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F-Isocrystals and De Rham Cohomology. I

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0. Introduction

This paper is devoted to a study of crystalline cohomology tensored with $\mathbb{Q}$. Its main theme is that, after tensoring with $\mathbb{Q}$, one obtains much stronger "analytic continuation" properties, in senses we shall try to make precise, than when working with modules over the Witt ring. For example, if V is a complete discrete valuation ring with perfect residue field k of characteristic p, we can think of Spec V as an analogue of a small disc in the classical case, whose size depends on the absolute index of ramification e. If X is a smooth proper V-scheme, a central "analytic continuation" result of crystalline cohomology asserts that the cohomology of the family $H^i_{\mathrm{DR}}(X/V)$ depends only on the closed fiber X_0, not on the lifting X, provided e is strictly less than p. If $p \leqq e < \infty$, we show here that this is still true "up to isogeny," i.e. $H^i_{\mathrm{DR}}(X/V) \otimes \mathbb{Q}$ depends only on X_0, and in fact is canonically isomorphic to $H^i_{\mathrm{cris}}(X_0/W) \otimes V \otimes \mathbb{Q}$, as had already been conjectured in [1].

The main idea in our approach is the systematic exploitation of the action of the absolute Frobenius endomorphism on crystalline cohomology - an idea which, in fact, goes back to Dwork [8, 9] and which was made very explicit for us in [13]. The technical key is (1.3), which asserts that, with suitably hypotheses, the relative Frobenius morphism $F_{X/S}$ induces an isogeny on the crystalline cohomology of a smooth family X/S of schemes in characteristic p.

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We in fact also show in (1.9) that, under suitable hypotheses, the F-crystal structure $F^*_{X/S}: F^*_{X/S} R^q f_* \mathcal{O}_{X/S} \to R^q f_* \mathcal{O}_{X/S}$ is nondegenerate in the sense of [21]: there is a morphism of crystals:

$$V_{X/S}: R^q f_* \mathcal{O}_{X/S} \to F^*_{X/S} R^q f_* \mathcal{O}_{X/S}$$

such that $V_{X/S} \circ F^*_{X/S}$ and $F^*_{X/S} \circ V_{X/S}$ are multiplication by a certain power of p.

The second section studies the crystalline cohomology of infinitesimal deformations. Using the results of §1, we prove that if $f: X \to S$ is a smooth morphism in characteristic p, if $S_0 \hookrightarrow S$ is a nilpotent immersion, and if $S \hookrightarrow T$ is a suitable divided power thickening, then $\mathbb{R} f_*(\mathcal{O}_{X/T}) \otimes^{\mathbb{L}} \mathbb{Q}$ depends functorially only on $X \times_S S_0$. This says that crystalline cohomology is, up to isogeny, invariant under deformations. (Note that no divided power structure on $S_0 \hookrightarrow S$ or on $S_0 \hookrightarrow T$ is required.) We easily deduce from this the existence of the isomorphism

$$\sigma_{\mathrm{cris}}: H^i_{\mathrm{DR}}(X/V) \otimes \mathbb{Q} \xrightarrow{\sim} H^i_{\mathrm{cris}}(X_0/W) \otimes V \otimes \mathbb{Q}$$

alluded to above, and verify several compatibilities. In addition, we give estimates on the relative positions of the two lattices $H^i_{\mathrm{DR}}(X/V)$ and $H^i_{\mathrm{cris}}(X_0/W) \otimes V$ in these vector spaces, and explain an explicit example.

The remaining sections of Part I of this paper are devoted to compatibilities and applications. In the third section, for example, we verify in (3.5) that σ_{cris} is compatible with formation of Chern classes of line bundles and show in (3.8) that if $L_0 \in \mathrm{Pic}(X_0)$, then a suitable (explicit) power of L_0 lifts to X iff $\sigma^{-1}_{\mathrm{cris}}(c_1(L_0))$ belongs to $F^1_{\mathrm{Hodge}} H^2_{\mathrm{DR}}(X/V)$. We also explain in (3.12) some conditions under which ramification is required for the lifting of line bundles and prove some analogous results for lifting morphisms of p-divisible groups and abelian varieties (3.15), and K3 surfaces (3.23).

In §4, we study the $W \otimes \mathbb{Q}$-module structure and action of Frobenius on $H^*_{\mathrm{DR}}(X/V) \otimes \mathbb{Q}$ induced by the isomorphism σ_{cris}. To keep track of these in a convenient way, we follow an idea of Deligne and introduce a group called the "crystalline Weil group" and a canonical semilinear action of this group on $H^*_{\mathrm{DR}}(X/V) \otimes \mathbb{Q}$ which summarizes these structures. Combining our results with some results of Messing and Gillet, we find that this entire package of data depends only upon the generic fiber of X/V. In the case of potentially good reduction, an interesting formula ((4.7)) expresses the inertial part of this action in terms of the geometric action of the inertia group on the closed fiber. This last result had in fact already been stated by Messing, in a somewhat different form. We end this section by conjecturing in (4.8) that an absolute Hodge cycle in De Rahm cohomology is invariant under the crystalline Weil group (which would be a consequence of the Hodge conjecture). We are able to verify this conjecture for abelian varieties of CM type with ordinary reduction.

Part I of this paper ends with a technical appendix illustrating the existence of torsion in divided power envelopes. This explains some of the technical difficulties we encounter in the early sections. Moreover, it leads to the apparently pathological appearance of torsion in $H^0_{\mathrm{cris}}(S/W)$ for certain (singular) schemes S, and to a counterexample to the full faithfulness of the Dieudonné crystal associated to p-divisible groups over a general base.

Part II of this paper will be devoted to a further elaboration of the results here. Its main ingredient will be the study of the F-isocrystals arising from families of varieties, enabling us to "analytically continue" F-isocrystals to certain p-adic neighborhoods. We shall especially study the F-isocrystals arising from families of abelian varieties and K3 surfaces, and shall use these techniques to verify Conjecture (4.8) in a very subtle case. For more details, we refer the reader to the introduction of part II.

Here are some conventions that will be in use throughout both parts of this paper.

We use k to denote a perfect field of characteristic $p>0$, $W(k)$ or W to denote its Witt ring, and V for a finite extension of W, i.e. a finite flat W-algebra which is also a discrete valuation ring (DVR). All morphisms of schemes are quasi-compact and quasi-separated. All formal V-schemes will have the p-adic topology, unless otherwise stated.

We shall have to deal systematically with crystalline cohomology in the limit, using the techniques of [3, § 7]. If $S \subseteq T$ is a divided power (PD) thickening and if X is an S-scheme, then $\mathrm{Cris}(X/T)$ will denote the site consisting of the T-PD thickenings of open subsets of X on which p is nilpotent. (This is the site denoted $\mathrm{Cris}(X/\hat{T})$ in [3, § 7].)

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1. The Relative Frobenius Morphism

(1.1) If X is a scheme in characteristic p, we let F_X denote its absolute Frobenius endomorphism. If $f: X \to S$ is a morphism in characteristic p, we have the familiar relative Frobenius diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[dr, "f"'] & X^{(p/S)} \arrow[r, "W_{X/S}"] \arrow[d, "f^{(p/S)}"] & X \arrow[d, "f"] \\ & S \arrow[r, "F_S"] & S. \end{tikzcd}$$

When there seems to be no confusion possible, we shall write $f': X' \to S$ instead of $f^{(p/S)}: X^{(p/S)} \to S$.

Let $\Sigma =: \mathrm{Spec}(\mathbb{Z}_p)$ endowed with its canonical PD-ideal (p). Our object is to study the morphism

(1.1.1) $$F^*_{X/S}: \mathbb{R}f'_{\mathrm{cris}*}\mathcal{O}_{X'/\Sigma} \to \mathbb{R}f_{\mathrm{cris}*}\mathcal{O}_{X/\Sigma}$$

in $(S/\Sigma)_{\mathrm{cris}}$ and the corresponding morphisms on the cohomology sheaves. Actually, we shall only do this indirectly. We shall first study the behavior of the morphism on crystalline cohomology relative to a "p-adic base" T, where T

is a formal scheme (with the p-adic topology), and $S \hookrightarrow T$ is a closed immersion defined by a PD-ideal $(\mathscr{I}, \gamma)$ compatible with the standard divided power structure on $(p) \subseteq W$. Note that if $\mathcal{O}_T$ is p-torsion free, as we shall henceforth assume, this last condition is automatic.

A key example is the following:

(1.2) **Example.** Let k be a perfect field of characteristic p, $W = W(k)$ the ring of Witt vectors with coefficients in k, V a finite extension of W of (arbitrary) ramification index e, and $S = \operatorname{Spec}(V/pV) \cong \operatorname{Spec}(k[t]/(t^e))$. Then we can regard $T = \operatorname{Spf}(V)$ as a PD-thickening of S. Note that if $e > 1$, the absolute Frobenius morphism of S cannot be lifted to an endomorphism of T.

(1.3) **Theorem.** *Suppose that X/S is smooth (of finite type), S is noetherian, and $S \hookrightarrow (T, \mathscr{I}, \delta)$ is a PD-immersion of S in a formal scheme. Let $u_{X/T}\colon (X/T)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ be the canonical projection, and*

$$f_{X/T}\colon (X/T)_{\mathrm{cris}} \xrightarrow{u_{X/T}} X_{\mathrm{Zar}} \xrightarrow{f_{\mathrm{Zar}}} T_{\mathrm{Zar}}$$

be the composed morphism. If $\mathcal{O}_T$ is p-torsion free, the natural map

$$F_{X/T} = : F_{X/S}^*\colon\ \mathbb{R} f'_{X'/T*} \mathcal{O}_{X'/T} \otimes^{\mathbb{L}} \mathbb{Q} \to \mathbb{R} f_{X/T*} \mathcal{O}_{X/T} \otimes^{\mathbb{L}} \mathbb{Q}$$

induced by $F_{X/S}$ is an isomorphism.

Proof. For any abelian sheaf F on X_{Zar}, the presheaf $U \mapsto F(U) \otimes \mathbb{Q}$ is a sheaf, since X_{Zar} is noetherian. This implies that the natural map $(f_{\mathrm{Zar}*} F) \otimes \mathbb{Q} \to f_{\mathrm{Zar}*}(F \otimes \mathbb{Q})$ is an isomorphism, and it follows that the same is true in the derived category: the morphism

$$(\mathbb{R} f_{\mathrm{Zar}*} F) \otimes^{\mathbb{L}} \mathbb{Q} \to \mathbb{R} f_{\mathrm{Zar}*}(F \otimes^{\mathbb{L}} \mathbb{Q})$$

is an isomorphism for any $F \in D(X_{\mathrm{Zar}})$. Thus, since $f_{X/T} = f_{\mathrm{Zar}} \circ u_{X/T}$, it suffices to prove that the arrow

(1.3.1) $$F_{X/S}^*\colon \mathbb{R}\, u_{X'/T*} \mathcal{O}_{X'/T} \otimes^{\mathbb{L}} \mathbb{Q} \to \mathbb{R}\, u_{X/T*} \mathcal{O}_{X/T} \otimes^{\mathbb{L}} \mathbb{Q}$$

is a quasi-isomorphism.

This statement is local on T and on X. We may assume, for instance, that X lifts to a smooth formal scheme Y over T. Suppose for a moment that the absolute Frobenius endomorphism F_S lifts to an endomorphism F_T - necessarily a PD morphism, since $\mathcal{O}_T$ is torsion free - and that F_X lifts to an F_T-endomorphism F_Y of Y. Then if $Y' = : Y \times_{F_T} T$, we get a lifting $F_{Y/T}\colon Y \to Y'$ of the relative Frobenius morphism $F_{X/S}$, and the action $F_{Y/T}^*$ of $F_{Y/T}$ on de Rham cohomology represents the action of $F_{X/S}$ on crystalline cohomology. Now, it is easy to check that in [3, (8.1)–(8.24)], one can replace the p-adic base by any p-torsion free formal scheme, so that these results remain valid in our setting. Then [3, (8.3)] tells us the precise image of $F_{Y/T}^*$, from which we can easily deduce the theorem.

To eliminate the hypothesis that F_S lift to T, we can try to choose a suitable lifting Y' of X' and a lifting $F_{Y/T}$ of $F_{X/S}$ for which [3, (8.3)] is still true. This is not difficult, and the only real difference occurs in the proof of the very first lemma [3, (8.5)], which we must generalize to the following:

(1.4) **Lemma.** *If X/S and T are as above, let Y/T be any smooth lifting of X/S, let $\bar{Y}/\bar{T}$ be its reduction mod p, and let $F_{Y/T}\colon Y\to Y'$ be any lifting of the relative Frobenius map: $F_{\bar{Y}/\bar{T}}\colon \bar{Y}/\bar{T}\to \bar{Y}'/\bar{T}$. Then the action of $F^*_{Y/T}$ on $\Omega^j_{Y'/T}$ is divisible by p^j, and there is a commutative diagram:*

$$\begin{tikzcd}
\Omega^j_{Y'/T} \arrow[r, "\varphi^j"] \arrow[d, "\pi^j"'] & \mathscr{H}^j(F_{\bar{Y}/\bar{T}*}\Omega^{\bullet}_{\bar{Y}/\bar{T}}) \arrow[r, "\sim"] & \mathscr{H}^j(F_{X/S*}\Omega^{\bullet}_{X/S}) \\
\Omega^j_{\bar{Y}'/\bar{T}} \arrow[r, "\sim"] \arrow[ur, "C^{-1}"] & \Omega^j_{X'/S} \arrow[ur, "C^{-1}"] &
\end{tikzcd}$$

where π^j is the natural projection, C^{-1} is the inverse Cartier operator, and $\varphi^j(\omega)$ is the image in $\mathscr{H}^j$ of $p^{-j}F^*_{Y/T}(\omega)$.

Proof. First of all, $\Omega^1_{\bar{Y}'/\bar{T}}=W^*_{\bar{Y}/\bar{T}}(\Omega^1_{\bar{Y}/\bar{T}})$, hence is generated as an $\mathcal{O}_{\bar{Y}'}$-module by elements of the form $W^*_{\bar{Y}/\bar{T}}(d\bar{\alpha})$, where $\bar{\alpha}$ is a local section of $\mathcal{O}_{\bar{Y}}$. For each $\bar{\alpha}$, choose a lifting α' of $W^*_{\bar{Y}/\bar{T}}(\bar{\alpha})$ in $\mathcal{O}_{Y'}$. Then the set of all these $d\alpha'$ generates $\Omega^1_{Y'/T}$ as an $\mathcal{O}_{Y'}$-module, since it does so mod p. Choose a lifting α of $\bar{\alpha}$ to $\mathcal{O}_Y$; then there is a $\beta\in\mathcal{O}_Y$ such that $F^*_{Y/T}(\alpha')=\alpha^p+p\,\beta$. Hence $F^*_{Y/T}(d\alpha')=p\,\alpha^{p-1}\,d\alpha+p\,d\beta$, and in particular $F^*_{Y/T}$ is divisible by p on $\Omega^1_{Y'/T}$. Moreover, $p^{-1}F^*_{Y/T}(d\alpha')=\alpha^{p-1}\,d\alpha+d\beta$, and $C^{-1}(d\bar{\alpha}')=C^{-1}(dW^*_{\bar{Y}/\bar{T}}(\bar{\alpha}))=$ the cohomology class of $\bar{\alpha}^{p-1}\,d\bar{\alpha}$. This proves the lemma with $j=1$, and the general case follows by taking exterior powers. $\square$

We can now proceed exactly as in [3] to obtain the following generalization of [(8.3)].

(1.5) **Proposition.** *With the notations of (1.4), let $N^{\bullet}_{Y/T}$ be the largest subcomplex of $F_{Y/T*}\Omega^{\bullet}_{Y/T}$ such that $N^k_{Y/T}$ is contained in $p^kF_{Y/T*}\Omega^k_{Y/T}$ for all k. Then $F^*_{Y/T}$ factors through a quasi-isomorphism: $\psi_{Y/T}\colon \Omega^{\bullet}_{Y'/T}\to \mathrm{N}^{\bullet}_{Y/T}$.*

Since the inclusion $N^{\bullet}_{Y/T}\subseteq F_{Y/T*}\,\Omega^{\bullet}_{Y/T}$ obviously becomes a quasi-isomorphism when tensored with $\mathbb{Q}$, this proves Theorem (1.3). $\square$

For the purposes of the remainder of this paper, Theorem (1.3) is adequate, and the reader may, if he wishes, proceed to Sect. 2. However, it still remains for us to return to the original problem of studying the morphism (1.1.1) in $(S/\Sigma)_{\mathrm{cris}}$; moreover, it is natural to ask for precise p-adic estimates for $F_{X/T}$ in Theorem (1.3). For these purposes, we shall construct an inverse, up to isogeny, to $F_{X/T}$.

(1.6) **Theorem.** *Under the hypothesis of (1.3), assume that the relative dimension of X/S is less than or equal to n. Then there exists a morphism*

$$V_{X/T}\colon \mathbb{R}f_{X/T*}\mathcal{O}_{X/T}\to \mathbb{R}f'_{X'/T*}\mathcal{O}_{X'/T}$$

such that $V_{X/T}\circ F_{X/T}$ and $F_{X/T}\circ V_{X/T}$ are multiplication by p^n. The morphism $V_{X/T}$ is functorial with respect to X/T: if X_1/S_1 and $S_1\hookrightarrow T_1$ satisfy the same hypothesis *as X/S, $S \hookrightarrow T$, then for any commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "g"'] & S_1 \arrow[r, hook] \arrow[d, "u"] & T_1 \arrow[d, "v"] \\ X \arrow[r, "f"] & S \arrow[r, hook] & T, \end{tikzcd}$$

the following square is commutative:

$$\begin{tikzcd} \mathbb{L}v^*\mathbb{R}f_{X/T*}\mathcal{O}_{X/T} \arrow[r] \arrow[d, "\mathbb{L}v^*(V_{X/T})"'] & \mathbb{R}f_{X_1/T_1*}\mathcal{O}_{X_1/T_1} \arrow[d, "V_{X_1/T_1}"] \\ \mathbb{L}v^*\mathbb{R}f'_{X'/T*}\mathcal{O}_{X'/T} \arrow[r] & \mathbb{R}f'_{X'_1/T_1*}\mathcal{O}_{X'_1/T_1}. \end{tikzcd}$$

Proof. Recall from [3, (8.19)] that the operation of replacing a bounded p-torsion free complex of sheaves $A^\bullet$ by the largest subcomplex $N(A^\bullet)$ such that $N^k \subseteq p^k A^k$ passes over to the derived category. Thus, in the above (global) setting, it makes sense to form $N\mathbb{R}u_{X/T*}\mathcal{O}_{X/T}$, and there is a canonical map: $N\mathbb{R}u_{X/T*}\mathcal{O}_{X/T} \to \mathbb{R}u_{X/T*}\mathcal{O}_{X/T}$. Using the method of [3], one finds easily that (1.5) globalizes to the following result.

(1.7) **Proposition.** *The map $F_{X/T}: \mathbb{R}u_{X'/T*}\mathcal{O}_{X'/T*} \to \mathbb{R}u_{X/T*}\mathcal{O}_{X/T}$ factors naturally through a quasi-isomorphism:*

$$\psi_{X/T}: \mathbb{R}u_{X'/T*}\mathcal{O}_{X'/T} \to N\mathbb{R}u_{X/T*}\mathcal{O}_{X/T}. \quad \square$$

To define $V_{X/T}$, it is necessary to use the calculus of gauges and cogauges [3, (8.7) ff.]. In this language, the functor N is the functor $\mathbb{L}\eta$, where η is the cogauge defined by $\eta(i)=i$. For any $k \geqq 0$, let $\eta_k = \inf(\eta, k)$. There is a natural transformation: $\mathbb{L}\eta \to \mathbb{L}\eta_k$. I claim that for X/T as above, and with $K_{X/T} =: \mathbb{R}u_{X/T*}\mathcal{O}_{X/T}$, the map $\mathbb{L}\eta K_{X/T} \to \mathbb{L}\eta_n K_{X/T}$ is an isomorphism. This can be checked locally, e.g. when X lifts to a smooth formal scheme of relative dimension n. In this case $K_{X/T}$ can be represented by a complex of length n, and the statement is obvious from the definitions.

It is easy to see from the caculus of cogauges that there are natural commutative diagrams:

$$\begin{tikzcd} \mathrm{id} \arrow[rr, "p^n"] \arrow[dr, "v_n"'] & & \mathrm{id} \\ & \mathbb{L}\eta_n \arrow[ur, "i_n"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L}\eta_n \arrow[rr, "p^n"] \arrow[dr, "i_n"'] & & \mathbb{L}\eta_n \\ & \mathrm{id}\ . \arrow[ur, "v_n"'] & \end{tikzcd}$$

Define $V_{X/T}: K_{X/T} \to K_{X'/T}$ to be v_n followed by the inverse of the composite:

$$K_{X'/T} \xrightarrow{\psi} \mathbb{L}\eta K_{X/T} \xrightarrow{\sim} \mathbb{L}\eta_n K_{X/T}.$$

It is clear from the diagrams that this map has the desired properties. The functoriality follows from the functoriality of each of the constructions involved. □

We now return to our original study of the morphism (1.1.1). Unfortunately we shall have to make some restrictive hypothesis on the base S in order for our proof to work. We begin with the following technical result:

(1.8) **Lemma.** *Let Σ'_1 be a perfect scheme in characteristic p, $\Sigma' = \mathrm{Spf}(W(\mathcal{O}_{\Sigma'_1}))$, and let S be a Σ'_1-scheme. If $j: S \hookrightarrow Y$, $j': S \hookrightarrow Y'$ are two closed immersions of S into smooth formal schemes over Σ', and T, T' the formal completions (along (p)) of the divided power neighborhoods of S in Y, Y', then $\mathcal{O}_{T'}$ is p-torsion free if and only if $\mathcal{O}_T$ is p-torsion free.*

Proof. Replacing the immersion $S \hookrightarrow Y'$ by the diagonal immersion, we may assume that there exists a smooth morphism of formal schemes $\pi: Y' \to Y$ such that $j = \pi \circ j'$. Since the assertion is local on S, we can reduce the problem to the two cases: Y' is étale over Y, or Y' is the formal affine line over Y. If $\mathcal{I}$ is the ideal of S in Y, and if Y' is étale over Y, we may assume that the ideal of S in Y' is $\mathcal{I}\mathcal{O}_{Y'}$ and that $\mathcal{O}_Y/\mathcal{I}^n \xrightarrow{\sim} \mathcal{O}_{Y'}/\mathcal{I}^n\mathcal{O}_{Y'}$ for all n. Since the divided power neighborhoods of S in the reductions Y_n, Y'_n of Y, Y' modulo p^n depend only upon the infinitesimal neighborhoods of S in Y_n, Y'_n [3, (3.20.7)], it follows that $T'_n \xrightarrow{\sim} T_n$ for all n. hence $T' \xrightarrow{\sim} T$. On the other hand, if Y' is the formal affine line over Y, we can find a section $s: Y \to Y'$ such that $j' = s \circ j$. If t generates the ideal of s, we get $\mathcal{O}_{Y'} \simeq \mathcal{O}_Y\{t\}$, and the ideal $\mathcal{I}'$ of S in Y' is $(\mathcal{I}, t)$. It is then immediate to check that $\mathcal{O}_{T'}$ is the algebra of formal power series $\sum_{q \in \mathbb{N}} a_q t^{[q]}$ where $a_q \in \mathcal{O}_T$ and $a_q \to 0$ if $q \to \infty$, which proves the lemma. □

(1.9) **Theorem.** *Let Σ'_1 be a perfect scheme in characteristic p, $\Sigma' = \mathrm{Spf}(W(\mathcal{O}_{\Sigma'_1}))$, and let S be a Σ'_1-scheme. Assume that, locally on S, there exists an embedding $S \hookrightarrow Y$ into a Σ'-smooth formal scheme such that the formal completion T of the divided power neighborhood of S in Y is p-torsion free. If $f: X \to S$ is smooth of relative dimension at most n, there exists for all i a canonical homomorphism*

$$V_{X/S}: R^i f_{\mathrm{cris}*}\mathcal{O}_{X/\Sigma} \to R^i f'_{\mathrm{cris}*}\mathcal{O}_{X'/\Sigma}$$

such that $V_{X/S} \circ F^*_{X/S}$ and $F^*_{X/S} \circ V_{X/S}$ are multiplication by p^n. The homomorphism $V_{X/S}$ is functorial in (Σ'_1, S, X).

Proof. Since the morphism $S \to \Sigma'_1$ induces an isomorphism of sites $\mathrm{Cris}(S/\Sigma) \xrightarrow{\sim} \mathrm{Cris}(S/\Sigma')$ [5, 1.1.13], we may replace Σ by Σ'. Let us choose a covering S_α of S and embeddings $S_\alpha \hookrightarrow Y_\alpha$, where Y_α is a smooth formal scheme over Σ'. Let T_α be the completed divided power neighborhood of S_α in Y_α, and $X_\alpha = X \times_S S_\alpha$. The smoothness of Y_α and the universal property of the divided power neighborhoods imply that, for any $(U, T', \delta) \in \mathrm{Cris}(S/\Sigma')$, there exists, locally on T', a Σ'-morphism $h_\alpha: T' \to T_\alpha$. If $X_U = : f^{-1}(U)$, let us observe first that there exists a base changing isomorphism

$$\mathbb{L} h^*_\alpha \mathbb{R} f_{X_\alpha/T_\alpha *} \mathcal{O}_{X_\alpha/T_\alpha} \xrightarrow{\sim} \mathbb{R} f_{X_U/T' *} \mathcal{O}_{X_U/T'}.$$

Indeed, h_α factors through the reduction $T_{\alpha,n}$ of T_α modulo p^n for some n, so that by the usual base changing theorem [3, (7.8)] we may assume that $T' = T_{\alpha,n}$, and the claim follows from [3, (7.24.2)]. (Note that T_α is not noetherian in general, so that the reference is slightly abusive; but one sees easily that, since are using the p-adic topology on $\mathcal{O}_{T_\alpha}$, all results from [3, (7.18)-(7.24)] remain valid except (7.24.3), which we shall not need.) On the other hand, there exists for any sheaf E on $\mathrm{Cris}(X/\Sigma')$ a canonical isomorphism

$$\mathbb{R}f_{\mathrm{cris}*}(E)_{(U,T',\delta)} \xrightarrow{\sim} \mathbb{R}f_{X_U/T'*}(E|_{(X_U/T')_{\mathrm{cris}}});$$

in particular, the value on T' of the morphism $F^*_{X/S}$ can be identified with the pull-back by h_α of the morphism

$$F_{X\alpha/T_\alpha}\colon \mathbb{R}f'_{X'_\alpha/T_\alpha*}\,\mathcal{O}_{X'_\alpha/T_\alpha} \to \mathbb{R}f_{X_\alpha/T_\alpha*}\,\mathcal{O}_{X_\alpha/T_\alpha}.$$

Now since $\mathcal{O}_{T_\alpha}$ is torsion free, Theorem (1.6) defines a morphism

$$V_{X_\alpha/T_\alpha}\colon \mathbb{R}f_{X_\alpha/T_\alpha*}\,\mathcal{O}_{X_\alpha/T_\alpha} \to \mathbb{R}f'_{X'_\alpha/T_\alpha*}\,\mathcal{O}_{X'_\alpha}.$$

We can take its pull-back by h_α, and thus we obtain by the above discussion a morphism

$$(\mathbb{R}f_{\mathrm{cris}*}\,\mathcal{O}_{X/\Sigma})_{(U,T,\delta)} \to (\mathbb{R}f'_{\mathrm{cris}*}\,\mathcal{O}_{X'/\Sigma})_{(U,T,\delta)}.$$

This morphism does not depend locally upon the choice of h_α or on the embedding $S_\alpha \hookrightarrow Y_\alpha$: if $x \in U$ belongs to $S_\alpha \cap S_\beta$, and if (Y_β, h_β) is another choice in a neighborhood of x, let $S_\alpha \cap S_\beta \to Y_\alpha \times_{\Sigma'} Y_\beta$ be the diagonal embedding, $T_{\alpha\beta}$ the corresponding completed divided power neighborhood, $h\colon T \to T_{\alpha\beta}$ a Σ'-PD-morphism extending $U \cap S_\alpha \cap S_\beta \to Y_\alpha \times_{\Sigma'} Y_\beta$. Thanks to (1.8), $\mathcal{O}_{T_{\alpha\beta}}$ is p-torsion free, and $V_{X_{\alpha\beta}/T_{\alpha\beta}}$ is defined. The functoriality assertion in (1.6) implies that

$$h^*_\alpha(V_{X_\alpha/T_\alpha}) = h^*_{\alpha\beta}(V_{X_{\alpha\beta}/T_{\alpha\beta}}) = h^*_\beta(V_{X_\beta/T_\beta}).$$

Thus we can glue the various maps $h^*_\alpha(V_{X_\alpha/T_\alpha})$ on the cohomology sheaves and obtain $V_{X/S}\colon R^i f_{\mathrm{cris}*}\mathcal{O}_{X/\Sigma} \to R^i f_{\mathrm{cris}*}\mathcal{O}_{X'/\Sigma}$. Its functoriality results from the functoriality of $V_{X/T}$ in (1.6). $\square$

(1.10) *Remarks.* (i) The hypothesis of (1.9) is satisfied if S is a complete intersection over Σ' [5, 2.3.3]. On the other hand, we shall show in an appendix that the divided power envelopes have some p-torsion in general, even for singularities as simple as $\mathrm{Spec}(k[X,Y]/(X^2, XY, Y^2))$.

(ii) If $S \hookrightarrow T$ is a PD-thickening such that p is nilpotent, and $u\colon X \to Y$ a finite, locally free morphism between two smooth S-schemes $f\colon X \to S$, $g\colon Y \to S$, one can define a trace morphism

$$u_*\colon \mathbb{R}f_{X/T*}\mathcal{O}_{X/T} \to \mathbb{R}\,g_{Y/T*}\mathcal{O}_{Y/T}$$

such that $u_* \circ u^*$ is multiplication by $\deg(u)$. In the case where $u = F_{X/S}$, this should give another construction of $V_{X/S}$, without any hypothesis on S. However, it is not known in general whether $F^*_{X/S} \circ F_{X/S*}$ is multiplication by $p^{\dim(X)}$.

Another method for constructing $V_{X/S}$ should be given by the formalism of the relative de Rham-Witt complex [Illusie, correspondance], but the construction of the relative de Rham-Witt complex seems itself to require the same hypothesis on the absence of torsion.

(iii) If $f: X \to S$ is smooth and each $R^i f_{\mathrm{cris}*} \mathcal{O}_{X/\Sigma}$ is flat, then it is a crystal on S, and its formation commutes with base change. The relative Frobenius map then induces an F-crystal structure:

$$\Phi_{X/S}: F_S^* R^i f_{\mathrm{cris}*} \mathcal{O}_{X/\Sigma} \to R^i f_{\mathrm{cris}*} \mathcal{O}_{X/\Sigma}.$$

Under the rather restrictive hypotheses of (1.9), we can conclude that there exists a

$$V_{X/S}: R^i f_{\mathrm{cris}*} \mathcal{O}_{X/\Sigma} \to F_S^* R^i f_{\mathrm{cris}*} \mathcal{O}_{X/\Sigma}$$

such that the compositions $V_{X/S} \circ \Phi_{X/S}$ and $\Phi_{X/S} \circ V_{X/S}$ are equal to multiplication by p^i. (Such F-crystals are called "nondegenerate" in [21].) We should also remark that satisfying finiteness results for the values of $R^i f_{\mathrm{cris}*} \mathcal{O}_{X/\Sigma}$ on non-noetherian objects of $\mathrm{Cris}(S/\Sigma)$ are not known, without further restrictive hypotheses (e.g. liftability of X or smoothness of S). $\square$

2. Infinitesimal Deformations

Let us consider the same situation as in (1.3): we denote by $f: X \to S$ a smooth morphism of noetherian schemes in characteristic p, and by $S \hookrightarrow T$ a PD-immersion of S in a formal scheme T such that $\mathcal{O}_T$ is p-torsion free. Let $S_0 \hookrightarrow S$ be a closed subscheme defined by a nilpotent ideal, $X_0 = X \times_S S_0$. We begin by applying (1.3) to show that the crystalline cohomology of X/T can be made "functorial up to isogeny" in X_0; in particular, it depends only, up to isogeny, upon X_0. An important consequence is the comparison theorem between the crystalline cohomology of a smooth scheme over a perfect field and the de Rham cohomology of a formal lifting over an arbitrarily ramified extension of the corresponding ring of Witt vectors.

(2.1) **Theorem.** *Under the previous hypothesis, let $f: X \to S$, $g: Y \to S$ be two smooth morphisms of noetherian schemes, $u: X_0 \to Y_0$ an S-morphism. There exist morphisms*

$$u^*: \mathbb{R} g_{Y/T*}(\mathcal{O}_{Y/T}) \otimes^{\mathbb{L}} \mathbb{Q} \to \mathbb{R} f_{X/T*}(\mathcal{O}_{X/T}) \otimes^{\mathbb{L}} \mathbb{Q}, \tag{2.1.1}$$

$$u^*: \mathbb{R}\Gamma_{\mathrm{cris}}(Y/T) \otimes^{\mathbb{L}} \mathbb{Q} \to \mathbb{R}\Gamma_{\mathrm{cris}}(X/T) \otimes^{\mathbb{L}} \mathbb{Q}, \tag{2.1.2}$$

which are canonical in the following sense: if $h: Z \to S$ is another smooth morphism of noetherian schemes, and $v: Y_0 \to Z_0$ an S_0-morphism, then

$$(v \circ u)^* = u^* \circ v^*. \tag{2.1.3}$$

If u is the reduction to S_0 of $\tilde{u}: X \to Y$, then

$$u^* = \tilde{u}^*. \tag{2.1.4}$$

For each i, we consequently obtain morphisms

$$u^*: R^i g_{Y/T*}(\mathcal{O}_{Y/T}) \otimes \mathbb{Q} \to R^i f_{X/T*}(\mathcal{O}_{X/T}) \otimes \mathbb{Q},$$
$$u^*: H^i_{\mathrm{cris}}(Y/T) \otimes \mathbb{Q} \to H^i_{\mathrm{cris}}(X/T) \otimes \mathbb{Q}.$$

Proof. As we have seen in the proof of (1.3), the noetherian hypothesis implies that (2.1.2) follows from (2.1.1) by taking global sections on T.

Let $X^{(1)}=X'$, $X^{(n)}=X^{(n-1)\prime}$, etc. It follows from (1.3) and induction that for each n we have a canonical isomorphism

(2.1.5) $$F_{X/S}^{(n)*}\colon \mathbb{R}f^{(n)}_{X^{(n)}/T*}(\mathcal{O}_{X^{(n)}/T})\otimes^{\mathbb{L}}\mathbb{Q}\xrightarrow{\sim}\mathbb{R}f_{X/T*}(\mathcal{O}_{X/T})\otimes^{\mathbb{L}}\mathbb{Q}$$

(canonical in the sense that it commutes with the functoriality morphisms induced by morphisms of S-schemes). Let $\mathscr{J}$ be the ideal of S_0 in S, $j\colon S_0\hookrightarrow S$. For large enough n, $\mathscr{J}^{p^n}=0$; hence F_S^n factors through a morphism $\rho^{(n)}\colon S\to S_0$, and the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "F_S^n"] \arrow[dr, "\rho^{(n)}"] & S \\ S_0 \arrow[u, "j"] \arrow[r, "F_{S_0}^n"'] & S_0 \arrow[u, "j"'] \end{tikzcd}$$

Pulling the morphism $u\colon X_0\to Y_0$ back through $\rho^{(n)}$, we get a morphism $\rho^{(n)*}(u)\colon X^{(n)}\to Y^{(n)}$. If $n'\geqq n$, $\rho^{(n')}=\rho^{(n)}\circ F_S^{n'-n}$, and hence $\rho^{(n')*}(u)=F_S^{n'-n*}(\rho^{(n)*}(u))$. This relation, together with the canonicity of $F_S^{n'-n*}$, implies that if we define u^* to be the composed morphism

$$\begin{tikzcd} \mathbb{R}g_{Y/T*}(\mathcal{O}_{Y/T})\otimes^{\mathbb{L}}\mathbb{Q} \arrow[r, "(F_{Y/S}^{(n)*})^{-1}", "\sim"'] \arrow[d, dashed, "u^*"'] & \mathbb{R}g^{(n)}_{Y^{(n)}/T*}(\mathcal{O}_{Y^{(n)}/T})\otimes^{\mathbb{L}}\mathbb{Q} \arrow[d, "\rho^{(n)*}(u)^*"] \\ \mathbb{R}f_{X/T*}(\mathcal{O}_{X/T})\otimes^{\mathbb{L}}\mathbb{Q} & \mathbb{R}f^{(n)}_{X^{(n)}/T*}(\mathcal{O}_{X^{(n)}/T})\otimes^{\mathbb{L}}\mathbb{Q}, \arrow[l, "F_{X/S}^{(n)*}"', "\sim"] \end{tikzcd}$$

u^* does not depend upon the choice of n.

Since $\rho^{(n)*}(v\circ u)=\rho^{(n)*}(v)\circ\rho^{(n)*}(u)$, the relation (2.1.3) follows from the functoriality of crystalline cohomology. The last assertion follows from the definition of u^* and the canonicity of $F_{X/S}^{(n)*}$. □

(2.2) **Corollary.** *Assume that X and X' are two deformations of X_0 over S. Then there exist canonical isomorphisms*

(2.2.1) $$\delta_{X,X'}\colon \mathbb{R}f_{X'/T*}(\mathcal{O}_{X'/T})\otimes^{\mathbb{L}}\mathbb{Q}\xrightarrow{\sim}\mathbb{R}f_{X/T*}(\mathcal{O}_{X/T})\otimes^{\mathbb{L}}\mathbb{Q},$$

(2.2.2) $$\delta_{X,X'}\colon \mathbb{R}\Gamma_{\mathrm{cris}}(X'/T)\otimes^{\mathbb{L}}\mathbb{Q}\xrightarrow{\sim}\mathbb{R}\Gamma_{\mathrm{cris}}(X/T)\otimes^{\mathbb{L}}\mathbb{Q},$$

such that:

(i) *if X'' is a third deformation of X_0 over S, then*

(2.2.3) $$\delta_{X,X''}=\delta_{X,X'}\circ\delta_{X',X''};$$

(ii) *if Y, Y' are two deformations of a smooth S_0-scheme Y_0, and $u\colon X_0\to Y_0$ an S_0-morphism, then*

(2.2.4) $$\delta_{X,X'}\circ u^*=u^*\circ\delta_{Y,Y'}.$$

Proof. The existence of $\delta_{X,X'}$ and the relation (2.2.3) follow from (2.1) applied to $u=v=\mathrm{Id}_{X_0}$. The relation (2.2.4) results from

$$\delta_{X,X'}\circ u^*=\mathrm{Id}_{X_0}^*\circ u^*=(u\circ \mathrm{Id}_{X_0})^*=(\mathrm{Id}_{Y_0}\circ u)^*=u^*\circ\delta_{Y,Y'}. \quad \square$$

(2.3) *Remarks.* (i) It is possible to extend the construction of u^* to the case where S, T vary. Let us consider a commutative diagram

$$\begin{tikzcd} S'_0 \arrow[r, hook] \arrow[d] & S' \arrow[r, hook] \arrow[d] & T' \arrow[d, "w"] \\ S_0 \arrow[r, hook] & S \arrow[r, hook] & T, \end{tikzcd}$$

where $S'_0\hookrightarrow S'\hookrightarrow T'$ satisfy the same hypothesis as $S_0\hookrightarrow S\hookrightarrow T$, and w is a PD-morphism. Let $f\colon X\to S'$, $g\colon Y\to S$ be two smooth morphisms, $X_0=:X\times_{S'}S'_0$, $Y_0=:Y_S S_0$, $Y'=:Y\times_S S'$, $Y'_0=:Y_0\times_{S_0}S'_0\simeq Y'\times_{S'}S'_0$, $g'\colon Y'\to S'$. There exists a functiorality morphism

$$\mathbb{L}w^*\,\mathbb{R}g_{Y/T*}(\mathcal{O}_{Y/T})\otimes^{\mathbb{L}}\mathbb{Q}\to\mathbb{R}g'_{Y'/T'*}(\mathcal{O}_{Y'/T'})\otimes^{\mathbb{L}}\mathbb{Q}. \tag{2.3.1}$$

If $u\colon X_0\to Y_0$ is a morphism over $S'_0\to S_0$, it admits a factorization $u'\colon X_0\to Y'_0$. We can then define

$$u^*\colon \mathbb{L}w^*\,\mathbb{R}g_{Y/T*}(\mathcal{O}_{Y/T})\otimes^{\mathbb{L}}\mathbb{Q}\to\mathbb{R}f_{X/T'*}(\mathcal{O}_{X/T'})\otimes^{\mathbb{L}}\mathbb{Q}$$

to be the composite of u'^* and (2.3.1), and it is easy to check that the properties of (2.1) and (2.2) remain valid.

(ii) At the cost of some notational complexity we can eliminate X and Y from Theorem (2.1) altogether in the following way. If $S_0\hookrightarrow S\subseteq T$ are as in the situation of (2.1) and if $f_0\colon X_0\to S_0$ is smooth, then for each $n\gg 0$ we get an object $D^{(n)}(X_0/T)$ in the derived category of $\mathcal{O}_T\otimes\mathbb{Q}$-modules. Each $D^{(n)}$ is functorial in X_0, and if $n'\geqq n$ there is a natural isomorphism: $D^{(n')}(X_0/T)\to D^{(n)}(X_0/T)$; moreover if $f\colon X\to S$ is a smooth lifting of $f_0\colon X_0\to S_0$, then there is a natural isomorphism

$$D^{(n)}(X_0/T)\to\mathbb{R}f_{X/T*}(\mathcal{O}_{X/T})\otimes^{\mathbb{L}}\mathbb{Q},\quad \text{for every } n\gg 0,$$

compatible with the "transition" isomorphisms above. Indeed, to construct $D^{(n)}$, just note that if $n\gg 0$, we can use diagram (2.1.6) to define a smooth $f^{(n)}\colon X^{(n)}\to S$ by pulling back $f_0\colon X_0\to S$ along $\rho^{(n)}\colon S\to S_0$; we then let $D^{(n)}(X_0/T)=:\mathbb{R}f^{(n)}_{X^{(n)}/T*}(\mathcal{O}_{X^{(n)}/T})\otimes^{\mathbb{L}}\mathbb{Q}$. If $n'\geqq n$, the relative Frobenius map $X^{(n)}\to X^{(n')}$ induces the transition isomorphism: $D^{(n')}\to D^{(n)}$. If $f\colon X\to S$ is a smooth lifting of f_0, (2.1.6) shows that $X^{(n)}$ is just the Frobenius pull-back of X/S, and hence relative Frobenius also induces the isomorphism: $D^{(n)}(X_0/T)\to\mathbb{R}f_{X/T*}(\mathcal{O}_{X/T})\otimes^{\mathbb{L}}\mathbb{Q}$. $\square$

(2.4) **Theorem.** *Let V be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic zero, and let* $\mathbf{X}$ *be a smooth formal V-scheme with special fiber X_0 over k. Then there are* *canonical isomorphisms:*

$$\sigma_{\mathrm{cris}}\colon \mathbb{R}\Gamma(\mathbf{X}, \Omega^{\cdot}_{\mathbf{X}/V}) \otimes^{\mathbb{L}}_{V} K \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{cris}}(X_0/W) \otimes^{\mathbb{L}}_{W} K, \tag{2.4.1}$$

$$\sigma_{\mathrm{cris}}\colon H^i_{\mathrm{DR}}(\mathbf{X}/V) \otimes_V K \xrightarrow{\sim} H^i_{\mathrm{cris}}(X_0/W) \otimes_W K. \tag{2.4.2}$$

Proof. The map $(W,(p),\gamma) \to (V,(p),\gamma)$ is of course a PD-morphism. Let $R =: V/(p)$; R has a unique structure of local artinian k-algebra, with residue field k. If $S =: \operatorname{Spec} R$, $T =: \operatorname{Spf} V$, $\Sigma =: \operatorname{Spf} W$, we can view T as a Σ-PD-thickening of S. Let $\Sigma_n =: \operatorname{Spec} W/p^n W$, $T_n =: \operatorname{Spec} V/p^n V$, $X =: \mathbf{X} \times_T S$, $\bar{X}_0 =: X_0 \times_{\Sigma_1} S$. The base changing theorem for crystalline cohomology [3, (7.8)] implies that there is a natural isomorphism

$$\mathbb{R}\Gamma_{\mathrm{cris}}(X_0/\Sigma_n) \otimes^{\mathbb{L}}_{W/p^nW} V/p^n V \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{cris}}(\bar{X}_0/T_n)$$

for all n. Since V is a flat W-module, we have an isomorphism of functors $\otimes^{\mathbb{L}}_{W/p^nW} V/p^n V \simeq \otimes^{\mathbb{L}}_{W} V$. Passing to the limit, we get an isomorphism:

$$\mathbb{R} \varprojlim (\mathbb{R}\Gamma_{\mathrm{cris}}(X_0/\Sigma_n) \otimes^{\mathbb{L}}_{W} V) \xrightarrow{\sim} \mathbb{R} \varprojlim \mathbb{R}\Gamma_{\mathrm{cris}}(\bar{X}_0/T_n).$$

Since V is a finitely generated free W-module, $\otimes^{\mathbb{L}}_{W} V$ commutes with $\mathbb{R} \varprojlim$ [3, (B2.3)], and so by [3, (7.22.2)] we obtain an isomorphism:

$$\mathbb{R}\Gamma_{\mathrm{cris}}(X_0/\Sigma) \otimes^{\mathbb{L}}_{W} V \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{cris}}(\bar{X}_0/T). \tag{2.4.3}$$

On the other hand, X is a smooth S-scheme and $\mathbf{X}$ is a smooth formal lifting of X to the S-PD-thickening T of S. By [3, (7.4)] and another limit argument, we obtain a canonical isomorphism:

$$\mathbb{R}\Gamma(\mathbf{X}, \Omega^{\cdot}_{\mathbf{X}/T}) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{cris}}(X/T). \tag{2.4.4}$$

Moreover, X and $\bar{X}_0$ are two deformations of X_0, so that (2.2) gives us an isomorphism:

$$\mathbb{R}\Gamma_{\mathrm{cris}}(X/T) \otimes^{\mathbb{L}} \mathbb{Q} \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{cris}}(\bar{X}_0/T) \otimes^{\mathbb{L}} \mathbb{Q}.$$

Thus, if we combine this with the isomorphisms deduced from (2.4.3) and (2.4.4) by tensoring with $\mathbb{Q}$, we obtain the isomorphism of the theorem.

The word "canonical" in the statement means the following. If $h\colon \mathbf{X} \to \mathbf{Y}$ is a morphism of smooth V-schemes, there is a commutative diagram:

$$\begin{array}{ccc} \mathbb{R}\Gamma(\mathbf{Y}, \Omega^{\cdot}_{\mathbf{Y}/V}) \otimes^{\mathbb{L}} K & \xrightarrow{\sim} & \mathbb{R}\Gamma_{\mathrm{cris}}(Y_0/W) \otimes^{\mathbb{L}} K \\ \downarrow{\scriptstyle \mathbb{R}\Gamma_{\mathrm{DR}}(h)} & & \downarrow{\scriptstyle \mathbb{R}\Gamma_{\mathrm{cris}}(h_0)} \\ \mathbb{R}\Gamma(\mathbf{X}, \Omega^{\cdot}_{\mathbf{X}/V}) \otimes^{\mathbb{L}} K & \xrightarrow{\sim} & \mathbb{R}\Gamma_{\mathrm{cris}}(X_0/W) \otimes^{\mathbb{L}} K. \end{array} \tag{2.4.5}$$

This follows from the canonicity of the maps used to construct σ_{cris}. $\square$

Applying Grothendieck's comparison theorem, we get the following corollary:

(2.5) **Corollary.** *If* $\mathbf{X}$ *is a smooth proper V-scheme with special fiber* X_0, *there is a canonical isomorphism:*

$$\sigma_{\mathrm{cris}}\colon H^i_{\mathrm{DR}}(\mathbf{X}/V)\otimes_V K \xrightarrow{\sim} H^i_{\mathrm{cris}}(X_0/W)\otimes_W K. \quad \square$$

(2.6) *Remarks.* (i) Let $X'_n \subseteq \mathbf{X}$ be the closed subscheme defined by $(\pi)^n$. Then for $n \geqq e/(p-1)$, the ideal $(\pi)^n \subseteq V$ has a (unique) PD structure, and we can define the crystalline cohomology $H^*_{\mathrm{cris}}(X'_n/V)$ with respect to this PD structure. In fact, for $n' \geqq n \geqq e/(p-1)$, we have a commutative diagram of canonical isomorphisms

$$\begin{tikzcd} H^*_{\mathrm{cris}}(X'_{n'}/V) \arrow[dr, "\sim"] \arrow[dd, "\wr"] & \\ & H^*_{\mathrm{DR}}(\mathbf{X}/V). \\ H^*_{\mathrm{cris}}(X'_n/V) \arrow[ur, "\sim"'] & \end{tikzcd}$$

Since $X'_e = X$. it follows from our construction of σ_{cris} that for $n \geqq e$, the following diagram commutes:

$$\begin{tikzcd} H^*_{\mathrm{cris}}(X'_n/V)\otimes K \arrow[r, "\sim"] \arrow[d, dash, "\wr"] & H^*_{\mathrm{DR}}(\mathbf{X}/V)\otimes K \arrow[d, dash, "\wr"] \\ H^*_{\mathrm{cris}}(X'_e/V)\otimes K \arrow[r, "\sim"] & H^*_{\mathrm{cris}}(X_0/W)\otimes K. \end{tikzcd}$$

(ii) Under the hypothesis of (2.4), let us assume that there exists a smooth formal W-scheme $\mathbf{Y}$ such that $\mathbf{X} \simeq \mathbf{Y} \otimes_W V$. There exist canonical isomorphisms:

$$\mathbb{R}\Gamma(\mathbf{Y}, \Omega^{\bullet}_{\mathbf{Y}/W}) \simeq \mathbb{R}\Gamma_{\mathrm{cris}}(X_0/W),$$
$$\mathbb{R}\Gamma(\mathbf{X}, \Omega^{\bullet}_{\mathbf{X}/V}) \simeq \mathbb{R}\Gamma(\mathbf{Y}, \Omega^{\bullet}_{\mathbf{Y}/W}) \otimes^{\mathbb{L}}_W V,$$

hence

(2.6.1) $$\mathbb{R}\Gamma(\mathbf{X}, \Omega^{\bullet}_{\mathbf{X}/V}) \simeq \mathbb{R}\Gamma_{\mathrm{cris}}(X_0/W) \otimes^{\mathbb{L}}_W V.$$

Then σ_{cris} is merely deduced from (2.6.1) by tensoring with $\mathbb{Q}$: indeed, with the notations of the proof of (2.4), there exists a canonical isomorphism $X \simeq \overline{X}_0$, which induces $\delta_{X, \overline{X}_0}$ by (2.1.4).

(2.7) **Proposition** (compatibility with base change). *Let* $V \to V'$ *be a local homomorphism of complete discrete valuation rings with residue fields* k, k', *let* $k \to k'$, $W(k) \to W(k')$, *and* $K \to K'$ *be the induced maps; let* $\mathbf{X}$ *be a smooth formal V-scheme and let* $\mathbf{X}' =: \mathbf{X} \times_{\mathrm{Spf}\, V} \mathrm{Spf}\, V'$. *Then the following diagram commutes:*

$$\begin{tikzcd} \mathbb{R}\Gamma(\mathbf{X}, \Omega^{\bullet}_{\mathbf{X}/V}) \otimes^{\mathbb{L}} K \arrow[r] \arrow[d, "\wr"] & \mathbb{R}\Gamma(\mathbf{X}', \Omega^{\bullet}_{\mathbf{X}'/V'}) \otimes^{\mathbb{L}} K' \arrow[d, "\wr"] \\ \mathbb{R}\Gamma_{\mathrm{cris}}(X_0/W(k)) \otimes^{\mathbb{L}} K \arrow[r] & \mathbb{R}\Gamma_{\mathrm{cris}}(X'_0/W(k')) \otimes^{\mathbb{L}} K'. \end{tikzcd}$$

Proof. As above, let $S =: \mathrm{Spec}(V/pV)$, $S' =: \mathrm{Spec}(V'/pV')$, $T =: \mathrm{Spf}\, V$, $T' =: \mathrm{Spf}\, V'$, $X =: \mathbf{X} \times_T S$, $X' =: \mathbf{X}' \times_{T'} S' \simeq X \times_S S'$. Applying (2.3) and (2.2.4) to the projection

$X' \to X$, we get a commutative diagram

$$\begin{array}{ccc} \mathbb{R}\Gamma_{\text{cris}}(X/T) \otimes^{\mathbb{L}} \mathbb{Q} & \longrightarrow & \mathbb{R}\Gamma_{\text{cris}}(X'/T') \otimes^{\mathbb{L}} \mathbb{Q} \\ \wr\downarrow & & \wr\downarrow \\ \mathbb{R}\Gamma_{\text{cris}}(\bar{X}_0/T) \otimes^{\mathbb{L}} \mathbb{Q} & \longrightarrow & \mathbb{R}\Gamma_{\text{cris}}(\bar{X}'_0/T') \otimes^{\mathbb{L}} \mathbb{Q}. \end{array}$$

Therefore, the proposition follows from the naturality of the base change isomorphism, and of the isomorphism $\mathbb{R}\Gamma_{\text{cris}}(X/T) \simeq \mathbb{R}\Gamma(\mathbf{X}, \Omega^{\cdot}_{\mathbf{X}/T})$. □

It is natural to ask how far the isomorphism σ_{cris} is from preserving the lattice structures. Our method, suitably refined, gives some information. Suppose $\mathbf{X}/V$ is a smooth proper V-scheme (or formal scheme), and use σ_{cris} to identify $H^i_{\text{DR}}(\mathbf{X}/V) \otimes K$ with $H^i_{\text{cris}}(X_0/W) \otimes K$. Then $L_{\text{DR}} =: H^i_{\text{DR}}(\mathbf{X}/V)$/torsion and $L_{\text{cris}} =: H^i_{\text{cris}}(X_0/W) \otimes V$/torsion become two V-lattices in the same K-vector space, and it makes sense to ask how far apart they are.

(2.8) **Theorem.** *With the above notations, let l be the smallest integer greater than or equal to $\log_p(e/p-1)$. Then:*

(2.8.1) *If h is the highest Hodge slope of X_0/k in degree i, $p^{lh} L_{\text{cris}} \subseteq L_{\text{DR}}$ and $p^{lh} L_{\text{DR}} \subseteq L_{\text{cris}}$.*

(2.8.2) *If t is the ordinate of the end point of the Hodge polygon of X_0/k in degree i, the lengths of $L_{\text{DR}}/L_{\text{DR}} \cap L_{\text{cris}}$ and of $L_{\text{cris}}/L_{\text{DR}} \cap L_{\text{cris}}$ are bounded by elt.*

Proof. Let n be the smallest integer greater than or equal to $e/p-1$. Then the ideal $(\pi^n) \subseteq V$ has divided powers (c.f. (3.9) below, for instance). Let

$$S =: \operatorname{Spec} V/(\pi^n), \qquad X =: \mathbf{X} \times_{\operatorname{Spec} V} S, \qquad X_0 =: \mathbf{X} \times_{\operatorname{Spec} V} \operatorname{Spec} k,$$

and $\bar{X}_0 =: X_0 \times_{\operatorname{Spec} k} S$. We have canonical isomorphisms:

$$H^i_{\text{DR}}(\mathbf{X}/V) \simeq H^i_{\text{cris}}(X/V), \qquad H^i_{\text{cris}}(X_0/W) \otimes V \simeq H^i_{\text{cris}}(\bar{X}_0/V),$$

just as before. Moreover, S is a scheme in characteristic p, and F^l_S factors through $\operatorname{Spec} k$, so we have a canonical isomorphism: $\bar{X}_0^{(l)} \simeq X^{(l)}$; hence a commutative diagram:

$$\begin{array}{ccccc} H^i_{\text{DR}}(\mathbf{X}/V) \otimes K & \longleftarrow & H^i_{\text{cris}}(X/V) & \xleftarrow{F^{(l)}_{X/V}} & H^i_{\text{cris}}(X^{(l)}/V) \\ \sigma_{\text{cris}} \downarrow \wr & & & & \downarrow \wr \\ H^i_{\text{cris}}(X_0/W) \otimes K & \longleftarrow & H^i_{\text{cris}}(\bar{X}_0/V) & \xleftarrow{F^{(l)}_{\bar{X}_0/V}} & H^i_{\text{cris}}(\bar{X}^{(l)}_0/V). \end{array}$$

Let L' denote the image of $H^i_{\text{cris}}(X^{(l)}/V) \simeq H^i_{\text{cris}}(\bar{X}^{(l)}_0/V)$ in

$$H^i_{\text{DR}}(\mathbf{X}/V) \otimes K \simeq H^i_{\text{cris}}(X_0/W) \otimes K,$$

so that $L' \subseteq L_{\text{DR}} \cap L_{\text{cris}}$. We shall prove that p^{lh} annihilates L_{cris}/L' and L_{DR}/L', and that the length of each of these V-modules is less than or equal to p^{elt}.

Clearly it suffices to prove these results for the cokernels of $F^{(l)}_{X_0/V}$ and $F^{(l)}_{X/V}$ respectively. The map $F^{(l)}_{X_0/V}$ is obtained from $F^{(l)}_{X/W}$ by tensoring with V, and hence the required estimates on its cokernel can easily be deduced from known results about Hodge polygons [3, (8.36)]. We can use the techniques of [3, §8] and the results of §1 to obtain similar estimates for $F^{(l)}_{X/V}$ (and to give a direct proof for $F^{(l)}_{X_0/V}$).

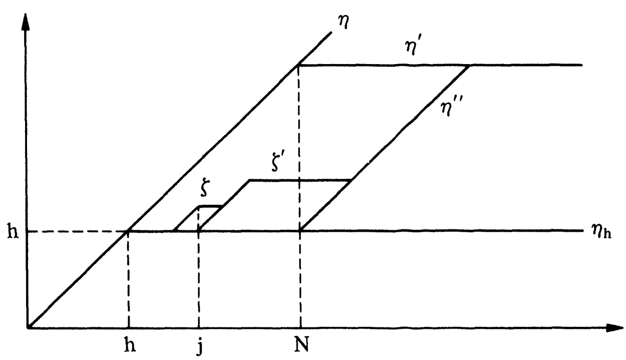
We shall prove that $p^h H^i_{\mathrm{cris}}(X/V)$ is contained in the image of $F_{X/V}$. (It is clear that we can then iterate this result and obtain the estimates necessary for (2.8.1).) We know by (1.7) that, in the derived category, $F_{X/V}$ factors through a quasi-isomorphism

$$\psi_{X/V}\colon \mathbb{R}\, u_{X'/V*}\mathcal{O}_{X'/V} \xrightarrow{\sim} \mathbb{L}\,\eta\,\mathbb{R}\, u_{X/V*}\mathcal{O}_{X/V},$$

where η is the cogauge: $\eta(i)=i$. This reduces us to a problem in the calculus of cogauges (c.f. [3, (8.22) ff.]). Let us abbreviate $\mathbb{R}\, u_{X/V*}\mathcal{O}_{X/V}$ by $K^{\boldsymbol{\cdot}}$; let h be the constant cogauge with value h and $\eta_h =: \mathrm{Inf}(\eta, h)$. We have a commutative diagram:

$$\begin{tikzcd} & \mathbb{L}\, h K^{\boldsymbol{\cdot}} \arrow[d] & K^{\boldsymbol{\cdot}} \arrow[l] \arrow[d, "p^h"] \\ \mathbb{L}\,\eta K^{\boldsymbol{\cdot}} \arrow[r] & \mathbb{L}\,\eta_h K^{\boldsymbol{\cdot}} \arrow[r] & K^{\boldsymbol{\cdot}}. \end{tikzcd}$$

After passing to cohomology, we see that it will suffice to prove that the map $H^i(\mathbb{L}\,\eta K^{\boldsymbol{\cdot}}) \to H^i(\mathbb{L}\,\eta_h K^{\boldsymbol{\cdot}})$ is surjective. It is clear that we may replace η and η_h by the gauges η' and η'' shown below, for $N \gg 0$, without changing anything. One can then pass from η'' to η' by a chain of simple augmentations. If ζ and ζ' lie between η'' and η' and if ζ is a simple augmentation of ζ' at j, then necessarily $j > h$.



We have a triangle [3, (8.23.6)]:

$$0 \to \mathbb{L}\,\zeta \to \mathbb{L}\,\zeta' \to \mathcal{H}^j(K^{\boldsymbol{\cdot}} \otimes \mathbb{F}_p)[-j] \to 0,$$

hence an exact sequence

$$\ldots \to H^i(\mathbb{L}\,\zeta) \to H^i(\mathbb{L}\,\zeta') \to H^{i-j}(\mathcal{H}^j(K^{\boldsymbol{\cdot}} \otimes \mathbb{F}_p)) \to \ldots.$$

Let $S'=:V/pV$ and Y' be the pull-back by $F_{S'}$ of $\mathbf{X}\times S'$. By the Cartier isomorphism, $\mathscr{H}^j(K^\bullet\otimes\mathbb{F}_p)\xrightarrow{\sim}\Omega^j_{Y'/S'}$, and so

$$H^{i-j}(\mathscr{H}^j(K^\bullet\otimes\mathbb{F}_p))\xrightarrow{\sim}H^{i-j}(\Omega^j_{Y'/S'}).$$

Since $j>h$, our hypothesis implies that $H^{i-j}(X'_0,\Omega^j_{X'_0/k})=0$, and it is easy to see that $H^{i-j}(Y',\Omega^j_{Y'/S})$ also vanishes. This implies that each $H^i(\mathbb{L}\,\zeta)\to H^i(\mathbb{L}\,\zeta')$ is surjective, hence also $H^i(\mathbb{L}\,\eta)\to H^i(\mathbb{L}\,\eta_h)$ is surjective, and (2.8.1) follows.

We shall leave the details of the proof of (2.8.2) to the reader. First one checks that the length of $H^q(Y',\Omega^p_{Y'/S'})$ as a V-module is less than or equal to $e\,h^{p,q}$. Then one follows the argument of [3, (8.38)] in the present context, obtaining the estimate as claimed. □

(2.9) *Remark.* Suppose S is a formally smooth formal W-scheme with the p-adic topology and (H,∇,F) is an F-crystal in the sense of Katz [13]. Then, as Katz explains, if θ_1 and θ_2 are two V-valued points of S which are congruent modulo the maximal ideal of V, the connection ∇ induces a canonical isomorphism $\varepsilon(\theta_2,\theta_1)$: $\theta_2^*H\otimes K\xrightarrow{\sim}\theta_1^*H\otimes K$. If $f\colon\mathbf{X}\to S$ is a smooth proper morphism of formal schemes and if the De Rahm cohomology sheaves $R^qf_*\Omega^\bullet_{\mathbf{X}/S}$ are locally free, then they inherit an F-crystal structure from crystalline cohomology, and we find ourselves in the above situation, with $H=R^qf_*\Omega^\bullet_{\mathbf{X}/S}$ and ∇ its Gauss-Manin connection. Suppose θ_1 factors through a W-valued point θ_0 of S, so that we have canonical isomorphisms: $\theta_0^*H\xrightarrow{\sim}H^q_{\mathrm{cris}}(X_0/W)$, $\theta_1^*H\xrightarrow{\sim}H^q_{\mathrm{cris}}(X_0/W)\otimes V$, where X_0 is the reduction of $\theta_i^*\mathbf{X}$ to k. Then if we combine $\varepsilon(\theta_2,\theta_1)$ with these isomorphisms, we construct a commutative diagram

$$\begin{array}{ccc}
K\otimes\theta_2^*H & \xrightarrow[\sim]{\varepsilon(\theta_2,\theta_1)} & K\otimes\theta_1^*H \\
\uparrow\wr & & \downarrow\wr \\
K\otimes H^q_{\mathrm{DR}}(\theta_2^*\mathbf{X}/V) & \longrightarrow & K\otimes H^q_{\mathrm{cris}}(X_0/W).
\end{array}$$

It is straightforward to check that the bottom arrow in the above square is exactly our isomorphism σ_{cris}. This was the method used by Deligne to contruct σ_{cris} in the special case of abelian varities.

(2.10) *Remark.* Using the above remark, it is easy to give an explicit example of σ_{cris} and hence to see that it really does not preserve the lattices L_{DR} and L_{cris}. Our example seems to show that the explicit bounds given in (2.8) are fairly sharp.

Let X_0/k be an ordinary elliptic curve, and let $\mathbf{X}/S$ be its universal formal deformation. From the theory of canonical coordinates [7] we know that there exist a basis $\{\eta,\omega\}$ for $H^1_{\mathrm{DR}}(\mathbf{X}/S)$ and a parameter t for S (so that $S\xrightarrow{\sim}\operatorname{Spf}W[[t]]$) such that $\nabla\eta=0$ and $\nabla\omega=(dt/1+t)\otimes\eta$. If $S(1)$ denotes the p-adic completion of the divided power envelope of the diagonal of S and if $\xi=:1\otimes t-t\otimes 1$, then the isomorphism $\varepsilon\colon p_2^*H\xrightarrow{\sim}p_1^*H$ on $S(1)$ is given by the usual rule:

$$\varepsilon(p_2^*(x))=\sum p_1^*(\nabla(d/dt)^n(x))\otimes\xi^{[n]}.$$

If we define $\theta_1 \in S(V)$ by $t \mapsto 0$ and $\theta_2 \in S(V)$ by $t \mapsto \tau$ (with τ any element of the maximal ideal of V), we easily calculate:

$$\varepsilon(\theta_2^*(\eta)) = \theta_1^*(\eta), \qquad \varepsilon(\theta_2^*(\omega)) = \theta_1^*(\omega) + \log(1+\tau) \cdot \theta_1^*(\eta).$$

If τ is a uniformizer of V and e is the absolute ramification index, $\log(1+\tau)$ need not be integral as soon as $e > p$.

3. Obstructions and the Hodge Filtration

Let k be a perfect field, $W = W(k)$, V a finite totally ramified extension of W, with fraction field K; let $\mathbf{X}$ be a smooth proper formal V-scheme, and let X_0 be its closed fiber. In this section we investigate the relationship between the obstruction to extending data from X_0 to $\mathbf{X}$ and the isomorphism (2.4.2)

$$\sigma_{\mathrm{cris}}: H^*_{\mathrm{DR}}(\mathbf{X}/V) \otimes_V K \to H^*_{\mathrm{cris}}(X_0/W) \otimes_W K.$$

We begin by reviewing the theory of obstructions and Chern classes of line bundles [11, 7.4; 2; 7, 2.2].

(3.1) To remain in the setting of §1, let T be a formal scheme, or a scheme on which p is nilpotent, endowed with a PD ideal $(\mathscr{I}, \gamma)$, and let $f: X \to T$ be a T-scheme on which p is nilpotent and such that γ extends to X (e.g. X is an S-scheme, where $S = V(\mathscr{I})$). There is an exact sequences of sheaves in $(X/T)_{\mathrm{cris}}$:

(3.1.1) $$0 \to \mathscr{I}_{X/T} \to \mathcal{O}_{X/T} \to i_* \mathcal{O}_X \to 0.$$

where $i: X_{\mathrm{Zar}} \to (X/T)_{\mathrm{cris}}$ is the canonical "immersion" [3, (5.19)]. Since $\mathscr{I}_{X/T}$ is a sheaf of nilideals, (3.1.1) has a multiplicative analog

(3.1.2) $$0 \to 1 + \mathscr{I}_{X/T} \to \mathcal{O}^*_{X/T} \to i_* \mathcal{O}^*_X \to 0.$$

Let δ be the canonical PD structure on $\mathscr{I}_{X/T}$, and x a local section of $\mathscr{I}_{X/T}$. Since $\mathscr{I}_{X/T}$ is a p-torsion sheaf, the series

$$\log(1+x) =: \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \, \delta_n(x)$$

is actually a finite sum, and defines a homomorphism of abelian sheaves

$$\log: 1 + \mathscr{I}_{X/T} \to \mathscr{I}_{X/T}.$$

Since i_* is exact, there exist canonical isomorphisms

$$R^1 f_* \mathcal{O}^*_X \simeq R^1 f_{X/T*} i_* \mathcal{O}^*_X, \qquad \mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^*_X) \simeq H^1(X/T, i_* \mathcal{O}^*_X),$$

and the coboundaries associated to (3.1.2) give homomorphisms

$$R^1 f_* \mathcal{O}^*_X \to R^2 f_{X/T*}(1 + \mathscr{I}_{X/T}), \qquad \mathrm{Pic}(X) \to H^2(X/T, 1 + \mathscr{I}_{X/T}),$$

Composing with log and the inclusion $\mathscr{I}_{X/T} \subset \mathcal{O}_{X/T}$, we obtain homomorphisms

$$c_{\mathrm{cris}}: R^1 f_* \mathcal{O}_X^* \to R^2 f_{X/T*} \mathcal{O}_{X/T}, \qquad c_{\mathrm{cris}}: \mathrm{Pic}(X) \to H^2_{\mathrm{cris}}(X/T)$$

which define the crystalline Chern class of a line bundle.

(3.2) Now suppose that $\mathbf{X}/T$ is a smooth formal scheme (resp. a smooth scheme if p is nilpotent on T); let $S =: V(\mathscr{I})$, and $X = \mathbf{X} \times_T S$. On $\mathbf{X}$ we have the multiplicative De Rham complex:

$$\Omega^{\times}_{\mathbf{X}/T}: \mathcal{O}^*_{\mathbf{X}} \xrightarrow{d\log} \Omega^1_{\mathbf{X}/T} \to \Omega^2_{\mathbf{X}/T} \to \ldots,$$

where $d\log(\alpha) = d\alpha/\alpha$. Let $\mathscr{K}^{\times}_{\mathbf{X}/T}$ (resp. $\mathscr{I}^{\bullet}_{\mathbf{X}/T}$) denote the kernel of the obvious map of complexes $\Omega^{\times}_{\mathbf{X}/T} \to \mathcal{O}^*_X$ (resp. $\Omega^{\bullet}_{\mathbf{X}/T} \to \mathcal{O}_X$). Using log in degree zero and the identity maps in higher degrees, we obtain a morphism of complexes:

$$\log^{\bullet}: \mathscr{K}^{\times}_{\mathbf{X}/T} \to \mathscr{I}^{\bullet}_{\mathbf{X}/T}. \tag{3.2.1}$$

The short exact sequences:

$$0 \to \mathscr{K}^{\times}_{\mathbf{X}/T} \to \Omega^{\times}_{X/T} \to \mathcal{O}^*_X \to 0,$$
$$0 \to \mathscr{K}^{0}_{\mathbf{X}/T} \to \mathcal{O}^*_{\mathbf{X}} \to \mathcal{O}^*_X \to 0,$$

give coboundaries which fit in a commutative diagram (as well as a similar diagram for the cohomology):

$$\begin{tikzcd}
R^1 f_* \mathcal{O}^*_X \arrow[r, "\partial"] \arrow[d, equal] & \mathbb{R}^2 f_* \mathscr{K}^{\times}_{\mathbf{X}/T} \arrow[r, "\log^{\bullet}"] \arrow[d] & \mathbb{R}^2 f_* \mathscr{I}^{\bullet}_{\mathbf{X}/T} \arrow[r] \arrow[d] & \mathbb{R}^2 f_* \Omega^{\bullet}_{\mathbf{X}/T} \arrow[d] \\
R^1 f_* \mathcal{O}^*_X \arrow[r, "\partial"] & R^2 f_* \mathscr{K}^{0}_{\mathbf{X}/T} \arrow[r, "\log"] & R^2 f_* \mathscr{I} \mathcal{O}_{\mathbf{X}} \arrow[r, "i"] & R^2 f_* \mathcal{O}_{\mathbf{X}}.
\end{tikzcd} \tag{3.2.2}$$

(3.3) **Lemma.** *Via the canonical isomorphism* $R^2 f_{X/T*} \mathcal{O}_{X/T} \simeq \mathbb{R}^2 f_* \Omega^{\bullet}_{\mathbf{X}/T}$ *(resp.* $H^2_{\mathrm{cris}}(X/T) \simeq H^2_{\mathrm{DR}}(\mathbf{X}/T)$*), the Chern class* $c_{\mathrm{cris}}(L)$ *of a line bundle* L *on* X *corresponds to the image of the class of* L *under the top horizontal arrow in* (3.2.2).

Proof. Let $L(\Omega^{\bullet}_{\mathbf{X}/T})$ be the complex on $\mathrm{Cris}(X/T)$ deduced from $\Omega^{\bullet}_{\mathbf{X}/T}$ by linearization; there exists a canonical homomorphism $\mathcal{O}_{X/T} \to L(\mathcal{O}_{\mathbf{X}})$, and $L(\Omega^{\bullet}_{\mathbf{X}/T})$ is a resolution of $\mathcal{O}_{X/T}$, by the Poincaré lemma [3, (6.12)]. There exists also a surjective homomorphism $L(\mathcal{O}_{\mathbf{X}}) \to i_* \mathcal{O}_X$; let $\mathscr{K}$ be its kernel, which is a PD-ideal in $L(\mathcal{O}_{\mathbf{X}})$ such that $(\mathcal{O}_{X/T}, \mathscr{I}_{X/T}) \to (L(\mathcal{O}_{\mathbf{X}}), \mathscr{K})$ is a PD-morphism. Let $L(\mathcal{O}_{\mathbf{X}})^*$ be the abelian sheaf of invertible sections of $L(\mathcal{O}_{\mathbf{X}})$, and $L(\Omega_{\mathbf{X}})^{\times}$ the complex:

$$L(\mathcal{O}_{\mathbf{X}})^* \xrightarrow{d\log} L(\Omega^1_{\mathbf{X}/T}) \xrightarrow{L(d)} L(\Omega^2_{\mathbf{X}/T}) \to \ldots;$$

let $\mathscr{K}^{\bullet}$, $\mathscr{K}^{\times}$ be the complexes:

$$\mathscr{K} \xrightarrow{L(d)} L(\Omega^1_{\mathbf{X}/T}) \xrightarrow{L(d)} L(\Omega^2_{\mathbf{X}/T}) \to \ldots,$$

$$1 + \mathscr{K} \xrightarrow{d\log} L(\Omega^1_{\mathbf{X}/T}) \xrightarrow{L(d)} L(\Omega^2_{\mathbf{X}/T}) \to \ldots.$$

Since $\mathcal{O}_{X/T} \to L(\mathcal{O}_{\mathbf{X}})$ is a PD-morphism, there exists a commutative diagram:

$$\begin{tikzcd}
1+\mathcal{J}_{X/T} \arrow[r, "\log"] \arrow[d] & \mathcal{J}_{X/T} \arrow[r] \arrow[d] & \mathcal{O}_{X/T} \arrow[d] \\
1+\mathcal{K} \arrow[r, "\log"] & \mathcal{K} \arrow[r] & L(\mathcal{O}_{\mathbf{X}}).
\end{tikzcd}$$

The commutative diagram of exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & 1+\mathcal{J}_{X/T} \arrow[r] \arrow[d] & \mathcal{O}^*_{X/T} \arrow[r] \arrow[d] & i_*\mathcal{O}^*_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}^\times \arrow[r] & L(\Omega_{\mathbf{X}/T})^\times \arrow[r] & i_*\mathcal{O}^*_X \arrow[r] & 0
\end{tikzcd}$$

gives a commutative diagram:

$$\begin{tikzcd}
R^1 f_*\mathcal{O}^*_X \arrow[r, "\partial"] \arrow[d, equal] & R^2 f_{X/T*}(1+\mathcal{J}_{X/T}) \arrow[r, "\log"] \arrow[d] & R^2 f_{X/T*}\mathcal{J}_{X\,T} \arrow[r] \arrow[d, "\wr"'] & R^2 f_{X/T*}\mathcal{O}_{X/T} \arrow[d, "\wr"'] \\
R^1 f_*\mathcal{O}^*_X \arrow[r, "\partial"] & \mathbb{R}^2 f_{X/T*}\mathcal{K}^\times \arrow[r, "\log^\bullet"] & \mathbb{R}^2 f_{X/T*}\mathcal{K}^\bullet \arrow[r] & \mathbb{R}^2 f_{X/T*}L(\Omega^\bullet_{\mathbf{X}/T}),
\end{tikzcd} \tag{3.3.1}$$

where the isomorphisms follow from the Poincaré lemma, and the top horizontal arrow is c_{cris}. On the other hand, the sheaves $i_*\mathcal{O}^*_X$, $L(\Omega^i_{\mathbf{X}/T})$, $\mathcal{K}$ are acyclic for the projection $u_{X/T*}$ on the Zariski topos, and so are $L(\mathcal{O}_{\mathbf{X}})^*$, $1+\mathcal{K}$ (by the same argument as in [3, proof of (7.1)]. Their direct images are respectively $\mathcal{O}^*_X$, $\Omega^i_{\mathbf{X}/T}$, $\mathcal{J}\mathcal{O}_{\mathbf{X}}$, $\mathcal{O}^*_{\mathbf{X}}$, $\mathcal{K}^0_{\mathbf{X}/T}$, and since $f_{X/T} = f \circ u_{X/T}$ we get a commutative diagram

$$\begin{tikzcd}
R^1 f_*\mathcal{O}^*_X \arrow[r, "\partial"] \arrow[d, equal] & \mathbb{R}^2 f_{X/T*}\mathcal{K}^\times \arrow[r, "\log^\bullet"] \arrow[d, "\wr"] & \mathbb{R}^2 f_{X/T*}\mathcal{K}^\bullet \arrow[r] \arrow[d, "\wr"] & \mathbb{R}^2 f_{X/T*}L(\Omega^\bullet_{\mathbf{X}/T}) \arrow[d, "\wr"] \\
R^1 f_*\mathcal{O}^*_X \arrow[r, "\partial"] & \mathbb{R}^2 f_*\mathcal{K}^\times_{\mathbf{X}/T} \arrow[r, "\log^\bullet"] & \mathbb{R}^2 f_*\mathcal{J}^\bullet_{\mathbf{X}/T} \arrow[r] & \mathbb{R}^2 f_*\Omega^\bullet_{\mathbf{X}/T}
\end{tikzcd} \tag{3.3.2}$$

where the bottom horizontal arrow is the top one in (3.2.2). This proves the lemma. □

Since the Chern class in De Rham cohomology c_{DR} is defined by the map $d\log\colon R^1 f_*\mathcal{O}^*_{\mathbf{X}} \to R^2 f_* F^1\Omega_{\mathbf{X}/T}$, where

$$F^i\Omega^\bullet_{\mathbf{X}/T} = : 0 \to \Omega^i_{\mathbf{X}/T} \xrightarrow{d} \Omega^{i+1}_{\mathbf{X}/T} \to \dots$$

with $\Omega^i_{\mathbf{X}/T}$ in degree i, and since this map is equal to the coboundary defined by the exact sequence

$$0 \to F^1\Omega^\bullet_{\mathbf{X}/T} \to \Omega^\times_{\mathbf{X}/T} \to \mathcal{O}^*_{\mathbf{X}} \to 0,$$

the lemma implies the following proposition:

(3.4) **Proposition** [2, Prop. 2.3]. *If $\mathscr{L}\in\mathrm{Pic}(\mathbf{X})$ lifts $L\in\mathrm{Pic}(X)$, then $c_{\mathrm{DR}}(\mathscr{L})$ and $c_{\mathrm{cris}}(L)$ correspond via the isomorphism*

$$\mathbb{R}^2 f_* \Omega^{\cdot}_{\mathbf{X}/T} \simeq R^2 f_{X/T_*} \mathcal{O}_{X/T} \qquad (\text{resp. } H^2_{\mathrm{DR}}(\mathbf{X}/T) \simeq H^2_{\mathrm{cris}}(X/T)).$$

In particular, $c_{\mathrm{cris}}(L)$ maps to zero in $R^2 f_* \mathcal{O}_{\mathbf{X}}$. □

We now check that the functoriality morphisms defined by (2.1) are compatible with Chern classes.

(3.5) **Proposition.** *Under the hypothesis of* (2.1), *let L be a line bundle on Y, L' a line bundle on X, L_0 and L'_0 their reductions on Y_0 and X_0, and let us assume that $L'_0 \simeq u^*(L_0)$. Then $u^*(c_{\mathrm{cris}}(L)\otimes 1) = c_{\mathrm{cris}}(L')\otimes 1$ in $\mathbb{R}^2 f_{X/T*}(\mathcal{O}_{X/T})\otimes\mathbb{Q}$.*

Proof. Let n be such that F^n_S factors through $\rho^{(n)}: S\to S_0$. Then u^* is defined by the commutative square

$$\begin{array}{ccc}
\mathbb{R} g_{Y/T*}(\mathcal{O}_{Y/T}) \overset{\mathbb{L}}{\otimes} \mathbb{Q} & \xrightarrow[\sim]{(F^{(n)*}_{Y/S})^{-1}} & \mathbb{R}\, g^{(n)}_{Y^{(n)}/T*}(\mathcal{O}_{Y^{(n)}/T}) \overset{\mathbb{L}}{\otimes} \mathbb{Q} \\
\Big\downarrow u^* & & \Big\downarrow \rho^{(n)*}(u)^* \\
\mathbb{R} f_{X/T*}(\mathcal{O}_{X/T}) \overset{\mathbb{L}}{\otimes} \mathbb{Q} & \xleftarrow{F^{(n)*}_{X/S}} & \mathbb{R} f^{(n)}_{X^{(n)}/T*}(\mathcal{O}_{X^{(n)}/T}) \overset{\mathbb{L}}{\otimes} \mathbb{Q}.
\end{array}$$

Let $L^{(n)}$, $L'^{(n)}$ be the inverse images of L, L' through the projections $W^{(n)}_{Y/S}: Y^{(n)}\to Y$, $W^{(n)}_{X/S}: X^{(n)}\to X$; we have

$$F^{(n)*}_{Y/S}(L^{(n)}) = F^{(n)*}_{Y/S} W^{(n)*}_{Y/S}(L) \simeq L^{p^n},$$

and similarly for L'. On the other hand, the factorization of F^n_S through S_0 defines similar factorizations for $W^{(n)}_{Y/S}$ and $W^{(n)}_{X/S}$, and the isomorphism $L'_0 \simeq u^*(L_0)$ gives an isomorphism $L'^{(n)} \simeq \rho^{(n)*}(u)^*(L^{(n)})$. The usual functoriality of Chern classes gives therefore

$$\begin{aligned}
p^n u^*(c_{\mathrm{cris}}(L)\otimes 1) &= F^{(n)*}_{X/S} \circ \rho^{(n)*}(u)^*(c_{\mathrm{cris}}(L^{(n)})\otimes 1) \\
&= F^{(n)*}_{X/S}(c_{\mathrm{cris}}(L'^{(n)})) \\
&= p^n c_{\mathrm{cris}}(L'). \quad \square
\end{aligned}$$

(3.6) **Corollary.** *Under the hypothesis of* (2.2), *let L and L' be two line bundles on X and X', such that their reductions L_0 and L'_0 on X_0 are isomorphic. Then*

$$\delta_{X,X'}(c_{\mathrm{cris}}(L')\otimes 1) = c_{\mathrm{cris}}(L)\otimes 1.$$

Proof. Apply (3.5) to $u = \mathrm{Id}_{X_0}$. □

(3.7) **Corollary.** *Under the hypothesis of* (2.4), *let $X\subset\mathbf{X}$ be the closed subscheme defined by the PD-ideal $(p)\subset V$. If $\mathscr{L}$ is a line bundle on $\mathbf{X}$ (resp. L on X) with reduction L_0 on X_0, the isomorphism*

$$\sigma_{\mathrm{cris}}: H^2_{\mathrm{DR}}(\mathbf{X}/V)\otimes_V K \xrightarrow{\sim} H^2_{\mathrm{cris}}(X_0/W)\otimes_W K$$

$$(resp.\quad \sigma'_{\mathrm{cris}}: H^2_{\mathrm{cris}}(X/V)\otimes_V K \simeq H^2_{\mathrm{cris}}(X_0/W)\otimes_W K)$$
takes $c_{\mathrm{DR}}(\mathcal{L})\otimes 1$ (*resp.* $c_{\mathrm{cris}}(L)\otimes 1$) *to* $c_{\mathrm{cris}}(L_0)\otimes 1$.

Proof. Since σ_{cris} is deduced from σ'_{cris} by composition with the canonical isomorphism $H^2_{\mathrm{DR}}(\mathbf{X}/V)\otimes K \simeq H^2_{\mathrm{cris}}(X/V)\otimes K$, the first assertion follows from the second thanks to (3.4), But σ'_{cris} is obtained by composition of (2.4.3) and (2.2.2) (for $X'=\bar{X}_0$), and the corollary follows from the functoriality of c_{cris} and from (3.6). □

(3.7.1) *Remark.* It is clear from the "splitting principle" that the results (3.4) to (3.7) extend to the Chern classes of vector bundles of arbitrary rank [2].

We now are ready for our first obstruction result. We keep the same hypothesis as in (2.4) (i.e., V is a complete discrete valuation ring of unequal characteristics, with perfect residue field k and fraction field K). We denote by $F^i H^*_{\mathrm{DR}}(\mathbf{X}/V) =: \mathrm{Im}(H^*(\mathbf{X}, F^i \Omega^{\cdot}_{\mathbf{X}/V}) \to H^*_{\mathrm{DR}}(\mathbf{X}/V))$ the Hodge filtration on De Rham cohomology. Recall that if $\mathbf{X}$ is a proper formal V-scheme, $H^2(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is a finitely generated V-module, and hence its p-torsion submodule has finite length.

(3.8) **Theorem.** *If* $\mathbf{X}$ *is a smooth proper formal* V*-scheme, let* t *be a nonnegative integer such that the* p*-torsion in* $H^2(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ *is killed by* p^t. *Define* l *as follows:*

$l =:$ *the smallest integer greater than* $\log_p(e/(p-1))$ *if* $p>2$;

$l =:$ *the smallest integer greater than or equal to* $\log_2(e)+1$ *if* $p=2$. *Then if* L_0 *is a line bundle on the special fiber* X_0 *of* $\mathbf{X}$, $L_0^{p^{(t+l)}}$ *lifts to a line bundle on* $\mathbf{X}$ iff $c_{\mathrm{cris}}(L_0) \in H^2_{\mathrm{cris}}(X_0/W)$ *corresponds, via* σ_{cris}, *to an element of* $F^1 H^2_{\mathrm{DR}}(\mathbf{X}/V)\otimes K$.

Proof. Recall that if n is a positive integer and π is a uniformizing parameter for V, $\mathrm{ord}_\pi(\pi^n/n!) = \dfrac{p-1-e}{p-1}n + e\dfrac{\Sigma(n)}{p-1}$, where $\Sigma(n)$ is the sum of the p-adic digits of n. From this one can easily deduce:

(3.9) **Lemma.** *The ideal* $(\pi)^n \subseteq V$ *has divided powers iff* $n \geqq e/(p-1)$, *and the divided powers are* π*-adically nilpotent iff* $n > e/(p-1)$. □

Let n be the smallest integer greater than $e/(p-1)$. Let $X' \subseteq \mathbf{X}$ be the closed subscheme defined by $I =: (\pi)^n$.

(3.10) **Lemma.** *If* L_0 *is any line bundle on* X_0, $L_0^{p^l}$ *extends to a line bundle* L' *on* X'.

Proof. First suppose that $p>2$. Then $p \in (\pi)^n$, so X' is a scheme of characteristic p. The closed subscheme $X_0 \subseteq X'$ is defined by the nilpotent ideal $(\pi)/(\pi^n)\otimes \mathcal{O}_{\mathbf{X}}$, which is generated by the image ε of $\pi\otimes 1$. Since p^l is an integer greater than $e/(p-1)$, $p^l \geqq n$, hence $(F_{X'})^l(\varepsilon)=0$, and there is a map $\rho: X' \to X_0$ such that $\rho\circ \mathrm{inc} = (F_{X_0})^l$. Then $\rho^*(L_0)$ is a line bundle on X' extending $L_0^{p^l}$.

Now suppose $p=2$. In this case the subscheme $X'' \subseteq \mathbf{X}$ defined by $(\pi)^e$ is in characteristic 2, and $2^{l-1} \geqq e$, so the above argument implies that $L_0^{2^{l-1}}$ extends to X''. The embedding $X'' \subseteq X'$ is defined by a square zero ideal I which is killed by multiplication by 2, so standard obstruction theory implies that the cokernel of $\mathrm{Pic}(X') \to \mathrm{Pic}(X'')$ is also killed by 2. Thus, $L_0^{2^l}$ extends to X'. □

Now suppose that $c_{\mathrm{cris}}(L_0)$ lies in $F^1 H^2_{\mathrm{DR}}(\mathbf{X}/V)\otimes K$, and let L' be a line bundle on X' lifting $L_0^{p^l}$. Then $c_{\mathrm{cris}}(L'_0)=p^l c_{\mathrm{cris}}(L_0)$, so it too lies in $F^1 H^2_{\mathrm{DR}}(\mathbf{X}/V)\otimes K$. We have a commutative diagram:

$$\begin{array}{ccc} H^*_{\mathrm{cris}}(X'/V) & \longrightarrow & H^*_{\mathrm{DR}}(\mathbf{X}/V) \\ \downarrow & & \downarrow \\ H^*_{\mathrm{cris}}(X_0/W)\otimes V\otimes K & \xleftarrow{\sigma_{\mathrm{cris}}} & H^*_{\mathrm{DR}}(\mathbf{X}/V)\otimes K. \end{array}$$

It follows that the image η of $c_{\mathrm{cris}}(L')$ via the map:

$$H^2_{\mathrm{cris}}(X'/V)\to H^2_{\mathrm{DR}}(\mathbf{X}/V)\to H^2(\mathbf{X},\mathcal{O}_{\mathbf{X}})$$

is torsion. By the commutativity of (3.2.2) we see that $i\circ\log\circ\partial(L')$ is torsion in $H^2(\mathbf{X},\mathcal{O}_{\mathbf{X}})$. The ideal $\mathcal{I}^0_{\mathbf{X}/T}=I\mathcal{O}_{\mathbf{X}}$ is principal, generated by π^n, and hence we can identify $H^2(\mathcal{I}^0_{\mathbf{X}/T})$ with $\mathcal{O}_{\mathbf{X}}$ and i with multiplication by π^n. Thus, $\log\circ\partial(L')\in H^2(\mathcal{O}_{\mathbf{X}})$ is torsion, hence killed by p^t. If $L''=:(L')^{p^t}$, $\log\circ\partial(L'')$ therefore maps to zero in $H^2(\mathcal{I}^0_{\mathbf{X}/T})$. Since $n>e/(p-1)$, the divided power structure γ on (π^n) is π-adically nilpotent, so the map log: $\mathcal{K}^{\cdot}_{\mathbf{X}/T}\to\mathcal{I}^0_{\mathbf{X}/T}$ is an isomorphism, with inverse exp: $x\mapsto\Sigma\,\gamma_n(x)$. Thus, $\partial(L'')=0$, and hence L'' lifts to an element of $H^1(\mathbf{X},\mathcal{O}^*_{\mathbf{X}})$. □

(3.11) **Corollary.** *With the assumptions of* (3.8), *the p-torsion part of the cokernel of the map* $\mathrm{Pic}(\mathbf{X})\to\mathrm{Pic}(X_0)$ *is killed by* p^{l+t}. □

By way of contrast, we offer a result showing that in some cases ramification is required to lift line bundles. Here we assume the residue field k is algebraically closed.

(3.12) **Proposition.** *Suppose* $\mathbf{X}/W$ *is smooth and proper with torsion free Hodge groups, and suppose that its Hodge to De Rham spectral sequence degenerates at* E_1. *Let* L_0 *be a line bundle on* X_0 *which is not a* p^{th} *power but whose Hodge Chern class in* $H^1(X_0,\Omega^1_{X_0/k})$ *vanishes, and suppose in addition that one of the following holds:*

a) X_0 *is a surface and* $\mathrm{ord}_p(L_0\cdot L_0)=1$.

b) $(F^{\cdot}H^2_{\mathrm{DR}}(\mathbf{X}/W),F^*_{X_0})$ *is strongly divisible, i.e.,* $F^*_{X_0}$ *acting on* $H^2_{\mathrm{cris}}(X_0/W)\cong H^2_{\mathrm{DR}}(\mathbf{X}/W)$ *is divisible by* p^2 *on* F^2_{Hodge}. *(Note: this is automatic if* $p\neq 2$.)

Then no non trivial power of L_0 *lifts to* $\mathbf{X}$.

Proof. Suppose on the contrary that L_0^n lifts to $\mathbf{X}$. Then $\sigma^{-1}_{\mathrm{cris}}[n\,c_{\mathrm{cris}}(L_0)]\in H^2_{\mathrm{cris}}(X_0/W)$ lies in $F^1_{\mathrm{Hodge}}H^2_{\mathrm{DR}}(\mathbf{X}/W)$, and since $H^2(\mathbf{X},\mathcal{O}_{\mathbf{X}})$ is torsion free, the same is true of $c_{\mathrm{cris}}(L_0)$. On the other hand, our assumption that the Hodge Chern class of L_0 vanishes implies that the image of $c_{\mathrm{cris}}(L_0)\bmod p$ lies in $F^2_{\mathrm{Hodge}}H^2_{\mathrm{DR}}(X_0/k)$, so $\sigma^{-1}_{\mathrm{cris}}[c_{\mathrm{cris}}(L_0)]=\xi+p\,\eta$, where $\xi\in F^2_{\mathrm{Hodge}}H^2_{\mathrm{DR}}(\mathbf{X}/W)$. Since $\sigma^{-1}_{\mathrm{cris}}[c_{\mathrm{cris}}(L_0)]\in F^1_{\mathrm{Hodge}}H^2_{\mathrm{DR}}(\mathbf{X}/W)$ and since $H^2(\mathbf{X},\mathcal{O}_{\mathbf{X}})$ is torsion free, $\eta\in F^1_{\mathrm{Hodge}}H^2_{\mathrm{DR}}(\mathbf{X}/W)$. Hence $(L_0\cdot L_0)=(\xi+p\,\eta\cdot\xi+p\,\eta)=p^2(\eta\cdot\eta)$, which contradicts a). Also $F^*_{X_0}c_{\mathrm{cris}}(L_0)=p\,c_{\mathrm{cris}}(L_0)$, so $F^*_{X_0}(\xi)+p\,F^*_{X_0}(\eta)=p(\xi+p\,\eta)$. Since $F^*_{X_0}(\xi)\in p^2H^2$ and $F^*_{X_0}(\eta)\in pH^2$, we see that

$p(\xi+p\eta)\in p^2H^2$, and hence $c_{\mathrm{cris}}(L_0)\in pH^2_{\mathrm{cris}}(X_0/W)$. By [18, (1.5)], this contradicts our assumption that L_0 is not a p^{th} power. □

(3.12.1) *Remark.* Situations as in the above proposition do arise in nature. For example let E be a supersingular elliptic curve and let $X_0=E\times E$. Then $H^1_{\mathrm{cris}}(X_0/W)$ has a basis $\{\omega_1,\omega_2,\eta_1,\eta_2\}$ which is transformed by $F^*_{X_0}$ into $\{p\eta_1,p\eta_2,\omega_1,\omega_2\}$. The map $c_{\mathrm{cris}}\colon \mathrm{NS}(X_0)\otimes\mathbb{Z}_p\to H^2_{\mathrm{cris}}(X_0/W)(1)^{F^*_{X_0}}$ is an isomorphism, and hence there is an element l_0 in $\mathrm{NS}(X_0)\otimes\mathbb{Z}_p$ such that $c_{\mathrm{cris}}(l_0)=\omega_1\wedge\omega_2+p\eta_1\wedge\eta_2$. Let L_0 be an element in $\mathrm{NS}(X_0)$ with the same image in $\mathrm{NS}(X_0)\otimes\mathbb{F}_p$ as l_0. Then it is clear that $c_{\mathrm{cris}}(L_0)$ maps to the image of $\omega_1\wedge\omega_2$ in $F^2H^2_{\mathrm{DR}}(X_0/k)$. Moreover, since it is easy to see that every lifting X of X_0 satisfies the strong divisibility property b), we see that there is no lifting X of X_0 to W to which any power of L_0 extends. On the other hand, one can choose liftings of X_0 defined over a ramified extension of W to which L_0 lifts - c.f. [18, 2.3] and [16].

(3.12.2) *Remark.* Let us also point out that once a lifting $\mathbf{X}$ of X_0 is given over V, and if V' is an extension of V and a power of some $L_0\in\mathrm{Pic}(X_0)$ lifts to $\mathbf{X}\times_{\mathrm{Spec}\,V}\mathrm{Spec}\,V'$, then already a power of L_0 lifts to $\mathbf{X}$. This follows immediately from (3.8).

(3.13) We shall now give a result similar to (3.8) for homomorphisms of abelian schemes. We can prove it for p-divisible groups as well, provided we extend (2.4) to p-divisible groups as follows.

Let $S=\mathrm{Spec}(V/pV)$, $S_0=\mathrm{Spec}(k)$, $\Sigma=\mathrm{Spec}(W)$, $T=\mathrm{Spec}(V)$, $\Sigma_n=\mathrm{Spec}(W/p^nW)$, $T_n=\mathrm{Spec}(V/p^nV)$ (so that $S_0=\Sigma_1$, $S=T_1$). If $\mathbf{G}$ is a p-divisible group on V, with reductions G on S and G_0 and S_0, the Dieudonné crystal $\mathbb{D}(\mathbf{G})$ of $\mathbf{G}$ is by definition the Dieudonné crystal $\mathbb{D}(G)$ of G [15; 5, 3.3.6]. Let

$$D(G_0)=:\Gamma(S_0/\Sigma,\mathbb{D}(G_0))\simeq\varprojlim_n\mathbb{D}(G_0)_{(S_0,\Sigma_n)},$$

$$D(\mathbf{G})=:\Gamma(S/T,\mathbb{D}(\mathbf{G}))\simeq\varprojlim_n\mathbb{D}(G)_{(S,T_n)}.$$

(Recall that $D(G_0)$ is semi-linearly isomorphic to the usual Dieudonné module $M(G_0)$.)

(3.14) **Proposition.** *With the above notations, there exists a canonical isomorphism*

$$\sigma_{\mathrm{cris}}\colon D(\mathbf{G})\otimes_V K\xrightarrow{\sim}D(G_0)\otimes_W K.$$

Proof. Let $\bar{G}_0$ be the pull-back of G_0 through the natural projection $S\to S_0$. Because the Dieudonné crystal commutes with base change, we have a canonical isomorphism

$$D(\bar{G}_0)\simeq D(G_0)\otimes_W V.$$

On the other hand, if n is such that F^n_S factors through S_0, we get an isomorphism $\bar{G}_0^{(n)}\xrightarrow{\sim}G^{(n)}$. We can then define an isomorphism $D(G)\otimes K\xrightarrow{\sim}D(\bar{G}_0)\otimes K$ by the commutative square

$$\begin{array}{ccc} D(G)\otimes K & \xrightarrow[\sim]{F_G^{(n)-1}} & D(G^{(n)})\otimes K \\ \big\downarrow \wr & & \big\downarrow \wr \\ D(\bar{G}_0)\otimes K & \xleftarrow[\sim]{F_{\bar{G}_0}^{(n)}} & D(G_0^{(n)})\otimes K, \end{array}$$

where $F_G^{(n)}$ and $F_{G_0}^{(n)}$ are isomorphisms because of the existence of the Verschiebung, and the proposition follows. □

(3.15) **Theorem.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be formal abelian schemes (resp. p-divisible groups) over V, and* $f_0: X_0 \to Y_0$ *a morphism. Use* σ_{cris} *to transport the Hodge filtration on De Rham cohomology (resp. Dieudonné crystals) of* $\mathbf{X}$ *and* $\mathbf{Y}$ *to crystalline cohomology (resp. Dieudonné crystals) of* X_0 *and* Y_0*, and suppose that* $H^1_{\text{cris}}(f_0)\otimes \mathrm{Id}_K$ *(resp.* $D(f_0)\otimes \mathrm{Id}_K$*) preserves the Hodge filtrations. Then if l is defined as in* (3.8), $p^l f_0$ *lifts to a morphism* $\mathbf{X}\to\mathbf{Y}$.

Note. Messing has pointed out to us that this result can be obtained fairly directly from the methods of his thesis, and, except for the explicit bound on l, has been known for a long time.

Proof. The proofs for abelian schemes and p-divisible groups are identical. We choose to explain the case of abelian schemes, although it follows in fact from the case of p-divisible groups by the Serre-Tate theorem [22, § 5, Th. 4].

Recall that if X, Y are smooth schemes over a scheme S, with reductions X_0, Y_0 over a subscheme $S_0\subset S$ defined by a square zero ideal $\mathscr{I}$, the obstruction to extending an S_0-morphism $f_0: X_0\to Y_0$ to an S-morphism $f: X\to Y$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^*\,\Omega^1_{Y_0/S_0}, \mathscr{I}\mathcal{O}_{X_0}) \simeq H^1(X_0, \mathscr{I}\otimes f_0^*\,\Omega^{1\,\vee}_{Y_0/S_0})$. If X, Y are abelian schemes, this obstruction is additive in f_0: this results immediately from its functoriality properties and the fact that $\Omega^1_{Y_0/S_0}\simeq \omega_{Y_0}\otimes_{\mathcal{O}_{S_0}}\mathcal{O}_{Y_0}$, where ω_{Y_0} is the sheaf of translation invariant differential forms (on which the addition of Y_0 acts as the diagonal map).

Returning to our situation, let n be the smallest integer greater than $e/(p-1)$, and let $X'\subset\mathbf{X}$, $Y'\subset\mathbf{Y}$ be the closed subschemes defined by (π^n).

(3.16) **Lemma.** *If* $f_0: X_0\to Y_0$ *is any homomorphism,* $p^l f_0$ *lifts to a homomorphism* $X'\to Y'$.

Proof. If $p\neq 2$, let $S'' =: V(\pi^n)$, $X'' =: X'$, $Y'' =: Y'$, and $m =: l$. If $p=2$, let $S'' =: S$, $X'' =: X$, $Y'' =: Y$, and $m =: l-1$. It is enough to show in each case that $p^m f_0$ lifts to a map $X''\to Y''$ (since when $p=2$ the obstruction to lift $p^m f_0$ to a map $X'\to Y'$ is killed by p).

We have relative Frobenius morphisms $F^{(m)}_{X''/S''}: X''\to X''^{(m)}$, $F^{(m)}_{Y''/S''}: Y''\to Y''^{(m)}$; let $V^{(m)}_{Y''/S''}: Y''^{(m)}\to Y''$ be the iterated Verschiebung. Since the morphism $F^m_{S''}: S''\to S''$ factors through $\rho^{(m)}: S''\to S_0$, $f_0^{(m)}$ extends to a map $g: X''^{(m)}\to Y''^{(m)}$. Let $f'' = V^{(m)}_{Y''/S''}\circ g\circ F^{(m)}_{X''/S''}: X''\to Y''$. Then
$f_0'' = V^{(m)}_{Y_0/S_0}\circ f_0^{(m)}\circ F^{(m)}_{X_0/S_0} = p^m f_0$. □

Now if $f': X'\to Y'$ lifts $p^l f_0$, we see that $H^1_{\text{cris}}(f')$: $H^1_{\text{cris}}(Y'/V)\to H^1_{\text{cris}}(X'/V)$ induces $p^l H^1_{\text{cris}}(f_0)$: $H^1_{\text{DR}}(\mathbf{Y}/V)\otimes\mathbb{Q}\to H^1_{\text{DR}}(\mathbf{X}/V)\otimes\mathbb{Q}$, and since $H^1(\mathbf{X},\mathcal{O}_{\mathbf{X}})$ is torsion free and $H^1_{\text{cris}}(f_0)$ preserves the Hodge filtration, so does $H^1_{\text{cris}}(f')$. We have chosen n large enough so that the divided power structure on $(\pi)^n$ is p-adically nilpotent. This implies that f' extends to a morphism $\mathbf{X} \to \mathbf{Y}$. To see this, it is enough to construct a compatible family of maps $f_i: X'_i \to Y'_i$ for all $i \geqq n$, where X'_i and Y'_i are defined by $(\pi)^i$, such that $f_n = f'$. But the ideal $(\pi^n)/(\pi^i) \subseteq V/(\pi^i) := V_i$ has *nilpotent* divided powers, and $H^1_{\mathrm{cris}}(f_n)$: $H^1_{\mathrm{cris}}(Y'_n/V_i) \to H^1_{\mathrm{cris}}(X'_n/V_i)$ preserves the Hodge filtrations defined by the liftings X'_i and Y'_i. Thus, the existence of f_i is guaranteed by the following result of Messing [15], which ends the proof of (3.15):

(3.17) **Theorem.** *If X and Y are abelian schemes (resp. p-divisible groups) defined over a nilpotent* PD *thickening $S \hookrightarrow T$ and if $f_0: X_0 \to Y_0$ is a homomorphism between their reductions to S such that $H^1_{\mathrm{cris}}(f_0)$ preserves the Hodge filtrations, then f_0 lifts uniquely to homomorphism $X \to Y$.*

Messing's proof is based on group theory, so it seems worthwhile in the case of abelian schemes to give a proof based on crystalline cohomology and deformation theory; this method has the advantage of being applicable to several other situations, c.f. (3.23). (The principles of this proof are also well known, but do not appear in the literature in any explicit form.) Prior to doing so, let us give an example showing that the value of l in (3.15) is the best possible; this example is an elaboration of a related one due to Serre [14, §1, e)].

(3.18) **Example** ("quasi canonical liftings"). Let X_0 be an ordinary elliptic curve over k, let X be its canonical lifting to W, and let $X^{(n)}$ be its pull back by $F^n_W: W \to W$; let $F^{(n)}: X \to X^{(n)}$ and $V^{(n)}: X^{(n)} \to X$ be the canonical liftings of the iterated Frobenius and Verschiebung. The group scheme $\mathrm{Ker}(p^n)$ on $X^{(n)}$ is canonically isomorphic to $\mathbb{Z}/p^n\mathbb{Z} \times \mu_{p^n}$, and $\mathbb{Z}/p^n\mathbb{Z} \times 0 = \mathrm{Ker}(V^{(n)})$, so $X \cong X^{(n)}/(\mathbb{Z}/p^n\mathbb{Z} \times 0)$.

Let V be a finite extension of W containing the $p^{n^{\text{th}}}$ roots of unity with $e = (p-1)p^{n-1}$. If ζ is a primitive $p^{n^{\text{th}}}$ root of unity, $\zeta \in \mu_{p^n}(V)$ defines a map of group schemes over V, $\zeta: \mathbb{Z}/p^n\mathbb{Z} \to \mu_{p^n}$, which specializes to the zero map over k. The graph of ζ is a subgroupscheme Γ of $\mathbb{Z}/p^n\mathbb{Z} \times \mu_{p^n} \cong {}_{p^n}X^{(n)}$, flat over V. Let $Y = X^{(n)}/\Gamma$. We have a diagram:

$$\begin{array}{ccccc} 0 \to (\mathbb{Z}/p^n\mathbb{Z}, 0) \to & X^{(n)} & \xrightarrow{V(n)} & X \to 0 \\ & \| & & \\ 0 \to \Gamma \to & X^{(n)} & \xrightarrow{\mathrm{proj}} & Y \to 0. \end{array}$$

Over k, Γ becomes $(\mathbb{Z}/p^n\mathbb{Z}, 0)$, so we can find an isomorphism $f_0: X_0 \to Y_0$ such that $f_0 \circ V_0^{(n)} = \mathrm{proj}_0$. Clearly $p^n f_0$ lifts to a map $X \to Y$. On the other hand, if $g: X \to Y$ lifted $p^{n-1} f_0$, then we would have: $(g \circ V^{(n)})_0 = p^{n-1} f_0 \circ V_0^{(n)} = p^{n-1}\,\mathrm{proj}_0$, so by rigidity $g \circ V^{(n)} = p^{n-1} \circ \mathrm{proj} = \mathrm{proj} \circ p^{n-1}$. Then $\mathrm{Ker}(V^{(n)})$ would be killed by $\mathrm{proj} \circ p^{n-1}$, and hence $p^{n-1}(\mathrm{Ker}\, V^{(n)}) = (p^{n-1})\mathbb{Z}/p^n\mathbb{Z} \times 0$ would be contained in Γ. But $((p^{n-1}), 0)$ is not in Γ, since ζ is a primitive $p^{n^{\text{th}}}$ root of unity. $\square$

(3.19) Our proof of (3.17) will result from a relation between the obstruction to extending f_0 and its action on the Hodge filtrations of X and Y. Let us consider the following general situation. Suppose $S \hookrightarrow T$ is a PD immersion of affine schemes defined by the PD ideal (I, γ), and suppose X and Y are smooth T-schemes with reductions X_0 and Y_0 to S. If $f_0: X_0 \to Y_0$ is a morphism, it induces a map $H^*_{\mathrm{cris}}(f_0)$: $H^i_{\mathrm{cris}}(Y_0/T) \to H^i_{\mathrm{cris}}(X_0/T)$. Using the canonical isomorphism between crystalline and DeRham cohomology, we can view this as a map: $H^i_{\mathrm{cris}}(f_0)$: $H^i_{\mathrm{DR}}(Y/T) \to H^i_{\mathrm{DR}}(X/T)$. (Warning: this map *depends* on the PD structure γ.)

Let us assume that the Hodge spectral sequences of X/T and Y/T degenerate at E_1 and are locally free, so that the Hodge and DeRham cohomology sheaves commute with base change. Then the map $H^i_{\mathrm{cris}}(f_0) \otimes \mathrm{id}_{\mathcal{O}_S}$ can be identified with $H^i_{\mathrm{DR}}(f_0)$, and hence it preserves the Hodge filtrations. It follows that $H^i_{\mathrm{cris}}(f_0)$ induces a map:
$F^1_{\mathrm{Hodge}} H^i_{\mathrm{DR}}(Y/T) \to \mathrm{gr}^0_F H^i_{\mathrm{DR}}(X/T) \otimes I = H^i(X, \mathcal{O}_X) \otimes I$. Assume also that I is a square zero ideal: then this map factors through a map:

$$\rho(f_0, \gamma): F^1 H^i_{\mathrm{DR}}(Y_0/S) \to H^i(X_0, \mathcal{O}_{X_0}) \otimes I.$$

On the other hand, the obstruction $\mathrm{ob}(f_0)$ to extending f_0 to an S-morphism $X \to Y$ is an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* \Omega^1_{Y_0/S}, \mathcal{O}_{X_0} \otimes I)$, and hence defines by cup-product a map:

$$\mathrm{ob}(f_0) \cup : H^{i-1}(Y_0, \Omega^1_{Y_0/S}) \to H^i(X_0, \mathcal{O}_{X_0}) \otimes I.$$

Consider the following diagram:

$$\begin{tikzcd} F^1 H^i_{\mathrm{DR}}(Y_0/S) \arrow[r, "\rho(f_0,\gamma)"] \arrow[d, "\mathrm{proj}"'] & H^i(X_0, \mathcal{O}_{X_0}) \otimes I \arrow[d, "\mathrm{proj}"] \\ H^{i-1}(Y_0, \Omega^1_{Y_0/S}) \arrow[r] \arrow[ur, "\mathrm{ob}(f_0)"] & H^i(X_0, \mathcal{O}_{X_0}) \otimes I/I^{[2]}. \end{tikzcd} \tag{3.19.1}$$

(3.20) **Proposition.** *After composition with* proj, *the above diagram becomes commutative. That is,* $\mathrm{proj} \circ \rho(f_0, \gamma) = \mathrm{proj} \circ \mathrm{ob}(f_0) \circ \mathrm{proj}$.

Proof. Recall that the morphism of ringed topoi $f_{0\,\mathrm{cris}}: (X_0/T)_{\mathrm{cris}} \to (Y_0/T)_{\mathrm{cris}}$ induces a map: $f^{-1}_{0\,\mathrm{cris}}(\mathcal{J}^{[k]}_{Y_0/T}) \to \mathcal{J}^{[k]}_{X_0/T}$ for all k, and that, thanks to [3, (7.2.1)] and our hypothesis, on the Hodge spectral sequence,
$H^i_{\mathrm{cris}}(X_0/T, \mathcal{J}^{[k]}_{X_0/T}) \cong \sum_{a+b=k} I^{[a]} F^b_X H^i_{\mathrm{DR}}(X/T)$, where $F^{\bullet}_X$ is the Hodge filtration of X/T. Moreover, the inclusion i: $X_0 \hookrightarrow X$ induces an exact functor $i_{\mathrm{cris}*}$, and $i_{\mathrm{cris}*} \mathcal{O}_{X_0/T} \cong \mathcal{O}_{X/T}$, so that we have an isomorphism:
$H^i_{\mathrm{cris}}(X/T, \mathcal{O}_{X/T}) \to H^i_{\mathrm{cris}}(X_0/T, \mathcal{O}_{X_0/T})$, and similarly for $Y_0 \subseteq Y$. Furthermore, there is an exact sequence: $0 \to \mathcal{J}_{X/T} \to i_{\mathrm{cris}*} \mathcal{J}_{X_0/T} \to i_{X/T*}(I \otimes \mathcal{O}_X) \to 0$, and the following diagram is commutative:

$$\begin{tikzcd} H^i(Y/T, \mathcal{J}_{Y/T}) \arrow[d] & F^1_Y H^i_{\mathrm{DR}}(Y/T) \arrow[l, "\cong"'] \arrow[r, "\rho(f_0,\gamma)"] & H^i(X, \mathcal{O}_X) \otimes I \arrow[d, "\wr"] \\ H^i(Y/T, i_* \mathcal{J}_{Y_0/T}) \arrow[r] \arrow[d] & H^i(X/T, i_* \mathcal{J}_{X_0/T}) \arrow[r] \arrow[d] & H^i(X, \mathcal{O}_X \otimes I) \arrow[d] \\ H^i(Y/T, \mathcal{O}_{Y/T}) \arrow[r, "f_0^*"] & H^i(X/T, \mathcal{O}_{X/T}) \arrow[r] & H^i(X, \mathcal{O}_X). \end{tikzcd}$$

This makes it clear that the map $\rho(f_0,\gamma)$ is induced by the following local map:

$$\begin{aligned}\mathbb{R}u_{Y/T*}\mathscr{I}_{Y/T} &\to \mathbb{R}u_{Y/T*}i_*\mathscr{I}_{Y_0/T} \cong \mathbb{R}u_{Y_0/T*}\mathscr{I}_{Y_0/T} \to \mathbb{R}f_{0*}\mathbb{R}u_{X_0/T*}\mathscr{I}_{X_0/T}\\ &\cong \mathbb{R}f_{0*}\mathbb{R}u_{X/T*}i_*\mathscr{I}_{X_0/T} \to \mathbb{R}f_{0*}\mathbb{R}u_{X/T*}i_{X/T*}(\mathscr{O}_X\otimes I) = \mathbb{R}f_{0*}(\mathscr{O}_X\otimes I).\end{aligned}$$

It is clear that there is a commutative diagram:

$$\begin{array}{ccc} \mathscr{I}_{Y_0/T} & \longrightarrow & \mathscr{I}_{Y_0/T}/\mathscr{I}^{[2]}_{Y_0/T} \\ \downarrow & & \downarrow \\ f_{0\,\mathrm{cris}*}\mathscr{I}_{X_0/T} & \longrightarrow & f_{0\,\mathrm{cris}*}\mathscr{I}_{X_0/T}/\mathscr{I}^{[2]}_{X_0/T}, \end{array}$$

and that the map $i_{\mathrm{cris}*}\mathscr{I}_{X_0/T} \to i_{X/T*}(\mathscr{O}_X\otimes I)$ induces a map $i_{\mathrm{cris}*}\,\mathscr{I}_{X_0/T}/\mathscr{I}^{[2]}_{X_0/T} \to i_{X/T*}(\mathscr{O}_X\otimes I/I^{[2]})$. Thus we obtain a commutative diagram:

$$\begin{array}{ccccc} \mathbb{R}u_{Y/T*}(\mathscr{I}_{Y/T}) & \longrightarrow & \mathbb{R}f_{0*}(\mathbb{R}u_{X_0/T*}\mathscr{I}_{X_0/T}) & \longrightarrow & \mathbb{R}f_{0*}(\mathscr{O}_X\otimes I) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}u_{Y/T*}(\mathscr{I}_{Y/T}/\mathscr{I}^{[2]}_{Y/T}) & \longrightarrow & \mathbb{R}f_{0*}(\mathbb{R}u_{X_0/T*}\mathscr{I}_{X_0/T}/\mathscr{I}^{[2]}_{X_0/T}) & \longrightarrow & \mathbb{R}f_{0*}(\mathscr{O}_X\otimes I/I^{[2]}), \end{array}$$

where the top horizontal arrow induces $\rho(f_0,\gamma)$. It follows from the proof of the filtered Poincaré lemma [3, (6.13) and (7.2)] that $\mathbb{R}u_{Y/T*}(\mathscr{I}_{Y/T}/\mathscr{I}^{[2]}_{Y/T}) \cong \Omega^1_{Y/T}[-1]$, so we have a morphism (in the derived category):

$$\tilde{\rho}(f_0,\gamma)\colon f_0^*\,\Omega^1_{Y_0/T}[-1] \to \mathscr{O}_{X_0}\otimes I/I^{[2]}.$$

This "is" an element of $H^0\mathbb{R}\operatorname{Hom}(f_0^*\,\Omega^1_{Y_0/S_0}[-1],\ \mathscr{O}_{X_0}\otimes I/I^{[2]}) \cong \operatorname{Ext}^1_{\mathscr{O}_{X_0}}(f_0^*\,\Omega^1_{Y_0/S_0}, \mathscr{O}_{X_0}\otimes I/I^{[2]})$. Thus it is clear that (3.20) reduces to the following:

(3.21) **Theorem.** *$\tilde{\rho}(f_0,\gamma)$ is the image $\bar{\xi}$ of the obstruction class $\xi = \mathrm{ob}(f_0)$.*

Proof. First of all we need a convenient expression for the obstruction class ξ. Let $Z =: X\times_T Y$, and regard X_0 as a closed subscheme of Z via the map $(\mathrm{inc}, \mathrm{inc}\circ f_0)$. Then to lift the morphism f_0 is the same as to lift X_0 to a closed subscheme of Z, flat over T (for then the projection to X will automatically be an isomorphism).

Let $\mathscr{I}$ be the ideal of X_0 in Z and $\mathscr{I}_0$ the ideal of X_0 in Z_0. As explained in [19, (2.7)], there is an exact sequence of $\mathscr{O}_{X_0}$-modules:

$$0 \to I\mathscr{O}_{X_0} \to \mathscr{I}/\mathscr{I}^2 \to \mathscr{I}_0/\mathscr{I}_0^2 \to 0. \tag{3.21.1}$$

Moreover, there is a natural one-one correspondence between splittings of (3.21.1) and flat liftings of X_0 in Z. Thus, the image ξ of (3.21.1) in $\operatorname{Ext}^1_{\mathscr{O}_{X_0}}(\mathscr{I}_0/\mathscr{I}_0^2, I\mathscr{O}_X)$ "is" the obstruction to extending f_0. Of course, $\mathscr{I}_0/\mathscr{I}_0^2$ in naturally isomorphic to $f_0^*\,\Omega^1_{Y_0/S}$.

Now let D be the PD envelope of X_0 in $X\times Y$, and let $\bar{\mathscr{I}}$ be the ideal of X_0 in D. We have:

$$\mathbb{R}u_{X_0/T*}\,\mathscr{I}^{[k]}_{X_0/T} \cong F^k_{X_0}\,\Omega^{\cdot}_{D/T},$$

where $F^k_{X_0}\Omega^\cdot_{D/T}$ is the complex: $\bar{\mathscr{J}}^{[k]}\to\bar{\mathscr{J}}^{[k-1]}\Omega^1_{D/T}\to\dots$. In particular, the map $\pi_X: D\to X$ induces a natural quasi-isomorphism: $\mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{X/T}\xrightarrow{\sim}\mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{D/T}$. It is clear that $\tilde{\rho}(f\circ\gamma)$ is the composite:

$$f_0^*(\mathrm{gr}^1_{F_{Y_0}}\Omega^\cdot_{Y_0/S})\xrightarrow{\pi_Y^*}\mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{D/T}\xleftarrow[\sim]{\pi_X^*}\mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{X/T}\to\mathcal{O}_{X_0}\otimes I/I^{[2]}.$$

Now we have a commutative diagram:

$$\begin{tikzcd}
\xi\colon 0 \arrow[r] & \mathcal{O}_{X_0}\otimes I \arrow[r] \arrow[d] & \mathscr{J}/\mathscr{J}^2 \arrow[r] \arrow[d] & f_0^*\Omega^1_{Y_0/S} \arrow[r] \arrow[d] & 0 \\
\bar{\xi}\colon 0 \arrow[r] & \mathcal{O}_{X_0}\otimes I/I^{[2]} \arrow[r] & \bar{\mathscr{J}}/\bar{\mathscr{J}}^{[2]} \arrow[r] & f_0^*\Omega^1_{Y_0/S}. &
\end{tikzcd}$$

Let $A^\cdot$ be the complex: $\bar{\mathscr{J}}/\bar{\mathscr{J}}^{[2]}\to f_0^*\Omega^\cdot_{Y_0/S}$. Then we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{X_0}\otimes I/I^{[2]} \arrow[r, "\sim"] & A^\cdot \\
 & f_0^*\Omega^\cdot_{Y_0/S}[-1]. \arrow[ul, dashed, "\bar{\xi}"] \arrow[u]
\end{tikzcd}$$

On the other hand, $\mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{D/T}$ is the complex: $\bar{\mathscr{J}}/\bar{\mathscr{J}}^{[2]}\to\Omega^\cdot_{X_0/S}\oplus f_0^*\Omega^1_{Y_0/S}$, which maps in an obvious way to $A^\cdot$. It is now easy to see that the following diagram commutes:

$$\begin{tikzcd}
f_0^*\Omega^1_{Y_0/S}[-1] \arrow[r] & \mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{D/T} \arrow[d] & \mathrm{gr}^1_{F_{X_0}}\Omega^\cdot_{X/T} \arrow[l, "\sim"'] \arrow[d] \\
 & A^\cdot & \mathcal{O}_{X_0}\otimes I/I^{[2]}. \arrow[l, "\sim"']
\end{tikzcd}$$

This implies that $\bar{\xi}=\tilde{\rho}(f_0,\gamma)$ in the derived category. □

(3.22) *Proof of* (3.17). Since the ideal of S in T is PD nilpotent, we can proceed inductively and assume that $I^{[2]}=0$. By rigidity of abelian schemes, the homomorphism lifting f_0 is unique, and we can therefore assume S affine. The homomorphism $H^1(Y_0,F^1\Omega^\cdot_{Y_0/S})\to H^0(Y_0,\Omega^1_{Y_0/S})$ is surjective, and the vanishing of $\rho(f_0,\gamma)$ implies that the cup product by $\mathrm{ob}(f_0)$: $H^0(Y_0,\Omega^1_{Y_0/S})\to H^1(X_0,\mathcal{O}_{X_0})\otimes I$ is zero. Since $\Omega^1_{Y_0/S}\simeq\omega_Y\otimes_{\mathcal{O}_S}\mathcal{O}_{Y_0}$, the theorem follows from the identification of the two groups $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^*\Omega^1_{Y_0/S},\mathcal{O}_{X_0}\otimes I)$ and

$$\mathrm{Hom}(H^0(Y_0,\Omega^1_{Y_0/S}),H^1(X_0,\mathcal{O}_{X_0})\otimes I)\quad\text{with}\quad\Gamma(S,\omega_Y)^\vee\otimes H^1(X_0,\mathcal{O}_{X_0})\otimes I.\quad\square$$

(3.23) *Remark.* It is straightforward to use the techniques of (3.17) in other contexts when "local Torelli" holds. For example, if X and Y are $K3$ surfaces over a *nilpotent* PD thickening $S\subseteq T$ and $f_0: X_0\to Y_0$ is an isomorphism between their reductions to S such that $H^2_{\mathrm{cris}}(f_0)$ preserves the Hodge fil-

trations, then f_0 extends to an isomorphism $X \to Y$. This provides a proof of a key step in the proof of [18, (2.5)] that an automorphism α of a K3 surface over a field of characteristic $p > 2$ acting trivially on its crystalline cohomology is necessarily the identity. (The references given there are rather vague and inconclusive.) We would like to take this opportunity to point out a subtlety in low characteristics. It is shown in [18, (2.3)] that if L is a polarization of X, there is a complete DVR V with $e(V) \leqq 2$ to which (X, L) can be lifted. Since $H^2_{\mathrm{cris}}(\alpha)$ automatically preserves the Hodge filtration of the lifting, we see that if the divided power structure on the maximal ideal of V is nilpotent, α will lift also, and hence be the identity by the well-known result in De Rham cohomology in characteristic zero. The necessary nilpotence will hold *unless* $p=3$ and $e=2$ – which means that X is a superspecial K3 surface in characteristic 3, by the results of [18, (2.3)]. Thus, the proof breaks down in this case. [Let us note that if Tate's conjecture is true, the Picard number of X is 22, and there will exist *another* polarization L' such that (X, L') lifts without ramification, so the same argument will again work.] However, we can argue as follows in this case. Let $V =: W(\sqrt{p})$, let $(\mathbf{X}, \mathscr{L})$ be a lifting of (X, L) to $T =: \operatorname{Spec} V$, and let $S =: \operatorname{Spec} k$, $S' =: \operatorname{Spec} V/pV$, $\mathbf{X}' = X \times_T S'$. Then $R =: V/pV \cong k[\varepsilon]/(\varepsilon^2)$, and the ideal (ε) can be endowed with the nilpotent PD structure δ, where $\delta_n(\varepsilon) = 0$ for all $n \geqq 2$. (This is *not* the divided power structure on (ε) compatible with the divided power structure on the maximal ideal of V.) There is an evident PD morphism $(W, \gamma) \to (R, \delta)$, and it follows from the base changing theorem for crystalline cohomology that we have a functorial isomorphism:
$R \otimes H^2_{\mathrm{cris}}(X/W) \xrightarrow{\cong} H^2_{\mathrm{cris}}(X/(R, \delta))$. Since $H^2_{\mathrm{cris}}(\alpha/W)$ is the identity, the same is true of $H^2_{\mathrm{cris}}(\alpha/(R, \delta))$. Now the canonical isomorphism:
$H^2_{\mathrm{cris}}(X/(R, \delta)) \cong H^2_{\mathrm{DR}}(X'/S')$ induces a Hodge filtration on $H^2_{\mathrm{cris}}(X/(R, \delta))$ which will be invariant under $\mathrm{id} = H^2_{\mathrm{cris}}(\alpha/(R, \delta))$, and hence the local Torelli theorem and (3.21) will imply that α lifts to an automorphism α' of X'/S'. We also have canonical isomorphisms: $H^2_{\mathrm{cris}}(X'/V) \xrightarrow{\cong} H^2_{\mathrm{cris}}(X/V) \xrightarrow{\cong} H^2_{\mathrm{cris}}(X/W) \otimes V$ under which the action of α' corresponds to that of α, and so it follows that α' acts trivially on $H^2_{\mathrm{cris}}(X'/V)$. But now $X' \subseteq \mathbf{X}$ is defined by the PD nilpotent ideal (3), and so we can conclude that α' lifts to $\mathbf{X}$. □

4. The Crystalline Weil Group

If X is a smooth proper V-scheme and X_k is its reduction $\bmod(\pi)$, the isomorphism $\sigma_{\mathrm{cris}}: H^*_{\mathrm{DR}}(X/V) \otimes \mathbb{Q} \to H^*_{\mathrm{cris}}(X_k/W) \otimes V \otimes \mathbb{Q}$ furnishes us with a $W \otimes \mathbb{Q}$-structure on the $V \otimes \mathbb{Q}$-vector space $H^*_{\mathrm{DR}}(X/V) \otimes \mathbb{Q}$. This structure, together with the action of Frobenius on crystalline cohomology, is in a sense analogous to the $\mathbb{Q}$-vector space structure on DeRham cohomology over the complex field furnished by singular cohomology. To keep track of these data, we follow an idea of Deligne by introducing the so-called "crystalline Weil group."

For the sake of simplicity, we shall restrict our attention to the following situation. Let k be a perfect field of characteristic $p > 0$, let $W(k)$ be its Witt ring, and let $K(k)$ be the fraction field of $W(k)$. We denote by $\bar{K}(k)$ or just $\bar{K}$ an algebraic closure of $K(k)$; the valuation of $K(k)$ prolongs uniquely to a valuation of $\bar{K}(k)$, and the residue field $\bar{k}$ of $\bar{K}(k)$ is an algebraic closure of k. Let $K_{nr}(k)$ or just K_{nr} denote the maximal unramified extension of $K(k)$ in $\bar{K}(k)$; its residue field is also $\bar{k}$. If we start with some finite extension K of $K(k)$, we will choose $\bar{K}(k)$ to be an algebraic closure of K.

(4.1) **Definition.** *The "crystalline Weil group of $\bar{K}(k)$", denoted $W_{\mathrm{cris}}(\bar{K}(k))$, is the group of automorphisms of $\bar{K}(k)$ covering some integral power of the Frobenius automorphism F_{nr} of $K_{nr}(k)$.*

Note that if $\psi \in W_{\mathrm{cris}}(\bar{K}(k))$, there is a unique integer $\deg(\psi)$ such that ψ acts as $F_{nr}^{\deg(\psi)}$ on $K_{nr}(k)$. Since F_{nr} preserves the valuation of $K(k)$, ψ is automatically continuous, hence acts on $\bar{k}$; evidently its action on $\bar{k}$ is $F_{\bar{k}}^{\deg(\psi)}$, which also determines $\deg(\psi)$ uniquely. Since $\bar{K}$ is algebraically closed, the map $\deg: W_{\mathrm{cris}}(\bar{K}) \to \mathbb{Z}$ is surjective, and we have an exact sequence:

$$1 \to I_{\mathrm{cris}}(\bar{K}) \to W_{\mathrm{cris}}(\bar{K}) \xrightarrow{\deg} \mathbb{Z} \to 1, \tag{4.1.1}$$

where $I_{\mathrm{cris}}(\bar{K}) =: \mathrm{Ker}(\deg)$ is just the inertial Galois group $\mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K}_{nr})$. Thus, $\bar{K}^{I_{\mathrm{cris}}(\bar{K})} = K_{nr}$ and $K^{W_{\mathrm{cris}}(\bar{K})} = K_{nr}^{F_{nr}} = \mathbb{Q}_p$. We can also let $W_{\mathrm{cris}}(\bar{K})$ act by continuity on the completion $\hat{\bar{K}}$ of $\bar{K}$; it is well-known [24] that $\hat{\bar{K}}^{I_{\mathrm{cris}}(\bar{K})} = \hat{K}_{nr} = W(\bar{k}) \otimes \mathbb{Q}_p$, so again we get $\hat{\bar{K}}^{W_{\mathrm{cris}}(\bar{K})} = \mathbb{Q}_p$.

If $\bar{K}'$ is another algebraic closure of $K(k)$, any $K(k)$-isomorphism $\sigma: \bar{K} \to \bar{K}'$ is continuous and induces an isomorphism: $K_{nr} \to K'_{nr}$. Since $F'_{nr} \circ \sigma = \sigma \circ F_{nr}$, we see that $\psi \mapsto \sigma \psi \sigma^{-1}$ defines an isomorphism:

$$W_{\mathrm{cris}}(\sigma): W_{\mathrm{cris}}(\bar{K}) \to W_{\mathrm{cris}}(\bar{K}')$$

compatible with deg. If $\tau: \bar{K}' \to \bar{K}''$ is another isomorphism,

$$W_{\mathrm{cris}}(\tau) \circ W_{\mathrm{cris}}(\sigma) = W_{\mathrm{cris}}(\tau\sigma).$$

If H is a $\bar{K}$-vector space, by a "semi-linear action of $W_{\mathrm{cris}}(\bar{K})$ on H" we mean a map $\rho: W_{\mathrm{cris}}(\bar{K}) \to \mathrm{Aut}_{\mathbb{Q}_p}(H)$ such that $\rho(\psi)(ax) = \psi(a)\rho(\psi(x))$ for $a \in \bar{K}$, $x \in H$, $\psi \in W_{\mathrm{cris}}(\bar{K})$. Let $\bar{K}(i)$ denote the $\bar{K}$-vector space $\bar{K}$ on which $W_{\mathrm{cris}}(\bar{K})$ acts by $\rho(\psi)(a) = p^{-\deg(\psi)i}\psi(a)$, and let $H(i) =: H \otimes_{\bar{K}} \bar{K}(i)$ with the tensor product action.

We shall say that a smooth proper $\bar{K}$-scheme X has "good reduction" iff there exist a finite extension K' of $K(k)$ in $\bar{K}$ with ring of integers V', a smooth proper V'-scheme $\mathbf{X}'$, and an isomorphism $\alpha: \mathbf{X}'_{\bar{K}} \xrightarrow{\sim} X$, where $\mathbf{X}'_{\bar{K}} =: \mathbf{X}' \times_{\mathrm{Spec}\, V'} \mathrm{Spec}\, \bar{K}$. In this situation let k' be the residue field of V' and $\mathbf{X}_{k'} =: \mathbf{X}' \times_{\mathrm{Spec}\, V'} \mathrm{Spec}\, k'$.

(4.2) **Theorem.** *Suppose X is a smooth proper $\bar{K}$-scheme with good reduction. There is a unique semi-linear action ρ_{cris} of $W_{\mathrm{cris}}(\bar{K})$ on $H^*_{\mathrm{DR}}(X/\bar{K})$ with the following properties.*

(4.2.1) *It is functorial in X. In fact, if $f: Y \circ\!\!\to X$ is a correspondence*[1] *of degree d, the map $H^*_{\mathrm{DR}}(f): H^*_{\mathrm{DR}}(Y/\bar{K}) \to H^{*+2d}_{\mathrm{DR}}(X/\bar{K})(+d)$ is compatible with the actions of $W_{\mathrm{cris}}(\bar{K})$.*

[1] i.e., a closed subset of $Y \times X$ of codimension $d + \dim(Y)$. Recall that a morphism $f: X \to Y$ induces a correspondence $Y \circ\!\!\to X$ of degree zero, viz. the transpose of the graph of f.

(4.2.2) *If* $\mathbf{X}'$ *is any smooth proper* V'*-scheme, the image of the map:*

$$H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes K_{nr}\subseteq H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes K' \xrightarrow{\sigma^{-1}_{\mathrm{cris}}} H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K')\subseteq H^*_{\mathrm{DR}}(\mathbf{X}'_{\bar{K}}/\bar{K})$$

consists precisely of the invariants of $I_{\mathrm{cris}}(\bar{K})$, *and the action of* $W_{\mathrm{cris}}(\bar{K})/I_{\mathrm{cris}}(\bar{K}) \cong \mathbb{Z}$ *on these invariants is given by the action of absolute Frobenius.*

Proof. In order to obtain the unicity and functoriality asserted, we shall have to use some results of a forthcoming paper [10] of Messing and Gillet on cycle classes in crystalline cohomology tensored with $\mathbb{Q}$. For the reader's convenience, we shall state here a summary of what we need.

(4.3) **Theorem** (Messing-Gillet). *If* X_0 *and* Y_0 *are smooth proper k-schemes, a correspondence* $z_0\colon Y_0 \circ\!\!\rightarrow X_0$ *of degree d induces a morphism*

$$z_0^*.\ H^i_{\mathrm{cris}}(Y_0/W)\otimes\mathbb{Q}\to H^{i+2d}_{\mathrm{cris}}(X_0/W)\otimes\mathbb{Q},$$

compatible with composition of correspondances. If $\mathbf{X}$ *and* $\mathbf{Y}$ *are smooth proper V-schemes with reductions* X_0 *and* Y_0 *and* $z\colon \mathbf{Y}_K \circ\!\!\rightarrow \mathbf{X}_K$ *is a correspondance of degree d with specialization* z_0, *we have a commutative diagram*

$$\begin{array}{ccc}
H^i_{\mathrm{DR}}(z)\colon H^i_{\mathrm{DR}}(\mathbf{Y}_K/K) & \longrightarrow & H^{i+2d}_{\mathrm{DR}}(\mathbf{X}_K/K)\\
\wr\downarrow & & \wr\downarrow\\
H^i_{\mathrm{cris}}(z_0)\otimes\mathrm{Id}\colon H^i_{\mathrm{cris}}(Y_0/W)\otimes K & \longrightarrow & H^{i+2d}_{\mathrm{cris}}(X_0/W)\otimes K\\
\uparrow & & \uparrow\\
H^i_{\mathrm{cris}}(z_0)\colon H^i_{\mathrm{cris}}(Y_0/W)\otimes\mathbb{Q} & \longrightarrow & H^{i+2d}_{\mathrm{cris}}(X_0/W)\otimes\mathbb{Q}.
\end{array}\quad\square$$

(4.4) We now begin the proof of (4.2). Choose a smooth proper $\mathbf{X}'$ over a suitable V' and an isomorphism $\alpha\colon \mathbf{X}'_{\bar{K}}\xrightarrow{\sim} X$. Using α^* and $\sigma_{\mathrm{cris}}\otimes\mathrm{id}_{\bar{K}}$ we get an isomorphism

$$\sigma_\alpha\colon H^*_{\mathrm{DR}}(X/\bar{K})\xrightarrow{\sim} H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes\bar{K}.$$

Notice that if $K'\subseteq K''\subseteq\bar{K}$, with K'' also finite over $K(k)$, and if $\mathbf{X}''=:\mathbf{X}'\times_{\mathrm{Spec}\,V'}\mathrm{Spec}\,V''$, we get by (2.7) a commutative diagram

$$\begin{tikzcd}
H^*_{\mathrm{DR}}(X/\bar{K}) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes\bar{K} \arrow[d, "\wr"] \\
& H^*_{\mathrm{cris}}(\mathbf{X}''_{k''}/W(k''))\otimes\bar{K}.
\end{tikzcd}\tag{4.4.1}$$

If now $\psi\in W_{\mathrm{cris}}(\bar{K})$ has degree d, ψ covers the d^{th} power of the Frobenius automorphism of $W(k')$. The absolute Frobenius automorphism $F_{\mathbf{X}'_{k'}}$ induces a semi-linear automorphism Φ of $H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes K(k')$, and hence $\Phi^d\otimes\psi$ makes sense and forms a ψ-linear automorphism of $H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes\bar{K}$. Using the isomorphism σ_α, we can transfer this to an automorphism of $H^*_{\mathrm{DR}}(X/\bar{K})$.

If we check that the action is functorial in $X/\bar{K}$, it will follow that it is independent of the choices. Notice first that (4.4.1) shows that it is independent of the choice of K', in the sense that we can replace K' by a finite extension K''. Now suppose that f: $Y \circ\!\!\to X$ is a correspondence over $\bar{K}$, where X and Y have good reduction. Suppose $\mathbf{X}'$ and $\mathbf{Y}'$ smooth and proper over V', V'', respectively, and that $\alpha: \mathbf{X}'_{\bar{K}} \xrightarrow{\sim} X$, $\beta: \mathbf{Y}'_{\bar{K}} \xrightarrow{\sim} Y$ are isomorphisms. By the previous remark, we can assume without loss of generality that $K' = K''$, and also that f is defined by a correspondence $f': \mathbf{Y}'_{K'} \circ\!\!\to \mathbf{X}'_{K'}$. Then we get by (4.3) a commutative diagram:

$$\begin{tikzcd} H^*_{\mathrm{DR}}(\mathbf{Y}'_{K'}/K') \arrow[r] \arrow[d, "\sigma_{\mathrm{cris}}"'] & H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') \arrow[d, "\sigma_{\mathrm{cris}}"] \\ H^*_{\mathrm{cris}}(\mathbf{Y}'_{k'}/W(k')) \otimes K' \arrow[r] & H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k')) \otimes K' \end{tikzcd}$$

which implies the functoriality we claimed. Property (4.4.2) is apparent from the definition, and the uniqueness is also clear. □

Perhaps it is worth remarking that if $\bar{K}'$ is another algebraic closure of $K(k)$ and $\sigma: \bar{K} \to \bar{K}'$ is a $K(k)$-isomophism, the map $H^*_{\mathrm{DR}}(X/\bar{K}) \to H^*_{\mathrm{DR}}(X_{\bar{K}'}/\bar{K}')$ induced by σ is compatible with the actions, via the map
$W_{\mathrm{cris}}(\sigma)$: $W_{\mathrm{cris}}(\bar{K}) \to W_{\mathrm{cris}}(\bar{K})$.

(4.5) *Remark.* In fact, we have an additional functoriality: if $(\mathbf{X}', \alpha)$ and $(\mathbf{Y}', \beta)$ are as above, and if f: $\mathbf{Y}'_{k'} \to \mathbf{X}'_{k'}$ is a correspondence between their closed fibers, then f induces a map: $H^*_{\mathrm{cris}}(\mathbf{Y}'_{k'}/W(k')) \to H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))$ which is compatible with the action of absolute Frobenius. Therefore, it induces a map:
$H^*_{\mathrm{DR}}(Y/\bar{K}) \to H^*_{\mathrm{DR}}(X/\bar{K})$ compatible with the actions of $W_{\mathrm{cris}}(\bar{K})$.

(4.6) *Remark.* In some situations, it is more convenient to work with finite extensions than with algebraically closed fields. To do this, we introduce the crystalline Weil category W_{cris}. An object in W_{cris} is a finite extension K' of $K(k)$; if its residue field is k', we have canonically $K(k') \subseteq K'$, (the maximal unramified extension of $K(k)$ in K'). A morphism $K' \to K''$ is a triple (d, a, b) where d is an integer, a: $K' \to K''$ is a K-linear map, and b: $K' \to K''$ is a map of fields inducing commutative diagrams:

$$\begin{tikzcd} K' \arrow[r, "b"] & K'' & \\ K(k') \arrow[u] \arrow[r, "F^d"] & K(k') \arrow[r, "a^*"] & K(k'') \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, "b"] & K'' & \\ K(k') \arrow[u] \arrow[r, "a^*"] & K(k'') \arrow[r, "F^d"] & K(k''). \arrow[ul] \end{tikzcd}$$

Composition in W_{cris} is defined in the evident fashion. If (d, a, b): $K' \to K''$ is a morphism, we have a commutative diagram:

$$\begin{tikzcd} H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') \arrow[r] \arrow[d, "\sigma_{\mathrm{cris}}"'] & K'' \otimes_a H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') \arrow[d, "\mathrm{id} \otimes \sigma_{\mathrm{cris}}"] \\ K' \otimes H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k')) \arrow[r] & K'' \otimes H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k')). \end{tikzcd}$$

Thus, the map $H^*_{\mathrm{cris}}(F_{\mathbf{X}'_{k'}})^d \otimes b$ defines a map:

$$\rho_{\mathrm{cris}}(d,a,b)\colon H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') \to K'' \otimes_a H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K').$$

If we fix an algebraic closure $\bar{K}$ of $K(k)$ and consider the "limit" over all $K' \hookrightarrow \bar{K}$, we can recover the entire action of $W_{\mathrm{cris}}(\bar{K})$ in this way.

(4.7) In the remainder of this section, we shall investigate a few properties of the action of W_{cris}, with the hope of developing some sort of feeling for what this structure means.

Suppose that K is a finite extension of $K(k)$ contained in $\bar{K}$ and that X is a smooth proper K-scheme. Then $H^*_{\mathrm{DR}}(X_{\bar{K}}/\bar{K}) \cong H^*_{\mathrm{DR}}(X/K) \otimes_K \bar{K}$, and hence there is an evident semi-linear action ρ_{DR} of $\mathrm{Gal}(\bar{K}/K)$, with invariants $H^*_{\mathrm{DR}}(X/K)$. We shall say that X/K has potentially good reduction if $X_{\bar{K}}$ has good reduction. In this case $W_{\mathrm{cris}}(\bar{K})$ also acts on $H^*_{\mathrm{DR}}(X_{\bar{K}}/\bar{K})$, and it will be instructive to compare the actions ρ_{DR} and ρ_{cris}. Before we do so, some preliminaries are necessary.

First of all, we had better establish a "sign" convention for Galois groups. If K'/K is a Galois extension, $\mathrm{Gal}(K'/K)$ is the group of automorphisms of K' over K, which operates on the *left* on K' and on the the *right* on $\operatorname{Spec} K'$. If $f\colon K \to K'$ is a map of rings and X is a K-scheme, we let fX denote the K'-scheme defined by base change using the map $\operatorname{Spec} f$; thus if $g\colon K' \to K''$ is another map, there is a canonical isomorphism $\theta_{g,f}\colon (g \circ f)(X) \to g(fX)$, satisfying the usual cocycle condition. In particular, if $f\colon K \to K'$ is a Galois extension, if X is a K-scheme, and $g \in \mathrm{Gal}(K'/K)$, we obtain in this way a K'-isomorphism $\gamma_g\colon X_{K'} \to g(X_{K'})$, where $X_{K'} =: fX$. If h is another element of $\mathrm{Gal}(K'/K)$, the cocycle condition implies that $h(\gamma_g) \circ \gamma_h = \gamma_{hg}\colon X_{K'} \to hgX_{K'}$.

For example, suppose that X/K has potentially good reduction. Then there exist a finite Galois extension K' of K with ring of integers V', a smooth proper V'-scheme $\mathbf{X}'$, and an isomorphism $\alpha\colon \mathbf{X}'_{K'} \to X_{K'}$. Thanks to α, we get an isomorphism $\gamma'_g\colon \mathbf{X}'_{K'} \to g\mathbf{X}'_{K'}$, so that the following diagram commutes:

$$\begin{tikzcd} X_{K'} \arrow[r, "\gamma_g"] & gX_{K'} \\ \mathbf{X}'_{K'} \arrow[u, "\alpha"] \arrow[r, "\gamma'_g"] & g\mathbf{X}'_{K'} \arrow[u, "g(\alpha)"'] \end{tikzcd} \tag{4.7.1}$$

The isomorphisms γ'_g also satisfy the cocycle conditions. Moreover, if $g_{k'}$ is the image of g in the residual Galois group $\mathrm{Gal}(k'/k)$, we can specialize γ'_g to a correspondence of degree zero $\gamma'_{g,k'}\colon g_{k'}\mathbf{X}'_{k'} \circ\!\!\to \mathbf{X}'_{k'}$. These specialized correspondences still satisfy the cocycle conditions. In particular, if g belongs to the inertia group $I(K'/K)$, $g_{k'} = \mathrm{id}$, $g_{k'}\mathbf{X}'_{k'}$ is canonically isomorphic to $\mathbf{X}'_{k'}$, and we can view $\gamma'_{g,k'}$ as an endomorphism of $\mathbf{X}'_{k'}$. The cocycle condition implies that the action is "on the left," that is, $\gamma'_{h,k'} \circ \gamma'_{g,k'} = \gamma_{hg,k'}$. By functoriality, we obtain a *linear* action of $I(K'/K)$ on $H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k')) \otimes \mathbb{Q}$, on the left. Using the isomorphisms σ_{cris} and α we can transfer this action to a (linear) action δ_{DR} of

$I(K'/K')$ on $H^*_{\mathrm{DR}}(X_{K'}/K')$, hence also on $H^*_{\mathrm{DR}}(X_{\bar{K}}/\bar{K})$, if $\bar{K}$ is an algebraic closure of K'. Note that by (4.3), the representations δ and ρ_{cris} commute with one another. We view δ_{DR} as a representation of $\mathrm{Gal}(\bar{K}/K)$ by the map $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(K'/K)$.

(4.8) **Theorem.** *If X/K is smooth, proper, and has potentially good reduction, and if $\bar{K}$ is an algebraic closure of K, let ρ_{cris}, ρ_{DR}, and δ_{DR} be the actions of $W_{\mathrm{cris}}(\bar{K})$, $\mathrm{Gal}(\bar{K}/K)$, and $I(\bar{K}/K)$ described above. Then if $g \in I(\bar{K}/K)$ is regarded as an element of $W_{\mathrm{cris}}(\bar{K})$, we have the following:*

$$\rho_{\mathrm{DR}}(g) = \delta_{\mathrm{DR}}(g) \circ \rho_{\mathrm{cris}}(g).$$

Proof. Suppose g is any member of $\mathrm{Gal}(K'/K)$, and use the notation of the discussion above. We have the following diagram:

$$\begin{tikzcd}
H^*_{\mathrm{DR}}(X/K)\otimes K' \arrow[r] & H^*_{\mathrm{DR}}(X_{K'}/K') \arrow[r, "\alpha^*"] & H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') \arrow[r, "\sigma_{\mathrm{cris}}"] & H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes K' \\
H^*_{\mathrm{DR}}(X/K)\otimes K' \arrow[r] \arrow[u, "\mathrm{id}"] & H^*_{\mathrm{DR}}(gX_{K'}/K') \arrow[r, "(g\alpha)^*"] \arrow[u, "\gamma^*_g"'] & H^*_{\mathrm{DR}}(g\mathbf{X}'_{K'}/\mathbf{K}') \arrow[r, "\sigma_{\mathrm{cris}}"] \arrow[u, "\gamma'^*_g"'] & H^*_{\mathrm{cris}}(g_k\mathbf{X}'_{k'}/W(k'))\otimes K' \arrow[u, "\gamma'^*_{g,k'}"'] \\
H^*_{\mathrm{DR}}(X/K)\otimes K' \arrow[r] \arrow[u, "\mathrm{id}\otimes g"] & H^*_{\mathrm{DR}}(X_{K'}/K') \arrow[r, "\alpha^*"] \arrow[u] & H^*_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') \arrow[r, "\sigma_{\mathrm{cris}}"] \arrow[u] & H^*_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes K'. \arrow[u, "\beta(g)"']
\end{tikzcd}$$

Let us analyze this diagram square by square, starting from the top left. Its horizontal arrows are the standard base change maps; it commutes because of the definition of $\gamma_g: X_{K'} \to gX_{K'}$. (Check it on the subspace $H^*_{\mathrm{DR}}(X/K)$ of $H^*_{\mathrm{DR}}(X/K)\otimes K'$.) The commutativity of the next square to the right is immediate from the definition of γ_g, and the last square on the top right commutes because of (4.3). In the second row, the horizontal arrows of the left most square are the base change maps again, and its right vertical map comes from the g-morphism: $gX_{K'} \to X_{K'}$. It commutes because the base change maps of DeRahm cohomology are natural. The next square on the right commutes for the same reason, and the last one is (2.7).

In particular, the left-most composed vertical arrow is the action of $\rho_{\mathrm{DR}}(g)$. If $g \in I(K'/K)$, then g_k is the identity and $\gamma'^*_{g,k'} = \delta_{\mathrm{DR}}(g)$, while $\beta(g)$ corresponds to $\rho_{\mathrm{cris}}(g)$. This proves the theorem. $\square$

If k is a finite field, we can generalize this somewhat. Namely, in this case let $W_{\mathrm{cris}}(\bar{K}/K) =: W_{\mathrm{cris}}(\bar{K}) \cap \mathrm{Gal}(\bar{K}/K)$, and note that $W_{\mathrm{cris}}(\bar{K}/K)$ is dense in $\mathrm{Gal}(\bar{K}/K)$. Just as above, we obtain an action of all of $W_{\mathrm{cris}}(\bar{K}/K)$ on $\mathbf{X}'_{\bar{k}'}/k'$ (by correspondences), and hence an action δ of $W_{\mathrm{cris}}(\bar{K}/K)$ on $H^*_{\mathrm{DR}}(X_{\bar{K}}/\bar{K})$. The same proof shows that (4.8) is still true, for any $g \in W_{\mathrm{cris}}(\bar{K}/K)$.

The following corollary, which in fact is essentially equivalent to the theorem, was first stated (independently) by Messing.

(4.9) **Corollary.** *With the above notations, assume that K'/K is Galois and totally ramified, and extend the action of $I(K'/K)$ on $H^*_{\mathrm{cris}}(X_0/W)\otimes\mathbb{Q}$ to*

$H^*_{\mathrm{cris}}(X_0/W)\otimes K'$ by letting it act (semi-linearly) on K' as usual. Then σ_{cris} induces an isomorphism

$$H^*_{\mathrm{DR}}(X/K) \xrightarrow{\sim} (H^*_{\mathrm{cris}}(X_0/W)\otimes K')^{I(K'/K)}. \quad \square$$

(4.10) Using the isomorphism σ_{cris} and the group W_{cris}, we can generalize the conjectures in [20] concerning the relationship of absolute Hodge cycles to crystalline cohomology. This approach was suggested by Deligne in his seminar in 1978 in the context of abelian varieties, for which the isomorphism σ_{cris} was already known.

Let $\bar{K}$ be as above and let $X_1, \ldots, X_r$ be a finite family of smooth proper $\bar{K}$-schemes with good reduction. If $i<0$, let

$$H^i_{\mathrm{DR}}(X_j/\bar{K}) =: \mathrm{Hom}[H^{-i}_{\mathrm{DR}}(X_j/\bar{K}), \bar{K}],$$

with the usual action of $W_{\mathrm{cris}}(\bar{K})$ on Hom. If $i_1, \ldots, i_r$ is a corresponding family of integers, we let X denote the pair $((X_1, \ldots, X_r), (i_1, \ldots, i_r))$, and

$$H_{\mathrm{DR}}(X/\bar{K}) =: \bigotimes_j H^{i_j}_{\mathrm{DR}}(X_j/\bar{K}),$$

on which $W_{\mathrm{cris}}(\bar{K})$ acts. Since $\bar{K}$ is a field of characteristic zero, it makes sense to speak of the absolute Hodge cycles of X [6, 20]; recall that these form a finite dimensional $\mathbb{Q}$-vector space $H_{\mathrm{AH}}(X)$ and that $H_{\mathrm{AH}}(X)\otimes_{\mathbb{Q}}\bar{K}\hookrightarrow H_{\mathrm{DR}}(X/\bar{K})$.

(4.11) **Conjecture.** *If $\xi\in H_{\mathrm{AH}}(X)$, its image ξ_{DR} in $H_{\mathrm{DR}}(X/\bar{K})$ is invariant under $W_{\mathrm{cris}}(\bar{K})$. In particular, if $\mathbf{X}/V'$ is smooth and $\xi\in H_{\mathrm{AH}}(\mathbf{X}_{K'}/K')$, then $\sigma_{\mathrm{cris}}(\xi_{\mathrm{DR}})\in H_{\mathrm{cris}}(\mathbf{X}_{k'}/W(k'))\otimes K'$ in fact lies in $H_{\mathrm{cris}}(\mathbf{X}_{k'}/W(k'))\otimes K(k')$ and is fixed by the action of absolute Frobenius.*

Suppose now that X is a finite family of smooth projective K-schemes with good reduction and indices, where K is a finite extension of $K(k)$. Then there is a natural representation ρ_{AH} of $\mathrm{Gal}(\bar{K}/K)$ in the finite dimensional $\mathbb{Q}$-vector space $H_{\mathrm{AH}}(X_{\bar{K}})$, and a morphism of $\mathrm{Gal}(\bar{K}/K)$-modules: $H_{\mathrm{AH}}(X_{\bar{K}})\to H_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_l)$. Moreover, if we let $\mathrm{Gal}(\bar{K}/K)$ act on $H_{\mathrm{AH}}(X_{\bar{K}})\otimes\bar{K}$ by the *semi-linear* tensor product action, the map: $H_{\mathrm{AH}}(X_{\bar{K}})\otimes_{\mathbb{Q}}\bar{K}\to H_{\mathrm{DR}}(X_{\bar{K}}/\bar{K})$ takes the action ρ_{AH} to the action ρ_{DR} [6]. Using these facts we can obtain a clearer understanding of the inertial part of the above conjecture. Choose as above a finite extension K' of K and a smooth projective family $\mathbf{X}'$ over V' with $\mathbf{X}'_{\bar{K}}\cong\mathbf{X}_{\bar{K}}$. Then it is well-known that via the map: $H(\mathbf{X}'_{\bar{k}}, \mathbb{Q}_l)\cong H(X_{\bar{K}}, \mathbb{Q}_l)$, the action of $I(\bar{K}/K)$ through its action δ_{DR} on $\mathbf{X}'_{\bar{k}}$ corresponds to its action ρ_l on $H(X_{\bar{K}}, \mathbb{Q}_l)$ via $I(\bar{K}/K)\hookrightarrow\mathrm{Gal}(\bar{K}/K)$. Conjectually [23], the character of this action is an integer independent of l, (including p if we use crystalline cohomology).

(4.12) **Proposition.** *The following are equivalent:*

i) *The image of $H_{\mathrm{AH}}(X_{\bar{K}})$ in $H_{\mathrm{DR}}(X_{\bar{K}})$ is (pointwise) fixed by $I(\bar{K}/K)\subseteq W_{\mathrm{cris}}(\bar{K})$.*

i)$^{\mathrm{bis}}$ *The image of $H_{\mathrm{AH}}(X_{\bar{K}})$ in $H_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes\bar{K}\cong H_{\mathrm{DR}}(X_{\bar{K}})$ in fact lies in $H_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k'))\otimes K_{nr}\cdot K$.*

ii) *The map $H_{\mathrm{AH}}(X_{\bar{K}})\to H_{\mathrm{DR}}(X_{\bar{K}})$ is compatible with the (linear) action ρ_{AH} of $I(\bar{K}/K)\subseteq\mathrm{Gal}(\bar{K}/K)$ on $H_{\mathrm{AH}}(X_{\bar{K}})$ and the (linear) action δ_{DR} of $I(\bar{K}/K)$ on $H_{\mathrm{DR}}(X_{\bar{K}})$.*

Proof. The equivalence of i) and i)$^{\text{bis}}$ is obvious. The equivalence of i) and ii) follows immediately from (4.8). □

(4.13) **Example.** If X/K has good reduction, $I(\bar{K}/K)$ acts trivially on $H_{\mathrm{AH}}(X_{\bar{K}}) \subseteq H_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_l)$ and on $H_{\mathrm{DR}}(X_{\bar{K}}/\bar{K})$. Thus, the image of $H_{\mathrm{AH}}(X_{\bar{K}}) \to H_{\mathrm{DR}}(X_{\bar{K}}/\bar{K})$ is contained in $H_{\mathrm{DR}}(X/K) \otimes K_{nr}$.

(4.14) **Proposition.** *Suppose that $\psi \in W_{\mathrm{cris}}(\bar{K})$ has degree $d \geqq 0$ and that there is a ψ-linear endomorphism f of $X/\bar{K}$ lifting the d^{th} power of its absolute Frobenius endomorphism over $\bar{k}$. Then $H_{\mathrm{AH}}(X)$ is fixed by ψ.*

Proof. Choose a finite extension K' of K with ring of integers V' such that there is a smooth proper $\mathbf{X}'$ over V' such that $\mathbf{X}'_{\bar{K}} \cong X$. Let K'' contain the compositum of K' and $\psi(K')$ in $\bar{K}''$, and let ψ' denote the map $K' \to K''$ induced by $\psi: \bar{K} \to \bar{K}$, or the corresponding map $V' \to V''$. Let $\mathbf{X}''$ be the V''-scheme obtained by base change and the "inclusion" $V' \subseteq V''$, and let $\mathbf{X}'''$ be the V''-scheme obtained by base change using the map $\psi': V' \to V''$. Then $\mathbf{X}'''_{\bar{K}} \cong \psi(\mathbf{X}''_{\bar{K}}) \cong \psi(X)$ in our previous notation. The ψ-linear morphism $f: X \to X$ corresponds to a $\bar{K}$-linear morphism $f_{/\bar{K}}: X \to \psi(X)$. By choosing K'' large enough, we may assume that $f_{/\bar{K}}$ descends to a K''-morphism $f_{/K''}: \mathbf{X}''_{K''} \to \mathbf{X}'''_{K''}$. Since the map of residue fields $\psi'_{k'}: k' \to k''$ corresponds to the inclusion followed by $F^d_{k''}$, we have $\mathbf{X}'''_{k''} \cong \mathbf{X}''^{(d)}_{k''}$, and the assumption on f is that the correspondence $f_{/k''}: \mathbf{X}'''_{k''} \circ\!\!\to \mathbf{X}''_{k'}$ is the d^{th} power of the relative Frobenius morphism. Hence by (2.7) and (4.3), the following diagram commutes:

$$\begin{array}{ccc}
H_{\mathrm{DR}}(\mathbf{X}'_{K'}/K') & \xrightarrow{\sigma_{\mathrm{cris}}} & H_{\mathrm{cris}}(\mathbf{X}'_{k'}/W(k')) \otimes K' \\
\Big\downarrow {\scriptstyle \text{base change}} & & \Big\downarrow {\scriptstyle \text{base change}} \\
H_{\mathrm{DR}}(\mathbf{X}'''_{K''}/K'') & \xrightarrow{\sigma_{\mathrm{cris}}} & H_{\mathrm{cris}}(\mathbf{X}'''_{k''}/W(k'')) \otimes K'' \\
\Big\downarrow {\scriptstyle f_{/K''}} & & \Big\downarrow {\scriptstyle F^{(d)*}_{X''_{k''}}} \\
H_{\mathrm{DR}}(\mathbf{X}''_{K''}/K'') & \xrightarrow{\sigma_{\mathrm{cris}}} & H_{\mathrm{cris}}(\mathbf{X}''_{k''}/W(k'')) \otimes K''.
\end{array}$$

After tensoring with $\bar{K}$, we see that $f^*: H_{\mathrm{DR}}(X/\bar{K}) \to H_{\mathrm{DR}}(X/\bar{K})$ is $\rho_{\mathrm{cris}}(\psi)$.

Now by the very definition of absolute Hodge cycles, the notion is functorial even for morphisms covering nontrivial automorphisms of the ground field, such as f. Thus, we have commutative diagrams:

$$\begin{array}{ccc}
H_{\mathrm{AH}}(X) & \hookrightarrow & H_{\mathrm{DR}}(X/\bar{K}) \\
\Big\downarrow {\scriptstyle f_{\mathrm{AH}}} & & \Big\downarrow {\scriptstyle f^*_{\mathrm{DR}}} \\
H_{\mathrm{AH}}(X) & \hookrightarrow & H_{\mathrm{DR}}(X/\bar{K})
\end{array}
\qquad
\begin{array}{ccc}
H_{\mathrm{AH}}(X) & \hookrightarrow & H_{\mathrm{et}}(X, \mathbb{Q}_l) \\
\Big\downarrow {\scriptstyle f_{\mathrm{AH}}} & & \Big\downarrow {\scriptstyle f_l^*} \\
H_{\mathrm{AH}}(X) & \hookrightarrow & H_{\mathrm{et}}(X, \mathbb{Q}_l).
\end{array}$$

But we also have a commutative diagram:

$$\begin{array}{ccc}
H_{\mathrm{et}}(X, \mathbb{Q}_l) & \xrightarrow{\sim} & H_{\mathrm{et}}(\mathbf{X}'_{\bar{k}}, \mathbb{Q}_l) \\
\Big\downarrow {\scriptstyle f_l^*} & & \Big\downarrow {\scriptstyle f_{\bar{k}}^*} \\
H_{\mathrm{et}}(X, \mathbb{Q}_l) & \xrightarrow{\sim} & H_{\mathrm{et}}(\mathbf{X}'_{\bar{k}}, \mathbb{Q}_l).
\end{array}$$

Since $f_{\bar{k}} = (F_{X'_{\bar{k}}})^d$ and *absolute* Frobenius acts *trivially* on l-adic cohomology, we see that $f_{AH} = \mathrm{id}$. □

(4.15) **Corollary.** *If each X_i is an abelian variety of* CM *type with ordinary reduction, then $H_{AH}(X)$ is fixed by $W_{\mathrm{cris}}(\bar{K})$.*

Proof. An abelian variety of CM type is defined over a number field and has potentially good reduction everywhere. Thus we may assume that each X_i has good reduction and that the residue field k is finite. Moreover, we may replace each X_i by any isogenous abelian variety, and in particular we may assume that each X_i is simple. It is clear from the preceeding result that it suffices to prove that X_i is isogenous to an abelian variety defined over $W(k)$ to which the absolute Frobenius endomorphism lifts, i.e., to the canonical lifting of its reduction. This is well-known, but here is a proof. Let $A = X_i$, let E be the CM field acting on A and assume that the full ring of integers $\mathcal{O}$ of E acts on A. Then $H^1(A_{\mathbb{C}}, \mathbb{Q})$ is a free E-vector space of rank one, and hence $H^1_{\mathrm{et}}(A_{\bar{K}'}, \mathbb{Q}_l)$ is a free $E \otimes \mathbb{Q}_l$-module of rank one. We have $E \hookrightarrow \mathrm{End}(A_k)$, and if $k = \mathbb{F}_{p^d}$, $\varphi =: (F_{A_k})^d$ is a k-linear endomorphism of A_k, i.e., belongs to $\mathrm{End}(A_k)$. This φ acts on $H^1(A_{\bar{k}}, \mathbb{Q}_l)$, and since $H^1(A_{\bar{k}}, \mathbb{Q}_l)$ is a free $E \otimes \mathbb{Q}_l$-module of rank one, there is an $e \in E \otimes \mathbb{Q}_l$ such that $H^1(e, \mathbb{Q}_l) = H^1(\varphi, \mathbb{Q}_l)$. But $\mathrm{End}(A_k)/E \hookrightarrow (\mathrm{End}(A_k)/E) \otimes \mathbb{Q}_l$ and $\mathrm{End}(A_k) \otimes \mathbb{Q}_l \subseteq \mathrm{End}\, H^1_{\mathrm{et}}(A_k, \mathbb{Q}_l)$, so φ in fact lies in E. Since φ is integral over $\mathbb{Z}$, it lies in $\mathcal{O}$. Then by [15], A is isogenous to the canonical lifting of A_k. □

Appendix: Torsion in PD Envelopes

(A.1) **Proposition.** *Let R be an algebra over the localization of $\mathbb{Z}$ at (p), let $I \subseteq R$ be an ideal, and let $\gamma_p: I \to I$ be a function satisfying:*

1) $\gamma_p(x+y) = \gamma_p(x) + \gamma_p(y) + \sum_{i=1}^{p-1} \frac{1}{i!} \frac{1}{(p-i)!} x^i y^{p-i}$.
2) $\gamma_p(ax) = a^p \gamma_p(x)$.
3) $p! \gamma_p(x) = x^p$.

Then there is a unique PD *structure $\{\gamma_n\}$ on I extending γ_p.*

Proof. The uniqueness is proved in [1, 1.2.5]. For the existence, suppose that N is a positive integer and that we have operators γ_n for all $n < N$ satisfying all the axioms, viz:

(i) $\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$ if $n < N$.

(ii) $\gamma_a(x)\gamma_b(x) = ((a,b))\gamma_{a+b}(x)$ if $a+b < N$ and $((a,b)) =: \frac{(a+b)!}{a!\,b!}$.

(iii) $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$ if $n < N$.

(iv) $\gamma_a(\gamma_b(x)) = C_{a,b}\gamma_{ab}(x)$ if $ab < N$ and $C_{a,b} = \frac{(ab)!}{a!(b!)^a}$.

Then we can define γ_N satisfying the same axioms, by the following rules:

Case 0: $N \leqq p$. If $N < p$, let $\gamma_N(x) = \frac{1}{N!} x^N$, if $N = p$, use γ_p. It is trivial to check the axioms.

Case 1: $p \nmid N$ and $N>p$. Set $\gamma_N(x)=N^{-1}x\gamma_{N-1}(x)$. Note that by (ii), $x\gamma_i(x)=(i+1)\gamma_{i+1}(x)$ for $0 \leqq i \leqq N-1$. Now let's check:

$$\text{(i)}\ \gamma_N(x+y)=N^{-1}(x+y)\sum_{i=0}^{N-1}\gamma_i(x)\gamma_{N-i-1}(y)$$
$$=N^{-1}\sum_{i=0}^{N-1}(i+1)\gamma_{i+1}(x)\gamma_{N-i-1}(y)+N^{-1}\sum_{i=0}^{N-1}(N-i)\gamma_i(x)\gamma_{N-i}(y)$$
$$=\gamma_N(x)+N^{-1}\sum_{i=1}^{N-1}i\gamma_i(x)\gamma_{N-i}(y)+N^{-1}\sum_{i=1}^{N-1}(N-i)\gamma_i(x)\gamma_{N-i}(y)+\gamma_N(y)$$
$$=\sum_{i+j=N}\gamma_i(x)\gamma_j(y).$$

(ii) We may assume that $a+b=N$, so that one of $\{a,b\}$ is not divisible by p; say $p \nmid a$. Then:

$$\gamma_a(x)\gamma_b(y)=a^{-1}x\gamma_{a-1}(x)\gamma_b(x)=a^{-1}((a-1,b))x\gamma_{a+b-1}(x)$$
$$=a^{-1}((a-1,b))(a+b)\gamma_{a+b}(x)=((a,b))\gamma_{a+b}(x).$$

(iii) Is trivial.

(iv) We may assume that $ab=N$, with $b>1$. Since $p \nmid a$ and $p \nmid b$,

$$\gamma_a(\gamma_b(x))=\gamma_a(b^{-1}x\gamma_{b-1}(x))=b^{-a}x^a\gamma_a(\gamma_{b-1}(x))$$
$$=b^{-a}(a!)\gamma_a(x)C_{a,b-1}\gamma_{ab-a}(x)$$
$$=b^{-a}(a!)C_{a,b-1}((a,ab-a))\gamma_{ab}(x)=C_{ab}\gamma_{ab}(x).$$

Case 2: $N=pm$ with $m>1$. In this case, one sees easily that $C_{m,p}$ is a p-adic unit, so we can define $\gamma_N(x)=C_{m,p}^{-1}\gamma_m(\gamma_p(x))$. Let us again check the axioms. The most difficult is (i), for which we shall need some notation. If $\mathbf{m}$ is a multi-index $(m_0,m_1,\ldots,m_p)$, let $|\mathbf{m}|=m_0+m_1+\ldots+m_p$, $a(\mathbf{m})=m_1+2m_2+\ldots+pm_p$, $b(\mathbf{m})=pm_0+(p-1)m_1+\ldots+m_{p-1}$.

Claim: There are universal constants $N_{\mathbf{m}}\in\mathbb{Z}$ (made explicit below) such that $\gamma_N(x+y)=C_{m,p}^{-1}\sum_{|\mathbf{m}|=m}N_{\mathbf{m}}\gamma_{a(\mathbf{m})}(x)\gamma_{b(\mathbf{m})}(y)$, (provided only that the γ_n's satisfy the axioms for $n<N$ and that γ_N is defined as above).

Granted the claim, let us prove property (i). Note that if $|\mathbf{m}|=m$, then $a(\mathbf{m})+b(\mathbf{m})=mp=N$. If $a+b=N$, let $N_{a,b}=:\sum\{N_{\mathbf{m}}\colon |\mathbf{m}|=m,\ a(\mathbf{m})=a,\ b(\mathbf{m})=b\}$. Then the claim can be rewritten:

$$\gamma_N(x+y)=C_{m,p}^{-1}\sum_{a+b=N}N_{a,b}\gamma_a(x)\gamma_b(y).$$

Thus, (i) reduces to the assertion that whenever $a+b=m$, $N_{a,b}=C_{m,p}$. Although this could be checked explicitly, we can also use the following argument. The ideal $(x,y)\subseteq\mathbb{Q}[x,y]$ admits a unique divided power structure, and it follows that the formula in the claim and formula (i) are both true in $\mathbb{Q}[x,y]$. Since $\{\gamma_a(x)\gamma_b(y)\}$ are linearly independent over $\mathbb{Q}$, we must have $N_{a,b}=C_{m,p}$ in $\mathbb{Q}$, hence also in $\mathbb{Z}$.

Perhaps it is unnecessary to make the ugly formula for $N_{\mathbf{m}}$ explicit, but here it is. If $(b_1,\ldots,b_k)$ is a sequence of positive integers, we let $((b_1,\ldots,b_k))=:\dfrac{(b_1+\ldots b_k)!}{b_1!\ldots b_k!}$. Then:

$$N_{\mathbf{m}}=:\left[\prod_{i=0}^{p}(m_i!)\,C_{m_i,i}\,C_{m_i,p-i}\right]((pm_0,(p-1)m_1,\ldots))((m_1,2m_2,\ldots,pm_p)).$$

Now let us prove the claim. We use formula (i) first for γ_p and then for γ_m to obtain

$$\gamma_N(x+y)=C_{m,p}^{-1}\gamma_m(\gamma_p(x+y))=C_{m,p}^{-1}\gamma_m\left(\sum_{i=0}^{p}\gamma_i(x)\gamma_{p-i}(y)\right)=C_{m,p}^{-1}\sum_{|\mathbf{m}|=m}\alpha_{\mathbf{m}},$$

$$\text{where }\alpha_{\mathbf{m}}=\prod_{i=0}^{p}\gamma_{m_i}(\gamma_i(x)\gamma_{p-i}(y)).$$

We now must calculate each term in the product. If $1 \leqq i \leqq p-1$, we have

$$\begin{aligned}\gamma_{m_i}(\gamma_i(x)\gamma_{p-i}(y)) &= (m_i)!\,\gamma_{m_i}(\gamma_i(x))\,\gamma_{m_i}(\gamma_{p-i}(y))\\ &= (m_i)!\,C_{m_i,i}\,\gamma_{im_i}(x)\,C_{m_i,p-i}\,\gamma_{(p-i)m_i}(y).\end{aligned}$$

For $i=0$ and $i=p$ we get, respectively:

$$\gamma_{m_0}(\gamma_p(y)) = C_{m_0,p}\,\gamma_{pm_0}(y) = (m_0)!\,C_{m_0,p}\,C_{m_0,0}\,\gamma_{pm_0}(y)$$

and

$$\gamma_{m_p}(\gamma_p(x)) = C_{m_p,p}\,\gamma_{pm_p}(x) = (m_p)!\,C_{m_p,0}\,C_{m_p,p}\,\gamma_{pm_p}(x),$$

which are of the same form. Thus we can write:

$$\alpha_{\mathbf{m}} = \prod_{i=0}^{p} (m_i!)\,C_{m_i,i}\,C_{m_i,p-i}\,\gamma_{im_i}(x)\,\gamma_{(p-i)m_i}(y).$$

Since $\prod_{i=0}^{p} \gamma_{im_i}(x)\,\gamma_{(p-i)m_i}(y) = ((pm_0,\dots,m_{p-1}))\,((m_1,2m_2,\dots,pm_p))\,\gamma_{a(\mathbf{m})}\,\gamma_{b(\mathbf{m})}$, the claim follows. This completes the proof of (i).

To prove (ii), we first suppose that $a=1$ and $b=N-1$. We have to check that $x\gamma_{N-1}(x) = N\gamma_N(x)$. We have

$$\begin{aligned}N\gamma_N(x) &= C_{m,p}^{-1}\,pm\gamma_m(\gamma_p(x)) = C_{m,p}^{-1}\,p\gamma_p(x)\,\gamma_{m-1}(\gamma_p(x))\\ &= C_{m,p}^{-1}\,x\gamma_{p-1}(x)\,C_{m-1,p}\,\gamma_{mp-p}(x)\\ &= C_{m,p}^{-1}\,C_{m-1,p}((p-1,mp-p))\,x\gamma_{mp-1}(x) = x\gamma_{mp-1}(x).\end{aligned}$$

Now suppose that one of $\{a,b\}$ is greater than 1 and not divisible by p - say a. Then $\gamma_a(x)\gamma_b(x) = a^{-1}x\gamma_{a-1}(x)\gamma_b(x) = a^{-1}((a-1,b))\,x\gamma_{N-1}(x) = a^{-1}((a-1,b))\,N\gamma_N(x)$ by the previous case. Since $a^{-1}((a-1,b))N = ((a,b))$, (ii) follows in this case also. Finally, if $a=pa'$ and $b=pb'$, we have

$$\begin{aligned}\gamma_a(x)\,\gamma_b(x) &= C_{a',p}^{-1}\,C_{b',p}^{-1}\,\gamma_{a'}(\gamma_p(x))\,\gamma_{b'}((\gamma_p(x))\\ &= C_{a',p}^{-1}\,C_{b',p}^{-1}((a',b'))\,\gamma_{a'+b'}(\gamma_p(x))\\ &= C_{a',p}^{-1}\,C_{b',p}^{-1}((a',b'))\,C_{a'+b',p}\;\gamma_N(x)\\ &= ((a,b))\,\gamma_N(x).\end{aligned}$$

Property iii) is trivial, and iv) is easy. □

(A.2) **Proposition.** *If $Z =: \operatorname{Spec} k[x,y]/(x^2,xy,y^2)$, then $H^0_{\text{cris}}(Z/W(k))$ contains p-torsion.*

Proof. We do the case $p \neq 2$ and leave the modifications when $p=2$ to the reader.

Let J be the ideal $(x^2,xy,y^2) \subseteq W[x,y]$, and let γ be the standard divided power structure on (p). The first step is to show that the PD envelope $(\mathscr{D},\bar{J},\gamma)$ of $\mathcal{O}_Z$ in $A =: W[x,y]$ (compatible with γ on (p)) contains p-torsion. Let $\tau \in \bar{J}$ be the element $\tau =: \gamma_p(x^2)\gamma_p(y^2) - [\gamma_p(xy)]^2$. We claim that $p\tau = 0$. Indeed, $p!\,\gamma_p(x^2)\gamma_p(y^2) = (x^2)^p\gamma_p(y^2) = \gamma_p(x^2y^2) = (xy)^p\gamma_p(xy) = (p!)\,[\gamma_p(xy)]^2$. The hard part is to show that $\tau \neq 0$. To do this, it suffices to find a PD-algebra (B,K,δ), and a homomorphism $\theta\colon A \to B$ mapping the ideal (x^2,xy,y^2) to K, such that $\delta_p\,\theta(x^2)\,\delta_p\,\theta(y^2) \neq [\delta_p\,\theta(xy)]^2$.

Take $B = k[x,y,u,v]/(x^3,y^3,x^2y,xy^2,u^2,v^2,xu,xv,yu,yv)$, and let $K \subseteq B$ be the ideal generated by (x^2,y^2,xy,u,v). As a k-vector space K has a basis: (x^2,xy,y^2,u,v,uv). Define a Frobenius linear map $\delta_p\colon K \to K$ by $\delta_p(ax^2+bxy+cy^2+du+ev+fuv) =: a^pu + c^pv$.

Claim: δ_p extends to a PD structure on K.

We use (A.1). If $\alpha_1 = a_1x^2 + b_1xy + \dots$ and $\alpha_2 = a_2x^2 + \dots$, then $\delta_p(\alpha_1+\alpha_2) = (a_1^p + a_2^p)u + (c_1^p + c_2^p)v$. On the other hand, $K^3 = 0$, so $\alpha_1^i\alpha_2^j = 0$ if $i+j=p$. This implies that the right hand side of 1) reduces to $\delta_p(\alpha_1) + \delta_p(\alpha_2)$, which checks 1). The other two properties are clear. Now with the obvious map $A \to B$ and the obvious notation, we see that in B, $\delta_p(x^2)\,\delta_p(y^2) = uv$, while $\delta_p(xy)^2 = 0$.

Note that since $pB = 0$, the map $\mathscr{D} \to B$ factors through $\mathscr{D} \otimes \mathbb{Z}/p\mathbb{Z}$, hence $\tau \notin p\mathscr{D}$ and hence τ maps to a nonzero element of the p-adic completion of $\mathscr{D}$.

It remains only to check that $d\tau=0$. Writing $\alpha^{[n]}$ for $\gamma_n(\alpha)$, we calculate:

$$\tau=(x^2)^{[p]}(y^2)^{[p]}-((xy)^{[p]})^2=(x^2)^{[p]}(y^2)^{[p]}-((p,p))(xy)^{[2p]}.$$

$$\begin{aligned}\partial\tau/\partial x&=(2x)(x^2)^{[p-1]}(y^2)^{[p]}-((p,p))\,y(xy)^{[2p-1]}\\&=\frac{2}{(p-1)!}x^{2p-1}(y^2)^{[p]}-((p,p))((p-1,p))^{-1}(xy)^{[p]}(xy)^{[p-1]}y\\&=2x^{[p-1]}x^p(y^2)^{[p]}-2(xy)^{[p]}x^{[p-1]}y^p\\&=2x^{[p-1]}(x^p(y^2)^{[p]}-(xy^2)^{[p]})\\&=0.\end{aligned}$$

Since a similar calculation implies that $\partial\tau/\partial y=0$, we find $d\tau=0$, hence τ induces a torsion element of $H^0_{\mathrm{cris}}(Z/W)$. □

(A.3) *Consequence: The absolute Frobenius endomorphism $F_Z^*\colon H^0_{\mathrm{cris}}(Z/W)\to H^0_{\mathrm{cris}}(Z/W)$ is not injective.*

Proof. Let $\varphi\colon W[x,y]\to W[x,y]$ send (x,y) to (x^p,y^p), and let $\bar\varphi\colon\mathscr{D}_J(A)\to\mathscr{D}_J(A)$ be the induced map. Then the map $\mathrm{Ker}(d)\to\mathrm{Ker}(d)$ induced by $\bar\varphi$ can be identified with F_Z^*, so it suffices to check that $\bar\varphi(\tau)=0$. But

$$\begin{aligned}\bar\varphi(\tau)&=\bar\varphi\{\gamma_p(x^2)\gamma_p(y^2)-[\gamma_p(xy)]^2\}\\&=\gamma_p(x^{2p})\gamma_p(y^{2p})-[\gamma_p(x^py^p)]^2\\&=\{(xy)^{2p-2}\}^p[\gamma_p(x^2)\gamma_p(y^2)-\gamma_p(xy)]^2\\&=0\end{aligned}$$

since $\{(xy)^{2p-2}\}^p$ is divisible by p. □

(A.4) **Consequence.** *Over the base Z, the Dieudonné functor $\mathbb{D}$ from the category of p-divisible groups to the category of (F,V)-crystals is not fully faithful.*

Proof. Let G be the p-divisible group associated to the constant deformation of an ordinary elliptic curve over Z. Its Dieudonné crystal $\mathbb{D}$ is determined by its value H on the object $D=\mathrm{Spf}\,\hat{\mathscr{D}}_J(A)$; it is a free $\mathcal{O}_D$-module of rank 2 with horizontal basis $\{\omega,\eta\}$ in which the (F,V)-crystal structure is given by $\Phi(\omega^\sigma)=p\omega$, $\Phi(\eta^\sigma)=\eta$, $V(\omega)=\omega^\sigma$, $V(\eta)=p\eta^\sigma$ (where the exponent σ denotes the pull-back by $\bar\varphi$). Consider the endomorphism ε of H given by $\omega\mapsto\tau\eta$, $\eta\mapsto 0$. Since $d\tau=0$, this endomorphism is horizontal, and it is clear that $\Phi\circ\varepsilon^\sigma=\varepsilon\circ\Phi$, $\varepsilon^\sigma\circ V=V\circ\varepsilon$. Since $\varepsilon(0)=0$ but $\varepsilon\neq 0$, ε cannot be induced by a morphism $G\to G$.

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