# p-adic / Étale / homology obstruction pts.

- We will always assume $X/K$ is finite type separated, $K$ a n.f. (f.ext$/\mathbb{Q}$).
  aka. a variety.

0. Prelim setup.
1. p adic homotopy obstruction.
2. Étale homotopy obstruction.
3. Étale homology obstruction.

- Our goal is to understand the construction of the following comm. dia.

$$\begin{tikzcd}
 & X(\mathbb{A})^{\text{ét},h} \arrow[r, hook] & \cdots \arrow[r, hook] & X(\mathbb{A})^{\text{ét},h,2} \arrow[r, hook] & X(\mathbb{A})^{\text{ét},h,1} \arrow[r, hook] & X(\mathbb{A}) \\
X(K) \arrow[r, hook] & X(\mathbb{A})^{h} \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[r, hook] & X(\mathbb{A})^{h,2} \arrow[r] & X(\mathbb{A})^{h,1} \arrow[u, hook] &
\end{tikzcd}$$

# 0. Prelim setup.

- $K$ has two type of places $v$ (Ostrowski)
  - infinite / arch  $(K \hookrightarrow \mathbb{R}$ or $K \overset{\sigma, \bar{\sigma}}{\hookrightarrow} \mathbb{C})$
  - finit / narch.  (Given by $|\cdot|_{\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$; $|\mathfrak{p}^n I| = q^{-n}$, $q = |\mathcal{O}_K/\mathfrak{p}|$, $I$ frac ideal.)

- We let $K_v$ be cpl. of $K$ at a narc. place $v$.

<u>Defn</u>: $\Gamma_v := \mathrm{Gal}(\bar{K}_v / K_v) \hookrightarrow \mathrm{Gal}(\bar{K}, K)$ given by restriction

- This is injective as $\bar{K}$ is dense in $\bar{K}_v$.

2. This is true as for any choice of embedding

$$\begin{tikzcd} \bar{K} \arrow[r, hook, "\bar{\iota}"] & \bar{K}_v \\ K \arrow[u, hook] \arrow[r, hook, "\iota"] & K_v \arrow[u, hook] \end{tikzcd}$$

- $\bar{K} \hookrightarrow \bar{K}_v$ has dense image
- Corresponds to $\mathfrak{q} | \mathfrak{p}$  $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_{\bar{K}})$, $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$.
  Can identify $\Gamma_v \simeq D(\mathfrak{q}|\mathfrak{p}) = \{\sigma \in \mathrm{Gal}(\bar{K}/K) : \sigma(\mathfrak{q}) = \mathfrak{q}\}$.
  - Induces ses
  $$1 \to I_v \to \Gamma_v \to \mathrm{Gal}(\mathcal{O}_{\bar{K}}/\mathfrak{q} \,/\, \mathcal{O}_K/\mathfrak{p}) \to 1$$

<u>Defn</u>: let $\Gamma_v^{ur} := \Gamma_v / I_v$. the unramified Galois gp.

<u>Q</u>. what is decomposition gp $\Gamma_v$ for $v$ infinite?

- $\mathbb{Z}/2 \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ or $\mathrm{Gal}(\mathbb{C}/\mathbb{C}) \simeq *$. of which both embeds in to $\mathrm{Gal}(\bar{K}/K)$.

## 1. Radic Étale homotopy.

- We have defined.

$$\acute{E}t^n_K : Var_K \longrightarrow Pro\ Ho(sSet_{\Gamma_K}).$$
$$X \longmapsto \acute{E}t^n_K X : I_C X \longrightarrow Ho\ sSet_{\Gamma_K}$$
$$\mathcal{U} \longrightarrow P_n(Ex^\infty \pi_0 \mathcal{U}_{\bar{K}})$$

for $0 \le n \le \infty$. if $n = \infty$ $P_n = id$.

- Further $\acute{E}t^n_K$ factors through the fib. obj. in str. mod. cat.
  ie. have LLP wrt. strict we. + cof.

  - The fgt ful functor $sSet_{\Gamma_K} \to sSet_{\Gamma_v}$ preserves str fib obj.
    This follows from LLP condition.
  - This induces fctr $Var_K \to Pro\ Ho(sSet_{\Gamma_v})$
    $X \mapsto \acute{E}t_K X$ regarded as $\Gamma_v$. obj.

Lemma: There is canical map $\acute{E}t^n_{K_v}(X_{K_v}) \to \acute{E}t^n_K(X)$. in Pro $sSet_{\Gamma_v}$.

Pf: • By def'n, a morphism

$$\varprojlim_{\mathcal{U} \in IC_K X} \ \varinjlim_{\mathcal{V} \in IC_{K_v} X_{K_v}} [(X_{K_v})_{\mathcal{V}_n}, X_{\mathcal{U}_n}]$$

- Simply let $\mathcal{V} = \mathcal{U}_{K_v}$.

Cor: We have diagram.

$$\begin{tikzcd}
Hom_K(Spec\,K, X) \arrow[r] \arrow[d] & {[\acute{E}t^n_K(Spec\,K), \acute{E}t^n_K X]} \arrow[dr] & \\
Hom_{K_v}(Spec\,K_v, X_{K_v}) \arrow[r] & {[\acute{E}t^n_{K_v} Spec\,K_v, \acute{E}t^n_{K_v} X_{K_v}]} \arrow[r] & {[\acute{E}t^n_{K_v} Spec\,K_v, \acute{E}t^n_K X]}.
\end{tikzcd}$$

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X^n(hK) \arrow[r] & X^n(hK_v) \\
X(K_v) \arrow[r] & {} &
\end{tikzcd}$$

- Combining $\forall$ places $v$.

[2.3.21, Pool?]

$$\begin{tikzcd} X(k) \arrow[r, "h_n"] \arrow[d, hook] & X^n(hK) \arrow[d] \\ \prod_v X(k_v) \arrow[r, "h_n"] & \prod_v X^n(hK_v). \end{tikzcd}$$

<u>Def'n</u>: The p-adic rationally homotopy set is given by pb. $\lrcorner$ denoted.

$$X(k) \hookrightarrow \left( \prod_v X(k_v)^{h, n} \right) \hookrightarrow \prod_v X(k_v).$$

## 2. Adelic homotopy fp.

- To put into perspective, we have $X(k) \hookrightarrow X(\mathbb{A}) \longrightarrow \prod_v X(k_v)$. So we want a homotopy fp. analogue.

## 2a. Recall'n of adelic pts.

<u>Def'n</u>: Let $v$ be nonarc. place. The unramified $\Gamma_{k_v}$ homotopy fix pts

$$X^{h\Gamma_v^{ur}} := (X^{I_v})^{h\Gamma_v^{ur}}.$$

## 1. Adelic pts

- let $S$ be a set. $I$ an index set. $(A_i, B_i)_{i \in I}$ s.t. $B_i \subseteq A_i$.

The restricted direct prodct denoted $R := \prod'_i (A_i, B_i) \subseteq \prod_i A_i$.

  - $(a_i) \in \prod a_i$ s.t. $a_i \in B_i$ all but finitely many $B_i$.

- For each finite set $S$, define $R^S := \prod_{i \in S} A_i \times \prod_{i \notin S} B_i$.

  Then $R \simeq \varinjlim R^S$, so we may equip $R$ with direct limit top.

- The Adele ring $\mathbb{A}_k := \prod'_{v \in \Omega_k} (k_v, \mathcal{O}_v)$. , $\Omega_k$ denotes set of all places of $k$.
- Ex: $K = \mathbb{Q}$. • $\{ (\mathbb{R}, \mathbb{R}), (\mathbb{Q}_p, \mathbb{Z}_{(p)})$ $p$ prime. $\}$ • $\frac{x}{y} \notin \mathbb{Z}_{(p)}$ iff $p | y$.
  - The condition guarantee that $\mathbb{Q} \hookrightarrow \prod_{v \in \Omega_k} (k_v, \mathcal{O}_v)$

## 2. Adelic top:

- We can show $X(\mathbb{A}) \simeq \prod'_{v \in \Omega_k} (X(k_v), \mathcal{X}(\mathcal{O}_v))$

- Denote $\mathcal{O}_{k,S} := \{ a \in K : v(a) \leq 1, \forall v \notin S \}$. the ring of $S$ integers.

Lemma: $\exists$ a finite set $S$ of places, and f.t. sep. $\mathcal{X}$ over $\mathcal{O}_{k,S}$, s.t. $\mathcal{X}_k \simeq X$.

Pf. This is a conseque of "Spreading out" in [3.2, Poo17]

- Now we equip $X(k_v)$ the analytic top.
  - If $X \hookrightarrow_{cl} \mathbb{A}^n$, then given $X(k_v)$ subspace top.
  - If $X$ is glued by affine pieces give it the lim topo.

  For nonarch. places $v \notin S$, $\mathcal{X}(\mathcal{O}_v) \subseteq \mathcal{X}(k_v) = X(k_v)$. subspace top.
- This gives the adelic top.

## 2b. Adelic fixed pts.

- First define for simplicial sets, then for Pro obj.

Defn let $S \subseteq \Omega_K$ a set of places of $K$.

$$X^{hA_S} := \underleftarrow{\lim}^{h} \left( \prod_{v \in T} X^{h\Gamma_v} \times \prod_{v \in S \setminus T} X^{h^{ur}\Gamma_v} \right)$$

Rmk:
- $\pi_0$ commutes with direct limit + products in $\mathcal{H}$,
  (this is bcos. $\pi_0$ is corepresented by a pt.)
- In general the cpt obj. in the homotopy cat. of spaces (($\infty$) split with homotopy class of maps)
  are retracts of finite (W) cpx.

- We define

$$X(hA_S) := \pi_0 \left( X^{hA_S} \right).$$

By rmk: it carries the restricted direct prod.

- Usually we take $S = \Omega_K$, we omit $S$, $X(hA) = \pi_0 X^{hA}$.
- If $S$ is the set of finite places we write. $X(hA_f) := X(hA^f)$.

- This induce. $Ho(sSet_{\Gamma_K}) \xrightarrow{hA_S} Ho(sSet_{\Gamma_K}) \xrightarrow{\pi_0} Set$

$$X \longmapsto X^{hA_S} \longmapsto X(hA_S)$$

- $$\text{Pro } Ho(sSet_{\Gamma_K}) \longrightarrow Set$$

$$X_I(hA_S) := \underleftarrow{\lim}_{\alpha \in I} X_\alpha(hA_S) = \underleftarrow{\lim}_{\alpha \in I} \pi_0 X_\alpha^{hA_S}$$

Def'n: $X^{\cdot}(hA) := Et^{n}_{K}(X)(hA)$.

## 2c. Étale homotopy obstr.

- We have the following diagram

$$\begin{tikzcd}
X(k) \arrow[r] \arrow[d, hook] & X^n(hK) \arrow[dr] \arrow[ddd] & \\
X(\mathbb{A}) \arrow[rr, dashed] \arrow[rrd, dashed] \arrow[dd] & & X^n(h\mathbb{A}) \arrow[ddl] \\
 & & X_{\mathcal{U},n}(h\mathbb{A}) \arrow[d] \\
\prod_v X(k_v) \arrow[r] & \prod_v X^n(hK_v) \arrow[r] & \prod_v X_{\mathcal{U},n}(hK_v) = X_{\mathcal{U},n}^{hT_v}.
\end{tikzcd}$$

## Existence of factorization.

### Step 1. Factorization of homotopic adelic pts.

- [3.11; HS18]. yields comm. of right square.

Let $K$ be nf. $X$ a str. bdd. $T_K$ simplicial set.

loc: $X(hK) \to \prod_v X(hK_v)$ factors through $X(hK) \to X(h\mathbb{A})$.

Pf: • $L/K$ be finite ext'n. $T_L :=$ places of $k$ that ramify $L$.

- For $v \notin T_L$, i.e. $v$ is unramified. $I_v < \Gamma_L := Gal(\bar{k}/L) \hookrightarrow Gal(\bar{k}/k)\ \Gamma_K$.

  (this follows from the ses for $\Gamma_v$)
- Thus we obtain a map.

$$f_L : X^{\Gamma_L} \longrightarrow \prod_{v \in T_L} X \times \prod_{v \notin T_L} X^{I_v}.$$

- first map is by inclusion,
- second map is by contrafunct. of $I_v \hookrightarrow \Gamma_v$ and fixed pt functor.

- Decomposing the action.
  - We have the following composite
$$\Gamma_v \to \Gamma_K \to Gal(L/K).$$

i) If $v \in T_L$
$$(X^{\Gamma_L})^{hGal(L/K)} \to (X^{\Gamma_L})^{h\Gamma_v} \to X^{h\Gamma_v}.$$

ii) If $v \notin T_L$
$$I_v \hookrightarrow \Gamma_v \hookrightarrow \Gamma_K \to Gal(L/K).$$ where $I_v$ is in the kernel.

Hence we obtain a map
$$\Gamma_v^{ur} \to Gal(L/K).$$
$$(X^{\Gamma_L})^{hGal(L/K)} \to (X^{I_v})^{hGal(L/K)} \to (X^{I_v})^{h\Gamma_v^{ur}}.$$

This yields a new map.

$$f : (X^{\Gamma_L})^{hG_L} \longrightarrow \prod_{v \in T_L} X^{h\Gamma_v} \times \prod_{v \notin T_L} (X^{I_v})^{h\Gamma_v^{ur}}.$$

Step 2: Some ramification theory.

'Lemma': Every finite set of primes is ramified for some ext.

- $\therefore$ The coll'n $\{T_L\}_{L/K \text{ finite}}$ is cofinal in $\{T\}$.

- So $\varinjlim (X^{\Gamma_L})^{hG_{L/K}} \simeq X^{hK} \longrightarrow \varinjlim^h_{T_L} \prod_{v \in T_L} X^{h\Gamma_v} \times \prod_{v \notin T_L} (X^{I_v})^{h\Gamma_v^{ur}} \simeq X(\mathbb{A}_S).$

Step 2.

- To prove cons, note

$$\prod_v X^n(hK_v) \simeq \lim_{\mathcal{U},n} \prod_v X_{\mathcal{U},n}(hK_v).$$
$$X^n(hA) \simeq \lim_{\mathcal{U},n} X_{\mathcal{U},n}(hA).$$

Hence green diagram.

- By def'n. $X_{\mathcal{U},n}(hA)$ is equipped canonically with restricted prod't top.

Indeed, $\pi_0 X^{h\Gamma_v} = X(hK_v)$ [remark 3.4] $X$ simplicial $T_k$ set.

So $X_{\mathcal{U},n}(hA) = \pi_0 X_{\mathcal{U},n}^{hA} \simeq \lim \prod_{v \in T} X(hK_v)_{\mathcal{U},n} \times \prod_{v \notin T} \pi_0((X_{\mathcal{U},n}^{I_v})^{h\Gamma_v^{ur}})$

- [3.12, HS18] shows bottom map is continuous.
  [3.13, HS18] shows we can reduce the problem to a problem of general topology, which is [Lem. 3.14].
- Continuity is not required until the very end. $\square$.

## 3. Étale homoty. fixed pts.

- Suppose $X$ is simplicial set.
- We have a monadic adjn

$$\mathrm{Ab} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathrm{Set}$$
$$\mathbb{Z}[X] \longleftarrow X$$

<u>Prop</u>: this induces a functor

$$\mathbb{Z}(-): \mathrm{sSet}_{\Gamma_K} \longrightarrow \mathrm{sSet}_{\Gamma_K}.$$

This induces a functor

$$\mathbb{Z}(-): \mathrm{Pro}\, \mathrm{Ho}_0^{st}(\mathrm{sSet}_{\Gamma_K}) \to \mathrm{Pro}\, \mathrm{Ho}_0^{st}(\mathrm{sSet}_{\Gamma_K}).$$

Pf.:
- $\mathbb{Z}[X]^\Lambda$ is fibrat $\forall\ \Lambda \triangleleft_{open} \Gamma_K$.
  This is true more generally, as all simplicial gp is fibrat. [Ker]
- $\mathbb{Z}(-)$ preserves simplicial homotopy.
- Recall that the obj in $\mathrm{Pro}\, \mathrm{Ho}_0^{st}(\mathrm{sSet}_{\Gamma_K})$ are given by
  $X$ st. $X^\Lambda$ is fibrat $\forall\ \Lambda \triangleleft_{open} \Gamma_K$.

The unit nat' transfm then des funs: for $0 \leq n \leq \infty$.

$$\acute{E}t_K \longrightarrow \acute{E}t^n_K \longrightarrow (\mathbb{Z}\acute{E}t_K)^n : \mathrm{Var}_K \to \mathrm{Pro}\, \mathrm{Ho}_0^{*}(\mathrm{sSet}_{\Gamma_K})$$

<u>Defn</u>: $X^{\mathbb{Z},n}(hK) := (\mathbb{Z}\acute{E}t^n_K)(X)(hK)$
$X^{\mathbb{Z},n}(hA) := (\mathbb{Z}\acute{E}t_K)^n(X)\, hA.$

- $$\begin{array}{ccccc} X(K) & \longrightarrow & X^n(hK) & \longrightarrow & X^{\mathbb{Z},n}(hK) \\ \downarrow & & \downarrow & & \downarrow \\ X(A) & \longrightarrow & X^n(hA) & \longrightarrow & X^{\mathbb{Z},n}(hA). \end{array}$$

Defn. Again we denote the pb. of + by $X(\mathbb{A})^{2Ch,n}$. also called étale homotopy obstruction.

- We thus obtain the following