The Local to Global Principle in Algebraic K-Theory

*Robert W. Thomason**

CNRS, U.F.R. de Mathématiques, Université de Paris 7, F-75251 Paris Cedex 05, France

In the early 70s, Quillen [Q] succeeded in finding a good definition of the higher algebraic K-groups, generalizing the usual Grothendieck group K_0. To a noetherian ring or scheme X, his theory associates two distinct series of K-groups: $K_*(X)$, the theory associated to the exact category of finitely generated projective modules or algebraic vector bundles on X; and $G_*(X)$ or $K'_*(X)$, the theory associated to the abelian category of all finitely generated modules or coherent sheaves on X. Quillen's localization theorem for the K-theory of abelian categories applies to $G_*(X)$, and enables him to prove many fundamental results for this theory, including invariance of G_* under replacing the ring R by the ring of polynomials $R[T]$ or the scheme X by $X[T]$, a Mayer-Vietoris exact sequence for a cover of X by two open subschemes, and a Brown-Gersten local-to-global spectral sequence which reduces many G-theory problems to the case of local rings. For regular rings or schemes X, the existence of finite projective resolutions for all finitely generated modules shows that $K_*(X) = G_*(X)$, so all these good results apply to $K_*(X)$ in the regular case. However if X has singularities, Quillen's methods give many fewer results about $K_*(X)$. Unfortunately, this case arises often: for example the integral group ring of a non-zero finite group is always singular. The essential difficulty was the lack of the key tool: a good localization theorem for K-theory.

The author and Thomas F. Trobaugh have remedied this lack by finding the good localization result, using the techniques of Waldhausen's "algebraic K-theory of spaces" [Wa2] and returning to some ideas of Grothendieck about defining K-theory using "perfect complexes" in place of single vector bundles [SGA6]. Using a characterization of perfect complexes, essentially defined as finite complexes of algebraic vector bundles, as being the finitely presented objects in the derived category of modules, we obtain the result that the K_0 class is the only obstruction to extension up to quasi-isomorphism of a perfect complex on an open subscheme of X to a perfect complex on all of X. This extension result and Waldhausen's theory give the localization theorem for K-theory, Theorem 2.1 below. This in turn unleashes a pack of new fundamental results for K-theory, including the Mayer-

* Author partially supported by NSF, The Johns Hopkins University and Université de Paris 7.

Proceedings of the International Congress
of Mathematicians, Kyoto, Japan, 1990

Vietoris exact sequence, the Brown-Gersten local-to-global spectral sequences for both the Zariski and Nisnevich topologies, the generalizations to schemes from rings of the results of Weibel, Ogle, and Goodwillie controlling the failure of K_* to be invariant under polynomial or nilpotent extension or to satisfy Mayer-Vietoris for closed covers, and the isomorphism between mod l algebraic K-theory localized by inverting the Bott element and etale topological K-theory. Recently Dongyuan Yao has obtained some similar localization results for the K-theory of non-commutative rings. Hiding in the background of all this is the fact that the Waldhausen K-theory of a category of complexes depends essentially only on the derived category; this idea has been much advanced by work in progress of Giffen and Neeman, which unfortunately I will not discuss below.

§ 1. Perfect Complexes and Waldhausen K-Theory

We consider schemes X which are quasi-compact and quasi-separated. *Quasi-compact* means that X is covered by a finite number of open subschemes which are affine, i.e. isomorphic to Spec(R) for some commutative rings R. *Quasi-separated* means that the intersection of any two such affines is again quasi-compact. These conditions are indeed very mild, and are met by all algebraic varieties over a field, by all schemes of finite type over a noetherian ring, and by Spec(R) for any commutative ring R.

Recall that a map between two chain complexes in an abelian category is a *quasi-isomorphism* if it induces an isomorphism on the homology of the complexes. Two complexes E and F are quasi-isomorphic if they are connected by a chain of quasi-isomorphisms

$$E \xleftarrow{\sim} A \xrightarrow{\sim} \cdots \xrightarrow{\sim} F$$

Definition (Grothendieck [SGA6]). *A perfect complex on a scheme X is a complex E of $\mathcal{O}_X$-modules, such that for every affine open $U =$ Spec(R) in X, the restriction $E|U$ is quasi-isomorphic to a finite complex of finitely generated projective R-modules.*

For E to be perfect, it suffices that $E|U$ be quasi-isomorphic on U to a finite complex of such projective modules for all the U in some affine open cover of X. In particular, for $X =$ Spec(R), the perfect complexes are those quasi-isomorphic to such complexes of projective R-modules.

The category of perfect complexes on X is a "category with cofibrations and weak equivalences" in the sense of [Wa2] 1.2, i.e., it is a Waldhausen category. The cofibrations are the maps of complexes which are degree-wise split monomorphisms; the weak equivalences are the quasi-isomorphisms. For Y a closed subspace of X, there is a full Waldhausen subcategory of those perfect complexes which are cohomologically supported on Y; i.e., those which are acyclic on $X - Y$.

The theory of Waldhausen [Wa2] associates higher algebraic K-groups to Waldhausen categories: these are the homotopy groups of a space, or even better, of an infinite loop space or spectrum in the sense of algebraic topology. Let $K_*(X)$ denote the Waldhausen algebraic K-groups associated to the category of perfect complexes on X, and $K(X)$ denote the associated K-theory spectrum. Let $K_*(X$ on $Y)$ and $K(X$ on $Y)$ denote the K-groups and spectrum associated to the Waldhausen subcategory of those perfect complexes on X which are acyclic on $X - Y$.

$K_0(X)$ is the Grothendieck group of [SGA6], generated by quasi-isomorphism classes $[E]$ of perfect complexes on X, modulo an Euler characteristic relation that $[E] = [E'] + [E'']$ whenever one can choose representatives of the quasi-isomorphism classes that fit in a short exact sequence of complexes $0 \to E' \to E \to E'' \to 0$, that is, whenever they are the vertices of an exact triangle in the derived category $D(X)$ of $\mathcal{O}_X$-modules.

Suppose X satisfies the mild hypothesis of having an "ample family of line bundles" in the sense of [SGA6] II.2.2.3. This condition is inherited by all subschemes of X and is satisfied when X is a commutative ring, or is quasi-projective over a commutative ring, or is a separated regular noetherian scheme. In particular, this condition holds for all classical algebraic varieties, since they are quasi-projective over a field. For a scheme X with an ample family of line bundles, a global resolution theorem of Illusie ([SGA6] II.2.2.8) says that every perfect complex on X is globally quasi-isomorphic to a finite complex of algebraic vector bundles defined on all of X. Then the $K_0(X)$ above is isomorphic to the "naive" Grothendieck group of algebraic vector bundles on X, and the Waldhausen $K_*(X)$ are isomorphic to the Quillen K-groups of X ([TT2] 3.10, 3.9).

We have an easy to verify but very useful characterization of perfect complexes in terms of the derived category $D(X)$. Recall that this category is obtained from the category of all chain complexes of $\mathcal{O}_X$-modules by inverting the quasi-isomorphisms (see Verdier [Ve]). $D(X)_{\text{qc}}$ denotes the full subcategory of those complexes whose homology sheaves are *quasi-coherent*, i.e., which locally on affine subschemes $\text{Spec}(R)$ are isomorphic to R-modules. $D^+(D)_{\text{qc}}$ is the further subcategory of complexes E such that the homology sheaves $H_k(E)$ are 0 for all k sufficiently large.

Proposition 1.1. *Let X be a quasi-compact and quasi-separated scheme. Let E be an object of $D^+(X)_{\text{qc}}$. Then the following are equivalent*:

a) *E is a perfect complex*

b) *For any direct system of complexes F_α with quasi-coherent homologies, the canonical map is an isomorphism of morphism sets*:

$$\operatorname{colim}_\alpha D(X)(E, F_\alpha) \xrightarrow{\sim} D(X)(E, \operatorname{colim}_\alpha F_\alpha)$$

c) *The functor $D(X)(E,\ )$ sends infinite direct sums in $D^+(X)_{\text{qc}}$ to direct sums in the category of abelian groups.*

Proof. [TT2] 2.4.3 yields that a) iff b). That b) implies c) is clear, and that c) implies b) follows from a demonstration dual to that of [T3] 2.6. □

Comparing condition b) with Grothendieck's characterization of finitely presented modules or algebras A over a ring or scheme as those such that $\text{Mor}(A,\ )$

preserves direct colimits ([EGA] IV 8.14), we see this proposition characterizes perfect complexes as finitely presented objects in the derived category. This suggests interesting generalizations. Thus, the "perfect complexes" in the stable homotopy category turn out to be the homotopy finite cell spectra: they have an associated K-theory, which turns out to be Waldhausen's $A(\mathrm{pt})$ ([Wa1] § 1). The proposition also suggests that we could apply the yoga for extending finitely presented objects from an open subscheme to all of X ([EGA] I.6.9). This suggestion leads to:

Key Proposition 1.2. *Let X be a quasi-compact and quasi-separated scheme, $j: U \to X$ the inclusion of a quasi-compact open subscheme, and F a perfect complex on U. Then F is quasi-isomorphic to the restriction j^*E of a perfect complex on X if and only if the class $[F]$ of F in $K_0(U)$ is in the image of $j^*: K_0(X) \to K_0(U)$.*

Proof. [TT2] 5.2.2 The basic idea is to write the complex with quasi-coherent homology $Rj_*(F)$ on X as quasi-isomorphic to a direct colimit of perfect complexes, $\operatorname{colim} E_\alpha \approx Rj_*F$. Then on U, F is quasi-isomorphic to $\operatorname{colim} j^*E_\alpha$. As F is perfect, Proposition 1.1 shows that this quasi-isomorphism factors through some j^*E_α. Thus F is quasi-isomorphic to a summand of j^*E_α, the restriction of a perfect complex E_α on X. This is the miracle for general U, although it would be trivial for affine U. The next step, more ordinary, is to show that given a morphism on U between restrictions of perfect complexes, $j^*E_1 \to j^*E_2$, one can extend this to a morphism $E_1 \to E_2$ on X after replacing the old E_1 by another perfect complex whose restriction to U is quasi-isomorphic to that of the old one. An induction and excision reduces this to the case where X has an ample family of line bundles. If $X - U$ is a divisor, one succeeds after replacing E_1 by its tensor product with a power of the line bundle of the divisor. For $X - U$ general, one succeeds after replacing E_1 by its tensor power with the positive degree part of a Koszul complex of line bundles for divisors whose intersection is $X - U$. From this extension result on morphisms, it follows that if 2 out of 3 terms in an exact triangle of perfect complexes in $D(U)$ extend, so does the third. We now adapt an idea of Grayson from cofinality theory: consider the abelian monoid of quasi-isomorphism classes of perfect complexes on U, and take the quotient monoid by setting equal to 0 the classes that are restrictions of perfect complexes on X; using the preceding sentence one sees that the class of a perfect complex on U is zero in this quotient if and only if it extends to a perfect complex on X. The miracle above shows this monoid has inverses, and so is an abelian group. We see from the above that this obstruction group to extension satisfies the Euler characteristic relation for exact triangles, and thus in fact that it is the quotient of $K_0(U)$ by the image of $K_0(X)$. □

In particular, this key Proposition gives a K_0-criterion for an extending up to quasi-isomorphism an algebraic vector bundle on U to a perfect complex on X, that is, to a finite complex of algebraic vector bundles on X in the usual case where X has an ample family. No similar criterion is possible for the question of extension to a single vector bundle on X: it is for this reason that it is necessary to go to the framework of perfect complexes and leave the framework of algebraic vector bundles to prove a good localization theorem for algebraic K-theory. The impossibility of such a criterion was noted by Serre ([Se] 5a), who observed that as the tangent vector bundle to the projective plane $\mathbb{P}^2$ is not a sum of line bundles, its preimage on $\mathbb{A}^3 - 0$ is a vector bundle which is not free, and which does not extend to a vector bundle on $\mathbb{A}^3$, even though $K_0(\mathbb{A}^3) \to K_0(\mathbb{A}^3 - 0) = \mathbb{Z}$ is an isomorphism, and though this vector bundle on $\mathbb{A}^3 - 0$ is the quotient of a monomorphism of free vector bundles on $\mathbb{A}^3 - 0$, which of course do extend. The vast literature on the Serre conjecture contains many negative and a few special positive results on the problem of extending to vector bundles on X for X regular: one may consult the articles of Horrocks, Swan, Murthy, Vasserstein, Bass, M. Kumar, Suslin, and Quillen on this conjecture. Very few have dared to say anything about the case where X is singular.

§ 2. The Localization Theorem and Its Basic Consequences

Theorem 2.1 (Localization Theorem) *Let X be a quasi-compact and quasi-separated scheme, and let Y be a closed subspace of X such that $X - Y$ is quasi-compact. Then there is a natural homotopy fibre sequence, and an associated long exact sequence of homotopy groups*:

$$K(X \text{ on } Y) \to K(X) \to K(X - Y)$$

$$\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(X - Y) \to K_{n-1}(X \text{ on } Y) \to$$

Proof. [TT2] 7.4. For $n \geq 0$, this results easily from the Key Proposition above and Waldhausen's fibration and approximation theorems [Wa2] 1.6.4 and 1.6.7. As the map $K_0(X) \to K_0(X - Y)$ is not always surjective in the singular case, it is necessary to construct new non-zero negative K-groups K_n for $n < 0$ to continue the exact sequence to the right, and to replace the spectrum $K(X)$ by a new non-connective spectrum of which the Waldhausen K-theory spectrum is a covering space. This is done by following an inductive procedure due to Bass [B], which depends on the part of the localization theorem already proved for higher n, and which uses the exact sequence of 2.2 to inductively define K_{n-1} for $n \leq 0$ given K_n. □

The special case of 2.1 where $X - Y$ is affine and Y is a Cartier divisor is due to Quillen [Gr] (at least in light of [TT2] 5.7 which identifies $K(X \text{ on } Y)$ with Quillen's third term $K(H_Y X)$ in this case). Painful efforts to squeeze a bit more out of Quillen's method have been made by Levine [Le3]. The counterexample of Deligne given in Gersten's paper ([Ge] § 7) shows the limits of an exact category approach to localization.

Corollary 2.2 (Bass Fundamental Theorem). *Let X be a quasi-compact and quasi-separated scheme. Denote by $X[T]$ the scheme $X \times \mathbb{A}^1 = X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, and similarly by $X[T^{-1}]$ another polynomial extension of X, and by $X[T, T^{-1}]$ the Laurent polynomial extension. Then there is a natural exact sequence for all integers $n \in \mathbb{Z}$*:

$$0 \to K_n(X) \to K_n(X[T]) \oplus K_n(X[T^{-1}]) \to K_n(X[T, T^{-1}]) \to K_{n-1}(X) \to 0$$

Similarly for $K_*(X$ on $Y)$.

Proof. [TT2] 7.5. This is proved by descending induction on n using the formula of Quillen ([Q] § 8) that $K_n(\mathbb{P}^1_X) \cong K_n(X) \oplus K_n(X)$, and the Mayer-Vietoris Corollary 2.3 to the localization theorem applied to the cover of $\mathbb{P}^1_X$ by $X[T]$ and $X[T^{-1}]$ with intersection $X[T, T^{-1}]$. For X affine and $n \leq 1$, the result is indeed due to Bass [B]; and for X affine and $n \geq 2$, it is due to Quillen [Gr]. □

Corollary 2.3 (Mayer-Vietoris Theorem). *Let U, V be two quasi-compact open subschemes of a quasi-separated scheme X. Then there is a homotopy cartesian square:*

$$\begin{array}{ccc} K(U \cup V) & \longrightarrow & K(U) \\ \downarrow & & \downarrow \\ K(V) & \longrightarrow & K(U \cap V) \end{array}$$

and an associated long exact Mayer-Vietoris sequence:

$$\cdots \to K_n(U \cup V) \to K_n(U) \oplus K_n(V) \to K_n(U \cap V) \to K_{n-1}(U \cup V) \to \cdots$$

Proof. [TT2] 8.1. The localization theorem calculates the homotopy fibres of the horizontal maps in the square, and an easy excision result shows they are equivalent, as U is an open neighborhood of $Y = (U \cup V) - V = U - (U \cap V)$ in $U \cup V$. □

Theorem 2.4 (Brown-Gersten or Local-to-Global Spectral Sequence, Spectral Sequence of Cohomological Descent). *Let X be a noetherian scheme of finite dimension. Then there is a strongly converging spectral sequence whose E^2 term is the cohomology of X for the Zariski topology with coefficients in the sheaf associated to the presheaf of K-groups:*

$$E_2^{p,-q} = H^p(X_{\mathrm{Zar}}; K_q) \Rightarrow K_{q-p}(X)$$

In fact, the augmentation map to the Zariski hypercohomology spectrum ([T1] § 1) *is a homotopy equivalence identifying this spectral sequence to the canonical hypercohomology spectral sequence.*

$$K(X) \xrightarrow{\sim} \mathbb{H}(X_{\mathrm{Zar}}; K)$$

One has the same results after replacing the Zariski topology by the slightly finer Nisnevich topology ([N] § 1, [TT2] Appendix E).

Proof. [TT2] 10.3, 10.8. In the Zariski case, Brown and Gersten showed this would follow from Mayer-Vietoris [BG], and proved the case for regular X or for G-theory. The argument for the Nisnevich topology is similar, but needs a slightly stronger excision result that $K(X$ on $Y)$ is unchanged by replacing X by its henselization along Y. Nisnevich [N] proved the Nisnevich case for regular X or for G-theory. Weibel [We4] proved the theorem in the case X has isolated singular points, using ideas of Collino and Pedrini. We note that the Nisnevich topology plays a big role in the work of Kato and Saito [KS], who call it the henselian topology. □

2.5 This spectral sequence reduces many problems to the case of local rings for the appropriate topology, since the stalks of the coefficient sheaves are the values of K_q at these local rings. For the Zariski topology, these are the local rings of X in the usual sense. For the Nisnevich topology, the local rings are the hensel local rings of X, the henselizations of the usual local rings. It is a theorem of Gabber [Ga], completing ideas of Suslin, that for $R_{\mathfrak{m}}$ a hensel local ring and n an integer such that $1/n \in R_{\mathfrak{m}}$, then the reduction map to the residue field, $R_{\mathfrak{m}} \to R/\mathfrak{m}$, induces an isomorphism on mod n K-theory, at least in positive degrees, $K/n_*(R_{\mathfrak{m}}) \cong K/n_*(R/\mathfrak{m})$. Thus many problems for mod n K-theory reduce to the case of fields. For example, this reduction plays an essential role in the proof of the étale cohomological descent Theorem 4.1 below.

2.6 In the previously known case where X is a regular scheme of finite type over a field k, this spectral sequence has interesting relations with classical intersection theory because of the formula for $A^p(X)$, the Chow group of codimension p algebraic cycles on X, inspired by Gersten, proved by Quillen ([Q] §7.5), and ascribed to Bloch (who in fact did prove the case $p = 2$, the case $p = 1$ being due to Cartier): $A^p(X) = H^p(X; K_p)$. As Nisnevich observed, this works either for the Zariski or the Nisnevich topology. It would be interesting to have a geometric interpretation of $H^p(X; K_p)$ and a relation to some kind of intersection theory in the singular case. Some work on this has been done by Collino [Co], Gillet [Gi], Levine [Le1, Le2, LeW], Pedrini [PW1, PW2], Weibel, and Barbieri Viale [Vi]. In the singular case, the Zariski and Nisnevich cohomologies are different, and there is some evidence that the Nisnevich cohomology works better.

2.7 The hard-core algebraist may have been wondering if all this scheme business gives anything new for the K-theory of ordinary commutative rings. The answer is that it does. For the proof of spectral sequence 2.4 for the case of a noetherian ring A, $X = \mathrm{Spec}(A)$, and thus the reduction of problems about $K(A)$ to the case of local rings, depends on an induction which requires considering arbitrary open subschemes U of $\mathrm{Spec}(A)$, not all of which will be affine when the dimension of A is greater than one. This is one of the reasons why Quillen's localization result with its hypothesis that $X - Y$ be affine did not suffice to prove Theorem 2.4 for noetherian rings. I also offer the following two new results:

Proposition 2.8. *Let R be a commutative ring, and let $r, s \in R$ be two elements such that the ideal generated by r and s is all R. Then there is a long exact Mayer-Vietoris sequence*:

$$\cdots \to K_n(R) \to K_n(R[1/r]) \oplus K_n(R[1/s]) \to K_n(R[1/rs]) \to K_{n-1}(R) \to \cdots.$$

Proof. This is a special case of 2.3 with $U = \mathrm{Spec}(R[1/r])$ and $V = \mathrm{Spec}(R[1/s])$. Since the ideal (r, s) is all of R, $\mathrm{Spec}(R) = U \cup V$. The result would follow from

Quillen's localization theorem [Gr] under the additional hypothesis that either r or s is not a zero-divisor in R, but this hypothesis is often embarrassing, e.g. in the case of group rings. □

Proposition 2.9 (Weibel). *Let R be a commutative ring, and $r \in R$. Denote by $R_r^\wedge$ the completion* $\lim_k R/r^k R$. *Then there is an exact Mayer-Vietoris sequence*:

$$\cdots \to K_n(R) \to K_n(R_r^\wedge) \oplus K_n(R[1/r]) \to K_n(R_r^\wedge[1/r]) \to K_{n-1}(R) \to \cdots$$

Proof. This follows from 2.1 and an excision result that $K(R$ on $R/r) \approx K(R_r^\wedge$ on $R_r^\wedge/r)$, which Weibel observed could be proved by the method of [TT2] 2.6.3 and 3.19. □

§ 3. Generalizations to Schemes of Some Results Known in the Affine Case

The Mayer-Vietoris Theorem 2.3 often permits one to generalize results known for affine schemes to general quasi-compact separated schemes X, by inducting on the number of affines in an open cover of X. (A second induction on the number of quasi-affines needed to cover X then relaxes the hypothesis of separation to that of quasi-separation). Work of Weibel [We1, We2, We3, We5], Ogle [Og, OW], Geller, Goodwillie [Go], Soulé [So], Staffeldt [Sta], van der Kallen [vdK], and Vorst [Vo] have produced many results controlling the failure of K-theory of rings to be invariant under polynomial or nilpotent extensions, and the failure of Mayer-Vietoris for closed covers of affine schemes. Their demonstrations depend on the use of projective objects and other ring-theoretic techniques, and do not readily generalize to schemes, where the locally projective algebraic vector bundles are not globally projective. However, our method generalizes their conclusions to the case of schemes. As a sample, I give the following results. The first two concern the mod n K-theory defined by Karoubi and Browder; these groups fit in a short exact universal coefficient sequence:

$$0 \to K_q(X) \otimes \mathbb{Z}/n \to K/n_q(X) \to \operatorname{Tor}_{\mathbb{Z}}(K_{q-1}(X), \mathbb{Z}/n) \to 0$$

Theorem 3.1 (Weibel) *Let X be a quasi-compact and quasi-separated scheme, and let $W \to X$ be the total space of a vector bundle over X (or even a torsor under a vector bundle). Let n be an integer such that $1/n \in \mathcal{O}_X$. Then $W \to X$ induces an isomorphism on* mod n *K-groups*:

$$K/n_*(X) \xrightarrow{\sim} K/n_*(W)$$

In particular, this is true when $W = X[T]$.

Proof. [TT2] 9.5. This is deduced by the method above, using the 5-lemma and the Mayer-Vietoris sequence 2.3, from the affine case due to Weibel [We1] 3.3, [We3]. The critical case is that of the polynomial extension $R[T]$ of a ring R. In general, the induced map $K_*(R) \to K_*(R[T])$ has a non-trivial cokernel, although it is obviously injective, split by the map induced by the ring homomorphism sending T to 0. Stienstra [Sti], following work of Almkvist, Grayson, and Bloch, showed that the cokernel was a module over the ring of Witt vectors of R. Weibel observed this shows the cokernel is uniquely n-divisible if $1/n \in R$, yielding the result in the critical case. □

Theorem 3.2 (Weibel). *Let X be a quasi-compact and quasi-separated scheme, with two closed subschemes Y, Z, such that $X = Y \cup Z$ as spaces. Give the intersection $Y \cap Z$ the scheme structure of the fibre product of Y and Z over X. Let n be an integer such that $1/n \in \mathcal{O}_X$. Then there is a homotopy cartesian square:*

$$\begin{array}{ccc} K/n(X) & \longrightarrow & K/n(Y) \\ \downarrow & & \downarrow \\ K/n(Z) & \longrightarrow & K/n(Y \cap Z) \end{array}$$

and an associated long exact Mayer-Vietoris sequence:

$$\cdots \to K/n_q(X) \to K/n_q(Y) \oplus K/n_q(Z) \to K/n_q(Y \cap Z) \to K/n_{q-1}(X) \to \cdots .$$

Proof. [TT2] 9.8. Again one reduces to the affine case, which is due to Weibel, essentially [We2] 1.3. The corresponding statement for K_* in place of K/n_* would not be true, but its failure can be analyzed locally using cyclic homology [OW], which result can be globalized by the method of 3.3 below. □

3.3 Much recent work has focused on cyclic homology HC_*, which serves as a sort of "linear approximation" to algebraic K-theory, and is more readily calculable. Discovered for operator algebras by Connes, cyclic homology has been extended to algebras over a field k by Loday-Quillen and Feigin-Tsygan, and has been extensively developed by J. Block, Brylinksi, Burghelea, Carlsson, Cathelineau, R. Cohen, Goodwillie, J. Jones, Karoubi, Kassel, Ogle, Staffeldt, Vigué-Poirier, Weibel, Wodzicki, and others in a torrent of articles [Lo].

Independently, several people, including J. Block, Loday, and Weibel have proved that HC_* has a Cech-Mayer-Vietoris spectral sequence for a cover of Spec(R) by affine open subschemes. This allows one to extend the definition of $HC_*(X)$ to schemes X over a field k, by taking homology of the total complex of the Cech complex of an affine cover of X, with coefficients in the cyclic homology complex HC. We deduce, as suggested by Weibel:

Theorem 3.4 (Goodwillie). *Let X' be a quasi-compact and quasi-separated scheme over* Spec($\mathbb{Q}$). *Let $i: X \to X'$ be a closed immersion defined by a sheaf of nil ideals. Then there is a natural isomorphism between the relative cyclic homology over $\mathbb{Q}$ and the relative algebraic K-theory:*

$$K_n(X \to X') \cong HC_{n-1}(X \to X').$$

Proof. Here $K(X \to X')$ is the homotopy fibre of the map $i^*: K(X') \to K(X)$, so that the relative K-groups $K_*(X \to X')$ fit in a long exact sequence with $K_*(X)$ and

$K_*(X')$; and similarly for the relative cyclic homology complex $HC(X \to X')$. For X' affine, the main theorem of [Go] gives a natural homotopy equivalence between $K(X \to X')$ and the complex $HC(X \to X')$ shifted one degree and considered as a generalized Eilenberg-MacLane spectrum. The result for general X then follows by Mayer-Vietoris, cf. [TT2] 9.10. □

Levine has remarked that this result is interesting when X is smooth over a field k of characteristic 0, and X' is the singular infinitesimal thickening of X to $X[\varepsilon]/\varepsilon^2$. Then $K_*(X \to X')$ is by Grothendieck's definition the "tangent space" to K_* at X, and the result shows that it indeed equals the tangent space to the "linear approximation" HC_*. One has the spectral sequence of tangent spaces given by applying 2.4 to the fibre of $K(X') \to K(X)$; using cyclic homology to calculate the E_2 term gives:

$$E_2^{p,-q} = H^p(X; \Omega_X^{q-1} \oplus \Omega_X^{q-3} \oplus \Omega_X^{q-5} \oplus \cdots)$$

where Ω_X are the absolute Kähler differentials over $\mathbb{Q}$, not the relative ones over k. This does suggest somewhat obscure relations between this tangent space, Hodge theory, and infinitesimal deformations of algebraic cycles.

§ 4. Comparison of Algebraic and Etale Topological K-Theory

4.0 Suppose now that X is a scheme of finite type over a ring k, where k is one of the following: an algebraically closed field, a separably closed field, a number field, a ring of integers in a number field, $\mathbb{Z}_p^\wedge$, $\mathbb{Q}_p^\wedge$, $\mathbb{F}_q[[t]]$, $\mathbb{F}_q((t))$, or $\mathbb{F}_q$. Fix a prime power $l^\nu \geq 3$. Suppose that $1/l \in \mathcal{O}_X$, and that if $l = 2$ then $\mathcal{O}_X$ contains a square root of -1.

If k contains all the l^ν-th roots of unity, this gives a torsion subgroup $\mathbb{Z}/l^\nu$ in the group of units k^*, and hence in $K_1(k)$. By the universal coefficient sequence, this corresponds to an element β in $K/l^\nu{}_2(k)$. One may then localize the ring $K/l^\nu{}_*(X)$ by inverting the image of this Bott element β. If k doesn't contain all the l^ν-th roots of unity, one can still form this $K/l^\nu(X)[\beta^{-1}]$: as Dwyer showed, essentially a power of β exists in $K/l^\nu{}_*(k)$, although β in degree 2 does not ([T1], Appendix A).

Theorem 4.1. *For X as in* 4.0, *there is a strongly converging spectral sequence from etale cohomolgy*:

$$E_2^{p,-q} \begin{cases} H^p(X_{\mathrm{et}}; \mathbb{Z}/l^\nu(i)) & q = 2i \\ 0 & q \text{ odd} \end{cases} \Rightarrow K/l^\nu{}_{q-p}(X)[\beta^{-1}]$$

The canonical map which forgets the algebraic structure on an algebraic vector bundle and remembers only the underlying topological vector bundle induces an equivalence with the Dwyer-Friedlander etale topological K-theory [DF]:

$$\varrho : K/l^\nu(X)[\beta^{-1}] \xrightarrow{\sim} K/l^{\nu\mathrm{Top}}(X)$$

Proof. [TT2] 11.5. The method of 2.5 reduces this to the case where X is replaced by its various residue fields, which case was done in [T1], as was the case where X is regular. □

One notes that the results of Suslin-Gabber-Gillet-Thomason [Ga, Su1, Su2, GiT] on the K-theory of strict hensel local rings, the local rings for the etale topology, show that the values of the sheaves $K/l^{\nu}{}_q$ in the etale topology are the i-times Tate-twisted cyclic groups $\mathbb{Z}/l^{\nu}(i)$ for $q = 2i$ even and non-negative, and are 0 for q odd and positive. Thus this spectral sequence is the spectral sequence of etale cohomological descent for $K/l^{\nu}{}_*$, at least in positive degrees. However, it converges to $K/l^{\nu}{}_*(X)[\beta^{-1}]$, and not to $K/l^{\nu}{}_*(X)$, which is in general different. I believe that $K/l^{\nu}{}_q(X)[\beta^{-1}]$ equals $K/l^{\nu}{}_q(X)$ for q sufficiently large (cf. the results of [T2]), but this is definitely false for q small, e.g. for $q = 0$ and X a $K3$ surface over $\mathbb{C}$ or $\mathrm{Spec}(R[1/l])$, where R is a ring of integers in a number field which has more than one prime lying over l ([T1] Example 4.5).

The relation between algebraic K-theory and etale cohomology remains a very active area. Leaving aside the infinity of baseless conjectures, there remain some other actual theorems in special cases: the surjectivity results of Soulé in the case of rings of integers, as perfected by Dwyer-Friedlander [DF]; complete results on K_2 of fields due to Merkurjev and Suslin [MS1]; less complete results about K_3 of fields due to Merkurjev-Suslin ([MS2], [MS3]), Levine [Le4], and Rost [R]; and the descent result of Carlsson for $\pi_* BGL_N{}^+$ with respect to finite Galois extensions of certain localized rings of integers. The K_2 result of Merkurjev-Suslin has been extensively employed to study algebraic cycles in codimension 2, notably in the hands of Colliot-Thélène, Coombes, M. Gros, Raskind, Salberger, Sansuc, and Soulé. M. Harada has proved a Riemann-Roch theorem for singular varieties without a quasi-projectivity hypothesis by using the homology version of 4.1 for G-theory to reduce Riemann-Roch to the projection formula ([H], [T4] § 15).

§ 5. The Case of Non-commutative Rings

D. Yao has begun the study of the localization theorem for non-commutative rings [Y]. This involves many new technical problems, and the construction of a sort of algebraic geometry of Grothendieck abelian categories. As a sample of his results, one has:

Theorem 5.1 (D. Yao) *Let R be a ring, not necessarily commutative. Suppose that s_1, $s_2, \ldots, s_n$ are elements of R such that the ideal they generate is all of R. Suppose there exist ring automorphisms ϕ_i for $i = 1, 2, \ldots, n$ such that:*

1) *for all $r \in R$, $\phi_i(r)s_i = s_i r$*
2) $\phi_i(s_j) = s_j$
3) $\phi_i\phi_j = \phi_j\phi_i$

(For example, the s_i could be central, and the $\phi_i = \mathrm{id}$.) Then there is a strongly converging Cech cohomology spectral sequence computing $K_*(R)$ from the K-groups of the localizations:

$$E_2^{p,-q} = H^p\left(\cdots \to \bigoplus_{i_0<i_1<\cdots<i_n} K_q(R[1/s_{i_0} . 1/s_{i_1} \cdot \ldots \cdot 1/s_{i_n}]) \to \cdots\right) \Rightarrow K_{q-p}(R).$$

References

[B] H. Bass: Algebraic K-theory. Benjamin 1968

[BG] K. Brown, S. Gersten: Algebraic K-theory as generalized sheaf homology. In: Higher K-theories. (Lecture Notes in Mathematics, vol. 341) Springer, Berlin Heidelberg New York 1973, pp. 266–292

[Co] A. Collino: Quillen's K-theory and algebraic cycles on almost nonsingular varieties Ill. J. Math. **25** (1981) 654–666

[DF] W. Dwyer, E. Friedlander: Etale K-theory and arithmetic. Trans. Amer. Math. Soc. **292** (1985) 247–280

[Ga] O. Gabber: K-theory of henselian local rings and henselian pairs Preprint 1985

[Ge] S. Gersten: The localization theorem for projective modules. Comm. Alg. **2** (1974) 307–350

[Gi] H. Gillet: On the K-theory of surfaces with multiple curves and a conjecture of Bloch. Duke Math. J. **51** (1984) 195–233

[GiT] H. Gillet, R. Thomason: The K-theory of strict hensel local rings and a theorem of Suslin. J. Pure Appl. Alg. **34** (1984) 241–254

[Go] T. Goodwillie: Relative algebraic K-theory and cyclic homology. Ann. Math. **124** (1986) 347–402

[Gr] D. Grayson: Higher algebraic K-theory II (after D. Quillen). In: Algebraic K-theory: Evanston 1976. (Lecture Notes in Mathematics, vol. 551) Springer, Berlin Heidelberg New York 1976, pp. 217–240

[H] M. Harada: A proof of the Riemann-Roch theorem. Preprint 1988

[KS] K. Kato, S. Saito: Global class field theory of arithmetic surfaces. In: Applications of algebraic K-theory to algebraic geometry and number theory, part I. (Contemporary Mathematics, vol. 55) AMS, 1986, pp. 255–331

[vdK] W. van der Kallen: Descent for K-theory of polynomial rings. Math. Z. **191** (1986) 405–415

[Lec] F. Lecomte: Rigidité des groupes de Chow. Duke J. Math. **53** (1986) 405–426

[Le1] M. Levine: Bloch's formula for singular surfaces. Topology **24** (1985) 165–174

[Le2] M. Levine: Zero cycles and K-theory on singular varieties. In: Algebraic geometry: Bowdoin 1985. (Proc. Symp. Pure Mathematics, vol. 46, part 2) AMS 1987, pp. 451–462

[Le3] M. Levine: Localization on singular varieties. Invent. math. **91** (1988) 423–464

[Le4] M. Levine: The indecomposable K_3 of fields. Ann. Sci. Ec. Norm. Sup. (4) **22** (1989) 255–344

[LeW] M. Levine, C. Weibel: Zero cycles and complete intersections on singular varieties. J. Reine Angew. Math. **359** (1985) 106–120

[Lo] J.-L. Loday: Cyclic homology: a survey. In: Geometric and algebraic topology. (Banach Center Publication, vol. 18) PWN 1986, pp. 285–307

[MS1] A.C. Меркурьев, A.A. Суслин: K-когомологий многооЬразий Севери-Брауэра и гомоморфизм норенного вычета. In: Изв. Акал. Наук С.С.С.Р. Сер. Мат. **46** (1982) 1011–1046 [English translation in Math. USSR Izv. **21** (1983) 307–340]

[MS2] A.S. Merkurjev, A.A. Suslin: On the norm residue homomorphism of degree three. LOMI preprint 1986

[MS3] A.S. Merkurjev, A.A. Suslin: On K_3 of a field. LOMI preprint 1987

[N] Y. Nisnevich: The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In: Algebraic K-theory: Connections with geometry and topology. Kluwer 1989, pp. 241–342

[Og] C. Ogle: On the K-theory and cyclic homology of a square-zero ideal. J. Pure Appl. Alg. **46** (1987) 233–248

[OW] C. Ogle, C. Weibel: Relative K-theory and cyclic homology (to appear)

[PW1] C. Perdrini, C. Weibel: K-theory and Chow groups on singular varieties. In: Applications of algebraic K-theory to algebraic geometry and number theory, part I. (Contemporary Mathematics, vol. 55) AMS 1986, pp. 339–370

[PW2] C. Pedrini, C. Weibel: Bloch's formula for varieties with isolated singularities. Comm. Alg. **14** (1986) 1895–1907

[Q] D. Quillen: Higher algebraic K-theory I. In: Higher K-theories. (Lecture Notes in Mathematics, vol. 341) Springer, Berlin Heidelberg New York 1973, pp. 85–147

[R] M. Rost: Hilbert's theorem 90 for ${K^M}_3$. Preprint 1986

[Se] J.-P. Serre: Prolongement des faisceaux analytiques cohérents. Ann. Inst. Fourier **16** (1966) 363–374

[So] C. Soulé: Rational K-theory of the dual numbers of a ring of algebraic integers. In: Algebraic K-theory, Evanston 1980. (Lecture Notes in Mathematics, vol. 854) Springer, Berlin Heidelberg New York 1981, pp. 402–408

[Sta] R. Staffeldt: Rational algebraic K-theory of certain truncated polynomial rings. Proc. Amer. Math. Soc. **95** (1985) 191–198

[Sti] J. Stienstra: Operations in the higher K-theory of endomorphisms. In: Current trends in algebraic topology, part 2. AMS 1982, pp. 59–115

[Su1] A.A. Suslin: On the K-theory of algebraically closed fields. Invent. math. **73** (1983) 241–245

[Su2] A.A. Suslin: On the K-theory of local fields. J. Pure Appl. Alg. **34** (1984) 301–318

[Su3] A.A. Suslin: Algebraic K-theory of fields. Proceedings of the International Congress of Mathematicians; Berkeley 1986, vol. 1. AMS, pp. 222–244

[T1] R.W. Thomason: Algebraic K-theory and etale cohomology. Ann. Sci. Ec. Norm. Sup. (4) **18** (1985) 437–552; erratum (4) **22** (1989) 675–677

[T2] R.W. Thomason: Bott stability in algebraic K-theory. In: Applications of algebraic K-theory to algebraic geometry and number theory, part I. (Contemporary Mathematics, vol. 55) AMS 1986, pp. 389–406

[T3] R.W. Thomason: The finite stable homotopy type of some topoi. J. Pure Appl. Alg. **47** (1987) 89–104

[T4] R.W. Thomason: Survey of algebraic vs. etale topological K-theory. In: Algebraic K-theory and algebraic number theory. (Contemporary Mathematics, vol. 83) AMS 1989, pp. 393–443

[TT1] R.W. Thomason, T.F. Trobaugh: Le théorème de localisation en K-théorie algébrique. C.R. Acad. Sci. Paris Ser. I **307** (1988) 829–831

[TT2] R.W. Thomason, T.F. Trobaugh: Higher algebraic K-theory of schemes and of derived categories. To appear in the Festschrift for Grothendieck (Progress in Math.) Birkhäuser, Basel

[Ve] J.-L. Verdier: Catégories dérivées. In: SGA4&1/2: Cohomologie étale. (Lecture Notes in Mathematics, vol. 569) Springer, Berlin Heidelberg New York 1977, pp. 262–311

[Vi] L. Barbieri Viale: Cohomology theories and algebraic cycles on singular varieties. Preprint 1989

[Vo] T. Vorst: Localization of the K-theory of polynomial extensions. Math. Ann. **244** (1979) 333–353

[Wa1] F. Waldhausen: Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy. In: Algebraic topology, Aarhus 1982. (Lecture Notes in Mathematics, vol. 1051) Springer, Berlin Heidelberg New York 1984, pp. 173–195

[Wa2] F. Waldhausen: Algebraic K-theory of spaces. In: Algebraic and geometric topology. (Lecture Notes in Mathematics, vol. 1126) Springer, Berlin Heidelberg New York 1985, pp. 318–419

[We1] C. Weibel: Mayer-Vietoris sequences and module structures on NK. In: Algebraic

K-theory, Evanston 1980 (Lecture Notes in Mathematics, vol. 854) Springer, Berlin Heidelberg New York 1981, pp. 466–493

[We2] C. Weibel: Mayer-Vietoris sequences and mod p K-theory. In: Algebraic K-theory, Oberwolfach 1980. (Lecture Notes in Mathematics, vol. 966) Springer, Berlin Heidelberg New York 1982, pp. 390–407

[We3] C. Weibel: Module structures on the K-theory of graded rings. J. Alg. **105** (1987) 465–483

[We4] C. Weibel: A. Brown-Gersten spectral sequence for the K-theory of varieties with isolated singularities. Adv. Math. **73** (1989) 192–203

[We5] C. Weibel: "Homotopy algebraic K-theory. In: Algebraic K-theory and number theory. (Contemporary Mathematics, vol. 83) AMS 1989, pp. 461–488

[Y] D. Yao: Thesis, The Johns Hopkins University 1990

[SGA6] P. Berthelot, A. Grothendieck, L. Illusie: SGA6: Théories des intersections et théorème de Riemann-Roch. (Lecture Notes in Mathematics, vol. 225) Springer, Berlin Heidelberg New York 1971

[EGA] A. Grothendieck, J. Dieudonné: Eléments de géométrie algébrique. (Publ. Math. I.H.E.S. Nos. 8, 11, 17, 20, 24, 28, 32) Presse Univ. France (1961–1967). Chapitre I refondu: (Grundlehren der mathematischen Wissenschaften, Bd. 166) Springer, Berlin Heidelberg New York 1971