A SPRINT TO THE KOSZUL SELF DUALITY OF E_n

CONNOR MALIN

ABSTRACT. We use pinch maps to verify the Koszul self duality of E_n with respect to a new model of Koszul duality recently introduced by Lurie.

Recently, Lurie introduced a particularly simple definition of Koszul duality in terms of presheaves on $\mathrm{Env}(O)$, the monoidal envelope of O [3]. I verify the equivalence of operads $\Sigma^\infty_+ E_n \simeq S_n \wedge K(\Sigma^\infty_+ E_n)$ with respect to this definition. Such a result was originally due to Fresse [2] in the algebraic case and Ching–Salvatore in the topological case [1]. All operads are reduced, i.e. $O(1) \simeq S^0$ and $O(0) = *$.

The Spec-enriched category $\mathrm{Fun}^{\mathrm{Spec}}(\mathrm{Env}(O)^{\mathrm{op}}, \mathrm{Spec})$ has a symmetric monoidal product $\otimes$ informally given by

$$(F \otimes G)(k) = \bigvee_{i+j=k} F(i) \otimes G(j) \wedge_{\Sigma_i \times \Sigma_j} \Sigma_{i+j}.$$

For a framed n-manifold M, the disk presheaf $E_M(i) = \mathrm{Emb}^{\mathrm{fr}}(\bigsqcup_i \mathbb{R}^n, M)$. These presheaves satisfy

$$E_M \otimes E_N \simeq E_{M \sqcup N}.$$

We write $\mathrm{CoEnd}(-)$ for the coendomorphism operad of an object $c \in (C, \otimes)$ given by

$$\mathrm{CoEnd}(c)(i) = \mathrm{C}(c, c^{\otimes i}).$$

Let 1 denote the presheaf on $\mathrm{Env}(O)$ for which $1(1) = S^0$ and is contractible otherwise.

Definition. The Koszul dual of O is $\mathrm{CoEnd}(1)$.

Definition. The n-sphere operad S_n is $\mathrm{CoEnd}(S^n)$.

The pinch map is the map

$$E_n(i) \wedge E_{(\mathbb{R}^n)^+}(j) \to E_{(\bigsqcup_i \mathbb{R}^n)^+}(j)$$

which takes a (framed) embedding $\bigsqcup_i \mathbb{R}^n \to \mathbb{R}^n$, applies the Pontryagin–Thom construction to get a map $(\mathbb{R}^n)^+ \to (\bigsqcup_i \mathbb{R}^n)^+$ and uses it to pushforward the j singularly embedded disks in $(\mathbb{R}^n)^+$ into $(\bigsqcup_i \mathbb{R}^n)^+$.

Theorem. *There is an equivalence $\Sigma^\infty_+ E_n \simeq S_n \wedge K(\Sigma^\infty_+ E_n)$.*

Proof. The equivalence is given by

$$\Sigma^\infty_+ E_n \xrightarrow{\simeq} \mathrm{CoEnd}(E_{(\mathbb{R}^n)^+}) \simeq \mathrm{CoEnd}(\Sigma^n 1) \simeq S_n \wedge K(\Sigma^\infty_+ E_n)$$

The first functor is the adjoint of the pinch map and is an equivalence by [4, Theorem 9.4]. The second map exists by the elementary equivalence $E_{(\mathbb{R}^n)^+} \simeq \Sigma^n 1$ [4, Lemma 7.2]. The third map is an equivalence by inspection of definitions.

□

References

[1] Michael Ching and Paolo Salvatore. Koszul duality for topological E_n operads. *Proceedings of the London Mathematical Society*, 125(1):1–60, 2022.

[2] Benoit Fresse. Koszul duality of E_n operads. *Selecta Mathematica*, 17(2):363–434. 2010.

[3] Jacob Lurie. Lie algebras in stable homotopy theory, 2023. Homotopy theory in honor of Paul Goerss.

[4] Connor Malin. Koszul self duality of manifolds, 2023.