Research Statement

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1 Introduction

My research addresses the interaction of algebraic and geometric topology, and specifically the application of homotopy theory to problems involving manifolds. Homotopy theory is the study of objects up to continuous deformation; consequently it is useful in the study of invariants which are immune to such changes, such as homology groups and homotopy groups. Manifolds, on the other hand, are naturally studied up to homeomorphism and diffeomorphism which do not allow for any deformation. As a result, the relation of homotopy theory and manifold theory is necessarily subtle. A particularly stark contrast is that injective functions play an important role in manifold theory, while injectivity is destroyed by continuous deformations. As a consequence, configuration spaces, embedding spaces, and related moduli spaces associated to manifolds are often functorial with respect to injective maps, but not with respect to arbitrary smooth maps.

These classical objects of study have seen a surge in activity recently due to advances in our understanding of cobordism categories, operad theory, and functor calculus. Functor calculus is a technique for understanding the homotopical behavior of functors. It was first developed by Goodwillie, in the case of spaces, and Weiss, in the case of manifolds [21,37]. The former goes by the name "Goodwillie calculus" and the latter by "embedding calculus." Functor calculus decomposes functors into simpler "polynomial approximations," analogous to the Taylor approximations of smooth functions. In this way, Goodwillie calculus and embedding calculus are forms of *differential* functor calculus. The theory of *integral* functor calculus of manifolds, or factorization homology, was developed by Ayala, Francis, Lurie, and Tanaka [7,10,29]. The primary goal of factorization homology is to construct invariants of manifolds from homotopy coherent algebras called E_n-algebras.

There are striking similarities between Goodwillie calculus, embedding calculus, and factorization homology, and their precise relationship is the subject of current research. Work of Arone–Ching, Boavida de Brito–Weiss, Ayala–Francis, and Turchin shows that certain operads govern the three versions of functor calculus [3,8,12,35]. Variants of the E_n operad govern embedding calculus and factorization homology, while the commutative and Lie operads govern Goodwillie calculus in Koszul dual ways. Koszul duality also has deep interactions with factorization homology, as demonstrated by Ayala–Francis.

Recently, Ching–Salvatore constructed a commutative diagram relating the operads lie, $\Sigma^\infty_+ E_n$, and com to their Koszul duals, extending work of Fresse in the algebraic case: [17,20]

$$\begin{tikzcd}
s_n\mathrm{lie} \arrow[r] \arrow[d, "\simeq"] & \Sigma^\infty_+ E_n \arrow[r] \arrow[d, "\simeq"] & \mathrm{com} \arrow[d, "\simeq"] \\
s_n K(\mathrm{com}) \arrow[r] & s_n K(\Sigma^\infty_+ E_n) \arrow[r] & K(\mathrm{lie})
\end{tikzcd}$$

In particular $\Sigma^\infty_+ E_n$ is *Koszul self dual.* In my work, I

- develop new methods to prove and utilize Koszul self duality results,
- produce geometric consequences of the Koszul self duality of $\Sigma^\infty_+ E_n$, and
- investigate the role of Koszul duality in functor calculus.

Using this program, I have solved conjectures of Ayala-Francis and Ching and have proven strong results in Goodwillie calculus, embedding calculus, and factorization homology. These new results progress programs of Arone–Ching, Ayala–Francis, and Knudsen.

2 Koszul duality

The primary tool in my research is Koszul duality. Koszul duality for operads in spectra was developed by Ching in order to explain why the homology of the Goodwillie derivatives of the identity is isomorphic to the Lie operad. Koszul duality fits into the following diagram [15]:

$$\begin{tikzcd} & \mathrm{Operad}(\mathrm{Spec}, \wedge) \arrow[d] \\ \mathrm{Operad}(\mathrm{Spec}, \wedge) \arrow[ur, "K"] \arrow[r, "B(1,-,1)^\vee"] & \mathrm{SymSeq}(\mathrm{Spec}) \end{tikzcd}$$

And for right O-modules:

$$\begin{tikzcd} & \mathrm{RMod}_{K(O)} \arrow[d] \\ \mathrm{RMod}_O \arrow[ur, "K"] \arrow[r, "B(-,O,1)^\vee"] & \mathrm{SymSeq}(\mathrm{Spec}) \end{tikzcd}$$

Let s_n denote operadic suspension. We say an operad O in spectra is Koszul self dual of dimension n, if there is an equivalence

$$O \simeq s_n K(O).$$

A right O-module pair (R, A), i.e a right O-module R with a chosen submodule A, is Koszul self dual of dimension (n, d) if there is compatible equivalence

$$R \simeq \Sigma^d s_n K(R/A).$$

3 Two approaches to Koszul self duality

Let E_n denote the little disks operad. The E_n operad is the first known nontrivial example of a Koszul self dual operad in spectra. This result is a spectral analog of algebraic self duality due to Fresse [19]. The self duality was proven by Ching–Salvatore only recently, after its initial conjecture 20 years ago. For M a framed n-manifold, let E_M denote the right E_n-module of disks in M. The Koszul self duality of E_M was conjectured by Ayala–Francis and Amabel [1, 8, 9].

In the process of solving this conjecture, I first demonstrated that Koszul duality could be organized using stable fibrations. Classically, Wall showed that a finite CW complex satisfies $\Sigma^\infty_+ X \simeq \Sigma^n(\Sigma^\infty_+ X)^\vee$, if and only if, the *Spivak normal fibration* of X is a trivial $(-n)$-spherical fibration [36]. For operads, the role of the n-sphere is replaced by the n-sphere operad S_n. The sphere modules over

S_n are $\Sigma^d S_n$, the levelwise d-fold suspension of S_n. These operads and modules are used to define n-trivializations of operads and (n,d)-trivializations of right modules in the category of parametrized spectra ParSp. I demonstrated the following:

Theorem 3.1 *Let O be an operad in $(\mathrm{Top}, \times)$ and (R, A) a module pair over O. There is an operad $\xi_O \in \mathrm{Operad}(\mathrm{ParSp}, \bar{\wedge})$ and a module $\xi_{(R,A)} \in \mathrm{RMod}_{\xi_O}$ such that the following are equivalent:*

1. *O is Koszul self dual of dimension n, and (R, A) is Koszul self dual of dimension (n, d).*
2. *There is a $(-n)$-trivialization of ξ_O and a $(-n, -d)$-trivialization of $\xi_{(R,A)}$.*

Let M be a framed n-manifold. The configurations of disks in M or M^+, the one point compactification of M, form modules over E_n called E_M, E_{M^+}. Using the theory of Koszul dualizing fibrations and Weiss cosheaves, I showed:

Theorem 3.2 (Koszul self duality via Koszul dualizing fibrations) *There is a zigzag of weak equivalences of operads*

$$\Sigma^\infty_+ E_n \simeq \cdots \simeq s_n K(\Sigma^\infty_+ E_n).$$

For any framed manifold M, there is a compatible zigzag of weak equivalences of right modules

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

These equivalences are constructed via trivializing Koszul dualizing fibrations.

While I expect the theory of Koszul dualizing fibrations to be useful in studying operad automorphisms in more general settings, it is also desirable to have a simpler description of the self duality of E_n and E_M. For this reason, I now employ alternative descriptions of K due to Lurie [30] and Espic [18]:

$$K(O) := \mathrm{CoEnd}^h_{\mathrm{RMod}_O}(1) = \mathrm{RMod}^h_O(1, 1 \otimes \cdots \otimes 1)$$

$$K(R) := \mathrm{CoEnd}^h_{\mathrm{RMod}_O}(R, 1) = \mathrm{RMod}^h_O(R, 1 \otimes \cdots \otimes 1)$$

In [34], I observed that the Pontryagin–Thom construction allows us to define a map

$$\Sigma^\infty_+ E_n \to \mathrm{CoEnd}^h_{\Sigma^\infty_+ E_n}(\Sigma^\infty E_{(\mathbb{R}^n)^+}).$$

Using the geometry of configuration spaces, I argued that this map was an equivalence of operads and the right side could be identified with $s_n K(\Sigma^\infty_+ E_n)$.

Theorem 3.3 (Koszul self duality via a Pontryagin–Thom construction) *There is a zigzag of weak equivalences of operads*

$$\Sigma^\infty_+ E_n \simeq \cdots \simeq s_n K(\Sigma^\infty_+ E_n).$$

For any framed manifold M, there is a compatible zigzag of weak equivalences of right modules

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

These equivalences are constructed via a Pontryagin–Thom construction.

Future Work

For $G \leq O(n)$, there is an operad E_n^G which acts upon manifolds with a G-tangential structure. In the future, I want to extend Koszul self duality to these modules. The method of proof for framed manifold , i.e. $G = *$, relies on the classification of Weiss cosheaves on framed manifolds in terms of Weiss cosheaves on $\mathbb{R}^n$. The major ingredient in proving the general case would be a classification Weiss cosheaves on manifolds with G-tangential structure in terms of G-equivariant Weiss cosheaves on $\mathbb{R}^n$.

4 Applications to embedding calculus

Embedding calculus was shown by Boavida de Brito–Weiss and Turchin to be expressible in terms of derived mapping spaces of modules over the unital framed little disks operad E_n^{fr} [12,35]. Arone–Lambrechts–Volic have investigated variants of the "E-homology embedding tower," which is the tower associated to a functor $M \to \Omega^\infty(\Sigma^\infty_+ \mathrm{Emb}(-, N) \wedge E)$, for a spectrum E [6]. This tower encodes approximations of the E-homology of embedding spaces for the homology theory associated to E. The E-homology embedding tower is easily seen to depend only on the stabilizations $\Sigma^\infty_+ E_M^{\mathrm{fr}}, \Sigma^\infty_+ E_N^{\mathrm{fr}}$ of the modules of embedded disks $E_M^{\mathrm{fr}}, E_N^{\mathrm{fr}}$.

By generalizing Knudsen's result on the homotopy invariance of stabilized configuration spaces to include tangential structures and projection maps [25,31,32], I demonstrated:

Theorem 4.1 *If N is a manifold, the layers of the embedding tower of any functor of the form $F \circ \Sigma^\infty_+ \mathrm{Emb}(-, N)$ are proper tangential homotopy invariants of N.*

This complements Arone's results on the layers of the stable orthogonal tower for embedding spaces [2].

For a framed manifold M, the modules E_M control a framed version of embedding calculus which has found use in the study of $\mathrm{BDiff}^\partial(D^n)$ [28]. Using the Koszul self duality of E_M, I constructed a Pontraygin–Thom type equivalence of mapping spaces over the nonunital E_n operad:

Theorem 4.2 *There is an equivalence*

$$T : \mathrm{Map}^h_{\Sigma^\infty_+ E_n}(\Sigma^\infty_+ E_M, \Sigma^\infty_+ E_N) \xrightarrow{\simeq} \mathrm{Map}^h_{\Sigma^\infty_+ E_n}(\Sigma^\infty E_{N^+}, \Sigma^\infty E_{M^+}).$$

Future Work

In the case where M, N are compact n-manifolds, this produces an equivalence

$$T : \mathrm{Map}^h_{\Sigma^\infty_+}(\Sigma^\infty_+ E_M, \Sigma^\infty_+ E_N) \xrightarrow{\simeq} \mathrm{Map}^h_{\Sigma^\infty_+}(\Sigma^\infty_+ E_N, \Sigma^\infty_+ E_M).$$

Conjecture 4.3 *If M, N are compact, framed n-manifolds with a map $f : E_M \to E_N$, then a stable homotopy inverse of f is given by $T(\Sigma^\infty_+ f)$. Consequently, if M, N are simply connected then all components of $\mathrm{Map}^h_{E_n}(E_M, E_N)$ are homotopy invertible.*

This conjecture, with tangential structures, seeks to complement recent work of Krannich–Kupers on the higher homotopy of classifying spaces of $\mathrm{Map}^h_{E_n^{\mathrm{fr}}}(E_M^{\mathrm{fr}}, E_M^{\mathrm{fr}})$ by stating the endomorphisms should form a group-like monoid [26].

A related technical question is to understand the relation between the unital and nonunital mapping spaces of right modules. In particular, are their limits equivalent? A related question was recently settled affirmatively by Horel–Krannich–Kupers in the case of endomorphisms of the E_n operad [22].

In the long term, I hope to develop surgery theory and cobordism theory internal to embedding calculus. Classically, Poincaré duality and Pontryagin–Thom constructions are two of the most important tools in these subjects. My results on the self duality of E_M and Pontryagin–Thom constructions for embedding calculus appear to be the first candidates for these statements in embedding calculus. As a consequence, I expect my work will be fundamental in progressing this program.

5 Applications to Goodwillie Calculus

In order to prove a chain rule for Goodwillie calculus, Arone–Ching equipped the derivatives of a functor $\mathrm{Top}_* \to \mathrm{Spec}$ with the structure of a right module over the nonunital Lie operad [3]. Using Koszul self duality, one constructs a map from the Lie operad to the shifted E_n operad:

$$\mathrm{lie} \to K(\Sigma^\infty_+ E_n) \simeq s_{-n}\Sigma^\infty_+ E_n$$

Verifying a conjecture of Ching [14], I proved:

Theorem 5.1 *If M is a framed n-manifold, there is an equivalence of* lie *modules*

$$\mathrm{res}_{\mathrm{lie}}(s_{(-n,-n)}\Sigma^\infty_+ E_M) \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

The proof follows from an explicit computation using the Koszul self duality of E_M. Using Arone–Ching's model of the Goodwillie tower of $\Sigma^\infty \mathrm{Map}(X, -)$ [5] and the above result, I construct comparisons:

Theorem 5.2 *For framed n-manifolds M, N and $0 \leq i \leq \infty$, there is a covariant comparison*

$$T_i(\Sigma^\infty_+ \mathrm{Emb}^{\mathrm{fr}}(M, N)) \to P_i(\Sigma^\infty \mathrm{Map}(M_+, N_+))$$

and a contravariant comparison

$$T_i(\Sigma^\infty_+ \mathrm{Emb}^{\mathrm{fr}}(M, N)) \to P_i(\Sigma^\infty \mathrm{Map}(N^+, M^+)).$$

In other words, an "approximate open embedding" has both an "approximate underlying map" and an "approximate one point compactification" in which the first is approximate in the sense of embedding calculus and the latter two are approximate in the sense of Goodwillie calculus.

Future work

The above comparison is closely related to work of Arone–Ching on classifying Goodwillie derivatives which occur as the restriction of a $K(\Sigma^\infty_+ E_n)$-module [4]. They found an answer in terms of the category of framed zero-pointed manifolds which is closely related to the construction of the module E_{M^+}. This paper was written before Koszul self duality of E_n and E_M was known, and it is worthwhile to revisit these results with this in mind.

6 Applications to factorization homology

An algebra A over an operad O consists of suitable maps $O(n) \otimes A^{\otimes n} \to A$. Of particular interest to homotopy theorists are algebras over the E_n operads. These were originally studied for their relevance in iterated loop space theory, and later because they have a rich theory of cohomology operations both in the stable and unstable settings.

For a framed n-manifold M and an E_n-algebra A, factorization homology can be defined as a coend $E_M \otimes_{E_n} A$. There is a dual theory of factorization cohomology which instead takes as input E_n-coalgebras. Knudsen used factorization homology combined with his theory of higher enveloping algebras to compute the rational cohomology of configuration spaces of surfaces [24, 25]. Similar techniques were used by Zhang to calculate torsion in the homology of configuration spaces of surfaces [38].

For a spectral Lie algebra A with bracket of degree $n-1$, the higher enveloping algebra $U_n(A)$ can be defined by inducing A along the map

$$s_n\mathrm{lie} \to s_n K(\Sigma^\infty_+ E_n) \simeq \Sigma^\infty_+ E_n.$$

With n-connectivity assumptions Ayala–Francis showed that, rationally, $\int_M U_n(A)$ is a homotopy invariant of M^+ via explicit calculation [7]. Idrissi extended the rational statement to all compact, simply-connected manifolds via an explicit rational model of E_M [23].

Using the resolution of Ching's conjecture, I proved the most general result:

Theorem 6.1 *If X is a spectral Lie algebra with bracket of degree $n-1$ and M is a framed n-manifold, then $\int_M U_n(X)$ is a homotopy invariant of M^+.*

My work on Koszul self duality of E_M has strong connections to the Poincaré–Koszul duality of Ayala–Francis. They construct a map comparing factorization homology with coefficients in A to factorization cohomology with coefficients in a shift of the Koszul dual coalgebra

$$\int_M A \to \int^{M^+} \Sigma^n B(A).$$

This map becomes an equivalence after applying P_∞ [8]. I extend this result to reduced left E_n-modules. A reduced left (co)module L over a (co)operad O is a symmetric sequence, concentrated in positive degrees, with a left (co)action of O. Ching defined factorization (co)homology of reduced left (co)modules using (co)bar constructions [16]. Using Koszul self duality, we define the $s_{(n,n)}(\Sigma^\infty_+ E_n)^\vee$ comodule $B(L)$ as the pullback of the $B(\Sigma^\infty_+ E_n)$-comodule of the same name. From Ching's work on bar constructions of operads and the self duality of E_M we have:

Theorem 6.2 *For M a framed n-manifold and L a reduced left $\Sigma^\infty_+ E_n$-module, there is an equivalence*

$$\int_M L \xrightarrow{\simeq} \int^{M^+} s_n B(L).$$

An algebra A has an associated reduced left module which is A in every nonzero degree. Applying the above result, yields a *new* form of Poincaré-Koszul duality for E_n-algebras which holds before applying P_∞. Recovering the map of Ayala-Francis is more subtle and I address it using a new approach to Goodwillie calculus in algebra categories.

Future Work

A comparison between Knudsen's higher enveloping algebras and the definition above is worthwhile. This would rely on a comparison of different definitions of the spectral Lie operad. This, in turn, requires a comparison of Lurie's Koszul duality and Ching's Koszul duality. Brantner made progress on this problem by showing that the operadic composites were homotopic as symmetric sequences [13]. I have work progress to show any reasonable definition of Koszul duality for operads and modules agrees by mapping to a universal construction of Koszul duality.

Once such a comparison is known, it is relatively easy to show that the two definitions of higher enveloping algebras agree. This derives from an observation I made in [33]: any reasonable definition of higher enveloping algebras implies the Koszul self duality of E_n. This would allow simultaneous use of

- the theoretical advantages Knudsen's definitions comes with, such as a Poincaré–Birkhoff–Witt theorem,
- compatibility with the Koszul self duality of E_n and E_M, and
- compatibility with techniques in Goodwillie calculus that we now describe.

7 Goodwillie calculus in algebra categories

Fix a reduced operad O in spectra. Let TAQ denote topological Andre–Quillen homology, i.e. derived indecomposables, of a nonunital O-algebra. Fix a functor $F : \mathrm{Alg}_O \to \mathrm{Spec}$. I define:

Definition 7.1 *The Koszul coderivatives* $\partial^* F$ *are* $\mathrm{Nat}(F, \mathrm{TAQ}(-)^{\wedge^*})$.

The Koszul coderivatives can be seen to form a right module over $K(O)$ as a result of the fact that $\mathrm{TAQ}(A)$ has a $K(O)$-coalgebra structure. This allows us to form an approximation to the Goodwillie tower of F:

Definition 7.2 *The ith fake Goodwillie approximation* $P_i^{\mathrm{fake}}(F)(A)$ *is*

$$\mathrm{Map}^h_{K(O)}(\partial^* F^{\le i}, \partial^*(\Sigma^\infty \mathrm{Map}(A, -))^{\le i}).$$

Via the enriched Yoneda lemma, one obtains a map

$$F(A) \to P_i(F)(A) \to P_i^{\mathrm{fake}}(F)(A)$$

If D_i and D_i^{fake} denote the fibers of the Goodwillie and fake Goodwillie towers, a computation with the Yoneda lemma shows:

Theorem 7.3 *If* $\partial_*(F)$ *are nonequivariantly finite spectra, the induced map on fibers*

$$D_i(F)(A) \to D_i^{\mathrm{fake}}(F)(A)$$

can be identified with the norm map

$$\partial_i(F) \wedge_{h\Sigma_i} \mathrm{TAQ}^{\wedge i}(A) \to \partial_i(F) \wedge^{h\Sigma_i} \mathrm{TAQ}^{\wedge i}(A).$$

In the process of proving this theorem I found a new model for the Goodwillie derivatives:

Theorem 7.4 *If $\partial_*(F)$ are finite as nonequivariant spectra, then*

$$\partial_*(F) \simeq K(\partial^*(F)).$$

As such, the derivatives of F are weakly equivalent to right modules over O.

In the case $\partial_i(F)$ is levelwise equivariantly finite, the norm maps will be equivalences, and the fake Goodwillie tower agrees with the Goodwillie tower. Taking $O = \Sigma^\infty_+ E_n$ and applying our theory to factorization homology, I demonstrate

Theorem 7.5 *For M a framed n-manifold and A a $\Sigma^\infty_+ E_n$-algebra, the map*

$$P_\infty(\int_M)(A) \to P_\infty^{\text{fake}}(\int_M)(A)$$

can be identified with a Koszul duality equivalence

$$P_\infty(\int_M)(A) \xrightarrow{\simeq} \int^{K(\Sigma^\infty_+ E_M)} B(A).$$

My work on Koszul self duality then recovers Poincaré–Koszul duality of E_n-algebras in spectra and can be extended to include spectral Lie algebras, which provides a computational proof of the homotopy invariance of factorization homology of higher enveloping algebras:

Theorem 7.6 *For M a framed n-manifold and A a $\Sigma^\infty_+ E_n$-algebra,*

$$P_\infty(\int_M)(A) \xrightarrow{\simeq} \int^{M^+} \Sigma^n B(A).$$

For a spectral Lie algebra X with bracket of degree $n-1$,

$$P_\infty(\int_M)U_n(X) \xrightarrow{\simeq} \int^{M^+} \Sigma^n B(X).$$

Future Work

In the future I would like to demonstrate a classification result for Goodwillie towers in O-algebra categories in terms of divided power modules, analogous to Arone–Ching's classification for functors $\mathrm{Top}_* \to \mathrm{Spec}$ [5]. They found that the norm data of the derivatives could be packaged into the structure of a "divided power right module" in order to enhance the Theorem 7.3 to be an equivalence in general.

When applied to generalized factorization homology for an operad O, I expect to recover Amabel's divided power Koszul duality map [1]. Such a result would allow us to understand divided power coalgebras in a new way. In particular, one could demonstrate Poincaré-Koszul duality for $\Sigma^\infty_+ E_\infty$-algebras in terms of divided power factorization cohomology over the spectral Lie operad, where T_∞ is taken in the category of $\Sigma^\infty_+ E_\infty$-algebras:

$$P_\infty(\int_M)(A) \xrightarrow{\simeq} \int_{\text{d.p.}}^{\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} E_{M^+}} B(A).$$

8 Work in progress on Orthogonal Calculus

I am working to adapt the techniques of the previous section to study stable orthogonal calculus. Let $\mathrm{Vect}^{\mathrm{inj}}$ denote the category of finite dimensional inner product spaces and maps preserving the inner product. Orthogonal calculus applies to functors $\mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$ and has been used to study objects of classical geometric interest such as $\mathrm{BTop}(n)$ and $\mathrm{Emb}(M, N)$, as well as homotopical objects such as the EHP spectral sequence [2, 11, 27].

I am working to upgrade the well-known analogy between Goodwillie calculus and orthogonal calculus to reflect the analogy between finite sets and finite dimensional vector spaces. In particular, Koszul duality explains that the Lie operad acts on Goodwillie derivatives precisely because the Koszul coderivatives of functors $\mathrm{Top}_* \to \mathrm{Spec}$ [3] can be upgraded to presheaves on the category of finite sets and surjections, the monoidal envelope of com. I aim to emulate this result in orthogonal calculus by replacing the role of finite sets and surjections by finite dimensional vector spaces and surjections.

Let $nS : \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$ denote the functor $V \to (\Sigma^\infty V^+)^{\wedge n}$. Fix a functor $F : \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$.

Definition 8.1 *The orthogonal sequence of Koszul coderivatives $\partial^* F$ is* $\mathrm{Nat}(F, *S)$.

Let $\mathrm{Vect}^{\mathrm{sur}}$ denote the category obtained by dualizing objects and maps of $\mathrm{Vect}^{\mathrm{inj}}$. By a module over $\mathrm{Vect}^{\mathrm{sur}}$, we mean a functor $(\mathrm{Vect}^{\mathrm{sur}})^{\mathrm{op}} = \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$. The Koszul coderivatives naturally have the structure of a $\mathrm{Vect}^{\mathrm{sur}}$-module. Under our analogy, this correponds to the fact the Koszul coderivatives of a functor $F : \mathrm{Top}_* \to \mathrm{Spec}$ form a $\mathrm{FinSet}^{\mathrm{sur}}$-module, i.e. a right com-module [3].

Preliminary definitions and computations suggest that with finiteness conditions on $\partial_i F$, there is an equivalence

$$\partial_i F \simeq \mathrm{TAQ}(\partial^* F)(i)^\vee \wedge D_{O(i)}.$$

Here TAQ is defined as the adjoint to $\mathrm{Triv} : \mathrm{OrthSeq}(\mathrm{Spec}) \to \mathrm{RMod}_{\mathrm{Vect}^{\mathrm{sur}}}$, and D_G is the dualizing spectrum of a topological group G. Using the Yoneda lemma, one constructs maps

$$F(V) \to T_i(F)(V) \to T_i^{\mathrm{fake}}(F)(V) := \mathrm{Map}_{\mathrm{Vect}^{\mathrm{sur}}}(\partial^* F^{\le i}, (\partial^* \Sigma^\infty_+ \mathrm{Vect}^{\mathrm{inj}}(V, -))^{\le i}).$$

As i varies, the rightmost spectra form the “fake orthogonal tower”. The induced maps on the layers of the towers take the form

$$\partial_i F \wedge_{hO(i)} iS(V) \to (\partial_i F \wedge D^\vee_{O(i)}) \wedge^{hO(i)} iS(V).$$

Conjecture 8.2 *This is the norm map for $\partial_i F \wedge D^\vee_{O(i)} \wedge iS(V)$.*

In order to express these constructions in terms of a module structure on $\partial_* F$ rather than $\partial^* F$, one needs a form of Koszul duality.

Definition 8.3 *The category $K(\mathrm{Vect}^{\mathrm{sur}})$ is the full subcategory of $\mathrm{RMod}_{\mathrm{Vect}^{\mathrm{sur}}}$ on $\mathrm{Triv}(O(i))$.*

For a levelwise nonequivariantly finite $\mathrm{Vect}^{\mathrm{sur}}$-module, $\mathrm{TAQ}(R)^\vee \wedge D_{O(*)}$ has the structure of a $K(\mathrm{Vect}^{\mathrm{sur}})$-module. So, under finiteness conditions, $\partial_* F$ forms a module over $K(\mathrm{Vect}^{\mathrm{sur}})$, analogous to how the Goodwillie derivatives of a functor $\mathrm{Top}_* \to \mathrm{Spec}$ form a module over $K(\mathrm{FinSet}^{\mathrm{sur}})$, or in other words, a right lie-module.

Joint with Niall Taggart, we hope to show the analog of Arone–Ching's classification of Goodwillie towers in terms of divided power lie-modules [5]:

Conjecture 8.4 *The orthogonal tower of $F : \mathrm{Vect}^{\mathrm{inj}} \to \mathrm{Spec}$ depends only on a canonical “divided power module structure” on $\partial_* F$ over the category $K(\mathrm{Vect}^{\mathrm{sur}})$.*

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