Research Statement

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Introduction

My research addresses the interaction of algebraic and geometric topology, and specifically the application of homotopy theory to problems involving manifolds. Homotopy theory is the study of objects up to continuous deformation; consequently it is useful in the study of invariants which are immune to such changes, such as homology groups and homotopy groups. Manifolds, on the other hand, are naturally studied up to homeomorphism and diffeomorphism which do not allow for any deformation. As a result, the relation of homotopy theory and manifold theory is necessarily subtle. A particularly stark contrast is that injective functions play an important role in manifold theory, while injectivity is destroyed by continuous deformations. As a consequence, configuration spaces, embedding spaces, and related moduli spaces associated to manifolds are often functorial with respect to injective maps, but not with respect to arbitrary smooth maps.

These classical objects of study have seen a surge in activity recently due to advances in our understanding of cobordism categories, operad theory, and functor calculus. Functor calculus is a technique for understanding the homotopical behavior of functors. It was first developed by Goodwillie, in the case of spaces, and Weiss, in the case of manifolds. I will call the former "Goodwillie calculus" and the latter "embedding calculus." Functor calculus decomposes functors into simpler "polynomial approximations," analogous to the Taylor approximations of smooth functions. In this way, Goodwillie calculus and embedding calculus are forms of *differential* functor calculus. The theory of *integral* functor calculus of manifolds, or factorization homology, was developed by Ayala, Francis, Lurie, and Tanaka. The primary goal of factorization homology is to construct invariants of manifolds from homotopy coherent algebras called E_n-algebras.

There are striking similarities between Goodwillie calculus, embedding calculus, and factorization homology, and their precise relationship is the subject of current research. Work of Arone–Ching, Boavida de Brito–Weiss, Ayala–Francis, and Turchin shows that certain operads govern the three versions of functor calculus. Variants of the E_n operad govern embedding calculus and factorization homology, while the commutative and Lie operads govern Goodwillie calculus in Koszul dual ways. Koszul duality also has deep interactions with factorization homology, as demonstrated by Ayala–Francis through their work on Poincaré/Koszul duality.

Recently, Ching–Salvatore constructed a commutative diagram relating the operads lie, $\Sigma^\infty_+ E_n$, and com to their Koszul duals, extending work of Fresse in the algebraic case:

$$\begin{tikzcd}
s_n\mathrm{lie} \arrow[r] \arrow[d, "\simeq"] & \Sigma^\infty_+ E_n \arrow[r] \arrow[d, "\simeq"] & \mathrm{com} \arrow[d, "\simeq"] \\
s_n K(\mathrm{com}) \arrow[r] & s_n K(\Sigma^\infty_+ E_n) \arrow[r] & K(\mathrm{lie})
\end{tikzcd}$$

In particular $\Sigma^\infty_+ E_n$ is *Koszul self dual.* In my work, I

- develop new methods to prove and utilize Koszul self duality results,

- produce geometric consequences of the Koszul self duality of $\Sigma^\infty_+ E_n$, and
- investigate the role of Koszul duality in functor calculus.

Using this program, I have solved conjectures of Ayala-Francis and Ching and proven strong results in Goodwillie calculus, embedding calculus, and factorization homology. Along the way, I produced a conceptual construction of the self duality of the spectral little disks operad, vastly simplifying the proof of Ching–Salvatore and connecting it to the work of Ayala–Francis. These results advance programs of Arone–Ching, Ayala–Francis, and Knudsen, and introduce new and fruitful questions about operad theory, functor calculus, and Koszul duality.

Past research

My Ph.D. research focused on the interaction of Koszul duality with Goodwillie calculus, embedding calculus, and factorization homology. In particular, I studied Koszul duality for right modules over operads and the applications to functor calculus. This section contains a selection of these results together with brief contextualizing comments.

The main result of my thesis regards the right E_n-module E_M of little disks in a framed n-manifold M. I compared E_M with $K(E_M)$, extending the work of Ching–Salvatore on the self duality of E_n. I made such a comparison in two different ways: first by constructing a theory of Koszul/Verdier duality for operads in $(\mathrm{Top}, \times)$ and, later, by studying Pontryagin–Thom constructions of configuration spaces.

Theorem 0.1 (The Koszul self duality of manifolds) *There is a zigzag of equivalences of operads*

$$\Sigma^\infty_+ E_n \simeq \cdots \simeq s_n K(\Sigma^\infty_+ E_n),$$

such that for any framed manifold M, there is a compatible zigzag of equivalences of right modules

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

The Koszul self duality of manifolds has many interesting consequences. Of particular interest, it produces an automorphism of stable embedding calculus:

Theorem 0.2 (The stable Pontryagin–Thom construction for E_M) *There is an equivalence*

$$T : \mathrm{Map}^h_{\Sigma^\infty_+ E_n}(\Sigma^\infty_+ E_M, \Sigma^\infty_+ E_N) \xrightarrow{\simeq} \mathrm{Map}^h_{\Sigma^\infty_+ E_n}(\Sigma^\infty E_{N^+}, \Sigma^\infty E_{M^+}).$$

I also applied the self duality of manifolds to solve a conjecture of Ching about the relation of the lie-module structures in Goodwillie calculus and the E_n-module structures in embedding calculus.

Theorem 0.3 (Ching's Conjecture) *If M is a framed n-manifold, there is an equivalence of* lie *modules*

$$\mathrm{res}_{\mathrm{lie}}(s_{(-n,-n)}\Sigma^\infty_+ E_M) \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

The Poincaré/Koszul duality of E_n-algebras has long been suspected to be related to the Koszul self duality of E_n and E_M. Using this idea, I extended Poincaré/Koszul duality to left E_n-modules:

Theorem 0.4 (Poincaré/Koszul duality for left E_n modules) *For M a framed n-manifold and L a reduced left $\Sigma^\infty_+ E_n$-module, there is an equivalence*

$$\int_M L \xrightarrow{\simeq} \int^{M^+} s_n B(L).$$

Using the Koszul self duality of E_n, it is possible to give an alternative definition of Knudsen's higher enveloping algebras of spectral Lie algebras as the induction along the map of operads $s_n\mathrm{lie} \to \Sigma^\infty_+ E_n$. The resolution of Ching's conjecture implies:

Theorem 0.5 (Homotopy invariance of factorization homology for spectral Lie algebras) *If X is a spectral Lie algebra with bracket of degree $n-1$ and M is a framed n-manifold, then $\int_M U_n(X)$ is a homotopy invariant of M^+.*

Using duality techniques, I studied the homotopy invariance of stable embedding calculus. Recall that a tangential homotopy equivalence is a homotopy equivalence which pulls back the tangent bundle.

Theorem 0.6 (Homotopy invariance of the layers of stable embedding calculus) *If N is a smooth manifold, the layers of the embedding tower of any functor of the form $F \circ \Sigma^\infty_+ \mathrm{Emb}(-, N)$ are proper tangential homotopy invariants of N.*

Current Research

Goodwillie calculus in algebra categories

My current research surrounds various extensions of the work of Arone–Ching on Goodwillie calculus to other settings. Primarily, I am interested in producing right module structures on the derivatives of functors with values in Spec and investigating the applications of this structure. Fix a reduced operad O in spectra. Let TAQ denote topological Andre–Quillen homology, i.e. derived indecomposables, of an O-algebra. Fix a functor $F : \mathrm{Alg}_O \to \mathrm{Spec}$. I define:

Definition 0.7 *The Koszul coderivatives $\partial^* F$ are* $\mathrm{Nat}(F, \mathrm{TAQ}(-)^{\wedge^*})$.

The Koszul coderivatives can be seen to form a right module over $K(O)$ as a result of the fact that $\mathrm{TAQ}(A)$ has a $K(O)$-coalgebra structure. This allows us to form an approximation to the Goodwillie tower of F:

Definition 0.8 *The ith fake Goodwillie approximation $P_i^{\mathrm{fake}}(F)(A)$ is*

$$\mathrm{Map}^h_{K(O)}(\partial^* F^{\le i}, \partial^*(\Sigma^\infty \mathrm{Map}(A, -))^{\le i}).$$

Via the enriched Yoneda Lemma, one obtains maps

$$F(A) \to P_i(F)(A) \to P_i^{\mathrm{fake}}(F)(A).$$

If D_i and D_i^{fake} denote the layers of the Goodwillie and fake Goodwillie towers, a computation with the Yoneda lemma shows:

Theorem 0.9 *If $\partial_*(F)$ are nonequivariantly finite spectra, the induced map on layers*

$$D_i(F)(A) \to D_i^{\mathrm{fake}}(F)(A)$$

can be identified with the norm map

$$\partial_i(F) \wedge_{h\Sigma_i} \mathrm{TAQ}^{\wedge i}(A) \to \partial_i(F) \wedge^{h\Sigma_i} \mathrm{TAQ}^{\wedge i}(A).$$

In the process of proving this theorem, I found a new model for the Goodwillie derivatives:

Theorem 0.10 *If $\partial_*(F)$ are finite as nonequivariant spectra, then*

$$\partial_*(F) \simeq K(\partial^*(F)).$$

As such, the derivatives of F are weakly equivalent to right modules over O.

In the case $\partial_*(F)$ is levelwise equivariantly finite, the norm maps for the layers will be equivalences, and the fake Goodwillie tower agrees with the Goodwillie tower. Taking $O = \Sigma^\infty_+ E_n$, I demonstrate

Theorem 0.11 *For M a framed n-manifold and A a $\Sigma^\infty_+ E_n$-algebra, the map*

$$P_\infty(\int_M)(A) \to P_\infty^{\text{fake}}(\int_M)(A)$$

can be identified with a Koszul duality equivalence

$$P_\infty(\int_M)(A) \xrightarrow{\simeq} \int^{K(\Sigma^\infty_+ E_M)} B(A).$$

My work on Koszul self duality then recovers the Poincaré–Koszul duality of E_n-algebras and an analogous statement for Lie algebras (note P_∞ is taken in $\Sigma^\infty_+ E_n$-algebras and Lie algebras, respectively):

Theorem 0.12 *For M a framed n-manifold and A a $\Sigma^\infty_+ E_n$-algebra,*

$$P_\infty(\int_M)(A) \xrightarrow{\simeq} \int^{M^+} \Sigma^n B(A).$$

For a spectral Lie algebra X with bracket of degree $n-1$,

$$P_\infty(\int_M U_n(-))(X) \xrightarrow{\simeq} \int^{M^+} \Sigma^n B(X).$$

As well, I am pursuing a classification result for Goodwillie towers in O-algebra categories in terms of divided power modules, analogous to Arone–Ching's classification of Goodwillie towers of functors $\mathrm{Top}_* \to \mathrm{Spec}$. They found that the norm data of the derivatives could be packaged into the structure of a "divided power right module" in order to enhance the comparison between the fake Goodwillie tower and the Goodwillie tower to be an equivalence in general.

When applied to generalized factorization homology for an operad O, I expect to recover Amabel's divided power Koszul duality map. Such a result would allow us to understand divided power coalgebras in a new way. In particular, one could establish Poincaré-Koszul duality for $\Sigma^\infty_+ E_\infty$-algebras, necessarily in terms of divided power factorization cohomology over the spectral Lie operad. Here P_∞ is taken in the category of $\Sigma^\infty_+ E_\infty$-algebras:

$$P_\infty(\int_M)(A) \xrightarrow{\simeq} \int_{\text{d.p.}}^{\text{res}_{s_n \text{lie}} E_{M^+}} \Sigma^n B(A).$$

Stable Orthogonal Calculus

I am working to adapt the techniques of the previous section to study stable orthogonal calculus. Let $\mathrm{Vect}^{\mathrm{inj}}$ denote the category of finite dimensional inner product spaces and maps preserving the inner product. Orthogonal calculus applies to functors $\mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$ and has been used to study objects of classical geometric interest such as $\mathrm{BTop}(n)$ and $\mathrm{Emb}(M, N)$, as well as homotopical objects such as the EHP spectral sequence.

I am using Koszul duality to upgrade the well-known analogy of the relation between Goodwillie calculus and orthogonal calculus and between finite sets and finite dimensional vector spaces. In particular, Koszul duality explains that the Lie operad acts on Goodwillie derivatives precisely because the Koszul coderivatives of functors $\mathrm{Top}_* \to \mathrm{Spec}$ can be upgraded from a symmetric sequence to a presheaf on the category of finite sets and surjections, the monoidal envelope of com. I emulate this result in orthogonal calculus by replacing the role of finite sets and surjections by finite dimensional vector spaces and surjections.

Let $nS : \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$ denote the functor $V \to (\Sigma^\infty V^+)^{\wedge n}$. Fix a functor $F : \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$.

Definition 0.13 *The orthogonal sequence of Koszul coderivatives $\partial^* F$ is $\mathrm{Nat}(F, *S)$.*

Let $\mathrm{Vect}^{\mathrm{sur}}$ denote the category obtained by dualizing objects and maps of $\mathrm{Vect}^{\mathrm{inj}}$. By a module over $\mathrm{Vect}^{\mathrm{sur}}$, we mean a functor $(\mathrm{Vect}^{\mathrm{sur}})^{\mathrm{op}} = \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$. The Koszul coderivatives naturally have the structure of a $\mathrm{Vect}^{\mathrm{sur}}$-module. Under our analogy, this correponds to the fact the Koszul coderivatives of a functor $F : \mathrm{Top}_* \to \mathrm{Spec}$ form a $\mathrm{FinSet}^{\mathrm{sur}}$-module, i.e. a right com-module.

Preliminary definitions and computations suggest that with finiteness conditions on $\partial_i F$, there is an equivalence

$$\partial_i F \simeq \mathrm{TAQ}(\partial^* F)(i)^\vee \wedge D_{O(i)}.$$

Here TAQ is defined as the adjoint to $\mathrm{Triv} : \mathrm{OrthSeq}(\mathrm{Spec}) \to \mathrm{RMod}_{\mathrm{Vect}^{\mathrm{sur}}}$, and D_G is the dualizing spectrum of a topological group G. Using the Yoneda lemma, one constructs maps

$$F(V) \to P_i(F)(V) \to P_i^{\mathrm{fake}}(F)(V) := \mathrm{Map}_{\mathrm{Vect}^{\mathrm{sur}}}(\partial^* F^{\leq i}, (\partial^* \Sigma_+^\infty \mathrm{Vect}^{\mathrm{inj}}(V, -))^{\leq i}).$$

As i varies, the rightmost spectra form the "fake orthogonal tower". The induced maps on the layers of the towers take the form

$$\partial_i F \wedge_{hO(i)} iS(V) \to (\partial_i F \wedge D^\vee_{O(i)}) \wedge^{hO(i)} iS(V).$$

Conjecture 0.14 *This is the norm map for $\partial_i F \wedge D^\vee_{O(i)} \wedge iS(V)$.*

In order to express these constructions in terms of a module structure on $\partial_* F$ rather than $\partial^* F$, one needs a form of Koszul duality.

Definition 0.15 *The category $K(\mathrm{Vect}^{\mathrm{sur}})$ is the opposite of the full subcategory of $\mathrm{RMod}_{\mathrm{Vect}^{\mathrm{sur}}}$ on $\mathrm{Triv}(\Sigma_+^\infty O(i))$, as i varies.*

For a levelwise nonequivariantly finite $\mathrm{Vect}^{\mathrm{sur}}$-module, $\mathrm{TAQ}(R)^\vee \wedge D_{O(*)}$ has the structure of a $K(\mathrm{Vect}^{\mathrm{sur}})$-module. So, under finiteness conditions, $\partial_* F$ forms a module over $K(\mathrm{Vect}^{\mathrm{sur}})$, analogous to how the Goodwillie derivatives of a functor $\mathrm{Top}_* \to \mathrm{Spec}$ form a module over $K(\mathrm{FinSet}^{\mathrm{sur}})$, or in other words – a right lie-module.

Jointly with Niall Taggart, I am working to show the analog of Arone–Ching's classification of Goodwillie towers in terms of divided power lie-modules:

Conjecture 0.16 *The orthogonal tower of $F : \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$ depends only on a canonical "divided power module structure" on $\partial_* F$ over the category $K(\mathrm{Vect}^{\mathrm{sur}})$.*