Adic and perfectoid spaces

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Abstract

The main goal of this course is to develop the foundations of the theory of perfectoid spaces, more precisely to prove the various tilting correspondences for perfectoid rings, the almost purity theorem, and almost vanishing theorems. We develop simultaneously what is needed from the theory of adic spaces.

Contents

Some notation

For the whole course we fix a prime number p. All rings are commutative. Given a ring A of characteristic p, we write $\varphi : A \to A$, $a \mapsto a^p$ for its Frobenius endomorphism (which is a ring homomorphism).

1 Integral perfectoid rings

This section is devoted to the foundations of the theory of integral perfectoid rings, which is largely of a commutative algebra flavour.

Let A be a topological ring. We say that A is *integral perfectoid*[1] if and only if there exists a non-zero-divisor $\pi \in A$ such that

(a) the topology on A is the π-adic topology, and A is complete for this topology (i.e., $A \to \varprojlim_n A/\pi^n A$ is an isomorphism of topological rings, where each $A/\pi^n A$ has the discrete topology);

(b) $p \in \pi^p A$;

(c) $\Phi : A/\pi A \to A/\pi^p A$, $a \mapsto a^p$, is an isomorphism

It is convenient for us, even though it is not exactly standard in the literature, to call any such element π a perfectoid pseudo-uniformiser (ppu).

Note that $\mathbb{Z}_p$ is "too small" to be perfectoid: it fails to satisfy condition (b).

Example 1.1. Here are some examples which furnish us with lots of integral perfectoid rings.

(i) Let C be an algebraic extension of $\mathbb{Q}_p$, and $\mathcal{O}_C$ the integral closure of $\mathbb{Z}_p$ in C; assume that every element of $\mathcal{O}_C/p\mathcal{O}_C$ has a p^{th}-root and that there exists a non-unit $\pi \in \mathcal{O}_C$ such that $p \in \pi^p\mathcal{O}_C$.

Check that $\Phi : \mathcal{O}_C/\pi\mathcal{O}_C \to \mathcal{O}_C/\pi^p\mathcal{O}_C$, $a \mapsto a^p$ is an isomorphism and deduce that the p-adic completion (equivalently the p-adic completion) $\widehat{\mathcal{O}_C}$ is an integral perfectoid ring.

For example, we can apply this to $C = \mathbb{Q}_p(p^{1/p^\infty})$ (in which case $\mathcal{O}_C = \mathbb{Z}_p[p^{1/p^\infty}]$), or $C = \mathbb{Q}_p(\zeta_{p^\infty})$ (in which case $\mathcal{O}_C = \mathbb{Z}_p[\zeta_{p^\infty}]$), or $C = \mathbb{Q}_p^{\text{alg}}$.

(ii) Let A be an integral perfectoid ring. Its algebra of "perfectoid polynomials" is

$$A\langle T^{1/p^\infty}\rangle := \text{the } \pi\text{-adic completion of } \bigcup_{i\geq 1} A[T^{1/p^i}],$$

where $\pi \in A$ is any perfectoid pseudo-uniformiser. Check that $A\langle T^{1/p^\infty}\rangle$ is an integral perfectoid ring, and more generally for several variables $T_1, \dots, T_d$.

(iii) Let A be an integral perfectoid ring and B any étale A-algebra. Show that the π-adic completion $\widehat{B}$ is an integral perfectoid ring. (You will need the following fact: given any ring of characteristic p, the diagram
$$\begin{tikzcd} k' \arrow[r, "\varphi"] & k' \\ k \arrow[u] \arrow[r, "\varphi"] & k \arrow[u] \end{tikzcd}$$
is a pushout, i.e., $k' \otimes_{k,\varphi} k \xrightarrow{\simeq} k'$.)

(iv) Let C be as in part (i) and let R be a smooth $\mathcal{O}_C$-algebra; suppose that there exists an étale morphism $\mathcal{O}_C[T_1, \dots, T_d] \to R$ (if R is equi-dimensional, this is always true locally on $\operatorname{Spec} R$). Now use (ii) and (iii) to construct an integral perfectoid ring R_∞

[1] La meilleure traduction de "integral perfectoid ring" n'est pas évidente, mais on va utiliser *anneau perfectoïde entier*.

Construction 1.2. Given any ring A we write

$$\varprojlim_{x\mapsto x^p} A := \{(a_0, a_1, \dots) \in A^{\mathbb{N}} : a_i^p = a_{i-1} \text{ for all } i \geq 1\}$$

for the set of compatible sequences of p-power roots in A. Note that we can multiply two such sequences, so $\varprojlim_{x\mapsto x^p} A$ forms a multiplicative monoid; if A has characteristic p then we can also add two such sequences, so then A is even a ring.

The construction is clearly functorial: a morphism of rings $A \to B$ induces a morphism of monoids $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} B$ (which is even a morphism of rings if A and B have characteristic p).

The following result is extremely important and will be used repeatedly: if $\pi \in A$ is an element such that (i) $p \in \pi A$ and (ii) A is π-adically complete, then the resulting map

$$\varprojlim_{x\mapsto x^p} A \longrightarrow \varprojlim_{x\mapsto x^p} A/\pi A$$

is actually a bijection (hence an isomorphism of monoids). We leave the details of the proof to the reader, but give the following recipe for the inverse of the map: given $b = (b_0, b_1, \dots) \in \varprojlim_{x\mapsto x^p} A/\pi A$, let $\widetilde{b_i} \in A$ be an arbitrary lift of $b_i \in A/\pi A$ for each $i \geq 0$; then set

$$a_i := \lim_{n\to\infty} \widetilde{b_{i+n}}^{p^n}$$

and check that $a := (a_0, a_1, \dots) \in \varprojlim_{x\mapsto x^p} A$ really is a well-defined lift of b.

Lemma 1.3. *Let A be integral perfectoid, and $\pi \in A$ a perfectoid pseudo-uniformiser. Then:*

(i) Every element of $A/\pi pA$ is a p^{th}-power (n.b., $A/\pi pA$ does not necessarily have characteristic p).

(ii) If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$, then $a \in A$.

(iii) After multiplying π by a unit it has a compatible sequence of p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots \in A$.

Proof. (i): Using the surjectivity of Φ, a simple induction lets us write any $a \in A$ as an infinite sum $a = \sum_{i\geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$; but this is $\equiv (\sum_{i\geq 0} a_i\pi^i)^p \bmod p\pi A$.

(ii): Let $l \geq 0$ be the smallest integer such that $\pi^l a \in A$. Assuming that $l > 0$, we get a contradiction by noting that $\pi^{pl}a^p \in \pi^{pl}A \subseteq \pi^p A$, whence $\pi^l a \in \pi A$ by condition (c), and so $\pi^{l-1}a \in A$.

(iii): Since the Frobenius is surjective on $A/\pi^p A$, there exists an element of $\varprojlim_{x\mapsto x^p} A/\pi^p A$ of the form $(\pi \bmod \pi^p A, ?, ?, \dots)$. Applying the exercise of Construction 1.2, we deduce that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/\pi^p A$ is a bijection. Hence there exists $a = (a_0, a_1, \dots) \in \varprojlim_{x\mapsto x^p} A$ such that $a_0 \equiv \pi \bmod \pi^p A$; therefore $a = u\pi$ for some $u \in 1 + \pi^{p-1}A \subseteq A^\times$ (the inclusion $\subseteq$ results from π-adic completeness of A). $\square$

Lemma 1.4. *Let A be integral perfectoid, and $\varpi \in A$ an element satisfying conditions (a) and (b). Then ϖ is a non-zero-divisor satisfying (c), i.e., it is a perfectoid pseudo-uniformiser.*

Proof. We must show that $\Phi : A/\varpi A \to A/\varpi^p A$ is an isomorphism. Let $\pi \in A$ be a perfectoid pseudo-uniformiser.

It follows from Lemma 3(i) that every element of A/pA is a p^{th}-power; hence every element of its quotient $A/\varpi^p A$ is a p^{th}-power, i.e., Φ is surjective.

The fact that π and ϖ define the same topology implies that a power of each is divisible by the other, whence ϖ is a non-zero-divisor and $A[\frac{1}{\varpi}] = A[\frac{1}{\pi}]$. If $a \in A$ satisfies $a^p \in \varpi^p A$, then $(a/\varpi) \in A[\frac{1}{\pi}]$ satisfies $(a/\varpi)^p \in A$, and it then follows from Lemma 3(ii) that in fact $a \in \varpi A$ as desired. $\square$

Lemma 1.5. *Suppose A is a complete topological ring such that $pA = 0$. Then A is integral perfectoid if and only if it is perfect and the topology is π-adic for some non-zero-divisor $\pi \in A$.*

Proof. Exercise. $\square$

1.1 The tilt of an integral perfectoid ring

Definition 1.6. The *tilt*[2] of an integral perfectoid ring A is $A^\flat := \varprojlim_{x\mapsto x^p} A/pA$, equipped with the inverse limit topology (A/pA is of course given the quotient topology, i.e., the π-adic topology for any choice of perfectoid pseudo-uniformiser for A).

Note that $A^\flat$ is a perfect ring of characteristic p; in fact, it is the initial object among all perfect rings of characteristic p mapping to A/pA.

Recalling from Construction 1.2 that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/pA$ is an isomorphism of monoids, we define the *untilting map* $\# : A^\flat \to A$, $b \mapsto b^\#$ to be projection to the 0^{th}-coordinate of $\varprojlim_{x\mapsto x^p} A$; explicitly, the map $\#$ is given by $\varprojlim_{x\mapsto x^p} A/pA \ni (b_0, b_1, \dots) \mapsto \lim_{i\to\infty} \widetilde{b_i}^{p^i}$, where $\widetilde{b_i} \in A$ are arbitrary lifts of the elements $b_i \in A/pA$.

The untilting map is multiplicative by generally not additive; in fact, given $b, c \in A^\flat$, it transforms under addition as follows:

$$(b+c)^\# = \lim_{i\to\infty} ((b^{1/p^i})^\# + (c^{1/p^i})^\#)^{p^i}.$$

However, note that the composition $A^\flat \overset{\#}{\longrightarrow} \overset{\text{mod } p}{} A/pA$ is a ring homomorphism: indeed, it is the surjective ring homomorphism given by projecting $A^\flat \cong \varprojlim_{x\mapsto x^p} A/pA$ to the 0^{th}-coordinate. Also, if A is of characteristic p, then the untilting map $\# : A^\flat \to A$ is an isomorphism of rings.

Lemma 1.7. *Let A be an integral perfectoid ring. Then:*

(i) the untilting map $\# : A^\flat \to A$ is continuous;

(ii) the isomorphisms of monoids $\varprojlim_{x\mapsto x^p} A \to A^\flat = \varprojlim_{x\mapsto x^p} A/pA \to \varprojlim_{x\mapsto x^p} A/\pi A$ are homeomorphisms, where $\pi \in A$ is any perfectoid pseudo-uniformiser;

(iii) $A^\flat$ is also an integral perfectoid ring.

Proof. Given $(1, \dots, 1, b_{n+1}, b_{n+2}, \dots) \in \varprojlim_{x\mapsto x^p} A/\pi A$, any chosen lifts $\widetilde{b_i}$ satisfy $\widetilde{b_i}^{p^{i-n}} \equiv 1 \mod \pi A$ for $i > n$, whence $\widetilde{b_i}^{p^i} \equiv 1 \mod \pi^n A$; taking the limit shows that the untilt is $\equiv 1 \mod \pi^n A$. This proves that the untilting map $\# : \varprojlim_{x\mapsto x^p} A/\pi A \to A$ is continuous (for the inverse limit of discrete topologies on the domain), from which (i) and (ii) easily follow. Filling in the details is left as an exercise.

(iii) We have already noted that $A^\flat$ is a perfect ring of characteristic, and the homeomorphism $A \cong \varprojlim_{x\mapsto x^p} A/\pi A$ shows that A is an inverse limit of discrete rings, whence A is a

[2] Le *basculé* de A en français.

complete topological ring. According to Lemma 1.5, it remains to prove the following: there exists a non-zero-divisor $\pi^\flat \in A^\flat$ such that the topology on $A^\flat$ is the $\pi^\flat$-adic topology.

Possibly after changing our perfectoid pseudo-uniformiser π, we may assume that it admits compatible p-power roots (by Lemma 3(iii)); let $\pi^\flat = (\pi, \pi^{1/p}, \dots,) \in A^\flat$ be the corresponding element of $A^\flat$, which satisfies $(\pi^\flat)^\# = \pi$.

We show first that $\pi^\flat$ is a non-zero-divisor. To do that we note that for each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow \pi^{1-1/p^n} A/\pi A \longrightarrow A/\pi A \xrightarrow{\times \pi^{1/p^n}} A/\pi A \xrightarrow{\varphi^n} A/\pi A \longrightarrow 0$$

Exactness is easy everywhere except possibly at the second term from the right: but if $a \in A$ satisfies $a^p \in \pi A$ then $a/\pi^{1/p^n} \in A[\frac{1}{\pi}]$ satisfies $(a/\pi^{1/p^n})^{p^n} \in A$, whence Lemma 3(ii) implies $a \in \pi^{1/p^n} A$ as desired.

These sequences are moreover compatible in n, with respect to the maps $0, \varphi, \varphi, \mathrm{id}$ respectively. Although it is not always the case that an inverse limit of exact sequences is still exact, in this case the transition map are either surjective (φ and id) or zero, and so taking the inverse limit does yield an exact sequence

$$0 \longrightarrow A^\flat \xrightarrow{\times \pi^\flat} A^\flat \xrightarrow{\sharp \bmod \pi} A/\pi A \longrightarrow 0.$$

That is, $\pi^\flat$ is a non-zero-divisor of $A^\flat$ and the untilting map induces an isomorphism of rings $A^\flat/\pi^\flat A^\flat \xrightarrow{\simeq} A/\pi A$.

Finally we check that the topology on $A^\flat$ is the $\pi^\flat$-adic topology. Since $A^\flat \cong \varprojlim_{x \mapsto x^p} A/\pi A$ is a homeomorphism by part (i), a basis of open neighbourhoods of $0 \in A^\flat$ is given by $\mathrm{Ker}(\mathrm{proj_n})$ for $n \geq 0$, where $\mathrm{proj}_n : \varprojlim_{x \mapsto x^p} A/\pi A \to A/\pi A$, $(b_0, b_1, \dots) \mapsto b_n$ denotes the n^{th}-projection map. Note that proj_0 is the untilting map. Since the composition $A^\flat \xrightarrow{\varphi^n \cong} A^\flat \xrightarrow{\mathrm{proj}_n} A/\pi A$ is proj_0, the basis of open neighbourhoods is given by

$$\mathrm{Ker}(\mathrm{proj_n}) = \varphi^n(\mathrm{Ker}(\mathrm{proj}_0)) = \varphi^n(\pi^\flat A^\flat) = \pi^{\flat p^n} A^\flat,$$

showing that the topology is indeed $\pi^\flat$-adic. □

We point out explicitly that we showed in the previous proof that the kernel of the surjective ring homomorphism $A^\flat \xrightarrow{\# \bmod \pi} A/\pi A$ (i.e., projection to the 0^{th}-coordinate of $A^\flat = \varprojlim_{x \mapsto x^p} A/\pi A$) is $\pi^\flat A^\flat$, i.e., untilting induces an isomorphism of rings

$$\# : A^\flat/\pi^\flat A^\flat \xrightarrow{\simeq} A/\pi A.$$

This will be used frequently.

Example 1.8. Let A be an integral perfectoid ring and $A\langle T^{1/p^\infty}\rangle$ its algebra of perfectoid polynomials from Example 1.1. Note that its tilt $A\langle T^{1/p^\infty}\rangle^\flat$ contains an element $T^\flat := (T, T^{1/p}, T^{1/p^2}, \dots)$. Construct an isomorphism of perfectoid $A^\flat$-algebras

$$A^\flat\langle U^{1/p^\infty}\rangle \xrightarrow{\simeq} A\langle T^{1/p^\infty}\rangle^\flat$$

which sends U to $T^\flat$, where the left side denotes the algebra of perfectoid polynomials over $A^\flat$.

1.2 Fontaine's map θ

In this section we introduce Fontaine's map $\theta : W(A^\flat) \to A$ for integral perfectoid rings, which will allow us to recover A from $A^\flat$ and will therefore play a fundamental role in untilting.

Remark 1.9 (Reminder on the ring of Witt vectors)**.** If k is any ring, let $W(k)$ denote its ring of p-typical Witt vectors. A classical reference is Serre's *Corps locaux*. Here are some reminders about this object:

(i) There is an identification of sets $W(k) = k^{\mathbb{N}}$. So each element of $W(k)$ may be written uniquely as $(a_0, a_1, \dots)$, with $a_i \in k$.

(ii) Additional and multiplication are given by certain polynomials with integer coefficients (which do not depend on k), for example

$$(a_0, a_1, a_2, \dots) + (b_0, b_1, b_2, \dots) = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i}, \dots)$$

$$(a_0, a_1, a_2, \dots) \cdot (b_0, b_1, b_2, \dots) = (a_0 b_0, a_0^p b_0 + b_0^p a_1 + p a_1 b_1, \dots)$$

(iii) There is a natural ring homomorphism called the "phantom" or "ghost" map

$$\text{phant} : W(k) \longrightarrow k^{\mathbb{N}}, \qquad (a_0, a_1, \dots) \mapsto (a_0, a_0^p + p a_1, a_0^{p^2} + p a_1^p + p^2 a_2, \dots).$$

In particular, its n^{th}-coordinate $\text{phant}_n : W(k) \to k$, $(a_0, a_1, \dots) \mapsto \sum_{i=0}^n p^i a_i^{p^{n-i}}$ is a ring homomorphism for each $n \geq 0$.

(iv) If $k \supseteq \mathbb{Q}$ then phant is an isomorphism of rings; if k is p-torsion-free then phant is injective.

(v) Given $a \in k$, its *Teichmüller lift* is $[a] := (a, 0, 0, 0, \dots) \in W(k)$; the map $[\cdot] : k \to W(k)$ is multiplicative but not additive.

(vi) Suppose now that $k \supseteq \mathbb{F}_p$. Then $W(k)$ is p-adically complete and

$$\sum_{i=0}^{\infty} [a_i] p^i = (a_0, a_1^p, a_2^{p^2}, \dots),$$

where $a_i \in k$. In particular, if k is perfect then one easily deduces the following: each element of $W(k)$ may be written uniquely as $\sum_{i=0}^{\infty} [a_i] p^i$ for some elements $a_i \in A$, the element $p \in W(k)$ is a non-zero-divisor, and $W(k)/pW(k) \overset{\simeq}{\to} k$, $(a_0, a_1, \dots) \mapsto a_0$.

(vii) Exercise: Continue to suppose that k is a perfect ring of characteristic p; also let $t \in k$ be a non-zero-divisor and $q \in W(k)$ an element such that $q \equiv p \bmod [t]W(k)$.

Check that $[t] \in W(k)$ is a non-zero-divisor (this does not use the hypotheses on k). Using that p is a non-zero-divisor of $W(k)$ and that t is a non-zero-divisor of $k = W(k)/pW(k)$, deduce that p is a non-zero-divisor of $W(k)/[t]W(k)$. Now deduce that q is a non-zero-divisor of $W(k)/[t]W(k)$, hence that $[t]$ is a non-zero-divisor of $W(k)/qW(k)$.

Now suppose further that k is t-adically complete. Prove by induction that $W(k)/p^n W(k)$ is $[t]$-adically complete for all $n \geq 1$, and take the limit to deduce that $W(k)$ is $[t]$-adically complete (even $(p, [t])$-adically complete). Next show that q is a non-zero-divisor of $W(k)$ (this is probably the hardest part of the exercise) and deduce that $W(k)/qW(k)$ is $[t]$-adically complete.

Theorem 1.10 (Fontaine)**.** *Let A be an integral perfectoid ring.*

(i) There is a unique ring homomorphism

$$\theta : W(A^\flat) \longrightarrow A$$

which satisfies $\theta([b]) = b^\#$ for all $b \in A^\flat$.

(ii) θ is surjective and its kernel is generated by a non-zero-divisor (usually denoted by $\xi \in W(A^\flat)$).

(iii) A given element $\chi \in \operatorname{Ker}\theta$ is a generator of this kernel if and only if its Witt vector expansion $\chi = (\chi_0, \chi_1, \dots)$ has the property that $\chi_1 \in A^{\flat\times}$.

Proof. (i): Since any element $b \in W(A^\flat)$ may be written uniquely as a p-adically convergent sum $b = \sum_{i\geq 0}[b_i]p^i$, the content of (i) is the assertion that the well-defined map

$$\theta : W(A^\flat) \longrightarrow A, \qquad b = \sum_{i=0}^{\infty}[b_i]p^i \mapsto \sum_{i=0}^{\infty} b_i^\# p^i$$

is actually a ring homomorphism (this map makes sense since A is p-adically complete). It is enough to check that this map is a ring homomorphism modulo any power of p (since A is p-adically separated), so fix $n \geq 0$.

We will use the phantom map from Remark 1.9(iii)

$$\mathrm{phant}_n : W(A) \xrightarrow{\mathrm{gh}} (A)^{\mathbb{N}} \xrightarrow{\mathrm{proj}_n} A, \qquad (a_0, a_1, \dots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n,$$

which is a ring homomorphisms. Note that if $a_i \equiv a_i'$ mod pA then $p^i a_i^{p^{n-i}} \equiv p^i a_i'^{p^{n-i}}$ mod $p^{n+1}A$, so in fact phant_n mod p^{n+1} only depends on the values of the Witt coordinates mod p, i.e., there is a commutative diagram

$$\begin{tikzcd}
W(A) \arrow[r, "\mathrm{phant}_n"] \arrow[d] & A \arrow[d] \\
W(A/pA) \arrow[r, "\overline{\mathrm{phant}}_n"'] & A/p^{n+1}A
\end{tikzcd}$$

in which $\overline{\mathrm{phant}}_n$ must also be a ring homomorphism. But the composition

$$W(A^\flat) \xrightarrow{W(\varphi^{-n})} W(A^\flat) \xrightarrow{W(\#\text{ mod }p)} W(A/pA) \xrightarrow{\overline{\mathrm{phant}}_n} A/p^{n+1}A$$

is exactly θ mod p^{n+1}:

$$\sum_{i=0}^{\infty}[b_i]p^i = (b_0, b_1^p, b_2^{p^2}, \dots) \mapsto (b_0^{p^n}, b_1^{p^{1-n}}, b_2^{p^{2-n}}, \dots) \mapsto (b_0^{p^n\#}, b_1^{p^{1-n}\#}, b_2^{p^{2-n}\#}, \dots) \mapsto \sum_{i=0}^{n} p^i b_i^{p^{i-n}\# p^{n-i}} = \sum_{i=0}^{n} p^i b_i^\#$$

Since the first two maps are ring homomorphisms (they are the maps on $W(-)$ induced by the ring homomorphisms $\varphi^{-n} : A^\flat \to A^\flat$ and $\# : A^\flat \to A/pA$), we deduce that θ mod p^{n+1} is a ring homomorphism, as required.

(ii): Since A and $W(A^\flat)$ are p-adically complete, to prove surjectivity of θ it is enough to show that it is surjective mod p. But this follows from the fact that $\#$ mod $p : A^\flat \to A/pA$ is surjective.

Now we construct a possible generator ξ of $\operatorname{Ker}\theta$. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power roots, and let $\pi^\flat = (\pi, \pi^{1/p}, \dots)$ be the associated perfectoid pseudo-uniformiser of $A^\flat$. Since $p \in \pi^p A$ and θ has been shown to be surjective, we may write $p = \pi^p\theta(-z)$ for some $z \in W(A^\flat)$, whence $\xi := p + [\pi^\flat]^p z \in \operatorname{Ker}\theta$. Note that ξ is a non-zero-divisor of $W(A^\flat)$, by applying Remark 1.9(vi) with $k = A^\flat$, $t = \pi^\flat$, $q = \xi$. We next show $\operatorname{Ker}\theta = \xi W(A^\flat)$. Since $W(A^\flat)$ is $[\pi^\flat]$-adically complete and A is $\theta([\pi^\flat]) = \pi$-torsion-free, one easily sees that $\theta : W(A^\flat)/\xi W(A^\flat) \to A$ is an isomorphism if and only if it becomes an isomorphism when we mod out by $[\pi^\flat]$; i.e., we must check that $\theta : W(A^\flat)/(\xi, [\pi^\flat]) \to A/\pi A$ is an isomorphism. But, using $\xi \equiv p \bmod [\pi^\flat]$, this map identifies with $A^\flat/\pi^\flat A^\flat \overset{\#\ \mathrm{mod}\ \pi}{\longrightarrow} A/\pi A$, which we saw was an isomorphism in the proof of Lemma 1.7(iii).

(iii): First note that the Witt vector expansion of our element ξ looks like

$$(\xi_0, \xi_1, \dots) = p + [\pi^\flat]^p x = (0, 1, 0, 0, \dots) + (\pi^{\flat p} x_0, \pi^{\flat p^2} x_1, \dots) = (\pi^{\flat p} x_0, 1 + \pi^{\flat p^2} x_1, \dots),$$

in particular $\xi_1 \in A^{\flat\times}$ (using $\pi^\flat$-adic completeness of $A^\flat$) and $\xi_0 \in \pi^\flat A^\flat$. Now let $\chi = (\chi_0, \chi_1, \dots) \in \operatorname{Ker}\theta$ be another element, and write $\chi = \beta\xi$ for some $\beta = (\beta_0, \beta_1, \dots) \in W(A^\flat)$. Expanding,

$$\chi = \beta\xi = (\beta_0, \beta_1, \dots)(\xi_0, \xi_1, \dots) = (\beta_0\xi_0, \beta_1\xi_0^p + \beta_0^p\xi_1, \dots).$$

Therefore:

$$\begin{aligned}
\operatorname{Ker}\theta = \xi W(A^\flat) &\iff \xi W(A^\flat) = \beta\xi W(A^\flat) \\
&\iff \beta \in W(A^\flat)^\times \text{ (using } \xi \text{ is a n-z-d)} \\
&\iff \beta_0 \in A^{\flat\times} \text{ (using that } W(A^\flat) \text{ is } p\text{-adically complete and } W(A^\flat)/pW(A^\flat) = A^\flat) \\
&\iff \beta_0^p\xi_1 \in A^{\flat\times} \text{ (since we already know } \xi_1 \in A^{\flat\times}) \\
&\iff \beta_1\xi_0^p + \beta_0^p\xi \in A^{\flat\times} \text{ (since } A^\flat \text{ is } \pi^\flat\text{-adically complete and } \xi_0 \in \pi^\flat A^\flat) \\
&\iff \xi \in A^{\flat\times},
\end{aligned}$$

completing the proof of part (iii). ☐

1.3 Tilting correspondence for integral perfectoid rings

We are now prepared to establish the easiest tilting correspondance,[3] namely that for integral perfectoid rings.

Given an integral perfectoid ring A and an A-algebra B, we always equip B with the canonical topology induced by A, i.e., we give B the π-adic topology where $\pi \in A$ is any perfectoid pseudo-uniformiser (this topology on B does not depend on the chosen π); we say that B is a *perfectoid A-algebra* if and only if B (equipped with the just-defined topology) is an integral perfectoid ring. Note that if B is a perfectoid A-algebra and $\pi \in A$ is any perfectoid pseudo-uniformiser, then the (image in B of) π is also a perfectoid pseudo-uniformiser of the perfectoid ring B; this follows from Lemma 1.4.

Theorem 1.11 (Tilting correspondence for integral perfectoid rings)**.** *Fix an integral perfectoid ring A. Then tilting induces an equivalence of categories*

$$\textit{perfectoid } A\textit{-algebras} \xrightarrow{\simeq} \textit{perfectoid } A^\flat\textit{-algebras}, \qquad B \mapsto B^\flat,$$

with inverse given by sending a perfectoid $A^\flat$-algebra C to $C^\# := W(C) \otimes_{W(A^\flat),\theta} A$.

[3] correspondance de basculement

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power-roots, and $\pi^\flat = (\pi, \pi^{1/\pi}, \dots) \in A^\flat$ the associated perfectoid pseudo-uniformiser of $A^\flat$; also let $\xi = p + [\pi^\flat]^p z \in W(A^\flat)$ be the generator of the ideal $\mathrm{Ker}(\theta : W(A^\flat) \to A)$ which we constructed in the proof of Theorem 1.10.

Step 1: Letting B be a perfectoid A-algebra, we show $(B^\flat)^\# = B$. We obviously have a commutative diagram (with surjective horizontal arrows by Theorem 1.10(ii))

$$\begin{tikzcd} W(B^\flat) \arrow[r, "\theta_B"] & B \\ W(A^\flat) \arrow[u] \arrow[r, "\theta=\theta_A"'] & A \arrow[u] \end{tikzcd}$$

and so the image of ξ in $W(B^\flat)$ lands in $\mathrm{Ker}\,\theta_B$ (this denotes the θ-map for the integral perfectoid ring B). But the first coordinate in the Witt vector expansion of ξ is a unit of $A^\flat$ (by Theorem 1.10(iii)), and so its image in $B^\flat$ is also a unit; therefore Theorem 1.10(iii) (this time for the ring B) implies that $\mathrm{Ker}\,\theta_B = \xi W(A^\flat)$. In other words, the above diagram is a pushout and so the induced map $(B^\flat)^\# = W(B^\flat) \otimes_{W(A^\flat),\theta} A \to B$ is an isomorphism, as required.

Step 2: Letting C be a perfectoid A-algebra, we show that $C^\#$ is a perfectoid $A^\flat$-algebra and that $(C^\#)^\flat = C$. Since θ is surjective with kernel $\xi W(A^\flat)$, we can write $C^\# = W(C)/\xi W(C)$ viewed as an A-algebra via the identification $\theta : W(A^\flat)/\xi W(A^\flat) \overset{\simeq}{\to} A$. Remark 1.9(vi) (with $k = C$, $t = \pi^\flat$, $q = \xi$) therefore shows that $C^\#$ is complete for the π-adic topology and that π is a non-zero-divisor of C. It remains to show that $\Phi : C^\#/\pi C^\flat \to C^\#/\pi^p C^\flat$ is an isomorphism. But again writing $C^\# = W(C)/\xi W(C)$ and recalling that $\xi \equiv p \bmod [\pi^\flat]^p$, this map may be rewritten as $\Phi : C/\pi^\flat C \to C/\pi^{\flat p} C$, which is indeed an isomorphism since $\pi^\flat$ is a perfectoid pseudo-uniformiser of C. This completes the proof that $C^\#$ is a perfectoid A-algebra.

Finally, as we already used in the previous paragraph, we have $C^\#/\pi C^\# = C/\pi^\flat C$. Tilting obtains

$$(C^\#)^\flat = \varprojlim_{x \mapsto x^p} C^\#/\pi C^\# = \varprojlim_{x \mapsto x^p} C/\pi^\flat C = C^\flat = C,$$

where the final equality is the fact that tilting an integral perfectoid ring of characteristic p has no effect. $\square$

1.4 Anneaux perfectoïdes entiers et basculement : exercices et exemples

(i) Soit C une extension algébrique de $\mathbb{Q}_p$, et $\mathcal{O}_C$ la clôture intégrale de $\mathbb{Z}_p$ dans C. Supposons que tout élément de $\mathcal{O}_C/p\mathcal{O}_C$ admette une racine p-ième et qu'il existe $\pi \in \mathcal{O}_C$ qui n'est pas une unité tel que $p \in \pi^p \mathcal{O}_C$ (par exemple $C = \mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ ou $\overline{\mathbb{Q}}_p$).

Démontrer que $\Phi : \mathcal{O}_C/\pi\mathcal{O}_C \to \mathcal{O}_C/\pi^p\mathcal{O}_C$, $a \mapsto a^p$ est un isomorphisme et en déduire que le complété p-adique $\widehat{\mathcal{O}_C}$ de $\mathcal{O}_C$ est un anneau perfectoïde entier.

(ii) Soit A un anneau perfectoïde entier. Son "algèbre de polynômes perfectoïde" est

$$A\langle T^{1/p^\infty}\rangle := \text{le complété } \pi\text{-adique de } \bigcup_{i \geq 1} A[T^{1/p^i}]$$

où π est n'importe quelle pseudo-uniformisante perfectoïde de A. Démontrer que $A\langle T^{1/p^\infty}\rangle$ est un anneau perfectoïde entier (en fait une A-algèbre perfectoïde).

On remarque que $A\langle T^{1/p^\infty}\rangle^\flat$ (càd le basculé de $A\langle T^{1/p^\infty}\rangle$) contient un élément $T^\flat := (T, T^{1/p}, T^{1/p^2}, \dots)$. Construire un isomorphisme de $A^\flat$-algèbres perfectoïdes

$$A^\flat\langle U^{1/p^\infty}\rangle \xrightarrow{\simeq} A\langle T^{1/p^\infty}\rangle^\flat$$

qui envoie U sur $T^\flat$, où le membre de gauche est une algèbre de polynômes perfectoïde sur $A^\flat$.

Démontrer les résultats analogues pour plusieurs variables $T_1, \dots, T_d$.

(iii) Soit A un anneau perfectoïde entier et B une A-algèbre étale. Démontrer que le complété π-adique $\widehat{B}$ est une A-algèbre perfectoïde. (Le fait suivant sera utile : si $k \to k'$ est un morphisme étale entre des anneaux de caractéristique p, alors le diagramme

$$\begin{tikzcd} k \arrow[r] & k' \\ k \arrow[r] \arrow[u, "\varphi"] & k' \arrow[u, "\varphi"'] \end{tikzcd}$$

est un pushout, càd $k' \otimes_{k,\varphi} k \xrightarrow{\simeq} k'$.)

(iv) Soit R une $\mathcal{O}_C$-algèbre lisse et supposons qu'il existe un morphisme étale $\mathcal{O}_C[T_1, \dots, T_d] \to R$ (si R est équi-dimensionnelle, cette hypothèse est toujours satisfaite localement sur Spec R). Utiliser (ii) et (iii) pour construire une $\widehat{\mathcal{O}_C}$-algèbre perfectoïde R_∞.

1.5 Treillis des sous-anneaux perfectoïdes entiers : exercices

Soit B un anneau perfectoïde entier.

(i) Soit $\pi \in B$ une pseudo-uniformisante perfectoïde qui admet une suit compatible de racines $\pi^{1/p}, \pi^{1/p^2}, \dots \in B$; soit $B^{\circ\circ}$ l'idéal des éléments topologiquement nilpotents de B. Montrer que $B^{\circ\circ} = \bigcup_{n\geq 0} \pi^{1/p^n} B$.

(ii) Étant donné un sous-anneau ouvert $A \subseteq B$, montrer que les assertions suivantes sont équivalentes :

(a) A est un anneau perfectoïde entier ;

(b) A est p-clos dans B (càd "$f \in B$ et $f^p \in A \Rightarrow f \in A$") ;

(c) $A \supseteq B^{\circ\circ}$ et $A/B^{\circ\circ}$ est p-clos dans $B/B^{\circ\circ}$.

En déduire que $A \mapsto A/B^{\circ\circ}$ définit une bijection de {sous-anneaux ouverts de B qui sont des anneaux perfectoïdes entiers} à {sous-anneaux p-clos de $B/B^{\circ\circ}$}. (De plus, que A est intégralement clos dans B ssi $A/B^{\circ\circ}$ est intégralement clos dans $B/B^{\circ\circ}$.)

(iii) Soit $A \subseteq B$ un sous-anneau ouvert qui est un anneau perfectoïde entier. Montrer que $A^\flat$ est un sous-anneau ouvert de $B^\flat$.

(iv) Montrer que l'application $\# : B^\flat \to B$ induit un isom. d'anneaux $B^\flat/B^{\flat\circ\circ} \xrightarrow{\simeq} B/B^{\circ\circ}$.

(v) ("Correspondance de basculement pour les sous-anneaux de B") Montrer que $A \mapsto A^\flat$ induit une bijection de {sous-anneaux ouverts de B qui sont des anneaux perfectoïdes entiers} à {idem. pour $B^\flat$} (De plus, que A est intégralement clos dans B ssi $A^\flat$ est intégralement clos dans $B^\flat$.)

2 Huber rings

The topological rings from which we will build adic spaces are Huber rings:

Definition 2.1. A *Huber ring* (or *f-adic ring* in the older terminology) is a topological ring R which satisfies the following: there exist an open subring $R_0 \subseteq A$ and a finitely generated ideal I of R_0 such that the topology on R_0 is the I-adic topology. (Warning: I is not usually an ideal of R!) Any such pair $I \subseteq R_0$ is called an *ideal of definition* and *subring of definition* of R.

Example 2.2. The simplest, but least interesting example, is as follows. Let R be a ring and $I \subseteq R$ a finitely generated ideal (even the case $I = 0$ is allowed). Then R is a Huber ring with ideal and subring of definition $I \subseteq R_0 := R$. See Example 2.5 for more interesting examples.

Note that the topology on R is uniquely determined by R_0 and I. However, the converse is more subtle. If R is a ring (without topology), $R_0 \subseteq R$ is a given subring, and I is an ideal of R_0, then there is obviously a uniquely linear topology on R for which R_0 is open in R and the induced topology on R_0 is the I-adic topology. Indeed, the unique linear topology with these properties has basis $f + I^m$, for $f \in R$ and $m \geq 1$. However, if we equip R with this topology, then it is not necessarily a topological ring (more precisely, multiplication is not necessarily continuous)! We leave it to the reader to construct such an example. Therefore the definition of a Huber ring involves a subtle compatibility between between the topology and the algebra.

However, as we will see in Proposition 2.4, this subtlety does not occur when constructing Tate rings.

Definition 2.3. A *Tate ring* R is a Huber ring with the following additional property: there exists a unit $\pi \in R^\times$ such that $\pi^n \to 0$ as $n \to \infty$. Any such element π is called a *pseudo-uniformiser* of R.

Proposition 2.4. *(i) Let A be a ring and $\pi \in A$ a non-zero-divisor; equip $A[\frac{1}{\pi}]$ with the topology having basis $f + \pi^m A$ for $f \in A[\frac{1}{\pi}]$, $m \geq 1$. Then $A[\frac{1}{\pi}]$ is a Tate ring with ideal and subring of definition $\pi A \subseteq A$.*

(ii) Conversely, let R be a Tate ring; chose a subring of definition R_0 and a pseudo-uniformiser $\pi \in R$. Then there exists $m \geq 1$ such that $\pi^m \in R_0$; moreover, then $\pi^m R_0$ is an ideal of definition, π^m is a non-zero-divisor of R_0, and $R = R_0[\frac{1}{\pi^m}]$ with topology as in (i).

Proof. (i): Exercise. (ii): Since R_0 is open and π is topologically nilpotent, there exists $m \geq 1$ such that $\pi^m \in R_0$. Since π^m is a unit of the topological ring R and R_0 is open, the ideal $\pi^m R_0$ is also open. Next let $I \subseteq R_0$ be an ideal of definition; since $\pi^m R_0$ is open, we have $I^n \subseteq \pi^m R_0$ for $n \gg 0$. Conversely, since π^m is topologically nilpotent, we also have $\pi^{mn} \in I$ for $n \gg 0$. This shows that $\pi^m R_0$ is also an ideal of definition.

Clearly π^m is a non-zero-divisor of R_0 (since it is a unit in the larger ring R). Moreover, given $f \in R$, continuity of multiplication implies $\pi^{mn} f \to 0$ as $n \to \infty$, so $\pi^{mn} f \in R_0$ for $n \gg 0$; this shows $R = R_0[\frac{1}{\pi^m}]$. □

Example 2.5. Here are some example of Tate rings.

(i) If A is an integral perfectoid ring and $\pi \in A$ is a perfectoid pseudo-uniformiser, then $A[\frac{1}{\pi}]$ is a Tate ring by Proposition 2.4(i). We will study such *perfectoid Tate rings* in Section 3.

(ii) $\mathbb{Q}_p$ is a Tate ring, with ideal and subring of definition $p\mathbb{Z}_p \subseteq \mathbb{Z}_p$.

(iii) More generally, if A is any flat $\mathbb{Z}_p$-algebra then we may equip it with the p-adic topology and form a Tate ring $A[\frac{1}{p}]$

(iv) $\mathbb{Q}_p\langle T\rangle := \{\sum_{i=0}^{\infty} a_i T^i : a_i \in \mathbb{Q}_p,\, a_i \to 0\} = \widehat{\mathbb{Z}_p[T]}[\frac{1}{p}]$ (where $\widehat{}$ denotes the p-adic completion).

2.1 Bounded sets

Definition 2.6. Let R be a Huber ring. A subset $S \subseteq R$ is called *bounded* if and only if for each open neighbourhood $0 \in U \subseteq R$ there exists an open neighbourhood $0 \in V \subseteq R$ such that $sv \in U$ for all $s \in S$ and $v \in V$.

We will always use the following notation: given subsets $T, S \subseteq A$, we write $ST \subseteq A$ for the subgroup generated by all products st, for $s \in S$, $t \in T$.

Lemma 2.7. *Let R be a Huber ring and $I \subseteq R_0 \subseteq R$ an ideal and a subring of definition. Then $S \subseteq R$ is bounded if and only if there exists $n \geq 1$ such that $SI^n \subseteq R_0$.*

Proof. $\Rightarrow$: Suppose $S \subseteq R$ is bounded. Since R_0 is open, there exists an open nhood $0 \in V \subseteq R$ such that $sV \subseteq R_0$ for all $s \in S$. But $0 \in I^n \subseteq V$ for $n \gg 0$, whence $sI^n \subseteq R_0$ for all $s \in S$ and so $SI^n \subseteq A_0$ since R_0 is closed under addition.

$\Leftarrow$: Suppose there exists $n \geq 1$ such that $SI^n \subseteq R_0$. Then, for any open $0 \in U \subseteq R$, pick $m \geq 1$ such that $0 \in I^m \subseteq U$; then $V := I^{n+m}$ works, since clearly $SI^{n+m} \subseteq R_0 I^m \subseteq I^m \subseteq U$. $\square$

Corollary 2.8. *Let R be a Huber ring. If $S, T \subseteq R$ are bounded, then so are $S+T$, $S \cup T$, and ST. Any finite subset of R is bounded. If R_0 is a subring of definition and $M \subseteq R$ is a finitely generated R_0-module, then M is bounded.*

Proof. This is all easy using Lemma 2.7. For example, pick $m \geq 1$ such that $I^m S \subseteq R_0$ and $I^m T \subseteq R_0$. Then $I^{2m} ST = I^m I^m ST \subseteq I^m R_0 T = I^m T \subseteq R_0$. For the finite set claim it is enough to show that singletons are bounded, i.e., given $f \in R$ there exists $m \geq 1$ such that $fI^m \subseteq R_0$; this follows from the fact that R_0 is an open neighbourhood of 0, that powers of I are an open nhood basis at 0, and that multiplication $R \times R \to R$ is continuous.

For the finitely generated module note that $M = SR_0$ for some finite set S, so we can apply the previous parts. $\square$

The following is the first result where we use the fact that the ideal of definition is required to be finitely generated:

Corollary 2.9. *Let R be a Huber ring and $f_1, \ldots, f_n \in R$ some elements which generate an open ideal, i.e., $f_1R + \cdots + f_nR$ is open. Let R_0 be a subring of definition. Then $f_1R_0 + \cdots + f_nR_0$ is open.*

Proof. Let $I \subseteq R_0$ be an ideal of definition. By hypothesis $I^m \subseteq f_1R + \cdots + f_nR$ for some $m \geq 1$. But I^m is a finitely generated R_0-module, so clearly there exists a finitely generated R_0-module $M \subseteq A$ such that $I^m \subseteq f_1R + \cdots + f_nR$. But M is bounded by the previous corollary, so $I^{m'} M \subseteq R_0$ for some $m' \geq 1$. Then $I^{m+m'} \subseteq f_1R_0 + \cdots + f_nR_0$. $\square$

The following is fundamental:

Proposition 2.10. *Let R be a Huber ring and $A \subseteq R$ a subring. Then A is a ring of definition if and only if it is bounded and open in R.*

Proof. $\Rightarrow$: Suppose A is a ring of definition. Then it is open by definition, and bounded by Lemma 2.7 (take $R_0 = A$ and $n = 1$).

$\Leftarrow$: Suppose A is bounded and open. Let $I \subseteq R_0 \subseteq A$ be an ideal and a subring of definition; let $T \subseteq R_0$ be a finite set such that $I = TR_0$. Since A is open and bounded, there exists (using Lemma 2.7) $m \geq 1$ such that $I^m \subseteq A$ and $I^m A \subseteq R_0$.

Let $J := T^m A$ be the ideal of A generated by the finite set $T^m \subseteq A$. Then $J^2 = T^m T^m A \subseteq T^m R_0 = I^m$ and $J = T^m A \supseteq T^m I^m = I^{2m}$.

Therefore powers of J and powers of I define the same topology on R, so $J \subseteq A$ are indeed an ideal and a subring of definition. □

Corollary 2.11. *Let R be a Huber ring.*

(i) If $R_0, R_0' \subseteq A$ are subrings of definition, then so are R_0R_0' and $R_0 \cap R_0'$ (note that these really are subrings; also note that $R_0R_0' \supseteq R_0, R_0'$); in particular, the subrings of definition form a filtered family.

(ii) Suppose that $A \subseteq B$ are subrings of R such that A is bounded and B is open. Then there exists a subring of definition R_0 such that $A \subseteq A_0 \subseteq B$.

Proof. (i) Easy exercise: check that R_0R_0' and $R_0 \cap R_0'$ are both open and bounded, then apply the previous proposition.

(ii) Let R_0 be any subring of definition; then R_0B is open and bounded, so $R_0B \cap C$ is also open and bounded, hence a subring of definition by the previous proposition. □

Definition 2.12. Let R be a Huber ring. An element $f \in R$ is called *power bounded* if and only if the set $f^{\mathbb{N}} := \{f^n : n \geq 0\}$ is bounded. Let $R^\circ \subseteq R$ be the subset of power bounded elements.

An element $f \in R$ is called *topologically nilpotent* if and only if $f^n \to 0$ as $n \to \infty$. Let $R^{\circ\circ} \subseteq R$ be the subset of topologically nilpotent element.

Lemma 2.13. *Let R be a Huber ring.*

(i) R° is an open subring, integrally closed in R.

(ii) R° is the union of all subrings of definition.

(iii) $R^{\circ\circ}$ is an open ideal of R°.

Proof. (i): Let $f, g \in R^\circ$. Then $f^{\mathbb{N}}g^{\mathbb{N}}$ is bounded by Corollary 2.8. But $f^{\mathbb{N}}g^{\mathbb{N}} \supseteq \{(f+g)^n, (fg)^n : n \geq 0\}$, and therefore $f+g$ and fg are also power bounded.

Next suppose that $x \in R$ is integral over R°, so that there are $a_0, \ldots, a_{d-1} \in R^\circ$ such that $x^d + a_{d-1}x^{d-1} + \cdots + a_0 = 0$. Then it is easy to see that $x^{\mathbb{N}} \subseteq a_0^{\mathbb{N}} \cdots a_{d-1}^{\mathbb{N}}\{1, x, \ldots, x^{d-1}\}$. The set on the right is bounded by Corollary 2.8, whence $f^{\mathbb{N}}$ is also bounded, i.e., $f \in R^\circ$.

Given any subring of definition A_0 and $f \in R_0$, we have $f^{\mathbb{N}} \subseteq R_0$ and so $f^{\mathbb{N}}$ is bounded (since R_0 is bounded); therefore $R_0 \subseteq R^\circ$, showing that R° is open.

(ii): We have just noted that any subring of definition R_0 is contained in R°. Conversely, supposing $f \in R^\circ$, we must find a subring of definition containing f. Let R_0 be any subring of definition, and note that $R_0f^{\mathbb{N}}$ is R_0-subalgebra of R generated by f; this is open (since it containes the open subring R_0) and bounded (by Corollary 2.8), hence is a subring of definition (by Proposition 2.10).

(iii): Exercise. □

It is important to note that R° is not necessarily bounded (e.g., Consider the Tate ring $R = \mathbb{Q}_p[\varepsilon]/\varepsilon^2$, with subring of definition $R_0 = \mathbb{Z}_p[\varepsilon]/\varepsilon^2$ equipped with the p-adic topology. Then any multiple of ε is nilpotent, hence power bounded, and so $R^\circ = \mathbb{Z}_p + \mathbb{Q}_p\varepsilon$; in particular $p^n R^\circ \not\subseteq R_0$ for all $n \geq 1$, whence R° is not bounded.). We say that the Huber ring R is *uniform* if and only if R° is bounded; this will turn out to be the case for perfectoid Tate rings.

Definition 2.14. Let R be a Huber ring. A subring R^+ is called a *subring of integral elements* if and only if it is open, integrally closed subring in R, and $R^+ \subseteq A^\circ$. For example, Lemma 2.13(i) clearly implies that R° is a subring of integral elements (moreover, it is the largest subring of integral elements).

A *Huber pair* (or *affinoid ring* in the older terminology) (R, R^+) is the data of a Huber ring R and a chosen subring of integral elements R^+.

3 Perfectoid Tate rings

As explained in Proposition 2.4, if we are given an integral perfectoid ring A we can construct a Tate ring $A[\frac{1}{\pi}]$, where π is any perfectoid pseudo-uniformiser of A. Note that $A[\frac{1}{\pi}]$ does not depend on the choice of π: given another π', the elements π and π' define the same topology on A, hence each divides a power of the other, and so $A[\frac{1}{\pi}] = A[\frac{1}{\pi}']$; in fact, $A[\frac{1}{\pi}]$ can be written without making any choices as

$$A[\tfrac{1}{\pi}] = A[\tfrac{1}{f} : f \in A \text{ and } fA \text{ is open in } A].$$

In any case, the Tate ring $A[\frac{1}{\pi}]$ is called the *generic fibre* of A, and so we have defined a functor

$$\text{integral perfectoid rings} \longrightarrow \text{Tate rings}, \qquad A \mapsto A[\tfrac{1}{\pi}]$$

The image of this functor is precisely the perfectoid Tate rings:

Definition 3.1. A Tate ring R is called *perfectoid* if and only if the following equivalent conditions are satisfied:

(i) R has a subring of definition which is an integral perfectoid ring;

(ii) R is in the image of the above functor;

(iii) the topological ring R° is integral perfectoid;

(iv) R is uniform and there exists a pseudo-uniformiser $\pi \in R$ such that $p \in \pi^p R^\circ$ and $\Phi : R^\circ/\pi R^\circ \to R^\circ/\pi^p R^\circ$, $f \mapsto f^p$ is an isomorphism (Fontaine's Bourbaki definition).

It is convenient to prove the equivalences at the same time as the following proposition:

Proposition 3.2. *Let R be a perfectoid Tate ring and $R_0 \subseteq R$ a subring of definition. Then R_0 is integral perfectoid if and only if it is p-closed in R (i.e., "$f \in R$ and $f^p \in R_0 \Rightarrow f \in R_0$"). In particular, every subring of integral elements $R^+ \subseteq R$ is integral perfectoid.*

Proof of equivalences and the proposition. (iv)⇒(i): If R is uniform then R° is a subring of definition; since $\pi \in R^\circ$, Proposition 2.4 shows that the topology on R° is the π-adic topology, and the other conditions in (iv) show that R° is integral perfectoid.

(i)⇒(ii): Suppose that $R_0 \subseteq R$ is an integral perfectoid subring of definition. Let $\pi \in R_0$ be a perfectoid pseudo-uniformiser and note that π is necessarily a pseudo-uniformiser of the Tate ring R (The proof is standard theory of Tate rings, similar to Proposition 2.4: Firstly, π is topologically nilpotent since it defines the topology on R_0; secondly, fixing a pseudo-uniformiser $\varpi \in R$, the openness of πR_0 implies $\varpi^n \in \pi R_0$ for $n \gg 0$, whence π is a unit in R.). Proposition 2.4 shows that $R = R_0[\frac{1}{\pi}]$, i.e., R is in the image of the above functor.

(iii)⇒(iv): If R° is integral perfectoid then it is an open subring whose topology is adic for a finitely generated ideal, whence it is a subring of definition; but any subring of definition is bounded, so this shows that R is uniform. Any choice of perfectoid pseudo-uniformiser π for R° is a pseudo-uniformiser for R (by the same argument as in the previous paragraph) with the desired properties in (iv).

To complete the proof of the equivalences, we must show (ii)⇒(iii), so suppose that $R = A[\frac{1}{\pi}]$, where A is an integral perfectoid ring and $\pi \in R$ is a pseudo-uniformiser. We note first that $R^{\circ\circ} \subseteq A$: indeed, if $f \in R$ is topologically nilpotent then $f^{p^n} \in A$ for $n \gg 0$, and so $f \in A$ by Lemma 3(ii). In particular this shows that $\pi R^\circ \subseteq A$; therefore R° is bounded, i.e., R is uniform.

Now let $R_0 \subseteq R$ be any p-closed subring of definition (e.g., $R_0 = R^\circ$, since we have just shown R° is bounded, hence is a subring of definition); we will prove that R_0 is integral perfectoid. Just as in the previous paragraph, p-closedness implies that $R^{\circ\circ} \subseteq A$; in particular $\pi \in R_0$, whence the topology on R_0 is the π-adic topology by Proposition 2.4 (this proves condition (a) in the definition of integral perfectoid).

We claim that every element of R_0 is a p^{th}-power modulo π (resp. $\pi^{1/2}$ if $p = 2$); let $f \in R_0$. Then $\pi f \in R^{\circ\circ} \subseteq A$ and so there exist $y, z \in A$ such that $\pi x = y^p + \pi^p z$; after multiplying π by a unit we may assume it admits a p^{th}-root in A, and we then deduce that $(y\pi^{-1/p})^p = x - \pi^{p-1}z \in R_0$ (note that $\pi^{p-1}z$ is topologically nilpotent, hence in R_0), whence $y\pi^{-1/p} \in R_0$ by topological nilpotence again. Since $\pi^{p-1}z \in \pi R_0$ (unless $p = 2$, in which case it is $\in \pi^{1/2}R_0$ since $\pi^{1/2}z \in R_0$), we have shown that x is a p^{th}-power modulo πR_0 (resp. $\pi^{1/2}R_0$), which proves the claim.

Next note that $p \in (\pi^{1/p})^p R_0$: indeed, we know that $p \in \pi^p A \subseteq \pi R^{\circ\circ}$ and that $R^{\circ\circ} \subseteq R_0$. Since the p-closedness of R_0 in R easily implies that $\Phi : R_0/\pi^{1/p}R_0 \to R_0/\pi R_0$ (resp. $R_0/\pi^{1/4}R_0 \to R_0/\pi^{1/2}R_0$) is injective, we have indeed proved that R_0 is integral perfectoid (with pseudo-uniformiser $\pi^{1/p}$, resp. $\pi^{1/4}$ if $p = 2$).

In conclusion, assuming condition (ii), we have proved that R° is integral perfectoid, and more generally that any p-closed subring of definition is integral perfectoid. This completes the proof that the conditions in Definition 3.1 are equivalent, and establishes the implication $\Leftarrow$ in the proposition; meanwhile, the implication $\Rightarrow$ is a consequence of (ii). For the final sentence in the proposition just note that, since R is uniform, any subring of integral elements is a subring of definition. □

Corollary 3.3. *Let R be a perfectoid Tate ring. Then any integral perfectoid subring of definition R_0 contains $R^{\circ\circ}$, and the resulting functor $R_0 \mapsto R_0/R^{\circ\circ}$ defines a bijection*

$$\{\textit{integral perfectoid subrings of definition of } R\} \xrightarrow{\simeq} \{p\textit{-closed subrings of } R^\circ/R^{\circ\circ}\}$$

This restricts to a bijection

$$\{\textit{subrings of integral elements of } R\} \xrightarrow{\simeq} \{\textit{integrally closed subrings of } R^\circ/R^{\circ\circ}\}$$

Proof. By basic commutative algebra, $A \mapsto A/R^{\circ\circ}$ defines a bijection

$$\{\text{subrings of } R^\circ \text{ containing } R^{\circ\circ}\} \xrightarrow{\simeq} \{\text{subrings of } R^\circ/R^{\circ\circ}\}.$$

Moreover, the reader can easily check that, given a ring A such that $R^{\circ\circ} \subseteq A \subseteq R^\circ$, then A is p-closed (resp. integrally closed) in R° if and only if $A/R^{\circ\circ}$ is p-closed (resp. integrally closed) in $R^\circ/R^{\circ\circ}$.

To complete the proof, it remains only to use Proposition 3.2 and the observation (which already appeared in the previous proof) that any open p-closed subring of R must contain $R^{\circ\circ}$. □

Remark 3.4. Similarly to Lemma 1.5, one can prove the following: Given a complete Tate ring R of characteristic p, then R is perfectoid if and only if it is perfect. By using Lemma 1.5, the only non-trivial part of the proof is the following: assuming that R is perfect, we will show that R is uniform. This was noticed first by Yves André, whose proof we give here.

Let $R_0 \subseteq R$ be a subring of definition, and $\pi \in R_0$ a pseduo-uniformiser of R. For each $n \geq 1$ set $R_n := \varphi^{-n}(R_0) = \{f \in R : f^{p^n} \in R_0\}$, which is a subring since R has characteristic p. The Frobenius morphism $\varphi : R \to R$ is a continuous bijection (since R is assumed to be perfect), hence is a homeomorphism by Banach's open mapping theorem (it is folklore that Banach's open mapping theorem holds in the generality of complete Tate rings; see, e.g., Henkel "An open mapping theorem..."). Therefore $\varphi(R_0)$ is an open subring of R, so there exists $m \geq 1$ such that $\pi^m R_0 \subseteq R_0$; applying φ^{-n} shows that $\pi^{m/p^n} A_n \subseteq A_{n-1}$ for all $n \geq 1$. By a trivial induction it this means that $\pi^{\sum_{i=1}^n m/p^i} A_n \subseteq A_0$, whence $\pi^m A_n \subseteq A_0$ for all $n \geq 0$ (since $m \geq \sum_{i=1}^n m/p^i$). Next, given $f \in R^\circ$, the set $f^{\mathbb{N}}$ is bounded and so there exists $n \geq 1$ such that $\pi^{p^n} f^{\mathbb{N}} \subseteq R_0$; in particular $\pi^{p^n} f^{p^n} \in R_0$, i.e., $\pi f \in R_n$ and so $\pi^{m+1} f \in \pi R_n \subseteq R_0$. This shows that $\pi^{m+1} R^\circ \subseteq R_0$, i.e., R° is bounded.

3.1 Aside: The language of almost mathematics

Before we can discuss tilting perfectoid Tate rings in the next subsection, it is useful to introduce some language of "almost mathematics".

Throughout this subsection we let A be an integral perfectoid ring, and $A^{\circ\circ} \subseteq A$ the open ideal consisting of topologically nilpotent elements. (Exercise: letting $\pi \in A$ be any perfectoid pseudo-uniformiser admitting compatible p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots$, check that $A^{\circ\circ} = \bigcup_{n \geq 0} \pi^{1/p^n} A$. In particular, this shows that the ideal $A^{\circ\circ}$ is its own square, which is key to the following theory.)

We say that an A-module M is *almost zero* if and only if $A^{\circ\circ} M = 0$; by the exercise, this is equivalent to saying that $\pi^{1/p^n} M = 0$ for all $n \geq 0$. Similarly, we say that a map of A-modules $M \to N$ is an *almost injection/surjection/isomorphism* if and only if the kernel/cokernel/both is almost zero.

The following should look surprising at first glance (it is not true if we replace $A^{\circ\circ}$ by an arbitrary ideal):

Lemma 3.5. *The category of almost zero A-modules is closed under the following operations: sub-modules, quotients, extensions, all limits, all colimits. Given a π-adically complete A-module M, then M is almost zero if and only if $M/\pi M$ is almost zero. Moreover, almost isomorphisms are closed under base change along an arbitrary module.*

Proof. The surprising fact is extensions: suppose that $0 \to M \to N \to P \to 0$ is a short exact sequence of A-modules such that M and P are killed by $A^{\circ\circ}$; the N is killed by the ideal $(A^{\circ\circ})^2$, but we have noted above that $(A^{\circ\circ})^2 = A^{\circ\circ}$. We leave it to the reader as an exercise in almost mathematics to check the other assertions. $\square$

Lemma 3.6. *Let B be a perfectoid A-algebra. Then $A \to B$ is an almost isomorphism if and only if $A^\flat \to B^\flat$ is an almost isomorphism.*

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting compatible p-power roots, and $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in A^\flat$ the corresponding perfectoid pseudo-uniformiser of $A^\flat$. Recall from the discussion before Theorem 1.11 that π is automatically a perfectoid pseduo-uniformiser for B (and similarly $\pi^\flat$ is a ppu for $B^\flat$).

$\Rightarrow$: $A \to B$ being an almost isomorphism means that it is injective (since A has no π-torsion), so we may view B as an extension of A, and that $\pi^{1/p^n} B \subseteq A$ for all $n \geq 0$. From the injectivity it is clear that

$$A^\flat = \varprojlim_{x \mapsto x^p} A \longrightarrow \varprojlim_{x \mapsto x^p} B = B$$

is injective. Moreover, given an element $b = (b_0, b_1, \dots) \in B^\flat$ and $n \geq 1$, we have

$$\pi^{\flat 1/p^n} b = (\pi^{1/p^n}, \pi^{1/p^{n+1}}, \dots)(b_0, b_1, \dots) = (b_0 \pi^{1/p^n}, b_1 \pi^{1/p^{n+1}}, \dots) \in A^\flat,$$

showing that $A\flat \to B^\flat$ is almost surjective.

$\Leftarrow$: $A^\flat \to B^\flat$ being an almost isomorphism means that $A^\flat/\pi^\flat A^\flat \to B^\flat/\pi^\flat B^\flat$ is almost an almost isomorphism (by the base change assertion in Lemma 3.5). But we know from the proof of Lemma 1.7 that $A^\flat/\pi^\flat A^\flat = A/\pi A$ and $B^\flat/\pi^\flat B^\flat \to B/\pi B$; so we have shown that $A \to B$ is an almost isomorphism modulo π. Since A and B are complete, (a modification of) Lemma 3.5 implies $A \to B$ is an almost isomorphism. $\square$

Lemma 3.7. *Let B be a perfectoid A-algebra. Then $A \to B$ is an almost isomorphism if and only if the induced map of perfectoid Tate rings $A[\frac{1}{\pi}] \to B[\frac{1}{\pi}]$ (i.e., the generic fibres) is an isomorphism.*

Proof. $\Rightarrow$: Suppose that $A \to B$ is an almost isomorphism. Then the kernel and cokernel are in particular killed by π, whence $A[\frac{1}{\pi}] \xrightarrow{\simeq} B[\frac{1}{\pi}]$ as desired.

$\Leftarrow$: Suppose that $A[\frac{1}{\pi}] \xrightarrow{\simeq} B[\frac{1}{\pi}]$. Denoting this common perfectoid Tate ring by R, we may therefore view $A \subseteq B$ as integral perfectoid subrings of definition of R. As we stated in Corollary 3.3, this implies $R^{\circ\circ} \subseteq A$; in particular, $\pi^{1/p^n} B \subseteq A$ for all $n \geq 1$, i.e., $A \to B$ is an almost isomorphism. $\square$

3.2 Tilting perfectoid Tate rings

The tilt of a perfectoid Tate ring R is defined to be the perfectoid Tate ring of characteristic p given by

$$\begin{aligned} R^\flat :=& \text{generic fibre of } R_0^\flat \\ =& R_0^\flat[\tfrac{1}{\pi^\flat}] \end{aligned}$$

where $R_0 \subseteq R$ is any integral perfectoid subring of definition and $\pi \in R_0$ is a perfectoid pseudo-uniformiser with compatible p-power roots. This does not depend on the chosen subring of definition (or perfectoid pseudo-uniformiser): indeed, obviously the two integral perfectoid subrings $R_0 \subseteq R^\circ$ have the same generic fibre, whence Lemmas 3.6 and 3.7 show that the two integral perfectoid rings $R_0^\flat \subseteq R^{\circ\flat}$ also have the same generic fibre. In other words, we could canonically define $R^\flat$ to be the generic fibre of $R^{\circ\flat}$; from this point of view, we have just shown that any other integral perfectoid subring of definition $R_0 \subseteq R$ tilts to an integral perfectoid subring of definition $R_0^\flat \subseteq R^\flat$.

Theorem 3.8 (Tilting correspondence – lattice of subrings)**.** *Let R be a perfectoid Tate ring. Then $R^{\circ\flat}$ is the subring of power bounded elements of R (i.e., $R^{\circ\flat} = R^{\flat\circ}$). Moreover, tilting $R_0 \mapsto R_0^\flat$ defines a bijection*

$$\{\textit{integral perfectoid subrings of definition of } R\} \xrightarrow{\simeq} \{\textit{integral perfectoid subrings of definition of } R^\flat\},$$

which restricts to a bijection

$$\{\textit{subrings integral elements of } R\} \xrightarrow{\simeq} \{\textit{subrings of integral elements of } R^\flat\}.$$

Proof. We begin by proving that $R^{\circ\flat} = R^{\flat\circ}$. Since $R^{\circ\flat}$ is a subring of definition for $R^\flat$, certainly we have $R^{\circ\flat} \subseteq R^{\flat\circ}$ and so we may view $R^{\flat\circ}$ as a perfectoid $R^{\circ\flat}$-algebra. By Theorem 1.11, untilting gives us a perfectoid R°-algebra B such that $B^\flat = R^{\flat\circ}$. Since $R^\circ \to B$ induces an isomorphism on generic fibres after tilting (namely $R \xrightarrow{=} R$), Lemma 3.7 tells us that it induces an isomorphism on generic fibres before tilting, i.e., we have $R^\circ \subseteq B \subseteq R$ where B is a subring of definition of R. But B being a subring of R implies $B \subseteq R^\circ$, i.e., $R^\circ = B$; tilting reveals $R^{\circ\flat} = R^{\flat\circ}$, as desired.

Let $\pi \in R^\circ$ be a perfectoid pseudo-uniformiser admitting p-power roots, and $\pi^\flat \in R^{\circ\flat} = R^{\flat\circ}$ the associated perfectoid pseudo-uniformiser of $R^{\flat\circ}$. We know that the untilting map $\#$ induces an isomorphism $R^{\flat\circ}/\pi^\flat R^{\flat\circ} \xrightarrow{\simeq} R^\circ/\pi R^\circ$. Since $R^{\circ\circ}/\pi R^\circ$ is the ideal of nilpotent elements of $R^\circ/\pi R^\circ$, and similarly on the tilted side, we also get an isomorphism $R^{\flat\circ}/R^{\flat\circ\circ} \xrightarrow{\simeq} R^\circ/R^{\circ\circ}$.

The desired bijections up subrings now follows by applying Corollary 3.3 to both R and $R^\flat$. $\square$

Finally we note that the tilting correspondence for integral perfectoid rings extends to the generic fibres:

Theorem 3.9 (Titling correspondence – perfectoid Tate rings)**.** *Let R be a perfectoid Tate ring. Then tilting $S \mapsto S^\flat$ defines an equivalence of categories*

$$\{\textit{perfectoid Tate rings over } R\} \simeq \{\textit{perfectoid Tate rings over } R^\flat\}$$

Proof. Exercice. $\square$

Exercices

Soit B un anneau perfectoïde entier.

(i) Étant donné un sous-anneau ouvert $A \subseteq B$, montrer que les assertions suivantes sont équivalentes :

  (a) A est un anneau perfectoïde entier ;

  (b) A est p-clos dans B (càd "$f \in B$ et $f^p \in A \Rightarrow f \in A$") ;

  (c) $A \supseteq B^{\circ\circ}$ et $A/B^{\circ\circ}$ est p-clos dans $B/B^{\circ\circ}$.

  En déduire que $A \mapsto A/B^{\circ\circ}$ définit une bijection entre {sous-anneaux ouverts de B qui sont des anneaux perfectoïdes entiers} et {sous-anneaux p-clos de $B/B^{\circ\circ}$}. (De plus, que A est intégralement clos dans B ssi $A/B^{\circ\circ}$ est intégralement clos dans $B/B^{\circ\circ}$.)

(ii) Soit $A \subseteq B$ un sous-anneau ouvert qui est un anneau perfectoïde entier. Montrer que $A^\flat$ est un sous-anneau ouvert de $B^\flat$.

(iii) Montrer que l'application $\# : B^\flat \to B$ induit un isom. d'anneaux $B^\flat/B^{\flat\circ\circ} \xrightarrow{\simeq} B/B^{\circ\circ}$.

(iv) ("Correspondance de basculement pour les sous-anneaux de B") Montrer que $A \mapsto A^\flat$ induit une bijection entre {sous-anneaux ouverts de B qui sont des anneaux perfectoïdes entiers} et {idem. pour $B^\flat$} (De plus, que A est intégralement clos dans B ssi $A^\flat$ est intégralement clos dans $B^\flat$.)

Soit R un anneau perfectoïde de Tate ; on va appliquer les résultats ci-dessus à l'anneau perfectoïde entier R°. Rappelons que le basculé de R est par définition la fibre générique $R^\flat$ de $R^{\circ\flat}$. On a montré le 8 mars que $R^{\circ\flat} = R^{\flat\circ}$.

(v) Étant donné un sous-anneau de définition qui est un anneau perfectoïde entier $R_0 \subseteq R$, montrer de (ii) que $R_0^\flat \subseteq R^\flat$ l'est aussi et en déduire que R est la fibre générique de $R_0^\flat$. (En deux mots, le basculé $R^\flat$ peut être défini d'être la fibre générique du basculé de n'importe quel sous-anneau de définition R_0 qui est un anneau perfectoïde.)

(vi) Déduire de (iv) que $R_0 \mapsto R_0^\flat$ induit des bijections

- {sous-anneaux de définition de R qui sont des anneaux perfectoïdes entiers} $\tilde{\to}$ {idem. pour $R^\flat$} ;
- {sous-anneaux ouverts et intégralement clos de R°} $\tilde{\to}$ {idem. pour $R^{\flat\circ}$}.

Utiliser la Correspondance de basculement pour les anneaux perfectoïdes entiers pour démontrer la Correspondance de basculement pour les anneaux perfectoïdes de Tate (Déf : un *anneau perfectoïde de Tate sur* R est un anneau perfectoïde de Tate S muni d'un morphisme continue $R \to S$) :

Theorem 3.10. *Le basculement* $S \mapsto S^\flat$ *induit une équivalence de catégories* {*anneaux perfectoïdes de Tate sur* R} $\tilde{\to}$ {*anneaux perfectoïdes de Tate sur* $R^\flat$}.

4 Perfectoid fields

Let K be a topological field. We say that K is a *perfectoid field* if and only if the following conditions are satisfied:

(a) the topology on K is induced by a valuation $|\cdot|: K \to \mathbb{R}_{\geq 0}$, and K is complete for this topology;

(b) there exists a non-unit $\pi \in \mathcal{O}_K$ such that $p \in \pi^p \mathcal{O}_K$;

(c) every element of $\mathcal{O}_K/p\mathcal{O}_K$ is a p^{th}-power.

Here $\mathcal{O}_K := \{f \in K : |f| \leq 1\}$ denotes the ring of integers of the valuation $|\cdot|$, which depends only on the topological field K and not on the chosen valuation: indeed, $f \notin \mathcal{O}_K$ iff $|f^{-1}| < 1$ iff $f^{-n} \to 0$ as $n \to \infty$ (See the exercises of the next section).

4.1 Exercices sur la fondation des corps perfectoïdes

Ces exercices établissent les fondations de la théorie des corps perfectoïdes, y compris leur correspondance de basculement de base, sans référence aux anneaux de Tate.

4.1.1 Corps valués

Soit K un corps topologique et supposons que

(a) la topologie est non discrète et induite par une valuation $|\cdot|: K \to \mathbb{R}_{\geq 0}$.

On pose

$$\mathcal{O}_K := \{f \in K : |f| \leq 1\} \qquad \mathfrak{m}_K := \{f \in K : |f| < 1\}.$$

(i) Montrer que l'anneau des entiers $\mathcal{O}_K$ est un sous-anneau de valuation de K tel que $\mathrm{Frac}(\mathcal{O}_K) = K$, et que $\mathfrak{m}_K$ est l'unique idéal maximal de $\mathcal{O}_K$.

(ii) Montrer que

$$K \setminus \mathcal{O}_K = \{f \in K : f^{-n} \to 0 \text{ quand } n \to \infty\}$$

et

$$\mathfrak{m}_K = \{f \in K : f^n \to 0 \text{ quand } n \to \infty\}.$$

En déduire que $\mathcal{O}_K$ et $\mathfrak{m}_K$ ne dépendent que du corps topologique K (pas du choix de la valuation $|\cdot|$).

(iii) Soit $0 \neq \pi \in \mathcal{O}_K$ non unité (pourquoi un tel élément existe-il ?). Montrer que la topologie sur $\mathcal{O}_K$ (induite par la topologie sur K) est la topologie π-adique.

Réciproquement, soit $\mathcal{O}$ un anneau de valuation muni d'une topologie et supposons que

(A) il existe $0 \neq \pi \in \mathcal{O}$ tel que la topologie sur $\mathcal{O}$ soit la topologie π-adique et $\bigcap_{n\geq 0} \pi^n \mathcal{O} = \{0\}$;

(B) $\mathcal{O}^{\circ\circ} := \{f \in \mathcal{O} : f^n \to 0 \text{ quand } n \to \infty\}$ est l'idéal maximal de $\mathcal{O}$.

On pose $F := \mathrm{Frac}(\mathcal{O})$.

(iv) Montrer que $F = \mathcal{O}[\frac{1}{\pi}]$. Étant donné un élément $f \in F^\times$, montrer que sa "norme spectrale"

$$|f| := \inf\{p^{-n/m} : n, m \in \mathbb{Z} \text{ t.q. } f^m \in \pi^n \mathcal{O}\} \in \mathbb{R}_{>0}$$

est bien définie et que $|\pi| = p^{-1}$.

(v) Montrer que $|\cdot| : F \to \mathbb{R}_{\geq 0}$ est une valuation telle que $\mathcal{O} = \{f \in F : |f| \leq 1\}$ et $\mathcal{O}^{\circ\circ} = \{f \in F : |f| < 1\}$.

(vi) Montrer que la topologie induite sur $\mathcal{O}$ par cette valuation $|\cdot|$ est la topologie π-adique.

En déduire que le foncteur

$$\{\text{corps topologiques } K \text{ satisfont (a)}\} \to \{\text{anneaux de valuations topologiques } \mathcal{O} \text{ satisfont (A) et (B)}\}$$

qui à K associe $\mathcal{O}_K$ est une équivalence de catégories, avec l'inverse $\mathcal{O} \mapsto \mathrm{Frac}(\mathcal{O})$ donné ci-dessus.

4.1.2 Le cas des corps perfectoïdes

Rappelons qu'un *corps perfectoïde* K est un corps topologique tel que la condition (a), K soit complet et

(b) tout élément de $\mathcal{O}_K/p\mathcal{O}_K$ admet une racine p-ième; et

(c) il existe $\pi \in \mathcal{O}_K$ non unité tel que $p \in \pi^p \mathcal{O}_K$.

(vii) Étant donné un corps topologique tel que la topologie soit induite par une valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$, montrer que K est un corps perfectoïde si et seulement si $\mathcal{O}_K$ (muni de la topologie induite par K) est un anneau perfectoïde entier.

(viii) Déduire de l'équivalence de catégories ci-dessus que le foncteur $K \mapsto \mathcal{O}_K$ induit une équivalence de catégories

$$\{\text{corps perfectoïdes } K\} \xrightarrow{\simeq} \{\text{anneaux perfds entiers } \mathcal{O} \text{ qui sont des anneaux de valuations satisfont (B)}\}$$

(ix) Soit $\mathcal{O}$ un anneau perfectoïde entier.

Montrer que $\# : \mathcal{O}^\flat/\mathcal{O}^{\flat\circ\circ} \to \mathcal{O}/\mathcal{O}^{\circ\circ}$ est un isomorphisme d'anneaux et en déduire que $\mathcal{O}^{\circ\circ}$ est un idéal maximal de $\mathcal{O}$ ssi $\mathcal{O}^{\flat\circ\circ}$ est un idéal maximal de $\mathcal{O}$.

On a vu pendant le cours que $\mathcal{O}$ est un anneau de valuation ssi $\mathcal{O}^\flat$ l'est. En déduire que $\mathcal{O}$ est un anneau de valuation satisfont (B) ssi $\mathcal{O}^\flat$ l'est.

Le basculé d'un corps perfectoïde K est $K^\flat := \mathrm{Frac}(\mathcal{O}_K^\flat) = \mathcal{O}_K^\flat[\frac{1}{\pi^\flat}]$. Déduire des résultats ci-dessus que $K^\flat$ est un corps perfectoïde avec anneau des entiers $\mathcal{O}_K^\flat$ et que le foncteur

$$\{\text{corps perfectoïdes sur } K\} \to \{\text{corps perfectoïdes sur } K^\flat\}, \qquad L \mapsto L^\flat$$

est une équivalence de catégories.

4.2 Aside: almost mathematics II

Let A be an integral perfectoid ring; recall from Section 3.1 that $A^{\circ\circ} = \bigcup_{n\geq 0} \pi^{1/p^n} A$, where as usual $\pi \in A$ denotes a perfectoid pseudo-uniformiser with compatible p-power roots.

We say that an A-module M is *almost free of rank d* if and only if for each element $\varepsilon \in A^{\circ\circ}$ there exists a morphism of A-modules $f_\varepsilon : A^d \to M$ whose kernel and cokernel are killed by ε. (It suffices to consider the elements $\varepsilon = \pi^{1/p^n}$ for each $n \geq 0$.) Note that the morphism depends on ε; in general the module M does not even need to be finitely generated.

Similarly, let's say that an $A/\pi A$-module N is *almost free of rank d* if and only if for each element $\varepsilon \in A^{\circ\circ}$ there exists a morphism of A-modules $(A/\pi A)^d \to N$ whose kernel and cokernel are killed by ε.

Lemma 4.1. *Let M be an A-module which is π-adically complete and without π-torsion; let $d \geq 0$. Then the A-module M is almost free of rank d if and only if the $A/\pi A$-module $M/\pi M$ is almost free of rank d.*

Proof. We suppose that $d = 1$ just to simplify the notation. For any $n \geq 1$ the hypothesis implies that there exists an element $e \in M$ such that the kernel and cokernel of $A/\pi A \to M/\pi M$, $a \mapsto ae$ are killed by π^{1/p^n}. It is then quite straightforward to check directly that the kernel and cokernel of $A \to M$, $a \mapsto am$ are also killed by π^{1/p^n}; the details are left as an exercise. □

The main reason we have introduced this notion is for the sake of the following rather subtle result:

Proposition 4.2. *Let K be a perfectoid field of characteristic p, and L/K a finite field extension. Then the $\mathcal{O}_K$-module $\mathcal{O}_L$ (= the integral closure of $\mathcal{O}_K$ inside L) is almost free of rank $|L : K|$.*

Proof. We begin with some elementary theory of field extensions. Since L/K is a separable extension (recall that K is perfect), the trace pairing $L \otimes_K L \xrightarrow{\text{mult}} L \xrightarrow{\mathrm{Tr}_{L/K}} K$ is non-degenerate. So let $e_1, \ldots, e_d \in L$ be a basis for L as a K-vector space, and let $e_1^*, \ldots, e_d^* \in L$ be the dual basis, i.e., $\mathrm{Tr}_{L/K}(e_i e_j^*) = \delta_{ij}$. By elementary linear algebra we have $b = \sum_{i=1}^d e_i \,\mathrm{Tr}_{L/K}(b_i e_i^*)$ for all $b \in B$, and by the theory of separable extensions the element $e := \sum_{i=1}^d e_i \otimes e_i \in L \otimes_K L$ is an idempotent (though this is not so easy to prove directly).

Let $\pi \in \mathcal{O}_K$ be a perfectoid pseudo-uniformiser, and fix $N \gg 0$ such that $\pi^N e_i \in \mathcal{O}_L$ for $i = 1, \ldots, N$ (note that $L = \mathcal{O}_L[\frac{1}{\pi}]$). We now use that the absolute Frobenius $\varphi : x \mapsto x^p$ is an automorphism of K, $\mathcal{O}_K$, L, $\mathcal{O}_L$ and $L \otimes_K L$; moreover it satisfies $\varphi(e) = e$ since e is an idempotent. Therefore, for any $n \geq 1$, we have

$$\pi^{2N/p^n} e = \varphi^{-n}(\pi^{2N} e) = \varphi^{-n}(\sum_{i=1}^d \pi^N e_i \otimes \pi^N e_i^*) = \sum_{i=1}^d \varphi^{-n}(\pi^N e_i) \otimes \varphi^{-n}(\pi^N e_i^*),$$

which is in the image of $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \to L \otimes_K L$; i.e., we have shown that e is almost in the image of $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \to L \otimes_K L$, which is the key idea of the proof. (To use the correct terminology, this shows that $\mathcal{O}_K \to \mathcal{O}_L$ is "almost étale".)

Now define morphisms

$$f : \mathcal{O}_K^d \to \mathcal{O}_L, \qquad (a_1, \ldots, a_d) \mapsto \sum_{i=1}^d \varphi^{-n}(\pi^N e_i) a_i$$

and

$$g : \mathcal{O}_L \to \mathcal{O}_K^d, \qquad b \mapsto (\mathrm{Tr}_{L/K}(b\,\varphi^{-n}(\pi^N e_1^*)), \ldots, \mathrm{Tr}_{L/K}(b\,\varphi^{-n}(\pi^N e_d^*)).$$

Using the properties of the basis, it is easy to check that the compositions fg and gf are both given by multiplication by π^{2N/p^n}. Hence the kernel and cokernel of f (and of g) are killed by π^{2N/p^n}; therefore $\mathcal{O}_L$ is indeed almost free of rank d. □

4.3 Tilting perfectoid fields

Let K be a perfectoid field; as explained in the previous exercises, the tilt of K is $K^\flat = \mathrm{Frac}(\mathcal{O}_K^\flat)$, where $\mathcal{O}_K^\flat$ is the tilt of the integral perfectoid ring $\mathcal{O}_K$. Our first goal is to prove:

Lemma 4.3. *Let K be a perfectoid field. Then K is algebraically closed if and only if $K^\flat$ is algebraically closed.*

Proof. We will only prove the implication $\Leftarrow$, since the implication $\Rightarrow$ is the same argument (and anyway will follow from the tilting correspondance below). So assume that $K^\flat$ is algebraically closed, and let $f(X) \in \mathcal{O}_K[X]$ be an irreducible monic polynomial of degree d; we must show that $f(X)$ has a root (whence it is linear qed). Fix a valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$ defining the topology.

Step 1 is a weak special case: given $x \in \mathcal{O}_K$, we show that there is $y \in \mathcal{O}_K$ such that $|y^d| = |x|$. Proof: we may write $x = \pi^m x'$ where $m \geq 0$ and $x' \in \mathcal{O}_K \setminus \pi\mathcal{O}_K$. Since $\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$ and $K^\flat$ is algebraically closed, there exists $y \in \mathcal{O}_K$ such that $y^d \equiv x' \bmod \pi\mathcal{O}_K$; since $x' \notin \pi\mathcal{O}_K$, the non-archimedean inequality implies $|y^d| = |x'|$. But $\pi^\flat \in K^\flat$ admits a d^{th} root and $\# : K^\flat \to K$ is multiplicative, so $\pi^{\flat 1/d\#}$ is a d^{th} root of π. In conclusion, the element $\pi^{\flat 1/d\# m} y'$ works.

Step 2: Given $a \in \mathcal{O}_K$ and $n \geq 0$ such that $|f(a)| \leq |\pi|^n$, there exists $\varepsilon \in \mathcal{O}_K$ such that $|\varepsilon| \leq |\pi|^{n/d}$ and $|f(a+\varepsilon)| \leq |\pi|^{n+1}$. Proof: By step 1 there is $y \in \mathcal{O}_K$ such that $|y^d| = |f(a)|$, whence $g(X) := y^{-d} f(a + yX)$ is a monic, irreducible polynomial in $K[X]$ whose constant coefficient $g(0) = y^{-d} f(a)$ lies in $\mathcal{O}_K$ (even in $\mathcal{O}_K^\times$). A clever application of Hensel's lemma shows that all coefficients of $g(X)$ lie in $\mathcal{O}_K$. (See, for example, Lemma 3.2.11 of Bhatt's notes.) Since $\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$ and $K^\flat$ is algebraically closed and $K^\flat$ is algebraically closed, there is therefore $b \in \mathcal{O}_K$ such that $g(b) \equiv 0 \bmod \pi\mathcal{O}_K$. We leave it to the reader to check (easily) that $\varepsilon := yb$ has the desired properties.

Step 3: $f(X)$ has a root. Proof: Use step 2 to successively approximate a root and take the limit. □

Remark 4.4. Let K be a field which is complete under a valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$, and $f(X) \in K[X]$ an irreducible monic polynomial such that $f(0) \in \mathcal{O}_K$. We will use Hensel's lemma to prove that $f(X) \in \mathcal{O}_K[X]$.

We argue by contradiction. Write $f(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0$, and let $i_0 \in \{0, \ldots, d-1\}$ be the largest index such that $|a_{i_0}| = \max_{0 \leq i \leq d-1} |a_i| > 1$. In other words, $|a_i| \geq |a_{i_0}|$ for $i = 0, \ldots, i_0$ and $|a_i| < |a_{i_0}|$ for $i = i_0 + 1, \ldots, d-1$. Set

$$f_0(X) := a_{i_0}^{-1} f(X), \qquad g(X) := X^{i_0} + a_{i_0}^{-1} a_{i_0 - 1} X^{i_0 - 1} + \cdots + a_{i_0}^{-1} a_0, \qquad h(X) := a_{i_0}^{-1} X^{d - i_0} + 1,$$

all of which are in $\mathcal{O}_K[X]$. Letting $\mathfrak{m}_K = \{a \in \mathcal{O}_K : |a| < 1\}$ be the maximal ideal of $\mathcal{O}_K$, we have $h(X) \equiv 1 \bmod \mathfrak{m}_K$, and $g(X) \equiv f_0(X) \equiv X(\cdots) \bmod \mathfrak{m}_K$. Therefore $f_0(X) \equiv g(X)h(X) \bmod \mathfrak{m}_K$, and the polynomials $g(X)$, $h(X)$ are coprime mod $\mathfrak{m}_K$. So (a strong form of) Hensel's lemma implies that $f_0(X)$ is reducible, whence $f(X)$ is also reducible, giving the desired contradiction.

Theorem 4.5 (Almost purity and the tilting correspondence)**.** *Let K be a perfectoid field. Then (i) any finite field extension L of K (topologised as a finite dimensional K-vector space) is also a perfectoid field, and (ii) $L \mapsto L^\flat$ defines a degree-preserving equivalence of categories*

$$\{\textit{finite field extensions of } K\} \xrightarrow{\cong} \{\textit{finite field extensions of } K^\flat\}$$

Corollary 4.6. *K a perfectoid field. Then there exists an isomorphism of absolute Galois groups* $\mathrm{Gal}_K \cong \mathrm{Gal}_{K^\flat}$.

Proof. Immediate from Galois theory and the correspondence of the previous theorem. □

Remark 4.7. Since K is complete with respect to a valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$, standard theory of valued fields implies the following facts: the valuation $|\cdot|$ extends uniquely to any finite extension L/K; the resulting topology on L is the same as its topology viewed as a finite dimensional K-vector space and in particular L is complete with respect to this topology; the ring of integers $\mathcal{O}_L$ with respect to this valuation is the same as the integral closure of $\mathcal{O}_K$ in L.

Therefore the only thing to prove for part (i) of Theorem 4.5 is the following: given a finite extension L/K, then every element of $\mathcal{O}_L/p\mathcal{O}_L$ is a p^{th}-power. It is remarkable that this is not "obvious"! Of course, if K has characteristic p then it is well-known that a finite extension of a perfect field is still perfect, and this proves (i) in this case (also, if K has characteristic p then part (ii) is trivial since tilting does nothing). The idea will be to deduce the result in characteristic zero by tilting into characteristic p, which will simultaneously prove the tilting correspondence.

Proof of Theorem 4.5. From the end of Exercise Section 4.1 we already know that we have an equivalence of categories

$$\{\text{perfectoid fields over } K\} \to \{\text{perfectoid fields } K^\flat\}, \qquad L \mapsto L^\flat,$$

but we know nothing about how it preserves finiteness.

Step 1: Let M be a finite extension of $K^\flat$ (we know M is automatically a perfectoid field by the previous remark); then its untilt $M^\#$ (which we know is a perfectoid field over K) is a finite extension of K of the same degree. Proof: Proposition 4.2 tells us that $\mathcal{O}_M/\pi^\flat$ is a almost free $\mathcal{O}_{K^\flat}/\pi^\flat$-module of rank $|M : K^\flat|$. But $\mathcal{O}_M/\pi^\flat = \mathcal{O}_{M^\#}/\pi$ and $\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$, so we may now apply Lemma 4.1 to deduce that $\mathcal{O}_{M^\#}$ is an almost free $\mathcal{O}_K$-module of rank $|M : K^\flat|$; inverting π tells us that $M^\#$ is a free K-module of rank $|M : K^\flat|$, as desired.

Thanks to step 1 we have fully faithful functors

$$\{\text{finite field extns of } K^\flat\} \xrightarrow{\#} \{\text{finite field extns of } K \text{ which are perfd}\} \subseteq \{\text{finite field extns of } K\}.$$

Step 2: The composition is surjective (which completes the proof of the theorem, since it shows that the two functors are equivalences). Proof: We use Krasner's Lemma

> *Let F be a field which is complete wrt a valuation $|\cdot| : F \to \mathbb{R}_{\geq 0}$, let $\alpha, \beta \in F^{\text{sep}}$, and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d \in F^{\text{sep}}$ be the conjugates of α; if $|\alpha - \beta| < |\alpha - \alpha_i|$ for $i = 2, \ldots, d$, then $\alpha \in F(\beta)$.*

and its corollary

> *Let F be a field which is complete wrt a valuation $|\cdot| : F \to \mathbb{R}_{\geq 0}$, and $F_0 \subseteq F$ a dense subfield. Then F is separably closed if and only if F_0 is separably closed.*

Now let Q be the completion of an algebraic closure of $K^\flat$. The corollary to Krasner's lemma implies that Q is an algebraically closed (to be precise, it is perfect and separably closed, hence algebraically closed) perfectoid field of characteristic p. By the previous lemma, its untilt $Q^\#$ is an algebraically closed perfectoid field over K. Moreover, for any finite subextension M of $Q/K^\flat$ we have $Q^\# \supseteq M^\# \supseteq K$; let $N := \bigcup_M M^\# \subseteq Q^\#$ be the union, where M runs over all finite subextensions of $Q/K^\flat$. So N is an algebraic extension of K. We claim that N is dense in $Q^\#$; indeed, at the level of rings of integers we have

$$\mathcal{O}_N/\pi = \varinjlim_M \mathcal{O}_{M^\#}/\pi = \varinjlim_M \mathcal{O}_M/\pi^\flat = \mathcal{O}_{K^{\flat \mathrm{alg}}}/\pi^\flat = \mathcal{O}_Q/\pi^\flat = \mathcal{O}_{Q^\#}/\pi,$$

which proves the density. The corollary to Krasner therefore implies that N is algebraically closed.

In particular, if L is any finite extension of K, it follows that $L \subseteq N$ and hence there exists a finite extension $M/K^\flat$ such that $L \subseteq M^\#$; we may as well replace $M/K^\flat$ by its Galois closure and so we can suppose that $M/K^\flat$ is a finite Galois extension. It remains only to prove that the fully faithful functor

$$\{\text{subextensions of } M/K^\flat\} \xrightarrow{\#} \{\text{subextensions of } M^\#/K\}$$

is surjective. But $|M^\# : K| = |M : K^\flat|$ by step 1 and $\mathrm{Gal}(M^\#/K) = \mathrm{Gal}(M/K^\flat)$ since the functor is fully faithful. Since $M/K^\flat$ is Galois, it follows from Galois theory that $M^\#/K$ is also Galois, and hence that the two categories have the same finite cardinality; so the injection is surjective, as required. $\square$

5 The adic spectrum of a Huber ring: fundamental properties

5.1 $\mathrm{Spa}(R, R^+)$

Definition 5.1. A *valuation* (or *absolute value*) x on a ring R is the data of a totally ordered abelian group Γ (written multiplicatively) and a map $x : R \to \Gamma \cup \{0\}$[4] which satisfies

- $x(0) = 0$, $x(1) = 1$;
- $x(fg) = x(f)x(g)$;
- $x(f + g) \leq \max(x(f), x(g))$.

The *value group* $\Gamma_x \subseteq \Gamma$ is defined to be the subgroup generated by the monoid $x(R) \setminus \{0\}$.

The *support* of a valuation x is the prime ideal $\mathfrak{p}_x := \{f \in A : x(f) = 0\}$. The associated *residue field* is $k(x) := \mathrm{Frac}\, R/\mathfrak{p}_x$, on which x obviously induces a valuation

$$x : k(x) \to \Gamma \cup \{0\}, \qquad f/g \mapsto x(f)/x(g)$$

($f, g \in R \setminus \mathfrak{p}_x$) with image $\Gamma_x \cup \{0\}$; let $\mathcal{O}_x := \{f \in k(x) : x(f) \leq 1\} \supseteq R/\mathfrak{p}_x$ be the associated valuation ring.

Lemma 5.2. *Let $x : R \to \Gamma$ and $y : A \to \Delta$ be valuations. Then the following are equivalent:*

(i) for any $f, g \in R$, we have $x(f) \leq x(g) \iff y(f) \leq y(g)$;

[4] By definition we have $0\gamma = 0$ and $0 < \gamma$ for all $\gamma \in \Gamma$.

(ii) $\mathfrak{p}_x = \mathfrak{p}_y$ *and* $\mathcal{O}_x = \mathcal{O}_y$*;*

(iii) there exists an isomorphism of ordered groups $\iota : \Gamma_x \cong \Delta_y$ *such that* $y = \iota \circ x$*.*

When this is true, we say that x and y are *equivalent* valuations.

Proof. (i)⇒(ii): We immediately deduce $\mathfrak{p}_x = \mathfrak{p}_y$ (since $x(f) = 0$ iff $x(f) \le x(0)$ and similarly for y). Next, if $f, g \in R \backslash \mathfrak{p}_x$ so that f/g defines an element of $k(x)$, then $f/g \in \mathcal{O}_x$ iff $x(f) \le x(g)$ iff $y(f) \le y(g)$ iff $f/g \in \mathcal{O}_y$.

(ii)⇒(iii): The valuation x induces an isomorphism $x : k(x)^\times/\mathcal{O}_x^\times \xrightarrow{\simeq} \Gamma_x$ such that, given $f, g \in k(x)^\times$, we have $x(f) \le x(g) \iff gf^{-1} \in \mathcal{O}_x$. This is also true for y, whence the isomorphism ι is given by

$$\Gamma_x \cong k(x)^\times/\mathcal{O}_x^\times = k(y)^\times/\mathcal{O}_y^\times \cong \Delta_y.$$

(using the correspondence between valuations on a field and the associated ring of integers).

(iii)⇒(i) is obvious. □

Definition 5.3 (The adic spectrum and its topology)**.** Let (R, R^+) be a Huber pair. Its associated *adic spectrum* $\mathrm{Spa}(R, R^+)$ is the set of equivalence classes of valuations $x : R \to \Gamma$ such that

- $x(f) \le 1$ for all $f \in R^+$;
- x is continuous with respect to the order topology on Γ; i.e., for each $\gamma \in \Gamma$, the set $\{f \in R : x(f) < g\}$ is open.

We give $\mathrm{Spa}(R, R^+)$ the coarsest topology for which the subsets

$$\mathrm{Spa}(R, R^+)(\tfrac{f}{g}) := \{x \in \mathrm{Spa}(R, R^+) : |f(x)| \le |g(x)| \ne 0\}$$

are open for all $f, g \in R$.

Example 5.4. To add.

Definition 5.5 (Rational subsets)**.** Let (R, R^+) be a Huber pair and $U \subseteq \mathrm{Spa}(R, R^+)$. Then we say that U is a *rational subset* of $\mathrm{Spa}(R, R^+)$ if and only if there exist $f_1, \dots, f_n \in A$ which generate an open ideal and $g \in R$ such that

$$U = \mathrm{Spa}(R, R^+)(\tfrac{f_1,\dots,f_n}{g}) := \{x \in \mathrm{Spa}(R, R^+) : |f_i(x)| \le |g(x)| \ne 0 \text{ for all } i = 1, \dots, n\}.$$

(Note that the elements $f_1, \dots, f_n, g$ are not unique.)

Remark 5.6 (Case of Tate–Huber pairs)**.** If (R, R^+) is a Tate–Huber pair, then the only open ideal of R is R itself. So in this case rational subsets are

$$\mathrm{Spa}(R, R^+)(\tfrac{f_1,\dots,f_n}{g}) := \{x \in \mathrm{Spa}(R, R^+) : |f_i(x)| \le |g(x)| \text{ for all } i = 1, \dots, n\}$$

for $f_1, \dots, f_n \in R$ generating the unit ideal. Note that we have omitted the condition "$|g(x)| \ne 0$", because now it is automatic: if $x(g) = 0$ then $x(f_i) = 0$ for all i, but this is impossible since the f_i generate the unit ideal.

We begin with the following basic lemma:

Lemma 5.7. *Let* (R, R^+) *be a Huber pair. The rational subsets of* $\mathrm{Spa}(R, R^+)$ *are open and closed under finite intersection.*

Proof. Let $U = \mathrm{Spa}(R, R^+)(\frac{f_1,\ldots,f_n}{g})$ be as in the definition. Clearly $U = \bigcap_{i=1}^n \mathrm{Spa}(R, R^+)(\frac{f_i}{g})$, which is open by definition of the topology.

Given another rational subset $U' = \mathrm{Spa}(R, R^+)(\frac{f'_1,\ldots,f'_m}{g'})$, write $f_0 = g$ and $f'_0 = g'$ for convenience; then it is not hard to use the axioms of a valuation to check

$$U \cap U' = \mathrm{Spa}(R, R^+)(\frac{f_i f'_j : i=0,\ldots,n,\, j=0,\ldots,m}{gg'})$$

Moreover, the ideal generated by $f_i f'_j$, $i = 1, \ldots, n$, $j = 1, \ldots m$ is open, since it is the product of two open ideals (note that in a Huber ring the product of two open ideals is again open; to prove this, fix an ideal of definition I and note that a subgroup is open if and only if it contains a power of I). □

Lemma 5.8. *Let* $\varphi : (R, R^+) \to (S, S^+)$ *be a morphism of Huber pairs; then there is an induced map*

$$\mathrm{Spa}(\varphi) : \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(S, S^+), \qquad x \mapsto x \circ \varphi,$$

which is continuous.

Proof. The existence and continuity of $\mathrm{Spa}(\varphi)$ are easy. □

Definition 5.9. A topological space X is said to be *spectral* if and only if it is quasi-compact, has a basis of quasi-compact opens closed under finite intersections, and is *sober* (:=any irreducible closed subset of of X admits a unique generic point).

The following theorem of Huber presents the main fundamental properties of the adic spectrum:

Theorem 5.10. *Let* (R, R^+) *be a Huber pair.*

(i) The topological space $\mathrm{Spa}(R, R^+)$ *is spectral.*

(ii) The rational subsets form a basis for the topology.

(iii) Any rational subset is quasi-compact.

(iv) $\mathrm{Spa}(R, R^+) = \emptyset$ *if and only if the topology on* R *is trivial (i.e., the only opens are* $\emptyset$ *and* R*).*

(v) $R^+ = \{f \in R : x(f) \leq 1 \, \forall x \in \mathrm{Spa}(R, R^+)\}$
$R^{\circ\circ} = \{f \in R : x(f) < 1 \, \forall x \in \mathrm{Spa}(R, R^+)\}$

Proof. Unfortunately we probably do not have time to prove this result – it is not so difficult but rather long. We recommend looking at Huber's paper *Continuous valuations* (the proof occupies section 2 and the first half of section 3). □

Corollary 5.11. *Let* (R, R^+) *be a complete Huber pair.*

(i) $R^\times = \{f \in R : x(f) \neq 0 \, \forall x \in \mathrm{Spa}(R, R^+)\}$.

(ii) If $I \subseteq R$ *is a proper ideal, then there exists* $x \in \mathrm{Spa}(R, R^+)$ *such that* $x(f) = 0$ *for all* $f \in I$.

Proof. (i): The inclusion $\subseteq$ is clear, so suppose that $f \in R$ is not a unit. Then there is a maximal ideal $\mathfrak{m} \subseteq R$ such that $f \in \mathfrak{m}$. We claim that $\mathfrak{m}$ is closed. Since the topological closure of $\mathfrak{m}$ is an ideal, hence equals $\mathfrak{m}$ or R, it is enough to check that $R \setminus \mathfrak{m}$ contains a non-empty open; the following claim shows that it contains the open $1 + R^{\circ\circ}$:

Claim: $1+R^{\circ\circ} \subseteq R^\times$. Proof: Let $I \subseteq R_0$ be an ideal and subring of definition. Given $g \in R^{\circ\circ}$ we have $g^m \in I$ for $m \gg 0$. Since R_0 is I^m-adically complete, certainly $1 - g^m \in R_0^\times \subseteq R^\times$. But $(1-g)(1+g+\cdots+g^{m-1}) = 1 - g^m$, so $1 - g \in R^\times$.

Set $K := R/\mathfrak{m}$ and let K^+ be the integral closure of R^+ in R. Since $\mathfrak{m}$ is closed, the quotient topology on the Huber ring K is separated, and hence Theorem 5.10(iv) implies that $\mathrm{Spa}(K, K^+) \neq \emptyset$. This pulls back to a point $x \in \mathrm{Spa}(R, R^+)$ satisfying $x(f) = 0$.

(ii): Next suppose that I is a proper ideal; let $\mathfrak{m} \supseteq I$ be a maximal ideal containing it. As above there exists $x \in \mathrm{Spa}(R, R^+)$ killing $\mathfrak{m}$. $\square$

Our next main goal is to construct a structure presheaf $\mathcal{O}_{\mathrm{Spa}(R,R^+)}$ on the adic spectrum $\mathrm{Spa}(R, R^+)$, mimicking the structure presheaf on the usual spectrum; given a rational subset $U \subseteq \mathrm{Spa}(R, R^+)$, the value $\mathcal{O}_{\mathrm{Spa}(R,R^+)}(U)$ will be defined to be a certain completed localisation of R. Therefore we must first discuss completion and localisation of Huber rings.

5.2 Completion of a Huber pair

Definition 5.12. We say that a Huber ring R is *complete* if and only if R is complete and Hausdorff in the usual topological sense.

In general, let $\widehat{R}$ be the Hausdorff completion of R. (Recall: if X is a topological space with a countable neighbourhood basis at every point, then we may define its Hausdorff completion to be $\widehat{R} = R^{\mathbb{N}}/\{\text{Cauchy sequences}\}$.

In practice, it is more convenient to define $\widehat{R}$ algebraically as $\widehat{R} := \widehat{R_0} \otimes_{R_0} R$, where $\widehat{R_0} := \varprojlim_r R_0/I^r$ is the I-adic completion of a chosen subring of definition R_0 with respect to an ideal of definition I. The next proposition shows that this process really gives the same result:

Proposition 5.13. *$\widehat{R}$ is a Huber ring. Moreover, letting $I \subseteq R_0 \subseteq R$ be any ideal and subring of definition:*

(i) $I\widehat{R_0} \subseteq \widehat{R_0}$ are an ideal and subring of definition of $\widehat{R}$, where $\widehat{R_0} = \varprojlim_n R_0/I^n$ is the I-adic completion of R_0;

(ii) the canonical map $\widehat{R_0} \otimes_{R_0} R \to \widehat{R}$ is an isomorphism.

Finally, if $R^+ \subseteq R$ is a subring of integral elements, then its Hausdorff completion $\widehat{R^+} \subseteq \widehat{R}$ is a subring of integral elements, whence $(\widehat{R}, \widehat{R^+})$ is a Huber pair (called the completion of the pair (R, R^+)).

Proof. The universal property of tensor product gives us a map $j : \widehat{R_0} \otimes_{R_0} R \to \widehat{R}$ making the diagram commute:

$$\begin{tikzcd}
\widehat{R} \arrow[dr, dashed, bend right, "h"'] & & R \arrow[ll, "i"'] \arrow[dl, "f"] \\
& \widehat{R_0} \otimes_{R_0} R \arrow[ul, "j"'] & \\
\widehat{R_0} \arrow[uu, hook] \arrow[ur, "g"] & & R_0 \arrow[ll] \arrow[uu, hook]
\end{tikzcd}$$

By elementary theory of topological completions, $i(R)$ is dense in $\widehat{R}$, and $\widehat{R_0}$ is open in $\widehat{R}$; therefore $\widehat{R} = i(R) + \widehat{R_0}$ and $i^{-1}(\widehat{R_0}) = R_0$. Algebraically, this means that the above square is cartesian in the category of R_0-modules, and therefore there exists a dotted arrow h making the top and left triangles commute (i.e., $hi = f$ and $h|_{\widehat{R_0}} = g$).

Therefore the R_0-linear map $hj : \widehat{R_0} \otimes_{R_0} R \to \widehat{R_0} \otimes_{R_0} R$ satisfies $hjf = f$ and $hjg = g$; so the universal property of tensor product implies $hj = \mathrm{id}$. But j is surjective (since $\widehat{R} = i(R) + \widehat{R_0}$), so therefore j is an isomorphism, proving (ii).

Since we already know that $\widehat{R_0}$ is an open subring of $\widehat{R}$, the only part left of (i) is that the topology on $\widehat{R_0}$ is the $I\widehat{R_0}$-adic topology, for which it is enough to check that $\widehat{R_0}/I^n\widehat{R_0} \cong R/I^n$ for all $n \geq 1$. For R Noetherian this is well-known commutative algebra; in general we refer to the Stacks project [Tag 05GG].

The assertions about R^+ are left as an exercise. □

The completion of a Huber pair obviously satisfies the following universal property:

Corollary 5.14. *Let (R, R^+) be a Huber pair. Then $(R, R^+) \to (\widehat{R}, \widehat{R^+})$ is an adic morphism with the following universal property: given any complete Huber pair (S, S^+) and morphism $\varphi : (R, R^+) \to (S, S^+)$, there exists a unique morphism $\widehat{\varphi} : (\widehat{R}, \widehat{R^+}) \to (S, S^+)$ such that*

$$\begin{tikzcd} (R, R^+) \arrow[r] \arrow[rr, bend right, "\varphi"'] & (\widehat{R}, \widehat{R^+}) \arrow[r, "\widehat{\varphi}"] & (S, S^+) \end{tikzcd}$$

commutes.

5.3 Behaviour of the adic spectrum under completion

Recall that if R is a Huber ring then we have defined its completion $\widehat{R}$; given a subring of definition R^+, we may also complete R^+ (as a topological ring) to get $\widehat{R}^+ \subseteq R^+$, which we showed was a subring of integer elements. In this subsection we prove the following:

Proposition 5.15. *Let (R, R^+) be a Huber pair. Then the canonical map* $\mathrm{Spa}(\widehat{R}, \widehat{R^+}) \to \mathrm{Spa}(R, R^+)$ *is a homemorphism identifying rational subsets.*

We need two lemmas:

Lemma 5.16. *Let (R, R^+) be a Huber pair, $Y \subseteq \mathrm{Spa}(R, R^+)$ a quasi-compact subset, and $g \in R$ an element such that $y(g) \neq 0$ for all $y \in Y$. Then there exists an open neighbourhood $0 \in V \subseteq R$ such that $y(f) < y(g)$ for all $f \in V$, $y \in Y$.*

Proof. Let $I \subseteq R_0 \subseteq R$ be an ideal and subring of definition; write $I = (t_1, \ldots, t_d)R_0$ for some $t_1, \ldots, t_d \in R_0$. Given $y \in Y$, continuity of y implies that $\{f \in R : x(f) < x(g)\}$ is open in R, whence it contains $(t_1^N, \ldots, t_d^N)R_0$ for $N \gg 0$. This shows that $Y \subseteq \bigcup_{N \geq 1} \mathrm{Spa}(R, R^+)(\frac{t_1^r, \ldots, t_d^r}{g})$, whence quasi-compactness of Y implies $Y \subseteq \mathrm{Spa}(R, R^+)(\frac{t_1^N, \ldots, t_d^N}{g})$ for some fixed $N \geq 1$; but then, for all $y \in Y$ and $t \in I$, we have

$$y(tt_i^N) = y(t)y(t_i^N) < y(t_i^N) \leq y(g),$$

and so $V := (t_1^N, \ldots, t_d^N)I$ works (which is open since it contains a power of I). □

Lemma 5.17. *Let (R, R^+) be a complete Huber ring; let $f_1, \ldots, f_n \in R$ generate an open ideal of R, and let $g \in R$. Then there exists an open neighbourhood $U \subseteq R$ of 0 such that: for all $g' \in g + U$ and all $f_i' \in f_i + U$, the elements $f_1', \ldots, f_n'$ also generate an open ideal of R^+ and*

$$\mathrm{Spa}(R, R^+)(\tfrac{f_1', \ldots, f_n'}{g'}) = \mathrm{Spa}(R, R^+)(\tfrac{f_1, \ldots, f_n}{g})$$

Proof. Let R_0 be a subring of definition. Then $(f_1, \ldots, f_n) \cap R_0$ is an open ideal of R_0, hence contains an ideal of definition $J \subseteq R_0$ (e.g., pick any ideal of definition J, then replace J by J^m for some $m \gg 0$). Let $r_1, \ldots, r_m$ be generators for J (as an ideal of R_0) and observe that $(r_1, \ldots, r_m)R_0 + J^2 = J$ (by Nakayama's lemma, since R_0 is J-adically complete and so $J \subseteq \mathrm{Jac}(R_0)$).

For convenience of notation, write $f_0 = g$ and set

$$X_i := \mathrm{Spa}(R, R^+)(\tfrac{f_0, f_1, \ldots, f_n}{f_i})$$

for $i = 0, \ldots, n$. Recall X_i is quasi-compact by Theorem 5.10(iii) and $x(f_i) \neq 0$ for all $x \in X_i$; therefore Lemma 5.16 shows that there exists $M \geq 1$ such that $x(f_i) > x(a)$ for all $x \in X_i$, all $a \in J^M$, and all $i = 0, \ldots, n$.

We claim that $U := J^M$ works. So let $f_i' \in f_i + J^M$. We must prove that

$$X_0 = \mathrm{Spa}(R, R^+)(\tfrac{f_1', \ldots, f_n'}{g'}).$$

$\subseteq$: Let $x \in X_0$. Since $f_i' - f_i \in J^M$ we have $x(f_i) > x(f_i' - f_i)$; also $x \in X_0$, so that $x(f_0) \geq x(f_i)$. Combining these inequalities we immediately get

$$x(f_i)' = x(f_i + (f_i' - f_i)) \leq x(f_0) = x(f_0 + (f_0' - f_0)) = x(f_0'),$$

i.e., $x \in \mathrm{Spa}(R, R^+)(\frac{f_1', \ldots, f_n'}{g'})$, as required.

$\supseteq$: Let $x \in \mathrm{Spa}(R, R^+) \setminus X_0$; we claim $x \notin \mathrm{Spa}(R, R^+)(\frac{f_1', \ldots, f_n'}{g'})$. If $x(f_i) = 0$ for all $i = 1, \ldots, n$ then the prime ideal $\mathfrak{p}_x$ is open in R (since it contains $f_1, \ldots, f_n$), hence contains J^M (since this consists of topologically nilpotent elements), in particular contains $f_0' - f_0$; therefore $x(f_0') = 0$ so $x \notin \mathrm{Spa}(R, R^+)(\frac{f_1', \ldots, f_n'}{g'})$. In the other case, i.e., if $x(f_i) \neq 0$ for some i, then the rest of the proof is quite easy. □

Now we can prove the main result of the subsection:

Proof of Proposition 5.15. To add (it is not difficult) – in the meantime see Prop. 3.9 of Huber's *Continuous valuations.* □

5.4 Localisation of a Huber pair

The correct way to localise a Huber pair turns out to be slightly subtle:

Definition 5.18. Let R be a Huber ring, let $g \in R$, and let $f_1, \ldots, f_n \in R$ be elements which generate an open ideal. Set

$$R[\tfrac{f_1, \ldots, f_n}{g}] := R[\tfrac{1}{g}]$$

with the following topology: fixing an ideal and subring of definition $I \subseteq R_0$, a neighbourhood basis of $R[\frac{f_1, \ldots, f_n}{g}]$ at 0 is given by $I^m R_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$ for $n \geq 1$.

Proposition 5.19. $R[\frac{f_1, \ldots, f_n}{g}]$ *is a Huber ring. Moreover:*

(i) It does not depend on the chosen ideal and subring of definition $I \subseteq R_0$.

(ii) $g \in R[\frac{f_1,\dots,f_n}{g}]^\times$ and $\frac{f_i}{g} \in R[\frac{f_1,\dots,f_n}{g}]^\circ$.

(iii) The canonical map $R \to R[\frac{f_1,\dots,f_n}{g}]$ is adic and satisfies the following universal property: if $\varphi : R \to S$ is a continuous map of Huber rings such that $\varphi(g) \in S^\times$ and $\frac{\varphi(f_i)}{\varphi(g)} \in S^\circ$ for all i, then φ extends uniquely to a continuous map of Huber rings $R[\frac{f_1,\dots,f_n}{g}] \to S$.

Proof. To show that $R[\frac{f_1,\dots,f_n}{g}]$ is a Huber ring, everything is obvious except for multiplication being continuous. We begin by checking the following:

Claim: For any $h \in R[\frac{1}{g}]$ there exists $m \geq 1$ such that $hI^m R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}] \subseteq R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}]$. Proof: Clearly we may reduce to the case $h = \frac{a}{g^s}$ for some $a \in R$ and $s \geq 0$; in fact, since $aI^m \subseteq R_0$ for $m \gg 1$, we even reduce to the case $h = \frac{1}{g_s}$. Now we use Corollary 2.9, which says that the set $f_1R_0 + \cdots + f_nR_0$ is open, hence contains I^m for some $m \geq 1$. Therefore $\frac{1}{g}I^m \subseteq R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}]$; since the right side is closed under multiplication, we also get $\frac{1}{g^s}I^{ms} \subseteq R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}]$, which is enough.

Now we can prove that multiplication is continuous. Let $h_1, h_2 \in R[\frac{1}{g}]$, and let $h_1h_2 + I^m R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}]$ be a typical neighbourhood of h_1h_2. By the claim there exist $m_1, m_2 \geq 1$ such that $h_1 I^{m_1} R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}] \subseteq R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}]$ and similarly for h_2. Then

$$(h_1 + I^{m+m_2} R_0[\tfrac{f_1}{g},\dots,\tfrac{f_n}{g}])(h_2 + I^{m+m_2} R_0[\tfrac{f_1}{g},\dots,\tfrac{f_n}{g}]) \subseteq h_1h_2 + I^m R_0[\tfrac{f_1}{g},\dots,\tfrac{f_n}{g}],$$

which proves continuity of multiplication.

(i): Easy exercise.

(ii): g is obviously a unit in $R[\frac{f_1,\dots,f_n}{g}]$, since the underlying ring is $R[\frac{1}{g}]$. By construction of the topology, each $\frac{f_i}{g}$ belongs to a subring of definition, namely $R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}]$, hence is power bounded.

(iii): The map $R \to R[\frac{f_1,\dots,f_n}{g}]$ is clearly continuous and adic by definition. We must check the universal property. Since $\varphi(g) \in S^\times$, there is obviously a unique map of rings $\widetilde{\varphi} : R[\frac{f_1,\dots,f_n}{g}] \to S$ which extends φ. We must show that $\widetilde{\varphi}$ is continuous; let $J \subseteq S_0 \subseteq S$ be an ideal and subring of definition, and fix $m \geq 1$. Since $\frac{\varphi(f_i)}{\varphi(g)}$ is power bounded for each i, Corollary 2.8 tells us that the set $S_0[\frac{\varphi(f_1)}{\varphi(g)},\dots,\frac{\varphi(f_n)}{\varphi(g)}]$ is bounded, so there exists $m' \geq 1$ such that $J^{m'} S_0[\frac{\varphi(f_1)}{\varphi(g)},\dots,\frac{\varphi(f_n)}{\varphi(g)}] \subseteq J^m$. Next, the continuity of φ implies the existence of $m'' \geq 1$ such that $\varphi(I^{m''}) \subseteq J^{m'}$. Now it is clear that

$$\widetilde{\varphi}(I^{m''} R_0[\tfrac{f_1}{g},\dots,\tfrac{f_n}{g}]) \subseteq J^{m'} S_0[\tfrac{\varphi(f_1)}{\varphi(g)},\dots,\tfrac{\varphi(f_n)}{\varphi(g)}] \subseteq J^m,$$

which proves continuity of $\widetilde{\varphi}$. □

Definition 5.20. Let (R, R^+) be a Huber pair, let $g \in R$, and let $f_1,\dots,f_n \in R$ be elements which generate an open ideal. We define $R[\frac{f_1,\dots,f_n}{g}]^+$ to be the integral closure of $R^+[\frac{f_1}{g},\dots,\frac{f_n}{g}]$ in $R[\frac{1}{g}]$. Thus we obtain a Huber pair $(R[\frac{f_1,\dots,f_n}{g}], R[\frac{f_1,\dots,f_n}{g}]^+)$. Let $(R\langle\frac{f_1,\dots,f_n}{g}\rangle, R\langle\frac{f_1,\dots,f_n}{g}\rangle^+)$ be its completion.

Corollary 5.21. *The canonical map of Huber pairs $(R, R^+) \to (R\langle\frac{f_1,\dots,f_n}{g}\rangle, R\langle\frac{f_1,\dots,f_n}{g}\rangle^+)$ is adic and satisfies the following universal property: given any complete Huber pair (S, S^+) and morphism $\varphi : (R, R^+) \to (S, S^+)$ such that $\varphi(g) \in S^\times$ and $\frac{\varphi(f_i)}{\varphi(g)} \in S^+$ for all i, then φ extends uniquely to a map of Huber pairs $\widetilde{\varphi} : (R\langle\frac{f_1,\dots,f_n}{g}\rangle, R\langle\frac{f_1,\dots,f_n}{g}\rangle^+) \to (S, S^+)$.*

Proof. Combine Corollary 5.14 and Proposition 5.19. $\square$

5.5 The structure presheaves on $\mathrm{Spa}(R, R^+)$

Goal: we construct presheaves of topological rings $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on $X = \mathrm{Spa}(R, R^+)$ with the following property: if U is a rational subset, given by $U = X(\frac{f_1,\dots,f_n}{g})$, then $\mathcal{O}_X(U) = R\langle\frac{f_1,\dots,f_n}{g}\rangle$ and $\mathcal{O}_X^+(U) = R\langle\frac{f_1,\dots,f_n}{g}\rangle^+$. To show that this does not depend on the chosen elements $f_1, \dots, f_n, g$ representing U we must argue via a universal property (which in turn implicitly depends on the long Theorem 5.10; thus the structure presehaf is surprisingly subtle).

Proposition 5.22. *Let (R, R^+) be a Huber pair and $U \subseteq X := \mathrm{Spa}(R, R^+)$ a rational subset. Then there exist a unique complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ and morphism $(R, R^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ with the following universal property:*

- *the induced map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to X$ has image in U;*
- *given any complete Huber pair (S, S^+) and morphism $(R, R^+) \to (S, S^+)$ such that the induced map $\mathrm{Spa}(S, S^+) \to X$ has image in U, then the morphism extends uniquely to $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (B, B^+)$.*

Moreover,

(i) the induced map $Y := \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to U$ is a homeomorphism inducing a bijection between the rational subsets of $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ and the rational subsets of X which are contained in U.

(ii) let $V \subseteq Y$ be a rational subset (which we identify with a rational subset of X contained inside U, by the previous part); then there is a unique map of Huber pairs $(\mathcal{O}_X(V), \mathcal{O}_X^+(V)) \to (\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$ such that the diagram commutes

$$\begin{tikzcd}
(\mathcal{O}_X(V), \mathcal{O}_X^+(V)) \arrow[r] & (\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \\
(R, R^+) \arrow[r] \arrow[u] & (\mathcal{O}_X(Y), \mathcal{O}_X^+(Y)) \arrow[u]
\end{tikzcd}$$

Moreover, the map is an isomorphism.

(iii) In fact, if we choose $f_1, \dots, f_n \in R$ generating an open ideal and $g \in R$ such that $U = X(\frac{f_1,\dots,f_n}{g})$, then $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (R\langle\frac{f_1,\dots,f_n}{g}\rangle, R\langle\frac{f_1,\dots,f_n}{g}\rangle^+)$.

Proof. We choose $f_1, \dots, f_n \in R$ generating an open ideal and $g \in R$ such that $U = X(\frac{f_1,\dots,f_n}{g})$, and we set

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (R\langle\tfrac{f_1,\dots,f_n}{g}\rangle, R\langle\tfrac{f_1,\dots,f_n}{g}\rangle^+).$$

We claim that this has the desired universal property. Firstly, if $y \in \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ has image $x \in X$, then $y(\frac{f_i}{g}) \leq 1$ (since $\frac{f_i}{g} \in \mathcal{O}_X^+(U)$) and $y(g) \neq 0$ (since $g \in \mathcal{O}_X(U)^\times$) so $x(f_i) \leq x(g) \neq 0$, i.e., $x \in U$; this shows that $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to X$ has image inside U.

Now suppose that $\varphi : (R, R^+) \to (S, S^+)$ is a morphism such that $\mathrm{Spa}(S, S^+) \to X$ has image inside U. Then every valuation $y \in \mathrm{Spa}(S, S^+)$ satisfies $y(\varphi(f_i)) \leq y(\varphi(g)) \neq 0$. Corollary 5.11 implies $\varphi(g) \in S^\times$, whence we can rewrite the previous inequality as $y(\frac{\varphi(f_i)}{\varphi(g)}) \leq 1$; now Theorem 5.10 (v) implies $\frac{\varphi(f_i)}{\varphi(g)} \in S^+$, and so finally the universal property of Corollary 5.21 implies the existence of a unique morphism $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (S, S^+)$ extending φ. This completes the proof of the universal property, and so shows that $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ does not depend (up to isomorphism) on the choice of $f_1, \dots, f_n, g$.

(i): We must show that $\mathrm{Spa}(R\langle \frac{f_1,\dots,f_n}{g}\rangle, R\langle \frac{f_1,\dots,f_n}{g}\rangle^+) \to X$ induces a bijection between the rational subsets of the first space and the rational subsets of X contained in U. By Proposition 5.15 we may replace the first space by $\mathrm{Spa}(R[\frac{f_1,\dots,f_n}{g}], R[\frac{f_1,\dots,f_n}{g}]^+)$; then it is not so hard (Huber2 Lemma 1.5(ii)).

(ii): This follows from the universal property. (iii): This is by definition. $\square$

The previous proposition has defined structure presheaves $\mathcal{O}_{\mathrm{Spa}(R,R^+)}$ and $\mathcal{O}^+_{\mathrm{Spa}(R,R^+)}$ on rational subsets of $\mathrm{Spa}(R, R^+)$ (which we recall from Theorem 5.10 form a basis of $\mathrm{Spa}(R, R^+)$); we formally extend these to all opens of $\mathrm{Spa}(R, R^+)$ in the usual way (indeed, the unique way if we hope to obtain sheaves):

Definition 5.23. Given a general open set $W \subseteq \mathrm{Spa}(R, R^+)$, set

$$\mathcal{O}_{\mathrm{Spa}(R,R^+)}(W) := \varprojlim_{U \subseteq W} \mathcal{O}_{\mathrm{Spa}(R,R^+)}(U),$$

where the inverse limit is taken over all rational subsets U of $\mathrm{Spa}(R, R^+)$ which are contained in W. Similarly $\mathcal{O}^+_{\mathrm{Spa}(R,R^+)} := \varprojlim_{U \subseteq V} \mathcal{O}^+_{\mathrm{Spa}(R,R^+)}(U)$.

An adic space is obtained by locally glueing adic specta (in fact, we this course we do not need the following definition, but it seems worth including):

Definition 5.24. An adic space is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ and a sheaf of subrings $\mathcal{O}_X^+ \subseteq \mathcal{O}_X$ such that, for each point $x \in X$, there exists an open neighbourhood $x \in U \subseteq X$, a Huber pair (R, R^+), and an isomorphism $(U, \mathcal{O}_X|_U, \mathcal{O}_X^+|_U) \cong (\mathrm{Spa}(R, R^+), \mathcal{O}_{\mathrm{Spa}(R,R^+)}, \mathcal{O}^+_{\mathrm{Spa}(R,R^+)})$.

It is a *perfectoid space* if we can choose each R to be a perfectoid Tate ring.

5.6 Sheafiness and stable uniformity

The theory of adic spaces suffers from a strange phenomena which does not appear in the theory of schemes: given a Huber pair (R, R^+), the presheaves $\mathcal{O}_{\mathrm{Spa}(R,R^+)}$, $\mathcal{O}^+_{\mathrm{Spa}(R,R^+)}$ might not be sheaves (in other words, $\mathrm{Spa}(R, R^+)$ might not be an adic space!). We say that the pair (R, R^+) is *sheafy* if and only if $\mathcal{O}_{\mathrm{Spa}(R,R^+)}$, $\mathcal{O}^+_{\mathrm{Spa}(R,R^+)}$ are sheaves (whence $\mathrm{Spa}(R, R^+)$ is indeed an adic space).

The following are classical conditions which ensure sheafifness and are sufficient for developing the theory of adic spaces of "reasonably finite type" spaces:

Theorem 5.25 (Tate, Bosch–Güntzer–Remmert, Huber)**.** *Let (R, R^+) be an adic space, and assume either that*

- *R is Tate and strongly Noetherian, i.e., the algebra of convergent polynomials $R\langle X_1, \dots, X_n\rangle$ is Noetherian for all $n \geq 0$,*

or that

- *R^+ has a subring of definition which is Noetherian.*

Then (R, R^+) is sheafy and $H^i(\mathrm{Spa}(R, R^+), \mathcal{O}_{\mathrm{Spa}(R,R^+)}) = 0$ for all $i > 0$.

We do not need the previous theorem, as it does not apply to perfectoid Tate rings. We will instead use the following condition for establishing sheafiness:

Definition 5.26. R Huber pair (R, R^+) is said to be *stably uniform* if and only if, for every rational subset $U \subseteq X := \mathrm{Spa}(R, R^+)$, the Huber ring $\mathcal{O}_X(U)$ is uniform.

Theorem 5.27 (Buzzard–Verberkmoes)**.** *Let (R, R^+) be a Tate–Huber pair which is stably uniform; set $X := \mathrm{Spa}(R, R^+)$. Then the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves, and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

To prove the theorem we must examine the behaviour of $\mathcal{O}_X$ on affine covers of $\mathrm{Spa}(R, R^+)$. This is done in two steps: first we formally reduce to particularly simple covers, secondly we do an explicit calculation for the simple covers. (We remark that the proof of Theorem 5.25 is similar.)

Definition 5.28. (R, R^+) a Tate–Huber pair, and $f_1, \dots, f_n \in R$ generating the unit ideal. Clearly the n rational subsets

$$\mathrm{Spa}(R, R^+)(\tfrac{f_1,\dots,f_n}{f_i}), \qquad i = 1, \dots, n$$

are an open cover of $\mathrm{Spa}(R, R^+)$. We call such a cover a *standard rational covering.* If moreover the elements $f_1, \dots, f_n$ are units, we call it a *standard rational covering generated by units.*

Given arbitrary elements $f_1, \dots, f_n \in R$ (not even assuming that they generate the unit ideal), and $\Lambda \subseteq \{1, \dots, n\}$, the subset

$$\begin{aligned} X_\Lambda :&= \{x \in \mathrm{Spa}(R, R^+) : x(f_i) \leq 1\, \forall i \in \Lambda,\ x(f_i) \geq 1\, \forall i \neq \Lambda\} \\ &= \bigcap_{i \in \Lambda} \mathrm{Spa}(R, R^+)(\tfrac{f_i, 1}{f_i}) \cap \bigcap_{i \notin \Lambda} \mathrm{Spa}(R, R^+)(\tfrac{1}{f_i}) \end{aligned}$$

is a rational subset (recall that an intersection of rational subsets is again a rational subset). Clearly the collection X_Λ, $\Lambda \subseteq \{1, \dots, n\}$ is an open cover of $\mathrm{Spa}(R, R^+)$; we call it a *Laurent cover.*

Lemma 5.29. *Let (R, R^+) be a Tate–Huber pair.*

(i) Every open cover of $\mathrm{Spa}(R, R^+)$ *may be refined to a standard rational covering.*

(ii) Given any standard rational covering $\mathcal{U}$, there exists a Laurent covering $\mathcal{V}$ such that, for each $V \in \mathcal{V}$, the open cover $\mathcal{U}|_V := \{U \cap V : U \in \mathcal{U}$ s.t. $U \cap V \neq \emptyset\}$ of V is a standard rational cover generated by units.

(iii) Every standard rational cover of $\mathrm{Spa}(R, R^+)$ *generated by units may be refined to a Laurent cover.*

Proof. (i) Since $\mathrm{Spa}(R, R^+)$ is quasi-compact and the rational subsets form a basis, we immediate reduce to the case of a finite cover by m rational subsets, say

$$X_j = \mathrm{Spa}(R, R^+)(\tfrac{f_{j,1},\dots,f_{j,n}}{g_j}), \qquad j = 1, \dots, m$$

where $f_{j,1}, \dots, f_{j,n} \in R$ generate the unit ideal.

Let S be the set of all products $s_1 \cdots s_m$, where $s_j \in \{f_{j,1}, \dots, f_{j,n}, g_j\}$ for all j and where $s_j = g_j$ for at least one value of j. We claim that the finite set S generates the unit ideal. By Corollary 5.11 it is enough to show the following: if $x \in \mathrm{Spa}(R, R^+)$, then $x(s) \neq 0$ for some

$s \in S$. But this is easy: pick j_0 such that $x \in X_{j_0}$, put $s_{j_0} := g_{j_0}$ (whose x valuation is $\neq 0$), and for $j \neq j_0$ let $s_j \in \{f_{j,1}, \dots, f_{j,n}\}$ satisfy $x(s_j) \neq 0$ (which exists since the set generates the unit ideal); then $x(s_1 \cdots s_n) \neq 0$.

(ii) Let $\mathcal{U}$ be the standard rational cover given by $f_1, \dots, f_n \in R$ (generating the unit ideal). Rfter rescaling $f_1, \dots, f_n$ by a unit (which does not change the rational cover they define), we claim that we can arrange the following: for each $x \in \mathrm{Spa}(R, R^+)$ there exists i such that $x(f_i) > 1$. Indeed, let π be a pseudo-uniformiser; since $f_1, \dots, f_n$ generate the unit ideal and $R = R^+[\frac{1}{\varpi}]$, we may write $\frac{1}{\pi} = \sum_{i=1}^n \frac{a_i}{\pi^m} f_i$ for some $m \geq 1$ and $a_i \in R^+$. Then $x(\pi^m) = x(\sum_i a_i \pi f_i) \leq \max_i x(\pi f_i) < \max_i x(f_i)$; so rescaling by π^{-m} does the trick.

Having rescaled in this way, simply let $\mathcal{V}$ be the Laurent covering given by $f_1, \dots, f_n$. We claim that this has the desired property, so let $V = X_\Lambda \in \mathcal{V}$ for some $\Lambda \subseteq \{1, \dots, n\}$. Note that if $\Lambda = \{1, \dots, n\}$ then $X_\Lambda = \emptyset$, so we can ignore this case. Otherwise we clearly have

$$X_\Lambda \cap \mathrm{Spa}(R, R^+)(\tfrac{f_1, \dots, f_n}{f_i}) = \begin{cases} \emptyset & i \in \Lambda \\ \{x \in X_\Lambda : x(f_j) \leq x(f_i) \, \forall j \notin \Lambda\} & i \notin \Lambda \end{cases}.$$

But the elements f_j, $j \notin \Lambda$ are all units of $\mathcal{O}_X(X_\Lambda)$ (since their valuations never vanish); so this shows that $\mathcal{U}|_{X_\Lambda}$ is the standard rational cover generated by the units f_j, $j \notin \Lambda$.

(iii) Let $\mathcal{U}$ be the standard rational cover generated by units $f_1, \dots, f_n \in R$. We leave it as an exercise to check that it is refined by the Laurent cover generated by the elements $f_i f_j^{-1}$, $1 \leq i < j \leq n$. $\square$

Proof of Theorem 5.27. Claim: To prove the theorem it is necessary and sufficient to prove the following: for each stably uniform Huber pair (R, R^+), and each $f \in R$, the sequence

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V) \longrightarrow 0 \qquad (\dagger)$$

is exact, where $U = \{x \in X : x(f) \leq 1\} = X(\frac{f,1}{1})$, $V := \{x \in X : x(f) \geq 1\} = X(\frac{1}{f})$, $U \cap V = \{x \in X : x(f) = 1\} = X(\frac{f^2,1}{f})$.

Proof: Let (R, R^+) be a stably uniform Tate–Huber pair and $X = \mathrm{Spa}(R, R^+)$. By a standard argument via Cech cohomology, sheafiness and acyclicity of $\mathcal{O}_X$ is equivalent to the following: for each rational subspace $V \subseteq \mathrm{Spa}(R, R^+)$, and each open cover by rational subspaces $V = \bigcup_i V_i$, the Cech complex

$$0 \longrightarrow \mathcal{O}_X(V) \longrightarrow \prod_i \mathcal{O}_X(V_i) \longrightarrow \prod_{i<j} \mathcal{O}_X(V_i \cap V_j) \longrightarrow \cdots$$

is exact. To simplify notation we may assume $X = V$. By the previous lemma we may assume that the open cover $\{V_i\}_i$ is a Laurent cover. Then argue by induction, with each inductive step using the case of a two element cover, i.e., exactness of $(\dagger)$ [I will add more details.] $\square_{\text{claim}}$

So, we have reduced to proving that $(\dagger)$ is exact. Consider the various rings which appear in the localisation process:

R with its usual topology
$B := R[\frac{f,1}{1}] = R$ with topology given by $\pi^m R^+[f]$, $m \geq 1$
$R[\frac{1}{f}]$ with topology given by $\pi^m R^+[\frac{1}{f}]$, $m \geq 1$
$B[\frac{1}{f}] := R[\frac{f^2,1}{f}] = R[\frac{1}{f}]$ with topology given by $\pi^m R^+[f, \frac{1}{f}]$, $m \geq 1$

Thus, by definition of $\mathcal{O}_X$ on rational subsets, the sequence (*) is given by

$$0 \longrightarrow R \longrightarrow \widehat{B} \oplus \widehat{R[\tfrac{1}{f}]} \longrightarrow \widehat{B[\tfrac{1}{f}]} \longrightarrow 0 \tag{‡}$$

We claim that the sequence

$$0 \longrightarrow R^+ \longrightarrow R^+[f] \oplus R^+[\tfrac{1}{f}] \longrightarrow R^+[f, \tfrac{1}{f}] \longrightarrow 0$$

is exact. Surjectivity on the right and injectivity on the left are trivial, as is the fact that the sequence is a complex; it remains to prove exactness at the middle, so suppose that $g \in R^+[f]$ and $h \in R^+[\frac{1}{f}]$ are elements with the same image in $R^+[f, \frac{1}{f}]$. In other words, $g \in R$ is an element in $R^+[f]$ (whence $x(g) \leq 1$ for all $x \in U$) whose image in $R[\frac{1}{f}]$ is $h \in R^+[\frac{1}{f}]$ (whence $x(g) \leq 1$ for all $x \in V$); thus $x(g) \leq 1$ for all $x \in \mathrm{Spa}(R, R^+)$ and so $g \in R^+$ (by Theorem 5.10(v)). This proves exactness at the middle.

Since the sequence consists of π-torsion-free modules, we also get freeness of

$$0 \longrightarrow R^+/\pi^m R^+ \longrightarrow R^+[f]/\pi^m R^+[f] \oplus R^+[\tfrac{1}{f}]/\pi^m R[\tfrac{1}{f}] \longrightarrow R^+[f, \tfrac{1}{f}]/\pi^m R^+[f, \tfrac{1}{f}] \longrightarrow 0$$

for each $m \geq 1$, and then taking the limit gives exactness of

$$0 \longrightarrow \widehat{R^+} \longrightarrow \widehat{R^+[f]} \oplus \widehat{R^+[\tfrac{1}{f}]} \longrightarrow \widehat{R^+[f, \tfrac{1}{f}]} \longrightarrow 0.$$

Finally, inverting π gives (‡) (using Proposition 5.13 to compute the completions). $\square$

6 The tilting correspondence for perfectoid spaces (analytic topology)

A Huber pair (R, R^+) is *perfectoid* if and only if R is a perfectoid Tate algebra (or, equivalently, R^+ is an integral perfectoid ring). In this section we fix a perfectoid Huber pair (R, R^+) and denote by $(R^\flat, R^{+\flat})$ its tilt. Let $\pi \in R^+$ be a perfectoid pseudo-uniformiser admitting a compatible sequence of p-power roots, and $\pi^\flat \in R^{+\flat}$ the corresponding perfectoid pseudo-uniformiser of the tilt.

Let

$$X := \mathrm{Spa}(R, R^+), \qquad X^\flat := \mathrm{Spa}(R^\flat, R^{+\flat})$$

be the corresponding adic spectra. Define the *tilting map*

$$\flat : X \to X^\flat, \qquad x \mapsto x^\flat$$

as follows: given a continuous valuation $x : R \to \Gamma \cup \{0\}$, then define $x^\flat : R^\flat \to \Gamma \cap \{0\}$ by $x^\flat(f) := x(f^\#)$. Here $\# : R^\flat \to R$ is a multiplicative untilting map; we only defined in previously on integral perfectoid rings, but it extends by multiplicativity to $R^\flat = R^{+\flat}[\frac{1}{\pi^\flat}] \to R = R^+[\frac{1}{\pi}]$ since it sends $\pi^\flat$ to π.

Lemma 6.1. *The tilting map is well-defined, i.e., each $x^\flat : R^\flat \to \Gamma \cap \{0\}$ really is a continuous valuation; the tilting map is moreover continuous.*

Proof. Since $f \mapsto f^\#$ is multiplicative and preserves $0, 1$, the only axiom for a valuation which is not obviously satisfied for $x^\flat$ is the non-archimedean inequality. But, if $f, g \in R^\flat$, then

$$\begin{aligned} x^\flat(f+g) &= x((f+g)^\#) \\ &= x(\lim_{n\to\infty}(f^{1/p^n\#} + g^{1/p^n\#})^{p^n}) \\ &= \lim_{n\to\infty} x(f^{1/p^n\#} + g^{1/p^n\#})^{p^n} \\ &\le \lim_{n\to\infty} \max(x(f^{1/p^n\#})^{p^n}, x(g^{1/p^n\#})^{p^n}) \\ &= \max(x(f^\#), x(g^\#)), \end{aligned}$$

which is exactly $\max(x^\flat(f), x^\flat(g))$ by definition. The fact that $\# : R^\flat \to R$ is continuous implies that $x^\flat$ is continuous.

It remains to show that $\flat : X \to X^\flat$ is continuous, for which it is enough to check that the pre-image of any rational subset $U \subseteq X^\flat$ is a rational subset of X. Pick elements $f_1, \ldots, f_n, g \in R^{+\flat}$, where f_n is a power of $\pi^\flat$ (we have implicitly picked a pseudo-uniformiser for R^+ having p-power roots), such that $U = X^\flat(\frac{f_1,\ldots,f_n}{g})$ (it is standard that any rational subset may be described by such elements). The untilts $f_1^\#, \ldots, f_n^\#, g^\# \in R^+$ define a corresponding rational subset $V := X(\frac{f_1^\#,\ldots,f_n^\#}{g^\#})$ of X (n.b., this really is a rational subset since $f_n^\#$ is a power of π), and it is tautological from the definition of the tiling map that $\flat^{-1}(U) = V$. □

The goal of the section is to prove the following:

Theorem 6.2 (Tilting correspondence – analytic topology of perfectoid spaces)**.** *The tilting map $\flat : X \to X^\flat$ is a homeomorphism which identifies rational subsets. Moreover, if $V \subseteq X$ and $U \subseteq X^\flat$ are corresponding rational subsets, then:*

(i) $\mathcal{O}_X(V)$ is a perfectoid Tate R-algebra;

(ii) $\mathcal{O}_{X^\flat}(U)$ is a perfectoid Tate $R^\flat$-algebra;

(iii) there is a unique continuous map of R-algebras $\mathcal{O}_X(V)^\flat \to \mathcal{O}_{X^\flat}(U)$; it is an isomorphism and restricts to an isomorphism of integral perfectoid $R^{+\flat}$-algebras $\mathcal{O}_X^+(V)^\flat \xrightarrow{\simeq} \mathcal{O}_{X^\flat}^+(U)$.

Example 6.3 (The case of a perfectoid field)**.** Suppose that $\mathcal{O}$ is an integral perfectoid ring which is a valuation ring, so that $F := \operatorname{Frac}\mathcal{O}$ is a perfectoid field (note that $\mathcal{O}$ might be smaller than the full ring of integers of F). In this example we prove the homeomorphism assertion of Theorem 6.2 for the $(F, \mathcal{O})$.

Indeed, since $\mathcal{O}$ is a valuation ring, we know that each point x of $\operatorname{Spa}(F, \mathcal{O})$ corresponds to a valuation subring of F of the form $\mathcal{O}_\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq \mathcal{O}$; continuity of x forces $\mathfrak{p}$ to be open, whence it contains $\mathcal{O}^{\circ\circ}$ and so corresponds to a prime ideal of $\mathcal{O}/\mathcal{O}^{\circ\circ}$. This gives a homeomorphism $\operatorname{Spa}(F, \mathcal{O}) \cong \operatorname{Spec}(\mathcal{O}/\mathcal{O}^{\circ\circ})$.

But in the same way we have $\operatorname{Spa}(F^\flat, \mathcal{O}^\flat) \cong \operatorname{Spec}(\mathcal{O}^\flat/\mathcal{O}^{\flat\circ\circ})$, so the proof is completed by recalling that $\# : \mathcal{O}^\flat/\mathcal{O}^{\flat\circ\circ} \xrightarrow{\simeq} \mathcal{O}/\mathcal{O}^{\circ\circ}$. (One should explicitly check that the homeomorphism $\operatorname{Spa}(F^\flat, \mathcal{O}^\flat) \simeq \operatorname{Spa}(F, \mathcal{O})$ we just constructed really is the one predicted by the theorem, but this is not hard: they are both given by pulling back along the untilting map.)

The proof of the theorem requires two key steps; the first of these is a careful study of rational subsets of $X^\flat$ and their pull-backs to X; we will eventually apply the following to $A = R^+$:

Proposition 6.4. *Let A be an integral perfectoid ring and $\pi \in A$ a ppu with p-power roots; let $f_1, \dots, f_n, g \in A^\flat$ where f_n is a power of $\pi^\flat$. Let C be the $A^\flat$-subalgebra of $A^\flat[1/g]$ generated by $f_i^{1/p^k}/g^{1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$; similarly, let B be the A-subalgebra of $A[1/g^\#]$ generated by $f_i^{\#1/p^k}/g^{\#1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$. Then:*

(i) The kernel of the surjection

$$\psi^\flat : A^\flat[\underline{X}^{1/p^\infty}] \longrightarrow C, \qquad X_i^{1/p^k} \mapsto f_i^{1/p^k}/g_i^{1/p^k}$$

is almost the ideal generated by $g_i^{1/p^k} X_i^{1/p^k} - f_i^{1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$.

(ii) Similarly, the kernel of the surjection

$$\psi : A[\underline{X}^{1/p^\infty}] \longrightarrow B, \qquad X_i^{1/p^k} \mapsto f_i^{\#1/p^k}/g_i^{\#1/p^k}$$

is almost the ideal generated by $g_i^{\#1/p^k} X_i^{1/p^k} - f_i^{\#1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$.

(iii) The $\pi^\flat$-adic completion $\widehat{C}$ is an integral perfectoid $A^\flat$-algebra.

Let $\widehat{C}^\#$ be its untilt (an integral perfectoid A-algebra), with corresponding generic fibre $\widehat{C}^\#[\frac{1}{\pi}]$ (a perfectoid Tate R-algebra).

(iv) There exists a unique continuous map of R-algebras $\widehat{B}[\frac{1}{\pi}] \to \widehat{C}^\#[\frac{1}{\pi}]$; it is an isomorphism (therefore $\widehat{B}[\frac{1}{\pi}]$ is a perfectoid Tate algebra, which is not obvious; however, we cannot show that $\widehat{B}$ is an integral perfectoid A-algebra), it restricts to an injective almost surjection $B \hookrightarrow \widehat{C}^\#$, and it sends sends $f_i^{\#1/p^k}/g^{\#1/p^k}$ to $(f_i^{1/p^k}/g^{1/p^k})^\#$ for all $i = 1, \dots, n$ and $k \geq 1$.

(v) $\widehat{C}^\#$ is integral over $\widehat{B}$.

(vi) $\widehat{C}$ is integral over its subring $\widehat{A^\flat[f_i/g]}$, and the difference between the rings is killed by a power of $\pi^\flat$; similarly B is integral over its subring $\widehat{A[f_i^\#/g^\#]}$ and the difference between the rings is killed by a power of π.

Proof. Although the above seems to be the clearest order in which the state the results, the proof proceeds differently:

(iii): C is obviously perfect and $\pi^\flat$ is a non-zero-divisor of it; therefore its completion $\widehat{C}$, equipped with the $\pi^\flat$-adic topology, is an integral perfectoid $A^\flat$-algebra by Lemma 1.5.

(i): Let $J \subseteq A[\underline{X}^{1/p^\infty}]$ be the ideal generated by the given elements; we obviously have an inclusion $J \subseteq \operatorname{Ker}\psi$, and we want to show that it is an almost equality. It is clear that the Frobenius acts isomorphically on both J and $\operatorname{Ker}\psi$, so it is enough to show that $\operatorname{Ker}\psi/J$ vanishes after inverting $\pi^\flat$, i.e., that

$$A^\flat[\tfrac{1}{\pi^\flat}][\underline{X}^{1/p^\infty}]/\langle g^{1/p^k} X_i^{1/p^k} - f_i^{1/p^k} : i, k\rangle \longrightarrow C[\tfrac{1}{\pi^\flat}]$$

is injective. We have assumed that f_N is a power of $\pi^\flat$, whence f_N is invertible on the left side, and so g is also invertible on both sides thanks to the relation $gX_N - f$. By rescalling the relations it is therefore equivalent to show that

$$A^\flat[\tfrac{1}{\pi^\flat}, \tfrac{1}{g}][\underline{X}^{1/p^\infty}]/\langle X_i^{1/p^k} - f_i^{1/p^k}/g^{1/p^k} : i, k\rangle \longrightarrow C[\tfrac{1}{\pi^\flat}, \tfrac{1}{g}] = A^\flat[\tfrac{1}{\pi^\flat}, \tfrac{1}{g}]$$

is injective; but it is clearly an isomorphism by elementary algebra.

(ii) & (iv): The untilts $(f_i^{1/p^k}/g^{1/p^k})^\# \in \widehat{C}^\#$ satisfy

$$g^{\#1/p^k}(f_i^{1/p^k}/g^{1/p^k})^\# = (g^{1/p^k}(f_i^{1/p^k}/g^{1/p^k}))^\# = f_i^{1/p^k\#} = f_i^{\#1/p^k},$$

and $g^\#$ is a non-zero-divisor in $\widehat{C}^\#$ (since it divides f_n, thanks to the previous line with $i = n$ and $k = 1$, which is a power of π). So there is a unique map of A-algebras $e : B \to \widehat{C}^\#[\frac{1}{\pi}]$; it sends $f_i^{\#1/p^k}/g^{\#1/p^k}$ to $(f_i^{1/p^k}/g^{1/p^k})^\#$ and has image in $\widehat{C}^\#$. Taking π-adic completion extends this map to $\widehat{e} : \widehat{B} \to \widehat{C}^\#$, and then we may invert π to obtain the desired map; conversely, since any continuous map of A-algebras $\widehat{B}[\frac{1}{\pi}] \to \widehat{C}^\#[\frac{1}{\pi}]$ is determined by its restriction to B, we have also proved uniqueness.

We next consider the composition

$$e\psi : A[\underline{X}^{1/p^\infty}]/J \longrightarrow B \longrightarrow \widehat{C}^\#,$$

whose reduction modulo π identifies with the reduction modulo $\pi^\flat$ of $\psi^\flat$. This latter map is an almost isomorphism by part (i), whence $e\psi$ is an almost isomorphism modulo π. So ψ is almost injective modulo π and surjective, whence it formally follows that it is almost injective (since its domain, resp. codomain, is π-adically separated, resp. π-torsion-free); this proves (ii).

Since $e\psi$ is an almost isomorphism modulo π and ψ is surjective, we also deduce that e is an almost isomorphism modulo π, hence also modulo any power of π by a trivial induction; taking the limit we deduce that $\widehat{e} : \widehat{B} \to \widehat{C}^\#$ is an almost isomorphism. But $\widehat{B}$ has no π-torsion, so the almost zero kernel is actually zero. This completes the proof of (iv).

(v): Let B' be the integral closure of $\widehat{B}$ in $\widehat{C}^\#$. Using that $\widehat{C}^\#$ is an open integral perfectoid subring of $\widehat{B}[\frac{1}{\pi}]$, it is easy to check that B' is also integral perfectoid. But B' contains $f_i^{\#1/p^k}/g^{\#1/p^k}$ for all $k \geq 0$, whence its tilt $B'^\flat$ (which is an integral perfectoid $A^\flat$-subalgebra of $\widehat{C}$) contains $f_i^{1/p^k}/g^{1/p^k}$ for all $k \geq 0$. But therefore $B'^\flat$ contains C, whence its completeness forces $B'^\flat = \widehat{C}$; the tilting correspondence now tells us that in fact $B' = \widehat{C}^\#$, as desired.

(vi): There is an obvious inclusion $A^\flat[f_i/g] \subseteq C$; we claim that the difference is killed by a power of $\pi^\flat$. Since f_n equals a power of $\pi^\flat$, it is enough to show that the difference is killed by f_n^n (the n both upstairs and downstairs is not a typo!); this follows from the nice observation that the fractional powers can be treated with the argument

$$f_n^n \prod_{i=1}^n \frac{f_i^{1/p^{k_i}}}{g^{1/p^{k_i}}} = \prod_{i=1}^n ((f_i^{1/p^{k_i}} g_i^{1-1/p^{k_i}} \frac{f_n}{g}) \in A[f_i/g\,\forall i]$$

for any $k_1, \dots, k_n \geq 0$. Having proved the claim, taking $\pi^\flat$-adic completions therefore yields an inclusion $\widehat{A^\flat[f_i/g]} \subseteq \widehat{C}$ such that $\widehat{A^\flat[f_i/g]} \supseteq \pi^{\flat N}\widehat{C}$ for $N \gg 0$ (this is the openness assertion); combined with the fact that C is clearly integral over $A^\flat[f_i/g]$, this shows that $\widehat{C}$ is integral over $\widehat{A^\flat[f_i/g]}$.

The verbatim argument works on the untilted side. □

Corollary 6.5. *Let $X := \operatorname{Spa}(R, R^+)$ and $X^\flat := \operatorname{Spa}(R^\flat, R^{+\flat})$ be as at the start of the section, and let $U \subseteq X^\flat$ be a rational subset. Then $V := \flat^{-1}(U)$ is a rational subset of X and the assertions of Theorem 6.2 are true in this case.*

Proof. Let $\pi \in R^+$ be our ppu with p-power roots, and pick elements $f_1, \dots, f_n, g \in R^{+\flat}$, where f_n is a power of $\pi^\flat$, such that $U = X^\flat(\frac{f_1,\dots,f_n}{g})$. As we already explained in the proof of Lemma 6.1, it follows that $V := \flat^{-1}(U)$ is the rational subset $X(\frac{f_1^\#,\dots,f_n^\#}{g^\#})$.

Let $B, \widehat{B}, C, \widehat{C}$ be as in the statement of the previous proposition. Part (vi) of the lemma gives equality of Tate algebras $\widehat{C}[\frac{1}{\pi^\flat}] = \mathcal{O}_{X^\flat}(U)$ and $\widehat{B}[\frac{1}{\pi}] = \mathcal{O}_X(V)$. Therefore parts (iii) and (iv) of the previous proposition show that $\mathcal{O}_{X^\flat}(V)$ and $\mathcal{O}_X(V)$ are perfectoid Tate algebras over $R^\flat$ and R respectively, that there is a unique continuous map of R-algebras $\mathcal{O}_X(V) \to \mathcal{O}_{X^\flat}(U)^\#$, and that this map is an isomorphism. Identifying these perfectoid R-algebras, we then have

$$\begin{aligned}
\mathcal{O}_X^+(V) &= \text{integral closure of } \widehat{R^+[f_i^\#/g^\#]} \text{ in } \mathcal{O}_X(U) && \text{(by def.)}\\
&= \text{integral closure of } \widehat{B} \text{ in } \widehat{B}[\tfrac{1}{\pi}] && \text{(by (vi) of prev. prop.)}\\
&= \text{integral closure of } \widehat{C}^\# \text{ in } \widehat{C}^\#[\tfrac{1}{\pi}] && \text{(by (v) of prev. prop.)}\\
&= (\text{integral closure of } \widehat{C} \text{ in } \widehat{C}[\tfrac{1}{\pi^\flat}])^\# && \text{(since tilting is compatible with int. closures)}\\
&= (\text{integral closure of } R^{+\flat}[f_i/g] \text{ in } \mathcal{O}_{X^\flat}(U))^\# && \text{(by (vi) of prev. prop.)}\\
&= \mathcal{O}_{X^\flat}^+(U)^\# && \text{(by def.)}
\end{aligned}$$

□

The second key step is a subtle approximation argument which will imply that all rational subsets of X are obtained by pulling back rational subsets of $X^\flat$:

Proposition 6.6 (Approximation lemma)**.** *Fix a perfectoid Tate K (I hope that it does not need to be a field!) and set $R := K\langle \underline{T}^{1/p^\infty}\rangle$; let $f \in R^\circ$ be homogeneous of degree $d \in \mathbb{Z}[\frac{1}{p}]$*[5] *and fix rational numbers $c \geq 0$ and $\varepsilon > 0$. Then there exists $g_{c,\varepsilon} \in R^{\flat\circ}$ homogeneous of degree d such that, for all $x \in \mathrm{Spa}(R, R^\circ)$,*

$$|f(x) - g_{c,\varepsilon}^\#(x)| \leq |\pi|^{1-\varepsilon} \max(|f(x)|, |\pi|^c)$$

Proof. See Scholze's *Perfectoid spaces.* □

The previous special case easily implies the approximation lemma in general:

Corollary 6.7. *R a perfectoid Tate algebra; let $f \in R$ and fix rational numbers $c \geq 0$ and $\varepsilon > 0$. Then there exists $g_{c,\varepsilon} \in R^\flat$ such that*

$$|f(x) - g_{c,\varepsilon}^\#(x)| \leq |\pi|^{1-\varepsilon} \max(|f(x)|, |\pi|^c)$$

for all $x \in \mathrm{Spa}(R, R^\circ)$.

Proof. See Scholze's *Perfectoid spaces.* □

Corollary 6.8. *Any rational subset of X has the form $\flat^{-1}(U)$ for some rational subset U of $X^\flat$.*

Proof. Let $V \subseteq X$ be a rational subset; pick $f_1, \ldots, f_n, g \in R^+$, where $f_n = \pi^N$ is a power of π, such that $U = X(\frac{f_1,\ldots,f_n}{g})$. Then $U = \bigcap_{i=1}^{n-1} X(\frac{f_i,\pi^N}{g})$, so it is enough to show that each $X(\frac{f_i,\pi^N}{g})$ is the preimage of a rational subset of $X^\flat$ (note that rational subsets are closed under finite intersections).

By two applications of the approximation lemma (first with $f = f_i$, $c = N$, and any $\varepsilon \in (0,1)$; secondly with $f = g$, $c = N$, and $\varepsilon = 1$), there exist $a, b \in R^\flat$ such that

$$\max(|f(x)|, |\pi(x)^N|) = \max(|a^\#(x)|, |\pi(x)^N|)$$

[5] i.e., in the completion of $\bigoplus_{k_1,\ldots,k_n \in \mathbb{Z}[\frac{1}{p}] : \sum_i k_i = d} \mathcal{O}_K T_1^{k_1} \cdots T_n^{k_n}$.

$$|g(x) - b^{\#}(x)| < \max(|g(x)|, |\pi(x)|^N).$$

Straightforward arguments with non-archimedean inequalities shows that $X(\frac{f,\pi^N}{g}) = X(\frac{a^{\#},\pi^N}{b^{\#}})$, which is we already know is $\flat^{-1}(X^\flat(\frac{a,\pi^{\flat N}}{b}))$. $\square$

We can now complete the proof of Theorem 6.2:

Proof of Theorem 6.2. We have proved that X has a basis of opens (namely the rational subsets) which are pull-backs from $X^\flat$; it follows from general topology that $\flat : X \to X^\flat$ is injective (since X is T_0) and then that $\flat$ is a homeomorphism onto its image.

Surjecitivity follows from Example 6.3 via the following functoriality argument. Given a point $y \in X^\flat$, its completed residue field $\widehat{k(x)}$ is a perfectoid field over R, hence may be untilted to a perfectoid field $\widehat{k(y)}^{\#}$ over R. The valuation y defines a valuation subring $\mathcal{O}_y \subseteq \widehat{k(y)}$, which may be similarly untilted to $\mathcal{O}_y^{\#} \subseteq \widehat{k(y)}^{\#}$. The map of Huber pairs $(R, R^+) \to (\widehat{k(y)}^{\#}, \mathcal{O}_y^{\#})$ gives rise to a commutative diagram by functoriality

$$\begin{tikzcd}
\mathrm{Spa}(\widehat{k(y)}^{\#}, \mathcal{O}_y^{\#}) \arrow[r, "\flat"] \arrow[d] & \mathrm{Spa}(\widehat{k(y)}, \mathcal{O}_y) \arrow[d] \\
X \arrow[r, "\flat"] & X^\flat
\end{tikzcd}$$

The point y is (by construction) in the image of the right vertical map; but the top horizontal map is bijective by Example 6.3. This proves that the bottom horizontal arrow is surjective, as required to complete the proof. $\square$

7 Tilting in the étale topology: The almost purity theorem

In the final section we give a rough sketch of how the theory developed so far can be used to prove the almost purity theorem. This is the fundamental result concerning perfectoids which allows étale cohomology of objects over $\mathbb{Q}_p$ to be studied via étale cohomology in characteristic p.

Theorem 7.1 (Almost Purity). *Let R be a perfectoid Tate algebra.*

(i) Let S be a finite étale R-algebra; give S the canonical topology (see the next remark). Then S is a perfectoid Tate algebra, and S° is almost finite étale over R°.

(ii) Tilting $S \mapsto S^\flat$ induces an equivalence of categories

$$R_{\text{fét}} := \{\textit{finite étale } R\textit{-algs}\} \overset{\simeq}{\to} \{\textit{finite étale } R^\flat\textit{-algs}\} =: R^\flat_{\text{fét}}$$

Remark 7.2 (Canonical topology). If R is a Tate ring and S is a finite étale R-algebra, then there is a unique way to topologise S as a Tate algebra so that $R \to S$ is continuous. Namely, letting R_0 be any subring of definition of R and $\pi \in R_0 \cap R^\times$ a pseudo-uniformiser, pick a finitely generated R_0-submodule $M \subseteq S$ such that $M[\frac{1}{\pi}] = S$ and give S the unique linear topology for which M is open and carries the π-adic topology. Equivalently, this is the weak R-linear topology, i.e., the weakest topology making all R-linear maps $S \to R$ continuous.

In other words, the category of finite étale R-algebras is equivalent to the category of Tate algebras which are finite étale over R.

We must explain "almost finite étale":

Definition 7.3. Let A be an integral perfectoid ring, and $B \to C$ a homomorphism of A-algebras. We that

- the B-module C is of *almost of finite presentation* if and only if for each $m \geq 0$ there is a B-module C_m of finite presentation and a morphism $C_m \to C$ with kernel and cokernel killed by π^{1/p^m};

- the B-module C is *almost projective* if and only if $\mathrm{Hom}_B(C, -)$ takes surjections to almost surjections;

- $B \to C$ is *almost unramified* if and only if the $C \otimes_B C$-module C (via multiplication) if almost projective;

- $B \to C$ is almost étale if and only if the three above properties are all true.

Remark 7.4 (Reminder on unramified)**.** A finite type morphism of rings $B \to C$ is said to be unramified of the following equivalent conditions hold:

(i) $\Omega^1_{C/B} = 0$;

(ii) the closed immersion $\operatorname{Spec} C \to \operatorname{Spec} C \otimes_B C$ is an open map;

(iii) the $C \otimes_B C$-module C is projective.

Condition (iii) amounts to the assertion that mult : $C \otimes_B C \to C$ admits a splitting which is $C \otimes_B C$-linear; this is equivalent to the existence of an idempotent $e \in C \otimes_B C$ which generates (as an ideal) the kernel of the map mult. In practice it is this idempotent which will be used in our arguments (c.f., the proof of Proposition 4.2).

Here is the strategy for proving the almost purity theory:

(A) Part (i) of the theorem is easy in characteristic p;

(B) Hence we will obtain an untilting functor $\# : (R^\flat)_{\text{fét}} \to R_{\text{fét}}$, and the theorem can be reformulated as the assertion that this functor is essentially surjective.

(C) We have already proved the theorem for perfectoid fields (Theorem 4.5).

(D) Finally we observe that the adic spectra X and $X^\flat$ are locally given by perfectoid perfectoid fields, so we use $X \simeq X^\flat$ to glue the results in the case of fields.

We begin with step (A):

Lemma 7.5. *Let T be a finite etale $R^\flat$-algebra. Then T is a perfectoid Tate algebra and T° is almost finite étale over $R^{\flat\circ}$.*

Proof. Since $R^\flat$ is perfect and $R^\flat \to T$ is étale, commutative algebra implies that T is also perfect. As we explained in Remark 7.2, T is automatically a Tate ring with its natural topology. It follows from Remark 3.4 that T is perfectoid.

Since $R^\flat \to T$ is unramified by hypothesis, we let $e \in T \otimes_{R^\flat} T$ be the idempotent from the previous remark. Since $T = T^\circ[\frac{1}{\pi^\flat}]$ and $R^\flat = R^{\flat\circ}[\frac{1}{\pi^\flat}]$, there exists $N \gg 0$ such that $\pi^{p^N} e \in T^\circ_{R^\flat\circ} T^\circ$. Now we argue exactly as in the proof of Proposition 4.2. Namely, the absolute

Frobenius φ is an automorphism of $R^{\flat\circ}$ and T°, hence of the tensor products, and fixed e (since it is an idempotent. Therefore $\pi^{\flat 1/p^m} e = \varphi^{-N-m}(\pi^{\flat p^N} e) \in T^\circ \otimes_{R^{\flat\circ}} T^\circ$. But this element shows that multiplication $T^\circ \otimes_{R^{\flat\circ}} T^\circ \to T^\circ$ is split up to obstructions killed by $\pi^{\flat 1/p^m}$; since this is true for all m, we have shown that $R^{\flat\circ} \to T^\circ$ is almost unramified.

To show that the $R^{\flat\circ}$-module T° is almost projective and almost of finite presentation, the idea is similar: these properties hold after inverting $\pi^\flat$, and φ is an automorphism of everything, hence they hold up to $\pi^{\flat 1/p^m}$ for any $m \geq 1$, i.e., they almost hold. We do not include the details. $\square$

Now we move to step (B): given a finite étale $R^\flat$-algebra T, we have just shown that T is perfectoid, hence we may untilt it to form a perfectoid algebra $T^\#$ over R. The key to the next step is the following:

Lemma 7.6. *$T^\#$ is finite étale over R; in fact, $T^{\#\circ}$ is almost finite étale over R°.*

Proof. By the previous lemma we know that $F^{\flat\circ} \to T^\circ$ is almost finite étale. Going mod $\pi^\flat$ this easily implies that $R^{\flat\circ}/\pi^\flat R^{\flat\circ} \to T^\circ/\pi^\flat$ is almost finite étale; but this may be rewritten $R^\circ/\pi R^\circ \to T^{\#\circ}/\pi T^{\#\circ}$. By a deformation argument which we omit (but which is of a similar spirt to Lemma 4.1), it follows that $R^\circ \to T^{\#\circ}$ is almost finite étale. $\square$

In other words, untilting defines a fully faithful untilting functor $\#$ from $(R^\flat)_{\text{fét}}$ to $R_{\text{fét}}$, whose image consists of perfectoid Tate algebras S such that S° is finite étale over R°. Therefore the almost purity theorem becomes equivalent to surjectivity of this functor.

Step (C) is to note that, in the case of a perfectoid field, this surjectivity is exactly what we showed to prove Theorem 4.5. Therefore we now turn to step (D), namely using adic spaces to glue these local results. We need two results from commutative algebra (neither of which are easy, but it would take us too far afield to discuss them):

Proposition 7.7. *(i) Let A be a ring which is Henselian along an ideal $tA \subseteq A$, where $t \in A$ is a non-zero-divisor. Then base change $A[\frac{1}{t}]_{\text{fét}} \to \widehat{A}[\frac{1}{t}]_{\text{fét}}$ is an equivalence of categories, where $\widehat{A}$ is the t-adic completion of A.*

(ii) Let $\varinjlim_i A_i$ be a filtered colimit of rings. Then $\varinjlim_i A_{i\,\text{fét}} \to (\varinjlim_i A_i)_{\text{fét}}$ is an equivalence of categories, where the left side is a filtered colimit of categories.

Next we need some remarks on the local structure of adic spaces, which we did not cover when discussing adic spaces:

Remark 7.8. Let (S, S^+) be a Tate pair, and $t \in S$ a pseudo-uniformiser. Let $x \in X := \mathrm{Spa}(S, S^+)$. We consider the stalks $\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U)$ and $\mathcal{O}^+_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U)^+$ of the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$, where the limit is taken over all rational subspaces of X containing x. The valuation x induces a valuation on $\mathcal{O}_{X,x}$ such that $\mathcal{O}^+_{X,x} = \{f \in \mathcal{O}_{X,x} : x(f) \leq 1\}$; one can also show that $\mathcal{O}_{X,x}$ and $\mathcal{O}^+_{X,x}$ are local rings with respective maximal ideals $\mathfrak{m}_{X,x} := \{f \in \mathcal{O}_{X,x} : x(f) = 0\}$ and $\{f \in \mathcal{O}_{X,x} : x(f) < 1\}$. It follows that $\mathfrak{m}_{X,x}$ is also an ideal of $\mathcal{O}^+_{X,x}$, and that $\mathcal{O}^+_{X,x}/\mathfrak{m}_{X,x}$ is the ring of integers of the valued field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.

We now take t-adic completions of these rings; the most important point is that $\mathfrak{m}_{X,x}$ is an ideal of $\mathcal{O}_{X,x}$ (in which t is a unit), and hence $\mathfrak{m}_{X,x}$ is t-divisible and so killed by t-adic completion (this does not really have an analogue in the theory of schemes):

$$\widehat{\mathcal{O}^+_{X,x}} = \widehat{\mathcal{O}^+_{X,x}/\mathfrak{m}_{X,x}}.$$

This is precisely the ring of integers of the completed residue field $\widehat{k(x)} = \widehat{\mathcal{O}^+_{X,x}}[\frac{1}{t}]$.

Finally we point out that $\mathcal{O}_{X,x}$ is Henselian along the ideal $t\mathcal{O}_{X,x}$, since it is a filtered colimit of t-adically complete rings. Putting all this together and appealing to the previous proposition, we deduce that

$$\widehat{k(x)}_{\text{fét}} = \widehat{\mathcal{O}^+_{X,x}}[\tfrac{1}{t}]_{\text{fét}} \simeq \mathcal{O}^+_{X,x}[\tfrac{1}{t}]_{\text{fét}} \simeq \varinjlim_{U \ni x} \mathcal{O}_X(U)_{\text{fét}}$$

Conclusion: any finite étale algebra $\widehat{k(x)}$-algebra spreads out to a finite étale $\mathcal{O}_X(U)$-algebra, for a sufficient small rational neighbourhood U of x; moreover, this spreading out is unique in the sense that, given another choice, the two agree on a smaller neighbourhood.

We are now prepared to complete our sketch of the proof of the almost purity theorem. Set $X = \mathrm{Spa}(R, R^\circ)$ and $X^\flat := \mathrm{Spa}(R^\flat, R^{\flat\circ})$. Let $x \in X$, with corresponding tilt $x^\flat \in X^\flat$. We will argue using the following commutative diagram of categories:

$$\begin{tikzcd}
\widehat{k(x^\flat)}_{\text{fét}} \arrow[d, equal] & \varinjlim_{U \ni x} \mathcal{O}_{X^\flat}(U)_{\text{fét}} \arrow[l, "\simeq"'] \arrow[d, equal] \\
\widehat{k(x)}^\flat{}_{\text{fét}} \arrow[d, "\#", "\simeq"'] & \varinjlim_{U \ni x} \mathcal{O}_X(U)^\flat{}_{\text{fét}} \arrow[l, "\simeq"'] \arrow[d, "\#"] \\
\widehat{k(x)}_{\text{fét}} & \varinjlim_{U \ni x} \mathcal{O}_X(U)_{\text{fét}} \arrow[l, "\simeq"']
\end{tikzcd}$$

The top horizontal arrow is the equivalence of the previous remark for $\mathrm{Spa}(R^\flat, R^{\flat\circ})$. The top vertical identifications are a consequence of Theorem 6.2. Hence we get the middle horizontal equivalence. The bottom horizontal arrow is the equivalence of the previous remark for $\mathrm{Spa}(R, R^\circ)$. The bottom left vertical map is almost purity equivalence for the perfectoid field $\widehat{k(x)}$.

Conclusion: Given a finite étale R-algebra S, the finite étale $\widehat{k(x)}$-algebra $S \otimes_R \widehat{k(x)}$ (which lies in the bottom left of the diagram) may be written as $T_x^\# \otimes_{\mathcal{O}_X(U)} \widehat{k(x)}$ for some finite $\mathcal{O}_X(U_x)^\flat$-algebra T_x, where U_x is some small enough rational neighbourhood of x.

Since $\mathcal{O}_X$, $\mathcal{O}_{X^\flat}$ are sheaves with no higher cohomology, one can glue (though it is somewhat subtle) the T_x as we vary x to show that $\# : R^\flat{}_{\text{fét}} \to R_{\text{fét}}$ hits S, as desired.