Non-archimedean analytic spaces

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Abstract: This paper provides an elementary introduction to Vladimir Berkovich's theory of analytic spaces over non-archimedean fields, focussing on topological aspects. We also discuss realizations of Bruhat-Tits buildings in non-archimedean groups and flag varieties.

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1 Introduction

About two decades ago, Vladimir Berkovich introduced a new approach to analytic geometry over non-archimedean fields. At this time, Tate's theory of rigid analytic spaces was quite well developed and established. Rigid analytic spaces, however, have rather poor topological properties. Berkovich's analytic spaces contain more points, which leads for example to better connectivity properties. Meanwhile, Berkovich's approach to non-archimedean analytic geometry has become an active area of research with important application to various branches of geometry.

This text aims at providing a very gentle introduction to the nice topological properties of Berkovich spaces. We hope to convey some of the fascination of non-archimedean geometry to researchers from other areas. Due to the introductory nature of this text, we focus on topological aspects.

We start with a discussion of fields with non-archimedean absolute values, pointing out features which are different from the probably better known archimedean situation. Then we discuss in detail the structure of the Berkovich unit disc. Whereas the rigid analytic unit disc is the set of maximal ideals in the Tate algebra, the Berkovich unit disc is a set of suitable seminorms on the Tate algebra. This includes seminorms whose kernels are not maximal ideals. We show how these additional points lead to good connectivity properties. We also discuss the Berkovich projective line.

Then we introduce affinoid algebras as the building blocks of general analytic spaces and we outline some features of the algebraization functor. Starting from an embedding of the Bruhat-Tits tree of SL_2 we explain how buildings can be embedded in analytic spaces. This article concludes with a few remarks about recent research contributions.

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2 Non-archimedean fields

Our ground field is a field K carrying a non-archimedean absolute value, i.e. K is endowed with a function $|\,| : K \to \mathbb{R}_{\geq 0}$ such that for all $a, b \in K$ the following conditions hold:

$$\text{i) } |a| = 0 \text{ if and only if } a = 0.$$

$$\text{ii) } |ab| = |a|\,|b|.$$

$$\text{iii) } |a+b| \leq \max\{|a|, |b|\}.$$

The last condition is the *ultrametric triangle inequality* which has very powerful consequences as we will see below. We endow K with the topology given by its absolute value. In the following we only consider complete fields, i.e. we assume that every Cauchy sequence has a limit in K. We can always place ourselves in this situation by embedding a field with a non-archimedean value in its completion, which is defined as the ring of Cauchy sequences modulo the ideal of zero sequences.

Examples: Here are some examples for fields which are complete with respect to a non-archimedean absolute value.

1) Let K be any field. Then the trivial absolute value

$$|a| = \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

is non-archimedean and K is complete with respect to this absolute value.

2) For every prime number p there is a p-adic absolute value $|\,|_p$ on the field of rational numbers $\mathbb{Q}$. It is defined by

$$|m/n|_p = p^{-v_p(m)+v_p(n)}.$$

Here m and n are non-zero integral numbers and v_p denotes the exponent of p in the prime factorization. The completion of $\mathbb{Q}$ with respect to this absolute value is denoted by $\mathbb{Q}_p$. It is a local field (as is any finite extension), i.e. it is locally compact in the topology given by the absolute value.

3) If K is a non-archimedean complete field and L/K is a finite field extension, then the absolute value $|\,|_K$ on K can be extended in a unique way to an absolute value $|\,|_L$ on L. Hence the absolute value on K extends uniquely to its algebraic closure. In particular, the field $\mathbb{C}_p$, which is defined as the completion of the algebraic closure of $\mathbb{Q}_p$, carries a non-archimedean absolute value. $\mathbb{C}_p$ is an algebraically closed and complete field, which can be regarded as a p-adic analog of the complex numbers.

4) Let k be any field and fix a real constant strictly between 0 and 1. Then, the field of formal Laurent series

$$k((X)) = \{\sum_{i \geq i_0} c_i X^i : c_i \in k \text{ and } i_0 \in \mathbb{Z}\}$$

is complete with respect to the X-adic absolute value

$$|\sum_{i \geq i_0} c_i X^i| = r^{i_0}, \text{ if } c_{i_0} \neq 0.$$

Let us now list some important consequences of the ultrametric triangle inequality which are fundamentally different from the more intuitive archimedean situation.

Properties 1 *Let K be a field which is complete with respect to a non-archimedean absolute value. Then*

i) The unit ball in K

$$\mathcal{O}_K = \{a \in K : |a| \leq 1\}$$

is a ring, since by the ultrametric triangle inequality it is closed under addition. It is called the ring of integers in K. Since the negative logarithmic absolute value defines a valuation on $\mathcal{O}_K$, it is a local ring with maximal ideal $\mathfrak{m}_K = \{a \in K : |a| < 1\}$ (the open unit ball). The quotient field $\widetilde{K} = \mathcal{O}_K/\mathfrak{m}_K$ is called the residue field of K.

ii) If $|a| \neq |b|$, then $|a+b| = max\{|a|,|b|\}$. Indeed, if $|b| < |a|$, the inequality $|a| \leq \max\{|a+b|, |-b|\}$ shows that $|a+b|$ cannot be strictly smaller than $|a|$.

Hence in every ultrametric triangle at least two of the three sides have the same length!

iii) For every a in K and $r \in \mathbb{R}_{\geq 0}$ we define the closed ball around a with radius r as

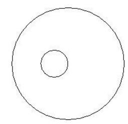
$$D(a,r) = \{x \in K : |x-a| \leq r\}$$

and the open ball around a with radius r as

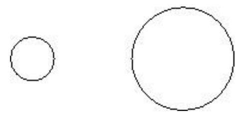
$$D^0(a,r) = \{x \in K : |x-a| < r\}.$$

Contrary to the archimedean situation, every closed ball is also open. A fortiori, every circle $\{x \in K : |x-a| = r\}$ is open in K, since it also contains the open ball $D^0(b,r)$ for each of its points b! This is an immediate consequence of ii).

iv) A similar argument as in iii) shows that two ultrametric balls are either nested



or disjoint



In other words, every point contained in a closed ball is its center.

v) *A popular error in the archimedean case becomes true in the non-archimedean world: An infinite sum* $\sum_{n=0}^{\infty} a_n$ *in* K *converges if and only if* $a_n \to 0$ *as* $n \to \infty$.

vi) *The topology on* K *is totally disconnected, i.e. it contains no non-trivial connected subset.*

For the rest of this paper we fix a field K which is complete with respect to a non-trivial non-archimedean absolute value. Sometimes we make additional assumptions.

3 The unit disc

Over the field of complex numbers one can define analytic functions as functions which are locally given by convergent power series. However, a similar definition over non-archimedean fields gives strange results as the next example shows.

Example 2 *The function* $f : D(0,1) \to \mathbb{R}$, *which is equal to* 0 *on the open unit disc* $D^0(0,1)$ *and equal to* 1 *on the unit circle* $\{x \in K : |x| = 1\}$ *has a local expansion in convergent power series since by Proposition 1 iii),* $D^0(0,1)$ *and the unit circle form an open covering of* $D(0,1)$.

In order to exclude such pathological functions, Tate [Ta66] defined his rigid analytic spaces by considering only so-called *admissible* open subsets and *admissible* open coverings. The technical tool here is to define a Grothendieck topology rather than look at the topology on K induced by the absolute value. Among the benefits of this approach is a good theory of sheaf cohomology.

In Tate's theory, the algebra of analytic functions on the closed unit ball $D(0,1)$ is the Tate algebra

$$T = \left\{ \sum_{n=0}^{\infty} c_n z^n : \sum_{n=0}^{\infty} c_n a^n \text{ converges for every } a \in D(0,1) \right\}.$$

Note that by Proposition 1 iv), an infinite series $\sum_n c_n z^n$ converges on every point of the unit disc $D(0,1)$ if and only if $|c_n| \to 0$ for $n \to \infty$. Hence we can define a norm on T setting

$$||\sum_n c_n z^n|| = \max_n |c_n|.$$

This norm is called the *Gauss norm.* It has the following properties which are easy to verify.

Lemma 3 *i) The Gauss norm is multiplicative, i.e.* $||fg|| = ||f||\,||g||$ *for all* f, g *in* T.

ii) It satisfies the ultrametric triangle inequality.

iii) T *is complete with respect to* $||\,||$*, hence a Banach algebra.*

A very important feature of the Gauss norm is that it coincides with the supremum norm on the unit disc over the algebraic closure. The following result is called the Maximum Modulus Principle, see [BGR84], section 5.1.4.

Lemma 4 *Let* K^a *denote the algebraic closure of* K*, which we endow with the absolute value extending the one on* K*, and denote by* $D_{K^a}(0,1) = \{x \in K^a : |x| \leq 1\}$ *the unit disc in* K^a*. Let* $f = \sum_{n=0}^{\infty} c_n z^n$ *be an element in* T*, and let* $a \in D_{K^a}(0,1)$*. Then, the sequence* $\sum_{n=0}^{\infty} c_n a^n$ *converges, since* $|c_n a^n| \leq |c_n| \to 0$*. We write* $|f(a)|$ *for the absolute value of its limit. We can express the Gauss norm as follows:*

$$||f|| = \sup_{a \in D_{K^a}(0,1)} |f(a)| = \max_{a \in D_{K^a}(0,1)} |f(a)|.$$

Proof: Note that for every $a \in D_{K^a}(0,1)$ the inequality $|f(a)| \leq \max_n |c_n a^n| \leq \max_n |c_n| = ||f||$ holds, so that the supremum of all $|f(a)|$ is also less than or equal to the Gauss norm.

It remains to show that for every element f in T there exists some $a \in D_{K^a}(0,1)$ such that $||f|| = |f(a)|$. We may assume that $f \neq 0$. Since there is an element b in K with $|b| = ||f||$ (namely, any coefficient c_n with maximal absolute value), we may replace f by $b^{-1}f$ and assume that $||f|| = 1$. Hence all coefficients c_n of f lie in the ring of integers $\mathcal{O}_K$. We denote by $c \mapsto \tilde{c}$ the quotient map from $\mathcal{O}_K$ to the residue field $\widetilde{K} = \mathcal{O}_K/\mathfrak{m}_K$. Then $\tilde{f} = \sum_n \tilde{c_n} z^n$ is a non-zero polynomial over the residue field.

The residue field of the algebraic closure K^a of K is defined as $\widetilde{K^a} = \mathcal{O}_{K^a}/\mathfrak{m}_{K^a}$, where

$$\mathcal{O}_{K^a} = \{x \in K^a : |x| \leq 1\} \text{ and } \mathfrak{m}_{K^a} = \{x \in K^a : |x| < 1\}.$$

Since $\widetilde{K^a}$ is an infinite field, it contains an element $\tilde{a}$ on which the polynomial $\tilde{f}$ does not vanish. Then any preimage $a \in \mathcal{O}_{K^a}$ of $\tilde{a}$ satisfies $|f(a)| = 1 = ||f||$. $\square$

Note that this lemma is in general not true if we only look at the supremum of $|f(a)|$ for points $a \in D(0,1)$, i.e. for K-rational points of the unit disc. We have to pass to the algebraic closure.

The rigid analytic version of the unit disc is defined as the space

$$\mathrm{Sp}(T) = \{\mathfrak{m} : \mathfrak{m} \subset T \text{ maximal ideal}\}$$

together with a sheaf of analytic functions on Tate's Grothendieck topology. If K is algebraically closed, the maximal spectrum $\mathrm{Sp}(T)$ coincides with the unit disc $D(0,1)$.

In the approach to non-archimedean analysis provided by rigid geometry one has to forfeit topological intuition to a certain extent. We will now see that Berkovich spaces provide topologically nice analytic spaces with good connectivity properties. We start by introducing the Berkovich version of the closed unit disc which we denote by $\mathcal{M}(T)$. Very roughly, Berkovich's idea is to add additional points to the classical unit disc $D(0,1)$.

To simplify the exposition *we assume that K is algebraically closed and complete* with respect to a non-trivial non-archimedean absolute value. Note that any field with a non-archimedean absolute value can be embedded into an algebraically closed and complete non-archimedean field.

Definition 5 *We define the Berkovich spectrum $\mathcal{M}(T)$ of T as the set of all non-trivial multiplicative seminorms on T bounded by the Gauss norm, i.e. as the set of all maps γ satisfying the following conditions:*

i) $\gamma \neq 0$ is a map from T to $\mathbb{R}_{\geq 0}$.

ii) γ is multiplicative, i.e. for all $f, g \in T$ we have $\gamma(fg) = \gamma(f)\gamma(g)$.

iii) γ satisfies the strong triangle inequality

$$\gamma(f+g) \leq \max\{\gamma(f), \gamma(g)\}.$$

iv) γ is bounded by the Gauss norm on T, i.e. for all f in T we have $\gamma(f) \leq ||f||$.

Note that by i) and ii) $\gamma(1) = 1$, which implies together with iv) that the restriction of γ to the field K (i.e. the constant functions) coincides with the absolute value on K.

Let us now show that the unit disc $D(0,1)$ is contained in the Berkovich unit disc $\mathcal{M}(T)$. Let a be a point in $D(0,1)$. We can associate to a the seminorm

$$\begin{aligned} \zeta_a : T &\rightarrow \mathbb{R}_{\geq 0} \\ f &\mapsto |f(a)|. \end{aligned}$$

It is easy to check that ζ_a satisfies properties i) to iv) in Definition 5. Since for $a \neq b$ we find $\zeta_a(z-a) = 0$ and $\zeta_b(z-a) = |b-a| \neq 0$, the association $a \mapsto \zeta_a$ is injective. We use it tacitly to identify $D(0,1)$ with a subset of $\mathcal{M}(T)$. Every point in the image, i.e. every seminorm of type ζ_a is called *a point of type* 1 in the Berkovich unit disc.

Note that $\mathcal{M}(T)$ is a subset of the set of all real valued functions on T. Hence, it can be endowed with a natural topology, namely the topology of pointwise convergence. This is the weakest topology such that for every $f \in T$ the evaluation map

$$\begin{aligned} \mathcal{M}(T) &\longrightarrow \mathbb{R} \\ \gamma &\mapsto \gamma(f) \end{aligned}$$

is continuous. Its restriction to the subset of points of type 1, i.e. to $D(0,1) \subset \mathcal{M}(T)$ is the topology induced by the absolute value on K which is totally disconnected. We will now show that $\mathcal{M}(T)$ contains many additional points which "fill up the holes" in the classical unit disc.

We have seen in Lemma 3 that the Gauss norm on T is multiplicative, hence it is a point in $\mathcal{M}(T)$. Recall that by Lemma 4, the Gauss norm is the supremum norm on the unit disc $D(0,1)$. More generally, we can look at supremum norms on other discs.

Definition 6 *Let $a \in D(0,1)$ and let r be a real number with $0 < r \leq 1$. For every $f \in T$ we define its supremum norm on $D(a,r)$ as*

$$\zeta_{a,r}(f) = \sup_{x \in D(a,r)} |f(x)|.$$

Every seminorm $\zeta_{a,r}$ is a point in the Berkovich spectrum $\mathcal{M}(T)$. Properties i), iii) and iv) of Definition 5 are obvious. In order to check multiplicativity it is useful to show that for $f = \sum_n c_n(z-a)^n \in T$ we have $\zeta_{a,r}(f) = \max_n(|c_n|r^n)$. If $r = |b|$ lies in the value group $K^\times$ this follows from Lemma 4 applied to $g(z) = f(bz+a)$. Otherwise, one can use a limit argument, since the value group $|K^\times|$ is dense in $\mathbb{R}_{>0}$.

If r is contained in $|K^\times|$, the point $\zeta_{a,r}$ is called a *point of type* 2. If r is not contained in the value group $|K^\times|$, then $\zeta_{a,r}$ is called a *point of type* 3.

Note that the Gauss norm is a point of type 2. It is equal to $\zeta_{0,1}$ in the notation of the the previous definition. We can extend this notation by allowing the radius r to be zero, and define

$$\zeta_{a,0}(f) = \sup_{x \in D(a,0)} |f(x)| = |f(a)|.$$

Then $\zeta_{a,0} = \zeta_a$ is the point of type 1 associated to a we have previously studied.

The difference between points of type 2 and 3 can be seen in the branching behaviour of paths in $\mathcal{M}(T)$. First of all, for every point ζ_a of type 1 there is a path $[\zeta_a, \zeta_{0,1}]$ from ζ_a to the Gauss point $\zeta_{0,1}$, which is given as the image of the map

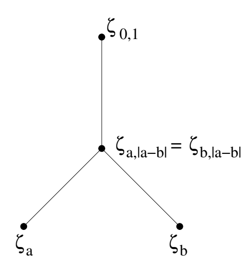
$$\begin{array}{ccc} [0,1] & \longrightarrow & \mathcal{M}(T) \\ r & \mapsto & \zeta_{a,r}. \end{array}$$

Recall that by Properties 1 iv), the Gauss point $\zeta_{0,1}$ is equal to $\zeta_{a,1}$. Apart from the starting point ζ_a, this path consists only of points of type 2 or 3. Moreover, the map is continuous, since it is continuous after evaluation on all functions in T.

Now we look at a second point ζ_b of type 1. Then

$$\zeta_{a,r} = \zeta_{b,r} \text{ if and only if } |a-b| \leq r.$$

Hence on $[0, |a-b|)$ the two paths $[\zeta_a, \zeta_{0,1}]$ and $[\zeta_b, \zeta_{0,1}]$ are disjoint. They meet in $\zeta_{a,|a-b|} = \zeta_{b,|a-b|}$ and travel together to the Gauss point from there on.



This example shows the good connectivity properties of Berkovich spaces: Two points from the totally disconnected unit disc $D(0,1)$ are connected by a path in $\mathcal{M}(T)$ which hits the unit disc $D(0,1)$ only at the starting point and the terminal point.

Note that by definition, $\zeta_{a,|a-b|}$ is a point of type 2. We have already seen that some kind of branching occurs at this point, since the paths from $\zeta_{a,|a-b|}$ to the three points $\zeta_{0,1}$, ζ_a and ζ_b share only the starting point.

We will now investigate the branches meeting at the point $\zeta_{a,|a-b|}$. Recall that the residue field $\widetilde{K}$ is defined as

$$\widetilde{K} = \mathcal{O}_K/\mathfrak{m}_K = \{x \in K : |x| \leq 1\}/\{x \in K : |x| < 1\}.$$

For simplicity, set $r = |a-b|$. Consider the map

$$\begin{array}{rcl} D(a,r) & \longrightarrow & \widetilde{K} \\ c & \longmapsto & \dfrac{a-c}{a-b} + \mathfrak{m}_K, \end{array}$$

where $\frac{a-c}{a-b} + \mathfrak{m}_K$ denotes the residue class in $\widetilde{K}$. Note that $\frac{a-c}{a-b}$ lies in $\mathcal{O}_K$, since $|a-c| \leq r = |a-b|$.

This map is obviously surjective, since for every $x \in \mathcal{O}_K$ the element $a - x(a-b)$ lies in $D(a,r)$ and maps to $x + \mathfrak{m}_K$. On the other hand, two points c and c' are mapped to the same residue class in $\widetilde{K}$ if and only if $|c - c'| < r$, which is equivalent to the fact that $\zeta_{c,s} = \zeta_{c',s}$ for all s in some interval $[t, |a-b|]$ of positive length. Hence c and c' are mapped to the same residue class in $\widetilde{K}$ if and only if the paths $[\zeta_{c,0}, \zeta_{0,1}]$ and $[\zeta_{c',0}, \zeta_{0,1}]$ meet in some point $\zeta_{c,t}$ for $t < r$ and travel together from there on, passing through $\zeta_{a,r}$ on their way to the Gauss point.

Hence we find a bijection between $\widetilde{K}$ and the set of equivalence classes of paths from $\zeta_{c,0}$ to $\zeta_{a,r}$ for $c \in D(a,r)$, where we call two paths equivalent if they coincide on an interval of non-zero length. These equivalences of paths are called branches.

If $\zeta_{a,r}$ is equal to the Gauss point, i.e. if $r = 1$, this gives a bijection between the set of branches in the Gauss point and the residue field $\widetilde{K}$. If $r < 1$, then there is one branch missing, which is the branch from $\zeta_{a,r}$ to the Gauss point. In this case we can identify the set of branches meeting in $\zeta_{a,r}$ with $\{\infty\} \cup \widetilde{K} = \mathbb{P}^1(\widetilde{K})$, i.e. with the projective line over the residue field.

Since we assumed that K is algebraically closed, the residue field $\widetilde{K}$ is infinite. Hence there is infinite branching in $\mathcal{M}(T)$ around every point of type 2.

We have not yet seen all the points in $\mathcal{M}(T)$. For every sequence of discs $D(a_n, r_n)$ in $D(0,1)$ such that

$$D(a_{n+1}, r_{n+1}) \subset D(a_n, r_n) \text{ for all } n \geq 1$$

we define

$$\zeta_{(a_n,r_n)_n}(f) = \inf_n \sup_{x \in D(a_n,r_n)} |f(x)| = \inf_n \zeta_{a_n,r_n}(f).$$

This is also a point in $\mathcal{M}(T)$. If the intersection of the discs $D(a_n, r_n)$ is not empty, it is a point or a disc, and then this limit seminorm is nothing new, but a seminorm of type

1, 2 or 3. However, if the intersection of the discs $D(a_n, r_n)$ is empty, the corresponding point in $\mathcal{M}(T)$ is new, and we call it a *point of type* 4.

Note that since the sequence of discs is decreasing, the sequence of radii $(r_n)_n$ is a decreasing sequence of non-negative real numbers. If $\inf_n r_n$ is equal to zero, then the sequence $(a_n)_n$ of centers is a Cauchy sequence, which implies that the intersection of the discs $D(a_n, r_n)$ is equal to the limit of $(a_n)_n$. Hence, if the intersection of the discs $D(a_n, r_n)$ is empty, leading to a point of type 4, the infimum of the radii must be positive! Points of type 4 only exist if the field K is not spherically complete. Here, spherically complete means that every nested sequence of closed discs has non-empty intersection.

We can now describe the topological space $\mathcal{M}(T)$ as follows.

Theorem 7 *i) Every point in $\mathcal{M}(T)$ is of type 1, 2, 3 or 4. If K is spherically complete only points of type 1, 2 or 3 occur.*

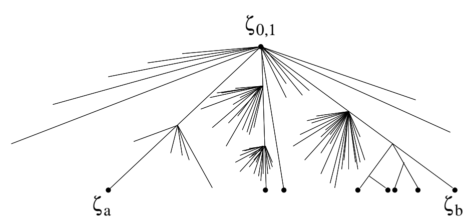
ii) The set of points of type 1 is dense in $\mathcal{M}(T)$, and the set of points of type 2 is dense.

iii) $\mathcal{M}(T)$ is a compact Hausdorff space and uniquely path-connected.

iv) We can visualize $\mathcal{M}(T)$ as a tree which has infinitely many branches growing out of every point contained in a dense subset of any line segment. Such a structure is an example of an $\mathbb{R}$-tree, not a combinatorial tree. More precisely, we can visualize the Gauss point as a root of the tree. The branches emanating from it are in bijection with the residue field $\widetilde{K}$ of K. The set of type 2 points is dense. At every point of type 2, the tree branches off again so that the set of branches in this point is in bijective correspondence with $\mathbb{P}^1(\widetilde{K})$. The leaves are the points of type 1 or of type 4.

A detailed discussion of the Berkovich unit disc where these statements are proved can be found in [Ba-Ru10], chapter 1.

We conclude this section with a picture of a part of the Berkovich unit disc in the situation of a spherically complete field. It is slightly misleading in the sense that it does not capture the effect of infinite branching occuring in a dense set of branch points.



4 The projective line

We can define the Berkovich projective line $(\mathbb{P}^1)^{an}$ as the result of glueing two unit discs together along the automorphism $\gamma \mapsto \gamma^{-1}$ of the annulus $\{\gamma \in \mathcal{M}(T) : \gamma(z) = 1\}$. Note that a point ζ_a of type 1 is contained in the complement $\{\gamma \in \mathcal{M}(T) : \gamma(z) < 1\}$ of the annulus if and only if $|a| < 1$. Hence the paths $[\zeta_a, \zeta_{0,1}]$ of such points all lie on the same branch emanating from the Gauss point.

Therefore we can think of $(\mathbb{P}^1)^{an}$ as the unit disc $\mathcal{M}(T)$ with an additional branch attached to the Gauss point.

There is an alternative description which is reminiscent of the Proj construction in algebraic geometry.

Proposition 8 *Two seminorms γ and δ on the polynomial ring $K[X,Y]$ are called equivalent if there exists a constant $C \in \mathbb{R}_{>0}$ such that for every homogeneous polynomial f of degree d we have $\gamma(f) = C^d \delta(f)$.*

The analytic projective line $\mathbb{P}^{1an}$ can be identified with the set of all equivalence classes of multiplicative seminorms on $K[X,Y]$ which extend the absolute value on K and do not vanish on the maximal ideal (X,Y).

Proof: Every point in the Berkovich unit disc $\mathcal{M}(T)$, i.e. every non-zero and bounded multiplicative seminorm γ on T induces a multiplicative seminorm γ^* on $K[X,Y]$ by setting

$$\gamma^* \left(\sum_{m,n} c_{m,n} X^m Y^n \right) = \gamma \left(\sum_{m,n} c_{m,n} z^m \right).$$

This map is injective on $\mathcal{M}(T)$, and its image consists of all equivalence classes of seminorms γ^* with $\gamma^*(X) \leq \gamma^*(Y)$. If we take another copy of the Berkovich unit disc and glue it to the given one along the unit circle, we can write down an analogous map which also captures the seminorms satisfying $\gamma^*(X) \geq \gamma^*(Y)$. More details can be found in [Ba-Ru10], chapter 2. $\square$

The analytic projective line $(\mathbb{P}^1)^{an}$ is compact and simply connected.

5 Berkovich spaces

In this section the ground field K is complete with respect to a non-archimedean non-trivial absolute value. We generalize the constructions of the previous sections from T to other Banach algebras. A commutative Banach K-algebra $(A, ||\,||)$ is a commutative K-algebra A together with a submultiplicative norm $||\,||$ on the K-vector space A such that A is complete with respect to the induced metric. Hence the norm map $||\,|| : A \to \mathbb{R}_{\geq 0}$ satisfies the following properties:

i) $||f|| = 0$ if and only if $f = 0$.

ii) $||a\,f|| = |a|\,||f||$ for $a \in K$ and $f \in A$.

iii) $||f + g|| \leq ||f|| + ||g||$ for all $f, g \in A$.

iv) $||fg|| \leq ||f||\,||g||$ for all $f, g \in A$.

Definition 9 *Let $(A, ||\,||)$ be a commutative Banach K-algebra. The Berkovich spectrum $\mathcal{M}(A)$ is defined as the set of non-zero multiplicative seminorms on A bounded by the norm $||\,||$. It is endowed with the topology of pointwise convergence, i.e. with the weakest topology such that for every element $a \in A$ the evaluation map $\gamma \mapsto \gamma(a)$ on A is continuous.*

In a natural way, every bounded morphism $A \to B$ between Banach algebras over K induces by composition a continuous map $\mathcal{M}(B) \to \mathcal{M}(A)$ of the associated spectra.

Note that for every multiplicative seminorm ρ on a commutative K-algebra over a non-archimedean field K, the ordinary triangle inequality implies the ultrametric triangle inequality. Namely, if $\rho(x) \leq \rho(y)$, we deduce for every natural number n

$$\rho(x+y)^n \leq \sum_{\nu=0}^{n} \rho\left(\begin{pmatrix} n \\ \nu \end{pmatrix}\right) \rho(x)^\nu \rho(y)^{n-\nu} \leq (n+1)\rho(y)^n,$$

which implies our claim after taking the limit of the n-th roots for $n \to \infty$.

Theorem 10 ([Ber90], Theorem 1.2.1) *If $(A, ||\,||)$ is a non-zero Banach algebra, its Berkovich spectrum $\mathcal{M}(A)$ is a nonempty compact Hausdorff space.*

Definition 11 *Fix $n \in \mathbb{N}$ and let $r = (r_1, \ldots, r_n)$ be a family of positive real numbers. Put $z = (z_1, \ldots, z_n)$, write $z^I = z_1^{i_1} \ldots z_n^{i_n}$ for any multi-index $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, and set $|I| = i_1 + \ldots + i_n$. We define the generalized Tate algebra as*

$$K\{r_1^{-1}z_1, \ldots, r_n^{-1}z_n\} = \left\{ f = \sum_{I=(i_1,\ldots,i_n)\in\mathbb{N}_0^n} c_I z^I : |c_I| r^I \to 0 \text{ as } |I| \to \infty \right\}.$$

We endow $K\{r_1^{-1}z_1, \ldots, r_n^{-1}z_n\}$ with the following variant of the Gauss norm:

$$||\sum_I c_I z^I|| = \max_I |c_I| r^I.$$

The algebra $K\{r_1^{-1}z_1, \ldots, r_n^{-1}z_n\}$ is a K-Banach algebra with respect to the multiplicative Gauss norm. If all r_i are equal to 1, we write

$$T_n = K\{z_1, \ldots, z_n\}$$

and call this algebra *Tate algebra* over K. Then the Tate algebra T discussed in Section 2 agrees with T_1. The Berkovich spectrum $\mathcal{M}(T_n)$ is the Berkovich version of the unit polydisc in n-space. Note that in higher dimensions there is no explicit description of $\mathcal{M}(T_n)$ in terms of types as for $\mathcal{M}(T)$.

A Banach algebra A is called a *K-affinoid algebra* if there exists a surjective K-algebra homomorphism

$$\varphi : K\{r_1^{-1}z_1, \ldots, r_n^{-1}z_n\} \longrightarrow A$$

for some n and $(r_1, \ldots, r_n)$, such that the residue norm $||f||_A = \inf_{\varphi(g)=f} ||g||$ on A is equivalent to its Banach algebra norm.

If we can take all $r_i = 1$, i.e. if A is a suitable quotient of a Tate algebra T_n, then A is called *strictly K-affinoid*. Berkovich spectra of affinoid algebras are the building blocks of Berkovich analytic spaces in a similar way as schemes are made up from spectra of rings.

Let us now define Berkovich affine space $(\mathbb{A}^k)^{an}$. If $r = (r_1, \ldots, r_k) \in \mathbb{R}^k_{>0}$ and $s = (s_1, \ldots, s_k) \in \mathbb{R}^k_{>0}$ satisfy $r_i < s_i$ for all $i = 1, \ldots, k$, the identity map

$$K\{s_1^{-1}z_1, \ldots, s_k^{-1}z_k\} \to K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\}$$

is bounded, hence it defines a continuous map

$$\mathcal{M}(K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\}) \to \mathcal{M}(K\{s_1^{-1}z_1, \ldots, s_k^{-1}z_k\}),$$

which is easily seen to be injective. The topological space $(\mathbb{A}^k)^{an}$ is then defined as the nested union of all $\mathcal{M}(K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\})$.

Lemma 12 *The space $(\mathbb{A}^k)^{an}$ can be identified with the set of all multiplicative seminorms on $K[z_1, \ldots, z_k]$ extending the absolute value on K, which is endowed with the topology of pointwise convergence.*

Proof: The restriction of a point in $\mathcal{M}(K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\})$ to the polynomial ring is a multiplicative seminorm which extends the absolute value on K. Conversely, given any such seminorm γ, we put $r_i = \gamma(z_i)$. Then γ can be extended to a bounded multiplicative seminorm on $K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\}$ in a natural way by writing an infinite series as a limit of polynomials. $\square$

Similarly, the analytification X^{an} of every affine algebraic variety

$$X = \mathrm{Spec} K[z_1, \ldots, z_k]/\mathfrak{a}$$

can be identified with the set of multiplicative seminorms on the coordinate ring $K[z_1, \ldots, z_k]/\mathfrak{a}$ extending the absolute value on K.

Since every scheme Z of finite type over K is glued together from affine schemes X as above, we can define the analytification Z^{an} by glueing the spaces X^{an}. The resulting GAGA functor associates to every scheme Z of finite type over K a topological space Z^{an}. It has the following properties.

Theorem 13 *[[Ber90], Theorem 3.4.8] i) Z is connected if and only if Z^{an} is path-connected.*

ii) Z is separated if and only if Z^{an} is Hausdorff.

iii) Z is proper if and only if Z^{an} is (Hausdorff and) compact.

Berkovich spaces also have a topological dimension which is compatible with the algebraic dimension under analytification.

The topological nature of Berkovich spaces is an important field of research. One-dimensional spaces are quite well understood (for details see [Ber90], section 4). In higher dimensions the situation is less clear. Unlike smooth complex analytic spaces, Berkovich spaces are in general not locally isomorphic to polydiscs. Nevertheless, Berkovich showed in [Ber99] and [Ber04] that smooth analytic spaces are locally contractible. Using tools from model theory, Hrushovski and Loeser [Hru-Loe10] proved that for any quasi-projective algebraic variety Z over K the analytification Z^{an} is locally contractible and admits a strong deformation retraction onto a closed subset which is homeomorphic to a simplicial complex.

So far, we have only discussed Berkovich spaces as topological spaces, without really discussing analysis. Berkovich's analytic spaces are also equipped with a K-affinoid atlas which is used to define the structure sheaf of analytic functions. Details can be found in [Ber93], in the Bourbaki talk [Du06] and in the introductory papers [Tem11] and [Con08].

6 Embedding buildings in analytic spaces

The goal of this section is to show how Bruhat-Tits buildings can be embedded in Berkovich spaces. In this section, we denote by K a *complete, discretely valued field with a perfect residue field* and a non-trivial absolute value. We fix a semisimple algebraic group G over K. For example, G could be a classical group like SL_n, PGL_n, Sp_{2n} or SO_n over K. Since K is not algebraically closed, there are also non-split groups to consider.

These are algebraic groups such that the maximal torus over the algebraic closure is not defined over the field K.

We can associate to G its Bruhat-Tits building $\mathfrak{B}(G, K)$. This is a metric space which is a product of simplicial complexes. Moreover, it carries a continuous $G(K)$-action.

The space $\mathfrak{B}(G, K)$ can be defined by glueing a family of real vector spaces which are called apartments. These apartments are the real cocharacter spaces of the maximal tori in G. The glueing process is based on deep results by Bruhat and Tits [Br-Ti72], [Br-Ti84]. A nice introduction to Bruhat-Tits buildings and their application in representation theory can be found in Schneider's survey paper [Schnei96].

Let us first consider the group SL_2 over K. The Bruhat-Tits building $\mathfrak{B}(SL_2, K)$ has an explicit description as a set of equivalence classes of norms on the K-vector space K^2 satisfying the ultrametric triangle inequality. In this guise, $\mathfrak{B}(SL_2.K)$ was investigated by Goldman and Iwahori [Go-Iw63] before Bruhat and Tits developed their general theory.

Definition 14 *i) A map $|| \, || : K^2 \to \mathbb{R}_{\geq 0}$ is a (non-archimedean) norm on K^2 if the following three conditions hold:*

- $||v|| = 0$ *if and only if* $v = 0$.
- $||av|| = |a| \, ||v||$ *for* $v \in K^2$ *and* $a \in K$.
- $||v + w|| \leq \max(||v||, ||w||)$ *for* $v, w \in K^2$.

ii) The norm $|| \, ||$ is called diagonalizable if there exists a basis v, w of K^2 such that for all $a, b \in K$ we have

$$||av + bw|| = \max(|a| \, ||v||, |b| \, ||w||).$$

In this case we say that $|| \, ||$ is diagonalizable with respect to the basis $\{v, w\}$.

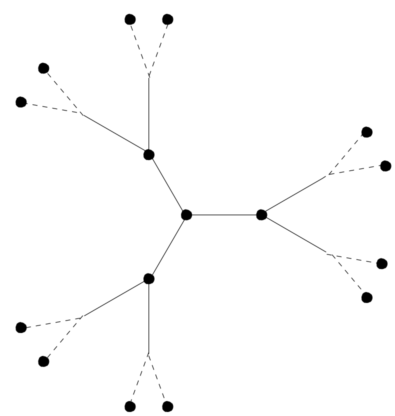
iii) Two norms $|| \, ||$ and $|| \, ||'$ on K^2 are equivalent, if there exists a constant $c \in \mathbb{R}_{\geq 0}$ such that $||v|| = c||v||'$ for all $v \in V$.

Note that if K is locally compact, then every norm on K^2 is diagonalizable with respect to a suitable basis.

Definition 15 *Assume that K is locally compact. The Bruhat-Tits building $\mathfrak{B}(SL_2, K)$ is defined as the set of all equivalence classes of norms on K^2.*

$\mathfrak{B}(SL_2, K)$ carries the topology of pointwise convergence on K^2 and a natural $SL_2(K)$-action given by $|| \, || \mapsto || \, || \circ g^{-1}$ for all $g \in SL_2(K)$.

The building $\mathfrak{B}(SL_2, K)$ can be seen as a non-archimedean analog of the complex upper half plane. It is a tree in the usual sense, i.e. a graph without cycles. Let q be the cardinality of the residue field of K. The tree $\mathfrak{B}(SL_2, K)$ is infinite and regular of valency $q+1$, i.e. $q+1$ edges meet in every vertex. If the residue field is $\mathbb{F}_2$, it looks like this (with branching infinitely continued):



The apartments of $\mathfrak{B}(SL_2, K)$ correspond to the doubly infinite geodesics in this tree. For each apartment A there exists a basis $\beta = \{v, w\}$ of K^2 such that $A = A(\beta)$ consists of all equivalence classes of norms which are diagonalizable with respect to β.

Using this description of the buildings in terms of norms, we can define an embedding of $\mathfrak{B} = \mathfrak{B}(SL_2, K)$ in the Berkovich projective line $(\mathbb{P}^1)^{an}$ over a non-archimedean extension field L of K which is complete and algebraically closed.

Let $\beta = \{e_1, e_2\}$ be the canonical basis of K^2. Then we define a map

$$\vartheta_A : A(\beta) \longrightarrow (\mathbb{P}^1)^{an}$$

as follows: If $|| \, ||$ is a representative of the norm class $x \in A(\beta)$, then we define a seminorm on $L[X, Y]$ by

$$\sum_{m,n} c_{m,n} X^m Y^n \mapsto \max_{m,n} |c_{m,n}| ||e_1||^m \, ||e_2||^n.$$

The point $\vartheta_A(x) \in (\mathbb{P}^1)^{an}$ is defined as the equivalence class of this multiplicative seminorm. This class is independent of the choice of the representative of x. Moreover, we can recover $|| \, ||$ from the above formula by looking at the value of the induced seminorm on X and Y, hence ϑ_A is injective on $A(\beta)$.

Now we extend ϑ_A in an equivariant way to the whole tree $\mathfrak{B}$. For every basis $\beta' = \{v, w\}$ of K^2, there exists an element $g \in GL_2(K)$ such that $g(e_1) = v$ and $g(e_2) = w$. Hence, the action of g^{-1} on $\mathfrak{B}$, which is given by $\|\,\| \mapsto \|\,\| \circ g$, maps the apartment $A(\beta')$ to the apartment $A(\beta)$. On the other hand, g acts in a natural way on $(\mathbb{P}^1)^{an}$, sending the equivalence class of the seminorm γ on $L[X,Y]$ to the class of $\gamma \circ g^{-1}$, where g^{-1} is the algebra automorphism of $L[X,Y]$ given by the matrix g^{-1} in terms of the basis (X,Y) of the degree one part.

Then we define $\vartheta_{A(\beta')} : A(\beta') \to (\mathbb{P}^1)^{an}$ as the composition

$$A(\beta') \xrightarrow{\|\,\|\circ g} A(\beta) \xrightarrow{\vartheta_A} (\mathbb{P}^1)^{an} \xrightarrow{\|\,\|\circ g^{-1}} (\mathbb{P}^1)^{an}.$$

This defines an $SL_2(K)$-equivariant injection of $\mathfrak{B}$ into $(\mathbb{P}^1)^{an}$.

Note that the image of the Bruhat-Tits tree only meets points of type 2 or 3, but none of the classical points of type 1.

The embedding in this example can be generalized. In fact, any Bruhat-Tits $\mathfrak{B}(G,K)$ building can be embedded in the analytic group G^{an} and also in suitable generalized analytic flag varieties. In the case of non-classical groups there is no explicit description of the building in terms of norms. Hence, a more intrinsic approach is necessary.

For split groups, it was shown by Berkovich in [Ber90], section 5 how to realize buildings in analytic group varieties and analytic flag spaces. The paper [RTW10] contains a generalization to the non-split case. Let us outline the construction in the general case.

Let G be a semisimple algebraic group over the field K. Under the hypotheses on the ground field K stated at the beginning of this section, the Bruhat-Tits building of all base changes of G to non-archimedean extension fields L exists. We denote it by $\mathfrak{B}(G,L)$. For every such extension field the group $G(L)$ acts continuously on $\mathfrak{B}(G,L)$.

Then, an embedding $\vartheta : \mathfrak{B}(G,K) \hookrightarrow G^{an}$ is defined as follows: First one shows [RTW10], Theorem 2.1, that for every point $x \in \mathfrak{B}(G,K)$ there exists a unique K-affinoid subgroup G_x of G^{an} satisfying the following condition: for every non-Archimedean field extension L/K, the group $G_x(L)$ is the stabilizer in $G(L)$ of the image of x under the injection $\mathfrak{B}(G,K) \to \mathfrak{B}(G,L)$. Secondly, $\vartheta(x)$ is defined as the (unique) Shilov boundary point of G_x. Hence, if $G_x = \mathcal{M}(A_x)$ for a K-affinoid algebra A_x (see Definition 9), the point $\vartheta(x)$ is maximal with respect to evaluation on functions of A_x. The existence of a Shilov boundary consisting of finitely many points follows from general results by Berkovich (see [Ber90], 2.4.5). It is a delicate fact that the Shilov boundary of the affinoid group G_x consists of one point only (this is proven in [RTW10], Proposition 2.4).

The embedding $\vartheta : \mathfrak{B}(G,K) \to G^{\mathrm{an}}$ is useful to compactify the Bruhat-Tits building $\mathfrak{B}(G,K)$. For this purpose, we choose a parabolic subgroup P of G. Then the flag variety

G/P is complete. By Theorem 13, the associated Berkovich space $(G/P)^{\mathrm{an}}$ is compact. Hence we can map the building to a compact space by the composition

$$\vartheta_P : \mathfrak{B}(G,K) \xrightarrow{\vartheta} G^{\mathrm{an}} \longrightarrow (G/P)^{\mathrm{an}}.$$

The map ϑ_P is by construction $G(K)$-equivariant and it depends only on the $G(K)$-conjugacy class of P: we have $\vartheta_{gPg^{-1}} = g\vartheta_P g^{-1}$ for any $g \in G(K)$.

However, ϑ_P may not be injective. By the structure theory of semisimple groups, there exists a finite family of normal closed subgroups G_i of G (each of them quasi-simple), such that the product morphism

$$\prod_i G_i \longrightarrow G$$

is a central isogeny. Then the building $\mathfrak{B}(G,K)$ can be identified with the product of all $\mathfrak{B}(G_i,K)$. If one of the factors G_i is contained in P, then the factor $\mathfrak{B}(G_i,K)$ is squashed down to a point in the analytic flag variety $(G/P)^{\mathrm{an}}$.

However, if we remove from $\mathfrak{B}(G,K)$ all factors $\mathfrak{B}(G_i,K)$ such that G_i is contained in P, then we obtain a product of buildings which is mapped injectively into $(G/P)^{\mathrm{an}}$ via ϑ_P.

Theorem 16 *Assume that the field K is locally compact, and that no almost simple factor G_i of G is contained in P. Then the closure $\overline{\mathfrak{B}}(G,K)$ of the image of $\mathfrak{B}(G,K)$ under ϑ_P is a compact space containing the building as an open dense subset. $\overline{\mathfrak{B}}(G,K)$ is a union of Bruhat-Tits buildings. The continuous $G(K)$-action on $\mathfrak{B}(G,K)$ extends in a natural way to a continuous $G(K)$-action on $\overline{\mathfrak{B}}(G,K)$.*

In [RTW10], Theorem 4.1 we describe in detail which Bruhat-Tits buildings appear on the boundary.

This approach to compactifications has applications to the structure theory of the group $G(K)$, as the following Theorem shows, which is proven in [RTW10], Proposition 4.20.

Theorem 17 *Fix an apartment A in $\mathfrak{B}(G,K)$ corresponding to the maximal split torus T of G. Let N be the normalizer of T in G. Moreover, denote by $\overline{A}$ the closure of A in $\overline{\mathfrak{B}}(G,K)$. For every $x \in \overline{A}$, we denote by P_x its stabilizer under the $G(K)$-action on $\overline{\mathfrak{B}}(G,K)$. Then for every choice of points x,y in the compactified apartment $\overline{A}$ we have the following generalized Bruhat decomposition of $G(K)$:*

$$G(K) = P_x N(K) P_y.$$

7 Some more applications

During the last two decades, Berkovich analytic spaces have become an ubiquitous tool in non-archimedean arithmetic geometry. We only name very few applications. In particular, he list of references we give here is far from complete.

Berkovich developed an étale cohomology theory for his analytic spaces and used it to prove a conjecture of Deligne on vanishing cycles. Harris and Taylor used étale cohomology of Berkovich spaces in their proof of the local Langlands conjecture for GL_n [Ha-Ta01]. Berkovich spaces also play a vital role in non-archimedean dynamics, see e.g. the book on the Berkovich projective line by Baker and Rumely [Ba-Ru10]. Moreover, they can be used to develop a non-archimedean substitute of the differential geometry at the infinite places in Arakelov theory. On curves, such a theory was developed by Thuillier [Thu05], and there are results in higher dimensions by Chambert-Loir and Ducros [Ch-Du12]. Berkovich spaces are also useful for questions of diophantine geometry over function fields, as in Gubler's proof of the Bogomolov conjecture [Gu07].

There exists also a related notion of analytic spaces incorporating all valuations on a Banach algebra, not only the ones of rank one. This was developed by Huber [Hu94]. Recently, Peter Scholze's theory [Scho11] of perfectoid spaces was formulated in the framework of Huber's analytic spaces. Perfectoid spaces provide a very useful framework for going back and forth between characteristic zero and positive characteristic. This leads to new results on the weight-monodromy conjecture.

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