Introduction.

Ref: L7: copen., v, em.

- The $\otimes$ product in $\mathcal{C}_{ab}$ is bad.

  Ex: $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{R}$, $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. ?

- What we will do is a Bousfield localisation.

⓪ We wish to find full subcats. (Thm 5.8.)

$$\mathrm{Solid} \underset{\perp}{\overset{\blacksquare}{\leftrightarrows}} \mathcal{C}_{ab}. \quad \text{Der: } D(\mathrm{Solid}) \underset{\perp}{\overset{L\blacksquare}{\leftrightarrows}} D(\mathcal{C}_{ab}).$$

Cor. ⓪ be mon. functors. Induce sym. mon. str. Thm 6.2.

$$M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}, \quad C \otimes^{L\blacksquare} C' := (C \otimes^{L} C')^{\blacksquare}$$

① Suffices: $\blacksquare$ on proj. gen. $\mathbb{Z}[S]$, $S \in$ Ed.

Require: $\mathbb{Z}^{\blacksquare} \simeq \mathbb{Z}$.

Cor: $\mathbb{Z}[[T]]$, $\mathbb{Z}_p$ are in Solid.

Pf: • Thm 5.8: + ① as $\mathbb{Z}^{\blacksquare} \simeq \mathbb{Z}$

② We know: $\forall S$ profin $S \simeq \varprojlim_{\substack{S \to S_i \\ S_i \text{ finite}}} S_i$ $\exists$ map.

$$\mathbb{Z}[S]^{\blacksquare} \longrightarrow \varprojlim \mathbb{Z}[S_i]^{\blacksquare} \underset{\text{additivity}}{\simeq} \varprojlim \mathbb{Z}[S_i].$$

Dustin: we define $\blacksquare$ in the "most completed" sense.

Defn: $\mathbb{Z}[S]^{\blacksquare} := \varprojlim \mathbb{Z}[S_i]$ $\forall S \simeq \varprojlim S_i$ S prof.

- we have defined this functor on cpt. proj. obj.
- $\mathcal{C}_{ab}^{cp} \xrightarrow{\blacksquare} \mathcal{C}_{ab}$, $\mathcal{C}_{ab}^{cp} \xrightarrow{i} \mathcal{C}_{ab} \xrightarrow{i_!\blacksquare} \mathcal{C}_{ab}$ (lke).

Notn.

$\mathcal{C}_{ab} := \mathrm{Cond}(\mathrm{Ab})$. $\mathcal{C} := \mathrm{Cond}(\mathrm{Set})$.

Top := 1-cat of top. spcs

$D(\mathcal{C}_{ab})$:= Derived (∞)-cat of $\mathcal{C}_{ab}$.

κ = a str limit. inn. card.

κ_{cg} := κ cpctly gen.

- Defn: $D \underset{i}{\overset{f}{\leftrightarrows}} \mathcal{C}$ let $if := L$. Then the ess'n img of L consists of the s-local obj. which are precisely those which are mapped to equivalences. 5.2.7, 5.5.4.14 HTT.
- 7.3. SAG.: Provided that we prove $(M \otimes N)^{\blacksquare} \simeq (M \otimes^{\blacksquare} N)^{\blacksquare}$ and similarly for $\otimes^{L\blacksquare}$. "Induced" is via forcing Localisation. str. monoidal. (Lurie.)
- Stone spc are profinite spcs.
- If $X \in \mathrm{Top}^{\mathrm{Stone}}$
- E1.4.1 SAG.:

$$\begin{array}{ccc} \mathrm{Pro}(\mathrm{Fin\,Set}) & \leftrightarrows & \mathrm{Top}^{\mathrm{Stone}} \\ (T)_{T \in \mathcal{C}_{X/}} & \longleftarrow & X \\ (S_i)_{i \in I} & \longmapsto & \varprojlim S_i. \end{array}$$

- $\mathcal{C}_{ab}^{cp}$:= cp. proj. obj. of $\mathcal{C}_{ab}$ = f. sums of $\mathbb{Z}[S]$, $S \in$ Ed.

Solid abelian Grps.

$\mathbb{Z}[S]^{■} := \varprojlim \mathbb{Z}[S_i] \quad \forall S \simeq \varprojlim S_i \quad S$ prof.

- determines our counit: $\exists\, \mathbb{Z}[S] \to \mathbb{Z}[S_i]$.
  over limits: $\mathbb{Z}[S] \to \varprojlim \mathbb{Z}[S_i] =: \mathbb{Z}[S]^{■}$

Defn: $A \in \mathcal{C}_{ab}$, $A' \in D(\mathcal{C}_{ab})$ is solid
iff it is local wrt

$$\{ \mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]^{■} \}_{S \text{ profinite}}$$

i.e.
$$\begin{cases} \mathrm{Hom}_{\mathcal{C}_{ab}}(\mathbb{Z}[S]^{■}, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_{ab}}(\mathbb{Z}[S], A). & \forall S \text{ profinite} \\ R\mathrm{Hom}_{\mathcal{C}_{ab}}(\mathbb{Z}[S]^{■}, A') \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{C}_{ab}}(\mathbb{Z}[S], A'). & \text{"} \end{cases}$$

- Regard. $X \in \mathcal{C}_{ab}$ in $D(\mathcal{C}_{ab})$ by $X[0]$.

Goal: • Understand $\mathbb{Z}[S]^{■}$:

- it is solid. "Measure space of S".
- It has a geometric Interpretation.

• Ideas behind the arguments.

- Reduction to finite case.
- Importance of Thm 1.7.

- S profinite $\underline{S} \in \mathcal{C}$
- $\mathcal{C}_{ab} \underset{\longrightarrow}{\xleftarrow{\mathbb{Z}}} \mathcal{C}$

Computation of $\mathbb{Z}[S]^{\blacksquare}$, $S \simeq \varprojlim S_i$ profinite.

last line p32

Prop: $\mathbb{Z}[S]^{\blacksquare} \simeq \underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\underline{C(S,\mathbb{Z})}, \underline{\mathbb{Z}})$.

Pf: Step 0: Reduce to finite case: $\varinjlim C(S_i,\mathbb{Z}) \simeq C(S,\mathbb{Z})$.

- In general: let A have disc. top.
- S is cHaus: let $f \in C(S,A)$. $\mathrm{Im}\, f$ is cpt, hence finite. So $f \in C(S_i,\mathbb{Z})$ for some i. Injectivity + topology: standard.

- rhs: $\underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(C(S,\mathbb{Z}),\mathbb{Z}) \simeq \varprojlim_i \underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\underline{C(S_i,\mathbb{Z})}, \underline{\mathbb{Z}})$
- lhs: $\simeq$ by def'n $\varprojlim \mathbb{Z}[S_i]$.

Step 1: disc: $\underset{\longrightarrow}{\bigoplus_I A} \simeq \bigoplus_I \underline{A}$. vertical map com. fint. coprodt. I finite.

$$\begin{tikzcd} \mathcal{C}_{ab} & \mathcal{C} \arrow[l, "\mathbb{Z}"'] \\ \text{disc. Ab.} \arrow[u] & \text{Disc.Set.} \arrow[l, "\mathbb{Z}"'] \arrow[u, "\text{Thm 1.7.}"'] \end{tikzcd}$$

T: Don't have to reduce to finite.

- We check at T-pts: $T \in \mathrm{ed}$. I finite.

NTS. $C(T, \bigoplus_I A) \simeq \bigoplus_I C(T,A)$. $A \in$ Dis Set/Ab.

We can first reduce to T fin:

- As Step 0. codomain is discrete, write both sides via $\varinjlim$,
- Filtered colim comm. fin. lim in Ab, Set.

- For finite set T, domain is disc. so

$$\prod_T \bigoplus_I A \simeq \bigoplus_I \prod_T A.$$ true in both Ab, Set

Step 2 Horizontal maps comm. with colimits.

- adj'n with ff.ful functor.

Step 3 $\mathbb{Z}[S] \simeq \underline{\bigoplus_S \mathbb{Z}} \simeq \underline{\mathbb{Z}[S]}$, for S finite.

$$\underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\mathbb{Z}[\underline{S}], A) \overset{\text{free ab.}}{\simeq} \underline{\mathrm{Hom}}_{\mathcal{C}}(\underline{S}, A) \overset{\text{Step 1.}}{\simeq} \prod_S \underline{\mathrm{Hom}}_{\mathcal{C}}(*, A).$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\bigoplus_S \mathbb{Z}, A) \overset{\text{Step 1.}}{\simeq} \prod_S \underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\underline{\mathbb{Z}}, A) \simeq \prod_S \underline{\mathrm{Hom}}_{\mathcal{C}}(*, A). \quad \square.$$

Step 3 $\Rightarrow$ $\mathbb{Z}[S] \simeq \underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\underline{C(S,\mathbb{Z})}, \underline{\mathbb{Z}})$. for S finite. $\square$

- There is an inclusion of sites. $\mathrm{Ed} \hookrightarrow \mathrm{ProFin} \hookrightarrow \mathrm{CHaus}$. whose associated topos are equivalent. 1. (Here the access. ones too).

- $\mathrm{Disc.Cpt}_\kappa \subset \mathrm{lc.Haus}_\kappa \subset \mathrm{cg.Haus}_\kappa$.

- $\mathrm{Top} \underset{U}{\overset{\mathrm{disc}}{\rightleftarrows}} \mathrm{Set}$, disc $\perp$ U.

$\therefore$ colimits of disc. spc. is disc.

- Colimits and limits in $\mathrm{Ab}(\mathrm{Shv}(\mathrm{Ed}_\kappa))$ are cpt ptwise.
- The shf condition is trivially satisfied. as cod. is ab. cat. $\therefore$ fin prod = coprd.

- There is a monadic adj. $\mathrm{Ab} \rightleftarrows \mathrm{Set}$
- Ab is an algebraic cat.
- U preserves
  - filtred colimt.
  - finite coprodts.

Here infinite coprodts.

Geometric Interpretation of $\underline{\mathrm{Hom}}_{\mathcal{C}ab}(\underline{C(S,\mathbb{Z})},\underline{\mathbb{Z}}) = \mathbb{Z}[S]^{\blacksquare}$

- Recall from Thm 4.2. there is the enriched emb.

$$LCA \hookrightarrow \mathcal{C}ab.$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}ab}(\underline{A},\underline{B}) \simeq \underline{C_{ab}(A,B)}.$$

- $C_{ab}(A,B)$ = cts gp hom with cpt open topoly.

- So the underlying set of $\mathbb{Z}[S]^{\blacksquare}(*) \simeq C_{ab}(C(S,\mathbb{Z}),\mathbb{Z})$.

- If $X \in \mathcal{C}ab$ not necess. true that $X(*) \in Ab$.

Prop: $C(S,\mathbb{Z}) \simeq \varinjlim C(S_i,\mathbb{Z})$, $S \simeq \varprojlim S_i$

- is gen. over $\mathbb{Z}$ by idempotents $\quad f = \sum_{s \in \mathbb{Z}} f(s)\, 1_{f^{-1}(s)}$
- By 1_U, U is clopen set. $U \subseteq S$.

clopen = closed + open.

Pf: • finite image. • Preimage on each value is clopen. □

Element in $\mathrm{Hom}_{Grp}(C(S,\mathbb{Z}),\mathbb{Z})$ is an add assignment

- If $U, V \hookrightarrow S$ disjoint clopen. $1_U \mapsto \alpha_U$. $1_V \mapsto \alpha_V$
Then $1_U + 1_V \mapsto \alpha_U + \alpha_V$.

- A measure is an add assis to the measurable set.

- Regard this set as space of mesure $\mathcal{M}(S,\mathbb{Z})$

- What does it mean if M is solid?

$$\mathrm{Hom}_{\mathcal{C}ab}(\mathbb{Z}[S]^{\blacksquare}, M) \simeq \mathrm{Hom}_{\mathcal{C}ab}(\mathbb{Z}[S], M).$$

Ex: Let us take S finite. M disc. ab. gp.

- $\mathcal{M}(S,\mathbb{Z}) \simeq$ an assignment $\mu : 1_s \mapsto \mu_s \in M$. weight

- an element rhs is a fuction.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & M \\ \mathbb{Z}[S] \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

$f(s) \in M$

$\sum \alpha_s e_s \mapsto \sum \alpha_s f(s)$. $\quad \alpha_s \in \mathbb{Z}$.

- An element lhs is integration.

$$\int \tilde{f} : \mu \longmapsto \int \tilde{f}\mu := \sum \alpha_s f(s)\, \mu_s \qquad □$$

Thm of Nöbling: $C(S,\mathbb{Z})$ is free abelian gp.

Fuc 73.

Thm. 97.2. Prop: let $R \in CAlg^{\heartsuit}$. identity 1. Additive gp R^+ torsion free.

If R is generated as a ring by a set E. of idempotents.

R^+ is a free abelian gp. gen. elements are fin. prod. of ele of E.

Pf: Step 1. Pick a well ordering of E. i.e. $E \simeq \lambda$ an ordinal

i.e. we given a well order $\{\varepsilon_0, \varepsilon_1, \ldots <$ λ, $= \{\varepsilon_\mu : \mu < \lambda\}$.

- E^* of all finite products of elements of E. $\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_n}$ is ordered lexicogrphicly

1 is minimal element.

Step 2: Claim: The products $\varepsilon_{\sigma_1} \ldots \varepsilon_{\sigma_k}$ that are not $\mathbb{Z}$-liner Combination of. Smaller products. form a basis.

under the lexicografic ordering.

Step 3: prove by transf. induction. Easy cases.

$\lambda = 0$: Then E is empty. so R is $\mathbb{Z}$ or 0.

λ is limit ordinal: filtered colimit by prepends of ordinals.

- The $R \xrightarrow{\sim} \varinjlim_{\sigma<\lambda} R_\sigma$, R_σ is genated by $\varepsilon_\rho \in E$, with. $\rho < \sigma$.

Since if $r \in R$, then. $\exists$ some finite. ordinal. $\tau < \lambda$. st. $r \in R_\tau$.

- Also $E_\lambda \simeq \varinjlim_{\sigma<\lambda} E_\sigma$, E_σ is basis defined by (*).

Step 4 λ is a successor ordinal: $\lambda = \rho + 1$ ρ is oridinal ; $\lambda = \rho \cup \{\rho\}$.

- let R_ρ subring gen. by. ε_σ, $\sigma < \rho$ Having basis E_ρ. $[\varepsilon_\rho]$ largest idempotent.
- let us extend basis of R_ρ. The other inducion is standard yoga
- let $\varepsilon_\rho = \varepsilon$. Then: $\varepsilon R_\rho = \varepsilon R$. is a ring with unit ε.

There is a gp sur $\varepsilon R \twoheadrightarrow R/R_\rho$.

Hence indeed $\frac{\varepsilon R}{R_\rho \cap \varepsilon R} \simeq R/R_\rho \simeq \bar{R}$.

4a: $\bar{R}$ is torsion free: this is proven by showing $R_\rho \cap \varepsilon R$ is pure in εR.

- Collect the facts.

R_ρ. is gen. by $(\varepsilon_\sigma)_{\sigma<\rho}$. Basis.: $\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_k}$.

$\bar{R}$ is gen. by $(\overline{\varepsilon\varepsilon_\sigma})_{\sigma<\rho}$. Basis: $\overline{\varepsilon\varepsilon_{\sigma_1}} \cdots \overline{\varepsilon\varepsilon_{\sigma_k}}$.

Inductive hypothesis applies. E_λ is precisely lift of bas of $\bar{R}$.

- We have exact seq. of ab. gps.

$$0 \longrightarrow R_\rho \longrightarrow R \longrightarrow \bar{R} \longrightarrow\!\!\!\!\!\rightarrow 0.$$

Strategy: apply inductive hypo. to R_ρ, $\bar{R} := A/B$ B is pure.

Key step in showing $\bar{R}$ is torsion free: we can replace a $\{\varepsilon_1, \ldots, \varepsilon_\alpha\}$ to a orthogonal idempotents $\{\varepsilon'_1, \ldots \varepsilon'_{2^k}\}$.

in a paragraph before proof.

Dachos.

Defn: A set T. is an ordinal.

i) if is transitive: every elmt of T is subset

ii) well-orded by $\in$.

- A liner order on a set P is well-ordered if any non-empty subset. P has a limit.

$\alpha < \beta \iff \alpha \in \beta$. a priori we also require to be a linear order.

Transfinite induction: let $\mathcal{C}$ be class of ord.

i) $0 \in \mathcal{C}$ ii) $\alpha \in \mathcal{C} \Rightarrow \alpha+1 \in \mathcal{C}$.

iii) if $\alpha \neq 0$ is limit ord then $\beta \in \mathcal{C}\ \forall \beta \in \alpha \Rightarrow \alpha \in \mathcal{C}$

Then $\mathcal{C}$ is class of all ordinals.

- Trans ind usually with axiom of choice.

General Procedure:

- $\varepsilon_{\sigma_1} \ldots \varepsilon_{\sigma_k} > \varepsilon_{\tau_1} \cdots \varepsilon_{\tau_j}$.

If $\sigma_1 > \tau_1$. othewise $\sigma_1 = \tau_1$.

- Then check $\sigma_2 > \tau_2$?. at one pt. $\sigma_k > \tau_k$.
- Example. $1 < \varepsilon_0 < \varepsilon_1 < \varepsilon_1\varepsilon_0 < \varepsilon_2 < \varepsilon_2\varepsilon_1 \varepsilon_2\varepsilon_1\varepsilon_0$

- A Successor ordinal $\alpha+1$ is defined as a set by $\alpha+1 = \alpha \cup \{\alpha\}$.

General thy:

Thm: Quotion of tf. gp. is tf. precisely when its subgp is pure.

Pure subgp Vol 1. of the Book.

Defn: a subgp $G < A$ is pure.

if $n|g$ in $A \Rightarrow n|g$ in G.

$g \in G$ $nx = g$ $x \in A \Rightarrow x \in G$

Ex: R_ρ is pure in R.

εR. is pure in R.

if $n\varepsilon = \sum \varepsilon_i$ $\varepsilon_i \in E$ multiply by (ε_ρ)

Lem: Intersection of pure subgps. are again pure. in tf.

Pf: soln is nice.

representation + resol'n.

Prop: For any profinite set S, $|I| \le 2^{|S|}$, there is an iso in $\mathcal{C}ab$,

$$\mathbb{Z}[S]^{\blacksquare} \simeq \underline{\mathrm{Hom}}_{\mathcal{C}ab}(\underline{C(S,\mathbb{Z})}, \underline{\mathbb{Z}}) \simeq \prod_I \mathbb{Z}.$$

Pf: Step 0. Choose an iso $C(S,\mathbb{Z}) \simeq \bigoplus_I \mathbb{Z}$ as ab gps.

Step 1. $\underline{\bigoplus_I \mathbb{Z}} \simeq \bigoplus_I \underline{\mathbb{Z}}$

- Same as page 2: inf coprod comm. with fin prod in ab grp.

Step 2: $\underline{C(S,\mathbb{Z})} \simeq \underline{\bigoplus_I \mathbb{Z}}$ as cds abe. grps.

- Check at T pts. $T \in \mathrm{Ed.}$; T arg comit with $\varinjlim$
- Can again reduce to when S is finite (hence $I = S$).

Step 3: $\underline{\mathrm{Hom}}_{\mathcal{C}ab}(\underline{C(S,\mathbb{Z})}, \underline{\mathbb{Z}}) \simeq \prod_I \underline{\mathrm{Hom}}_{\mathcal{C}ab}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \simeq \prod_I \underline{\mathbb{Z}}$. $\square$.

- Thm 5.8 ⟹ these are the cpt proj. obj in Solid. when S is ed.

- "Behave algebraicly".

- $C(T, C(S,\mathbb{Z})) \simeq C(T \times S, \mathbb{Z})$.

- $\underline{\mathrm{Hom}}_{\mathcal{C}ab}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(*, \underline{\mathbb{Z}}) \simeq \underline{\mathbb{Z}}$

Prop: let $S_\bullet \to S$ be hyp cover. of $S \in \mathrm{Pro}(\mathrm{Fin\ Set})$.

$$\cdots \to \mathbb{Z}[S_2]^{\blacksquare} \to \mathbb{Z}[S_1]^{\blacksquare} \to \mathbb{Z}[S]^{\blacksquare} \to 0.$$

is an exact cplx in $\mathcal{C}ab$.

Pf: Step 0. Recoll'n of Thm 3.2 $H^i_{sf}(S,\mathbb{Z}) \simeq H^i_{cond}(S,\mathbb{Z})$.

- $R\Gamma_{cond}(S,\mathbb{Z}) := (0 \to C(S_0,\mathbb{Z}) \to C(S_1,\mathbb{Z}) \to \cdots)$
- $H^i_{cond}(S,\mathbb{Z})$ ⟹ cplx is exact. in $\mathcal{C}ab$.

Step 1. Applying $\underline{R\mathrm{Hom}}(\ ,\mathbb{Z})$:

Lm.: $0 \to A \to B \to C \to 0$ exact in $\mathcal{A} \simeq D^{\heartsuit}(\mathcal{C}ab)$

then we obtain fibe seqne. ?: we get dist. triangle!

$$\underline{R\mathrm{Hom}}(C,\mathbb{Z}) \to \underline{R\mathrm{Hom}}(B,\mathbb{Z}) \to \underline{R\mathrm{Hom}}(A,\mathbb{Z}). \in D^-(\mathcal{C}ab)$$

Induces les

$$\cdots \to \pi_i \underline{R\mathrm{Hom}}(C,\mathbb{Z}) \to \pi_i \underline{R\mathrm{Hom}}(B,\mathbb{Z}) \to \pi_i \underline{R\mathrm{Hom}}(A,\mathbb{Z})$$
$$\pi_{i-1} \underline{R\mathrm{Hom}}(C,\mathbb{Z}) \to \cdots$$

- We thus obtain cpt. proj. resol'n by finite set.

- 1.3.4.4 HA let $\mathcal{A}$ be ab. cat enogh proj. $\mathcal{A}[W^{-1}] \simeq D^-(\mathcal{C}ab)$. $A = \mathrm{Ch}^-(\mathcal{A})$, W = quasi iso.

- $D(\mathcal{C}ab)$ is equipped with a t-str $(D_{\ge 0}, D_{\le 0})$.

- 1.3.3.2 HA: The functor on LHS ∃ as $\underline{\mathrm{Hom}}(-,\mathbb{Z}): \mathcal{C}ab \to \mathcal{C}ab$. is a right exact fctr.

1.3.2.19.: $\mathcal{A}$ ab cat. with enogh proj. Then $D^-(\mathcal{A})$ has t-str.

$D^-(\mathcal{A})_{\ge 0} := H_n(A) \simeq 0 \quad \forall n < 0$

$D^-(\mathcal{A})_{\le 0} := H_n(A) \simeq 0 \quad \forall n > 0$

$D^-(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$.

T: K-projective machinery (?).

1b. Apply to our case:

$$\begin{tikzcd} 0 \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & \cdots \\ & 0 \arrow[r] & F_{1/0} \arrow[ur] & F_{2/1} \arrow[ur] & F_{3/2} \arrow[r] \arrow[ur] & 0 \end{tikzcd}$$

$0 \to F_{2/1}$

• Each term F_i is free ab gp by Nöbeling.

• all the other terms vanish as there are no higher homotopy grps.

$\forall i \geq 0$. we have

$$0 \to \pi_0 \underline{RHom}_{\mathcal{C}ab}(F_{i+1/i}, \mathbb{Z}) \to \underline{Hom}_{\mathcal{C}ab}(F_{i+1}, \mathbb{Z}) \to \underline{Hom}_{\mathcal{C}ab}(F_i, \mathbb{Z}) \to \pi_1 \underline{RHom}_{\mathcal{C}ab}(F_{i+1/i}, \mathbb{Z}) \to 0.$$

of exact seque in $\mathcal{C}ab$

• Warning: This does not fit into les: we need $\pi_1 \underline{RHom}_{\mathcal{C}ab}(F_{i+1/i}, \mathbb{Z})$.

• Timo: A subgroup of a free group is free! This is also the Nielson-Schrier theorem. $\square$.

Prop: $\mathbb{Z}[S]^{■}$ is solid both in $\mathcal{C}ab$ and $D(\mathcal{C}ab)$

NTS: $RHom(\mathbb{Z}[T]^{■}, \mathbb{Z}[S]^{■}) \xrightarrow{\sim} RHom(\mathbb{Z}[T], \mathbb{Z}[S]^{■})$ $\forall T$ profinite spc.

Pf: Step 0: The derived case generalizes

Step 1: May suppe $\mathbb{Z}[S]^{■} = \mathbb{Z}$. by $\mathbb{Z}[S]^{■} \simeq \prod_I \mathbb{Z}$.

$$RHom(\mathbb{Z}[T]^{■}, \mathbb{Z}) \simeq RHom(\mathbb{Z}[T], \mathbb{Z}).$$

• rhs is $H^i RHom(\mathbb{Z}[T], \mathbb{Z}) \simeq H^i_{cond}(T; \mathbb{Z}) \simeq \begin{cases} 0 & \text{for } i > 0 \\ C(T, \mathbb{Z}) & \text{for } i = 0. \end{cases}$

• $\pi_0 RHom(P, \mathbb{Z}) \simeq Hom(P, \mathbb{Z})$ P is proj. obj.

$R\mathrm{Hom}(\prod_y \mathbb{T}, \mathbb{Z})$.

Step 2: lhs. Close ibo $\prod_y \mathbb{Z} \cong \mathbb{Z}[S]^{\blacksquare}$

- We use our knowledge of $\mathbb{R}, \mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$,
- Exists strict exact seque.

$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$

• From Thm 4.2.

lemma: The embedding $LCA \hookrightarrow \mathcal{C}_{ab}$ sends strictly exact seques to exact seq.

• pf in F. Déglise notes on condensed maths, 4.8

$0 \to \prod_y \mathbb{Z} \to \prod_y \mathbb{R} \to \prod \mathbb{T} \to 0$, $\prod_y$ are exact in $\mathcal{C}_{ab} \leftarrow$ m:

- again obtain fibre seque

$$R\mathrm{Hom}(\prod_y \mathbb{T}, \mathbb{Z}) \leftarrow \underbrace{R\mathrm{Hom}(\prod_y \mathbb{R}, \mathbb{Z})}_{\text{computed daily in 4.3.}} \leftarrow R\mathrm{Hom}(\prod_y \mathbb{Z}).$$ $\leftarrow$ desired obj.

2a   2b

2a • We derive the adj:

$$\mathcal{C}_{ab} \simeq \mathcal{C}_{\mathbb{Z}} \underset{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{R}, -)}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{C}_{\mathbb{R}}$$

- $R\mathrm{Hom}_{\mathbb{Z}}(\prod_y \mathbb{R}, \mathbb{Z}) \simeq R\mathrm{Hom}_{\mathbb{R}}(\prod_y \mathbb{R}, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z}))$
$\simeq 0$

• Th 4.3 $\underline{R\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z}) \simeq 0$.

2b. $R\mathrm{Hom}(\prod_y \mathbb{T}, \mathbb{Z}) \simeq \bigoplus_y \mathbb{Z}[-1]$

• Thm 4.3 $\underline{R\mathrm{Hom}}_{\mathbb{Z}}(\prod_y \mathbb{T}, \mathbb{Z}) \simeq 0$.

- Now apply les. which shifts degree $\square$.

The embeddings.

In particular: we have commuting diagram.

$$\begin{tikzcd}
\mathrm{Top}^{T1} \arrow[rr, "2.15"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] \arrow[drr, hook] & & \mathcal{C} \\
k\text{-}cg \arrow[u, shift right] \arrow[rr, hook, "1.7"'] & & \mathcal{C}^{qc} \arrow[u] \arrow[ull, "\perp"]
\end{tikzcd}$$

- Also have ff. embedding for gp, rings, ab grps.

of sps of interest. $cg^{T2} \xhookrightarrow{ff.} \mathcal{C}_{ab}$

In fact $\underline{\mathrm{Hom}}_{\mathcal{C}_{ab}}(\underline{A}, \underline{B}) \simeq_{\mathcal{C}_{ab}} \underline{C^0(A,B)}$ 4.2.

- A condensed set. is $T \in \mathcal{C} \simeq \varinjlim_{\kappa} \mathcal{C}_\kappa$.
  $\mathcal{C}_\kappa := \mathrm{Shv}(\mathrm{ProFin}_\kappa)$
- It is a shf on profinite site. that is like on κ-profinite site. for some κ. see BH19.
- $\mathrm{Top} \xrightarrow{\text{yoneda}} \mathrm{Shv}(\mathrm{Pro}(\mathrm{Fin}))$.
  but $\not\longrightarrow$ $\uparrow$ $\mathcal{C}$!
- In CHaus to proj obj. are Ed sets.
- Fullness of $\mathcal{C}_{ab}(\underline{A}, \underline{B})$ can be checked at $*$.
- all most all spcs of interest: loc. cpt. or cplex. are cg^{T2}.
- Ex 1.9. explains how $\mathcal{C}_{ab}$ "sees" κ-LCAs.