_{\mathcal{T}}(nk) \simeq kY_{\mathcal{T}}(n) \xrightarrow{k\epsilon} kA \simeq EY_{\mathcal{S}}(k) \xrightarrow{E\bar{\epsilon}} E\bar{A} \xrightarrow{i^{-1}} Y_{\mathcal{T}}(n)$$

then there exist unique morphisms $e\colon nk \to n$ and $m\colon n \to nk$ in $\mathcal{T}$ which $Y_{\mathcal{T}}$ maps on the above composites. One immediately checks that $Y_{\mathcal{T}}(e \cdot u^k \cdot m) = \mathrm{id}$, that is $e \cdot u^k \cdot m = \mathrm{id}$. Thus u is pseudoinvertible in $\mathcal{T}^{[n]}$.
To complete the proof, we construct an equivalence functor $\bar{E}\colon \mathcal{S} \to u\mathcal{T}^{[n]}u$. It is the identity map on objects. If $f\colon p \to q$ is a morphism in $\mathcal{S}$, $\bar{E}f$ is the unique morphism $np \to nq$ in $\mathcal{T}$ such that

$$\begin{tikzcd} qY_{\mathcal{T}}(n) \simeq Y_{\mathcal{T}}(nq) \arrow[r, "Y_{\mathcal{T}}(\bar{E}f)"] \arrow[d, "q\epsilon"'] & Y_{\mathcal{T}}(np) \simeq pY_{\mathcal{T}}(n) \\ qA \simeq E(Y_{\mathcal{S}}(q)) \arrow[r, "E(Y_{\mathcal{S}}(f))"'] & E(Y_{\mathcal{S}}(p)) \simeq pA \arrow[u, "p\eta"'] \end{tikzcd}$$

commutes. Using once again $Y_{\mathcal{T}}(u) = \eta \cdot \epsilon$ and the faithfulness of $Y_{\mathcal{T}}$, one easily checks that $u^p \cdot \bar{E}f \cdot u^q = \bar{E}f$, so that $\bar{E}f$ is a morphism $p \to q$ in $u\mathcal{T}^{[n]}u$. The proof that $\bar{E}$ is a well defined, full and faithful functor is analogous to that in Theorem 15.6 and is left to the reader. $\square$

15.8 Example. All one-sorted theories of *Set*. These are, up to equivalence of categories, precisely the theories $\mathcal{T}_k$ of 15.1. In fact, it is easy to see that $\mathcal{T}_k \simeq \mathcal{T}_1^{[k]}$ is the matrix theory for every $k \geq 1$. Moreover, given an idempotent $u\colon 1 \to 1$ of $\mathcal{T}_k$, then the function $u\colon k \to k$ in *Set* is pseudoinvertible iff it is invertible, thus $u = \mathrm{id}$. Consequently, there are no other one-sorted theories of *Set*.

15.9 Example. Let R be a ring with unit. Following 11.22, we can describe a one-sorted theory $\mathcal{T}_R$ of R-*Mod* : $\mathcal{T}_R$ is the full subcategory of R-*Mod*^{op} of the finitely generated free R-modules R^n ($n \in \mathbb{N}$). Every one-sorted algebraic theory of R-*Mod* is equivalent to $\mathcal{T}_S$ for some ring S which is Morita equivalent to R. Indeed, the two constructions of 15.3 fully correspond to the two constructions of 15.4:

1. $\mathcal{T}_{(R^{[k]})}$ is equivalent to $(\mathcal{T}_R)^{[k]}$;
2. given an idempotent element $u \in R$, the corresponding module homomorphism $\bar{u}\colon R \to R$ with $\bar{u}(x) = ux$ fulfils: uRu is equivalent to $\bar{u}(\mathcal{T}_R)\bar{u}$.

15.10 Example. For every monoid M, consider the category M-*Set* (cf. 13.14). Two monoids M and N are called *Morita equivalent* if M-*Set* and N-*Set* are equivalent categories. Here we need just one operation on monoids: if N is Morita equivalent to M, then N is isomorphic to an idempotent modification uMu for some pseudoinvertible idempotent u of M.
In contrast with the situation of 15.9, M-*Set* has, in general, many one-sorted theories not connected to any Morita equivalent monoid. (This is true even for $M = \{*\}$, since M-*Set* = *Set* has infinitely many theories which are not equivalent as categories, see 15.1.) However, all *unary theories* of M-*Set* have the form which correspond to Morita equivalent monoids. By a unary theory we mean a one-sorted algebraic theory $\mathcal{T}$ on objects T^n ($n \in \mathbb{N}$) which is a free completion of its full subcategory on $\{T\}$ (i.e., of the endomorphism monoid of T) under finite products. The category M-*Set* has an obvious one-sorted theory $\mathcal{T}_{[M]}$: the theory of free M-sets $M + \ldots + M$ on n generators ($n \in \mathbb{N}$) as a full subcategory of $(M\text{-}Set)^{op}$. Consequently, for every Morita equivalent monoid N we have a unary theory $\mathcal{T}_{[N]}$ for the category M-*Set*. And these are, up to categorical equivalence, all unary theories. In fact, let $\mathcal{T}$ be a unary theory with $Alg\,\mathcal{T}$ equivalent to M-*Set*. For the monoid $N = \mathcal{T}(T, T)$, there is an obvious categorical equivalence between $Alg\,\mathcal{T}$ and N-*Set* : every N-set $A\colon \mathcal{T}(T,T) \to$ *Set* has an essentially unique extension to a $\mathcal{T}$-algebra $A'\colon \mathcal{T} \to$ *Set*, and $(-)'$ is the desired equivalence functor. Therefore, N is Morita equivalent to M, and $\mathcal{T}$ is equivalent to $\mathcal{T}_{[N]}$.

15.11 Remark. Another approach to classical Morita theory for rings is based on the following result, due to Eilenberg and Watts (see [19], Theorem 2.3):
Let R, S be unitary rings; the assignement

$$M \;\mapsto\; M \otimes_S (-) : S\text{-}Mod \to R\text{-}Mod$$

induces a bijection between isomorphism classes of R-S-bimodules and isomorphism classes of colimit preserving functors.
Using Eilenberg-Watts Theorem one can prove that R and S are Morita equivalent iff there are bimodules M, N and bimodule isomorphisms

$$M \otimes_S N \simeq R \quad \text{and} \quad N \otimes_R M \simeq S$$

These facts are easy to generalize to algebraic theories. The generalization of Eilenberg-Watts Theorem is given by the following lemma, which is a consequence of 4.15.

15.12 Lemma. *Let $\mathcal{T}$ be an algebraic theory. The functor*

$$Y_{\mathcal{T}} : \mathcal{T}^{op} \to Alg\,\mathcal{T}$$

is a free completion of $\mathcal{T}^{op}$ conservative with respect to finite coproducts. This means that

1. *$Alg\mathcal{T}$ is cocomplete and $Y_{\mathcal{T}}$ preserves finite coproducts*

and

2. *for every functor $F\colon \mathcal{T}^{op} \to \mathcal{B}$ preserving finite coproducts, where $\mathcal{B}$ is a cocomplete category, there exists an essentially unique functor $F^*\colon Alg\mathcal{T} \to \mathcal{B}$ preserving colimits with F naturally isomorphic to $F^* \cdot Y_{\mathcal{T}}$.*

15.13 Definition. Let $\mathcal{T}, \mathcal{S}$ be algebraic theories. A *bimodule*

$$M\colon \mathcal{T} \Rightarrow \mathcal{S}$$

is a finite coproduct preserving functor $M\colon \mathcal{T}^{op} \to Alg\mathcal{S}$.

15.14 Remark.

1. The functor $Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to Alg\mathcal{T}$ is a bimodule $\mathcal{T} \Rightarrow \mathcal{T}$. More generally, every morphism of theories $\mathcal{T} \to \mathcal{S}$ induces a bimodule $\mathcal{T} \Rightarrow \mathcal{S}$ by composition with $Y_{\mathcal{S}}$.
2. Bimodules compose: given bimodules $M\colon \mathcal{T} \Rightarrow \mathcal{S}$ and $N\colon \mathcal{S} \Rightarrow \mathcal{R}$ we define $N \circ M\colon \mathcal{T} \Rightarrow \mathcal{R}$ by $N^* \cdot M$, where $N^*\colon Alg\mathcal{S} \to Alg\mathcal{R}$ is the colimit preserving extension of N (15.12). This composition is associative up to isomorphism and $Y_{\mathcal{S}} \circ M \simeq M \simeq M \circ Y_{\mathcal{T}}$. (In other words, the 2-category Th_{Bim} defined below is in fact a *bicategory* in the sense of [22].)

15.15 Definition.

1. We define the *2-category* Th_{Bim} to have

   objects: algebraic theories,

   1-cells: bimodules,

   2-cells: natural transformations.

2. We define the *2-category* ALG_{colim} to have

   objects: algebraic categories,

   1-cells: colimit preserving functors,

   2-cells: natural transformations.

15.16 Corollary. *Let $\mathcal{T}$ and $\mathcal{S}$ be algebraic theories.*

1. *The assignement*

$$M\colon \mathcal{T} \Rightarrow \mathcal{S} \;\mapsto\; M^*\colon Alg\mathcal{T} \to Alg\mathcal{S}$$

   extends to a biequivalence $Th_{Bim} \simeq ALG_{colim}$.

2. *$\mathcal{T}$ and $\mathcal{S}$ are Morita equivalent if and only if there exist two bimodules $M\colon \mathcal{T} \Rightarrow \mathcal{S}$ and $N\colon \mathcal{S} \Rightarrow \mathcal{T}$ such that $N \circ M \simeq Y_{\mathcal{T}}$ and $M \circ N \simeq Y_{\mathcal{S}}$.*

In fact, $Th_{Bim} \to ALG_{colim}$ is an equivalence on hom-categories by 15.12. The rest of the proof is obvious.

Historical Remarks for Chapter 15

The classical results concerning equivalences for categories of modules were proved by K. Morita in [76]. Thirty years later J.J. Dukarm proved a generalization to one-sorted algebraic theories in [40].

For one-sorted theories an approach to Morita equivalence via bimodules is due to F. Borceux and E. M. Vitale [28]. Theorem 15.7, a many-sorted version of Morita equivalence, is due to J. Adámek, M. Sobral and L. Sousa [12]. Example 15.10 is due to B. Banaschewski, see [14].

The Eilenberg-Watts theorem quoted in 15.11 was independently proved by S. Eilenberg in [43] and C. E. Watts in [96]. An exhaustive treatment of Morita theory for rings in terms of bimodules appears in the monograph of H. Bass [19].

Chapter 16

Free exact categories

We know that every algebraic category is an exact category having enough regular projective objects (see 3.18 and 5.15). In the present chapter, we study *free exact completions* and prove that every algebraic category is a free exact completion of its full subcategory of all regular projectives. This will be used in the next chapter to characterize algebraic categories among exact categories, and to describe all finitary localizations of algebraic categories. The trouble with regular projective objects in an algebraic category is namely that they are not closed under finite limits. Luckily they have weak finite limits. Recall that *weak limits* are defined as limits except that the uniqueness of the factorization is not requested (see 16.7). The main point is that the universal property of a free exact completion is based on *left covering functors.* These are functors which play, for categories with weak finite limits, the role that functors preserving finite limits play for finitely complete categories.

We will be concerned with regular epimorphisms (3.4) in an exact category (3.16). For the comfort of the reader, we start by listing some of their (easy but) important properties. In diagrams, regular epimorphisms are denoted by $\twoheadrightarrow$.

16.1 Lemma. *Let $\mathcal{A}$ be an exact category.*

1. *Any morphism factorizes as a regular epimorphism followed by a monomorphism.*

2. *Consider a morphism $f\colon X \to Z$. The following conditions are equivalent:*

   (a) *f is a regular epimorphism;*

   (b) *f is a strong epimorphism;*

   (c) *f is an extremal epimorphism.*

Proof. 1: Consider a morphism $f\colon X \to Z$ and its factorization through the coequalizer of its kernel pair

$$\begin{tikzcd}
N(f) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[r, "f"] \arrow[d, two heads, "e"'] & Z \\
& I \arrow[ur, "m"'] &
\end{tikzcd}$$

We have to prove that m is a monomorphism. For this, consider the following diagram, where each square is a pullback

$$\begin{tikzcd}
N(f) \arrow[r, two heads] \arrow[d] \arrow[dr, dotted, "q"] & {} \arrow[r] \arrow[d] & X \arrow[d, "e"] \\
{} \arrow[r, two heads] \arrow[d] & N(m) \arrow[r, "m_1"] \arrow[d, "m_2"'] & I \arrow[d, "m"] \\
X \arrow[r, two heads, "e"'] & I \arrow[r, "m"'] & Z
\end{tikzcd}$$

Since in $\mathcal{A}$ regular epimorphisms are pullback stable, the diagonal q is an epimorphism. Now, $m_1 \cdot q = e \cdot f_1 = e \cdot f_2 = m_2 \cdot q$, so that $m_1 = m_2$. This means that m is a monomorphism.
2: a $\Rightarrow$ b: Let u, v and m be morphisms such that $v \cdot f = m \cdot u$. If f is the coequalizer of a pair (x, y), and m a monomorphism, then u also coequalizes x and y.
b $\Rightarrow$ c: If $f = m \cdot u$ with m a monomorphism, we can write $v \cdot f = m \cdot u$ with $v = \mathrm{id}$. By condition (b) m is a split epimorphism, but a monomorphism which is also a split epimorphism is an isomorphism.
c $\Rightarrow$ a: Just take a regular epi-mono factorization $f = m \cdot e$ (which exists by 1.); if condition (c) holds, then m is an isomorphism and therefore f is a regular epimorphism. $\square$

16.2 Corollary. *Let $\mathcal{A}$ be an exact category.*

1. *The factorization stated in Lemma 16.1.1 is essentially unique;*
2. *The composite of two regular epimorphisms is a regular epimorphism;*
3. *If the triangle*

$$\begin{tikzcd}
X \arrow[rr, two heads, "f"] \arrow[dr, "g"'] & & Z \\
& A \arrow[ur, "h"'] &
\end{tikzcd}$$

   commutes and f is a regular epimorphism, then h is a regular epimorphism;

4. *If a morphism is a regular epimorphism and a monomorphism, then it is an isomorphism.*

In fact, everything follows easily from condition 2.b of Lemma 16.1.

16.3 Lemma. *Every exact category has the following properties:*

1. *The product of two regular epimorphisms is a regular epimorphism;*

2. *Consider the following diagram*

$$\begin{tikzcd} A_0 \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[d, "f_0"'] & A_1 \arrow[d, "f_1"] \\ B_0 \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & B_1 \end{tikzcd}$$

   with $f_1 \cdot a_i = b_i \cdot f_0$ for $i = 1, 2$. If f_0 is a regular epimorphism and f_1 is a monomorphism, then the unique extension to the equalizers is a regular epimorphism;

3. *Consider the following commutative diagram*

$$\begin{tikzcd} A_0 \arrow[r, "a_1"] \arrow[d, "f_0"'] & A \arrow[d, "f"'] & A_1 \arrow[l, "a_2"'] \arrow[d, "f_1"] \\ B_0 \arrow[r, "b_1"'] & B & B_1 \arrow[l, "b_2"] \end{tikzcd}$$

   If f_0 and f_1 are regular epimorphisms and f is a monomorphism, then the unique extension to the pullbacks is a regular epimorphism.

Proof. 1: Observe that $f \times \mathrm{id}$ is the pullback of f along the suitable projection, and the same holds for $\mathrm{id} \times g$. Now $f \times g = (f \times \mathrm{id}) \cdot (\mathrm{id} \times g)$.
2: Since f_1 is a monomorphism, the pullback of the equalizer of (b_1, b_2) along f_0 is the equalizer of (a_1, a_2).
3: This follows from 1. and 2., using the usual construction of pullbacks via products and equalizers. ☐

For the sake of generality, let us point out that in 16.1, 16.2 and 16.3 we do not need that in $\mathcal{A}$ equivalence relations are effective.

16.4 Remark. In 16.3.2 if f_1 is any morphism (not necessarily a monomorphism) it is no longer true that the pullback of an equalizer $e\colon E \to B_0$ of (b_1, b_2) along f_0 is an equalizer of (a_1, a_2). What remains true (in any category with finite limits) is the following fact: let $e'\colon E' \to A_0$ be a pullback of e along f_0, let $k_1, k_2\colon N(f_1) \rightrightarrows A_1$ be a kernel pair of f_1, and $n\colon E' \to N(f_1)$ the unique morphism such that $k_i \cdot n = a_i \cdot e'$ $(i = 1, 2)$. Then the following is a limit diagram

$$\begin{tikzcd}
 & E' \arrow[dl, "e'"'] \arrow[dr, "n"] & \\
A_0 \arrow[d, "a_1"'] \arrow[drr, "a_2" near start] & & N(f_1) \arrow[d, "k_2"] \arrow[dll, "k_1"' near start] \\
A_1 & & A_1
\end{tikzcd}$$

We will use this fact in the proof of Theorem 16.24.

From Propositions 3.18 and 5.15 we know that an algebraic category is an exact category having enough regular projective objects. In fact, each algebra is a regular quotient of a regular projective algebra. In the following we study categories having enough regular projectives; we introduce the concept of a regular projective cover for a subcategory of regular projectives in case there is "enough of them".

16.5 Definition. Let $\mathcal{A}$ be a category. A *regular projective cover* of $\mathcal{A}$ is a full and faithful functor $I\colon \mathcal{P} \to \mathcal{A}$ such that

1. for every object P of $\mathcal{P}$, IP is regular projective in $\mathcal{A}$;
2. for every object A of $\mathcal{A}$, there exists an object P in $\mathcal{P}$ and a regular epimorphism $P \to A$ (we write P instead of IP and we call $P \to A$ a $\mathcal{P}$-*cover* of A).

16.6 Definition. A functor is *exact* if it preserves finite limits and regular epimorphisms.

The present chapter is devoted to the study of exact functors defined on an exact category $\mathcal{A}$ having a regular projective cover $\mathcal{P} \to \mathcal{A}$. First of all, observe that regular projective objects are not closed under finite limits, so that we cannot hope that $\mathcal{P}$ inherits finite limits from $\mathcal{A}$. Nevertheless, a "trace" of finite limits remains in $\mathcal{P}$. In fact, $\mathcal{P}$ has weak finite limits.

16.7 Definition. A *weak limit* of a diagram $D\colon \mathcal{D} \to \mathcal{A}$ is a cone $p_X\colon W \to DX$ ($X \in obj\mathcal{D}$) such that for every other cone $a_X\colon A \to DX$ there exists a morphism a : $A \to W$ such that $p_X \cdot a = a_X$ for all X.

Observe that, unlike limits, weak limits are very much "non-unique". For example, any non-empty set is a weak terminal object in the category *Set*.

16.8 Lemma. *If $\mathcal{P} \to \mathcal{A}$ is a regular projective cover of a finitely complete category $\mathcal{A}$, then $\mathcal{P}$ has weak finite limits.*

Proof. Consider a finite diagram $D\colon \mathcal{D} \to \mathcal{P}$. If

$$\langle\, \pi_X\colon L \to DX \,\rangle_{X \in \mathcal{D}}$$

is a limit of D in $\mathcal{A}$, then we choose a $\mathcal{P}$-cover $l\colon P \to L$. The resulting cone

$$\langle\, \pi_X \cdot l\colon P \to DX \,\rangle_{X\in\mathcal{D}}$$

is a weak limit of D in $\mathcal{P}$. □

In the situation of the previous lemma, apply an exact functor $G\colon \mathcal{A} \to \mathcal{B}$. Since G preserves finite limits, the factorization of the cone

$$\langle\, G(\pi_X \cdot l)\colon GP \to G(DX) \,\rangle_{X\in\mathcal{D}}$$

through the limit in $\mathcal{B}$ is $Gl\colon GP \to GL$, which is a regular epimorphism because G is exact.

We can formalize this property in the following definition.

16.9 Definition. Let $\mathcal{B}$ be an exact category and let $\mathcal{P}$ be a category with weak finite limits. A functor $F\colon \mathcal{P} \to \mathcal{B}$ is *left covering* if, for any finite diagram $D\colon \mathcal{D} \to \mathcal{P}$ with weak limit W, the canonical comparison morphism $FW \to \mathit{lim}\, F \cdot D$ is a regular epimorphism.

16.10 Remark. To avoid any ambiguity in the previous definition, let us point out that if the comparison $w\colon FW \to \mathit{lim}\, F \cdot D$ is a regular epimorphism for a certain weak limit W of D, then the comparison $w'\colon FW' \to \mathit{lim}\, F \cdot D$ is a regular epimorphism for any other weak limit W' of D. This follows from Corollary 16.2 because w factorizes through w'.

16.11 Example.

1. If a finite diagram $D\colon \mathcal{D} \to \mathcal{A}$ has a limit L, then the weak limits of D are precisely the objects W such that L is a retract of W. Therefore any functor preserving finite limits is left covering.

2. If $\mathcal{P} \to \mathcal{A}$ is a regular projective cover of an exact category $\mathcal{A}$, then it is a left covering functor.

3. The composition of a left covering functor with an exact functor is a left covering functor.

16.12 Example. Let $\mathcal{P}$ be a category with weak finite limits, and consider the (possibly illegitimate) functor category $[\mathcal{P}^{op}, \mathit{Set}]$. The canonical Yoneda embedding $Y_{\mathcal{P}^{op}}\colon \mathcal{P} \to [\mathcal{P}^{op}, \mathit{Set}]$ is a left covering functor.

Proof. Consider a finite diagram $D\colon \mathcal{D} \to \mathcal{P}$ in $\mathcal{P}$, a weak limit W of D and a limit L of $Y_{\mathcal{P}^{op}} \cdot D$. The canonical comparison $\tau\colon Y_{\mathcal{P}^{op}}(W) \to L$ is a regular epimorphism whenever, for all $Z \in \mathcal{P}$, $\tau_Z\colon Y_{\mathcal{P}^{op}}(W)(Z) \to LZ$ is surjective. Since the limit L is computed pointwise in Set, an element of LZ is a cone from Z to $\mathcal{L}$, so that the surjectivity of τ_Z is just the weak universal property of W. □

16.13 Remark. In the main result of this chapter (16.24) we show that an exact category with enough regular projective objects is a free exact completion of any of its regular projective covers. This is one of the results that request working with the left covering property (instead of the seemingly more natural condition of preservation of weak finite limits). In fact, the basic example 16.11.2 would not be true otherwise. This can be illustrated by the category of rings: the inclusion of the full subcategory $\mathcal{P}$ of all regular projective rings does not preserve weak finite limits. For example, the ring $\mathbb{Z}$ of integers is a weak terminal object in $\mathcal{P}$, but it is not a weak terminal object in $\mathcal{A}$ because the unique morphism from $\mathbb{Z}$ to the one-element ring does not have a section.

A remarkable fact about left covering functors is that they classify exact functors. Before stating this in a precise way, see 16.24, we need some facts about left covering functors and pseudoequivalences. A pseudoequivalence is defined "almost" as an equivalence relation, but (a) using a weak pullback instead of a pullback to express the transitivity, and (b) without the assumption that the graph be jointly monic.

16.14 Definition. Let $\mathcal{P}$ be a category with weak pullbacks. A *pseudoequivalence* is a parallel pair

$$X' \underset{x_2}{\overset{x_1}{\rightrightarrows}} X$$

which is

1. reflexive, i.e., there exists $r\colon X \to X'$ such that $x_1 \cdot r = \mathrm{id}_X = x_2 \cdot r$,
2. symmetric, i.e., there exists $s\colon X' \to X'$ such that $x_1 \cdot s = x_2$ and $x_2 \cdot s = x_1$,

and

3. transitive, i.e., in an arbitrary weak pullback

$$\begin{tikzcd} P \arrow[r, "x_1'"] \arrow[d, "x_2'"'] & X' \arrow[d, "x_2"] \\ X' \arrow[r, "x_1"'] & X \end{tikzcd}$$

there exists $t\colon P \to X'$ such that $x_1 \cdot t = x_1 \cdot x_1'$ and $x_2 \cdot t = x_2 \cdot x_2'$. The morphism t is called a *transitivity morphism* of x_1 and x_2.

16.15 Remark.

1. Observe that the existence of a transitivity morphism of x_1 and x_2 does not depend on the choice of a weak pullback of x_1 and x_2.
2. Recall that a regular factorization of a morphism is a factorization as a regular epimorphism followed by a monomorphism. In a category with

binary products, we speak about *regular factorization of a parallel pair* $p, q\colon A \rightrightarrows B$. What we mean is a factorization of (p, q) as in the following diagram, where e is a regular epimorphism and (p', q') is a jointly monomorphic parallel pair

$$\begin{tikzcd}
A \arrow[rr, shift left, "p"] \arrow[rr, shift right, "q"'] \arrow[drr, two heads, "e"'] & & B \\
& & I \arrow[u, shift left, "p'"] \arrow[u, shift right, "q'"']
\end{tikzcd}$$

obtained by a regular factorization of $\langle p, q\rangle\colon A \to B \times B$. Since jointly monomorphic parallel pairs are also called relations, we call (p', q') the *relation induced* by (p, q).

3. If $\mathcal{P}$ has finite limits, then equivalence relations are precisely those parallel pairs which are, at the same time, relations and pseudoequivalences. The next result, which is the main link between pseudoequivalences and left covering functors, shows that any pseudoequivalence in an exact category is a composition of an equivalence relation with a regular epimorphism. (The converse is not true: if we compose an equivalence relation with a regular epimorphism in general we do not obtain a pseudoequivalence. Consider the category of rings, the unique equivalence relation on the one-element ring 0, and the unique morphism $\mathbb{Z} \to 0$. The parallel pair $\mathbb{Z} \rightrightarrows 0$ is not reflexive, because there are no morphisms from 0 to $\mathbb{Z}$.)

16.16 Lemma. *Let $F\colon \mathcal{P} \to \mathcal{B}$ be a left covering functor. For every pseudoequivalence $x_1, x_2\colon X' \rightrightarrows X$ in $\mathcal{P}$, the relation in $\mathcal{B}$ induced by (Fx_1, Fx_2) is an equivalence relation.*

Proof. Consider a regular factorization in $\mathcal{B}$

$$\begin{tikzcd}
FX' \arrow[rr, shift left, "Fx_1"] \arrow[rr, shift right, "Fx_2"'] \arrow[drr, two heads, "p"'] & & FX \\
& & I \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"']
\end{tikzcd}$$

Since the reflexivity and transitivity are obvious, we only check the transitivity of (i_1, i_2). The pullback of $i_1 \cdot p$ and $i_2 \cdot p$ factorizes through the pullback of i_1 and i_2, and the factorization, v, is a regular epimorphism (because p is a regular epimorphism and $\mathcal{B}$ is an exact category):

$$\begin{tikzcd}
W \arrow[rr, "j_1"] \arrow[dd, "j_2"'] \arrow[dr, two heads, "v"] & & FX' \arrow[d, two heads, "p"] \\
& Q \arrow[r, "i_1'"] \arrow[d, "i_2'"] & I \arrow[d, "i_2"] \\
FX' \arrow[r, two heads, "p"'] & I \arrow[r, "i_1"'] & FX
\end{tikzcd}$$

Consider also a transitivity morphism $t\colon P \to X'$ of (x_1, x_2) as in Definition 16.14. Since $F\colon \mathcal{P} \to \mathcal{B}$ is left covering, the factorization $q\colon FP \to W$ such that $j_1 \cdot q = Fx_1'$ and $j_2 \cdot q = Fx_2'$ is a regular epimorphism. Finally, we have the following commutative diagram

$$\begin{tikzcd}
FP \arrow[r, two heads, "v\cdot q"] \arrow[d, "p\cdot Ft"'] & Q \arrow[d, "\langle i_1\cdot i_1', i_2\cdot i_2'\rangle"] \arrow[dl, dotted, "\tau"'] \\
I \arrow[r, "\langle i_1, i_2\rangle"'] & FX \times FX
\end{tikzcd}$$

Since $v \cdot q$ is a regular epimorphism and $\langle i_1, i_2\rangle$ is a monomorphism, there exists $\tau\colon Q \to I$ such that $\langle i_1, i_2\rangle \cdot \tau = \langle i_1 \cdot i_1', i_2 \cdot i_2'\rangle$. This implies that (i_1, i_2) is transitive. $\square$

16.17 Remark. Generalizing the fact that functors preserve finite limits iff they preserve finite products and equalizers, we are going to prove the same for left coverings. We use the phrase "left covering with respect to weak finite products" for the restriction of 16.9 to discrete categories $\mathcal{D}$. Observe that this is equivalent to being left covering with respect to weak binary products and weak terminal objects. Analogously, we use "left covering with respect to equalizers".

16.18 Lemma. *A functor $F\colon \mathcal{P} \to \mathcal{B}$, where $\mathcal{P}$ has weak finite limits and $\mathcal{B}$ is exact, is left covering if and only if it is left covering with respect to weak finite products and weak equalizers.*

Proof. 1. Using Lemma 16.3 and working by induction, one extends the left covering character of F to joint equalizers of parallel n-tuples, and then to multiple pullbacks.
2. Consider a finite diagram $D\colon \mathcal{D} \to \mathcal{P}$. We can construct a weak limit of D using a weak product $\Pi_{X\in\mathcal{D}} DX$, weak equalizers E_d, one for each morphism $d\colon X \to X'$ in $\mathcal{D}$, and a weak multiple pullback E as in the following diagram

$$\begin{tikzcd}
 & E_d \arrow[dr, "e_d"] & & DX' \\
E \arrow[ur, "e_d'"] \arrow[dr, "e_c'"'] & \vdots & \Pi_{X\in\mathcal{D}} DX \arrow[ur, shift left, "Dd\cdot\pi_X"] \arrow[ur, shift right, "\pi_{X'}"'] \arrow[dr, shift left, "Dc\cdot\pi_Y"] \arrow[dr, shift right, "\pi_{Y'}"'] & \vdots \\
 & E_c \arrow[ur, "e_c"'] & & DY'
\end{tikzcd}$$

Perform the same constructions in $\mathcal{B}$ to get limits as in the following diagrams

$$
\begin{tikzcd}
 & L_d \arrow[dr, "l_d"] & & FDX' \\
L \arrow[ur, "l'_d"] \arrow[dr, "l'_c"'] & \vdots & \Pi_{X\in\mathcal{D}}FDX \arrow[ur, shift left, "FDd\cdot\tilde{\pi}_X"] \arrow[ur, shift right, "\tilde{\pi}_{X'}"'] \arrow[dr, shift left, "FDc\cdot\tilde{\pi}_Y"] \arrow[dr, shift right, "\tilde{\pi}_{Y'}"'] & \vdots \\
 & L_c \arrow[ur, "l_c"'] & & FDY'
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S_d \arrow[dr, "s_d"] & & FDX' \\
S \arrow[ur, "s'_d"] \arrow[dr, "s'_c"'] & \vdots & F(\Pi_{X\in\mathcal{D}}DX) \arrow[ur, shift left, "FDd\cdot F\pi_X"] \arrow[ur, shift right, "F\pi_{X'}"'] \arrow[dr, shift left, "FDc\cdot F\pi_Y"] \arrow[dr, shift right, "F\pi_{Y'}"'] & \vdots \\
 & S_c \arrow[ur, "s_c"'] & & FDY'
\end{tikzcd}
$$

By assumption, the canonical factorization $q_d\colon FE_d \to S_d$ is a regular epimorphism. By Lemma 16.3, this gives rise to a regular epimorphism $q\colon Q \to S$, where Q is the multiple pullback of the Fe_d. By part 1., the canonical factorization $t\colon FE \to Q$ is a regular epimorphism. Finally, a diagram chase shows that the pullback of $l_d \cdot l'_d$ along the canonical factorization $p\colon F(\Pi_{X\in\mathcal{D}}DX) \to \Pi_{X\in\mathcal{D}}FDX$ is $s_d \cdot s'_d$. By part 1., p is a regular epimorphism, so that we get a regular epimorphism $p'\colon S \to L$. The regular epimorphism $p' \cdot q \cdot t\colon FE \to Q \to S \to L$ shows that F is left covering. $\square$

16.19 Lemma. *A left covering functor $F\colon \mathcal{P} \to \mathcal{B}$ preserves finite jointly monomorphic sources.*

Proof. A family of morphisms $(f_i\colon A \to A_i)_{i\in I}$ is jointly monomorphic if and only if the span formed by $\mathrm{id}_A, \mathrm{id}_A$ is a limit of the corresponding diagram:

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "f_i"] \arrow[ddr, "f_j"] & \\
A \arrow[r, "f_i"] \arrow[drr, "f_j"'] & A_i & \ddots \\
 & & A_j
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "x"] \arrow[d, "y"'] & FA \arrow[d, "Ff_i"] \arrow[ddr, "Ff_j"] & \\
FA \arrow[r, "Ff_i"] \arrow[drr, "Ff_j"'] & FA_i & \ddots \\
 & & FA_j
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & FA \arrow[d, "Ff_i"] \arrow[ddr, "Ff_j"] & \\
FA \arrow[r, "Ff_i"] \arrow[drr, "Ff_j"'] & FA_i & \ddots \\
 & & FA_j
\end{tikzcd}
$$

Now apply F and consider the canonical factorization $q\colon FA \to L$, where L is a limit in $\mathcal{B}$ of the corresponding diagram. By assumption, q is a regular epimorphism. It is also a monomorphism, because $x\cdot q = \mathrm{id}$, and so it is an isomorphism. This implies that $\mathrm{id}_{FA}, \mathrm{id}_{FA}$ is a limit, thus the family $(Ff_i\colon FA \to FA_i)_{i\in I}$ is jointly monomorphic. $\square$

16.20 Lemma. *Consider a functor $F\colon \mathcal{P} \to \mathcal{B}$. Assume that $\mathcal{P}$ has finite limits and $\mathcal{B}$ is exact. Then F is left covering if and only if it preserves finite limits.*

Proof. One implication is clear, see 16.11.1. Thus, let us assume that F is left covering and consider a finite non-empty diagram $D\colon \mathcal{D} \to \mathcal{P}$. Let $(\pi_X\colon L \to DX)_{X\in\mathcal{D}}$ be a limit of D and $(\tilde{\pi}_X\colon \tilde{L} \to FDX)_{X\in\mathcal{D}}$ a limit of $F\cdot D$. Since the family $(\pi_X)_{X\in\mathcal{D}}$ is jointly monomorphic, by Lemma 16.19 also the family $(F\pi_X)_{X\in\mathcal{D}}$ is monomorphic. This implies that the canonical factorization $p\colon FL \to \tilde{L}$ is a monomorphism. But it is a regular epimorphism by assumption, so that it is an isomorphism.
The argument for the terminal object T is different. In $\mathcal{P}$, the product of T with itself is T with the identity morphisms as projections. Then the canonical factorization $FT \to FT\times FT$ is a (regular) epimorphism. This implies that the two projections $\pi_1, \pi_2\colon FT\times FT \rightrightarrows FT$ are equal. But the pair (π_1,π_2) is the kernel pair of the unique morphism q to the terminal object of $\mathcal{B}$, so that q is a monomorphism. Since F is left covering, q is a regular epimorphism, thus an isomorphism. □

Let us point out that in 16.16 and 16.18 we do not need to assume that equivalence relations are effective in $\mathcal{B}$. Moreover, if in Definition 16.9 we replace *regular epimorphism* by *strong epimorphism*, then 16.19 and 16.20 hold for all categories $\mathcal{B}$ with finite limits.

16.21 Definition. Let $\mathcal{P}$ be a category with weak finite limits. A *free exact completion* of $\mathcal{P}$ is an exact category $\mathcal{P}_{ex}$ with a left covering functor

$$\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$$

such that for every exact category $\mathcal{B}$ and for every left covering functor $F\colon \mathcal{P} \to \mathcal{B}$, there exists an essentially unique exact functor $\hat{F}\colon \mathcal{P}_{ex} \to \mathcal{B}$ with $\hat{F}\cdot\Gamma$ naturally isomorphic to F.

Note that, since a free exact completion is defined via a universal property, it is determined uniquely up to equivalence.

16.22 Remark. Since the composition of the left covering functor $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$ with an exact functor $\mathcal{P}_{ex} \to \mathcal{B}$ clearly gives a left covering functor $\mathcal{P} \to \mathcal{B}$, the previous universal property can be restated in the following way: Composition with Γ induces an equivalence functor

$$-\cdot\Gamma\colon \mathit{Ex}[\mathcal{P}_{ex},\mathcal{B}] \to \mathit{Lco}[\mathcal{P},\mathcal{B}]$$

where $\mathit{Ex}[\mathcal{P}_{ex},\mathcal{B}]$ is the category of exact functors from $\mathcal{P}_{ex}$ to $\mathcal{B}$ and natural transformations, and $\mathit{Lco}[\mathcal{P},\mathcal{B}]$ is the category of left covering functors from $\mathcal{P}$ to $\mathcal{B}$ and natural transformations.

16.23 Remark. In order to prepare the proof of Theorem 16.24, let us explain how an exact category with enough regular projective objects can be reconstructed using any of its regular projective covers. Let $\mathcal{P} \to \mathcal{A}$ be a regular projective cover of an exact category $\mathcal{A}$. Fix an object A in $\mathcal{A}$ and consider a $\mathcal{P}$-cover $a\colon X \to A$, its kernel pair $a_1, a_2\colon N(a) \rightrightarrows X$, and again a $\mathcal{P}$-cover $x\colon X' \to N(a)$. In the resulting diagram

$$\begin{tikzcd} X' \arrow[r, shift left, "a_1\cdot x"] \arrow[r, shift right, "a_2\cdot x"'] & X \arrow[r, two heads, "a"] & A \end{tikzcd}$$

the left-hand part is a pseudoequivalence in $\mathcal{P}$ (not in $\mathcal{A}$!) and A is its coequalizer. Consider a morphism $\varphi\colon A \to B$ in $\mathcal{A}$ and the following diagram

$$\begin{tikzcd} X' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, shift right, "a_1\cdot x"'] \arrow[d, shift left, "a_2\cdot x"] & Z' \arrow[d, shift right, "b_1\cdot z"'] \arrow[d, shift left, "b_2\cdot z"] \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[ur, dotted, "\Sigma"] \arrow[d, two heads, "a"'] & Z \arrow[d, two heads, "b"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

Using the regular projectivity of X and X' and the universal property of the kernel pair of b, we get a pair (f', f) such that $\varphi \cdot a = b \cdot f$ and $f \cdot a_i \cdot x = b_i \cdot z \cdot f'$ for $i = 1, 2$. Conversely, a pair (f', f) such that $f \cdot a_i \cdot x = b_i \cdot z \cdot f'$ for $i = 1, 2$ induces a unique extension to the quotient. Moreover, two such pairs (f', f) and (g', g) have the same extension if and only if there is a morphism $\Sigma\colon X \to Z'$ such that $b_2 \cdot z \cdot \Sigma = f$ and $b_2 \cdot z \cdot \Sigma = g$.

16.24 Theorem. *Let $I\colon \mathcal{P} \to \mathcal{A}$ be a regular projective cover of an exact category $\mathcal{A}$. Then $\mathcal{A}$ is a free exact completion of $\mathcal{P}$.*

Proof. 1. Extension of a left covering functor $F\colon \mathcal{P} \to \mathcal{B}$ to a functor $\hat{F}\colon \mathcal{A} \to \mathcal{B}$. To define $\hat{F}$ on objects $A \in \mathcal{A}$, construct the coequalizer

$$\begin{tikzcd} X' \arrow[r, shift left, "x_1 = a_1\cdot x"] \arrow[r, shift right, "x_2 = a_2\cdot x"'] & X \arrow[r, two heads, "a"] & A \end{tikzcd}$$

as in 16.23. By 16.16, the relation (i_1, i_2) induced by $Fx_1, Fx_2\colon FX' \rightrightarrows FX$ in $\mathcal{B}$ is an equivalence relation. Since $\mathcal{B}$ is exact, we can define $\hat{F}A$ to be a coequalizer of (i_1, i_2)

$$\begin{tikzcd} FX' \arrow[r, shift left, "Fx_1"] \arrow[r, shift right, "Fx_2"'] \arrow[dr, "p"'] & FX \arrow[r, two heads, "\alpha"] & \hat{F}A \\ & R \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] & \end{tikzcd}$$

Let now $\varphi\colon A \to B$ be a morphism in $\mathcal{A}$. We can costruct a pair $f\colon X \to Z$, $f'\colon X' \to Z'$ as in 16.23 and define $\hat{F}\varphi$ to be the unique extension to the quotients as in the following diagram

$$\begin{tikzcd}
FX' \arrow[r, "Ff'"] \arrow[d, shift right, "Fx_1"'] \arrow[d, shift left, "Fx_2"] & FZ' \arrow[d, shift right, "Fz_1"'] \arrow[d, shift left, "Fz_2"] \\
FX \arrow[r, "Ff"'] \arrow[d, "\alpha"'] & FZ \arrow[d, "\beta"] \\
\hat{F}A \arrow[r, "\hat{F}\varphi"'] & \hat{F}B
\end{tikzcd}$$

The discussion in 16.23 shows that this definition does not depend on the choice of the pair f, f'. The preservation of composition and identity morphisms by $\hat{F}$ comes from the uniqueness of the extension to the quotients. It is clear that $\hat{F} \cdot I$ is naturally isomorphic to F and that a different choice of $\mathcal{P}$-covers X and X' for a given object $A \in \mathcal{A}$ produces a functor naturally isomorphic to $\hat{F}$.
2. $\hat{F}$ is the essentially unique exact functor such that $\hat{F} \cdot I$ is naturally isomorphic to F. Indeed, using once again the notations of 16.23, $(\hat{F}a_1, \hat{F}a_2)$ is the kernel pair of $\hat{F}a$, and $\hat{F}x$ and $\hat{F}a$ are regular epimorphisms:

$$\begin{tikzcd}
FX' \simeq \hat{F}X' \arrow[r, two heads, "\hat{F}x"] & \hat{F}N(a) \arrow[r, shift left, "\hat{F}a_1"] \arrow[r, shift right, "\hat{F}a_2"'] & \hat{F}X \simeq FX \arrow[rr, two heads, "\hat{F}a"] & & \hat{F}A
\end{tikzcd}$$

This implies that $\hat{F}A$ is necessarily a coequalizer of (Fx_1, Fx_2). In a similar way one shows that $\hat{F}$ is uniquely determined on morphisms.
3. The extension $\hat{F}\colon \mathcal{A} \to \mathcal{B}$ preserves finite limits. In fact, it is sufficient to show that $\hat{F}$ is left covering with respect to the terminal object, binary products and equalizers of pairs, see Lemma 16.18 and Lemma 16.20.
3a. Products: let A and B be objects in $\mathcal{A}$. Working as in 16.23 we get coequalizers

$$\begin{tikzcd}
X' \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & X \arrow[r, two heads, "a"] & A
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Z' \arrow[r, shift left, "z_1"] \arrow[r, shift right, "z_2"'] & Z \arrow[r, two heads, "b"] & B
\end{tikzcd}$$

Consider the following diagram, where both lines are products in $\mathcal{A}$ and $c\colon R \to X \times Z$ is a $\mathcal{P}$-cover

$$\begin{tikzcd}
& R \arrow[d, "c"] & \\
X \arrow[d, "a"'] & X \times Z \arrow[l, "\pi_X"'] \arrow[r, "\pi_Z"] \arrow[d, "a \times b"] & Z \arrow[d, "b"] \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

By 16.3.1 $a \times b$ is a regular epimorphism, so that $(a \times b) \cdot c\colon R \to A \times B$ is a $\mathcal{P}$-cover. Moreover, by 16.8 $(R, \pi_X \cdot c, \pi_Z \cdot c)$ is a weak product of X and Z in

$\mathcal{P}$. Applying $\hat{F}$, we have the following diagram in $\mathcal{B}$

$$\begin{tikzcd}
FX \arrow[d, "\alpha"'] & FR \arrow[l, "F(\pi_X \cdot c)"'] \arrow[r, "F(\pi_Z \cdot c)"] \arrow[d, "\gamma"'] & FZ \arrow[d, "\beta"] \\
\hat{F}A & \hat{F}(A \times B) \arrow[l, "\hat{F}\pi_A"] \arrow[r, "\hat{F}\pi_B"'] & \hat{F}B
\end{tikzcd}$$

from which we get the following commutative square

$$\begin{tikzcd}
FR \arrow[r, two heads, "\langle F(\pi_X \cdot c), F(\pi_Z \cdot c)\rangle"] \arrow[d, "\gamma"'] & FX \times FZ \arrow[d, "\alpha \times \beta"] \\
\hat{F}(A \times B) \arrow[r, "\langle \hat{F}\pi_A, \hat{F}\pi_B\rangle"'] & \hat{F}A \times \hat{F}B
\end{tikzcd}$$

The top morphism is a regular epimorphism because F is left covering, and the right-hand morphism is a regular epimorphism by 16.3.3, so that the bottom morphism also is a regular epimorphism, as requested.
3b. Equalizers: let

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & B
\end{tikzcd}$$

be an equalizer in $\mathcal{A}$. The idea is once again to construct a $\mathcal{P}$-cover of E using $\mathcal{P}$-covers of A and B. For that, consider the following diagram:

$$\begin{tikzcd}
R \arrow[d, "c"'] & & Z' \arrow[d, "z"] \\
E'' \arrow[d, "z'"'] \arrow[urr, "n'"] & & N(b) \arrow[d, shift right, "b_1"'] \arrow[d, shift left, "b_2"] \\
E' \arrow[r, "e'"'] \arrow[d, "a'"'] \arrow[urr, "n"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "a"'] & Z \arrow[d, "b"] \\
E \arrow[r, "e"'] & A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & B
\end{tikzcd}$$

where $a\colon X \to A$ and $b\colon Z \to B$ are $\mathcal{P}$-covers, E' is a pullback of a and e, the equations $b \cdot f = \varphi \cdot a$ and $b \cdot g = \psi \cdot a$ hold, $N(b)$ is a kernel pair of b, n is the unique morphism such that $b_1 \cdot n = f \cdot e'$ and $b_2 \cdot n = g \cdot e'$, E'' is a pullback of z and n, and $z\colon Z' \to N(b)$ and $c\colon R \to E''$ are $\mathcal{P}$-covers. From 16.4 we know that the following is a limit diagram in $\mathcal{A}$

$$
\begin{tikzcd}
& E' \arrow[dl, "e'"'] \arrow[dr, "n"] & \\
X \arrow[d, "f"'] \arrow[drr, "g" near start] & & N(b) \arrow[dll, "b_1"' near start] \arrow[d, "b_2"] \\
Z & & Z
\end{tikzcd}
$$

Clearly, it remains a limit diagram if we paste it with the pullback E'' and, by 16.8, we get a weak limit in $\mathcal{P}$ by covering it with $c\colon R \to E''$

$$
\begin{tikzcd}
& R \arrow[dl, "e' \cdot z' \cdot c"'] \arrow[dr, "n' \cdot c"] & \\
X \arrow[d, "f"'] \arrow[drr, "g" near start] & & Z' \arrow[dll, "b_1 \cdot z"' near start] \arrow[d, "b_2 \cdot z"] \\
Z & & Z
\end{tikzcd}
$$

Consider the following diagram in $\mathcal{B}$:

$$
\begin{tikzcd}
& & & FZ' \arrow[d, "q"] \\
& V'' \arrow[urr, "m'"] \arrow[d, "q'"'] & & N(\beta) \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] \\
FR \arrow[ur, dotted, "k"] \arrow[d, "\gamma"'] & V' \arrow[r, "v'"'] \arrow[urr, "m"] \arrow[d, "\alpha'"'] & FX \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] \arrow[d, "\alpha"'] & FZ \arrow[d, "\beta"] \\
\hat{F}E \arrow[r, "h"] \arrow[rr, bend right, "\hat{F}e"'] & V \arrow[r, "v"] & \hat{F}A \arrow[r, shift left, "\hat{F}\varphi"] \arrow[r, shift right, "\hat{F}\psi"'] & \hat{F}B
\end{tikzcd}
$$

where V is an equalizer of $\hat{F}\varphi$ and $\hat{F}\psi$, $\alpha\colon FX \to \hat{F}A$, $\beta\colon FZ \to \hat{F}B$ and $\gamma\colon FR \to \hat{F}E$ are the coequalizers defining $\hat{F}A, \hat{F}B$ and $\hat{F}E$ (as explained in the first part of the proof), V' is a pullback of α and v, $N(\beta)$ is a kernel pair of β, m is the unique morphism such that $\beta_1 \cdot m = Ff \cdot v'$ and $\beta_2 \cdot m = Fg \cdot v'$, q is the unique morphism such that $\beta_i \cdot q = F(b_i \cdot z)$ $(i = 1, 2)$, V'' is a pullback of q and m, and $h\colon \hat{F}E \to V$ is the unique morphism such that $v \cdot h = \hat{F}e$. We have to prove that h is a regular epimorphism. By 16.4 the following is a limit diagram:

$$\begin{tikzcd}
& V'' \arrow[dl, "v'\cdot q'"'] \arrow[dr, "m'"] & \\
FX \arrow[d, "Ff"'] \arrow[drr, "Fg" near start] & & FZ' \arrow[d, "F(b_2\cdot z)"] \arrow[dll, "F(b_1\cdot z)"' near start] \\
FZ & & FZ
\end{tikzcd}$$

Since F is left covering, the unique morphism $k\colon FR \to V''$ such that $m' \cdot k = F(n' \cdot c)$ and $v' \cdot q' \cdot k = F(e' \cdot z' \cdot c)$ is a regular epimorphism. As $\mathcal{B}$ is exact and F is left covering, by 16.16 (β_1, β_2) is the equivalence relation induced by (z_1, z_2), so that q is a regular epimorphism, and then q' also is a regular epimorphism. Since α' also is a regular epimorphism, it remains to check that $\alpha' \cdot q' \cdot k = h \cdot \gamma$. By composing with the monomorphism v, this is an easy diagram chasing.
3c. Terminal object: let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be terminal objects of $\mathcal{A}$ and $\mathcal{B}$, and $T \to 1_{\mathcal{A}}$ a $\mathcal{P}$-cover. Applying $\hat{F}$ we get a commutative diagram

$$\begin{tikzcd}
FT \arrow[r] \arrow[d] & \hat{F}1_{\mathcal{A}} \arrow[dl] \\
1_{\mathcal{B}} &
\end{tikzcd}$$

Since T is weak terminal in $\mathcal{P}$ and F is left covering, $FT \to 1_{\mathcal{B}}$ is a regular epimorphism. Therefore, $\hat{F}1_{\mathcal{A}} \to 1_{\mathcal{B}}$ also is a regular epimorphism.
4. The extension $\hat{F}\colon \mathcal{A} \to \mathcal{B}$ preserves regular epimorphisms. This is obvious: if $\varphi\colon A \to B$ is a regular epimorphism in $\mathcal{A}$ and $a\colon X \to A$ is a $\mathcal{P}$-cover, we can choose as a $\mathcal{P}$-cover of B the morphism $\varphi \cdot a\colon X \to B$. Applying $\hat{F}$ we get a commutative diagram

$$\begin{tikzcd}
FX \arrow[r, "\mathrm{id}"] \arrow[d, two heads, "\alpha"'] & FX \arrow[d, two heads, "\beta"] \\
\hat{F}A \arrow[r, "\hat{F}\varphi"'] & \hat{F}B
\end{tikzcd}$$

which shows that $\hat{F}\varphi$ is a regular epimorphism. $\square$

16.25 Corollary.

1. *Let $\mathcal{A}$ and $\mathcal{B}$ be exact categories and $I\colon \mathcal{P} \to \mathcal{A}$ a regular projective cover. Given exact functors $G, G'\colon \mathcal{A} \rightrightarrows \mathcal{B}$ with $G \cdot I \simeq G' \cdot I$, then $G \simeq G'$.*

2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be exact categories, $\mathcal{P} \to \mathcal{A}$ a regular projective cover of $\mathcal{A}$ and $\mathcal{P}' \to \mathcal{A}'$ one of $\mathcal{A}'$. Any equivalence $\mathcal{P} \simeq \mathcal{P}'$ extends to an equivalence $\mathcal{A} \simeq \mathcal{A}'$.*

16.26 Remark. For later use, let us point out a simple consequence of the previous theorem. Consider the free exact completion $I\colon \mathcal{P} \to \mathcal{A}$ as in 16.24 and a functor $K: \mathcal{A} \to \mathcal{B}$, with $\mathcal{B}$ exact. If K preserves coequalizers of equivalence relations and $K \cdot \Gamma$ is left covering, then K is exact.

16.27 Corollary. *Let $\mathcal{A}$ be an algebraic category and $\mathcal{P}$ its full subcategory of regular projective objects. The inclusion $I: \mathcal{P} \to \mathcal{A}$ is a free exact completion of $\mathcal{P}$.*

In fact, this follows from Theorem 16.24 because $\mathcal{A}$ is exact (see 3.18) and $\mathcal{P}$ is a projective cover of $\mathcal{A}$ (see 5.15).

Historical Remarks for Chapter 16

Following a suggestion of A. Joyal, the exact completion of a category with finite limits was presented by A. Carboni and R. Celia Magno in [32]. The more general approach working with categories with weak finite limits is due to A. Carboni and E. M. Vitale [33].

The connection between the exact completion and the homotopy category of topological spaces (see 17.4) was established by M. Gran and E. M. Vitale in [49].

Chapter 17

Exact completion and reflexive-coequalizer completion

The present chapter is devoted to elementary constructions of two free completions described in Chapters 16 and 7, respectively, in a different manner: a free exact completion (of categories with weak finite limits) and a free reflexive-coequalizer completion (of categories with finite coproducts). The reader may decide to skip this chapter without losing the connection to other chapters.

In Chapter 17 we have seen that if $I\colon \mathcal{P} \to \mathcal{A}$ is a regular projective cover of an exact category $\mathcal{A}$, then $\mathcal{P}$ has weak finite limits and I is a free exact completion of $\mathcal{P}$. We complete here the study of the exact completion showing that for any category $\mathcal{P}$ with weak finite limits it is possible to construct a free exact completion $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$. Moreover, Γ is a regular projective cover of $\mathcal{P}_{ex}$.

The following construction of $\mathcal{P}_{ex}$ is suggested by 16.23.

17.1 Definition. Given a category $\mathcal{P}$ with weak finite limits, we define the category $\mathcal{P}_{ex}$ as follows:

1. Objects of $\mathcal{P}_{ex}$ are pseudoequivalences $x_1, x_2\colon X' \rightrightarrows X$ in $\mathcal{P}$ (we sometimes denote such an object by X/X').
2. A *premorphism* in $\mathcal{P}_{ex}$ is a pair of morphisms (f', f) as in the diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & Z' \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

   such that $f \cdot x_1 = z_1 \cdot f'$ and $f \cdot x_2 = z_2 \cdot f'$.

3. A *morphism* in $\mathcal{P}_{ex}$ is an equivalence class $[f', f]\colon X/X' \to Z/Z'$ of premorphisms, where two parallel premorphisms (f', f) and (g', g) are equivalent if there exists a morphism $\Sigma\colon X \to Z'$ such that $z_1 \cdot \Sigma = f$ and $z_2 \cdot \Sigma = g$.

4. Composition and identities are obvious.

17.2 Notation. We denote by $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$ the embedding of $\mathcal{P}$ into $\mathcal{P}_{ex}$ assigning to a morphism $f\colon X \to Z$ the following morphism

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] & Z \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

17.3 Remark.

1. The fact that the above relation among premorphisms is an equivalence relation can be proved (step by step) using the assumption that the codomain $z_1, z_2\colon Z' \rightrightarrows Z$ is a pseudoequivalence. Observe also that the class of (f', f) depends on f only (compose f with a reflexivity morphism of (z_1, z_2) to show that (f', f) and (f'', f) are equivalent); for this reason, we often write $[f]$ instead of $[f', f]$.

2. The fact that composition is well-defined is obvious.

3. Γ is a full and faithful functor. This is easy to verify.

4. Observe that if $\mathcal{P}$ is small (or locally small), then $\mathcal{P}_{ex}$ also is small (or locally small, respectively).

17.4 Remark. The above equivalence relation among premorphisms in $\mathcal{P}_{ex}$ can be thought of as a kind of "homotopy" relation. And in fact, this is the case in a particular example: let X be a topological space and $X^{[0,1]}$ the space of continuous maps from the interval $[0, 1]$ to X; the evaluation maps $ev_0, ev_1\colon X^{[0,1]} \rightrightarrows X$ constitute a pseudoequivalence. This gives rise to a functor $\mathcal{E}\colon \mathbf{Top} \to \mathbf{Top}_{ex}$. Now two continuous maps $f, g\colon X \to Z$ are homotopic in the usual sense precisely when $\mathcal{E}(f)$ and $\mathcal{E}(g)$ are equivalent in the sense of Definition 17.1. More precisely, $\mathcal{E}$ factorizes through the homotopy category, and the factorization $\mathcal{E}'\colon \mathbf{HTop} \to \mathbf{Top}_{ex}$ is full and faithful (and left covering).

We are going to prove that the above category $\mathcal{P}_{ex}$ is exact and the functor $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$ is a regular projective cover. For this, it is useful to have an equivalent description of $\mathcal{P}_{ex}$ as a full subcategory of the functor category $[\mathcal{P}^{op}, Set]$.

17.5 Lemma. *Let $\mathcal{P}$ be a category with weak finite limits, and let*

$$Y_{\mathcal{P}^{op}}\colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$$

be the Yoneda embedding. The following properties of a functor $A\colon \mathcal{P}^{op} \to Set$ are equivalent:

1. *A is a regular quotient of a representable object modulo a pseudoequivalence in $\mathcal{P}$, i.e., there exists a pseudoequivalence $x_1, x_2 \colon X' \rightrightarrows X$ in $\mathcal{P}$ and a coequalizer*

$$\begin{tikzcd} Y_{\mathcal{P}^{op}}(X') \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & Y_{\mathcal{P}^{op}}(X) \arrow[r, two heads] & A \end{tikzcd}$$

   in $[\mathcal{P}^{op}, Set]$;

2. *A is a regular quotient of a representable object modulo a regular epimorphism $a \colon Y_{\mathcal{P}^{op}}(X) \to A$ such that $N(a)$, the domain of a kernel pair of a, is also a regular quotient of a representable object:*

$$\begin{tikzcd} Y_{\mathcal{P}^{op}}(X') \arrow[r, two heads, "x"] & N(a) \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & Y_{\mathcal{P}^{op}}(X) \arrow[r, two heads, "a"] & A \end{tikzcd} \qquad \textit{(for some } X' \textit{ in } \mathcal{P}).$$

Proof. Consider the diagram displayed in 2. Since a is the coequalizer of $(a_1 \cdot x, a_2 \cdot x)$, we have to prove that $(a_1 \cdot x, a_2 \cdot x)$ is a pseudoequivalence in $\mathcal{P}$. Let us check the transitivity: consider the following diagram

$$\begin{tikzcd}
Y_{\mathcal{P}^{op}}(W) \arrow[dr, "v"] \arrow[drrr, "x_1'"] \arrow[dddr, "x_2'"'] & & & \\
& P' \arrow[rr, "u_1"] \arrow[dr, "u"] \arrow[dd, "u_2"'] & & Y_{\mathcal{P}^{op}}(X') \arrow[d, two heads, "x"] \\
& & P \arrow[r, "a_1'"] \arrow[d, "a_2'"'] & N(a) \arrow[d, "a_2"] \\
& Y_{\mathcal{P}^{op}}(X') \arrow[r, two heads, "x"'] & N(a) \arrow[r, "a_1"'] & Y_{\mathcal{P}^{op}}(X)
\end{tikzcd}$$

where P and P' are pullbacks, and W is a weak pullback. Since $Y_{\mathcal{P}^{op}}(W)$ is regular projective and x is a regular epimorphism, the transitivity morphism $t \colon P \to N(a)$ of (a_1, a_2) extends to a morphism $t' \colon Y_{\mathcal{P}^{op}}(W) \to Y_{\mathcal{P}^{op}}(X')$ such that $t \cdot u \cdot v = x \cdot t'$. This morphism t' is a transitivity morphism for $(a_1 \cdot x, a_2 \cdot x)$. The converse implication follows from Lemma 16.16, since $Y_{\mathcal{P}^{op}} \colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$ is left covering (see 16.12). $\square$

Note that the fact that (x_1, x_2) is a pseudoequivalence in $\mathcal{P}$ does not mean that $(Y_{\mathcal{P}^{op}}(x_1), Y_{\mathcal{P}^{op}}(x_2))$ is a pseudoequivalence in $[\mathcal{P}^{op}, Set]$ because $Y_{\mathcal{P}^{op}}$ does not preserve weak pullbacks.

17.6 Remark. The full subcategory of $[\mathcal{P}^{op}, Set]$ of all objects satisfying 1. or 2. of the above lemma is denoted by ${\mathcal{P}'}_{ex}$. In the next lemma, the codomain restriction of the Yoneda embedding $Y_{\mathcal{P}^{op}} \colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$ to $\mathcal{P}'_{ex}$ is again denoted by $Y_{\mathcal{P}^{op}}$, and Γ is the functor from 17.2.

17.7 Lemma. *There exists an equivalence of categories $\mathcal{E}\colon \mathcal{P}_{ex} \to \mathcal{P}'_{ex}$ such that $\mathcal{E} \cdot \Gamma = Y_{\mathcal{P}^{op}}$.*

Proof. Consider the functor $\mathcal{E}\colon \mathcal{P}_{ex} \to \mathcal{P}'_{ex}$ sending a morphism $[f]\colon X/X' \to Z/Z'$ to the corresponding morphism φ between the coequalizers as in the following diagram

$$\begin{tikzcd}
Y_{\mathcal{P}^{op}}(X') \arrow[r, "f'"] \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & Y_{\mathcal{P}^{op}}(Z') \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\
Y_{\mathcal{P}^{op}}(X) \arrow[r, "f"'] \arrow[d, "a"'] & Y_{\mathcal{P}^{op}}(Z) \arrow[d, "b"] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

The functor $\mathcal{E}$ is well-defined because a is an epimorphism and b coequalizes y_0 and y_1. Moreover, $\mathcal{E}$ is essentially surjective by definition of $\mathcal{P}'_{ex}$. Let us prove that $\mathcal{E}$ is faithful: if $\mathcal{E}[f] = \mathcal{E}[g]$, then the pair (f, g) factorizes through the kernel pair $N(b)$ of b, which is a regular factorization of (y_0, y_1). Since $Y_{\mathcal{P}^{op}}(X)$ is regular projective, this factorization extends to a morphism $Y_{\mathcal{P}^{op}}(X) \to Y_{\mathcal{P}^{op}}(Z')$, which shows that $[f] = [g]$.
$\mathcal{E}$ is full: given $\varphi\colon A \to B$, we get $f\colon Y_{\mathcal{P}^{op}}(X) \to Y_{\mathcal{P}^{op}}(Z)$ by regular projectivity of $Y_{\mathcal{P}^{op}}(X)$. Since $b \cdot f \cdot x_1 = b \cdot f \cdot x_2$, we get $\overline{f}\colon Y_{\mathcal{P}^{op}}(X') \to N(b)$. Since $N(b)$ is the regular factorization of (z_1, z_2) and $Y_{\mathcal{P}^{op}}(X')$ is regular projective, $\overline{f}$ extends to $f'\colon Y_{\mathcal{P}^{op}}(X') \to Y_{\mathcal{P}^{op}}(Z')$. Clearly, $\mathcal{E}[f', f] = \varphi$. □

17.8 Proposition. *For every category $\mathcal{P}$ with weak finite limits, the functor*

$$\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$$

of 17.2 is a left covering functor into an exact category. Moreover, this is a regular projective cover of $\mathcal{P}_{ex}$.

Proof. 1. $\mathcal{P}_{ex}$ has finite limits. Since the construction of the other basic types of finite limits is completely analogous, we explain in details the case of equalizers, mentioning the construction for binary products and terminal object just briefly.
1a. Equalizers: Consider a parallel pair in $\mathcal{P}_{ex}$ together with what we want to be their equalizer

$$\begin{tikzcd}
E' \arrow[r, "e'"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] & X' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & Z' \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\
E \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

This means that we need the following equations: $x_1 \cdot e' = e \cdot e_1$ and $x_2 \cdot e' = e \cdot e_2$. Moreover, we request $f \cdot e$ and $g \cdot e$ being equivalent, that is, we need a morphism

$\varphi\colon E \to Z'$ such that $z_1 \cdot \varphi = f \cdot e$ and $z_2 \cdot \varphi = g \cdot e$. Let us take E and E' to be the following weak limits

$$\begin{tikzcd}
& E \arrow[dl, "e"'] \arrow[dr, "\varphi"] & \\
X \arrow[d, "f"'] \arrow[drr, "g", near start] & & Z' \arrow[d, "z_2"] \arrow[dll, "z_1"', near start] \\
Z & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[d, "e"'] & E' \arrow[l, "e_1"'] \arrow[r, "e_2"] \arrow[d, "e'"'] & E \arrow[d, "e"] \\
X & X' \arrow[l, "x_1"] \arrow[r, "x_2"'] & X
\end{tikzcd}$$

(i) It is straightforward to check that (e_1, e_2) is a pseudoequivalence in $\mathcal{P}$ (just use the fact that (x_1, x_2) is a pseudoequivalence).

(ii) To show that $[e]$ equalizes $[f]$ and $[g]$, use the morphism $\varphi\colon E \to Z'$.

(iii) The morphism $[e]$ is a monomorphism: in fact, consider two morphisms in $\mathcal{P}_{ex}$

$$\begin{tikzcd}
A' \arrow[r, shift left, "h'"] \arrow[r, shift right, "k'"'] \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] & E' \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_2"] \\
A \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & E
\end{tikzcd}$$

such that $[e] \cdot [h] = [e] \cdot [k]$. This means that there is a morphism $\Sigma\colon A \to X'$ such that $x_1 \cdot \Sigma = e \cdot h$ and $x_2 \cdot \Sigma = e \cdot k$. By the weak universal property of E', we have a morphism $\Sigma'\colon A \to E'$ such that $e_1 \cdot \Sigma' = h$ and $e_2 \cdot \Sigma' = k$. This means that $[h] = [k]$.

(iv) We prove that every morphism

$$\begin{tikzcd}
A' \arrow[r, "h'"] \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] & X' \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] \\
A \arrow[r, "h"'] & X
\end{tikzcd}$$

in $\mathcal{P}_{ex}$ such that $[f] \cdot [h] = [g] \cdot [h]$ factorizes through $[e]$. We know that there is $\Sigma\colon A \to Z'$ such that $z_1 \cdot \Sigma = f \cdot h$ and $z_2 \cdot \Sigma = g \cdot h$. The weak universal property of E yields then a morphism $k\colon A \to E$ such that $e \cdot k = h$ and $\Sigma \cdot k = \varphi$. Now, $x_1 \cdot h' = e \cdot k \cdot a_1$ and $x_2 \cdot h' = e \cdot k \cdot a_2$. The weak universal property of E' yields a morphism $k'\colon A' \to E'$ such that $e_1 \cdot k' = k \cdot a_1$ and $e_2 \cdot k' = k \cdot a_2$. Finally, the needed factorization is $[k', k]\colon A/A' \to E/E'$.

1b. Products: Consider two objects $x_1, x_2\colon X' \rightrightarrows X$ and $z_1, z_2\colon Z' \rightrightarrows Z$ in $\mathcal{P}_{ex}$. Their product is given by

$$\begin{tikzcd}
X' \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & P' \arrow[l, "x'"'] \arrow[r, "z'"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & Z' \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\
X & P \arrow[l, "x"] \arrow[r, "z"'] & Z
\end{tikzcd}$$

where

$$X \xleftarrow{x} P \xrightarrow{z} Z$$

is a weak product of X and Z in $\mathcal{P}$, and P' is the following weak limit

$$\begin{tikzcd}
 & & P' \arrow[dll, "x'"'] \arrow[dl, "p_1"] \arrow[dr, "p_2"'] \arrow[drr, "z'"] & & \\
X' \arrow[d, "x_1"'] \arrow[dr, "x_2"' near end] & P \arrow[dl, "x" near start] \arrow[drr, "z" near start] & & P \arrow[dll, "x"' near start] \arrow[dr, "z"' near start] & Z' \arrow[dl, "z_1" near end] \arrow[d, "z_2"] \\
X & X & & Z & Z
\end{tikzcd}$$

1c. Terminal object: For any object T of $\mathcal{P}$, the projections from a weak product $\pi_1, \pi_2 \colon T \times T \rightrightarrows T$ form a pseudoequivalence. If T is a weak terminal object in $\mathcal{P}$, then (π_0, π_1) is a terminal object in $\mathcal{P}_{ex}$.
2. $\mathcal{P}_{ex}$ is closed under finite limits in $[\mathcal{P}^{op}, Set]$. In fact, by Lemma 17.7, we can identify $\mathcal{P}_{ex}$ with $\mathcal{P}'_{ex}$. We prove that the full inclusion of $\mathcal{P}_{ex}$ into $[\mathcal{P}^{op}, Set]$ preserves finite limits. Because of Lemma 16.20, it is enough to prove that the inclusion is left covering. We give the argument for equalizers, since that for products and terminal object is similar (and easier). With the notations of part 1., consider the following diagram, where ϵ, α and β are extensions to the coequalizers, the triangle on the right is a regular factorization, and the triangle at the bottom is the factorization through the equalizer

$$\begin{tikzcd}
Y_{\mathcal{P}^{op}}(E') \arrow[r, "e'"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & Y_{\mathcal{P}^{op}}(X') \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & Y_{\mathcal{P}^{op}}(Z') \arrow[dr, "z"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \\
Y_{\mathcal{P}^{op}}(E) \arrow[r, "e"] \arrow[d, "c"'] & Y_{\mathcal{P}^{op}}(X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "a"'] & Y_{\mathcal{P}^{op}}(Z) \arrow[d, "b"] & N(b) \arrow[l, shift right, "b_1"'] \arrow[l, shift left, "b_2"] \\
C \arrow[r, "\epsilon"] \arrow[d, "\epsilon'"'] & A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B & \\
L \arrow[ur, "l"'] & & &
\end{tikzcd}$$

We have to prove that ϵ' is a regular epimorphism. Using $\varphi \colon E \to Z'$, we check that $\alpha \cdot a \cdot e = \beta \cdot a \cdot e$, so that there is $p \colon Y_{\mathcal{P}^{op}}(E) \to L$ such that $l \cdot p = a \cdot e$, and then $p = \epsilon' \cdot c$. So, it is enough to prove that p is a regular epimorphism, that is, the components $p(P) \colon Y_{\mathcal{P}^{op}}(E)(P) \to L(P)$ are surjective. This means that, given a morphism $u \colon Y_{\mathcal{P}^{op}}(P) \to A$ such that $\alpha \cdot u = \beta \cdot u$, we need a morphism $\hat{u} \colon P \to E$ with $l \cdot p \cdot \hat{u} = u$. First of all, observe that, since a is a regular epimorphism and $Y_{\mathcal{P}^{op}}(P)$ is regular projective, there is $u' \colon P \to X$ such that $a \cdot u' = u$. Now, $b \cdot f \cdot u' = b \cdot g \cdot u'$, so that there is $u'' \colon Y_{\mathcal{P}^{op}}(P) \to N(b)$ such that $b_1 \cdot u'' = f \cdot u'$ and $b_2 \cdot u'' = g \cdot u'$. Moreover, since z is a regular epimorphism

and $Y_{\mathcal{P}^{op}}(P)$ is regular projective, there is $\tilde{u}\colon P \to Z'$ with $z \cdot \tilde{u} = u''$. Finally, $z_1 \cdot \tilde{u} = f \cdot u'$ and $z_2 \cdot \tilde{u} = g \cdot u'$, so that there is $\hat{u}\colon P \to E$ such that $\varphi \cdot \hat{u} = \tilde{u}$ and $e \cdot \hat{u} = u'$. This last equation implies that $l \cdot p \cdot \hat{u} = u$.
3. $\mathcal{P}_{ex}$ is closed in $[\mathcal{P}^{op}, Set]$ under coequalizers of equivalence relations. In fact, consider an equivalence relation in $\mathcal{P}_{ex}$, with its coequalizer in $[\mathcal{P}^{op}, Set]$

$$\begin{tikzcd} B \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & A \arrow[r, two heads, "c"] & C \end{tikzcd}$$

We have to prove that C lies in $\mathcal{P}_{ex}$. For this, consider the following diagram:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & Y_{\mathcal{P}^{op}}(X) \arrow[d, "a"] \\ B' \arrow[r] \arrow[d] & B \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "c"] \\ Y_{\mathcal{P}^{op}}(X) \arrow[r, two heads, "a"'] & A \arrow[r, two heads, "c"'] & C \end{tikzcd}$$

with each square except, possibly, the right-hand bottom one, a pullback. The remaining square is, then, also a pullback because $[\mathcal{P}^{op}, Set]$ is exact, $X \in \mathcal{P}$ and a is a regular epimorphism. Since A, B and $Y_{\mathcal{P}^{op}}(X)$ are in $\mathcal{P}_{ex}$, which is closed in $[\mathcal{P}^{op}, Set]$ under finite limits (see part 2.), also K lies in $\mathcal{P}_{ex}$. So, K is a regular quotient of a representable object. But K is also the kernel pair of the regular epimorphism $c \cdot a\colon Y_{\mathcal{P}^{op}}(X) \to C$. By Lemma 17.5, this means that C is in $\mathcal{P}_{ex}$. $\square$

17.9 Corollary. *For every category $\mathcal{P}$ with weak finite limits, the functor*

$$\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$$

of 17.2 is a free exact completion of $\mathcal{P}$.

In fact, this follows from 16.24 and 17.8.

17.10 Remark. Let $\mathcal{A}$ be an algebraic category. From Chapter 7 we know that there are equivalences

$$\mathcal{A} \simeq Ind(\mathcal{A}_{fp}) \text{ and } \mathcal{A}_{fp} \simeq Rec(\mathcal{A}_{pp})$$

where $\mathcal{A}_{pp}$ and $\mathcal{A}_{fp}$ are the full subcategories of $\mathcal{A}$ of perfectly presentable objects and of finitely presentable objects, respectively, and *Rec* is the free completion under finite colimits conservative with respect to finite coproducts (see 17.11). An analogous situation holds with the exact completion. In fact, there are equivalences

$$\mathcal{A} \simeq (\mathcal{A}_{rp})_{ex} \simeq (FCSum(\mathcal{A}_{pp}))_{ex} \text{ and } \mathcal{A}_{rp} \simeq Ic(FCSum(\mathcal{A}_{pp}))$$

where $\mathcal{A}_{rp}$ is the full subcategory of regular projective objects and *FCSum* is the free completion under coproducts conservative with respect to finite coproducts.

The second part of this chapter is devoted to an elementary description of the free completion under reflexive coequalizers

$$E_{Rec} : \mathcal{C} \to Rec\,\mathcal{C}$$

of a category $\mathcal{C}$ with finite coproducts, already studied in Chapter 7. Let us start by observing that 7.5 can be restated as follows.

17.11 Proposition. *Let $\mathcal{T}$ be an algebraic theory. The functor*

$$Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to (Alg\,\mathcal{T})_{fp}$$

is a free completion of $\mathcal{T}$ under finite colimits, conservative with respect to finite coproducts. This means that

1. *$(Alg\,\mathcal{T})_{fp}$ is finitely cocomplete and $Y_{\mathcal{T}}$ preserves finite coproducts*

and

2. *for every functor $F \colon \mathcal{T}^{op} \to \mathcal{B}$ preserving finite coproducts, where $\mathcal{B}$ is a finitely cocomplete category, there exists an essentially unique functor $F^* \colon Alg\,\mathcal{T} \to \mathcal{B}$ preserving finite colimits with F naturally isomorphic to $F^* \cdot Y_{\mathcal{T}}$.*

We pass now to the elementary description of $Rec\,\mathcal{C}$.

17.12 Definition. Given a category $\mathcal{C}$ with finite coproducts, we define the category $Rec\,\mathcal{C}$ as follows:

1. Objects of $Rec\,\mathcal{C}$ are reflexive pairs $x_1, x_2 \colon X_1 \rightrightarrows X_0$ in $\mathcal{C}$ (that is, parallel pairs for which there exists $d \colon X_0 \to X_1$ such that $x_1 \cdot d = \mathrm{id}_{X_0} = x_2 \cdot d$, see 3.12).

2. Consider the following diagram in $\mathcal{C}$

$$\begin{tikzcd} & Z_1 \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\ V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z_0 \end{tikzcd}$$

with z_1, z_2 a reflexive pair. We write

$$h \colon f \mapsto g$$

if there exists a morphism $h \colon V \to Z_1$ such that $z_1 \cdot h = f$ and $z_2 \cdot h = g$. This is a reflexive relation in the hom-set $\mathcal{C}(V, Z_0)$. We write $f \sim g$ if f and g are in the equivalence relation generated by this reflexive relation.

3. A premorphism in $Rec\mathcal{C}$ from (x_1, x_2) to (z_1, z_2) is a morphism f in $\mathcal{C}$ as in the diagram

$$\begin{tikzcd} X_1 \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & Z_1 \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\ X_0 \arrow[r, "f"'] & Z_0 \end{tikzcd}$$

   such that $f \cdot x_1 \sim f \cdot x_2$.

4. A morphism in $Rec\mathcal{C}$ from (x_1, x_2) to (z_1, z_2) is an equivalence class $[f]$ of premorphisms with respect to the equivalence $\sim$ of 2.

5. Composition and identities in $Rec\mathcal{C}$ are the obvious ones.

6. The functor $E_{Rec} : \mathcal{C} \to Rec\mathcal{C}$ is defined by

$$P(X \xrightarrow{f} Z) \qquad = \qquad \begin{tikzcd} X \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] & Z \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] \\ X \arrow[r, "{[f]}"'] & Z \end{tikzcd}$$

17.13 Remark.

1. Consider

$$\begin{tikzcd} & Z_1 \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\ V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z_0 \end{tikzcd}$$

   as in 17.12. Explicitly, $f \sim g$ means that there exists a zig-zag

$$\begin{tikzcd}[column sep=small] & f_1 \arrow[dl, mapsto, "h_1"'] \arrow[dr, mapsto, "h_2"] & & & & f_n \arrow[dl, mapsto, "h_n"'] \arrow[dr, mapsto, "h_{n+1}"] & \\ f & & f_2 & \cdots\cdots & f_{n-1} & & g \end{tikzcd}$$

2. Using the explicit description of $f \sim g$, it is straightforward to prove that $Rec\mathcal{C}$ is a category and $P\colon \mathcal{C} \to Rec\mathcal{C}$ is a full and faitful functor.

3. The above description of $P\colon \mathcal{C} \to Rec\mathcal{C}$ does not depend on the existence of finite coproducts in $\mathcal{C}$

17.14 Lemma. *Let $\mathcal{C}$ be a category with finite coproducts. The category $Rec\mathcal{C}$ of 17.12 has finite colimits and $E_{Rec} : \mathcal{C} \to Rec\mathcal{C}$ preserves finite coproducts.*

Proof. (1) Finite coproducts in $Rec\mathcal{C}$ are computed componentwise, i.e., if $x_1, x_2 \colon X_1 \rightrightarrows X_0$ and $z_1, z_2 \colon Z_1 \rightrightarrows Z_0$ are objects of $Rec\mathcal{C}$, their coproduct is

$$\begin{tikzcd} X_1 \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & X_1 \coprod Z_1 \arrow[d, shift right, "x_1 \coprod z_1"'] \arrow[d, shift left, "x_2 \coprod z_2"] & Z_1 \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\ X_0 \arrow[r, "{[i_{X_0}]}"'] & X_0 \coprod Z_0 & Z_0 \arrow[l, "{[i_{Z_0}]}"] \end{tikzcd}$$

(2) Reflexive coequalizers in $Rec\mathcal{C}$ are depicted in the following diagram

$$\begin{tikzcd} X_1 \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & Z_1 \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] & X_0 \coprod Z_1 \arrow[d, shift right, "\langle f, z_1 \rangle"'] \arrow[d, shift left, "\langle g, z_2 \rangle"] \\ X_0 \arrow[r, shift left, "{[f]}"] \arrow[r, shift right, "{[g]}"'] & Z_0 \arrow[r, "{[\mathrm{id}]}"'] & Z_0 \end{tikzcd}$$

□

17.15 Lemma. *Consider the diagram*

$$\begin{tikzcd} & Z_1 \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] \\ V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z_0 \end{tikzcd}$$

as in 17.12. If a morphism $w \colon Z_0 \to W$ is such that $w \cdot z_1 = w \cdot z_2$ and $f \sim g$, then $w \cdot f = w \cdot g$.

Proof. Clearly if $h \colon f \mapsto g$, then $w \cdot f = w \cdot g$. The claim now follows from the fact that to be coequalized by w is an equivalence relation in $\mathcal{C}(V, Z_0)$. □

17.16 Remark. For every reflexive pair $x_1, x_2 \colon X_1 \rightrightarrows X_0$ in $\mathcal{C}$, the diagram

$$E_{Rec}X_1 \underset{E_{Rec}x_2}{\overset{E_{Rec}x_1}{\rightrightarrows}} E_{Rec}X_0 \xrightarrow{[\mathrm{id}_{X_0}]} (X_1 \underset{x_2}{\overset{x_1}{\rightrightarrows}} X_0)$$

is a reflexive coequalizer in $Rec\mathcal{C}$. Therefore, if two functors $F, G \colon Rec\mathcal{C} \to \mathcal{B}$ preserve reflexive coequalizers and $F \cdot E_{Rec} \simeq G \cdot E_{Rec}$, then $F \simeq G$.

17.17 Proposition. *Let $\mathcal{C}$ be a category with finite coproducts. The functor*

$$E_{Rec} : \mathcal{C} \to Rec\mathcal{C}$$

of 17.12 is a free completion of $\mathcal{C}$ under finite colimits, conservative with respect to finite coproducts. This means that

1. *$Rec\mathcal{C}$ has finite colimits and E_{Rec} preserves finite coproducts*

and

2. *for every functor $F\colon \mathcal{C} \to \mathcal{B}$ preserving finite coproducts, where $\mathcal{B}$ is a finitely cocomplete category, there exists an essentially unique functor $F^*\colon Rec\mathcal{C} \to \mathcal{B}$ preserving finite colimits with F naturally isomorphic to $F^* \cdot E_{Rec}$.*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "E_{Rec}"] \arrow[dr, "F"'] & \simeq & Rec\mathcal{C} \arrow[dl, "F^*"] \\ & \mathcal{B} & \end{tikzcd}$$

Proof. Given $F\colon \mathcal{C} \to \mathcal{B}$ as above, we define $F^*\colon Rec\mathcal{C} \to \mathcal{B}$ on objects by the following coequalizer in $\mathcal{B}$:

$$\begin{tikzcd} FX_1 \arrow[r, shift left, "Fx_1"] \arrow[r, shift right, "Fx_2"'] & FX_0 \arrow[r] & F^*(x_1, x_2) \end{tikzcd}$$

Lemma 17.15 makes it clear how to define F^* on morphisms. The argument for the essential uniqueness of F^* is stated in 17.16. The rest of the proof is straightforward. □

The previous universal property allows us to give a different prove of the equation $Sind\mathcal{C} \simeq Ind(Rec\mathcal{C})$ already established in 7.4.

17.18 Corollary. *Let $\mathcal{C}$ be a small category with finite coproducts. There exists an equivalence of categories*

$$Ind(Rec\mathcal{C}) \simeq Sind\mathcal{C}\,.$$

Proof. Let $\mathcal{B}$ be a cocomplete category. By 4.18, the functors $Ind(Rec\mathcal{C}) \to \mathcal{B}$ preserving colimits correspond to the functors $Rec\mathcal{C} \to \mathcal{B}$ preserving finite colimits and then, by 17.17, to the functors $\mathcal{C} \to \mathcal{B}$ preserving finite coproducts. On the other hand, the functors $\mathcal{C} \to \mathcal{B}$ preserving finite coproducts correspond, by 15.12, to the functors $Sind\mathcal{C} \to \mathcal{B}$ preserving colimits. Since both $Ind(Rec\mathcal{C})$ and $Sind\mathcal{C}$ are cocomplete (4.16 and 4.5), we can conclude that $Ind(Rec\mathcal{C})$ and $Sind\mathcal{C}$ are equivalent categories. □

Historical Remarks for Chapter 17

The reflexive coequalizer completion of a category with finite coproducts is due to A. M. Pitts (unpublished notes [82]). It appeared in press in [31]. The connection between the exact completion and the reflexive coequalizer completion was established by M. C. Pedicchio and J. Rosický in [79], see also [87].

Chapter 18

Finitary localizations of algebraic categories

In 6.9, we characterized algebraic categories among cocomplete categories by the existence of a suitable generator. In this chapter, we will analogously characterize algebraic categories among exact categories. We also study the relationship between algebraic categories and exact, locally finitely presentable categories.

18.1 Theorem. *Let $\mathcal{E}$ be an exact category with sifted colimits, $\mathcal{A}$ a category with finite limits and sifted colimits, and $F\colon \mathcal{E} \to \mathcal{A}$ a functor preserving finite limits and filtered colimits. Then F preserves reflexive coequalizers if and only if it preserves regular epimorphisms.*

Proof. Necessity is evident, because every regular epimorphism is a reflexive coequalizer of its kernel pair. For sufficiency, let F preserve finite limits, filtered colimits and regular epimorphisms.
1. Since every equivalence relation in $\mathcal{E}$ is a kernel pair of its coequalizer and since every regular epimorphism is a coequalizer of its kernel pair, F preserves coequalizers of equivalence relations. Since every pseudoequivalence in $\mathcal{E}$ can be decomposed as a regular epimorphism followed by an equivalence relation (cf. 16.16), F preserves coequalizers of pseudoequivalences.
2. Consider a reflexive and symmetric pair $r = (r_1, r_2\colon X' \rightrightarrows X)$ of morphisms in $\mathcal{E}$. We construct a pseudoequivalence $\overline{r}$ containing r (the *transitive hull* of r) as a (filtered) colimit of the chain of compositions

$$r \circ r \circ \ldots \circ r \qquad n\text{-times}$$

(the composition $r \circ r$ is depicted in the following diagram, where the square is a pullback)

$$\begin{tikzcd}[column sep=small]
 & & r \circ r \arrow[dl, "r'_1"'] \arrow[dr, "r'_2"] & & \\
 & X' \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & X' \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & X & & X
\end{tikzcd}$$

Since F preserves filtered colimits and finite limits, we have $\overline{Fr} = F\overline{r}$. The pseudoequivalence $\overline{r}$ has a coequalizer, which is preserved by F. But a coequalizer of $\overline{r}$ is also a coequalizer of r, and so F preserves coequalizers of reflexive and symmetric pairs of morphisms.
3. If $r = (r_1, r_2)$ is just a reflexive pair, then a reflexive and symmetric pair containing r is given by $r \circ r^{-1}$, that is

$$\begin{tikzcd}[column sep=small]
 & & r \circ r^{-1} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & \\
 & X' \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & X' \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & \\
X & & X & & X
\end{tikzcd}$$

Once again, a coequalizer of $r \circ r^{-1}$ is also a coequalizer of r, so that F preserves reflexive coequalizers. □

18.2 Corollary. *Let $\mathcal{E}$ be a cocomplete exact category, $\mathcal{A}$ a category with finite limits and sifted colimits and $F\colon \mathcal{E} \to \mathcal{A}$ a functor preserving finite limits. Then F preserves sifted colimits if and only if it preserves filtered colimits and regular epimorphisms.*

In fact, this follows from 7.7 and 18.1.

We can now generalize 5.16.

18.3 Corollary. *In a cocomplete exact category, perfectly presentable objects are precisely finitely presentable regular projectives.*

Proof. One implication is established in 5.4. For the converse implication, apply 18.2 to the hom-functor $\hom(G, -)$ of a finitely presentable regular projective object G. □

18.4 Corollary. *A category is algebraic if and only if it is cocomplete, exact and has a strong generator consisting of finitely presentable regular projectives.*

Proof. Necessity follows from 3.18 and 6.9. Sufficiency follows from 18.3 and 6.9. □

In the previous corollary, the assumption of cocompleteness can be reduced to asking for the existence of coequalizers of kernel pairs, which is part of the exactness of the category, and the existence of coproducts of objects from the generator. In fact, we have the following result (recall that a category is *well-powered* if, for a fixed object A, the subobjects of A constitute a set (not a proper class):

18.5 Lemma. *Let $\mathcal{A}$ be a well-powered exact category with a regular projective cover $\mathcal{P} \to \mathcal{A}$. If $\mathcal{P}$ has coproducts, then $\mathcal{A}$ is cocomplete.*

Proof. 1. The functor $\mathcal{P} \to \mathcal{A}$ preserves coproducts. Indeed, consider a coproduct

$$s_i \colon P_i \to \coprod_I P_i$$

in $\mathcal{P}$, and a family of morphisms $\langle x_i \colon P_i \to X \rangle_I$ in $\mathcal{A}$. Let $q \colon Q \to X$ be a regular epimorphism with $Q \in \mathcal{P}$. For each $i \in I$, consider a morphism $y_i \colon P_i \to Q$ such that $q \cdot y_i = x_i$. Since Q is in $\mathcal{P}$, there is $y \colon \coprod_I P_i \to Q$ such that $y \cdot s_i = y_i$, and then $q \cdot y \cdot s_i = x_i$, for all $i \in I$.
As far as the uniqueness of the factorization is concerned, consider a pair of morphisms $f, g \colon \coprod_I P_i \rightrightarrows X$ such that $s_i \cdot f = s_i \cdot g$ for all i. Consider also $f', g' \colon \coprod_I P_i \rightrightarrows Q$ such that $q \cdot f' = f$ and $q \cdot g' = g$. Since $q \cdot f' \cdot s_i = q \cdot g' \cdot s_i$, there is $t_i \colon P_i \to N(q)$ such that $q_1 \cdot t_i = f' \cdot s_i$ and $q_2 \cdot t_i = g' \cdot s_i$, where $q_1, q_2 \cdot N(q) \rightrightarrows Q$ is a kernel pair of q. From the first part of the proof, we obtain a morphism $t \colon \coprod_I P_i \to N(q)$ such that $t \cdot s_i = t_i$ for all i. Moreover, $q_1 \cdot t \cdot s_i = f' \cdot s_i$ for all i, so that $q_1 \cdot t = f'$ because Q is in $\mathcal{P}$. Analogously, $q_2 \cdot t = g'$. Finally, $f = q \cdot f' = q \cdot q_1 \cdot t = q \cdot q_2 \cdot t = q \cdot g' = g$.
2. Recall that $Sub_{\mathcal{A}}(A)$ is the ordered class of subobjects of A. For every category $\mathcal{A}$, we denote by $\theta(\mathcal{A})$ its "ordered reflection", i.e., the ordered class obtained from the preorder on the objects of $\mathcal{A}$ given by $A \leq B$ iff $\mathcal{A}(A, B)$ is nonempty. We are going to prove that for any object A of $\mathcal{A}$, $Sub_{\mathcal{A}}(A)$ and $\theta(\mathcal{P}/A)$ are isomorphic ordered classes. In fact, given a monomorphism $m \colon X \to A$, we consider a $\mathcal{P}$-cover $q \colon Q \to X$ and we get an element in $\theta(\mathcal{P}/A)$ from the composition $m \cdot q$. Conversely, given an object $f \colon Q \to A$ in $\mathcal{P}/A$, the monomorphic part of its regular factorisation gives an element in $Sub_{\mathcal{A}}(A)$.
3. $\mathcal{A}$ has coequalizers. Consider a parallel pair (a, b) in $\mathcal{A}$ and its regular factorization

$$\begin{tikzcd} B \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr, "r"'] & A \\ & R \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] \end{tikzcd}$$

Consider now the equivalence relation $a_1, a_2 \colon A' \rightrightarrows A$ generated by (i_1, i_2), that is the intersection of all the equivalence relations on A containing (i_1, i_2). Such an intersection exists: by part 2., $Sub_{\mathcal{A}}(A)$ is isomorphic to $\theta(\mathcal{P}/A)$, which is cocomplete because $\mathcal{P}$ has coproducts. Since, by assumption, $\mathcal{A}$ is well-powered, $Sub_{\mathcal{A}}(A)$ is a set, and a cocomplete ordered set is also complete. Since $\mathcal{A}$ is exact, (a_1, a_2) has a coequalizer, which is also a coequalizer of (i_1, i_2) and then of (a, b).
4. $\mathcal{A}$ has coproducts. Consider a family of objects $(A_i)_I$ in $\mathcal{A}$. Each of them can

be seen as a coequalizer of a pseudoequivalence in $\mathcal{P}$ as in the following diagram, where the first and the second columns are coproducts in $\mathcal{P}$ (and then in $\mathcal{A}$, see 1.), x_0 and x_1 are the extensions to the coproducts, the bottom row is a coequalizer (which exists by Part 3.), and σ_i is the extension to the coequalizer.

$$\begin{tikzcd} P_i' \arrow[r, shift left, "x_1^i"] \arrow[r, shift right, "x_2^i"'] \arrow[d, "s_i'"'] & P_i \arrow[r, two heads, "a_i"] \arrow[d, "s_i"] & A_i \arrow[d, "\sigma_i"] \\ \coprod_I P_i' \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & \coprod_I P_i \arrow[r, two heads, "q"'] & Q \end{tikzcd}$$

Since coproducts commute with coequalizers, the third column is a coproduct of the family $(A_i)_I$. ☐

18.6 Corollary. *A category is algebraic if and only if it is exact and has a strong generator $\mathcal{G}$ consisting of finitely presentable regular projectives such that coproducts of objects of $\mathcal{G}$ exist.*

Proof. Let $\mathcal{A}$ be an exact category and $\mathcal{G}$ a strong generator consisting of regular projectives. Since a coproduct of regular projectives is regular projective, the full subcategory $\mathcal{P}$ consisting of coproducts of objects from $\mathcal{G}$ is a regular projective cover of $\mathcal{A}$. Following 18.5, it remains to prove that $\mathcal{A}$ is well-powered. Consider an object A and the map

$$\mathcal{F}\colon \Omega(\mathcal{G} \downarrow A) \to Sub_{\mathcal{A}}(A)$$

assigning to a subset $\mathcal{M}$ of $\mathcal{G} \downarrow A$ the subobject of A represented by the monomorphism $s_{\mathcal{M}}\colon S_{\mathcal{M}} \to A$, where

$$e_{\mathcal{M}}\colon \coprod_{(G,g)\in\mathcal{M}} G \to A$$

is the canonical morphism whose (G,g)-component is g, and

$$\textstyle\coprod_{(G,g)\in\mathcal{M}} G \xrightarrow{q_{\mathcal{M}}} S_{\mathcal{M}} \xrightarrow{s_{\mathcal{M}}} A$$

is the regular factorization of $e_{\mathcal{M}}$. We are to prove that $\mathcal{F}$ is surjective, so that $Sub_{\mathcal{A}}(A)$ is a set. For this, consider a monomorphism $m\colon S \to A$ and let $\mathcal{M}(s)$ be the set of those $(G,g) \in \mathcal{G} \downarrow A$ such that g factorizes through s:

$$\begin{tikzcd} G \arrow[rr, "g"] \arrow[dr, "h"'] & & A \\ & S \arrow[ur, "s"'] & \end{tikzcd}$$

We get the commutative diagram

$$\begin{tikzcd} \coprod_{(G,h)\in\mathcal{G}\downarrow S} G \arrow[r, "\hat{s}"] \arrow[dr, "e_S"'] & \coprod_{(G,g)\in\mathcal{M}(s)} G \arrow[r, "q_{\mathcal{M}(s)}"] \arrow[d, "\sigma"] & S_{\mathcal{M}(s)} \arrow[d, "s_{\mathcal{M}(s)}"] \arrow[dl, dotted, "d"] \\ & S \arrow[r, "s"'] & A \end{tikzcd}$$

where the (G,h)-components of e_S and $\hat{s}$ are h and g, respectively, and the (G,g)-component of σ is h. By diagonal fill-in there exists $d\colon S_{\mathcal{M}(s)} \to S$ such that $d \cdot q_{\mathcal{M}(s)} = \sigma$ and $s \cdot d = s_{\mathcal{M}(s)}$. Such a d is an isomorphism: it is a monomorphism because $s_{\mathcal{M}(s)}$ is, and an extremal epimorphism because e_S is. Thus, $\mathcal{F}(\mathcal{M}(s)) = s$ and the proof is complete. □

From Propositions 3.18 and 6.22, we know that an algebraic category is exact and locally finitely presentable. The converse is not true because of the lack of projectivity of the generator. In the remaining part of this chapter we want to state in a precise way the relationship between algebraic categories and exact, locally finitely presentable categories.

18.7 Definition. Given a category $\mathcal{B}$, by a *localization* of $\mathcal{B}$ is meant a reflective subcategory $\mathcal{A}$ whose reflector preserves finite limits. And $\mathcal{A}$ is called a *finitary localization* if, moreover, it is closed in $\mathcal{B}$ under filtered colimits.

18.8 Remark. More loosely, we speak about localizations of $\mathcal{B}$ as categories equivalent to full subcategories having the above property. We use the notation

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{B}$$

that is, R is left adjoint to I and I is full and faithful.

Let us start with a general lemma.

18.9 Lemma. *Consider a reflection*

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{B}$$

1. *If I preserves filtered colimits and an object $P \in \mathcal{B}$ is finitely presentable, then $R(P)$ is finitely presentable;*

2. *If the reflection is a localization and $\mathcal{B}$ is exact, then $\mathcal{A}$ is exact.*

Proof. 1. Same argument as in the proof of 6.16.1.
2. Let $r_1, r_2\colon A' \rightrightarrows A$ be an equivalence relation in $\mathcal{A}$. Its image in $\mathcal{B}$ is an equivalence relation, so that it has a coequalizer Q and it is the kernel pair of its coequalizer (because $\mathcal{B}$ is exact)

$$\begin{tikzcd} IA' \arrow[r, shift left, "Ir_1"] \arrow[r, shift right, "Ir_2"'] & IA \arrow[r, two heads, "q"] & Q \end{tikzcd}$$

If we apply the functor R to this diagram, we obtain a coequalizer (because R is a left adjoint) and a kernel pair (because R preserves finite limits)

$$\begin{tikzcd} RIA' \simeq A' \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \simeq RIA \arrow[r, two heads, "Rq"] & RQ \end{tikzcd}$$

and this means that (r_1, r_2) is effective. It remains to prove that regular epimorphisms are stable under pullbacks. For this, consider a pullback

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in $\mathcal{A}$, with f a regular epimorphism. Form its image in $\mathcal{B}$, computed as a two-step pullback of Ig along the regular factorization $m \cdot e$ of If

$$\begin{tikzcd} IP \arrow[r, "e'"] \arrow[d, "Ig'"'] & Q \arrow[r, "m'"] \arrow[d, "h"] & IC \arrow[d, "Ig"] \\ IA \arrow[r, "e"'] & E \arrow[r, "m"'] & IB \end{tikzcd}$$

so that e' is a regular epimorphism. If we apply the functor R to the second diagram, we come back to the original pullback, computed now as a two-step pullback (because R preserves finite limits)

$$\begin{tikzcd} P \simeq RIP \arrow[r, "Re'"] \arrow[d, "g'"'] & RQ \arrow[r, "Rm'"] \arrow[d, "Rh"] & RIC \simeq C \arrow[d, "g"] \\ A \simeq RIA \arrow[r, "Re"'] & RE \arrow[r, "Rm"'] & RIB \simeq B \end{tikzcd}$$

Now observe that Rm is a monomorphism (because R preserves finite limits) and also a regular epimorphism (because f is a regular epimorphism, and $Rm \cdot Re = f$), so that it is an isomorphism. It follows that Rm' is an isomorphism. Moreover, Re' is a regular epimorphism (because R, being a left adjoint, preserves regular epimorphisms). Finally, f' is a regular epimorphism because $f' = Rm' \cdot Re'$. □

18.10 Theorem. *Finitary localizations of algebraic categories are precisely the exact, locally finitely presentable categories.*

Proof. Since an algebraic category is exact and locally finitely presentable, necessity follows from 6.16.1 and 18.9. For the sufficiency, let $\mathcal{A}$ be an exact and locally finitely presentable category. Following 6.26, $\mathcal{A}$ is equivalent to $Lex\mathcal{T}$, where $\mathcal{T} \simeq \mathcal{A}_{fp}^{op}$, and $Lex\mathcal{T}$ is a reflective subcategory of $Alg\mathcal{T}$ closed under filtered colimits (see 6.29). Consider the full subcategory $\mathcal{P}$ of $Alg\mathcal{T}$ consisting of regular projective objects. Such an object P is a retract of a coproduct of representable algebras (5.14.2). Since every coproduct is a filtered colimit of its finite subcoproducts, and a finite coproduct of representable algebras is representable (1.5), P is a retract of a filtered colimit of representable algebras. Following 4.3, we have that $\mathcal{P}$ is contained in $Lex\mathcal{T}$. Moreover, $\mathcal{P}$ is a regular projective cover of $Alg\,\mathcal{T}$ (5.15). Since, by 16.27, the full inclusion of $\mathcal{P}$ into $Alg\,\mathcal{T}$ is a free exact completion of $\mathcal{P}$ and, by assumption, $Lex\,\mathcal{T}$ is exact, it remains just to prove that the inclusion $\mathcal{P} \to Lex\,\mathcal{T}$ is left covering. Once this is done, we can apply 16.26 to the following situation

$$\begin{tikzcd} \mathcal{P} \arrow[rr] \arrow[dr] & & Alg\,\mathcal{T} \arrow[dl, "R"] \\ & Lex\,\mathcal{T} & \end{tikzcd}$$

where R is the reflector, and we conclude that R is an exact functor. But the inclusion $\mathcal{P} \to Alg\,\mathcal{T} \simeq \mathcal{P}_{ex}$ is left covering, and $Lex\,\mathcal{T}$ is closed in $Alg\,\mathcal{T}$ under limits, so that also the inclusion $\mathcal{P} \to Lex\,\mathcal{T}$ is left covering. $\square$

18.11 Remark. To finish this chapter, we observe that Corollary 18.2 can be used to prove the characterization of varieties established in 10.24 without using Birkhoff's Variety Theorem 10.22. We sketch the argument. Let $\mathcal{T}$ be an algebraic theory and

$$I\colon \mathcal{A} \to Alg\,\mathcal{T}\ ,\quad R \dashv I$$

a regular epireflective subcategory closed under regular quotients and directed unions. From the closedness under regular quotients it immediately follows that I preserves regular epimorphisms and that in $\mathcal{A}$ equivalence relations are effective, so that $\mathcal{A}$ is exact. To prove that I preserves filtered colimit, consider a functor $F\colon \mathcal{D} \to \mathcal{A}$ with $\mathcal{D}$ filtered and let $\langle \sigma_d\colon Fd \to B\rangle_{d\in\mathcal{D}}$ be its colimit cocone in $Alg\,\mathcal{T}$. Let

$$Fd \xrightarrow{e_d} Gd \xrightarrow{m_d} B$$

be the regular factorization of σ_d. For any morphism $f\colon d \to d'$ in $\mathcal{D}$ there exists a unique $Gf\colon Gd \to Gd'$ such that $m_{d'} \cdot Gf = m_d$ (use diagonal fill-in, cf. 0.16). This defines a new functor $G\colon \mathcal{D} \to \mathcal{A}$ (indeed $Gd \in \mathcal{A}$ because it is a regular quotient of $Fd \in \mathcal{A}$) and B is the directed union of the Gd's, so that $B \in \mathcal{A}$. Following 18.2 I preserves sifted colimits and then by 6.18 $\mathcal{A}$ is algebraic. Moreover, an algebraic theory $\mathcal{T}_{\mathcal{A}}$ of $\mathcal{A}$ can be described as follows (cf. 6.16):

$$\mathcal{T}_{\mathcal{A}}^{op} = \{R(\mathcal{T}(X,-)) \mid X \in \mathcal{T}\}$$

The functor

$$\mathcal{T} \to \mathcal{T}_{\mathcal{A}}\ ,\quad X \mapsto R(\mathcal{T}(X,-))$$

preserves finite products (by 1.5) and is surjective on objects. It remains to prove that it is also full: let η be the unit of the adjunction $R \dashv I$ and consider $f\colon R(\mathcal{T}(X,-)) \to R(\mathcal{T}(Z,-))$. Since $\eta_{\mathcal{T}(Z,-)}$ is a regular epimorphism and $\mathcal{T}(X,-)$ is regular projective, there exists $g\colon \mathcal{T}(X,-) \to \mathcal{T}(Z,-)$ such that $f \cdot \eta_{\mathcal{T}(X,-)} = \eta_{\mathcal{T}(Z,)} \cdot g$. Since η is natural and $\eta_{\mathcal{T}(X,-)}$ is an epimorphism, it follows that $Rg = f$. By 10.13 $\mathcal{T}_{\mathcal{A}}$ is a quotient of $\mathcal{T}$. Finally

$$\mathcal{A} \simeq Alg\,\mathcal{T}_{\mathcal{A}} \simeq Alg(\mathcal{T}/\sim)$$

so that $\mathcal{A}$ is a variety of $\mathcal{T}$-algebras.

Historical Remarks for Chapter 18

The first systematic study of localizations is due to P. Gabriel [47] who also together with N. Popesco characterized Grothendieck categories, see [83]. This is an ancestor of Theorem 18.10.

Arbitrary (i.e., non necessarily finitary) localizations of one-sorted algebraic categories and, more in general, of monadic categories over *Set* are characterized in [93] and in [94]. Essential localizations are studied in [10], which generalizes the original result for module categories due to J. E. Roos [88].

One of the results of Lawvere's thesis [63] is a characterization of one-sorted algebraic categories, compare with Corollary 18.6. The only difference is that in Lawvere's original result the generator is required to be abstractly finite, a notion that without the other conditions of the characterization theorem is weaker than finitely presentability.

Appendix A

Monads

An important aspect of algebraic categories which has not yet been treated in this book are monads. The aim of this appendix is to give a short introduction to monads on a category $\mathcal{K}$, and then to explain how finitary monads for $\mathcal{K} = \mathit{Set}$ precisely yield one-sorted algebraic theories, and for $\mathcal{K} = \mathit{Set}^S$ the S-sorted ones.

The word "monad" stems from monoid: recall that a monoid in a category $\mathcal{K}$ is an object M together with a morphism $m\colon M \times M \to M$ which

1. is associative, that is, the square

$$\begin{tikzcd} M \times M \times M \arrow[r, "m \times \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times m"'] & M \times M \arrow[d, "m"] \\ M \times M \arrow[r, "m"'] & M \end{tikzcd}$$

   commutes, and

2. has a unit, that is a morphism $e\colon 1 \to M$ such that the triangles

$$\begin{tikzcd} & M & \\ M = 1 \times M \arrow[ur, "\mathrm{id}_M"] \arrow[r, "e \times \mathrm{id}_M"'] & M \times M \arrow[u, "m"] & M \times 1 = M \arrow[ul, "\mathrm{id}_M"'] \arrow[l, "\mathrm{id}_M \times e"] \end{tikzcd}$$

   commute.

A.1 Definition.

1. A *monad* $\mathbb{M}$ on a category $\mathcal{K}$ consists of an endofunctor M on $\mathcal{K}$ and natural transformations

   (a) $\mu\colon MM \to M$ (monad multiplication) and

(b) $\eta\colon \mathrm{Id}_{\mathcal{K}} \to M$ (monad unit)

such that the diagrams

$$\begin{tikzcd} MMM \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & MM \arrow[d, "\mu"] \\ MM \arrow[r, "\mu"'] & M \end{tikzcd} \qquad [A.1]$$

and

$$\begin{tikzcd} & M & \\ M \arrow[ur, "M"] \arrow[r, "\eta M"'] & M \cdot M \arrow[u, "\mu"] & M \arrow[ul, "M"'] \arrow[l, "M\eta"] \end{tikzcd} \qquad [A.2]$$

commute.

2. The monad is called *finitary* if M is a finitary functor.

A.2 Example.

1. The functor $M\colon \mathit{Set} \to \mathit{Set}$ given by

$$MX = X + 1$$

carries an obvious structure of a monad: the unit is the coproduct injection $\eta_X\colon X \to X+1$ and the multiplication $\mu_X\colon X+1+1 \to X+1$ merges the two copies of 1 to one.

2. The *word monad* on *Set* assigns to every set X the set

$$MX = X^*$$

of all words on it, that is, the (underlying set of the) free monoid on X. This yields an endofunctor on *Set* together with natural transformations $\eta_X\colon X \to X^*$, the formation of one-letter words, and $\mu_X\colon (X^*)^* \to X^*$ given by concatenation of words.

3. Recall that for every finitary endofunctor H of *Set* free H-algebras exist, giving a left adjoint $F_H\colon \mathit{Set} \to H\text{-}\mathit{Alg}$ (see 12.7). The corresponding monad H^* on *Set* is called the *free monad on H*. For example, if H is the polynomial functor of a signature Σ, the free monad is the monad of Σ-terms.
In the notation of 12.6 we have two natural transformations

$$\eta\colon \mathrm{Id} \to U_H F_H = H^*$$

and

$$\varphi\colon HH^* \to H^*$$

yielding a natural transformation

$$\psi = \varphi \cdot H\eta \colon H \to H^* .$$

We will see in A.28 that it has the universal property explaining the name free monad.

A.3 Example. The basic example of a monad is that induced by any adjoint situation

$$\mathcal{A} \underset{U}{\overset{F}{\leftrightarrows}} \mathcal{K} \qquad \text{where} \quad F \dashv U .$$

Let $\eta \colon \mathrm{Id}_{\mathcal{K}} \to UF$ and $\varepsilon \colon FU \to \mathrm{Id}_{\mathcal{A}}$ denote the unit and counit of the adjunction. Recall the equalities

$$U = U\varepsilon \cdot \eta U \ \text{ and } \ F = \varepsilon F \cdot F\eta \tag{$A.3$}$$

characterizing adjoint situations. Then for the endofunctor

$$M = UF \colon \mathcal{K} \to \mathcal{K}$$

we have the natural transformation

$$\mu = U\varepsilon F \colon MM = UFUF \to UF = M \tag{$A.4$}$$

which, together with the unit $\eta \colon \mathrm{Id}_{\mathcal{K}} \to M$, forms a monad on $\mathcal{K}$. In fact, the commutativity of the two triangles [A.2] follows from [A.3] and the square [A.1] follows from the naturality of ε:

$$\varepsilon \cdot FU\varepsilon = \varepsilon \cdot \varepsilon FU \tag{$A.5$}$$

yielding

$$\mu \cdot M\mu = U(\varepsilon \cdot FU\varepsilon)F = U(\varepsilon \cdot \varepsilon FU)F = \mu \cdot \mu M .$$

Observe that whenever U is a finitary functor, this monad is finitary because F, being a left adjoint, always preserves filtered colimits.

A.4 Example.

1. Every one-sorted algebraic category $U \colon \mathcal{A} \to \mathit{Set}$ defines a monad on Set assigning to every set X the free algebra generated by it. In other words, this is the monad induced by the adjunction $F \dashv U$ as in A.3, where F is the free-algebra functor of 11.21.
   Since U preserves filtered colimits by 11.8, all these monads are finitary.

2. Analogously, every S-sorted algebraic category defines a finitary monad on Set^S.

A.5 Remark. Recall from 12.1 the category M-Alg of M-algebras for the endofunctor M of $\mathcal{K}$. If $\mathbb{M}$ is the monad induced by an adjoint situation $F \dashv U$

as in A.3, then every object A of $\mathcal{A}$ yields a canonical M-algebra on $X = UA$: put

$$x = U\varepsilon_A \colon MX = UFUA \to UA = X\,.$$

This algebra has the property that the triangle

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr, "\mathrm{id}_X"] & \\ MX \arrow[r, "x"'] & X \end{tikzcd} \qquad [A.6]$$

commutes: see [A.3]. Also the square

$$\begin{tikzcd} MMX \arrow[r, "\mu_X"] \arrow[d, "Mx"'] & MX \arrow[d, "x"] \\ MX \arrow[r, "x"'] & X \end{tikzcd} \qquad [A.7]$$

commutes, see [A.5]. This leads to the following

A.6 Definition. An *Eilenberg-Moore algebra* for a monad $\mathbb{M} = (M, \mu, \eta)$ on $\mathcal{K}$ is an algebra (X, x) for M such that [A.6] and [A.7] commute. The full subcategory of M-*Alg* formed by all Eilenberg-Moore algebras is denoted by

$$\mathcal{K}^{\mathbb{M}}\,.$$

A.7 Remark. The Eilenberg-Moore category $\mathcal{K}^{\mathbb{M}}$ is considered a concrete category on $\mathcal{K}$ via the faithful functor

$$U_{\mathbb{M}} \colon \mathcal{K}^{\mathbb{M}} \to \mathcal{K}\,, \quad (X, x) \mapsto X$$

It is easy to verify that this concrete category is uniquely transportable (same argument as in 13.17.3).

A.8 Example.

1. For every category $\mathcal{K}$ we have the trivial monad $\mathbb{I}\mathrm{d} = (\mathrm{Id}_{\mathcal{K}}, \mathrm{id}, \mathrm{id})$. The only Eilenberg-Moore algebras are $\mathrm{id}_X \colon X \to X$. Thus, $\mathcal{K}^{\mathbb{I}} \simeq \mathcal{K}$.

2. For the monad $MX = X + 1$ of A.2.1 an Eilenberg-Moore algebra is a pointed set: given $x \colon X + 1 \to X$ with [A.6], the left-hand component of x is id_X, thus, x just chooses an element $1 \to X$. Here [A.7] always commutes. Homomorphisms are functions preserving the choice of element. Shortly: $\mathit{Set}^{\mathbb{M}}$ is the category of pointed sets.

3. For the word monad A.2.2 the category $\mathit{Set}^{\mathbb{M}}$ is essentially the category of monoids. In fact, given an Eilenberg-Moore algebra $x \colon X^* \to X$, then [A.6] states that the response to one-letter words is trivial: $x(a) = a$. And

[A.7] states that for words of length larger than 2 the response is given by the binary operation
$$a_1 * a_2 = x(a_1 a_2)\,.$$
In fact, for example in length 3 we get
$$x(a_1a_2a_3) = x(a_1(a_2a_3)) = a_1 * (a_2 * a_3)$$
as well as
$$x(a_1a_2a_3) = x((a_1a_2)a_3) = (a_1 * a_2) * a_3\,.$$
Thus, $*$ is an associative operation. And [A.7] also states that the response of x to the empty word is a unit for x.
Conversely: every monoid defines an Eilenberg-Moore algebra: see Remark A.5. The monoid homomorphisms are easily seen to be precisely the homomorphisms in $Set^{\mathbb{M}}$. Thus, $Set^{\mathbb{M}}$ is isomorphic to the category of monoids.

A.9 Example. Free Eilenberg-Moore algebras. For every monad $\mathbb{M}$, the M-algebra
$$(MX, \mu_X \colon MMX \to MX)$$
is an Eilenberg-Moore algebra: [A.6] and [A.7] follow from the definition of monad. This algebra is free with respect to $\eta_X \colon X \to MX$. In fact, given an Eilenberg-Moore algebra (Z, z) and a morphism $f \colon X \to Z$ in $\mathcal{K}$, the unique homomorphism extending f is $\overline{f} = z \cdot Mf$:

(a) $\overline{f} = z \cdot Mf$ is a homomorphism:

$$\begin{tikzcd}
MMX \arrow[r, "\mu_X"] \arrow[d, "MMf"'] & MX \arrow[d, "Mf"] \\
MMZ \arrow[r, "\mu_Y"] \arrow[d, "Mz"'] & MZ \arrow[d, "z"] \\
MZ \arrow[r, "z"'] & Z
\end{tikzcd}$$

due to [A.5] and the naturality of μ.

(b) Conversely, if $\overline{f} \colon (MX, \mu_X) \to (Z, z)$ is a homomorphism, then $f = \overline{f} \cdot \eta_X$ implies $\overline{f} = z \cdot Mf$:

$$\begin{tikzcd}
MMX \arrow[rr, "\mu_X"] \arrow[dd, "M\overline{f}"'] & & MX \arrow[dd, "\overline{f}"] \\
& MX \arrow[ul, "M\eta_X"'] \arrow[ur, "\mathrm{id}"'] \arrow[dl, "Mf"] & \\
MZ \arrow[rr, "z"'] & & Z
\end{tikzcd}$$

A.10 Corollary. *Every monad is induced by some adjoint situation.*

In fact, given a monad $\mathbb{M}$ on a category $\mathcal{K}$, we have the above adjoint situation

$$\mathcal{K}^{\mathbb{M}} \underset{U_{\mathbb{M}}}{\overset{F_{\mathbb{M}}}{\rightleftarrows}} \mathcal{K}$$

where $F_{\mathbb{M}}$ is the free-algebra functor

$$F_{\mathbb{M}} X = (MX, \mu_X)\,.$$

It is defined on morphisms by $F_{\mathbb{M}} f = Mf$. Thus, the monad induced by the adjunction $F_{\mathbb{M}} \dashv U_{\mathbb{M}}$ has the underlying endofunctor $U_{\mathbb{M}} \cdot F_{\mathbb{M}} = M$, and the unit η (recall the universal arrows η_X above). We need to verify that for the counit ε of the adjoint situation we have

$$\mu = U_{\mathbb{M}}\, \varepsilon\, F_{\mathbb{M}}\,.$$

In fact, the component of ε at an Eilenberg-Moore algebra (X, x) is the unique morphism

$$\varepsilon_{(X,x)} \colon (MX, \mu_X) \to (X, x)$$

with $U_{\mathbb{M}} \varepsilon_{(X,x)} \cdot \eta_X = \mathrm{id}_X$. But due to [A.6] and [A.7] the morphism x carries such a homomorphism. Therefore, $U_{\mathbb{M}} \varepsilon_{(X,x)} = x$ for all algebras (X, x). In particular,

$$(U_{\mathbb{M}}\, \varepsilon\, F_{\mathbb{M}})_X = U_{\mathbb{M}} \varepsilon_{(MX,\mu_X)} = \mu_X\,.$$

A.11 Definition. For every adjoint situation

$$\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{K} \qquad \text{with} \quad F \dashv U$$

let $\mathbb{M}$ be the monad of A.3. The *comparison functor* is the functor

$$K \colon \mathcal{A} \to \mathcal{K}^{\mathbb{M}}$$

which assigns to every object A the Eilenberg-Moore algebra

$$KA = (UA, U\varepsilon_A)$$

of A.5. The definition of K on morphisms $f \colon A \to B$ uses the naturality of ε which shows that $Kf = Uf$ is a homomorphism:

$$\begin{tikzcd}
MUA \arrow[r, "U\varepsilon_A"] \arrow[d, "MUf"'] & UA \arrow[d, "Uf"] \\
MUB \arrow[r, "U\varepsilon_B"'] & UB
\end{tikzcd}$$

A.12 Remark. The comparison functor $K\colon \mathcal{A} \to \mathcal{K}^{\mathbb{M}}$ of A.11 is the unique functor such that $U_{\mathbb{M}} \cdot K = U$ and $K \cdot F = F_{\mathbb{M}}$.

A.13 Example.

1. For the concrete category of monoids $U\colon \mathit{Mon} \to \mathit{Set}$ the comparison functor $K\colon \mathit{Mon} \to \mathit{Set}^{\mathbb{M}}$ is an isomorphism. The inverse K^{-1} was described in A.8.2 on objects, and acts trivially on morphisms: $K^{-1}f = f$.

2. The concrete category $U\colon \mathit{Pos} \to \mathit{Set}$ of partially ordered sets yields the trivial monad $\mathbb{I}\mathrm{d} = (\mathrm{Id}_{\mathit{Set}}, \mathrm{id}, \mathrm{id})$ of A.8: recall that the left adjoint of U assigns to every set X the discrete order on the same set. For this monad we have an isomorphism between Set and $\mathit{Set}^{\mathbb{M}}$, and the comparison functor is then simply the forgetful functor U.

3. For the free monad on H, see A.2.3, the Eilenberg-Moore category is concretely isomorphic to H-Alg. Indeed, the functor $J\colon \mathit{Set}^{H^*} \to H\text{-}\mathit{Alg}$ taking an Eilenberg-Moore algebra $x\colon H^*X \to X$ to the H-algebra obtained by composing with ψ_X (see A.2.3) is easily seen to be a concrete isomorphism.

A.14 Definition. A concrete category $(\mathcal{A}, U)$ on $\mathcal{K}$ is *monadic* if U has a left adjoint F and the comparison functor $K\colon \mathcal{A} \to \mathcal{K}^{\mathbb{M}}$ is an isomorphism.

A.15 Remark.

1. The fact that $(\mathcal{A}, U)$ is monadic does not depend on the choice of the left adjoint of U. Indeed, if F' is another left adjoint, then the canonical natural isomorphism $F \simeq F'$ induces an isomorphism of monads $\mathbb{M} \simeq \mathbb{M}'$ (see A.24 for the notion of monad morphism), where $\mathbb{M}'$ is the monad induced by the adjunction $U \dashv F'$. As we will see in A.26, this implies that $\mathcal{K}^{\mathbb{M}}$ and $\mathcal{K}^{\mathbb{M}'}$ are concretely isomorphic.

2. In other words, monadic concrete categories are precisely those which, up to concrete isomorphism, have the form $\mathcal{K}^{\mathbb{M}}$. It is not surprising, then, that monoids are an example and posets are not.

A.16 Definition. A coequalizer in a category $\mathcal{K}$ is called *absolute* if every functor with domain $\mathcal{K}$ preserves it.

A.17 Example. For every Eilenberg-Moore algebra (X, x) we have an absolute coequalizer

$$MMX \underset{\mu_X}{\overset{Mx}{\rightrightarrows}} MX \xrightarrow{x} X$$

in $\mathcal{K}$. In fact, x merges the parallel pair by [A.7] and, moreover, the morphisms η_X and η_{MX} are easily seen to fulfil the following equations:

(i) $\mu_X \cdot \eta_{MX} = \mathrm{id}_{MX}$,

(ii) $\eta_X \cdot x = \mathrm{id}_X$,

and

(iii) $Mx \cdot \eta_{MX} = \eta_X \cdot x$.

It is easy to derive from (i)–(iii) that x is a coequalizer of Mx and μ_X. Since every functor G preserves the equations (i)–(iii), it follows that Gx is a coequalizer of GMx and $G\mu_X$.

A.18 Beck's Theorem. *(Characterization of monadic categories) A concrete category $(\mathcal{A}, U)$, with U a right adjoint, is monadic if and only if*

(a) it is uniquely transportable,

and

(b) $\mathcal{A}$ has coequalizers of all reflexive pairs f, g such that Uf, Ug have an absolute coequalizer, and U preserves these coequalizers.

A proof of A.18 can be found in [71], Chapter 6, Section 7. The reader has just to observe that the parallel pairs of morphisms used in that proof are reflexive, and U creates the coequalizers involved in condition (b) since it is amnestic and conservative.

A.19 Proposition. *Every equational category is monadic.*

Proof. Condition (a) of A.18 follows from 13.17 and (b) from 13.11 and 11.8: the forgetful functor is algebraic, thus it preserves reflexive coequalizers. ☐

A.20 Example.

1. Pointed sets, monoids, groups, abelian groups, ..., with their forgetful functors are monadic.

2. For a one-sorted algebraic theory $(\mathcal{T}, T)$, the concrete category $(Alg\,\mathcal{T}, Alg\,T)$ in general is not monadic, as Example 11.7 shows: $Alg\,T_{ab}$ is not amnestic whereas $U_{\mathbb{M}}$ always is. What remains true is that $(Alg\,\mathcal{T}, Alg\,T)$ is *pseudo-monadic*, as we will see in C.4,

A.21 Theorem. *Equational categories are up to concrete isomorphism precisely the categories $Set^{\mathbb{M}}$ of Eilenberg-Moore algebras for finitary monads $\mathbb{M}$ on Set.*

Proof. In fact, every equational category is, by A.19, concretely isomorphic to $Set^{\mathbb{M}}$ where $\mathbb{M}$ is the monad of its free algebras.
Conversely, given a finitary monad $\mathbb{M} = (M, \mu, \eta)$ on *Set*, we know from 13.23 that *M-Alg* is concretely isomorphic to an equational category. Therefore, it is sufficient to prove that $Set^{\mathbb{M}}$ is closed in *M-Alg* under products, subobjects, and regular quotients. Then the result follows from Birkhoff's Variety Theorem in the form 13.22.

(a) Products: let $(X, x) = \prod_{i \in I}(X_i, x_i)$ where each (X_i, x_i) is an Eilenberg-Moore algebra. (X, x) satisfies [A.6] because the projections $(\pi_i)_{i \in I}$ are a limit cone, thus collectively monomorphic, and the diagram

$$
\begin{tikzcd}
 & & X \arrow[dll, "\pi_i"'] \arrow[dr, "\mathrm{id}_X"] & & \\
X_i \arrow[d, "\eta_{X_i}"'] & & MX \arrow[r, "x"] \arrow[dll, "M\pi_i"] & X \arrow[dr, "\pi_i"] & \\
MX_i \arrow[rrrr, "x_i"'] & & & & X_i
\end{tikzcd}
$$

commutes for every i. Thus $x_i \cdot \eta_{X_i} = \mathrm{id}$ implies $x \cdot \eta_X = \mathrm{id}$. (X, x) satisfies [A.7] for similar reasons:

$$
\begin{tikzcd}
MMX_i \arrow[rrr, "\mu_{X_i}"] \arrow[ddd, "Mx_i"'] & & & MX_i \arrow[ddd, "x_i"] \\
 & MMX \arrow[ul, "MM\pi_i"'] \arrow[r, "\mu_X"] \arrow[d, "Mx"'] & MX \arrow[ur, "M\pi_i"'] \arrow[d, "x"] & \\
 & MX \arrow[r, "x"'] \arrow[dl, "M\pi_i"] & X \arrow[dr, "\pi_i"'] & \\
MX_i \arrow[rrr, "x_i"'] & & & X_i
\end{tikzcd}
$$

(b) Subalgebras $m\colon (X, x) \to (Z, z)$ of Eilenberg-Moore algebras (Z, z). In the following diagram

$$
\begin{tikzcd}
 & & Z \arrow[dddll, bend right, "\eta_Z"'] \arrow[dddrr, bend left, "\mathrm{id}_Z"] & & \\
 & & X \arrow[u, "m"] \arrow[dl, "\eta_X"'] \arrow[dr, "\mathrm{id}_X"] & & \\
 & MX \arrow[rr, "x"'] \arrow[dl, "Mm"] & & X \arrow[dr, "m"'] & \\
MZ \arrow[rrrr, "z"'] & & & & Z
\end{tikzcd}
$$

the outward triangle commutes and all parts except the middle triangle also commute. Thus so does the middle triangle since m is a monomorphism. Therefore, (X, x) satisfies [A.6]. The proof of [A.7] is analogous.

(c) Regular quotients $e\colon (Z, z) \to (X, x)$ of Eilenberg-Moore algebras (Z, z). In the following diagram

$$\begin{tikzcd}
 & & Z \arrow[d, "e"] \arrow[dddll, bend right, "\eta_Z"'] \arrow[dddrr, bend left, "\mathrm{id}_Z"] & & \\
 & & X \arrow[dl, "\eta_X"'] \arrow[dr, "\mathrm{id}_X"] & & \\
 & MX \arrow[rr, "x"'] & & X & \\
MZ \arrow[ur, "Me"'] \arrow[rrrr, "z"'] & & & & Z \arrow[ul, "e"']
\end{tikzcd}$$

again all parts except the middle triangle commute. Since e is an epimorphism, so does the middle triangle. Therefore, (X, x) satisfies [A.6]. For [A.7] use the analogous argument plus the fact that M preserves epimorphisms (because they split in *Set*). ☐

A.22 Corollary. *One-sorted algebraic categories are up to concrete equivalence precisely the categories* $\mathit{Set}^{\mathbb{M}}$ *of Eilenberg-Moore algebras for finitary monads* $\mathbb{M}$ *on Set.*

In fact, this follows from A.21 and 13.11.

A.23 Corollary. *For every finitary monad* $\mathbb{M}$ *on Set the category* $\mathit{Set}^{\mathbb{M}}$ *is cocomplete and the forgetful functor* $U_{\mathbb{M}} \colon \mathit{Set}^{\mathbb{M}} \to \mathit{Set}$ *preserves sifted colimits.*

In fact, the category $\mathit{Set}^{\mathbb{M}}$ is equational by A.21 and then one-sorted algebraic by 13.11. Use now 4.5 and 11.9.

A.24 Definition. Let $\mathbb{M} = (M, \mu, \eta)$ and $\mathbb{M}' = (M', \mu', \eta')$ be monads on a category $\mathcal{K}$. A *monad morphism* from $\mathbb{M}$ to $\mathbb{M}'$ is a natural transformation $\rho \colon M \to M'$ such that the diagrams

$$\begin{tikzcd}
 & \mathrm{Id}_{\mathcal{K}} \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\
M \arrow[rr, "\rho"'] & & M'
\end{tikzcd}$$

and

$$\begin{tikzcd}
MM \arrow[r, "M\rho"] \arrow[d, "\mu"'] & MM' \arrow[r, "\rho M'"] & M'M' \arrow[d, "\mu'"] \\
M \arrow[rr, "\rho"'] & & M'
\end{tikzcd}$$

commute.

A.25 Example. Let M be any endofunctor of *Set*. Following 4.11 and 4.13 there exists a finitary endofunctor M^f and a natural transformation $\epsilon^M : M^f \to M$ universal among the natural transformations $N \to M$ with N finitary.
If M is part of a monad $\mathbb{M} = (M, \mu, \eta)$, then M^f becomes part of a finitary monad and ϵ^M a monad morphism: the monad multiplication for M^f is

$$M^f \cdot M^f \xrightarrow{\ i\ } (M \cdot M)^f \xrightarrow{\ \mu^f\ } M^f$$

where μ^f is the unique natural transformation with $\epsilon^M \cdot \mu^f = \mu \cdot \epsilon^{MM}$ and i is the unique natural isomorphism with $i_n = \mathrm{id}$ for all $n \in \mathcal{N}$, and analogously for the monad unit.

A.26 Proposition. *Every monad morphism $\rho\colon \mathbb{M} \to \mathbb{M}'$ induces a concrete functor*

$$H_\rho \colon \mathcal{K}^{\mathbb{M}'} \to \mathcal{K}^{\mathbb{M}}$$

assigning to every Eilenberg-Moore algebra (X, x) for $\mathbb{M}'$ the algebra $(X, x \cdot \rho_X)$ for $\mathbb{M}$. Conversely, given a concrete functor

$$\begin{tikzcd}
\mathcal{K}^{\mathbb{M}'} \arrow[rr, "H"] \arrow[dr, "U_{\mathbb{M}'}"'] & & \mathcal{K}^{\mathbb{M}} \arrow[dl, "U_{\mathbb{M}}"] \\
& \mathcal{K} &
\end{tikzcd}$$

there exists a unique monad morphism $\rho\colon \mathbb{M} \to \mathbb{M}'$ with $H = H_\rho$.

In fact, for a free Eilenberg-Moore algebra $(M'A, \mu'_A)$, the algebra $H(M'A, \mu'_A)$ has the form $(M'A, \sigma_A \colon MM'A \to M'A)$. We get a monad morphism

$$\rho\colon \mathbb{M} \to \mathbb{M}'\ , \quad \rho_A \colon\ MA \xrightarrow{M\eta'_A} MM'A \xrightarrow{\sigma_A} M'A\,.$$

A full proof can be found in [27], Vol. 2, Proposition 4.5.9.

A.27 Corollary. *The category of finitary monads on Set and monad morphisms is dually equivalent to the category of finitary monadic categories on Set and concrete functors.*

In fact, this follows from A.26.

A.28 Corollary. *The free monad on a finitary endofunctor (see A.2.3) is indeed free on H: for every finitary monad $\mathbb{M} = (M, \mu, \eta)$ and every natural transformation $\alpha\colon H \to M$ there exists a unique monad morphism $\alpha^* \colon H^* \to M$ with $\alpha = \alpha^* \cdot \psi$.*

In fact, recall from A.13.3 that H-*Alg* is concretely isomorphic to Set^{H^*} and use, in place of α^*, the concrete functor from $Set^{\mathbb{M}}$ to H-*Alg* assigning to every Eilenberg-Moore algebra $x\colon MX \to X$ the algebra $x \cdot \alpha_X \colon HX \to X$.

A.29 Remark. We observed above that every monad $\mathbb{M}$ is induced by an adjoint situation using the Eilenberg-Moore algebras. There is another way to induce $\mathbb{M}$: the construction of the Kleisli category of $\mathbb{M}$. This is, as we note below, just the full subcategory of $\mathcal{K}^{\mathbb{M}}$ on all free algebras:

A.30 Definition. The *Kleisli category* of a monad $\mathbb{M}$ is the category

$$\mathcal{K}_{\mathbb{M}}$$

with the same objects as $\mathcal{K}$ and with morphisms from X to Z given by morphisms $f\colon X \to MZ$ in $\mathcal{K}$:

$$\mathcal{K}_{\mathbb{M}}(X,Z) = \mathcal{K}(X, MZ)\,.$$

The identity morphisms are η_X and the composition of two morphisms $f \in \mathcal{K}_{\mathbb{M}}(X,Z)$ and $g \in \mathcal{K}_{\mathbb{M}}(Z,W)$ is given by the following composition in $\mathcal{K}$:

$$X \xrightarrow{f} MZ \xrightarrow{Mg} MMW \xrightarrow{\mu_W} MW$$

A.31 Example. For the monad $MX = X + 1$ of A.2.1 the Kleisli category is the category of sets and partial functions. A partial function from X to Z is represented as a (total) function from X to $Z + 1$.

A.32 Notation. For every monad $\mathbb{M}$ we denote

1. by $K_{\mathbb{M}}\colon \mathcal{K}_{\mathbb{M}} \to \mathcal{K}^{\mathbb{M}}$ the functor which assigns to $X \in \mathcal{K}_{\mathbb{M}}$ the free Eilenberg-Moore algebra (MX, μ_X) and to $f \in \mathcal{K}_{\mathbb{M}}(X,Z)$ the morphism

$$MX \xrightarrow{Mf} MMZ \xrightarrow{\mu_Z} MZ$$

2. by $J_{\mathbb{M}}\colon \mathcal{K} \to \mathcal{K}_{\mathbb{M}}$ the functor which is the identity map on objects and which to every morphism $u\colon X \to Z$ of $\mathcal{K}$ assigns

$$X \xrightarrow{u} Z \xrightarrow{\eta_Z} MZ$$

A.33 Lemma. *The functor $J_{\mathbb{M}}$ is a left adjoint of*

$$\mathcal{K}_{\mathbb{M}} \xrightarrow{K_{\mathbb{M}}} \mathcal{K}^{\mathbb{M}} \xrightarrow{U_{\mathbb{M}}} \mathcal{K}$$

and $\mathbb{M}$ is the monad induced by the adjunction $J_{\mathbb{M}} \dashv U_{\mathbb{M}} \cdot K_{\mathbb{M}}$. The functor $K_{\mathbb{M}}$ is the corresponding comparison functor; it is full and faithful.

Proof. $J_{\mathbb{M}}$ is a left adjoint of $U_{\mathbb{M}} \cdot K_{\mathbb{M}}$ with unit

$$\eta_X\colon X \to MX = U_{\mathbb{M}} K_{\mathbb{M}} J_{\mathbb{M}} X$$

and counit given by the morphism in $\mathcal{K}_{\mathbb{M}}(J_{\mathbb{M}} U_{\mathbb{M}} K_{\mathbb{M}} X, X)$ which is $\varepsilon = \mathrm{id}_{MX}$ in $\mathcal{K}$. The two axioms

$$\varepsilon J_{\mathbb{M}} \cdot J_{\mathbb{M}} \eta = J_{\mathbb{M}} \quad \text{and} \quad U_{\mathbb{M}} K_{\mathbb{M}} \varepsilon \cdot \eta U_{\mathbb{M}} K_{\mathbb{M}} = U_{\mathbb{M}} K_{\mathbb{M}}$$

are easy to check. □

A.34 Theorem. *Given monads $\mathbb{M}$ and $\mathbb{M}'$ on a category $\mathcal{K}$, there is a bijective correspondence between monad morphisms $\rho\colon \mathbb{M} \to \mathbb{M}'$ and functors $G\colon \mathcal{K}_{\mathbb{M}} \to \mathcal{K}_{\mathbb{M}'}$ for which the triangle*

$$\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "J_{\mathbb{M}}"'] \arrow[dr, "J_{\mathbb{M}'}"] & \\
\mathcal{K}_{\mathbb{M}} \arrow[rr, "G"'] & & \mathcal{K}_{\mathbb{M}'}
\end{tikzcd}$$

commutes.

We will see in the proof that the bijective correspondence assigns to every monad morphism $\rho\colon \mathbb{M} \to \mathbb{M}'$ the functor $\hat{\rho}\colon \mathcal{K}_{\mathbb{M}} \to \mathcal{K}_{\mathbb{M}'}$ which is the identity map on objects, and assigns to $p\colon X \to MZ$ in $\mathcal{K}_{\mathbb{M}}(X, Z)$ the value

$$X \xrightarrow{p} MZ \xrightarrow{\rho_Z} M'Z$$

Proof. (a) $\hat{\rho}$ is well-defined: Preservation of identity morphisms follows from $\rho_X \cdot \eta_X = \eta'_X$. Preservation of composition follows from the commutativity of the diagram below, where $p\colon X \to MW$ and $q\colon W \to MZ$ are arbitrary morphisms:

$$\begin{tikzcd}
 & & & MMZ \arrow[r, "\mu_Z"] \arrow[ddd, "\rho_{MZ}"] & MZ \arrow[dr, "\rho_Z"] & \\
X \arrow[r, "p"] & MW \arrow[urr, "Mq"] \arrow[dr, "\rho_W"'] & & & & M'Z \\
 & & M'W \arrow[dr, "M'q"'] & & & \\
 & & & M'MZ \arrow[r, "M'\rho_Z"'] & M'M'Z \arrow[uur, "\mu'_Z"'] &
\end{tikzcd}$$

(b) The equality $\hat{\rho} \cdot J_{\mathbb{M}} = J_{\mathbb{M}'}$ follows from $\rho \cdot \eta = \eta'$.
(c) If ρ, σ are different monad morphisms, then $\hat{\rho} \neq \hat{\sigma}$ due to the fact that the component ρ_X is obtained from $\hat{\rho}$ by $\hat{\rho}(\mathrm{id}_{MX}) = \rho_X$.
(d) Let $G\colon \mathcal{K}_{\mathbb{M}} \to \mathcal{K}_{\mathbb{M}'}$ be a functor such that $G \cdot J_{\mathbb{M}} = J_{\mathbb{M}'}$. Observe that this condition tells us that G is the identity map on objects. The identity morphism $\mathrm{id}_{MX}\colon MX \to MX$ can be seen as a morphism $\varepsilon_X\colon MX \to X$ in $\mathcal{K}_{\mathbb{M}}$. Applying G we get a morphism $G\varepsilon_X\colon MX \to X$ in $\mathcal{K}_{\mathbb{M}'}$, that is, a morphism $MX \to M'X$ in $\mathcal{K}$ that we denote by ρ_X. We claim that ρ_X is the component at X of a monad morphism $\rho\colon \mathbb{M} \to \mathbb{M}'$ with $\hat{\rho} = G$.
The fact that G is a functor means that

(i) $G\eta_X = \eta'_X$ (preservation of identity morphisms)

(ii) given $p\colon X \to MW$ and $q\colon W \to MZ$ in $\mathcal{K}$ we have

$$G(\ X \xrightarrow{p} MW \xrightarrow{Mq} MMZ \xrightarrow{\mu_Z} MZ\ ) =$$

$$= (\ X \xrightarrow{Gp} M'W \xrightarrow{M'Gq} M'M'Z \xrightarrow{\mu'_Z} M'Z\ )$$

(preservation of composition)

and

(iii) for every $u\colon X \to W$ in $\mathcal{K}$ we have

$$G(\ X \xrightarrow{u} W \xrightarrow{\eta_W} MW\ ) = (\ X \xrightarrow{u} W \xrightarrow{\eta'_W} M'W\ )$$

(due to $G \cdot J_{\mathbb{M}} = J_{\mathbb{M}'}$).

This implies for all $u\colon X \to W$ in $\mathcal{K}$ the equation

(iv) $GMu = Gu \cdot \rho_X$

since we apply (ii) to $p = \mathrm{id}_{MX}$ and $q = \eta_W \cdot u$ (thus the previous objects X, W and Z are now MX, X and W, respectively, and $Gq = \eta'_W \cdot u$ by (iii)) and use [A.2] for M and M'. Also for every $q\colon W \to MZ$ we have

(v) $G(q \cdot u) = Gq \cdot u$

by applying (ii) and (iii) to $p = \eta_W \cdot u$. From this we derive the naturality of ρ:

(vi) $\rho_W \cdot Mv = M'v \cdot \rho_X$ for all $v\colon X \to W$

since (v) yields for $q = \mathrm{id}_{MW}$ and $u = Mv$

$$\rho_W \cdot Mv = GMv$$

and then we apply (iv). The equality $\rho \cdot \eta = \eta'$ follows from (v) by $u = \eta_W$ and $q = \mathrm{id}_{MW}$, and the equality $\rho \cdot \mu = \mu' \cdot \rho M' \cdot M\rho$ follows from (ii) by $p = \mathrm{id}_{MMX}$ and $q = \mathrm{id}_{MX}$; this proves that the right-hand side is equal to $G\mu_X$, whereas the left-hand side is $G\mu_X$ by (v) applied to $q = \mu_X$ and $u = \mathrm{id}_{MMX}$. Thus,

$$\rho\colon \mathbb{M} \to \mathbb{M}'$$

is a monad morphism. Finally we need to prove $\hat{\rho} = G$, that is, for every $p_0\colon X \to MW$

$$Gp_0 = G\varepsilon_X \cdot p_0$$

and for this apply (v) to $u = p_0$ and $q = \mathrm{id}_{MW}$. □

A.35 Remark. Let $\mathbb{M}$ be a finitary monad on *Set*. Then the functor M is essentially determined by its domain restriction to $\mathcal{N}^{op}$ (the full subcategory of natural numbers). Also the natural transformations η and μ are uniquely determined by their components η_n and μ_n for natural numbers n. In fact, this follows from 4.11 applied to $Y_{\mathcal{N}}\colon \mathcal{N}^{op} \to$ *Set*.

This leads us to the following restriction of the Kleisli category:

A.36 Notation. For every finitary monad $\mathbb{M}$ on *Set* we denote by

$$Set^f_{\mathbb{M}}$$

the full subcategory of the Kleisli category $\mathcal{K}_{\mathbb{M}}$ on all natural numbers, and by

$$J^f_{\mathbb{M}} \colon \mathcal{N}^{op} \to Set^f_{\mathbb{M}}$$

the domain-codomain restriction of $J_{\mathbb{M}}$.

A.37 Corollary. *Given finitary monads $\mathbb{M}$ and $\mathbb{M}'$ on Set, there is a bijective correspondence between monad morphisms $\rho \colon \mathbb{M} \to \mathbb{M}'$ and those functors $G \colon Set^f_{\mathbb{M}} \to Set^f_{\mathbb{M}'}$ for which the triangle*

$$\begin{tikzcd} & \mathcal{N}^{op} \arrow[dl, "J^f_{\mathbb{M}}"'] \arrow[dr, "J^f_{\mathbb{M}'}"] & \\ Set^f_{\mathbb{M}} \arrow[rr, "G"'] & & Set^f_{\mathbb{M}'} \end{tikzcd}$$

commutes. It assigns to ρ the functor

$$\hat{\rho}^f \colon (\, n \xrightarrow{p} Mk \,) \mapsto (\, n \xrightarrow{p} Mk \xrightarrow{\rho_k} M'k \,)$$

The proof is completely analogous to that of A.34. All we need to notice is that a monad morphism $\rho \colon \mathbb{M} \to \mathbb{M}'$ is uniquely determined by its components $\rho_n, n \in \mathcal{N}^{op}$. This follows once again from 4.11 applied to $Y_{\mathcal{N}} \colon \mathcal{N}^{op} \to Set$.

In 11.36 we defined the 2-category Th^1 of one-sorted algebraic theories. In the following theorem we consider it as a category, that is, we forget the 2-cells.

A.38 Theorem. *The category Th^1 of one-sorted algebraic theories is equivalent to the category of finitary monads on Set.*

Proof. Denote by *FMon* the category of finitary monads and monad morphisms. Every object $\mathbb{M}$ defines a one-sorted theory by dualizing $J^f_{\mathbb{M}}$ of A.36:

$$(J^f_{\mathbb{M}})^{op} \colon \mathcal{N} \to (Set^f_{\mathbb{M}})^{op}\,.$$

In fact, since $J_{\mathbb{M}} \colon Set \to Set_{\mathbb{M}}$ is a left adjoint, it preserves coproducts. In other words, coproducts are the same in *Set* and in $Set_{\mathbb{M}}$.The same, then, holds for finite coproducts in the full subcategories $\mathcal{N}^{op}$ and $Set^f_{\mathbb{M}}$, respectively. Therefore, the identity-on-objects functor $(J^f_{\mathbb{M}})^{op}$ preserves finite products and we obtain a one-sorted theory

$$E(\mathbb{M}) = \Big((Set^f_{\mathbb{M}})^{op}, (J^f_{\mathbb{M}})^{op} \Big)\,.$$

This yields a functor

$$E \colon FMon \to Th^1$$

which to every monad morphism $\rho\colon \mathbb{M} \to \mathbb{M}'$ between finitary monads assigns

$$E(\rho) = (\hat{\rho}^f)^{op} .$$

The commutative triangle of A.37 tells us that this is a morphism of one-sorted theories. It is easy to verify that E is a well-defined functor.

Next E is full and faithful due to the bijection in A.37. Finally, E is essentially surjective. Recall from 11.21 that for every one-sorted theory $(\mathcal{T}, T)$ we have a left adjoint F_T to the forgetful functor $Alg\,T$ (of evaluation at 1) such that

$$F_T(n) = \mathcal{T}(n, -) \ \text{ and } \ F_T(\pi_i^n) = - \cdot T\pi_i^n .$$

Moreover, the naturality square for $\eta\colon \mathrm{Id} \to Alg\,T \cdot F_T$ applied to π_i^n yields

$$\eta_n(i) = T\pi_i^n \ \text{ for all } \ i = 0, \ldots, n-1 .$$

Let $\mathbb{M}$ be the monad corresponding to this adjoint situation. We know that $\mathbb{M}$ is finitary and the values at $n \in \mathcal{N}$ are

$$Mn = \mathcal{T}(n, 1) .$$

The theory $(Set^f_{\mathbb{M}})^{op}$ thus has as morphisms from n to k precisely all functions

$$p\colon k \to \mathcal{T}(n, 1) .$$

This k-tuple of morphisms defines a unique morphism

$$Ip \in \mathcal{T}(n, k)$$

characterized by

$$T\pi_i^k \cdot Ip = p(i) \ \ (i = 0, \ldots, k-1) .$$

We obtain an isomorphism of categories

$$I\colon (Set^f_{\mathbb{M}})^{op} \to \mathcal{T}$$

and it remains to prove that the triangle

$$\begin{tikzcd}
 & \mathcal{N} \arrow[dl, "(J^f_{\mathbb{M}})^{op}"'] \arrow[dr, "T"] & \\
(Set^f_{\mathbb{M}})^{op} \arrow[rr, "I"'] & & \mathcal{T}
\end{tikzcd}$$

commutes. Given a morphism $u \in \mathcal{T}(n, k)$, then Tu is the unique morphism with

$$T\pi_i^k \cdot Tu = T\pi_{u(i)}^n$$

for all i. And we have

$$T\pi_i^n \cdot (I \cdot (J^f_{\mathbb{M}})^{op}(u)) = T(\pi_i^k) I(\eta_n \cdot u) = \eta_n \cdot u(i) = \pi_{u(i)}^n .$$

$\square$

A.39 Remark. The situation with S-sorted theories and S-sorted algebraic categories is entirely analogous: all the above results translate without problems from *Set* to Set^S, only Theorem A.21 needs some work:

A.40 Theorem. *S-sorted equational categories are up to concrete isomorphism precisely the categories $Set^{\mathbb{M}}$ of Eilenberg-Moore algebras for finitary monads $\mathbb{M}$ on Set^S.*

Proof. The only difference with respect to the proof of A.21 is that in the S-sorted case we have to check also that $(Set^S)^{\mathbb{M}}$ is closed in M-*Alg* under directed unions. Let $k_i\colon (X_i, x_i) \to (X, x)$ $(i \in I)$ be a colimit cocone of a filtered colimit in M-*Alg*, where each (X_i, x_i) is an Eilenberg-Moore algebra. Then (X, x) satisfies [A.6] because the cocone (k_i) is collectively epimorphic:

$$\begin{tikzcd}
 & & X_i \arrow[d, "k_i"] \arrow[ddll, bend right, "\eta_{X_i}"'] \arrow[ddrr, bend left, "\mathrm{id}_{X_i}"] & & \\
 & & X \arrow[dl, "\eta_X"'] \arrow[dr, "\mathrm{id}_X"] & & \\
 & MX \arrow[rr, "x"'] & & X & \\
MX_i \arrow[ur, "Mk_i"] \arrow[rrrr, "x_i"'] & & & & X_i \arrow[ul, "k_i"']
\end{tikzcd}$$

And it satisfies [A.7] because the cocone (MMk_i) is collectively epimorphic being a colimit cocone (since MM preserves filtered colimits):

$$\begin{tikzcd}
MMX_i \arrow[rrr, "\mu_{X_i}"] \arrow[dr, "MMk_i"] \arrow[ddd, "Mx_i"'] & & & MX_i \arrow[dl, "Mk_i"'] \arrow[ddd, "x_i"] \\
 & MMX \arrow[r, "\mu_X"] \arrow[d, "Mx"'] & MX \arrow[d, "x"] & \\
 & MX \arrow[r, "x"'] & X & \\
MX_i \arrow[ur, "Mk_i"] \arrow[rrr, "x_i"'] & & & X_i \arrow[ul, "k_i"']
\end{tikzcd}$$

□

A.41 Theorem. *The category Th^S of S-sorted algebraic theories is equivalent to the category of finitary monads on Set^S.*

The proof is completely analogous to the proof in the one-sorted case. We just observe that the functor $J\colon Set^S \to (Set^S)^{\mathbb{M}}$ can, in case of M finitary, be restricted to $J^f\colon (S^*)^{op} \to (Set^S)^f_{\mathbb{M}}$ analogously to A.36.

Appendix B

Abelian categories

The categories *R-Mod* of left modules over a unitary ring R are algebraic and abelian. The aim of the present appendix is to prove that these are the only one-sorted abelian algebraic categories. We also prove the many-sorted generalization of this result.

B.1 Remark. In the following we use the standard terminology of the theory of abelian categories:

1. A *zero object* is an object 0 which is initial as well terminal. For two objects A, B the composite $A \to 0 \to B$ is denoted by $0\colon A \to B$.
2. A *biproduct* of objects A and B is a product $A \times B$ with the property that the morphisms

   $$\langle \mathrm{id}_A, 0\rangle\colon A \to A \times B \quad \text{and} \quad \langle 0, \mathrm{id}_B\rangle\colon B \to A \times B$$

   form a coproduct of A and B.
3. A category is called *preadditive* if it is enriched over the category Ab of abelian groups, i.e., if every hom-set carries the structure of an abelian group such that composition is a group homomorphism.
4. In a preadditive category, an object is a zero object iff it is terminal, and a product of two objects is a biproduct. A preadditive category with finite products is called *additive*.
5. A functor $F\colon \mathcal{A} \to \mathcal{A}'$ between preadditive categories is called *additive* if it is enriched over Ab, i.e., the derived functions $\mathcal{A}(A, B) \to \mathcal{A}'(FA, FB)$ are group homomorphisms. In case of additive categories this is equivalent to F preserving finite products.
6. Finally, a category is called *abelian* if it is exact and additive.

B.2 Example. Just as one-object categories are precisely the monoids, one-object preadditive categories are precisely the unitary rings R. Every left R-module M defines an additive functor $\overline{M}\colon R \to Ab$ with $\overline{M}(*) = M$ and $\overline{M}r = r \cdot -\colon M \to M$ for $r \in R$. Conversely, every additive functor $F\colon R \to Ab$ is naturally isomorphic to $\overline{M}$ for $M = F(*)$.

For a small, preadditive category $\mathcal{C}$, we denote by $Add\,[\mathcal{C}, Ab]$ the abelian category of all additive functors into Ab (and all natural transformations). The previous example implies that R-*Mod* is equivalent to $Add\,[R, Ab]$.

B.3 Theorem. *Abelian algebraic categories are precisely those equivalent to*

$$Add\,[\mathcal{C}, Ab]$$

for a small additive category $\mathcal{C}$.

Proof. Sufficiency. For every small additive category $\mathcal{C}$, we prove that the category $Add\,[\mathcal{C}, Ab]$ is equivalent to $Alg\,\mathcal{C}$. We denote by $\hom(C, -)\colon \mathcal{C} \to Set$ and $\mathrm{Hom}(c, -)\colon \mathcal{C} \to Ab$ the hom-functors. Consider the forgetful functor $U\colon Ab \to Set$. Since U preserves finite products, it induces a functor

$$\widehat{U} = U \cdot - \,:\, Add\,[\mathcal{C}, Ab] \to Alg\,\mathcal{C}$$

Let us prove that $\widehat{U}$ is an equivalence functor.
(a) $\widehat{U}$ is faithful. This is obvious, because U is faithful.
(b) $\widehat{U}$ is full. In fact, we first observe that $\widehat{U}$ preserves sifted colimits. This follows from the fact that sifted colimits commute in Ab (as in any algebraic category, see 2.7) with finite products, and the functor $U = \hom(\mathbb{Z}, -)$ preserves sifted colimits.We first verify that $\widehat{U}$ is full for morphisms (natural transformations)

$$\alpha\colon \widehat{U}(\mathrm{Hom}(C, -)) \to \widehat{U}G$$

where $C \in obj\mathcal{C}$ and $G\colon \mathcal{C} \to Ab$ is additive. By the Yoneda Lemma, for all $X \in \mathcal{C}$ and for all $x\colon C \to X$, we have $\alpha_X(x) = Gx(a)$, where $a = \alpha_C(\mathrm{id}_C)$, thus, α_X is a group homomorphism. Consequently, α lies in the image of $\widehat{U}$. The general case of a morphism $\beta\colon \widehat{U}F \to \widehat{U}G$ reduces to the previous one by using the fact that F is a sifted colimit of representables and that $\widehat{U}$ preserves sifted colimits. To see that F is a filtered colimit of representables, observe that, following 4.2, $\widehat{U}F$ is a sifted colimit of representables. Now

$$\widehat{U}F = colim\,\hom(C_i, -) = colim\,(U \cdot \hom(C_i, -)) =$$

$$= colim\,\widehat{U}(\mathrm{Hom}(C_i, -)) = U(colim\,\mathrm{Hom}(C_i, -))\,.$$

This implies $F = colim\,\mathrm{Hom}(C_i, -)$ because $\widehat{U}$ reflects sifted colimits (since it preserves sifted colimits and reflects isomorphisms).
(c) $\widehat{U}$ is essentially surjective. We first take the representable functors $\hom(C, -)$

for $C \in obj\mathcal{C}$, which are objects of $Alg\,\mathcal{C}$. Since $\mathcal{C}$ is preadditive, for every object C' the hom-set $\hom(C, C')$ is an abelian group, and $\hom(C, -)$ factorizes as

$$\mathcal{C} \xrightarrow{\operatorname{Hom}(C,-)} Ab \xrightarrow{U} Set$$

with $\operatorname{Hom}(C, -)\colon \mathcal{C} \to Ab$ additive. Therefore, $\widehat{U}(\operatorname{Hom}(C, -)) \simeq \hom(C, -)$. Once again, the general case follows from the previous one by using the fact that any $\mathcal{C}$-algebra is a sifted colimit of representable $\mathcal{C}$-algebras and that $\widehat{U}$ preserves sifted colimits.

Necessity. Let $\mathcal{T}$ be an algebraic theory, and assume that $Alg\,\mathcal{T}$ is abelian. Since $\mathcal{T}^{op}$ embeds into $Alg\,\mathcal{T}$, $\mathcal{T}$ is preadditive (with finite products), and then it is a small additive category. Following the first part of the proof, $Alg\,\mathcal{T}$ is equivalent to $Add\,[\mathcal{T}, Ab]$. ☐

B.4 Corollary. *Abelian algebraic categories are precisely the additive cocomplete categories with a strong generator consisting of perfectly presentable objects.*

In fact, this follows from 6.9 and B.3.

B.5 Remark. In B.3 the condition that $\mathcal{C}$ is additive can be weakened: preadditivity is enough. In fact, let $\mathcal{C}$ be a small preadditive category. We can construct the small and preadditive category $Mat\,(\mathcal{C})$ of matrices over $\mathcal{C}$ as follows:

- Objects are finite (possibly empty) families $(X_i)_{i\in I}$ of objects of $\mathcal{C}$;
- Morphisms from $(X_i)_{i\in I}$ to $(Z_j)_{j\in J}$ are matrices $M = (m_{i,j})_{(i,j)\in I\times J}$ of morphisms $m_{i,j}\colon X_i \to Z_j$ in $\mathcal{C}$;
- The matrix multiplication, the identity matrices, and matrix addition, as well known from Linear Algebra, define the composition, the identity morphisms and the preadditive structure, respectively.

This new category $Mat\,(\mathcal{C})$ is additive. Indeed, it has a zero object given by the empty family, and biproducts $\oplus$ given by disjoint unions. Let us check that the obvious embedding $\mathcal{C} \to Mat\,(\mathcal{C})$ induces an equivalence between $Add\,[Mat\,(\mathcal{C}), Ab]$ and $Add\,[\mathcal{C}, Ab]$. Indeed, given $F \in Add\,[\mathcal{C}, Ab]$, we get an extension $F' \in Add\,[Mat\,(\mathcal{C}), Ab]$ in the following way: $F'(M)$ is the unique morphism such that the following square

$$\begin{tikzcd} \bigoplus_I F(X_i) \arrow[r, "F'(M)"] & \bigoplus_J F(Z_j) \arrow[d] \\ F(X_i) \arrow[u] \arrow[r, "F(m_{i,j})"'] & F(Z_j) \end{tikzcd}$$

commutes for all $(i, j) \in I \times J$, where the vertical morphisms are injections in the coproduct and projections from the product, respectively. It is easy to verify that the functor $F \mapsto F'$ is an equivalence from $Add\,[\mathcal{C}, Ab]$ to $Add\,[Mat\,(\mathcal{C}), Ab]$.

B.6 Remark. Observe that an object G of an additive, cocomplete category $\mathcal{A}$ is perfectly presentable iff its enriched hom-functor $\mathrm{Hom}(G, -)\colon\ \mathcal{A} \to Ab$ preserves colimits. (Compare with the absolutely presentable objects of 5.8.) In fact, if G is perfectly presentable, then $\mathrm{Hom}(G, -)$ preserves finite coproducts (because they are finite products) and reflexive coequalizers (because $U\colon Ab \to Set$ reflects them). This implies that $\mathrm{Hom}(G, -)$ preserves finite colimits. Indeed, given a parallel pair $a, b\colon X \rightrightarrows Z$ in $\mathcal{A}$, its coequalizer is precisely the coequalizer of the reflexive pair $(a, \mathrm{id}_Z), (b, \mathrm{id}_Z)\colon X + Z \rightrightarrows Z$. Finally, $\mathrm{Hom}(G, -)$ preserves arbitrary colimits because they are filtered colimits of finite colimits.

B.7 Example. The group $\mathbb{Z}$ is perfectly presentable in Ab. Indeed,

$$\mathrm{Hom}(\mathbb{Z}, -)\colon Ab \to Ab$$

is naturally isomorphic to the identity functor. Observe that $\mathbb{Z}$ is of course not absolutely presentable.

B.8 Corollary. *One-sorted abelian algebraic categories are precisely the categories equivalent to R-Mod for a unitary ring R.*

Proof. Following B.3, a one-sorted abelian algebraic category $\mathcal{A}$ is of the form $Add\,[\mathcal{T}, Ab]$ for $\mathcal{T}$ a one-sorted additive algebraic theory with objects T^n ($n \in \mathbb{N}$). Any $F \in Add\,[\mathcal{T}, Ab]$ restricts to an additive functor $\mathcal{T}(T,T) \to Ab$, where the ring $\mathcal{T}(T,T)$ is seen as a preadditive category with a single object. Moreover, F is uniquely determined by such a restriction, because each object of $\mathcal{T}$ is a finite product of T. Finally, $Add\,[\mathcal{T}(T,T), Ab]$ is equivalent to $\mathcal{T}(T,T)$-*Mod*. □

B.9 Corollary. *Finitary localizations of abelian algebraic categories are precisely the abelian locally finitely presentable categories.*

Proof. Let $\mathcal{A}$ be an abelian, locally finitely presentable category. Following the proof of 18.10, we have that $\mathcal{A} = Lex\,\mathcal{T}$ is a finitary localization of $Alg\,\mathcal{T}$, with $\mathcal{T}$ an additive algebraic theory. Due to B.3, $Alg\,\mathcal{T}$ is equivalent to $Add\,[\mathcal{T}, Ab]$. □

Appendix C

More about dualities for one-sorted algebraic categories

Throughout our book we took the "strict view" of what a theory morphism or a concrete functor or a monadic functor etc. should be. That is, the condition put on the functor in question was formulated as equality between two functors. There is a completely natural "non-strict view" where the conditions are formulated as natural equivalences between functors. This has a number of advantages. For example, we can present a characterization of one-sorted algebraic categories (see Theorem C.6 below) for which we know no analogous result in the strict variant. Also the duality between one-sorted algebraic theories and uniquely transportable one-sorted algebraic categories can be directly derived from the non-strict version of the biduality 11.39 without using monads. In the present appendix we shortly mention the non-strict variants of some conepts in our book.

C.1 Definition.

1. Given concrete categories $U: \mathcal{A} \to \mathcal{K}$ and $V: \mathcal{B} \to \mathcal{K}$ by a *pseudo-concrete functor* between them we mean a functor $F: \mathcal{A} \to \mathcal{B}$ such that $V \cdot F$ is naturally isomorphic to U.

2. Given concrete categories $U: \mathcal{A} \to \mathcal{K}$ and $V: \mathcal{B} \to \mathcal{K}$ by a *pseudo-concrete equivalence* between them we mean a functor $F: \mathcal{A} \to \mathcal{B}$ which is at the same time an equivalence and pseudo-concrete. (Note that any quasi-inverse of F is necessarily pseudo-concrete.) We then say that $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$ are *pseudo-concretely equivalent*.

C.2 Definition. A concrete category $(\mathcal{A}, U)$ on $\mathcal{K}$ is *pseudo-monadic* if there exists a monad $\mathbb{M}$ on $\mathcal{K}$ and a pseudo-concrete equivalence $\mathcal{A} \to \mathcal{K}^{\mathbb{M}}$.

C.3 Beck's Theorem. *(Characterization of pseudo-monadic categories) A concrete category $(\mathcal{A}, U)$, is pseudo-monadic if and only if*

(a) U has a left adjoint,

(b) U is conservative,

and

(c) $\mathcal{A}$ has coequalizers of all reflexive pairs f, g such that Uf, Ug have an absolute coequalizer, and U preserves these coequalizers.

A proof of C.3 can be found in [27], Chapter 4, Section 4. The reader needs just to observe that the parallel pairs of morphisms used in that proof are reflexive.

C.4 Proposition. *For every one-sorted algebraic theory $(\mathcal{T}, T)$, the concrete category $(Alg\,\mathcal{T}, Alg\,T)$ is pseudo-monadic.*

Proof. The functor $Alg\,T$ is algebraic and conservative (11.8), and algebraic functors are right adjoint and preserve reflexive coequalizers. Following C.3, $(Alg\,\mathcal{T}, Alg\,T)$ is pseudo-monadic. □

C.5 Definition. A *pseudo-one-sorted algebraic category* is a concrete category over *Set* which is pseudo-concretely equivalent to $Alg\,T\colon Alg\,\mathcal{T} \to Set$ for a one-sorted algebraic theory $(\mathcal{T}, T)$.

C.6 Theorem. *(Characterization of pseudo-one-sorted algebraic categories) The following conditions on a concrete category $(\mathcal{A}, U)$ over Set are equivalent:*

1. *$(\mathcal{A}, U)$ is pseudo-one-sorted algebraic;*
2. *$\mathcal{A}$ is cocomplete and U is a conservative right adjoint preserving sifted colimits.*

More detailed: let $\mathcal{A}$ be a cocomplete category. Given a faithful functor

$$U\colon \mathcal{A} \to Set$$

with $\mathcal{A}$ cocomplete, there exists a one-sorted algebraic theory $(\mathcal{T}, T)$ and an equivalence functor

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "E"] \arrow[dr, "U"'] & \arrow[d, phantom, "\simeq"] & Alg\,\mathcal{T} \arrow[dl, "Alg\,T"] \\ & Set & \end{tikzcd}$$

making the above diagram commutative up to natural isomorphism iff U is a conservative right adjoint preserving sifted colimits.

Proof. The conditions are necessary by 11.8. Let us prove that they are sufficient: let $U\colon \mathcal{A} \to Set$ be as above, with F a left adjoint of U. The set $\mathcal{F} = \{Fn : n \in \mathbb{N}\}$ is closed in $\mathcal{A}$ under finite coproducts and, by 6.16, it is a strong generator formed by perfectly presentable objects. Following the proof of 6.9 (implication 3 ⇒ 1), the functor

$$E\colon \mathcal{A} \to Alg(\mathcal{F}^{op})\,, \quad K \mapsto \mathcal{A}(-,K)\colon \mathcal{F}^{op} \to Set$$

is an equivalence. Moreover, the codomain restriction $T\colon \mathcal{N} \to \mathcal{F}^{op}$ of the functor $F^{op} \cdot Y_{\mathcal{N}}\colon \mathcal{N} \to \mathcal{A}^{op}$ is a morphism of theories. Let us check that there exists a natural isomorphism $AlgT \cdot E \simeq U$: for every $A \in \mathcal{A}$ and $n \in \mathbb{N}$ the functor $AlgT \cdot E(A)$ assigns to $n \in \mathbb{N}$ the set $\mathcal{A}(Tn, A)$, and since $Tn = F(Y_{\mathcal{N}}(n))$, the adjunction $F \dashv U$ clearly yields a natural isomorphism $\mathcal{A}(Tn, A) \simeq UA(n)$. Factorize T as a functor $T'\colon \mathcal{N} \to \mathcal{T}$ which is the identity on objects and equal to T on arrows, followed by a functor $T''\colon \mathcal{T} \to \mathcal{F}^{op}$ which is the identity on arrows and equal to T on objects. Therefore, T'' is an equivalence (because T is surjective on objects) and $(\mathcal{T}, T')$ is a one-sorted theory. The following diagram concludes the proof:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "E"] \arrow[dr, "U"'] & Alg(\mathcal{F}^{op}) \arrow[r, "Alg\,T''"] \arrow[d, "Alg\,T"] \arrow[dl, phantom, "\simeq"] & Alg\mathcal{T} \arrow[dl, "Alg\,T'"] \\
 & Set &
\end{tikzcd}$$

□

C.7 Corollary. *Pseudo-one-sorted algebraic categories are up to pseudo-concrete equivalence precisely the categories $Set^{\mathbb{M}}$ of Eilenberg-Moore algebras for finitary monads $\mathbb{M}$ on Set.*

In fact, this follows from C.4, C.6 and A.23.

It is easy to extend the biequivalence of Theorem 11.39 to pseudo-one-sorted algebraic categories.

C.8 Definition. Given one-sorted algebraic theories $(\mathcal{T}_1, T_1)$ and $(\mathcal{T}_2, T_2)$, a *pseudo-morphism*

$$M\colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$$

is a functor $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ with $M \cdot T_1$ naturally isomorphic to T_2.

C.9 Remark. Pseudo-morphisms preserve finite products.

C.10 Theorem. *(Non-strict one-sorted algebraic duality) The 2-category of*

objects: one-sorted algebraic theories,

1-cells: pseudo-morphisms,

2-cells: natural transformations,

is dually biequivalent to the 2-category of *objects: pseudo-one-sorted algebraic categories,*

1-cells: pseudo-concrete functors,

2-cells: natural transformations.

Proof. The proof is analogous to that of 11.39, just observe that from a pseudo-concrete functor

$$(G,\varphi)\colon (Alg\,\mathcal{T}_2, Alg\,T_2) \to (Alg\,\mathcal{T}_1, Alg\,T_1)$$

we get a natural isomorphism $\psi\colon F_{T_1} \to F \cdot F_{T_2}$ between the left adjoints. The rest of the proof of 11.39 can now be repeated with no other changes. □

C.11 Remark. In Appendix A we have obtained the duality between the category of one-sorted algebraic theories and the category of uniquely transportable one-sorted algebraic categories using finitary monads. We are going to derive such a duality from the biequivalence of Theorem C.10. In order to perform this we need several preliminary steps. We start by an observation explainin why when we restrict our attention to uniquely transportable one-sorted algebraic categories we do not need a non-strict version.

C.12 Lemma. *Let $(\mathcal{A}_1, U_1), (\mathcal{A}_2, U_2)$ be concrete categories over $\mathcal{K}$.*

1. *If $(\mathcal{A}_2, U_2)$ is transportable, then for every pseudo-concrete functor*

$$G\colon (\mathcal{A}_1, U_1) \to (\mathcal{A}_2, U_2)$$

there exists a concrete functor $H\colon (\mathcal{A}_1, U_1) \to (\mathcal{A}_2, U_2)$ naturally isomorphic to G.

2. *If $(\mathcal{A}_1, U_1)$ and $(\mathcal{A}_2, U_2)$ are transportable, then they are pseudo-concretely equivalent if and only if they are concretely equivalent.*

Proof. 1. Let $\varphi\colon U_1 \to U_2 \cdot G$ be a natural isomorphism. For every object $A \in \mathcal{A}_1$, consider the isomorphism

$$\varphi_A\colon U_1 A \to U_2 G A\,.$$

Since $(\mathcal{A}_2, U_2)$ is transportable, there exists an object HA in $\mathcal{A}_2$ and an isomorphism

$$\psi_A\colon HA \to GA$$

such that $U_2 \psi_A = \varphi_A$. This gives a map on objects $H\colon \mathcal{A}_1 \to \mathcal{A}_2$. For $f\colon A \to B$ in $\mathcal{A}_1$, put $Hf = \psi_B^{-1} \cdot Gf \cdot \psi_A$. In this way H is a functor such that $U_2 \cdot H = U_1$, and $\psi\colon H \to G$ is a natural isomorphism.
2. follows immediately from 1. □

C.13 Corollary. *Every transportable pseudo-one-sorted algebraic category is one-sorted algebraic.*

Recall from 11.36 the 2-categories Th^1 of one-sorted algebraic theories, and ALG^1 of one-sorted algebraic categories. In the remaining part of this appendix we consider Th^1 and ALG^1 as categories, that is, we forget the 2-cells. Explicitely:

C.14 Definition. We define

1. the category Th^1 of one-sorted theories to have

   objects: one-sorted algebraic theories,

   morphisms: morphisms of one-sorted algebraic theories,

2. the category ALG^1 of one-sorted algebraic categories to have

   objects: one-sorted algebraic categories,

   morphisms: concrete functors.

3. the category ALG^1_u to be the full subcategory of ALG^1 of all uniquely transportable one-sorted algebraic categories.

C.15 Lemma. *The category Th^1 (seen as a 2-category with only identity 2-cells) is biequivalent to the 2-category $PsTh^1$ having*

objects: one-sorted algebraic theories $(\mathcal{T}, T)$,

1-cells from $(\mathcal{T}_1, T_1)$ to $(\mathcal{T}_2, T_2)$: pairs (M, μ) with $M\colon \mathcal{T}_1 \to \mathcal{T}_2$ a pseudomorphism of one-sorted theories and $\mu\colon M \cdot T_1 \to T_2$ a natural isomorphism,

2-cells from (M, μ) to (N, ν): natural transformations $\alpha\colon M \to N$ which are coherent, i.e., such that $\nu \cdot \alpha T_1 = \mu$.

Proof. (1) Let us start by observing that the coherence condition $\nu \cdot \alpha T_1 = \mu$ on a 2-cell α of $PsTh^1$ immediately implies that α is invertible and that between two parallel 1-cells of $PsTh^1$ there is at most one 2-cell.
(2) The inclusion $Th^1 \to PsTh^1$ is a biequivalence: since Th^1 and $PsTh^1$ have the same objects, we have to prove that the induced functor

$$Th^1((\mathcal{T}_1, T_1), (\mathcal{T}_2, T_2)) \to PsTh^1((\mathcal{T}_1, T_1), (\mathcal{T}_2, T_2))$$

is full and essentially surjective (it is certainly faithful because Th^1 has only identity 2-cells).
(2a) Full: let $M, N\colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ be 1-cells in Th^1 and $\alpha\colon (M, =) \to (N, =)$ a 2-cell in $PsTh^1$. The coherence condition gives $\alpha_n = \mathrm{id}$ for every $n \in \mathcal{N}$, and then $M = N$ by naturality of α.
(2b) Essentially surjective: consider a 1-cell $(M, \mu)\colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ in $PsTh^1$. We define a functor

$$N\colon \mathcal{T}_1 \to \mathcal{T}_2\,, \quad N(f\colon x \to y) = \mu_y \cdot Mf \cdot \mu_x^{-1}\colon x \to y\,.$$

It is easy to check that $N \cdot T_1 = T_2$ and that $\mu\colon (M, \mu) \to (N, =)$ is a 2-cell in $PsTh^1$. $\square$

C.16 Lemma. *The category ALG^1_u (seen as a 2-category with only identity 2-cells) is biequivalent to the 2-category $PsALG^1$ having*

> *objects: pseudo-one-sorted algebraic categories $(\mathcal{A}, U)$,*
>
> *1-cells from $(\mathcal{A}_1, U_1)$ to $(\mathcal{A}_2, U_2)$: pseudo-concrete with $G\colon \mathcal{A}_1 \to \mathcal{A}_2$ a functor and $\varphi\colon U_1 \to U_2 \cdot G$ a natural isomorphism,*
>
> *2-cells from (G, φ) to (H, ψ): natural transformations $\alpha\colon G \to H$ which are coherent, i.e., such that $U_2\alpha \cdot \varphi = \psi$.*

Proof. (1) Let us start by observing that, since U_2 is conservative, the coherence condition $U_2\alpha \cdot \varphi = \psi$ implies that α is invertible and, since U_2 is faithful, it implies that between two parallel 1-cells of $PsALG^1$ there is at most one 2-cell.
(2) The inclusion $ALG^1_u \to PsALG^1$ is essentially surjective (in the sense of the 2-category $PsALG^1$) : let $(\mathcal{A}, U)$ be an object in $PsALG^1$. We are going to construct the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "E_1"] \arrow[dr, "U"'] & \mathcal{A}_1 \arrow[r, "E_2"] \arrow[d, "U_1"] & \mathcal{A}_2 \arrow[dl, "U_2"] \\ & Set & \end{tikzcd}$$

where E_1 and E_2 are pseudo-concrete equivalences, $(\mathcal{A}_1, U_1)$ is transportable, and $(\mathcal{A}_2, U_2)$ is uniquely transportable. Therefore, $(\mathcal{A}_2, U_2)$ is a uniquely transportable pseudo-one-sorted algebraic category. By Corollary C.13 we conclude that $(\mathcal{A}_2, U_2)$ is an object of ALG^1_u.
(2a) Objects in $\mathcal{A}_1$ are triples $(A, \pi_{A,X}\colon UA \to X, X)$ with $A \in \mathcal{A}$, X a set and $\pi_{A,X}$ an isomorphism. A morphism from $(A, \pi_{A,X}, X)$ to $(A', \pi_{A',X'}, X')$ is a pair of morphism $a\colon A \to A', x\colon X \to X'$ such that $x \cdot \pi_{A,X} = \pi_{A',X'} \cdot Ua$. Clearly the forgetful functor

$$U_1\colon \mathcal{A}_1 \to Set\,, \quad U_1(A, \pi_{A,X}, X) = X$$

is transportable. Moreover, we have an equivalence

$$E_1\colon \mathcal{A} \to \mathcal{A}_1\,, \quad E_1 A = (A, \mathrm{id}_{UA}, UA)$$

with quasi-inverse the forgetful functor

$$E_1'\colon \mathcal{A}_1 \to \mathcal{A}\,, \quad E_1'(A, \pi_{A,X}, X) = A\,.$$

Note that E_1 is a concrete functor, whereas E_1' is pseudo-concrete: a natural isomorphism $\pi\colon U \cdot E_1' \to U_1$ is given by

$$\pi_{(A,\pi_{A,X},X)} = \pi_{A,X}\,.$$

(2b) Consider any concrete category $U_1\colon \mathcal{A}_1 \to Set$. Objects of $\mathcal{A}_2$ are equivalence classes of objects of $\mathcal{A}_1$, with X equivalent to X' if there exists a U_1-identity $i\colon X \to X'$, that is, an isomorphism i such that $U_1 i = \mathrm{id}_{U_1X}$. We denote by

$[X]$ the equivalence classe of X. The hom-set $\mathcal{A}_2([X][Z])$ is the quotient of the disjoint union of the $\mathcal{A}_1(X', Z')$ for $X' \in [X], Z' \in [Z]$, with $f\colon X \to Z$ equivalent to $f'\colon X' \to Z'$ if there exist U_1-identites $i\colon X \to X'$ and $j\colon Z \to Z'$ and $U_1 f = U_1 f'$. The composition of $[f]\colon [X] \to [Z]$ and $[g]\colon [Z] \to [W]$ is $[gjf]\colon [X] \to [W]$, where $j\colon Z \to Z'$ is any U_1-identity. The functors E_2 and U_2 are defined by

$$E_2\colon \mathcal{A}_1 \to \mathcal{A}_2\,, \quad E_2(f\colon X \to Z) = [f]\colon [X] \to [Z]\,,$$

$$U_2\colon \mathcal{A}_2 \to \mathit{Set}\,, \quad U_2([f]\colon [X] \to [Z]) = Ff\colon FX \to FZ\,.$$

Clearly, $U_2 \cdot E_2 = U_1$ and E_2 is full and surjective on objects, so that it is an equivalence (because U_1 is faithful). Finally, it is easy to check that U_2 is amnestic if U_1 is conservative, and that U_2 is transportable if U_1 is transportable.
(3) The induced functor

$$ALG^1_u((\mathcal{A}_1, U_1), (\mathcal{A}_2, U_2)) \to PsALG^1((\mathcal{A}_1, U_1), (\mathcal{A}_2, U_2))$$

is an equivalence:
(3a) Full: let $G, H\colon (\mathcal{A}_1, U_1) \to (\mathcal{A}_2, U_2)$ be 1-cells in ALG^1_u and $\alpha\colon (G, =) \to (H, =)$ a 2-cell in $PsALG^1$. The coherence condition gives $U_2(\alpha_A) = \mathrm{id}$ for every $A \in \mathcal{A}$. Since U_2 is amnestic, α_A is the identity.
(3b) Faithful: obvious because ALG^1 has only identity 2-cells.
(3c) Essentially surjective: let $(\mathcal{A}_1, U_1)$, $(\mathcal{A}_2, U_2)$ be objects in ALG^1_u and $(G, \varphi)\colon (\mathcal{A}_1, U_1) \to (\mathcal{A}_2, U_2)$ a 1-cell in $PsALG^1$. As in the proof of C.12 we get a concrete functor $H\colon (\mathcal{A}_1, U_1) \to (\mathcal{A}_2, U_2)$ and a natural isomorphism $\psi\colon H \to G$. To end the proof observe that $\psi\colon (H, =) \to (G, \varphi)$ is a 2-cell in $PsALG^1$. Indeed, the condition $U_2(\psi_A) = \varphi_A$ is precisely the coherence condition on ψ. ☐

Recall the 2-fucntor $Alg^1\colon (Th^1)^{op} \to ALG^1$ from 11.38. There is an obvious version of this 2-functor in the present context: all we need to observe is that given a coherent natural transformation α in $PsTh^1$, then $Alg\,\alpha$ is also coherent. By a slight abuse of notation we denote this 2-functor by Alg^1 again:

C.17 Notation. We denote by

$$Alg^1\colon (PsTh^1)^{op} \to PsALG^1$$

the 2-functor assigning to every one-sorted theory $(\mathcal{T}, T)$ the one-sorted algebraic category $(Alg\,\mathcal{T}, Alg\,T)$, to every 1-cell $(M, \mu)\colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$ the 1-cell $(Alg\,M, Alg\,\mu^{-1})$, and to every 2-cell $\alpha\colon (M, \mu) \to (N, \nu)$ the 2-cell $Alg\,\alpha$ whose component at a $\mathcal{T}_2$-algebra A is $A \cdot \alpha\colon A \cdot M \to A \cdot N$.

C.18 Theorem. *(One-sorted algebraic duality) The category ALG^1_u of uniquely transportable one-sorted algebraic categories is equivalent to the dual of the category Th^1 of one-sorted algebraic theories.*

Proof. All that remains to be proved after C.15 and C.16 is that the 2-functor

$$Alg^1\colon (PsTh^1)^{op} \to PsALG^1$$

of C.17 is a biequivalence.
(1) The 2-functor Alg^1 is well-defined by 11.8.
(2) The 2-functor Alg^1 is essentially surjective (in the sense of the 2-category $PsALG^1$): following C.6, for every object $(\mathcal{A}, U)$ of $PsALG^1$ there exists a pseudo-concrete equivalence $E\colon \mathcal{A} \to Alg\mathcal{T}$ with natural isomorphism

$$\varphi\colon AlgT \cdot E \to U\,.$$

Recall from 0.3 that it is possible to choose a quasi-inverse $E'\colon Alg\mathcal{T} \to \mathcal{A}$ and natural isomorphisms

$$\eta\colon \mathrm{Id}_{Alg\mathcal{T}} \to E \cdot E' \quad \text{and} \quad \varepsilon\colon E' \cdot E \to \mathrm{Id}_{\mathcal{A}}$$

such that

$$E\varepsilon \cdot \eta E = E \quad \text{and} \quad \varepsilon E' \cdot E'\eta = E'\,.$$

We get a natural isomorphism

$$\psi = \varphi^{-1}E' \cdot AlgT\eta\colon AlgT \to U \cdot E'\,.$$

It follows that

$$\eta\colon (\mathrm{Id}_{Alg\mathcal{T}}, =) \to (E \cdot E', \varphi E' \cdot \psi) \quad \text{and} \quad \varepsilon\colon (E' \cdot E, \psi E \cdot \varphi) \to (\mathrm{Id}_{\mathcal{A}}, =)$$

are 2-cells in $PsALG^1$. Indeed, the coherence condition on η is just the definition of ψ, and the coherence condition on ε follows from the equation $E\varepsilon \cdot \eta E = E$:

$$U\varepsilon \cdot \psi E \cdot \varphi = U\varepsilon \cdot \varphi^{-1}E'E \cdot AlgT\eta E \cdot \varphi = \varphi^{-1} \cdot AlgTE\varepsilon \cdot AlgT\eta E \cdot \varphi = \varphi^{-1} \cdot \varphi = U\,.$$

We conclude that $(\mathcal{A}, U)$ and $(Alg\mathcal{T}, AlgT)$ are equivalent objects in $PsALG^1$.
(3) We will prove that for two one-sorted algebraic theories $(\mathcal{T}_1, T_1)$ and $(\mathcal{T}_2, T_2)$ the functor

$$PsTh^1((\mathcal{T}_1, T_1), (\mathcal{T}_2, T_2)) \xrightarrow{Alg^1_{(\mathcal{T}_1,T_1),(\mathcal{T}_2,T_2)}} PsALG^1((Alg\mathcal{T}_2, AlgT_2), (Alg\mathcal{T}_1, AlgT_1))$$

is an equivalence of categories.
(3a) Full and faithful: the proof as in Theorem 9.15, indeed $\alpha\colon M \to N$ is coherent iff $Alg\alpha\colon AlgM \to AlgN$ is coherent.
(3b) Essentially surjective: we follow the proof of 11.39. Let

$$(G, \varphi)\colon (Alg\mathcal{T}_2, AlgT_2) \to (Alg\mathcal{T}_1, AlgT_1)$$

be a 1-cell in $PsALG^1$ and $\psi\colon F_{T_2} \to F \cdot F_{T_1}$ the induced natural isomorphism on the adjoint functors. As in 11.39 we get a 1-cell

$$(M, =)\colon (\mathcal{T}_1, T_1) \to (\mathcal{T}_2, T_2)$$

in $PsTh^1$ and we have to construct a 2-cell

$$\alpha\colon (AlgM, =) \to (G, \varphi)$$

in $PsALG^1$. Since $Alg\,T_1 \cdot Alg\,M = Alg\,T_2$, there exists a natural isomorphism $i\colon F_{T_2} \to M^* \cdot F_{T_1}$. Let

$$\psi_| \colon Y_{\mathcal{T}_2} \cdot M^{op} \to F \cdot Y_{\mathcal{T}_1}\ , \quad i_| \colon Y_{\mathcal{T}_2} \cdot M^{op} \to M^* \cdot Y_{\mathcal{T}_1}$$

be the restrictions of ψ and i to $\mathcal{T}_1$. By 9.3, there exists a natural isomorphism $\alpha^*\colon M^* \to F$. Moreover, α^* is unique with the condition $\alpha^* Y_{\mathcal{T}_1} \cdot i_| = \psi_|$ (apply 4.11 to $Y_{\mathcal{T}_1}$). Since F_{T_2} and $F \cdot F_{T_1}$ preserve colimits, the previous equation gives $\alpha^* F_{T_1} \cdot i = \psi$ (apply 4.11 to $Y_{\mathcal{N}}$). Passing to the right adjoints, we get a natural isomorphism $\alpha\colon Alg\,M \to G$ such that $(Alg\,T_1)\alpha = \varphi$, as required. $\square$

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