Koszul duality and manifold calculus

Advances in Homotopy Theory IV

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The rational story

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It is straightforward to check that

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where $|x| = 2k, |y| = 4k - 1$ with $dy = x^2$.

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The generators and differentials conspire to allow for a map $S^{4k-1}_{\mathbb{Q}} \to S^{2k}_{\mathbb{Q}}$ which can be seen to generate $\pi_{4k-1}(S^{2k}) \otimes \mathbb{Q} \cong \mathbb{Q}$.

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Definition
An operad O in a symmetric monoidal category $(C, \otimes)$ is a collection of objects $O(I)$ with actions of Σ_I, for all nonempty finite sets I, together with maps for all I, J and $a \in I$

$$O(I) \otimes O(J) \to O(I \cup_a J)$$

which satisfy associativity and equivariance conditions.

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The commutative operad com in $(\mathrm{dgVect}_{\mathrm{k}}, \otimes)$ is given by

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Proposition
The cup product on $\bar{C}^*(X; k)$ naturally extends to an E_∞-algebra structure.

Theorem (Mandell)
For simply connected spaces X, Y there is a homotopy equivalence $X \simeq Y$, if and only if, there is an equivalence of E_∞-algebras $C^*(X;\mathbb{Z}) \simeq C^*(Y;\mathbb{Z})$.

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The composite

$$\mathrm{Top}_*^{\geqslant 2} \xrightarrow{\bar{C}^*} \mathrm{Alg}_{E_\infty}(\mathrm{DGVect}_{\mathbb{F}_p}, \otimes) \xrightarrow{K} \mathrm{Alg}_{L_\infty}(\mathrm{DGVect}_{\mathbb{F}_p}, \otimes)$$

takes values in contractible L_∞-algebras.

Higher algebra

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Given a pointed space X, the function spectrum $F(\Sigma^\infty X, S^0)$ obtains an E_∞-algebra structure from the diagonal of X.

This E_∞-structure is a lift of the E_∞-structure on $\bar{C}^*(X)$ to spectra.

Spectral lie algebras

A combination of results of Ching, Francis, Gaitsgory, Harper, and Lurie establish that theories of Koszul duality for algebras and operads in spectra exist, and these lift algebraic Koszul duality in some sense.

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With this in mind, one defines the spectral lie operad $\mathrm{lie} := K(\mathrm{com})$, and one obtains a functor

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Unfortunately, the negative results of Mandell still imply that this functor destroys all p-torsion information when applied to $F(\Sigma^\infty X, S^0)$.

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The layers of this tower can be "destabilized" by work of Bousfield and Heuts to produce analogs of rational homotopy theory that detect p-torsion.

$$\mathrm{Top}_{v_0} = \mathrm{Top}_{\mathbb{Q}} \qquad \mathrm{Top}_{v_1} \qquad \dots \qquad \mathrm{Top}_{v_n} \qquad \dots$$

Higher algebra models

Theorem (Heuts)
If $n > 0$, there is an equivalence $\mathrm{Alg}_{\mathrm{lie}}(\mathrm{Spec}_{v_n}) \simeq \mathrm{Top}_{v_n}$.

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Theorem (Behrens-Rezk, Heuts)
If the Goodwillie tower for the identity of $X \in \mathrm{Top}_{v_n}$ converges, the Koszul dual of $F(X, S^0_{v_n})$ is the v_n lie algebra model of X. In particular, the Goodwillie tower converges for the spheres $S^d_{v_n}$.

Goodwillie calculus

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 & P_\infty(F)(X) \arrow[d] & \\
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 & P_n(F)(X) \arrow[d] & \partial_n(F) \wedge_{h\Sigma_n} \Sigma^\infty X^{\wedge n} \arrow[l] \\
 & \cdots \arrow[d] & \\
 & P_1(F)(X) \arrow[d] & \partial_1(F) \wedge \Sigma^\infty X \arrow[l] \\
F(X) \arrow[r] \arrow[ur] \arrow[uuur] \arrow[uuuuur] & P_0(F)(X) &
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Definition
A right module R over an operad O is a collection of objects $R(I)$ with actions of Σ_I, for all nonempty finite sets I, together with maps for all I, J and $a \in I$

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Theorem (Arone-Ching)
Let $F : \mathrm{Top}_* \to \mathrm{Spec}$*. If* $\partial_* F$ *is levelwise a finite, free* Σ_n*-spectrum, then*

$$P_i(F)(X) \simeq \mathrm{Map}^h_{\mathrm{lie}}((\partial_* \Sigma^\infty \mathrm{Map}(X, -))^{\leqslant i}, (\partial_* F)^{\leqslant i}).$$

In fact, we can compute the derivatives of $\Sigma^\infty \mathrm{Map}_*(X, -)$ explicitly. Define the com module

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in order to form the "fake Goodwillie tower" of Arone-Ching.

The little disks operad

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Examples of algebras over the E_n operad are n-fold loop spaces. If we apply Σ^∞_+ we obtain a family of operads of spectra whose algebras define increasing levels of commutativity for ring spectra.

(framed) Manifold calculus

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By restricting F to the subcategory of disjoint unions of disks, F determines a right E_n module. Using some nonstandard notation, we call this right E_n module $\partial_*(F)$.

Proposition (Boavida de Brito-Weiss)
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Let E_M denote $\partial_* \mathrm{Emb}^{\mathrm{fr}}(-, M)$. This can be identified with the collection of configuration spaces of disks in M. By "collapsing disks", one can show that the com module $(M_+)^\wedge$ that we saw earlier is the induction of $(E_M)_+$ along $(E_n)_+ \to \mathrm{com}$.

Comparison of manifold and Goodwillie calculus

By taking inductions and applying Koszul duality, one obtains a map of towers

$$\begin{tikzcd}
T_\infty(\Sigma^\infty_+ \mathrm{Emb}(-, M))(N) \arrow[r] \arrow[d] & P_\infty(\Sigma^\infty \mathrm{Map}(N_+, -)))(M_+) \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] \\
T_n(\Sigma^\infty_+ \mathrm{Emb}(-, M))(N) \arrow[r] \arrow[d] & P_n(\Sigma^\infty \mathrm{Map}(N_+, -)))(M_+) \arrow[d] \\
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Koszul self duality

Theorem (Arone-Ching)
There is a commutative diagram of operads

$$\begin{tikzcd} s_n\mathrm{lie} \arrow[r] \arrow[d, dash, "\simeq"] & \Sigma^\infty_+ E_n \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{com} \arrow[d, dash, "\simeq"] \\ s_n K(\mathrm{com}) \arrow[r] & s_n K(\Sigma^\infty_+ E_n) \arrow[r] & K(\mathrm{lie}) \end{tikzcd}$$

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Conjecture (Ching)
There is an equivalence

$$\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} \Sigma^\infty_+ E_M \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

Koszul dual comparison

By instead restricting from E_n to $s_n\mathrm{lie}$, subject to the conjecture, we obtain a **contravariant** comparison

$$\begin{tikzcd}
T_\infty(\Sigma^\infty_+\mathrm{Emb}(-,M))(N) \arrow[r] \arrow[d] & P_\infty(\Sigma^\infty\mathrm{Map}(M^+,-)))(N^+) \arrow[d] \\
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Main Results

Theorem (M.)
For a framed n-manifold M, there is an equivalence

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In the process, we prove a stronger result extending the Koszul self duality of E_n. Let E_{M^+} denote the module of configurations of disks in M^+.

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In the process, we prove a stronger result extending the Koszul self duality of E_n. Let E_{M^+} denote the module of configurations of disks in M^+.

Theorem (M.)
There is a zigzag of equivalences of operads

$$\Sigma^{\infty}_{+} E_n \simeq \cdots \simeq s_n K(\Sigma^{\infty}_{+} E_n)$$

and a compatible zigzag of equivalences of modules

$$\Sigma^{\infty}_{+} E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^{\infty} E_{M^+}).$$

These equivalences are natural with respect to framed embeddings.

Normal bundles of operads

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Proposition (M.)
There is a factorization

$$\begin{tikzcd} & \mathrm{Operad}(\mathrm{ParSp}, \bar{\wedge}) \arrow[d, "\mathrm{Th}(-)"] \\ \mathrm{Operad}(\mathrm{Top}, \times) \arrow[ur, dashed, "\xi_{(-)}"] \arrow[r, "K(\Sigma^\infty_+ -)"'] & \mathrm{Operad}(\mathrm{Sp}, \wedge) \end{tikzcd}$$

Here, $\mathrm{Th}(-)$ *denotes the Thom complex of a parametrized spectrum, i.e. the spectrum obtained by collapsing the zero section.*

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This assignment of the normal bundle can be seen to be covariant with respect to maps $E_N \to E_M$ induced by $N \subset M$.

Corollary
If $\xi^c_{\mathbb{R}^n}$ has a trivialization as a module over ξ_{E_n}, then for any $U \subset \mathbb{R}^n$ we may choose a module trivialization of ξ^c_U, natural with respect to inclusion.

Koszul self duality inside $\mathbb{R}^n$

Using the Koszul duality of the operad E_n, it is not difficult to deduce the local case

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Such an equivalence implies that $\xi^c_{E_{\mathbb{R}^n}}$ is trivial.

Corollary
If M is a codimension 0 submanifold of $\mathbb{R}^n$, there is a zigzag of equivalences

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

These equivalences are natural with respect to inclusion.

Globalizing the result

It turns out that the functors

$$M \to \Sigma^\infty_+ E_M$$

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form Weiss cosheaves, i.e. are sheaves with respect to multilocal covers.

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form Weiss cosheaves, i.e. are sheaves with respect to multilocal covers. Using ∞-categorical techniques due to Ayala-Francis and Lurie, to show these functors agree on $\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}}$, it suffices to check on $\mathrm{Open}(\mathbb{R}^n)$.

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It turns out that the functors

$$M \to \Sigma^\infty_+ E_M$$

$$M \to s_{(n,n)} K(\Sigma^\infty E_{M^+})$$

form Weiss cosheaves, i.e. are sheaves with respect to multilocal covers. Using ∞-categorical techniques due to Ayala-Francis and Lurie, to show these functors agree on $\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}}$, it suffices to check on $\mathrm{Open}(\mathbb{R}^n)$.

Theorem
For a framed n-manifold M, there is a natural zigzag of equivalence of modules

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}),$$

compatible with the Koszul self duality of E_n.

Resolving Ching's conjecture

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Corollary
There is an equivalence

$$\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} \Sigma^\infty_+ E_M \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

Questions?

Verdier duality

Definition
The Verdier dualizing parametrized spectrum $P(X, A)$ of a pair (X, A) is the parametrized spectrum over X such that

$$P(X, A)|_x := \Gamma^c(\Sigma^\infty_X \mathrm{Path}(x, -))$$

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Theorem (Klein)
Suppose $A \to X$ is a cofibration such that A and X each have the homotopy type of a compact CW complex and X is connected, then for any $q \in \mathrm{Sp}_X$ there is a zigzag,

$$\mathrm{Th}(P(X, A) \wedge q) \leftarrow \mathrm{Th}(P(X, A) \wedge F_X(P, q)) \to \Gamma^A(q)$$

where the rightmost map is given on the fiber over $x \in X$ by the composition

$$F^A_X(\Sigma^\infty_X X, \Sigma^\infty_X \mathrm{Path}(x, -)) \wedge F_X(\Sigma^\infty_X \mathrm{Path}(x, -), q) \to F^A_X(\Sigma^\infty_X X, q) = \Gamma^A(q)$$

If q is level-fibrant, then after deriving $\wedge, \mathrm{Th}$ these are equivalences