Lecture 1: Overview

September 28, 2018

Let X be an algebraic curve over a finite field $\mathbf{F}_q$, and let K_X denote the field of rational functions on X. Fields of the form K_X are called *function fields*. In number theory, there is a close analogy between function fields and *number fields*: that is, fields which arise as finite extensions of $\mathbf{Q}$. Many arithmetic questions about number fields have analogues in the setting of function fields. These are typically much easier to answer, because they can be connected to algebraic geometry.

Example 1 (The Riemann Hypothesis)**.** Recall that the *Riemann zeta function* $\zeta(s)$ is a meromorphic function on $\mathbf{C}$ which is given, for $\mathrm{Re}(s) > 1$, by the formula

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \sum_{n>0} \frac{1}{n^s},$$

where the product is taken over all prime numbers p. The celebrated *Riemann hypothesis* asserts that $\zeta(s)$ vanishes only when $s \in \{-2, -4, -6, \ldots\}$ is negative even integer or when $\mathrm{Re}(s) = \frac{1}{2}$.

To every algebraic curve X over a finite field $\mathbf{F}_q$, one can associate an analogue ζ_X of the Riemann zeta function, which is a meromorphic function on $\mathbf{C}$ which is given for $\mathrm{Re}(s) > 1$ by

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{1 - |\kappa(x)|^{-s}} = \sum_{D \subseteq X} \frac{1}{|\mathcal{O}_D|^s}$$

here $|\kappa(x)|$ denotes the cardinality of the residue field $\kappa(x)$ at the point x, D ranges over the collection of all effective divisors in X and $|\mathcal{O}_D|$ denotes the cardinality of the ring of regular functions on D. The *Riemann hypothesis for* X asserts that all zeroes of the function $\zeta_X(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$. This is equivalent to the more concrete assertion that the size of the set $X(\mathbf{F}_{q^n})$ of $\mathbf{F}_{q^n}$-valued points of X satisfies an estimate of the form

$$|X(\mathbf{F}_{q^n})| = q^n + O(q^{n/2}).$$

This was originally conjectured by Artin, proved by Hasse in the case of elliptic curves, and proved by Weil for algebraic curves in general (and later generalized to arbitrary algebraic varieties by Deligne). Weil's proof proceeds by studying intersection theory on the algebraic surface $X \times_{\mathrm{Spec}(\mathbf{F}_q)} X$ (in modern language, it is a consequence of the Hodge index theorem for $X \times_{\mathrm{Spec}(\mathbf{F}_q)} X$, together with some elementary linear algebra). By contrast, the classical Riemann hypothesis (for the usual Riemann zeta function $\zeta(s)$) remains an open question.

The algebraic curve associated to a function field has an analogue in the number field setting. To every number field K, one can associate the affine scheme $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of algebraic integers in

K. There is a close analogy between schemes of this form and algebraic curves over finite fields:

Function Field Arithmetic	Number Field Arithmetic
Function Field K_X	Number Field $\mathbf{Q}$
Algebraic curve X	Affine Scheme $\mathrm{Spec}(\mathbf{Z})$
Closed points $x \in X$	Prime numbers p
Residue field $\kappa(x)$	Residue field $\mathbf{Z}/p\mathbf{Z}$

But there is a fundamental weakness in this analogy. The affine scheme $\mathrm{Spec}(\mathbf{Z})$ behaves very much like an algebraic curve over a finite field if we consider its points *as a topological space*. However, it behaves very differently when we think about its points *as a functor*. If X is a scheme and R is a commutative ring, we let $X(R) = \mathrm{Hom}(\mathrm{Spec}(R), X)$ denote the set of R-valued points of X. In the case where $X = \mathrm{Spec}(\mathbf{Z})$, this is just the set of ring homomorphisms from $\mathbf{Z}$ into R: that is, it always consists of exactly one element. It follows that the product $\mathrm{Spec}(\mathbf{Z}) \times \mathrm{Spec}(\mathbf{Z})$, formed in the category of schemes, is just $\mathrm{Spec}(\mathbf{Z})$. In particular, it is a very poor replacement for the algebraic surface $X \times_{\mathrm{Spec}(\mathbf{F}_q)} X$ of Example 1. This motivates the following:

Question 2. Can one replace the affine scheme $\mathrm{Spec}(\mathbf{Z})$ by some other mathematical object S, for which the product $S \times S$ is more interesting? (And, in particular, not isomorphic to S?)

Several years ago, Scholze proposed an answer to Question 2, at least after completing at some prime number p. To explain the idea, we need a brief digression.

Definition 3. Let K be a field. A *non-archimedean absolute value* on K is a map

$$| \, |_K : K \to \mathbf{R}_{\geq 0} \qquad x \mapsto |x|$$

which satisfies the following conditions:

$$|0|_K = 0 \qquad |1|_K = 1$$

$$|xy|_K = |x|_K |y|_K \qquad |x+y|_K \leq \sup\{|x|_K, |y|_K\}.$$

If K is equipped with a non-archimedean absolute value, then the subset

$$\mathcal{O}_K = \{x \in K : |x|_K \leq 1\}$$

is a subring of K, which we call the *valuation ring of* K. This is a local ring, whose unique maximal ideal is given by $\mathfrak{m}_K = \{x \in K : |x|_K < 1\}$. We let k denote the quotient field $\mathcal{O}_K / \mathfrak{m}_K$ and refer to it as the *residue field of* k.

We say that an absolute value $| \, |_K$ on K is *nontrivial* if $\mathcal{O}_K \neq K$: equivalently, if there exists an element $\pi \in K$ satisfying $0 < |\pi|_K < 1$. Any such element is called a *pseudo-uniformizer*. We say that K is *complete* with respect to a nontrivial absolute value $| \, |_K$ if it is complete with respect to the metric given by $d(x, y) = |x - y|_K$. Equivalently, K is complete if the valuation ring $\mathcal{O}_K$ is π-adically complete, with respect to any choice of pseudo-uniformizer $\pi \in K$.

A *completely valued field* is a field K equipped with a nontrivial non-archimedean absolute value $| \, |_K$ such that K is complete with respect to $| \, |_K$.

Construction 4 (Tilting)**.** Fix a prime number p. For any field K, we let $K^\flat$ denote the set given by the inverse limit of the system

$$\cdots \to K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K.$$

In other words, $K^\flat$ is the collection of all sequences $\{x_n\}_{n \geq 0}$ in K which satisfy $x_n^p = x_{n-1}$ for $n > 0$.

The set $K^\flat$ comes equipped with an obvious multiplication law, given by

$$\{x_n\}_{n\geq 0} \cdot \{y_n\}_{n\geq 0} = \{x_n y_n\}_{n\geq 0}.$$

This is well-defined since the operation $x \mapsto x^p$ is multiplicative. One cannot define an addition law in the same way (unless K has characteristic p), because the map $x \mapsto x^p$ is not additive. Nevertheless, we have the following:

Proposition 5. *Let K be a completely valued field which is algebraically closed, and suppose that the residue field of K has characteristic p. Then $K^\flat$ can be equipped with the structure of a field of characteristic p, with multiplication defined as above and addition given by*

$$\{x_n\}_{n\geq 0} + \{y_n\}_{n\geq 0} = \{\lim_{m\to\infty} (x_{n+m} + y_{n+m})^{p^m}\}_{n\geq 0}.$$

We will refer to $K^\flat$ as the tilt of K.

We will prove Proposition 5 in the next lecture.

Notation 6. Let K be a completely valued field which is algebraically closed. For each $x = \{x_n\}_{n\geq 0} \in K^\flat$, we set $x^\sharp = x_0$, so that $x \mapsto x^\sharp$ defines a map of sets $\sharp : K^\flat \to K$. If K has characteristic p, then this map is an isomorphism. If K has characteristic zero, then it is multiplicative but not additive (note that it cannot be a ring homomorphism, since $K^\flat$ and K have different characteristics). Instead we have

$$(x+y)^\sharp = \lim_{m\to\infty} ((x^{1/p^m})^\sharp + (y^{1/p^m})^\sharp)^{p^m}.$$

We define a map $|\,|_{K^\flat} : K^\flat \to \mathbf{R}_{\geq 0}$ by the formula $|x|_{K^\flat} = |x^\sharp|_K$. We will see in the next lecture that this is a non-archimedean absolute value on $K^\flat$, and endows $K^\flat$ with the structure of a completely valued field of characteristic p. We denote the valuation ring of $K^\flat$ by

$$\mathcal{O}_K^\flat = \{x \in K^\flat : |x^\sharp|_K \leq 1\} = \varprojlim(\cdots \to \mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K).$$

One can show that if K is algebraically closed, then $K^\flat$ is also algebraically closed. Consequently, tilting provides a construction

$$\begin{array}{c}\{\text{Algebraically closed completely valued fields of residue characteristic } p\} \\ \Big\downarrow{\scriptstyle \flat} \\ \{\text{Algebraically closed completely valued fields of characteristic } p\}.\end{array}$$

Beware that this construction is not invertible.

Definition 7. Let C be an algebraically closed completely valued field of characteristic p. An *untilt* of C is a pair (K, ι), where K is an algebraically closed completely valued field with residue characteristic p and $\iota : C \simeq K^\flat$ is an isomorphism of fields which carries the valuation ring $\mathcal{O}_C$ to the valuation ring $\mathcal{O}_K^\flat$.

Example 8. Up to isomorphism, there is a unique untilt of C which has characteristic p (given by the pair (C, id_C)). However, we will see that there exist many other untilts (K, ι), where K has characteristic zero. In this case, one can think of K as a "characteristic zero incarnation" of the field C, which is an algebra over the field $\mathbf{Q}_p$ of p-adic rational numbers.

Let A be a commutative ring. Then the affine scheme $\mathrm{Spec}(A)$ can be understood in terms of its "functor of points," which assigns to another commutative ring R the set $\mathrm{Hom}(A, R)$ of all A-algebra structures on R. In the cases $A = \mathbf{Z}$ and $A = \mathbf{Q}$, this functor is not very interesting: every commutative ring admits a unique $\mathbf{Z}$-algebra structure, and at most one $\mathbf{Q}$-algebra structure.

Heuristic Idea 9 (Scholze)**.** Let C be an algebraically closed completely valued field of characteristic p. Then

$$\{\text{Untilts of } C\}/\simeq$$

is a good replacement for the set of C-valued points of $\mathrm{Spec}(\mathbf{Z})$. Similarly,

$$\{\text{Characteristic zero untilts of } C\}/\simeq$$

is a good replacement for the set of C-valued points of $\mathrm{Spec}(\mathbf{Q})$.

One virtue of this idea is that it allows us to make sense of products like $\mathrm{Spec}(\mathbf{Z}) \times \mathrm{Spec}(\mathbf{Z})$ in a nontrivial way. Rather than taking a product in the category of schemes (which gives nothing interesting), we instead apply Heuristic 9: if C is an algebraically closed completely valued field of characteristic p, then we should think of "C-valued points" of $\mathrm{Spec}(\mathbf{Z}) \times \mathrm{Spec}(\mathbf{Z})$ as corresponding to *pairs* of untilts of C. This is different from the interpretation that Heuristic 9 suggests for $\mathrm{Spec}(\mathbf{Z})$ itself: while C has only one $\mathbf{Z}$-algebra structure, it has many different untilts.

To exploit this idea effectively, we need to address the following:

Question 10. Let C be an algebraically closed completely valued field of characteristic p. What can we say about the collection of all untilts of C? How can they be classified?

We observed above that, up to isomorphism, C has only one untilt of characteristic p (namely, the field C itself). Let us therefore confine our attention to the classification of *characteristic zero* untilts of C. Note that since C is a perfect field of characteristic p, the construction $x \mapsto x^p$ induces an automorphism of C called the *Frobenius map*, which we denote by $\varphi_C : C \to C$. If (K, ι) is any untilt of C, then we can construct a family of untilts given by $\{(K, \varphi_C^n \circ \iota)\}_{n \in \mathbf{Z}}$. When K has characteristic zero, these untilts are pairwise non-isomorphic but should nevertheless be considered "the same" in some sense. Let us therefore rephrase Question 10 as follows:

Question 11. Let C be an algebraically closed completely valued field of characteristic p. What can one say about the quotient

$$\{\text{Isomorphism classes of characteristic zero untilts of } C\}/\varphi_C^{\mathbf{Z}}?$$

Question 11 has a very beautiful answer, which is the subject of this course.

Theorem 12. *Let C be an algebraically closed completely valued field of characteristic p. There exists a Dedekind scheme X equipped with a bijection*

$$\begin{array}{c} \{\ \text{Closed points } x \in X\ \} \\ \big\downarrow \sim \\ \{\text{Isomorphism classes of characteristic zero untilts of } C\}/\varphi_C^{\mathbf{Z}}. \end{array}$$

For each closed point $x \in X$, the corresponding untilt of C can be identified with the residue field $\kappa(x)$ of X.

The scheme X of Theorem 12 is called the *Fargues-Fontaine curve*. It has many amazing properties:

- There is a canonical isomorphism $\mathbf{Q}_p \simeq \mathrm{H}^0(X, \mathcal{O}_X)$. In particular, X can be regarded as a $\mathbf{Q}_p$-scheme.

The structural morphism $\pi : X \to \mathrm{Spec}(\mathbf{Q}_p)$ is not of finite type (for example, the residue fields of X at its closed points are not finite extensions of $\mathbf{Q}_p$). Nevertheless, X behaves in many respects like a *complete* algebraic curve of genus zero:

- For every rational function f on X, there is a degree formula

$$\sum_{x \in X} \deg_x(f) = 0.$$

  Note that if X were an algebraic curve, then this would imply that X is projective: that is, it does not have any "missing" points.

- Every line bundle $\mathscr{L}$ on X has a well-defined degree $\deg(\mathscr{L})$, any any two line bundles of the same degree are isomorphic (if X were an algebraic curve, this would say that the genus of X is equal to zero).

- The cohomology group $\mathrm{H}^1(X; \mathcal{O}_X)$ vanishes (if X were an algebraic curve, this would also be equivalent to saying that X has genus zero).

- Every vector bundle $\mathscr{E}$ on X has a canonical Harder-Narasimhan filtration (as if $\mathscr{E}$ were a vector bundle on an algebraic curve), which is non-canonically split (as if the curve were of genus zero).

- The fibers of the map $\pi : X \to \mathrm{Spec}(\mathbf{Q}_p)$ are simply connected. More precisely, pullback along π induces an equivalence

$$\{ \text{ Representations of } \mathrm{Gal}(\mathbf{Q}_p) \ \} \to \{ \text{ Local systems on } X \ \}$$

  (for various definitions of the right and left hand side).

Motivated by Heuristic 9, Fargues and Scholze have proposed a program for applying geometric ideas to the study of the local Langlands program. The philosophy of their approach can be roughly summarized by the slogan

$$\{\text{Local Langlands for } \mathbf{Q}_p\} \simeq \{\text{Geometric Langlands for the curve } X\}.$$

The Fargues-Fontaine curve X is also closely connected other ideas in arithmetic: p-adic Hodge theory, the classification of p-divisible groups, Banach-Colmez spaces, Tate local duality, ...