Homology of Iterated Loop Spaces

Author(s): Eldon Dyer and R. K. Lashof

Source: *American Journal of Mathematics*, Jan., 1962, Vol. 84, No. 1 (Jan., 1962), pp. 35-88

Published by: The Johns Hopkins University Press

Stable URL: https://www.jstor.org/stable/2372804

HOMOLOGY OF ITERATED LOOP SPACES.*

By Eldon Dyer [1] and R. K. Lashof.[1]

One of the impotrant problems of topology is to determine the stable homotopy groups $S_n(X)$ of a space X; i.e., $\varinjlim \pi_{i+n}(s^iX)$, where s^iX denotes the i-th reduced suspension of X and $\pi_{i+n}(s^iX) \to \pi_{i+1+n}(s^{i+1}X)$ is the Freudenthal Suspension Homomorphism. The Freudenthal Suspension Theorem asserts that this homomorphism is an isomorphism for $n < i-1$; and so, the group $S_n(X)$ is isomorphic to $\pi_{i+n}(s^iX) = \pi_n(\Omega^i s^i X)$ for $n < i-1$, where Ω^i denotes the i-th iterated loop space. As there is a natural imbedding of $\Omega^i s^i X$ in $\Omega^{i+1}s^{i+1}X$, it is clear that if $Q(X)$ denotes the direct limit of the spaces $\Omega^i s^i X$ under this imbedding, then

$$S_n(X) = \pi_n(Q(X)).$$

The space $Q(X)$ is in a well-defined sense an "infinite loop space." Araki and Kudo found [1] that iterated loop spaces have additional structure beyond the H-space structure to be found in all loop spaces. This structure affords a measure of the lack of commutativity of the H-space multiplication. In terms of this structure they defined mod 2 "homology operations" much analogous to the Steenrod Squares and were able to compute in terms of these operations the mod 2 homology structure of $\Omega^i S_n$, $i < n$, where S_n denotes the n-sphere.

Difficulties arise in attempting to extend their definitions to the mod p case, for p an odd prime. Browder in his thesis [3] defined a type of mod p homology operation and computed the mod p homology of $\Omega^n s^n X$, where X is arcwise connected and $p > n/2$.

In this paper the structure on iterated loop spaces found by Araki and Kudo is further developed into a form which appears dual to the diagonal structure used by Steenrod in defining cohomology reduced powers. In terms of this structure operations on homology classes are defined, *homology extended powers*. These operation have properties analogous to those of the Steenrod operations. Also these operations are in a certain sense "stable" homology operations, as distinct from Browder's.

* Received January 3, 1961; revised December 11, 1961.

[1] This research was supported in part by the National Science Foundation under contract number G-10369.

As applications, we compute the mod p homology of $\Omega^i S_n$, $i < n$, and $Q(sX)$, for X an arcwise connected space. Also, partial results on $\Omega^i s^i X$ are obtained. Unlike the mod 2 case, $H_*(\Omega^i s^n X; Z_p)$ is not generated by homology operations (as defined herein) on $\operatorname{im} H_*(s^{n-i}X; Z_p)$. However, $Q(sX)$ is, roughly speaking, the smallest space containing sX for which the homology operations are all defined, and, in fact, the homology of $Q(sX)$ is generated by homology operations on the homology classes of sX. This may be compared with the result of James [8] that ΩsX is in some sense the smallest H-space containing X. Also, we show that in a space $\Omega^i s^n X$, $i < n$, a number of cohomology operations are trivial.

I. H^n-spaces.

1. *Definition of H^n-spaces.* Let Σ_p denote the group of permutations on p symbols and let $J^n\Sigma_p$ denote the n-th-join of Σ_p with itself (in the sense of Milnor [9]). Briefly, a point of $J^n\Sigma_p$ is determined by a sequence $t_0, \cdots, t_{n-1}$ of real numbers such that $t_i \geqq 0$ and $t_0 + \cdots + t_{n-1} = 1$, and an element $\sigma_i \in \Sigma_p$ for each i such that $t_i \neq 0$. Such a point is denoted by $(t_0\sigma_0 \oplus \cdots \oplus t_{n-1}\sigma_{n-1})$. The set of these points is given the strong topology.

A *space X is an $H^n{}_p$-space,* $n \geqq 0$, provided it has an associative multiplication with unit e and there is a map (where X^p is the Cartesian product of X with itself p times)

$$\theta^n{}_p\colon J^{n+1}\Sigma_p \times X^p \to X$$

which is

1.1. Σ_p-*equivariant*; i.e., for each $\sigma \in \Sigma_p$

$$\theta^n{}_p(t_0\sigma_0 \oplus \cdots \oplus t_n\sigma_n; x_1, \cdots, x_p) = \theta^n{}_p(t_0\sigma_0\sigma^{-1} \oplus \cdots \oplus t_n\sigma_n\sigma^{-1}; x_{\sigma(1)}, \cdots, x_{\sigma(p)})$$

and

1.2. *normalized*; i. e., for each $\sigma \in \Sigma_p$

$$\theta^n{}_p(0 \oplus \cdots \oplus 0 \oplus 1 \cdot \sigma; x_1, \cdots, x_p) = x_{\sigma(1)} \cdot \cdots \cdot x_{\sigma(p)}.$$

Let $J^\infty\Sigma_p = \varinjlim J^n\Sigma_p$, where $J^n\Sigma^p \subset J^{n+1}\Sigma_p$ is given by

$$(t_1\sigma_1 \oplus \cdots \oplus t_n\sigma_n) \to (0 \oplus t_1\sigma_1 \oplus \cdots \oplus t_n\sigma_n).$$

A *space X is an $H^\infty{}_p$-space* if there is a map $\theta^\infty{}_p\colon J^\infty\Sigma_p \times X^p \to X$ such that for each n, $\theta^\infty{}_p | J^{n+1}\Sigma_p \times X^p$ makes X an $H^n{}_p$-space.

The H_n-spaces defined by Araki and Kudo [1] are $H^n{}_2$-spaces in the above sense. It is also clear that an H^{n+1}-space is *a fortiori* an $H^n{}_p$-space. Araki and Kudo showed that an $(n+1)$-st loop space is an $H^n{}_2$-space. The corresponding theorem is true here also.

A *space X is a special $H^n{}_p$-space* if it is an $H^n{}_p$-space and

1.3. the map $\theta^n{}_p$ is *projective*; i.e.,

$$\theta^n{}_p(w; e, \cdots, e, \overset{i}{x}, e, \cdots, e) = x$$

for all i, $1 \leqq i \leqq p$, $w \in J^{n+1}\Sigma_p$ and $x \in X$.

THEOREM 1.1. *An H-space* (in the usual sense = associative multiplication with unit) *is a special $H^0{}_p$-space for all p. If X is a special $H^n{}_p$-space, then ΩX, the loop space of X, is a special $H^{n+1}{}_p$-space.*

In particular, we note that *an $(n+1)$-st-loop space is an $H^n{}_p$-space for all p.*

Proof. We use the "Moore loop space." This is of the same homotopy type as the ordinary loop space and is more convenient for our purposes. Recall that a point of $\Omega(X, e)$ is a pair (w, r), $r \geqq 0$ a real number and w a map: $w: [0, r] \to X$, such that $w(0) = w(r) = e$, and if (w, r) and (w', r') are points of $\Omega(X, e)$, then

$$(w, r) \cdot (w', r') = (w'', r'')$$

is defined by

$$\begin{aligned} w''(t) &= w(t), 0 \leqq t \leqq r \\ &\quad w'(t - r), r \leqq t \leqq r + r'. \end{aligned}$$

For X a special $H^n{}_p$-space, we define the loop $(t_i \geqq 0, \sum t_i = 1, 0 \leqq s \leqq 1)$

1.4. $\theta^{n+1}{}_p((1-s)t_0\sigma_0 \oplus \cdots \oplus (1-s)t_n\sigma_n \oplus s\epsilon;\ (w_1, r_1), \cdots (w_p, r_p))(t)$ for $0 \leqq t \leqq r = r_1 + \cdots + r_p$, in the space ΩX to be

$$\theta^n{}_p(t_0\sigma_0 \oplus \cdots \oplus t_n\sigma_n; w^s{}_1(t), \cdots, w^s{}_p(t)),$$

where

$$\begin{aligned} w^s{}_i(t) &= \bar{w}_i(t - s(r_1 + \cdots + r_{i-1})) \text{ and} \\ \bar{w}_i(t) &= w_i(0), t < 0 \\ &\quad w_i(t), 0 \leqq t \leqq r_i \\ &\quad w_i(r_i), r_i < t. \end{aligned}$$

This defines $\theta^{n+1}{}_p$ for $\sigma_{n+1} = \epsilon$, the unit element of Σ_p; for σ_{n+1} different from ϵ, we define $\theta^{n+1}{}_p$ by equivariance.

It follows immediately from this definition that $\theta^{n+1}{}_p$ satisfies conditions 1.1, 1.2 and 1.3.

A map $f\colon X \to \bar{X}$ of $H^n{}_p$-spaces will be called an *$H^n{}_p$-map* if

$$\begin{array}{ccc} J^{n+1}\Sigma_p \times X^p & \xrightarrow{\theta^n{}_p} & X \\ \Big\downarrow 1\times f^p & & \Big\downarrow f \\ J^{n+1}\Sigma_p \times \bar{X}^p & \xrightarrow{\bar{\theta}^n{}_p} & \bar{X}, \end{array}$$

commutes up to a Σ_p-equivariant homotopy. In particular, if $X = \Omega^{n+1}Y$, $\bar{X} = \Omega^{n+1}\bar{Y}$ and f is the map induced by a map $g\colon Y \to \bar{Y}$, then f is an $H^n{}_p$-map.

Iterated loop spaces have additional structure beyond that of being $H^n{}_p$-spaces. This structure is necessary to establish "Adem Relations" for the homology operations.[2] We now describe it.

Recall that the *wreath product* $\Sigma_m \int \Sigma_l$ of Σ_m by Σ_l is the subgroup of Σ_{ml} described as follows: consider Σ_{ml} as the permutation group on the ml objects $x_1, \cdots, x_{ml}$. $\Sigma_m \int \Sigma_l$ is the subgroup of Σ_{ml} generated by $(\Sigma_m)^l$ and Σ_l, where the i-th factor of $(\Sigma_m)^l$ acts on the i-th block of m objects (leaving the others fixed), $i = 1, \cdots, l$, and Σ_l acts by permuting the l such blocks. Explicitly, if $\sigma \in \Sigma_l$ and $(\tau_1, \cdots, \tau_l) \in (\Sigma_m)^l$, then $\sigma[\tau_1, \cdots, \tau_l]$ denotes an element of $\Sigma_m \int \Sigma_l$ and multiplication is given by

$$\sigma[\tau_1, \cdots, \tau_l] \cdot \bar{\sigma}[\bar{\tau}_1, \cdots, \bar{\tau}_l] = \sigma\bar{\sigma}[\tau_{\bar{\sigma}(1)}\bar{\tau}_1, \cdots, \tau_{\bar{\sigma}(l)}\bar{\tau}_l].$$

It will be convenient for our purposes to have an explicit description of the simplicial structures of the complexes $J^r\Sigma_l$ and $J^s\Sigma_m \times J^t\Sigma_n$. A *vertex* v of $J^r\Sigma_l$ is a point of the form

$$v = v(i) = (0 \oplus \cdots \oplus 0 \oplus \sigma \oplus 0 \oplus \cdots \oplus 0),$$

$0 \leqq i \leqq r-1$. We call i the *weight* of v, and write $w(v) = i$. *A q-simplex* in $J^r\Sigma_l$ is of the form $(\tau_0(i_0), \cdots, \tau_q(i_q))$, $\tau_i \in \Sigma_l$, $i_0 < \cdots < i_q < r$, and consists of all points $(a_0\sigma_0 \oplus \cdots \oplus a_{r-1}\sigma_{r-1})$ in $J^r\Sigma_l$ whose coordinates are all zero except in the i_j places and $\sigma_{i_j} = \tau_j$, $j = 0, \cdots, q$.

For u a vertex of $J^s\Sigma_m$ and v a vertex of $J^t\Sigma_n$, (uv) will denote the corresponding vertex of $J^s\Sigma_m \times J^t\Sigma_n$. Let $w(uv) = w(u) + w(v)$. Given p- and q-simplices of $J^s\Sigma_m$ and $J^t\Sigma_n$, respectively, we simplicially subdivide their direct product in the usual way; i.e., an r-simplex of their product is a

[2] The Adem relations are not explicitly used in the examples we compute in this paper, so that the remainder of this section may be omitted on first reading.

sequence $(u_0v_0,\cdots,u_rv_r)$, where u_i and v_i are vertices of the p and q-simplices, respectively, satisfying

(i) $w(u_iv_i) < w(u_{i+1}v_{i+1})$ and

(ii) $w(u_i) \leqq w(u_{i+1})$ and $w(v_i) \leqq w(v_{i+1})$.

(Note that $w(v_i) = w(v_{i+1})$ implies $v_i = v_{i+1}$.)

$J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l$ is a $\Sigma_m \int \Sigma_l$-free complex; i.e. the i-th factor of $(\Sigma_m)^l$ acts on the i-th factor of $(J^{r+1}\Sigma_m)^l$ and leaves the other factors as well as $J^{s+1}\Sigma_l$ fixed and Σ_l acts by permuting the factors in $(J^{r+1}\Sigma_m)^l$ and acts on $J^{s+1}\Sigma_l$ in the usual way.

Corresponding to the inclusion map of $\Sigma_m \int \Sigma_l$ in Σ_{ml}, there is a $\Sigma_m \int \Sigma_l$-equivariant map

$$\psi\colon J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l \to J^{t+1}\Sigma_{ml},\quad t = s + rl,$$

which we describe explicitly. A vertex in $J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l$ is of the form

$$v = \sigma(i)\tau_1(j_1)\cdots\tau_l(j_l),\ w(v) = i + j_1 + \cdots + j_l = k.$$

Let $\psi(v) = [\tau_1,\cdots,\tau_l](k)$. The condition (i) above implies that ψ is simplicial. Equivariance is immediate from the definitions.

An *H^t-space* is an H-space X which is an $H^t{}_p$-space for every positive integer p and if $t = s + rl$, then

$$\begin{CD} J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m \times X^m)^l @>{i \times (\theta^r{}_m)^l}>> J^{s+1}\Sigma_l \times X^l \\ @V{\bar\psi}VV @VV{\theta^s{}_l}V \\ J^{t+1}\Sigma_{ml} \times X^{ml} @>{\theta^t{}_{ml}}>> X \end{CD}$$

commutes up to a $\Sigma_m \int \Sigma_l$-equivariant homotopy, where $\bar\psi$ is the composition

$$J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m \times X^m)^l \to J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l \times X^{ml}$$
$$\to J^{t+1}\Sigma_{ml} \times X^{ml}.$$

THEOREM 1.2. *Every $(t+1)$-st loop space is an H^t-space.*

Proof. Let $t = s + rl$. We first define a map

$$\phi\colon J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l \to J^{t+2}\Sigma_l \times (J^{t+2}\Sigma_m)^l$$

which is $\Sigma_m \int \Sigma_l$-equivariantly homotopic to the standard inclusion of $J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l$ in $J^{t+2}\Sigma_l \times (J^{t+2}\Sigma_m)^l$. For each vertex $v = \sigma(t)\tau_1(j_1)\cdots\tau_l(j_l)$, let $\phi(v) = \sigma(k)\tau_1(k)\cdots\tau_l(k)$, where $k = i + j_1 + \cdots + j_l$.

This map is simplicial, as above, and since $\dim(J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l) = t$ and $J^{t+2}G$, G any finite group, is $(t+1)$-connected, any two $\Sigma_m \int \Sigma_l$-equivariant maps of $J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m)^l$ into $J^{t+2}\Sigma_l \times (J^{t+2}\Sigma_m)^l$ are $\Sigma_m \int \Sigma_l$-equivariantly homotopic.

It is this sufficient to show that the diagram

$$\begin{array}{ccc}
J^{s+1}\Sigma_l \times (J^{r+1}\Sigma_m \times X^m)^l & \xrightarrow{\bar\phi} & J^{t+2}\Sigma_l \times (J^{t+2}\Sigma_m \times X^m)^l \\
 & & \Big\downarrow {\scriptstyle i \times (\theta)^l} \\
\Big\downarrow {\scriptstyle \bar\psi} & & J^{t+2}\Sigma_l \times X^l \\
 & & \Big\downarrow {\scriptstyle \theta} \\
J^{t+1}\Sigma_{ml} \times X^{ml} & \xrightarrow{\theta} & X
\end{array}$$

is commutative. We will show this by an induction on dimension of cells of $J^{t+1}\Sigma_{ml}$ in the image of ψ.

Since

$$\theta^r{}_n(s_0\sigma_0 \oplus \cdots \oplus s_{r-2}\sigma_{r-2} \oplus 0; \lambda_1, \cdots, \lambda_n)(t)$$
$$= \theta^{r-1}{}_n(s_0\sigma_0 \oplus \cdots \oplus s_{r-2}\sigma_{r-2}; \lambda_1(t), \cdots, \lambda_n(t)),$$

the result is immediate for 0-cells. Assume then that the diagram commutes for p-simplices in the image of ψ; i. e., a simplex

$$(\sigma^0[\tau^0{}_1, \cdots, \tau^0{}_l](k^0), \cdots, \sigma^p[\tau^p{}_1, \cdots, \tau^p{}_l](k^p)),$$

where

$$\sigma^a[\tau^a{}_1, \cdots, \tau^a{}_l](k^a) = \psi(\sigma^a(i^a)\tau^a{}_1(j^a{}_1) \cdots \tau^a{}_l(j^a{}_l))$$

and $k^a = i^a + j^a + \cdots + j^a{}_l$. Thus, we assume

$$\theta^t{}_{ml}(\bigoplus_{a=0}^{p} s_a\sigma^a[\tau^a{}_1, \cdots, \tau^a{}_l](k^a); \lambda_1, \cdots, \lambda_{ml})$$
$$= \theta^{t+1}{}_l(\bigoplus_{a=0}^{p} s_a\sigma^a(k^a); \theta^{t+1}{}_m(\bigoplus_{a=0}^{p} s_a\tau^a{}_1(k^a); \lambda_1, \cdots, \lambda_m), \cdots,$$
$$\theta^{t+1}{}_m(\bigoplus_{a=0}^{p} s_a\tau a_l(k^a); \lambda_{m(l-1)+1}, \cdots, \lambda_{ml})).$$

We shall use the formula (1.4) to reduce the case of the $(p+1)$-simplex to that of the p-simplex. It is sufficient to prove the formula when the $(p+1)$-st vertex is the unit element of $\Sigma_m \int \Sigma_l$ as the action of the group on the two sides of the equation corresponds.

$$\theta^t{}_{ml}(\bigoplus_{a=0}^{p} s_a(1-s)\sigma^a[\tau^a{}_1,\cdots,\tau^a{}_l](k^a)\oplus s\epsilon(k^{p+1});\lambda_1,\cdots,\lambda_{ml})(t)$$

$$=\theta^{t-1}{}_{ml}(\bigoplus_{a=0}^{p} s_a\sigma^a[\tau^a{}_1,\cdots,\tau^a{}_l](k^a);\lambda^s{}_1(t),\cdots,\lambda^s{}_{ml}(t))$$

$$=\theta^t{}_l(\bigoplus_{a=0}^{p} s_a\sigma^a(k^a);\theta^t{}_m(\bigoplus_{a=0}^{p} s_a\tau^a{}_1(k^a);\lambda_1(t),\cdots\lambda_m(t-s\sum_{i=1}^{m-1}u_i)),\cdots,$$
$$\theta^t{}_m(\bigoplus_{a=0}^{p} s_a\tau^a{}_l(k^a);\lambda_{m(l-1)+1}(t-s\sum_{i=1}^{m(l-1)}u_i),\cdots,$$
$$\lambda_{ml}(t-s\sum_{i=1}^{ml-1}u_i)))$$

$$=\theta^t{}_l(\bigoplus_{a=0}^{p} s_a\sigma^a(k^a);\theta^{t+1}{}_m(\bigoplus_{a=0}^{p} s_a(1-s)\tau^a{}_1(k^a)\oplus s\epsilon(k^{p+1});$$
$$\lambda_1,\cdots,\lambda_m)(t),\cdots,$$
$$\theta^{t+1}{}_m(\bigoplus_{a=0}^{p} s_a(1-s)\tau^a{}_l(k^a)\oplus s\epsilon(k^{p+1});\lambda_{m(l-1)+1},\cdots,\lambda_{ml})(t-s\sum_{i=1}^{m(l-1)}u_i))$$

$$=\theta^{t+1}{}_l(\bigoplus_{a=0}^{p} s_a(1-s)\sigma^a(k^a)\oplus s\epsilon(k^{p+1});\theta^{t+1}{}_m(\bigoplus_{s=0}^{p} s_a(1-s)\tau^a{}_1(k^a)\oplus s\epsilon(k^{p+1});$$
$$\lambda_1,\cdots,\lambda_m),\cdots,$$
$$\theta^{t+1}{}_m(\bigoplus_{s=0}^{p} s_a(1-s)\tau^a{}_l(k^a)\oplus s\epsilon(k^{p+1});\lambda_{m(l-1)+1},\cdots,\lambda_{ml}))(t).$$

$$=\theta^{t+1}{}_l(\iota\times(\theta^{t+1}{}_m)^l)\bar{\phi}(\bigoplus_{s=0}^{p} s_a(1-s)\sigma^a(i^a)\tau^a{}_1(j^a{}_1)\cdots\tau^a{}_l(j^a{}_l)$$
$$\oplus s\epsilon(i^{p+1})\epsilon(j^{p+1}{}_1)\cdots\epsilon(j^{p+1}{}_l);\lambda_1,\cdots,\lambda_{ml})(t).$$

Thus, $\theta^t{}_{ml}\circ\bar{\psi}=\theta^{t+1}{}_l\circ(\iota\times(\theta^{t+1}{}_m)^l)$ on $(p+1)$-simplices of $J^{t+1}\Sigma_{ml}$ in the image of ψ. The induction is complete.

II. Definition and first properties of homology operations. Our definitions are closely patterned on those of Steenrod ([15], [16]) for cohomology operations. The homology operations will have many properties similar to those of cohomology operations.

For any space A, let $C(A)$ denote the normalized cubical chains of A. An action of the symmetric group Σ_p on A induces an action of Σ_p *on* $C(A)$. Now let X be an $H^n{}_p$-space and $\theta=\theta^n{}_p: J^{n+1}\Sigma_p\times X^p\to X$ be the Σ_p-equivariant map defined in Section 1. The induced chain map $\theta: C(J^{n+2}\Sigma_p\times X^p)\to C(X)$ is then an equivariant chain map. Further, let $h: C(J^{n+1}\Sigma_p)\oplus C(X^p)\to C(J^{n+1}\Sigma_p\times X^p)$, $h(w\otimes a)=w^*a$, $w\in C(J^{n+1}\Sigma_p)$, $a\in C(X^p)$, be the standard chain equivalence (Eilenberg-Zilber Theorem). Then

$$\theta\circ h: C(J^{n+1}\Sigma_p)\otimes C(X^p)\to C(X)$$

is again clearly equivariant. If $\pi\subset\Sigma_p$, then $J^{n+1}\pi\subset J^{n+1}\Sigma_p$ and every Σ_p-equivariant map is π-equivariant. Summarizing we have:

LEMMA 2.1. *For any subgroup π of Σ_p, the equivariant map $\theta\colon J^{n+1}\pi \times X^p \to X$ induces an equivariant chain group $\theta\circ h\colon C(J^{n+1}\pi)\otimes C(X^p)\to C(X)$ on normalized cubical chains, where $h(w\otimes a)=w^*a$, $w\in C(J^{n+1}\pi)$, $a\in C(X^p)$.*

Let $C(X)^p=C(X)\otimes\cdots\otimes C(X)$, p times, and let Σ_p act on $C(X)^p$ in the usual way by permitting factors (with the appropriate sign [13]). In Lemma 2.1, we would like to substitute $C(X)^p$ for $C(X^p)$ as it is easier to handle. Although one may write down an explicit equivariant chain map from $C(X)^p$ to $C(X^p)$, we would like to know how unique the result is and whether we are losing any information by this substitution. The answer is given by the following theorem and corollary

Let K and L be covariant functors on a category $\mathcal{A}$ with values in the category $\partial\mathcal{G}$ of chain complexes and let $f\colon K\to L$ be any map. Let π be a finite group, and assume that K and L are π-functors; i.e., π is a group of natural transformations. If W and V are any π-chain complexes (assumed free over the integers) we make $W\otimes K$ and $V\otimes L$ into π functors by having π act on both factors. For a complex W we will write $W^{(n)}$ for the n-skeleton: i.e. $W^{(n)}=\sum_{i=0}^{n}W_i$. (See [5] for notation.)

THEOREM 2.1. *If W is π-free, if K is representable and L is acyclic (for some set of models $\mathfrak{M}\subset\mathcal{A}$, and if f is equivariant in dimension zero; then given any π-equivariant chain map $t\colon W\to V$.*

(a) *$\exists$ a π-equivariant map $F\colon W\otimes K\to V\otimes L$, satisfying*

(1) *$F(W^{(n)}\otimes K(X))\subset V^{(n)}\otimes L(X)$, all n,*

(2) *$F(w\otimes a)=t(w)\otimes f(a)$, $w\in W$, $a\in K_0(X)$.*

(b) *If t, $t^1\colon W\to V$ are π-equivariantly chain homotopic, and F, F^1 are any two maps satisfying (1) and (2) above for t, t^1 respectively, then F and F^1 are π-equivariantly chain homotopic.*

(c) *We may further choose F so that given any two dimensional π-generator e_0 of W, $F(e_0\otimes a)=t(e_0)\otimes f(a)$, $a\in K(X)$.*

Proof. Choose a set S of π-free generators of W. Define $F(e_0\otimes a)=t(e_0)\otimes f(a)$, $a\in K(X)$, all o-dim π generators $e_0\in S$; and extend to all of $W_0\otimes K(X)$ by equivariance. Since f is a natural chain map, F will be a natural chain map, equivariant by definition, and clearly satisfying (1) and (2).

Assume F has been defined on $W^{(i)}\otimes K(X)$, then define it on $W_{i+1}\otimes K_0(X)$

by (2). Now assume F is defined on $W^{(i+1)} \otimes K(X)^{(r)}$; we show that F may be extended to $W^{(i+1)} \otimes K(X)^{(r+1)}$. Thus F will be extended to all of $W \otimes K(X)$ by induction.

For any $(i+1)$ dim π-generator $e_{i+1} \in S$, and any $m_{r+1} \in K_{r+1}(M)$, $M \subset \mathfrak{M}$, consider $e_{i+1} \otimes m_{r+1}$. Then

$$F(\partial(e_{i+1} \otimes m_{r+1})) = F(\partial e_{i+1} \otimes m_{r+1}) + (-1)^{i+1} F(e_{i+1} \otimes \partial m_{r+1})$$

is defined by the induction assumption, and is a cycle. Further, it is of the form $\sum_\rho v^\rho \otimes b^\rho$, $\dim v \leqq i+1$, $v \in V$, $b^\rho \in L(M)$. Let $\{v^\lambda{}_{i+1}\}$, be a set of generators of V_{i+1} over the integers. Then $\sum_\rho v^\rho \otimes b^\rho = \sum_\lambda v^\lambda{}_{i+1} \otimes b^\lambda + \sum_\sigma v^\sigma \otimes b^\sigma$, $\dim v^\sigma \leqq i$. Since $\sum_\rho v^\rho \otimes b^\rho$ is a cycle, and

$$\partial(v^\lambda{}_{i+1} \otimes b^\lambda) = \partial v_{i+1} \otimes b^\lambda + (-1)^{i+1} v^\lambda{}_{i+1} \otimes \partial b^\lambda;$$

it follows (since the $v^\lambda{}_{i+1}$ are independent) that $v^\lambda{}_{i+1} \otimes \partial b^\lambda = 0$ and hence $\partial b^\lambda = 0$. But $b^\lambda \in L_r(M)$, and $H_r(L(M)) = 0$ if $r > 0$, so that $b^\lambda = \partial c^\lambda$, $c^\lambda \in L_{r+1}(M)$. On the other hand, if $r = 0$,

$$F(e_{i+1} \otimes \partial m_{r+1}) = t(e_{i+1}) \otimes f(\partial m_{r+1}) = t(e_{i+1}) \otimes \partial f(m_{r+1})$$

by (2), and the form is the same as above, taking $c^\lambda = f(m_{r+1})$.

Consider now $F\partial(e_{i+1} \otimes m_{r+1}) - \partial(\sum v^\lambda{}_{i+1} \otimes e^\lambda)$. This is again a cycle and is of the form $\sum v^\rho \otimes b^\rho$, $\dim v^\rho \leqq t$. Proceeding as above, we got finally:

$$F(\partial(e_{i+1} \otimes m_{r+1})) = \partial(\sum_\tau v^\tau \otimes c^\tau), \dim v^\tau \leqq i+1.$$

Hence we may set $F(e_{i+1} \otimes m_{r+1}) = \sum_\tau v \otimes c^\tau$. Since K is representable, and hence any $x_{r+1} \in K_{r+1}(X)$ is of the form $K(\phi)(m_{r+1}) = x_{r+1}$, where $\phi\colon M \to X$ and (ϕ, m_{r+1}) is the chosen representative; we can define

$$F(e_{i+1} \otimes x_{r+1}) = \otimes L(\phi)(F(e_{i+1} \otimes m_{r+1})),$$

$i\colon V \to V$ the identity. F is then extended to all of $W^{(i+1)} \otimes K(X)^{(r+1)}$ by equivariance. It is easy to check that F has all the desired properties, including (c). This completes the induction argument.

The proof of (b) is entirely analogous, using condition (2) and the fact that t and t^1 are π equivariantly chain homotopic, to get the induction started. We leave the details to the reader.

COROLLARY. *Let W be any π-free chain complex, $\pi \subset \Sigma_p$; then*

(a) *Given any π-equivariant chain map $t\colon W \to C(J^{n+1}\pi)$, there exist a natural π-equivariant chain map $F\colon W \otimes C(X)^p \to C(J^{n+1}\pi) \otimes C(X^p)$ such that*

1) $F(w \otimes x_1 \otimes \cdots \otimes x_p) = t(w) \otimes x_1 * \cdots * x_p$, $\dim x_j = 0$, $j = 1, \cdots, p$

2) $F(W^{(i)} \otimes C(X)^p) \subset C(J^{n+1}\pi)^{(i)} \otimes C(X^p)$

(b) *If $W^{(i)} = 0$ for $i > n$ (e.g., if W is the n-skeleton of any π-free complex), then a map $t\colon W \to C(J^{n+1}\pi)$ always exists, and any two maps F, F^1 with equivariant chain maps t, t^1 resp., satisfying* 1), 2), *of* (a), *are equivariantly chain homotopic when restricted to $W^{(n-1)} \otimes C(X)^p$.*

If further, $H_0(W) = Z$, $H_i(W) = 0$, $0 < i < n$, then $\exists$ a map G: $C(J^{n+1}\pi)^{(n-1)} \otimes C(X^p) \to W \otimes C(X)^p$, so that $GF\colon W^{(n-1)} \otimes C(X)^p \to W \otimes C(X)^p$ and $FG\colon C(J^{r+1}\pi)^{(n-1)} \otimes C(X^p) \to C(J^{n+1}\pi) \otimes C(X^p)$ are equivariantly chain homotopic to the respective inclusion maps.

(c) *Any two maps F, F^1 for equivariantly chain homotopic maps t, t^1, satisfying* 1), 2) *of* (a), *are equivariantly chain homotopic; if t is an equivariant chain equivalence, F is an equivariant chain equivalence. In particular if $W = C(J^{n+1}\pi)$ and t is the identity, then any F satisfying* 1) *and* 2) *of* (a) *is an equivarient chain equivalence.*

(d) *If e_0 is any zero dimensional π-generator of W, then we may further choose F so that $F(e_0 \otimes x_1 \cdots \otimes x_p) = t(e_0) \otimes x_1 * \cdots * x_p$, $x_i \in C(X)$. Also, t may be chosen so that $t(e_0)$ is any given zero cell in $C(J^{n-1}\pi)$.*

From the comparison theorem applied to the Cartan-Leray spectral sequence for covering spaces, one gets immediately the following two Lemmas:

LEMMA. *If W is a π-free complex, $\mu\colon C \to D$ a map of π-complexes such that $\mu_*\colon H(C) \cong H(D)$, then ($\pi$ acting trivially on G):*

$$\mu_{\pi *}\colon H(W \otimes_\pi C; G) = H(W \otimes_\pi D; G)$$

LEMMA. *If W is a π-free complex, C is any π-complex, G is a field of coefficients or the cycles of $C \otimes G$ are a direct summand, then (trivial boundary on $H(C; G)$)*

$$H(W \otimes_\pi C; G) \cong H(W \otimes_\pi H(C; G))$$

under a natural isomorphism.

Applying these lemmas to our situation we get:

LEMMA 2.3. *If $\mu\colon X \to Y$ induces $\mu_*\colon H(X) \cong H(Y)$, then*

$$\mu_{\pi *}\colon H(W^{(n)} \otimes_\pi C(X)^p; G) \cong H(W^{(n)} \otimes C(Y)^p; G).$$

LEMMA 2.4. *If $G=K$ is a field or if $G=Z$, $H(X,Z)$ has no torsion and the cycles of $C(X)$ are a direct summand, then*

$$H(W^{(n)}\otimes_\pi C(X)^p;G)\cong H(W^{(n)}\otimes_\pi H(X;G)^p)$$

under a natural isomorphism.

Definition 2.1. Now consider the case where p is an odd prime, π is the cyclic group of order p with generator α. Then we take for W the complex:

$$\cdots\to Z\pi\xrightarrow{\Gamma}Z\xrightarrow{\Delta}Z\pi\to\cdots\to Z\pi\xrightarrow{\Gamma}Z\pi\xrightarrow{\Delta}Z\pi\xrightarrow{\epsilon}Z.$$

where $Z\pi$ is the group ring of π; i.e., W_i has a single π-generator e_i, $i\geqq 0$, and

$$\begin{aligned}
&\partial e_{2i+1}=\Delta e_{2i}, && \Delta=\alpha-1,\quad i\geqq 0\\
&\partial e_{2i+2}=\Gamma e_{2i+1}, && \Gamma=1+\alpha+\cdots+\alpha^{p-1},\quad i\geqq 0.\\
&\epsilon e_1=1.
\end{aligned}$$

From this and Lemma 2.4 we get:

PROPOSITION 2.2. *Let π be the cyclic group on p elements, p an odd prime, W the free π-resolution of Z defined above, and $K=H(X;Z_p)$ a mod p chain complex with trivial boundary. Then if $x_1,x_2,\cdots$ is a vector basis of homogeneous elments for K, the homology classes of the following cycles form a vector basis for $H(W\otimes_\pi K^p)$:*

$$e_i\otimes_\pi x^p{}_j,\ \textit{all } j,\ i>0,\ x^p{}_j=x_j\otimes\cdots\otimes x_j\ (p\ \text{times})$$
$$e_0\otimes_\pi x_{j_1}\otimes\cdots\otimes x_{j_p},$$

where we choose one representative from each class obtained by cyclic permutation of the indices.

Proof. In the proof we will represent chains in $W\otimes_\pi K^p$ by chains in $W\otimes K^p$ but consider them as identified under π.

Let $c_n=\sum_{k+l=n}e_k\otimes d_l$, $d_l\in K^p$, be any chain of $W\otimes K^p$. Then $\partial c_n=\sum\partial e_k\otimes d_l$ and hence $\partial c_n=0$ implies

(a) K even (and not zero)

$$\partial e_k\otimes d_l=\Gamma e_{k-1}\otimes d_l=e_{k-1}\otimes\Gamma d_l=0$$

for each l, and therefore

$$\Gamma d_l=0 \bmod p.$$

(b) K odd

$$\partial e_k \otimes d_l = \Delta e_{k-1} \otimes d_l = e_{k-1} \otimes \Delta d_l = 0$$

for each l, and

$$\Delta d_l = 0 \bmod p.$$

Further, all chains of the following forms are cycles:

(a) $e_k \otimes d_l$, k even $(\neq 0)$, $\Gamma d_l = 0 \bmod p$

(b) $e_k \otimes d_l$, k odd, $\Delta d_l = 0 \bmod p$

(c) $e_0 \otimes d_l$, d_l arbitrary.

We proceed to examine these cases:

(a) $\Gamma d_l = 0 \bmod p$, k even

d_l is a linear combination of terms of the form $x_{j_1} \otimes \cdots \otimes x_{j_p}$.

Divide the terms up into transitivity classes under π; i. e. $d_l = d^1{}_l + \cdots + d^t{}_l$ with $\Gamma d^r{}_l$ linearly independent from the other $\Gamma d^s{}_l$, $s \neq r$. Hence $\Gamma d^r{}_l = 0$. But $d^r{}_l = P(\alpha)(x_{j_1} \otimes \cdots \otimes x_{j_p})$ for some fixed term $x_{j_1} \otimes \cdots \otimes x_{j_p}$, where $P(\alpha) \in Z_p(\pi)$; and either $x_{j_1} \otimes \cdots \otimes x_{j_p} = x^p{}_j$ or the representation $Z_p(\pi) \to Z_p(\pi)(x_{j_1} \otimes \cdots \otimes x_{j_p})$ is faithful. In the first case, $e_k \otimes d^r{}_l$ is simply a multiple of $e_k \otimes x^p{}_j$ as demanded by the proposition. In the second case, we must have $\Gamma P(\alpha) = 0$ in $Z_p(\pi)$, and it follows (by the well-known argument for the acyclicity of W) that $P(\alpha) = \Delta Q(\alpha)$, for some $Q(\alpha) \in Z_p(\pi)$, and $e_k \otimes d^r{}_l = \partial(Q(\alpha) e_{k+1} \otimes d^r)$.

(b) $\Delta d_l = 0 \bmod p$, k odd

Writing $d_l = d^1{}_l + \cdots + d^t{}_l$ as in (a), we have

$$\Delta d^r{}_l = \Delta P(\alpha)(x_{j_1} \otimes \cdots \otimes x_{j_p}) = 0.$$

Again either $x_{j_1} \otimes \cdots \otimes x_{j_p} = x^p{}_j$ or $\Delta P(\alpha) = 0$. Now $\Delta P(\alpha) = 0$ implies $P(\alpha) = \Gamma Q(\alpha)$ and $e_k \otimes d^r{}_l = \partial(Q(\alpha) e_{k+1} \otimes d^r{}_l)$. Hence again either $e_k \otimes d^r{}_l$ is a boundary or it is a multiple of $e_k \otimes x^p{}_j$.

(c) Since

$$\partial(e_1 \otimes x_{j_1} \otimes \cdots \otimes x_{j_p}) = \Delta e_0 \otimes x_{j_1} \otimes \cdots \otimes x_{j_p} = e_0 \otimes \Delta(x_{j_1} \otimes \cdots \otimes x_{j_p}),$$

and $\Delta = \alpha - 1$; we see that two cycles of the form $e_0 \otimes x_{j_1} \otimes \cdots \otimes x_{j_p}$ are homologous if and only if they differ by a cyclic permutation of indices.

Finally we note that no cycle of the form $e_i \otimes x^p{}_j$ can be a boundary; since it only could be a boundary of something of the form $P(\alpha)e_{i+1} \otimes x^p{}_j$. But $\partial(P(\alpha)e_{i+1} \otimes x^p{}_j) = P(\alpha)\partial e_{i+1} \otimes x^p{}_j = 0$.

This proves the proposition.

Remark. If instead of $W \otimes_\pi K^p$ we consider $W^{(n)} \otimes_\pi K^p$, $n > 0$, then it is clear from the above proof, that we get the same result except that for the first class of cycles in Proposition 2.2 we restrict i so that $0 < i \leqq n$, and in addition we get all cycles of the form

$$e_n \otimes P(\alpha)(x_{j_1} \otimes \cdots \otimes x_{j_p}), \qquad P(\alpha) = 0,\ n \text{ even}$$
$$\Delta P(\alpha) = 0,\ n \text{ odd}$$

Consider now the map: $\Theta_\pi = \theta_\pi h_\pi F_\pi \colon H(W^{(n)} \otimes_\pi C(X)^p, G) \to H(X; G)$.

Definition 2.2. Let X be an $H^n{}_p$-space. Let $x \in H_j(X; Z_p)$, For $0 \leqq i \leqq n$, define $Q_i(x) = Q_i^{(p)}(x) \in H_{pj+i}(X, Z_p)$ by

$$Q_i(x) = \Theta_\pi(e_i \otimes_\pi x^p),$$

where we write again $e_i \otimes_\pi x^p$ for the homology class in $H(W^{(n)} \otimes_\pi C(X)^p; Z_p)$ represented by this cycle. $Q_i(x)$ is called the *i-th extended p-th power of x.*

Remarks.

1. It follows from (d) of Theorem 2.1, that

$$\Theta_\pi(e_0 \otimes_\pi x_{j_1} \otimes \cdots \otimes x_{j_p}) = \theta h F(e_0 \otimes x_{j_1} \otimes \cdots \otimes x_{j_p}) = x_{j_1} x_{j_2} \cdots x_{j_p},$$

the Pontrjagin product of the homology classes. In particular,

$$Q_0(x) = x \cdot x \cdots x = x^p.$$

2. It follows from (b) of Theorem 2.1, that $Q_i(x)$, $0 \leqq i < n$, is uniquely defined (i.e. independent of the choice of F). However, $Q_n(x)$ is not determined until a choice of $t \colon W^{(n)} \to J^{(n+1)}\Sigma_p$ has been made. See Lemma 2.5 below.

3. The classes $e_n \otimes P(\alpha)(x_{j_1} \otimes \cdots \otimes x_{j_p})$ can be thought of as defining 'non-stable' operations in the sense that they disappear under an $H^n{}_p$-map of X into an H^{n+1}-space.

4. Using different coefficients in $H(W^{(n)} \otimes_\pi C(X)^p; G)$, other stable

and non-stable operations may be defined. In the non-stable case, Browder [3] has studied non-stable operations for $p=2$, corresponding to the cycles

$$e_n \otimes \Delta(x \otimes y),\ n \text{ even}$$
$$e_n \otimes \Gamma(x \otimes y),\ n \text{ odd},$$

where x and y are cycles in X, arbitrary coefficients. In particular,

$$\begin{aligned} \text{let } \psi_n(x,y) = \psi^{(p)}{}_n(x,y) &= \Theta_\pi(e_n \otimes_\pi \Delta(x \otimes y)),\ n \text{ even} \\ &= \Theta_\pi(e_n \otimes_\pi \Gamma(x \otimes y)),\ n \text{ odd}, \end{aligned}$$

$x \in H_i(X;Z_p)$, $y \in H_j(X,Z_p)$. Then $\psi_n(x,y) \in H_{i+j+n}(X;Z_p)$.

In this paper, we will consider only the extended p-th powers, and the operations ψ_n defined above.

Note that $\psi_0(x,y) = \Theta_\pi(e_0 \otimes_\pi \Delta(x \otimes y)) = xy - (-1)^{ij}yx$, for an $H^0{}_p$-space X.

5. If X is connected and is a special $H^n{}_p$-space, and $1 \in H_0(X;Z_p)$ is the unit element, then $Q_i(1) = 0$, $i > 0$. In fact the extra condition implies that the inclusion of $(e) \to X$ is an $H^n{}_p$-map, (e) the space consisting of the unit element of X only. Hence $Q_i(1)$ is in the image of $H_i((e);Z_p) = 0$, $i > 0$.

LEMMA 2.5. *Let W be as in Definition 2.1 and consider $J^\infty\pi$ as a simplicial complex then there exists an equivariant chain map $S\colon W \to J^\infty\pi$, $\pi = Z_p$, such that $S(W^{(n)}) \subset J^{n+1}\pi$.*

Proof. Given a simplex $(\sigma_0(i_0),\cdots,\sigma_r(i_r))$ in $J^\infty Z_p$, we will let $(\sigma_0(i_0),\cdots,\sigma_r(i_r)) \circ \sigma_{r+1}(i_{r+1}) = (\sigma_0(i_0),\cdots,\sigma_r(i_r),\sigma_{r+1}(i_{r+1}))$,

and extend this operation linearly to chains.

Now set $S(e_0) = (\epsilon(0))$, ϵ the unit of Z_p

$$S(e_1) = (\epsilon(0),\epsilon(1)) - (\alpha(0),\epsilon(1)) = -(\Delta S e_0) \circ \epsilon(1)$$

and in general

$$S(e_{2i+1}) = (-\Delta S e_{2i}) \circ \epsilon(2i+1)$$
$$S(e_{2i+2}) = (\Gamma S e_{2i+1}) \circ \epsilon(2i+2)$$

Then
$$\begin{aligned} S(e_{2i+1}) &= (-\partial\Delta S e_{2i}) \circ \epsilon(2i+1) + \Delta(e_{2i}) \\ &= (-\Delta\partial S e_{2i}) \circ \epsilon(2i+1) + \Delta(e_{2i}) \\ &= (-\Delta\Gamma S e_{2i}) \circ \epsilon(2i+1) + \Delta(e_{2i}) \\ &= \Delta S(e_{2i}). \end{aligned}$$

Similarly $\partial S(e_{2i+2}) = \Gamma S(e_{2i+1})$.

COROLLARY. *Let $t\colon W \to C(J^\infty Z_p)$ be the equivariant chain map defined by sending W into the simplicial complex of $J^\infty Z_p$ as described in the proof above, then the simplicial complex into the simplicial singular complex, and finally the simplicial singular complex into the cubical singular complex by the canonical maps described in* ([5]). *Then $t\colon W^{(n)} \to C(J^{n+1}Z_p)$ is an equivariant chain map, and if F of Theorem* 2.1, *Corollary, satisfies* 1) *and* 2) *for this t,*

$$F\colon W^{(n)} \otimes C(X)^p \to C(J^{n+1}\pi) \otimes C(X^p),$$

Any two maps F, F^1 with equivariant chain map t, are equivariantly chain homotopic.

If we define the operation $Q^{(p)}{}_n$ using this t and such an F, it will be uniquely defined in an $H^n{}_p$-space. Likewise for the Browder operations $\psi^{(p)}{}_n$.

THEOREM 2.2. *Let X be an $H^n{}_p$-space. Then*

a) *$Q^{(p)}{}_i$ is a homomorphism for $i \leqq n-1$.*

b) *$Q^{(p)}{}_0$ is the Pontrjagin p-th power.*

c) *If ∂_p is the homology Bockstein operator induced by the sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$, then $Q^{(p)}{}_{2i-1} = \partial_p Q^{(p)}{}_{2i}$, $2i \leqq n-1$.*

d) *For p an odd prime, $x \in H_r(X; Z_p)$, $Q^{(p)}{}_{2i}(x) = 0$ unless the change in dimension, $2i + pr - r$ is an even multiple of $p-1$.*

e) *The operations are natural under $H^n{}_p$-maps.*

Proof.

(a) For $i < n$, we show that for $x, y \in H_r(X; Z_p)$,

$$T = (e_i \otimes_\pi (x+y)^p) - (e_i \otimes_\pi x^p) - (e_i \otimes_\pi y^p),$$

is a boundary in $W^{(n)} \otimes_\pi H(X; Zp)^p$. This will give (a) above.

Since $(x+y)^p - x^p - y^p$ is invariant under π, the terms in its expansion are divided into transivity classes. But after cancelling out x^p and y^p all remaining terms in $(x+y)^p$ involve both x's and y's, and cannot be left fixed by any cyclic permutation. Consequently, $T = \Gamma S$, where S is obtained by choosing one representative from each transitivity class in T. Since $\Delta\Gamma = 0$ and $\Gamma\Gamma = p\Gamma = 0 \bmod p$, it follows as in proof of Proposition 2.2, that T is a boundary. Explicitly, since $\Gamma = \Delta^{p-1} \bmod p$,

$$\partial(e_{i+1} \otimes S) = T \quad (i \text{ odd}), \qquad \partial(e_{i+1} \otimes \Delta^{p-2} S) = T \quad (i \text{ even})$$

(b) This follows from the definition (see Remark 1 on Definition 2.3).

(c) In $W^{(n)} \otimes_\pi C(X)^p$, consider $e_{2i} \otimes_\pi x^p$ where x is a mod p cycle; i. e. $\partial x = py$. Then

$$\begin{aligned}
\partial(e_{2i} \otimes_\pi . x^p) &= e_{2i-1} \otimes_\pi x^p + p \sum_k (-1)^{\dim x(k-1)} e_{2i} \otimes_\pi x^{k-1} y^{p-k} \\
&= p e_{2i-1} \otimes_\pi x^p + p e_{2i} \otimes_\pi \Gamma(y x^{p-1}) \\
&= p\{e_{2i} \otimes_\pi x^p + \partial(e_{2i+1} \otimes_\pi x^{p-2}(y x^{p-1}))\} \bmod p^2.
\end{aligned}$$

The result follows.

(d) From the definitions we have the commutative diagram:

$$\begin{tikzcd}
W^{(n)} \otimes C(X)^p \arrow[r, "t \otimes i^p"] & J^{n+1}\pi \otimes C(X)^p \arrow[r, "\theta \circ F"] \arrow[d, "j \otimes i^p"] & C(X) \\
& J^{n+1}\Sigma_p \otimes C(X)^p \arrow[ur, "\theta \circ \bar{F}"'] &
\end{tikzcd}$$

where $i^p \colon C(X)^p \to C(X)^p$ is the identity, $j \colon J^{n+1}\pi \to J^{p+1}\Sigma_p$ is the inclusion induced by the inclusion of π in Σ_p, F, $\bar{F}$ are the maps of Proposition 2.1, Corollary for π, Σ_p, respectively, and θ is the defining map for the $H^n{}_p$ structure of X. All maps are π equivariant, but $\theta \circ \bar{F}$ is Σ_p equivariant as well.

From the above we get

$$\begin{tikzcd}
(W^{(n)} \otimes Z_p) \otimes_\pi C(X)^p \otimes Z_p \arrow[r, "(\theta \circ F)_\pi"] \arrow[d, "j \circ t \otimes_\pi i^p"] & C(X) \otimes Z_p \otimes_{\Sigma_p} Z_p \\
(J^{n+1}\Sigma_p \otimes Z_p) \otimes_{\Sigma_p} (C(X)^p \otimes Z_p) \arrow[ur, "(\theta \circ \bar{F})_{\Sigma_p}"'] &
\end{tikzcd}$$

commutes in the following cases:

i) Σ_p acts trivially on both coefficient groups Z_p

ii) Each $\sigma \in \Sigma_p$ multiplies both coefficients by the sign of σ.

In both cases, π acts trivially on the coefficients since cyclic permutation, p odd prime, are even. Also $Z_p \otimes_{\Sigma_p} Z_p \cong Z_p$ with trivial action. Hence in either case we have:

$$\begin{tikzcd}
H(W^{(n)} \otimes_\pi C(X)^p; Z_p) \arrow[r, "\theta_*"] \arrow[dr, "(j \circ t \otimes_\pi i^p)"'] & H(X; Z_p) \\
& H(J^{p+1}\Sigma_p \otimes Z_p) \otimes_{\Sigma_p} (C(X)^p \otimes Z_p)) \arrow[u, "(\theta \circ \bar{F})_{\Sigma_p}"']
\end{tikzcd}$$

commutes.

Let x is a cycle in $C(X)$, thus $j \circ t \otimes_\pi i^p(e_{2i} \otimes_\pi x^p) = j \circ t(e_{2i}) \otimes_{\Sigma p} x^p$. Let $\bar{e}_{2i} = j \circ t(e_{2i})$. Thus since e_{2i} represents a mod p cycle in $W^{(n)}/\pi$, $\bar{e}_{2i}$ represents a mod p cycle in $J^{n+1}\Sigma p/\Sigma p$. Consider

i) x even dimensional

If $\bar{e}_{2i} = \partial c \bmod p$ in $J^{n+1}\Sigma p/\Sigma p$, then

$$\bar{e}_{2i} \otimes_{\Sigma p} x^p = \partial(c \otimes_{\Sigma p} x^p) \bmod p,$$

Since $\partial c \otimes_{\Sigma p} x^p$ depends only on the class of ∂c under Σ_p as the action of Σp on x^p is trivial. It will then follow from the above diagram, case i, that $\Theta(e_{2i} \otimes_\pi x^p)$ is homologous to zero and hence $Q^{(p)}{}_{2i}(x) = 0$.

ii) odd dimensional

If $\bar{e}_{2i} = \partial c$ in $J^{n+1}\Sigma p \otimes_{\Sigma p} Z_p$ acting in Z_p by sign of permutation; then $\bar{e}_{2i} \otimes_{\Sigma p} x^p = \partial(c \otimes x^p)$ in $(J^{n+1}\Sigma_p \otimes Z_p) \otimes_{\Sigma p} (C(X)^p \otimes Z_p)$, as the action of Σp on x^p is trivial in $C(X)^p \otimes Zp$; Σ_p acting on Z_p by the sign of the permutation. Hence again, $Q^{(p)}{}_{2i}(x) = 0$. Now observe that $H_i(\pi) = H_i(W^{(r)}/\pi)$, $i < n$; and $H_i(\Sigma_p) = H_i(J^{n+1}\Sigma p/\Sigma p)$, $i < n$; and further that

$$H_i(W^{(n)}/\pi) \xrightarrow{(J \circ t)_{\Sigma p}} H_i(J^{n+1}\Sigma_p/\Sigma_p)$$

corresponds to the inclusion map of π in Σp. Hence our result will follow from:

LEMMA. *Let $j_{2i}: H_{2i}(\pi; Zp) \to H_{2i}(\Sigma_p; ZZ_p)$ be induced by the inclusion map; then*

i) *If Σp acts trivially on Zp, $j_{2i} = 0$ if $2i$ is not an even multiple of $p-1$.*

ii) *If Σ_p acts on Z_p by the sign of the permutation, $j_{2i} = 0$ if $2i$ is not an odd multiple of $p-1$.* (For proof see [15]).

(e) This follows from the definitions.

Remark. If x is an integral cycle, then $e_{2i-1} \otimes x^p$ is an integral cycle, and we may consider Q_{2i-1} as an integral operation. However, letting ∂_* be the Bockstein for the sequence $0 \to Z \to Z \to Zp \to 0$, we get again as in (c) that:

$$Q_{2i-1} = \partial_* Q_{2i}.$$

We now derive a Cartan formula for operations on products:

For two $H^n{}_p$-spaces X and Y, $X \times Y$ has an obvious $H^n{}_p$-structure given by:

$$\theta\colon J^{n+1}\Sigma_p \times (X \times Y)^p \to X \times Y,$$

$$\theta(w\,; (x_1, y_1), \cdots, (x_p, y_p)) = (\theta_x(w\,; x_1, \cdots, x_p), \theta_y(w\,; y_1, \cdots, y_p)),$$

where θ_x and θ_y are the defining maps for X and Y respectively. The properties for θ follow immediately from those for θ_x, θ_y. The same formula holds if we restrict to $J^{n+1}\pi$, $\pi \subset \Sigma_p$. We may write θ as the composition of maps:

$$J^{n+1}\pi \times (X \times Y)^p \xrightarrow{\Delta \times i} J^{n+1}\pi \times J^{n+1}\pi \times (X \times Y)^p$$

$$\xrightarrow{\tau} J^{n+1}\pi \times X^p \times J^{n+1}\pi \times Y^p \xrightarrow{\theta_x \times \theta_y} X \times Y,$$

where Δ is the diagonal map on $J^{n+1}\pi$, τ the permutation of factors.

In the following diagram we write simply J for $J^{n+1}\pi$; the bottom line is the chain map induced by θ.

$$\begin{tikzcd}
C(J) \otimes C((X \times Y)^p) \arrow[r, "\Delta \otimes i"] \arrow[d] & C(J \times J) \otimes C((X \times Y)^p) \arrow[r, "G_1 \times G_2"] \arrow[d] & C(J) \otimes C(J) \otimes C(X^p) \otimes C(Y^p) \arrow[r] & {} \\
C(J \times (X \times Y)^p) \arrow[r, "\Delta \times i"] & C(J \times J \times (X \times Y)^p) \arrow[rr] & & {}
\end{tikzcd}$$

$$\begin{tikzcd}
C(J) \otimes C(X^p) \otimes C(J) \otimes C(Y^p) \arrow[r, "\theta_x \otimes \theta_y"] \arrow[d] & C(X) \otimes C(Y) \arrow[d] \\
C(J \times X^p \times J \times Y^p) \arrow[r, "\theta_x \times \theta_y"] & C(X \times Y).
\end{tikzcd}$$

Here G_1, G_2 are the natural maps from direct product to tensor product given by the Eilenberg-Zilber theorem, and the vertical maps are the natural chain maps from tensor product to direct product. The fact that the above squares all commute up to π-equivariant chain homotopies, follows from the fact that they all commute up to natural chain homotopies and π operates on each factor separately. (I. e. π induces a map in the category).

Consider next the diagram

$$\begin{tikzcd}[column sep=small]
C(J)\otimes(C(X)\otimes C(Y))^p \arrow[r, "G_1\Delta\otimes i"] \arrow[d, "i\otimes f^p"] & C(J)\otimes C(J)\otimes(C(X)\otimes C(Y))^p \arrow[r, "i\otimes\sigma"] & C(J)\otimes C(J)\otimes C(X)^p\otimes C(Y)^p \arrow[d, "\tau^{-1}F_x\otimes F_y\tau"] \\
C(J)\otimes C(X\times Y)^p \arrow[r, "F_{x\times y}"] & C(J)\otimes C((X\otimes Y)^p) \arrow[r, "(G_1\otimes G_2)\circ(\Delta\otimes i)"] & C(J)\otimes C(J)\otimes C(X^p)\otimes C(Y^p)
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r, "\tau"] & C(J)\otimes C(X)^p\otimes C(J)\otimes C(Y)^p \arrow[d, "F_x\otimes F_y"] \\
\arrow[r, "\tau"] & C(J)\otimes C(X^p)\otimes C(J)\otimes C(Y^p).
\end{tikzcd}$$

The maps F_x and F_y are chosen to satisfy Theorem 2.1 for the standard maps $C(X)^p\to C(X^p)$ and $C(Y)^p\to C(Y^p)$ respectively. The second square then commutes by definition. Also $F_{x\times y}$ is chosen to satisfy Theorem 2.1 for the standard map $C(X\times Y)^p\to C((X\times Y)^p)$. Further, $f\colon C(X)\otimes C(Y)\to C(X\times Y)$ is the standard map. Finally, σ is the permutation of factors. The first square commutes up to a π-equivariant homotopy by Theorem 2.1, because in going around either way in the square we get a map satisfying Theorem 2.1 for the map: $(x_1\otimes y\ )\cdot\cdot\cdot(x_p\otimes y_p)\to(x_1{*}\cdot\cdot\cdot{*}y_p)\otimes(y_1{*}\cdot\cdot\cdot{*}y_p)$ where x_i and y_i, $i=1,\cdot\cdot\cdot,p$, are zero dimensional.

Combining the two diagrams we have:

$$\begin{tikzcd}[column sep=small]
C(J)\otimes(C(X)\otimes C(Y))^p \arrow[r, "G_1\Delta\otimes i"] \arrow[d, "i\otimes f^p"] & C(J)\otimes C(J)\otimes(C(X)\otimes C(Y))^p \arrow[r, "\tau(i\otimes\sigma)"] & C(J)\otimes C(X)^p\otimes C(J)\otimes C(Y)^p \arrow[d, "\Theta_x\otimes\Theta_y"] \\
C(J)\otimes C(X\times Y)^p \arrow[r, "\Theta"] & C(X\times Y) & C(X)\otimes C(Y) \arrow[l, "f"']
\end{tikzcd}$$

commutes up to a π-equivariant homotopy.

Now let W by the π-free acyclic complex defined previously (Definition 2.1). If we restrict to $W^{(n+1)}$, any two π-homotopic, since $C(J^{n+1}\pi)\otimes C(J^{n+1}\pi)$ is acyclic in dimensions less than n.

Consequently, we have

LEMMA 2.6.

$$\begin{tikzcd}[column sep=small]
W^{(n-1)}\otimes(C(X)\otimes C(Y))^p \arrow[r, "d\otimes i"] \arrow[d, "\Theta\circ i\otimes f^p"] & W^{(n-1)}\otimes W^{(n-1)}\otimes(C(X)\,C(Y))^p \arrow[r, "\tau(i\otimes\sigma)"] & W^{(n-1)}\otimes C(X)^p\otimes W^{(n-1)}\otimes C(Y)^p \arrow[d, "\Theta_x\otimes\Theta_y"] \\
C(X\times Y) & & C(X)\otimes C(Y) \arrow[ll, "f"']
\end{tikzcd}$$

commutes up to a π-equivariant homotopy.

We are now in a position to prove the following:

THEOREM 2.3. (a) *For p an odd prime, X and Y $H^n{}_p$-spaces, $x_r \in H_r(X;Z_p)$, $y_s \in H_s(Y;Z_p)$, $2i < n$, the following holds in $X \times Y$:*

$$Q^{(p)}{}_{2i}(x_r \otimes y_s) = (-1)^{rs(p-1/2)} \sum_{j=0}^{i} Q^{(p)}{}_{2j}(x_r) \otimes Q^{(p)}{}_{2i-2j}(y_s)$$

(b) *For $p = 2$, $i < n$*

$$Q^{(2)}{}_i(x_r \otimes y_s) = \sum_{j=0}^{i} Q^{(2)}{}_j(x_r) \otimes Q^{(2)}{}_{i-j}(y_s).$$

COROLLARY 1. *If X is the n-th loop space of an H-space, then the following holds where multiplication indicates Pontrjagin product:*

$$Q^{(p)}{}_{2i}(x_r \cdot y_s) = (-1)^{rs(p-1/2)} \sum_{j=0}^{i} Q^{(p)}{}_{2j}(x_r) \cdot Q^{(p)}{}_{2i-2j}(y_s), \quad 2i < n$$

$$Q^{(2)}{}_i(x_r \cdot y_s) = \sum_{j=0}^{i} Q^{(2)}{}_j(x_r) \cdot Q^{(2)}{}_{i-j}(y_s), \quad i < n$$

where $x_r \in H_r(X;Z_p)$, $y_s \in H_s(X;Z_p)$.

Proof. Let π be the cyclic group on p elements with generator α, W the π-free acyclic complex of Definition 2.1; let as in [13], $d\colon W \to W \otimes W$ be defined by,

$$de_{2i} = \sum_{j=0}^{i} e_{2j} \otimes e_{2i-2j} + \sum_{j=0}^{i-1} \Lambda e_{2j+1} \otimes e_{2i-2j-1}$$

and

$$de_{2i+1} = \sum_{j=0}^{i} (e_{2j} \otimes e_{2i-2j+1} + e_{2j+1} \otimes \alpha e_{2i-2j}),$$

where $\Lambda = \sum \alpha^k \times \alpha^l$, $0 \leqq k < l \leqq p-1$. Then d is an equivariant chain map.

Using the commutative diagram of Lemma 2.6 and mod p coefficients, we have:

$$\Theta_\pi(e_{2i} \otimes_\pi (x_r \otimes y_s)^p) = f \circ (\Theta_x)\, {}_\pi\!\otimes (\Theta_y)_\pi \circ \tau \circ (i \otimes \sigma)(de_{2i} \otimes_\pi (x_r \otimes y_s)^r)$$

$$= f \circ (\Theta_x)\, {}_\pi\!\otimes (\Theta_y)_\pi \{ (-1)^{p(p-1/2)rs} \sum_{j=0}^{i} (e_{2j} \otimes_\pi x^p{}_r) \otimes (e_{2i-2j} \otimes_\pi y^p{}_s)$$

$$+ \sum_{j=0}^{i-1} (-1)^{(2i-2j-1)rp} \sum_{k<l} (\alpha^k e_{2j+1} \otimes_\pi x^p{}_r) \otimes (\alpha^l e_{2i-2j-1} \otimes_\pi y^p{}_s).$$

But $x^p{}_r$ and $y^p{}_s$ are invariant under π and hence for p odd prime:

$$\sum_{k<l} (\alpha^k e_{2j+1} \otimes_\pi x^p{}_r) x (\alpha^l e_{2i-2j-1} \otimes_\pi y^p{}_s)$$
$$= p(p-1/2)(e_{2j+1} \otimes_\pi x^p{}_r) \otimes (e_{2i-2j-1} \otimes_\pi y^p{}_s) = 0$$

Hence

$$\Theta_\pi(e_{2i}\otimes_\pi(x_r\otimes y_s)^p)$$
$$=(-1)^{(p-1/2)r\cdot s}f\sum_{j=0}^{i}(\Theta_x)_\pi(e_{2j}\otimes_\pi x^p{}_r)\otimes(\Theta_y)_\pi(e_{2i-2j}\otimes_\pi y^p{}_s)$$

or

$$Q^{(p)}{}_{2i}(x_r\otimes y_s)=(-1)^{(p-1/2)rs}\sum_{j=0}^{i}Q^{(p)}{}_{2j}(x_r)\otimes Q^{(p)}{}_{2i-2j}(y_s).$$

Similarly, for $p=2$, using the above result and the corresponding one for e_{2i+1}, we get for i odd or even:

$$\Theta_\pi(e_i\otimes_\pi(x_r\otimes y_s)^p)=f\sum_{j=0}^{i}(\Theta_x)_\pi(e_j\otimes_\pi x^p{}_r)\otimes(\Theta_y)_\pi(e_{i-j}\otimes_\pi y^p{}_s)$$

or

$$Q^{(2)}{}_i(x_r\otimes y_s)=\sum_{j=0}^{i}Q^{(2)}{}_j(x_r)\otimes Q^{(2)}{}_{i-j}(y_s).$$

This proves the theorem.

Proof of Corollary 1. Now assume $X=\Omega^nZ$, where Z is an H-space, $h\colon Z\times Z\to Z$ the multiplication. This induces $\Omega^nh\colon\Omega^n(Z\times Z)\to\Omega^nZ$. On the other hand, the H-structure in X is defined by considering $X=\Omega(\Omega^{n-1}Z)$, and adding loops to define the multiplication $\mu\colon X\times X\to X$.

We wish to apply the theorem to the case where $Y=X$. Since X is an $H^n{}_p$-space, $X\times X$ has an $H^n{}_p$-structure as defined previously. But, $\Omega^n(Z\times Z)$ has an H^{n-1}-structure, and it is easy to check that the canonical map $i\colon\Omega^n(Z\times Z)\to\Omega^nZ\times\Omega^nZ$ is an $H^{n-1}{}_p$-map. Further, it is well known and trivial to prove that

$$\begin{tikzcd}
\Omega^n(Z\times Z) \arrow[rr, "i"] \arrow[dr, "\Omega^n h"'] & & \Omega^nZ\times\Omega^nZ \arrow[dl, "\mu"] \\
& \Omega^nZ &
\end{tikzcd}$$

commutes up to homotopy. Since $i_*\colon H_*(\Omega^n(Z\times Z);Z_p)\cong H_*(\Omega^nZ\times\Omega^nZ)$ and Ω^nh is an $H^{n-1}{}_p$-map, it follows that μ_* commutes with the homology operations:

$$\mu_*Q^p{}_{2i}=Q^{(p)}{}_{2i}\mu_*\quad 2i<n;\qquad \mu_*Q^{(2)}{}_i=Q^{(2)}{}_i\mu_*,\quad i<n$$

the corollary now follows immediately by applying μ_* to both sides of the equations of the theorem.

COROLLARY 2. *If X is an $H^n{}_p$-space, and $x_r \in H_r(X;Z_p)$ is a primitive homology class; then $Q_i(x_r)$ is primitive, $i < n$, p any prime.*

Proof. The diagonal map $X \to X \times X$ is an $H^n{}_p$-map, and hence commutes with the operations. The result now follows from the theorem.

III. Further properties of homology operations. As seen in Theorem 2.2(d) many of the homology operations $Q^{(p)}{}_i$ are trivial. It is thus convenient to define

$$S^j{}_{(p)}(x) = Q^{(p)}{}_{(2j-n)(p-1)}(x), \quad n = \dim x.$$

We shall find it convenient to introduce also a numerical coefficient.

THEOREM 3.1. *Let $\sigma_*: H_{n-1}(\Omega X;Z_p) \to H_n(X;Z_p)$ be the usual homology suspension and suppose for $y \in H_{n-1}(\Omega X;Z_p)$ both $S^j{}_{(p)}(y)$ and $S^j{}_{(p)}(\sigma_* y)$ are defined (this requires X to be an $H^m{}_p$-space, where*

$$m + 1 \geqq (2j - n + 1)(p - 1)),$$

then

$$\sigma_* S^j{}_{(p)}(y) = (-1)^{(p-1/2)} \cdot \gamma(n) \cdot S^j{}_{(p)}(\sigma_* y),$$

where $\gamma(n) \equiv (-1)^{(n-1)(p-1)/2} \cdot ((p-1)/2\,!) \bmod p$.

Proof. Let PX denote the space of paths over X and $\pi: PX \to X$ denote the end point projection. Suppose given classes

$$x \in H_n(X;Z_p),\ y \in H_{n-1}(\Omega X;Z_p) \text{ and } z \in H_n(PX, \Omega X;Z_p)$$

such that

$$\pi_*(z) = x \in H_n(X, e;Z_p) = H_n(X;Z_p) \text{ and } \partial_*(z) = y.$$

We must show there is a chain $c \in C_{pn+i}(PX)$, where $i = (2j - n)(p - 1)$, such that

$$\begin{aligned} &\{c\} \in H_{pn+i}(PX, \Omega X;Z_p), \\ &\pi_*\{c\} = (-1)^{(p-1)/2} \cdot \gamma(n) \cdot S^j{}_{(p)}(x), \text{ and} \\ &\partial_*\{c\} = S^j{}_{(p)}(y). \end{aligned}$$

Let (z, y), $y = \partial z$, denote the subchain complex of $C(PX)$, the normalized cubical singular complex of PX, generated by a chain representing z (which we also denote by z) and by ∂z, which represents y.

As X is an $H^m{}_p$-space, we have the equivariant maps

$$\theta^m{}_p: J^{m+2}\Sigma_p \times X^p \to X \text{ and}$$
$$\theta^{m+1}{}_p: J^{m+2}\Sigma_p \times (\Omega X)^p \to \Omega X,$$

the latter being defined by formula (1.4).

Let $f\colon J^{m+1}\Sigma_p \to J^{m+2}\Sigma_p$ be defined by

$$f(t_0\sigma_0 \oplus \cdots \oplus t_m\sigma_m) = (t_0\sigma_0 \oplus \cdots \oplus t_m\sigma_m \oplus 0)$$

and let

$$\widetilde{\theta^m}_p\colon J^{m+2}\Sigma_p \times (\Omega X)^p \to \Omega X$$

be the composition $\theta^{m+1}{}_p \circ (f \times (\text{id.})^p)$.

Define

$$\bar{\theta}^m{}_p\colon J^{m+1}\Sigma_p \times (PX)^p \to PX$$

by

$$\bar{\theta}^m{}_p(t_0\sigma_0 \oplus \cdots \oplus t_m\sigma_m; w_1, \cdots, w_p)(t) \\ = \theta^m{}_p(t_0\sigma_0 \oplus \cdots \oplus t_m\sigma_m; w_1(t), \cdots, w_p(t))$$

and define

$$\hat{\theta}^m{}_p\colon (J^{m+1}\Sigma_p \circ \epsilon) \times (\Omega X)^{p-1} \times (PX) \to PX$$

by

$$\hat{\theta}^m{}_p((1-s)t_0\sigma_0 \oplus \cdots \oplus (1-s)t_m\sigma_m \oplus s\epsilon; l_1, \cdots, l_{p-1}, w)(t) \\ = \theta^m{}_p(t_0\sigma_0 \oplus \cdots \oplus t_m\sigma_m; l^s(t), \cdots, l^s{}_{p-1}(t), w^s(t)),$$

where, as before,

$$l^s{}_i(t) = l_i(t - s(r_1 + \cdots + r_{i-1})) \text{ and } w^s(t) = w(t - s(r_1 + \cdots + r_{p-1})).$$

It is clear that

$$\pi \circ \hat{\theta}^m{}_p(t_0\sigma_0 \oplus \cdots \oplus t_m\sigma_m \oplus t_{m+1}\epsilon; l_1, \cdots, l_{p-1}, w) = \pi(w).$$

As each two of the maps $\tilde{\theta}^m{}_p$, $\bar{\theta}^m{}_p$ and $\hat{\theta}^m{}_p$ agree on the common part of their domains of definition, we have the commutative diagram

$$\begin{array}{ccc}
J^{m+2}\Sigma_p \times (\Omega X)^p & \longrightarrow & \Omega X \\
\downarrow & & \downarrow \\
[J^{m+2}\Sigma_p \times (\Omega X)^p] \cup [(J^{m+1}\Sigma_p) \circ \epsilon \times (\Omega X)^{p-1} \times PX] \cup [J^{m+1}\Sigma_p \times (PX)^p] & \longrightarrow & PX \\
\uparrow & & | \\
J^{m+1}\Sigma_p \times (PX)^p & & | \\
\downarrow & & \downarrow \\
J^{m+1}\Sigma_p \times X^p & \longrightarrow & X
\end{array}$$

Let $W^{(k)}$ be the k-skeleton of the standard acyclic complex for Z_p, and let $(W^{(m+1)}, e_{m+2})$ denote the subcomplex of $W^{(m+2)}$ having only the generator e_{m+2} in dimension $m+2$ (not $T^i e_{m+2}, 1 < i < p$).

Then by the Corollary of Theorem 2.1 the diagram

$$\begin{array}{ccc}
W^{(m+2)} \otimes C(\Omega X)^p & \longrightarrow & C(\Omega X) \\
\downarrow & & \downarrow \\
[W^{(m+2)} \otimes C(\Omega X)^p] \oplus [(W^{(m+1)}, e_{m+2}) \otimes C(\Omega X)^{p-1} \otimes C(PX)] \oplus [W^{(m+1)} \otimes C(PX)^p] & \longrightarrow & C(PX) \\
\uparrow & & \Big| \\
W^{(m+1)} \otimes C(PX)^p & & \Big| \\
\downarrow & & \downarrow \\
W^{(m+1)} \otimes C(X)^p & \longrightarrow & C(X)
\end{array}$$

commutes.

The chain complex

$$\{W^{(m+2)} \otimes y^p\} \oplus \{(W^{(m+1)}, e_{m+2}) \otimes y^{p-1} \otimes z\} \oplus \{W^{(m+1)} \otimes (z, y)^p\}$$

is a subcomplex of the chain complex

$$[W^{(m+2)} \otimes C(\Omega X)^p] \oplus [(W^{(m+1)}, e_{m+2}) \otimes C(\Omega X)^{p-1} \otimes C(PX)] \oplus [W^{(m+1)} \otimes C(PX)^p]$$

and the image of any term of this subcomplex involving at least one factor y projects under π into a degenerate chain in $C(X)$ since $y \in C(\Omega X)$.

We shall define a chain c' in this subcomplex such that modulo residual chains and with coefficients Z_p

$$c' = (-1)^{p-1/2} \cdot \gamma(n) \cdot e_i \otimes z^p + \text{terms involving at least one } y,$$

and

$$\partial c' = e_{i+p-1} \otimes y^p.$$

Then if c is the image of this chain in $C(PX)$, it follows from the above that c has the required properties.

To define c' it is convenient when p is an odd prime to replace $(z, y)^p$ by a smaller chain complex, and to this end we use a lemma of R. G. Swan [17].

LEMMA. *Let K be the chain complex*

$$0 \to Z_p \xrightarrow{\Gamma} KZ_p \xrightarrow{\Delta} \cdots \xrightarrow{\Gamma} KZ_p \xrightarrow{\Delta} KZ_p \xrightarrow{n} Z_p \to 0,$$

where KZ_p denotes the Z_p group ring of Z_p, $p \neq 2$, n is augmentation, $\Delta = \alpha - 1$, $\Gamma = \alpha^{p-1} + \cdots + \alpha + 1$, and dimensions range from $(n-1)p$ to np. (α acts as left multiplication in KZ_p and trivially on the two end groups Z_p; this action commutes with boundary.) Then there is an equivariant equivalence $\phi\colon K \to (z, y)^p$ such that

$$\phi(\epsilon) = y^p, \ \epsilon \in Z_p = K_{(n-1)p}$$

$$\phi(\epsilon) = y^{p-1} \otimes z, \ \epsilon \in KZ_p = K_{(n-1)p+1}, \text{ and}$$

$$\phi(\epsilon) = \gamma(n) \cdot z^p, \ \epsilon \in Z_p = K_{np}, \text{ where}$$

$$\gamma(n) \equiv (-1)^{(n-1)(p-1)/2} \cdot ((p-1)/2)! \bmod p.$$

We shall define a chain c'' in $W^{(n)} \otimes K$ such that its image in $W^{(n)} \otimes (z,p)^p$ under the map $1 \otimes \phi$ is the desired chain c'.

Since $i = (2j - n)(p-1)$ and p is odd, i is even. We let

$$\begin{aligned} c'' = {} & e_{i+p-1} \otimes \epsilon_{(n-1)p+1} + e_{i+p-2} \otimes \Delta^{p-2}\epsilon_{(n-1)p+2} \\ & - e_{i+p-3} \otimes \epsilon_{(n-1)p+3} - e_{i+p-4} \otimes \Delta^{p-2}\epsilon_{(n-1)p+4} + \cdots \\ & + (-1)^{p-1/2} e_i \otimes \epsilon_{np}. \end{aligned}$$

Then

$$\begin{aligned} \partial c'' = e_{i+p-1} \otimes \epsilon_{(n-1)p} & + \Gamma e_{i+p-2} \otimes {}_{(n-1)p+1} - e_{i+p-2} \otimes \Delta^{p-1}\epsilon_{(n-1)p+1} \\ & + \Delta e_{i+p-3} \otimes \Delta^{p-2}\epsilon_{(n-1)p+2} - e_{i+p-3} \otimes \Gamma\epsilon_{(n-1)p+2} \\ & - \Gamma e_{i+p-4} \otimes \epsilon_{(n-1)+3} + \cdots \\ & + (-1)^{p-1/2}\Gamma e_{i-1} \otimes \epsilon_{np} \end{aligned}$$

$\equiv e_{i+p-1} \otimes \epsilon_{(n-1)p}$, modulo residual chains and with coefficients modulo p. (Note that in $W \otimes K$, $\alpha a \otimes b \equiv a \otimes \alpha b$, mod residual chains; hence $\Delta a \otimes b = a \otimes \Delta b$ and $\Gamma a \otimes b = a \otimes \Gamma b$. Also, $\Gamma \equiv \Delta^{p-1} \bmod p$ and $\Gamma\epsilon_{np} = 0 \bmod p$.) It is clear that c'' has the property described above.

For the case $p = 2$, it is sufficient simply to let

$$c' = e_i \otimes z^2 + e_{i+1} \otimes z \otimes y.$$

Then

$$\partial c' \equiv e_{i+1} \otimes y^2 \bmod 2 \text{ and mod residual chains.}$$

As is the case with the Steenrod operations, it is also convenient here to multiply the homology operations $S^j{}_{(p)}$ by suitably selected numerical coefficients so that they will commute precisely with the homology suspension homomorphism. That is, we wish to determine numbers $f(j,n)$ such that if $Q^j{}_{(p)}(w) = f(j,n) \cdot S^j{}_{(p)}(w)$, $\dim w = n$, then

(1) $Q^k{}_{(p)}(w) = w^p$, if $n = 2k$, and

(2) $\sigma_* Q^j{}_{(p)}(w) = Q^j{}_{(p)}(\sigma_* w)$.

The conditions (1) and (2) may be translated into conditions on $f(j,n)$ as follows:

(1) $Q^k(w_{(2k)}) = w^p = S_0(w_{(2k)}) = S^k(w_{(2k)})$.

Thus, $f(k, 2k) = 1$.

(2) $$\begin{aligned} \sigma_* Q^j(y) = \sigma_* f(j,n-1) S^j(y) = f(j,n-1) \cdot (-1)^{p-1/2} \cdot \gamma(n) \cdot S^j(\sigma_* y), \\ \text{and } \sigma_* Q^j(y) = Q^j(\sigma_* y) = f(j,n) S^j(\sigma_* y). \end{aligned}$$

Thus, $f(j, n-1)(-1)^{p-1/2}\gamma(n) = f(j, n)$, where

$$\gamma(n) = (-1)^{(n-1)(p-1)/2} \cdot (p-1/2)! \bmod p.$$

A moment's calculation reveals that

$$f(j, n) \equiv (-1)^{n+j(p-1)/2+(n(n-1)/2)(p-1/2)} \cdot (p-1/2!)^{2j-n} \bmod p.$$

In terms of the operations $Q^j{}_{(p)}$, the Corollary 1 of Theorem 2.3 becomes,

$$Q^n{}_{(p)}(x_r \cdot y_s) = \sum_{i=0}^{n} Q^i{}_{(p)}(x_r) Q^{n-c}{}_{(p)}(y_s).$$

The following is proved in [3], we give our proof for completeness.

THEOREM (Browder). *Let $\sigma_*: H_{r-1}(\Omega X; Z_p) \to H_r(X; Z_p)$ be the usual homology suspension. Let X be an $H^n{}_p$-space. Let $u \in H_{i-1}(\Omega X; Z_p)$, $v \in H_{j-1}(\Omega X; Z_p)$, then*

$$\sigma_*\psi_{n+1}(u, v) = (-1)^{i+n+1}\psi_n(\sigma_* u, \sigma_* v).$$

Proof. Let $\sigma_* u = x \in H_i(X; Z_p)$, $\sigma_* v = y \in H_j(X; Z_p)$. Also let $\bar{x} \in H_i(PX, \Omega X; Z_p)$, $\bar{y} \in H_j(PX, \Omega X; Z_p)$ be such that $\pi_*(\bar{x}) = x \in H_n(X, e; Z_p) = H(X; Z_p)$ and $\partial_*(\bar{x}) = u$, $\pi_*(\bar{y}) = y \in H_n(X, e; Z_p) = H(X; Z_p)$ and $\partial_*(\bar{y}) = v$. We must show there is a chain $c \in C_{i+j+n}(PX)$, such that: c is a relative cycle; i.e. $\{c\} \in H_{i+j+n}(PX, \Omega X; Z_p)$,

(1) $\pi_*\{c\} = (-1)^{i+n+1}\psi_n(x, y)$, and

(2) $\partial_*\{c\} = \psi_{n+1}(u, v)$.

For n even,

let c' be the chain in $W^{(n+1)} \otimes C(PX)^2$

$$c' = e_n \otimes \Delta(\bar{x} \otimes \bar{y}) - e_{n+1} \otimes (u \otimes \bar{y}) + (-1)^{ij} e_{n+1} \otimes (v \otimes \bar{x}).$$

Then $\partial c' = (-1)^{i-1} e_{n+1} \otimes \Gamma(u \otimes v)$.

For n odd,

$$\text{let } c' = e_n \otimes \Gamma(\bar{x} \otimes \bar{y}) + e_{n+1} \otimes (u \otimes \bar{y}) + (-1)^{ij} e_{n+1} \otimes (v \otimes \bar{x}).$$

Then $\partial c' = (-1)^{i} e_{n+1} \otimes \Delta(u \otimes v)$.

Now as in Theorem 3.1, c' is contained in the special complex

$$[(W^{(n)}, e_{n+1}) \otimes C(\Omega X) \otimes C(PX)] + [W^{(n)} \otimes C(PX)^2]$$

which maps into $C(PX)$. By the same reasoning as in Theorem 3.1, the image c of $(-1)^{i+n+1}c'$ satisfies (1) and (2) above. The result follows.

Remark. From Theorem 1.2 it may be shown that certain "Adem Relations" hold for the homology operations in an iterated loop space. This may be done (following Adem) using the homology of the symmetric group. However, the device used by Adem to find the effect on odd classes from that on even classes does not apply here. Consequently, it appears necessary to compute the homology of the symmetric group with twisted coefficients to obtain the rest of the relations for the homology operations. (Note also that no universal space, such as a $K(\pi, n)$ exists in the case of homology operations.)

IV. Some theorems on Hopf algebras and spectral sequences. The object of this section is to present certain results, largely of an algebraic nature, which are useful in making explicit computations in Sevtion V. We use the notation and results of *The Structure of Hopf Algebras* by Milnor and Moore [11] and assume acquaintance with that paper on the part of the reader.

Our first theorem is a slight strengthening of the standard comparison theorem for spectral sequences essentially due to Zeeman [19].

THEOREM 4.1. *Suppose* $\{h^r\}: \{'E^r\} \to \{E^r\}$ *is a homomorphism of homology spectral sequences and*

$$'\chi_{a,b}: {'E^2_{a,b}} \to {'E^2_{a,0}} \otimes {'E^2_{0,b}} \text{ and } \chi_{a,b}: E^2_{a,b} \to E^2_{a,0} \otimes E^2_{0,b}$$

are isomorphisms such that the diagram

$$\begin{array}{ccc} 'E^2_{a,b} & \xrightarrow{'\chi_{a,b}} & 'E^2_{a,0} \otimes 'E^2_{0,b} \\ \big\downarrow h^2_{a,b} & & \big\downarrow h^2_{a,0} \otimes h^2_{0,b} \\ E^2_{a,b} & \xrightarrow{\chi_{a,b}} & E^2_{a,0} \otimes E^2_{0,b} \end{array}$$

commutes

(α_n) $h^2_{a,0}$ *is an isomorphism for* $0 \leqq a \leqq n$,

and

(γ_n) $h^\infty_{a,b}$ *is an isomorphism if* $0 \leqq a+b \leqq n$ *and* $b \leqq n-2$.

Then

(β_{n-2}) $h^2_{0,b}$ *is an isomorphism for* $0 \leqq b \leqq n-2$, *and*

$h^r_{a,b}$ is an epimorphism if either $a+b \leqq n-1$, $a \leqq r-2$, and *$b \leqq n-2$ or $b \leqq n-r$ and $a \leqq n$,*

and

$h^r_{a,b}$ is a monomorphism if $b \leqq n-2$ and either $a+b \leqq n-1$ and *$n-(r-1) \leqq a$ or $a \leqq n-(r-1)$.*

Proof. (i) α_n and β_m for $m \leqq n-2$ imply

$(a_{r-1})\, h^{r-1}{}_{a,b}$ is *epic* on $Z('E^{r-1}{}_{a,b})$ into $Z(E^{r-1}{}_{a,b})$

if either $a+b \leqq m+1$, $a \leqq r-2$, $b \leqq m$

or $b \leqq m-(r-2)$, $a \leqq n$

$(b_{r-1})\, h^{r-1}{}_{a,b}$ is *monic* on $Z('^{r-1}{}_{a,b})$ and is an *isomorphism*

from $B('E^{r-1}{}_{a,b})$ onto $B(E^{r-1}{}_{a,b})$

if either $a+b \leqq n-1$, $n-(r-1) \leqq a$, $b \leqq m$

or $a \leqq n-(r-1)$, $b \leqq m$

$(c_r)\, h^r{}_{a,b}$ is *epic* if either $a+b \leqq m+1$, $a \leqq r-2$, $b \leqq m$
or $b \leqq m-(r-2)$, $a \leqq n$, and

$(d_r)\, h^r{}_{a,b}$ is *monic* if either $a+b \leqq n-1$, $n-(r-1) \leqq a$, $b \leqq m$
or $a \leqq n-(r-1)$, $b \leqq m$.

This argument is by induction on r. Clearly (c_r) and (d_r) are true for $r=2$.

It is easy to see that (a_{r-1}) implies (c_r) and (b_{r-1}) implies (d_r).

We now show that (c_r) and (d_r) imply (a_r). For $a+b \leqq m+1$, $a \leqq r-1$, $b \leqq m$, every element of $'E^r{}_{a,b}$ and $E^r{}_{a,b}$ is a cycle. Thus, (c_r) alone is sufficient to imply (a_r) in this case. For $\alpha \in Z(E^r{}_{a,b})$ with $b \leqq m-(r-1)$ and $a \leqq n$, by (c_r) there is an $'\alpha \in {'E^r{}_{a,b}}$ such that $h^r('\alpha) = \alpha$. We wish to show $'\alpha$ is a cycle. But $'d^r('\alpha) \in {'E^r{}_{a-r,b+r-1}}$ and $a-r \leqq n-r < n-(r-1)$ and $b+r-1 \leqq m$; and so, h^r is monic on $'d^r('\alpha)$. But $'d^r(h^r{}'\alpha) = 0$. Hence, $'d^r('\alpha) = 0$ and $a_r)$ is proved in this case also.

It is clear that (d_r) implies (b_r) for the cycles in $'E^r{}_{a,b}$ and also that h^r is monic on $B('E^r{}_{a,b})$ in the range stated. We need to show only that it is epic.

Let $\beta \in B(E^r{}_{a,b})$. There is a $\gamma \in E^r{}_{a+r,b-r+1}$ such that $d^r(\gamma) = \beta$.

If $a+b\leqq n-1$, $n-r\leqq a$ and $b\leqq m$, then $n-r+b\leqq n-1$ and $b-r+1\leqq 0$. For $b-r+1<0$, $\gamma=0$ and so $\beta=0$. For $b-r+1=0$, $b=r-1$ and $a+r-1\leqq n-1$; i.e., $a+r\leqq n$. But $n-r\leqq a$. Thus, $\gamma\in E^r{}_{n,0}$. Also, $b\leqq m$ and $b-r+1=0$ implies $0\leqq m-r+1\leqq m-(r-2)$. By (c_r) there is a $'\gamma\in {}'E^r{}_{n,0}$ such that $h^r('\gamma)=\gamma$. Hence, if $'\beta={}'d^r\gamma\in{}'E^r{}_{a,b}$, then $h^r('\beta)=\beta$.

If $a\leqq n-r$ and $b\leqq m$, we have $\gamma\in E^r{}_{a+r,b-r+1}$ with $a+r\leqq n$ and $b-r+1\leqq m-(r-1)\leqq m-(r-2)$. Thus, by (c_r) there is a $'\gamma\in{}'E^r{}_{a+r,b-r+1}$ such that $h^r('\gamma)=\gamma$; and so, $h^r('d^{r\prime}\gamma)=\beta$. Thus, h^r is epic as stated in (b_r).

(ii) α_n, γ_n, and β_m imply $h^{r+s}{}_{r,m-r+2}$ is an isomorphism for $r\geqq 2$ and $s\geqq 1$ if either

(a) $m+2\leqq n-1$, $n-s\leqq 2r$ or

(b) $2r\leqq n-s$.

The argument is by induction downwards on $r+s$ and follows from the diagram

$$\begin{array}{ccccccccc}
 & & & & {}'E^{r+s}_{r,m-r+2} & & & & \\
 & & & & \downarrow\cong & & & & \\
0 & \longleftarrow & {}'E^{r+s+1}_{r,m-r+2} & \longleftarrow & Z({}'E^{r+s}_{r,m-r+2}) & \longleftarrow & B({}'E^{r+s}_{r,m-r+2}) & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & {}'E^{r+s+1}_{r,m-r+2} & \longleftarrow & Z({}'E^{r+s}_{r,m-r+2}) & \longleftarrow & B({}'E^{r+s}_{r,m-r+2}) & \longleftarrow & 0. \\
 & & & & \uparrow\cong & & & & \\
 & & & & E^{r+s}_{r,m-r+2} & & & &
\end{array}$$

The isomorphism on the boundaries follows from (b_{r+s}) as

$$m+2=r+m-r+2\leqq n-1, n-(r+s)\leqq r,\ m-r+2\leqq m$$

in the case (a) and

$$r\leqq n-(r+s), m-r+2\leqq m \text{ in the case (b).}$$

That the isomorphism holds for large $r+s$ is implied by (γ_n). The induction then follows by the "five lemma"

(iii) $h^r{}_{r,m-r+2}$ is an isomorphism on cycles for $r\geqq 2$ if either

(a) $m+2\leqq n-1\leqq 2r$

or

(b) $2r\leqq n-1$.

This follows from the diagram

$$\begin{array}{ccccccccc} 0 & \longleftarrow & 'E^{r+1}_{r,m-r+2} & \longleftarrow & Z('E^r_{r,m-r+2}) & \longleftarrow & B('E^r_{r,m-r+2}) & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & E^{r+1}_{r,m-r+2} & \longleftarrow & Z(E^r_{r,m-r+2}) & \longleftarrow & B(E^r_{r,m-r+2}) & \longleftarrow & 0. \end{array}$$

(iv) α_n, γ_n, β_m imply β_{m+1} if $m+2 \leqq n-1$.

That $h^r{}_{0,m+1}$ is an isomorphism, $r \geqq 2$, is shown by induction down on r with the following diagram

$$\begin{array}{ccccccc} 0 \longleftarrow 'E^{0,m+1} & \longleftarrow & 'E^r_{0,m+1} & \longleftarrow & 'E^r_{r,m-r+2} & \longleftarrow & Z('E^r_{r,m-r+2}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \longleftarrow E^{r+1}_{0,m+1} & \longleftarrow & E^r_{0,m+1} & \longleftarrow & E^r_{r,m-r+2} & \longleftarrow & Z(E^r_{r,m-r+2}). \end{array}$$

That $h^r{}_{r,m-r+2}$ is monic follows from (d_r) since $r+(m-r+2)=m+2 \leqq n-1$ and $m-r+2 \leqq m$. That it is epic follows from (c_r) since the only cases in which the groups are not zero are those in which $m-r+2 \geqq 0$;

Finally the second part of the conclusion of the theorem is simply the statements (c_r) and (d_r) with $m=n-2$.
i.e., $r \leqq m+2 \leqq n-1$. Of course, $m-r+2 \leqq m-(r-2)$.

The next theorem is from [11].

THEOREM 4.2. *If A is a connected Hopf algebra of finite type over a field of non-zero characteristic p, then A is primitively generated if and only if A is cocommutative and coassociative and for each positive dimensional element $u \in A^*$, $u^p = 0$.*

In our applications A will be the mod p homology algebra of an H-space; and so, the conditions "cocommutative" and "coassociative" will be automatically satisfied (as they are equivalent to commutativity and associativity of cup products in cohomolgy).

Recall (see [11]) that if B is a connected, commutative associative Hopf algebra and A is a sub-Hopf algebra, then letting $\bar{A}$ denote the module of positive dimensional elements of A and $C = B/\!\!/A = B/B\cdot\bar{A}$, C is a Hopf algebra and the projection map $\pi\colon B \to C$ is a map of Hopf algebras. Further, if there is a Hopf algebra map $j\colon C \to B$ such that $\pi \circ j$ is the identity, then $A \otimes C \cong B$ as Hopf algebras. Finally, recall also that if A is a sub-Hopf algebra of the connected commutative Hopf algebra B, then the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow P(A) & \longrightarrow & P(B) & \longrightarrow & P(B/\!\!/A) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ Q(A) & \longrightarrow & Q(B) & \longrightarrow & Q(B/\!\!/A) & \longrightarrow & 0 \end{array}$$

is commutative and each of the horizontal rows is exact.

THEOREM 4.3. *Suppose B is a free, primitively generated, connected, commutative, assaciative Hopf algebra of finite type over a field of characteristic $p \neq 0$ and $f: A \to B$ is a map of Hopf algebras such that both f and $Q(f)$ are monomorphisms.*

Then A and $B/\!\!/A$ are primitively generated and

$$B \cong A \otimes (B/\!\!/A) \text{ as Hopf algebras.}$$

Proof. It is clear from the above diagram that $C = B/\!\!/A$ is primitively generated. Since $f^*: B^* \to A^*$ is an epimorphism, and B^* is commutative and associative and has all p-th powers zero (by Theorem 4.2), A^* has the same properties. Again by Theorem 4.2 it follows that A is primitively generated.

We shall define inductively direct sequences $\{A^{(n)}, f^{(n)}, g^{(n)}\}$ such that (1) $A^{(0)} = A$, (2)

$$\begin{tikzcd} A^{(n)} \arrow[rr, "g^{(n)}"] \arrow[dr, "f^{(n)}"'] & & A^{(n+1)} \arrow[dl, "f^{(n+1)}"] \\ & B & \end{tikzcd}$$

is a commutative diagram of maps of Hopf algebras, (3) $A^{(n+1)} = A^{(n)} \otimes H(x^{(n+1)})$ as Hopf algebras, where $H(x^{(n+1)})$ is the free monogenic Hopf algebra generated by $x^{(n+1)}$, and (4) both $f^{(n)}$ and $Q(f^{(n)})$ are monomorphisms. Finally, we shall require that

$$(5) \qquad \varinjlim (f^n) : \varinjlim (A^{(n)}, g^{(n)}) \to B$$

be an isomorphism of Hopf algebras.

Before proceeding to the construction, let us observe that this is indeed sufficient to prove the theorem. It follows from (3) and (5) that $A^{(n+1)} \cong A^{(n)} \otimes (A^{(n+1)}/\!\!/A^{(n)})$ as Hopf algebras, that

$$A^{(n+1)} \cong A \otimes (A^{(1)}/\!\!/A) \otimes \cdots \otimes (A^{(n+1)}/\!\!/A^{(n)})$$

as Hopf algebras, and finally that $B \cong A \otimes \{A^{(1)}/\!\!/A \otimes A^{(2)}/\!\!/A^{(1)} \otimes \cdots\}$ as Hopf algebras. Letting $C' = \{A^{(1)}/\!\!/A \otimes A^{(2)}/\!\!/A^{(1)} \otimes \cdots\}$, we then have

$$C = B/\!\!/A \cong C'; \text{ and so,}$$

$$B \cong A \otimes C \text{ as Hopf algebras.}$$

Assume $g^{(i-1)}$, $f^{(i)}$, $A^{(i)}$ defined and having the stated properties for $i \leqq n$. If there is an element of $Q(B)$ not in the image of $f^{(n)}$ (if not the induction is complete), let $x^{(n+1)}$ be a primitive element of B projecting onto one of least dimension. Let $A^{(n+1)} = A^{(n)} \otimes H(x^{(n+1)})$, let $g^{(n)}$ be the inclusion $A^{(n)} \rightarrow A^{(n)} \otimes 1 \rightarrow A^{(n+1)}$, and let $f^{(n+1)}$ be the extension of $f^{(n)}$ defined by sending $x^{(n+1)}$ into $x^{(n+1)}$. Clearly (1), (2) and (3) are true. To show that $f^{(n+1)}$ is a monomorphism, it is sufficient to show that $P(f^{(n+1)})$ is. As $P(f^{(n)})$ is a monomorphism, it is thus sufficient to show that no x^{p^g} (letting $x = x^{(n+1)}$) lies in the image of $f^{(n)}$. Suppose x^{p^g} is in the image of $f^{(n)}$. As x^{p^g} is primitive and decomposable and $Q(f^{(n)})$ is a monomorphism, there is an element $a \in A^{(n)}$ such that

$$x^{p^g} = f^{(n)}(a^p).$$

Thus, $(x^{p^{g-1}} - f^{(n)}(a))^p = 0$, and since B is free, $x^{p^{g-1}} = f^{(n)}(a)$. Since x is not in the image of $f^{(n)}$, this leads to a contradiction. To see that $Q(f^{(n+1)})$ is a monomorphism, it is sufficient to abserve thot

$$\dim Q(A^{(n+1)}) = \dim Q(A^{(n)}) + 1$$

and that by construction

$$\dim f^{(n+1)}Q(A^{(n+1)}) = \dim f^{(n)}Q(A^{(n)}) + 1.$$

But by assumption $Q(f^{(n)})$ is a monomorphism. Thus, (4) is proved. The statement (5) follows from the fact that B is of finite type and $f^{(n)}$ is a monomorphism together with the method of choosing $x^{(n+1)}$ (by exhaustion!).

A commutative Hopf algebra homology spectral sequence $\{E^r, d^r\}$ (over Z_p) is a homology spectral sequence in which each E^r is a commutative Hopf algebra over Z_p, d^r is a derivative, and $E^{r+1} \cong H_*(E^r)$ as Hopf algebras.

Definition. A commutative Hopf algebra homology spectral sequence $\{E^r, d^r\}$ over Z_p is a *model spectral sequence* provided either

(a) $E^2{}_{*,0} \cong E(x, n)$ and $E^2{}_{0,*} \cong P(y, n-1)$, these isomorphisms being as Hopf algebras, where n is odd and $d^n(x \otimes y^i) = y^{i+1}$ for all $i \geqq 0$, ($E(x, n)$ denotes the monogenic exterior Hopf algebra with generator x in dimension x and $P(y, n-1)$ the monogenic polynomial Hopf algebra with generator y in dimension $n-1$) or

(b) $E^2{}_{*,0} \cong P(x, n)$ and $E^2{}_{0,*} \cong \bigotimes_{k \geqq 0} E(y_k, p^k n - 1) \otimes \bigotimes_{j > 0} P(z_j, p^j n - 2)$, these isomorphisms also being as Hopf algebras, where n is even and

$$d^{np^k}(x^{p^k}) = y_k \text{ for } k \geqq 0 \text{ and}$$
$$d^{np^j(p-1)}((x^{p^j})^{p-1} \otimes y_j) = z_{j+1} \text{ for } j \geqq 0.$$

(The spectral sequence (a) is that of the mod p homology ($p \neq z$) of the fibration $\Omega S_n \to PS_n \to S_n$, n odd, and (b) is the spectral sequence of the mod p homology of the fibration $\Omega^2 S_n \to P\Omega S_n \to \Omega S_n$, n odd (see Moore [12]). In the latter case it is also true that $\partial *_p(y_k) = + z_k$, where $\partial *_p$ denotes the homology Bockstein of the sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$.)

It is known [11] that if A is a connected, free, associative, commutative, primitively generated mod p Hopf algebra of finite type, then A is isomorphic, as Hopf algebra, to a tensor product of free monogenic Hopf algebras; i.e., to a tensor product of exterior and polynomial Hopf algebras each having one generator. Associated with each of these free monogenic Hopf algebras is a model spectral sequence with that Hopf algebra as the term $E^2{}_{*,0}$. The tensor product of these model spectral sequences, one for each monogenic factor in the decomposition of A, is a *canonical spectral sequence for A*. It will be shown later in this section that two canonical spectral sequences for A corresponding to different decompositions of A into monogenic factors are isomorphic as spectral sequences of Hopf algebras. We shall denote such a spectral sequence by $\{C^r(A)\}$.

THEOREM 4.4. *Suppose $\{E^r, d^r\}$ is an associative, commutative Hopf algebra homology spectral sequence over Z_p, $p \neq 2$, such that $E^\infty = 0$ and if $x \in E^2{}_{2n,0}$ such that $d^r x = 0$ for all $r < 2n$, then*

(i) $d^r(x^p) = 0$ *for all* $r < 2np$,

(ii) $d^r(x^{p-1} \otimes \tau(x)) = 0$ *for all* $r < 2n(p-1)$, *and*

(iii) *if* $d^{2n(p-1)}(x^{p-1} \otimes \tau(x)) \neq 0$, *then* $d^{2np}(x^p) \neq 0$.

Then

(a) *if $E^2{}_{*,0}$ is transgressively generated, it is primitively generated and free,*

(b) *if $E^2{}_{*,0}$ is primitively generated and free, and*

$$\bigotimes_{i \in I} E(x_i, n_i) \otimes \bigotimes_{j \in J} P(y_j, m_j)$$

is a Hopf algebra decomposition of $E^2{}_{*,0}$ into its monogenic parts, and there do not exist n_i, m_j, m_k and integers f and g, $f \geqq 1$ and $g \geqq 1$, such that either

(A) $m_j = 2 + (n_i - 1) \cdot p^f$, *or*

(B) $m_j = 2 + (m_k \cdot p^g - 2) \cdot p^f$,

then $E^2{}_{*,0}$ is trangressively generated, and

(c) *if* $E^2{}_{*,0}$ *is transgressively generated, then* $\{E^r\}$ *is isomorphic to* $\{C^r(E^2{}_{*,0})\}$ *as spectral sequences of algebras.*

In proving part (b) of this theorem we shall need

LEMMA 4.5. *Suppose A and B are connected, associative, commutative, Hopf algebras of finite type, A is free, n is a positive integer, and $f\colon A \to B$ is an algebra homomorphism which is an isomorphism in dimensions less than n. Then if $Q(A)_n = 0$ and $n \neq p^k \cdot m$ for some integer k and even integer m such that $Q(A)_m \neq 0$, then f is a monomorphism in dimension n.*

Proof of Lemma. As f is an algebra isomorphism in dimensions less than n, $Q(f)$ is an isomorphism in dimensions less than n and, in particular, $\dim Q(A)_i = \dim Q(B)_i$ for $i < n$. As an algebra $B \cong \otimes E(\zeta_i) \otimes P^{f_j}(\eta_j)$, where $P^{f_j}(\eta_j)$ denotes the truncated polynomial algebra in η_j of height p^{f_j}. Further as B is free in dimensions less than n, $(\dim \bar{\eta}_j) \cdot p^{f_j} > n$. Let $\bar{B} \equiv \otimes E(\bar{\zeta}_i) \otimes \bigotimes P(\bar{\eta}_j)$ and $\pi\colon \bar{B} \to B$ be the unique algebra extension of $\pi(\bar{\zeta}_i) = \zeta_i$ and $\pi(\bar{\eta}_j) = \eta_j$, where the tensor product is over just those $\bar{\zeta}_i$ and $\bar{\eta}_j$ of dimension less than n. The map π is a map of algebras, is an isomorphism in dimensions less than n, and is a monomorphism in dimension n. There is a sub-Hopf algebra $\bar{A} = \bigotimes E(x_i) \otimes \bigotimes P(y_j)$ of A such that $Q(\bar{A})_i = 0$ for $i \geqq n$ and the inclusion map $\iota\colon \bar{A} \to A$ is an isomorphism in dimensions less than or equal to n. Define $\bar{f}\colon \bar{A} \to \bar{B}$ to be the unique algebra extension of $\bar{f}(x) = \pi^{-1} f \iota(x)$ for each $x = x_i$ or $x = y_j$. Then

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\quad \bar{f} \quad} & \bar{B} \\ {\scriptstyle \iota}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ A & \xrightarrow[\quad f \quad]{} & B \end{array}$$

is a commutative diagram of algebra homomorphisms; and so, $Q(\pi \circ \bar{f}) = Q(f \circ \iota)$. In dimension $i < n$,

$$\dim Q(f \circ \iota)(Q(\bar{A})_i) = \dim Q(\bar{A})_i = \dim Q(A)_i \text{ and}$$
$$\dim Q(\pi)(Q(\bar{B})_i) = \dim Q(\bar{B})_i = \dim Q(B)_i.$$

Were $Q(\bar{f})_i$ not an epimorphism, then

$$\dim Q(\pi \circ \bar{f})(Q(\bar{A})_i) < \dim Q(\bar{B}_i) = \dim Q(f \circ \iota)(Q(\bar{A})_i).$$

Thus, $Q(\bar{f})$ is an epimorphism; and since $\bar{f}$ is a map of algebras, $\bar{f}$ is an epimorphism. A counting argument then shows that $\bar{f}$ is a monomorphism, and the conclusion follows immediately.

Proof of Theorem 4.4.

(a) As any transgressive element in a Hopf algebra spectral sequence is primitive, it is immediate that $E^2{}_{*,0}$ is primitively generated. To show it is free, it will suffice to show that if x is an even dimensional generator in $E^2{}_{*,0}$, then $x^{p^i} \neq 0$ for all i. Suppose $x^{p^f} \neq 0$ and $x^{p^{f+1}} = 0$. Let $y = x^{p^f}$. Both y and y^p $(=0)$ transgress, by hypothesis. Let $w = \tau(y)$ and let $\bar{w}$ be an element in $E^{0,*}{}_2$, the dual cohomology spectral sequence, such that $\langle \bar{w}, w \rangle \neq 0$. As $\langle \bar{w}, d^n y \rangle \neq 0$, $\langle \delta_n \bar{w}, y \rangle \neq 0$ and $\bar{y} = \delta_n \bar{w} \neq 0$, where $n = \dim y$. Since $y^p = 0$, by hypotheses (ii) and (iii), all d^r on $y^{p-1} \otimes w$ are 0. Thus, $\bar{y}^{p-1} \otimes \bar{w}$ is not a coboundary in E_r. Since $E^2{}_{*,0}$ is primitively generated, by Theorem 4.2 we have $\bar{y}^p = 0$. But $\delta_i(\bar{y}^{p-1} \otimes \bar{w}) = 0$ for $i < n$ and $\delta_n(\bar{y}^{p-1} \otimes \bar{w}) = \bar{y}^p = 0$. Thus, all δ_r on $\bar{y}^{p-1} \otimes \bar{w}$ are 0; and so, $y^{p-1} \otimes w$ is not a boundary in any E^r. The class $y^{p-1} \otimes w$ thus leads to a non-zero class in E^∞, which is a contradiction.

(b) Let $\mathfrak{X}$ be the set of all generators of $E^2{}_{*,0}$ which transgress. $\mathfrak{X} \neq \phi$, as the least dimensional generator of $E^2{}_{*,0}$ transgresses.

Suppose $\mathfrak{X}$ does not generate $E^2{}_{*,0}$. Let z be a least dimensional generator of $E^2{}_{*,0}$ not in the algebra generated by $\mathfrak{X}$. Among all generators of the dimension of z not in that algebra let ν be one which is maximally transgressive; i.e.,

$$d^s \nu = 0, \ s < r$$
$$\mu = d^r \nu \neq 0, \ \text{and}$$

if $\tilde{\nu}$ is any other generator of the dimension of z not in the algebra generated by $\mathfrak{X}$ and

$$d^t \tilde{\nu} = 0, \ t < \tilde{r},$$

then $\tilde{r} \leqq r$. Further, if there is such an element ν which is primitive, choose it. Let $n = \dim z = \dim \nu$.

For each $x \in \mathfrak{X}$, let $\{C^r(x)\}$ be the model spectral sequence for $E(x)$ or $P(x)$, as the case may be. It follows as a routine verification from the hypothesis (i), (ii) and (iii) that there is an algebra homomorphism of spectral sequences

$$\{h^r(x)\} : \{C^r(x)\} \to \{E^r\}.$$

Let $'E = \bigotimes_{x \in \mathfrak{X}} C(x)$ and

$$\{h^r\} : \{'E^r\} \to \{E^r\}$$

be the map induced by the $\{h^r(x)\}$. $\{h^r\}$ is an algebra homomorphism of spectral sequences and $h^2{}_{i,0}$ is an isomorphism for $i \leqq n-1$.

Let μ_2 be an element of $E^2{}_{n-r,r-1}$ projecting onto μ. There is an element

$'\mu_2 \in {}'E^2{}_{n-r,r-1}$ such that $h^2('\mu_2) = \mu_2$. Since μ_2 projects onto μ, the class of μ_2 in E^s, $s < r$, is not a boundary under d^s. Thus, the same is true for $'\mu_2$. Since $h^s{}_{a,b}$ is a onomorphism for $a + b \leqq n - 2$ and $b \leqq n - 3$, no $'d^s$ on the class of $'\mu_2$ is non-trivial for $s < r$, unless possibly some $'d^s('\mu_2) \in {}'E^s{}_{0,n-2}$. We shall return in the next paragraph to show this cannot occur. Thus, $'\mu_2$ projects onto a class $'\mu$ such that $h^r('\mu) = \mu$. If there were no element $'\gamma \in {}'E^2{}_{n,0}$ such that $'d^r('\gamma) = {}'\mu$, then $'\mu$ would give rise to a non-zero term of $'E^\infty{}_{n-r,r-1}$. But $'E^\infty = 0$. Thus, there is a $'\gamma \in {}'E^2{}_{n,0}$ such that $'d^r('\gamma) = {}'\mu$.

Hence,

$$d^r\{\nu - h^r('\gamma)\} = \mu - h^r('\mu) = 0,$$

and $\nu - h^2('\gamma)$ is more transgressive than ν. As $h^2('\gamma)$ is in the algebra generated by $\mathfrak{X}$, $\nu - h^2('\gamma)$ is a generator of $E^2{}_{*,0}$ not in that algebra; and so, the assumption that $\mathfrak{X}$ does not generate $E^2{}_{*,0}$ leads to a contradiction; i. e., $E^2{}_{*,0}$ is transgressively generated.

We have only to show that it is impossible for $'\lambda_{n-r} = {}'d^{n-r}('\mu_2)$ to be non-zero. Since $h^2{}_{i,0}$ is an isomorphism for $i \leqq n - 1$ and h^∞ is an isomorphism (both $'E^\infty$ and E^∞ are trivial), $h^{n-r}{}_{0,i}$ is an isomorphism for $i \leqq n - 3$. If $'\lambda_{n-r} \in {}'E^{n-r}{}_{0,n-2}$ were non-zero, then since $h^{n-r}('\lambda_{n-r}) = 0$, by Lemma 4. 5 either it is indecomposable in $'E^{n-r}{}_{0,*}$ or $n - 2 = p^k \cdot m$, where m is the dimension of some even dimensional generator of $'E^{n-r}{}_{0,*}$. If $'\lambda_{n-r}$ were indecomposable, then since $'E^2{}_{0,*} \to {}'E^{n-r}{}_{0,*}$ is an epimorphism of algebras, there is a generator $'\lambda_2 \in {}'E^2{}_{0,*}$ which projects onto $'\lambda_{n-r}$. In the canonical spectral sequence $'E$ any odd dimensional generator in $'E^2{}_{0,*}$ is the transgression of an element in $'E^2{}_{*,0}$ and thus cannot be the image under $'d^{n-r}$ of some element. The even dimensional generators are either transgressions of odd dimensional generators in $'E^2{}_{*,0}$, and cannot be in the image of $'d^{n-r}$, or are the images of elements of the form $('x^{p-1} \otimes \tau(x'))$, where $'x = {}'z^{p^i}$ for some even dimensional generator $'z$. Let $m = \dim {}'x$ and $'y = \tau('x)$. Let x and y denote the images of $'z$. Let $m = \dim {}'x$ and $'y = \tau('x)$. Let x and y denote the images of $'x$ and $'y$ under h. In E^m both x and y are primitive and $(x^{p-1} \otimes y)$ is *not primitive*. For $'\lambda_{n-r}$ to be the differential of $('x^{p-1} \otimes \tau('x))$ we must have $\mu = (x^{p-1} \otimes y)$ and $d^r\nu = \mu$, $r = m$. Since ν is a generator of $E^2{}_{*,0}$, it is also one in $E^m{}_{*,0}$. As $E^m{}_{*,0}$ is a sub-Hopf algebra of $E^2{}_{*,0}$, it is primitively generated by the first part of Theorem 4. 3. Thus, $\nu = w + \gamma$ in $E^m{}_{*,0}$, where w is a primitive generator in $E^m{}_{*,0}$ and γ is decomposable. Since ν is a generator in $E^2{}_{*,0}$ and γ is decomposable, w is a primitive generator in $E^2{}_{*,0}$. It is then at least as transgressive as ν; and so, by our initial assumption on ν, ν is primitive. It is impossible for $d^r\nu$ to be μ, which is not primitive in E^m. The only case remaining is for $n-2=p^k\cdot m$, where m is the dimension of some even dimensional generator of $'E^{n-r}{}_{0,*}$, $k>0$. Such a generator is the image of a generator y in $'E^2{}_{0,*}$. Such generators arise in one of two ways:

(1) $y=\tau(x)$, where x is an odd dimensional generator of $'E^2{}_{*,0}$, or

(2) $y=\tau(w)$, where $w=(x^{p^f})^{p-1}\otimes\tau(x^{p^f})$ and x is an even dimensional generator of $'E^2{}_{*,0}$.

In case (1), $n-2=p^k\cdot\dim y$; i.e., $n=2+(\dim x-1)\cdot p^k$, which is excluded by hypothesis. In case (2), $'\lambda=\tau(w\otimes y^{p^k-1})$; and so, $'\mu=w\otimes y^{p^k-1}$. If $k\geqq 1$, then

$$\begin{aligned}\dim\nu&=1+\dim w+\dim(y^{p^k-1})\\&=2+p^k\cdot\{mp^{f+1}-2\},\text{ where}\end{aligned}$$

$m=\dim x$ and $f\geqq 0$. This is also excluded by hypothesis.

(c) One picks a multiplicative base of transgressive generators and for each of them maps a model spectral sequence into $\{E^r\}$. The tensor product of these maps (as in part (b)) is an algebra map of spectral sequences and induces isomorphisms on $E^2{}_{*,0}$ and E^∞. By Theorem 4.1 the conclusion follows.

In view of the complicated technical nature of the hypotheses, it is appropriate to discuss them. The standing hypotheses (i), (ii) and (iii) are vacuously true if $E^2{}_{*,0}$ is an exterior Hopf algebra.

One can see that conditions (i) and (ii) hold in a loop space $X=\Omega Y$, by considering the canonical map $s\Omega X\to X$, which in homology is onto all transgressive elements. Also, since $\Omega Y\to\Omega s\Omega Y\to\Omega Y$ is a retraction, the diagram

$$\begin{array}{ccccc}\Omega s^2\Omega X & \to & \Omega sX & \to & X\\ & & \uparrow & & \\ & & X & & \end{array}$$

shows that if conditions (i) and (ii) hold in mod p homology spectral sequence of the fibration

$$\Omega^2s^2W\to P\Omega s^2W\to\Omega s^2W$$

for every space W, then they hold for the fibration

$$\Omega X\to PX\to X$$

for every loop space X. Browder, in his thesis, has determined explicitly the structure of the mod p homology spectral of the fibration

$$\Omega^2s^2W\to P\Omega s^2W\to\Omega s^2W$$

and the conditions (i) and (ii) hold. He has asserted elsewhere that condition (iii) also is valid. We shall give later (Theorem 4.7) a homology analogue of Kudo's Transgression Theorem which gives more precise information about the conditions (i), (ii) and (iii) in case X is a special $H^{(p-2)}{}_p$-space. Regarding the hypothesis in (b), there are examples of non-transgressively generated spectral sequences satisfying the other hypotheses which do not satisy (A) and also examples which do satisfy (B). Finally an inspection of the proof of (b) shows that the only place in which the Hopf algebra structure is used is in showing $h^s{}_{0,n-2}$ is a monomorphism, where n is the dimension of some generator in $E^2{}_{*,0}$. If $E^2{}_{*,0}$ is an exterior algebra, in each of the canonical spectral sequences, $'E^2{}_{0,*}$ is a polynomial algebra and has no odd dimensional elements. Thus, $h^s{}_{0,n-2}$ is vacuously a monomorphism. Hence, we have the following dual of Borel's Transgression Theorem:

THEOREM 4.6. *Suppose $\{E^r, d^r\}$ is an associative, commutative, algebra homology spectral sequence over Z_p such that $E^\infty = 0$ and $E^2{}_{*,0}$ is an exterior algebra of finite type. Then each generator of $E^2{}_{*,0}$ is transgressive and $\{E^r\} \cong \{C^r(E^2{}_{*,0})\}$ as algebra spectral sequences. In particular, $E^2{}_{0,*}$ is the polynomial alegebra generated by the transgressions of the generators of $E^2{}_{*,0}$.*

As for the standing hypotheses (i), (ii), and (iii) of Theorem 4.4, we have

THEOREM 4.7. *If X is a special $H_p{}^{(p-2)}$-space, p odd,* (see Theorem 1.1), *$\{E^r\}$ is the mod p homology spectral sequence of the fibration $\Omega X \to PX \to X$, and $x \in E^2{}_{2n,0}$ is a transgressive class and $y \in E^2{}_{0,2n-1}$ is a class such that $\tau(x) = y$ in E^{2n}, then*

$$\tau(x^p) = Q^n{}_{(p)}(y) \text{ in } E^{2np}$$

and

$$\tau(x^{p-1} \otimes y) = + \partial^*{}_p Q^n{}_{(p)}(y) \text{ in } E^{2n(p-1)},$$

where $\partial^*{}_p$ is the homology Bockstein homomorphism associated with the exact coefficient sequence

$$0 \to Z_p \to Z_{p^2} \to Z_p \to 0.$$

Proof. Since $\sigma_*(y) = x$, it follows from Theorem 3.1 that $\sigma_* Q^n{}_{(p)}(y) = Q^n{}_{(p)}(x)$, but $Q^n{}_{(p)}(x) = x^p$. Thus, x^p is transgressive and $\tau(x^p) = Q^n{}_{(p)}(y)$ in E^{2np}. More specifically, the class c'' for $i = 0$ in the proof of Theorem 3.1 is relevant.

$$c'' = e_{p-1} \otimes \epsilon_{(2n-1)p+1} + e_{p-2} \otimes \Delta^{p-2}\epsilon_{(2n-1)p+2} + \cdots + (-1)^{p-1/2} e_0 \otimes \epsilon_{2np}.$$

$$\partial c'' = e_{p-1} \otimes \epsilon_{(2n-1)p} + (\Gamma - \Delta^{p-1})\{e_{p-2} \otimes \epsilon_{(2n-1)p+1} - e_{p-3} \otimes \epsilon_{(2n-1)p+2} + \cdots + (-1)^{p-1/2} e_0 \otimes \epsilon_{(2np)-1}\}.$$

The polynomial $\Gamma - \Delta^{p-1}$ is congruent to zero mod p; i.e., $\Gamma - \Delta^{p-1} = pR(\alpha)$. This implies that $R(1) = 1$; and so, the sum of the coefficients of $R(\alpha)$ is 1. Under the map ϕ of the proof of Theorem 3.1, we have

$$\begin{aligned}
&\phi(\epsilon_{(2n-1)p}) = (\partial z)^p,\\
&\phi(\epsilon_{(2n-1)p+1}) = z \otimes (\partial z)^{p-1},\\
&\quad\vdots\\
&\phi(\epsilon_{2np-1}) = P(\alpha) \cdot z^{p-1} \otimes (\partial z), \text{ and}\\
&\phi(\epsilon_{2np}) = \gamma \cdot z^p,
\end{aligned}$$

where $z \in C(PX)$ such that $\partial z = y + p \cdot C$, $y \in C(\Omega X)$, $\pi_*(z) = \{x\} \bmod p$, and $P(\alpha)$ is some polynomial in α. We can determine $P(\alpha)$ as follows:

$$\begin{aligned}
\Gamma \cdot P(\alpha) \cdot z^{p-1} \otimes (\partial z) &= \phi(\Gamma \epsilon_{2np-1}) = \phi(\partial \epsilon_{2np})\\
&= \partial \phi(\epsilon_{2np}) = \gamma \cdot \partial z^p = \gamma \cdot \Gamma \cdot z^{p-1} \otimes (\partial z).
\end{aligned}$$

Thus, $\Gamma \cdot P(\alpha) = \gamma \cdot \Gamma$ and so $P(\alpha) = \gamma + \Delta \cdot Q(\alpha)$. (Actually, $Q(\alpha)$ is a polynomial in Δ, but this will not be needed.) Letting c denote the image of c'' in $C(PX)$, we have

$$\begin{aligned}
&c = a_1 + \cdots + a_p \text{ and}\\
&\partial c = b_0 + p \cdot R(\alpha)\{b_1 - b_2 + \cdots + (-1)^{p-1/2} b_{p-1}\},
\end{aligned}$$

where $a_p = (-1)^{p-1/2} \cdot \gamma \cdot z^p$, $b_0 = (-1)^{p-1/2} \cdot \gamma \cdot Q^{(n)}{}_p(y) \bmod p$ and $\bmod p^2$ (since $(\partial z)^p = (y + pc)^p = y^p \bmod p^2$) and

$$p \cdot b_{p-1} = p \cdot \{\gamma + \Delta \cdot Q(\alpha)\}(z^{p-1} \cdot (y + pc)) = p\{\gamma + \Delta \cdot Q(\alpha)\}(z^{p-1} \cdot y) \bmod p^2.$$

Thus, modulo p^2 we have

$$\begin{aligned}
&(-1)^{p-1/2} \cdot \gamma \cdot Q^{(n)}{}_p(y)\\
&\qquad = \partial c - R(\alpha) \cdot \{pb_1 - pb_2 + \cdots + (-1)^{p-1/2} p \cdot [\gamma + \Delta Q(\alpha)](z^{p-1} \cdot y).
\end{aligned}$$

Since Δe_0 is a boundary, $\Delta(z^{p-1} \cdot y)$ is a boundary and since PX is homotopy commutative and the dimension of z is even, for any i we have that $(\alpha^i - 1)(z^{p-1} \cdot y)$ is a boundary. Since the sum of the coefficients in $R(\alpha)$ is 1, it follows that $(R(\alpha) - 1)(z^{p-1} \cdot y)$ is a boundary. Hence, we have

$$\begin{aligned}
&(-1)^{p-1/2} \cdot \gamma \partial^*{}_p Q^{(n)}{}_p(y)\\
&\qquad = \partial d - (-1)^{p-1/2} p \cdot \gamma \cdot (z^{p-1} \cdot y) - R(\alpha) \cdot \{pb_1 - pb_2 + \cdots + pb_{p-2}\}.
\end{aligned}$$

mod p^2; and so,

$$(-1)^{p-1/2}\cdot\gamma\partial *_p Q^{(n)}{}_p(y)$$
$$=+\gamma\cdot(-1)^{p-1/2}(z^{p-1}\cdot y)+R(\alpha)\cdot\{b_1-b_2+\cdots+b_{p-2}\}.$$

The filter degrees of the terms b_1 to b_{p-2} are such that they all give zero in $E^2{}_{2n(p-1),2n-1}$. Hence,

$$\partial *_p Q^{(n)}{}_p(y)=+\tau(z^{p-1}\cdot y).$$

We next show that a canonical spectral sequence for A is independent of the particular monogenic decomposition of A used in its construction. To this end the follownig lemma suffices:

LEMMA 4.8. *If* $\{'E^r\}$ *and* $\{E^r\}$ *are canonical spectral sequences* ($\mathrm{mod}\, p$, $p\neq 2$) *and* $f\colon 'E^2{}_{*,0}\to E^2{}_{*,0}$ *is a Hopf algebra homomorphism, then there is a Hopf algebra spectral sequence homomorphism* $\{f^r\}:\{'^r\}\to\{E^r\}$ *such that* $f^2{}_{*,0}=f$.

Proof. It suffices to prove this for the case where $'E^2{}_{*,0}$ is a monogenic Hopf algebra, $H(u)$. $f(u)$, being a primitive element of $E^2{}_{*,0}$, is either $\sum a_iv_i$, where the v_i are odd dimensional generators of the particular decomposition of $E^2{}_{*,0}$ used in describing $\{E^r\}$ or $\sum b_iw_i^{p^{k_i}}$, where the w_i are even dimensional generators. In the former case, each of the v_i transgresses into a primitive element $y_i\in E^2{}_{0,*}$. We let $f^2{}_{0,*}('\tau(u))=\sum a_iy_i$. This induces a Hopf algebra map $f^2\colon 'E^2\to E^2$. There is only one non-trivial differential d^r, $r=\dim u$, and the induced f^r commutes with it. Hence, f^r is defined as a Hopf algebra homomorphism of spectral sequences. The latter case is slightly more troublesome. Let $z_i=b_iw_i^{k_i}$. Then z_i is primitive and transgresses into the primitive element t_i. Let $f^2{}_{0,*}('\tau(u))=\sum t_i$. As $(\sum z_i)^p=\sum(z_i)^p$, we let $f^2{}_{0,*}('\tau(u^p))=\sum\tau(z_i)^p$. We must show that $(\sum z_i)^{p-1}\otimes(\sum t_i)$ transgresses. The terms $z_{i_1}{}^{j_1}\cdots z_{i_k}{}^{j_k}\cdot t_l$ with at least two non-zero j_s's as exponents are boundaries under d^r, $r=\dim u$, of $z_{i_1}{}^{j_1}\cdots z_{i_k}{}^{j_k}\cdot z'_l$ since $j_1+\cdots+j_k+1=p$, two j_s's are non-zero and so no z_s has exponent p. The terms $z^{p-1}{}_i\otimes t_j$ for $i\neq j$ are boundaries of $z^{p-1}{}_i\cdot z_j$. Thus, in E^r, $(\sum z_i)^{p-1}\otimes(\sum t_i)-(\sum z^{p-1}{}_i\otimes t_i)$ is a boundary. As $(\sum z^{p-1}{}_i\otimes t_i)$ transgresses, in fact to a primitive class, it follows that $(\sum z_i)^{p-1}\otimes(\sum t_i)$ also does. Let

$$f^2{}_{0,*}('\tau(u^{p-1}\otimes '\tau(u)))=\tau(\{f^2{}_{*,0}(u)\}^{p-1}\otimes f^*{}_{*,0}'\tau(u)).$$

Clearly, a similar argument is valid for terms of the form u^{p^i} and $(u^{p^i})^{p-1}\otimes '\tau(u^{p^i})$. Thus, there is described a Hopf algebra map $f^2\colon 'E^2\to E^2$ which commutes with differentials.

In particular, we have that if $\{'C^r(A)\}$ and $\{C^r(A)\}$ are canonical spectral sequences induced by different monogenic decompositions of A, then the extension of the identity map of A into itself is a Hopf algebra homomorphism of spectral sequences, and since it induces isomorphisms on $'C^2{}_{*,0}(A)$ and $'C^\infty$ levels, it is then a Hopf algebra isomorphism of spectral sequences.

THEOREM 4.9. *Suppose $\{Y^r\}$ is a canonical spectral sequence, $\{X^r\}$ is a spectral sequence of algebras, $X^2{}_{*,0}$ is a Hopf algebra, and $\{f^r\}: \{X^r\} \to \{Y^r\}$ is an algebra homomorphism of spectral sequences such that $f^2{}_{*,0}$ and $Q(f^2{}_{*,0})$ are monomorphism, $f^2{}_{*,0}$ is a map of Hopf algebras and $X^\infty = 0$. Then*

(a) *$\{X^r\}$ is isomorphic as algebra spectral sequence to a canonical spectral sequence*

(b) *$f^2{}_{0,*}$ and $Q(f^2{}_{0,*})$ are monomorphisms*

(c) *if $\{X^r\}$ is a spectral sequence of Hopf algebras and $\{f^r\}$ is a Hopf algebra homomorphism of spectral sequences, then $\{X^r\}$ is isomorphic as Hopf algebra spectral sequence to a canonical spectral sequence.*

Proof. By Theorem 4.3 there is a Hopf algebra D and a homomorphism $g: D \to Y^2{}_{*,0}$ such that $X^2{}_{*,0} \otimes D \xrightarrow{(f^2{}_{*,0} \otimes g)} Y^2{}_{*,0}$ is a Hopf algebra isomorphism. By Lemma 4.8 there is an extension

$$\{(fg)^r\}: \{C^r(X^2{}_{*,0} \otimes D)\} \to \{Y^r\}.$$

$\{(fg)^r\}$ is a Hopf algebra isomorphism of spectral sequences. But there is an isomorphism

$$\{h^r\}: \{C^r(X^2{}_{*,0})\} \otimes \{C^r(D)\} \to \{C^r(X^2{}_{*,0} \otimes D)\}$$

as Hopf algebra spectral sequences.

Thus, we have the sequence of maps

$$\{X^r\} \xrightarrow{\{f^r\}} \{Y^r\} \xrightarrow{\{k^r\}} \{C^r(X^2{}_{*,0})\} \otimes \{C^r(D)\} \xrightarrow{\{l^r\}} \{C^r(X^2{}_{*,0})\}$$

in which $\{k^r\}$ and $\{l^r\}$ are Hopf algebra maps and $l^2{}_{*,0} \circ k^2{}_{*,0} \circ f^2{}_{*,0}$ is the identity.

The conclusions (a) and (c) are then immediate from Theorem 4.1; (b) is clear since Q(identity) is the identity on Q.

V. Homology of iterated loop spaces.

Definition 5.1. A *spectrum* $S=\{X_i,f_i\}$ (see [18]) is a sequence of spaces $\{X_i\}$, $i=0,1,2,\cdots$, and maps $f_i\colon X_i\to\Omega X_{i+1}$. Our spaces are assumed to have base points and the maps are base point preserving. Also in this definition we take only loops of length one. The suspension below is the reduced suspension.

By the well-known natural isomorphism $\rho\colon\{X,\Omega Y\}\to\{sX,Y\}$, ($\{A,B\}$ = set of maps from A to B), the last is equivalent to being given maps $g_i\colon sX_i\to X_{i+1}$, where $g_i=\rho(f_i)$.

Example 1. Let X be any topological space and let $S(X)$ be the spectrum defined by the sequence of spaces $\{s^iX\}$, $i=0,1,2,\cdots$ with maps $g_i\colon s(s^iX)\to s^{i+1}X$, $g_i=$ identity.

Definition 5.2. The homology groups $H_n(S)$ are defined by:

$$H_n(S)=\varinjlim H_{n+i}(X_i),\ H_{n+i}(X_i)\to H_{n+i+1}(X_{i+1})$$

is the composite $H_{n+i}(X_i)\xrightarrow{\sigma}H_{n+i+1}(sX_i)\xrightarrow{g_i}H_{n+i+1}(X_{i+1})$, where σ is the suspension isomorphism.

The homotopy groups $\pi_n(S)$ are defined by:

$$\pi_n(S)=\varinjlim \pi_{n+i}(X_i),\ \pi_{n+i}(X_i)\to\pi_{n+i+1}(X_{i+1})$$

is the composite $\pi_{n+i}(X_i)\xrightarrow{\sigma}\pi_{n+i+1}(sX_i)\xrightarrow{g_i}\pi_{n+i+1}(X_{i+1})$, where σ is the suspension homomorphism.

Example 2. If $S(X)$ is as in Example 1, then

$$H_n(S(X))=H_n(X),\ n>0,$$
$$H_0(S(X))=H_0(X)\ \text{(reduced homology group)}$$
$$\pi_n(S(X))=n\text{-th stable homotopy group.}$$

Definition 5.3. The *associated spectrum* $Q(S)$ *to the spectrum* S is defined by the sequence of spaces $\{s^i\Omega^iX_i\}$, $i=0,1,2,\cdots$, and maps

$$s^{i+1}\Omega^if_i\colon s(s^i\Omega^iX_i)\to s^{i+1}\Omega^{i+1}X_{i+1}.$$

Example 3. Let $S(X)$ be as in Example 1, then $Q(S(X))$ is defined by the sequence $\{s^i\Omega^is^iX\}$ and maps $s^{i+1}\Omega^if_i\colon s^{i+1}\Omega^is^iX\to s^{i+1}\Omega^{i+1}s^{i+1}X$, where $f_i\colon s^iX\to\Omega s^{i+1}X$, $f_i=\rho^{-1}(g_i)$. The map f_i may be considered as an inclusion, and then $\Omega^i f^i\colon \Omega^i s^i X \to \Omega^{i+1} s^{i+1} X$ is an inclusion. Consequently, we may define $Q(X) = \operatorname{Lim} \Omega^i s^i X$. Then $Q(X)$ satsfies

$$H_n(Q(X)) = H_n(Q(S(X))$$
$$\pi_n(Q(X)) = \pi_n(S(X))$$
$$\Omega Q(sX) = Q(X).$$

Also, one may consder the spectrum S' defined by $\{Q(s^i X)\}$ with maps $g'_i\colon Q(s^i X) \to \Omega Q(s^{i+1} X) = Q(s^i X)$, g'_i the identity. Then

$$H_n(Q(S')) = H_n(Q(X)) = H_n(Q(S(X))).$$

Example 4. Suppose that X is a space and we are given a map $f\colon X \to \Omega^k X$. Then we may form a spectrum S defined by the sequence: $X, \Omega^{k-1}X, \Omega^{k-2}X, \cdots, \Omega X, X, \Omega^{k-1}X, \cdots$, with every k-th map being $f\colon X \to \Omega(\Omega^{k-1}X)$, the other maps being identities.

If further, f is a homotopy equivalence (or at least induces isomorphisms of the homology groups), then

$$H_n(Q(S)) \cong H_n(X)$$

(*Remark.* This holds true of Bott's infinite Lie groups $U(\infty)$ and $SO(\infty)$.)

Let S *and* $Q(S)$ be as in Definitions 5.1 and 5.3. Then $H_n(Q(S)) = \underrightarrow{\operatorname{Lim}}\, H_{n+i}(s^i \Omega^i X_i) = \underrightarrow{\operatorname{Lim}}\, H_n(\Omega^i X)$, where in the last limit the maps are induced by $\Omega^i f_i\colon \Omega^i X_i \to \Omega^i(\Omega X_{i+1})$.

Since the inclusion of the subspace of loops of length one into the Moore loop space is natural and induces isomorphisms on homotopy and homology groups, we may substitute the Moore loop space in the last limit. Then $\Omega^i f_i$ is a map of H^{i-1}-spaces. This enables us to introduce homology operations $Q^{(p)}{}_j$, for arbitrarily large$j$, in $H_*(Q(S); Z_p)$.

Definition 5.4. For $\chi \in H_n(Q(S); Z_p)$, define $Q^{(p)}{}_j(\chi) = \operatorname{Lim} Q^{(p)}{}_j(\chi_i)$, where $\chi = \operatorname{Lim} \chi_i \in H_n(\Omega^i X_i)$, i sufficiently large so that $Q^{(p)}{}_j(\chi_i)$ defined.

This rich structure in $H_*(Q(S); Z_p)$ will enable us to compute $H_*(Q(S(X)); Z_p) = H_*(Q(X), Z_p)$ for any connected space X. We will show that it is the free graded algebra generated by all 'allowable words' in the homology operations, acting an a vector basis of $H^+{}_*(X; Z_p) \subset H_*(Q(X); Z_p)$. Explicitly,

Definition 5.5. A word mod p, p odd, is a formal product of Bocksteins $\partial *_p$ and extended p-th powers $Q_{j(p-1)}$, $j > 0$. A word w acting on a class x of $\dim r$ is *allowable* if:

1. w is empty
2. $w = Q_{j(p-1)}w'$, w' allowable, and one of the following holds
   (a) w' empty, j and r same parity
   (b) w' begins with $Q_{i(p-1)}$, $j \leqq i$, j and i same parity
   (c) w' begins with $\partial*_p Q_{i(p-1)}$, $j < i$, j and i same parity
3. $w = \partial*_p w'$, w' allowable, w' begins with $Q_{i(p-1)}$.

For any graded vector space V, we designate by $\mathfrak{F}^\infty(V)$ the free commutative, associative, graded algebra generated by all allowable words 'acting' on a basis of V^+.

We will show that there exists a natural inclusion

$$\phi_*\colon H_*(X; Z_p) \to H_*(Q(X); Z_p).$$

Since all operations are defined in this last homology algebra, ϕ_* extends to a homomorphism $\phi^\infty{}_*\colon \mathfrak{F}^\infty(H_*(X; Z_p)) \to H_*(Q(X; Z_p)$. Then it will remain to be shown that $\phi^\infty{}_*$ is an isomorphism.

Definition 5.6. A map $\phi\colon S \to S'$ of spectra, $S = \{X_i, f_i\}$, $S' = \{X'_i, f'_i\}$ is a sequence of maps $\phi_i\colon X_i \to X'_i$, such that $(\Omega, \phi_{i+1}) \circ f_i = f'_i \circ \phi_i$. In terms of the maps $g_i = \rho^{-1}f_i$, $g'_i = \rho^{-1}f'_i$, this condition is equivalent to $\phi_{i+1} \circ g_i = g'_i \circ (s\phi_i)$.

LEMMA 5.1. *The map* $\phi\colon S(X) \to Q(S(X))$ *defined by*

$$\phi_i\colon s^iX \to s^i\Omega^i s^iX,\ \phi_i = s^i\rho^{-i}(s^i\lambda),\ \lambda\colon X \to X \text{ the identity,}$$

induces a monomorphism $\phi_*\colon H_*(S(X)) \to H_*(Q(S(X)))$.

Proof. The map

$$s^iX \xrightarrow{\phi_i} s^i\Omega^i s^iX \xrightarrow{\psi_i} s^iX,\ \psi_i = \rho^i(\Omega^i s^i\lambda),$$

is the identity. In fact,

$$\begin{aligned}\psi_i \circ \phi_i &= \rho^i(\Omega^i s^i\lambda) \circ s^i\rho^{-i}(s^i\lambda) = \rho^i(\Omega^i s^i\lambda \circ \rho^{-i}(s^i\lambda))\\ &= \rho^i\rho^{-i}(s^i\lambda \circ s^i\lambda) = s^i\lambda = \text{identity}.\end{aligned}$$

Consequently, $\phi^*_i\colon H_*(s^iX) \to H_*(s^i\Omega^i sX)$ is a monomorphism for every i. It follows that ϕ_* is a monomorphism.

Remark. We had $H_*(X) \cong H_*(S(X))$ and $H_*(Q(X)) \cong H_*(Q(S(X)))$ and it is clear that the map $\phi_*\colon H_*(X) \to H_*(Q(X))$ is the monomorphism induced by the topological inclusion $X \to Q(X)$, defined by the sequence of inclusions $\rho^{-i}(s^i\lambda)\colon X \to \Omega^i s^iX$.

LEMMA 5.2. *Let X be connected. For p an odd prime, $H_*(\Omega^k s^i X; Z_p)$, $1 < k \leqq i$, is isomorphic in dimensions $< 3i - 2k$, to the free commutative associative graded algebra generated by*:

(1) *The allowable words in the extended p-th powers and the Bockstein, acting on a vector basis (x_l) of $H^+_*(s^{i-k}X; Z_p) \subset H_*(\Omega^k s^i X; Z_p)$; and*

(2) *The classes $\psi_{k-1}(x_l, x_m)$, $l < m$, where (x_l) is ordered by dimension (arbitrary order for elements of the same dimension); and $\psi_{k-1}(x_l, x_l)$,* $\dim x_l$ *same parity as k.*

Remark. If $x \in H_p(s^{i-k}X; Z_p)$ and $\dim Q_{j(p-1)}(x) < 3i - 2k$, then $j(p-1) < k$, and the operation is defined. In fact, since $r \geqq i - k + 1$, we have

$$3i - 2k > pr + j(p-1) \geqq pi - pk + j(p-1)$$

or

$$k > j(p-1) + (p-3)(i-k) + p > j(p-1).$$

Hence all allowable words in the operations within the dimension restriction are defined.

Proof. For $k = 2$, this is contained in the result of Browder [3]. (He obtains the complete algebra $H_*(\Omega^2 s^i X; Z_p)$, but his argument is quite long; we give a short argument in the proposition below for he weaker result required here.)

We now proceed by induction using the partial comparison theorem (4.1). That is, we can map the canonical spectral sequence into the spectral sequence for the acyclic fibre space $\Omega^{k+1}s^i X \subset P\Omega^k s^i X \to \Omega^k s^i X$ for terms of base degree $< 3i - 2k$; since for $k < i$, $H^+_*(s^{i-k}X; Z_p) \subset H_*(\Omega^k s^i X; Z_p)$ is transgressive, and hence the same is true for all operations on such classes. Thus, the base is transgressively generated in dimensions $< 3i - 2k$, and since $\Omega^k s^i X$ is an $H^{k-1}{}_p$-space, the hypothesis of Theorem (4.4) is fulfilled (see Theorem 4.7). It follows that $H_*(\Omega^{k+1}s^i X; Z_p)$ is isomorphic to the fibre of the canonical spectral sequence in dimensions $< 3i - 2k - 2 = 3i - 2(k+1)$.

Explicitly, $H_*(\Omega^{k+1}s^i X; Z_p)$ is generated by the transgressive images of the generators of $H_*(\Omega^k s^i X; Z_p)$, the transgressive images of the p-th powers of the generators of $H_*(\Omega^k s^i X; Z_p)$, and the Bocksteins of the latter. The words acting on a basis of $H^+_*(s^{i-k}X; Z_p)$ transgress into ($\pm$) the corresponding words acting on the corresponding basis of

$$H^+_*(s^{i-k-1}X; Z_p) \subset H_*(\Omega^{k+1}s^i X; Z_p);$$

i.e., $w(x_l)$ transgresses into $w'(\tau(x_l))$ where each $Q_{j(p-1)}$ in $w(x_l)$ is replaced in $w'(\tau(x_l))$ by $(\pm)Q_{(j+1)(p-1)}$, and Bocksteins transgress into Bocksteins. Also, $(w(x_l))^{p^f}$ transgresses into $Q^f{}_1 w'(\tau x_l)$, and in addition we get the generators $\partial*_p Q^f{}_1 w'(\tau x_l)$. The words $w'(\tau(x_l))$, $Q^f{}_1(w'\tau x_l))$, $\partial*_p Q^f{}_1(w'(\tau x_l))$ give all allowable words (under the given restriction) in $H_*(\Omega^{k+1}s^iX;Z_p)$. Finally, each $\psi_{k-1}(x_l,x_m)$ transgresses into $\psi_k(\tau x_l,\tau x_m)$. (Notes that the p-th power of $\psi_{k-1}(x_l,x_m)$ is too large a dimension to enter the given range.)

This completes the induction and establishes the Lemma.

PROPOSITION 5.1. *Let X be connected. For p an odd prime, $H_*(\Omega^2 s^iX;Z_p)$, $i \geqq 2$, is isomorphic in dimensions $\leqq 3i-4$, to the free commutative associative graded algebra generated by:*

(1) *The allowable words in the extended p-th power Q_{p-1} and the Bockstein, acting on a vector basis (x_l) of $H^+{}_*(s^{i-2}X;Z_p) \subset H_*(\Omega^2 s^iX;Z_p)$; and*

(2) *The classes $\psi_1(x_l,x_m)$, $l<m$, (x_l) order by dimension; and $\psi_1(x_l,x_l)$,* $\dim x_l$ *even.*

Proof. $H_*(\Omega s^iX;Z_p)$ is well known (Cartan Seminar, 1954-55) to be the free associative (non-commutative) algebra generated by a basis (y_l) of $H^+{}_*(s^{i-2}X;Z_p) \subset H_*(\Omega s^iX;Z_p)$. Any such algebra may be written *additively* as a tensor product of polynomial algebras with new generators the 'basic products,' (b_r), $r=1,2,\cdots$, of the original generators (see Hilton or Browder [3] for details). If (y_l) is the original basis ordered by dimension, then the basic products are $y_1, y_2, \cdots$; $[y_l,y_m]$, all $l<m$; $[y_l,[y_m,y_n]]$, etc. Here $[y_l,y_m]=y_ly_m-(-1)^{\dim y_l \dim y_m}y_my_l=\psi_0(y_l,y_m)$. For any b_r of odd dimension we may further write the polynomial algebra $P(b_r)\cong E(b_r)\otimes P(b^2{}_r)$ (additive isomorphism), $E(b_r)$ the exterior algebra on b_r. The $H_*(\Omega s^iX;Z_p)$ will be additively isomorphic to the tensor product of exterior algebras generated by odd dim classes and polynomial algebras generated by even dimensional clases. Further, all these generators are transgressive. In fact, all y_l are transgressive and hence $[x_l,x_m]=\psi_0(x_l,x_m)$ is transgressive, etc. Also $\psi_0(b_r,b_r)=2b^2{}_r$, if b_r odd dimensional, and hence $b^2{}_r$ is transgressive.

Since Ωs^iX is an H-space, the differential in the spectral sequence of the fibration $\Omega^2s^iX \subset P\Omega s^iX \to \Omega s^iX$ is a derivation. Also

$$E^2 \cong H_*(\Omega s^iX;Z_p)\otimes H_*(\Omega^2 s^iX;Z_p)$$

as algebras (in particular, any class in $H_*(\Omega e^iX;Z_p)$ commutes with any class in $H_*(\Omega^2s^iX;Z_p)$.

For dimensions $< 3i - 2$, $H_*(\Omega s^i X; Z_p)$ is additively isomorphic to the tensor product of exterior algebras and polynomial algebras generated by the classes (y_l); $\psi_0(y_l, y_m)$, $l < m$; and $\psi_0(y_l, y_l)$, $\dim y_l$ odd. Further, since $\dim y_l \geqq i$, all p-th powers of the above classes are of dimension $\geqq 3i$. It follows that we may again map in our canonical spectral sequence (additively) for base degree $< 3i - 2$. That is the images of the canonical sequence $C(b_S)$ for each basic product above are multiplied together in the given order in E. Since the differential commutes with the map on each $C^r(b_S)$ and the differential is a derivation in E^r, the map on $\otimes C(b_S)$ is a spectral sequence map (even though it is not an algebra map). Moreover the map is an algebra map on the fibre. The conclusion now follows by Theorem 4.1.

Remark. We have considered only p an odd prime. For $p = 2$, the problem is much simpler because the operation $Q^{(2)}{}_1$ already exists in $H_*(\Omega^2 s^i X; Z_2)$, and hence all 2^f powers of transgressive classes in $H_*(\Omega s^i X; Z_2)$ again transgress. Similarly, the needed operations in the higher loop spaces exist. Consequently, one may apply the ordinary comparison theorem (and the canonical spectral sequence) to obtain the complete algebras $H_*(\Omega^k s^i X; Z_2)$ $k \leqq i$. Since this result is already given in Browder [3], we do not repeat it here.

Returning to Lemma 5.2, we see that for $k = i$, it states that $H_*(\Omega^i s^i X; Z_p)$ is isomorphic in dimensions $< i$ to the free commutative associative graded algebra generated by the allowable words acting on a vector basis of $H^+{}_*(X; Z_p) \subset H_*(\Omega^i s^i X; Z_p)$. (Note that $\dim \psi_{i-1}(x_l, x_m) > i$.) It follows that $\Omega^i \rho^{-1}(s^{i+1}\lambda): \Omega^i s^i X \to \Omega^{i+1} s^{i+1} X$ induces a homomorphism $\Omega^i \rho^{-1}(s^{i+1}\lambda)_*: H_r(\Omega^i s^i X; Z_p) \to H_r(\Omega^{i+1} s^{i+1} X; Z_p)$, $r < i$. Taking the direct limit we get:

THEOREM 5.1. *$H_*(Q(X); Z_p)$ is isomorphic to the free commutative associative graded algebra generated by the allowable words in the extended p-th powers and the Bockstein acting on a vector basis of $H^+{}_*(X; Z_p) \subset H_*(Q(X); Z_p)$.*

Remark. This result holds for $p = 2$ as well as p an odd prime, as follows directly from the results of Browder discussed above.

COROLLARY. *$H_*(Q(sX); Z_p)$ is isomorphic as a Hopf algebra to the tensor product of monogenic Hopf algebras, these are exterior or polynomial algebras according as the generator is odd or even; the generators being the allowable words in the extended p-th powers and the Bockstein, acting on a vector basis of $H^+{}_*(sX; Z_p) \subset H_*(Q(sX); Z_p)$.*

Proof. This follows from the fact that all classes in $H_*(sX; Z_p)$ are primitive (in fact transgressive in $H_*(Q(sX); Z_p)$) and consequently the same is true of the words acting on these classes.

THEOREM 5.2. *Let S^n be a sphere of dimension n, n odd. Then $H_*(\Omega^k S^n, Z_p)$, $k < n$, is the tensor product as a Hopf algebra of monogenic algebras; exterior algebras for odd dim generators, polynomial algebras for even dim generators. Further there is one generator for each allowable word in the extended p-th powers $Q_{j(p-1)}$, $j < k$, and the Bockstein 'acting' on the generator of $H_{n-k}(\Omega^k S^n; Z_p)$.* (Note that we do not claim that these operations are actually defined in $\Omega^k S^n$.)

Proof. Consider the map $\Omega^k S^n \to \Omega^k Q(S^n) = Q(S^{n-k})$. We have $H_*(Q(S^{n-k}); Z_p)$ is transgressively generated for $k < n-1$, and primitively generated for $k < n$. Further the map $\Omega S^n \to Q(S^{n-1})$ induces a monomorphism $H_*(\Omega S^n; Z_p) \to H_*(Q(S^{n-1}); Z_p)$ since it takes generator into generator and both are free. Since $H_*(\Omega S^n; Z_p)$ is a polynomial algebra, the only indecomposable element is the generator, so that this map is also a monomorphism on the space I of indecomposable elements. By Theorem (4.9), it follows inductively, that

(1) $0 \to H_*(\Omega^k S^n; Z_p) \to H_*(Q(S^{n-k}; Z_p)$ exact, $k < n$

(2) $0 \to I(H_*(\Omega^k S^n; Z_p) \to I(H_*(Q(S^{n-k}; Z_p))$ exact, $k < n$

(3) $H_*(\Omega^k S^n; Z_p)$ is transgressively generated, $k < n-1$.

From (1) and (2) we conclude that $H_*(\Omega^k S^n; Z_p)$ is primitively generated and free, $k < n$. From (3) we see that the acyclic fibre space over $\Omega^k S^n$, $k < n-1$, has a canonical spectral sequence. Consequently, we can compute the number of generators in each successive loop space inductively. But it is easy to check that the number of generators must be the same as the number of allowable words and have the same dimensions.

COROLLARY. *$H_*(\Omega^k S^n; Z_p)$, $k < n$, n even, p an odd prime, is the tensor product as Hopf algebras of polynomial and exterior monogenic algebras. In fact, as Hopf algebras:*

$$H_*(\Omega^k S^n; Z_p) \cong H_*(\Omega^{k-1} S^{n-1}; Z_p) \otimes H_*(\Omega^k S^{2n-1}; Z_p).$$

Proof. The generator of $\pi_{2n-1}(S^n)$ defines a map $S^{2n-1} \to S^n$, from which we get a map of $S^{n-1} \times \Omega S^{2n-1} \to \Omega S^n$, by multiplying the inclusion $S^{n-1} \to \Omega S^n$ by the induced map $\Omega S^{2n-1} \to \Omega S^n$. This induces an isomorphism

$$\pi_i(S^{n-1} \times \Omega S^{2n-1}) \cong \pi_i(\Omega S^n)$$

in suitable C-theories (Serre [13]) and consequently the isomorphism of the Corollary.

THEOREM 5.3. *Let X be a homotopy commutative H-space, which is n-connected. If $H_*(\Omega^k X; Z_p)$ is primitively generated and all primitive elements transgress, $k \leqq n-1$; then $P^i u_q = 0$ for $i > (q-n)/2$, $u_q \in H^q(X; Z_p)$, P^i the Steenrod reduced power.*

Proof. The proof is by induction on q. If u_q is an element of lowest (non-zero) dimension in $H^* = H^*(X, Z_p)$. Then u_q is primitive, and likewise $P^i u_q$ is primitive. Since H_* is primitively generated, the p-th power of every element in H^* is zero, and a non-zero primitive element of H^* is indecomposable i. e.

$$0 \to P^* \to Q^*$$

is exact.

Since every primitive element in $H_*(\Omega^k X, Z_p)$ is the image of a primitive element in $H_*(\Omega^{k+1} X, Z_p)$, for $k < n-1$, $Q(H^*(\Omega^* X, Z_p))$ is mapped monomorphically into $Q(H^*(\Omega^{k+1} X, Z_p))$ by the suspension homomorphism. Also $Q(H^*(\Omega^{n-1} X, Z_p))$ is mapped monomorphically into $H^*(\Omega^n X, Z_p)$, since $H_*(\Omega^{n-1} X, Z_p)$ is transgressively generated. Hence if $s^n P^i u_q = 0$, then $P^i u_q = 0$. But $s^n P^i u_q = P^i s^n u_q = 0$ if $i > (q-n)/2$.

Now assume the theorem is true for all elements of dim less that q. Then if $u_q = v_r \cdot w_{q-r}$, $0 < r < q$, $P^i u_q = \sum_j P^j v_r \cdot P^{i-j} w_{q-r}$. Now by the induction assumption

$$P^j v_r \neq 0 \text{ implies } j < (r-n)/2$$

$$P^{i-j} w_{q-r} \neq 0 \text{ implies } i-j < (q-r-n)/2$$

and $\quad P^j v_r \cdot P^{i-j} w_{q-r} \neq 0$ implies $i < (q-n)/2 + (-n)/2 < (q-n)/2$.

It follows that the assertion holds for any decomposable element. On the other hand, even if u_q is indecomposable we claim that $P^i u_q$ is primitive; and hence, zero by the same argument as in the first two paragraphs. To see that $P^i u_q$ is primitive, suppose $\phi^* u_q = u_q \otimes 1 + 1 \otimes u_q +$ terms of the form $v_r \otimes w_{q-r}$, $0 < r < q$ (ϕ^* the diagonal homomorphism in H^*). Again by the Cartan formula, and the argument above applied to $v_r \otimes w_{q-r}$, $P^i(v_r \otimes w_{q-r}) = 0$ if $i > (q-n)/2$. Hence $P^i u_q$ is primitive if $i > (q-n)/2$, and $P^i u_q = 0$.

The theorem now follows by induction.

COROLLARY 1. *In $H^*(Q(s^n X; Z_p)$, $P^i u_q = 0$ for $i > (q-n)/2$, X an arbitrary connected space.*

COROLLARY 2. *In* $H^*(\Omega^l S^{n+l+1}; Z_p)$; $P^i u_q = 0$ *for* $i > (q-n)/2$.

Proof. For $n+l+1$ odd, this follows from the fact that $H_*(\Omega^l S^{n+l+1}; Z_p)$ maps monomorphically onto $H_*(Q^{n+1}(S^{n+1}); Z_p)$ and hence the cohomology map coming back is onto. For even spheres it follows by Corollary to Theorem 5.2.

Remark. The above result may be sharpened slightly to give: In $H^*(\Omega^l S^{n+1}; Z_p)$, $P^i u_q = 0$ for $i > (q-n)/2$. The following is a brief outline of the argument:

All stable cohomology operations on $\gamma^*_n \in H^n(\Omega^l s^{n+l}; Z_p)$ are trivial since this is true in $H^*(S^{n+1}; Z_p)$. Hence the same is true of $\gamma^*_n \in H^n(Q(S^n); Z_p)$. On the other hand, all homology operations on $\gamma_n \in H_n(Q(S^n); Z_p)$ transgress to $H_*(Q(S^0); Z_p)$ (see Lemma 5.3 below); and hence as in the proof of Theorem 5.3, the result follows from the primitive cohomology classes and then for all classes by the Cartan formula. Hence as in Corollary 2, the result holds for $H^*(\Omega^l S^{n+l}; Z_p)$.

We wish to prove that certain cohomology operations are not zero in $H^*(\Omega^k S^n; Z_p)$. For this we need the:

LEMMA 5.3. *Let X be a k-th loop space and suppose $x \in H_1(X; Z_p)$. Then there exist a map $\alpha_\pi: \dfrac{J^k\pi}{\pi} \to \Omega_0 X$, $\Omega_0 X$ the connected component of the identity loop, $\pi = Z_p$. If $\rho(\alpha_\pi): s(J^k\pi/\pi) \to X$ is the corresponding map, then $(\rho\alpha_\pi)_*(s\bar{e}_{i+p-1}) = Q_i(x)$, $i+p-1 < k$, where $\bar{e}_{i+p-1} \in H_*(J^k\pi/\pi)$ is $t_{\pi*}(e_{i+p-1})$, $t_\pi: W^{(k-1)}/\pi \to J^k\pi/\pi$.*

Proof. Since X is an $H^{k-1}{}_p$-space, we can define an $H^{k-1}{}_p$-structure on the space of paths PX by:

$$\bar{\theta}^{k-1}(w, \bar{x}_1, \cdots, \bar{x}_p)(t) = \theta^{k-1}(w, \bar{x}_1(t), \cdots, \bar{x}_p(t)),$$

$\bar{x}_i \in PX$, θ^{k-1} the defining map for X, $w \in J^{k-1}\pi$. This map restricted to ΩX gives an H^{k-1}-structure to ΩX which is the same as that given by the H^k-structure on ΩX defined previously (1.4). For any loop $w \in \Omega X$,

$$\alpha: J^k\pi \xrightarrow{\times w^p} J^k\pi \times w^p \xrightarrow{\theta^{k-1}} \Omega X \xrightarrow{w^{-p_0}} \Omega X$$

is equivariant, and the image of $J^k\pi$ will be in the identity component of ΩX. On the other hand, since $x \in H_1(X; Z_p)$ is spherical, it is represented by a loop w, and we define α as above for this w.

Now let X^* be the simply connected covering space of X, then the map $\theta^{k-1}{}_p: J^k\pi \times X^p \to X$ induces a unique lifted map $\theta^{k-1}{}_p: J^k\pi \times (X^*)^p \to X^*$, such that

$$\begin{tikzcd} J^k\pi \times (PX)^p \arrow[r, "\bar{\theta}"] \arrow[d, "1\times\lambda_*"] & PX \arrow[r, "w^{-p_0}"] \arrow[d, "\lambda^*"] & PX \arrow[d, "\lambda^*"] \\ J^k\pi \times (X^*)^p \arrow[r, "\theta^*"] \arrow[d, "1\times p^p"] & X^* \arrow[r, "w^{-p_0}"] \arrow[d, "p"] & X^* \arrow[dl, "p"] \\ J^k\pi \times X^p \arrow[r, "\theta"] & X & \end{tikzcd}$$ commutes.

The loop $w: [o, r] \to X$ defines a singular 1-cube in X, which we again denote by $x, x(t) = w(tr)$, $0 \leqq t \leqq 1$. Let $\bar{x}$ be the singular 1-cube in PX defined by $\bar{x}(t)(s) = w(ts)$, $0 \leqq t \leqq 1$, $0 < s \leqq r$. Then $\bar{x}$ projects on x by the end point map λ: $PX \to X$.

Consider $e_i \otimes \bar{x}^p$; this projects onto $e_i \otimes x^p$. We know from Section (III), that there exists a chain $\bar{c}$ in $W^{(n)} \otimes C(PX)^p$ of the form

$$e_i \otimes \bar{x}^p \pm \cdots \pm e_{i+p-1} \otimes (\partial\bar{x})^{p-1} \otimes \bar{x},$$

all terms except the first involving $\partial\bar{x}$, and such that $\partial\bar{c} = \pm e_{i+p-1} \otimes (\partial\bar{x})^p$. Now $\partial\bar{x} = w - \bar{c}$, so that the projection of the terms involving $\partial\bar{x}$ are terms involving $\partial x = 0$, and hence $(1 \otimes \lambda^p)_*(\bar{c}) = e_i \otimes x^p$. For then, $\bar{\theta}_*(\bar{c})$ is a chain in PX whose ∂ is in ΩX; hence $\partial\lambda^*\bar{\theta}_*(\bar{c})$ is degenerate in X^*; i. e. is a cycle, and $p\lambda^*(\bar{c}) = \lambda\bar{\theta}_*(\bar{c}) = \theta_*(\bar{c}) = Q_i(x)$.

On the other hand, we claim that $\bar{\theta}_*(e_{i+p-1} \otimes (\partial\bar{x})^p) = \bar{\theta}_*(e_{i+p-1} \otimes w^p)$ in ΩX. Once this is shown, we have $w_*{}^{-p}\bar{\theta}_*(e_{i+p-1} \otimes w^p) = \alpha_*(\bar{e}_{i+p-1})$, and our result will follow from the commutativity of the above diagram.

To show this last, note that the map $\theta^{k-1}{}_p$ as defined inductively for a k-th loop space satisfies the additional property.

$$\theta^{k-1}{}_p(w; x_1, \cdots, x_p) = \theta^{k-1}{}_p(w; e, \cdots, e, x_{i_1}, \cdots, x_{i_p}),$$

where $x_j \in X$, $j = 1, \cdots, p$, are such that $p - r$ of them are the base point e, the rest being $x_{i_1}, \cdots, x_{i_r}$ written in their given order; i. e., $i_1 < i_2 < \cdots < i_r$. This property then also holds of $\bar{\theta}^{k-1}{}_p$: $J^k\pi \times (\Omega_0 X)^p \to \Omega_0 X$. For chains we claim we may then assume

$$\bar{\theta}(w \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_p) = \bar{\theta}(w \otimes \bar{l} \otimes \cdots \otimes \bar{l} \otimes x_{i_1} \otimes \cdots \otimes \bar{x}_{i_p}),$$

where $\bar{x}_j \in C(\Omega_0 X)$, $j = 1, \cdots, p$, $p - r$ of them are the unit element $\bar{l}$, the rest being $\bar{x}_{i_1}, \cdots, \bar{x}_{i_r}$ written in their given order. It follows immediately from this that all terms in the expansion of $\bar{\theta}_*(e_{i+p-1} \otimes (w - \bar{l})^p)$ involving the same number of $\bar{l}$'s are equal; and hence

$$\bar{\theta}_*(e_{i+p-1} \otimes (w - \bar{l})^p) = \bar{\theta}_*(e_{i+p-1} \otimes w^p) - \bar{\theta}_*(e_{i+p-1} \otimes \bar{l}^p), \bmod p.$$

But as remarked in (II. 2), $\bar{\theta}_*(e_{i+p-1} \otimes \bar{l}^p) = 0$, since $i + p - 1 > 0$.

It remains to show that $\bar{\theta}$ has the desired property. To see this, for any space Y, take the explicit Σ_p-equivariant chain map $\phi\colon C(Y)^p \to C(Y^p)$ defined as follows: Let $g\colon C(Y) \to S(X)$ be the natural chain equivalence [5], $f\colon S(Y)^p \to S(Y^p)$ the explicit Eilenberg-Zilber map as given in [6], and $h\colon C(Y^p) \to S(X^p)$ be the natural chain equivalence inverse to g. Then since f is Σ_p-equivariant, the same is true of

$$\phi\colon C(Y)^p \xrightarrow{g^p} S(Y)^p \xrightarrow{f} S(Y^p) \xrightarrow{h} C(Y^p)$$

Then the map $t \otimes \phi\colon W^{(k-1)} \otimes C(Y)^p \to J^k\pi \otimes C(Y^p)$ satisfies the conditions of Theorem (2.1), and hence for $Y = \Omega_0 X$ we may take $\bar{\theta} = \bar{\theta} \circ (t \otimes \phi)$. That $\bar{\theta}$ has the above property follows from the fact that ϕ is Σ_p-equivariant and is such that $\phi(e \otimes X^{p-1}) \subset C(e \times X^{p-1})$, by the naturality of its component maps.

Corollary 1. *Let $\gamma_{n-k} \in H_{n-k}(\Omega^k S^n; Z_p)$ be a generator. There exists a map $\rho^{n-k}(\alpha_\pi)\colon s^{n-k}(J^{n-1}\pi/\pi) \to \Omega^k S^n$ such that*

$$\rho^{n-k}(\alpha_\pi)_*(s^{n-k}\bar{e}_{i+(n-k)(p-1)}) = Q_i(\gamma_{n-k}),\ i < n - 1 - (n-k)(p-1).$$

Corollary 2. *Let $Q(S^0) = \varinjlim \Omega^i{}_0 S^i$. $H_*(Q(S^0); Z_p)$ is the free associative commutative graded algebra with generators in* 1-1 *correspondence to all allowable words 'acting' on the unit element* $1 \in H_*(Q(s^0); Z_p)$. (Note that we do not say $H_*(Q(s^0); Z_p)$ is generated by these words acting on 1, as all such operations are trivial.)

Proof. $Q(S^0) = \Omega_0 Q(S^1) = \Omega(Q(S^1)^*)$, $Q(S^1)^* =$ simply connected covering space of $Q(S^1)$. $\pi_1(Q(S^1)) = Z$, and hence by a well-known spectral sequence arguments we have the exact sequence of $H_*(Q(S^1)^*; Z_p)$ modules:

$$0 \to H_*(Q(S^1)^*; Z_p) \to H_*(Q(S^1); Z_p) \to H_1(K(Z,1); Z_p) \otimes H_*(Q(S^1)^*; Z_p) \to 0.$$

It follows that $H_*(Q(S^1)^*; Z_p)$ is isomorphic to the subalgebra of $H_*(Q(S^1); Z_p)$ generated by all the non-empty allowable words acting on $H_*(S^1; Z_p) \subset H_*(Q(S^1); Z_p)$, but *not* the class γ_1 itself ($\gamma_1 =$ generator of $H_*(S^1; Z_p)$).

Now the Lemma implies that all words of the form $Q_{j(p-1)}\gamma_1$ transgress, even though γ_1 no longer exists in $H_*(Q(S^1)^*; Z_p)$. Since all other allowable words are operations on these words, $H_*(Q(S^1)^*; Z_p)$ is transgressively generated. The Corollary now follows immediately.

As a consequence of Corollary 1 of Lemma 5.3, we show that there are a number of non-trivial ways in which the Steenrod Algebra can act on the cohomology of iterated loop spaces of spheres.

THEOREM 5.4. *In $H^*(\Omega^l S^{n+l}; Z_p)$, if t is an even integer, $t \geqq n$ and $(t-n)(p-1) < l$, for which there is an integer j such that $(t/2(p-1), j) \not\equiv 0 \bmod p$ and $(t+2j)(p-1) < n+l-1$, then, letting $q = n + t(p-1)$, there is a class $u_q \in H^q(\Omega^l S^{n+l}; Z_p)$ such that $P^j(u_q) \neq 0$. In particular, if $2j$ is an even multiple of $(p-1)$, $2jp < n+l-1$ and $0 \leqq 2j - n(p-1) < l$, then $P^j(u_{n+2j}) \neq 0$.*

Proof. By the cited Corollary there exists a map

$$\rho = \rho^{n-k}(\alpha_\pi) : s^n(J^{n+l-1}Z_p/Z_p) \to \Omega^l S^{n+1}$$

such that $\rho_*(s^n e_{i+n(p-1)}) = Q_i(\gamma_n)$, in mod p homology, provided $i < l$ and $i + n(p-1) < n+l-1$. Then if t is even and $t \geqq n$,

$$\rho_*(s^n e_{t(p-1)}) = Q_{(t-n)(p-1)}(\gamma_n) \neq 0.$$

Let $v \in H^2(J^{n+l-1}Z_p/Z_p; Z_p)$ denote the dual of (e_2). Then $v^{(t/2)(p-1)}$ is the dual of $e_{t(p-1)}$. An easy computation shows that

$$P^j(v^{t/2(p-1)}) = (t/2(p-1), j)\, v^{(t/2+j)(p-1)},\ (t/2+j)(p-1) < n+l-1.$$

Thus, $P^j(s^n v^{t/2(p-1)}) = (t/2(p-1, j)\, s^n v^{(t/2+j)(p-1)}$. In $H^*(\Omega^l S^{n+1}; Z_p)$, letting u_q be non-zero on $Q_{(t-n)(p-1)}(\gamma_n)$, we have $P^j(u_q) \neq 0$; provided

(1) $q = n + t(p-1)$,

(2) t is even and $t \geqq n$,

(3) $(t/2(p-1), j) \not\equiv 0 \bmod p$,

(4) $(t-n)(p-1) < l$, and

(5) $(t+2j)(p-1) < n+l-1$.

Letting $j = t/2(p-1)$, (3) is satisfied, (1) becomes $q = n + 2j$, and (2), (4) and (5) are equivalent to

$$2jq < n+l-1, \text{ and } 0 \leqq 2j - n(p-1) < l.$$

Remark. Comparing Theorem 5.4 with Corollary 2 of Theorem 5.3 and the remark following it, we see that is is the best possible result.

REFERENCES.

[1] S. Araki and T. Kudo "Topology of H_n-spaces and H-squaring operations," *Memoirs of the Faculty of Science, Kyusyu University, Ser. A*, vol. 10 (1956), pp. 85-120.

[2] R. Bott, "The space of loops on a Lie group," *Michigan Mathematical Journal*, vol. 5 (1958), pp. 35-61.

[3] W. Browder, *Homology of loop spaces*, Thesis, Princeton, 1958.

[4] ———, "Suspensions in loop spaces," Abstract No. 553-79, *Notices of the American Mathematical Society*, vol. 5, no. 7 (1958), p. 820.

[5] S. Eilenberg and S. MacLane, "Acyclic models," *American Journal of Mathematics*, vol. 75 (1953), pp. 189-199.

[6] S. Eilenberg and J. Zilber, "On products of complexes," *ibid.*, pp. 200-204.

[7] Edward Halpern, *Twisted polynomial hyperalgebras*, Memoir No. 29, American Mathematical Society, 1958.

[8] I. M. James, "Reduced product spaces," *Annals of Mathematics*, vol. 62 (1955), pp. 170-197.

[9] J. Milnor, "Universal bundles II," *ibid.*, vol. 63 (1956), pp. 430-436.

[10] ———, "Geometric realization of a semi-simplicial complex," *ibid.*, vol. 65 (1957), pp. 357-362.

[11] J. Milnor and J. Moore, "The structure of Hopf algebrais," *to appear.*

[12] J. Moore, "The double suspension and p-primary components of homotopy groups of spheres," *Bol. de la Soc. Math. Mex.*, vol. 1 (1956), pp. 28-37.

[13] J. P. Serre, "Groupes d'homotopie et classes de groupes abélians," *Annals of Mathematics*, vol. 58 (1953), pp. 258-294.

[14] N. E. Steenrod, *The Topology of Fiber Bundles*, Princeton University Press, 1951.

[15] ———, "Homology groups of symmetric groups and reduced power operations," *Proceedings of the National Academy of Sciences, U.S.A.*, vol. 39 (1953), pp. 213-223.

[16] ———, "Cohomology operations derived from the symmetric group," *Commentarii Mathematici Helvetici*, vol. 31 (1957), pp. 195-218.

[17] R. G. Swan, "The homology of cyclic products," *Transactions of the American Mathematical Society*, vol. 95 (1960), pp. 27-68.

[18] G. W. Whitehead, "Homology theories and duality," *Proceedings of the National Academy of Sciences, S.A.*, vol. 46 (1960), pp. 554-556.

[19] E. C. Zeeman, "A proof of the comparison theorem for spectral sequences," *Proceedings of the Cambridge Philosophical Society*, vol. 53 (1957), pp. 57-62.