Generic Representations of the Finite General Linear Groups and the Steenrod Algebra: I

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GENERIC REPRESENTATIONS OF THE FINITE GENERAL LINEAR GROUPS AND THE STEENROD ALGEBRA: I

By Nicholas J. Kuhn

Introduction to the Series. Ever since they were constructed by N. Steenrod in the late 1940's [S1,2], the reduced pth-power operations have been powerful tools for studying homotopy theory. A great body of algebra has been developed by topologists over this forty-year span to make the most effective use of these. The 1980's were no exception: underlying the solutions to both the Segal and Sullivan conjectures, and thus all subsequent applications, were some wonderful new theorems about the Steenrod algebra.

The primary goal of this series of three papers is to develop this recent work as part of the representation theory of the general linear groups over finite fields. This is intended as a direct assault upon compartmentalization in mathematics, and, in the course of this series, I will exhibit rewards to topologists willing to think like representation theorists, and vice versa.

Schematically, the three papers do the following:

$$\left\{\begin{array}{c}\text{Steenrod}\\ \text{algebra}\end{array}\right\} \underset{III}{\overset{I}{\substack{\Longleftarrow\\ \Longrightarrow}}} \left\{\begin{array}{l}\text{generic}\\ \text{representation}\\ \text{theory}\end{array}\right\} \overset{II}{\Longleftrightarrow} \left\{\begin{array}{c}GL_n(\mathbb{F}_q)\text{-}\\ \text{modules}\end{array}\right\}.$$

Thus, the first paper will introduce "generic representation theory" and then use it to develop the Steenrod algebra. A generic embedding theorem is proved which, via a generalized Morita theorem, is then used to derive much of the "Sullivan conjecture algebra" in a unified way.

Part II will justify the terminology "generic representation theory" by relating this to the modular representation theory of the finite general linear groups. The main tools here will be "recollement" and the "prolongement intermédiaire" which were used in the study of perverse sheaves over stratified spaces [BBD].

Finally, Part III will use results about the Steenrod algebra to prove theorems in generic representation theory. For example, a recent theorem of J. Lannes and L. Schwartz [LS] will be shown to be equivalent to a theorem in representation theory that formally looks very similar to "Mumford's conjecture" [H]. As

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another example, a new bigraded algebra will be computed, closely related to G. Carlsson's original algebra [C], that encodes all natural maps from symmetric invariants to coinvariants.

I wish to stress that the only prerequisites for most of my arguments are elementary linear algebra and category theory. No specialized knowledge of either the Steenrod algebra or algebraic groups is assumed.

This project arose from my desire to understand recent work of H.-W. Henn, Lannes, and Schwartz [HLS] relating unstable modules over the Steenrod algebra to a category of "analytic functors." I have been aided in this by insights into representation theory gained through my collaborations with Dave Carlisle and John Harris [CK1,2], [HK]. Conversations with my "algebraic" colleagues at Virginia–Len Scott, Brian Parshall, and Amnon Neeman–have been enlightening. Finally, I would particularly like to thank Lionel Schwartz for an almost constant correspondence on these topics dating from early 1987.

Some of the results here were announced in [K].

1. Introduction to Part I. Let V and W be finite-dimensional vector spaces over $\mathbb{F}_p$, the field with a prime p elements. Let $H^*(V)$ denote the cohomology of the classifying space BV with $\mathbb{F}_p$ coefficients. Underlying the solutions to both the Segal and Sullivan conjectures are some wonderful properties of $H^*(V)$ viewed as an object of $\mathcal{U}_p$, the category of unstable modules over the Steenrod algebra A_p. For example, we have:

(1.1) $H^*(V)$ is injective in $\mathcal{U}_p$;

(1.2) The natural map $\mathbb{F}_p[\mathrm{Hom}(V, W)] \to \mathrm{Hom}_{A_p}(H^*(W), H^*(V))$ is an isomorphism;

(1.3) For all $M, N \in \mathcal{U}$, the natural map from the completed tensor product $\mathrm{Hom}_{A_p}(M, H^*(V)) \hat{\otimes} \mathrm{Hom}_{A_p}(N, H^*(V)) \to \mathrm{Hom}_{A_p}(M \otimes N, H^*(V))$ is an isomorphism.

Statement (1.1) is due to [C] and [M] (see also [LZ1]), (1.2) first appeared in [AGM], and (1.3) was proved in [LZ2, appendix]. All of these have strong generalizations needed for the strongest topological applications (see [AGM] and [L])–but these weaker statements can be used in proofs of the stronger ones (besides providing plausibility).

For our purposes, it is much more natural to work "without Bocksteins" and over a general finite field $\mathbb{F}_q$–in the category $\mathcal{U}(q)$ of unstable modules over $A(q)$, the algebra of Steenrod reduced q^{th} power operations. The object in $\mathcal{U}(q)$ corresponding to $H^*(V)$ is then $S^*(V)$, the symmetric (polynomial) algebra on a finite $\mathbb{F}_q$ vector space V, and the analogues of (1.1)–(1.3) hold. The statements for $\mathcal{U}(p)$ are well known to easily lead to the statements in $\mathcal{U}_p$ (see e.g., [LZ1, appendix A.1]).

The published proofs of (1.1)–(1.3) all have a mysterious *ad hoc* feeling to them. A major purpose of this paper is to give new unified proofs of all of these that answer the question: "Why are they true?"

The answer comes from representation theory. We first establish an appropriately general version of classical Morita equivalence–this theorem was essentially known to Gabriel and Popesco by 1965 [GP], and its antecedents go back to the 1950's. We then prove a "generic" version of the representation theoretic fact that any finitely generated $\mathbb{F}_q[GL(V)]$-module embeds in the sum of the form $\oplus_{i=1}^k S^{d_i}(V)$. Via our Morita theorem, this generic embedding theorem becomes equivalent to our q-analogues of (1.1) and (1.2) with $\mathcal{U}(q)$ replaced by a certain representation category $\operatorname{Rep}(\mathcal{S}^*)$. An elementary calculation then verifies that $\mathcal{U}(q) = \operatorname{Rep}(\mathcal{S}^*)$. Everything needed about the Steenrod algebra certainly predates Milnor's 1957 article [Mn]. As an added bonus, we also establish one of the main theorems of [HLS]:

(1.4) There is an isomorphism of abelian categories

$$\mathcal{U}(q)/\mathcal{N}(q) \simeq \mathcal{F}_\omega(q).$$

Here $\mathcal{N}(q)$ is the Serre-subcategory of $\mathcal{U}(q)$ with objects M such that $\operatorname{Hom}_{\mathcal{U}}(M, S^*(V)) = 0$ for all V, and $\mathcal{F}_\omega(q)$ is a certain representation-theoretic category.

Quite remarkably, (1.1), (1.2), and (1.4) are established without using the existence of a tensor product in $\mathcal{U}(q)$. In fact, in just constructing a tensor product in $\operatorname{Rep}(\mathcal{S}^*)$, we simultaneously prove the q-analogue of (1.3): both are based on the natural isomorphism of graded algebras

$$S^*(V \oplus W) \simeq S^*(V) \otimes S^*(W).$$

The author can see no hint of this in previous proofs of (1.3).

The organization of the rest of the paper is as follows.

In §2 we establish our Morita theorem (Theorem 2.1). Because the literature in this area seems so scattered and poorly cross referenced, we include some historical remarks.

The next two sections define and establish what we need about generic representation theory. Intuitively, a generic representation over $\mathbb{F}_q$ will be a family of $GL_n(\mathbb{F}_q)$-modules compatible as n varies. Featured here is the category of $\mathcal{F}_\omega(q)$ of locally finite representations.

In §5 we prove our generic embedding result (Theorem 5.1): any finite generic representation embeds in a sum of the form $\oplus_{i=1}^k S^{d_i}$.

The rewards of the previous sections come in §6 with the definition of $\mathcal{U}(q)$.

Statements (1.1), (1.2), and (1.4) are immediately established.

§7 deals with tensor products in $\mathcal{U}(q)$. The tensor product theorem (1.3) is established.

In the last section, we use the tensor product to give a representation-theoretic construction of the suspension in $\mathcal{U}(q)$ and the Hopf algebra $A(q)$.

Appendix A sketches a proof that the locally finite representations are precisely the analytic functors of [HLS], reconciling our two very different definitions of $\mathcal{F}_\omega(q)$.

Appendix B contains some categorical properties of $\mathcal{F}_\omega(q)$ not needed in the main body of the paper.

2. One-Sided Morita Equivalence. We first recall the "classical" Morita context. Let R be a ring and P a right R-module. If we then define S to be the ring $\mathrm{End}_R(P)$, P becomes a left S-module, and there are adjoint functors

$$\text{Right } S\text{-modules} \underset{r}{\overset{\ell}{\rightleftarrows}} \text{Right } R\text{-modules}$$

defined by $\ell(M) = M \otimes_S P$ and $r(N) = \mathrm{Hom}_R(P, N)$.

Morita theorems are typically of the following form: good properties of P as a right R-module imply that ℓ and r have good properties, and conversely. For example, if P is a finitely generated projective generator for right R-modules, then both ℓ and r are exact and fully faithful, and $R \simeq \mathrm{End}_S(P)$ [CR, p. 60].

We wish to focus on the hypothesis that P is a generator, and greatly generalize the situation using the concepts of additive and abelian categories.

An additive category is a generalization of a ring (with unit): a ring R is an additive category with one object. Similarly, a left (respectively, right) R-module is nothing more than an additive covariant (contravariant) functor from this one object category to abelian groups. Thus, if $\mathcal{A}$ is an additive category, the generalization of the category of left R-modules is $\mathrm{Rep}(\mathcal{A})$, where an object in $\mathrm{Rep}(\mathcal{A})$ is an additive functor

$$M : \mathcal{A} \to \text{Abelian groups},$$

and morphisms are natural transformations. Dually, $\mathrm{Rep}(\mathcal{A}^{op})$ generalizes right R-modules.

It is illuminating to view $M \in \mathrm{Rep}(\mathcal{A})$ as a graded group M_* with extra structure: for each object A_i in $\mathcal{A}$, $M_i = M(A_i)$ is an abelian group, and for each map $a : A_i \to A_j$ in $\mathcal{A}$, there is a corresponding map $a : M_i \to M_j$ with appropriate properties. We use the notation $m \in M$ to mean that $m \in M_i$ for some i. When needed, this index i will be designated $i(m)$. For $a : A_i \to A_j$ in $\mathcal{A}$ and $m \in M_i$, $a \cdot m$ is the corresponding element in M_j. (Dually, for $m \in M \in \mathrm{Rep}(\mathcal{A}^{op})$, the notation $m \cdot a$ will be used).

$\operatorname{Rep}(\mathcal{A})$ becomes an abelian category in the obvious way, e.g., $M_* \to N_*$ is monic if each $M_i \to N_i$ is.

Finally, we will always be working over a fixed commutative ring k: the Hom-sets in a k-additive category $\mathcal{A}$ are k-modules, an object in $k - \operatorname{Rep}(\mathcal{A})$ is a k-additive functor to the category of k-modules, etc. For example, a k-additive category with one object is just a k-algebra. We will generally omit "k" from our notation.

Two general references for these concepts are B. Mitchell's treatise on "rings with several objects" [Mt] and Popescu's book [P].

We now describe our generalized Morita context. Let $\mathcal{C}$ be an abelian category having arbitrary direct (inductive) limits. Let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$ with a set of objects $\{S_i\}$. Then there are functors

$$\operatorname{Rep}(\mathcal{S}^{op}) \underset{r}{\overset{\ell}{\rightleftarrows}} \mathcal{C}$$

with $r(C)_* = \operatorname{Hom}_{\mathcal{C}}(S_*, C)$, and ℓ left adjoint to r. Here ℓ exists because $\mathcal{C}$ has direct limits [Mt, §6], and it is illuminating to write $\ell(M_*) = M_* \otimes_{\mathcal{S}} S_*$.

Our "one-sided" Morita theorem is as follows.

THEOREM 2.1. *Let $\mathcal{C}$ be an abelian category with exact direct limits, and let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$ with a set of objects $\{S_i\}$. Then, with notation as above, the following are equivalent.*

(1) *$\mathcal{S}$ generates $\mathcal{C}$.*

(2) (i) *ℓ is exact;*
(ii) *r is fully faithful.*

(3) *$\mathcal{C}$ has enough injectives, and for all injectives I and J in $\mathcal{C}$:*

(i) *$r(I)$ is injective in* $\operatorname{Rep}(\mathcal{S}^{op})$;
(ii) *$r_{I,J} : Hom_{\mathcal{C}}(I,J) \to Hom_{\operatorname{Rep}}(r(I), r(J))$ is an isomorphism.*

We refer to this result as a "one-sided" Morita theorem because (2)(ii) holds if and only if the natural map

$$\varepsilon_C : \ell r(C) \to C$$

is an isomorphism for all $C \in \mathcal{C}$.

In the quotient category language of [G, chap. III], condition (2) implies that $\operatorname{Ker}(\ell)$ is a localizing subcategory of $\operatorname{Rep}(\mathcal{S}^{op})$, and that ℓ and r induce an equivalence

$$\operatorname{Rep}(\mathcal{S}^{op}) / \operatorname{Ker}(\ell) \simeq \mathcal{C}.$$

Furthermore, $\ell(M) = 0$ if and only if $\mathrm{Hom}_{\mathrm{Rep}}(M, r(I)) = 0$ for all $\mathcal{C}$-injectives I, and the natural map

$$\eta_M : M \to r\ell(M)$$

can be interpreted as "localization away from Ker(ℓ)" ("Ker(ℓ)-closure" in [G, chap. III]).

Historical Remarks 2.2. I discovered this theorem (in the setting used in §6) in 1989. Not surprisingly, a search eventually yielded variations on this result scattered throughout the ring theory and category theory literature.

The earliest version of a one-sided Morita theorem that I have found occurs in the appendix of a 1960 paper by Auslander and Goldman [AG].

The closest I have found to the theorem as stated above is a 1964 theorem of Gabriel and Popescu [GP]: if $\mathcal{S}$ has just one object, then (1) $\Rightarrow$ (2). The proof I discovered differs significantly from both [GP] and the improved 1973 version in [P, p. 111], and my first draft of this paper included this proof. However, an August 1990 Math Review (MR 90h: 18002) alerted me to the elegant 1971 proof of the Gabriel-Popescu theorem by Takeuchi [T], and (swallowing my pride) it is the "multiobject" version of his proof that is sketched below.

Finally, a change of viewpoint relates our theorem to the work of Kent Fuller in ring theory. Suppose we define $\mathrm{Gen}(\mathcal{S}) \subseteq \mathcal{C}$ to be the full subcategory with objects generated by $\mathcal{S}$. Then ℓ and r restrict to an adjoint pair

$$\mathrm{Rep}(\mathcal{S}^{op}) \underset{r}{\overset{\ell}{\rightleftarrows}} \mathrm{Gen}(\mathcal{S}).$$

Obviously $\mathrm{Gen}(\mathcal{S})$ has arbitrary direct limits and is generated by $\mathcal{S}$. Why can we not apply the theorem? The problem is that $\mathrm{Gen}(\mathcal{S})$ need no longer be abelian: the kernel of a map between objects in $\mathrm{Gen}(\mathcal{S})$ need not be in $\mathrm{Gen}(\mathcal{S})$. Let $\overline{\mathrm{Gen}(\mathcal{S})}$ be the smallest full abelian subcategory of $\mathcal{C}$ containing $\mathrm{Gen}(\mathcal{S})$ and closed under direct limits. Then the theorem can be restated as $(1') \Leftrightarrow (2')$:

$(1')$ $\mathrm{Gen}(\mathcal{S}) = \overline{\mathrm{Gen}(\mathcal{S})}$;

$(2')$ (i) ℓ' is exact,
(ii) r' is fully faithful.

In the classical module situation, the theorem in this form implies many of the results in the 1974 paper [F].

We begin the proof of our Morita theorem by observing that $\mathcal{S}$ generates $\mathcal{C}$ if and only if r is faithful. Thus, it is obvious that (2) implies (1). If $\mathcal{C}$ has a set of generators and has exact direct limits, it is known to have enough injectives [Gr, Thm. 1.10.1]. (The reader who wishes to avoid this argument is welcome to assume from the beginning that $\mathcal{C}$ has enough injectives–this will be clear in the examples of interest.) Then it is formal that (2)(i) implies (3)(i) and (2)(ii) implies (3)(ii).

Of the remaining implications, (3) $\Rightarrow$ (2) is also quite straightforward, while (1) $\Rightarrow$ (2) is the heart of the matter.

Proof that (3) $\Rightarrow$ (2). We first show that (3)(i) implies (2)(i). Supposing that $f : M \to N$ is monic in $\operatorname{Rep}(\mathcal{S}^{op})$, we need to show that $\ell(f) : \ell(M) \to \ell(N)$ is monic. Let $\ell(M)$ embed in an injective I. Then $\ell(f) : \ell(M) \to \ell(N)$ is monic

$$\Leftrightarrow \ell(f)^* : \operatorname{Hom}(\ell(N), I) \to \operatorname{Hom}(\ell(M), I) \text{ is epic}$$
$$\Leftrightarrow f^* : \operatorname{Hom}(N, r(I)) \to \operatorname{Hom}(M, r(I)) \text{ is epic.}$$

But this last map is epic, since F is monic and $r(I)$ is injective.

It remains to show that, assuming (3), the natural map

$$r_{C,D} : \operatorname{Hom}(C, D) \to \operatorname{Hom}(r(C), r(D))$$

is an isomorphism for all C and D in $\mathcal{C}$.

Given C, let $0 \to C \to I_0 \to I_1$ be exact, with I_0, I_1 injective. If J (and thus $r(J)$) is injective, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{Hom}(I_1, J) & \to & \operatorname{Hom}(I_0, J) & \to & \operatorname{Hom}(C, J) & \to & 0 \\
\downarrow r_{I_1,J} & & \downarrow r_{I_0,J} & & \downarrow r_{C,J} & & \\
\operatorname{Hom}(r(I_1), r(J)) & \to & \operatorname{Hom}(r(I_0), r(J)) & \to & \operatorname{Hom}(r(C), r(J)) & \to & 0.
\end{array}$$

Thus, $r_{I_0,J}$ and $r_{I_1,J}$ being isomorphisms implies that $r_{C,J}$ is also. A similar argument using a resolution $0 \to D \to J_0 \to J_1$ then shows that if r_{C,J_0} and r_{C,J_1} are isomorphisms, so is $r_{C,D}$.

Proof that (1) $\Rightarrow$ (2). As commented above, this is basically a multiobject version of Takeuchi's proof.

LEMMA 2.3. *Given $j : M \subseteq r(C)$, the adjoint $\tilde{j} : \ell(M) \to C$ is monic.*

Proof. We begin with some generalities.

For any $M \in \operatorname{Rep}(\mathcal{S}^{op})$, there is a canonical map

$$\pi : \oplus_{m \in M} S_{i(m)} \to \ell(M)$$

constructed as follows. The adjunction $\eta : M \to r\ell(M) = \operatorname{Hom}_{\mathcal{C}}(S_*, \ell(M))$ has components $\eta_i : M_i \to \operatorname{Hom}_{\mathcal{C}}(S_i, \ell(M))$. Then π is defined by letting its m^{th} component be $\eta_{i(m)}(m)$. It is a formal exercise in adjunctions to check that π is epic.

Suppose given a map $\tilde{f} : \ell(M) \to D$ with adjoint $f : M \to r(D) = \operatorname{Hom}(S_*, D)$. Given $m \in M$, we regard $f(m)$ as a map $f(m) : S_{i(m)} \to D$. Again, it is formal that there is a commutative diagram

$$\begin{tikzcd} \bigoplus_{m\in M} S_{i(m)} \arrow[r, "\Sigma f(m)"] \arrow[d, "\pi"'] & D \\ \ell(M) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Armed with these generalities, we start to prove the lemma. To show $\tilde{j}$ is monic, we will show that given any $\tilde{f} : \ell(M) \to D$, $\mathrm{Ker}\,(\tilde{j}) \subseteq \mathrm{Ker}\,(\tilde{f})$. Letting $\tilde{f}$ have adjoint $f : M \to r(D)$, there is a commutative diagram

$$\begin{tikzcd} \bigoplus_{m\in M} S_{i(m)} \arrow[dr, "\pi"] \arrow[drr, "\sum f(m)"] \arrow[ddr, "\sum j(m)"'] & & \\ & \ell(M) \arrow[r, "\tilde{f}"'] \arrow[d, "\tilde{j}"] & D \\ & C & \end{tikzcd}$$

with π a quotient map. Thus we wish to show that

$$\mathrm{Ker}\left(\sum_{m\in M} j(m)\right) \subseteq \mathrm{Ker}\left(\sum_{m\in M} f(m)\right).$$

Since $\mathcal{C}$ has exact direct limits, it suffices to show that $\sum_{m\in N} f(m)(K_N) = 0$ where $K_N = \mathrm{Ker}\,(\sum_{m\in N} j(m))$ and N is a *finite* subset of M. Since $\mathcal{S}$ generates $\mathcal{C}$, it suffices to show that $(\sum_{m\in N} f(m)) \circ \alpha = 0$ for all $\alpha : S_i \to K_N$, i.e., that

$$\left(\sum_{m\in N} j(m)\right) \circ \alpha = 0 \text{ implies } \left(\sum_{m\in N} f(m)\right) \circ \alpha = 0.$$

Letting $S_i \xrightarrow{\alpha} K_N \subseteq \oplus_{m\in N} S_{i(m)}$ have components $a(m) : S_i \to S_{i(m)}$, this can be written as

$$\sum_{m\in N} j(m) \cdot a(m) = 0 \text{ implies } \sum_{m\in N} f(m) \cdot a(m) = 0.$$

Recalling that j and f are maps in $\mathrm{Rep}\,(\mathcal{S}^{op})$, i.e., maps of "right $\mathcal{S}$-modules," we have

$$\sum j(m) \cdot a(m) = j\left(\sum m \cdot a(m)\right) \text{ and } \sum f(m) \cdot a(m) = f\left(\sum m \cdot a(m)\right).$$

But since j is monic, $j(\sum m \cdot a(m)) = 0$ implies that $\sum m \cdot a(m) = 0$, and thus that $f(\sum m \cdot a(m)) = 0$, as needed.

COROLLARY 2.4. *$\varepsilon_C : \ell r(C) \to C$ is an isomorphism, i.e., r is fully faithful.*

Proof. Letting $M = r(C)$ in the lemma shows that ε_C is monic. Since $\mathcal{S}$ generates $\mathcal{C}$, ε_C is epic.[1]

To prove exactness of ℓ, we need to define free objects in $\mathrm{Rep}(\mathcal{S}^{op})$. Let $F(i) = r(S_i)$. By Yoneda's lemma, $\mathrm{Hom}_{\mathrm{Rep}}(F(i), M) = M_i$ and $\ell(F(i)) = S_i$. Thus, the $F(i)$ form a set of projective generators for $\mathrm{Rep}(\mathcal{S}^{op})$. Call an object F *free* if it is of the form $\oplus_{k\in K} F(i_k)$.

LEMMA 2.5. *Given $i : M \subseteq F$ with F free, $\ell(i) : \ell(M) \to \ell(F)$ is monic.*

Proof. Let $F = \oplus_{k\in K} F(i_k)$. Since direct limits are exact in both $\mathcal{C}$ and $\mathrm{Rep}(\mathcal{S}^{op})$, and ℓ commutes with direct limits,[2] it suffices to prove the lemma with $i : M \subseteq F$ replaced by $i_J : M_J \subseteq F_J$, where $J \subset K$ is *finite*, $F_J = \oplus_{k\in J} F(i_k)$, and $M_J = M \cap F_J$. Letting $S_J = \oplus_{k\in J} S_{i_k}$, we have $F_J = r(S_J)$ and $\ell(F_J) = S_J$, so the lemma in this case is a special case of Lemma 2.3.

COROLLARY 2.6. *ℓ is exact.*

Proof. Being a left adjoint, ℓ is right exact. If $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ is a free resolution of $M \in \mathrm{Rep}(\mathcal{S}^{op})$, then $0 \leftarrow \ell(M) \leftarrow \ell(F_0) \leftarrow \ell(F_1) \leftarrow \cdots$ is exact by the last lemma. Thus the left derived functors of ℓ are all 0.

Remark 2.7. If the objects in $\mathcal{S}$ are assumed to be *small*, i.e., $\oplus_\alpha \mathrm{Hom}(S_i, C_\alpha) \simeq \mathrm{Hom}(S_i, \oplus_\alpha C_\alpha)$, then the "finiteness" arguments in Lemmas 2.3 and 2.5 are not needed. This is the case in our example of interest in §6.

3. Generic Representation Theory. For the remainder of this paper, all vector spaces are over the characteristic p field $\mathbb{F}_q$ with q elements, "additive" will mean $\mathbb{F}_q$-additive, and V, W will denote finite-dimensional vector spaces.

Definition 3.1. Let $\mathcal{F}(q)$ be the category with objects the functors

$$F : \text{finite vector spaces} \to \text{vector spaces},$$

and with the natural transformations as morphisms.

This is an abelian category in the obvious way, e.g., F is a subpoint of G means that $F(V) \subseteq G(V)$ for all V. In fact, $\mathcal{F}(q)$ is a representation category as follows: let $\mathcal{L}(q)$ be the category with objects V and with

$$\mathrm{Hom}_{\mathcal{L}(q)}(W, V) = \mathbb{F}_q[\mathrm{Hom}(W, V)],$$

[1] $\mathcal{S}$ generates $\mathcal{C}$ if and only if r is faithful. By adjunction, r is faithful if and only if ε_C is epic for all C.
[2] Left adjoints always commute with direct limits.

the linear space with the set $\mathrm{Hom}(W, V)$ as basis. Then

$$\mathcal{F}(q) = \mathrm{Rep}\,(\mathcal{L}(q)).$$

Note that if $F \in \mathcal{F}(q)$, then, for all $V, F(V)$ is a representation of the group $GL(V)$ (and the semigroup $\mathrm{End}\,(V)$). For this reason, we sometimes refer to the objects of $\mathcal{F}(q)$ as *generic representations*. (Part II of this series will study this observation more thoroughly.)

In the rest of this section, we discuss the most basic structure in $\mathcal{F}(q)$.

3.2. Tensor Product. Given F and G in $\mathcal{F}(q)$, $F \otimes G$ is defined by

$$(F \otimes G)(V) = F(V) \otimes G(V).$$

3.3. Scalar Decomposition. This is the finite field version of MacDonald's eigenspace (degree) decomposition [MacD, appendix to chap. 1]. Scalar multiplication on V induces a functorial $\mathbb{F}_q^{\mathrm{mult}}$-module structure on $F(V)$, for all $F \in \mathcal{F}(q)$, where $\mathbb{F}_q^{\mathrm{mult}}$ denotes $\mathbb{F}_q$ viewed as a multiplicative semigroup. It is easy to verify that the semigroup ring $\mathbb{F}_q[\mathbb{F}_q^{\mathrm{mult}}]$ is isomorphic to $\mathbb{F}_q \times \mathbb{F}_q[\mathbb{F}_q^{\times}]$, and is thus semisimple. It follows that each $F(V)$ will decompose into the sum of its isotypical pieces:

$$F(V) = F_0(V) \oplus \cdots \oplus F_{q-1}(V),$$

where $F_i(V) = \{x \in F(V) | F(\lambda)(x) = \lambda^i x \text{ for all } \lambda \in \mathbb{F}_q\}$. This splitting is natural, so there is a decomposition of abelian categories:

$$\mathcal{F}(q) \simeq \mathcal{F}(q)_0 \times \cdots \times \mathcal{F}(q)_{q-1}.$$

The subcategory $\mathcal{F}(q)_0$ is the category of *constant* functors and is isomorphic to the category of vector spaces. A functor F with $F_0 = 0$ will be called *constant free*. Finally we note that the tensor product induces pairings

$$\otimes : \mathcal{F}(q)_i \times \mathcal{F}(q)_j \to \mathcal{F}(q)_{i+j}$$

(with indices taken mod q).

3.4. Duality. Given $F \in \mathcal{F}(q)$, we define the *dual* of F, DF, by

$$(DF)(V) = F(V^*)^*.$$

Note that this is still covariant in V, but contravariant in F. For all F and G in $\mathcal{F}(q)$, D induces a natural isomorphism

$$\mathrm{Hom}_{\mathcal{F}(q)}(F, DG) \simeq \mathrm{Hom}_{\mathcal{F}(q)}(G, DF).$$

To put this construction in context, we observe that D corresponds to "transpose duality" of $GL_n(\mathbb{F}_q)$-modules: $(DF)(\mathbb{F}_q^n)$ is the left $GL_n(\mathbb{F}_q)$-module obtained from the right $GL_n(\mathbb{F}_q)$-module $F(\mathbb{F}_q^n)^*$ by letting matrices act via their transposes. There is a generalization of this to arbitrary reductive algebraic groups where it plays a basic role in representation theory (see, e.g., [J, II.1.16, II.2.12–2.16]).

The most significant property of D will be proved in Part II: all simple functors are self dual. We will not need this result in Part I.

3.5. Projectives. Given a finite vector space W, let P_w be defined by

$$P_W(V) = \mathbb{F}_q[\mathrm{Hom}(W, V)].$$

By Yoneda's lemma, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{F}(q)}(P_W, F) \simeq F(W)$$

for all $F \in \mathcal{F}(q)$. Thus the P_W form a set of projective generators for $\mathcal{F}(q)$.

By inspection, there is an isomorphism

$$P_V \otimes P_W \simeq P_{V \oplus W}.$$

Thus the tensor product of two projectives is again projective. It follows that the functors $P_{\mathbb{F}_q}^{\otimes n}$, $n = 0, 1, 2, \ldots$, form a set of projective generators for $\mathcal{F}(q)$.

For a description of the projective indecomposable functors, we refer the reader to Part II. However, we do need one example,

LEMMA 3.6. *Under the scalar decomposition (3.3), $P_{\mathbb{F}_q}$ splits into a sum of q indecomposable functors.*

Proof. By standard arguments (Fitting's lemma), direct sum decompositions of $F \in \mathcal{F}(q)$ will correspond to orthogonal idempotent decompositions of $1 \in \mathrm{End}_{\mathcal{F}(q)}(F)$. In our case, $\mathrm{End}_{\mathcal{F}(q)}(P_{\mathbb{F}_q}) \simeq \mathbb{F}_q[\mathbb{F}_q^{\mathrm{mult}}]$ as rings.

3.7. Injectives. Let $I_W = DP_W$. Then there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{F}(q)}(F, I_W) \simeq (DF)(W)$$

for all $F \in \mathcal{F}(q)$. Thus the I_W form a set of injective cogenerators for $\mathcal{F}(q)$.

Dualizing our results about projectives, we have, e.g.,

$$I_V \otimes I_W \simeq I_{V \oplus W},$$

the functors $I_{\mathbb{F}_q}^{\otimes n}$, $n = 0, 1, 2, \ldots$, form a set of injective cogenerators for $\mathcal{F}(q)$, and $I_{\mathbb{F}_q}$ is the sum of precisely q indecomposable injectives.

4. Finite and Locally Finite Functors. Since $\mathcal{F}(q)$ is an abelian category, we have the following standard terminology (compare with [P, chap. 5]).

Definition 4.1.

(1) $F \in \mathcal{F}(q)$ is *simple* if it has only 0 and F subobjects.

(2) F is *finite* if it has a finite composition series with simple subquotients.

(3) F is *locally finite* if it is the union of its finite subfunctors. Let $\mathcal{F}_\omega(q)$ denote the full subcategory of $\mathcal{F}(q)$ with the locally finite functors as objects.

The subscript "ω" is chosen to agree with [HLS]: when p is a prime, our category $\mathcal{F}_\omega(p)$ is the same as their category of "analytic" functors, although this is far from obvious from the definitions. See Appendix A.

Some remarks are also in order here, to explain why locally finite objects are intrinsically interesting. Representation theorists seem to be well acquainted with naturally arising locally finite categories and their advantages—see, e.g., [J, II.1.20] or [CPS, §3]. For topologists, we offer the following analogy: finite objects have good properties similar to those of compact spaces, and locally finite objects are then the analogues of compactly generated spaces.

A formal consequence of being a category of locally finite objects is the following

PROPOSITION 4.2. *A direct sum of injectives in $\mathcal{F}_\omega(q)$ is again injective. Every injective in $\mathcal{F}_\omega(q)$ is uniquely the direct sum of indecomposable injectives.*

Here "uniquely" means in the usual Krull-Schmidt sense. See [G, IV.2].

Definition 4.3. Let T^d, S^d, $S_d \in \mathcal{F}(q)$ be defined, for $d \geq 1$, by $T^d(V) = V^{\otimes d}$, $S^d(V) = (V^{\otimes d})/\Sigma_d$, and $S_d(V) = (V^{\otimes d})^{\Sigma_d}$ respectively. We let $T^0 = S^0 = S_0$ be the constant functor of dimension one.

We are now aiming to show that the functors T^d (and their subquotients) are finite, and from this deduce that the injectives I_W are locally finite. We would like to thank Lionel Schwartz for the key "derivative" idea (Definition 4.6, below) in our proof of the first fact (which replaces our original more computational proof). For more use of this idea, see [S].

LEMMA 4.4. *If F is finite, then $F(V)$ is finite dimensional for all V.*

Proof. It suffices to prove this when F is simple. But any simple functor will be a quotient of P_W for some W, and $P_W(V)$ is finite dimensional for all V.

Definition 4.5. If $F(V)$ is finite dimensional for all V, let

$$d_F(n) = \dim_{\mathbb{F}_q} F(\mathbb{F}_q^n).$$

Thus the last lemma says that d_F is defined for all finite functors F.

Definition 4.6.

(1) For $F \in \mathcal{F}(q)$, let $F^{(1)}$ be defined by $F^{(1)}(V) = F(V \oplus \mathbb{F}_q)/F(V)$. (Note that $F(V)$ is a natural retract of $F(V \oplus \mathbb{F}_q)$.) Let $F^{(k)} = (F^{(k-1)})^{(1)}$ for $k = 2, 3, \ldots$.

(2) For $d : \mathbb{N} \to \mathbb{Z}$, let $d^{(1)}$ be defined by $d^{(1)}(n) = d(n+1) - d(n)$, and let $d^{(k)} = (d^{(k-1)})^{(1)}$ for $k = 2, 3, \ldots$.

The next lemma is clear.

LEMMA 4.7. (1) *If $G \subseteq F$, then $G^{(k)} \subseteq F^{(k)}$, and $d_G(n) \leq d_F(n)$.*

(2) $d_{F^{(k)}} = (d_F)^{(k)}$.

Slightly less obvious is

LEMMA 4.8. *$d : \mathbb{N} \to \mathbb{Z}$ is polynomial of degree $\leq r$ if and only if $d^{(r+1)} \equiv 0$.*

Proof. Since

$$\binom{k}{k} = 1$$

and

$$\binom{n}{k} = 0$$

if $n < k$, it is easy to see that d has a *unique* description as

$$d(n) = \sum_{k=0}^{\infty} c_k \binom{n}{k}.$$

Since

$$\binom{n}{k}^{(1)} = \binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1},$$

it follows that

$$d^{(r+1)}(n) = \sum_{k=0}^{\infty} c_{r+1+k} \binom{n}{k},$$

and the result follows easily.

Combining the last two lemmas yields

COROLLARY 4.9. *If $G \subseteq F$ and d_F is polynomial of degree $\leq r$, then so is d_G.*

Using this, we get our first criterion for finiteness.

PROPOSITION 4.10. *If d_F is polynomial of degree $\leq r$, then the lattice of subobjects of F is isomorphic to the lattice of sub-$M_r(\mathbb{F}_q)$-modules of $F(\mathbb{F}_q^r)$. In particular, F is finite.*

Proof. Let $\mathcal{L}_F$ and $\mathcal{L}_{F(\mathbb{F}_q^r)}$ denote the two lattices. We define lattice maps

$$\mathcal{L}_F \underset{e}{\overset{\ell}{\rightleftarrows}} \mathcal{L}_{F(\mathbb{F}_q^r)}$$

as follows. If $G \subseteq F$, let $e(G) = G(\mathbb{F}_q^r)$. If $M \subseteq F(\mathbb{F}_q^r)$ is an $M_r(\mathbb{F}_q)$-submodule, define $\ell(M)$ by

$$\ell(M)(V) = \operatorname{Im}\{\varepsilon : \mathbb{F}_q[\operatorname{Hom}(\mathbb{F}_q^r, V)] \otimes_{\mathbb{F}_q[M_r(\mathbb{F}_q)]} M \to F(V)\}.$$

Clearly $e\ell(M) = M$. We need to check that $\ell e(G) = G$. By construction $\ell e(G) \subseteq G$ and $d_{\ell e(G)}(n) = d_G(n)$ for $n \leq r$. By the corollary, both $d_{\ell e(G)}$ and d_G are polynomials of degree $\leq r$. Since they agree on $r+1$ points, they are equal, i.e., $\ell e(G) = G$.

COROLLARY 4.11. *T^d is finite, and, if F is any nonzero subquotient, then $F(\mathbb{F}_q^d) \neq 0$.*

We now turn to the injectives I_W. The following elementary lemma is one of the keys to the whole paper. Note that

$$I_{\mathbb{F}_q}(V) = \mathbb{F}_q[V^*]^* = \{\text{set maps } f : V^* \to \mathbb{F}_q\}.$$

Thus $I_{\mathbb{F}_q}(V)$ is naturally a commutative algebra using pointwise addition and multiplication. The obvious inclusion $V \hookrightarrow I_{\mathbb{F}_q}(V)$ thus extends to an algebra map

$$S^*(V) \to I_{\mathbb{F}_q}(V).$$

Because $\lambda^q = \lambda$ in $\mathbb{F}_q$, the kernel contains the ideal generated by $(v^q - v)$, $v \in V$.

LEMMA 4.12. *The induced map $S^*(V)/(v^q - v) \longrightarrow I_{\mathbb{F}_q}(V)$ is an isomorphism.*

Proof. The map is onto: this is just the fact that any function can be made to agree with a polynomial on any finite number of points, but V^* has only a finite number of points. Now the map is seen to be an isomorphism by checking dimensions (filtering the domain by degree yields an associated graded $S^*(V)/(V^q)$).

THEOREM 4.13. *I_W is locally finite for all W.*

Proof. If W is n dimensional, $I_W \simeq I_{\mathbb{F}_q}^{\otimes n}$. By the last lemma, $I_{\mathbb{F}_q}$ is visibly a quotient of a sum of T^d's. Since $T^i \otimes T^j = T^{i+j}$, the same is true for $I_{\mathbb{F}_q}^{\otimes n}$. By Corollary 4.11, we are done.

THEOREM 4.14. *For $F \in \mathcal{F}(q)$, the following are equivalent:*

(1) *F is finite;*

(2) *F is a subquotient of a sum of the form $\oplus_{i=1}^k T^{d_i}$;*

(3) *d_F is defined and is polynomial.*

Proof. We have already shown that (2) $\Rightarrow$ (3) $\Rightarrow$ (1). To show (1) $\Rightarrow$ (2), suppose F is finite. Then F embeds in a finite sum of I_W's, thus, reasoning as in the proof of Theorem 4.13, F is a subquotient of a sum of the form $\oplus_{i=1}^{\infty} T^{d_i}$. Since F is finite, it is noetherian, and it easily follows that F is a subquotient of $\oplus_{i=1}^k T^{d_i}$ for some k.[3]

COROLLARY 4.15. (1) *If F and G are finite, so is $F \otimes G$.*

(2) *If F and G are locally finite, so is $F \otimes G$.*

We end this section with two further consequences of our work here.

PROPOSITION 4.16. *Duality D takes finite functors to finite functors.*

Proof. Although this is an obvious consequence of Theorem 4.14, this is overkill: the proposition even follows from the more elementary Lemma 4.4. Note that it is obvious that if F is *not* finite, neither is DF, since the dual of a nonzero composition factor will be a nonzero composition factor. Stated equivalently, if DF is finite, so is F. Similarly, if DDF is finite, so is DF. Finally, if F is finite, so is DDF since the natural map $F \longrightarrow DDF$ is then an isomorphism, courtesy of Lemma 4.4.

PROPOSITION 4.17. *Every injective in $\mathcal{F}_\omega(q)$ is uniquely the direct sum of indecomposable summands of the I_W.*

[3]Given $F \hookrightarrow Q \overset{\pi}{\leftarrow} \oplus_{i=1}^{\infty} T^{d_i}$, let π_k be π restricted to $\oplus_{i=1}^k T^{d_i}$, let $Q_k = \operatorname{Im}(\pi_k)$, and let $F_k = F \cap Q_k$. Then $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ and converge to F, so $F = F_k$ for some k.

This is a formal consequence of the locally finite category $\mathcal{F}_\omega(q)$ containing the cogenerating family of I_W's.

We remark that $\mathcal{F}_\omega(q)$ has no nonconstant projectives. See Appendix B.

5. The Embedding Theorem The goal of this section is to prove the following embedding theorem.

THEOREM 5.1. *Any finite $F \in \mathcal{F}(q)$ embeds in a sum of the form $\oplus_{i=1}^k S^{d_i}$. Equivalently, the family S_d, $d \geq 0$, generate $\mathcal{F}_\omega(q)$.*

The second statement comes from duality: S^d is dual to S_d.[4]

Once again, we offer some remarks to put this theorem in context. A quite standard way to construct representations of a finite group G is to perform natural constructions on a faithful G-module V. For example, in [A, p. 45], Alperin shows that $\mathbb{F}_q[G]$ embeds in $S^*(V)$. It is easy to see (letting $G = GL(V)$) that this is equivalent to

(5.2) Any finitely generated $GL(V)$-module embeds in a sum $\oplus_{i=1}^k S^{d_i}(V)$.

In Part II, we will show that Theorem 5.1 is equivalent to (5.2) with "$GL(V)$-module" replaced by "End (V)-module" (and thus, in fact, *implies* (5.2) as stated). We note, however, that Alperin's elementary proof, using Galois theory and the normal basis theorem, has no hope of generalizing to the semigroup setting.

We prove Theorem 5.1 by first reducing the case of a general finite F to a family of specific examples. These are then embedded explicitly.

Collecting together (3.7) and Lemma 4.12 yields

LEMMA 5.3. (1) *$I_{\mathbb{F}_q} \simeq I_0 \oplus \cdots \oplus I_{q-1}$, where each I_k is an indecomposable injective.*

(2) *$I_k(V) = \oplus_{m=0}^{\infty} S^{k+m(q-1)}(V)/(v^q - v)$, for $1 \leq k \leq q-1$, and $I_0 = S^0$.*

(3) *I_k is the injective envelope of S^k.*

Our family of examples comes from taking the obvious filtration of the I_k's.

Definition 5.4. For $1 \leq k \leq q-1$ and $n \geq 0$, define $\bar{S}^{k,n} \in \mathcal{F}(q)$ by

$$\bar{S}^{k,n}(V) = \oplus_{m=0}^{n} S^{k+m(q-1)}(V)/(v^q - v).$$

The next lemma records basic properties of these finite functors.

LEMMA 5.5. (1) *The natural map $S^{k+n(q-1)} \longrightarrow \bar{S}^{k,n}$ is onto.*

[4]By Proposition 4.16, any finite functor embeds in a sum of the form $\oplus_{i=1}^k S^{d_i}$ if and only if any finite functor is a quotient of a sum of the form $\oplus_{i=1}^k S_{d_i}$. By definition, $\mathcal{F}_\omega(q)$ is generated by the finite functors.

(2) *There are inclusions*

$$S^k = \bar{S}^{k,0} \subseteq \bar{S}^{k,1} \subseteq \bar{S}^{k,2} \subseteq \cdots \subseteq \cup_{n=0}^{\infty} \bar{S}^{k,n} = I_k.$$

(3) *I_k is the injective envelope of $\bar{S}^{k,n}$ for all n. Thus a map $f : \bar{S}^{k,n} \to F$ will be monic if and only if its restriction to $\bar{S}^{k,0} = S^k$ is monic.*

(4) *Any constant free finite functor F embeds in a finite sum of generic representations of the form $\bar{S}^{k_1,n_1} \otimes \cdots \otimes \bar{S}^{k_\ell,n_\ell}$.*

It follows from (4) that to prove Theorem 5.1, it suffices to embed the functors $\bar{S}^{k_1,n_1} \otimes \cdots \otimes \bar{S}^{k_\ell,n_\ell}$. We now further reduce to the case $\ell = 1$. To do this we need to introduce the two basic operations involving the S^d's. Standard multiplication of polynomials yields maps

$$S^i \otimes S^j \to S^{i+j}, \tag{5.6}$$

and, since we are working over $\mathbb{F}_q$, there are q^{th} power maps

$$S^d \to S^{dq}. \tag{5.7}$$

The q^{th} power maps (and their iterates) are monic. In fact, the following lemma is easily checked.

LEMMA 5.8. *If $q^r > j$, then the composite*

$$S^i \otimes S^j \to S^{iq^r} \otimes S^j \to S^{iq^r+j}$$

is monic.

COROLLARY 5.9. *If F and G embed as in Theorem 5.1, so does $F \otimes G$.*

Combining all of our above reductions, we have shown that to prove Theorem 5.1, it suffices to embed $\bar{S}^{k,n}$ for $1 \leq k \leq q-1$, $n \geq 0$. Statement (3) of the next theorem asserts that this can be done.

THEOREM 5.10. *For $1 \leq k \leq q-1$ and $n \geq 0$, there exist*

$$\Theta_{k,n} : \bar{S}^{k,n} \to S^{kq^n}$$

with the following properties:

(1) $\Theta_{k,0} = 1 : S^k \to S^k$.

(2) *There are commutative diagrams*

$$\begin{array}{ccc} \bar{S}^{k,n-1} & \hookrightarrow & \bar{S}^{k,n} \\ \Theta_{k,n-1} \downarrow & & \Theta_{k,n} \downarrow \\ S^{kq^{n-1}} & \xrightarrow[q^{\text{th}}\ \textit{power}]{} & S^{kq^n} \end{array}.$$

(3) $\Theta_{k,n}$ *is monic.*

(4) *The* $\Theta_{k,n}$ *induce an inclusion* $\Theta_k : I_k \hookrightarrow \varinjlim_n S^{kq^n}$.

Proof. Statement (3) follows from (1) and (2), recalling Lemma 5.5(3). Statement (4) is also clear, given (1) and (2).

Since $\bar{S}^{k,n}$ is a quotient of $S^{k+n(q-1)}$, to construct $\Theta_{k,n}$ it suffices to construct natural maps $S^{k+n(q-1)}(V) \to S^{kq^n}(V)$ factoring through the equivalence relation induced by $v^q - v = 0$. Equivalently, we need to define maps

$$\bar{\Theta}_{n,k} : V^{k+n(q-1)} \to S^{kq^n}(V)$$

such that

(a) $\bar{\Theta}_{k,n}$ is natural in V;

(b) $\bar{\Theta}_{k,n}$ is $(k + n(q-1))$-linear.

(c) $\bar{\Theta}_{k,n}$ is symmetric under the action of symmetric group $\Sigma_{k+n(q-1)}$;

(d) $\bar{\Theta}_{k,0} : V^k \to S^k(V)$ is the canonical map;

(e) there are commutative diagrams

$$\begin{array}{ccc} V^{k+(n-1)(q-1)} & \xrightarrow{\Delta \times 1} & V^{k+n(q-1)} \\ \bar{\Theta}_{k,n-1} \downarrow & & \bar{\Theta}_{k,n} \downarrow \\ S^{kq^{n-1}}(V) & \xrightarrow[q^{\text{th}}\ \text{power}]{} & S^{kq^n}(V) \end{array}$$

where $\Delta : V \to V^q$ is the diagonal.

Formula 5.11. Let $d = k + n(q-1)$. Define $\bar{\Theta}_{k,n}$ by

$$\bar{\Theta}_{k,n}(x_1, \ldots, x_d) = \sum_{q^{i_1}+\cdots+q^{i_d}=kq^n} x_1^{q^{i_1}} \ldots x_d^{q^{i_d}}.$$

Clearly properties (a)–(d) hold. We now check that so does property (e), thus completing our proofs of Theorems 5.10 and 5.1.

We are trying to verify the equation

$$\bar{\Theta}_{k,n}(\overbrace{x_1,\dots,x_1}^{q},x_2,\dots,x_{k+(n-1)(q-1)}) \tag{5.12}$$
$$= \bar{\Theta}_{k,n-1}(x_1,\dots,x_{k(n-1)(q-1)})^q.$$

We introduce some notation. Let $\mathcal{J}_{k,n}$ be the indexing set in formula 5.11, i.e.,

$$\mathcal{J}_{k,n} = \{I = (i_1,\dots,i_d) | q^{i_1} + \cdots + q^{i_d} = kq^n\}.$$

We write this as a disjoint union:

$$\mathcal{J}_{k,n} = \mathcal{J}^0_{k,n} \coprod \mathcal{J}^1_{k,n} \coprod \mathcal{J}^2_{k,n},$$

where

$$\begin{aligned}
\mathcal{J}^0_{k,n} &= \{I | i_1 = \cdots = i_q \text{ and } i_j \geq 1,\ \forall j > q\},\\
\mathcal{J}^1_{k,n} &= \{I | i_1 = \cdots = i_q \text{ and } \exists j > q \text{ with } i_j = 0\},\\
\mathcal{J}^2_{k,n} &= \{I | \exists 2 \leq j \leq q \text{ and } i_j \neq i_1\}.
\end{aligned}$$

Clearly, $\mathcal{J}_{k,n-1} \simeq \mathcal{J}^0_{k,n}$ via the correspondence

$$(i_1,\dots,i_{k+(n-1)(q-1)}) \mapsto (i_1,\dots,i_1,1+i_2,\dots,1+i_{k+(n-1)(q-1)}),$$

and these indices give the terms on the right side of (5.12). Thus the next two lemmas complete the verification of (5.12).

LEMMA 5.13. $\mathcal{J}^1_{k,n} = \emptyset$.

LEMMA 5.14. $\sum_{I \in \mathcal{J}^2_{k,n}} x_1^{q^{i_1}} \dots x_2^{q^{i_{q+1}}} \dots x_{k+(n-1)(q-1)}^{q^{i_d}} = 0.$

Proof of Lemma 5.13. Suppose 0 appears in $I = (i_1,\dots,i_{k+n(q-1)}) \in \mathcal{J}_{k,n}$. Deleting this index yields a decomposition of $kq^n - 1$ into the sum of $(k-1) + n(q-1)$ powers of q. But

$$kq^n - 1 = (k-1)q^n + (q^n - 1) = (k-1)q^n + \sum_{j=0}^{n-1}(q-1)q^j$$

is clearly the unique such decomposition (recall that $k < q$). Thus if I contains a 0, then it has precisely q 0's, $(q-1)$ j's for $1 \leq j \leq n-1$, and $(k-1)$ n's. It follows that if $i_1 = \cdots = i_q$, then these must be *all* the 0's, so that $I \in \mathcal{J}^0_{k,n}$.

Proof of Lemma 5.14. This is proved by cancellation. A term appears $[\Sigma_q : \Sigma_I]$ times where Σ_I is the stabilizer of $\{i_1, \ldots, i_q\}$ under the Σ_q-action (such a subgroup is called *parabolic*). If $I \in \mathcal{J}^2_{k,n}$, then Σ_I is a proper parabolic subgroup of Σ_q, and it is well known that then p divides the index $[\Sigma_q : \Sigma_I]$ (i.e., no proper parabolic subgroup of Σ_{p^s} contains a p-Sylow subgroup).

Remarks 5.15. (1) Formula 5.11 was discovered experimentally for small n (with $q = 2$). It is just the sum of the "obvious" natural symmetric d-linear maps $V^d \to S^{kq^n}(V)$, and thus the induced map

$$\Theta_{k,n} : S^d \to S^{kq^n}$$

is just the sum of the "obvious" morphisms. As will become clear in the next section, these morphisms are precisely those induced by the Milnor basis for $A(q)$ in dimension $kq^n - d$. For example, when $q = 2$, $\Theta_{1,2} = Sq(1) : S^3 \to S^4$ and $\Theta_{1,3} = Sq(4) + Sq(1,1) : S^4 \to S^8$. Rather amazingly, I have recently learned that both J. F. Adams and V. Franjou have independently discovered the operations $\Theta_{k,n}$ and used them in new proofs of the $\mathcal{U}(p)$-injectivity of $S^*(V)$. Adams' unpublished handwritten notes [Ad] should be appearing in his collected works. For Franjou's work, see [BF]. These rather distinct "2nd generation" proofs suggest that it is the combinatorics of the operations $\Theta_{k,n}$ that somehow drive "$\mathcal{U}(q)$-technology."

(2) Theorem 5.10 shows that the injective I_k embeds, and thus is a summand of $\varinjlim_n S^{kq^n}$. Those familiar with Carlsson's paper [C] may recall that he showed that $H^*(\mathbb{Z}/2)$ is a summand of an unstable $A(2)$-module $K(1)$ which was obviously injective. In Part III, we will connect these two results: a q-analogue of work by Lannes and Schwartz in [LS] will be shown to be equivalent to the injectivity of $\mathcal{F}_\omega(q)$ of the functors $\varinjlim_n S^{kq^n}$, for all k. From this we will deduce that $\varinjlim_n S^{kq^n} \in \mathcal{F}_\omega(q)$ corresponds to the generalized Carlsson modules $K(k) \in \mathcal{U}(q)$ under the functors ℓ and r of the next section.

6. The Category $\mathcal{U}(q)$. The category $\mathcal{F}_\omega(q)$, being a subcategory of $\mathcal{F}(q)$, has exact direct limits. Theorems 2.1 and 5.1 suggest the following definitions.

Definition 6.1.

(1) Let $\mathcal{S}_*$ be the full subcategory of $\mathcal{F}(q)$ with objects S_d, $d \geq 0$.

(2) Let $\mathcal{S}^*$ be the full subcategory of $\mathcal{F}(q)$ with objects S^d, $d \geq 0$.

(3) Let $\mathcal{U}(q) = \mathrm{Rep}\,(\mathcal{S}^*)$.

Thus an object $M \in \mathcal{U}(q)$ is a non-negatively graded vector space together with maps $a : M_n \to M_m$ for every $a : S^n \to S^m$, etc. The notation $\mathcal{U}(q)$ will be justified below.

By duality $\mathcal{S}^* \simeq \mathcal{S}_*^{op}$, so that $\mathcal{U}(q) = \operatorname{Rep}(\mathcal{S}_*^{op})$. Theorem 5.1 says that $\mathcal{S}_*$ generates $\mathcal{F}_\omega(q)$. We are set up to use our Morita theorem.

Let $\mathcal{U}(q) \underset{r}{\overset{\ell}{\rightleftarrows}} \mathcal{F}_\omega(q)$ be the adjoint pair defined by letting ℓ be left adjoint to r, where

$$r(F)_* = \operatorname{Hom}_{\mathcal{F}(q)}(S_*, F).$$

For example, we compute that

$$r(I_V) = \operatorname{Hom}_{\mathcal{F}(q)}(S_*, I_V) = \operatorname{Hom}_{\mathcal{F}(q)}(P_V, S^*) = S^*(V).$$

Note also that

$$\operatorname{Hom}_{\mathcal{F}(q)}(I_V, I_W) = \operatorname{Hom}_{\mathcal{F}(q)}(P_W, P_V) = P_V(W) = \mathbb{F}_q[\operatorname{Hom}(V, W)].$$

Theorems 2.1 and 5.1 immediately imply the next three theorems.

THEOREM 6.2. *ℓ is exact and r is faithful. They induce an equivalence of abelian categories*

$$\mathcal{U}(q)/\mathcal{N}(q) \simeq \mathcal{F}_\omega(q),$$

where $\mathcal{N}(q)$ is the full subcategory with objects M such that $Hom_{\mathcal{U}(q)}(M, S^*(V)) = 0$ for all V.

THEOREM 6.3. *$S^*(V)$ is injective in $\mathcal{U}(q)$, for all V.*

THEOREM 6.4. *The natural map*

$$\mathbb{F}_q[Hom(V, W)] \longrightarrow Hom_{\mathcal{U}(q)}(S^*(V), S^*(W))$$

is an isomorphism for all V and W.

To relate these theorems to topology as quickly as possible, we make the following definitions. (The reader unfamiliar with the Steenrod algebra should trust that the author will elaborate on this definition, both below and in §8.)

Definition 6.5.

(1) Let $A(p)$ be the algebra of Steenrod reduced p^{th} powers (no Bocksteins), with grading divided by 2 if $p > 2$. Thus $A(p)$ is a Hopf algebra with dual isomorphic to a polynomial algebra $\mathbb{F}_p[\xi_1, \xi_2, \ldots]$ with $|\xi_i| = p^i - 1$ [Mn].

(2) If $q = p^s$, let $A(q) \subseteq A(p) \otimes_{\mathbb{F}_p} \mathbb{F}_q$ be the sub Hopf algebra dual to the quotient Hopf algebra $\mathbb{F}_q[\xi_1, \xi_2, \ldots]/(\xi_i | i \neq 0 \bmod s)$.

(3) Given $r = (r_1, \ldots, r_\ell)$, let $d(\underline{r}) = \sum_{i=1}^{\ell} r_i(q^i - 1)$. Using the monomial basis for $A(q)^*$, let $\mathcal{P}(q; \underline{r}) \in A(q)_{d(\underline{r})}$ be dual to $\xi_s^{r_1} \ldots \xi_{\ell s}^{r_s}$.

(4) Given $\underline{r} = (r_1, \ldots, r_\ell)$, let $e(\underline{r}) = r_1 + \cdots + r_\ell$. Call a graded left $A(q)$-module M *unstable* if $\mathcal{P}(q; \underline{r}) \cdot M_n = 0$ whenever $e(\underline{r}) > n$.

As an example $S^*(V)$ is an unstable $A(q)$-module: $H^*(BV; \mathbb{F}_p)$ is an unstable $A(p)$-module and $S^*(V)$ is then a sub $A(q)$-module of $H^*(BV^*; \mathbb{F}_q)$. Since this is natural in V, any $M \in \mathcal{U}(q)$ will inherit an unstable $A(q)$-structure. In fact, we make the following observation.

PROPOSITION 6.6. *$\mathcal{U}(q)$ is the category of unstable left $A(q)$-modules.*

This proposition is making two assertions: firstly, every natural map $S^n(V) \to S^m(V)$ is induced by an element of $A(q)$, and secondly, every element of $A(q)$ is detected by its action on $S^n(V)$ subject to the excess condition. This latter fact has been basically known since the early 1950's—this leads to one derivation of the Adem relations. Below, we will prove both assertions by explicitly computing $\mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^m)$.

Because of this proposition, Theorems 6.2, 6.3, and 6.4 correspond to statements (1.4), (1.1), and (1.2) of §1. To summarize, our generic embedding theorem is equivalent to the main properties of the unstable $A(q)$-module $S^*(V)$. The only points of our arguments of a computational nature are the proofs of Theorem 5.10 and Proposition 6.6.

Remarks 6.7. (1) Suitably interpreted, the left adjoint ℓ is given by the formula

$$\ell(M)(V) = M \otimes_{\mathcal{S}_*} S_*(V), \tag{6.8}$$

and the exactness of ℓ becomes the statement

$$S_*(V) \text{ is flat in the category } \mathrm{Rep}\,(\mathcal{S}_*) = \mathrm{Rep}\,(\mathcal{S}^{*op}). \tag{6.9}$$

Projectivity implies flatness, but $S_*(V)$ is *not* projective in $\mathrm{Rep}\,(\mathcal{S}^{*op})$. It *is* projective in $\mathrm{Rep}\,(\mathcal{S}^*)^{op}$, however (carefully dualize Theorem 6.3). These categories have the following interpretation: $\mathrm{Rep}\,(\mathcal{S}^{*op})$ is the category of unstable right $A(q)$-modules, while $\mathrm{Rep}\,(\mathcal{S}^*)^{op}$ is isomorphic to the category of "profinite" unstable right $A(q)$-modules (each vector space M_d has a profinite topology). The difference between these categories explain the "finite type" hypotheses that pervade §6 of H. Miller's paper [M].

(2) Note that for any two $\mathbb{F}_q$-vector spaces B and C, there is a natural isomorphism

$$(B \otimes C)^* \simeq \mathrm{Hom}(B, C^*),$$

and thus, taking profinite duals (denoted "$'$")

$$B \otimes C \simeq \mathrm{Hom}(B, C^*)'.$$

Applying this to (6.8) yields the alternative formula for ℓ:

$$\ell(M)(V) = \mathrm{Hom}_{\mathcal{U}(q)}(M, S^*(V^*))', \tag{6.10}$$

where the profinite topology comes from filtering M by its "finitely generated" subobjects. (As in §2, M is formally a quotient of a sum of $F(d)$'s, where $F(d) = r(S_d) = \mathrm{Hom}_{\mathcal{F}(q)}(S^d, S^*)$, and $\mathrm{Hom}_{\mathcal{U}(q)}(F(d), M) = M_d$.) Formula 6.10 is the one used by [HLS].

To prove Proposition 6.6, we need to better understand $\mathrm{Rep}\,(\mathcal{S}^*)$, and thus $\mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^d)$. Since S^n is the symmetric quotient of T^n, we first compute $\mathrm{Hom}_{\mathcal{F}(q)}(T^n, S^d)$. We use polynomial multiplication and the q^{th} power operation to define basic maps.

Definition 6.11. If $I = (i_1, \ldots, i_n)$ and $q^{i_1} + \cdots + q^{i_n} = d$, let

$$b(I) : T^n \to S^d$$

be defined as the composite $T^n \to S^{q^{i_1}} \otimes \cdots \otimes S^{q^{i_n}} \to S^d$. Explicitly,

$$b(I)(x_1 \otimes \cdots \otimes x_n) = x_1^{q^{i_1}} \ldots x_n^{q^{i_n}}.$$

Call such a map *basic*.

LEMMA 6.12. *The basic maps form a basis for $Hom_{\mathcal{F}(q)}(T^n, S^d)$.*

Assuming this, one can immediately read off a basis for $\mathrm{Hom}_{\mathcal{F}(q)}(S^n, S^d)$ by summing over orbits of the action of Σ_n on the basic maps. To compare with standard notation, we make the following definition.

Definition 6.13. Given $\underline{r} = (r_1, \ldots, r_\ell)$, define

$$\mathcal{P}(q; \underline{r}; n) : S^n \to S^{n+d(\underline{r})}$$

as follows. If $e(\underline{r}) > n$, $\mathcal{P}(q; \underline{r}; n) = 0$. If $e(\underline{r}) \leq n$,

$$\mathcal{P}(q; \underline{r}; n)(x_1 \ldots x_n) = \sum x_1^{q^{i_1}} \ldots x_n^{q^{i_n}},$$

with the sum ranging over $(i_1, \ldots, i_n)$ with precisely r_i of the i_j equal to i for $i = 1, \ldots, \ell$. (Thus $n - e(\underline{r})$ of the i_j equal 0.)

COROLLARY 6.14. *$\{\mathcal{P}(q;\underline{r};n) | e(\underline{r}) \leq n$ and $d(\underline{r}) = d\}$ is a basis for $Hom_{\mathcal{F}(q)}(S^n, S^{n+d})$.*

Proposition 6.6 follows immediately: by the very construction of the generators $\xi_j \in A(q)^*$, $\mathcal{P}(q;\underline{r})$ acting on $S^n(V)$ induces the map $\mathcal{P}(q;\underline{r};n)$. (See, e.g., [Sw, Proposition 18.18].)

It remains to prove Lemma 6.12. We prove a stronger result.

LEMMA 6.15. *If $\dim V \geq n+1$, the basic maps form a basis for $Hom_{\mathrm{End}\,(V)}(V^{\otimes n}, S^d(V))$.*

Proof. The proof is elementary, but computational. Let V have a basis $t_1, \ldots, t_k$ with $k \geq n+1$, and let $\alpha : V^{\otimes n} \to S^d(V)$ be an End(V)-map. We first claim:

$$\text{(6.16)} \quad \begin{cases} \text{If } \alpha(t_1 \otimes \cdots \otimes t_n) = \sum_{j_1+\cdots+j_k=d} a(j_1,\ldots,j_k) t_1^{j_1} \ldots t_k^{j_k}, \\ \text{then, for all } x_1,\ldots,x_n \in V, \\ \alpha(x_1 \otimes \cdots \otimes x_n) = \sum_{j_1+\cdots+j_k=d} a(j_1,\ldots,j_k) x_1^{j_1} \ldots x_k^{j_k}, \\ \text{where } x_r = 0 \text{ if } r > n. \end{cases}$$

To see this, let $A : V \to V$ be defined by $A(t_r) = x_r$. Then

$$\begin{aligned} \alpha(x_1 \otimes \cdots \otimes x_n) &= \alpha(A(t_1 \otimes \cdots \otimes t_n)) \\ &= A(\alpha(t_1 \otimes \cdots \otimes t_n)) \\ &= \Sigma a(j_1,\ldots,j_k) A(t_1^{j_1} \ldots t_k^{j_k}) \\ &= \Sigma a(j_1,\ldots,j_k) x_1^{j_1} \ldots x_k^{j_k}. \end{aligned}$$

From the special case of (6.16) when $x_r = t_r$ for $r \leq n$, we see that $a(j_1,\ldots,j_k) = 0$ unless $j_r = 0$ whenever $r > n$. Thus there exist coefficients $a(j_1,\ldots,j_n)$ such that, for all $x_1,\ldots,x_n \in V$,

$$\text{(6.17)} \qquad \alpha(x_1 \otimes \cdots \otimes x_n) = \sum_{j_1+\cdots+j_n=d} a(j_1,\ldots,j_n) x_1^{j_1} \ldots x_n^{j_n}.$$

Thus far, we have just used that $\dim V \geq n$. Now assuming that $\dim V \geq n+1$, we show that $a(j_1,\ldots,j_n) = 0$ unless $j_r = q^{i_r}$ for all r, thus proving the lemma. To see this, we expand both sides of the identity

$$\text{(6.18)} \qquad \begin{gathered} \alpha(t_1 \otimes \cdots \otimes t_{n-1} \otimes (t_n + t_{n+1})) \\ \alpha(t_1 \otimes \cdots \otimes t_n) + \alpha(t_1 \otimes \cdots \otimes t_{n-1} \otimes t_{n+1}). \end{gathered}$$

The left side $= \Sigma a(j_1,\ldots,j_n) t_1^{j_1} \ldots t_{n-1}^{j_{n-1}} (t_n + t_{n+1})^{j_n}$. The right side $= \Sigma a(j_1,\ldots,j_n) t_1^{j_1} \ldots t_{n-1}^{j_{n-1}} (t_n^{j_n} + t_{n+1}^{j_n})$. So if $a(j_1,\ldots,j_n) \neq 0$, then $(t_n + t_{n+1})^{j_n} = t_n^{j_n} + t_{n+1}^{j_n}$, forcing $j_n = p^i$ for some i. By symmetry, $j_1,\ldots,j_{n-1}$ are also powers of p. Finally, the fact that α commutes with scalar multiplication forces these powers of p to actually be powers of q.

Remarks 6.19. (1) A corollary of Lemma 6.15 is a strengthened version of Corollary 6.14: if $\dim V \geq n+1$, every $\mathrm{End}(V)$-module map $S^n(V) \to S^d(V)$ is induced by a Steenrod operation. Dave Carlisle first showed me, in early 1987, that this could be given an elementary proof—inspiring the above proof of Lemma 6.15.

(2) In general $\dim V = n$ does not suffice. For example, when $q = 2$, $S^1(\mathbb{F}_2) \simeq S^d(\mathbb{F}_2)$ as $M_1(\mathbb{F}_2)$-modules, for all $d > 0$.

(3) A similar, but simpler, calculation shows that, if $\dim V \geq n$,

$$\mathrm{Hom}_{\mathrm{End}\,(V)}(V^{\otimes n}, V^{\otimes d}) = \begin{cases} \mathbb{F}_q[\Sigma_n], & \text{if } d = n \\ 0, & \text{otherwise.} \end{cases}$$

7. Tensor Products in $\mathcal{U}(q)$. As topologists well know, the category $\mathcal{U}(q)$ has an internal tensor product. However, starting from our representation theoretic definition, $\mathcal{U}(q) = \mathrm{Rep}\,(\mathcal{S}^*)$, this is not obvious. Here we develop this product representation theoretically, based on the two natural isomorphisms

$$\bigoplus_{i+j=k} S^i(V) \otimes S^j(W) \approx S^k(V \oplus W), \tag{7.1}$$

$$I_V \otimes I_W \approx I_{V \oplus W}. \tag{7.2}$$

As a biproduct, we prove the q-analogue of Lannes' tensor product theorem (1.3).

Definition 7.3. Let F and G be locally finite functors. Define

$$\mu(F,G) : \oplus_{i+j=k}\mathrm{Hom}(S_i, F) \otimes \mathrm{Hom}(S_j, G) \to \mathrm{Hom}(S_k, F \otimes G)$$

as follows. Given $\alpha : S_i \to F$ and $\beta : S_j \to G$, $\mu(F,G)(\alpha \otimes \beta)$ is the composite $S_k \xrightarrow{\psi} S_i \otimes S_j \xrightarrow{\alpha \otimes \beta} F \otimes G$.

Here ψ is dual to polynomial multiplication.

Proposition 7.4. *For all $F, G \in \mathcal{F}_\omega(q)$, $\mu(F,G)$ is an isomorphism.*

Proof. Using (7.2), $\mu(I_V, I_W)$ becomes the isomorphism (7.1). Since S_i is finite, $\mu(F,G)$ commutes with direct sums in either variable. Thus $\mu(I,J)$ is an isomorphism for all injectives I and J in $\mathcal{F}_\omega(q)$. A 5-lemma argument applied to an injective resolution $0 \to F \to I_0 \to I_1$ shows that $\mu(F,J)$ is an isomorphism for all F, with J injective. Resolving the second variable then gives the result.

Definition 7.5. Given M and N in $\mathcal{U}(q)$, define $M \otimes N \in \mathcal{U}(q)$ as follows. First let $(M \otimes N)_k = \oplus_{i+j=k}(M_i \otimes N_j)$. Given $a : S_n \to S_k$ and $i+j=k$, the proposition implies that there exist, for $r+s=n$, $a'_r : S_r \to S_i$ and $a''_s : S_s \to S_j$ with $a'_r \otimes a''_s$ unique such that the diagram

$$\begin{array}{ccc} S_n & \xrightarrow{a} & S_k \\ \downarrow \psi & & \downarrow \psi \\ \oplus_{r+s=n} S_r \otimes S_s & \xrightarrow{a'_r \otimes a'_s} & S_i \otimes S_j \end{array}$$

commutes. Then given $x \in M_i$ and $y \in N_j$, let

$$(x \otimes y) \cdot a = \sum_{r+s=n} (x \cdot a'_r) \otimes (y \cdot a''_s).$$

It is formal, but tedious, to check that this yields an object in $\mathcal{U}(q)$, i.e., that $(x \otimes y) \cdot (ab) = ((x \otimes y) \cdot a) \cdot b$.

Recall the adjoint pair

$$\mathcal{U}(q) \underset{r}{\overset{\ell}{\rightleftarrows}} \mathcal{F}_\omega(q)$$

defined by $r(F)_* = \mathrm{Hom}_{\mathcal{F}(q)}(S_*, F)$ and

$$\ell(M)(V) = \mathrm{Hom}_{\mathcal{U}(q)}(M, S^*(V^*))'.$$

COROLLARY 7.6. *$\mu(F,G) : r(F) \otimes r(G) \to r(F \otimes G)$ is an isomorphism in $\mathcal{U}(q)$.*

This is just Proposition 7.4 restated, except we are asserting that $\mu(F,G)$ is a map in $\mathcal{U}(q)$. Once again, $\mu(F,G)((x \otimes y) \cdot a) = (\mu(F,G)(x \otimes y)) \cdot a$ follows formally from the definitions.

Definition 7.7. Given M and N in $\mathcal{U}(q)$, define

$$\mu'(M,N) : \ell(M \otimes N) \to \ell(M) \otimes \ell(M)$$

to be adjoint to the composite

$$M \otimes N \xrightarrow{\eta_M \otimes \eta_N} r\ell(M) \otimes r\ell(N) \xrightarrow{\mu(\ell(M),\ell(N))} r(\ell(M) \otimes \ell(N)).$$

THEOREM 7.8. *$\mu'(M,N)$ is an isomorphism. Explicitly, the natural map*

$$Hom_{\mathcal{U}(q)}(M, S^*(V)) \hat{\otimes} Hom_{\mathcal{U}(q)}(M, S^*(V)) \to Hom_{\mathcal{U}(q)}(M \otimes N, S^*(V))$$

induced by $S^*(V) \otimes S^*(V) \to S^*(V)$, is an isomorphism for all M and N in $\mathcal{U}(q)$.

Proof. With a bit of care, this becomes a formal consequence of Theorem 6.2 and Corollary 7.6. We first need a lemma.

LEMMA 7.9. *If $N \in \mathcal{N}(q)$, then $N \otimes M \in \mathcal{N}(q)$ for all $M \in \mathcal{U}(q)$.*

Assuming this for the moment, we deduce Theorem 7.8 as follows. Theorem 6.2 implies that the kernel and cokernel of $\eta_M : M \to r\ell(M)$ are both in $\mathcal{N}(q)$. Consider the composite

$$M \otimes N \xrightarrow{\eta_M \otimes 1} r\ell(M) \otimes N \xrightarrow{1 \otimes eta_N} r\ell(M) \otimes r\ell(N) \xrightarrow{\mu(\ell(M),\ell(N))} r(\ell(M) \otimes \ell(N)).$$

By the lemma, the first two maps have kernel and cokernel in $\mathcal{N}(q)$, and the last map is an isomorphism by Corollary 7.6. Applying ℓ thus yields an isomorphism

$$\ell(M \otimes N) \xrightarrow{\sim} \ell r(\ell(M) \otimes \ell(N)).$$

Composing this with $\varepsilon_{\ell(M)\otimes\ell(N)} : \ell r(\ell(M) \otimes \ell(N)) \xrightarrow{\sim} \ell(M) \otimes \ell(N)$ yields our map $\mu'(M, N)$.

To prove Lemma 7.9, we use a handy characterization of objects in $\mathcal{N}(q)$.

LEMMA 7.10. *$N \in \mathcal{N}(q)$ if and only if there exists $F \xrightarrow{\alpha} G \longrightarrow 0$ exact in $\mathcal{F}_\omega(q)$ with $N \simeq coker(r(\alpha))$.*

Proof. Since ℓ is exact and $\varepsilon : \ell r \to 1$ is an equivalence, if $n \equiv \text{coker}(r(\alpha))$, then $\ell(N) \simeq (\ell r(\alpha)) = \text{coker}(\alpha) = 0$, i.e., $N \in \mathcal{N}(q)$. For the converse, any $N \in \mathcal{U}(q)$ has a resolution

$$\oplus F(d_j) \xrightarrow{f} \oplus F(d_i) \longrightarrow N \longrightarrow 0$$

where $F(d) = r(S_d)$. If $\ell(N) = 0$, then $N \simeq \text{cok}(r(\alpha))$, where $\alpha = \ell(f) : \oplus S_{d_j} \to \oplus S_{d_i}$ is epic.

Proof of Lemma 7.9. Suppose $N \in \mathcal{N}(q)$. It suffices to show that $N \otimes F(d) \in \mathcal{N}(q)$ for all d. By the last lemma, there exists an epic map $\alpha : F \to G$ with $N \simeq \text{coker}(r(\alpha))$. Then $\alpha \otimes 1 : F \otimes S_d \to G \otimes S_d$ is also epic. By Corollary 7.6, $N \otimes F(d) \simeq \text{coker}(r(\alpha)) \otimes r(S_d) \simeq \text{coker}(r(\alpha \otimes 1))$, and thus $N \otimes F(d) \in \mathcal{N}(q)$ by Lemma 7.10.

8. Suspension in $\mathcal{U}(q)$. In this section, we use the tensor product to define the suspension in $\mathcal{U}(q)$. This is used to recover the graded Hopf-algebra $A(q)$ as the ring of "stable" operations. Topologists reading this should be reminded of standard proofs which use the Cartan formula to show that the Steenrod operations are stable.

Definition 8.1.

(1) Let $\Sigma^n\mathbb{F}_q \in \mathcal{U}(q)$ denote the object with

$$(\Sigma^n\mathbb{F}_q)_d = \begin{cases} \mathbb{F}_q, & \text{if } d = n; \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $\Sigma^n : \mathcal{U}(q) \to \mathcal{U}(q)$, the n^{th} *suspension*, be defined by $\Sigma^n M = \Sigma^n\mathbb{F}_q \otimes M$.

The reader is now asked to forget our previous definition of the algebra $A(q)$ (Definition 6.5). Instead, we give a new one below.

It is convenient to define S^d to the zero functor for $d < 0$, and to let $\mathcal{S}^*$ have objects S^d, $d \in \mathbb{Z}$. We then require $M \in \operatorname{Rep}(\mathcal{S}^*)$ to have $M_d = 0$ for $d < 0$. Yoneda's lemma then implies that $\Sigma : \mathcal{U}(q) \to \mathcal{U}(q)$ is induced by an additive functor

$$\Sigma : \mathcal{S}^* \to \mathcal{S}^*,$$

with $\Sigma(S^{d+1}) = S^d$.

Definitions 8.2.

(1) Let $\tilde{\mathcal{S}}^* \varprojlim\{\cdots \xrightarrow{\Sigma} \mathcal{S}^* \xrightarrow{\Sigma} \mathcal{S}^*\}$. Explicitly, $\tilde{\mathcal{S}}^*$ has

(a) objects S^d, $d \in \mathbb{Z}$,
(b) $\operatorname{Mor}_{\tilde{\mathcal{S}}}(S^n, S^{n+d}) = \varprojlim_m \operatorname{Hom}_{\mathcal{F}(q)}(S^{n+m}, S^{n+m+d})$,
(c) an equivalence $\Sigma : \tilde{\mathcal{S}}^* \longrightarrow \tilde{\mathcal{S}}^*$ with $\Sigma(S^{d+1}) = S^d$.

(2) Let $A(q)$ be the graded algebra with $A(q)_d = \operatorname{Mor}_{\tilde{\mathcal{S}}}(S^n, S^{n+d})$ for any $n \in \mathbb{Z}$, and with multiplication induced by composition in $\tilde{\mathcal{S}}^*$.

Formally, we then have

PROPOSITION 8.3. (1) *$\operatorname{Rep}(\tilde{\mathcal{S}}^*)$ is the category of graded left $A(q)$-modules.*

(2) *The projection $\tilde{\mathcal{S}}^* \to \mathcal{S}^*$ induces an inclusion*

$$\mathcal{U}(q) \subseteq \textit{left } A(q)\textit{-modules},$$

compatible with suspension.

(3) *The tensor product in $\mathcal{U}(q)$ extends to a tensor product in the category of $A(q)$-modules, inducing a Hopf algebra structure on $A(q)$.*

We now construct a basis for $A(q)_d$, using the bases for $\operatorname{Hom}_{\mathcal{F}(q)}(S^n, S^{n+d})$ constructed in §6.

LEMMA 8.4. *Let* $\underline{r} = (r_1, \ldots, r_\ell)$ *and* $d = d(\underline{r})$. *Under the suspension map*

$$\Sigma : Hom_{\mathcal{F}(q)}(S^{n+1}, S^{n+d+1}) \to Hom_{\mathcal{F}(q)}(S^n, S^{n+d}),$$

we have $\Sigma\mathcal{P}(q;\underline{r};n+1) = \mathcal{P}(q;\underline{r};n)$.

Proof. Unraveling our definitions, we are asserting that the diagram

$$\begin{array}{ccc} S^1(\mathbb{F}_q) \otimes S^n(V) & \xrightarrow{1\otimes\mathcal{P}(q;\underline{r};n)} & S^1(\mathbb{F}_q) \otimes S^{n+d}(V) \\ i \downarrow & & \pi \uparrow \\ S^{n+1}(\mathbb{F}_q \oplus V) & \xrightarrow{\mathcal{P}(q;\underline{r};n+1)} & S^{n+d+1}(\mathbb{F}_q \oplus V) \end{array}$$

commutes. This follows immediately from Definition 6.13.

COROLLARY 8.5. $\mathcal{U}(q)$ *is a full subcategory of* $A(q)$*-modules.*

Proof. This is a consequence of the fact that $\tilde{\mathcal{S}}^* \to \mathcal{S}^*$ is onto on Hom-sets.

Definition 8.6. Given $\underline{r} = (r_1, \ldots, r_\ell)$, define $\mathcal{P}(q;\underline{r}) \in A(q)_{d(\underline{r})}$ to be $\varprojlim_n \mathcal{P}(q;n;r)$.

COROLLARY 8.7. $\{\mathcal{P}(q;\underline{r}) | d(\underline{r}) = d\}$ *is a basis for* $A(q)_d$.

This reconciles our two definitions of $A(q)$.

Appendix A: Analytic and Locally Finite Functors. Given a prime p [HLS], define $\mathcal{F}_\omega(p) \subseteq \mathcal{F}(p)$ to be the full subcategory of what they call "analytic functors," after first generalizing the notion of a "polynomial functor" (as in I. G. MacDonald's book [MacD, appendix to chapter 1] to prime fields of positive characteristic. Here we sketch a proof that their category is the same as our category of locally finite functors.

Recall that the constant functor I_0 is a direct summand of $I_{\mathbb{F}_p}$. Let I be the complementary summand. It is quite easy to check that the definition of a polynomial functor from [HLS, Part I, §6] is equivalent to (1) below.

Definition A.1.

(1) F is a *polynomial of degree* $\leq r$ if $\mathrm{Hom}_{\mathcal{F}(p)}(F, I^{\otimes k}) = 0$ for $k > r$.

(2) F is *analytic* if it is the union of its polynomial subfunctors.

Our goal here is to prove

PROPOSITION A.2. *F is analytic if and only if it is locally finite.*

Proof. It clearly suffices to show that polynomial functors are locally finite and that finite functors are polynomial. Any functor $F \in \mathcal{F}(p)$ is the union of the images of maps $P_V \to F$, and if G is such an image, $\dim G(V) < \infty$ for all V. Thus it suffices to show that if $\dim F(V) < \infty$ for all V, then F is polynomial if and only if F is finite. By Theorem 4.14, we will be done with

LEMMA A.3. *Suppose* $\dim F(V) < \infty$. *Then F is polynomial of degree $\leq r$ if and only if $d_F(n)$ is polynomial of degree $\leq r$.*

Proof. Note that

$$I_{\mathbb{F}_p^n} = I_{\mathbb{F}_p}^{\otimes n} = (I_0 + I)^{\otimes n} = \oplus_{k=0}^{n} \binom{n}{k} I^{\otimes k}.$$

Then

$$\begin{aligned} d_F(n) &= d_{DF}(n) \\ &= \dim DF(\mathbb{F}_p^n) \\ &= \dim \operatorname{Hom}_{\mathcal{F}(q)}(F, I_{\mathbb{F}_p^n}) \\ &= \sum_{k=0}^{n} \binom{n}{k} \dim \operatorname{Hom}_{\mathcal{F}(q)}(F, I^{\otimes k}). \end{aligned}$$

The lemma immediately follows.

Similar arguments were made in [CK2, §4].

COROLLARY A.4. *F is polynomial of degree $\leq r$ if and only if $F^{(r+1)} = 0$.*

Appendix B: Some Categorical Properties of $\mathcal{F}_\omega(q)$. In this appendix we list some categorical properties of $\mathcal{F}_\omega(q)$ that seem to be particularly fundamental. Since these properties hold for many locally finite A.B.5 categories (abelian categories with exact direct limits [P]), throughout this appendix we let $\mathcal{C}$ denote such a category.

The first property is a slight strengthening of [CPS, Lemma 3.8(a)].

PROPOSITION B.1. (1) *For any $H \in \mathcal{C}$ and finite F, the natural map*

$$\varinjlim i_{\alpha *} : \varinjlim \operatorname{Ext}^*_{\mathcal{C}}(F, H_\alpha) \to \operatorname{Ext}^*_{\mathcal{C}}(F, H)$$

is an isomorphism, where $i_\alpha : H_\alpha \subseteq H$ ranges over the finite subobjects of H.

(2) *For any finite F and H, the natural map*

$$\operatorname{Ext}^*_{\mathcal{C}_f}(F, H) \to \operatorname{Ext}^*_{\mathcal{C}}(F, H)$$

is an isomorphism, where $\mathcal{C}_f$ is the full subcategory of finite objects.

Here Ext-groups should be defined via equivalence classes of extensions. We remark that (1) is the algebraic analogue of the following topological fact: if

X is compact and Y is compactly generated then there is a natural bijection of homotopy sets

$$\varinjlim [X, Y_\alpha] \to [X, Y],$$

where Y_α ranges over the compact subspaces of Y.

We begin the proof with a simple lemma.

LEMMA B.2. *A diagram in $\mathcal{C}$ with F' finite,*

$$\begin{array}{ccccccccc} & & & & & & F' & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\pi} & F & \longrightarrow & 0, \end{array}$$

can be extended to a diagram

$$\begin{array}{ccccccccc} 0 & \to & H' & \to & G' & \to & F' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H & \to & G & \to & F & \to & 0, \end{array}$$

with H' and G' finite.

Proof. Choose a finite $G' \subseteq \pi^{-1}(F')$ large enough so that $\pi(G') = F'$. Then let $H' = \ker(\pi|_{G'})$.

Call an extension $E : 0 \to H \to G_s \to \cdots \to G_1 \to F \to 0$ *finite* if F, H, and G_j are finite for all j.

COROLLARY B.3. *Given an extension $E : 0 \to H \to G_s \to \cdots \to G_1 \to F \to 0$ with F finite, there exists a finite subobject H' of H, a finite extension $E' : 0 \to H' \to G'_s \to \cdots \to G'_1 \to F \to 0$, and an inclusion $E' \hookrightarrow E$ extending $i : H' \hookrightarrow H$.*

Proof. Recalling that E can be viewed as s short exact sequences spliced together, the result follows by induction on s and repeated use of the last lemma.

Proof of Proposition B.1(1). It is easy to see that the map is onto: given $[E] \in \operatorname{Ext}^s(F, H)$, choose $i : H' \hookrightarrow H$, E' as in the corollary. Then $i_*([E']) = [E]$. The argument that $\varinjlim i_{\alpha *}$ is monic is as in [CPS]: given $x \in \operatorname{Ext}^s(F, H_\alpha)$, if $i_{\alpha *}(x) = 0$, then there exists $y \in \operatorname{Ext}^{s-1}(F, H/H_\alpha)$ such that $\delta(y) = x$. Applying the part of the proposition proved thus far to H/H_α, there exits $H_\alpha \subseteq H_\beta \subseteq H$ and $z \in \operatorname{Ext}^{s-1}(F, H_\beta/H_\alpha)$ such that $i_{\beta *}(z) = y$. By naturality, $\delta(z) = x$ so that $i_{\alpha\beta *}(x) = 0$ where $i_{\alpha\beta} : H_\alpha \hookrightarrow H_\beta$ is the inclusion. Thus x maps to 0 in the direct limit, as needed.

Proof of Proposition B.1(2). Corollary B.3 immediately implies that every equivalence class of extension in $\operatorname{Ext}^s_{\mathcal{C}}(F, H)$ is represented by a finite extension, implying that the map is onto. To show that it is monic, recall from [MacL, Prop. III.5.2] that two finite s-fold exact sequences E and E' will represent the same element in $\mathrm{Ext}^s_{\mathcal{C}}(F,H)$ if and only if there exist $2k+1$ extensions $E_0,\ldots,E_{2k}$ and maps

$$E = E_0 \to E_1 \leftarrow E_2 \to \cdots \leftarrow E_{2k-2} \to E_{2k-1} \leftarrow E_{2k} = E'. \tag{B.4}$$

To show that then E and E' are also equivalent in $\mathrm{Ext}^s_{\mathcal{C}}(F,H)$, it suffices to show that the E_i's can be chosen to be finite in (B.4). By Corollary B.3, the E_{2i} can be assumed to be finite: just precompose with $E'_{2i} \to E_{2i}$ chosen as in the corollary. To replace E_{2i-1} by a finite extension we use a variant of Corollary B.3:

Lemma B.5. *Given maps $E_1 \xrightarrow{f} E \xleftarrow{g} E_2$ between three extensions, with E_1 and E_2 finite, there exists a finite extension $E' \hookrightarrow E$ such that* $\mathrm{Im}(f) + \mathrm{Im}(g) \subseteq E'$.

Proof. This follows by induction on the length of the extension, as in Lemma B.2 and Corollary B.3, making sure that the finite objects in E' are chosen to contain $\mathrm{Im}(f) + \mathrm{Im}(g)$ at each stage.

The next results assume that $\mathcal{C}$ has a nice generating (or cogenerating) set, with properties analogous to the S_d's (or S^d's) in $\mathcal{F}_\omega(q)$.

Proposition B.6. *Suppose $\mathcal{C}$ has a set of objects $\{S_\alpha | \alpha \in J\}$ such that*

(i) *every finite object is a quotient of a sum $\oplus S_{\alpha_i}$.*

(ii) *for all $\alpha, \beta \in J$, there exists $\gamma \in J$ with $Hom_{\mathcal{C}}(S_\beta, S_\gamma) = 0$, and an epic map $S_\gamma \to S_\alpha$.*

Then $\mathcal{C}$ has no nonzero projectives.

Proof. By (i), the S_α generate $\mathcal{C}$. Thus if $P \in \mathcal{C}$ is projective, there exists an epic map $\oplus S_{\alpha_i} \to P$. Choose a nonzero map $S_\beta \xrightarrow{f} P$. For each i, choose $S_{\gamma_i} \twoheadrightarrow S_{\alpha_i}$ with $\mathrm{Hom}_{\mathcal{C}}(S_\beta, S_{\gamma_i}) = 0$. Consider the diagram

$$\begin{tikzcd} & & S_\beta \arrow[d, "f"'] \\ & & P \arrow[dll] \arrow[d, equal] \\ \oplus S_{\gamma_i} \arrow[r] & \oplus S_{\alpha_i} \arrow[r] & P \end{tikzcd}$$

The indicated lift exists if P is projective. But then f factors through a map in $\mathrm{Hom}_{\mathcal{C}}(S_\beta, \oplus S_{\gamma_i}) = 0$, and is thus zero, contradicting our choice of f.

Corollary B.7 (Lionel Schwartz). *$F_\omega(q)$ has no nonconstant projectives.*

Proof. With $\mathcal{C}$ = constant free locally finite functors, the family $\{S_d | d \geq 1\}$ satisfy the hypotheses of the proposition: use the q^{th} powers $S_{q^N d} \twoheadrightarrow S_d$ for the needed epis, and recall that $\mathrm{Hom}(S_i, S_j) = 0$ if $j > i$.

PROPOSITION B.8. *Suppose $\mathcal{C}$ has a set of objects $\{S^\alpha | \alpha \in J\}$ such that*

(i) *every finite object embeds in a sum $\oplus S^{\alpha_i}$,*

(ii) *for all $\alpha, \beta \in J$, there exists $\gamma \in J$ with $Hom_{\mathcal{C}}(S_\gamma, S_\beta) = 0$ and a monic map $S_\alpha \to S_\gamma$.*

Then if I is injective and F is finite, $Hom_{\mathcal{C}}(I, F) = 0$.

Proof. By (i), it suffices to assume $F = S^\beta$. Since I is locally finite, a map $f : I \to S^\beta$ will be zero if and only if all composites $G \xrightarrow{g} I \xrightarrow{f} S^\beta$ are zero, with G finite. Now using (i) and (ii) construct a diagram

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[d, "g"] & \oplus S^{\alpha_i} \arrow[r, hook] & \oplus S^{\gamma_i} \arrow[dll] \\
I \arrow[d, "f"] & & \\
S^\beta & &
\end{tikzcd}$$

with $\mathrm{Hom}_{\mathcal{C}}(S^{\gamma_i}, S^\beta) = 0$ for all i. The indicated extension then exists since I is injective, showing that $f \circ g$ factors through the zero map.

COROLLARY B.9. *If $F \in \mathcal{F}(q)$ is finite and constant free, then*

$$0 = Hom_{\mathcal{F}(q)}(I_V, F) = Hom_{\mathcal{F}(q)}(F, P_V).$$

Proof. The first equality follows from the proposition. Now use duality (DF is again finite).

Finally, we state a couple of homological finiteness results holding in $\mathcal{F}_\omega(q)$. Since the study of such properties is an ongoing joint project with L. Schwartz, we omit the proofs.

PROPOSITION B.10. *If F is finite and $G \in \mathcal{F}_\omega(q)$ has finite socle, then $\mathrm{Ext}^s_{\mathcal{F}_\omega(q)}(F, G)$ is finite for all s.*

PROPOSITION B.11. *Given $F \in \mathcal{F}_\omega(q)$, F is finite if and only if $r(F) = Hom_{\mathcal{F}(q)}(S_*, F)$ is a finitely generated $A(q)$-module.*

With Schwartz, we conjecture

CONJECTURE B.12. *I_V is an Artinian object.*

This can be viewed as a significant strengthening of Proposition B.10, which it easily implies. For a prime p, it is easy to show that the p indecomposable summands of $I_{\mathbb{F}_p}$ are each uniserial, thus are Artinian. We know the conjecture for no other V.

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