# Bousfield Localization (Lecture 20)

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Let $\mathcal{C}$ be a full subcategory of the category Sp of spectra, which is closed under shifts and homotopy colimits and satisfies the following technical condition:

$(*)$ There exists a small subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ which generates $\mathcal{C}$ under homotopy colimits.

In this case, the inclusion $\mathcal{C} \subseteq \mathrm{Sp}$ preserves homotopy colimits; using a version of the adjoint functor theorem one deduces that this inclusion admits a right adjoint $G$ (at the level of homotopy categories). We can think of $G$ as a functor from Sp to itself, which takes values in $\mathcal{C}$.

**Remark 1.** Roughly speaking, if $X$ is a spectrum then we want to define $G(X)$ to be the homotopy colimit of all objects $Y \in \mathcal{C}$ with a map to $X$. Condition $(*)$ is used to make this homotopy colimit sensible (that is, to replace it by a homotopy colimit indexed by a small category).

For every spectrum $X$, we have a counit map $v : G(X) \to X$. We let $L(X)$ denote the cofiber of $v$, so that we have a cofiber sequence

$$G(X) \to X \to L(X).$$

By construction, for every object $Y \in \mathcal{C}$, the map of function spectra $G(X)^Y \to X^Y$ is a homotopy equivalence; it follows that $L(X)^Y \simeq 0$.

**Definition 2.** A spectrum $X$ is $\mathcal{C}$-*local* if every map $Y \to X$ is nullhomotopic when $Y \in \mathcal{C}$. We denote the category of $\mathcal{C}$-local spectra by $\mathcal{C}^{\perp}$.

**Remark 3.** The full subcategory $\mathcal{C}^{\perp} \subseteq \mathrm{Sp}$ is stable under shifts and homotopy limits.

The above analysis shows that for every $X$, the spectrum $L(X)$ is $\mathcal{C}$-local. Moreover, for every $\mathcal{C}$-local spectrum $Z$, we have $Z^{G(X)} \simeq 0$, so that the map $Z^{L(X)} \to Z^X$ is a homotopy equivalence. It follows that $L$ can be viewed as a left adjoint to the inclusion $\mathcal{C}^{\perp} \subseteq \mathrm{Sp}$.

**Example 4** (Bousfield)**.** Fix a spectrum $E$. We say that another spectrum $X$ is $E$-*acyclic* if the smash product $X \otimes E$ is zero. The collection $\mathcal{C}_E$ of $E$-acyclic spectra is clearly stable under shifts and homotopy colimits, and one can show that it satisfies $(*)$. We say that a spectrum $X$ is $E$-*local* if every map $Y \to X$ is nullhomotopic whenever $Y$ is $E$-acyclic. The above analysis shows that every spectrum $X$ sits in an essentially unique cofiber sequence

$$G_E(X) \to X \to L_E(X)$$

where $G(X)$ is $E$-acyclic and $L_E(X)$ is $E$-local. The functor $L_E$ is called *Bousfield localization* with respect to $E$. The map $X \to L_E(X)$ is characterized up to equivalence by two properties:

$(a)$ The spectrum $L_E(X)$ is $E$-local.

$(b)$ The map $X \to L_E(X)$ is an $E$-equivalence: that is, it induces an isomorphism on $E$-homology groups $E_*(X) \simeq E_* L_E(X)$.

**Example 5.** Let $E$ be a ring spectrum. If $X$ is an $E$-module spectrum, then $X$ is $E$-local. Indeed, suppose that $Y$ is $E$-acyclic and we are given a map $f : Y \to X$. Then $f$ can be written as the composition

$$Y \xrightarrow{f} X \to E \otimes X \to X.$$

The composition of the first pair of morphisms factors as a composition

$$Y \to E \otimes Y \xrightarrow{\mathrm{id} \otimes f} E \otimes X,$$

and is therefore nullhomotopic since $E \otimes Y \simeq 0$.

**Remark 6.** Let $E$ be an $A_\infty$-ring spectrum and $X$ an arbitrary spectrum, and let $X^\bullet$ be the cosimplicial spectrum given by $X^n = E^{\otimes n+1} \otimes X$. Each $X^n$ is $E$-local, so the totalization $\varprojlim X^\bullet$ of $X^\bullet$ is $E$-local. It follows that the canonical map $X \to \varprojlim X^\bullet$ factors through a map $\alpha : L_E X \to \varprojlim X^\bullet$. In many cases, one can show that $\alpha$ is a homotopy equivalence: that is, the cosimplicial object $X^\bullet$ is a means of computing the $E$-localization of $X$.

**Example 7.** Let $E$ be the Eilenberg-MacLane spectrum $H\,\mathbf{Q}$. Then a spectrum $X$ is $E$-acyclic if and only if the homotopy groups $\pi_* X$ consist entirely of torsion. A spectrum $X$ is $E$-local if and only if the homotopy groups $\pi_* X$ are rational vector spaces.

**Example 8.** The theory of Bousfield localization works in a very general context. For example, rather than working with spectra, we can work with chain complexes of abelian groups. Fix a prime number $p$. We say that a projective chain complex $A_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-acyclic if $A_\bullet \otimes \mathbf{Z}/p\mathbf{Z}$ is nullhomotopic: equivalently, $A_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-acyclic if each homology group $\mathrm{H}_n(A_\bullet)$ is a $\mathbf{Z}[\frac{1}{p}]$-module. We say that $A_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-local if every map from a projective $\mathbf{Z}/p\mathbf{Z}$-acyclic chain complex into $A_\bullet$ is nullhomotopic.

For any projective chain complex $A_\bullet$, we define its completion $\widehat{A}_\bullet$ to be the homotopy limit

$$\varprojlim_n A_\bullet \otimes \mathbf{Z}/p^n\mathbf{Z}.$$

As a homotopy limit of $\mathbf{Z}/p\mathbf{Z}$-local chain complexes, we conclude that $\widehat{A}_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-local. On the other hand, a simple calculation shows that the map $A_\bullet \to \widehat{A}_\bullet$ induces a quasi-isomorphism modulo $p$, so that $\widehat{A}_\bullet$ can be identified with the $\mathbf{Z}/p\mathbf{Z}$-localization of $A_\bullet$.

In general, it is good to think of Bousfield localization as involving a mix of Examples 7 and 8. In algebro-geometric terms, it can behave sometimes like restriction to an open subscheme (as in Example 7) and sometimes like completion along a closed subscheme (Example 8). Our next goal is to describe Bousfield localizations of the first type more precisely.

**Lemma 9.** *Let $\mathcal{C}$, $\mathcal{C}^\perp$, $G$, and $L$ be as above. The following conditions are equivalent:*

(1) *The subcategory $\mathcal{C}^\perp \subseteq \mathrm{Sp}$ is stable under homotopy colimits.*

(2) *The functor $L$ preserves homotopy colimits.*

(3) *The functor $G$ preserves homotopy colimits.*

(4) *The functor $L$ has the form $L(X) = K \otimes X$ for some spectrum $K$.*

*Proof.* We first prove (1) $\Rightarrow$ (2). Assume $\mathcal{C}^\perp$ is stable under homotopy colimits. For any diagram of spectra $\{X_\alpha\}$, we have canonical maps

$$\varinjlim X_\alpha \xrightarrow{\gamma} \varinjlim L(X_\alpha) \xrightarrow{\beta} L \varinjlim X_\alpha.$$

The fiber of $\gamma$ belongs to $\mathcal{C}$ (since $\mathcal{C}$ is stable under homotopy colimits), and $\varinjlim L(X_\alpha) \in \mathcal{C}^\perp$ by (1). It follows that $\beta$ is an equivalence.

To prove that (2) $\Rightarrow$ (1), we note that if $\{X_\alpha\}$ is a diagram in $\mathcal{C}^\perp$, then $L(\varinjlim X_\alpha) \simeq \varinjlim L(X_\alpha) \simeq \varinjlim X_\alpha$ so that $\varinjlim X_\alpha \in \mathcal{C}^\perp$.

The equivalence of (2) and (3) follows from the cofiber sequence of functors

$$G \to \mathrm{id} \to L.$$

Finally, the equivalence of (2) and (4) follows from from the following observation: every functor $F : \mathrm{Sp} \to \mathrm{Sp}$ which preserves homotopy colimits has the form $F(X) \simeq K \otimes X$, for some spectrum $K$. $\square$

We say that a Bousfield localization $L$ is *smashing* if it satisfies the equivalent conditions of Lemma 9.

**Remark 10.** In the situation of Lemma 9, the spectrum $K$ can be recovered as the image $L(S)$ of the sphere spectrum $S$ under the localization functor $L$.

**Remark 11.** Let $\mathcal{C} \subseteq \mathrm{Sp}$ be a subcategory satisfying the conditions of Lemma 9. Then a spectrum $X$ belongs to $\mathcal{C}$ if and only if $L(X) = L(S) \otimes X \simeq 0$. In other words, $\mathcal{C}$ can be identified with the collection of $L(S)$-acyclic spectra, so that $L = L_E$ for $E = L(S)$.

**Example 12.** Let $\mathcal{C} \subseteq \mathrm{Sp}$ be a subcategory which is stable under shifts and homotopy colimits, which is generated under homotopy colimits by a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ consisting of *finite* spectra. Then it is easy to see that $\mathcal{C}$ satisfies condition (1) of Lemma 9, so that $\mathcal{C}$ determines a smashing localization functor.