COMPOSITIO MATHEMATICA

DORETTE A. PRONK

Etendues and stacks as bicategories of fractions

Compositio Mathematica, tome 102, nº 3 (1996), p. 243-303
<http://www.numdam.org/item?id=CM_1996__102_3_243_0>

Compositio Mathematica **102:** 243-303, 1996.

Etendues and stacks as bicategories of fractions

DORETTE A. PRONK
Department of Mathematics, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands

Received 8 October 1993; accepted in final form 31 March 1995

Abstract. This paper presents a generalization for bicategories of the Gabriel–Zisman theory of categories of fractions. Subsequently, this theory is applied to show that étendues and stacks (among others) arise as bicategories of fractions from appropriate categories of groupoids.

Key words: algebraic stacks, category of fractions, 2-categories, étendues, étale groupoids.

Introduction

The main purpose of this paper is to give the construction of a bicategory of fractions, as a generalization of the Gabriel–Zisman notion of a category of fractions (see (Gabriel–Zisman, 1967)). In other words: for a bicategory $\mathcal{C}$ and a class of 1-arrows W which satisfy certain conditions (which form a generalization of those in (Gabriel–Zisman, 1967), see Section 2.1) we construct a bicategory $\mathcal{C}[W^{-1}]$ and a homomorphism $U: \mathcal{C} \to \mathcal{C}[W^{-1}]$. This homomorphism sends the 1-arrows in W to equivalences and it is universal in the sense that composition with U induces an equivalence of bicategories

$$\mathrm{Hom}\,(\mathcal{C}[W^{-1}], \mathcal{D}) \simeq \mathrm{Hom}_W\,(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Hom}_W\,(\mathcal{C}, \mathcal{D})$ is the bicategory of homomorphisms and transformations which invert the elements of W in a suitable sense (see Section 3.2).

The motivation and inspiration for this construction come from the study of étendues and topological groupoids. Etendues form a special kind of topos, examples of which locally look like a topological space. They were introduced by Grothendieck in SGA4 as a sort of generalized quotient space for foliations. The relation between étendues and foliations, is further studied in (Moerdijk, 1991) and (Moerdijk, 1993).

The category of toposes and isomorphism classes of geometric morphisms can be viewed as a category of fractions in the Gabriel–Zisman sense of a specific category of groupoids with respect to the class of weak equivalences (see (Moerdijk, 1988b)). This equivalence restricts to the following

$$[\mathit{Etendues}] \simeq [\mathit{Etale\ Groupoids}]\,[W^{-1}],$$

where [*Etendues*] is the category of étendues and isomorphism classes of geometric morphisms, and [*Etale Groupoids*] is the category of étale groupoids in the category of sober topological spaces and isomorphism classes of continuous maps (see Section 1) and W is the class of weak equivalences. We want to understand this equivalence also on the level of 2-cells. One approach which is totally independent of the category of fractions theory is presented in (Moerdijk, 1990). A similar result is obtained in (Bunge, 1990). Our construction works to get the following theorem.

THEOREM 1. *There is a canonical equivalence of bicategories*

$$(T_1\text{-}Etendues) \simeq_{bi} (T_1\text{-}Etale\ Groupoids)\,[W^{-1}].$$

Here (T_1-*Etendues*) is the 2-category of toposes which roughly speaking locally look like a T_1-space (for the precise definition, see Section 4), and (T_1-*Etale Groupoids*) is the 2-category of étale groupoids in the category of T_1-spaces. W denotes here and in the following the class of weak equivalences of groupoids (see Section 1.3). The equivalence in the theorem above is an equivalence of bicategories (and therefore denoted by $\simeq_{bi}$), because in general the category of fractions of a 2-category will turn out to be a bicategory and is called a *bicategory of fractions*. For the difference between 2-categories and bicategories, see (Bénabou, 1967), Section 2.1.

Algebraic stacks were also introduced as a generalized quotient: of an étale equivalence relation in the category of schemes (see (Deligne–Mumford, 1969) and (Artin, 1974)). They form a generalization of the algebraic spaces as defined by Artin and Knutson in (Artin, 1971) and (Knutson, 1971). The bicategory of fractions construction can be applied to give the following:

THEOREM 2. *There is a canonical equivalence of bicategories*

$$(Algebraic\ Stacks) \simeq_{bi} (Algebraic\ Groupoids)\,[W^{-1}].$$

Here (*Algebraic Groupoids*) is the 2-category of étale groupoids in the category of schemes. This theorem is proved using a special kind of topos, which we call an 'algebraic étendue'. However: an algebraic étendue is not a special kind of étendue, but it is defined in an analogous way and:

THEOREM 3. *There is an equivalence of 2-categories*

$$(Algebraic\ Stacks) \simeq_2 (Algebraic\ Etendues).$$

Completely analogous to algebraic stacks we can define topological stacks and differentiable stacks over the categories of sober topological spaces and differentiable manifolds respectively. In the topological case we find:

THEOREM 4. *There is an equivalence of 2-categories*

$$(Etendues) \simeq_2 (Topological\ Stacks).$$

and therefore

COROLLARY 5. *There is a canonical equivalence of bicategories*

$$(\textit{Topological Stacks}) \simeq_{bi} (T_1\textit{-Etale Groupoids})\,[W^{-1}].$$

In the differentiable case we find:

THEOREM 6. *There is an equivalence of 2-categories*

$$(\textit{Differentiable Etendues}) \simeq_2 (\textit{Differentiable Stacks}).$$

And these are also bicategories of fractions:

COROLLARY 7. *There is a canonical equivalence of bicategories*

$$(\textit{Differentiable Stacks}) \simeq_{bi} (\textit{Differentiable Groupoids})\,[W^{-1}].$$

Here differentiable groupoids are étale groupoids in the category of differentiable manifolds.

The first section of this paper gives an overview of the results on étendues which will be used in this paper. There are also references to find more details. Those who are just interested in the bicategory of fractions can start with Section 2 which gives the conditions on the class of arrows to be inverted and the construction of the bicategory of fractions $\mathcal{C}[W^{-1}]$. Section 3 shows that $\mathcal{C}[W^{-1}]$ has indeed the required universal property and gives conditions on a bicategory $\mathcal{D}$ to be equivalent to $\mathcal{C}[W^{-1}]$. Finally Sections 4 to 7 present the applications by proving the Theorems 1 to 7 above. There is an appendix giving some details about the coherence axioms for $\mathcal{C}[W^{-1}]$.

1. Overview of étendues

1.1. Etendues and groupoids

In this section we will give the facts about étendues, which we will use in the rest of this paper.

DEFINITION 8. A Grothendieck topos $\mathcal{E}$ is called an *étendue* if there exists an object $U \twoheadrightarrow 1$ in $\mathcal{E}$ such that $\mathcal{E}/U$ is equivalent to $\mathrm{Sh}(X)$ for some topological space X.

Etendues can also be described in terms of topological groupoids. A *topological* (or: *continuous*) groupoid is an internal groupoid in the category of topological spaces and continuous maps. Such a groupoid

$$\mathcal{G} = \left(\begin{tikzcd} G_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & G_0 \arrow[l, "i" description] \end{tikzcd} \right)$$

is called *étale* when both d_0 and d_1 are étale maps. The main theorem of this section is the following result from (Grothendieck et al., 1972), p. 481, 482:

THEOREM 9. *A Grothendieck topos $\mathcal{E}$ is an étendue if and only if there exists an étale groupoid $\mathcal{G}$ such that $\mathcal{E} \simeq B\mathcal{G}$.*

Proof. Recall that for an arbitrary topological groupoid $\mathcal{G}$ we have the topos $B\mathcal{G}$ of $\mathcal{G}$-equivariant sheaves on G_0. (For more details see (Moerdijk, 1988a) or (Moerdijk, 1991).) If $\mathcal{G}$ is an étale groupoid, then $B\mathcal{G}$ is an étendue. In this case U is the étale space $G_1 \overset{d_0}{\to} G_0$ with action by composition $g \bullet g_1 = m\,(g, g_1)$.

When we start with an étendue $\mathcal{E}$, the corresponding groupoid $\mathcal{G}$ can be found as follows: $\mathrm{Sh}(G_0) = \mathcal{E}/U$ and $\mathrm{Sh}(G_1) = \mathcal{E}/\,(U \times U)$. We claim that $\mathcal{E} \simeq B\mathcal{G}$. This follows from the fact that $\mathcal{E} \simeq \mathrm{Des}(u)$, since $u \colon U \twoheadrightarrow 1$ is an effective descent morphism in the category $\mathcal{E}$ (see example (8) in Section 1 of (Moerdijk, 1988a)). (Information on descent theory can be found in (Moerdijk, 1989).) Recall that objects of $\mathrm{Des}(u)$ consist of arrows $p \colon V \to U$ with descent data, i.e. a morphism

$$\theta \colon V \times_1 U \to V,$$

satisfying the unit and cocycle conditions. By the equivalence $\mathrm{Sh}(G_0) \simeq \mathcal{E}/U$, this corresponds to a map

$$\bar{\theta} \colon \bar{V} \times_{G_0} G_1 \to \bar{V}.$$

This map $\bar{\theta}$ satisfies precisely the conditions for being a right G_1-action on $\bar{V}$ and we conclude that

$$\mathcal{E} \simeq \mathrm{Des}(u) \simeq B\mathcal{G}.$$

Remark 10. Etendues can also be described in terms of sites, see (Rosenthal, 1981).

1.2. Morphisms between Étendues

By Theorem 9 we can write up to equivalence every étendue as $B\mathcal{G}$ for an étale groupoid $\mathcal{G}$. In this section we will describe the geometric morphisms

$$B\mathcal{G} \to B\mathcal{H},$$

between étendues in terms of groupoid morphisms

$$\mathcal{G} \to \mathcal{H}.$$

Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{H} = (H_1 \rightrightarrows H_0)$ be étale groupoids and let $\varphi\colon \mathcal{G} \to \mathcal{H}$ be a groupoid morphism. Let $E \xrightarrow{p} H_0$ be an $\mathcal{H}$-equivariant sheaf with a right H_1-action $\theta\colon E \times_{H_0} H_1 \xrightarrow{\sim} H_1 \times_{H_0} E$. Now define $(B\varphi)^*E$ to be the étale space given by the following pullback

$$\begin{tikzcd}
(B\varphi)^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
G_0 \arrow[r, "\varphi_0"'] & H_0
\end{tikzcd}$$

We can define a right G_1-action on $(B\varphi)^*E$ by

$$\varphi_1^*\,\theta\colon \varphi_1^*\,d_1^*E = d_1^*\,\varphi_0^*E \to \varphi_1^*\,d_0^*E = d_0^*\,\varphi_0^*E.$$

It is not difficult to see that $B\varphi$ thus defined preserves finite limits and arbitrary colimits. So it is the inverse image of a geometric morphism

$$B\varphi\colon B\mathcal{G} \to B\mathcal{H}.$$

EXAMPLE 11. The inclusion of groupoids

$$\begin{tikzcd}
G_0 \arrow[r, "i"] \arrow[d, shift right, "I_{G_0}"'] \arrow[d, shift left, "I_{G_0}"] & G_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
G_0 \arrow[r, "I_{G_0}"'] & G_0
\end{tikzcd}$$

induces a geometric morphism, denoted

$$G_0 \xrightarrow{\pi_{\mathcal{G}}} B\mathcal{G}.$$

1.3. WEAK EQUIVALENCES

Now we want to describe those morphisms of groupoids which induce an equivalence of étendues. (This shows also to what extent the choice of the groupoid $\mathcal{G}$ is unique for a given étendue $\mathcal{E}$.)

DEFINITION 12. Let $f: \mathcal{G} \to \mathcal{H}$ be a morphism of continuous groupoids.

(i) f is called *open* if f_1 and (hence) f_0 are open maps.

(ii) f is called *essentially surjective* if the map $d_0 \pi_2: G_0 \times_{H_0} H_1 \to H_0$ is an open surjection. (Here the pullback is along $d_1: H_1 \to H_0$; the condition is of course equivalent to the condition that $d_1 \pi_1: H_1 \times_{H_0} G_0 \to H_0$ is an open surjection, where the pullback is along d_0.)

(iii) Consider the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & H_1 \arrow[d, "(d_0,d_1)"] \\ G_0 \times G_0 \arrow[r, "f_0 \times f_0"'] & H_0 \times H_0 \end{tikzcd}$$

f is called *faithful* (resp. *full, fully faithful*) if the map $((d_0, d_1), f_1): G_1 \to P$ is an inclusion (resp. an open surjection, an isomorphism) of spaces.

(iv) f is called a *weak equivalence* if f is essentially surjective and fully faithful.

LEMMA 13. *For a weak equivalence between étale groupoids* $f = (f_0, f_1): \mathcal{G} \to \mathcal{H}$, *the maps* $f_0: G_0 \to H_0$ *and* $f_1: G_1 \to H_1$ *are étale.*

Proof. Consider the diagram

$$\begin{tikzcd} G_0 \times_{H_0} H_0 \arrow[r, tail, "G_0 \times_{H_0} i"] \arrow[dr, "\pi_1"'] & G_0 \times_{H_0} H_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & H_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & H_0 \\ & G_0 \arrow[r, "f_0"'] & H_0 & \end{tikzcd}$$

Since f is essentially surjective, $d_0 \circ \pi_2$ is an open surjection. Since i is a section of an étale map, i is itself étale and therefore open. So $d_0 \circ \pi_2 \circ (G_0 \times_{H_0} i)$ is open. Now note that $d_0 \circ \pi_2 \circ (G_0 \times_{H_0} i) = d_1 \circ \pi_2 \circ (G_0 \times_{H_0} i) = f_0$, so f_0 is open.

To prove that f_0 is étale, it remains to show that the diagonal $\Delta_{f_0}: G_0 \to G_0 \times_{H_0} G_0$ is open. Therefore consider the diagram

$$
\begin{tikzcd}
& G_0 \times_{H_0} G_0 \arrow[dl, "!"'] \arrow[dd] \arrow[dr] & \\
G_1 \arrow[rr] \arrow[dd, "\varphi_1"'] & & G_0 \times G_0 \arrow[dd, "f_0 \times f_0"] \\
& H_0 \arrow[dl, "i"'] \arrow[dr, "\Delta"] & \\
H_1 \arrow[rr, "{(d_0,d_1)}"'] & & H_0 \times H_0
\end{tikzcd}
$$

The front face is a pullback since f is assumed to be fully faithful. $G_0 \times_{H_0} G_0 \xrightarrow{!} G_1$ is the unique map induced by the universality of this pullback. Now since the front face and the right back face are pullbacks and everything commutes, the left back face is a pullback too. So $G_0 \times_{H_0} G_0 \xrightarrow{!} G_1$ is étale since $i\colon H_0 \to H_1$ is. Now consider the following triangle

$$
\begin{tikzcd}
& G_0 \arrow[dl, "i"'] \arrow[d, "\Delta_{f_0}"] \\
G_1 & G_0 \times_{H_0} G_0 \arrow[l, "!"']
\end{tikzcd}
$$

It is clear that this triangle commutes and Δ_{f_0} is étale since i and ! are. So f_0 is étale and since f_1 is the pullback of $f_0 \times f_0$ along (d_0, d_1), it is étale too.

In (Moerdijk, 1988b), theorem 3 it is shown that weak equivalences of continuous groupoids induce equivalences of toposes.

1.4. Localization theorem

It is also shown in (Moerdijk, 1988b) that for étale complete groupoids this is the universal way to 'invert' the class of weak equivalences W in the sense that the functor B induces an equivalence of categories

$$B\colon [\textit{Etale-Compl.-Groupoids}]\,[W^{-1}] \xrightarrow{\sim} [\textit{S-toposes}].$$

Here [*S-toposes*] is the category of S-toposes with isomorphism classes of morphisms, whereas [*Etale-Compl.-Groupoids*] denotes the category of étale complete groupoids, i.e. groupoids $\mathcal{G}$ for which

$$\begin{tikzcd} G_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] & G_0 \arrow[d, "\pi_{\mathcal{G}}"] \\ G_0 \arrow[r, "\pi_{\mathcal{G}}"'] & B\mathcal{G} \end{tikzcd}$$

is a pullback of toposes, with isomorphism classes of morphisms. And [*Etale-Compl.-Groupoids*] $[W^{-1}]$ is the category of fractions with respect to W (as in (Gabriel–Zisman, 1967)).

Remark that it is clear from the proof of Theorem 9, that every étale groupoid is étale complete. We will now show that the equivalence above restricts to an equivalence

$$B\colon [\textit{Etale-Groupoids}][W^{-1}] \xrightarrow{\sim} [\textit{Etendues}].$$

Here [*Etale-Groupoids*] is the category of étale groupoids with isomorphism classes of morphisms. So we have to check:

(i) $B\colon [\textit{Etale-Groupoids}] \to [\textit{Etendues}]$ is essentially surjective on objects.

(ii) When $f, g\colon \mathcal{G} \rightrightarrows \mathcal{H}$ are parallel arrows with $Bf = Bg$, there is a weak equivalence $w\colon \mathcal{K} \to \mathcal{G}$ such that $f \circ w = g \circ w$.

(iii) For any geometric morphism $\varphi\colon B\mathcal{G} \to B\mathcal{H}$ in *Etendues* there exist a weak equivalence $w\colon \mathcal{K} \to \mathcal{G}$ and a map $f\colon \mathcal{K} \to \mathcal{H}$ such that $\varphi \circ Bw = Bf$. (Cf. (Gabriel–Zisman, 1967) or (Moerdijk, 1988b).)

Part (i) is established in Section 1.1. For (ii): I. Moerdijk has shown that for étale complete groupoids $\mathcal{G}, \mathcal{H}$ and a natural isomorphism $\alpha\colon Bf \to Bg$ there exists a natural transformation $\bar{\alpha}\colon G_0 \to H_1$ between f and g. Since étale groupoids are étale complete we are done. Finally for (iii), we have to do some work: we must show that $\mathcal{K}$ as constructed in (Moerdijk, 1988b) is étale when $\mathcal{G}$ and $\mathcal{H}$ are étale. So we recall that construction and give the necessary remarks.

Let $\varphi\colon B\mathcal{G} \to B\mathcal{H}$ be a geometric morphism, where $\mathcal{G}$ and $\mathcal{H}$ are étale groupoids. The space of objects K_0 of the groupoid $\mathcal{K}$ is obtained as the pullback

$$\begin{tikzcd} K_0 \arrow[rr, "f_0"] \arrow[d, "w_0"'] & & H_0 \arrow[d, "\pi_H"] \\ G_0 \arrow[r, "\pi_G"'] & B\mathcal{G} \arrow[r, "\varphi"'] & B\mathcal{H}. \end{tikzcd}$$

It follows from Lemma 15 below that

$$(K_0 \overset{w_0}{\to} G_0) = \pi_G^* f^* (H_1 \overset{d_0}{\to} H_0),$$

so K_0 is indeed a topological space and w_0 is étale. The space of arrows K_1 with the structure maps d_0' and d_1' for $\mathcal{K}$ are defined as the pullback

$$\begin{tikzcd} K_1 \arrow[r, "w_1"] \arrow[d, "(d_0',d_1')"'] & G_1 \arrow[d, "(d_0,d_1)"] \\ K_0 \times K_0 \arrow[r, "w_0 \times w_0"'] & G_0 \times G_0, \end{tikzcd}$$

which assures that $\mathcal{K} \overset{w}{\to} \mathcal{G}$ is a weak equivalence. From the fact that w_0, d_0 and d_1 are étale it follows that $d_0, d_1\colon K_1 \rightrightarrows K_0$ are étale maps too. So $\mathcal{K}$ is indeed an étale groupoid. (The map $f_1\colon K_1 \to H_1$ can be constructed from the étale completeness of $\mathcal{H}$, as in (Moerdijk, 1988b)

We conclude:

THEOREM 14. *The functor B as defined above induces an equivalence of categories*

$$[\textit{Etale-Groupoids}]\,[W^{-1}] \simeq [\textit{Etendues}].$$

LEMMA 15. *The following diagram of toposes is a pullback square*

$$\begin{tikzcd} \mathcal{F}/\chi^* E \arrow[r] \arrow[d] & \mathcal{E}/E \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{E}. \end{tikzcd}$$

Proof. To prove this, we will use the following correspondence for topos morphisms:

$$\frac{\varphi\colon \mathcal{D} \to \mathcal{E}/E}{\psi\colon \mathcal{D} \to \mathcal{E}, \alpha\colon 1 \to \psi^*E}. \tag{1}$$

Recall that this goes as follows: given the morphism φ, let ψ be the composition $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}/E \to \mathcal{E}$, and

$$\alpha = \varphi^* \left(\begin{tikzcd} E \arrow[rr, "\Delta"] \arrow[dr] & & E \times E \arrow[dl] \\ & E & \end{tikzcd} \right)$$

For the other direction: $\varphi^*(F \to E)$ is computed as the pullback

$$\begin{tikzcd} \varphi^*(F \to E) \arrow[r] \arrow[d] & \psi^*F \arrow[d] \\ 1 \arrow[r, "\alpha"'] & \psi^*E. \end{tikzcd}$$

Now, to establish the lemma, let $\mathcal{D}$, η_1 and η_2 be as in

$$\begin{tikzcd} \mathcal{D} \arrow[dr, dashed, "\eta_3"] \arrow[drr, "\eta_2"] \arrow[ddr, "\eta_1"'] & & \\ & \mathcal{F}/\chi^*E \arrow[r] \arrow[d] & \mathcal{E}/E \arrow[d] \\ & \mathcal{F} \arrow[r] & \mathcal{E} \end{tikzcd} \tag{2}$$

Assume that η_2 corresponds to $\psi\colon \mathcal{D} \to \mathcal{D}$ and $\alpha\colon 1 \to \psi^*E \cong \eta_1^*\chi^*E$, by the correspondence (1) and commutativity of the diagram (2). Then $\eta_3\colon \mathcal{D} \to \mathcal{F}/\chi^*E$ is determined by $\eta_1\colon \mathcal{D} \to \mathcal{F}$ and $\alpha\colon 1 \to \eta_1^*\chi^*E$. It is clear that η_3 is uniquely (up to 2-isomorphism) determined by commutativity of the diagram. This proves the lemma.

Recall that [*Etale-Groupoids*] (respectively [*Etendues*]) is the category of étale groupoids (respectively étendues) and isomorphism classes of geometric morphisms (resp. groupoid morphisms). In this article we want to investigate the rôle of the 2-cells. Therefore we will need the notion of a bicategory of fractions. We will define it in such a way that it is a generalization of the category of fractions in the Gabriel and Zisman sense, has the required universal property and such that we have the following equivalence of bicategories:

$$(\textit{Etale-Groupoids})\,[W^{-1}] \simeq_{bi} (\textit{Etendues}),$$

where (*Etale-Groupoids*) and (*Etendues*) are the usual 2-categories.

2. Construction of bicategories of fractions

Given a bicategory $\mathcal{C}$ and a class W of arrows, which satisfy certain conditions (see Subsection 2.1), we will construct a bicategory of fractions of $\mathcal{C}$ with respect to W. That is, a bicategory $\mathcal{C}[W^{-1}]$, and a homomorphism

$$U\colon \mathcal{C} \to \mathcal{C}[W^{-1}],$$

with the following properties:

(i) U sends elements of W to equivalences,
(ii) U is universal with this property, i.e. composition with U gives an *equivalence of bicategories*:

$$\mathrm{Hom}\,(\mathcal{C}[W^{-1}],\mathcal{D}) \longrightarrow \mathrm{Hom}_W\,(\mathcal{C},\mathcal{D})$$

for each bicategory $\mathcal{D}$. Here Hom denotes the bicategory of homomorphisms and Hom_W denotes the subbicategory of those cells which send the elements of W to equivalences.

(Note that a morphism of bicategories is a homomorphism if it preserves compositions and units up to 2-isomorphism, see (Bénabou, 1967), p. 31. The 1-cells of a bicategory $\mathrm{Hom}\,(\mathcal{A},\mathcal{B})$ are described in Section 8 of this paper. There it is shown that we can view them as morphisms $\mathcal{A} \to \mathrm{Cyl}(\mathcal{B})$, into the bicategory of cylinders on $\mathcal{B}$. So a transformation $\alpha\colon f \to g$, where $f, g \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$, is represented by a morphism $K_\alpha\colon \mathcal{A} \to \mathrm{Cyl}(\mathcal{B})$ (and we only consider those which are again represented by homomorphisms), such that $d_0 \circ K_\alpha = f$ and $d_1 \circ K_\alpha = g$ in the notation of (Bénabou, 1967), p. 60. (These K_α's are analogous to homotopies between continuous maps of topological spaces and the cylinders play the rôle of the path space.) Similarly modifications between transfromations (i.e. 2-cells in $\mathrm{Hom}\,(\mathcal{A},\mathcal{B})$) are represented by homomorphisms $\mathcal{A} \to \mathrm{Cyl}(\mathrm{Cyl}(\mathcal{B}))$.)

Recall:

DEFINITION 16. A 1-cell $w\colon A \to B$ in a bicategory $\mathcal{C}$ is called an *equivalence* when there exist a 1-cell $v\colon B \to A$ and invertible 2-cells $\eta\colon w \circ v \Rightarrow I_B$ and

$\varepsilon\colon I_A \Rightarrow v \circ w$, which satisfy the triangle identities (see (Maclane, 1971), p. 83). We will call v a *quasi inverse* for w.

We will denote a bicategory $\mathcal{C}$ as an eight-tuple $(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, c, \bar{c}, I, a, l, r)$, where $\mathcal{C}_0$ denotes the class of objects and $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is the class of 1-cells (resp. 2-cells), c is the horizontal composition on both 1- and 2-cells (also denoted by $\circ$), $\bar{c}$ is the vertical composition on 2-cells (also denoted by $\bullet$). Vertical composition is strictly associative, but horizontal composition is only associative up to the natural associativity isomorphism a. The identities I are not strict identities with respect to horizontal composition either, only up to the natural isomorphisms l (for left) and r (for right). All these data have to satisfy certain coherence conditions which can be found in (Bénabou, 1967), where the reader can also find more information on bicategories. We remark that we will use the composition symbols to denote 'apply after' (so $f \circ g$ means: apply f after g) contrary to what is done by Bénabou.

2.1. CONDITIONS

Let $\mathcal{C}$ be a bicategory as above. A subset W of $\mathcal{C}_1$ is said *to admit a right calculus of fractions* if it satisfies the following conditions:

BF1. All equivalences are in W.

BF2. When $f\colon A \to B$ and $g\colon B \to C$ are in W, then $g \circ f\colon A \to C$ is in W too.

BF3. For every pair $w\colon A \to B$, $f\colon C \to B$ with $w \in W$ there exists a 2-isomorphism as in the square

$$\begin{tikzcd} D \arrow[r, "v"] \arrow[d, "h"'] \arrow[dr, phantom, "\alpha\nearrow"] & C \arrow[d, "f"] \\ A \arrow[r, "w"'] & B \end{tikzcd}$$

with $v \in W$.

BF4. If $\alpha\colon w \circ f \Rightarrow w \circ g$ is a 2-cell and $w \in W$, then there exist a 1-cell $v \in W$ and a 2-cell $\beta\colon f \circ v \Rightarrow g \circ v$ such that $\alpha \circ v = w \circ \beta$. Moreover: when α is an isomorphism, we require β to be an isomorphism too; when v' and β' form another such pair, there exist 1-cells u, u', such that $v \circ u$ and $v' \circ u'$ are in W and a 2-isomorphism $\varepsilon\colon v \circ u \Rightarrow v' \circ u'$ such that the following diagram commutes:

$$\begin{tikzcd} f \circ v \circ u \arrow[r, "\beta \circ u"] \arrow[d, "g \circ \varepsilon"'] & g \circ v \circ u \arrow[d, "g \circ \varepsilon"] \\ f \circ v' \circ u' \arrow[r, "\beta' \circ u'"'] & g \circ v' \circ u'. \end{tikzcd}$$

BF5. When $w \in W$ and there is a 2-isomorphism $\alpha: v \Rightarrow w$, then $v \in W$.

Remark 17. These conditions form a generalization of those in (Gabriel–Zisman, 1967). When we have an ordinary 1-category and we make a 2-category out of it by just adding the identity 2-cells, our conditions hold in the 2-category if and only if the Gabriel-Zisman conditions hold in the original category.

Now we are ready for the construction of the bicategory of fractions, which we will denote by $\mathcal{C}[W^{-1}]$. In this section we give a description of the 0-, 1- and 2-cells and we also define composition (of 1-cells) and pasting (of 2-cells). However, we will not prove that this construction satisfies the coherence axioms now. This will be done in the appendix.

2.2. Construction of $\mathcal{C}[W^{-1}]_0$ and $\mathcal{C}[W^{-1}]_1$

Let the objects of $\mathcal{C}[W^{-1}]$ be those of $\mathcal{C}$. The 1-cells of $\mathcal{C}[W^{-1}]$ are formed by pairs

$$(w, f): A \longrightarrow B,$$

such that

$$w: C \longrightarrow A$$

is in W and

$$f: C \longrightarrow B$$

is an arbitrary 1-cell in $\mathcal{C}$. To define the composition of two of these 1-cells, we must first *choose* for every pair of 1-cells in $\mathcal{C}$

$$\begin{tikzcd} & D \arrow[d, "u"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

with u in W, morphisms v and g, and a 2-isomorphism $\alpha: f \circ v \Rightarrow u \circ g$ as in the following square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha \nearrow"] & D \arrow[d, "u"] \\ C \arrow[r, "f"'] & B. \end{tikzcd}$$

And when $f = I$ we choose

$$\begin{tikzcd}
D \arrow[r, "I"] \arrow[d, "u"'] \arrow[dr] \arrow[dr, phantom, shift left=3ex, "\overset{l}{\Nearrow}"] \arrow[dr, phantom, shift right=3ex, "\overset{r^{-1}}{\Nearrow}"] & D \arrow[d, "u"] \\
B \arrow[r, "I"'] & B
\end{tikzcd}$$

and analogously when $u = I$.

Now define $(w_2, f_2) \circ (w_1, f_1)$ as in the following picture

$$\begin{tikzcd}
 & & F \arrow[dl, "u"'] \arrow[dr, "g"] & & \\
 & D \arrow[dl, "w_1"'] \arrow[dr, "f_1"] \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] & & E \arrow[dl, "w_2"'] \arrow[dr, "f_2"] & \\
A & & B & & C
\end{tikzcd}$$

where α is a chosen square. So $(w_2, f_2) \circ (w_1, f_1) := (w_1 \circ u \,,\, f_2 \circ g)$. (Remark that $\circ$ on the left hand side is the one to be defined, whereas the $\circ$ on the right hand side is the old composition in $\mathcal{C}$.)

Remark 18. From the universal property of $\mathcal{C}[W^{-1}]$ we will see that our construction does not really depend on the choices made above. That is: other choices will give an equivalent bicategory.

2.3. Construction of $\mathcal{C}[W^{-1}]_2$

In this subsection we will give a description of the 2-cells of $\mathcal{C}[W^{-1}]$ and we will define both the horizontal and vertical composition of them.

Let $w: C \to A$ and $v: D \to A$ be in W and let $f: C \to B$ and $g: D \to B$ be arbitrary 1-cells in $\mathcal{C}$. A 2-cell $\alpha: (w, f) \Rightarrow (v, g)$ in $\mathcal{C}[W^{-1}]$ is represented by a quadruple $(u_1, u_2, \alpha_1, \alpha_2)$ such that $w \circ u_1: E \to A$ and $v \circ u_2: E \to A$ are in W and $\alpha_1: w \circ u_1 \overset{\sim}{\Rightarrow} v \circ u_2$ and $\alpha_2: f \circ u_1 \Rightarrow g \circ u_2$ are 2-cells in $\mathcal{C}$, as in the following picture:

$$\begin{tikzcd}
& C \arrow[dr, "w"] \arrow[drr, "f"] & & \\
E \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1"] & & A \arrow[r, phantom, "\Downarrow\alpha_2"] & B \\
& D \arrow[ur, "v"'] \arrow[urr, "g"'] & &
\end{tikzcd}$$

We have the following equivalence relation on these quadruples: For $(u_1, u_2, \alpha_1, \alpha_2)$ as above and another $(s_1, s_2, \beta_1, \beta_2)$ as in the picture

$$\begin{tikzcd}
& C \arrow[dr, "w"] \arrow[drr, "f"] & & \\
E' \arrow[ur, "s_1"] \arrow[dr, "s_2"'] \arrow[rr, phantom, "\Downarrow\beta_1"] & & A \arrow[r, phantom, "\Downarrow\beta_2"] & B \\
& D \arrow[ur, "v"'] \arrow[urr, "g"'] & &
\end{tikzcd}$$

we define

$$(u_1, u_2, \alpha_1, \alpha_2) \sim (s_1, s_2, \beta_1, \beta_2)$$

if there exist 1-cells $r_1: F \to E$ and $r_2: F \to E'$, such that $w \circ s_1 \circ r_2$ and $w \circ u_1 \circ r_1$ are in W and 2-isomorphisms $\gamma_1: s_1 \circ r_2 \overset{\sim}{\Rightarrow} u_1 \circ r_1$ and $\gamma_2: u_2 \circ r_1 \overset{\sim}{\Rightarrow} s_2 \circ r_2$ in $\mathcal{C}$ as in the following diagram:

$$\begin{tikzcd}
& & & C \arrow[dr, "w"] \arrow[drr, "f"] & & \\
E' \arrow[urrr, "s_1"] \arrow[drrr, "s_2"'] & F \arrow[l, "r_2"'] \arrow[r, "r_1"] \arrow[r, phantom, shift left=3, "\Downarrow\gamma_1"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma_2"'] & E \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1"] & & A \arrow[r, phantom, "\Downarrow\alpha_2"] & B \\
& & & D \arrow[ur, "v"'] \arrow[urr, "g"'] & &
\end{tikzcd}$$

such that α_1 pasted with γ_1 and γ_2 gives $\beta_1 \circ r_2$ and α_2 pasted with γ_1 and γ_2 gives $\beta_2 \circ r_2$. It follows from our conditions BF2 to BF5 that this is indeed an equivalence relation. (For transitivity, 'compose' $(r_1, r_2, \gamma_1, \gamma_2)$ with $(r_3, r_4, \gamma_3, \gamma_4)$ via a square of BF3 for

$$\bullet \xrightarrow[w\circ s_1\circ r_3]{} \bullet \xleftarrow[w\circ s_1\circ r_2]{} \bullet$$

and then apply BF4 to $w \circ s_1$ to get a square for

$$\bullet \xrightarrow[r_3]{} \bullet \xleftarrow[r_2]{} \bullet$$

Before we can define pastings of these new 2-cells, we need some more choices of special 1- and 2-cells. (Note that the Remark 18 above applies to these choices too.) For every 2-cell $\alpha: v \circ f \Rightarrow v \circ g$, with $v \in W$ we choose a 1-arrow $w \in W$ and a 2-cell $\beta: f \circ w \Rightarrow g \circ k$ as in condition BF4. We do this such that w is the identity and $\beta = v^{-1} \circ \alpha$ when v is an isomorphism, and such that β is an isomorphism whenever α is.

Vertical composition of 2-cells is defined as in the following picture

$$\begin{tikzcd}
 & & C_1 \arrow[dr, "v_1"] \arrow[ddrrr, "f_1"] \arrow[dd, phantom, "\Downarrow\alpha_1"] & & & \\
 & E_1 \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[dd, phantom, "\Downarrow\gamma"] & & A \arrow[r, phantom, "\Downarrow\alpha_2"] & {} & \\
F \arrow[ur, "w_1"] \arrow[dr, "w_2"'] \arrow[r, phantom, "{[1]}"] & {} & D \arrow[ur, "v_2"] \arrow[dr, "v_2"] \arrow[rrr, "f_2"] \arrow[dd, phantom, "\Downarrow\beta_1"] & & & B \\
 & E_2 \arrow[ur, "u_3"] \arrow[dr, "u_4"'] & & A \arrow[r, phantom, "\Downarrow\beta_2"] & {} & \\
 & & C_2 \arrow[ur, "v_3"] \arrow[uurrr, "f_3"'] & & &
\end{tikzcd}$$

Here [1] is a chosen square. So

$$\begin{aligned}
&[(u_3, u_4, \beta_1, \beta_2)] \bullet [(u_1, u_2, \alpha_1, \alpha_2)] \\
&\quad = [(u_1 \circ w_1, u_4 \circ w_2, \\
&\quad (\beta_1 \circ w_2) \bullet (v_2 \circ \gamma) \bullet (\alpha_1 \circ w_1), \\
&\quad (\beta_2 \circ w_2) \bullet (f_2 \circ \gamma) \bullet (\alpha_2 \circ w_1))]
\end{aligned}$$

(with notation as in (Maclane, 1971), p. 43). With a straightforward but lengthy computation one can verify that this composition is well defined on equivalence classes and strictly associative.

The identity 2-cell $i_{(w,f)}$ at a given 1-cell (w, f) can now be defined as

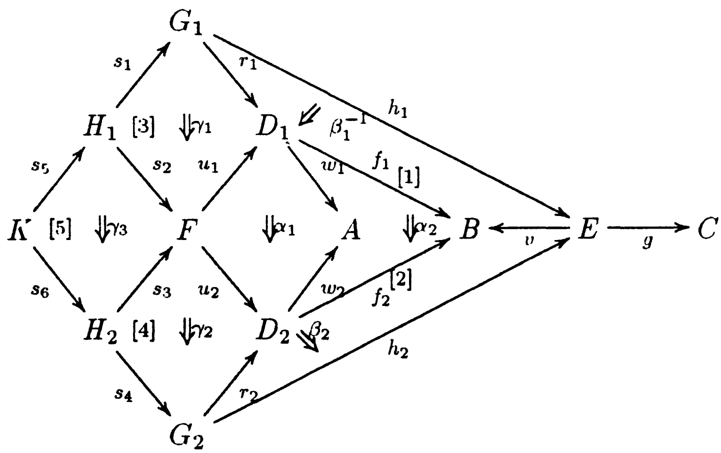
$$i_{(w,f)} = [(I_{\mathrm{dom}(f)}, I_{\mathrm{dom}(f)}, i_{I \circ w}, i_{I \circ f})],$$

where I gives the identity 1-cells and i the identity 2-cells in C. (We leave it to the reader to verify that this is indeed a strict identity for the vertical composition.)

We define the horizontal composition of 2-cells in two steps to keep the pictures simple. First we form

$$\begin{tikzcd} A \arrow[r, bend left, "{(w_1,f_1)}"] \arrow[r, bend right, "{(w_2,f_2)}"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "{(v,g)}"] & C \end{tikzcd}$$

with the following picture



In this picture the squares [1], [2], [3], [4] and [5] are all chosen squares (chosen in this order). So $(w_i \circ r_i, g \circ h_i) = (v, g) \circ (w_i, f_i)$, for $i \in \{1, 2\}$. And we see that

$$(\beta_2 \circ s_4 \circ s_6) \bullet (f_2 \circ \gamma_2 \circ s_6) \bullet (\alpha_2 \circ \gamma_3) \bullet (f_1 \circ \gamma_1 \circ s_5) \bullet (\beta_1^{-1} \circ s_1 \circ s_5), (3)$$

is a 2-cell $v \circ h_1 \circ s_1 \circ s_5 \Rightarrow v \circ h_2 \circ s_4 \circ s_6$ up to associativity in $\mathcal{C}$. So let $t: T \to K$ and $\eta: h_1 \circ s_1 \circ s_5 \circ t \Rightarrow h_2 \circ s_4 \circ s_6 \circ t$ be our chosen 1- and 2-cell such that $v \circ \eta = (3) \circ t$ as in condition BF4. Now

$$\begin{aligned}(v, g) \circ [(u_1, u_2, \alpha_1, \alpha_2)] = {} & [(s_1 \circ s_5 \circ t, s_4 \circ s_6 \circ t, \\ & (w_2 \circ \gamma_2 \circ s_6) \bullet (\alpha_1 \circ \gamma_3) \\ & \bullet (w_1 \circ \gamma_1 \circ s_5) \circ t, g \circ \eta)].\end{aligned}$$

We define the composition

$$\begin{tikzcd} A \arrow[r] & B \arrow[r, bend left, "{(v_1,g_1)}"] \arrow[r, bend right, "{(v_2,g_2)}"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

with the following pasting diagram (for simplicity we do not draw w in the picture).

$$
\begin{tikzcd}
 & & & G_1 \arrow[r, "h_1"] \arrow[ddr, "w_1" near start] \arrow[d, phantom, "\Downarrow\gamma_1"] & E_1 \arrow[ddr, "v_1"] \arrow[ddrr, "g_1"] \arrow[d, phantom, "\Swarrow\beta_1^{-1}"] & & \\
 & & H_1 \arrow[ur, "s_1"] \arrow[dr, "k_1"] & & & & \\
R' \arrow[r, "r'"] & R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] \arrow[rr, phantom, "\Downarrow\eta"] & & F \arrow[uur, "u_1" near start] \arrow[ddr, "u_2" near start] \arrow[r, phantom, "\Downarrow\alpha_1"] \arrow[d, phantom, "\Swarrow\gamma_2^{-1}"] & D \arrow[r, "f"] \arrow[d, phantom, "\Swarrow\beta_2"] & B \arrow[r, phantom, "\Downarrow\alpha_2"] & C \\
 & & H_2 \arrow[ur, "k_2"] \arrow[dr, "s_2"'] & & & & \\
 & & & G_2 \arrow[r, "h_2"'] \arrow[uur, "w_2"' near start] & E_2 \arrow[uur, "v_2"'] \arrow[uurr, "g_2"'] & &
\end{tikzcd}
\tag{4}
$$

Here the squares $\beta_i: f \circ w_i \Rightarrow v_i \circ h_i$ $(i \in \{1,2\})$ are chosen squares. The squares $\gamma_i: h_i \circ s_i \Rightarrow u_i \circ k_i$ $(i \in \{1,2\})$ can be constructed in the following way: we have chosen squares

$$
\begin{tikzcd}
\bullet \arrow[r, "\overline{k_i}"] \arrow[dd, "\overline{s_i}"'] & F \arrow[d, "u_i"] \\
\arrow[r, phantom, "\delta_i \Nearrow"] & E_i \arrow[d, "v_i"] \\
D \arrow[r, "f"'] & B
\end{tikzcd}
$$

with $\overline{s_i} \in W$ (since $v_i \circ u_i \in W$). We also have chosen squares

$$
\begin{tikzcd}
\bullet \arrow[r, "k'_i"] \arrow[d, "s'_i"'] \arrow[dr, phantom, "\varepsilon \Nearrow"] & \bullet \arrow[d, "\overline{s_i}"] \\
G_i \arrow[r, "w_i"'] & D
\end{tikzcd}
$$

and with the same method as in the proof of Lemma 53 below we get the morphism $\tilde{k}_i$ and the 2-isomorphism γ_i in the following picture

$$
\begin{tikzcd}
\bullet \arrow[rrrrdd, "\overline{k_i}"] \arrow[rrrrddddd, bend right, "\overline{s_i}"'] & & & & & \\
& H_i \arrow[lu, "k'_i"'] & & & & \\
& & \bullet \arrow[lu, "\tilde{k}_i"'] \arrow[rd, "\tilde{k}_i"] \arrow[ld, phantom, "\Uparrow\varepsilon"] & & F \arrow[rdd, "u_i"] \arrow[dd, phantom, "\Uparrow\gamma_i"] & \\
& & & H_i \arrow[rd, "s_i"] & & \\
& & & & G_i \arrow[r, "h_i"] \arrow[d, "w_i"'] \arrow[rd, phantom, "\Uparrow\beta_i"] & E_i \arrow[d, "v_i"] \\
& & & & D \arrow[r, "f"'] & B
\end{tikzcd}
$$

such that the resulting pasting is equal to $\delta_i \circ k'_i \circ \tilde{k}_i$. Now $k_i := \overline{k_i} \circ k'_i \circ \tilde{k}_i\, s_i = s'_i \circ \tilde{k}_i$ and to find η and r' in (4), let $\overline{\alpha_1}\colon w_1 \circ s_1 \circ r_1 \Rightarrow w_2 \circ s_2 \circ r_2$ be a chosen square and r_1 and r_2 are in W. Then we get a 2-cell $v_1 \circ w_1 \circ k_1 \circ r_1 \Rightarrow v_1 \circ w_1 \circ k_2 \circ r_2$ as in the following picture

$$
\begin{tikzcd}
& & F \arrow[r, "u_1"] & E_1 \arrow[rd, bend left, "v_1"] & \\
& H_1 \arrow[ru, "k_1"] \arrow[r, "s_1"'] \arrow[rd, phantom, "\Downarrow\overline{\alpha_1}"] & G_1 \arrow[ru, "h_1"'] \arrow[r, "w_1"'] \arrow[u, phantom, "\Downarrow\gamma_1^{-1}"] & D \arrow[r, "f"'] \arrow[u, phantom, "\Downarrow\beta_1^{-1}"] \arrow[d, phantom, "\Downarrow\beta_2"] & B \\
R \arrow[ru, "r_1"] \arrow[rd, "r_2"'] & & G_2 \arrow[ru, "w_2"] \arrow[r, "h_2"'] \arrow[d, phantom, "\Downarrow\gamma_2"] & E_2 \arrow[ru, "v_2"'] \arrow[r, phantom, "\Downarrow\alpha_1^{-1}"] & \\
& H_2 \arrow[ru, "s_2"] \arrow[r, "k_2"'] & F \arrow[ru, "u_2"'] \arrow[r, "u_1"'] & E_1 \arrow[ruu, bend right, "v_1"'] &
\end{tikzcd}
$$

Remark that $v_1 \circ u_1$ is in W, so we can apply condition BF4 and $R' \xrightarrow{r'} R$ and η above are the chosen 1- and 2-cell for this case. We define

$$
\begin{aligned}
[(u_1, u_2, \alpha_1, \alpha_2)] \circ (w, f) = {} & [(s_1 \circ r_1 \circ r', s_2 \circ r_2 \circ r', w \circ \overline{\alpha_1} \circ r', \\
& (g_2 \circ \gamma_2^{-1} \circ r_2 \circ r') \bullet (\alpha_2 \circ \eta) \\
& \bullet (g_1 \circ \gamma_1 \circ r_1 \circ r'))].
\end{aligned}
$$

It can be verified that this composition is well defined on equivalence classes and that the identity 2-cell as defined before is an identity with respect to this horizontal composition too.

2.4. THE UNIVERSAL HOMOMORPHISM U

Now we will define a homomorphism $U\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ (cf. (Bénabou, 1967)).

(i) U is defined on objects as: $U(A) = A$ for each $A \in \mathcal{C}_0$.

(ii) For each pair of objects A, B, the functor $U(A,B): \mathcal{C}(A,B) \to \mathcal{C}[W^{-1}](A,B)$ is defined as:
On 1-cells in $\mathcal{C}$: $U(f) = (I_A, f)$;
On 2-cells in $\mathcal{C}$: for $\alpha: f \Rightarrow g$, $U(\alpha): (I_A, f) \Rightarrow (I_A, g)$ is represented by the quadruple $(I_A, I_A, i_{I_A \circ I_A}, l^{-1}(g) \circ \alpha \circ l(f))$.

$$
\begin{tikzcd}
& A \arrow[dr, "I_A"] \arrow[drrr, "f"] & & & \\
A \arrow[ur, "I_A"] \arrow[dr, "I_A"'] \arrow[rr, phantom, "\Downarrow i_{I_A \circ I_A}"] & & A \arrow[rr, phantom, "\Downarrow l^{-1}(g)\circ\alpha\circ l(f)"] & & B \\
& D \arrow[ur, "I_A"'] \arrow[urrr, "g"'] & & &
\end{tikzcd}
$$

(iii) $I_{U(A)} = U(I_A)$ so let $v_A = i_{I_A}$.

(iv) For each triple of objects A, B, C in $\mathcal{C}$, define a family of natural isomorphisms relating the horizontal compositions in $\mathcal{C}$ and $\mathcal{C}[W^{-1}]$ (cf. (Bénabou, 1967), p. 29)

$$
\begin{aligned}
v(A,B,C)&: c(UA, UB, UC) \circ (U(A,B) \\
&\times U(B,C)) \Rightarrow U(A,C) \circ c(A,B,C)
\end{aligned}
$$

as follows: for 1-cells $f: A \to B$, $g: B \to C$

$$
\begin{aligned}
&v(A,B,C)_{(f,g)} \\
&\quad = (I_A, I_A, l(I_A), i_{I_A} \circ i_f \circ i_g): (I_A, g) \circ (I_A, f) \to (I_A, g \circ f).
\end{aligned}
$$

$$
\begin{tikzcd}
& A \arrow[dr, "I_A"] \arrow[rrr, "f"] & & & B \arrow[ddr, "g"] & \\
& & A \arrow[dr, "I_A"] & & & \\
A \arrow[uur, "I_A"] \arrow[ddr, "I_A"'] \arrow[rrr, phantom, "\Downarrow l(I_A)"] & & & A \arrow[rr, phantom, "\Downarrow i_{I_A}\circ i_f\circ i_g"] & & C \\
& & & & B \arrow[ur, "g"'] & \\
& A \arrow[uurr, "I_A"'] \arrow[urrr, "f"'] & & & &
\end{tikzcd}
$$

We leave it to the reader to verify that this construction satisfies the coherence axioms for bifunctors.

3. Properties of U

In this section we will prove that $U: \mathcal{C} \to \mathcal{C}[W^{-1}]$ has indeed the required properties: it sends elements of W to equivalences and it has a universal property which implies that $\mathcal{C}[W^{-1}]$ is unique up to equivalence of bicategories, i.e. when $V: \mathcal{C} \to \mathcal{D}$ is another homomorphism with these properties, $\mathcal{C}[W^{-1}]$ is equivalent to $\mathcal{D}$.

3.1. The image of W

Note that v, η and ε in Definition 16 of equivalences and quasi inverses are not necessarily unique. However, when we have two inverses $(v_1, \eta_1, \varepsilon_1)$ and $(v_2, \eta_2, \varepsilon_2)$, then there is a canonical 2-isomorphism $v_1 \Rightarrow v_2$:

LEMMA 19. *When both v_1 and v_2 are quasi inverses of w with 2-isomorphisms η_i and ε_i with $i \in \{1, 2\}$ as in Definition 16, there is a unique canonical isomorphism $\omega((\eta_1, \varepsilon_1), (\eta_2, \varepsilon_2)): v_1 \Rightarrow v_2$ induced by these isomorphisms.*

Proof. Define the isomorphism $\omega((\eta_1, \varepsilon_1), (\eta_2, \varepsilon_2))$ as

$$\begin{tikzcd}
{} \arrow[rrr, "v_1"'] & {} \arrow[d, phantom, "\Downarrow r^{-1}"] & & {} \\
{} \arrow[r, "v_1"'] \arrow[d, phantom, "\parallel"] & {} \arrow[rr, "I_A"'] & {} \arrow[d, phantom, "\Downarrow \varepsilon_2"] & {} \\
{} \arrow[r, "v_1"'] \arrow[d, phantom, "\Downarrow \eta_1"] & {} \arrow[r, "w"'] & {} \arrow[r, "v_2"'] \arrow[d, phantom, "\parallel"] & {} \\
{} \arrow[rr, "I_C"'] & {} \arrow[d, phantom, "\Downarrow l"] & {} \arrow[r, "v_2"'] & {} \\
{} \arrow[rrr, "v_2"'] & & & {}
\end{tikzcd}$$

By the triangle equalities this is the only canonical way to define an isomorphism $v_1 \Rightarrow v_2$ (the other constructions give the same isomorphism).

PROPOSITION 20. *U sends elements of W to equivalences.*

Proof. Let $(w: A \to B) \in W$, then $U(w) = (I_A, w)$. We claim that (w, I_A) is a quasi inverse for (I_A, w) with the following 2-cells $\eta: (I_A, w) \circ (w, I_A) \Rightarrow (I_B, I_B)$ and $\varepsilon: (I_A, I_A) \Rightarrow (w, I_A) \circ (I_A, w)$:

$(I_A, w) \circ (w, I_A) = (w \circ I_A, w \circ I_A)$ and η can be represented by

$$\eta = [(I_A, w, r \circ l^{-1} \circ l^{-1}, r \circ l^{-1} \circ l^{-1})].$$

Let

$$\begin{tikzcd}
C \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\alpha\Nearrow"] & A \arrow[d, "w"] \\
A \arrow[r, "w"'] & B
\end{tikzcd}$$

be a chosen square. Then $(w, I_A) \circ (I_A, w) = (I_A \circ v_1, I_A \circ v_2)$. To define the third coordinate of a representing element for ε, consider the following pasting

$$\begin{tikzcd}
C \arrow[r, "v_1"] \arrow[d, "v_2"'] & A \arrow[r, "I_A"] \arrow[ddr, "w"'] \arrow[dr, phantom, "\Leftarrow l^{-1}"] & A \arrow[dd, "w"] \\
A \arrow[d, "I_A"'] \arrow[drr, "w"] \arrow[r, phantom, "\Swarrow\alpha"] & {} & \\
A \arrow[rr, "w"'] \arrow[r, phantom, "\Downarrow l"] & {} & B
\end{tikzcd}$$

Let $u: D \to C$ and $\beta: I_A \circ v_1 \circ u \Rightarrow I_A \circ v_2 \circ u$ be the choice on account of condition BF4 for w and this 2-cell. Now ε can be represented by

$$(v_2 \circ u, u, \beta^{-1}, a\,(D, C, A, A)_{(u, v_2, I_A)}),$$

where $a\,(D, C, A, A)_{(u, v_2, I_A)})$ is the associativity 2-cell. It is not difficult to verify that this η and ε satisfy the triangle equalities.

3.2. UNIVERSALITY OF U

The main aim of this subsection is to prove the following theorem:

THEOREM 21. *Composition with U gives an equivalence of bicategories*

$$\mathrm{Hom}\,(\mathcal{C}[W^{-1}], \mathcal{D}) \longrightarrow \mathrm{Hom}_W\,(\mathcal{C}, \mathcal{D}).$$

Here $\mathrm{Hom}(-,-)$ is the bicategory of homomorphisms, transformations and modifications. Recall that a transformation $\alpha: F \Rightarrow G$ between homomorphisms $F, G: \mathcal{C} \rightrightarrows \mathcal{D}$ can be represented by a homomorphism $K_\alpha: \mathcal{C} \to \mathrm{Cyl}\,(\mathcal{D})$. Now

$\mathrm{Hom}_W(\mathcal{C},\mathcal{D})$ is the subbicategory whose objects (homomorphisms) and 1-arrows (transformations) are homomorphisms which send the elements of W to equivalences.

To prove that composition with U is *essentially surjective*, let $F:\mathcal{C} \to \mathcal{D}$ be an element of $\mathrm{Hom}_W(\mathcal{C},\mathcal{D})$. Now we will define a homomorphism $\tilde{F}:\mathcal{C}[W^{-1}] \to \mathcal{D}$ such that there is an invertible 1-cell $\alpha:\mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ from $\tilde{F} \circ U$ to F which sends the elements of W to equivalences:

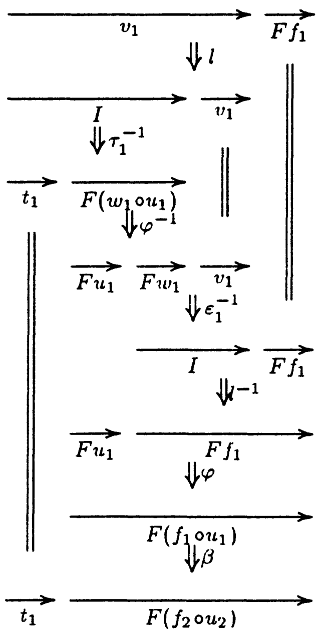
- on 0-cells: $\tilde{F}(A) = F(A)$ for all $A \in \mathcal{C}[W^{-1}]_0 = \mathcal{C}_0$.
- to define $\tilde{F}$ on 1- and 2-cells first choose quasi inverses for all elements of $F[W]$ and 2-cells as in Definition 16. For the identities we choose: as quasi inverse of $F(I_A)$: I_{FA} with the following 2-cells

$$\eta_A := \left\{ \begin{array}{c} \xrightarrow[I_{FA}]{} \xrightarrow[F(I_A)]{} \\ \Downarrow l^{-1} \\ \xrightarrow[F(I_A)]{\qquad} \\ \Downarrow \varphi_A \\ \xrightarrow[I_{FA}]{\qquad} \end{array} \right.$$

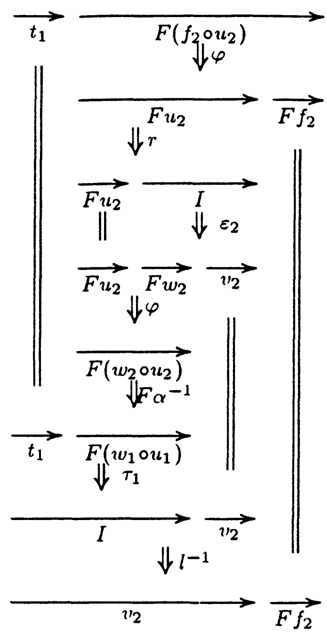
and

$$\varepsilon_A := \left\{ \begin{array}{c} \xrightarrow[I_{FA}]{\qquad} \\ \Downarrow \varphi_A^{-1} \\ \xrightarrow[F(I_A)]{\qquad} \\ \Downarrow r \\ \xrightarrow[F(I_A)]{} \xrightarrow[I_{FA}]{} \end{array} \right.$$

where $\varphi_A: F(I_A) \Rightarrow I_{FA}$ is the 2-cell belonging to F as in (Bénabou, 1967). It follows from the coherence conditions on bicategories, that these 2-cells satisfy the triangle equalities (see (Kelly, 1964)). Now define $\tilde{F}((w,f)) = F(f) \circ v$, where v is a chosen quasi inverse for $F(w)$.
- Let $(u_1,u_2,\alpha,\beta):(w_1,f_1) \Rightarrow (w_2,f_2)$ represent a 2-cell in $\mathcal{C}[W^{-1}]$. Let $\eta_i: Fw_i \circ v_i \Rightarrow I, \varepsilon_i: I \Rightarrow v_i \circ Fw_i$ (for $i \in \{1,2\}$) and $\tau_i: F(w_i \circ u_i) \circ t_i \Rightarrow I$, $\sigma_i: I \Rightarrow t_i \circ F(w_i \circ u_i)$ $(i \in \{1,2\})$ be the 2-cell isomorphisms for the chosen quasi inverses r_i and t_i. We define $\tilde{F}([u_1,u_2,\alpha,\beta])$ to be the following composition of 2-cells in $\mathcal{D}$:



composed with



By drawing some diagrams and using coherence and Lemma 19, one can show that this is well defined on equivalence classes of 2-cells.

The definition of the 2-cells $\tilde{\varphi}_A$ (for $A \in \mathcal{C}_0$) and $\tilde{\varphi}_{ABC}$ (for $A, B, C \in \mathcal{C}_0$) for $\tilde{F}$ follows in the evident way from φ_A and φ_{ABC} from F. We leave it to the reader to verify this and the fact that $\tilde{F}$ satisfies the coherence axioms, which follows from the fact that F satisfies them.

It remains to show that $\tilde{F}$ is indeed the homomorphism we were looking for, i.e. to construct a homomorphism $K\psi\colon \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ which 'inverts' the elements of W, and represents a $\psi\colon F \overset{\sim}{\Rightarrow} \tilde{F} \circ U$. Let us first compute $\tilde{F} \circ U$:

- $\tilde{F} \circ U(A) = \tilde{F}(A) = F(A)$ for all $A \in \mathcal{C}_0$.
- $\tilde{F} \circ U(f) = \tilde{F}((I, f)) = F(f) \circ I$
- $\tilde{F} \circ U(\alpha) = \tilde{F}([I, I, i \circ i, l^{-1} \circ \alpha \circ l])$, which, by some computation, can be seen to be the following composition of 2-cells

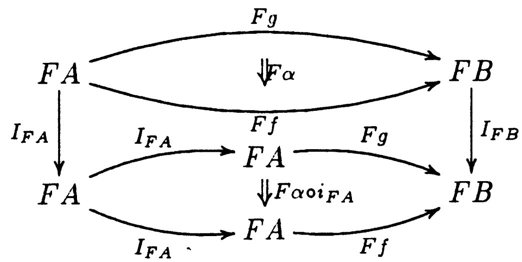
$$\begin{array}{c}
\xrightarrow[\quad I \quad]{} \;\; \xrightarrow[\quad F(f) \quad]{} \\
\Downarrow\varphi_A^{-1} \qquad\qquad \| \\
\xrightarrow[\quad FI \quad]{} \;\; \xrightarrow[\quad F(f) \quad]{} \\
\Downarrow\varphi \\
\xrightarrow[\qquad F(f\circ I) \qquad]{} \\
\Downarrow Fl^{-1} \\
\xrightarrow[\qquad F(f) \qquad]{} \\
\Downarrow F\alpha \\
\xrightarrow[\qquad F(g) \qquad]{} \\
\Downarrow Fl \\
\xrightarrow[\qquad F(g\circ I) \qquad]{} \\
\Downarrow\varphi^{-1} \\
\xrightarrow[\quad FI \quad]{} \;\; \xrightarrow[\quad F(g) \quad]{} \\
\Downarrow\varphi_A \qquad\qquad \| \\
\xrightarrow[\quad I \quad]{} \;\; \xrightarrow[\quad F(g) \quad]{}
\end{array}$$

This composition is symmetric, so we can define $K\psi : \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ as:

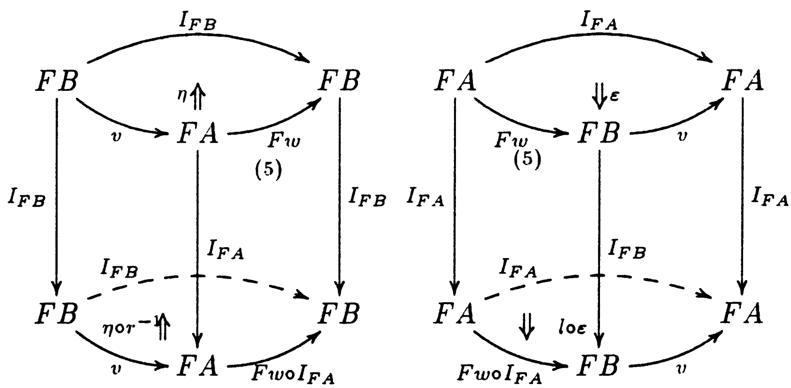
- $K\psi(A) = I_{FA}$
- $K\psi(f : A \to B)$ is the following square

$$\begin{tikzcd}
FA \arrow[rr, "Ff"] \arrow[d, "I_{FA}"'] \arrow[dr, "F(I_A)"] \arrow[drr, bend left=10, "F(f\circ I)"'] \arrow[drr, bend left=30, "Ff"] \arrow[drr, phantom, bend left=45, "\Downarrow r^{-1}"] \arrow[drr, phantom, bend left=20, "\Downarrow Fl"] & & FB \arrow[d, "I_{FB}"] \\
FA \arrow[r, "I_{FA}"'] \arrow[ur, phantom, "\Downarrow l\cdot\varphi_A"] & FA \arrow[r, "Ff"'] \arrow[ur, phantom, "\Downarrow\varphi^{-1}"] & FB
\end{tikzcd} \tag{5}$$

- $K\psi(\alpha : f \Rightarrow g)$ is the following cylinder

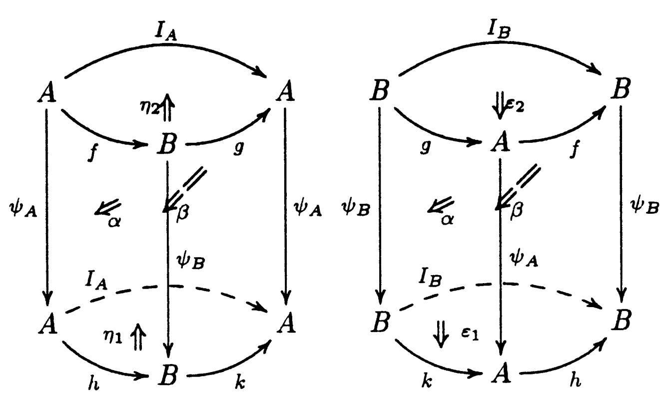


where the front and the back face are as (5) above. The reader may check that this is indeed a homomorphism of bicategories and induces an isomorphism $\psi: F \Rightarrow \tilde{F} \circ U$. It remains to be shown that $K\psi$ quasi inverts the elements of W (and thus is an arrow in the bicategory $\mathrm{Hom}_W(\mathcal{C}, \mathcal{D})$). So let $w \in W$ and let v be a quasi inverse of Fw in $\mathcal{D}$ and choose η and ε as before. Now use the following lemma to find the quasi inverse for $K\psi(w)$ as in the cylinders



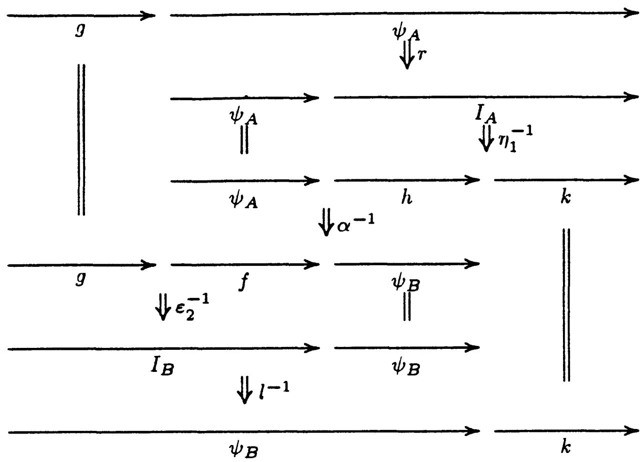
(Note that since the 2-cells are just the old ones, the triangle equalities automatically hold.)

LEMMA 22. *Given two cylinders*



in any bicategory, such that all 2-cells above are isomorphisms and (η_1, ε_1) *and* (η_2, ε_2) *both satisfy the triangle equalities, then there is a unique 2-isomorphism* $\beta: \psi_A \circ g \Rightarrow k \circ \psi_B$ *making both cylinders commute.*

Proof. Define β as the following composition



Using the triangle equalities it is easily shown that this β makes both cylinders commute.

Now it is clear that composition with U gives a bifunctor which is essentially surjective, and essentially full since 1-cells can be represented by homomorphisms $\mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$. But from Lemma 22 above we see that once we have chosen $\tilde{F}$ and $\tilde{G}$ and we have $K\psi: F \Rightarrow G$, there is only one choice left for $\tilde{K\psi}: \tilde{F} \Rightarrow \tilde{G}$. (Remark: when $K\psi(w)$ for $w \in W$ is the following square

$$\begin{tikzcd}
FA \arrow[r, "Fw"] \arrow[d, "K\psi(A)"'] \arrow[dr, phantom, "\alpha\Nearrow"] & A \arrow[d, "K\psi(B)"] \\
GA \arrow[r, "Gw"'] & GB
\end{tikzcd}$$

α must be a 2-isomorphism, since $K\psi(w)$ 'inverts' the elements of W. So we can apply Lemma 22 to see that there is a unique choice which corresponds with the right domain and codomain.) So composition with U is fully faithful on 1- and 2-cells.

3.3. Unicity of $\mathcal{C}[W^{-1}]$

The category $\mathcal{C}[W^{-1}]$ is determined, up to equivalence of bicategories, by the universality Theorem 21 above. Let $V: \mathcal{C} \to \mathcal{E}$ be a homomorphism of bicategories 'inverting' W which is universal in the sense defined above. By universality of $U: \mathcal{C} \to \mathcal{C}[W^{-1}]$, V induces a homomorphism $\tilde{V}: \mathcal{C}[W^{-1}] \to \mathcal{E}$ and $\psi: V \Rightarrow \tilde{V} \circ U$; whereas by universality of $V: \mathcal{C} \to \mathcal{E}$, U induces $\tilde{U}: \mathcal{E} \to \mathcal{C}[W^{-1}]$ and $\varphi: U \Rightarrow \tilde{U} \circ V$. Now ψ and φ induce a 2-isomorphism $\tilde{U} \circ \tilde{V} \circ U \Rightarrow U$

$$\begin{tikzcd}
\cdot \arrow[r, "U"] \arrow[d, equal] & \cdot \arrow[r, "\tilde{V}"] \arrow[d, phantom, "\Downarrow\psi^{-1}"] & \cdot \arrow[r, "\tilde{U}"] \arrow[d, equal] \arrow[dr, phantom, "\|"] & \cdot \arrow[d, equal] \\
\cdot \arrow[rr, "V"] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\varphi^{-1}"] & \cdot \arrow[r, "\tilde{U}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[rrr, "U"] & & & \cdot
\end{tikzcd}$$

which we will call ϑ for short. So $K\vartheta: \mathcal{C} \to \mathrm{Cyl}\,(\mathcal{C}[W^{-1}])$ inverts W and factorizes in a unique way to $\tilde{K}\vartheta: \mathcal{C}[W^{-1}] \to \mathrm{Cyl}\,(\mathcal{C}[W^{-1}])$, which represents a 2-isomorphism $\tilde{U} \circ \tilde{V} \Rightarrow I_{\mathcal{C}[W^{-1}]}$. So $\mathcal{C}[W^{-1}] \simeq \mathcal{E}$. We conclude:

THEOREM 23. *For each homomorphism* $V: \mathcal{C} \to \mathcal{E}$ *which induces for each bicategory* $\mathcal{D}$ *an equivalence of bicategories* $\mathrm{Hom}\,(\mathcal{E}, \mathcal{D}) \simeq \mathrm{Hom}_W\,(\mathcal{C}, \mathcal{D})$ *by composition, there is a canonical equivalence of bicategories*

$$\mathcal{E} \simeq_{bi} \mathcal{C}[W^{-1}].$$

3.4. CONDITIONS ON $\mathcal{D}$ TO BE EQUIVALENT TO $\mathcal{C}[W^{-1}]$

Let $F: \mathcal{C} \to \mathcal{D}$ be a homomorphism of bicategories, which sends the elements of W to equivalences. Then F corresponds to a homomorphism $\tilde{F}: \mathcal{C}[W^{-1}] \to \mathcal{D}$ by Theorem 21 above. Now we want to know when $\tilde{F}$ is an equivalence of bicategories. So $\tilde{F}$ must be essentially surjective, essentially full and fully faithful on 2-cells.

PROPOSITION 24. *A homomorphism* $F: \mathcal{C} \to \mathcal{D}$ *which sends the elements of the class* W *as above to equivalences, induces an equivalence of bicategories* $\tilde{F}: \mathcal{C}[W^{-1}] \xrightarrow{\sim} \mathcal{D}$ *if and only if the following conditions hold:*

EF1. F *is essentially surjective on objects.*

EF2. *For every* 1*-cell* f *in* $\mathcal{D}$ *there exists a* $w \in W$ *such that* $Fg \overset{\sim}{\Rightarrow} f \circ Fw$ *for some* g *in* $\mathcal{C}_1$.

EF3. F *must be fully faithful on* 2*-cells.*

Proof. Necessity of condition EF1 and EF3 is clear.

Necessity of condition EF2: let v be the chosen quasi inverse for Fw, then a 2-cell $\alpha: Fg \circ v \Rightarrow f$ induces a 2-cell $Fg \Rightarrow f \circ Fw$ as follows

$$\begin{tikzcd}
{} \arrow[rrr, "Fg"'] \arrow[drrr, phantom, "\Downarrow l"] & & & {} \\
{} \arrow[rr, "I"'] \arrow[drr, phantom, "\Downarrow \varepsilon"] & & {} \arrow[r, "Fg"'] \arrow[dr, phantom, "\parallel"] & {} \\
{} \arrow[r, "Fw"'] \arrow[dr, phantom, "\parallel"] & {} \arrow[r, "v"'] \arrow[drr, phantom, "\Downarrow \alpha"] & {} \arrow[r, "Fg"'] & {} \\
{} \arrow[r, "Fw"'] & {} \arrow[rr, "f"'] & & {}
\end{tikzcd}$$

For sufficiency of these conditions: It is clear that EF1 and EF2 imply that $\tilde{F}$ is essentially surjective on objects and essentially full. We will prove that EF3 implies that $\tilde{F}$ is fully faithful on 2-cells.

To show that $\tilde{F}$ is full on 2-cells, let $\alpha: \tilde{F}(u_1, f) \Rightarrow \tilde{F}(u_2, g)$ be a 2-cell, i.e. $\alpha: Ff \circ v_1 \Rightarrow Fg \circ v_2$ where v_i is the chosen quasi inverse for Fu_i with 2-cells η_i and ε_i as in Definition 16 (where $i \in \{1, 2\}$). Let

$$\begin{tikzcd}
\bullet \arrow[r, "u_3"] \arrow[d, "u_4"'] \arrow[dr, phantom, "\gamma \Nearrow"] & \bullet \arrow[d, "u_1"] \\
\bullet \arrow[r, "u_2"'] & \bullet
\end{tikzcd}$$

be a chosen square, then we have the following 2-cell $F(f \circ u_3) \Rightarrow F(g \circ u_4)$

$$\begin{array}{cccc}
\multicolumn{4}{c}{\xrightarrow[\qquad\qquad\qquad F(f\circ u_3)\qquad\qquad\qquad]{}} \\
\multicolumn{4}{c}{\Downarrow \varphi(u_3,f)} \\
\xrightarrow[Fu_3]{} & \multicolumn{3}{c}{\xrightarrow[\qquad\qquad Ff \qquad\qquad]{}} \\
\| & \multicolumn{3}{c}{\Downarrow l} \\
\xrightarrow[Fu_3]{} & \multicolumn{2}{c}{\xrightarrow[\qquad I \qquad]{}} & \xrightarrow[Ff]{} \\
\| & \multicolumn{2}{c}{\Downarrow \varepsilon_1} & \| \\
\xrightarrow[Fu_3]{} & \xrightarrow[Fu_1]{} & \xrightarrow[v_1]{} & \xrightarrow[Ff]{} \\
\multicolumn{2}{c}{\Downarrow \varphi(u_1,u_3)} & \multicolumn{2}{c}{\Downarrow \alpha} \\
\multicolumn{2}{c}{\xrightarrow[\qquad F(u_1\circ u_3)\qquad]{}} & \xrightarrow[v_2]{} & \xrightarrow[Fg]{} \\
\multicolumn{2}{c}{\Downarrow F\gamma} & \| & \| \\
\multicolumn{2}{c}{\xrightarrow[\qquad F(u_2\circ u_4)\qquad]{}} & \xrightarrow[v_2]{} & \xrightarrow[Fg]{} \\
\multicolumn{2}{c}{\Downarrow \varphi(u_4,u_2)} & \| & \| \\
\xrightarrow[Fu_4]{} & \xrightarrow[Fu_2]{} & \xrightarrow[v_2]{} & \xrightarrow[Fg]{} \\
\| & \multicolumn{2}{c}{\Downarrow \eta_2} & \| \\
\xrightarrow[Fu_4]{} & \multicolumn{2}{c}{\xrightarrow[\qquad I \qquad]{}} & \xrightarrow[Fg]{} \\
\| & \multicolumn{3}{c}{\Downarrow l^{-1}} \\
\xrightarrow[Fu_4]{} & \multicolumn{3}{c}{\xrightarrow[\qquad\qquad Fg \qquad\qquad]{}} \\
\multicolumn{4}{c}{\Downarrow \varphi(u_4,g)} \\
\multicolumn{4}{c}{\xrightarrow[\qquad\qquad\qquad F(g\circ u_4)\qquad\qquad\qquad]{}}
\end{array}$$

Since F is full, there is a 2-cell $\beta\colon f \circ u_3 \Rightarrow g \circ u_4$ such that $F\beta$ is the 2-cell above. Now $\tilde{F}([u_3, u_4, \gamma, \beta]) = \alpha$, and we conclude that $\tilde{F}$ is full on 2-cells.

To show that $\tilde{F}$ is faithful on 2-cells, remark that once we have chosen u_3, u_4 and γ, it is clear that β in the construction above is uniquely determined since F is faithful. Now suppose we have chosen arbitrary v_1, v_2 and a 2-isomorphism δ such that the following square commutes

$$\begin{tikzcd}
\bullet \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\delta\nearrow"] & \bullet \arrow[d, "u_1"] \\
\bullet \arrow[r, "u_2"'] & \bullet
\end{tikzcd}$$

When there exists a $\bar{\beta}$, such that $\tilde{F}([v_2, v_1, \delta, \bar{\beta}]) = \alpha$, it can be found by the construction above and is unique. With essentially the same proof as that of Lemma 53 it can be shown that $(u_3, u_4, \gamma, \beta)$ and $(v_2, v_1, \delta, \bar{\beta})$ are equivalent 2-cells. And we conclude that $\tilde{F}$ is faithful on 2-cells as required.

4. Etendues as a bicategory of fractions

In this section we want to give a sharper version of Theorem 14 as promised before. Therefore we will have to check that the class W of weak equivalences satisfies the conditions BF1 to BF5; and under what conditions the functor B:(*Etale-Groupoids*) $\rightarrow$ (*Etendues*) satisfies the conditions EF1 to EF3. We will see that this is the case when we consider the 2-category (*2-Iso-Etendues*), i.e. the category with only those 2-cells which are isomorphisms. We will see that when we consider T_1-étendues, that is étendues $\mathcal{E}$ for which there exists an object $U \twoheadrightarrow 1$ in $\mathcal{E}$, such that $\mathcal{E}/U \simeq \mathrm{Sh}(X)$ with X a T_1-space, all 2-cells are isomorphisms. Remark: when $\mathcal{E}$ is a T_1-étendue, each X as in Definition 8 is a T_1-space.

4.1. Weak equivalences

We will now check that the class W of weak equivalences as defined in Section 1.3 satisfies the conditions BF1 to BF5 of Section 2.1.

BF1: isomorphisms are clearly weak equivalences.

BF2: it is also clear that they are closed under composition.

BF3: this condition was already checked in (Moerdijk, 1988b).

BF4: let $\alpha: \eta \circ \varphi \Rightarrow \eta \circ \psi$ be a 2-cell, where $\eta: \mathcal{H} \to \mathcal{K}$ is a weak equivalence, and φ, $\psi: \mathcal{G} \to \mathcal{H}$, and $\alpha: G_0 \to K_1$. Since η is a weak equivalence the square

$$\begin{tikzcd} H_1 \arrow[r, "\eta_1"] \arrow[d, "{(d_0,d_1)}"'] & K_1 \arrow[d, "{(d_0,d_1)}"] \\ H_0 \times H_0 \arrow[r, "\eta_0 \times \eta_0"'] & K_0 \times K_0 \end{tikzcd} \tag{6}$$

is a pullback, and the maps $(\varphi_0, \psi_0): G_0 \to H_0 \times H_0$ and $\alpha: G_0 \to K_1$ induce a unique map $\tilde{\alpha}: G_0 \to H_1$. We claim that $\tilde{\alpha}$ gives the required 2-cell $\varphi \Rightarrow \psi$. Indeed, it is clear that $d_0 \circ \tilde{\alpha} = \varphi_0$, and $d_1 \circ \tilde{\alpha} = \psi_0$. To see that $m \circ (\psi_1, \tilde{\alpha} \circ d_0) = m \circ (\tilde{\alpha} \circ d_1, \varphi_1)$ consider the following diagram:

$$\begin{tikzcd}
G_1 \arrow[r, "(\psi_1, \tilde{\alpha} \circ d_0)"] \arrow[d, "(\tilde{\alpha} \circ d_1, \varphi_1)"'] \arrow[dr, phantom, "(1)"] & H_1 \times_{H_0} H_1 \arrow[rr, "(\eta_1, \eta_1)"] \arrow[d, "m"] & & K_1 \times_{K_0} K_1 \arrow[dd] \\
H_1 \times_{H_0} H_1 \arrow[r, "m"'] \arrow[d, "(\eta_1, \eta_1)"'] & H_1 \arrow[drr, "\eta_1"] & & \\
K_1 \times_{K_0} K_1 \arrow[rrr, "m"'] & & & K_1
\end{tikzcd}$$

Commutativity of the outer square follows from the fact that $\alpha\colon \eta \circ \varphi \Rightarrow \eta \circ \psi$ is a 2-cell. So $\eta_1 \circ m \circ (\psi_1, \tilde{\alpha} \circ d_0) = \eta_1 \circ m \circ (\tilde{\alpha} \circ d_1, \varphi_1)$. It is clear that $(d_0, d_1) \circ m \circ (\psi_1, \tilde{\alpha} \circ d_0) = (d_0, d_1) \circ m \circ (\tilde{\alpha} \circ d_1, \varphi_1)$ too, so from the pullback (6) above commutativity of square (1) follows. It is clear that $\eta \circ \tilde{\alpha} = \alpha$.

BF5: is clearly satisfied.

4.2. THE FUNCTOR B

In this subsection we will see under what conditions the functor $B\colon (\textit{Etale-Groupoids}) \to (\textit{Etendues})$ as defined in Section 1.2 satisfies the conditions EF1 to EF3 as in Section 3.4 above. In Section 1 we saw already that B is essentially surjective and condition EF2 was checked in Section 1.4. For condition EF3 we will use the following lemmas:

LEMMA 25. *For sober groupoids $\mathcal{G}$ and $\mathcal{H}$ and a pair of morphisms*

$$\varphi, \psi : \mathcal{G} \rightrightarrows \mathcal{H}$$

the functor B induces an isomorphism between the set of 2-cells $\mathrm{Hom}(\varphi, \psi)$ *and the set of 2-isomorphisms* $\mathrm{Isom}(B\varphi, B\psi)$.

Proof. We show first that B is surjective. So let $\alpha\colon B\varphi \Rightarrow B\psi$ be an invertible 2-cell. In our notation this corresponds with a natural transformation, also called $\alpha\colon B\psi^* \Rightarrow B\varphi^*$. Since $(d_0\colon H_1 \to H_0) \in B\mathcal{H}$ we have $\alpha_{H_1} := \alpha_{H_1 \xrightarrow{d_0} H_0} : H_1 \times_{H_0, d_0, \psi} G_0 \to H_1 \times_{H_0, d_0, \varphi} G_0$ over G_0. Now define $\beta\colon G_0 \to H_1$ as

$$\beta(x) = \pi_1 \circ \alpha_{H_1}(i \circ \psi_0(x), x).$$

We see that $\beta(x)\colon \varphi_0(x) \to ?(x)$ and we want to show that $?(x) = \psi_0(x)$ and $B\beta = \alpha$. When we define $\eta\colon \mathcal{G} \to \mathcal{H}$ as

$$\eta_0\colon x \mapsto d_1 \circ \pi_1 \circ \alpha_{H_1}(i \circ \psi_0(x), x),$$

and

$$\begin{aligned}&\eta_1 : g \mapsto \\ &\quad (\pi_1 \circ \alpha_{H_1}(i \circ \psi_0(d_0(g)), d_0(g))) \bullet (\varphi_1(g))^{-1} \\ &\quad \bullet(\pi_1 \circ \alpha_{H_1}(i \circ \psi_0(d_1(g)), d_1(g)))^{-1},\end{aligned}$$

we see that $?(x) = \eta_0(x)$. We will use this in the rest of this proof.

When $U \subset H_0$ is an open subset we will write $H_1(-, U)$ for $d_1^{-1}(U)$. Note that $H_1(-, U) \overset{d_0}{\to} H_0$ is an object of $B\mathcal{H}$. We will write $\alpha_{H_1(-,U)}$ for $\alpha_{H_1(-,U)\overset{d_0}{\to}H_0}$. For every pair $U_1 \subset U_2$ of open subsets of H_1 containing $\psi_0(x)$ we have by naturality of α the following commutative square

$$\begin{tikzcd} \psi^* H_1(-, U_1) \arrow[r, tail] \arrow[d, "\alpha_{H_1(-,U_1)}"'] & \psi^* H_1(-, U_2) \arrow[d, "\alpha_{H_1(-,U_2)}"] \\ \varphi^* H_1(-, U_1) \arrow[r, tail] & \varphi^* H_1(-, U_2) \end{tikzcd}$$

Since $i(\psi_0(x)) \in H_1(-, U_1)$ for every U containing $\psi_0(x)$, we find that $\psi_0(x) \in \overline{\{?(x)\}}$. Now we can define a 2-cell $\rho: B\psi \Rightarrow B\eta$ as follows: Let E be an $\mathcal{H}$-equivariant sheaf. Note that $((B\psi)^* E)_x \cong E_{\psi_0(x)}$ and $((B\eta)^* E)_x \cong E_{\eta_0(x)}$. So it is enough to define $\rho_x: E_{\psi_0(x)} \to E_{\eta_0(x)}$. Let $\sigma: U \to E$ be a representing element of $E_{\psi_0(x)}$. Since $\psi_0(x) \in U$, also $\eta_0(x) \in U$ and σ is a representative of an element of $E_{\eta_0(x)}$ too. Define $\rho_x(\sigma) = \sigma$.

It is not difficult to see that $\rho \circ \alpha = B\beta$ and since α and $B\beta$ are isomorphisms, so is ρ. And we claim that $\overline{\{\psi_0(x)\}} = \overline{\{\eta_0(x)\}}$. For suppose that there exists a neighbourhood V of $\eta_0(x)$ not containing $\psi(x)$, then consider $d_0: d_1^{-1}(V) \to H_0$. This is an $\mathcal{H}$-equivariant sheaf and $i: V \to d_1^{-1}(V)$ is a section representing an element of the stalk $d_1^{-1}(V)_{\eta(x)}$. But this section clearly cannot be extended to a section over a neighbourhood of $\psi_0(x)$. This contradicts the fact that ρ is an isomorphism.

Now by sobriety it follows that $\psi_0(x) = \eta_0(x)$ and ρ is the identity 2-cell. So $B\beta = \alpha$ as required.

To show that B is also injective, consider two 2-cells $\beta_1, \beta_2: G_0 \to H_1, \varphi \Rightarrow \psi$, with the same image under B. Recall that

$$(B\beta_i)_E(x, e) = (x, e \bullet \beta_i(x)).$$

So in particular, taking $E = H_1$ and $e = s(\psi_0(x))$ we find that:

$$\beta_1(x) = i(\psi_0(x)) \circ \beta_1(x) = s(\psi_0(x)) \circ \beta_2(x) = \beta_2(x),$$

for every $x \in G_0$, so $\beta_1 = \beta_2$.

LEMMA 26. *With the same notation as in the previous lemma, when $\mathcal{H}$ is a T_1-groupoid, B: Hom $(\mathcal{G}, \mathcal{H}) \to$ Hom $(B\mathcal{G}, B\mathcal{H})$ is fully faithful.*

Proof. This follows immediately from the proof of the previous proposition, for now the fact that $\psi_0(x) \in \overline{\{?(x)\}}$ implies that $\psi_0(x) = ?(x)$. So we don't need ρ to find that $B\beta = \alpha$.

We conclude:

THEOREM 27. *The functor B induces an equivalence of bicategories*

$$(\textit{2-Iso-Etendues}) \simeq_{bi} (\textit{Etale Groupoids})\,[W^{-1}].$$

THEOREM 28. *The functor B induces an equivalence of bicategories*

$$(T_1\textit{-Etendues}) \simeq_{bi} (T_1\textit{-Etale Groupoids})\,[W^{-1}].$$

5. Topological stacks and étendues

5.1. Topological stacks

In this section we will define a special kind of stacks over the category of topological spaces with the usual Grothendieck topology of covers. We will call them *topological stacks*, since they are analogous to algebraic stacks over the category of schemes. We will first recall the definition of a stack over an arbitrary category $\mathcal{C}$ with a (subcanonical) Grothendieck topology.

Let $\mathcal{S}$ be a category over $\mathcal{C}$ via a functor $p: \mathcal{S} \to \mathcal{C}$. One calls $\mathcal{S}$ a *fibered category* over $\mathcal{C}$ when

(i) For each morphism
$$f: X \to Y,$$
in $\mathcal{C}$ and each object $y \in p^{-1}(Y)$ there is a map
$$\varphi: x \to y,$$
in $\mathcal{S}$ with $p(\varphi) = f$, which is universal in the following sense:

(ii) Given a diagram

$$x \xrightarrow{\varphi} y \xleftarrow{\psi} x'$$

in $\mathcal{S}$ with

$$X \xrightarrow{f} Y \xleftarrow{g} X'$$

as image under p. Then for all $h: X \to X'$ such that $g \circ h = f$ there exists a unique $\chi: x \to x'$ such that $p(\chi) = h$ and $\psi \circ \chi = \varphi$. It follows that x in (i) is unique up to isomorphism and we denote it by f^*y, or by $y\,|\,X$ when the map f is clear from the context. A fibered category $p: \mathcal{S} \to \mathcal{C}$ is called a *stack* when for every covering family of morphisms $\mathcal{U} = \{U_i \to X, i \in I\}$, the canonical map $p^{-1}(X) = \mathcal{S}(X) \to \mathrm{Des}(\mathcal{U})$ is an equivalence of categories. Here $\mathrm{Des}(\mathcal{U})$ denotes the category of descent data relative to the family $\{U_i \to X; i \in I\}$. In other words: $\mathcal{S}$ is a stack iff the following two conditions hold:

(a) (arrows) For any object X in $\mathcal{C}$ and any objects $x, y \in \mathcal{S}(X)$ the functor $\mathcal{C}/X \to$ *Sets* which with any $f: U \to X$ associates $\mathrm{Hom}_{\mathcal{S}(U)}(f^*x, f^*y)$ is a sheaf. Here $\mathrm{Hom}_{\mathcal{S}(U)}(-,-)$ denotes the set of morphisms which are sent to Id_U by p.

(b) (objects) If $\varphi_i: U_i \to X, i \in I$ is a covering family in $\mathcal{C}$, any descent datum relative to the φ_i, for objects in $\mathcal{S}$, is effective; i.e. for each set of objects $u_i \in \mathcal{S}(U_i)$, such that for all $i, j \in I$ there exist isomorphisms $\alpha_{ij}: u_i \,|\, (U_i \times_X U_j) \xrightarrow{\sim} u_j \,|\, (U_i \times_X U_j)$ satisfying the usual cocycle conditions, there exists an object $u \in \mathcal{S}(X)$, which is unique up to isomorphism, such that $u \,|\, U_i \cong u_i$, and these last isomorphisms must be compatible with the α_{ij}.

When all morphisms in the fibers $\mathcal{S}(X), X \in \mathrm{Ob}(\mathcal{C})$, are isomorphisms we call $\mathcal{S}$ a *stack in groupoids*.

Remark 29. In what follows, 'stack' will always mean 'stack in groupoids'.

Note that, since the topology is subcanonical, for each $X \in \mathrm{Ob}(\mathcal{C})$ the Yoneda embedding $\mathrm{y}(X)$ gives a stack with discrete fibers whose objects are the morphisms into X.

Morphisms of stacks $F: \mathcal{S}_1 \to \mathcal{S}_2$ are cartesian functors over $\mathcal{C}$, i.e. for an object $x \in \mathcal{S}_1(X)$ and a morphism $f: Y \to X$ in $\mathcal{C}$ we have $F(f^*(x)) \cong f^*(F(x))$ in the fiber $p^{-1}(Y)$. Note that morphisms $\mathrm{y}(X) \to \mathcal{S}$ correspond to objects of $\mathcal{S}(X)$.

From now on we will assume that $\mathcal{C} =$ *Top*. A stack $\mathcal{S}$ over *Top* is called *topological* when it satisfies the following conditions:

(i) The diagonal $\Delta: \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is *representable*, i.e. for each pair of morphisms $x: \mathrm{y}(X) \to \mathcal{S}$, $y: \mathrm{y}(Y) \to \mathcal{S}$ the pullback $\mathrm{y}(X) \times_{\mathcal{S}} \mathrm{y}(Y)$ is representable (in other words: up to equivalence of stacks over $\mathcal{C}$, of the form $\mathrm{y}(Z)$ for some space Z).

(ii) There exists a 1-morphism $x: \mathrm{y}(X) \to \mathcal{S}$, such that for all $y: \mathrm{y}(Y) \to \mathcal{S}$, the projection morphism $\mathrm{y}(X) \times_{\mathcal{S}} \mathrm{y}(Y) \to \mathrm{y}(Y)$ is surjective and étale. This makes sense, since by (i) this projection comes from a map of spaces $Z \to Y$. (Then x itself is called étale and surjective too.)

Remark that this definition is an analogue of the definition of an 'algebraic stack' (cf. (Deligne–Mumford, 1969) for example) and we will prove analogous results about both structures.

We can make a 2-category of topological stacks (*Top-Stacks*) by defining 2-cells to be the natural isomorphisms of cartesian functors between these stacks. We will spend the rest of this section to prove the following equivalence of 2-categories:

$$(\textit{2-Iso-Etendues}) \simeq (\textit{Top-Stacks}).$$

(Recall that (*2-Iso-Etendues*) is the category of étendues with just the isomorphic 2-cells.)

5.2. THE STACK $S(\mathcal{E})$

Let $\mathcal{E}$ be an étendue. Define a stack $S(\mathcal{E})$ over *Top* as follows: For X a topological space the objects in the fiber $S(\mathcal{E})(X)$ over X are geometric morphisms

$$\mathrm{Sh}(X) \to \mathcal{E}.$$

Morphisms from $\mathrm{Sh}(X) \to \mathcal{E}$ to $\mathrm{Sh}(Y) \to \mathcal{E}$ are of the form:

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[rr, "\tilde{\alpha}"] \arrow[dr] \arrow[rr, phantom, shift right=4, "\alpha \Nearrow"] & & \mathrm{Sh}(Y) \arrow[dl] \\ & \mathcal{E} & \end{tikzcd}$$

where $\tilde{a}$ is the map induced by the morphism $a: X \to Y$ of topological spaces, and α is a natural isomorphism of geometric morphisms. (Recall that α is automatically an isomorphism when $\mathcal{E}$ is a T_1-étendue.) We define $p(\tilde{a}, \alpha) = a$.

THEOREM 30. *Let $\mathcal{E}$ be an étendue and let $S(\mathcal{E})$ be defined as above, then $S(\mathcal{E})$ is a topological stack.*

Proof. It is not difficult to see that $S(\mathcal{E})$ is a stack. For example the condition on descent data holds since $\mathrm{Sh}(X)$ is the lax colimit of the $\mathrm{Sh}(U_i)$ for a cover U_i of X. So descent data with respect to this cover give rise to a unique (up to isomorphism) arrow $\mathrm{Sh}(X) \to \mathcal{E}$. To prove that this stack is topological, we verify the conditions (i) and (ii) above.

(i) To see that the diagonal $\Delta: S(\mathcal{E}) \to S(\mathcal{E}) \times S(\mathcal{E})$ is representable, let $x: \mathrm{y}(X) \to S(\mathcal{E})$ and $y: \mathrm{y}(Y) \to S(\mathcal{E})$ be two stack morphisms corresponding to objects $x: \mathrm{Sh}(X) \to \mathcal{E}$ and $y: \mathrm{Sh}(Y) \to \mathcal{E}$ with the same names. We claim that the fibered product $\mathrm{y}(X) \times_{S(\mathcal{E})} \mathrm{y}(Y)$ of stacks over $S(\mathcal{E})$ is isomorphic to $\mathrm{y}(Z)$, where

$$\begin{tikzcd} \mathrm{Sh}(Z) \arrow[r] \arrow[d] & \mathrm{Sh}(Y) \arrow[d, "y"] \\ \mathrm{Sh}(X) \arrow[r, "x"'] & \mathcal{E} \end{tikzcd}$$

is a pullback of toposes. (This pullback is of this form by Lemma 32 below.) The fiber of $\mathrm{y}(X) \times_{S(\mathcal{E})} \mathrm{y}(Y)$ over a space U consists of triples (f, g, α) where $f: U \to X$ and $g: U \to Y$ are maps and $\alpha: f^*(x) \xrightarrow{\sim} g^*(y)$ is an element of $\mathrm{Hom}_{S(\mathcal{E})(U)}(f^*(x), g^*(y))$. It is clear that such triples correspond precisely to morphisms $\mathrm{Sh}(U) \to \mathrm{Sh}(Z)$ by the universal property of the pullback above. So Z represents the pullback and the diagonal is representable.

(ii) Let $\mathcal{E} \simeq B\mathcal{G}$ where $\mathcal{G}$ is an étale groupoid. Then we claim that the morphism

$$\mathrm{y}(G_0) \to S(\mathcal{E})$$

induced by

$$\varphi: \mathrm{Sh}(G_0) \to \mathcal{E},$$

where φ^* is just the forgetful functor, is the required étale surjection. So let $x: \mathrm{y}(X) \to S(\mathcal{E})$ be another morphism of stacks, where x is induced by $x: \mathrm{Sh}(X) \to \mathcal{E}$ and consider the pullback

$$\begin{tikzcd} \mathrm{y}(P) \arrow[r] \arrow[d] & \mathrm{y}(G_0) \arrow[d, "\varphi"] \\ \mathrm{y}(X) \arrow[r, "x"'] & S\mathcal{E} \end{tikzcd}$$

where P comes from the pullback

$$\begin{tikzcd} \mathrm{Sh}(P) \arrow[r] \arrow[d] & \mathrm{Sh}(G_0) \arrow[d, "\varphi"] \\ \mathrm{Sh}(X) \arrow[r, "x"'] & \mathcal{E}. \end{tikzcd}$$

Now we see that $P = x^*(G_1 \xrightarrow{d_0} G_0)$ and therefore it is an étale surjection over X. This proves our claim.

Remark 31. The fact that $S(\mathcal{E})$ is a stack (not necessarily topological) for any topos $\mathcal{E}$ was shown in (Bunge, 1990). Moreover, if $\mathcal{G}$ is an étale complete and open groupoid, $S(B\mathcal{G})$ is shown therein to be the stack completion of $\mathcal{G}$ for the class of open surjections.

LEMMA 32. *Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{E} \simeq B\mathcal{G}$. Then the fibred product of two geometric morphisms $x: \mathrm{Sh}(X) \to \mathcal{E}$ and $y: \mathrm{Sh}(Y) \to \mathcal{E}$ over $\mathcal{E}$, where X and Y are topological spaces, is again of the form* $\mathrm{Sh}(Z)$ *for some topological space Z.*

Proof. First remark that when $X = G_0$, then $Z = y^*(G_1 \overset{d_0}{\to} G_0)$, so it is a topological space. For the general case, consider the following diagram

$$\begin{tikzcd}
? \arrow[d] \arrow[ddr] & & Sh(V \times_{G_0} U) \arrow[ll] \arrow[d] \arrow[ddr] & & Sh(V \times_Y V \times_{G_1} U \times_X U) \arrow[ll, shift right, "(\pi_2,\pi_4)"'] \arrow[ll, shift left, "(\pi_1,\pi_3)"] \arrow[d, "(\pi_1,\pi_2)"'] \arrow[ddr] & \\
Sh(Y) \arrow[ddr] & & Sh(V) \arrow[ll] \arrow[ddr] & & Sh(V \times_Y V) \arrow[ll, shift right] \arrow[ll, shift left] \arrow[ddr] & \\
& Sh(X) \arrow[d, "x"'] & & Sh(U) \arrow[ll] \arrow[d] & & Sh(U \times_X U) \arrow[ll, shift right, "\pi_2"'] \arrow[ll, shift left, "\pi_1"] \arrow[d] \\
& \mathcal{E} & & Sh(G_0) \arrow[ll] & & Sh(G_1) \arrow[ll, shift right, "d_1"'] \arrow[ll, shift left, "d_0"]
\end{tikzcd}$$

Here all commuting squares are pullbacks. We claim that

$$V \times_Y \; V \times_{G_0} U \times_X U \underset{(\pi_2,\pi_4)}{\overset{(\pi_1,\pi_3)}{\rightrightarrows}} V \times_{G_0} U$$

defines an equivalence relation and moreover both maps are étale as pullbacks of étale maps. Therefore their coequalizer exists in the category of spaces and ? above is the topos of sheaves on this coequalizer.

Proof of the claim: The map

$$(\pi_1, \pi_3, \pi_2, \pi_4)\colon V \times_Y V \times_{G_0} U \times_X U \to V \times_{G_0} U \times V \times_{G_0} U,$$

is clearly a monomorphism. And the diagonal

$$\Delta\colon V \times_{G_0} U \to V \times_{G_0} U \times V \times_{G_0} U,$$

factors through $(\pi_1, \pi_3, \pi_2, \pi_4)$ via $\Delta_V \times \Delta_U$, so the relation is reflexive. To check that this relation is symmetric define

$$\tau\colon V \times_Y V \times_{G_0} U \times_X U \to V \times_Y V \times_{G_0} U \times_X U,$$

as

$$\tau = (\pi_2, \pi_1, \pi_4, \pi_3).$$

It is clear that $(\pi_1, \pi_3) \circ \tau = (\pi_2, \pi_4)$ and $(\pi_2, \pi_4) \circ \tau = (\pi_1, \pi_3)$. Finally consider the pullback

$$\begin{tikzcd}
T \arrow[r, "p_2"] \arrow[d, "p_1"'] & V \times_Y V \times_{G_0} U \times_X U \arrow[d, "(\pi_1,\pi_3)"] \\
V \times_Y V \times_{G_0} U \times_X U \arrow[r, "(\pi_2,\pi_4)"'] & V \times_{G_0} U
\end{tikzcd}$$

The condition that $((\pi_1, \pi_3) \circ p_1, (\pi_2, \pi_4) \circ p_2)$ factors through $((\pi_1, \pi_3), (\pi_2, \pi_4))$ is trivially satisfied. This proves the lemma.

Remark 33. Viewing topological spaces as discrete groupoids, i.e. as groupoids with only identity arrows, we have a prestack $\mathrm{Hom}(-, \mathcal{G})$. $S(B\mathcal{G})$ is the stack completion of this prestack.

5.3. THE FUNCTOR S

A morphism of étendues $f: \mathcal{E} \to \mathcal{F}$ induces a morphism $S(f): S(\mathcal{E}) \to S(\mathcal{F})$ of stacks by composition:

- On objects: $S(f)(X)(\varphi: \mathrm{Sh}(X) \to \mathcal{E}) = (f \circ \varphi: \mathrm{Sh}(X) \to \mathcal{F})$
- On morphisms: the image of a triangle

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\varphi"'] \arrow[r, phantom, "\alpha \Nearrow" near start] & & \mathrm{Sh}(Y) \arrow[dl, "\psi"] \\
& \mathcal{E} &
\end{tikzcd}$$

under $S(f)$ becomes

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "f \circ \varphi"'] \arrow[r, phantom, "f\alpha \Nearrow" near start] & & \mathrm{Sh}(Y) \arrow[dl, "f \circ \psi"] \\
& \mathcal{F} &
\end{tikzcd}$$

It is clear that $S(f)$ is a cartesian functor over *Top*. Furthermore let $\eta: f \Rightarrow g: \mathcal{E} \to \mathcal{F}$ be a 2-isomorphism between two étendue morphisms. We define a 2-cell $S(\eta): S(f) \Rightarrow S(g)$ as follows: let $\varphi: \mathrm{Sh}(X) \to \mathcal{E}$ be an object of $S(\mathcal{E})$, then $S(\eta)_\varphi: f \circ \varphi \overset{\sim}{\Rightarrow} g \circ \varphi$ is the pair $(I_{Sh(X)}, \eta \circ \varphi)$. Now our aim is to prove the following theorem:

THEOREM 34. *S defines an equivalence of 2-categories*

$$(2\text{-}\mathit{Iso}\text{-}\mathit{Etendues}) \simeq (\mathit{Top}\text{-}\mathit{stacks})$$

COROLLARY 35. *There exists an equivalence of bicategories*

$$(\mathit{Top}\text{-}\mathit{Stacks}) \simeq (\mathit{Etale}\text{-}\mathit{Groupoids})\,[W^{-1}]$$

The hardest part of the proof is to show that S is essentially surjective, which we will do in the next subsection. The other parts will be proved in the last subsection.

5.4. THE GROUPOID $X_{\mathcal{T}}$

Let $\mathcal{T}$ be a topological stack and choose an étale surjective chart $x: \mathrm{y}(X) \to \mathcal{T}$ of $\mathcal{T}$. We define the groupoid $X_{\mathcal{T}}$ as follows:

The space of objects is X and the space of morphisms is $X \times_{\mathcal{T}} X$. Domain and codomain are given by π_1 and π_2, whereas $i: X \to X \times_{\mathcal{T}} X$ is Δ, the diagonal. The composition $\mu: X_1 \times_{d_0, X_0, d_1} X_1 \to X_1$ is the unique map in the following diagram

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[dr, dashed, "\mu"] \arrow[drr, "d_1 \circ \pi_2"] \arrow[ddr, "d_0 \circ \pi_1"'] & & \\
& X_1 \arrow[r, "d_1"'] \arrow[d, "d_0"] & X_0 \arrow[d, "x"] \\
& X_0 \arrow[r, "x"'] & \mathcal{T}
\end{tikzcd}$$

We claim that $\mathcal{T}$ is equivalent to $S(B\,X_{\mathcal{T}})$ as categories over *Top* (note that $B\,X_{\mathcal{T}}$ does not really depend on the chart X that was chosen: when $y: \mathrm{y}\,Y \to \mathcal{T}$ is another chart, there is a common refinement $X \times_{\mathcal{T}} Y$ and we have weak equivalences of groupoids $X_{\mathcal{T}} \leftarrow (X \times_{\mathcal{T}} Y)_{\mathcal{T}} \to Y_{\mathcal{T}}$). To prove this claim we construct a functor

$$G: \mathcal{T} \to S(B\,X_{\mathcal{T}}).$$

To define G on objects, let $\mathrm{y}(Y) \xrightarrow{y} \mathcal{T}$ represent an object $y \in \mathcal{T}(Y)$. Consider the pullback cube

$$\begin{tikzcd}
& X \times_{\mathcal{T}} Y \times_{\mathcal{T}} X \arrow[rr, "(\pi_3, \pi_2)"] \arrow[dl, "(\pi_1, \pi_2)"'] \arrow[dd, "f_1"] & & X \times_{\mathcal{T}} Y \arrow[dl, "\pi_2"'] \arrow[dd, "f_0"] \\
X \times_{\mathcal{T}} Y \arrow[rr, "\pi_2"] \arrow[dd, "f_0"'] & & Y \arrow[dd, "y"] & \\
& X \times_{\mathcal{T}} X \arrow[rr, "\pi_2"'] \arrow[dl, "\pi_1"] & & X \arrow[dl, "x"] \\
X \arrow[rr, "x"'] & & \mathcal{T} &
\end{tikzcd}$$

(In this cube all faces are pullback squares.) Now remark that since (π_1, π_2), (π_3, π_2) and π_2 in the top face are all étale surjections, the left morphism of groupoids in the following diagram is a weak equivalence:

$$
\begin{tikzcd}
Y \arrow[d, shift right, "I_Y"'] \arrow[d, shift left, "I_Y"] & X \times_{\mathcal{T}} Y \times_{\mathcal{T}} X \arrow[l, "w_1:=\pi_2\circ(\pi_3,\pi_2)"'] \arrow[r, "f_1"] \arrow[d, shift right, "(\pi_1,\pi_2)"'] \arrow[d, shift left, "(\pi_3,\pi_2)"] & X \times_{\mathcal{T}} X \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\
Y & X \times_{\mathcal{T}} Y \arrow[l, "w_0:=\pi_2"] \arrow[r, "f_0"'] & X
\end{tikzcd}
$$

(Notation: we will denote a groupoid of the same form as the middle one by $\mathrm{Gr}\,(\mathcal{T}, X, x, Y, y)$ and the left one will be denoted by Y_{dis}.) So

$$Bf \circ (Bw)^{-1}: \mathrm{Sh}\,(Y) \to B\,X_{\mathcal{T}}, \tag{7}$$

is an object of $S\,(B\,X_{\mathcal{T}})\,(Y)$. Let $G\,(y)$ be this object.

Remark that to define G on morphisms, it is sufficient to define G on fiber morphisms by the conditions on a fibered category. So let $\alpha \in \mathrm{Hom}_{\mathcal{T}\,(Y)}\,(y_1, y_2)$. The morphisms y_1 and y_2 give rise to the following diagram of étale groupoids

$$
\begin{tikzcd}
Gr\,(\mathcal{T}, X, x, Y, y_1) \arrow[dr] \arrow[drr] & & \\
 & Y_{\mathrm{dis}} & X_{\mathcal{T}} \\
Gr\,(\mathcal{T}, X, x, Y, y_2) \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Now consider the groupoid $\mathrm{Gr}\,(\mathcal{T}, X, x, Y, y_1, y_2)$

$$X \times_{\mathcal{T},x,x} X \times_{\mathcal{T},x,y_1} Y \times_{\mathcal{T},y_2,x} X \times_{\mathcal{T},x,x} X \underset{(\pi_2,\pi_3,\pi_4)}{\overset{(\pi_1,\pi_3,\pi_4)}{\rightrightarrows}} X \times_{\mathcal{T},x,y_1} Y \times_{\mathcal{T},y_2,x} X.$$

There are evident projection morphisms

$$\mathrm{Gr}\,(\mathcal{T}, X, x, Y, y_1, y_2) \overset{p_1}{\to} \mathrm{Gr}\,(\mathcal{T}, X, x, Y, y_1),$$

and

$$\mathrm{Gr}\,(\mathcal{T}, X, x, Y, y_1, y_2) \overset{p_2}{\to} \mathrm{Gr}\,(\mathcal{T}, X, x, Y, y_2),$$

making the left-hand square commute in the following diagram

$$\begin{tikzcd}
 & \mathrm{Gr}(\mathcal{T},X,x,Y,y_1) \arrow[dr] \arrow[drr] & & \\
\mathrm{Gr}(\mathcal{T},X,x,Y,y_1,y_2) \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & Y_{\mathrm{dis}} & X_{\mathcal{T}} \\
 & \mathrm{Gr}(\mathcal{T},X,x,Y,y_2) \arrow[ur] \arrow[urr] & &
\end{tikzcd} \tag{8}$$

To define a 2-cell $G(y_1) \stackrel{\sim}{\Rightarrow} G(y_2)$, it is sufficient to define one for the large square in this diagram. So we must construct an appropriate morphism $X \times_{\mathcal{T},x,y_1} Y \times_{\mathcal{T},y_2,x} X \to X \times_{\mathcal{T},x,x} X$. Note that $X \times_{\mathcal{T},x,y_1} Y \times_{\mathcal{T},y_2,x} X \simeq (X \times_{\mathcal{T},x,y_1} Y) \times_Y (Y \times_{\mathcal{T},y_2,x} X) \simeq (X \times_{\mathcal{T},x,y_1} Y) \times_Y (X \times_{\mathcal{T},x,y_2} Y) \simeq \mathrm{Hom}_{\mathcal{T}(X\times Y)} (X \times Y, \pi_1^* x, \pi_2^* y_1) \times_Y \mathrm{Hom}_{\mathcal{T}(X\times Y)} (X \times Y, \pi_1^* x, \pi_2^* y_2)$. So a point of this space corresponds to a pair (β_1, β_2) with $\beta_i \colon \pi_1^* x \stackrel{\sim}{\Rightarrow} \pi_2^* y_i$, $(i = 1,2)$. Composition of these isomorphisms with $\pi_2^*\alpha \colon \pi_2^* y_1 \stackrel{\sim}{\Rightarrow} \pi_2^* y_2$ gives an isomorphism $\pi_1^* x \stackrel{\sim}{\Rightarrow} \pi_3^* x$, i.e. a 2-cell in the following square:

$$\begin{tikzcd}
X \times_{\mathcal{T},x,y_1} Y \times_{\mathcal{T},y_2,x} X \arrow[r, "\pi_3"] \arrow[d, "\pi_1"'] & X \arrow[d, "x"] \\
X \arrow[r, "x"'] & \mathcal{T}
\end{tikzcd}$$

And this induces a unique (up to 2-isomorphism) map

$$X \times_{\mathcal{T},x,y_1} Y \times_{\mathcal{T},y_2,x} X \xrightarrow{\bar{\alpha}} X \times_{\mathcal{T},x,x} X,$$

which defines the required 2-cell in diagram (8). This finishes our definition of G.

To show that G induces an equivalence of categories we must prove the following two facts (cf. (Maclane, 1971), p. 91):

- G is fully faithful;
- each object $\mathrm{Sh}(Z) \xrightarrow{\varphi} B X_{\mathcal{T}}$ is isomorphic to $G(z)$ for some $z \in \mathcal{T}(Z)$.

To establish that G is fully faithful, we construct an inverse on the hom sets $\mathrm{Hom}_{\mathcal{S}(BX_{\mathcal{T}})(Y)} (G(y_1), G(y_2)) \to \mathrm{Hom}_{\mathcal{T}(Y)}(y_1, y_2)$. A 2-cell $\varphi \colon G(y_1) \stackrel{\sim}{\Rightarrow} G(y_2)$ is an isomorphism between geometric morphisms of the form (7). So it can be represented by a diagram of the form (8), with an arbitrary groupoid $\mathcal{H}$ instead of $\mathrm{Gr}(\mathcal{T}, X, x, Y, y_1, y_2)$. Therefore φ induces a map $H_0 \to X \times_{\mathcal{T}} X$ and étale maps $p_0^i\colon H_0 \to X \times_{\mathcal{T},x,y_i} Y$ such that $\pi_2 \circ p_0^i$ is an étale surjection and $\pi_i \circ \varphi = f_0^i \circ p_0^i$, where f_0^i is the pullback of y_i along x. So we have the following diagram

$$\begin{tikzcd}
 & H_0 \arrow[rr, "p_0^2"] \arrow[dl, "p_0^1"'] \arrow[dd, "\varphi"] & & X \times_{\mathcal{T},x,y_2} Y \arrow[dl, "\pi_2^2"'] \arrow[dd, "f_0^2"] \\
X \times_{\mathcal{T},x,y_1} Y \arrow[rr, "\pi_2^1"] \arrow[dd, "f_0^1"'] & & Y \arrow[dd, shift right, "y_1"'] \arrow[dd, shift left, "y_2"] & \\
 & X \times_{\mathcal{T}} X \arrow[rr, "\pi_2"'] \arrow[dl, "\pi_1"] & & X \arrow[dl, "x"] \\
X \arrow[rr, "x"'] & & \mathcal{T} &
\end{tikzcd}$$

We see that $y_1 \circ \pi_2^1 \circ p_0^1 \cong x \circ f_0^1 \circ p_0^1 \cong x \circ \pi_1 \circ \varphi \cong x \circ \pi_2 \circ \varphi \cong x \circ f_0^2 \circ p_0^2 \cong y_2 \circ \pi_2^2 \circ p_0^2 \cong y_2 \circ \pi_2^1 \circ p_0^1$ and since $\pi_2^1 \circ p_0^1$ is an étale surjection, this induces an isomorphism $y_1 \cong y_2$.

Now we will show that G is essentially surjective. So let $\varphi\colon \mathrm{Sh}(Z) \to B(X_{\mathcal{T}})$. This morphism 'corresponds to' a diagram

$$\begin{tikzcd}
Z \arrow[d, shift right, "I_z"'] \arrow[d, shift left, "I_z"] & G_1 \arrow[l, two heads, "w_1"'] \arrow[r, "f_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & X \times_{\mathcal{T}} X \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\
Z & G_0 \arrow[l, two heads, "w_0"] \arrow[r, "f_0"'] & X
\end{tikzcd}$$

i.e. $Bf \circ (Bw)^{-1} \cong \varphi$ (w is a weak equivalence). But this gives precisely a descent datum on Z via composition with $x\colon X \to \mathcal{T}$. Choose an amalgamation $z\colon Z \to \mathcal{T}$ and it is clear that $G(z)$ is isomorphic to φ. This ends the proof that S is essentially surjective.

5.5. Properties of S

In this section we prove the remaining part of the main theorem. We saw already that S is essentially surjective, so we must show that S is essentially full and fully faithful on 2-cells.

PROPOSITION 36. *S is essentially full.*

Proof. Let $S(\mathcal{E}) \xrightarrow{F} S(\mathcal{F})$ be a morphism of topological stacks, where $\mathcal{E} \simeq B\mathcal{G}$ and $\mathcal{F} \simeq B\mathcal{H}$. Then G_0 (resp. H_0) is an étale chart of $S(\mathcal{E})$ (resp. $S(\mathcal{F})$). The following diagram shows that $G_0 \times_{S(\mathcal{F})} H_0$ is another étale surjective chart of $S(\mathcal{E})$

$$\begin{tikzcd}
G_0 \times_{S(\mathcal{F})} H_0 \arrow[r] \arrow[d] & S(\mathcal{E}) \times_{S(\mathcal{F})} H_0 \arrow[r] \arrow[d] & H_0 \arrow[d] \\
G_0 \arrow[r] & S(\mathcal{E}) \arrow[r, "F"'] & S(\mathcal{F})
\end{tikzcd} \tag{9}$$

and moreover the induced groupoid $(G_0 \times_{S(\mathcal{F})} H_0)_{S(\mathcal{E})}$ is weakly equivalent to $\mathcal{G}$, so $B(\mathcal{G}) \simeq B((G_0 \times_{S(\mathcal{F})} H_0)_{S(\mathcal{E})})$. Let $\overline{F}$ be the composition of the upper morphisms in (9). This $\overline{F}$ gives a morphism of groupoids $(G_0 \times_{S(\mathcal{F})} H_0)_{S(\mathcal{E})} \to \mathcal{H}$, and therefore a geometric morphism:

$$\mathcal{E} \simeq B((G_0 \times_{S(\mathcal{F})} H_0)_{S(\mathcal{E})}) \to \mathcal{F}.$$

It is clear that the S-image of this morphism is isomorphic to F.

LEMMA 37. *S is fully faithful on 2-cells.*

Proof. This follows immediately from Lemma 25.

6. Differentiable stacks and differentiable étendues

In this and the next section we will see that the construction of the previous section also works in the context of differentiable manifolds and schemes instead of topological spaces. This will give us results about differentiable stacks (to be defined below) and differentiable étendues. And, in the next section, about étale groupoids in schemes (algebraic groupoids) and algebraic stacks.

6.1. Differentiable Étendues

Let us first recall the definition of a *differentiable étendue* (cf. (Grothendieck *et al.*, 1972), p. 484).

DEFINITION 38. A ringed Grothendieck topos $(\mathcal{E}, R)$ is called a *differentiable étendue* when there exists an object $U \twoheadrightarrow 1$ in $\mathcal{E}$ such that $(\mathcal{E}/U, \pi_U^* R) \simeq (\mathrm{Sh}(M), C^\infty(M))$ for a differentiable (not necessarily Hausdorff) manifold M. Here $C^\infty(M)$ is the sheaf of germs of smooth functions.

It is not difficult to see that the correspondence between étendues and étale groupoids restricts to a correspondence between differentiable étendues and étale groupoids in the category of differentiable manifolds (differentiable étale groupoids for short). Here and in the rest of this paper a manifold need not be Hausdorff.

We will give a sketch of this correspondence: Let $\mathcal{E}$ be a differentiable étendue, then the groupoid is defined by

$$M \times_{\mathcal{E}} M \rightrightarrows M,$$

as before. $M \times_{\mathcal{E}} M$ gets its manifold structure by computing it as a pullback of ringed toposes (which gives the right structure, since $M \times_{\mathcal{E}} M \to M$ is étale).

When we start with a differentiable étale groupoid

$$G_1 \rightrightarrows G_0,$$

the corresponding étendue is just $(B\mathcal{G}, C^{\infty}(G_0))$. Note that $C^{\infty}(G_0)$ is a $\mathcal{G}$-equivariant sheaf by the following action of G_1: let $f: U \to \mathcal{R}$ be a differentiable map representing an element of $C^{\infty}(G_0)_x, x \in U$, and let $g \in d_1^{-1}(x) \subset G_1$. Since d_0 and d_1 are local homeomorphisms there exists an open neighbourhood $V \subset G_1$ of g such that $d_i: V \xrightarrow{\sim} d_i(V)$ for both $i = 1$ and $i = 2$. Let $W = U \cap d_1(V)$ and $\overline{f} := f \mid W$, then $\overline{f}$ represents the same element of $C^{\infty}(G_0)_x$. Now $d_1 \circ d_0^{-1}: d_0(d_1^{-1}(W)) \to W$ is a differentiable map and composition with $\overline{f}$ gives the element $f \bullet g := \overline{f} \circ d_1 \circ d_0^{-1}: d_0(d_1^{-1}(W)) \to \mathcal{R}$ in $C^{\infty}(G_0)_{d_0(g)}$, which defines the action of G_1. (It is not difficult to prove that this is well defined and satisfies the conditions on an action.)

THEOREM 39. *A ringed Grothendieck topos $(\mathcal{E}, R)$ is a differentiable étendue if and only if there exists a differentiable étale groupoid $\mathcal{G}$ such that $(\mathcal{E}, R) \simeq (B\mathcal{G}, C^{\infty}(G_0))$.*

Proof. This can be established in the same way as Theorem 9, when we use the following results by Godement on quotients and pullbacks of manifolds (cf. (Serre, 1965)):

First: for a manifold X and an equivalence relation $R \subset X \times X$ the following are equivalent

1. X/R is a manifold, that is, R is regular.
2. (a) R is a submanifold of $X \times X$.
   (b) $\pi_2: R \to X$ is a submersion.

Second: let $f_i: Y_i \to X, i = 1, 2$ be a pair of differentiable maps, where one of them is a submersion. Then f_1 and f_2 are everywhere transversal and therefore $Y_1 \times_X Y_2$ is a submanifold of $Y_1 \times Y_2$.

It is clear that the functor B sends differentiable maps of differentiable étale groupoids to morphisms of ringed toposes, and that it sends weak equivalences to equivalences of toposes. And the only difficulty in proving the next theorem (the rest is the same as for Theorem 28) is to make K_0 in the following diagram a manifold by taking a pullback of ringed toposes (and then it becomes $(K_0, w_0^*(C^\infty(G_0)))$. This is possible since w_0 is étale.

$$\begin{tikzcd} K_0 \arrow[rr, "\varphi_0"] \arrow[d, "w_0"'] & & H_0 \arrow[d, "\pi_H"] \\ G_0 \arrow[r, "\pi_G"'] & BG \arrow[r, "f"'] & BH \end{tikzcd}$$

Since manifolds are automatically T_1-spaces we have:

THEOREM 40. *The functor B induces an equivalence of 2-categories*

$$(\textit{Differentiable-Etendues}) \simeq (\textit{Differentiable-Etale-Groupoids})\,[W^{-1}].$$

6.2. Differentiable stacks

To come to the main subject of this section let us describe a differentiable stack. A *stack in groupoids* $\mathcal{S}$ over the category of differentiable manifolds is called *differentiable* when the following condition holds:

6.2.0.1. There exists a 1-morphism $x: \mathrm{y}(X) \to \mathcal{S}$, such that for all $y: \mathrm{y}(Y) \to \mathcal{S}$ the pullback $\mathrm{y}(X) \times_{\mathcal{S}} \mathrm{y}(Y)$ is representable and the second projection $\mathrm{y}(X) \times_{\mathcal{S}} \mathrm{y}(Y) \to \mathrm{y}(Y)$ is differentiable, surjective and étale.

6.2.0.2. Remark that we do not require the diagonal to be representable now. (This would not even be true for representable stacks, since the category of differentiable manifolds is not closed under pullbacks.) We just want the pullbacks along the chart x to be representable.

We will show that we have again an equivalence of 2-categories:

THEOREM 41. *The following 2-categories are equivalent*

$$(\textit{Differentiable-Stacks}) \simeq (\textit{Differentiable-Etendues}). \tag{10}$$

In particular we can view the 2-category of differentiable stacks as a bicategory of fractions

COROLLARY 42. *There is a canonical equivalence of bicategories*

$$(\textit{Differentiable-Stacks}) \simeq (\textit{Differentiable-Etale-Groupoids})\,[W^{-1}].$$

To prove the equivalence (10), we define a functor

$$S: (\textit{Differentiable-Etendues}) \to (\textit{Differentiable-Stacks}),$$

analogous to Section 5.2. So let $(\mathcal{E}, R)$ be a differentiable étendue. Define

$$S(\mathcal{E}, R) \xrightarrow{P} (\textit{Differentiable-manifolds}),$$

as follows:

objects are morphisms of ringed toposes:

$$(\varphi, f)\colon (\mathrm{Sh}(M), C^\infty(M)) \to (\mathcal{E}, R),$$

(so $f\colon \varphi^*(R) \to C^\infty(M)$ is a morphism of sheaves over M);

arrows are triangles

$$\begin{tikzcd}
(\mathrm{Sh}(M), C^\infty(M)) \arrow[rr, "(\beta,b)"] \arrow[dr, "(\varphi,f)"'] \arrow[rr, phantom, shift right=4ex, "\alpha\Uparrow"] & & (\mathrm{Sh}(N), C^\infty(N)) \arrow[dl, "(\psi,s)"] \\
& (\mathcal{E}, R) &
\end{tikzcd}$$

where (β, b) comes from a differentiable map of manifolds $\bar{\beta}\colon M \to N$. Now

$$P((\mathrm{Sh}(M), C^\infty(M)) \xrightarrow{(\varphi,f)} (\mathcal{E}, R)) = M,$$

$$P((\beta, b)) = \bar{\beta}.$$

It is clear that this is a stack, and to see that it is differentiable, let $\mathcal{E} \simeq B\mathcal{G}$, where $\mathcal{G}$ is a differentiable étale groupoid. And consider

$$\varphi\colon (\mathrm{Sh}(G_0), C^\infty(G_0)) \to (\mathcal{E}, R),$$

with φ^* the forgetful functor, as before. As above we can make the pullback P in

$$\begin{tikzcd}
\mathrm{Sh}(P) \arrow[r] \arrow[d] & (\mathrm{Sh}(G_0), C^\infty(G_0)) \arrow[d] \\
(\mathrm{Sh}(X), C^\infty(X)) \arrow[r] & (\mathcal{E}, R)
\end{tikzcd}$$

a differentiable manifold since the projection to X is étale. And it is clear that $\mathrm{y}(P)$ and $\mathrm{y}(G_0) \times_{S(\mathcal{E})} \mathrm{y}(X)$ are equivalent as stacks. We conclude that $S((\mathcal{E}, R))$ as defined above is a differentiable stack. The definition of S on 1- and 2-cells is by composition, completely analogous to that in the topological case.

The proof that S induces an equivalence of 2-categories goes precisely as in the topological context, since all pullbacks and quotients which were used there, satisfy the conditions in the proof of Theorem 39 above, so they exist in the category of differentiable manifolds. Furthermore note that β as constructed in the proof of Lemma 25 is a differentiable map when α is a 2-cell between morphisms of ringed toposes, and vice versa. So S remains fully faithful on 2-cells. This finishes our proof of Theorem 41.

7. Algebraic stacks and étendues

In the case of algebraic stacks over the category of schemes, the previous construction can be used to get a more explicit description, which uses toposes, of the stack associated to an étale groupoid of schemes; and to prove that the category of algebraic stacks is 'the' bicategory of fractions of the 2-category of these groupoids with respect to weak equivalences. To do this we first introduce another special kind of Grothendieck toposes:

7.1. Algebraic Étendues

Fix a base scheme S. Let $\mathcal{T}$ denote the topos of sheaves on the site of all schemes over S with the étale topology.

DEFINITION 43. A Grothendieck topos $\mathcal{E}$ over $\mathcal{T}$ is an *algebraic étendue* when there exists an object $U \twoheadrightarrow 1$ in $\mathcal{E}$ such that $\mathcal{E}/U$ is equivalent to $\mathrm{Sh}\,(X_{\mathrm{et}})$ (where X_{et} is the site of étale schemes over X with the étake topology, see (Milne, 1980)) and $\mathcal{E}/(U \times U)$ is equivalent to $\mathrm{Sh}\,(Y_{\mathrm{et}})$ over $\mathcal{T}$ for some schemes X and Y over S, and the induced projections $Y \rightrightarrows X$ are étale separated surjections. (We call X a *chart* of $\mathcal{E}$.) We define (*Alg. Etendues*) to be the 2-category of algebraic étendues, geometric morphisms and natural transformations.

Remark 44. Algebraic étendues are *not* a special kind of étendue!

DEFINITION 45. We will call an étale separated groupoid in the category of S-schemes an *algebraic groupoid.* Notation: (*Alg. Groupoids*) will denote the 2-category of algebraic groupoids.

It is not difficult to see that $Y \rightrightarrows X$ in the definition above is an algebraic groupoid. Conversely an algebraic groupoid $\mathcal{G} = G_1 \rightrightarrows G_0$ gives rise to a site $\mathcal{G}_{\mathrm{et}}$ which objects are étale schemes over G_0, i.e. étale maps $P\colon E \to G_0$, with a right G_1-action: $\theta\colon E \times_{p,G_0,d_1} G_1 \xrightarrow{\sim} G_1 \times_{d_0,G_0,p} E$; arrows of $\mathcal{G}_{\mathrm{et}}$ are morphisms over G_0 which respect the actions. The topology on this category is again the étale topology. We define $B\,\mathcal{G}_{\mathrm{et}}$ to be the topos of sheaves on this site.

PROPOSITION 46. *$B\,\mathcal{G}_{\mathrm{et}}$ is an algebraic étendue.*

Proof. Let U in the definition above be the object $\mathbf{y}(G_1 \xrightarrow{d_0} G_0)$ with an action by composition. Then

$$B\,\mathcal{G}_{\mathrm{et}}/\mathbf{y}\,(G_1 \overset{d_0}{\to} G_0) \simeq \mathrm{Sh}\,(\mathcal{G}_{\mathrm{et}}/G_1 \overset{d_0}{\to} G_0),$$
$$\simeq \mathrm{Sh}\,((G_0)_{\mathrm{et}}),$$

and

$$B\mathcal{G}_{\mathrm{et}}/(\mathbf{y}(G_1 \overset{d_0}{\to} G_0) \times \mathbf{y}(G_1 \overset{d_0}{\to} G_0))$$
$$\simeq B\mathcal{G}_{\mathrm{et}}/\mathbf{y}(G_1 \overset{d_0}{\to} G_0) \times_{B\mathcal{G}_{\mathrm{et}}} B\mathcal{G}_{\mathrm{et}}/\mathbf{y}(G_1 \overset{d_0}{\to} G_0)$$
$$\simeq \mathrm{Sh}((G_1)_{\mathrm{et}}).$$

It is straightforward to extend the definition of B on arrows so as to get a functor

$$B\colon (\mathit{Alg.\ Groupoids}) \to (\mathit{Alg.\ Etendues}).$$

Since a morphism of groupoids $\varphi\colon \mathcal{G} \to \mathcal{H}$ induces a morphism of sites $\mathcal{H}_{\mathrm{et}} \to \mathcal{G}_{\mathrm{et}}$ (or $\mathcal{G}_{\mathrm{et}} \to \mathcal{H}_{\mathrm{et}}$ in the notation of (Milne, 1980)) by pullback and therefore a morphism of toposes $B\varphi\colon B\,\mathcal{G}_{\mathrm{et}} \to B\,\mathcal{H}_{\mathrm{et}}$.

DEFINITION. A map $\varphi = (\varphi_0, \varphi_1)\colon \mathcal{G} \to \mathcal{H}$ between algebraic groupoids is a *weak equivalence* when φ_0 and φ_1 are étale surjections and the square

$$\begin{tikzcd} G_1 \arrow[r, "\varphi_1 \times \varphi_1"] \arrow[d, "{(d_0,d_1)}"'] & H_1 \arrow[d, "{(d_0,d_1)}"] \\ G_0 \times G_0 \arrow[r, "\varphi_0 \times \varphi_0"'] & H_0 \times H_0 \end{tikzcd}$$

is a pullback.

THEOREM 47. *This functor* $B\colon (\mathit{Alg.\ Groupoids}) \to (\mathit{Alg.\ Etendues})$ *induces an equivalence of bicategories*

$$(\mathit{Alg.\ Groupoids})\,[W^{-1}] \simeq (\mathit{Alg.\ Etendues}),$$

where W is the class of weak equivalences of groupoids.

Proof. The fact that W admits a calculus of fractions, i.e. that the category (*Alg. Groupoids*) $[W^{-1}]$ is well defined, can be proved in the same way as before. It is not difficult to see that B sends weak equivalences to equivalences of toposes.

The first difficulty is in checking condition EF2. So let $\varphi\colon B\mathcal{G}_{\mathrm{et}} \to B\mathcal{H}_{\mathrm{et}}$ be a morphism of algebraic étendues. Consider the pullback

$$\begin{tikzcd}
\mathrm{Sh}((G_0)_{\mathrm{et}})/F \arrow[rr] \arrow[d] & & \mathrm{Sh}((H_0)_{\mathrm{et}}) \arrow[d] \\
\mathrm{Sh}((G_0)_{\mathrm{et}}) \arrow[r] & B\mathcal{G}_{\mathrm{et}} \arrow[r, "\varphi"'] & B\mathcal{H}_{\mathrm{et}}
\end{tikzcd}$$

where $F \simeq \pi^*\varphi^*(\mathbf{y}\,(H_1 \overset{d_0}{\to} H_0))$. A priori F need not be a representable sheaf, but it can be covered by a representable one $\mathbf{y}\,(\overline{F}) \twoheadrightarrow F$, and we have a diagram

$$\begin{tikzcd}
 & Sh((H_1)_{et}) \arrow[rr, "d_1"] \arrow[d, "d_0"] & & Sh((H_0)_{et}) \arrow[d] & \\
 & Sh((H_0)_{et}) \arrow[rr] & & B\mathcal{H}_{et} & \\
Sh(((\overline{F} \times_{G_0} \overline{F}) \times_{\overline{F}} (H_1 \times_{H_0} \overline{F}))_{et}) \arrow[rr, shift left, "\mu"] \arrow[rr, shift right, "\pi_2"'] \arrow[d] & & Sh((H_1 \times_{H_0} \overline{F})_{et}) \arrow[rr] \arrow[d] \arrow[uul] & & Sh((G_0)_{et})/F \arrow[d] \arrow[uul] \\
Sh((\overline{F} \times_{G_0} \overline{F})_{et}) \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] & & Sh(\overline{F}_{et}) \arrow[rr] \arrow[uul] & & Sh((G_0)_{et}) \arrow[uul]
\end{tikzcd}$$

where the lefthand square is a pullback along π_2. The action μ is obtained in the following way: there is a canonical map (induced by the pullbacks) $\eta\colon \overline{F} \times_{G_0} \overline{F} \to H_1$ such that the following square commutes for $i = 1, 2$

$$\begin{tikzcd}
X \times_{G_0} X \arrow[r, "\pi_i"] \arrow[d] & X \arrow[d] \\
H_1 \arrow[r, "d_{i-1}"'] & H_0
\end{tikzcd}$$

and $\mu = (\pi_1, \mathrm{comp}(\varphi_1 \circ (\pi_1, \pi_2), \pi_4))$. Now

$$(\bar{F} \times_{G_0} (\bar{F} \times_{H_0} H_1)) \overset{\mu}{\underset{\pi_2}{\rightrightarrows}} (H_1 \times_{H_0} \bar{F})$$

is an étale equivalence relation in the category of schemes over S and its quotient F can be represented by an algebraic space (see (Knutson, 1971), p. 93), which is étale separated over G_0 and therefore a scheme itself (see loc cit., p. 138). Now we can make an algebraic groupoid $\mathcal{F}$ with scheme of objects F in precisely the same way as before; and there are morphisms of groupoids

$$\mathcal{G} \xleftarrow{w} \mathcal{F} \xrightarrow{f} \mathcal{H},$$

such that $\varphi \circ Bw \cong Bf$.

The second difficulty is in proving that B is fully faithful on 2-cells. First remark that a 2-cell $\alpha\colon B\varphi \Rightarrow B\psi\colon B\mathcal{G}_{\mathrm{et}} \to B\mathcal{H}_{\mathrm{et}}$ gives rise to a map $\alpha_{H_1}\colon G_0 \times_{\varphi_0,H_0,d_0} H_1 \to G_0 \times_{\psi_0,H_0,d_0} H_1$ and since the étale topology is sober we can use the same arguments as in Lemma 25 (note that all 2-cells are isomorphisms).

7.2. Algebraic stacks

In this section our aim is to prove the following theorem:

THEOREM 48. *The 2-categories of algebraic stacks and algebraic étendues are equivalent.*

This equivalence provides us with a more precise description of the stack associated to a groupoid and of the relation between algebraic stacks and algebraic groupoids. Let us first recall the definition of an algebraic stack:

DEFINITION 49. An *algebraic stack* is a stack $\mathcal{S}$ over the category of *Schemes* such that the following conditions hold:

(i) The diagonal $\mathcal{S} \xrightarrow{\Delta} \mathcal{S} \times \mathcal{S}$ is representable and separated;
(ii) There exists an étale surjective morphism of stacks $x\colon \mathbf{y}(X) \to \mathcal{S}$. We call X a *chart* of $\mathcal{S}$.

To prove Theorem 48 above we first construct an algebraic stack to an algebraic étendue. So let $\mathcal{E}$ be such an étendue. Define a stack $\mathcal{S}$ over $\mathcal{T}$ with fibers

$$\mathcal{S}(X)_0 = \mathrm{Hom}(\mathrm{Sh}(X_{\mathrm{et}}), \mathcal{E})$$

and morphisms

$$\begin{tikzcd} \mathrm{Sh}(X_{\mathrm{et}}) \arrow[rr, "a"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\alpha \Nearrow"] & \mathrm{Sh}(Y_{\mathrm{et}}) \arrow[dl, "\psi"] \\ & \mathcal{E} & \end{tikzcd}$$

where a comes from a morphism $\overline{a}\colon X \to Y$ of schemes. The functor $P\colon \mathcal{S} \to \mathcal{T}$ is then defined by $P(\varphi\colon \mathrm{Sh}(X_{\mathrm{et}}) \to \mathcal{E}) = X$ and $P(a,\alpha) = \overline{a}$. It is not difficult to see that this is a stack. To show that it is algebraic, let X_0 be a chart of $\mathcal{E}$ and let

$\mathbf{y}(X_1) = \mathbf{y}(X_0) \times_{\mathcal{E}} \mathbf{y}(X_0)$. Then the object $\mathrm{Sh}((X_0)_{\mathrm{et}}) \xrightarrow{\pi_U} \mathcal{E}$ of $\mathcal{S}$ induces an étale map $\mathbf{y}(X_0) \to \mathcal{E}$, as follows from the proof of Theorem 47 above. Finally to prove that the diagonal is representable, let $y_1 \colon \mathbf{y}(Y_1) \to \mathcal{S}$ and $y_2 \colon \mathbf{y}(Y_2) \to \mathcal{S}$ be maps of stacks represented by objects $\psi_i \colon \mathrm{Sh}(Y_i) \to \mathcal{E}$. Consider the diagram

$$\begin{tikzcd}
Sh(Z_{et}) \arrow[d] \arrow[ddr] & & Sh(F_1 \times_{X_0} F_2) \arrow[ll] \arrow[d] \arrow[ddr] & & Sh(F_1 \times_{Y_1} F_1 \times_{X_1} F_2 \times_{Y_2} F_2) \arrow[ll, shift right, swap, "(\pi_2,\pi_4)"] \arrow[ll, shift left, "(\pi_1,\pi_3)"] \arrow[d, "(\pi_1,\pi_2)"'] \arrow[ddr] & \\
Sh(Y_1) \arrow[ddr, "\psi_1"'] & & Sh(F_1) \arrow[ll] \arrow[ddr] & & Sh(F_1 \times_{Y_1} F_1) \arrow[ll, shift left] \arrow[ll, shift right] \arrow[ddr] & \\
& Sh(Y_2) \arrow[d, "\psi_2"] & & Sh(F_2) \arrow[ll] \arrow[d] & & Sh(F_2 \times_{Y_2} F_2) \arrow[ll, shift right, swap, "\pi_2"] \arrow[ll, shift left, "\pi_1"] \arrow[d] \\
& \mathcal{E} & & Sh(X_0) \arrow[ll] & & Sh(X_1) \arrow[ll, shift right, swap, "d_1"] \arrow[ll, shift left, "d_0"]
\end{tikzcd}$$

where Z is an algebraic space by (Knutson, 1971), p. 93. Now remark that the diagonal $\Delta \colon \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is unramified, since we have an étale map $\mathbf{y}(X_0) \to \mathcal{S}$; and we can write Z also as the pullback

$$\begin{tikzcd}
\mathrm{Sh}(Z_{\mathrm{et}}) \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\
\mathrm{Sh}((Y_1 \times Y_2)_{\mathrm{et}}) \arrow[r] & \mathcal{S} \times \mathcal{S}
\end{tikzcd}$$

and we find that Z is unramified over $Y_1 \times Y_2$, so Z is a scheme itself (see (Knutson, 1971), p. 138). It is clear that a morphism of toposes between algebraic étendues induces a map of stacks by composition. So we have a functor $S \colon$ *(Alg. Etendues)* $\to$ *(Alg. Stacks)* with $S(\mathcal{E}) = \mathcal{S}$ as above, and we can make the Theorem 48 more precise:

THEOREM 50. *The functor $S \colon$ (Alg. Etendues) $\to$ (Alg. Stacks), is an equivalence of 2-categories.*

Proof. To see that S is essentially surjective on objects, let $\mathcal{R}$ be an algebraic stack with chart $X_0 \to \mathcal{R}$ and $X_1 := X_0 \times_{\mathcal{R}} X_0$. Then $X_{\mathcal{R}} := X_1 \rightrightarrows X_0$ is an étale groupoid. Let $\mathcal{E} := B(X_{\mathcal{R}})_{\mathrm{et}}$. Then it is not difficult to see that $S(\mathcal{E}) \simeq \mathcal{R}$. The rest of the proof is completely analogous to that in the topological case.

Remark 51. It follows immediately that the associated stack of an algebraic groupoid $\mathcal{G}$ can be described as $S(B\mathcal{G}_{\mathrm{et}})$.

COROLLARY 52. *There is an equivalence of bicategories*

$$(Alg.\ Groupoids)\,[W^{-1}] \simeq (Alg.\ Stacks).$$

Appendix

A. Appendix associativities and identities

A.1. Generalities on pasting

In this section we will prove some lemmas on pasting with respect to the conditions BF3 and BF4. The consequence of these lemmas is that in certain cases we can first do some pasting before applying condition BF4. We will need this to verify the associativity coherence.

LEMMA 53. *When w_1 and w_2 are 1-arrows in W, any squares*

$$\begin{tikzcd} \bullet \arrow[r, "u_2"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\alpha \Nearrow"] & \bullet \arrow[d, "w_2"] \\ \bullet \arrow[r, "w_1"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\beta \Nearrow"] & \bullet \arrow[d, "w_2"] \\ \bullet \arrow[r, "w_1"'] & \bullet \end{tikzcd}$$

give rise to equivalent 2-cells $(u_1, u_2, \alpha, \alpha) \sim (v_1, v_2, \beta, \beta)\colon (w_1, w_1) \Rightarrow (w_2, w_2)$ in $C[W^{-1}]$.

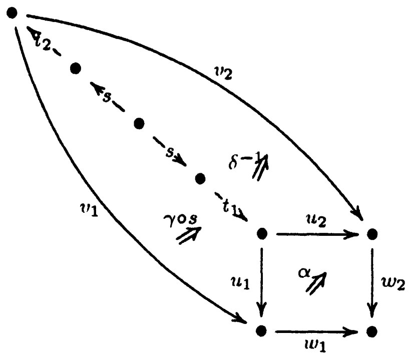
Proof. Consider the following diagram

$$\begin{tikzcd} \bullet \arrow[rrr, bend left, "v_2"] \arrow[ddr, bend right, "v_1"'] & & & \\ & \bullet \arrow[ul, "t_2"'] \arrow[dr, phantom, "\gamma \Nearrow"] & & \\ & & \bullet \arrow[ul, "t_1"'] \arrow[r, "u_2"] \arrow[d, "u_1"] \arrow[dr, phantom, "\alpha \Nearrow"] & \bullet \arrow[d, "w_2"] \\ & & \bullet \arrow[r, "w_1"'] & \bullet \end{tikzcd}$$

where $t_2 \in W$, t_1 and γ exist by condition BF3 for $\bullet \xrightarrow{v_1} \bullet \xleftarrow{u_1} \bullet$. Now we also want to fill out the upper part such that the resulting pasting is something like β. We have a 2-cell from $w_2 \circ v_2 \circ t_2$ to $w_2 \circ u_2 \circ t_1$

$$\begin{tikzcd} & \bullet \arrow[r, "v_2"] \arrow[dr, "v_1"] & \bullet \arrow[dr, "w_2"] \arrow[d, phantom, "\Downarrow\beta"] & \\ \bullet \arrow[ur, "t_2"] \arrow[dr, "t_1"'] \arrow[r, phantom, "\Downarrow\gamma"] & {} & \bullet \arrow[r, "w_1"'] \arrow[d, phantom, "\Downarrow\alpha"] & \bullet \\ & \bullet \arrow[ur, "u_1"] \arrow[r, "u_2"'] & \bullet \arrow[ur, "w_2"'] & \end{tikzcd} \tag{11}$$

So there is a chosen pair (s,δ) such that $\delta\colon v_2 \circ t_2 \circ s \Rightarrow u_2 \circ t_1 \circ s$ and $(11) \circ s = w_2 \circ \delta$ on account of condition BF4. The diagram above becomes



and some elementary calculation shows that this pasting is equal to $\beta \circ t_2 \circ s$. So we get

$$(u_1, u_2, \alpha, \alpha) \sim (v_1, v_2, \beta, \beta).$$

LEMMA 54. *Suppose we have a pair of* 2*-cells* $\alpha_1\colon w_1 \circ u_1 \Rightarrow w_2 \circ u_2$, $\alpha_2\colon v \circ f \circ u_1 \Rightarrow v \circ g \circ u_2$ *in* C *with* w_1, w_2, $w_1 \circ u_1$, $w_2 \circ u_2$ *and* v *in* W *and* f *and* g *arbitrary* 1*-cells.*

$$\begin{tikzcd}
 & C \arrow[rr, "f"] \arrow[dr, "w_1"] & & B \arrow[dr, "v"] & \\
E \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1"] & & A \arrow[rr, phantom, "\Downarrow\alpha_2"] & & B' \\
 & D \arrow[ur, "w_2"'] \arrow[rr, "g"'] & & B \arrow[ur, "v"'] &
\end{tikzcd}$$

Let (s_1, β_1) *and* (s_2, β_2) *be two different choices on account of condition* BF4 *such that* $\alpha_2 \circ s_i = v \circ \beta_i$*. Then* $(u_1 \circ s_1, u_2 \circ s_1, \alpha_1 \circ s_1, \beta_1)$ *and* $(u_1 \circ s_2, u_2 \circ s_2, \alpha_1 \circ s_2, \beta_2)$ *are equivalent 2-cells from* (w_1, f) *to* (w_2, g) *in* $C[W^{-1}]$.

Proof. Let

$$\begin{tikzcd}
\bullet \arrow[r, "t_1"] & \bullet \arrow[d, "s_1"] \\
\bullet \arrow[from=1-1, "t_2"'] \arrow[r, "s_2"'] \arrow[ur, phantom, "\gamma \Nearrow"] & \bullet
\end{tikzcd}$$

be a square with $s_1 \circ t_1$ and $s_2 \circ t_2$ in W as in condition BF3, then

$$\begin{aligned}
(u_1 \circ s_1, u_2 \circ s_1, \alpha_1 \circ s_1, \beta_1) &\sim (u_1 \circ s_1 \circ t_1, u_2 \circ s_1 \circ t_1, \alpha_1 \circ s_1 \circ t_1, \beta_1 \circ t_1) \\
&\sim (u_1 \circ s_2 \circ t_2, u_2 \circ s_2 \circ t_2, \alpha_1 \circ s_2 \circ t_2, \beta_2 \circ t_2) \\
&\sim (u_1 \circ s_2, u_2 \circ s_2, \alpha_1 \circ s_2, \beta_2)
\end{aligned}$$

Remark 55. Given two 2-cells in $\mathcal{C}[W^{-1}]$ as defined in Section 3:

$$(\alpha_1, \alpha_2, u_1, u_2) \colon (w_1, w \circ f) \Rightarrow (w_2, w \circ g),$$

$$\begin{tikzcd}
& C \arrow[rr, "f"] \arrow[dr, "w_1"] & & B \arrow[dr, "w"] & \\
E \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1"] & & A \arrow[rr, phantom, "\Downarrow\alpha_2"] & & B' \\
& D \arrow[rr, "g"'] \arrow[ur, "w_2"'] & & B \arrow[ur, "w"'] &
\end{tikzcd} \tag{12}$$

and

$$(\overline{\alpha}_1, \overline{\alpha}_2, \overline{u}_1, \overline{u}_2) \colon (w_2, w \circ g) \Rightarrow (w_3, w \circ h),$$

$$\begin{tikzcd}
& C \arrow[rr, "g"] \arrow[dr, "w_2"] & & B \arrow[dr, "w"] & \\
E \arrow[ur, "\bar{u}_1"] \arrow[dr, "\bar{u}_2"'] \arrow[rr, phantom, "\Downarrow\bar{\alpha}_1"] & & A \arrow[rr, phantom, "\Downarrow\bar{\alpha}_2"] & & B' \\
& D \arrow[ur, "w_3"'] \arrow[rr, "h"'] & & B \arrow[ur, "w"'] &
\end{tikzcd}$$

where $w \in W$, there are two ways of using vertical composition of 2-cells and applying our choices for condition BF4 to get a 2-cell $(w_1, f) \Rightarrow (w_3, h)$. We can first apply our choices for BF4 two times to remove the w's in the 2-cells above and then take the vertical composition of the resulting 2-cells; or first take the vertical composition and then apply BF4 to remove w. (Applying our choices for BF4 to (12) gives for example

$$\begin{tikzcd}
& & C \arrow[ddr, "w_1"] \arrow[ddrr, "f"] & & \\
& E \arrow[ur, "u_1"] & & & \\
E' \arrow[ur, "\overline{w}"] \arrow[dr, "\overline{w}"'] \arrow[rrr, phantom, "\Downarrow \alpha_1 \circ \overline{w}"] & & & A \arrow[r, phantom, "\Downarrow\beta_2"] & B.) \\
& E \arrow[dr, "u_2"'] & & & \\
& & D \arrow[uur, "v"] \arrow[uurr, "g"'] & &
\end{tikzcd}$$

From the lemmas above it follows (with some computation) that these operations will give equivalent 2-cells $(w_1, f) \Rightarrow (w_3, h)$. We will use this fact in the next section.

A.2. Associativity

Let $(w_1, f_1): A \to B$, $(w_2, f_2): B \to C$ and $(w_3, f_3): C \to D$ be 1-morphisms in $\mathcal{C}[W^{-1}]$. We want to define an associativity 2-cell $a: (w_3, f_3) \circ ((w_2, f_2) \circ (w_1, f_1)) \overset{\sim}{\Rightarrow} ((w_3, f_3) \circ (w_2, f_2)) \circ (w_1, f_1)$. We use the following pictures:

first way of composing

$$
\begin{tikzcd}
 & & & & T_1 \arrow[dl, "u_1"'] \arrow[ddrr, "h_1"] & & & \\
 & & & F_1 \arrow[dl, "v_1"'] \arrow[dr, "g_1"] \arrow[rr, phantom, "\delta_1\Nearrow"] & & & & \\
 & & E_1 \arrow[dl, "w_1"] \arrow[dr, "f_1"] \arrow[rr, phantom, "\gamma_1 \Rightarrow"] & & E_2 \arrow[dl, "w_2"] \arrow[dr, "f_2"] & & E_3 \arrow[dl, "w_3"] \arrow[dr, "f_3"] & \\
 & A & & B & & C & & D
\end{tikzcd}
$$

second way of composing

$$
\begin{tikzcd}
 & & & & T_2 \arrow[ddll, "u_2"'] \arrow[dr, "h_2"] & & & \\
 & & & \arrow[d, phantom, "\delta_2\Searrow"] & & F_2 \arrow[dl, "v_2"'] \arrow[dr, "g_2"] & & \\
 & & E_1 \arrow[dl, "w_1"] \arrow[dr, "f_1"] & & E_2 \arrow[dl, "w_2"] \arrow[dr, "f_2"] \arrow[rr, phantom, "\gamma_2 \Rightarrow"] & & E_3 \arrow[dl, "w_3"] \arrow[dr, "f_3"] & \\
 & A & & B & & C & & D
\end{tikzcd}
$$

Now we take some chosen squares

$$
\begin{tikzcd}
S \arrow[r, "s_2"] \arrow[d, "s_1"'] \arrow[dr, phantom, "\alpha\Nearrow"] & F_2 \arrow[d, "v_2"] \\
F_1 \arrow[r, "g_1"'] & E_2
\end{tikzcd}
$$
with $s_1 \in W$,

$$
\begin{tikzcd}
U_1 \arrow[r, "q_1"] \arrow[d, "t_1"'] \arrow[dr, phantom, "\beta_1\Nearrow"] & S \arrow[d, "s_1"] \\
T_1 \arrow[r, "u_1"'] & F_1
\end{tikzcd}
$$
with $t_1 \in W$ and

$$
\begin{tikzcd}
U_2 \arrow[r, "q_2"] \arrow[d, "t_2"'] \arrow[dr, phantom, "\beta_2\Nearrow"] & T_2 \arrow[d, "u_2"] \\
S \arrow[r, "v_1 \circ s_1"'] & E_1
\end{tikzcd}
$$
with $t_2 \in W$.

So we get pasting squares

$$
\begin{tikzcd}
T_1 \arrow[rrrr, "h_1"] \arrow[ddd, "u_1"'] & & & & E_3 \arrow[ddd, "w_3"] \\
& U_1 \arrow[ul, "t_1"'] \arrow[dr, "q_1"] \arrow[dl, phantom, "\beta_1\Rightarrow"] & & & \\
& & S \arrow[r, "s_2"] \arrow[dll, "s_1"'] \arrow[d, phantom, "\alpha\Rightarrow"] & F_2 \arrow[uur, "g_2"] \arrow[dl, "v_2"'] \arrow[dr, phantom, "\gamma_2\Rightarrow"] & \\
F_1 \arrow[rr, "g_1"'] & & E_2 \arrow[rr, "f_2"'] & & C
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
T_2 \arrow[rrrr, "u_2"] \arrow[ddd, "h_2"'] & & \arrow[d, phantom, "\beta_2\Uparrow"] & & E_1 \arrow[ddd, "f_1"] \\
& U_2 \arrow[ul, "q_2"'] \arrow[dr, "t_2"] & & & \\
& & S \arrow[r, "s_1"] \arrow[dll, "s_2"'] \arrow[d, phantom, "\alpha^{-1}\Rightarrow"] & F_1 \arrow[uur, "v_1"] \arrow[dl, "g_1"'] \arrow[dr, phantom, "\gamma_1\Rightarrow"] & \\
F_2 \arrow[rr, "v_2"'] & & E_2 \arrow[rr, "w_2"'] & & B
\end{tikzcd}
$$

With the same method as in Lemma 53 above we can find $\overline{U_1} \xrightarrow{r_1} U_1$ and $\overline{U_2} \xrightarrow{r_2} U_2$, $\varepsilon_1\colon h_1 \circ t_1 \circ r_1 \Rightarrow g_2 \circ s_2 \circ q_1 \circ r_1$ and $\varepsilon_2\colon s_2 \circ t_2 \circ r_2 \Rightarrow h_2 \circ q_2 \circ r_2$, filling the empty places in such a way that the pastings become $\delta_1 \circ t_1 \circ r_1$ and $\delta_2 \circ q_2 \circ r_2$. Now the associativity 2-cell a can be defined as

Third coordinate

$$
\begin{tikzcd}
& & & T_1 \arrow[dr, "u_1"] \arrow[dd, phantom, "\Downarrow\beta_1"] & & & \\
& \overline{U}_1 \arrow[r, "r_1"] & U_1 \arrow[ur, "t_1"] \arrow[dr, "q_1"] \arrow[dd, phantom, "\Downarrow\eta"] & & F_1 \arrow[dr, "v_1"] & & \\
R \arrow[ur] \arrow[dr] & & & S \arrow[ur, "s_1"'] \arrow[drr, phantom, "\Downarrow w_1\circ\beta_2"] & & E_1 \arrow[dr, "w_1"] & \\
& \overline{U}_2 \arrow[r, "r_2"'] & U_2 \arrow[ur, "t_2"'] \arrow[ddrr, "q_2"'] & & & & A \\
& & & & & E_1 \arrow[ur, "w_1"'] & \\
& & & & T_2 \arrow[ur, "u_2"'] & &
\end{tikzcd}
$$

where η is a chosen square $(\bar{U}_2 \xrightarrow{r_2} U_2 \xrightarrow{t_2} S \in W)$.

Fourth coordinate

$$\begin{tikzcd}
 & & U_1 \arrow[r, "t_1"] & T_1 \arrow[drr, "h_1"] & & & \\
 & \overline{U}_1 \arrow[ur, "r_1"] \arrow[r, "r_1"] & U_1 \arrow[dr, "q_1"] & & & & \\
R \arrow[ur] \arrow[dr] \arrow[rrr, phantom, "\Downarrow\eta"] & & & S \arrow[r, "s_2"'] \arrow[ur, phantom, "\Swarrow\varepsilon_1"] \arrow[d, phantom, "\Searrow\varepsilon_2"] & F_2 \arrow[r, "g_2"'] & E_3 \arrow[r, "f_3"'] & D \\
 & \overline{U}_2 \arrow[r, "r_2"'] \arrow[dr, "r_2"'] & U_2 \arrow[ur, "t_2"'] & & & & \\
 & & U_2 \arrow[r, "q_2"'] & T_2 \arrow[uur, "h_2"'] & & &
\end{tikzcd}$$

With the last remark of the previous section it is possible to prove the associativity coherence axiom: first you do all the pasting (such that you can cancel a lot of things) and then you apply the choices for condition BF4. At the end of the proof we have to apply the procedure from the proof of Lemma 53 several times.

A.3. Identities

Let A be an object of $\mathcal{C}[W^{-1}]$, the *identity* 1-*cell* $I'_A \in \mathcal{C}[W^{-1}](A, A)$ is given by the pair (I_A, I_A) with I_A the identity 1-cell on A in $\mathcal{C}$.

Let A and B be two objects of $\mathcal{C}[W^{-1}]$ and $(v, f) \in \mathcal{C}[W^{-1}](A, B)$, then we define the isomorphism

$$l(A, B)(v, f) \colon (v, f) \circ I'_A \Rightarrow (v, f),$$

as in the following picture

$$\begin{tikzcd}
 & C \arrow[dr, "v"] \arrow[drrr, "I_C"] & & & & \\
 & & A \arrow[dr, "I_A"] & & C \arrow[dr, "f"] & \\
C \arrow[uur, "I_C"] \arrow[ddr, "I_C"'] \arrow[rrr, phantom, "\Downarrow r_v \circ I_C"] & & & A \arrow[rr, phantom, "\Downarrow l_f \circ I_C"] & & B \\
 & & & & & \\
 & C \arrow[uurr, "v"] \arrow[uurrrr, "f"'] & & & &
\end{tikzcd}$$

Recall that the composite is defined as

$$\begin{tikzcd}
& & C \arrow[dl, "v"'] \arrow[dd, "v"] \arrow[dr, "I_C"] & & \\
& A \arrow[dl, "I_A"'] \arrow[dr, "I_A"] \arrow[r, phantom, "\overset{r_v}{\Rightarrow}"] & & C \arrow[dl, "v"'] \arrow[dr, "f"] \arrow[l, phantom, "v\ \overset{l_v}{\Rightarrow}"] & \\
A & & A & & B.)
\end{tikzcd}$$

The isomorphism

$$r(A,B)(v,f)\colon I'_B \circ (v,f) \Rightarrow (v,f),$$

is defined by

$$\begin{tikzcd}
& C \arrow[dr, "I_C"] \arrow[rr, "f"] & & B \arrow[ddr, "I_B"] & \\
& & C \arrow[d, "v"] & & \\
C \arrow[uur, "I_C"] \arrow[dr, "I_C"'] & \Downarrow l_v \circ I_C & A & \Downarrow r_f \circ I_C & B \\
& C \arrow[ur, "v"] \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

It is left to the reader to verify that the above defined isomorphisms a, l and r are natural in their arguments and satisfy the identity coherence axioms.

Acknowledgements

I am much indebted to Ieke Moerdijk for suggesting the subject to me, and for his continuing encouragement and reading of several earlier drafts of this paper. I would also like to thank Mike Johnson for his careful reading of and remarks on an earlier version of this paper.

References

Artin, M.: *Algebraic Spaces*, Yale Math. Monographs 3, Yale University Press, 1971.

Artin, M.: Versal deformations and algebraic stacks, *Inventiones Mathematicae* 27 (1974) 165–189.

Bénabou, J.: Introduction to bicategories, in *Reports of the Midwest Category Seminar*, L.N.M. 47, 1–77, Springer-Verlag, New York, 1967.

Bunge, M.: An application of descent to a classification theorem for toposes, *Math. Proc. Camb. Phil. Soc.* 107–59 (1990) 59–79.

Deligne, P. and Mumford, D.: The irreducibility of the spaces of curves of given genus, *Publ. Math. I.H.E.S.* 36 (1969) 75–109.

Faltings, G. and Chai, C. L.: *Degeneration of Abelian Varieties*, Springer-Verlag, New York, 1990.

Gabriel, P. and Zisman, M.: *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, New York, 1967.

Grothendieck, A. and Verdier, J. L.: *Séminaire de Géometrie Algébrique* IV, L.N.M. 296, Springer-Verlag, New York, 1972.

Johnstone, P. T.: *Topos Theory*, Academic Press, London, 1977.

Joyal, A. and Tierney, M.: An extension of the Galois theory of Grothendieck, *Memoirs of the A.M.S.*, Vol. 51, Nr. 309, 1984.

Kelly, G. M.: On Maclane's conditions for coherence of natural associativities, commutativities, etc., *Journal of Algebra* 1 (1964) 397–402.

Knutson, D.: *Algebraic Spaces*, L.N.M. 203, Springer-Verlag, New York, 1971.

Maclane, S.: *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971.

Milne, J. S.: *Etale Cohomology*, Princeton University Press, Princeton, New Jersey, 1980.

Moerdijk, I.: Toposes and groupoids, in *Categorical Algebra and its Applications* (ed. F. Borceux), L.N.M. 1348, Springer-Verlag, New York, 1988, 280–298.

Moerdijk, I.: The classifying topos of a continuous groupoid. I, *Transactions of the A.M.S.* 310–2 (1988) 629–668.

Moerdijk, I.: Descent theory for toposes, *Bulletin de la Société Mathématique de Belgique* (série A) tome XLI Fasc. 2 (1989) 373–391.

Moerdijk, I.: The classifying topos of a continuous groupoid. II, *Cah. de top. et Géom. Diff. Cat.* 31–2 (1990) 137–168.

Moerdijk, I.: Classifying toposes and foliations, *Ann. Inst. Fourier* 41, Grenoble (1991) 189–209.

Moerdijk, I.: Foliations, groupoids and Grothendieck étendues, *Publ. del Sem. Mat. Garcia de Galdeano*, serie II, no. 17 (1993), Universidad de Zaragoza.

Rosenthal, K.: Etendues and categories with monic maps, *J. Pure and Appl. Algebra* 22 (1981) 193–212.

Serre, J.-P.: *Lie Algebras and Lie Groups*, reprinted as L.N.M. 1500, Springer-Verlag, New York, 1992.