# Koszul duality and manifold calculus

Advances in Homotopy Theory IV

Connor Malin June 22, 2023

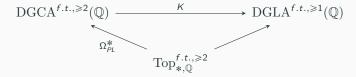
University of Notre Dame

## The rational story

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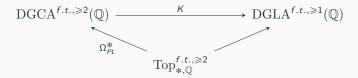
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The functor  $\bar{\Omega}_{PL}^*$  is a strictly commutative model of reduced singular cochains,

#### Example

It is straightforward to check that

$$\bar{\Omega}^*_{PL}(S^{2k}) \simeq \Lambda\langle x,y \rangle$$

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The generators and differentials conspire to allow for a map  $S^{4k-1}_{\mathbb{Q}} \to S^{2k}_{\mathbb{Q}}$  which can be seen to generate  $\pi_{4k-1}(S^{2k}) \otimes \mathbb{Q} \cong \mathbb{Q}$ .

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#### **Definition**

An operad O in a symmetric monoidal category  $(C, \otimes)$  is a collection of objects O(I) with actions of  $\Sigma_I$ , for all nonempty finite sets I, together with maps for all I, J and  $a \in I$ 

$$O(I) \otimes O(J) \rightarrow O(I \cup_a J)$$

which satisfy associativity and equivariance conditions.

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An algebra A over an operad O in  $(C, \otimes)$  is and object  $A \in C$  with maps

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#### Proposition

The cup product on  $\bar{C}^*(X;k)$  naturally extends to an  $E_{\infty}$ -algebra structure.

**Theorem (Mandell)** For simply connected spaces X,Y there is a homotopy equivalence  $X \simeq Y$ , if and only if, there is an equivalence of  $E_{\infty}$ -algebras  $C^*(X; \mathbb{Z}) \simeq C^*(Y; \mathbb{Z}).$ 

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### Theorem (Mandell)

The composite

$$\operatorname{Top}_*^{\geqslant 2} \xrightarrow{\bar{\mathcal{C}}^*} \operatorname{Alg}_{E_\infty}(\operatorname{DGVect}_{\mathbb{F}_p}, \otimes) \xrightarrow{K} \operatorname{Alg}_{L_\infty}(\operatorname{DGVect}_{\mathbb{F}_p}, \otimes)$$

takes values in contractible  $L_{\infty}$ -algebras.

Consider the category  $(\operatorname{Spec},\wedge).$ 

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By setting  $com(I) = S^0$ , we can define  $E_{\infty}$ -algebras as algebras over a cofibrant replacement of com.

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Given a pointed space X, the function spectrum  $F(\Sigma^{\infty}X, S^0)$  obtains an  $E_{\infty}$ -algebra structure from the diagonal of X.

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#### **Example**

Given a pointed space X, the function spectrum  $F(\Sigma^{\infty}X, S^0)$  obtains an  $E_{\infty}$ -algebra structure from the diagonal of X.

This  $E_{\infty}$ -structure is a lift of the  $E_{\infty}$ -structure on  $\bar{C}^*(X)$  to spectra.

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# Spectral lie algebras

A combination of results of Ching, Francis, Gaitsgory, Harper, and Lurie establish that theories of Koszul duality for algebras and operads in spectra exist, and these lift algebraic Koszul duality in some sense.

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With this in mind, one defines the spectral lie operad lie := K(com), and one obtains a functor

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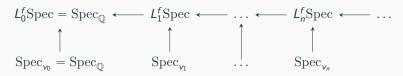
$$\mathrm{Alg}_{E_{\infty}}(\mathrm{Spec}, \otimes) \xrightarrow{K} \mathrm{Alg}_{L_{\infty}}(\mathrm{Spec}, \otimes).$$

Unfortunately, the negative results of Mandell still imply that this functor destroys all p-torsion information when applied to  $F(\Sigma^{\infty}X, S^0)$ .

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The layers of this tower can be "destabilized" by work of Bousfield and Heuts to produce analogs of rational homotopy theory that detect *p*-torsion.

$$\operatorname{Top}_{v_0} = \operatorname{Top}_{\mathbb{Q}}$$
  $\operatorname{Top}_{v_1}$  ...  $\operatorname{Top}_{v_n}$  ...

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# Higher algebra models

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#### Theorem (Behrens-Rezk, Heuts)

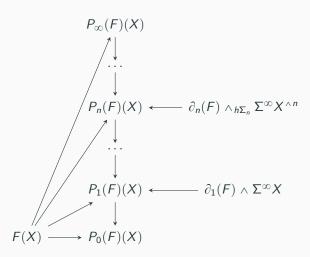
If the Goodwillie tower for the identity of  $X \in \operatorname{Top}_{v_n}$  converges, the Koszul dual of  $F(X, S_{v_n}^0)$  is the  $v_n$  lie algebra model of X. In particular, the Goodwillie tower converges for the spheres  $S_{v_n}^d$ .

### Goodwillie calculus

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A right module R over an operad O is a collection of objects R(I) with actions of  $\Sigma_I$ , for all nonempty finite sets I, together with maps for all I, J and  $a \in I$ 

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### Theorem (Arone-Ching)

Let  $F: \mathrm{Top}_* \to \mathrm{Spec}$ . If  $\partial_* F$  is levelwise a finite, free  $\Sigma_n$ -spectrum, then

$$P_i(F)(X) \simeq \operatorname{Map}_{\mathrm{lie}}^h((\partial_* \Sigma^\infty \operatorname{Map}(X,-))^{\leqslant i}, (\partial_* F)^{\leqslant i}).$$

In fact, we can compute the derivatives of  $\Sigma^\infty\mathrm{Map}_*(X,-)$  explicitly. Define the  $\mathrm{com}$  module

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in order to form the "fake Goodwillie tower" of Arone-Ching.

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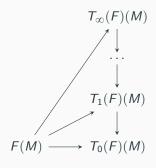
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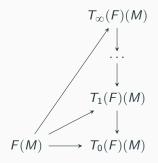
Examples of algebras over the  $E_n$  operad are n-fold loop spaces. If we apply  $\Sigma_+^{\infty}$  we obtain a family of operads of spectra whose algebras define increasing levels of commutativity for ring spectra.

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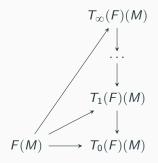


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By restricting F to the subcategory of disjoint unions of disks, F determines a right  $E_n$  module. Using some nonstandard notation, we call this right  $E_n$  module  $\partial_*(F)$ .

#### Proposition (Boavida de Brito-Weiss)

At a framed n-manifold M

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Let  $E_M$  denote  $\partial_* \mathrm{Emb}^{\mathrm{fr}}(-,M)$ . This can be identified with the collection of configuration spaces of disks in M. By "collapsing disks", one can show that the  $\mathrm{com}$  module  $(M_+)^{\wedge}$  that we saw earlier is the induction of  $(E_M)_+$  along  $(E_n)_+ \to \mathrm{com}$ .

#### Comparison of manifold and Goodwillie calculus

By taking inductions and applying Koszul duality, one obtains a map of towers

# Koszul self duality

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# Conjecture (Ching)

There is an equivalence

$$\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} \Sigma^{\infty}_{+} E_{M} \simeq \partial_{*}(\Sigma^{\infty} \mathrm{Map}_{*}(M^{+},-)).$$

#### Koszul dual comparison

By instead restricting from  $E_n$  to  $s_n$  lie, subject to the conjecture, we obtain a **contravariant** comparison

$$T_{\infty}(\Sigma_{+}^{\infty}\mathrm{Emb}(-,M))(N) \longrightarrow P_{\infty}(\Sigma^{\infty}\mathrm{Map}(M^{+},-)))(N^{+})$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

#### Main Results

**Theorem (M.)** For a framed n-manifold M, there is an equivalence

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In the process, we prove a stronger result extending the Koszul self duality of  $E_n$ . Let  $E_{M^+}$  denote the module of configurations of disks in  $M^+$ .

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#### Theorem (M.)

There is a zigzag of equivalences of operads

$$\Sigma_+^\infty E_n \simeq \cdots \simeq s_n K(\Sigma_+^\infty E_n)$$

and a compatible zigzag of equivalences of modules

$$\Sigma^{\infty}_{+}E_{M} \simeq \cdots \simeq s_{(n,n)}K(\Sigma^{\infty}E_{M^{+}}).$$

These equivalences are natural with respect to framed embeddings.

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# **Proposition (M.)** *There is a factorization*

$$\begin{array}{c} \operatorname{Operad}(\operatorname{ParSp},\bar{\wedge})) \\ \xrightarrow{\xi_{(-)}} & \bigvee^{\operatorname{Th}(-)} \\ \operatorname{Operad}(\operatorname{Top},\times) \xrightarrow{K(\Sigma_+^{\infty}-)} \operatorname{Operad}(\operatorname{Sp},\wedge) \end{array}$$

Here,  $\operatorname{Th}(-)$  denotes the Thom complex of a parametrized spectrum, i.e. the spectrum obtained by collapsing the zero section.

#### **Local Koszul self duality**

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This assignment of the normal bundle can be seen to be covariant with respect to maps  $E_N \to E_M$  induced by  $N \subset M$ .

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There is a similar construction of normal bundles  $\xi_{E_M}^c$ . Taking Thom complexes, we have

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This assignment of the normal bundle can be seen to be covariant with respect to maps  $E_N \to E_M$  induced by  $N \subset M$ .

#### **Corollary**

If  $\xi_{\mathbb{R}^n}^c$  has a trivialization as a module over  $\xi_{E_n}$ , then for any  $U \subset \mathbb{R}^n$  we may choose a module trivialization of  $\xi_U^c$ , natural with respect to inclusion.

#### Koszul self duality inside $\mathbb{R}^n$

Using the Koszul duality of the operad  $E_n$ , it is not difficult to deduce the local case

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Such an equivalence implies that  $\xi^c_{E_{\mathbb{R}^n}}$  is trivial.

#### Corollary

If M is a codimension 0 submanifold of  $\mathbb{R}^n$ , there is a zigzag of equivalences

$$\Sigma_+^{\infty} E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^{\infty} E_{M^+}).$$

These equivalences are natural with respect to inclusion.

It turns out that the functors

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#### **Theorem**

For a framed n-manifold M, there is a natural zigzag of equivalence of modules

$$\Sigma_+^{\infty} E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^{\infty} E_{M^+}),$$

compatible with the Koszul self duality of  $E_n$ .

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#### Corollary

There is an equivalence

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**Questions?** 

#### **Verdier duality**

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The Verdier dualizing parametrized spectrum P(X,A) of a pair (X,A) is the parametrized spectrum over X such that

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$$P(X,A)|_{x} := \Gamma^{c}(\Sigma_{X}^{\infty} \operatorname{Path}(x,-))$$

#### Theorem (Klein)

Suppose  $A \to X$  is a cofibration such that A and X each have the homotopy type of a compact CW complex and X is connected, then for any  $q \in \operatorname{Sp}_X$  there is a zigzag,

$$\operatorname{Th}(P(X,A) \wedge q) \leftarrow \operatorname{Th}(P(X,A) \wedge F_X(P,q)) \rightarrow \Gamma^A(q)$$

where the rightmost map is given on the fiber over  $x \in X$  by the composition

$$F_X^A(\Sigma_X^\infty X, \Sigma_X^\infty \mathrm{Path}(x,-)) \wedge F_X(\Sigma_X^\infty \mathrm{Path}(x,-),q) \to F_X^A(\Sigma_X^\infty X,q) = \Gamma^A(q)$$

If q is level-fibrant, then after deriving  $\wedge$ , Th these are equivalences