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Rigid analytic spaces

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RIGID ANALYTIC SPACES (*)

by Marius VAN DER PUT

1. Tate-algebras.

(1.1) Notations. - k is a complete non-archimedean valued field. For a Banach-algebra A over k (always commutative and with 1) and indeterminates

$$T_1, \ldots, T_n,$$

we define

$$A\langle T_1, \ldots, T_n\rangle = \{\sum a_\alpha T^\alpha ; \; a_\alpha \in A \text{ and } \lim a_\alpha = 0\} .$$

This is a new Banach-algebra over k with respect to (w. r. t.) the norm $\|\sum a_\alpha T^\alpha\| = \max\|a_\alpha\|$. A free Tate-algebra is a ring of the type $k\langle T_1, \ldots, T_n\rangle$.

(1.2) PROPOSITION (Weierstrass preparation and division). - Let $f \in k\langle T_1, \ldots, T_n\rangle$ be non-zero. There exists an automorphism σ of $k\langle T_1, \ldots, T_n\rangle$ (of the form $X_i \to X_i + X_n^{e_i}$ $(e_i \geq 1, \; i < n)$; $X_n \to X_n$) such that $\sigma(f)(0, - 0, T_n)$ has order d.

Moreover $k\langle T_1, \ldots, T_n\rangle/\sigma(f)$ is a free finitely generated $k\langle T_1, \ldots, T_{n-1}\rangle$-module of rank d.

Proof. - See [7] GRAUERT-REMMERT.

(1.3) Consequences.

(1.3.1) Every $k\langle T_1, \ldots, T_n\rangle$ is noetherean.

(1.3.2) $k\langle T_1, \ldots, T_n\rangle$ is a unique factorisation domain.

Proof. - Induction on n and (1.2).

(1.4) LEMMA. - Let M be a Banach-module over A, (i. e. A Banach-algebra and M is a complete normed A-module s. t. $\|am\| \leq \|a\| \|m\|$, $\forall a \in A$, $\forall m \in M$).

The following are equivalent

(a) M is noetherean.

(b) Every A-submodule of M is closed.

(*) Survey of the works done by J. TATE, H. GRAUERT, R. REMMERT, L. GERRITZEN, R. KIEHL, L. GRUSON, M. RAYNAUD and al.

Proof. – (b) $\Rightarrow$ (a) : Let $M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \dots$ be an infinite chain of submodules of A. Then one can easily see that $\bigcup_{i \geq 1} M_i$ is not closed. Contradiction.

(a) $\Rightarrow$ (b) : Let N be a maximal non closed submodule of M. Then $N \subset \overline{N}$ has no intermediate A-modules. Hence $\overline{N}/N \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Since $\mathfrak{m}$ is closed in A it follows that N is also closed. Contradiction.

(1.5) Every ideal I in $k\langle T_1, \dots, T_n\rangle$ is closed according to (1.4) and (1.3.1). A Tate-algebra is an algebra of the type $k\langle T_1, \dots, T_n\rangle/I$ provided with the quotient norm.

Easy consequences are :

(1.5.1) Any k-homomorphism of Tate-algebra is continuous.

(1.5.2) Any finitely generated module over a Tate-algebra A has a unique structure as Banach-module. A linear map between those modules is automatically continuous.

(1.6) From (1.2), it follows :

For every Tate-algebra A, there exists a map $K\langle T_1, \dots, T_d\rangle \xrightarrow{\alpha} A$ with α injective and finite. Moreover $d = \text{Krull-dim} A$.

In particular, for every maximal ideal $\mathfrak{m}$ of A, we have $[(A/\mathfrak{m}) : k] < \infty$. On $A/\mathfrak{m}$, we put the unique valuation extending the valuation of k.

(1.7) Some notations.

$X = \operatorname{Sp} A$ = the set of maximal ideals of A.

For $x \in X$, we put $k(x) = A/x$. For $f \in A$, we denote by $f(x)$ the image of f into A/x. The spectral semi-norm $\|f\|_{sp}$ is defined by $\|f\|_{sp} = \sup_{x\in X} |f(x)|$.

For $A = k\langle T_1, \dots, T_n\rangle$ one easily checks $\|f\|_{sp} = \| \ \|$ and the norm is multiplicative.

(1.8) Properties of the spectral norm.

(1.8.1) $|f(x)| <$ for all $x \in X \iff \lim \|f^n\| = 0$,

(1.8.2) $\|f\|_{sp} = \lim \|f^n\|^{1/n}$,

(1.8.3) $|f(x)| \leq 1$ for all $x \in X \iff \sup\{\|f^n\| ; n \geq 0\} < \infty$,

(1.8.4) A k-algebra homomorphism $\varphi : A\langle T_1, \dots, T_n\rangle \to B$ is uniquely determined by φ/A and $\varphi(T_i) = f_i \in B$ $(i = 1, \dots, n)$. A φ with prescribed φ/A and f_i $(1 \leq i \leq n)$ exists if, and only if, $|f_i(x)| \leq 1$ for all $x \in \operatorname{Sp}(B)$ and $i = 1, \dots, n$.

(1.8.5) If A is reduced (i. e. has no nilpotents elements) then $\| \ \|_{sp}$ is equivalent with $\| \ \|$.

(1.8.6) <u>There is</u> $x_0 \in X = \mathrm{Sp}\,A$ <u>with</u> $|f(x_0)| = \max_{x\in X}|f(x)|$.

<u>Proof</u>.- (1.8.1) : The ideal $(1 - Tf)\,A\langle T\rangle$ in $A\langle T\rangle$ must be improper because of (1.6) and $|f(x)| < 1$ for all $x \in X$. Hence $(1 - Tf)$ has an inverse in $A\langle T\rangle$. That inverse must be $\sum_{n\geq 0} f^n T^n$. So $\lim \|f^n\| = 0$.

On the other hand, if $\lim \|f^n\| = 0$, then $|f(x)| \leq \|f^n\|^{1/n}$ is < 1 for all x and $n \gg 0$.

(1.8.2) : "$\leq$" is trivial. If $\|f\|_{sp} < \lim \|f^n\|^{1/n}$, then we can arrange things such that $\|f\|_{sp} < 1 \leq \lim \|f^n\|^{1/n}$. But this contradicts (1.8.1).

(1.8.3) : The implication "$\Longleftarrow$" follows from (1.8.2). The implication "$\Longrightarrow$" is more complicated :

Suppose that $k\langle T_1 , \dots , T_d\rangle \hookrightarrow A$ is injective and finite. If we can show that $f \in A$ is integral over $V\langle T_1 , \dots , T_d\rangle$ (V the valuation-ring of k), then clearly $\{\|f^n\|/n \geq 0\}$ is a bounded set. For show the integral dependence of A , it suffices to consider the case where A has no zero-divisors.

Let L be the least <u>normal</u> field extension of $K = QE(k\langle T_1 , \dots , T_d\rangle)$ containing A , and let $G = \mathrm{Aut}(L/K)$. Then $B = \underset{\sim}{Z}[A^\sigma \,;\, \sigma \in G]$ is also integral over $k\langle T_1 , \dots , T_d\rangle$ and the minimum polynomial of f over K divides

$$P = \prod_{\sigma\in G}(X - f^\sigma)^q \quad (\,q = \text{some power of the characteristic})\ .$$

Since $k\langle T_1 , \dots , T_d\rangle$ is normal, P has coefficients in $k\langle T_1 , \dots , T_d\rangle$. Since $|f^\sigma(x)| \leq 1$ for all maximal ideal of B , the coefficients of P have spectral norms ≤ 1 . So $P \in V\langle T_1 \dots T_d\rangle[X]$.

(1.8.4) : Easy consequence of (1.8.3).

(1.8.5) : This is more complicated (proved by L. GERRITZEN). We only sketch a proof. As in (1.8.3), we may suppose that A has no zero-divisors. Let $f \in A$ have minimum polynomial $X^d + a_1 X^{d-1} + \dots + a_d$ $(= 0)$ over $k\langle T_1 , \dots , T_d\rangle$. Then $\|f\|_{sp} = \max_{1\leq i\leq s} \|a_i\|^{1/i}$. The hard part is to show with the aid of this formula that A is complete w. r. t. $\|\ \|_{sp}$. Then it follows from the open mapping theorem that $\|\ \|_{sp}$ and $\|\ \|$ are equivalent on A (See R. REMMERT [14]).

(1.8.6) : By the formula of (1.8.5) one sees that, after replacing f by λf^e $(e \geq 1 \,,\ \lambda \in k^*)$, we may work with $\|f\|_{sp} = 1$.

If $|f(x)| < 1$ for all $x \in X$ then, from (1.8.1), it follows that $\|f^n\| < 1$ for $n \gg 0$. So $\|f\|_{sp} < 1$. This contradiction shows the existence of $x_0 \in X$ with $|f(x_0)| = \|f\|_{sp}$.

(1.9) <u>Further structure theorems on Tate-algebras</u>.

(1.9.1) (GERRITZEN) : <u>If</u> k <u>is</u> (quasi-)<u>complete then any Tate-algebra</u> A/k <u>is japanese</u> (i. e. integral extensions of A in a finite field extension are finite modules over A).

(1.9.2) (KIEHL-KUNZ-BERGER-NASTOLD) : *If* k *is* (quasi-)*complete then* A *is an excellent ring* (*in the sense of GROTHENDIECK*). (See : KIEHL-KUNZ-BERGER- NASTOLD [1])

2. Affine holomorphic spaces.

(2.1) Let A be a Tate-algebra, defined over a field k. Let $X = Sp(A)$ denote the collection of all maximal ideals of A. For every $x \in X$, the residue field $k(x) = A/x$ is a finite extension of k and has therefore a unique valuation, always denoted by $|\ |$, extending the valuation of k. For $x \in X$ and $f \in A$, we denote by $f(x)$ the image of f in $k(x)$.

The topology on X is generated by the subsets $\{x \in X ; |f(x)| \leq 1\}$ with $f \in A$. A base for this topology is the set of the so-called *Weierstrass-domains*

$$W(f_1 , \dots , f_n) = \{x \in X ; |f_i(x)| \leq 1 \text{ for all } i\} .$$

A more general class of open (and closed) subsets of X are the *rational domains*

$$R = R(f_0 , \dots , f_n) = \{x \in X ; |f_i(x)| \leq |f_0(x)| \text{ for all } i\} ,$$

where we have supposed that $f_0 , \dots , f_n$ have no common zero on X. With R, we associate a Tate-algebra B, $B = A\langle T_1 , \dots , T_n\rangle/(f_1 - T_1 f_0 , \dots , f_n - T_n f_0)$.

(2.2) PROPOSITION.

(2.2.1) *The map* $A \xrightarrow{\varphi} B$ *induces a continuous map* $Sp(\varphi) : Sp(B) \to Sp(A)$. *The image is* R *and* $Sp(\varphi) : Sp(B) \to R$ *is a homeomorphism*.

(2.2.2) *For every* (k-*algebra homomorphism*) $\psi : A \to C$ *of Tate-algebras with* $Sp(\psi)(Sp(C)) \subseteq R$ *there is a unique* $\chi : B \to C$ *with* $\chi\varphi = \psi$.

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi"'] & C \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Proof.

(2.2.1) : For any k-algebra homomorphism φ, the induced map $Sp(\varphi)$ is continuous. For the given B, one easily verifies that $Sp(\varphi) : Sp(B) \to R$ is a homeomorphism.

(2.2.2) : The map $\chi : B \to C$ is uniquely determined by $\chi(T_i)$ $(i = 1 , \dots , n)$ and $\chi(T_i) = \psi(f_i)/\psi(f_0)$ must hold. The existence of χ follows from § 1 (1.8.4). Namely, the elements $g_i = \psi(f_i)/\psi(f_0)$ in C satisfy :

$$|g_i(x)| \leq 1 \text{ for all } x \in Sp(C) .$$

Hence, the set $\{\| g_1^{\alpha_1} \cdots g_n^{\alpha_n} \| ; \alpha_1 , \dots , \alpha_n \geq 0\}$ is bounded and the map

$$\tilde{\chi} : A\langle T_1 , \dots , T_n\rangle \to C ,$$

given by

$$\sum a_\alpha T_1^{\alpha_1} \dots T_n^{\alpha_n} \to \sum \varphi(a_\alpha)\, g_1^{\alpha_1} \dots g_n^{\alpha_n} \text{ , (with } a_\alpha \in A \text{ , } \lim a_\alpha = 0)$$

is a k-algebra homomorphism. The kernel of $\tilde{\chi}$ contains

$$(f_1 - T_1 f_0 , \dots , f_n - T_n f_0)$$

and $\tilde{\chi}$ induced the required $\chi : B \to C$.

(2.3) For every rational domain $R = R(f_0 , \dots , f_n)$, we define

$$P(R) = A\langle T_1 , \dots , T_n\rangle/(f_i - T_i f_0)_{i=1}^n .$$

According to (2.2.2), $P(R)$ does not depend on the choice of $\{f_0 , \dots , f_n\}$ and moreover $R \to P(R)$ is a pre-sheaf defined on the base $\{R ; R \text{ rational}\}$. Let us denote by H_X the sheaf on X (with the usual topology) associated with P .

(2.4) Results.

(2.4.1) For $x \in X$, the stalk $H_{X,x}$ is a local analytic ring (i. e. a finite extension or a ring of convergent power series over k).

(2.4.2) The natural map of the localisation of A at x : $A_x \to H_{X,x}$, induces an isomorphism for the completions of those local rings, $\hat{A}_x \xrightarrow{\sim} \hat{H}_{X,x}$.

(2.4.3) For a rational domain R with $B = P(R)$, the map $\varphi : A \to B$ induces an isomorphism of ringed spaces $(\mathrm{Sp}\, B , H_{\mathrm{Sp}\, B}) \xrightarrow{\sim} (R , H_X/R)$.

Proof. - For $X = \mathrm{Sp}(k\langle T_1 , \dots , T_n\rangle) = \{(t_1 , \dots , t_n) \in k^n , \text{ all } |t_i| \leq 1\}$ all this is easily verified. All the operations : completion, localisation, forming of H , commute with taking residues w. r. t. an ideal $I \subset k\langle T_1 , \dots , T_n\rangle$. From this obeervation the general case follows.

(2.5) Definition. - An open subset $Y \subset X = \mathrm{Sp}\, A$ is called affine if there exists a Tate-algebra B and a morphism $\varphi : A \to B$ which induces an isomorphism of ringed spaces $(\mathrm{Sp}\, B , H_{\mathrm{Sp}\, B}) \xrightarrow{\sim} (Y , H_{X/Y})$.

(2.6) Remarks. - The ringed space (X , H_X) is an example of what H. CARTAN and S. ABHYANKAR would call a k-analytic space. Since X is totally disconnected, the sheaf H_X is very big. In particular, $\Gamma(X , H_X) \supsetneq A$.

Note that $A \to \Gamma(X , H_X)$ is injective, since the map

$$A \to \Gamma(X , H_X) \to \prod_{x\in X} H_{X,x} \to \prod_{x\in X} \hat{H}_{X,x} \xrightarrow{\sim} \prod_{x\in X} \hat{A}_x$$

is injective.

To get something interesting, we have to consider on X a Grothendieck-topology instead of the ordinary topology. For this purpose, we have introduced open affine subsets of X . Our definition is (with a slight modification) the one of GERRITZEN-GRAUERT ([6], p. 162). Afterwards, we will show that Y determines the algebra B (this is of course clear for rational domains Y). It follows that Y is an affine open subset in the sense of J. TATE ([16], p. 270). (It is immediate that an affine open subset in the sense of J. TATE is also an affine open set in the sense of (2.5)).

In order to see what this Grothendieck topology on X should be, we have to find "gluing-properties" for the pre-sheaf P.

(2.7) LEMMA.

(2.7.1) *If* $Y_1, Y_2 \subset X$ *are rational domains, then so is* $Y_1 \cap Y_2$. *Moreover* $P(Y_1 \cap Y_2) = P(Y_1) \hat{\otimes}_A P(Y_2)$.

(2.7.2) *If* $Y_1 \subset Y_2 \subset X$ *are open subsets such that* Y_2 *is rational in* X *and* Y_1 *is rational in* Y_2, *then* Y_1 *is rational in* X.

Proof.

(2.7.1) : Let $Y_1 = R(f_0, \dots, f_n)$ and $Y_2 = R(g_0, \dots, g_m)$ then

$$Y_1 \cap Y_2 = R(f_0 g_0, f_1 g_1, \dots, f_1 g_m, \dots, f_n g_1, \dots, f_n g_m).$$

Moreover

$$P(Y_1 \cap Y_2) = A\langle T_{ij} ; 1 \leq i, j \leq n, m\rangle/(f_i g_j - T_{ij} f_0 g_0)$$

is easily seen to be isomorphic with

$$A\langle T_i\rangle/(f_i - T_i f_0) \hat{\otimes} A\langle S_j\rangle/(g_j - S_j g_0) \cong \frac{A\langle T_1, \dots, T_n, S_1, \dots, S_m\rangle}{(f_i - T_i f_0, g_j - S_j g_0)_{ij}}.$$

(2.7.2) : Let $Y_2 = R(g_0, \dots, g_m)$ and let

$$f_0, \dots, f_n \in A\langle S_1, \dots, S_m\rangle/(g_j - S_j g_0)$$

define Y_1 as a rational subset of Y_2. Elements $f'_0, \dots, f'_n \in P(Y_2)$ such that the $\|f'_i - f_i\|$ are very small define the same rational subset of Y_2. So we may suppose that $f_0, \dots, f_n$ are represented by elements in $A[S_1, \dots, S_m]$ of total degree $\leq N$. We may replace $f_0, \dots, f_n$ by $g_0^N f_0, \dots, g_0^N f_n$. Hence, we may suppose that $f_0, \dots, f_n \in A$. For suitable constants $\lambda_0, \dots, \lambda_m \in k^*$ we have on Y_1:

$$|f_0(x)| \geq |\lambda_i g_i(x)| \quad \text{for all } i \text{ and } x \in Y_1.$$

And thus $Y_1 = Y_2 \cap R(f_0, \dots, f_m, \lambda_0 g_0, \dots, \lambda_m g_m)$ is rational in X.

(2.8) THEOREM. – *For any finite covering* $\mathfrak{X} = (X_i)$ *of* X *by rational domains, the Čech-complex* $\mathcal{C}_{\mathfrak{X}} : 0 \to P(x) \to \bigoplus P(X_i) \to \bigoplus P(X_i \cap X_j) \to \dots$ *is universally acyclic (i. e.* $\mathcal{C}_{\mathfrak{X}} \hat{\otimes}_A M$ *is acyclic for every normed A-module* M *).*

Proof. – We follow J. TATE ([16], p. 272). First two special cases of coverings.

(2.8.1) LEMMA. – *Let* $f \in A$ *and put*

$$X_1 = \{x \in X ; |f(x)| \leq 1\} \text{ and } X_2 = \{x \in X ; |f(x)| \geq 1\}.$$

Then the covering $\{X_1 , X_2\}$ of X is u. a. (universally acyclic).

(2.8.2) LEMMA. – Let $f_0 , \dots , f_n \in A$ satisfy $\max_i |f_i(x)| = 1$ for all $x \in X$. Then the covering of X by $X_i = \{x \in X ; |f_i(x)| = 1\}$ $(i = 0 , \dots , n)$ is u. a.

Proof. – J. TATE ([16] lemma 8.3 and 8.4) shows that both coverings have a continuous A-linear homotopy $\mathcal{C}_{\mathfrak{X}} \xrightarrow{\partial} \mathcal{C}_{\mathfrak{X}}$. This induces a homotopy $\partial \hat{\otimes} 1_M$ on $\mathcal{C}_{\mathfrak{X}} \hat{\otimes}_A M$. Now we need some general hocus pocus to do the general case :

(2.8.3) LEMMA. – Let $\mathfrak{X}$ and $\mathfrak{Y}$ be coverings of X (by finitely many affine open subsets). Suppose that $\mathfrak{X}/Z$ is u. a. for every Z which is an intersection of elements in $\mathfrak{Y}$.

If $\mathfrak{Y}$ is u. a. then $\mathfrak{X}$ is u. a.

We consider the double complex $\mathcal{C}_{\mathfrak{X}} \hat{\otimes}_A \mathcal{C}_{\mathfrak{Y}}$. It is given that

1° $\mathcal{C}_{\mathfrak{X}} \hat{\otimes}_A P(Z)$, for Z an intersection of elements in $\mathfrak{Y}$, is exact,

2° $\mathcal{C}^i_{\mathfrak{X}} \hat{\otimes}_A \mathcal{C}_{\mathfrak{Y}}$, for $i = -1 , 0 , \dots , r$, is exact.

So, all rows and columns, except possibly $\mathcal{C}_{\mathfrak{X}} \hat{\otimes}_A \mathcal{C}^{-1}_{\mathfrak{Y}} = \mathcal{C}_{\mathfrak{X}}$, are exact. Hence $\mathcal{C}_{\mathfrak{X}}$ is exact. The same reasoning holds for $\mathcal{C}_{\mathfrak{X}} \hat{\otimes}_A M$.

(2.8.4) Continuation of the proof of (2.8). – First we observe : If $\mathfrak{X}$ and $\mathfrak{Y}$ are u. a., then so is $\mathfrak{X} \cap \mathfrak{Y} = \{X \cap Y ; X \in \mathfrak{X} , Y \in \mathfrak{Y}\}$. Indeed, by (2.8.3) applied to $\mathfrak{X}' = \mathfrak{X} \cap \mathfrak{Y}$ and $\mathfrak{Y}' = \mathfrak{Y}$ this follows.

Let us start with any finite covering $\mathfrak{X} = \{R(f_0^{(i)} , \dots , f_n^{(i)}\}$ by rational domains. Choose $\varepsilon > 0$ such that $|f_0^{(i)}(x)| > \varepsilon$ for all $x \in R(f_0^{(i)} , \dots , f_n^{(i)})$. Let $\{g_1 , \dots , g_s\}$ denote the set $\{f_j^{(i)}\}$, and let, for every subset σ of $\{1 , \dots , s\}$,

$$Y_\sigma = \{x \in X ; |g_i(x)| \leq \varepsilon \text{ for } i \in \sigma \text{ and } |g_i(x)| \geqslant \varepsilon \text{ for } i \notin \sigma\} .$$

The covering $\mathfrak{Y} = \{Y_\sigma\}_{\text{all } \sigma}$ is the intersection of s coverings of the type in (2.8.1). Hence $\mathfrak{Y}$ is u. a. In order to show that $\mathfrak{X}$ is u. a., it suffices to see that $\mathfrak{X}/Z$ is u. a. for any Z which is an intersection of elements of $\mathfrak{Y}$.

This new covering $\mathfrak{X}' = \mathfrak{X}/Z$ consist of Weierstrass-domains in Z, i. e. sets of the type $\{x \in Z ; |f_i(x)| \leq 1$ for some i's$\}$. Let $\{h_1 , \dots , h_t\}$ denote the set of all functions occuring in those inequalities, and let $\mathfrak{Y}' = (Y'_\sigma)$ denote the covering of Z given by

$$Y'_\sigma = \{x \in Z ; |h_i(x)| \leq 1 \text{ for } i \in \sigma \text{ and } |h_i(x)| \geqslant 1 \text{ for } i \notin \sigma\} .$$

Again $\mathfrak{Y}'$ is u. a. and in order to show that $\mathfrak{X}'$ is u. a., we have to show $\mathfrak{X}'/Z'$, Z' any intersection of elements of $\mathfrak{Y}'$, is u. a. This last covering however is of the type mentioned in (2.8.2), and the proof is finished.

(2.9) THEOREM (GERRITZEN-GRAUERT [6] p. 178). – An open affine subset of $X = Sp(A)$ is a finite union of rational domains.

Proof. – The proof is quite long. The eesential part is a result on Runge embeldings (There seems to be a gap in the proof.).

(2.10) COROLLARY. – The open affine subset Y of X determines uniquely the morphism of Tate-algebra's $A \xrightarrow{\varphi} B$ for which $(\mathrm{Sp}\, B\, ,\, H_{\mathrm{Sp}\, B}) \to (Y\, ,\, H_X/Y)$ is an isomorphism.

Proof. – Put $Y = \bigcup_{i=1}^{n} X_i$ where the X_i are rational domains in X. Then the X_i are also rational in Y and (2.8) implies $B = \ker(\bigoplus P(X_i) \to P(X_i \cap X_j))$.

(2.11) COROLLARY. – Any finite covering of X by affine open subsets is universally acyclic.

Proof. – Follows from (2.9), (2.8) and (2.8.3).

(2.12) Remarks. – A morphism $\mathrm{Sp}(\varphi)$: $Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ is called a Runge-map when $\varphi : A \to B$ has a dense image. The proof of (2.9) relies on the following proposition :

Let $u = \mathrm{Sp}(\varphi)$; $Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ be given, and let $f_0\, , \dots ,\, f_n \in A$ be given such that $(f_0\, , \dots ,\, f_n)A = A$. Put

$$X_\varepsilon = \{x \in X\ ;\ \ |f_i(x)| \leq \varepsilon |f_0(x)| \ \text{ for all } x\} \ \text{ and } \ Y_\varepsilon = u^{-1}(X_\varepsilon)\ .$$

If $u : Y_1 \to X_1$ is Runge then for ε close to 1, $u : Y_\varepsilon \to X_\varepsilon$ is also a Runge-map.

(2.13) For our purpose, we define a Grothendieck-topology on a topological space X as follows

1° A family $\mathfrak{F}$ of open subsets of X such that

$$\Phi\, , X \in \mathfrak{F}\ ;\ \ U_1\ V \in \mathfrak{F} \Longrightarrow U \cap V \in \mathfrak{F}\ .$$

2° For every $U \in \mathfrak{F}$ a set $\mathrm{Cov}(U)$ of coverings by elts in $\mathfrak{F}$, i. e. any

$$\mathcal{U} = (U_i) \in \mathrm{Cov}(U)$$

satisfies : all $U_i \in \mathfrak{F}$ and $\bigcup U_i = U$.

3° $\{U \to U\} \in \mathrm{Cov}(U)$ for all $U \in \mathfrak{F}$.

4° $\mathcal{U} \in \mathrm{Cov}(U)$ and $V \subseteq U\, , V \in \mathfrak{F}$ then $\mathcal{U}/V \in \mathrm{Cov}(V)$.

5° $\mathcal{U}_i \in \mathrm{Cov}(U_i)$ and $(U_i) \in \mathrm{Cov}(U)$ then $\bigcup \mathcal{U}_i \in \mathrm{Cov}(U)$.

We remark that the object defined above is in fact a special case of a pre-topology in the sense of Grothendieck. So we can use the whole machinery of sheaves and cohomology for a Grothendieck-topology.

(2.14) An affine holomorphic space $(X\, ,\, \mathfrak{F}\, , \mathcal{O}_X)$ is the following :

1) $X = \mathrm{Sp}\, A$ for some Tate-algebra A.

2) $\mathfrak{T}$ consists of all open affine subsets of X.

3) For all $U \in \mathfrak{T}$, $\mathrm{Cov}(U)$ consists of all coverings of U by elements in $\mathfrak{T}$ which have a finite subcovering.

4) $\mathcal{O}_X$ is the sheaf (for $\mathfrak{T}$) of rings defined by $\mathcal{O}_X(U)$ = the unique Tate-algebra B for which $A \to B$ with an immersion $U = \mathrm{Sp}\, B \hookrightarrow \mathrm{Sp}\, A$.

$\mathcal{O}_X$ is a sheaf according to (2.11).

(2.15) A holomorphic space $(X, \mathfrak{T}, \mathcal{O}_X)$ is a topological space X with a Grothendieck-topology $\mathfrak{T}$ and a sheaf of rings $\mathcal{O}_X$ such that $\exists\, (U_i) \in \mathrm{Cov}(X)$ with $(U_i, \mathfrak{T}/U_i, \mathcal{O}_X/U_i)$ is an affine holomorphic space for all i.

[Note. – $U \in \mathfrak{T}$ is called affine if $(U, \mathfrak{T}/U, \mathcal{O}_X/U)$ is an affine holomorphic space. If U is affine and $V \in \mathfrak{T}$ then $U \cap V$ is an affine open subset of U.]

(2.16) Some properties of affine holomorphic spaces (see [10]).

(2.16.1) $\mathrm{Hom}_{k\text{-alg}}(A, B) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Sp}\, B, \mathrm{Sp}\, A)$.

(2.16.2) Definition. – An $\mathcal{O}_X$-module M on $X = \mathrm{Sp}\, A$ is called coherent if there exists a finitely generated A-module N such that the sheaf M is isomorphic with the sheaf $U \to \mathcal{O}_X(U) \otimes_A N$ (U open affine $\subseteq X$).

(2.16.3) Proposition. – An $\mathcal{O}_X$-module M is coherent if there exists a

$$(U_i) \in \mathrm{Cov}(X)$$

such that M/U_i is coherent for each i.

If M is coherent, then

$$H^i(X, M) = 0, \quad i > 0$$

$H^0(X, M) = N$, and M is associated with the A-module N.

Proof. – The second part of the proposition follows directly from (2.11). The first part is a property of "descent" for $A \to B = \oplus\, \mathcal{O}_X(U_i)$, i. e. consider $A \to B \rightrightarrows B \hat{\otimes}_A B$ (note $B \hat{\otimes}_A B = \oplus_{(i,j)} \mathcal{O}_X(U_i \cap U_j)$), then :

(i) A B-module M(f g) is isomorphic with some $N \otimes_A B$ if there exists a $B \hat{\otimes}_A B$-module isomorphism

$$M \otimes_B (B \hat{\otimes}_A B) \xrightarrow{\sim} M \otimes_B (B \hat{\otimes}_A B).$$

(ii) For fg A-modules N_1 and N_2, the sequence

$$\mathrm{Hom}_A(N_1, N_2) \to \mathrm{Hom}_B(N_1 \otimes_A B, N_2 \otimes_A B) \rightrightarrows \mathrm{Hom}_{B \hat{\otimes}_A B}(N_1 \otimes_A (B \hat{\otimes}_A B), N_2 \otimes_A (B \hat{\otimes}_A B)).$$

This "descent"-property is proved by R. KIEHL.

3. Global properties of holomorphic spaces.

(3.1) (Quasi-)Stein spaces.

Definition. - A holomorphic space X is called a quasi-Stein space if

$$\exists\ (X_i)_{i\in \underset{\sim}{N}} \in \mathrm{Cov}(X)\ ,$$

an affine covering with

1) $X_i \subset X_{i+1}$ for all i.

2) $\mathcal{O}_X(X_{i+1}) \to \mathcal{O}_X(X_i)$ has dense image.

X is called a Stein-space if a more restrictive property holds :

$$\exists\ f_1, \dots, f_r \in \mathcal{O}_X(X_{i+1})$$

with

(a) $X_i = \{x \in X_{i+1}\ ;\ |f_j(x)| \leq 1$ for all $j\}$.

(b) $f_1/a, \dots, f_r/a$ (for some $a \in k^*$) are topological generators of $\mathcal{O}_X(X_{i+1})$

(3.1.1) THEOREM (R. KIEHL [10]). - If M is a coherent $\mathcal{O}_X$-module (i. e. M/U coherent for every open affine $U \subset X$) and X is quasi-Stein, then

1° $M(X) \to M(X_i)$ has dense image.

2° $H^i(X, M) = 0$ for $i > 0$.

3° M_x is generated over $\mathcal{O}_{X,x}$ by $M(X)$.

Proof. - Easy consequence of (2.16.3) + definition (3.1).

(3.1.2) THEOREM (KIEHL [10] ; LÜTKEBOHMERT [11]). - Let X be a Stein-space of dimension n, which can locally be embedded in a N-dimensional space $/k$. Then X has an embedding into k^{N+n+1}.

(3.1.3) Examples. - k^n and $G = k^{*n}$ are Stein-spaces.

The structure of G can be given by :

$$G = \cup X_m\ ;\ X_m = \{(x_1, \dots, x_n) \in k^{*n}\ ;\ |\pi|^m \leq |x_i| \leq |\pi|^{-m} \text{ all } i\}.$$

(Here $\pi \in k^*$ and $0 < |\pi| < 1$).

An open subset $U \subset G$ is called open affine is U is open affine in some X_n.

For an open affine $U \subset G$, it is clear what $\mathrm{Cov}(U)$ is. For G, $\mathrm{Cov}(G)$ consists of the coverings (U_i) be open affine sets such that $(U_i)/U \in \mathrm{Cov}(U)$ for every open affine $U \subset G$.

With $(X_n) \in \mathrm{Cov}(G)$, one calculates :

$$\mathcal{O}(G) = \varprojlim \mathcal{O}(X_n) = \{\sum_{\alpha\in \underset{\sim}{Z}^n} a_\alpha\, x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \text{convergent on all of } G\}.$$

More generally, any algebraic variety has a unique structure of holomorphic space. If the variety is affine then the holomorphic space is a Stein-space.

(3.2) Proper mappings. – A morphism $f : X \to Y$ of holomorphic spaces is called proper if the following holds.

(a) f is separated, i. e. $\Delta : X \to X \times_Y X$ is a closed embedding.

(f) There is $(Y_i)_{i\in I} \in \mathrm{Cov}(Y)$, with each Y_i affine open, and for each $i \in I$ there are two finite coverings $(U_{ij})_{j=1}^{ni}$, $(V_{ij})_{j=1}^{ni}$ of $f^{-1}(Y_i)$ by affine sets such that $U_{ij} \ll V_{ij}$ (all i, j).

Here $U \ll V$ for affine open sets U, V, means the following; there is an ε, $0 < \varepsilon < 1$, and an embedding $V \subset \{(\lambda_1 \ldots \lambda_n) \in k^n ;\ \text{all } |\lambda_i| \leq 1\}$ such that $U \subseteq \{(\lambda_1 \ldots \lambda_n) \in k^n ;\ \text{all } |\lambda_i| \leq \varepsilon\}$.

A holomorphic space X is called compact (or complete) if "$X \to$ point" is proper.

(3.2.1) THEOREM (R. KIEHL [9]). – $f : X \to Y$ proper, M a coherent $\mathcal{O}_X$-module then all $R^i f_* M$ are coherent $\mathcal{O}_Y$-modules.

COROLLARY. – If X is compact and M is a coherent $\mathcal{O}_X$-module, then

$$\dim H^i(X, M) < \infty \text{ for all } i.$$

(3.3) Projective spaces. – $\underset{\sim}{P}_n(k)$ is a compact holomorphic space. The well known GAGA-properties hold:

1° 1.1 Correspondance between algebraic coherent sheaves N and the coherent $\mathcal{O}_X$-modules M.

2° $H^i_{alg}(X, N) \simeq H^i_{anal}(X, M)$.

3° Any analytic subset of $\underset{\sim}{P}^n(k)$ is algebraic.

(3.4) The sheaves $\mathcal{O}^*$, $\mathfrak{M}$, $\mathfrak{M}^*$, Div.

(3.4.1) $\mathcal{O}^*$ is defined by $U \to \mathcal{O}_X(u)^*$ (* = invertible elements). This is a sheaf since $\mathcal{O}(U) \to \bigoplus \mathcal{O}(U_i) \rightrightarrows \bigoplus \mathcal{O}(U_i \cap U_j)$ is exact for every $(U_i) \in \mathrm{Cov}(U)$.

(3.4.2) $\mathfrak{M}$ = the sheaf of meromorphic functions is defined by $U \to \mathrm{Qt}(\mathcal{O}_X(U))$ for every affine open U (Qt = total quotient ring).

Proof. – We have to verify that this is in fact a sheaf on every affine open space $U \leq X$. Let $(U_i) \in \mathrm{Cov}(U)$ and let $(t_i/n_i)_i \in \bigoplus \mathrm{Qt}(\mathcal{O}_X(U_i))$ satisfy $t_i/n_i = t_j/n_j$ in $\mathrm{Qt}(\mathcal{O}_X(U_i \cap U_j))$ (all i, j). Then we have to show the existence of $t/n \in \mathrm{Qt}(\mathcal{O}_X(U))$ with image t_i/n_i in every $\mathrm{Qt}(\mathcal{O}(U_i))$.

One proceeds as follows: let

$$I(U_i) = \{s \in \mathcal{O}(U_i) ;\ st_i \in n_i\, \mathcal{O}(U_i)\}.$$

Then

$$I(U_i) \otimes \mathcal{O}_X(U_i \cap U_j) \simeq I(U_j) \otimes \mathcal{O}_X(U_i \cap U_j).$$

By (2.16.3), there is an ideal $I \subset \mathcal{O}_X(U)$ with $I/U_i = I(U_i)$ for all i. I contains a non-zero divisor, otherwise $Iz = 0$ for some $z \in \mathcal{O}_X(\mathfrak{U})$, $z \neq 0$. And also $I(U_i)\, z = 0$, $\forall$ i. But each $I(U_i)$ contains a non-zero divisor. Hence $z/U_i = 0$, $\forall$ i and so $z = 0$. Take $n \in I$, $n \neq 0$, n a non-zero-divisor. Then $t_i/n_i = s_i/n$, $\forall$ i and the s_i satisfy $s_i/U_i \cap U_j = s_j/U_i \cap U_j$. So the s_i glue to an element $t \in \mathcal{O}_X(U)$.

(3.4.3) $\mathcal{M}^*$ is defined by $\mathcal{M}^*(U) = Qt(\mathcal{O}(U))^* = \mathcal{M}(U)^*$ for every open affine $U \subset X$. As in (3.4.2) this is a sheaf.

(3.4.4) The sheaf of divisors Div is defined by an exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathrm{Div} \to 0 .$$

(3.4.5) As in the classical case,

$$H^1(X , \mathcal{O}^*) \cong \text{invertible sheaves on } X/\text{isomorphism}.$$

Proof. - The usual one

$$H^1(X , \mathcal{O}^*) = \varinjlim_{\mathfrak{U}\in \mathrm{Cov}(X)} \check{H}^1(\mathfrak{U} , \mathcal{O}^*) .$$

(3.4.6) If $X = \mathrm{Sp}\, A$ is affine, then there is a 1.1 correspondance between invertible sheaves on X and projective rank 1 modules over A. Hence

$$H^1(X , \mathcal{O}_X^*) = \text{rank } 1 \text{ projective } A\text{-modules / isomorphism} \quad [2].$$

(3.4.7) Suppose $X = \mathrm{Sp}\, A$, and A is regular, then $H^1(X , \mathcal{O}_X^*)$ = Class groups of A. In particular,

$$A \text{ is a unique factorisation domain} \iff H^1(X , \mathcal{O}^*) = 0 .$$

(3.4.8) PROPOSITION (L. GRUSON [8]). - <u>Let</u> $X = \mathrm{Sp}\, A$, <u>and let</u> A <u>be regular</u>. <u>If</u> A <u>has unique factorisation then also</u> $A\langle T\rangle$ <u>and</u> $A\langle T , T^{-1}\rangle$ <u>have unique factorisation</u>.

(3.4.9) CONSEQUENCE. - <u>Let</u> $G = k^{*n}$ <u>then</u> $H^1(G , \mathcal{O}_G^*) = 0$.

<u>Proof</u>. - It suffices to consider

$$X_n = \{(x_1 , \dots , x_n) \in k^n ;\ |\pi| \leqslant |x_i| \leqslant |\pi|^{-1} \text{ for all } i\} ,$$

where $\pi \in k$, $0 < |\pi| < 1$. We want to show that any invertible sheaf $\mathcal{L}$ on X_n is trivial (i. e. $\simeq \mathcal{O}_{X_n}$). Let $\mathcal{L}_0$ be the structure sheaf on

$$X_{n-1} \times \{x_n \in k ;\ |x_n| \leqslant |\pi|\} .$$

Then

$$(\mathcal{L}_0/X_{n-1}) \times \{x_n \in k ;\ |x_n| = |\pi|\} \cong (\mathcal{L}/X_{n-1}) \times \{x_n \in k ;\ |x_n| = |\pi|\}$$

because of (3.4.8). Hence by (2.16.3), $\mathcal{L}$ and $\mathcal{L}_0$ glue together to form an invertible sheaf

$$\mathcal{L}' \text{ on } X_{n-1} \times \{x_n \in k ;\ |x_n| \leqslant |\pi|^{-1}\} .$$

But $\mathcal{L}'$ is trivial by (3.4.8). Hence also $\mathcal{L}$ is trivial.

4. Analytic tori and abelian varieties.

The results of this sections are mainly due to L. GERRITZEN ([2] , [4]).

(4.1) A subgroup Γ of $G = k^{*n}$ is called <u>discrete</u> if

$$\Gamma \cap \{x \in G \; ; \quad \varepsilon \leq |x_i| \leq \varepsilon^{-1} \; , \; \forall \, i\} \text{ is finite for all } \varepsilon \leq 1 \; .$$

The map $\ell : G \to \mathbb{R}^n$ defined by

$$\ell(x_1 , \dots , x_n) = (-\log |x_1| , \dots , -\log |x_n|)$$

is a group homomorphism. It is easily seen that

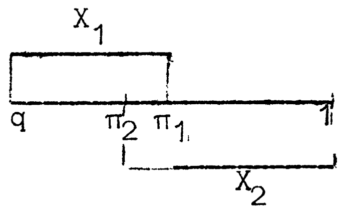
$$\Gamma \text{ is discrete} \iff \ell(\Gamma) \subset \mathbb{R}^n \text{ is discrete and } \ker \ell/\Gamma = \text{finite} \; .$$

We are interested in the case : Γ has maximal rank $(= n)$, and Γ has no torsion elements. Hence $\Gamma \simeq \ell(\Gamma)$ and $\ell(\Gamma)$ is a lattice in $\mathbb{R}^n$.

PROPOSITION. – <u>The quotient</u> G/Γ <u>is called a holomorphic torus</u> ; G/Γ <u>has a unique structure of holomorphic space over</u> k <u>such that</u> $\pi : G \to G/\Gamma$ <u>is a holomorphic map. Moreover</u> G/Γ <u>is "compact".</u>

<u>Proof</u>. – For convenience, we do only $n = 1$; $n > 1$ can be done in the same way. Then $\Gamma = \langle q \rangle$, and we may suppose $0 < |q| < 1$. The topological space G/Γ can be covered by the images $\tilde{X}_1 , \tilde{X}_2$ under π of

$$X_1 = \{x \in G \; ; \quad |q| \leq |x| \leq |\pi_1| < 1\}$$
$$X_2 = \{x \in G \; ; \quad |\pi_2| \leq |x| \leq 1\}$$



where $|q| < |\pi_2| < |\pi_1| < 1$.

Of course, $\pi/X_i : X_i \to \tilde{X}_i$ is a homeomorphism. Further $\tilde{X}_1 \cap \tilde{X}_2$ is the disjoint union of the images (under π) of

$$\{x \in k \; ; \quad |x| = 1\} \text{ and } \{x \in k \; ; \quad |\pi_2| \leq |x| \leq |\pi_1|\} \; .$$

So $\tilde{X}_1$ and $\tilde{X}_2$ are glued in a nice way, and G/Γ becomes a holomorphic space. One can make another covering of G/Γ by Y_1 , Y_2 such that $Y_i \ll X_i$. Hence G/Γ is compact.

(4.2) <u>Let</u> $T = G/\Gamma$ <u>have dimension</u> n . <u>Then</u>

$$H^1(G/\Gamma , \mathcal{O}^*) = \mathbb{Z}^n$$

$$H^1(T , C) = C \quad \underline{\text{for any constant sheaf}} \; C \; .$$

<u>Proof</u>. – Again we consider only $n = 1$. Then $H^1(G/\Gamma , \mathcal{O}^*)$ is given by the exact sequence

$$0 \to \mathcal{O}^*(G/\Gamma) \to \mathcal{O}^*(\tilde{X}_1) \oplus \mathcal{O}^*(\tilde{X}_2) \to \mathcal{O}^*(\tilde{X}_1 \cap \tilde{X}_2) \to H^1(G/\Gamma , \mathcal{O}^*) \to 0 \; ,$$

because $H^1(Z, \mathcal{O}^*) = 0$ for $Z = \tilde{X}_1, \tilde{X}_2$ or $\tilde{X}_1 \cap \tilde{X}_2$. The same covering can be used to calculate $H^1(T, C)$.

(4.3) Our aim is to calculate the field of meromorphic functions on G/Γ, $\mathcal{M}(G/\Gamma)$.

(4.3.1) PROPOSITION. – $\mathcal{M}(G)$ = <u>the quotient field of</u>

$$\mathcal{O}(G) = \{\sum_{\alpha\in \underset{\sim}{Z}^n} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \text{ everywhere convergent}\}.$$

<u>Proof.</u> – $\mathcal{M}(G) = \varprojlim \mathcal{M}(X_i)$ with

$$X_i = \{(z_1, \dots, z_n) \in k^n ; \quad |\pi|^i \leq |z_j| \leq |\pi|^{-i} \text{ for all } j\}.$$

Given a projective system (a_i/b_i) in $\varprojlim \mathcal{M}(X_i)$, we can make ideals

$$I_i = \{t \in \mathcal{O}(X_i) ; \quad t(a_i/b_i) \in \mathcal{O}(X_i)\} ; \quad I_{i+1}|X_i = I_i.$$

So we find a coherent sheaf of ideals $\mathfrak{J} \subset \mathcal{O}$. Since G is a Stein-space, we have $\mathfrak{J}(G) \neq 0$. Take $n \in \mathfrak{J}(G)$ and $n \neq 0$. Then $t_i/n_i = a_i/b_i$ in $Qt(\mathcal{O}(X_i))$ for suitable $t_i \in \mathcal{O}(X_i)$. Since $t_{i+1}/U_i = t_i$, we find an element $t \in \mathcal{O}(G)$ with $t/U_i = t_i$, $\forall$ i. Hence $t/n = \text{"}\varprojlim\text{"}(a_i/b_i)$.

Using further $H^1(G, \mathcal{O}^*) = 0$, we can choose t and n such that

$$\text{g. c. d. } (t_x, h_x) = 1 \text{ in } \mathcal{O}_{G,x} \text{ for every point } x \in G.$$

(4.3.2) PROPOSITION. – <u>The group</u> Γ <u>acts on</u> G <u>and</u> $\mathcal{M}(G)$. <u>For this action, we</u> have $\mathcal{M}(G)^\Gamma = \mathcal{M}(G/\Gamma)$.

<u>Proof.</u> – More or less clear.

(4.3.3) DEFINITION. – <u>An holomorphic function</u> $f : G \to k$ <u>is called a theta-function for</u> (G, Γ) <u>if for every</u> $\gamma \in \Gamma$ <u>there exists a function</u> $\mathfrak{z}_\gamma \in \mathcal{O}(G)$ <u>with</u>

$$f(z) = \mathfrak{z}_\gamma(z)\, f(\gamma z).$$

It follows easily that $\mathfrak{z}_\gamma$ has no zero's in G and hence $\mathfrak{z}_\gamma$ must be an element of the group

$$A = \{\lambda z_1^{\alpha_1} \cdots z_n^{\alpha_n} ; \; \lambda \in k^* ; \; \alpha_1, \dots, \alpha_n \in \underset{\sim}{Z}\} = \mathcal{O}(G)^*.$$

(4.3.4) PROPOSITION. – <u>Any</u> $f \in \mathcal{M}(G/\Gamma)$ <u>can be written as</u> $f = \theta_1/\theta_0$, <u>where</u> θ_0, θ_1 <u>are theta-functions with the same "multiplicator"</u> $\mathfrak{z}_\gamma$.

<u>Proof.</u> – Write $f = \theta_1/\theta_0$ with $\theta_i \in \mathcal{O}(G)$ and θ_i relatively prime. Then

$$f(\gamma z) = \frac{\theta_1(\gamma z)}{\theta_0(\gamma z)} = f(z).$$

Since θ_0, θ_1 are relatively prime, we find

$$\theta_i(z) = \mathfrak{z}_\gamma(z)\, \theta_i(\gamma z) \quad (i = 0, 1) \text{ for some } \mathfrak{z}_\gamma \in \mathcal{O}(G).$$

(4.4) <u>Construction of p-adic theta-functions</u>. - In order to compute $\mathfrak{M}(G/\Gamma)$ = the meromorphic functions on G/Γ, we have to construct theta functions with a given "multiplicator" $\gamma \to z_\gamma$.

(4.4.1) LEMMA.

1° <u>The multiplicator</u> $\gamma \to z_\gamma$ <u>is a 1-cocycle in</u> $H^1(\Gamma, A)$, <u>i. e.</u>

$$z_{\gamma'\gamma}(z) = z_{\gamma'}(\gamma z)\, z_\gamma(z) \quad (\text{for all } \gamma, \gamma' \in \Gamma ;\ z \in G).$$

2° <u>Any 1-cocycle</u> $\gamma \to z_\gamma$ (<u>in</u> $H^1(\Gamma, A)$) <u>has the form</u> ($d(\gamma) \in k^*$)

$$z_\gamma(z) = d(\gamma)\, \sigma(\gamma)(z) \quad \text{where } \sigma : \Gamma \to H = \{z_1^{\alpha_1} \cdots z_n^{\alpha_n} ;\ \alpha \in \underset{\sim}{Z}^n\}$$

<u>is a group homomorphism</u>(H = <u>all analytic characters on</u> G).

Moreover $d(\gamma\gamma')\, d(\gamma)^{-1}\, d(\gamma')^{-1} = \sigma(\gamma')(\gamma)$.

Define $q : \Gamma \times H \to h^*$ by $q(\gamma, h) = h(\gamma)$ then $\sigma(\gamma')(\gamma) = q(\gamma, \sigma(\gamma'))$ and $\Gamma \times \Gamma \to h^*$ given by $(\gamma, \gamma') \to q(\gamma, \sigma(\gamma'))$ is bilineair symmetric.

3° <u>After possibly a finite field extension of</u> k <u>there is a symmetric bilineair from</u> $p : \Gamma \times \Gamma \to k^*$ <u>and a group homomorphism</u> $c : \Gamma \to k^*$ <u>such that</u>

$$z_\gamma = c(\gamma)\, p(\gamma, \gamma)\, \sigma(\gamma)$$

$$p(\gamma, \gamma')^2 = q(\gamma, \sigma(\gamma')).$$

<u>Proof</u>. - 1° and 2° are clear if one uses $A = k^* H$.

3° Choose a base $\gamma_1, \cdots, \gamma_n$ of Γ and elements $p(\gamma_i, \gamma_j)$ satisfying

$$p(\gamma_i, \gamma_j) = p(\gamma_j, \gamma_i) \quad \text{and} \quad p(\gamma_i, \gamma_j)^2 = q(\gamma_i, \sigma(\gamma_j)).$$

The bilineair extension of p is symmetric and satisfies

$$p(\gamma, \gamma')^2 = q(\gamma, \sigma(\gamma')).$$

Moreover $z_\gamma = c(\gamma)\, p(\gamma, \gamma)\, \sigma(\gamma)$ for some function $c : \Gamma \to k^*$.

Substitution in 1° guilds that c is a homomorphism.

(4.4.2) <u>Definition</u>. - Given a 1-cocycle z, we want to determine $L(z)$ = the vectorspace of theta-functions with multiplicator z, i. e. the holomorphic function on G satisfying

$$f(z) = z_\gamma(z)\, f(\gamma z) \qquad (\gamma \in \Gamma,\ z \in G).$$

To simplify matters, we introduce M = all formal expressions $\sum_{h \in H} a_h h$ with coefficients $a_h \in k$. M is a vector space over k with some extra structure :

$$\text{action of } \Gamma : \ (\textstyle\sum a_h h)^\gamma := \sum a_h\, q(\gamma, h)\, h$$

multipl. by elts in

$$H : \ h'(\textstyle\sum a_h h) := \sum a_h h' h .$$

$L^0(z)$ = the elements of M satisfying $f = z_\gamma f^\gamma$

= the <u>formal θ-functions with cocycle</u> .

(4.4.3) LEMMA.

1° $L^0(z) \neq 0$ if and only if there is $h \in H$ such that $z_\gamma = q(\gamma, h)$ for all $\gamma \in \ker \sigma$.

2° If $L^0(z) \neq 0$, then $\dim L^0(z) \leq \#$ (torsion elements of $H/\sigma(\Gamma)$).

Equality holds if σ is injective.

3° $L(z) \neq 0$ if and only if $L^0(z) \neq 0$ and $|q(\gamma, \sigma(\gamma))| < 1$ as soon as $\sigma(\gamma) \neq 1$.

4° If $L(z) \neq 0$, then $L(z) = L^0(z)$.

<u>Proof</u>. – We introduce the following notations : sub groups H' , H'' of H and Γ' of Γ such that $H' \oplus H'' = H$; $\sigma(\Gamma) \leq H'$ and $H'/\sigma(\Gamma)$ is a finite group with representatives $w_1, \dots, w_t$; $\Gamma' \oplus \ker \sigma = \Gamma$.

Any $f \in M$ has uniquely the form

$$f = \sum_{i=1,\dots,t,\nu\in\Gamma',h''\in H''} a_{i,\nu,h''} z_\nu w_i h'' \qquad (a_{i,\nu,h''} \in k^*) .$$

Since $z_\gamma(z) f(\gamma z) = \sum a_{i,\nu,h''} q(\gamma, w_i h'') z_{\nu\gamma} w_i h''$; the condition $f \in L^0(z)$ is equivalent with

$$\begin{cases} a_{i,\nu,h''} q(\gamma, w_i h'') = a_{i,\nu\gamma,h''} & \text{for all } \gamma \in \Gamma' \\ a_{i,\nu,h''} q(\gamma, w_i h'') z_\gamma = a_{i,\nu,h''} & \text{for all } \gamma \in \ker\sigma . \end{cases}$$

In another form, for some $a_{i,h''} \in k$, we have

$$\begin{cases} a_{i,\gamma,h''} = q(\gamma, w_i h'') a_{i,h''} \\ a_{i,h''} \neq 0 \Longleftrightarrow z_\gamma = q(\gamma, (w_i h'')^{-1}) & \text{for all } \gamma \in \ker\sigma . \end{cases}$$

From this 1° follows immediately; 2° also follows because

$$H_0 = \{h \in H ;\ q(\gamma, h) = 1 \text{ for all } \gamma \in \ker\sigma\}$$

is contained in H' . So there is at most one h'' with $a_{i,h''} \neq 0$.

Further explication : since q is non-degenerate, the group H_0 has

$$\text{rank} = n - \text{rank}(\ker\sigma) = \text{rank}\,\sigma(\Gamma) .$$

Further since $q(\gamma, \sigma(\gamma'))$ is symmetric one has $q(\ker\sigma, \sigma(\Gamma)) = 1$ and $H_0 \supseteq \sigma(\Gamma)$. Hence $H_0 \subset H'$.

3° and 4° : We have to estimate the absolute values of the coefficients of $f \in L^0(z)$.

$$a_{i,\nu,h''} z_\nu w_i h'' = a_{i,h''} q(\nu, w_i h'') c(\nu) p(\nu, \nu) \sigma(\nu) w_i h'' .$$

Suppose $a_{i,h''} \neq 0$ and $\nu \neq 0$. Convergence of the subsequence

$$\sum_{n\geq 1} a_{i,h''}\, q(n\nu , w_i h'')\, c(n\nu)\, p(n\nu , n\nu)\, \sigma(n\nu)\, w_i h'' \quad (\text{of } f)$$

on all of G implies clearly $|p(\nu , \nu)| < 1$.

On the other hand if $|p(\nu , \nu)| < 1$ for all $\nu \in \Gamma'$, $\nu \neq 0$, then

$$\langle \nu , \nu' \rangle = -\log|q(\nu , \sigma(\nu'))|$$

is a positive definite symmetric bilinear from on $\Gamma' \times \Gamma'$. So $\langle \nu , \nu' \rangle$ is an inner product on $\Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$ and

$$\langle \nu , \nu \rangle \geq c \sum \nu_i^2 \quad (\nu = (\nu_i - \nu_n) \text{ and } c > 0).$$

From this one easily sees that $f \in \mathbb{L}(\mathcal{Z})$.

(4.5) Algebraicity of G/Γ.

THEOREM. – The following conditions are equivalent

(1) G/Γ is algebraic,

(2) G/Γ is projective algebraic,

(3) G/Γ is an abelian variety,

(4) There is a group homomorphism $\sigma : \Gamma \to H$ such that

(a) $q(\gamma , \sigma(\gamma')) = q(\gamma' , \sigma(\gamma))$ for all $\gamma , \gamma' \in \Gamma$

(b) $\langle \gamma , \gamma' \rangle = -\log|q(\gamma , \sigma(\gamma'))|$ is positive definite.

Proof. – (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are obvious.

(1) $\Longrightarrow$ (4) the transcendence degree of $\mathfrak{M}(G/\Gamma)$ over k is at least n. Take algebraic independent elts $f_1 , \cdots , f_n \in \mathfrak{M}(G/\Gamma)$ and write them as

$$f_1 = \frac{\theta_1}{\theta_0} , \ldots , \quad f_n = \frac{\theta_n}{\theta_0} \quad \text{with "g. c. d. } (\theta_0 , \cdots , \theta_n) = 1\text{"},$$

$\theta_0 , \cdots , \theta_n$ holomorphic functions. Then $\theta_0 , \cdots , \theta_n$ are theta functions with the same multiplicator $\mathcal{Z}$.

The algebraic independence of $f_1 , \cdots , f_n$ implies that

$$\{\theta_0^{r_0}\, \theta_1^{r_1} \ldots \theta_n^{r_n} ; \ \sum r_i = \ell\}$$

are algebraically independent over k. Hence $\dim \mathbb{L}(\mathcal{Z}^{\ell}) \geq \binom{\ell+n}{n}$. On the other hand,

$$\dim \mathbb{L}(\mathcal{Z}^{\ell}) = |H/\sigma(\Gamma)|_{\text{torsion}}^{\ell^r} \quad \text{where } r = \text{rank}\, \sigma(\Gamma).$$

Hence $\text{rank}\, \sigma(\Gamma) = n$, and we have proved (4).

(2) $\Longrightarrow$ (3). The multiplicator of $G/\Gamma \subseteq \underset{\sim}{P}^n$: $G/\Gamma \times G/\Gamma \to G/\Gamma$ is an analytic map. By GAGA, it is also an algebraic map.

The hard part is to show (4) $\Longrightarrow$ (2) :

(4.5.1) LEMMA. – Let $\mathcal{Z}$ be a cocycle with a positive definite σ (as in (4)). Then

(1) For every $z \in G$, there exists a $\theta \in L(\mathcal{Z}^3)$ with $\theta(z) \neq 0$.

(2) Let $\theta_0, \dots, \theta_t$ be a base of $L(\mathcal{Z}^3)$. Suppose that $z_1 , z_2 \in G$ and $z_1 \not\equiv z_2 \mod \Gamma$. Then the vectors $(\theta_0(z_1), \dots, \theta_L(z_1))$ and $(\theta_0(z_2), \dots, \theta_L(z_2))$ in k^{t+1} are linearly independent over k .

Proof.

(1) For $\theta \in L(\mathcal{Z})$ and $a , b \in G$ the functions

$$\theta_3 = \theta(za^{-1})\,\theta(zb^{-1})\,\theta(zab)$$

belong to $L(\mathcal{Z}^3)$. Let $\theta \neq 0$, then the zero set X of θ in G has codimension 1 . One can find a , b with $a^{-1} , b^{-1} , ab \notin z^{-1} X$. Hence $\theta_3(z) \neq 0$.

(2) Suppose that the vectors $(\theta_0(z_1), \dots, \theta_t(z_1))$ and $(\theta_0(z_2), \dots, \theta_t(z_2))$ are linearly dependent over k . For any $F \in L(\mathcal{Z})$ one has for any $z , b \in G$ and a fixed constant $c \in k^*$:

$$F(z_1 z^{-1})\, F(z_1 b^{-1})\, F(z_1 zb) = c\, F(z_2 z^{-1})\, F(z_2 b^{-1})\, F(z_2 zb) .$$

Hence the meromorphic function (of z) $(F(z_1 zz^{-1}))/(F(z_2 zz^{-1}))$ has no zero's and no poles. So

$$\frac{F(z_1 z^{-1})}{F(z_2 z^{-1})} \in A = \mathcal{O}^*(G) .$$

That means $F(z\nu) = a(z)\,F(z)$ with $\nu = z_1 z_2^{-1}$ and $a \in A$. The explicit formula for the F's in $L(\mathcal{Z})$ given in (4.4.3) implies $\nu \in \Gamma$.

(4.5.2) LEMMA. – Let $\mathcal{Z}$ be a positive definit 1-cocycle and let $\theta_0, \dots, \theta_t$ be a base of $L(\mathcal{Z}^3)$. The holomorphic map φ : $G/\Gamma \to \underset{\sim}{P}_t(k)$ given by

$$\varphi(z) = [\theta_0(z), \dots, \theta_t(z)]$$

has the properties

1° $X = \mathrm{im}(\varphi)$ is an algebraic subspace of $\underset{\sim}{P}_t(k)$ of dimension n .

2° $\varphi : G/\Gamma \to X$ is an isomorphism of holomorphic spaces.

Proof.

1° $\varphi : G/\Gamma \to \underset{\sim}{P}_t(k)$ is well defined and injective according to (4.5.1) part (1) and (2). Since G/Γ is "compact", the map φ is proper. By the proper mapping theorem, $X = \mathrm{im}(\varphi)$ is a closed analytic subset of $\underset{\sim}{P}_t(k)$.

By GAGA, $X = \mathrm{im}(\varphi)$ is also an algebraically closed subset of $\underset{\sim}{P}_t(k)$. Since $\varphi : G/\Gamma \to X$ is bijective, we have

$$n = \dim G/\Gamma = \dim X + \dim(\text{fibre}) \quad \text{and} \quad \dim(\text{fibre}) = 0 .$$

(2) A covering Y_i $(i = 0 , \dots , t)$ by affine open pieces is given by

$$Y_i = \{[a_0,\dots,a_t] \in \underset{\sim}{P}_t(k) \; ; \; |a_j| \leq |a_i| \text{ for all } j\} \simeq \{(\lambda_1,\dots,\lambda_t) \in k^t \; ; \text{ all } |\lambda_j| \leq 1\}.$$

Put $X_i = Y_i \cap X$; then $(X_i) \in \mathrm{Cov}(X)$, and one can verify that

$$(\varphi^{-1}(X_i))_{i=0}^{t} \in \mathrm{Cov}(G/\Gamma) .$$

The map $\varphi_i : \varphi^{-1}(X_i) \to X_i$ is bijective, and after a calculation of derivatives and finds, for every $x \in X_i$,

$$\hat{\mathcal{O}}_{X_i,x} \to \hat{\mathcal{O}}_{G(\Gamma,\varphi^{-1}(x))} .$$

By methods of the type, explained in (2.10), it follows that $\varphi_i^{-1} : X_i \to \varphi^{-1}(X_i)$ is also holomorphic. Hence $\varphi : G/\Gamma \to X$ has an holomorphic inverse.

(4.6) <u>Final remarks</u>.- Now every abelian variety over Q_p can be obtained as a holomorphic torus G/Γ. One can only parametrize those abelian varieties by a G/Γ, which degenerate over the residue field $\underset{\sim}{F}_p$ of Q_p.

In particular, only those elliptic curves over k can be parametrized which split into projective lines over the residue field of k (Equivalently, the j-invariant has absolute value > 1). (See [15]). In [12], D. MUMFORD has shown that also degenerating curves of genus $g > 1$, over a local field, have a nice non-archimedean representation.

REFERENCES

[1] BERGER (R.), KIEHL (R.), HUNZ (E.), NASTOLD (H.-J.). - Differentialrechnung in der analytischen Geometrie. - Berlin, Sptinger-Verlag, 1967 (Lecture Notes in Mathematics, 38).

[2] GERRITZEN (L.). - Über Endomorphismen nichtarchimedischer holomorpher Tori, Invent. Math., t. 11, 1970, p. 27-36.

[3] GERRITZEN (L.). - On multiplication algebras of Riemann matrices, Math. Annalen, t. 194, 1971, p. 109-122.

[4] GERRITZEN (L.). - On non-archimedean representations of abelian varieties, Math. Annalen, t. 196, 1971, p. 323-346.

[5] GERRITZEN (L.). - Periode und Index eines principal-homogenen Raumes über gewissen Abelschen Varietäten, Manuscripta Math., t. 8, 1973, p. 131-142.

[6] GERRITZEN (L.) und GRAUERT (H.). - Die Azyklizität der affinoiden Überdeckungen ; "Global analysis, Papers in honor of K. Kodaira", p. 159-184. - Princeton, University of Tokyo Press and Princeton University Press, 1970 (Princeton mathematical Series, 29).

[7] GRAUERT (H.) und REMMERT (R.). - Nichtarchimedische Funktionentheorie, "Festschr. Gedächtnisfeier K. Weierstrass", p. 393-476. - Köln, Westdeutscher Verlag, 1966.

[8] GRUSON (L.). - Fibrés vectoriels sur un polydisque ultramétrique, Annales scient. Ec. Norm. Sup., Série 2, t. 1, 1968, p. 45-89.

[9] KIEHL (R.). - Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionen theorie, Invent. Math., t. 2, 1967, p. 191-214.

[10] KIEHL (R.). - Theorem A und theorem B in der nichtarchimedischen Funktionentheorie, Invent. Math., t. 2, 1967, p. 256-273.

[11] LÜTKEBOHMERT (W.). - Steinsche Räume in der nichtarchimedischen Funktionentheorie, Schriftenreihe Math. Inst. Univ. Münster, Série 2, Heft 6, 1973, 54 p.

[12] MUMFORD (D.). - An analytic construction of degenerating curves over complete local rings, Compositio Math., Groningen, t. 24, 1972, p. 129-174.

[13] RAYNAUD (M.). - Géométrie analytique rigide, d'après Tate, Kiehl, ..., "Table ronde d'analyse ultramétrique [1972].

[14] REMMERT (R.). - Algebraische Aspekte in der nichtarchimedischen Analysis, "Proceedings of the conference on local fields [1966. Driebergen]", p. 86-117. - Berlin, Springer-Verlag, 1967.

[15] ROQUETTE (P.). - Analytic theory of elliptic functions over local fields. - Göttingen, Vandenhoeck and Ruprecht, 1970 (Hamburger mathematische Einzelschriften, 1).

[16] TATE (J.). - Rigid analytic spaces, Invent. Math., t. 12, 1971, p. 257-289.

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