p-adic Repns. Intro 16/11/20.

Goal: Understand $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

- its cont. repn.
- other Galois grps.
  - $\mathrm{Gal}(F/\mathbb{Q})$ F fin. galois $/\mathbb{Q}$.

"locally".
  - Decomp. grp $G_{\mathbb{Q}_p} =: G_p \hookrightarrow G_{\mathbb{Q}}$. p prime.

Prop: L/K Galois ext. (alg. ext. nor + sep).

- $G \simeq \mathrm{Gal}(L/K)$, given profinite top / krull top. [Con GNT 20]
- $G \simeq \varprojlim \mathrm{Gal}(L'/K)$. L'/K is fin. gal.
- let V/F with disc. top.

prof. gps. [RV99, Fourier analysis on number fields, 2]

Cont repn. of G on V $\sim$ $\exists$ a factorization.,

$$\begin{tikzcd} G \arrow[r] \arrow[d, two heads] & \mathrm{Aut}_F(V) \\ \mathrm{Gal}(L'/K) \arrow[ur] & \end{tikzcd}$$

for some L'/K fin. Gal. ext.

Rmk: kernel actn of G on V is an open normal subgp of G.

$$1 \to \underbrace{\mathrm{Gal}(L/L')}_{\text{kernel}} \hookrightarrow \mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(L'/K) \to 1$$

Goals of this talk.

- Decomp. gp. of $G_{\mathbb{Q}}$.

Ber 01 : (p-adic thy) = (Understand $G_{\mathbb{Q}}$ from G_p of $\mathbb{Q}_p$-vs.).
(l-adic thy) = (" " of $\mathbb{Q}_\ell$ vs. $\ell \neq p$).

↳ more alg. because no compatibility in the topology.

- General thy of B-rep. 11.1. FO.

- C_E when E char $p > 0$. $E/\mathbb{Q}_p$ char 0 ← case of interest. 11.2-4. FO.

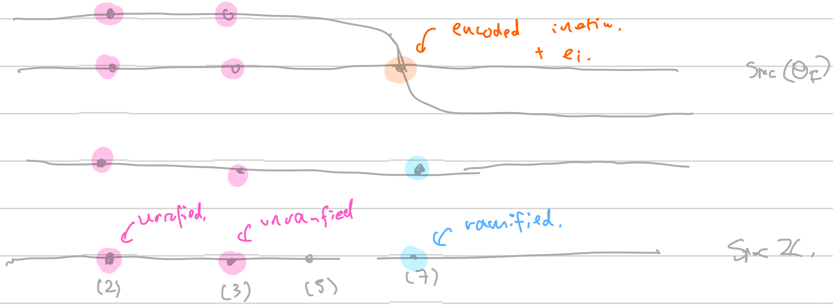
Decom Grps I   Review of fin. Galois ext.

- $F/\mathbb{Q}$ is fin. ext. (not. necss. Galois) (disc. hold genly of $\mathbb{Q}$ repl. by K nf.)
- Setup.

$$\begin{tikzcd} \mathcal{O}_F \arrow[r] & F \\ \mathbb{Z} \arrow[u, "\text{int. clos.}"] \arrow[r] & \mathbb{Q} \arrow[u, "\text{fin. sep.}"'] \end{tikzcd}$$

1. $\mathcal{O}_F$ is Dedekind domain. free $\mathbb{Z}$-mod.
2. p prime of $\mathbb{Z}$.
   $q \mid p$ primes in $\mathcal{O}_F$ living above p.
   ($q \cap \mathbb{Z} = p$.).

- This diagram. is just for splitting of primes. $p\mathcal{O}_F = \prod_{i=1}^{g} q_i$.



- $\mathcal{O}_F$ Ded. impl. $p\mathcal{O}_F = \prod_{i=1}^{g} q_i^{e_i}$

Rmk. : this picture is slightly simpler we have Galois ext. as $p\mathcal{O}_F = (\prod_{i=1}^{g} q_i)^e$

- e_i = ramification index g = # of lines.
- $F/\mathbb{Q}$ is ramified at p. if $\exists$ some i, $e_i > 1$.
  ~~e_i in diagm = # of crossing~~.
- A: there's the inertia. picture. (i'll look at).

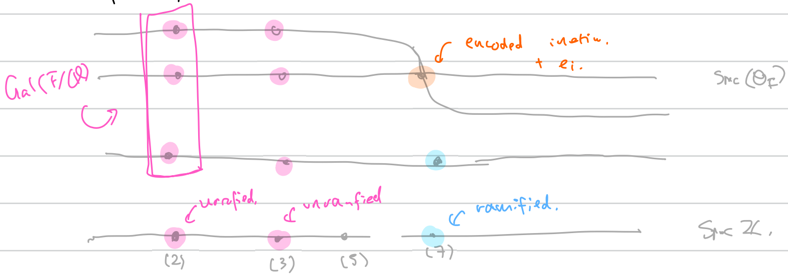
Prop: $F/\mathbb{Q}$ is unramified all but fintely may pts.

Pf: • Invariant called Δ_F. p ramifies $\Leftrightarrow$ $(p) \supseteq (\Delta_F)$ or $p \mid \Delta_F$.

- Δ_F is an ideal of $\mathbb{Z}$. $\square$.

Decomp gp. local study.

• Fix a prime p.



Obs: $\mathrm{Gal}(F/\mathbb{Q})$ acts on fibres $\{\mathfrak{q} \mid p\}$.

Defn Decom gp. Fix $\mathfrak{q} \mid p. \to$ for any other choice $\mathfrak{q}' \mid p$. $G(\mathfrak{q}) \overset{\text{conj}}{\sim} G(\mathfrak{q}')$.

$$G(\mathfrak{q}) := \{\sigma \in \mathrm{Gal}(F/\mathbb{Q}) : \mathfrak{q}^\sigma = \mathfrak{q}\}. \simeq \mathrm{Gal}(F_\mathfrak{q}/\mathbb{Q}_p).$$

•

$$\begin{array}{ccc} \mathcal{O}_\mathfrak{q} & \longrightarrow & F_\mathfrak{q} \\ \uparrow & & \uparrow \\ \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p \end{array}$$

$\sigma \in G(\mathfrak{q})$ act by isomety on $F_\mathfrak{q}$. hence on $\mathcal{O}_\mathfrak{q}/\mathfrak{q}\mathcal{O}_\mathfrak{q}$. fin. field.

$$1 \to I(\mathfrak{q}) \longrightarrow G(\mathfrak{q}) \longrightarrow \mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{q}) \to 1$$

$$\cong \mathrm{Gal}(\mathcal{O}_\mathfrak{q}/\mathfrak{q}\mathcal{O}_\mathfrak{q} \,/\, \mathbb{Z}_p/p\mathbb{Z}_p).$$

Defn. • $\overline{\mathbb{Q}}/\mathbb{Q}$. Fix $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

• $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$. (restrict the action)

$$1 \to I_p \longrightarrow G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1.$$

$$\simeq \hat{\mathbb{Z}} \simeq \varprojlim \mathbb{Z}/n. \quad \mathbb{Z}/m \to \mathbb{Z}/n \quad n \mid m.$$

• If I_p is trivial action of $G_{\mathbb{Q}_p}$. is pretty much determined. $\hat{\mathbb{Z}}$. (Frob.)

Fontaine's Approach. FO.

- Goal. $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\mathrm{Gal}(\bar{E}^s/E)$ local fields of char. 0
  - $G = G_{\mathbb{Q}_p}$ $\quad G_E$, $E/\mathbb{Q}_p$ is fin. ext
  - $\mathbb{Q}_p$-repns of G.

Fontaine:
- constructs period rings, $\mathbb{Q}_p$ algs B. $\mathbb{Q}_p^F \hookrightarrow B$.
  - cont. actn of G_E
  - compatibility with other str. maps.
    ex. frobenius, ...

B is a $(\mathbb{Q}_p, G)$-ring

- we consider B-repns. (not $\mathbb{Q}_p$-reps. of G)
  For thy: B to be G_E regular.
  hence, B^G is a field.

- Given a $\mathbb{Q}_p$-repns of G_E. $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$

$$D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_E}$$ is a B^G module.

Obs: $\dim_{B^G} D_B V \leq \dim_{\mathbb{Q}_p} V = d$. $\quad (*)$

Defn: when equal say $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$ is B-admissible

Q: what are the B-admissible reps of G?

$$\{\text{period rings}\}^{B} \quad \text{induces} \quad \mathrm{Rep}^{B}_{\mathbb{Q}_p}(G) \hookrightarrow \mathrm{Rep}_{\mathbb{Q}_p}(G) \quad \text{full subcat.}$$

Period rings examples.

Ctx. $E/\mathbb{Q}_p$ fin. ext.

Ex: • $B = \bar{E}^s$, $B^{G_E} = E$. Rule: alg ext / $\mathbb{Q}_p$ ext. val'n. (cplt. unit.)

• $B = \mathbb{C} := \widehat{\bar{E}^s}$, $B^{G_E} = E$ (thm of Ax-Sen-Tate).

• $B = B_{dR}$ $B^{G_E} = E$.

Defn: $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$ is de Rham if

• $D_{B_{dR}}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_E}$, $\dim_E D_{B_{dR}} V = \dim_{\mathbb{Q}_p} V$

i.e. V is B_{dR}-admissible.

$$\mathrm{Rep}^{B_{dR}}_{\mathbb{Q}_p}(G) \to \mathrm{Rep}_{\mathbb{Q}_p}(G).$$

$$D_{B_{dR}}: \mathrm{Rep}_{\mathbb{Q}_p}(G) \xrightarrow{B_{dR} \otimes_{\mathbb{Q}_p} -} \{B\text{-repns}\} \xrightarrow{(\,)^G} \mathrm{Vect}_E$$

$$V \in \mathrm{Rep}^{B_{dR}}_{\mathbb{Q}_p}(G) = \{V \in \mathrm{Rep}_{\mathbb{Q}_p}(G) : \dim_E D_{B_{dR}}(V) = \dim_{\mathbb{Q}_p} V\}.$$

• If $V = H^i_{et}(X_{\bar{E}}; \mathbb{Q}_p)$. is a $\mathbb{Q}_p$ repn. of G_E.

1. $D_{B_{dR}}(V) = H^i_{dR}(X/E)$.
2. V is B_{dR}-admissible. Hence de Rham.
3. Filtration on both sides.

i.e. this is the invariant mentioned.

- by defining diff'n period rings, B, we classify certain classes of $\mathbb{Q}_p$-repns.

$\mathbb{Q}_p$-reprs. / p-adic repn.		invariants attached.
	all.	(φ, Γ)-modules.
B_{HT}.	Hodge Tate	Hodge Tate weights
B_{dR}:	de Rham rep's	p-adic diff'n + filtration of E.
	$\vdots$	

B-repns. FO.11.1

Ctx. 1. G is top. gp $B \in$ TopRng.

B has cont. G-action with rings str.

- $g(x+y) = gx + gy$. $gxy = gx\, gy$. $x, y \in B$ $g \in G$.

Defn: A B-repn X of G is

1) B-module X of finite type.
2) G-acts semi linearly

i.e. $g(v_1 + v_2) = gv_1 + gv_2$ $g(bv_1) = gb \cdot gv_1$. $b \in B$ $v_1, v_2 \in X$.

Goal: understand. $\mathrm{Rep}_F(G)$ (Take $F = \mathbb{Q}_p$).

- a priori F has no relation with B.
- use langage of B repns, to understand. F-repns
  B to be a (F,G)-ring.

(F,G) regular repn 11.1 .FO.

Ctx 2 • Ctx 1. • B^G is a field.

• $F \hookrightarrow B^G$ is closed subfield of B^G.

Defn: B is a (F,G) ring + prop.

Ex: B is a field.