Perf Fields.

- Last time: Huber Pairs, Tate rings, Tate Algebras. (← Basic building block ⇝ Spcs)
- Perf Fields are special class ↪ Prefectoid tate rings. [Weil7] (Adic Spc / Perf fields → Perf rings).

Today:
- Give motivation. Perf fields,
- Tilting ~~and untilting~~. ~~Correspondence~~.

[Bhal7], [Lur18]

Next time: " " rings ⇝ spcs. need more about val'n thy. [Mor19] [Wed].

Next time:
1. Adic spcs. / more val'n thy. [Bhal7, Mor.16] → Perfectoid spc.
2. Tilting + untilting correspondence. [Lur18]

Motivations.

$v_p : K_X \to \mathbb{R}_{\geq 0}$.

- Hodge Thy [Mat18]
- Fargue-Fontaine. Today. [I, Lur18]

Ctx. • $X/\mathbb{F}_q$. alg curv. $K_X :=$ function field.

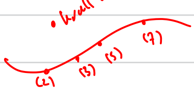
N: points on right looks the same in a formal nhood.

$\lim_{\leftarrow} \mathbb{Z}_x \cong \mathbb{F}_q[[x]]$.

Arith. on nf. $K/\mathbb{Q}$ fin. ext.	Alg. Fun field.
$\mathbb{Q}$ has arc. val'n. $\mathbb{Q} \hookrightarrow \mathbb{R}, \mathbb{C}$ nonarc val'n ↔ primes.	K_X
Spec $\mathbb{Z}$. + ∞-place.	X
p prime.	$x \in K$ cls pts.
$\mathbb{Z}/p\mathbb{Z}$	$K(x)$. res. field.

$(0,\infty)$ in $E \subset \mathbb{P}^2_{\mathbb{C}}$.

imagine as a curve. • Krull dim 1.



$|6|_2 |6|_3 = \frac{1}{2}\cdot\frac{1}{3}\cdot 1 \cdots |6|_\infty$

$\prod |x|_p = 1$.

Riem sfce. $X/\mathbb{C}$.
- An anlytic nhd looks anywhere like a disk.
- # of poles + zeros add up to 0.
- ⇒ product of all valuation.

$\prod_p v_p f = 1$.

- Often arith. q. has alg. analogue.
- "Topologically". X and spec $\mathbb{Z}$ has similar pts
  "Categorically" they don't. Spec. $\mathbb{Z}$ terminal.

Also need Γ factr. in the funtional eqn.

- Γ factor accounts of infinite place.
- $\zeta(s) = c\,\Gamma(1-s)\,\zeta(1-s)$

$\xi(s) := c\,\Gamma(s/2)\,\zeta(s)$, $\Rightarrow$ $\xi(s) = \xi(1-s)$.

• $X \times_{\mathrm{Spec}\,\mathbb{F}_q} X$. $\longleftrightarrow$ $\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z}$.

$\downarrow$

Reason we interested is this is used in the pf. of

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{1-|k(x)|^{-s}} = \sum_{D \leq X} \frac{1}{|\mathcal{O}_D|^s}$$

Arakelov geometry w/ artificially force cplx & p-adic.

Hasse-weil Zeta fn.
also need to take ∞-place acct.

• D eff. divisor. • $|\mathcal{O}_D|$ cardinality of its ring of functions.

• Want to find a good arithmetic replacement of "$\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z}$."
• What does this mean? To describe the K-valued pts

Scholze's answer: He describes the K-valued pts for. K.
• K be cpl. nonarc. ($|\cdot|_K : K \to \mathbb{R}_{\geq 0} \cup \{0\}$).
• $K^\flat$ tilting of K.

$$K^\flat := \varprojlim \left(\cdots K \xrightarrow{(\ )^p} K \xrightarrow{(\ )^p} K \right). \quad \text{as a set.}$$

Rmk: • K is not necess. char p. $(\ )^p$ are not ring homo.
• $K^\flat$ is a mult. monoid. no addition str.

Thm: K a cpl. nonarc. field. 1. alg clo. (AC). 2. res p. (RP).
Then $K^\flat$ can be equipped with a str of a field. (which of char p (CP)).
More general statement: K is a perf. field.

• $\{$ AC. CV. RP fields $\}$ ** "char 0" K can be char 0 OR p.

$$\downarrow (-)^\flat$$

$\{$ AC. CV. CP fields $\}$ * "char p".

• Goal: 1. To make sure $(-)^\flat$ is well-defined.
2. To classify the untilts.
} [2 lectures].

Defn: • let C be AC. CV. CP. ($C \in$ *)

• An untilt of C is a pair (K, ι) where K is AC, CV. RP. ($K \in$ **) and an iso $\iota : C \xrightarrow{\sim} K^{\flat}$ (Sends $\mathcal{O}_C$ into $\mathcal{O}_K^{\flat}$ ring of integers $K^{\flat}$)

$\mathcal{O}_C := \{x \in C : |x|_C \leq 1\}$.

Recall: R-valued pts of $\operatorname{Spec} A$. $\simeq$ $\underset{(\text{ring})}{A \to R}$ $\simeq$ A-structures on R.

1. $A = \mathbb{Z}$. There is just one unique.
2. $A = \mathbb{Q}$. There $\exists$ at most one.

iso. of class of untile.

(K_1, ι_1) (K_2, ι_2) of C.

char 0.

$$\begin{tikzcd} & K_2^{\flat} \\ C \arrow[ur] \arrow[r, "\iota_1"'] & K_1^{\flat} \arrow[u] \end{tikzcd}$$

$K_2 \leftarrow K_1$

there is no frobenius so you don't have iso.

Schulze: let C be AC. CV. CP field.

char p.

• $\{ C\text{-valued pts of } \mathbb{Z} \}$ $\longleftrightarrow$ $\{\text{untilts of } C\} / \sim$.

• $\{ C\text{-valued pts of } \mathbb{Q} \}$ $\longleftrightarrow$ $\{\text{char } 0. \text{ untilts of } C\} / \sim$ ***

iso class of untilt.

$$\begin{array}{ccc} C & \xrightarrow{\varphi_C} & C \\ \uparrow & & \uparrow \\ \mathcal{O}_C & \xrightarrow{\varphi_C} & \mathcal{O}_C \end{array}$$

Rmk: (K, ι) is an untilt of C. $(\iota : C \xrightarrow{\sim} K^{\flat})$ $\left(C \xrightarrow{\varphi_C} C \xrightarrow{\iota} K\right)$

• C is AC. CP. $\Rightarrow \varphi_C : C \to C$ is an iso of rings. (elements have pth root).

• $\{(K, \varphi_C^n \circ \iota)\}_{n \in \mathbb{Z}}$ is a coll'n of untilts of C.

• The obj for *** to consider is $\{\text{iso classes of untilts of } C\} / \varphi_C^{\mathbb{Z}}$.

Thm: Let C be AC, CV. CP. There X^{FF} inducing a bij.

$$\{ \text{cl. pt. } x \in X^{FF} \}$$
$$\Big\updownarrow \wr$$
$$\{ \text{iso. class. of untilts of } C \} / \varphi_C^{\mathbb{Z}}.$$

• You get this curves mle X^{FF} enjoys a lot of nice property. [FS, Lur18]

How does this relate to local langland.

1. Tilting.

Thm: K a cpl. (CV) nonarc. field. 1. alg clo. (AC). 2. res p. (RP). (K to be perfectoid field).
Then $K^\flat$ can be equipped with a str of a field. (which of char p (P)).

Step 1: Construct ring of integers $\mathcal{O}_K^\flat \hookrightarrow K^\flat$. ← begins its life as a mult. monoid.
• This involves giving $K^\flat$ a norm.

$\mathcal{O}_K := \varprojlim (\cdots \to \mathcal{O}_K \xrightarrow{(\cdot)^p} \mathcal{O}_K)$. ← proj'n onto first component.

Need a norm on $K^\flat$. : $K^\flat \xrightarrow{\#} K \xrightarrow{|\cdot|} \mathbb{R}_{>0} \cup \{0\}$

(*) Step 2: $\mathcal{O}_K^\flat$ admits a structure of ring.

AIM 1. Step 3: $\mathcal{O}_K^\flat [\frac{1}{\pi}] \simeq K^\flat$ as mult. mon.

$K^\flat \simeq \varprojlim (\mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \to \cdots) = \bigcup_{n>0} \pi^{-n} \mathcal{O}_K^\flat \quad \pi \in \mathcal{O}_K$

This gives $K^\flat$ a ring str.

(nonarc $|x+y| \le \max(|x|,|y|)$, mult $|x||y| = |xy|$)

Defn: A perf field. a field K, equipped $|\cdot|_K : K \to \mathbb{R}_{>0} \cup \{0\}$

- RP — A1. Residue field $k := \mathcal{O}_K / \mathfrak{m}_K$ has char p. $\Leftrightarrow$ $|p| < 1$. ($\bar{p} = 0 \in k$, $p \in \mathfrak{m}_K \Leftrightarrow |p| < 1$).
- CV — A2. Field is cplt wrt $|\cdot|_K$.
- SP — A3. Semi perfect. $\mathcal{O}_K / p \xrightarrow{(\cdot)^p} \mathcal{O}_K / p =: (\mathcal{O}_K / p\mathcal{O}_K)$ is surj.
- DC (Discreteness Condition.) — A4a The max. ideal is not gen by p.

  $\mathfrak{m}_K := \{ x \in K : |x|_K < 1 \}$.

  A4b. $\exists$ some $x \in \mathfrak{m}_K$ s.t. $|p|_K < |x|_K < 1$.

  A4c. The val'n gp $|K^\times| \subset \mathbb{R}_{>0}$ is not discrete.

Rmk on A4. There are few equivalent formulation.
$\Leftrightarrow$ $\exists$ some $x \in K$ st. $|p|_K < |x| < 1$.
1. A4 $\Rightarrow$ $|\cdot|_K$ is nontrivial. i.e. $\exists$ some y st. $0 < |y| < 1$.
2. $|a| \le |b| \overset{b \neq 0}{\Rightarrow} a = by$ $y \in \mathcal{O}_K$. $\because$ $|a/b| \le 1$, $a/b \in \mathcal{O}_K$.
So if $\mathfrak{m}_K$ is not gen'ed by p 2. implies, $\exists x$, st. $|p|_K < |x| < 1$.

Ex: AC. CV. RP. $\Rightarrow$ Perf field, A1, A2 are satisfied. AC $\Rightarrow$ A3, A4. (because take pth (nontrivial). root of unit of some $0 < |\pi| < 1$.)

(nontrivial iff $\exists$ some π s.t. $0 < |\pi| < 1$.).

1. Ring str of $\mathcal{O}_K^\flat$. [2, Lur18]

Prop: The canonical map $\mathcal{O}_K \longrightarrow \mathcal{O}_K/p\mathcal{O}_K$ induces a bij. map of mult monoids.

$$\mathcal{O}_K^\flat \longrightarrow \mathcal{O}_{K/p}^\flat := \varprojlim\left(\cdots \to \mathcal{O}_K/p \xrightarrow{(\,)^p} \mathcal{O}_K/p\right).$$

• This obj has a ring str.

Cor: $\mathcal{O}_K^\flat$ inherits str. of a ring.

Rmk: We can replace $\mathcal{O}_K$ here by any R, which p-adically complete.

Pf: • The stat. clearly true when char $K = p$. ($\mathcal{O}_K^\flat$ has a ring str.)

• Assume $|p| \neq 0$. $\Rightarrow$ $|p| < 1$ $\because$ K has RP.

Step 0: Recoll'n

• K cplt $\Rightarrow$ $\mathcal{O}_K \simeq \varprojlim \mathcal{O}_K/p\mathcal{O}_K$ (SP / Defn).

$$= \varprojlim\left(\cdots \mathcal{O}_K/p^{n+1} \to \mathcal{O}_K/p^n \to \cdots \to \mathcal{O}_K/p\right). \quad ***.$$

Step 1: Reindex diagm.

• Let $Z(n) := \varprojlim\left(\cdots \to \mathcal{O}_K/p^n \xrightarrow{(\,)^p} \mathcal{O}_K/p^n\right)$

$$\mathcal{O}_K^\flat = \varprojlim_n \Big(\cdots \underbrace{\mathcal{O}_K}_{***} \xrightarrow{(\,)^p} \underbrace{\mathcal{O}_K}_{***}\Big).$$

$$= \varprojlim_n\Big(\varprojlim_m \big(\mathcal{O}_K/p^m\mathcal{O}_K\big)\Big)$$

$$\begin{tikzcd}
\arrow[r] & \mathcal{O}_K/p^2K \arrow[r] \arrow[d] & \mathcal{O}_K/pK \arrow[d] & Z(2) \arrow[d, "?"] \\
\cdots & \mathcal{O}_K/pK \arrow[r] & \mathcal{O}_K/pK & Z(1)
\end{tikzcd}$$

lim

iso.

• Switch limit

$$= \varprojlim_m \big(Z(m) \big)$$

$Z(m) \to Z(m-1)$ is quotient.

NTS: $\mathcal{O}_K^\flat \simeq \varprojlim_m Z(m) \xrightarrow{\sim} Z(1)$. is an iso.

we'll show $Z(m) \xrightarrow{\sim} Z(m-1)$ is iso. $\forall m$.

Step 2 Consider diagm.

$$\begin{tikzcd}
\mathcal{O}/p^m \arrow[r, "(\,)^p"] \arrow[d, "q"] & \mathcal{O}_K/p^m \arrow[r, "(\,)^p"] \arrow[d, "q"] & \mathcal{O}_K/p^m \arrow[d, "q"] \\
\mathcal{O}/p^{m-1} \arrow[r, "(\,)^p"] \arrow[ur, dashed, "g"] & \mathcal{O}_K/p^{m-1} \arrow[r] \arrow[ur, dashed, "g"] & \mathcal{O}_K/p^{m-1}
\end{tikzcd}$$

induce map of iso limit.

Claim: $\exists$ a map g. making diagram commutes.

$\Rightarrow$ In limit. we have an iso. $\mathbb{N}_{>0} \hookrightarrow \mathbb{N}_{\geq 0}$ is final.

(omitted the K)

$$\begin{tikzcd}
\mathcal{O}/p^{m-1} \arrow[rr, "(\ )^p"] \arrow[d, "(\ )^p"'] & & \mathcal{O}/p^{m-1} \arrow[d, "(\ )^p"] \\
\mathcal{O}/p^{m-1} \arrow[r, "g"] \arrow[rr, bend right, "(\ )^p"'] & \mathcal{O}/p^{m} \arrow[r] & \mathcal{O}/p^{m-1}
\end{tikzcd}$$

at limit. this an iso. in limit.

$\Rightarrow \varprojlim \xrightarrow{\sim} \varprojlim$

Step 3: NTS. $x \equiv y \ (p^{m-1}) \Rightarrow x^p \equiv y^p \ (p^m)$.

$x = y + t p^{m-1}$. $\quad x^p = y^p + \underbrace{\sum_{i=1}^{p-1} \binom{p}{i} y^i (t p^{m-1})^{p-i}}_{\text{also divisible by } p^m} + \underbrace{(t p^{m-1})^p}_{p^m |}$.

Hence $x^p \equiv y^p \ (p^m)$. $\square$