INTRODUCTION TO ALGEBRAIC STACKS

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ABSTRACT. In these notes, we give an introduction to stacks with an eye toward moduli spaces of elliptic curves. The goal is to give a full definition of a Deligne-Mumford stack.

CONTENTS

In beginning of algebraic geometry, one starts with varieties over the complex numbers, a set of points with a Zariski topology in which all points are closed points. In generalizing this to schemes, one asks for a locally ringed topological space equipped with a structure sheaf, allowing for closed and nonclosed points. To generalize this further, we define an object called a *stack* which will allow "points" equipped with nontrivial automorphisms: it will be a category with a Grothendieck topology.

1. WHY STACKS?

Stacks are a natural class of objects to consider in many situations.

First, stacks arise in the context of moduli spaces. For example, over $\mathbb{C}$, two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$. Therefore, the j-line $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[j]$ is a natural 'coarse moduli space' for elliptic curves over $\mathbb{C}$. But this is *too coarse* in some sense. For one, every elliptic curve has an automorphism $[-1]$, therefore the j-line cannot 'distinguish' between an elliptic curve and its negative. Worse still, the curves with j invariants $j = 0$ and $j = 1728$ have extra automorphisms, and this information is forgotten by the j-line. The existence of these automorphisms mean that there is no 'universal elliptic curve', but such an object exists as a stack. (For more on this, read on!)

If we would like to not just parametrize algebraic objects but also families of algebraic objects, we are led to allow the base to be any scheme. In this context,

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one can prove for example that the moduli space of smooth curves of genus g is irreducible in any characteristic.

Second, stacks arise when one takes quotients. If X is a variety and G a group acting on X, the quotient X/G may not exist as a variety or a scheme. Quotients are much more natural as stacks. As a novelty example, given positive integers p, q, r we may ask for all coprime integer solutions to the Diophantine problem $x^p + y^q = z^r$. This equation defines a scheme over $\operatorname{Spec} \mathbb{Z}$ of relative dimension 2. Being 'weighted homogeneous', it has an action by the scheme $\mathbb{G}_m$. Removing the zero section and dividing out by this action gives a stack of dimension 1, and its 'rational points' are the desired coprime solutions!

Finally, stacks are a more basic object than sheaves, topological spaces, or even groups!

2. Fibered categories

We will build up slowly to the definition of a stack. Let S be a scheme, and let Sch/S be the category of S-schemes.

Definition. A *category over S* is a category $\mathcal{F}$ equipped with a functor

$$p : \mathcal{F} \to \mathsf{Sch}/S.$$

We will often leave the "projection" functor p implicit in the notation, and will refer to $\mathcal{F}$ as an *S-category.*

Definition. A category $\mathcal{F}$ over S is *fibered in groupoids* if:

(i) (Arrow lifting) For all arrows $\phi : U \to V$ in Sch/S and all $y \in p^{-1}(V)$, there exists an arrow $f : x \to y$ in $\mathcal{F}$ such that $p(f) = \phi$;

(ii) (Diagram lifting) For all diagrams

$$\begin{tikzcd} x \arrow[dr, "f"] & \\ & z \\ y \arrow[ur, "g"] & \end{tikzcd}$$

in $\mathcal{F}$, with image

$$\begin{tikzcd} U \arrow[dr, "\phi"] & \\ & W \\ V \arrow[ur, "\psi"] & \end{tikzcd}$$

under p, and for all $\chi : U \to V$ such that $\phi = \psi \circ \chi$, there exists a unique $h : x \to y$ in $\mathcal{F}$ such that $f = g \circ h$ and $p(h) = \chi$.

An element $y \in p^{-1}(V)$ is referred to as a *lift* of V; the element f is a *lift* of ϕ, or *lies above* ϕ.

Remark. Condition (ii) implies that the $f : x \to y$ in (i) is unique up to unique isomorphism. Given $f : x \to y$ and $f' : x' \to y$ two such, we have a diagram

$$\begin{tikzcd} x \arrow[dr, "f"] & \\ & y \\ x' \arrow[ur, "f'"] & \end{tikzcd}$$

so by (ii) there exists a unique map $x \to x'$, which by swapping x and x' we see to be an isomorphism.

Remark. Condition (ii) implies that an arrow f of $\mathcal{F}$ is an isomorphism if and only if $p(f)$ is an isomorphism. The sufficiency is trivial (apply p); in the other direction, suppose we have $f : x \to y$ in $\mathcal{F}$ with image $\phi : U \to V$ an isomorphism. Let $\psi : V \to U$ be a left inverse, so we have a diagram

$$\begin{tikzcd} V \arrow[dd, "\psi"] \arrow[dr, "\mathrm{id}"] & \\ & V \\ U \arrow[ur, "\phi"] & \end{tikzcd}$$

so by (ii) there exists a lift to a diagram

$$\begin{tikzcd} y \arrow[dd, "g"] \arrow[dr, "\mathrm{id}"] & \\ & y \\ x \arrow[ur, "f"] & \end{tikzcd}$$

which implies that f has a left inverse as well. This also applies to the right inverse, so f is an isomorphism.

Definition. Let U be a scheme over S. Then $\mathcal{F}(U)$, the *fiber of $\mathcal{F}$ over U*, is the category whose objects are $p^{-1}(U)$ and whose arrows $f : x \to y$ with $x, y \in p^{-1}(U)$ have $p(f) = \mathrm{id}$.

From the preceding remark, we see that the fiber $\mathcal{F}(U)$ is a *groupoid*, i.e. all arrows in $\mathcal{F}(U)$ are isomorphisms.

Almost always, these lifting properties follow automatically by base-change or by fibered products.

3. Elliptic curves

In order to introduce our main example, we recall in this section the definition of an elliptic curve over a general base scheme.

Definition. Let S be a scheme. An *elliptic curve* E over S is an S-scheme $p : E \to S$ where:

(i) p is proper and smooth of relative dimension 1;
(ii) The geometric fibres of p are connected curves of genus 1; and
(iii) There exists a section $O \in E(S) = \operatorname{Hom}(S, E)$ (i.e. $p \circ O = \mathrm{id}_S$).

A morphism $f : X \to S$ is *smooth* if and only if:

- f is flat;
- f is locally of finite presentation; and
- For all $s \in S$, the fiber $X_s = X \times_S \operatorname{Spec} k(s)$ over the residue field $k(s)$ is geometrically regular.

A scheme X over a field k is *geometrically regular* if $X_{\overline{k}}$ is nonsingular (every local ring is a regular local ring). Therefore condition (i) implies that all fibers of p are smooth (nonsingular) curves. Thus one can think of $p : E \to S$ as a "continuous family of curves over the scheme S".

Condition (ii) says the following: if $k = \overline{k}$ is an algebraically closed field and $\operatorname{Spec} k \to S$ a morphism, then in the pullback (fibre square)

$$\begin{tikzcd} E_k = E \times_S \operatorname{Spec} k \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

we have that the curve E_k satisfies $\dim_k H^1(E_k, \mathscr{O}_{E_k}) = 1$, i.e. E_k has (arithmetic or geometric) genus 1. (Note that the curve is smooth and connected, therefore integral.)

Remark. Note that there are no elliptic curves over $\operatorname{Spec} \mathbb{Z}$. Also, there is no elliptic curve over $\operatorname{Spec} \mathbb{C}[t]$ with $\Delta(E) \notin \mathbb{C}$, since then $\Delta(E)$ has a zero in $\mathbb{C}$. We can recover in each of these examples by removing finitely many points, i.e. those points in the support of $\Delta(E)$.

Let Ell be the category whose objects are elliptic curves (over schemes) and whose morphisms are pullback diagrams

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

which are *cartesian squares* compatible with the section O, i.e. E' is a fiber product of E and S' over S, so there is a unique isomorphism $E' \to E \times_S S'$.

We would like to find a final object in Ell, since then we could realize every elliptic curve over a scheme as a unique pullback of this "universal" elliptic curve over a universal base. Unfortunately, it does not exist: the morphism $E' \to E$ can always be changed by the automorphism $[-1]$ on E, so the pullback cannot be unique.

However, the forgetful map $p : \mathsf{Ell} \to \mathsf{Sch}/\mathbb{Z}$ by $(p : E \to S) \mapsto S$ gives Ell the structure of a category over $\mathbb{Z}$. The condition (i) follows immediately by pullback: If $E \to S$ is an elliptic curve, and $\phi : T \to S$ a morphism, then we have the fiber square,

$$\begin{tikzcd} E \times_S T \arrow[r] \arrow[d] & E \arrow[d] \\ T \arrow[r, "\phi"] & S \end{tikzcd}$$

which is a morphism in Ell whose image is ϕ.

Condition (ii) follows similarly: Given cartesian squares

$$\begin{tikzcd} E'S' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

and

$$\begin{tikzcd} E''S'' \arrow[r] \arrow[d] & E \arrow[d] \\ S'' \arrow[r] & S \end{tikzcd}$$

and a connecting arrow

$$\begin{tikzcd} S' \arrow[dd] \arrow[dr] & \\ & S \\ S'' \arrow[ur] & \end{tikzcd}$$

on the base, we see that $E'' \times_{S''} S'$ is a fiber product for $E \to S$ and $S' \to S$: Given morphisms $Z \to E$ and $Z \to S'$ which are equal on S, we obtain a unique morphism $Z \to E \times_S S'' \cong E''$ by the extension $Z \to S' \to S''$, and hence a unique morphism $Z \to E'' \times_{S''} S'$.

Therefore Ell is fibered in groupoids over $\mathbb{Z}$. The fiber $\mathsf{Ell}(S)$ is the category of elliptic curves over S with morphisms the isomorphisms among them.

4. Stacks

Recall the definition of a Grothendieck topology: it makes sense to say whether $\{S_i \to S\}$ is an open cover (isomorphisms are an open cover, an open cover of an open cover is an open cover, and a base extension of an open cover is an open cover). We assume that our base category Sch/S is equipped with a Grothendieck topology, the étale topology if not specified.

Furthermore, we assume a choice of lift for each arrow $\phi : U \to V$ in Sch/S; this will be the fiber product in our examples, and we denote it $x = y|_U = \phi^* y$.

Definition. A *stack* over a scheme S is a category $\mathcal{F}$ fibered in groupoids over S such that the assignment

$$\mathsf{Sch}/S \to \mathsf{Set}$$
$$U \mapsto \mathcal{F}(U) = p^{-1}(U)$$

is a *sheaf of groupoids*, i.e.:

(i) (Isomorphisms are a sheaf) For all $U \in \mathsf{Sch}/S$ and all $x, y \in \mathcal{F}(U)$, the functor

$$\begin{aligned} &\mathbf{Isom}_U(x,y) : \mathsf{Sch}/U \to \mathsf{Set} \\ &\qquad V \mapsto \{\alpha : x|_V \xrightarrow{\sim} y|_V \text{ an isomorphism in } \mathcal{F}(V)\} \end{aligned}$$

is a sheaf (in the (étale) topology), i.e. for all $U \in \mathsf{Sch}/S$, all $x, y \in \mathcal{F}(U)$, all open covers $\{U_i \to U\}$ of U, and all isomorphisms $\alpha_i : x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ such that $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$ where $U_{ij} = U_i \times_U U_j$, there exists a unique isomorphism $\alpha : x \to y$ such that $\alpha|_{U_i} = \alpha_i$;

(ii) (Descent datum is effective) For all open covers $\{U_i \to U\}$, all $x_i \in \mathcal{F}(U_i)$, and all $\alpha_{ij} : x_i|_{U_{ij}} \xrightarrow{\sim} x_j|_{U_{ij}}$ such that $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ over U_{ijk}, there exists an $x \in \mathcal{F}(U)$ and $\alpha_i : x|_{U_i} \xrightarrow{\sim} x_i$ in $\mathcal{F}(U_i)$ such that $\alpha_{ij} = \alpha_j|_{U_{ij}} \circ (\alpha_i|_{U_{ij}})^{-1}$.

Example. Let Bund_r/S be the category of vector bundles of rank r, whose objects are line bundles $\mathscr{L}$ of rank r on an S-scheme U and morphisms pullbacks, i.e. there exists a morphism $\mathscr{L}' \to \mathscr{L}$ if and only if there exists a map $\phi : U' \to U$ of S-schemes and an isomorphism $\mathscr{L}' \cong \phi^*\mathscr{L}$, which is necessarily unique. The forgetful functor (to the underlying scheme of the bundle) gives Bund_r/S the structure of an S-category: a fiber $\mathsf{Bund}_r(T)$ is the category of vector bundles of rank r over T with isomorphisms between them.

This category is fibered in groupoids via the pullback: given $\phi : U \to V$ and $\mathscr{L}$ a vector bundle on V, we have the vector bundle $\phi^*\mathscr{L}$ on U, and given vector bundles $\mathscr{L}'$ and $\mathscr{L}''$ over U' and U'', respectively, a commuting morphism $\chi : U' \to U''$ gives the pullback $\chi^*\mathscr{L}''$ which, since the pullback is unique up to unique isomorphism, and $(fg)^* = g^*f^*$, there is a unique map $\mathscr{L}' \cong \chi^*(\mathscr{L}'')$.

Condition (i) in the definition of a stack says that isomorphisms between bundles on the same scheme can be defined locally on an open cover and glued in a unique way if they agree on overlaps.

Condition (ii) says that line bundles can be glued. You cannot reconstruct a vector bundle $\mathscr{L}$ from its restrictions $\mathscr{L}_i$ over an open cover $\{U_i \to U\}$ (every bundle is trivialized over some open cover!); however, we have induced isomorphisms $\alpha_{ij} : \mathscr{L}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{L}_j|_{U_{ij}}$ which satisfy the cocycle condition, and by glueing, $\mathscr{L}$ can be recovered from $\mathscr{L}_i$ and the α_{ij}.

Example. Let $g \geq 1$ be an integer. Let $\mathcal{M}_g$ be the category over $\operatorname{Spec} \mathbb{Z}$ of smooth curves of genus g whose objects are smooth, proper morphisms $p : C \to S$ of relative dimension 1 whose geometric fibres are connected curves of genus g and whose morphisms are cartesian squares. (If $g = 1$, then $\mathcal{M}_g$ differs from Ell in that an elliptic curve comes equipped with a point.) Then $\mathcal{M}_g$ is a stack.

5. Schemes and functors as stacks

A functor is a stack, and a scheme is a stack via its functor of points. More explicitly, let X be an S-scheme. Then

$$p : \mathsf{Sch}/X \to \mathsf{Sch}/S$$

given by composition with the structure morphism $X \to S$ gives Sch/X the structure of an S-category: indeed, an X-morphism $Y \to Z$ becomes an S-morphism via composition with $X \to S$.

Moreover, Sch/X is fibered in groupoids: in fact, the fibers are just $\mathcal{F}(U) = \mathrm{Hom}_S(U, X)$ as a set. The objects of $\mathcal{F}(U)$ are X-schemes Z such that $Z \to X \to S$ is the map $U \to S$, so $Z = U$, and an X-morphism $U \to U$ in $\mathcal{F}$ projects to the S-morphism $U \to U$, which if the identity then so was the original map. Therefore the only morphisms in the category $\mathcal{F}(U)$ are the identity morphisms, and in this way we identify the fiber (a category) as a set.

Condition (i) is clear: if $f, g : U \to X \in \mathrm{Hom}_S(U, X)$ are elements of $\mathcal{F}(U)$, then from the above $\mathbf{Isom}_U(f, g)(V)$ has one element or none, depending on if $f|_V = g|_V$ or not. Therefore locally $\mathbf{Isom}_U(f, g)$ is the constant sheaf or the empty sheaf.

Condition (ii) is nontrivial, and follows from the statement that the functor of points is a sheaf in the étale topology. It is true in the Zariski topology and the étale topology for general schemes. To prove it in general, we appeal to the following lemma:

Lemma. *A presheaf F of sets on X in the étale topology is a sheaf if and only if the 'condition for sheaves' holds for coverings of the following types: $\{U_i \to U\}$ is a surjective family of open immersions; $\{V \to U\}$ is a single surjective morphism of affine schemes.*

Corollary. *For each S-scheme X, the functor* $\mathrm{Hom}_S(-, X)$ *is a* sheaf *in the étale topology on S.*

Proof. Let U be an S-scheme. Given an open Zariski covering $U = \bigcup_i U_i$, a morphism $\phi : U \to X$ is uniquely determined by its restrictions $\phi|_{U_i} : U_i \to X$, and conversely, given $\phi_i : U_i \to Z$ such that $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$, there exists a unique morphism $\phi : U \to Z$ such that $\phi|_{U_i} = \phi_i$.

If $V \to U$ is a surjective S-morphism of affine schemes, étale over S, then such a morphism is faithfully flat and quasi-compact, therefore we quote the following general result from descent theory: A faithfully flat, quasi-compact morphism of schemes is a universal effective epimorphism in the category of schemes. □

Therefore, by the 'sheaf condition', we obtain not only one but a unique lift as required by the descent datum in this case.

We will denote by X both the scheme X and the stack X given above in the étale topology.

6. Morphisms of stacks

Definition. A *morphism of stacks* is a functor $F : \mathcal{F} \to \mathcal{G}$ such that $p_{\mathcal{F}} = p_{\mathcal{G}} \circ F$.

Example. Let X, Y be S-schemes. Then a morphism $F : X \to Y$ of stacks is determined by a morphism $f : X \to Y$ of S-schemes, and vice versa. This is the statement of Yoneda's lemma, which we reprove for illustrative purposes.

Given $f : X \to Y$, composition with f gives a functor $\mathsf{Sch}/X \to \mathsf{Sch}/Y$ which commutes with projections: the projection of the X-morphism $U \to V$ is the S-morphism $U \to V$, which is the same as the projection of the Y-morphism $U \to V$.

Conversely, given $F : \mathsf{Sch}/X \to \mathsf{Sch}/Y$, let $f = F(X \xrightarrow{\mathrm{id}} X) = Z \to Y$. Then since F commutes with projections, we have

$$p_X(X \xrightarrow{\mathrm{id}} X) = X \to S = (p_Y \circ F)(X \xrightarrow{\mathrm{id}} X) = Z \to Y \to S.$$

Therefore $X = Z$, and $(f : X \to Y) \in \mathrm{Hom}_S(X, Y)$. It is easy to see that F is determined by f. Let $g : U \to X$ be any X-scheme. Consider the morphism

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "g"'] & & X \arrow[dl, "\mathrm{id}"] \\ & X & \end{tikzcd}$$

of X schemes. Then the image of this diagram under F is the morphism of Y-schemes

$$\begin{tikzcd} U \arrow[rr, "F(g)"] \arrow[dr, "F(U)"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

so the equality of projections says that $F(U) = f \circ g$.

Example. Let $\mathcal{M}_{g,1}$ be the stack of smooth pointed curves of genus g, whose objects are curves $C \to S$ as above together with a section $S \to C$ and whose morphisms are cartesian squares which respect the sections. It is easy to see that the map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is a morphism of stacks.

Example. The natural map $\mathsf{Ell} \to \operatorname{Spec} \mathbb{Z}[j]$ is a morphism of stacks.

Let $E \to S$ be an elliptic curve. There exists an affine open cover $S_i = \operatorname{Spec} R_i$ of S on which the restriction E_i of E to R_i is given by a Weierstrass equation with coefficients in R_i. We define the map $\operatorname{Spec} R_i \to \operatorname{Spec} \mathbb{Z}[j]$ by $j \mapsto j(E_i)$, given by the usual rational expression for j; this is well-defined since $\Delta(E_i) \in R_i^*$, as E_i is an elliptic curve over R_i. Now $S_{ij} = S_i \cap S_j$ also has an affine open cover $\operatorname{Spec} R_{ijk}$ such that $E_i|_{S_{ijk}} \cong E_j|_{S_{ijk}}$, hence $j(E_i|_{S_{ijk}}) = j(E_j|_{S_{ijk}}) \in R_{ijk}$, so by glueing these morphisms we obtain a well-defined morphism $S \to \operatorname{Spec} \mathbb{Z}[j]$.

This gives a commutative diagram

$$\begin{tikzcd} \mathsf{Ell} \arrow[rr] \arrow[dr] & & \mathsf{Sch}/\operatorname{Spec} \mathbb{Z}[j] \arrow[dl] \\ & \mathsf{Sch}/\operatorname{Spec} \mathbb{Z} & \end{tikzcd}$$

by

$$\begin{tikzcd} (E \to S) \arrow[rr, mapsto] \arrow[dr, mapsto] & & (S \to \operatorname{Spec} \mathbb{Z}[j]) \arrow[dl, mapsto] \\ & (S \to \operatorname{Spec} \mathbb{Z}) & \end{tikzcd}$$

so the functor commutes on the level of objects. Let

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

be a cartesian square and let S be covered by $\operatorname{Spec} R_i$, with $f^{-1}(\operatorname{Spec} R_i)$ covered by $\operatorname{Spec} R_i j'$ such that E and E' are given by Weierstrass equations on $\operatorname{Spec} R_i$ and $\operatorname{Spec} R_i'$, respectively. Since $E' \cong E \times_S S'$, we see that $E'|_{\operatorname{Spec} R_i j'}$ is isomorphic to the curve defined by the Weierstrass equation for E over R_i under the map $f|^{\#}_{\operatorname{Spec} R_i} : R_i \to R'_{ij}$, hence $f(j(E_{ij'})) = j(E_i) \in R_i$. Hence as above these morphisms glue, and we have a morphism $S' \to S$ over $\operatorname{Spec} \mathbb{Z}[j]$.

Remark. Note if $\mathcal{F}$ and $\mathcal{G}$ are moduli stacks (i.e. 'families' of objects over schemes), then a morphism of stacks is simply a mapping between families which commutes with base change.

Definition. A stack $\mathcal{F}$ is *representable* if there exists an S-scheme X and an isomorphism $\mathcal{F} \to X$.

Remark. Note that a morphism $\mathcal{F} \to X$ is a functor, so an isomorphism of stacks is an equivalence of categories: we *do not* insist that the composition of the two functors is the identity functor spot on. In fact, the morphisms from two stacks $\mathcal{F}$ and $\mathcal{G}$ form a category: the arrows in $\operatorname{Hom}_S(\mathcal{F}, \mathcal{G})$ are natural transformations of functors. Therefore we say that stacks form a 2-*category.*

7. Fibered products and representability

Definition. Let $F : \mathcal{F} \to \mathcal{H}$ and $G : \mathcal{G} \to \mathcal{H}$ be morphisms of stacks over S. The *fiber product* $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is the category whose objects are triples (x, y, α) where $\alpha : F(x) \to G(y)$ is a morphism in a fiber of $\mathcal{H}$ (i.e. $p_{\mathcal{F}}(x) = p_{\mathcal{G}}(y)$ and $\alpha : F(x) \to G(y)$ is an arrow in $\mathcal{H}$ such that $p_{\mathcal{H}}(\alpha) = \mathrm{id}$) and whose morphisms $(x, y, \alpha) \to (x', y', \alpha')$ are pairs $(\phi : x \to x', \psi : y \to y')$ in fibers of $\mathcal{F}$ and $\mathcal{G}$ such that

$$G(\psi) \circ \alpha = \alpha' \circ F(\phi) : F(x) \to G(y').$$

Remark. The fiber of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ over U are pairs $(x, y) \in \mathcal{F}(U) \times \mathcal{G}(U)$ such that $F(x)$ is isomorphic to $G(y)$ in $\mathcal{H}(U)$.

Proposition. *The category $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is a stack over S. Given a commutative diagram of stacks*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{H} \end{tikzcd}$$

there is a morphism $\mathcal{A} \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ which is unique up to unique isomorphism.

Definition. Let $F : \mathcal{F} \to \mathcal{G}$ be a morphism of stacks. Then F is *representable* if for all S-schemes X and all morphisms $X \to \mathcal{G}$, the fiber product $\mathcal{F} \times_{\mathcal{G}} X$ is a scheme.

Definition. Let $\mathbf{P}$ be a property of morphisms which is stable under base change and local on the target, and let $F : \mathcal{F} \to \mathcal{G}$ be a representable morphism. Then F *has* $\mathbf{P}$ if for all S-schemes X and all morphisms $X \to \mathcal{G}$, the induced morphism of schemes $\mathcal{F} \times_{\mathcal{G}} X \to X$ has $\mathbf{P}$.

Example. Any morphism of schemes is representable.

Example. A morphism $\mathcal{F} \to X$ where X is a scheme is representable if and only if $\mathcal{F}$ is a representable stack. (Take the identity map $X \to X$.) Therefore the map $\mathsf{Ell} \to \operatorname{Spec} \mathbb{Z}[j]$ is not representable.

Example. If $g \geq 1$, the morphism $F : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is representable, proper, and smooth. If S is a scheme and $G : S \to \mathcal{M}_g$ a morphism (think: G is a family of genus g curves over S), then it is represented by a family of curves $p : C \to S$

(the image of the identity, commutes with projection implies the base is S), and $\mathcal{M}_{g,1} \times_{M_g} S \cong C$.

Note: $\mathcal{M}_{g,1} \times_{\mathcal{M}_g} S$ has objects triples $(C' \to S', S'' \to S, \alpha)$ with $p(C' \to S') = S' = p(S'' \to S) = S''$ (over $\operatorname{Spec} \mathbb{Z}$), so $S'' = S'$. Now α is an arrow in $\mathcal{M}_g$ (i.e. a Cartesian diagram) with $p_{\mathcal{M}_g}(\alpha) = \mathrm{id}$, so

$$\begin{tikzcd} F(C' \to S') = C' \arrow[r] \arrow[d] & C'' = G(S') \arrow[d] \\ S' \arrow[r, equal] & S' \end{tikzcd}$$

but G is a functor, so the image of the diagram (morphism)

$$\begin{tikzcd} S' \arrow[rr] \arrow[dr] & & S \arrow[dl, "\mathrm{id}"] \\ & S & \end{tikzcd}$$

is the cartesian square

$$\begin{tikzcd} C'' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

which implies that

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is Cartesian as well. Thus we have a map

$$\begin{aligned} \mathcal{M}_{g,1} \times_{\mathcal{M}_g} S &\longrightarrow C \\ (C', S', \alpha) &\longmapsto S' \\ (C \times_S T, T, \mathrm{id}) &\longleftarrow\!\shortmid T \end{aligned}$$

which shows this is an equivalence of categories, hence an isomorphism. (One can also prove this using the universal property of the fibered product.)

It is proper and smooth because $C \to S$ is so.

8. Algebraic stacks

Let $\mathcal{F}$ be a stack, and consider the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ given by the two identity morphisms. Let X be an S-scheme and let $X \to \mathcal{F} \times_S \mathcal{F}$ be a morphism. By projection, we get two maps $X \to \mathcal{F}$, i.e. two objects $x, y \in \mathcal{F}(X)$.

Lemma. *The fiber product $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X$ is (isomorphic to) the sheaf (functor) $\mathbf{Isom}_X(x, y)$.*

Proof. The sections over an S-scheme U are pairs $(u, v) \in \mathcal{F}(U) \times X(U)$ such that

$$\operatorname{img}(u \in \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}) = (u, u, \mathrm{id}) \cong \operatorname{img}(v \in X \to \mathcal{F} \times_S \mathcal{F}) = (x|_U, y|_U, \alpha),$$

which is what we need to show. $\square$

Proposition. *The diagonal Δ is representable if and only if every morphism from a scheme to $\mathcal{F}$ is representable.*

Proof. Suppose Δ is representable. Let $F : X \to \mathcal{F}$ be a morphism. We must show that for all $G : Y \to \mathcal{F}$, $X \times_{\mathcal{F}} Y$ is a scheme. We have a cartesian diagram

$$\begin{tikzcd}
X \times_{\mathcal{F}} Y \arrow[r] \arrow[d] & X \times_S Y \arrow[d, "(f,g)"] \\
\mathcal{F} \arrow[r, "\Delta"] & \mathcal{F} \times_S \mathcal{F}
\end{tikzcd}$$

since it is easy to see that $X \times_{\mathcal{F}} Y$ satisfies the universal property. Now Δ representable implies that the fiber product is a scheme, which by definition says that G is representable.

Conversely, suppose every morphism from a scheme to $\mathcal{F}$ is representable. We want to show that for all morphisms $h : X \to \mathcal{F} \times_S \mathcal{F}$ that the fiber product $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X$ is a scheme. Now $h = (f, g) \circ \Delta_X$ for maps $F, G : X \to \mathcal{F}$, i.e. the diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, "G"] \arrow[ddr, "F"'] & & \\
& \mathcal{F} \times_S \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
& \mathcal{F} \arrow[r] & S
\end{tikzcd}$$

commutes. Then we have a cartesian diagram

$$\begin{tikzcd}
\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\
X \times_{\mathcal{F}} X \arrow[r] \arrow[d] & X \times_S X \arrow[d, "(f,g)"] \\
\mathcal{F} \arrow[r, "\Delta_{\mathcal{F}}"] & \mathcal{F} \times_S \mathcal{F}
\end{tikzcd}$$

so $X \times_{\mathcal{F}} X$ is a scheme. This says that

$$\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X \cong (X \times_{\mathcal{F}} X) \times_{X \times_S X} X$$

is a scheme as well. □

Definition. A stack $\mathcal{F}$ is *Deligne-Mumford (DM)* if the two conditions hold:

(i) The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact, and separated;
(ii) There is a scheme U and an étale surjective morphism $U \to \mathcal{F}$.

A stack $\mathcal{F}$ is *Artin* if (i) holds and with (ii) replaced by: There is a scheme U and a smooth surjective morphism $U \to \mathcal{F}$.

The scheme U in (ii) is called an *atlas*.

Example. A functor $F : \mathsf{Sch}/S \to \mathsf{Set}$ is a DM stack if and only if F is represented by a *locally separated algebraic space*.

Example. The stacks $\mathcal{M}_g$ and $\mathcal{M}_g$ are DM provided $g \geq 2$.

Example. The stack $\mathsf{Bund}_r(X)$ is not a DM stack if $r \geq 1$, because (in general) the groups of automorphisms of a nonzero vector bundle is not finite. Under some hypotheses on X, it is Artin.

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