p1-2 BMS1.
Mar 16. A Arf cohomology.

Intro to BMS 1.

- $K/\mathbb{Q}_p$ fin ext. $k := \mathcal{O}_K/\mathfrak{m}_K$ res. field. Fix p. prime. $\mathbb{C}_p := \hat{\bar{\mathbb{Q}}}_p$.

Question 1: As a variety "degen." from K to k how does coho. change.

Ctx: • Let $\mathfrak{X}$ be a proper smooth for. sch $/\mathcal{O}_K$.
• Diff coho. The best under. l-adic, $l \neq p$.

$$\underbrace{H^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Z}_l)}_{\text{generic fibre}} \underbrace{\simeq}_{\mathcal{O}_{\bar{K}}} \underbrace{H^i_{et}(\mathfrak{X}_{\bar{k}}, \mathbb{Z}_l)}_{\text{Special fibre}}$$ "Sing-sort".

Growth: $l = p$ • $H^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Z}_p)$ is still good.
• Special fibre is too small.
(N. look at the case of elliptic curves).
$H^i_{crys}(\mathfrak{X}_{\bar{k}}/W(k))$ "de-Rham-sort"

Question 2: Is there a comparison functor: $H^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Z}_p)$ and $H^i_{crys}(\mathfrak{X}_k/W(k))$.

$I \to Gal(\bar{K}/K) \to Gal(\bar{k}/k)$.
generic special

• Diff $/\mathbb{C}_p$ "not much".
• Fontained defined. $W(k)[\frac{1}{p}]$-alg. B_{crys}. • has action of $\underbrace{\text{Frob}, \text{Galois}}_{\text{"relation?"}}$.

Thm: $H^i_{et}(\mathfrak{X}_{\bar{K}}, \underline{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} B_{crys} \simeq H^i_{crys}(\mathfrak{X}_k/W(k))[\frac{1}{p}] \otimes_{W(k)[\frac{1}{p}]} B_{crys}$. (Tsuji). (*)
scholze's is $\mathbb{Z}_p$.

- One can recover $H^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p)$ from $H^i_{crys}(\mathfrak{X}_k/W(k))[\frac{1}{p}]$ by considering the Hodge filtration (from $H^i_{crys}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} K = H^i_{dR}(\mathfrak{X}_K)$.
- $H^i_{crys}(\mathfrak{X}_k/W(k))[\frac{1}{p}] \simeq (*)^{G_K}$. $\leftrightarrow$
In FO. There are equivalence of categories $\{G_K \text{ reps} / (-)\} \simeq \{\text{Modules with } \varphi \text{ frobenius}\}$.

2. Main Result.

Thm 1.1 • K as before • $C = \hat{\bar{K}}$. $\mathfrak{X}$ proper smooth f.d. $/ \mathcal{O}_C$. Fix $i \geq 0$.

• $\mathfrak{X}_C$ as the rigid analytic fibre of $\mathfrak{X}$ → have to define this.

i) $H^i_{ét}(\mathfrak{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{crys} \simeq H^i_{crys}(\mathfrak{X}_k / W(k)) \otimes_{W(k)} B_{crys}$.
integral

compatible with actions. i) Frob. ii) Galois iii) Filtration.

ii) don't really care.

iii) If $H^i_{crys}, H^{i+1}_{crys}(\mathfrak{X}_k / W(k))$ are p-torsion free. Then we recover.

($H^i_{crys}(\mathfrak{X}_k / W(k))$ + φ-action) from ($H^i_{ét}(\mathfrak{X}_C, \mathbb{Z}_p)$ with G_K action).

p5. BMS1.

• Strategy of pf.

Thm 1.8. $\mathcal{O}$ ring of integers of $C = \hat{\bar{\mathbb{Q}}}_p$.

• $\exists$ perfect cplx of A_{inf}-modules. $R\Gamma_{A_{inf}}(\mathfrak{X})$ + φ-semi-linear map. $\in D(A_{inf})$.

• $A_{inf}(\mathcal{O}) = W(\mathcal{O}^\flat)$. • M is A_{inf}-module. $\varphi: M \to M$.

$\lambda \in A_{inf}$. A_{inf} equipped φ: i) $\varphi(x+y) = \varphi x + \varphi y$ ii) $\varphi(\lambda x) = \varphi(\lambda)\varphi(x)$.

In $D(A_{inf})$.

i) Crys of Special fibre : $R\Gamma_A(\mathfrak{X}) \otimes_A W(k) \simeq R\Gamma_{crys}(\mathfrak{X}_k / W(k))$

ii) de Rham of $\mathfrak{X}$: $R\Gamma_A(\mathfrak{X}) \otimes_A \mathcal{O} \simeq R\Gamma_{dR}(\mathfrak{X})$

iii) Crys of $\mathfrak{X} / \mathcal{O}/p$: $R\Gamma_A(\mathfrak{X}) \otimes_A A_{crys} \simeq R\Gamma_{crys}(\mathfrak{X}_{\mathcal{O}/p} / A_{crys})$.

iv + v.

• To construct $R\Gamma_A(\mathfrak{X})$ we need a complex $A\Omega_{\mathfrak{X}}$ of shvs of A_{inf}-modules on $\mathfrak{X}_{Zar}$: an object in $D(\mathfrak{X}_{Zar}, A)$.

Then $R\Gamma_A(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{Zar}, A\Omega_{\mathfrak{X}})$.

3.BMS.

3. Fontaine's Period Ring Ainf.

- Fix prime p. $S \in \mathrm{CAlg}^{\heartsuit}$. π-adically cpl + sep., $\pi | p$.
- $\varphi: S/pS \to S/pS$ to denote Frobenius. $S^\flat := \varprojlim(\cdots \to S/pS \to S/pS)$.

3.2 i) The canonical map.

$$\varprojlim_{x \mapsto x^p} S \longrightarrow S^\flat = \varprojlim_{\varphi} S/pS \longrightarrow \varprojlim_{\varphi} S/\pi S.$$

is an iso of monoids, rings resp.

Notn: we let $x = (x_0, x_1, \ldots) \in S^\flat$. $x_i \in S/pS$ (or $S/\pi S$)

write it as $(x^{(0)}, x^{(1)}, \ldots) \in \varprojlim_{x \mapsto x^p} S$. $x^{(i)} \in S$.

- $x_{i+1}^p = x_i$, $(x^{(i+1)})^p = x^{(i)}$.

- Witt vectors. • Rabl4. • 2_Bha17. Speciali.. char. 0 to char p.
- $A_{\inf}(S) = W(S^\flat)$. These are the p-typical witt vectors.
  $\cong (S^\flat)^{\mathbb{N}_{\geq 0}}$ set theoretically.

3.2 (iii)-(v)

$$\begin{tikzcd}
W(S^\flat) \simeq \varprojlim_R W_r(S^\flat) & & \varprojlim_F W_r(S^\flat) \arrow[ll, "\varphi^\infty"'] \arrow[d, "(iv)"] \\
\varprojlim_F W_r(S) & & \varprojlim_F W_r(S/\pi S) \arrow[ll, "(v)"']
\end{tikzcd}$$

φ^∞: $W_r(S^\flat) \xrightarrow{\varphi^r} W_r(S^\flat)$, ($\varphi: S^\flat \to S^\flat$ is Frobenius. induces $W_r(S^\flat) \xrightarrow{\varphi} W_r(S^\flat)$)

$S^\flat$ is perf char p.

(

3.2 (iii)-(v) $$\begin{tikzcd} W(S^\flat) \simeq \varprojlim_R W_r(S^\flat) & \varprojlim_F W_r(S^\flat) \arrow[l, "\varphi^\infty", "\sim"'] \arrow[d, "S", "(iv)"'] \\ \varprojlim_F W_r(S) \arrow[r, "(v)", "\sim"'] & \varprojlim_F W_r(S/\pi S) \end{tikzcd}$$

As W_r is functorial: the maps are induced by.

iv) $S^\flat \to S/pS \to S/\pi S$.

v) $S \longrightarrow S/\pi S$.

All maps are isomorphisms.

Pf: i) We have the equality $\varphi R = R\varphi = F$. If K is a perfect $\mathbb{F}_p$-ring, then φ^r is an iso.

- By UP,

A1. $$\begin{tikzcd} W_{r+1}(K) \arrow[rr, "\varphi"] \arrow[dr, "F"'] & & W_{r+1}(K) \arrow[dl, "R"] \\ & W_r(K). & \end{tikzcd}$$

A2. $$\begin{tikzcd} W_{r+1}(K) \arrow[r, "\varphi"] \arrow[d, "R"'] & W_{r+1}(K) \arrow[d, "R"] \\ W_r(K) \arrow[r, "\varphi"] & W_r(K) \end{tikzcd}$$

$\Rightarrow$ $$\begin{tikzcd} W_{r+1}(K) \arrow[rr, "\varphi^{r+1}"] \arrow[d, "F"'] \arrow[dr, "\varphi^r"] & & W_{r+1}(K) \arrow[d, "R"] \\ W_r(K) \arrow[rr, "\varphi^r"] \arrow[dr, "\varphi^{r-1}"'] & W_{r+1}(K) \arrow[ur, "\varphi"] \arrow[d, dashed] & W_r(K) \\ & W_r(K) \arrow[ur] & \end{tikzcd}$$

- Take limit of diagram. ∎

3.2 (iii)-(v)

$$\begin{tikzcd} W(S^\flat) \simeq \varprojlim_{R} W_r(S^\flat) & \varprojlim_{F} W_r(S^\flat) \arrow[l, "\varphi^\infty"', "\sim"] \arrow[d, "S", "(iv)"'] \\ \varprojlim_{F} W_r(S) \arrow[r, "(v)", "\sim"'] & \varprojlim_{F} W_r(S/\pi S) \end{tikzcd}$$

As W_r is functorial: the maps are induced by.

iv) $S^\flat \to S/pS \to S/\pi S$.

v) $S \longrightarrow S/\pi S$.

All maps are isomorphisms.

- $W(S^\flat) \xrightarrow{\sim} \varprojlim_{F} W_r(S)$.

Rmk: topology of Witt vectors, see 4. Rab14. And any element $x \in W(S^\flat)$ has a Teichmüller expansion given by $x = \sum_0^\infty V^i[x_i]$,

- let $x \in S^\flat$. $(x_0, x_1, \dots) \in \varprojlim S/pS \simeq \varprojlim S/\pi S$. or $(x^{(0)}, \dots) \in \varprojlim S$.
- $(\varphi^\infty)^{-1}$: $[x] \mapsto ([x], [x]^{1/p}, [x]^{1/p^2}, \dots) \in \varprojlim_{F} W_r(S^\flat)$. $[-]$ abusively $\forall\ S \to W_r(S)$, $r \geq 1$.

As we have com. dia. of monoids

$$\begin{tikzcd} W_{r+1}(S^\flat) \arrow[r, "F"] & W_r(S^\flat) \\ S^\flat \arrow[u, "{[-]}"] \arrow[r, "(-)^p"] & S^\flat \arrow[u, "{[-]}"'] \end{tikzcd}$$

- iv: $([x], [x]^{1/p}, \dots) \mapsto ([x_0], [x_1], \dots) \in \varprojlim_{F} W_r(S/\pi S)$, $x_i \in S/\pi S$, $x_{i+1}^p = x_i$.
- v: $([x_0], [x_1], \dots) \mapsto ([x^{(0)}], [x^{(1)}], \dots) \in \varprojlim_{F} W_r(S)$, $x^{(i)} \in S$, image $x_i \in S/\pi S$.

- $W(S^\flat) \xrightarrow{\sim} \varprojlim_{F} W_r(S)$ $[x] \mapsto ([x^{(0)}], [x^{(1)}], \dots)$. is formula for identifi.

3.4 BMS.

Fontaine maps θ_r and $\tilde{\theta}_r$.

- S as above. $r \geq 1$

Defn: $\tilde{\theta}_r : W(S^\flat) \xrightarrow{\sim} \varprojlim_F W_r(S) \xrightarrow{p_r} W_r(S)$. $\Big\}$ just proj.

$\theta_r : \tilde{\theta}_r \circ \varphi^r : W(S^\flat) = A_{inf}(S) \longrightarrow W_r(S)$

Cor 3.3 BMS, 5.2 FO. $\forall x \in S^\flat$. we have.

$$\theta_r[x] = [x^{(0)}] \in W_r(S).$$
$$\tilde{\theta}_r[x] = [x^{(r)}] \in W_r(S).$$

Pf: • By formula in 3.2 compute $\tilde{\theta}_r$: $[x] \overset{\sim}{\longmapsto} ([x^{(0)}], \ldots, [x^{(r)}], \ldots) \longmapsto [x^{(r)}]$

- $\varphi^r : [x] \overset{\varphi^r}{\longmapsto} [x^{p^r}] = [x]^{p^r} \in W(S^\flat)$.
- $\tilde{\theta}_r$ is ring hom. $\tilde{\theta}_r([x]^{p^r}) = [x^{(r)}]^{p^r} = [\underbrace{(x^{(r)})^{p^r}}_{\text{by defn.}}] = [x^{(0)}]$.

When $r=1$, we obtain Fontaine θ-map. (5.2, FO)

- $$W(S^\flat) \longrightarrow S$$
$$[x] \longmapsto x^{(0)}.$$

3.4. The following diagram commutes.

$$\begin{tikzcd} A_{inf}(S) \arrow[r, "\theta_{r+1}"] \arrow[d, "A"'] & W_{r+1}(S) \arrow[d, "B"] \\ A_{inf}(S) \arrow[r, "\theta_r"] & W_r(S) \end{tikzcd} \qquad \begin{tikzcd} A_{inf}(S) \arrow[r, "\theta_{r+1}"] & W_{r+1}(S) \\ A_{inf}(S) \arrow[r, "\theta_r"] \arrow[u, "\lambda_{r+1}\varphi^{-1}"] & W_r(S) \arrow[u, "V"'] \end{tikzcd}$$

$(A, B) = (id, F), (\varphi, F)$

$\lambda_{r+1} \in A_{inf}(S)$ s.t. $\theta_{r+1}(\lambda_{r+1}) = V(1) \in W(S^\flat) = A_{inf}$.

and a similar diagram with $\tilde{\theta}_r$.

- $\theta_r = \tilde{\theta}_r \circ \varphi^r$.

m2. BMS.

Commuting of $\theta_r \Rightarrow \tilde{\theta}_r$ and vice versa.

$$\begin{tikzcd} A_{inf}(S) \arrow[r, "\varphi^{-(r+1)}"] \arrow[d, "id"'] & A_{inf} \arrow[r, "\theta_r"] \arrow[d, "\varphi"] & W_{r+1}(S) \arrow[d, "F"] \\ A_{inf}(S) \arrow[r, "\varphi^{-(r)}"'] & A_{inf} \arrow[r, "\theta_r"'] & W_{r+1}(S) \end{tikzcd} \Rightarrow \varphi' \quad \begin{tikzcd} A_{inf}(S) \arrow[r, "\tilde{\theta}_r"] \arrow[d] & W_{r+1}(S) \arrow[d, "R"] \\ A_{inf}(S) \arrow[r, "\tilde{\theta}_r"'] & W_r(S) \end{tikzcd}$$

An analys of Frobenius map.

- S is π-adicly cpl., $\pi^p | p$: Ex: $S = \mathcal{O}_C$, $C = \hat{\bar{\mathbb{Q}}}_p$.

3.9. Surj. of Frob. Thm.

i) Every el. of $S/\pi_p S$ is a pth power.
ii) " S/pS "
iii) " $S/\pi^p S$ "
iv) $F: W_{r+1}(S) \to W_r(S)$ is surj $\forall r \geq 1$.
v) $\theta_r : A_{inf}(S) \to W_r(S)$ "

Pf: i) $\Rightarrow$ ii) $\Rightarrow$ iii) as $\pi^p | p | \pi p$.

iii) $\Rightarrow$ i) use π-adic completeness. Let $y \in S$.

$y = x_0^p + \pi^p y_1$ using surjectivity. apply same thing for $y_1, \ldots$ inductively.

$y = \sum_0^\infty x_i^p \pi^{pi} \equiv (\sum x_i \pi^i)^p \; (\pi p)$ ▢

v) $\Rightarrow$ ii) Uses the formula. $\theta[x] = x^{(0)}$, $(x^{(1)})^p = x^{(0)}$ mod p.

$x^{(1)}, x^{(0)} \in S$.

3. p27. BMS1.

Perfectoid Rings.

- S as above.

Defn: $\xi \in \ker(\theta: A_{inf}(S) \to W_?(S) \simeq S)$ is distinguished.
if $\xi \in W(S^\flat)$ $\xi = (\xi_0, \xi_1, \dots)$, ξ_1 is unit in $S^\flat$.

3.10. Injectivity of Frob. $\varphi: S/\pi S \to S/\pi^p S$, $x \mapsto x^p$. is surj.
i) $\ker \theta$ is principal then.
a) φ is an iso.
b) $\xi \in \ker\theta$ is gen. iff ξ is distinguished.

3.5 S is perf iff
i) π-ad. cpl. $\pi^p | p$ for some $\pi \in S$.
ii) $\varphi: S/pS \to S/pS$ is surj. i.e. semi perfect.
iii) $\ker(\theta: A_{inf}(S) \to S)$ is principal.
$A_{inf}(S) = W(S^\flat)$.

- Next time: can define B_{crys}, B_{dR} via case $S = \mathcal{O}_K$, K is perf, field, char 0.

Some generalities on Witt vectors.

Defn: $P \subseteq \mathbb{N}$ is dss: if $P \neq \emptyset$. $\forall n \in P$ all proper divisors of n are in P.

2.6 Rabl4 : If P dss. $\exists!$ covariant functor. $W_P : CAlg^{\heartsuit} \to CAlg^{\heartsuit}$

1) $W_P(A) = \prod_{n \in P} A = A^P$. $f: A \to B$, $f_* : (x_n)_{n\in P} \mapsto (f(x_n))_{n\in P}$.

2) The $w_n : W_P(A) \to A$ are homo. $\forall n \in P$.

3). zero el. is $(0, \dots)$ unit is $(1, 0, \dots)$

- we will be applying to $P = \{p^0, p^1, p^2, p^3, \dots\}$. $P_{\le(n)} := \{p^0, \dots, p^n\}$.

Denote $W_P(A) =: W(A)$,

$W_{P(n)}(A) =: W_n(A)$.

These are called the p-typical witt vectors.

- Write elemt $x \in W(A)$ as $(x_{p^0}, x_{p^1}, \dots)$, abusively $(x_0, x_1, \dots)$.
- The maps: $\forall r \geq 1$

$w_* : W_r(S) = S^r \longrightarrow S^r$

we have ring homo. $(x_0, \dots, x_{r-1}) \longmapsto (w_0(x_0), \dots, w_{r-1}(x_0, \dots, x_{r-1}))$.

$$w_n(x_0, \dots, x_{n-1}) = \sum_i p^i x_i^{p^{n-i}}$$ are the ghost maps

- Restrictions: $R : W_r(S) \to W_{r-1}(S)$ $(x_0, \dots, x_{r-1}) \mapsto (x_0, \dots, x_{r-2})$.
- Frobenius : $W_r(S) \xrightarrow{F} W_{r-1}(S)$
- Ver : $V : W_{r-1}(S) \to W_r(S)$. $(x_0, \dots, x_{r-2}) \mapsto (0, x_0, \dots, x_{r-2})$.
- Teich: $[-] : A \to W_r(A)$ $a \mapsto (a, 0, \dots, 0]$.

Rmk: • $F[a] = [a^p]$. • we have $[-] : A \to W(S)$. Since
$$\begin{tikzcd} & W_{r+1}(S) \arrow[d, "R"] \\ A \arrow[ur] \arrow[r] & W_r(S) \end{tikzcd}$$.

• $W(S) \xrightarrow{\sim} \varprojlim_R W_r(S)$.