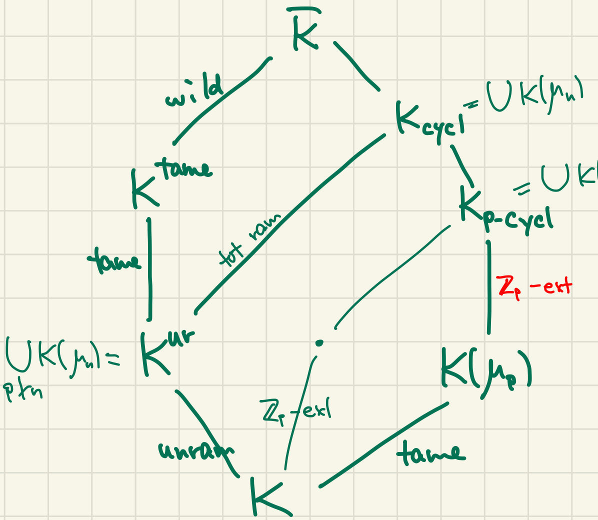
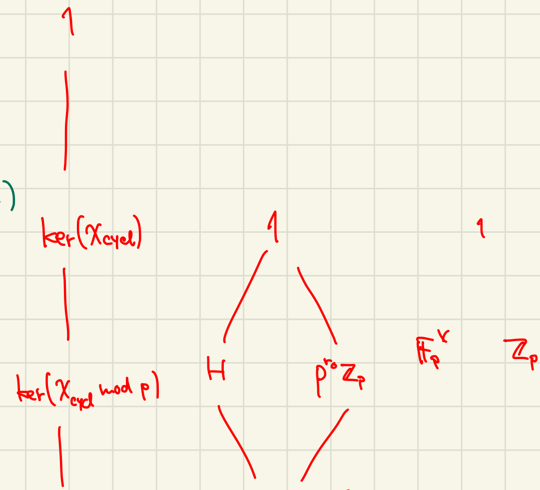
Cheat Sheet

$$
\begin{tikzcd}
\bar{k} & \mathcal{O}_{\bar{K}} \arrow[l, two heads] \arrow[r, phantom, "\subset"] & \bar{K} \arrow[d, dash, "\text{wild ram}"] & & 1 \arrow[d, dash, "\text{pro-}p"] & 1 \\
& & K^{tr} = \bigcup_{p \nmid n} K^{ur}(\sqrt[n]{\pi}) \arrow[d, dash, "\text{tame ram}"'] & & P_K \arrow[d, dash, "\prod_{\ell \neq p} \mathbb{Z}_\ell"] & \\
\bar{\mathbb{F}}_q \cong \bar{k} \arrow[d, dash] & & K^{ur} = \bigcup_{p \nmid n} K(\mu_n) \arrow[d, dash, "\text{unram}"] & & I_K \arrow[d, dash, "\hat{\mathbb{Z}}"] & 1 \\
\mathbb{F}_q \cong k \arrow[d, equal] & \mathcal{O}_K \arrow[l, two heads, "/\mathfrak{m}_K = (\pi)"'] \arrow[r, phantom, "\subset"] \arrow[d, dash, "(p) = (\pi^e)"'] & K \arrow[d, dash, "e"', "\text{tot. ram}"] & 1 \arrow[d, dash] & G_K \arrow[r, two heads] & \mathrm{Gal}(\bar{k}/k) \arrow[d, equal] \\
k \arrow[d, dash] & W(k) \arrow[l, two heads, "/(p)"] \arrow[d, dash] & K_0 \arrow[d, dash, "\text{unram}"] & I \arrow[d, dash] & & 1 \in \hat{\mathbb{Z}} \\
\mathbb{F}_p & \mathbb{Z}_p \arrow[l, two heads, "/(p)"] \arrow[r, phantom, "\subset"] & \mathbb{Q}_p & \mathrm{Gal}(K/\mathbb{Q}_p) & & 1 = \mathrm{Frob}_q
\end{tikzcd}
$$

Δ

residue field (char p) | ring of integers | local field of char 0 | Gal of extensions of $\mathbb{Q}_p$ | Gal of residue fields





$$\begin{tikzcd} G_K \arrow[r, two heads, "res"'] \arrow[rr, bend right, "\chi_{cycl}"'] & Gal(K_{p\text{-}cycl}/K) \arrow[r, hook, "\chi"'] & \mathbb{Z}_p^\times \end{tikzcd}$$

χ_{cycl} the cyclotomic character

$$\mathbb{Z}_p^\times = \mu_p \times (1 + p\mathbb{Z}_p) \xrightarrow{pr \times \log} \mathbb{F}_p^\times \times p\mathbb{Z}_p, \quad p\mathbb{Z}_p \cong \mathbb{Z}_p$$

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{F}_p^\times \arrow[r, "{[-]}"'] & \mathbb{Z}_p^\times \arrow[r, two heads, "\log"'] \arrow[l, dashed, bend right, "pr"'] & p\mathbb{Z}_p \arrow[r] & 1 \\
1 \arrow[r] & H \arrow[r] \arrow[u, hook] & Gal(K_{p\text{-}cycl}/K) \arrow[r] \arrow[u, hook, "\chi"] \arrow[l, dashed, bend left] & p^{r_0}\mathbb{Z}_p \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

$p^{r_0}\mathbb{Z}_p \cong \mathbb{Z}_p$

Hodge - Tate Representation

$$B_{dR} \twoheadrightarrow B_{HT} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)$$

filtration   gr.   as G_K-rep.

Hodge - Tate decomp

<u>Thm</u> (Faltings) X : smooth proper scheme $/K$

$\Longrightarrow$ G_K-equiv isom

$$H^k_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{a+b=k} H^b(X, \Omega^a_{X/K}) \otimes_K \mathbb{C}_p(-a)$$

Tate twist

(cf. analog for $/\mathbb{C}$, X : cpt Kähler : Hodge decomp)

<u>Recall</u> E : top field, G. top gp B : top E-alg

G acts on B, restricts to trivial on E

$$\mathrm{Rep}_B(G) = \{ V : B\text{-mod}, \; G \curvearrowright V : B\text{-semilin} \}$$

$B \otimes_E -$ $\left(g(b\overset{x \in V}{x}) = g(b)\, g(\overset{x}{x}) \right)$

$$\mathrm{Rep}_E(G) = \{ E\text{-linear rep of } G \}$$

$G \curvearrowright$ triv.

$\mathrm{Rep}_B(G)$: abelian. $\underbrace{\otimes_B, \ \mathrm{Hom}_B(-,-)}_{\text{mon. closed}}$

$$\mathrm{Hom}_B(-,-)^G = \mathrm{Hom}_{\mathrm{Rep}_B(G)}(-,-) \quad \left(\begin{tikzcd} \cdot \arrow[r, "g\varphi"] & \cdot \\ \cdot \arrow[u, "g"] \arrow[r, "\varphi : V \to W"'] \arrow[ur, phantom, "\circlearrowleft"] & \cdot \arrow[u, "g"'] \end{tikzcd} \right)$$

B^n : free B-mod with componentwise G-action is called trivial

$$\mathrm{Rep}_E^{B\text{-adm}}(G) \underset{\text{full}}{\subset} \mathrm{Rep}_E(G) \qquad B\text{-semilin rep}$$

$$\| $$

$$\{ V : \text{fin. dim} /E, \ B \otimes_E V \text{ is trivial} \}$$

closed under $\oplus$, $\otimes$, $(-)^\vee$

Assumptions :
- (H1) B : domain
- (H2) $(\mathrm{Frac}\, B)^G = B^G$
- (H3) $b \in B \setminus \{0\}$, $Eb \subset B$ is G-inv $\Rightarrow b \in B^\times$

$\Leftarrow$ [FO's book Def 2.8 (F, G) regular] follows:

B^G: field $\quad b \in B^G \setminus \{0\}$

$\Rightarrow Eb$ G inv $\Rightarrow b \in B^\times$

$(b^{-1} \in B^G)$ ┘

Prop Assume (H1) – (H3)

1.4.1 $V \in \mathrm{Rep}_E(G)$ fin dim $/E$. $W = B \otimes_E V$] TFAE:

(i) W: trivial (i.e. $V \in \mathrm{Rep}_E^{B\text{-adm}}(G)$)

(ii) $\alpha_W : B \otimes_{B^G} W^G \longrightarrow W$ is an iso

(injective: automatic)

(iii) $\dim_{B^G} W^G = \dim_E V$

proof elementary $\square$

Cor From characterization (iii) it's easy to prove

$\mathrm{Rep}_E^{B\text{-adm}}(G) \subset \mathrm{Rep}_E(G)$ closed under sub quotient.

$$\left(\begin{array}{l} 0 \to V_1 \to V \to V_2 \to 0 \\ V : B\text{-adm} \\ \Rightarrow V_1, V_2 : B\text{-adm.} \end{array} \right)$$

$0 \to (B \otimes_E V_1)^G \to (B \otimes_E V)^G \to (B \otimes_E V_2)^G$

(4) Cyclotomic character, Tate twist

<u>Setting</u>

$$\begin{array}{c} \mathbb{C}_p \\ | \text{ compl.} \\ \overline{K} \\ | \\ K \\ | \text{ finite} \\ \mathbb{Q}_p \end{array}$$

- $G = G_K = \mathrm{Gal}(\overline{K}/K) \overset{\text{conti}}{\curvearrowright} \mathbb{C}_p$
  $\supseteq$ triv.
- $E = \mathbb{Q}_p$

$\mu_{p^n}(K) \xrightarrow{\simeq} \mathbb{Z}/p^n)^\times$
$\circlearrowleft \qquad\qquad \circlearrowleft x_n(\sigma)$ mult.

ζ_n prim. p^nth root of 1

$$\chi_n : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(K(\mu_{p^n})/K) \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times$$
$$\sigma \longmapsto \left(\zeta_n \mapsto \zeta_n^{\chi_n(\sigma)}\right) \qquad \chi_n(\sigma)$$

$$\begin{tikzcd} & \mathbb{Z}_p^\times \arrow[d, dotted] \\ & (\mathbb{Z}/p^2\mathbb{Z})^\times \arrow[d] \\ \mathrm{Gal}(\overline{K}/K) \arrow[uur, bend left, "\chi"] \arrow[ur, "\chi_2"'] \arrow[r, "\chi_1"'] & (\mathbb{Z}/p\mathbb{Z})^\times \end{tikzcd}$$

$\rightsquigarrow \chi : G_K \longrightarrow \mathbb{Z}_p^\times$ cyclotomic character

$$\mathbb{Z}_p^\times = GL_1(\mathbb{Z}_p) \hookrightarrow GL_1(\mathbb{Q}_p)$$

1-dim rep'n $\mathbb{Q}_p(1)$ of G_K / $\mathbb{Q}_p$

$$\underbrace{\mathbb{Q}_p(n)}_{\uparrow} = \begin{cases} \mathbb{Q}_p(1)^{\otimes n} & (n \geq 0) \\ (\mathbb{Q}_p(1)^\vee)^{\otimes(-n)} & (n < 0) \end{cases}$$

Tate twist

repn corr. char χ^n

$$V(n) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}(n)$$

§1.1.3

<u>Aside</u>: characters of $G_{\mathbb{Q}_p}$ are

$$\chi^a \cdot \underset{\uparrow}{\omega^b} \cdot \lambda \quad \text{unramified} \quad \hat{\mathbb{Z}} \to \mathbb{Z}_p^\times$$

$\hat{\mathbb{Z}}$ = ⟨Frob⟩ ↦ λ

$$\begin{cases} a \in \mathbb{Z}_p \\ b \in \mathbb{F}_p^{\times\vee} \end{cases} \left(G_K \to \mathbb{Z}_p^\times \xrightarrow[\text{mod } p]{} \mathbb{F}_p^\times \xrightarrow[{[\cdot]}]{} \mathbb{Z}_p^\times \right)$$

$$1 \to 1+p\mathbb{Z}_p \to \mathbb{Z}_p^\times \to \mathbb{F}_p^\times \to 1$$

$$1+p\mathbb{Z}_p \cong \mathbb{Z}_p, \qquad \mathbb{F}_p^\times \cong \mu_p$$

$$B_{HT} := \mathbb{C}_p[t^{\pm 1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_p(n), \qquad \mathbb{C}_p(n) = \mathbb{C}_p \cdot t^n$$

<u>G_K-mod</u> $\quad g \cdot (a t^n) = \underbrace{ga}_{\uparrow} \cdot \chi(g)^n \cdot t^n$

semilinearity

③ Back to Fontaine's formalism.

① $\mathbb{C}_p$-admissibility $\quad \begin{cases} \cdot\ B = \mathbb{C}_p \\ \cdot\ E = \mathbb{Q}_p \\ \cdot\ G = G_K \end{cases}$

Thm V: $\mathbb{Q}_p$-linear, fin dim rep of G_K is $\mathbb{C}_p$-admissible iff

$$\begin{array}{ccccc} I_K & \hookrightarrow & G_K & \longrightarrow & \mathrm{Aut}_{\mathbb{Q}_p}(V) \\ \downarrow & & & & \\ \left(\begin{array}{c}\text{finite}\\ \text{quotient}\end{array}\right) & & & \dashrightarrow & \end{array}$$

"potentially unramified"

i.e. $\exists\, L/K^{ur}$ finite

s.t. $\mathrm{Gal}(\overline{K}/L) \hookrightarrow I_K$
$\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad \mathrm{Gal}(\overline{K}/K^{ur})$

acts trivially

$$\begin{array}{ccc} \overline{K} & & 1 \\ | & & | \\ L & & \mathcal{O} \\ | & & \| \leftarrow \text{finite quotient} \\ K^{ur} & & I_K \\ | & & \\ K & & G_K \end{array}$$

G_K by χ^n

$\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$

proof hard

(B) B_{HT}-admissibility

Prop 1. V as above. is B_{HT}-adm. iff

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V = \mathbb{C}_p(n_1) \oplus \cdots \oplus \mathbb{C}_p(n_d)$$

(such V is called a Hodge-Tate repn)

Prop. 2 $n_1, \dots, n_d$: Uniquely determined by V
called Hodge - Tate weights of V

Prop 3 B_{HT} satisfy (H1) – (H3)
$B'_{HT} = \mathbb{C}_p((t))$

Prop 2 : enough to check $\dim_K \mathrm{Hom}_{\mathrm{Rep}_{\mathbb{C}_p}(G_K)}(\mathbb{C}_p(n), \mathbb{C}_p(m)) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$$\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{C}_p}(G_K)}(\mathbb{C}_p(n), \mathbb{C}_p(m))$$

$$= \mathrm{Hom}_{\mathbb{C}_p}(\mathbb{C}_p(n), \mathbb{C}_p(m))^{G_K}$$

$$= \mathbb{C}_p(m-n)^{G_K}$$

$$\dim_K \left(\mathbb{C}_p(m-n)^{G_K}\right)$$

$m = n$ $\quad \mathbb{C}_p^{G_K} = K$ (Ax - Sen - Tate)

$m \neq n$ enough to show $\mathbb{C}_p(i)$ is not $\mathbb{C}_p$-admissible

$\left(\rightsquigarrow \underset{\substack{\parallel \\ \mathbb{C}_p^{G_K}}}{\dim_K} \overset{\substack{0 \\ \parallel}}{\mathbb{C}_p(i)^{G_K}} < \overset{\substack{1 \\ \parallel}}{\dim_{\mathbb{Q}_p}(\mathbb{Q}_p(i))} \right)$

[biproduct: $\mathbb{C}_p(n)^{G_K} = 0$, $n \neq 0$]

by the Thm this is equiv. to

$I_K \curvearrowright \mathbb{C}_p(i)$ does not factor through a fin. quotient

χ^i

$$\begin{tikzcd}
I_K \arrow[r, hook] \arrow[d] & G_K \arrow[r, "\chi^i"'] & \mathbb{Z}_p^\times \\
I_K/I_K \cap \ker \chi^i \arrow[urr] & &
\end{tikzcd}$$

$$\begin{tikzcd}[arrows=dash]
& \overline{K} \arrow[d] & & & 1 & \\
& K_{cycl} \arrow[dl] \arrow[dr] & & & \ker\chi \cap I_K \arrow[dl] \arrow[dr, "\text{can't be finite}"] & \\
K_{p\text{-}cycl.} \arrow[dr] & & K^{ur} \arrow[dl] & \ker \chi \arrow[dr, "\text{infinitely ramified}"'] & & I_K \arrow[dl] \\
& K & & & G_K &
\end{tikzcd}$$

Prop 1. Prop 3 : exercise

follows easily from $(\mathbb{C}_p(n))^{G_K} = 0$