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Prerequisites for S.BMS1.

Goal of Today:

i) Pro étale Site.
ii) Adic Spaces.
iii) Define the infinite period shf: , $\mathbb{A}_{\inf, X}$

We will define: $X_{\text{proét}}$,
$\mathbb{A}_{\inf, X}$ would be a shf on this site.

Ref:
[Sch13, p-adic Hodge theory for rigid ana. var.]. ← all the arguments are here.

3, Sch13, PR.

The Pro-Étale Site.

- The foll. holds when
  - X is loc. Noeth. sch.
  - X is loc noeth. adic spc.

- The disc. is formal : so you don't need to know the obj.

- Idea: $X_{f\acute{e}t} \hookrightarrow X_{\acute{e}t}$.
  In $X_{f\acute{e}t}$. the structures $U \xrightarrow{f\acute{e}t} X$ are finite étale.
- 1.11.1 $X_{f\acute{e}t} \xrightarrow{\sim} \pi_1(X, \bar{x})$ - FinSets.
  $\pi_1(X, \bar{x})$ is étale fundamental gp of X.

Rmk: $\pi_1(X, \bar{x})$: $X = \operatorname{Spec} k$. $\pi_1(X, \bar{x}) := \operatorname{Gal}(k^s/k)$.
$\bar{x} := \operatorname{Spec} k^s$.

- Goal of defining $X_{pro\acute{e}t}$.
  i) limit of shvs of well defined. $\mathbb{Z}_\ell \simeq \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$.
  ii) Properties of adic spcs are nice :
  Every adic spc. + cd is pro-étale loc perfectid.

- I'll explain, inclusion of cat.
  $\operatorname{Pro}(X_{f\acute{e}t}) \hookrightarrow X_{pro\acute{e}t} \hookrightarrow \operatorname{Pro}(X_{\acute{e}t})$.

$X_{\acute{e}t}$: obj: $U \xrightarrow{\acute{e}t} X$ $U \in \mathrm{Sch}$.
cov: étale cov.
$\{f_i : \mathcal{U}_i \to \mathcal{U}\}$.
$|\mathcal{U}| = \bigcup f_i |\mathcal{U}_i|$
$|\mathcal{U}|$ is underlying space of $\mathcal{U}$.

Lenstra: Galois theory of schemes.

Rmk: if $f: C \to D$ is morphism of sites ($\exists$ a funct. $f^{-1}: D \to C$) SP.

3.3 RV : $X_{pro\acute{e}t}$
- Under' Cat $\operatorname{Pro}(X_{f\acute{e}t})$.
- There is an 'underlying spc.' for such an obj.

$$\begin{aligned} \operatorname{Pro}(X_{f\acute{e}t}) &\longrightarrow \mathrm{Top} \qquad (*) \\ \text{“}\varprojlim_i\text{”} \mathcal{U}_i &\longmapsto |\mathcal{U}| \simeq \varprojlim |\mathcal{U}_i|. \end{aligned}$$

- Cov. family: $\{f_i : V_i \to V\}$
  i) Each f_i is open "from (*)"
  ii) jointly surj. i.e. $|V| = \bigcup_i f_i(V_i)$

$\operatorname{Pro}(\mathcal{C})$
- obj: $F: I \to \mathcal{C}$.
  I is a small cofiltered diagr.
  "$\varprojlim$" F_i.
- $\operatorname{Hom}(\text{“}\varprojlim_i\text{”} F_i, \text{“}\varprojlim_j\text{”} G_j) \simeq \varprojlim_j \varinjlim_i \operatorname{Hom}_{\mathcal{C}}(F_i, G_j)$

3.3 RV $G \in \operatorname{Pro}(\mathrm{FinGrp})$.
- G-FinSet := obj: Fin sets with G-action. ; morph : G-equi
  cov: $\{f_i : S_i \to S\}$. s.t. $S = \bigcup f_i(S_i)$.
- G-Pro(FinSet). := obj: fin set with cts G-action.
  cov: $\{f_i : S_i \to S.\}$ are open cts. G. st. $S = \bigcup f_i(S_i)$

3.5.RV Let X be conn. loc. noeth. sch / adic spc. Then.

we $X_{profét} \xrightarrow{\sim} \pi_1(X,\bar{x})$ - Pro(fin sets).

pf: Step 1: Describe the funtr. lhs to rhs.

- $(Y \simeq \text{"}\varprojlim\text{"} Y_i \to X.) \mapsto (S_Y := \varprojlim S_i.)$

where $S_i :=$ fibre of Y_i at $\bar{x}$ i.e.
$$\begin{tikzcd} S_i \arrow[r] \arrow[d] & Y_i \arrow[d] \\ \bar{x} \arrow[r] & X \end{tikzcd}$$

- Each S_i carries a cts. $\pi_1(X,\bar{x})$-action.

By the 1.1L $X_{fét} \xrightarrow{\sim} \pi_1(X,\bar{x})$ - Fin Sets.

$\rightsquigarrow$ induces an action (by taking $\varprojlim_i$) on S_Y.

Step 2: Equivalence of cat.

Is formal and boils down to Thm 1.1L.

Remark on case of field.

$$(\mathrm{Spec}\, k)_{fét} \xrightarrow{\sim} \pi_1(X,\bar{x}) := \mathrm{Gal}(k^s/k) - \text{Fin Sets}.$$

Sketch:

Step 0: Setup.

- lhs is equivalent to cat. of finite. seperable k-algebras.
- $F: B \mapsto \mathrm{Alg}_k(B, k^s)$? admits an action of $\pi_1 := \mathrm{Gal}(k^s/k)$

Step 1: Com. alg.

$B \simeq \prod_{i=1}^{t} k_i$. st. each k_i/k is a finite sep. ext.

- $\mathrm{Alg}_k(B, k^s) \simeq \coprod_{i=1}^{t} \mathrm{Alg}_k(k_i, k^s)$
- $k_i \simeq (k^s)^{\pi_i}$, π_i is open subgp of π_1.

$\rightsquigarrow$ we can write $\coprod_{i=1}^{t} \mathrm{Alg}_k(k_i, k^s) \simeq \coprod_{i=1}^{t} \pi_1/\pi_i \leftarrow$ each of these are finite with π_1-action.

In general case? No idea yet. $\square$.

Step 3: Equivalence of Site:

NTS: $Y \to Z$ is $X_{profét}$ is open iff $S_Y \to S_Z$ is open.

($\Rightarrow$). easier direction

($\Leftarrow$) In paper is more difficult.

- $X_{fét} \xrightarrow{\sim} \pi_1(X,\bar{x})$ - fin sets.
- $\pi_1(X,\bar{x})$ for a scheme.

for a adic spc?

- 2.9L.

($\Rightarrow$) Boils down to

1. $\varprojlim_i$ comts with pb. in Spcs.
2. $|\bar{x} \times_X Y| \simeq |Y_{\bar{x}}| \simeq |\bar{x}| \times_{|X|} |Y|$

"underlying space functor" comts with fibre products.

• I can similarly define a site on $X_{\text{proét}}$.

3.9 BV Notions on morphisms: $\mathcal{U} \to V$ in $\text{Pro}(X_{\text{ét}})$.

i) (fin) étale. iff it is induced by (fin) étale in $X_{\text{ét}}$.

• (*) $\Rightarrow$ $X_{\text{ét}} \hookrightarrow \text{Pro}(X_{\text{ét}})$.

and $\mathcal{U} \to V$ is (fin) étale if $\exists$ a diagram in $\text{Pro}(X_{\text{ét}})$

pb.
$$\begin{array}{ccc} \mathcal{U} & \to & V \\ \downarrow & \lrcorner & \downarrow \\ U_0 & \to & V_0 \end{array}$$
$U_0 \to V_0$ is a (fin) étale in $X_{\text{ét}}$.

ii) pro étale if $\exists$ a presentation, $\mathcal{U} \simeq \varprojlim_i \mathcal{U}_i$.

i) $\mathcal{U}_i \to V$ is étale. in $\text{Pro}(X_{\text{ét}})$.

ii) $\mathcal{U}_i \to \mathcal{U}_j$ $\forall$ $i \geq j$.

is finite étale + surj.

Defn in 5.BMS1, defined $\mathcal{U} \to V$ to be pro étale.

$\exists$ $\mathcal{U} \simeq \varprojlim_{\mu < \lambda} \mathcal{U}_\mu$, where λ is an ordinal. st.

i) $\mathcal{U}_0 \to V$ is étale.

ii) $\forall \mu > 0$, $\mathcal{U}_\mu \to \mathcal{U}_{<\mu} := \varprojlim_{\mu' < \mu} \mathcal{U}_{\mu'}$ is finite étale and surjective.

• These two notions coincides for presentation when λ is countable.

• Q: Should by λ?

• Notes of Morrow:

$$\begin{array}{c} \vdots \\ \mathcal{U}_3 \\ \downarrow \text{fin étale} \\ \mathcal{U}_2 \\ \downarrow \text{fin étale} \\ \mathcal{U}_1 \\ \downarrow \\ V \end{array}$$

Defn: $X_{\text{proét}}$: obj: full subcat of $\text{Pro}(X_{\text{ét}})$ spanned those which are pro étale over X.

cov: $\{ f_i : \mathcal{U}_i \to \mathcal{U} \}$ f_i is pro étale.

st. $|\mathcal{U}| = \bigcup f_i |\mathcal{U}_i|$.

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(*) $\text{Pro}(\mathcal{C}) \subseteq \text{Fun}(\mathcal{C}, \text{Set})^{\text{op}}$

f. sub spanned those which

cofil. of representables.

$\mathcal{C} \longrightarrow \text{Pro}(\mathcal{C})$ (coyoneda)

• ordinal:

i) transitive set

ii) well-ordered set by $\in$.

• $\mathcal{U} := \text{"}\varprojlim_i\text{"} \mathcal{U}_i$ $\in \text{Pro}(X_{\text{ét}})$

$|\mathcal{U}| := \varprojlim_i |\mathcal{U}_i|$

- There is a comparison map, a morphism of sites
  $\nu : X_{\text{proét}} \longrightarrow X_{\text{ét}}$.
  $(\cdots U \to U_i \to U \to \cdots \to U \to X) \longleftrightarrow (U \to X)$

  map $(\nu^*, \nu_*) : \mathrm{Shv}(X_{\text{proét}}) \rightleftarrows \mathrm{Shv}(X_{\text{ét}})$.

3.16 RV. • $\mathcal{F}_{\text{ét}} \in \mathrm{Ab}(X_{\text{ét}})$ then
$$H^j(\mathcal{U}, \nu^* \mathcal{F}_{\text{ét}}) \simeq \varinjlim_{i \in I} H^j(\mathcal{U}_i, \mathcal{F}_{\text{ét}}), \quad j \geq 0.$$
where $\mathcal{U} = ``\varprojlim" \mathcal{U}_i \in X_{\text{proét}}$

- This says, these sheaves on $X_{\text{proét}}$ induced from pb. don't really give new info.

5.4. BMS1. They construct 6 shvs on $X_{\text{proét}}$.

- Idea: On $X_{\text{ét}}$: (X is an adic spc).
  - Can defines shvs $\mathcal{O}_{X_{\text{ét}}}, \mathcal{O}^+_{X_{\text{ét}}}$.
  - $\rightsquigarrow \underbrace{\mathcal{O}_X}_{\text{rat.}}, \underbrace{\mathcal{O}^+_X}_{\text{int.}} := \nu^* \mathcal{O}_{X_{\text{ét}}}, \nu^* \mathcal{O}^+_{X_{\text{ét}}}$.
  - $\rightsquigarrow$ We can cstr. formally new shvs.

5.4 iii) Cpl. int. shvs. $\hat{\mathcal{O}}^+_X := \varprojlim_{r} \mathcal{O}^+_X / p^r$, completion (along p cotext)
iv) Cpl. str shf $\hat{\mathcal{O}}_X := \hat{\mathcal{O}}^+_X [\frac{1}{p}]$
v) tilt. $\hat{\mathcal{O}}^{+}_{X^\flat} := \varprojlim \mathcal{O}^+_X / p$.
vi) $A_{\inf, X} :=$ the $\underline{\text{derived } p\text{-adic completion}}$ of $W(\hat{\mathcal{O}}^+_{X^\flat})$

- W is functorial on perfect $\mathbb{F}_p$-algebras.

Rmk: Morrow notes: We can apply Witt functor on $\hat{\mathcal{O}}^+_{X^\flat}$ pt wise,
$$W\hat{\mathcal{O}}^{+,\flat}_X(\mathcal{U}) := W(\hat{\mathcal{O}}^+_{X^\flat}(\mathcal{U})).$$
- and this gives the correct answer ??.

Adic Spaces. Sch12, SP, Bha17, BP.

Defn: $A \in \mathrm{TopRng}$ is **Tate** if
1. $\exists$ open subring A_0. ← of defn.
2. induced top on A_0 is t-adic. for some $t \in A_0$ which is unit in A. t is uniformizer,

• We say (A_0, t) as a couple of defn.

• Conventn. In all discussion K, na + cplt. field. $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$

• "everything is determined by unit ball".
1) (A_0, t) couple of defn.
$t^n A_0$ is nbhd basis of A_0 but also of A as $A_0 \subset_{open} A$.
2) $A \xrightarrow{\cdot f} A$ is cts. $\exists n$ st. $t^n f \in A_0$. i.e. $f \in A_0[\frac{1}{t}]$
$\leadsto$ $A \cong A_0[\frac{1}{t}]$.

Notn: A° := pbdd elents of A.

7.2.1 BP • $A \in \mathrm{CAlg}^{Tate}$
• A^+ as a ring of int. elements is any open + int. clos. subring of A°.
• An affinoid tate ring is a pair (A, A^+).
• Mapsn: $(A, A^+) \to (B, B^+)$ cts rng $f: A \to B$. St. $f(A^+) \subset B^+$.
$\leadsto$ a cat. $\mathrm{CAlg}^{Aff, Tat}$.

7.2.2 BP. $(A, A^+) \in \mathrm{CAlg}^{Aff,Tat}$ ($\to \mathrm{CAlg}^{Tat}$). is complete if A is complete (wrt to the topology).

Prop: $\mathrm{CAlg}^{Tat,cpl} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{CAlg}^{Tat}$.
A + property of knowing (A_0, t).

Pf: Step 0. General thing. let $A \in \mathrm{CAlg}^{Tat}$.
• 05GG, SP. M is an R-module. $\hat{M} := \varprojlim M/I^n M$.
~ gives the right defn for completion.
• Apply to our case: fix a couple of defn. (A_0, t) of A.
A is an A_0-module.

$$\hat{A} := \varprojlim A/t^n A_0. \simeq \varprojlim A_0[\tfrac{1}{t}]/t^n A_0[\tfrac{1}{t}]$$
$$\simeq \dots$$
$$\simeq (\varprojlim \hat{A}_0/t^n \hat{A}_0)[\tfrac{1}{t}].$$

where $\hat{A}_0 := \varprojlim A_0/t^n A_0$.
$\leadsto$ $\hat{A}$ is in fact a cplt tate ring with couple of defn $(\hat{A}_0, t)$. □

• all rings comm. + ord.

Huber rings / f-adic:
1. the same.
2. A_0 is I-adic, for some I f.g. A_0.

Defn: R is a Banach k-algebra.
1) $|\cdot|: R \to \mathbb{R}_{\geq 0}$.
i) norm. $|f| = 0 \iff f = 0$.
ii) submultipl: $|fg| \leq |f||g|$.
iii) na. $|f+g| \leq \max(|f|, |g|)$.
2) Cpl. with to the topology induced from norm.

(Can define a metric on R. $d(x,y) := |x - y|$.).

• $f \in A$ is pbdd iff $\{f^n : n \in \mathbb{N}_{\geq 0}\}$, this is bdd.
In our ctx: (A_0, t)
$\leadsto$ f is pbdd iff $\exists\, c > 0$, st. $t^c f^{\mathbb{N}} \subseteq A_0$.

• It is not true then $\hat{M}$ I-adically cplt. (cplt wrt I-adic topology). Is true, if I is f.g.

References are all to Bhatt.

7.2.5. $\mathrm{CAlg}_g^{\mathrm{cpl.\,Aff.\,tate}} \underset{\iota}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{CAlg}_g^{\mathrm{Aff.\,tat}}$.

Descrip
- $(A, A^+) \in$ rhs
- cpl. is given by $(\hat{A}, \hat{A}^+)$.
  We defined $\hat{A}$ already ($\hat{A} = \varprojlim A/t^n A_0$, for some (A_0, t)).
- There is canonical map $A^+ \to A \to \hat{A}$,
  $\hat{A}^+$ is int closure of $\mathrm{im}(A^+ \to \hat{A})$.

7.1.1 $(A, A^+) \in \mathrm{CAlg}_g^{\mathrm{aff.\,tat.}}$ The adic spectrum.

$$(\mathrm{CAlg}_g^{\mathrm{aff.\,tat}})^{op} \longrightarrow \mathrm{Top}^{\mathrm{Spec}}$$
$$(A, A^+) \mapsto \mathrm{Spa}(A, A^+).$$

$\mathrm{Spa}(A, A^+) := \{ x : A \to \Gamma \cup \{0\}$ cts val; $x(f) := |f(x)| = |f|_x \leq 1$ $\forall f \in A^+ \}$.

- 2.2 ECD One char. $X = \mathrm{Spec}\, A$ $A \in \mathrm{CAlg}_g^{?}$.

- $\Gamma \cup \{0\}$ has the order topology. $x^{-1}(\Gamma_{<\gamma} \cup \{0\})$ is open in A, $\gamma \in \Gamma$

7.3.10. The unit map of completion.
$(A, A^+) \to (\hat{A}, \hat{A}^+)$ induces equivalence of Top-spc.
$\mathrm{Spa}(A, A^+) \xleftarrow{\sim} \mathrm{Spa}(\hat{A}, \hat{A}^+)$.

7.4.10. Special spc Up basis given by qc. rational subsets.

$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) := \{ x \in \mathrm{Spa}(A, A^+) : |f_i(x)| \leq |g(x)| \ \forall i \}$

7.5.1 Thm: $\forall\, U \underset{\text{open}}{\subseteq} \mathrm{Spa}(A, A^+)$ is rational subset.
$\exists!$ cpl. aff. tat. $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. satisfying:

i) It is terminal of all cpl. aff. tate rings. satisfying

$$\begin{tikzcd}
\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \arrow[rr] \arrow[dr, "\wr"] & & \mathrm{Spa}(A, A^+) \\
\mathrm{Spa}(B, B^+) \arrow[u, dashed, "\exists!"] \arrow[r] \arrow[urr] & U \arrow[ur] &
\end{tikzcd}$$

$\forall (B, B^+) \in \mathrm{CAlg}_g^{\mathrm{aff.\,tat.\,cpl.}}$

ii) $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \cong U$. homeomorphic.

7.5.11. • $(A, A^+) \in \mathrm{CAlg}^{\mathrm{Taf.\,Aff.}}$ $X := \mathrm{Spa}\,(A, A^+)$.

rational presheaf $\mathcal{O}_X$
int. presheaf $\mathcal{O}^+_X$.
defined on basis by 7.5.1.

• In general if $W \subseteq_{\text{open}} X$.

$$\mathcal{O}_X(W) = \varprojlim_{\substack{U \subset W \\ \text{rat. subsets}}} \mathcal{O}_X(U). \qquad \text{(via rke)}.$$

7.5.12. An aff. Tate ring (A, A^+) sheafy if $\mathcal{O}_X$ is a sheaf.
Rem. 7.2 $\Rightarrow$ $\mathcal{O}^+_X$ is also a sheaf.

Next time: affinoid adic space.
an adic space is a space locally