4 Lecture 4: Specialization constructions

4.1 Introduction

We recall that in order to prove Tate's Acyclicity Theorem (i.e., for an affinoid space X, the presheaf $\mathscr{O}_X$ on the category of affinoid subdomains of X – or just rational subdomains – with inclusions as morphisms satisfies the sheaf conditions with respect to *finite* coverings), one reduces to the case of Laurent coverings (in fact, Laurent coverings generated by a single function), on which the statement is checked "by hand". A proof of an analogue of Tate's Acyclicity for adic affinoids satisfying some mild finiteness hypotheses will be absolutely essential, and in order to do so we need to get a handle on the topology of the valuation spectrum of commutative rings A, which will contain adic spectra $\mathrm{Spa}(A, A^+)$ (to be defined in a few weeks).

The reader may refer to [H2, Thm. 2.5] for Huber's proof of the adic version of Tate's acyclicity, under strong finiteness assumptions on the rings involves. In [BV], K. Buzzard and A. Verberkmoes give a proof for very general adic spaces, and in both cases the proof is based on a concrete understanding of specializations. We will address this in a later lecture.

We have already encountered a simple and concrete construction encoding specialization relations, which we shall call *vertical*. Namely, when $A = K$ a field, recall that $\mathrm{Spv}(K)$ is just the Riemann-Zariski space $\mathrm{RZ}(K, \mathbf{Z})$, and a point $v \in \mathrm{Spv}(K)$ is a specialization of $w \in \mathrm{Spv}(K)$, which is to say

$$v \in \overline{\{w\}},$$

if and only if we have inclusions of their respective valuation rings:

$$R_v \subset R_w \subset K$$

where both R_v and R_w have field of fractions K. Given w, the specialization v arises from a valuation on the residue field of R_w, as explained in the construction in the Appendix to Lecture 2.

The goal of this lecture is to describe yet another procedure providing specialization/generization relations, which we shall call *horizontal*, and explain how general specializations/generizations are decribed in terms of these two basic processes.

4.2 Review of vertical generizations

We recall a few facts from the Appendix in Lecture 2.

Given an inclusion of valuation rings

$$R_v \subset R_w \subset K$$

whose fraction fields coincide with K, the value groups

$$\Gamma_v = v(K^\times) = K^\times/R_v^\times \quad \text{and} \quad \Gamma_w = w(K^\times) = K^\times/R_w^\times$$

fit in the short exact sequence

$$1 \to v(R_w^\times) \to \Gamma_v \to \Gamma_w \to 1$$

where $H := v(R_w^\times)$ is a *convex* subgroup of Γ_v (meaning that if $\gamma \in \Gamma_v$ satisfies $h \leq \gamma \leq h'$ for some $h, h' \in H$ then $\gamma \in H$).

The value group Γ_w is endowed with the "quotient ordering" from Γ_v in the following sense. If we are given a totally ordered abelian group Γ and a convex subgroup $H \subset \Gamma$, then a total ordering on $\overline{\Gamma} := \Gamma/H$ is induced by the assignment

$$\overline{\Gamma}_{\leq 1} = (\Gamma_{\leq 1} \cdot H)/H;$$

that is, for all $\overline{\gamma}, \overline{\gamma}' \in \overline{\Gamma}$,

$$\overline{\gamma} \leq \overline{\gamma}' \text{ if and only if } \gamma/\gamma' \in \Gamma_{\leq 1} \cdot H.$$

The above indeed induces a total ordering relation because H is convex, which ensures antisimmetry (i.e., if $\overline{\gamma} \leq \overline{\gamma}'$ and $\overline{\gamma}' \leq \overline{\gamma}$ then $\overline{\gamma} = \overline{\gamma}'$).

Example 4.2.1 Recall that in Example 2.3.4 in Lecture 2 we found a higher-rank point in

$$\mathrm{RZ}(k(\!(x)\!)(\!(y)\!), k(\!(x)\!)[\![y]\!])$$

whose value group is $\mathbf{Z} \times \mathbf{Z}$ with the lexicographic ordering; i.e.,

$$(m,n) < (m',n') \text{ if and only if either } m < m' \text{ or } m = m' \text{ with } n < n'.$$

An example of a convex subgroup is $\{0\} \times \mathbf{Z}$. On the other hand, $\mathbf{Z} \times \{0\}$ is not convex (e.g., $(0,0) < (1,-5) < (1,0)$ but $(1,-5) \notin \mathbf{Z} \times \{0\}$), nor is $\mathbf{Z}$ diagonally embedded in $\mathbf{Z} \times \mathbf{Z}$ (check!). We'll get back to this example later.

Definition 4.2.2 Let $v : A \to \Gamma \cup \{0\}$ be a valuation on A, and call Γ_v its value group; that is, Γ_v is the subgroup of Γ generated by $v(A - \mathfrak{p}_v)$. Let $H \subset \Gamma$ be a convex subgroup. We define

$$v_{/H} : A \to (\Gamma/H) \cup \{0\}$$

by

$$v_{/H}(a) := \begin{cases} v(a) \bmod H & \text{if } v(a) \neq 0, \\ 0 & \text{if } v(a) = 0. \end{cases}$$

Remark 4.2.3 Note that $\mathrm{supp}(v_{/H}) = \mathrm{supp}(v) = \mathfrak{p}_v$. We call $v_{/H}$ a *vertical generaization* because the generization procedure takes place in the fiber of $\mathrm{Spv}(A) \to \mathrm{Spec}(A)$ over $\mathfrak{p}_v$, which is just $\mathrm{RZ}(\kappa(\mathfrak{p}_v), \mathbf{Z})$. Basically, we return to the old picture "$R_v \subset R_{v_{/H}} \subset \kappa(\mathfrak{p}_v)$" fiberwise.

Remark 4.2.3 suggests an exhaustive description of such specializations/generizations:

Proposition 4.2.4 *Let $v : A \to \Gamma \cup \{0\}$ be a valuation, and assume $\Gamma = \Gamma_v$; i.e., Γ is generated by $v(A - \mathfrak{p}_v)$. Then we have a bijective correspondence*

$$\{\text{convex subgroups } H \subset \Gamma\} \leftrightarrow \{\text{vertical generizations of } v\}$$

assigned by $H \mapsto v_{/H}$.

Proof of Proposition 4.2.4. The proof amounts to the previous discussion and the Appendix to Lecture 2. See [Wed, Prop. 2.14] for further details. □

Notice that $H \subset H'$ if and only if $v_{/H'}$ generizes $v_{/H}$. We shall write the specialization relation as

$$v_{/H'} \rightsquigarrow v_{/H}.$$

Remark 4.2.5 Consider the special case when $A = K$ is a field, so v is just a valuation ring R with fraction field K. In this case every prime ideal $\mathfrak{p}$ of R gives rise to a valuation ring $R_{\mathfrak{p}}$ between R and K, and by [Mat, Thm. 10.1(i), (ii)] this construction gives rise to *every* valuation ring between R and K without repetition. So another way to describe Proposition 4.2.4 is as a bijection between $\mathrm{Spec}(R)$ and the set of convex subgroups of the value group $\Gamma_R = K^\times/R^\times$.

4.3 Horizontal specialization

We want to visualize the general process by which a point v in $\mathrm{Spv}(A)$ can specialize to another point w. The picture of $\mathrm{Spv}(A)$ lying over $\mathrm{Spec}(A)$ will guide our visualization. The case when v and w lie in a common fiber has already been discussed, and now we want to explore cases when the supports are distinct. Adapting Definition 4.2.2, we give the following:

Definition 4.3.1 Let $H \subset \Gamma$ be a subgroup, and let $v : A \to \Gamma \cup \{0\}$ be a valuation on A. We define the function

$$v_{|_H} : A \to \Gamma \cup \{0\}$$

by

$$v_{|_H}(a) := \begin{cases} v(a) & \text{if } v(a) \in H \\ 0 & \text{otherwise} \end{cases}$$

We will later call $v_{|_H}$ a *horizontal* specialization of v (once we figure out under what conditions on H this is a valuation, and when so then when it is a specialization of v). We shall often use the notation

$$v' := v_{|_H}.$$

Later we will see that this construction produces unique generic points in (the non-empty) fibers of $\overline{\{v\}}$ over $\mathrm{Spec}(A)$ when $v(A) \not\subset \Gamma_{\leq 1}$ (see Corollary 4.5.5).

To give some motivation for this construction, consider the following example.

Example 4.3.2 Let $A = \mathbf{Z}$. We study the fiber of $\mathrm{Spv}(\mathbf{Z}) \to \mathrm{Spec}(\mathbf{Z})$ over (0). The fiber over $(0) \in \mathrm{Spec}(\mathbf{Z})$ is $\mathrm{Spv}(\mathbf{Q}) = \mathrm{RZ}(\mathbf{Q}, \mathbf{Z})$, which consists of the trivial valuation and the (normalized, say) p-adic valuations for primes p. Let $v_p \in \mathrm{Spv}(\mathbf{Q})$ be the p-adic valuation on $\mathbf{Q}$, p prime. We have $\Gamma_{v_p} = p^{\mathbf{Z}} \simeq \mathbf{Z}$, and the only convex subgroups of $\mathbf{Z}$ are $\{0\}$ and $\mathbf{Z}$ itself. If $H = (0) \subset \mathbf{Z} \simeq \Gamma_{v_p}$, then we have that $v_{/_H}$ is v_p itself. Instead, if $H = \Gamma$, then $v_{/_H}$ is the trivial valuation v_{triv}, and we see that

$$v_{\mathrm{triv}} \rightsquigarrow v_p$$

for all primes p. The trivial valuation is the unique generic point in $\mathrm{Spv}(\mathbf{Q})$. (Topologically, $\mathrm{Spv}(\mathbf{Q}) = \mathrm{Spec}(\mathbf{Z})$.)

Explicitly, we have $\Gamma_{v_p} = p^{\mathbf{Z}}$ and $\Gamma_{v_{\mathrm{triv}}} = (1)$ fit into the following short exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & v_p(R_{v_{\mathrm{triv}}}^{\times}) & \longrightarrow & \Gamma_{v_p} & \longrightarrow & \Gamma_{v_{\mathrm{triv}}} & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & p^{\mathbf{Z}} & \longrightarrow & p^{\mathbf{Z}} & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

We consider now the remaining valuations on $\mathbf{Z}$. Recall that by Definition 2.4.3 in Lecture 2, to specify a point v in $\mathrm{Spv}(\mathbf{Z})$ we assign a support $\mathfrak{p}_v$ and a valuation on its residue field. The only possible supports are (0), which we have just studied, and (p) for p prime. The residue field of this latter is $\mathbf{F}_p$, and the only possible valuation on $\mathbf{F}_p$ is the trivial valuation. We call $\overline{v}_p$ the corresponding valuation on $\mathbf{Z}$, and get that $\mathrm{Spv}(\mathbf{F}_p) = \{\overline{v}_p\}$.

Let us step back to the p-adic valuation v_p. If $x \notin (p)$, then $v_p(x) = 1$, and it is not 1 otherwise. If we choose $H = (1) = p^{(0)} \subset p^{\mathbf{Z}}$ then

$$\overline{v}_p = (v_p)_{|_H};$$

in this particular case, Definition 4.3.1 indeed yelds a valuation on $\mathbf{Z}$.

It is also easy to check that the p-adic valuation specializes to $\overline{v}_p$, and this is the case for each prime p. In fact, we have the following commutative square encoding the specialization relations encoutered so far:

$$\begin{tikzcd}
\mathrm{Spv}(\mathbf{Z}) \arrow[ddd] & \mathrm{Spv}(\mathbf{Q}) & \mathrm{Spv}(\mathbf{F}_p) \\
 & v_{\mathrm{triv}} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \overline{v}_p \arrow[d, equal] \\
 & v_p \arrow[r, rightsquigarrow] & \overline{v}_p \\
\mathrm{Spec}(\mathbf{Z}) & (0) & (p)
\end{tikzcd}$$

The very same picture, notationally updated, can be obtained for any Dedekind domain whose residue fields at maximal ideals are finite.

We now discuss a less trivial example.

Example 4.3.3 Recall Example 2.3.4 in Lecture 2. We are going to build a new example on it. Let $K = k(\!(y)\!)(\!(x)\!) = \mathrm{Frac}(k(\!(y)\!)[\![x]\!])$ for a field k.

Consider $\mathrm{Spv}(k(\!(y)\!)[\![x]\!])$. This is actually a huge space since it includes lots of valuations that are nontrivial on k (when k isn't algebraic over a finite field), but we'll study some valuations that are trivial on k. Note that $k(\!(y)\!)[\![x]\!]$ is a discrete valuation ring, and hence $\mathrm{Spec}(k(\!(y)\!)[\![x]\!])$ is just $\{(0),(x)\}$. We first consider the fiber of $\mathrm{Spv}(k(\!(y)\!)[\![x]\!]) \to \mathrm{Spec}(k(\!(y)\!)[\![x]\!])$ over the generic point (0). From Example 2.3.4 in Lecture 2 we know three points in $\mathrm{Spv}(K)$: the trivial valuation η, the x-adic valuation v, and the valuation w arising from a procedure we briefly recall now.

The valuation ring R_v of the x-adic valuation is $k(\!(y)\!)[\![x]\!]$, and we generate a specialization w of v with the same support via a nontrivial valuation on the residue field of R_v as follows. The maximal ideal $\mathfrak{m}_v \subset R_v$ is xR_v. We equip $R_v/\mathfrak{m}_v \simeq k(\!(y)\!)$ with the evident y-adic valuation $\overline{w}$ (trivial on k), and w corresponds to the valuation ring preimage $R_w = k[\![y]\!] + xk(\!(y)\!)[\![x]\!] \subset K$, so $\Gamma_v = x^{\mathbf{Z}}$, $\Gamma_{\overline{w}} = y^{\mathbf{Z}}$, and there is a natural exact sequence of abelian groups

$$1 \to \Gamma_{\overline{w}} \to \Gamma_w \to \Gamma_v \to 1$$

which splits to give

$$\Gamma_w \simeq \Gamma_v \times \Gamma_{\overline{w}} \simeq x^{\mathbf{Z}} \times y^{\mathbf{Z}} \simeq \mathbf{Z} \times \mathbf{Z}$$

in which $\mathbf{Z} \times \mathbf{Z}$ is totally ordered with the *lexicographical ordering* (left factor comes first!).

Convex subgroups H of Γ_w include Γ_w (corresponding to the trivial valuation η), $1 \times y^{\mathbf{Z}}$ corresponding to $v = \eta_{/_{1\times y^{\mathbf{Z}}}}$ (see Example 4.2.1), and 1×1 corresponding to $w = v_{/_{1\times 1}}$.

The reader may wish to have a look at the picture at the end of the example, before going on.

Now we discuss valuations on $k(\!(y)\!)[\![x]\!]$ whose support is (x), using the construction in Definition 4.3.1. We certainly have the trivial valuation $\overline{\eta}$ on $k(\!(y)\!)$, which arises from η by "horizontal specialization" with respect to the trivial group $H = (1)$. On the other hand, if we consider the x-adic valuation v on $k(\!(y)\!)[\![x]\!]$ then we may choose $H = 1 \times 1 \subset \Gamma_v = x^{\mathbf{Z}} \times 1$ and (according to Definition 4.3.1) get again the trivial valuation $\overline{\eta} = v_{|_{1\times 1}}$ on $k(\!(y)\!)$. On the other hand, we may also choose $H = 1 \times y^{\mathbf{Z}} \subset x^{\mathbf{Z}} \times y^{\mathbf{Z}} = \Gamma_w$, and $w_{|_{1\times y^{\mathbf{Z}}}}$ is the y-adic valuation on $k(\!(y)\!)$ that we previously called $\overline{w}$. The situation is now like in the picture below.

$$
\begin{tikzcd}
\mathrm{Spv}(k(\!(y)\!)[\![x]\!]) \arrow[dddd] & \mathrm{Spv}(K) & \mathrm{Spv}(k(\!(y)\!)) \\
 & \eta \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \overline{\eta} = \eta_{|_{(1)}} \arrow[d, equal] \\
 & v \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \overline{\eta} = v_{|_{1\times 1}} \arrow[d, rightsquigarrow] \\
 & w \arrow[r, rightsquigarrow] & \overline{w} = w_{|_{1\times y^{\mathbf{Z}}}} \\
\mathrm{Spec}(k(\!(y)\!)[\![x]\!]) & (0) & (x)
\end{tikzcd}
$$

We remark that in specializing v, a choice like $x^{\mathbf{Z}} \times 1 = \Gamma_v$ was not allowed because we wanted any such specialization of v to have support (x). For the same reason, in specializing w we could not choose H to be $x^{\mathbf{Z}} \times y^{\mathbf{Z}} = \Gamma_w$ or $x^{\mathbf{Z}} \times 1$. We observe that the latter is not a convex subgroup of Γ_w.

For a valuation $v : A \to \Gamma \cup \{0\}$ and subgroup $H \subset \Gamma$ we now address the issue of finding suitable *general* conditions on H so that $v_{|_H}$, defined in Definition 4.3.1 is indeed a valuation on A that moreover lies in the closure of $\{v\}$.

Proposition 4.3.4 *A necessary condition for $v' := v_{|_H}$ to be a valuation is*

$$\Gamma_{\geq 1} \cap v(A) \subseteq H;$$

that is, for all $a \in A$, if $v(a) \geq 1$ then $v(a) \in H$.

Proof. Suppose not. We have $1 < v(a)$ for some $a \in A$ with $v(a) \notin H$. It follows $v'(a) = 0$. However, $v(a+1) > 1$, and so $v'(a+1) = 0$. Hence, a and $a+1$ are in the support of v', so this support contains 1. Contradiction. $\square$

Example 4.3.5 Let $R' \subset R$ be an inclusion of valuation rings with the same fraction field K, with v the valuation attached to R, and v' the one attached to R'. Recall that the couple (R', v') is induced by a valuation $\overline{v}'$ on the residue field $k = R/\mathfrak{m}_R$ of R, having valuation ring denoted $\overline{R}' \subset k$. Thus, we have the following diagram:

$$
\begin{tikzcd}
\mathfrak{m}_R \arrow[r, hook] \arrow[d, hook] & R \arrow[d, equal] & \\
R' \arrow[r, hook] \arrow[d, two heads] & R \arrow[r, hook] \arrow[d, two heads] & K \\
\overline{R}' \arrow[r, hook] & k &
\end{tikzcd}
$$

The above diagram induces, in turn, the following:

$$\begin{tikzcd}
R'^{\times} \arrow[r, hook] \arrow[d, two heads] & R^{\times} \arrow[r, hook] \arrow[d, two heads] & K^{\times} \\
\overline{R}'^{\times} \arrow[r, hook] & k^{\times} &
\end{tikzcd}$$

The top row yelds the inclusion

$$(*) \quad R^{\times}/R'^{\times} \subset K^{\times}/R'^{\times} = \Gamma_{v'},$$

as well as a surjection

$$(**) \quad R^{\times}/R'^{\times} \twoheadrightarrow k^{\times}/\overline{R'}^{\times} = \Gamma_{\overline{v}'},$$

which is seen to be an isomorphism by inspection. By $(*)$, we can view v' as a valuation on R (after restriction from K) in the sense of Definition from Lecture 2. By $(**)$, we can do the same with $\overline{v}'$. Observe that $\mathfrak{p}_{v'} = (0)$ and $\mathfrak{p}_{\overline{v}'} = \mathfrak{m}_R$.

(The reader may wish to turn back to Example 4.3.2 and consider respectively

$$R = \mathbf{Z}_{(p)} \subset \mathbf{Q} = K, \quad R' = R$$

and $v' = v_p$, $\overline{v}' = \overline{v}_p$, as well as turn back to Example 4.3.3 and consider

$$R = k(\!(y)\!)[\![x]\!] \subset k(\!(y)\!)(\!(x)\!) = K, \quad R' = \{f(x,y) \in R \mid f(0,y) \in k[\![y]\!]\},$$

$v' = w$, and $\overline{v}' = y$-adic valuation on $k(\!(y)\!)$. In those examples, we saw $\overline{v}'$ is a specialization of v'.)

Continuing with our example, we claim that

$$\overline{v}' \in \overline{\{v'\}}$$

in $X := \mathrm{Spv}(R)$. We have to show that there is no open neighborhood U of $\overline{v}'$ which does not contain v'. It is sufficient to show this for all *rational* open neighbourhoods of $\overline{v}'$; that is, for all

$$X\left(\frac{f}{s}\right) = \{w \in X \mid w(f) \leq w(s) \neq 0\},$$

where $f, s \in R$, we must prove that if $\overline{v}' \in X(f/s)$ then $v \in X(f/s)$. We define

$$\overline{f} := f \bmod \mathfrak{m}_R, \quad \text{and} \quad \overline{s} := s \bmod \mathfrak{m}_R.$$

We have to show that for all $f, s \in R$ such that

$$\overline{v}'(\overline{f}) \leq \overline{v}'(\overline{s}) \neq 0,$$

necessarily

$$v'(f) \leq v'(s) \neq 0.$$

Since $\overline{v}'(\overline{s}) \neq 0$, s is a unit in R. We are reduced to show that as soon as $\overline{f}/\overline{s} \in \overline{R}'$ then $f/s \in R'$. This is equivalent to show that if $\overline{f} \in \overline{s}\overline{R}'$, then $f \in sR'$. But the former assumption implies

$$f \in sR' + \mathfrak{m}_R = sR' + s \cdot \mathfrak{m}_R \subset sR'$$

(since $\mathfrak{m}_R = s \cdot \mathfrak{m}_R$ because $s \in R^{\times}$). It follows $f \in sR'$, as required.

Now we return to a valuation v on a general ring A and a subgroup $H \subset \Gamma_v$. Example 4.3.5 motivates interest in the case when H is convex. For such H, if $v' := v_{|_H}$ is to be a valuation then H has to contain the minimal convex subgroup $c\Gamma_v$ of Γ_v containing $\Gamma_{\geq 1} \cap v(A)$ (as this is the intersection of all convex subgroups of Γ_v containing $\Gamma_{\geq 1} \cap v(A)$); we call $c\Gamma_v$ the *characteristic subgroup* of v.

Proposition 4.3.6 *Let H be a convex subgroup of Γ_v. Then $v' := v_{|_H}$ is a valuation on A if and only if $c\Gamma_v \subset H$, in which case it is a specialization of v.*

Proof. The necessity has already been shown. For the sufficiency (which is stated without proof in [HuKne, Lemma 1.2.1 (2)], where the terminology "primary" is used for "horizontal" and likewise "secondary" is used for "vertical"), suppose H contains $c\Gamma_v$. To show that v' is multiplicative, we have to prove that if $v(a), v(b) \notin H$ then $v(ab) \notin H$. Necessarily $0 \leq v(a), v(b) < 1$ since $\Gamma_{\geq 1} \cap v(A) \subset H$, and by swapping the labels if necessary we can assume $v(a) \leq v(b)$. If $v(a) = 0$ then all is clear, so we can assume $0 < v(a) \leq v(b) < 1$. Since $v(ab) < v(b) < 1$ and $v(b) \notin H$ but $1 \in H$, convexity of H forces $v(ab) \notin H$ as desired. Some easy case-checking shows that $v'(a + b) \leq \max(v'(a), v'(b))$, so v' is a valuation.

To see that v' is a specialization of v, we have to check that any open set around v' in $X := \mathrm{Spv}(A)$ must contain v. In view of the definition of the topology, it is equivalent to show that if $a, b \in A$ with $v' \in X(a/b)$ then $v \in X(a/b)$, which is to say that if $v'(a) \leq v'(b) \neq 0$ then $v(a) \leq v(b) \neq 0$. If $v'(a) \neq 0$ then necessarily $v(a), v(b) \in H$ and all is clear (by definition of v'). If $v'(a) = 0$ then $v(a) \notin H$ and $v(b) \in H$, so we just need to check that $v(a) \leq v(b)$. Suppose to the contrary that $v(b) < v(a)$. But $\Gamma_{\geq 1} \cap v(A) \subset H$, so the condition $v(a) \notin H$ forces $v(a) < 1$. Hence, $v(a)$ lies between the elements $v(b), 1 \in H$ yet $v(a) \notin H$, contradicting the convexity of H. Thus, indeed, v' is a specialization of v. $\square$

For a convex subgroup $H \subset \Gamma$ satisfying $c\Gamma_v \subset H$ we can reconstruct the specialization $v_{|_H}$ of v from knowledge of its support. To express this efficiently, it is convenient to introduce some terminology.

Definition 4.3.7 We say that a subset $\Sigma \subset A$ is *v-convex* if any $a \in A$ satisfying $v(s) \leq v(a) \leq v(s')$ for $s, s' \in \Sigma$ necessarily lies in Σ.

Note that a v-convex set is the preimage of its image under v, and a subset Σ of A satisfying $\Sigma = v^{-1}(v(\Sigma))$ is v-convex precisely when $v(\Sigma)$ is convex in $v(A)$; if $0 \in v(\Sigma)$ then this says exactly that any $\gamma \in v(A)$ satisfying $\gamma \leq v(s)$ for some $s \in \Sigma$ necessarily lies in $v(\Sigma)$.

Note that any v-convex subset containing 0 must contain $v^{-1}(0) = \mathfrak{p}_v$; in particular, v-convex primes contain $\mathfrak{p}_v$ and so can be viewed as prime ideals of $A/\mathfrak{p}_v$.

Proposition 4.3.8 *Let $H \subset \Gamma_v$ be a convex subgroup containing $c\Gamma_v$. The prime ideal $\mathrm{supp}(v_{|_H}) = v^{-1}((\Gamma_v \cup \{0\}) - H)$ is v-convex.*

Proof. Clearly $\mathrm{supp}(v_{|_H})$ is the v-preimage of its image under v, and it contains 0, so we just have to check that if $a \in A$ and $v(a) \leq v(s)$ for some $s \in A$ satisfying $v(s) \notin H$ then $v(a) \notin H$. Note that $v(s) < 1$ since $v(A)_{\geq 1} \subset H$. Hence, $v(a) \leq v(s) < 1$, so by convexity of H the hypothesis $v(s) \notin H$ rules out the possibility that $v(a) \in H$. $\square$

We finally give the official definition of horizontal specialization.

Definition 4.3.9 Let $v : A \to \Gamma \cup \{0\}$ be a valuation on A. A *horizontal specialization* of v is a valuation on A of the form $v_{|_H}$ for a convex subgroup $H \subset \Gamma_v$ containing $c\Gamma_v$.

4.4 Comparison of several horizontal specializations

It is generally difficult to compute the exact value group of a horizontal specialization, essentially because (i) the value group Γ_v of a valuation v on A is merely generated by $v(A - \mathfrak{p}_v)$ (so elements of Γ_v need not have the form $v(a)$ for $a \in A$, in contrast with the case when A is a field) and (ii) convexity is sensitive to the ambient ordered group. Thus, it is not clear from the definitions if $v_{|_H}$ determines H for convex subgroups $H \subset \Gamma_v$ containing $c\Gamma_v$, and in fact this can fail (so for horizontal specialization we don't have an analogue of Proposition 4.2.4 or Remark 4.2.5). The following example was suggested by John Pardon.

Example 4.4.1 Let $A' = k[x, y]$ for a field k and let $\Gamma = c^{\mathbf{Z}} \times c^{\mathbf{Z}}$ for your favorite $0 < c < 1$, where Γ is given the lexicographical ordering (left factor comes first); we write Γ in multiplicative notation since our axioms for valuations are expressed in multiplication form (e.g., $\Gamma \cup \{0\}$ might lead to confusion if we write Γ additively). Note that $1 \times c^{\mathbf{Z}}$ is a convex subgroup of Γ (see Example 4.2.1).

Define the rank-2 valuation $v' : A' \to \Gamma \cup \{0\}$ by the requirement of triviality on k and $v'(x^i y^j) = (c^i, c^j)$, so $v'(A') = \Gamma_{\geq 1} \cup \{0\}$ and hence $v'(A')_{\leq 1} = 1$. Thus, for any convex subgroup $H \subset \Gamma$ we may form the horizontal specialization $v'_{|_H}$. Clearly $v'_{|_1}$ is the trivial valuation with support at the origin and for $H = 1 \times c^{\mathbf{Z}}$ (convexity seen in Example 4.2.1) we have that $v'_{|_H}$ has support equal to xA' (i.e., it is the line $x = 0$, which is to say the y-axis).

Now comes the geometric idea: let $A = k[x, z]$ for $z = xy$, so $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is the "x is a generator" chart of the blow-up of the affine plane $\mathrm{Spec}(A)$ at the origin $(x, z) = (0, 0)$. In particular, A and A' have the same fraction field. The line $x = 0$ in $\mathrm{Spec}(A')$ is the exceptional divisor, so it is crushed onto the origin in $\mathrm{Spec}(A)$ (in algebraic terms, $A \to A'/xA'$ kills (x, z)), as is the origin of $\mathrm{Spec}(A')$, so the restriction $v : A \to \Gamma \cup \{0\}$ of v' to the subring $A \subset A'$ has the same value group as v' (via the equality of fraction fields) and satisfies $\mathrm{supp}(v_{|_1}) = \mathrm{supp}(v_{|_H})$. By inspection, we see that $v_{|_1}$ and $v_{|_H}$ each coincide with the trivial valuation on A supported at the origin.

(To give some geometric motivation, in Theorem 4.4.3 we will record the general fact that a horizontal specialization is *determined* by its support. Hence, the equality of supports implies that $v_{|_1} = v_{|_H}$ in $\mathrm{Spv}(A)$, which we saw directly.)

Proposition 4.4.2 *If v' is a horizontal specialization of v in* $\mathrm{Spv}(A)$ *and w is a horizontal specialization of v' then w is a horizontal specialization of v.*

Proof. I am grateful to Kęstutis Česnavičius and John Pardon for (independently) suggesting the following simple argument. The main point is to reformulate horizontal specialization essentially without reference to the value group. Consider a valuation

$$v : A \to \Gamma \cup \{0\}$$

(with Γ_v possibly a proper subgroup of Γ), and for any subgroup $H \subset \Gamma$ clearly $v_{|_H} = v_{|_{H \cap \Gamma_v}}$ as functions from A into $H \cup \{0\}$. More importantly, if H is convex in Γ_v and if G denotes the order-theoretic convex hull

$$G = \{\gamma \in \Gamma \mid h_1 \leq \gamma \leq h_2 \text{ for some } h_1, h_2 \in H\}$$

of H in Γ (so G is obviously a subgroup of Γ) then $G \cap \Gamma_v = H$, so $v_{|_H} = v_{|_G}$. Note also that $v(A)_{\geq 1}$ lies in H if and only if it lies in G, and that in such cases $(v_{|_H})(A)_{\geq 1} = v(A)_{\geq 1}$.

Now we may write $v' = v_{|_G}$ for a convex $G \subset \Gamma$ containing $v(A)_{\geq 1}$, and likewise we may write $v'' = v'_{|_{G'}}$ for the convex hull G' in G of some subgroup of $\Gamma_{v'}$ containing $v'(A)_{\geq 1} = v(A)_{\geq 1}$. But then G' is convex in Γ since it is convex in the convex subgroup G of Γ, so $v_{|_{G'}}$ makes sense as a horizontal specialization of v and it clearly coincides with v''. $\square$

Theorem 4.4.3 *Let $v : A \to \Gamma \cup \{0\}$ be a valuation. The support of a horizontal specialization of v is a v-convex prime ideal and there is a natural bijection*

$$\{\text{horizontal specializations of } v\} \leftrightarrow \{v\text{-convex primes}\},$$

defined by $w \mapsto \mathfrak{p}_w := \mathrm{supp}(w)$. Moreover, for horizontal specializations w and w' of v we have that w' is a specialization of w in $\mathrm{Spv}(A)$ if and only if $\{\mathfrak{p}_{w'}\}$ is a specialization of $\{\mathfrak{p}_w\}$ in $\mathrm{Spec}(A)$ (equivalently, $\mathfrak{p}_w \subset \mathfrak{p}_{w'}$).

An inverse map is given explicitly by assigning to any v-convex prime $\mathfrak{q}$ in $\mathrm{Spec}(A)$ the valuation

$$w_{\mathfrak{q}} : a \mapsto \begin{cases} v(a) & \text{if } a \notin \mathfrak{q}, \\ 0 & \text{if } a \in \mathfrak{q} \end{cases}$$

which is equal to $v|_{H_{\mathfrak{q}}}$ for the minimal convex subgroup $H_{\mathfrak{q}}$ of Γ_v containing $v(A - \mathfrak{q})$.

Proof. See [Wed, Prop. 4.18] for a proof of the bijectivity, and see the *proof* of [HuKne, Lemma 1.2.3] for the explicit description of the inverse (including the recipe for $H_{\mathfrak{q}}$). □

The meaning of Theorem 4.4.3 is conveyed by the following picture:

$$\begin{tikzcd}
\mathrm{horz}(v) \arrow[r] \arrow[d, "\simeq"'] & \overline{\{v\}} \arrow[r] \arrow[d] & \mathrm{Spv}(A) \arrow[d] \\
\mathrm{conv}(v) \arrow[r] & \mathrm{Spec}(A/\mathfrak{p}_v) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

where $\mathrm{horz}(v)$ is the set of horizontal specializations of v and $\mathrm{conv}(v)$ is the set of v-convex prime ideals of A.

The following corollary is in the spirit of the going-up theorem (which holds for flat maps of affine schemes), but concerns the map of topological spaces $\mathrm{Spv}(A) \to \mathrm{Spec}(A)$:

Corollary 4.4.4 ("going-up") *Let v be a valuation on A, and $\mathfrak{p}_v$ its support. Let $\mathfrak{p} \in \mathrm{Spec}(A)$ be a point which specializes to $\mathfrak{p}_v$ in $\mathrm{Spec}(A)$. Then there exists a point $w \in \mathrm{Spv}(A)$ such that $\mathfrak{p}_w = \mathfrak{p}$, and w specializes to v in $\mathrm{Spv}(A)$.*

Proof. The proof can be found in [Wed, Cor. 4.20] (it rests on Proposition 4.5.1(1) below, but we present it here first purely for expository reasons, since we are not providing the proof of either anyway), and is also proved in [HuKne, Lemma 1.2.6]. □

We will see below that as long as $v(A) \not\subset \Gamma_{\leq 1}$ (a very typical situation; see Remark 4.5.3), $\mathrm{conv}(v)$ is the *exact image* of $\overline{\{v\}}$ in $\mathrm{Spec}(A)$ and the horizontal specializations of v are precisely the "most generic" points in the fibers of $\overline{\{v\}}$ over its image in $\mathrm{Spec}(A)$ (see Corollary 4.5.5); this is generally false if $v(A) \subset \Gamma_{\leq 1}$ (e.g., the trivial valuation in a domain that is not a field).

4.5 Robustness of the two specialization constructions

Before we state the general result (our main goal today) which expresses the ubiquity of horizontal and vertical specialization to describe *arbitrary* specializations inside $\mathrm{Spv}(A)$, we first discuss how these two operations interact with each other.

Proposition 4.5.1 ("Exchange Lemma")

(1) *If $v \rightsquigarrow w$ is a horizontal specialization and $w \rightsquigarrow w'$ is a vertical specialization, then there exists a vertical specialization $v \rightsquigarrow v'$ that admits w' as a horizontal specialization; that is,*

$$\begin{tikzcd}
v \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow, "\exists"'] & w \arrow[d, rightsquigarrow] \\
v' \arrow[r, rightsquigarrow, "\text{such that}"'] & w'
\end{tikzcd}$$

(2) *If $v \rightsquigarrow w$ is a horizontal specialization and $v \rightsquigarrow v'$ is a vertical specialization, then there exists a* unique *horizontal specialization $v' \rightsquigarrow w'$ such that w' is a vertical specialization of w; that is,*

$$\begin{tikzcd}
v \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & w \arrow[d, rightsquigarrow, "\text{such that}"] \\
v' \arrow[r, rightsquigarrow, "\exists!"'] & w'
\end{tikzcd}$$

That is, among all horizontal specializations of v', there is exactly one which is a vertical specialization of w.

Proof. The proof consists of bare-hands constructions with valuations on fields. The reader is referred to [Wed, Lemma 4.19 (2), (1)] and [HuKne, Lemma 1.2.5 (ii), (iii)] for proofs. □

Now comes the main result which provides exhaustiveness of the two specialization constructions:

Theorem 4.5.2 *Suppose $v \rightsquigarrow w$ is an arbitrary specialization in* $\mathrm{Spv}(A)$*. Then always there exists a vertical specialization $v \rightsquigarrow v'$ that admits w as a horizontal specialization:*

$$\begin{tikzcd}
v \arrow[d, rightsquigarrow, "\exists"'] \arrow[dr, rightsquigarrow] & \\
v' \arrow[r, rightsquigarrow, "\text{such that}"'] & w
\end{tikzcd}$$

Also, there exists a horizontal specialization $v \rightsquigarrow w'$ admitting w as a vertical specialization, which is to say

$$\begin{tikzcd}
v \arrow[r, rightsquigarrow, "\exists"] \arrow[dr, rightsquigarrow] & w' \arrow[d, rightsquigarrow, "\text{such that}"] \\
& w
\end{tikzcd}$$

except for possibly when $v(A) \subset \Gamma_{\leq 1}$ and $v(A - \mathfrak{p}_w) = 1$.

Proof. We refer the reader to [Wed, Prop. 4.21], or [HuKne, Prop. 1.2.4]. Note that neither of these formulates the exceptional case (i.e., $v(A) \subset \Gamma_{\leq 1}$ and $v(A - \mathfrak{p}_w) = 1$) in the way that we do, instead speaking in terms of triviality of some valuations; it is equivalent to the version above upon unraveling some definitions. Also, the proof of the second assertion produces w' as a succession of two horizontal specializations so it is essential to know via Proposition 4.4.2 that horizontal specialization is a transitive relation. □

Remark 4.5.3 The situation $v(A) \subset \Gamma_{\leq 1}$ never happens if A contains a field k with $v|_k$ a nontrivial valuation.

Corollary 4.5.4 *If $v(A) \not\subset \Gamma_{\leq 1}$ then the horizontal specialization $v_{|c\Gamma_v}$ has* only *vertical specializations.*

Proof. Let w be a specialization of $v_{|c\Gamma_v}$. By the second assertion in Theorem 4.5.2, there is a horizontal specialization w' of $v_{|c\Gamma_v}$ such that w is a vertical specialization of w'. But $v \rightsquigarrow v_{|c\Gamma_v} \rightsquigarrow w'$ is a chain of horizontal specializations, so it is also a horizontal specialization (see Proposition 4.4.2). That is, $w' = v_{|H}$ for a convex subgroup $H \subset \Gamma_v$ containing $c\Gamma_v$. But w' is a specialization of $v_{|c\Gamma_v}$, forcing $c\Gamma_v \supset H$, so $H = c\Gamma_v$. Thus, w is a vertical specialization of $w' = v_{|c\Gamma_v}$. □

Intuitively, Corollary 4.5.4 amounts to the fact that $v|_{c\Gamma_v}$ is the "most generic" horizontal specialization of v, so it cannot have further horizontal specializations of its own. (It is instructive to compute this for v as in Example 4.3.3 subject to the hypothesis $v(A) \not\subset \Gamma_{\leq 1}$.)

The conclusion in this corollary is generally false when the hypothesis on $v(A)$ is not satisfied. For example, consider the trivial valuation v on the fraction field of a domain A that is not a field. Then $c\Gamma_v = 1$ and $v_{|c\Gamma_v} = v$. This admits as a vertical specialization a valuation $v_{\mathfrak{p}}$ dominating the local ring $A_{\mathfrak{p}}$ for any nonzero prime ideal $\mathfrak{p}$ of A. Since $v_{\mathfrak{p}}(a) \leq 1$ for all $a \in A$, we have $c\Gamma_{v_{\mathfrak{p}}} = \{1\}$. The resulting horizontal specialization $(v_{\mathfrak{p}})_{|1}$ is the trivial valuation associated to the residue field at $\mathfrak{p}$, so v admits specializations with support equal to any prime ideal of A.

Corollary 4.5.5 *If $v(A) \not\subset \Gamma_{\leq 1}$ then the image of $\overline{\{v\}}$ in* $\mathrm{Spec}(A)$ *coincides with the set* $\mathrm{conv}(v)$ *of v-convex primes and the topological fiber of $\overline{\{v\}} \twoheadrightarrow \mathrm{conv}(v)$ over a v-convex prime $\mathfrak{q}$ contains a unique generic point, namely the corresponding horizontal specialization.*

Proof. Since $v(A) \not\subset \Gamma_{\leq 1}$, the second option in Theorem 4.5.2 applies: every $w \in \overline{\{v\}}$ is the vertical specialization of a horizontal specialization $w' = v|_H$ for a convex subgroup $H \subset \Gamma_v$ containing $c\Gamma_v$. Thus, $\mathrm{supp}(w) = \mathrm{supp}(w')$ is a v-convex prime. Hence, $\mathrm{conv}(v)$ is the image of $\overline{\{v\}}$ in $\mathrm{Spec}(A)$.

For a v-convex prime $\mathfrak{q}$, consider a specialization w of v with support $\mathfrak{q}$. By the same reasoning, w is the vertical specialization of a horizontal specialization w' of v, so w' has to be the unique horizontal specialization $v_{\mathfrak{q}}$ of v with support equal to the v-convex prime $\mathrm{supp}(w) = \mathfrak{q}$. Hence, we conclude that all points in the $\mathfrak{q}$-fiber of $\overline{\{v\}}$ are specializations of $v_{\mathfrak{q}}$, so $v_{\mathfrak{q}}$ is the unique generic point of the topological fiber of $\overline{\{v\}}$ over $\mathfrak{q}$. □

The preceding corollary is generally false if we drop the hypothesis on $v(A)$. Indeed, if v is the trivial valuation on a domain A that is not a field then the only v-convex prime is $\{0\}$ but we have seen that v admits specializations in $\mathrm{Spv}(A)$ with any desired support in $\mathrm{Spec}(A)$.

Despite the robustness of horizontal and vertical specialization for probing the topology of the valuation spectrum, when we restrict attention to distinguished subspaces of $\mathrm{Spv}(A)$ (such as an adic spectrum when A is in a reasonable class of topological rings) then we will need to revisit the above results in order to study specialization relations with such subspaces since the "intermediate" valuations produced by the Exchange Lemma and Theorem 4.5.2 might not lie in the subspace of interest.

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