Picard Groups of Moduli Problems

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The purpose of this lecture is to describe a single specific calculation which gives a modern formulation of an old fact. However, I want to devote a large part of this lecture to the explanation of the machinery which has been developed to give a new and, I think, enlightening setting to a whole group of old questions.

Severi, for one, raised the question: look at maximal families of (irreducible) space curves—is the parameter space of such families rational [10]? A more intrinsic question is whether the moduli variety for nonsingular curves of genus g is rational; in other words, look at the parameter space of the universal family of nonsingular curves of genus g and ask whether this is rational;† this question

† Actually, there is no such family. But if $g \geq 3$, then almost all such curves admit no automorphisms, and there is a universal family of the automorphism-free nonsingular curves.

may be very difficult. However, it can be approximated by any number of weaker questions: is this space unirational, or is it regular in the sense that its function field does not admit everywhere regular differential forms (cf. [5], Chapter 7, §2)? Or, still weaker, is the Picard variety of its function field trivial (cf. [6], Chapter 6, §1)? One of the principal results of our theory is that the last statement is true in characteristic 0. In the same line, can we determine various cohomology groups of this moduli variety?

Now all these questions, especially the last two, suffer from a certain vagueness because of our uncertainty about

1. Whether to look only at birational invariants of the function field,
2. Or, if we want to look at invariants of a definite model, which model to select (since there is no universal family of nonsingular curves),
3. If we settle for the usual moduli variety (i. e., the *coarse* one, cf. [7]), it has singularities (cf. [9]) and is not compact.

If we want an answer which has some pretense of being a basic fact, or of being more than idle, we certainly need to start with the *correct* variety, that is, the one which is most relevant to the set of all nonsingular curves with whatever structure is contained therein. Now the real clue here, I contend, is that we must not ask for the cohomology or the Picard group simply of a variety; there is a much better object, which is much more intrinsically related to the moduli problem and which possesses equally (a) a function field, (b) a Picard group, and (c) both étale and coherent cohomology theories. The invariants of this object—call it X—are the basic pieces of information.

In the first section, I want to describe the whole class of objects of which our X is an example. These objects, "topologies," were discovered by Grothendieck, and are the basic concept on which his theory of étale cohomology is constructed. In fact, it was chiefly in order to better understand this important concept that I made the calculations described in this paper. In the second section, I want to describe the étale topology proper, and its relation to the Zariski and the classical topology. In the third section, I want to introduce the topologies relevant to the problem of moduli. All this is nothing but definitions, and I hope that they possess enough intrinsic symmetry and interest to make the reader bear with their mounting abstractness. In the fourth section, I try to alleviate this abstractness by giving the full gory details of the topology relevant to the computations described later. In the fifth section, I describe precisely in two different ways the Picard groups associated to the moduli problem. In the last two sections, for $g = 1$, we give two separate computations of this group.

§1. TOPOLOGIES

In the classical definition of a topology, we start with a basic set X, the space, and we are given a collection A of subsets of X, called the open subsets. Suppose we try to eliminate the set X from our description and develop the theory from A alone: then we will have to endow A with extra structure to compensate for the loss of X. First of all, we make A into a category $\mathbf{A}$ by defining:

$$\begin{aligned} \mathrm{Hom}(U, V) &= \text{set with one element } f_{U,V}, \text{ if } U \subset V \\ &= \text{empty set, if } U \not\subset V \\ &\quad\ (\text{all } U, V \in A). \end{aligned}$$

Notice that the operation of intersecting two open sets U, V can be defined in terms of this category:

1.1.

$U \cap V$ is the product of U and V in $\mathbf{A}$, that is, it fits into a diagram

$$\begin{tikzcd} & U \cap V \arrow[dl] \arrow[dr] & \\ U & & V \end{tikzcd}$$

and has the universal mapping property: for all $W \in A$, and for all maps f, g as below, there is a unique h making the diagram commute:

$$\begin{tikzcd} W \arrow[r, dashed, "h"] \arrow[d, "f"'] \arrow[dr, "g"] & U \cap V \arrow[dl] \arrow[d] \\ U & V \end{tikzcd}$$

Similarly, arbitrary unions of open sets can be defined as *sums* in the category $\boldsymbol{A}$:

1.2.

If $U = \bigcup_{i \in I} U_i$, then with respect to the inclusions

$$\begin{tikzcd} U_1 \arrow[dr] & \\ U_2 \arrow[r] & U \\ & \vdots \end{tikzcd}$$

U is the categorical sum of the U_i's.

Moreover, the whole space X as an object of A—but not as a set—can be recovered as the final object of $\boldsymbol{A}$; X is the unique element Y of A such that for all other $U \in A$, there is one and only one map from U to Y.

Now suppose that we want to define the concept of a sheaf $\mathfrak{F}$ (of sets) on X purely in terms of $\boldsymbol{A}$. This goes as follows: first of all we must have a presheaf. This will be a collection of sets $\mathfrak{F}(U)$, one for each $U \in A$; and a collection of restriction maps, that is, if $U \subset V$, or if, equivalently, there is an element $f_{U,V} \in \mathrm{Hom}(U, V)$, then we must have a map

$$\mathrm{res}_{U,V} : \mathfrak{F}(V) \longrightarrow \mathfrak{F}(U).$$

This is nothing more than a contravariant functor $\mathfrak{F}$ from $\boldsymbol{A}$ to the category (Sets). In order to be a sheaf, it must have an additional property:

1.3.

If U_α is a *covering* of U, that is, each U_α is contained in U and

$$\bigcup_\alpha U_\alpha = U,$$

then an element x of $\mathfrak{F}(U)$ is determined by its restrictions to the subsets U_α; and every set of elements $x_\alpha \in \mathfrak{F}(U_\alpha)$, such that x_α and x_β always have the same restriction to $U_\alpha \cap U_\beta$, come from such an x.

To define sheaves, it is now clear that we may as well start with any category $\mathfrak{C}$, instead of A, and call its objects the open sets, provided that:

a. If U, V are open sets, $\mathrm{Hom}(U, V)$ contains at most one element.
b. Finite products and arbitrary sums of objects in $\mathfrak{C}$ exist; $\mathfrak{C}$ has a final object X.
   Also this turns out to be essential:
c. $$V \cap \Big[\bigcup_{i \in I} U_i\Big] = \bigcup_{i \in I} (V \cap U_i)$$
   where $\cap$, $\cup$ denote products and sums.

Then sheaves are simply contravariant functors $\mathfrak{F}$ from $\mathfrak{C}$ to (Sets) such that, whenever $U = \bigcup_{i \in I} U_i$, the following diagram of sets is exact:

$$\mathfrak{F}(U) \to \prod_{i \in I} \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathfrak{F}(U_i \cap U_j).$$

Moreover, the "global sections" $\Gamma(\mathfrak{F})$ of a sheaf $\mathfrak{F}$ are nothing but the elements of the set $\mathfrak{F}(X)$. If we look at sheaves of abelian groups instead of sheaves of Sets, then we can define the higher cohomology groups as well as Γ ($= H^0$). Namely, we verify in the standard way:

a. The category of abelian sheaves is an abelian category with lots of injective objects.
b. Γ is a left-exact functor from this category to the category (abelian groups).

Hence, as usual, if $\mathfrak{F}$ is an abelian sheaf, put $H^i(\mathfrak{F})$ ($i \geq 0$) equal to the ith derived functor of Γ (cf. [4], §3.2; [1], Ch. 2, §2).

So far, the theory is essentially trivial: it is nothing more than an exercise in avoiding the explicit mention of points. Grothendieck's fantastic idea is to enlarge the set of possibilities by dropping the assumption that $\mathrm{Hom}(U, V)$ contains at most one element; for example, open sets may even have nontrivial automorphisms. Notice first of all that then it is no longer sufficient to say simply that open sets U_α cover the open set U: it will be necessary to specify particular maps

$$p_\alpha : U_\alpha \to U$$

with respect to which U is covered by the U_α's. Moreover, it is generally not enough to say that the U_α's cover U only when U is the categorical sum of the U_α's: usually there are other collections of maps $\{p_\alpha\}$ which we will want to call coverings. The concept which emerges from these ideas is the following:

Definition. A "topology" T is a category $\mathcal{C}$ whose objects are called open sets and a set of "coverings." Each covering is a set of morphisms in $\mathcal{C}$, where all the morphisms have the same image; that is, it is a set of the form:

$$\{U_\alpha \xrightarrow{p_\alpha} U\}.$$

The axioms are:

a. Fibred products† $U_1 \times_V U_2$ of objects in $\mathcal{C}$ exist.
b. $\{U' \xrightarrow{p} U\}$ is a covering if p is an isomorphism; if $\{U_\alpha \xrightarrow{p_\alpha} U\}$ is a covering and if, for all α,

$$\{U_{\alpha,\beta} \xrightarrow{q_{\alpha,\beta}} U_\alpha\}$$

is a covering, then the whole collection

$$\{U_{\alpha,\beta} \xrightarrow{p_\alpha \circ q_{\alpha,\beta}} U\}$$

is a covering.
c. To generalize property (c) under 1.3, if $\{U_\alpha \xrightarrow{p_\alpha} U\}$ is a covering, and $V \longrightarrow U$ is any morphism, then

$$\{V \times_U U_\alpha \xrightarrow{q_\alpha} V\}$$

is a covering (q_α being the projection of the fibre product on its first factor).

† In any category, given morphisms $p : X \longrightarrow Z$ and $q : Y \longrightarrow Z$, a fibre product is a commutative diagram:

$$\begin{tikzcd} & W \arrow[dl, "u"'] \arrow[dr, "v"] & \\ X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

such that for all objects W' and morphisms $u' : W' \longrightarrow X$, $v' : W' \longrightarrow Y$ such that $p \circ u' = q \circ v'$, there is a unique morphism $t : W' \longrightarrow W$ such that $u' = u \circ t$, $v' = v \circ t$. This object W is usually written

$$X \times_Z Y$$

and referred to alone as the fibre product of X and Y over Z.

In general, we want to assume that $\mathcal{C}$ possesses a final object X, but this is not necessary. We want to generalize the concept of a sheaf to an arbitrary topology:

Definition. A sheaf (of sets) on T is a contravariant functor $\mathfrak{F}$ from $\mathcal{C}$ to the category (Sets) such that, for all coverings $U_\alpha \xrightarrow{p_\alpha} U$ in T, the following diagram of sets is exact:

$$\mathfrak{F}(U) \longrightarrow \prod_\alpha \mathfrak{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathfrak{F}(U_\alpha \times_U U_\beta)$$

(the arrows being the usual maps given by the functor $\mathfrak{F}$, contravariant to p_α and to the projections of $U_\alpha \times_U U_\beta$ to U_α and to U_β).

Exactly as before, each sheaf $\mathfrak{F}$ of abelian groups has a group of global sections:

$$\Gamma(\mathfrak{F}) = \mathfrak{F}(X)$$

(X the final object) and hence, by the method of derived functors, higher cohomology groups $H^i(T, \mathfrak{F})$.

A topology in the classical sense gives a topology in an obvious way. To give the theory some content, consider the following example:

Let a group π act freely and discontinuously on a topological space X; that is, for all $x \in X$, there is an open neighborhood U of x such that $U \cap U^\sigma = \emptyset$ for all $\sigma \in \pi$, $\sigma \neq e$. For every set S and action of π on S, we can construct the topological space $\mathcal{S} = (X \times S)/\pi$ (endowing S with the discrete topology). With two π sets S and T and a π-linear map $f : S \longrightarrow T$, we obtain a local homeomorphism

$$\begin{array}{ccc} (X \times S)/\pi & \xrightarrow{\tilde{f}} & (X \times T)/\pi \\ \| & & \| \\ \mathcal{S} & & \mathcal{T} \end{array}$$

that makes $\mathcal{S}$ into a covering space of $\mathcal{T}$. Let the category $\mathcal{C}$ consist in the set of such spaces $\mathcal{S}$ and such maps $\tilde{f}$; let the coverings consist of maps $\tilde{f}_\alpha : \mathcal{S}_\alpha \longrightarrow \mathcal{T}$ such that, equivalently, $\mathcal{T} = \bigcup_\alpha \tilde{f}_\alpha(\mathcal{S}_\alpha)$ or $T = \bigcup_\alpha f_\alpha(S_\alpha)$. The final object in this topology is the topological space X/π, since every other open set has a unique projection

$$\mathcal{S} = (X \times S)/\pi \longrightarrow X/\pi$$

in the category. In other words, what has happened is that the open sets are no longer subsets of X/π; they are covering spaces of X/π.

If X is simply connected and connected, then X is just the universal covering space of X/π, and the topology consists in fact in *all* covering spaces $\mathfrak{S}$ of X/π, and all continuous maps $\mathfrak{S} \to \mathfrak{T}$ making the following diagram commute:

$$\begin{tikzcd} \mathfrak{S} \arrow[rr, "\tilde{f}"] \arrow[dr] & & \mathfrak{T} \arrow[dl] \\ & X/\pi & \end{tikzcd}$$

On the other hand, this topology is actually independent of X: we may as well "call" the π-sets S themselves the open sets, and call the π-linear maps $f : S \to T$ the morphisms. Then the space X corresponds to the π-set π, (say with left multiplication as the action of π on itself), and the final object X/π corresponds to the π-set $\{0\}$, with trivial action of π. We shall call this topology T_π.

In this form, it is easy to give an explicit description of a sheaf $\mathfrak{F}$ on the topology. Let π, considered only as a set with the *group* π acting on the left, be denoted $\langle\pi\rangle$. Then the right action of π on $\langle\pi\rangle$ makes π into a group of automorphisms of the π-set $\langle\pi\rangle$. But the group of automorphisms of $\langle\pi\rangle$ obviously acts on the set $\mathfrak{F}(\langle\pi\rangle)$ for every sheaf $\mathfrak{F}$. Let $M = \mathfrak{F}(\langle\pi\rangle)$. Then M itself becomes a π-set. I claim that $\mathfrak{F}$ is canonically determined by the π-set M.

a. Let S be a π-set on which π acts transitively. Then there is a π-linear surjection

$$\langle\pi\rangle \xrightarrow{p} S$$

making $\langle\pi\rangle$ into a covering of S. By applying the sheaf axiom to this covering, we check that $\mathfrak{F}(S)$ is isomorphic to the subset M^h of M of elements, left fixed by $h \subset \pi$, where h is the stabilizer of $p(e)$.

b. If S is any π-set, then S is the disjoint union of π-subsets S_α on which π acts transitively. If i_α is the inclusion of S_α in S, apply the sheaf axiom to the covering

$$\{S_\alpha \xrightarrow{i_\alpha} S\}.$$

We check that, via $\mathfrak{F}(i_\alpha)$,

$$\mathfrak{F}(S) \cong \prod_\alpha \mathfrak{F}(S_\alpha).$$

Conversely, given the π-set M, the isomorphisms in (a) and (b) define a sheaf $\mathfrak{F}$: hence to give a sheaf (of sets) $\mathfrak{F}$ in this topology and to give a π-set M are one and the same thing. In particular, a sheaf of abelian groups $\mathfrak{F}$ is the same thing as a π-module M. As the π-set $\{0\}$ is the final object in T_π, we find by means of (a) that the global sections $\Gamma(\mathfrak{F})$ of the sheaf $\mathfrak{F}$ are just the invariant elements M^π of M. Now it is well known that the category of π-modules is an abelian category, and that

$$M \longrightarrow M^\pi$$

is a left-exact functor on this category. Its derived functors are known as the cohomology groups of π with coefficients in M:

$$H^i(\pi, M)$$

(cf. [8], §10.6). Therefore, we find:

$$H^i(T_\pi, \mathfrak{F}) \cong H^i(\pi, M).$$

One final set of concepts: if T_1 and T_2 are two topologies with final object, a *continuous map* F from T_1 to T_2 consists in a functor from the category of open sets of T_2 to the category of open sets of T_1 such that:

a. It takes the final object to the final object.
b. It takes fibre products in T_2 to fibre products in T_1.
c. It takes coverings in T_2 to coverings in T_1.

For the sake of tradition, if U is an open set in T_2, we let $F^{-1}(U)$ denote the open set in T_1 associated to U by this functor; in other words, requirement (b) means:

$$F^{-1}(U \underset{U}{\times} U_2) \cong (F^{-1}(U_1) \underset{F^{-1}(U)}{\times} F^{-1}(U_2)).$$

If F is a continuous map, then F induces a map F_* from sheaves on T_1 to sheaves on T_2: let $\mathfrak{F}$ be a sheaf on T_1. Define

$$F_*(\mathfrak{F})(U) = \mathfrak{F}(F^{-1}(U))$$

for all open sets U in T_2. This is clearly a sheaf. By standard techniques (cf. [4] and [1], Ch. 2, §4), we find that there is a canonical homomorphism:

$$H^i(T_2, F_*(\mathfrak{F})) \longrightarrow H^i(T_1, \mathfrak{F}).$$

Moreover, let U be an open set in a topology T. Then "U with its induced topology" is a topology T_U defined as follows:

a. Its open sets are morphisms $V \to U$ in T.
b. Its morphisms are commutative diagrams:

$$\begin{tikzcd} V_1 \arrow[rr] \arrow[dr] & & V_2 \arrow[dl] \\ & U & \end{tikzcd}$$

c. A set of morphisms

$$\begin{tikzcd} V_\alpha \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & U & \end{tikzcd}$$

is a covering, if the set of morphisms $\{V_\alpha \to V\}$ is a covering in T.

Then there is a canonical continuous "inclusion" map:

$$i : T_U \to T,$$

that is, to the open set V in T, associate the open set $i^{-1}(V)$ which is the projection:

$$p_2 : V \times U \to U.$$

§2. ÉTALE AND CLASSICAL TOPOLOGIES

From now on, we will be talking about schemes. For the sake of simplicity, we will work over an algebraically closed field k, and all schemes will be assumed separated and of finite type over k, *without further mention.*

Definition. Let $f : X \to Y$ be a morphism. Then if, for all closed points $y \in Y$, $f^{-1}(y)$ is a finite set and for all $x \in f^{-1}(y)$, the induced homomorphism

$$f^* : \mathfrak{o}_y \to \mathfrak{o}_x$$

gives rise to an isomorphism of the completions of these rings

$$\hat{f}^* : \hat{\mathfrak{o}}_y \xrightarrow{\sim} \hat{\mathfrak{o}}_x,$$

then f is *étale.*

As an exercise, the reader might prove that this is equivalent to assuming:

a. f is flat, that is, for all $x \in f^{-1}(y)$, $\mathfrak{o}_x$ is a flat $\mathfrak{o}_y$-module,
b. The scheme-theoretic fibre $f^{-1}(y)$ is a reduced finite set, that is, $f^{-1}(y)$ is a finite set, and for all $x \in f^{-1}(y)$, $\mathfrak{m}_x = f^*(\mathfrak{m}_y) \cdot \mathfrak{o}_x$.

Clearly, "étale" is the scheme-theoretic analog of "local homeomorphism" for topological spaces. Now let X be a scheme.

Definition. The étale topology $X_{\text{ét}}$ of X consists of

a. The category whose objects are étale morphisms $p : U \to X$, and whose morphisms are arbitrary X-morphisms; in other words, given $U \xrightarrow{p} X$, $V \xrightarrow{q} X$, then $\operatorname{Hom}(p, q)$ is the set of commutative diagrams

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "p"'] & & V \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

(For simplicity, we shall refer to the objects of this category as schemes U, the étale morphism p to X being understood).

b. The coverings consist in arbitrary sets of morphisms $\{U_\alpha \xrightarrow{p_\alpha} U\}$ provided that

$$U = \bigcup_\alpha p_\alpha(U_\alpha).$$

Let X_{Zar} be the Zariski topology on X: its category consists in the open *subsets* of X and the inclusion maps between them; and a set of inclusion maps $p_\alpha : U_\alpha \subset U$ is said to be a covering if

$$U = \bigcup_\alpha U_\alpha.$$

These two topologies are related by a continuous map

$$\sigma : X_{\text{ét}} \to X_{\text{Zar}}.$$

Namely, if $U \subset X$ is an open subset, the inclusion morphism i of U in X is obviously étale, so that i is an open set in $X_{\text{ét}}$. The reader should check to see that the map from U to i extends to a functor from the category of X_{Zar} to the category of $X_{\text{ét}}$, which takes coverings to coverings and fibre products to fibre products.

If $k = \mathbf{C}$, the field of complex numbers, we can compare $X_{\text{ét}}$ with the classical topology too. The set of closed points of X forms an analytic set $X_{\mathbf{C}}$; and has an underlying topology inherited from the usual topology on $\mathbf{C}$: call this X_{cx}. Unfortunately, there is no continuous map in either direction between $X_{ét}$ and X_{cx}. However, there is a third topology related to both: let open sets consist in analytic sets U and holomorphic maps

$$f : U \to X_{\mathbf{C}},$$

which are local homeomorphisms—as usual, coverings are just sets of maps

$$\begin{tikzcd} U_\alpha \arrow[r, "f_\alpha"] \arrow[dr] & U \arrow[d] \\ & X_{\mathbf{C}} \end{tikzcd}$$

such that $U = \bigcup_\alpha f_\alpha(U_\alpha)$. Call this topology X^*_{cx}. Then there are continuous maps

$$\begin{tikzcd} & X^*_{cx} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X_{cx} & & X_{ét}, \end{tikzcd}$$

since

a. An open set in X_{cx} is an "open" subset $U \subset X_{\mathbf{C}}$; and this defines the inclusion map

$$i : U \to X_{\mathbf{C}},$$

which is a holomorphic local homeomorphism.

b. An open set in $X_{ét}$ is an étale morphism $f : U \to X$ of a scheme U to the scheme X; and this defines the holomorphic local homeomorphism

$$f_{\mathbf{C}} : U_{\mathbf{C}} \to X_{\mathbf{C}}$$

of the corresponding analytic sets.

On the other hand, although a is not an isomorphism of topologies, it is very nearly one in the following sense:

Definition. Let $f : T_1 \to T_2$ be a continuous map of topologies. f is an *equivalence* of T_1 and T_2 if

a. The functor f^{-1} from the category of open sets of T_2 to that of T_1 is fully faithful,

b. Every open set U in T_1 admits a covering in T_1 of the form $\{f^{-1}(V_\alpha) \xrightarrow{g_\alpha} U\}$, with suitable open sets V_α in T_2,

c. A collection of maps $\{V_\alpha \xrightarrow{g_\alpha} V\}$ in T_2 is a covering, if the collection of maps $\{f^{-1}(V_\alpha) \xrightarrow{f^{-1}(g_\alpha)} f^{-1}(V)\}$ in T_1 is a covering.

We leave it to the reader to check several simple points: $\boldsymbol{a}$ is an equivalence of topologies; if $f: T_1 \to T_2$ is an equivalence of topologies, f_* defines an equivalence between the category of sheaves on T_1 and the category of sheaves on T_2; hence if $\mathfrak{F}$ is a sheaf of abelian groups on T_1, the canonical homomorphism:

$$H^i(T_1, \mathfrak{F}) \rightleftarrows H^i(T_2, f_*\mathfrak{F}),$$

is an isomorphism. In fact, there is no significant difference between equivalent topologies. For this reason, we often speak of "the continuous map" from X_{cx} to $X_{\mathrm{\acute{e}t}}$, although strictly speaking this does not exist. Finally, there is a very nice result of M. Artin: let $\mathbf{Z}/n$ denote the sheaf on X^*_{cx}

$$\mathbf{Z}/n(U) = \bigoplus_{\substack{\text{connected components of} \\ U \text{ in complex topology}}} \mathbf{Z}/n;$$

(this is the same as the sheaf associated to the presheaf which simply assigns the group $\mathbf{Z}/n$ to every open set U.) We shall denote $b_*(\mathbf{Z}/n)$ simply by $\mathbf{Z}/n$; since the connected components of a scheme U in its complex and in its Zariski topologies are the same, we have:

$$b_*(\mathbf{Z}/n)(U) = \bigoplus_{\substack{\text{connected components} \\ \text{of } U \text{ in Zariski topology}}} \mathbf{Z}/n.$$

If X is nonsingular, M. Artin has proven that the canonical homomorphism

$$H^i(X_{\mathrm{\acute{e}t}}, \mathbf{Z}/n) \longrightarrow H^i(X^*_{\mathrm{cx}}, \mathbf{Z}/n) \rightleftarrows H^i(X_{\mathrm{cx}}, \mathbf{Z}/n)$$

is an isomorphism. This result assures us that, at least for nonsingular varieties, the étale topology, defined purely in terms of schemes, captures much of the topological information contained in the a priori finer complex topology.

To complete this comparison of the topologies associated to a scheme X, we must mention three other topologies, defined over any k, which are interesting. The idea behind these topologies is to enlarge the category as much as you want, but to keep the coverings relatively limited. In all of them, an open set is an *arbitrary* morphism

$$f : U \to X,$$

and a map between two open sets f_1 and f_2 is a commutative diagram:

$$\begin{tikzcd} U_1 \arrow[rr] \arrow[dr, "f_1"'] & & U_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

The restriction on the coverings involves new classes of morphisms, defined as follows:

Definition. A morphism $f : X \to Y$ is *flat* if for all $x \in X$, the local ring $\boldsymbol{o}_x$ is a flat module over $\boldsymbol{o}_{f(x)}$. Moreover, f is *smooth* if it is flat and if the scheme-theoretic fibres of f are nonsingular varieties (not necessarily connected).

To understand smoothness better, the reader might check that it is equivalent to requiring, for all $x \in X$, that the completion $\hat{\boldsymbol{o}}_x$ is isomorphic, as $\hat{\boldsymbol{o}}_{f(x)}$-algebra, to

$$\hat{\boldsymbol{o}}_{f(x)}[[X_1, \ldots, X_n]]$$

for some n.

For the purposes of §3, it is very important to know that smooth morphisms are also characterized by the following property (cf. [3], exposé 3, Theorem 3.1).

2.1.

Let A be a finite-dimensional commutative local k-algebra, and let $I \subset A$ be an ideal. Let a commutative diagram of solid arrows be given:

$$\begin{tikzcd} & \operatorname{Spec}(A/I) \arrow[dl] \arrow[dr] & \\ \operatorname{Spec}(A) \arrow[rr, dashed] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

Then there exists a morphism denoted by the dotted arrow filling in the commutative diagram.

This should be understood as a kind of "homotopy lifting property," so that smooth morphisms are somewhat analogous to fibre spaces.

We can now define three topologies,

$$X^*_{\text{ét}},\ X^*_{\text{smooth}},\ \text{and } X^*_{\text{flat}},$$

by defining a covering as a collection of morphisms $\{U_\alpha \xrightarrow[f_\alpha]{} U\}$ such that $U = \bigcup_\alpha f_\alpha(U_\alpha)$ and f_α is étale; f_α is smooth; f_α is flat, respectively. You can check to see that all our topologies are related by continuous maps as follows:

$$X^*_{\text{flat}} \to X^*_{\text{smooth}} \to X^*_{\text{ét}} \to X_{\text{ét}} \to X_{\text{Zar}}.$$

The important fact about these maps is that, in particular, they set up isomorphisms between the cohomology of X^*_{smooth}, $X^*_{\text{ét}}$ and $X_{\text{ét}}$. Therefore, as far as cohomology is concerned, any one of these three topologies is just as good as the others.

§3. MODULI TOPOLOGIES

For this entire section, fix a nonnegative integer g. We first recall the basis of the moduli problem for curves of genus g:

Definition. A "curve" (over k, of genus g) is a connected, reduced, one-dimensional scheme X, such that

$$\dim H^1(X, o_X) = g.$$

Definition. A "family of curves" over S (or, parametrized by S) is a flat, projective morphism of schemes

$$\pi : \mathfrak{X} \to S,$$

whose fibres over all closed points are curves.

Definition. A "morphism" F of one family $\pi_1 : \mathfrak{X}_1 \to S_1$ to another $\pi_2 : \mathfrak{X}_2 \to S_2$ is a diagram of morphisms of schemes:

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r] & S_2 \end{tikzcd}$$

making $\mathfrak{X}_1$ into the fibre product of S_1 and $\mathfrak{X}_2$ over S_2. F is smooth/flat/étale if the morphism from S_1 to S_2 is smooth/flat/étale.

Definition. Given a family of curves $\pi : \mathfrak{X} \to S$ and a morphism $g : T \to S$, the "induced family of curves" over T is the projection:

$$p_2 : \mathfrak{X} \times_S T \to T.$$

The most natural problem is to seek a universal family of curves, that is, one such that every other one is induced from it by a unique morphism of the parameter spaces. As indicated in the introduction, the usual compromises made in order that this existence problem has a solution are exactly what we want to avoid now. Instead, we want to define a topology; in the ideal case, if a universal family of curves had existed, this would be one of the standard topologies on the universal parameter space. Inasmuch as such a family does not exist (unless stringent conditions on the curves in our families are adopted), this topology is a new object.

Definition (Provisional Form). The moduli topologies $\mathfrak{M}^*_{\text{ét}}$, $\mathfrak{M}^*_{\text{smooth}}$, and $\mathfrak{M}^*_{\text{flat}}$ are as follows:

a. Their open sets are families of curves.
b. Morphisms between open sets are morphisms between families.
c. A collection of such morphisms

$$\begin{tikzcd} \mathfrak{X}_\alpha \arrow[r] \arrow[d, "\pi_\alpha"'] & \mathfrak{X} \arrow[d, "\pi"] \\ S_\alpha \arrow[r, "g_\alpha"'] & S \end{tikzcd}$$

is called a covering, if $S = \bigcup_\alpha g_\alpha(S_\alpha)$ and if each g_α is étale, smooth, or flat, respectively.

The first thing to check is that this is a topology and, in particular, that fibre products exist in our category. However, unlike the examples considered in §2, there is not necessarily a final object in our category. Such a final object would be a universal family of curves. A second point is that, if $\pi : \mathfrak{X} \to S$ is any family of curves, the topology induced on the open set π is equivalent to the topology $S^*_{\text{ét}}$, S^*_{smooth}, or S^*_{flat} on S.

A less trivial fact is that absolute products exist in our category. Let $\pi_i : \mathfrak{X}_i \to S_i$ $(i = 1, 2)$ be two families of curves: I shall sketch the construction of the product family. First, over the scheme

$S_1 \times S_2$, we have two induced families of curves,

$$\begin{tikzcd}
\mathfrak{X}_1 \times S_2 \arrow[dr] & & S_1 \times \mathfrak{X}_2 \arrow[dl] \\
& S_1 \times S_2. &
\end{tikzcd}$$

Now suppose $\mathfrak{Y} \to T$ is a third family of curves, and that the following morphisms are morphisms of families:

$$\begin{tikzcd}
& \mathfrak{Y} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathfrak{X}_1 \arrow[dd] & & \mathfrak{X}_2 \arrow[dd] \\
& T \arrow[dl] \arrow[dr] & \\
S_1 & & S_2
\end{tikzcd} \tag{a}$$

To have such morphisms is obviously equivalent to having (1) a morphism $T \to S_1 \times S_2$, and (2) isomorphisms *over* T of the three families of curves:

$$\begin{tikzcd}
{[\mathfrak{Y}]} \arrow[dr] & {[(\mathfrak{X}_1 \times S_2) \times_{(S_1 \times S_2)} T]} \arrow[d] & {[(S_2 \times \mathfrak{X}_2) \times_{(S_1 \times S_2)} T]} \arrow[dl] \\
& T &
\end{tikzcd} \tag{b}$$

But now we must digress for a minute; consider, á la Grothendieck, the following class of universal mapping properties which can be used to define auxiliary schemes. Let S be a scheme, and let

$$\begin{tikzcd}
X_1 \arrow[dr] & & X_2 \arrow[dl] \\
& S &
\end{tikzcd}$$

be two morphisms. Look at all pairs (T, Φ) consisting of schemes T over S (i. e., with given morphism to S) and isomorphisms over T:

$$\begin{tikzcd}
X_1 \times_S T \arrow[rr, "\sim", "\Phi"'] \arrow[dr] & & X_2 \times_S T. \arrow[dl] \\
& T &
\end{tikzcd}$$

If there is one such pair (T, Φ) such that for every other such pair (T', Φ'), there is a unique morphism (over S)

$$f : T' \to T$$

making the following diagram commute:

$$\begin{array}{ccc} X_1 \times_S T' & \xrightarrow{\Phi'} & X_2 \times_S T' \\ {\scriptstyle 1_{X_1} \times f} \downarrow & & \downarrow {\scriptstyle 1_{X_2} \times f} \\ X_1 \times_S T & \xrightarrow{\Phi} & X_2 \times_S T \end{array}$$

then (T, Φ) is uniquely determined up to canonical isomorphism by this property. And T is denoted

$$\mathrm{Isom}_S(X_1, X_2).$$

Now returning to our families of curves, suppose that the scheme

$$I = \mathrm{Isom}_{S_1 \times S_2}(\mathfrak{X}_1 \times S_2, S_1 \times \mathfrak{X}_2)$$

exists. Then, in situation (a), we get not only a canonical morphism from T to $S_1 \times S_2$ but even a canonical morphism from T to I. Now over I, the two families of curves induced from $\mathfrak{X}_1/S_1$ and $\mathfrak{X}_2/S_2$ are canonically identified: call this family $\mathfrak{X}/I$. Then the situation (a) is obviously equivalent with a morphism from the family Y/T to the family $\mathfrak{X}/I$. In other words, $\mathfrak{X}/I$ is the only possible product of the families $\mathfrak{X}_1/S_1$ and $\mathfrak{X}_2/S_2$. Fortunately, I does exist in our case. This is a consequence of a general result of Grothendieck's (cf. [2], exposé 221), and we will pass over this point completely.

Definition. The product of the families $\pi_i : \mathfrak{X}_i \to S_i$ $(i = 1, 2)$ will be denoted:

$$\pi : (\mathfrak{X}_1, \mathfrak{X}_2) \to \mathrm{Isom}(\pi_1, \pi_2).$$

Since products do exist in the common category of the topologies $\mathfrak{M}^*$, there is no reason not to add a final object M to this category in a perfectly formal way. In order to enlarge the topology, though, we have to define the coverings of the final object M. The point is, however, that if $\pi : \mathfrak{X} \to S$ is part of a covering of M, and if $\pi' : \mathfrak{X}' \to S'$ is any other family of curves whatsoever, then the morphism from the product family

$$(\mathfrak{X}, \mathfrak{X}') \to \mathrm{Isom}(\pi, \pi')$$

to the family π' must be part of a covering of π'. In particular, the projection from $\mathrm{Isom}(\pi, \pi')$ to S' must be étale, smooth, or flat according to the case involved. This leads to:

Definition. A family of curves $\pi : \mathcal{X} \to S$ is *étale*, *smooth*, or *flat over* M if, for all other families $\pi' : \mathcal{X}' \to S'$, the projection from $\mathrm{Isom}(\pi, \pi')$ to S' is étale, smooth, or flat.

If we use criterion 2.1 for smoothness given at the end of §2 (p. 46), the condition that π is smooth over M can be reformulated. In fact, after unwinding all the definitions by various universal mapping properties, this condition comes out as follows.

3.1.

Let A be a finite-dimensional commutative local k-algebra, and let $I \subset A$ be an ideal. Suppose we are given a diagram of solid arrows:

$$\begin{tikzcd}
\mathcal{Y} \arrow[rr, dashed] \arrow[dd] & & \mathcal{X} \arrow[dd] \\
& \mathcal{Y}_0 \arrow[ul] \arrow[ur] \arrow[dd] & \\
\mathrm{Spec}(A) \arrow[rr, dashed] & & S \\
& \mathrm{Spec}(A/I) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where $\mathcal{Y}/\mathrm{Spec}(A)$ and $\mathcal{Y}_0/\mathrm{Spec}(A/I)$ are families of curves, and where the two solid squares are morphisms of families of curves. Then there should be a morphism of families denoted by the dotted arrows filling in the commutative diagram.

Such families have been considered already: compare, especially, the thesis of M. Schlessinger. The important thing is that plenty of such families exist. In particular, if C is any curve over k, we certainly want C to be part of such a family. This can be proven by the method of "linear rigidifications" (cf. [7], §5.2 and §7.2). A fortiori, plenty of families π flat over M exist too.

With families étale over M, it is another matter. In fact, unless we stick to curves C without global vector fields (i. e., everywhere finite derivations), such families do not exist. Let us analyze what it means for $\pi : \mathfrak{X} \longrightarrow S$ to be étale over M. Let C be any curve over k, and let

$$p : C \longrightarrow \mathrm{Spec}(k)$$

be the trivial family given by C. Then if π is étale over M, $\mathrm{Isom}(\pi, p)$ must be étale over $\mathrm{Spec}(k)$; that is, $\mathrm{Isom}(\pi, p)$ must consist in a finite set of reduced points. But the points of $\mathrm{Isom}(\pi, p)$ represent isomorphisms of C with the fibres $\pi^{-1}(s)$ of the family π. Therefore, if π is étale over M, the following is satisfied.

3.2.

For all curves C over k, C occurs only a finite number of times in the family $\pi : \mathfrak{X} \longrightarrow S$, that is C is only isomorphic to a finite number of curves $\pi^{-1}(s)$. Moreover, if C occurs at all in π, the group of automorphisms of C is finite.

Now conversely, the smoothness of π and (3.2) (i.e., 3.1 and 3.2), guarantee the étaleness of π. To see this, let $\pi' : \mathfrak{X}' \longrightarrow S'$ be any other family of curves. Assume (3.1) and (3.2). Then we know that $\mathrm{Isom}(\pi, \pi')$ is smooth over S'; for it also to be étale over S' means simply that $\mathrm{Isom}(\pi, \pi')$ has only a finite number of closed points over every closed point $s' \in S'$. But let C be the curve $\pi'^{-1}(s')$. There is an isomorphism between the set of closed points of $\mathrm{Isom}(\pi, \pi')$ over s' and the set of isomorphisms of C with the curves $\pi^{-1}(s)$ (s a closed point of S). Therefore, the finiteness of this set follows from 3.2.

Definition. A family $\pi : \mathfrak{X} \longrightarrow S$ of curves satisfying (3.1) and (3.2) will be called a "modular" family of curves.

Modular families have the following very nice property. Let $\pi_i : \mathfrak{X}_i \longrightarrow S_i$ be two modular families of curves, and suppose the curve C occurs in π_1 over the point $s_1 \in S_1$ and in π_2 over the point $s_2 \in S_2$, that is,

$$\pi_1^{-1}(s_1) \cong C \cong \pi_2^{-1}(s_2).$$

I claim that S_1 and S_2 are formally isomorphic at the points s_1, s_2; in other words, the complete local rings $\hat{o}_{s_1}$ and $\hat{o}_{s_2}$ are isomorphic. To see this, fix an isomorphism τ of $\pi_1^{-1}(s_1)$ and $\pi_2^{-1}(s_2)$. Then τ determines a point

$$t \in \mathrm{Isom}(\pi_1, \pi_2)$$

lying over s_1 and s_2. But since both π_1 and π_2 are modular, $\mathrm{Isom}(\pi_1, \pi_2)$

is étale over both S_1 and S_2. Therefore, via the projections, we get isomorphisms:

$$\hat{o}_{s_1} \cong \hat{o}_t \cong \hat{o}_{s_2}.$$

More precisely, two modular families containing the same curve are related by an étale correspondence at the point where this curve occurs. As a consequence, for example, either all or none of such families are nonsingular, and they all have the same dimension.

Another very important property of modular families is that any morphism between two such families is necessarily étale. Assume that

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "\varpi"'] & \mathcal{X} \arrow[d, "\pi"] \\ T \arrow[r] & S \end{tikzcd}$$

is a morphism of modular families. This morphism defines an isomorphism of $\mathcal{Y}/T$ and the family induced by $\mathcal{X}/S$ over T: so it defines a morphism of T to $\mathrm{Isom}(\varpi, \pi)$ by the universal mapping property defining Isom. We get the diagram

$$\begin{tikzcd} & \mathrm{Isom}(\varpi, \pi) \arrow[dl, shift left, "p_1"] \arrow[dr, "p_2"] & \\ T \arrow[ur, shift left, "\sigma"] \arrow[rr] & & S \end{tikzcd}$$

where $p_1 \circ \sigma = 1_T$, and $p_2 \circ \sigma = g$. Since p_1 is étale, a section, such as σ, of p_1 defines an isomorphism of T with an open component

$$I_0 \subset \mathrm{Isom}(\varpi, \pi).$$

Since p_2 is étale, the restriction of p_2 to I_0 is étale; hence g is étale.

We now return to our topologies.

Definition (*Final Form*). The moduli topologies $\mathfrak{M}^*_{\text{smooth}}$, and $\mathfrak{M}^*_{\text{flat}}$ are as follows:

a. Their open sets are families of curves, and a final object M.
b. Their morphisms are morphisms of families of curves, and projections onto the final object M,
c. A collection of such morphisms with image a family $\pi : \mathcal{X} \to S$ is called a covering exactly as before; a collection of projections of families $\pi_\alpha : \mathcal{X}_\alpha \to S_\alpha$ onto the final object M is called a covering

of M if: (1) each family π_α is smooth, or flat over M, and (2) every curve C occurs in one of the families π_α.

It is clear that a topology $\mathfrak{M}^*_{\text{ét}}$ could be defined in the same way, but then the final object M would have no coverings at all. This is because some curves have an infinite group of automorphisms, and hence do not occur in any modular families. One result is that sheaves on this topology would not be sufficiently restricted; the topology is too loosely tied together and would not be useful.

However, suppose that we happen to be interested only in nonsingular curves. This is perhaps short sighted, but never mind. By considering only families of nonsingular curves, we can modify $\mathfrak{M}^*_{\text{smooth}}$, for example, and get a smaller topology. Now if the genus g is at least 2, it is well known that such nonsingular curves have only a finite group of automorphisms. It is to be expected that they belong to modular families, and indeed this is the case. Therefore, we can define an étale moduli topology by looking only at nonsingular curves and modular families. We make the definition in analogy to the scheme topology $X_{\text{ét}}$ rather than $X^*_{\text{ét}}$:

Definition. The moduli topology $\mathfrak{M}_{\text{ét}}$ is as follows:

a. Open sets are modular families of nonsingular curves, and a final object M.
b. Morphisms are morphisms of families of curves, and projections onto the final object M.
c. A collection of morphisms:

$$\begin{array}{ccc} \mathfrak{X}_\alpha & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ S_\alpha & \xrightarrow[g_\alpha]{} & S \end{array}$$

is a covering if $S = \bigcup_\alpha g_\alpha(S_\alpha)$; a collection of projections of families $\pi_\alpha : \mathfrak{X}_\alpha \to S_\alpha$ onto the final object M is a covering if every curve C occurs in one of the families π_α.

In the rest of this paper, this is the topology we will be interested in; therefore, we will refer to it simply as $\mathfrak{M}$, *rather than* $\mathfrak{M}_{\text{ét}}$.

It is, I think, a very important topology. At a future occasion, I hope to give some deeper results about this topology and compute some of its cohomology groups. For the present, I just want to mention a few nice facts about it:

a. The induced topology on an open set $\pi : \mathfrak{X} \longrightarrow S$ is equivalent to the étale topology $S_{\text{ét}}$ on S.
b. If $\pi : \mathfrak{X} \longrightarrow S$ is an open set, S is a nonsingular $3g - 3$-dimensional variety.
c. The so-called "higher level moduli schemes" form (for $n \geq 3$, n prime to the characteristic of k) modular families

$$\pi_n : \mathfrak{X}_n \longrightarrow S_n$$

each of which is, by itself, a covering over M. Moreover, $\text{Isom}(\pi_n, \pi_n)$ is a finite Galois covering of S_n.

§4. THE ELLIPTIC TOPOLOGY

The last topology that I want to define is the one which we shall study closely in §§6 and 7. It is essentially the topology $\mathfrak{M}_{\text{ét}}$ in the case $g = 1$, except that certain modifications are necessary to extend the definition given in §3 when $g \geq 2$. With this topology everything can be made very explicit, and hopefully the abstractness of all our definitions will be enlivened by this case. This topology is the classical proving ground for all notions of moduli, and, as such, it is found in various forms in hundreds of places.

The difficulty in using the definitions of §3 when $g = 1$ is that a nonsingular curve of genus 1 admits a structure of a group scheme, and therefore it has an infinite group of automorphisms. But by a minor modification, we can make everything go through. The key is to consider not curves, but *pointed* curves, that is, curves with a distinguished base point.

Definition. A nonsingular pointed curve of genus 1 is an "elliptic curve."

Definition. A "family of pointed curves" is a family of curves $\pi : \mathfrak{X} \longrightarrow S$ with a given section $\varepsilon : S \longrightarrow \mathfrak{X}$ (i. e., $\pi \circ \varepsilon = 1_s$). If $g = 1$, and the curves are nonsingular, this is called a "family of elliptic curves".

We can define a modular family of elliptic curves just as before.

Since modular families of elliptic curves do exist, it makes sense to state the next definition.

Definition. The topology $\mathfrak{M}$ is as follows:

a. Its open sets are modular families of elliptic curves, and a final object M,
b. Its morphisms are (étale) morphisms of families of elliptic curves, and projections of every open set to M.
c. Coverings of a family $\pi : \mathfrak{X} \to S$ are collections of morphisms of families:

$$\begin{array}{ccc} \mathfrak{X}_\alpha & \longrightarrow & \mathfrak{X} \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi} \\ S_\alpha & \xrightarrow{f_\alpha} & S \end{array}$$

such that $S = \bigcup_\alpha f_\alpha(S_\alpha)$; coverings of M are collections of projections of families π_α to M, provided that every elliptic curve occurs in one of the families π_α.

We want to describe this topology explicitly. First, we shall outline the basic facts about elliptic curves, and then indicate step by step, without complete proofs, how this leads to our final description. We shall assume from now on that the characteristic of k is not 2 or 3, so as to simplify the situation.

The basic fact is that elliptic curves are exactly the curves obtained as double coverings of the line ramified in four distinct points. Therefore, they are the curves C described by equations

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4).$$

Since the group of automorphisms acts transitively on the curve C, we can assume that the distinguished point e on C is the point $x = \alpha_4$, $y = 0$. By a projective transformation in the coordinate x, we can put α_4 at ∞, and the equation becomes:

$$y^2 = (x - \alpha_1')(x - \alpha_2')(x - \alpha_3'),$$

where e is now the unique point of this curve over $x = \infty$.

In the language of schemes, the conclusion is that every elliptic curve is isomorphic as *pointed* curve to the subscheme of $\mathbf{P}_2$ defined by homogeneous ideal

$$\mathfrak{A} = (X_2^2X_0 - (X_1 - \alpha_1'X_0)(X_1 - \alpha_2'X_0)(X_1 - \alpha_3'X_0))$$

together with the distinguished point

$$X_0 = 0;\ X_1 = 0;\ X_2 \neq 0.$$

It can be shown that this representation is essentially unique; in fact, the triple $(\alpha_1', \alpha_2', \alpha_3')$ is uniquely determined by the curve up to permutations and to affine substitutions of the form

$$\beta_i' = A\alpha_i' + B.$$

It follows easily from this that elliptic curves are *classified* by the number:

$$j = -64\left[\frac{(\lambda - 2)\cdot(2\lambda - 1)\cdot(\lambda + 1)}{\lambda\cdot(\lambda - 1)}\right]^2 \tag{1}$$

where $\lambda = \dfrac{\alpha_3' - \alpha_1'}{\alpha_2' - \alpha_1'}$.

Why is this? First, λ determines the triple $(\alpha_1', \alpha_2', \alpha_3')$ up to affine transformations. And, if we permute the α_i''s, λ is transformed into one of six numbers:

$$\lambda,\ 1 - \lambda,\ \frac{1}{\lambda},\ \frac{\lambda - 1}{\gamma},\ \frac{1}{1 - \lambda},\ \frac{\lambda}{\lambda - 1}.$$

Also, the values $\lambda = 0$ and $\lambda = 1$ are excluded, since the three numbers α_1', α_2', and α_3' are distinct. It can be checked that j is unchanged by any of these substitutions in λ, and, conversely, that only λ's related by these substitutions give the same j. The factor -64 arose historically, and turns out to be crucial if we specialize to characteristic 2. In characteristics other than 2, it is obviously harmless!

How about automorphisms of elliptic curves as *pointed* curves? Every elliptic curve C obviously possesses the automorphism

$$\begin{aligned} x &\longrightarrow x \\ y &\longrightarrow -y \end{aligned}$$

corresponding to its being a double covering of the x-line. We will call this the *inversion* ρ of C. A very important fact is that if $\pi : \mathfrak{X} \longrightarrow S$, $\varepsilon : S \longrightarrow \mathfrak{X}$ is any family of elliptic curves, then the inversions of all the fibres piece together to an automorphism $P : \mathfrak{X} \longrightarrow \mathfrak{X}$, of the family π; we will also call this the inversion of π. A related fact is that ρ commutes with any other automorphism α of C. Since $k(x)$ is the field of functions on C fixed by the inversion, such an α will take $k(x)$ into itself; that is, it will be given by a projective transformation in x. Also since α leaves e fixed, it leaves $x = \infty$ fixed; and it must permute the other three branch points $\alpha_1', \alpha_2', \alpha_3'$. It is now an elementary result that such an α occurs only in two cases:

a. $j = 0$; $\lambda = 2, \frac{1}{2}$, or -1; $\alpha_1', \alpha_2', \alpha_3'$ of the form $\beta, \beta + \gamma, \beta + 2\gamma$.
b. $j = 12^3$; $\lambda = -\omega$ or $-\omega^2$ (ω a cube root of 1); $\alpha_1', \alpha_2', \alpha_3'$ of the form $\beta + \gamma, \beta + \omega\gamma, \beta + \omega^2\gamma$.

Now normalizing the first case by choosing $\alpha_1' = -1$, $\alpha_2' = 0$, $\alpha_3' = 1$, we find that C possesses the automorphism σ of order 4:

$$\begin{aligned} x^\sigma &= -x \\ y^\sigma &= i \cdot y \end{aligned}$$

such that σ^2 is the inversion. Normalizing the second case by choosing $\alpha_1' = 1$, $\alpha_2' = \omega$, $\alpha_3' = \omega^2$, we find that C possesses the automorphism τ of order 6:

$$\begin{aligned} x^\tau &= \omega \cdot x \\ y^\tau &= -y \end{aligned}$$

such that τ^3 is the inversion. These are the only automorphisms.

Now, what about modular families. Since only one parameter j is involved, it is natural to expect that modular families are always parametrized by nonsingular curves S. This is true. The most natural thing would be to look for a modular family parametrized by j itself. The following is an example of such a family:

$$y^2 = x^3 + A \cdot (x + 1)$$

where

$$A = \frac{27}{4} \cdot \frac{12^3 - j}{j}.$$

We check that if $j \neq 0, 12^3$, then A is finite and the roots of $x^3 + A(x + 1)$ are all distinct—so we have an elliptic curve. And we can compute its j-invariant in an elementary way, and it is the j we had at the start.

In the language of schemes, let

$$\begin{aligned} \mathbf{A}_j &= \operatorname{Spec} k[j] \\ S &= \mathbf{A}_j - (0, 12^3), \end{aligned}$$

and let $\mathfrak{X}$ be the closed subscheme of $\mathbf{P}_2 \times S$ defined by the vanishing of the section

$$X_2^2 \cdot X_0 - X_1^3 - \frac{27}{4} \cdot \frac{12^3 - j}{j} \cdot (X_1 X_0^2 + X_0^3)$$

of the sheaf $\mathit{o}(3)$. Let ϵ be the morphism

$$S \xrightarrow{\sim} (0, 0, 1) \times S \subset \mathfrak{X}.$$

Then a rigorous analysis of the infinitesimal deformations of an elliptic curve shows that this is a modular family.

Can we extend this family π to cover the points $j = 0$ and 12^3? For the value $j = 0$, A is infinite; and for $j = 12^3$, our equation degenerates. But even a priori it is clear that there has to be trouble. If π is a modular family, then $\mathrm{Isom}(\pi, \pi)$ must be étale over S. Now for each closed point $t \in S$, the closed points of $\mathrm{Isom}(\pi, \pi)$ over t stand for: (a) closed points $t' \in S$ such that $\pi^{-1}(t)$ and $\pi^{-1}(t')$ are isomorphic, plus (b) isomorphisms of $\pi^{-1}(t)$ and $\pi^{-1}(t')$. If S is an open set in the j-line, $\pi^{-1}(t)$ and $\pi^{-1}(t')$ can never be isomorphic unless $t = t'$. Therefore, the number of points in $\mathrm{Isom}(\pi, \pi)$ over t equals the order of the group of automorphisms of $\pi^{-1}(t)$. For $j \neq 0$, 12^3, this is 2, so $\mathrm{Isom}(\pi, \pi)$ is a double covering of S; and $\mathrm{Isom}(\pi, \pi)$ could not have four or six points over $j = 0$ or $j = 12^3$. The real problem here is that j is not the "right" parameter at $j = 0$ and 12^3. At $j = 0$, $\sqrt{j}$ or something analytically equivalent is needed; at $j = 12^3$, $\sqrt[3]{j - 12^3}$ is needed. This works out as follows. Let $\pi : \mathfrak{X} \to S$ be *any* modular family. In particular, S is a nonsingular curve. Suppose we define a function on the closed points of S by assigning to the point $s \in S$ the j-invariant of the curve $\pi^{-1}(s)$. It can be proven that this function is a morphism:

$$S \xrightarrow{j} \mathbf{A}_j.$$

We can then prove the following.

4.1.

Each component of S dominates $\mathbf{A}_j$ and the ramification index of j at a closed point $s \in S$ is 1, 2, or 3 according to whether $j(s) \neq 0$ and 12^3, $j(s) = 0$, or $j(s) = 12^3$.

We now want to return to the problem of giving an explicit description of the topology $\mathfrak{M}$. The morphism j is one invariant which we can attach to the family $\pi : \mathfrak{X} \to S$. Unfortunately, a given j may correspond to more than one family π. A second invariant is needed. The key is to use more strongly the particular modular family over $\mathbf{A}_j - (0, 12^3)$ which we have constructed. With this as a reference point, so to speak, we will get the second invariant. Let $\pi_0 : \mathfrak{X}_0 \to S_0$ denote this one family. We use the diagram:

$$\begin{tikzcd}
& \mathrm{Isom}(\pi, \pi_0) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
S \arrow[dr, "j"'] & & S_0 \arrow[dl, hook'] \\
& \mathbf{A}_j &
\end{tikzcd}$$

The first thing to notice is that this is commutative: let t be a closed point of $\mathrm{Isom}(\pi, \pi_0)$. If $s = p_1(t)$ and $s_0 = p_2(t)$, then t represents an isomorphism of $\pi^{-1}(s)$ and $\pi_0^{-1}(s_0)$. Therefore, $\pi^{-1}(s)$ and $\pi_0^{-1}(s_0)$ have the same j-invariant, that is, $p_1(t)$ and $p_2(t)$ have the same image in $\mathbf{A}_j$.

Now what is $\mathrm{Isom}(\pi, \pi_0)$? Over a closed point $s \in S$, its points represent isomorphisms of $\pi^{-1}(s)$ with curves $\pi_0^{-1}(s_0)$, $s_0 \in S_0$. In other words, $\mathrm{Isom}(\pi, \pi_0)$ has no points over s if $j(s) = 0$ or 12^3; two points otherwise. $\mathrm{Isom}(\pi, \pi_0)$ is a double étale covering of the open set:

$$j^{-1}(S_0) \subset S.$$

This covering extends uniquely to a covering I of all of S (not necessarily étale!)†. The covering I/S is the second invariant. I claim that j and I/S determine the modular family π uniquely.

Indication of Proof. The first step is to check that there is at most

† By a double covering T/S, I mean a second nonsingular curve T, and a finite, flat, surjective morphism $f : T \to S$ of degree 2, étale over an open dense subset $S' \subset S$. Now either Isom is the disjoint union of two copies of $j^{-1}(S_0)$; and then I is the disjoint union of two copies of S; or Isom is the normalization of $j^{-1}(S_0)$ in a quadratic extension of its function field, and then I is the normalization of S in this field.

one family $\mathfrak{X}/S$ extending the restriction of this family to the open subset $j^{-1}(S_0)$. After this, we may assume $j(S) \subset S_0$. Let $\mathfrak{Y}$ be the given family of elliptic curves over $I = \mathrm{Isom}(\pi, \pi_0)$. Then we have a diagram of morphisms of families:

$$\begin{tikzcd} & \mathfrak{Y} \arrow[dl] \arrow[dd] \arrow[dr] & \\ \mathfrak{X} \arrow[dd, "\pi"'] & & \mathfrak{X}_0 \arrow[dd, "\pi_0"] \\ & I \arrow[dl] \arrow[dr] & \\ S & & S_0 \end{tikzcd}$$

The family $\mathfrak{Y}/I$ is determined by j and I/S, because it is just the family induced over I by the base extension

$$I \longrightarrow S \xrightarrow{j} S_0$$

from the standard family $\mathfrak{X}_0$. On the other hand, $\mathfrak{Y}$ is also induced from $\mathfrak{X}$ via the double étale covering I/S. Therefore, $\mathfrak{Y}$ is a double étale covering of $\mathfrak{X}$. We could recover $\mathfrak{X}$ from $\mathfrak{Y}$ if we knew the involution ι of $\mathfrak{Y}$ interchanging the two sheets of this covering. But let P_0 be the inversion of the family π_0: this is an involution of $\mathfrak{X}_0$ over S_0. Let $\bar{\iota}$ be the involution of I corresponding to the covering I/S: this is an automorphism of I over S_0 too. Since the diagram sets up an identification

$$\mathfrak{Y} = I \times_{S_0} \mathfrak{X}_0,$$

$\bar{\iota}$ and P_0 induce an involution $\bar{\iota} \times P_0$ of $\mathfrak{Y}$. We check that $\iota = \bar{\iota} \times P_0$.

Q.E.D.

The next question is whether there are any restrictions on j and I/S for these to come from a modular family. Besides the restriction (4.1) on j mentioned above, it turns out that the following is the only further restriction.

4.2.
I is ramified over all points s of S where $j(s) = 0$ or 12^3.

Turning all this around, we can make it into a second definition of the topology $\mathfrak{M}$:

Definition. The topology $\mathfrak{M}$ is as follows:

a. Its open sets are morphisms j of nonsingular curves S to $\mathbf{A}_j$ satisfying restriction (4.1), plus double coverings I/S satisfying restriction (4.2); and a final object M.

b. Its morphisms are commutative diagrams:

$$\begin{tikzcd} I_1 \arrow[r] \arrow[d] & I_2 \arrow[d] \\ S_1 \arrow[r] \arrow[dr, "j_1"'] & S_2 \arrow[d, "j_2"] \\ & \mathbf{A}_j \end{tikzcd}$$

making I_1 into the fibre product $S_1 \times_{S_2} I_2$; and projections of every open set onto M,

c. Coverings of $(j, I/S)$ are collections of morphisms

$$\begin{tikzcd} I_\alpha \arrow[r] \arrow[d] & I \arrow[d] \\ S_\alpha \arrow[r, "f_\alpha"] & S \end{tikzcd}$$

such that $S = \bigcup_\alpha f_\alpha(S_\alpha)$; coverings of M are collections of projections of open sets $(j_\alpha, I_\alpha/S_\alpha)$ onto M such that $\mathbf{A}_j = \bigcup_\alpha j_\alpha(S_\alpha)$.

Note that, because of restriction (4.1), given a morphism of open sets:

$$\begin{tikzcd} I_1 \arrow[r] \arrow[d] & I_2 \arrow[d] \\ S_1 \arrow[r, "f"] & S_2 \end{tikzcd}$$

the morphism f is necessarily étale.

Let us work out (absolute) products in these terms to see how it all fits together. Say $(j_1, I_1/S_1)$ and $(j_2, I_2/S_2)$ are two open sets. Suppose we want to map a third open set $(j, I/S)$ to both:

$$
\begin{tikzcd}
 & I \arrow[dl] \arrow[d] \arrow[dr] & \\
I_1 \arrow[d] & S \arrow[dl] \arrow[dr] \arrow[dd, "j"] & I_2 \arrow[d] \\
S_1 \arrow[dr, "j_1"'] & & S_2 \arrow[dl, "j_2"] \\
 & \mathbf{A}_j &
\end{tikzcd}
$$

Then first we get a morphism f_1 from S to $S_1 \times_{\mathbf{A}} S_2$. But S is non-singular, and f maps each component of S to an open subset of $S_1 \times_{\mathbf{A}} S_2$; therefore, f_1 factors through the normalization of $S_1 \times_{\mathbf{A}} S_2$. Denote this normalization by T, and let $f_2 : S \to T$ be the morphism that factors f_1. Let I_1' and I_2' be the double coverings of T induced by I_1/S_1 and I_2/S_2. Then pulling these double coverings all the way back to S, we get isomorphisms of both with I/S, hence an isomorphism between them. Exactly as in §3, we get a factorization of f_2 via $S \xrightarrow{f_3} \mathrm{Isom}_T(I_1', I_2')$. But what is this Isom? At points where I_1' and I_2' are unramified, it is just the "quotient" double covering; that is, if I_1' is defined by extracting $\sqrt{f_1}$, and I_2' by $\sqrt{f_2}$, then Isom is the double covering given by $\sqrt{f_1/f_2}$. Since I_1' and I_2' are ramified over exactly the same points of T, this "quotient" covering extends to an étale double covering I_3' over all of T. It turns out that I_3' is a closed subscheme of $\mathrm{Isom}_T(I_1', I_2')$ and f_3 factors via

$$S \xrightarrow{f_4} I_3'.$$

This I_3' is the S of the product open set. Over I_3', I_1 and I_2 can be canonically identified to the I of the product open set.

§5. THE PICARD GROUPS

Now we come to the Picard groups, which are one of the interesting invariants of our topologies. There are two quite different ways to define these groups. One is a direct method going back to the moduli problem itself; the other is a cohomological method using our topologies. We will first explain the direct method:

Fix, as before, the genus g.

Definition. An "invertible sheaf" L (on the moduli problem itself) consists in two sets of data:

a. For all families of nonsingular curves (of genus g) $\pi : \mathfrak{X} \to S$, an invertible sheaf $L(\pi)$ on S.
b. For all morphisms F between such families:

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "f"] & S_2 \end{tikzcd}$$

an isomorphism $L(F)$ of $L(\pi_1)$ and $f^*(L(\pi_2))$.†

The second set of data should satisfy a compatibility condition with respect to composition of morphisms:

Let

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_2 \arrow[r] \arrow[d, "\pi_2"'] & \mathfrak{X}_3 \arrow[d, "\pi_3"'] \\ S_1 \arrow[r, "f"'] & S_2 \arrow[r, "g"'] & S_3 \end{tikzcd}$$

be a composition of the morphism F from π_1 to π_2, and G from π_2 to π_3. Then the diagram:

$$\begin{tikzcd} & f^*(L(\pi_2)) \arrow[dr, "f^*(L(G))"] & \\ L(\pi_1) \arrow[ur, "L(F)"] \arrow[dr, "L(G \circ F)"'] & & f^*(g^*(L(\pi_3))) \\ & (g \circ f)^*(L(\pi_3)) \arrow[ur, equal] & \end{tikzcd}$$

should commute.

This definition has an obvious translation into the language of fibred categories, which is left to the reader who has a taste for that approach. Loosely speaking, an invertible sheaf is simply a pro-

† Note that the morphism F is the whole diagram, while f is simply the morphism from S_1 to S_2. In the sequel, we will denote morphisms of families by capital letters and the component morphism of base spaces by the same small letters.

cedure for attaching canonically a one-dimensional vector space to every nonsingular curve of genus g: Start with L as above. If C/k is such a curve, let

$$\pi : C \longrightarrow \mathrm{Spec}(k)$$

be the projection. Then $L(\pi)$ is a one-dimensional vector space (over k) attached to C. Conversely, if this procedure is "canonical" enough, then given a family $\pi : \mathfrak{X} \longrightarrow S$, the one-dimensional vector spaces attached to the curves $\pi^{-1}(s)$ ($s \in S_k$) should form a line bundle over S; and its sections then form an invertible sheaf $L(\pi)$.

Example. Given any $\pi : \mathfrak{X} \longrightarrow S$ as above, let

$$E(\pi) = R^1\pi_*(o_X).$$

This is known to be a locally free sheaf on S of rank g. Let

$$L(\pi) = \Lambda^g\{R^1\pi_*(o_X)\}.$$

This is an invertible sheaf on S. Moreover, for all morphisms of families:

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "f"] & S_2 \end{tikzcd}$$

there is a canonical identification of $E(\pi_1)$ and $f^*(E(\pi_2))$, hence of $L(\pi_1)$ and $f^*(L(\pi_2))$. This is, therefore, an invertible sheaf on the moduli problem. It corresponds to attaching to each curve C the one-dimensional vector space

$$\Lambda^g\{H^1(C, o_C)\}.$$

It is clear what is meant by an isomorphism of two invertible sheaves.

Definition. The set of isomorphism classes of such invertible sheaves is called $\mathrm{Pic}(\mathfrak{M})$.

As usual, $\mathrm{Pic}(\mathfrak{M})$ is an abelian group. Given L and M, two invertible sheaves, define $L \otimes M$ by

$$\begin{aligned} (L \otimes M)(\pi) &= L(\pi) \otimes M(\pi) \\ (L \otimes M)(F) &= L(F) \otimes M(F). \end{aligned}$$

This induces the product on the set of isomorphism classes $\mathrm{Pic}(\mathfrak{M})$.

Now we give the second definition of $\mathrm{Pic}(\mathfrak{M})$. Recall that, by definition, a scheme X is a particular type of topological space, together with a given sheaf of rings $\boldsymbol{o}_X$. Now that we have generalized the concept of a topological space, it is clear that an important type of object to look at will be a topology T, together with a given sheaf of rings $\boldsymbol{o}_T$. This combination is known as a "site." For example, if X is a scheme it is not only the Zariski topology which comes with the sheaf of rings $\boldsymbol{o}_X$. Recall the five topologies on X and the continuous maps:

$$X^*_{\text{flat}} \longrightarrow X^*_{\text{smooth}} \longrightarrow X^*_{\text{ét}} \longrightarrow X_{\text{ét}} \longrightarrow X_{\text{Zar}}.$$

Let $\pi : U \longrightarrow X$ be a morphism, that is, an open set in X^*_{flat}. Then define† a sheaf $\boldsymbol{o}$ on X_{flat} by

$$\boldsymbol{o}(U \longrightarrow X) = \Gamma(U, \boldsymbol{o}_U).$$

By taking direct images, this also defines a sheaf $\boldsymbol{o}$ on X^*_{smooth}, $X^*_{\text{ét}}$, $X_{\text{ét}}$ and X_{Zar}; on X_{Zar} this is just the original sheaf $\boldsymbol{o}_X$. Thus each of these topologies is a site.

What is more important now is that the topologies $\mathfrak{M}$ are also sites. Let $\pi : \mathfrak{X} \longrightarrow S$ be an open set in $\mathfrak{M}$, that is, a modular family of nonsingular curves. Let

$$\boldsymbol{o}(X \xrightarrow{\pi} S) = \Gamma(S, \boldsymbol{o}_S).$$

This defines a sheaf of rings $\boldsymbol{o}$ on $\mathfrak{M}$, except for the ring $\boldsymbol{o}(M)$: this is simply determined by the sheaf axiom. Fix a covering of $\mathfrak{M}$ by open sets $\{\mathfrak{X}_\alpha \xrightarrow{\pi_\alpha} S_\alpha\}$. Let the product of π_α and π_β be the open set

$$\mathfrak{X}_{\alpha,\beta} \xrightarrow{\pi_{\alpha,\beta}} S_{\alpha,\beta}.$$

Then $\boldsymbol{o}(M)$ is the kernel of the usual homomorphism

$$\prod_\alpha \boldsymbol{o}(\mathfrak{X}_\alpha \longrightarrow S_\alpha) \longrightarrow \prod_{\alpha,\beta} \boldsymbol{o}(\mathfrak{X}_{\alpha,\beta} \longrightarrow S_{\alpha,\beta}).$$

In fact, if $g \geq 3$, it is known that $\boldsymbol{o}(M)$ is just k. In any case, this defines $\boldsymbol{o}$, and it brings $\mathfrak{M}$ into a familiar context: we can now develop a theory of coherent sheaves, and their cohomology on $\mathfrak{M}$,

† This is not obviously a sheaf; it is so as a consequence of the theory of descent (cf. [3], exposé 8).

as well as a general theory of (étale) cohomology. Moreover, in addition to $\boldsymbol{o}$ we get the two auxiliary sheaves:

a. $\boldsymbol{o}^*$, defined by $\boldsymbol{o}^*(\mathfrak{X} \xrightarrow{\pi} S)$ = group of units in $\boldsymbol{o}(\mathfrak{X} \xrightarrow{\pi} S)$.
b. $\boldsymbol{K}$, defined as the sheaf associated to the presheaf $\tilde{\boldsymbol{K}}(\mathfrak{X} \xrightarrow{\pi} S)$ = total quotient ring of $\boldsymbol{o}(\mathfrak{X} \xrightarrow{\pi} S)$.

The ring of global sections of $\boldsymbol{K}$ is, so to speak, the function field of the moduli problem. Now the second definition of $\mathrm{Pic}(\mathfrak{M})$ is simply the cohomology group:

$$H^1(\mathfrak{M}, \boldsymbol{o}^*).$$

Sketch of Proof of Isomorphism. The first thing to do is to set up a map between these groups. The map goes like this: let L be an invertible sheaf on the moduli problem. Then we will associate to L an element:

$$\lambda \in H^1(\mathfrak{M}, \boldsymbol{o}^*).$$

First choose any collection of families $\pi_\alpha : \mathfrak{X}_\alpha \to S_\alpha$ which is a covering of the final object M. Then $L(\pi_\alpha)$ is an invertible sheaf on S_α. By replacing S_α with a suitable set of (Zariski) open subsets and replacing π_α by the set of induced families over these subsets, we can assume that for each α there is an isomorphism:

$$L(\pi_\alpha) \underset{\phi_\alpha}{\xrightarrow{\sim}} \boldsymbol{o}_{S_\alpha}$$

For each α, choose such an isomorphism. For all α, β, let

$$\pi_{\alpha,\beta} : (\mathfrak{X}_\alpha, \mathfrak{X}_\beta) \longrightarrow \mathrm{Isom}(\pi_\alpha, \pi_\beta) = I_{\alpha,\beta}$$

be the product of the families π_α and π_β. Let p_1 and p_2 denote the projections of $\mathrm{Isom}(\pi_\alpha, \pi_\beta)$ onto S_α and S_β. By definition of an invertible sheaf, we are given isomorphisms of $p_1^*(L(\pi_\alpha))$ and $p_2^*(L(\pi_\beta))$ with $L(\pi_{\alpha,\beta})$. Now look at the composite isomorphism:

$$\begin{aligned} \boldsymbol{o}_{I_{\alpha,\beta}} &= p_1^*(\boldsymbol{o}_{S_\alpha}) \\ &\xrightarrow{\sim} p_1^*(L(\pi_\alpha)) && \text{via } \phi_\alpha \\ &\cong L(\pi_{\alpha,\beta}) \\ &\cong p_2^*(L(\pi_\beta)) \\ &\xleftarrow{\sim} p_2^*(\boldsymbol{o}_{S_\beta}) && \text{via } \phi_\beta \\ &= \boldsymbol{o}_{I_{\alpha,\beta}}. \end{aligned}$$

This isomorphism is set up by multiplication by a unit:

$$\sigma_{\alpha,\beta} \in \Gamma(I_{\alpha,\beta}, o^*_{I_{\alpha,\beta}}) = o^*(\pi_{\alpha,\beta}).$$

I claim that, for the covering $\{\pi_\alpha\}$ of $\mathfrak{M}$, $\{\sigma_{\alpha,\beta}\}$ forms a 1-Czech cocycle in the sheaf o^*. This is checked using the compatibility property for the invertible sheaf L (cf., last part of the definition of an invertible sheaf). Then this cocycle induces an element λ_1 in the first Czech cohomology group for this covering, hence an element λ_2 of $H^1(\mathfrak{M}, o^*)$.

Now suppose the isomorphisms ϕ_α are varied? The only possible change is to replace ϕ_α by $\phi'_\alpha = \sigma_\alpha \cdot \phi_\alpha$, where σ_α means multiplication by the unit:

$$\sigma_\alpha \in \Gamma(S_\alpha, o^*_{S_\alpha}) = o^*(\pi_\alpha)$$

But then $\sigma_{\alpha,\beta}$ is replaced by the homologous cocycle:

$$\sigma'_{\alpha,\beta} = p_1^*(\sigma_\alpha) \cdot p_2^*(\sigma_\beta^{-1}) \cdot \sigma_{\alpha,\beta}.$$

Therefore even λ_1 is unaltered. Now suppose the covering $\{\pi_\alpha\}$ is changed. Any two coverings are dominated by a finer covering, so we can assume that the new covering is finer. It is immediate that the new λ_1 is just the element of the new Czech cohomology group induced by the old λ_1 under restriction. Therefore, λ_2 is unaltered.

This defines a map from $\mathrm{Pic}_1(\mathfrak{M})$ (the first group) to $\mathrm{Pic}_2(\mathfrak{M})$ (the second group). To show that this is a surjective, we first use the fact that (for any sheaf F),

$$H^1(\mathfrak{M}, F) = \varinjlim_{\text{coverings } \mathfrak{A}} H^1(\mathfrak{A}, F)$$

where $H^1(\mathfrak{A}, -)$ denotes the first Czech cohomology group for the covering $\mathfrak{A}$. Now suppose $\lambda_2 \in H^1(\mathfrak{M}, o^*)$ is given. Then λ_2 is induced by a $\lambda_1 \in H^1(\mathfrak{A}, o^*)$ for some covering $\mathfrak{A}$. And λ_1 is defined by some cocycle $\{\sigma_{\alpha,\beta}\}$ in o^*, (if $\mathfrak{A}$ is the covering $\pi_\alpha : \mathfrak{X}_\alpha \to S_\alpha$). Now suppose $\pi : \mathfrak{X} \to S$ is any modular family of nonsingular curves. Then

$$\{I_\alpha = \mathrm{Isom}(\pi, \pi_\alpha) \to S\}$$

is an étale covering of S. Moreover, via the natural projection

$$I_\alpha \times_S I_\beta \longrightarrow \mathrm{Isom}(\pi_\alpha, \pi_\beta),$$

the cocycle $\{\sigma_{\alpha,\beta}\}$ induces a cocycle $\{\tau_{\alpha,\beta}\}$ for the covering $\{I_\alpha \longrightarrow S\}$ of S and the sheaf $\mathbf{o}_S^*$. We then require a theorem of Grothendieck:

Theorem 90 (Hilbert-Grothendieck). Let $\{U_\alpha \xrightarrow{q_\alpha} X\}$ be a flat covering of X; for all α, let L_α be an invertible sheaf on U_α; and for all α, β, let $\phi_{\alpha,\beta}$ be an isomorphism on $U_\alpha \times_X U_\beta$ of the sheaves $p_1^*(L_\alpha)$ and $p_2^*(L_\beta)$. Assume an obvious compatibility of isomorphisms on $U_\alpha \times_X U_\beta \times_X U_\gamma$ (for all α, β, γ). Then there is an invertible sheaf L on X, and for all α, isomorphisms ψ_α on U_α of L_α and $q_\alpha^*(L)$ such that, on $U_\alpha \times_X U_\beta$, the diagram:

$$\begin{tikzcd}
p_1^*(L_\alpha) \arrow[r, "\sim", "\phi_{\alpha,\beta}"'] \arrow[d, "p_1^*(\psi_\alpha)"', "\wr"] & p_2^*(L_\beta) \arrow[d, "p_2^*(\psi_\alpha)", "\wr"'] \\
p_1^*(q_\alpha^*(L)) \arrow[r, equal] & p_2^*(q_\beta^*(L))
\end{tikzcd}$$

commutes. Moreover, L and ψ_α are uniquely determined, up to canonical isomorphisms. (cf. [3], exposé 8, Theorem 1.1).

There is a shorthand which is used in connection with this theorem: given the L_α, the isomorphisms $\{\phi_{\alpha,\beta}\}$ are called "descent data" for $\{L_\alpha\}$. The L obtained is said to be gotten by "descending" the sheaves L_α to X (that is, reversing the process, the L_α are gotten by lifting L to U_α).

Apply this theorem with $U_\alpha = I_\alpha$, $X = S$, $L_\alpha = \mathbf{o}_{I_\alpha}$, and $\phi_{\alpha,\beta}$ given by $\sigma_{\alpha,\beta}$. The L constructed is to be our $L(\pi)$. We leave it to the reader to construct the isomorphisms $L(F)$ required for an invertible sheaf; and to check that this L does induce λ_2 when the process is reversed.

Finally, why is the map injective? If λ_2 were 0, then for a suitable covering λ_1 would be 0, and for suitable choices of the ϕ_α's, the cocycle $\sigma_{\alpha,\beta}$ itself would come out 1. The question is then, if $\sigma_{\alpha,\beta} = 1$ for all α, β show that $L = \mathbf{o}$ (the trivial invertible sheaf). What we need to do is to construct, for every family $\pi : \mathfrak{X} \longrightarrow S$, an isomorphism

$$\psi(\pi) : L(\pi) \xrightarrow{\sim} \mathbf{o}_S,$$

such that, for every morphism F of families:

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r] \arrow[d, "\pi_1"'] & \mathfrak{X}_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "f"] & S_2 \end{tikzcd}$$

the diagram:

$$\begin{tikzcd} L(\pi_1) \arrow[r, "L(F)", "\sim"'] \arrow[d, "\psi(\pi_1)", "\wr"'] & f^*(L(\pi_2)) \arrow[d, "f^*(\psi(\pi_2))", "\wr"'] \\ o_{S_1} \arrow[r, equal] & f^*(o_{S_2}) \end{tikzcd}$$

commutes. Exactly as before, we use the induced étale covering

$$\{I_\alpha = \mathrm{Isom}(\pi, \pi_\alpha) \xrightarrow{q_\alpha} S\}.$$

The family of curves $\mathcal{Y}_\alpha$ over I_α induced via q_α from $\mathfrak{X}/S$ is canonically isomorphic to the family induced from $\mathfrak{X}_\alpha/S_\alpha$. But we are given an isomorphism of $L(\pi_\alpha)$ and o_{S_α}. This induces an isomorphism of $L(\mathcal{Y}_\alpha/I_\alpha)$ and o_{I_α}; hence an isomorphism

$$q_\alpha^*(L(\pi)) \underset{\psi_\alpha}{\xrightarrow{\sim}} o_{I_\alpha}.$$

The fact that $\sigma_{\alpha,\beta} = 1$ can be easily seen to imply that the diagram of sheaves on $I_\alpha \times_S I_\beta$:

$$\begin{tikzcd} p_1^*(q_\alpha^*(L(\pi))) \arrow[r, "p_1^*(\psi_\alpha)", "\sim"'] \arrow[d, equal] & p_1^*(o_{I_\alpha}) \arrow[d, equal] \\ p_2^*(q_\beta^*(L(\pi))) \arrow[r, "\sim", "p_2^*(\psi_\beta)"'] & p_2^*(o_{I_\beta}) \end{tikzcd}$$

commutes. In order words, both L and o_S satisfy the conclusions of Theorem 90 for the setup $U_\alpha = I_\alpha$, $X = S$, $L_\alpha = o_{I_\alpha}$, and $\phi_{\alpha,\beta} = 1$. Therefore, the uniqueness half of that theorem states that there is a canonical isomorphism of L and o_S. This is to be $\psi(\pi)$. We omit the rest of the details.

§6. COMPUTATIONS: DIRECT METHOD

We return to the case $g = 1$, and its topology $\mathfrak{M}$. In this section, for $\mathrm{char}(k) \neq 2, 3$, we shall give a direct computation of $\mathrm{Pic}(\mathfrak{M})$. In the next section, for $k = \mathbf{C}$, we shall give a transcendental computation of this same group.

Let L be an invertible sheaf on the moduli problem. First of all, let us try to extract some numerical invariants directly from L. Start with a family of curves $\pi : \mathfrak{X} \to S$. Any such family has one nontrivial automorphism: the inversion ρ of order 2. By definition of an invertible sheaf, the morphism of families:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\rho} & \mathfrak{X} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow[1_S]{} & S \end{array}$$

induces an automorphism $L(\rho)$ of $L(\pi)$. Since ρ has order 2, so does $L(\rho)$. But $L(\rho)$, as any automorphism of an invertible sheaf, is given by multiplication by an element $\alpha \in \Gamma(S, o_S^*)$. Therefore, $\alpha^2 = 1$; hence on each connected component S_i of S, α equals $+1$ or -1. In particular, suppose $S = \mathrm{Spec}(k)$, and $\mathfrak{X} = C$ is an elliptic curve. Then we have defined a number:

$$\alpha(C) = \pm 1.$$

Moreover, if $\pi : \mathfrak{X} \to S$ is any family, then the fact that the inversion ρ for π induces the inversion on each fibre $\pi^{-1}(s)$ of the family implies that the function $\alpha \in \Gamma(S, o_S^*)$ has value $\alpha(\pi^{-1}(s))$ at the point $s \in S$. This shows that α is a "continuous" function of C; that is, if we have a family $\pi : \mathfrak{X} \to S$ with connected base S, then α is constant on the set of curves $\pi^{-1}(s)$ occuring as fibres in the family π. Actually this shows that α is constant on all curves; either $\alpha(C) = +1$ for all C, or $\alpha(C) = -1$. Namely, it is easy to exhibit a family π with connected base S, such that every C occurs in π. For example, take the family of all nonsingular cubic curves; or take the modular family of cubic curves

$$y^2 = x(x - 1)(x - \lambda),$$

where $\lambda \neq 0, 1, \infty$. Therefore, in fact, we have defined one number $\alpha(L)$ equal to ± 1. And, quite clearly, this gives a homomorphism

$$\mathrm{Pic}(\mathfrak{M}) \xrightarrow{\alpha} \mathbf{Z}/2.$$

In fact, this same method goes further. After all, there are two elliptic curves with bigger groups of automorphisms. Let C_A be the curve with a group of automorphisms of order 4 (i. e., $j = 0$); let C_B be the curve with a group of order 6 (i. e., $j = 12^3$). Pick generators σ and τ of $\mathrm{Aut}(C_A)$ and $\mathrm{Aut}(C_B)$. Note that σ^2 is the inversion of C_A and τ^3 is the inversion of C_B. Let

$$\pi_A : C_A \longrightarrow \mathrm{Spec}(k)$$
$$\pi_B : C_B \longrightarrow \mathrm{Spec}(k)$$

be the trivial families. Then L gives us one-dimensional vector spaces $L(\pi_A)$ and $L(\pi_B)$, *and* L gives us an action of $\mathrm{Aut}(C_A)$ on $L(\pi_A)$ and of $\mathrm{Aut}(C_B)$ on $L(\pi_B)$. In particular, σ acts on $L(\pi_A)$ by multiplication by a fourth root of 1: call it $L(\sigma)$; and τ acts on $L(\pi_B)$ by multiplication of a sixth root of 1: call it $L(\tau)$. Clearly,

$$L(\sigma)^2 = \alpha(L)$$
$$L(\tau)^3 = \alpha(L).$$

If we also fix a primitive twelfth root of 1, ζ, then we can determine uniquely an integer β mod 12 by the equations:

$$\zeta^{6\beta} = \alpha(L);\ \zeta^{3\beta} = L(\sigma);\ \zeta^{2\beta} = L(\tau).$$

Then this associates an invariant $\beta(L) \in \mathbf{Z}/12$ to each invertible sheaf L. It is easy to see that this is a homomorphism:

$$\mathrm{Pic}(\mathfrak{M}) \xrightarrow{\beta} \mathbf{Z}/12.$$

Actually, β is not quite as nice as α, in that to define β we had to make three arbitrary choices, namely, σ, τ, and ζ. Our next step is to simultaneously make β more canonical and to prove that β is surjective. Recall the invertible sheaf Λ on $\mathfrak{M}$ given as an example in §3:

$$\Lambda(\mathfrak{X} \xrightarrow{\pi} S) = R^1\pi_*(o_{\mathfrak{X}})$$

(with the obvious compatibility morphisms for each morphism of families). The interesting fact is that $\beta(\Lambda)$ is a generator of $\mathbf{Z}/12$. To verify this, all we have to check is that $\Lambda(\sigma)$ [resp. $\Lambda(\tau)$] is a *primitive* fourth root (resp. a sixth root) of 1. But this means simply that $\mathrm{Aut}(C_A)$ [resp. $\mathrm{Aut}(C_B)$] acts faithfully on $\Lambda(\pi_\alpha)$ [resp. $\Lambda(\pi_B)$]. Now by definition:

$$\begin{aligned} \Lambda(\pi_A) &= H^1(C_A, o_{C_A}) \\ \Lambda(\pi_B) &= H^1(C_B, o_{C_B}). \end{aligned}$$

We could say, at this point, that it is a classical fact that these actions are faithful. But this is not hard to check:

Proof of Faithfulness.

a. By Serre duality, for any curve C, $H^1(C, o_C)$ is canonically dual to the vector space of regular differentials on C.

b. If C is the elliptic curve:

$$y^2 = x^3 + Ax + B,$$

then the differential dx/y is regular, and is a basis of the space of such differentials.

c. C_A is the curve

$$y^2 = x^3 - x = x(x+1)(x-1)$$

and σ may be taken to be

$$\begin{aligned} x &\longmapsto -x \\ y &\longmapsto iy. \end{aligned}$$

Then $dx/y \longmapsto i(dx/y)$, so the action of $\mathrm{Aut}(C_A)$ is faithful.

d. C_B is the curve

$$y^2 = x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$$

and τ may be taken to be

$$\begin{aligned} x &\longmapsto \omega \cdot x \\ y &\longmapsto -y. \end{aligned}$$

Then $dx/y \longmapsto -\omega(dx/y)$, so the action of $\mathrm{Aut}(C_B)$ is faithful.

Q.E.D.

Therefore β is indeed surjective. But also β can be normalized by the requirement:

$$\beta(\Lambda) \equiv 1 \pmod{12}.$$

Then β becomes completely canonical.

The last step is that β is injective, completing the proof of:

Main Theorem. If char(k) $\neq$ 2, 3, and if g = 1, then there is a canonical isomorphism

$$\mathrm{Pic}(\mathfrak{M}) \cong \mathbf{Z}/12.$$

Sketch of Rest of Proof. Let L be an invertible sheaf on the moduli problem such that $\beta(L) \equiv 0$ (mod12). Then all automorphisms of all elliptic curves C induce trivial automorphisms of the corresponding vector spaces $L(C/\mathrm{Spec}(k))$. We must set up consistent isomorphisms of all the invertible sheaves $L(\pi)$ with the sheaves $\boldsymbol{o}_S$. But say $\pi : \mathfrak{X} \to S$ is a modular family of curves containing every elliptic curve as a fibre. Then according to the results of §5, it is sufficient to set up an isomorphism ϕ of this one $L(\pi)$ and $\boldsymbol{o}_S$ provided that the compatibility property written out in §5 is satisfied.

Look at the diagram:

$$\begin{tikzcd}
\mathrm{Isom}(\pi, \pi) \arrow[d, "f"] \\
\{\text{Normalization of } S \times_{\mathbf{A}} S\} \arrow[d, "g"] \\
S \times_{\mathbf{A}_j} S \arrow[d, bend right, "p_1"'] \arrow[d, bend left, "p_2"] \\
S
\end{tikzcd}$$

(cf. §4, last part). Recall that f is an étale double covering. Let $q_i = p_i \circ g \circ f$. By definition of an invertible sheaf, we are given an isomorphism ψ of $q_1^*(L)$ and $q_2^*(L)$. We can use the fact that $\beta(L) \equiv 0$ to show that there is actually an isomorphism ψ_0 of $p_1^*(L)$ and $p_2^*(L)$ which induces ψ via $f^* \circ g^*$. Set theoretically, we see this is as follows: let t and t' be two closed points of $\mathrm{Isom}(\pi, \pi)$ over the same point $\bar{t}$ of $S \times_{\mathbf{A}} S$. Let $\bar{L}_1$ and $\bar{L}_2$ be the one-dimensional vector spaces obtained by restricting the invertible sheaves $p_1^*(L)$ and $p_2^*(L)$ to the one-point subscheme $\{\bar{t}\}$. If $s_i = p_i(\bar{t})$, then $\bar{L}_i = L(\pi^{-1}(s_i))$. Moreover, t and t' define two isomorphisms τ and τ' of $\pi^{-1}(s_1)$ and $\pi^{-1}(s_2)$. By hypothesis, $L(\tau' \circ \tau^{-1})$ is the identity! Therefore, $L(\tau) = L(\tau')$. But $L(\tau)$ and $L(\tau')$ are just the isomorphisms of $\bar{L}_1$ and $\bar{L}_2$ given by looking at the action of ψ at the points t and t'. Therefore, ψ induces a *unique* isomorphism ψ_0 of $\bar{L}_1$ and $\bar{L}_2$ at $\bar{t}$. One must still check that this isomorphism ψ_0 is given by functions in the local rings of $S \times_{\mathbf{A}} S$ (this scheme is not normal, so this is not obvious). We omit this technical point.

Now the compatibility property of ψ shows immediately that ψ_0 is descent data for the invertible sheaf $L(\pi)$ on S with respect to the morphism $j : S \to \mathbf{A}_j$. Also j is clearly a flat covering of $\mathbf{A}_j$. Therefore, we can apply Theorem 90 of §5! In other words, we find an invertible sheaf L_0 on $\mathbf{A}_j$, and an isomorphism ϕ of $L(\pi)$ with $j^*(L_0)$ such that the following commutes:

$$\begin{array}{ccc} p_1^*(L(\pi)) & \underset{\psi}{\xrightarrow{\sim}} & p_2^*(L(\pi)) \\ {\scriptstyle p_1^*(\phi)}\downarrow\wr & & \downarrow\wr{\scriptstyle p_2^*(\phi)} \\ p_1^*(j^*(L_0)) & = & p_2^*(j^*(L_0)) \end{array} \tag{2}$$

But now every invertible sheaf on the affine line is trivial, that is, $L_0 \cong o_{\mathbf{A}}$. Use this isomorphism to set up an isomorphism of L with o_S. Finally, the compatibility property follows immediately from (2).

Q.E.D.

§7. COMPUTATIONS: TRANSCENDENTAL METHOD

Now assume $k = \mathbf{C}$. We shall give a completely different approach to $\mathrm{Pic}(\mathfrak{M})$ which has the virtue of generalizing to higher genus in various ways. This approach is based on:

Definition. An "analytic family of elliptic curves" is a morphism $\pi : \mathfrak{X} \to S$ of analytic spaces, which is proper and flat, plus a section $\varepsilon : S \to \mathfrak{X}$ of π, such that the fibres of π are elliptic curves.

We can now define a modular analytic family in two ways: either by the same properties used to define an (algebraic) modular family;† or else by defining the j-morphism from the base S to the complex j-plane and requiring that

a. S is a nonsingular one-dimensional complex space.
b. j is open.
c. j has ramification index 1, 2, 3 at $x \in S$ according to $j(x) \neq 0, 12^3$, $j(x) = 0$, or $j(x) = 12^3$.

† The lifting property goes over verbatim. But instead of asking that each elliptic curve only occur a finite number of times in a modular family $\pi : \mathfrak{X} \to S$, we should ask that it occur only over the points of a discrete subset $\Delta \subset S$.

A morphism of families is defined exactly as before, using analytic maps rather than algebraic ones.

Definition. The topology $\mathfrak{M}_{cx}$ is as follows:

a. Its open sets are analytic modular families of elliptic curves $\pi : \mathfrak{X} \to S$, and a final object M,
b. Its morphisms and coverings are exactly as in $\mathfrak{M}$.

We check to see that products exist in this topology and that they have exactly the same interpretation as before. Moreover, we get a continuous map of topologies:

$$\mathfrak{M}_{cx} \xrightarrow{\alpha} \mathfrak{M},$$

just as, in §2, we found a continuous map from the complex topology to the étale topology on a scheme. For all integers n, define a sheaf $\mathbf{Z}/n$ on $\mathfrak{M}_{cx}$ by

$$\mathbf{Z}/n[\mathfrak{X} \xrightarrow{\pi} S] = \bigoplus_{\substack{\text{topological components} \\ S\alpha \text{ of } S}} \mathbf{Z}/n$$

The direct image $\alpha_*(\mathbf{Z}/n)$ of this sheaf is simply the "same" sheaf:

$$\mathbf{Z}/n[\mathfrak{X} \xrightarrow{\pi} S] = \bigoplus_{\substack{\text{topological components} \\ \text{(in Zariski topology) of } S}} \mathbf{Z}/n.$$

An immediate extension of Artin's result tells us that the canonical homomorphism

$$H^i(\mathfrak{M}, \mathbf{Z}/n) \to H^i(\mathfrak{M}_{cx}, \mathbf{Z}/n)$$

is an isomorphism.†

This gives us a transcendental approach to the cohomology groups $H^i(\mathfrak{M}, \mathbf{Z}/n)$. These are related to the Picard group by virtue of the standard exact sequences of sheaves:

$$0 \to \mathbf{Z}/n \to \boldsymbol{o}^* \xrightarrow{n} \boldsymbol{o}^* \to 0,$$

where n indicates the homomorphism $f \mapsto f^n$ (cf. [1], p. 102). The

† The stronger form in which Artin gave his result is that if $g : X_{cx} \to X_{\acute{e}t}$ is the canonical map, then:

$$R^i g_*(\mathbf{Z}/n) = (0), \qquad i > 0.$$

This gives $R^i\alpha_*(\mathbf{Z}/n) = (0)$, $(i > 0)$ as a corollary because $\mathfrak{M}$ (respectively, $\mathfrak{M}_{cx}$) induces on an open set $\mathfrak{X} \xrightarrow{\pi} S$ the topology $S_{\acute{e}t}$ (respectively, S_{cx}).

cohomology sequence tells us:

$$0 \to H^0(\mathfrak{M}, \mathbf{Z}/n) \to H^0(\mathfrak{M}, o^*) \xrightarrow{n} H^0(\mathfrak{M}, o^*) \to H^1(\mathfrak{M}, \mathbf{Z}/n) \to \mathrm{Pic}(\mathfrak{M}) \to \mathrm{Pic}(\mathfrak{M}) \to H^2(\mathfrak{M}, \mathbf{Z}/n).$$

Via these sequences, we can work out the structure of Pic($\mathfrak{M}$), given that of $H^i(\mathfrak{M}, \mathbf{Z}/n)$. This is because we can prove by general arguments that

a. $H^0(\mathfrak{M}, o^*)$ has the subgroup $\mathbf{C}^*$ of constant functions, with factor group isomorphic to $\mathbf{Z}^a$.
b. Pic($\mathfrak{M}$) has a subgroup $\mathrm{Pic}^\tau(\mathfrak{M})$ of the type $\mathbf{R}^b/\mathbf{Z}^c$, where the lattice $\mathbf{Z}^c$ spans $\mathbf{R}^b$ (it need not be discrete), with finitely generated factor group.

Corollary. If there is a prime p such that $H^1(\mathfrak{M}, \mathbf{Z}/p) = (0)$, then $H^0(\mathfrak{M}, o^*) = \mathbf{C}^*$, and Pic($\mathfrak{M}$) is finitely generated.

Corollary. If there is a prime p such that $H^1(\mathfrak{M}, \mathbf{Z}/p) = H^2(\mathfrak{M}, \mathbf{Z}/p) = (0)$, then Pic($\mathfrak{M}$) is a finite group, and

$$\mathrm{Pic}(\mathfrak{M}) = \varinjlim H^1(\mathfrak{M}, \mathbf{Z}/n),$$

where the limit is taken with respect to the ordering:

$$n_1 \geq n_2 \quad \text{if} \quad n_2 \mid n_1,$$

and the maps

$$\mathbf{Z}/n_2 \xrightarrow{n_1/n_2} \mathbf{Z}/n_1.$$

(The proofs are obvious.)

We now go on to consider the topology $\mathfrak{M}_{\mathrm{cx}}$. The point is that there is one open set in $\mathfrak{M}_{\mathrm{cx}}$ which is very well known:

Let $\mathfrak{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$.

Let $\mathbf{Z} \oplus \mathbf{Z}$ act on the analytic space $\mathbf{C} \times \mathfrak{H}$ so that the generators act by:

$$(x, z) \mapsto (x + 1, z)$$
$$(x, z) \mapsto (x + z, z).$$

Let $\mathfrak{X} = (\mathbf{C} \times \mathfrak{H}/\mathbf{Z} \oplus \mathbf{Z})$.

Let $\pi : \mathfrak{X} \to \mathfrak{H}$ be induced by $p_2 : \mathbf{C} \times \mathfrak{H} \to \mathfrak{H}$.

Let $\epsilon : \mathfrak{H} \to \mathfrak{X}$ be induced by the section $\mathfrak{H} \xrightarrow{\sim} (0) \times \mathfrak{H} \subset \mathbf{C} \times \mathfrak{H}$.

Then π (and ϵ) define a modular analytic family of elliptic curves. Moreover, every elliptic curve occurs in π, so it is a covering of M.

Let $\Gamma = SL(2; \mathbf{Z})$

$$= \text{group of integral } 2 \times 2 \text{ matrices } \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $ad - bc = 1$.

Let Γ act on $\mathfrak{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times (z) \longmapsto \left(\frac{az + b}{cz + d}\right)$$

Call this morphism $\tau_0 : \Gamma \times \mathfrak{H} \to \mathfrak{H}$.

Let Γ act on $\mathbf{C} \times \mathfrak{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times (x, z) \longmapsto \left(\frac{x}{cz + d}, \frac{az + b}{cz + d}\right)$$

Then we check that the action of Γ normalizes the action of $\mathbf{Z} \oplus \mathbf{Z}$, hence it induces an action of Γ on $\mathfrak{X}$. Denote by $\tau : \Gamma \times \mathfrak{X} \to \mathfrak{X}$ the morphism giving this action. Clearly, π and ϵ commute with this action of Γ, so that we have made Γ into a group of automorphisms of the family $\mathfrak{X}/\mathfrak{H}$. This action of Γ has the following interpretation: via the diagram

$$\begin{tikzcd}
 & \Gamma \times \mathfrak{X} \arrow[dl, "\tau"'] \arrow[dd, "1_\Gamma \times \pi"] \arrow[dr, "p_2"] & \\
\mathfrak{X} \arrow[dd, "\pi"'] & & \mathfrak{X} \arrow[dd, "\pi"] \\
 & \Gamma \times \mathfrak{H} \arrow[dl, "\tau_0"'] \arrow[dr, "p_2"] & \\
\mathfrak{H} & & \mathfrak{H}
\end{tikzcd} \tag{3}$$

the family of elliptic curves $\Gamma \times \mathfrak{X}/\Gamma \times \mathfrak{H}$ is made into the product of $\mathfrak{X}/\mathfrak{H}$ with itself; in particular,

$$\Gamma \times \mathfrak{H} = \mathrm{Isom}(\pi, \pi).$$

The effect of this is to make a connection between the topology $\mathfrak{M}_{\mathrm{cx}}$ and the topology T_Γ of the discrete group Γ (cf. §1). We recall:

Definition. The topology T_Γ is as follows:

a. Its open sets are Γ-sets S, that is, sets plus action of Γ.
b. Its morphisms are Γ-linear maps between Γ-sets.
c. Its coverings are collections $S_\alpha \xrightarrow{p_\alpha} S$ of morphisms such that

$$S = \bigcup_\alpha p_\alpha(S_\alpha).$$

For our purpose, we need a slight modification of this topology.

Definition. Let Γ_n be the subgroup of Γ of matrices such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod n$$

Definition. Let T'_Γ be the following topology:

a. Its open sets are Γ-sets S such that, for all $x \in S$, the subgroup of Γ of elements leaving x fixed is contained in Γ_3; and a final object M.
b. Its morphisms are Γ-linear maps of Γ-sets, and projections of Γ-sets to M.
c. Its covering are collections $S_\alpha \xrightarrow{p_\alpha} S$ of morphisms such that

$$S = \bigcup_\alpha p_\alpha(S_\alpha);$$

and any collections of morphisms to M.

There is a continuous map:

$$T_\Gamma \xrightarrow{\beta} T'_\Gamma$$

such that β^{-1} of a Γ-set S is S; and $\beta^{-1}(M)$ is the Γ-set $\{e\}$ with one element. It is easy to check that β_* sets up an equivalence between the category of abelian sheaves on T_Γ and the category of abelian sheaves on T'_Γ. Therefore, cohomologically T_Γ and T'_Γ are identical. In fact, as we saw in §1, these categories of sheaves are equivalent to the category of Γ-modules (where the group of global sections of a sheaf is equal to the subgroup of Γ-invariants of the corresponding module). Therefore, the cohomology of T_Γ and T'_Γ is also the same as the cohomology of the group Γ.

Finally, there is a continuous map

$$\mathfrak{M}_{cx} \xrightarrow{\gamma} T'_\Gamma$$

which is as follows:

Definition. Let S be a Γ-set in T'_Γ; give S the discrete topology. Then $\gamma^{-1}(S)$ is the family:

$$\begin{array}{c} \mathfrak{X} \times S/\Gamma \\ \downarrow \\ \mathfrak{H} \times S/\Gamma \end{array}$$

where Γ acts on $\mathfrak{X} \times S$ and $\mathfrak{H} \times S$ by a product of τ and τ_0 with the given action on S.

This makes sense only provided that Γ acts freely on $\mathfrak{H} \times S$ (hence on $\mathfrak{X} \times S$). But if an element $a \in \Gamma$ leaves fixed some element of S, then by definition of T'_Γ, $a \in \Gamma_3$; then it is easy to check that a acts on $\mathfrak{H}$ without fixed points. Therefore, the action is free. Of course, $\gamma^{-1}(M)$ is to be M. The key point to check is that fibre products in T'_Γ go into fibre products in $\mathfrak{M}_{cx}$. We omit the proof, except to say that this fact follows readily from the fact that diagram (3) makes $\Gamma \times \mathfrak{X}/\Gamma \times \mathfrak{H}$ into the product of $\mathfrak{X}/\mathfrak{H}$ with itself.

Recapitulating, we have unwound the structure of M by the following continuous maps:

$$\begin{tikzcd} & \mathfrak{M} \\ \mathfrak{M}_{cx} \arrow[ur, "\alpha"] \arrow[dr, "\gamma"] & \\ & T'_\Gamma. \\ T_\Gamma \arrow[ur, "\beta"] & \end{tikzcd}$$

The final step is to prove that, via γ, we get an isomorphism:

$$H^i(T'_\Gamma, \mathbf{Z}/n) \xrightarrow{\sim} H^i(\mathfrak{M}_{cx}, \mathbf{Z}/n)$$

This follows from the Leray spectral sequence, once we know that:

$$R^i\gamma_*(\mathbf{Z}/n) = (0), \qquad i > 0;$$

and this is equivalent to knowing that $\mathbf{Z}/n$ has no higher cohomology in the induced topology on the open set $\mathfrak{X}/\mathfrak{H}$ in $\mathfrak{M}_{cx}$. But this is just the classical topology on $\mathfrak{H}$; and since $\mathfrak{H}$ is homeomorphic to a cell, $\mathbf{Z}/n$ has no higher cohomology in this topology.

Corollary. There are canonical isomorphisms:

$$H^i(\mathfrak{M}, \mathbf{Z}/n) = H^i(\Gamma, \mathbf{Z}/n) \qquad \text{for all } i.$$

Now it is well known that

a. $H^i(\Gamma, M) = (0)$, $i \geq 2$, for any Γ-module M which is p-torsion, $p \neq 2, 3$,

b. $H^1(\Gamma, M) = \mathrm{Hom}(\mathbf{Z}/12, M)$ for any abelian group M with trivial Γ action.

Putting all the results of this section together, we have proven again that $\mathrm{Pic}(\mathfrak{M}) \cong \mathbf{Z}/12$.

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