Connor Malin
Notre Dame

Recently, Lurie introduced a particularly simple definition of Koszul duality in terms of presheaves on $\mathrm{Env}(O)$, the monoidal envelope of O [3]. I verify the equivalence of operads $\Sigma^\infty_+ E_n \simeq S_n \wedge K(\Sigma^\infty_+ E_n)$ with respect to this definition. Such a result was originally due to Fresse [2] in the algebraic case and Ching–Salvatore in the topological case [1]. All operads are reduced, i.e. $O(1) \simeq S^0$ and $O(0) \simeq *$.

Denote the Spec-enriched category $\mathrm{Fun}^{\mathrm{Spec}}(\mathrm{Env}(O)^{\mathrm{op}}, \mathrm{Spec})$ by $\mathrm{RMod}(O)$. Day convolution yields a symmetric monoidal product $\otimes$ computed by

$$(F \otimes G)(k) = \bigvee_{i+j=k} F(i) \otimes G(j) \wedge_{\Sigma_i \times \Sigma_j} \Sigma_{i+j}.$$

For a framed n-manifold M, the disk module E_M is defined by $E_M(i) = \mathrm{Emb}^{\mathrm{fr}}(\bigsqcup_i \mathbb{R}^n, M)$ (see [4, Definition 2.14] for the case of a zero-pointed manifold). These modules satisfy

$$E_M \otimes E_N \simeq E_{M \sqcup N}.$$

We write $\mathrm{CoEnd}_C(-)$ for the coendomorphism operad of an object $c \in (C, \otimes)$ given by

$$\mathrm{CoEnd}_C(c)(i) = \mathrm{C}(c, c^{\otimes i}).$$

Let 1 denote the right O-module for which $1(1) = S^0$ and is contractible otherwise.

Definition. The Koszul dual of O is $\mathrm{CoEnd}_{\mathrm{RMod}(O)}(1)$.

Definition. The n-sphere operad S_n is $\mathrm{CoEnd}_{\mathrm{Spec}}(S^n)$.

The pinch map is the map

$$E_n(i) \wedge E_{(\mathbb{R}^n)^+}(j) \to E_{(\bigsqcup_i \mathbb{R}^n)^+}(j)$$

which takes a (framed) embedding $\bigsqcup_i \mathbb{R}^n \to \mathbb{R}^n$, applies the Pontryagin–Thom construction to get a map $(\mathbb{R}^n)^+ \to (\bigsqcup_i \mathbb{R}^n)^+$ and uses it to pushforward the j singularly embedded disks in $(\mathbb{R}^n)^+$ into $(\bigsqcup_i \mathbb{R}^n)^+$.

There is a Pontryagin-Thom type equivalence [4, Theorem 9.4]

Theorem. *For framed n-manifolds M, N there is an equivalence*

$$\mathrm{RMod}(\Sigma^\infty_+ E_n)(\Sigma^\infty_+ E_M, \Sigma^\infty_+ E_N) \simeq \mathrm{RMod}(\Sigma^\infty_+ E_n)(\Sigma^\infty E_{N^+}, \Sigma^\infty E_{M^+}).$$

Applying this theorem to $\mathrm{RMod}(\Sigma^\infty_+ E_n)(\Sigma^\infty_+ E_{\bigsqcup_i \mathbb{R}^n}, \Sigma^\infty_+ E_{\mathbb{R}^n}) \simeq \Sigma^\infty_+ E_n(i)$, the above map coincides with the adjoint of the pinch map.

Theorem. *There is an equivalence $\Sigma^\infty_+ E_n \simeq S_n \wedge K(\Sigma^\infty_+ E_n)$.*

Proof. The equivalence is given by

$$\Sigma^\infty_+ E_n \xrightarrow{\simeq} \mathrm{CoEnd}_{\mathrm{RMod}(\Sigma^\infty_+ E_n)}(E_{(\mathbb{R}^n)^+}) \simeq \mathrm{CoEnd}_{\mathrm{RMod}(\Sigma^\infty_+ E_n)}(\Sigma^n 1) \simeq S_n \wedge K(\Sigma^\infty_+ E_n)$$

The first functor is the adjoint of the pinch map, hence, an equivalence by the previous remark. The second map exists by the elementary equivalence $E_{(\mathbb{R}^n)^+} \simeq \Sigma^n 1$ [4, Lemma 7.2]. The third map is an equivalence by inspection of definitions.

□

References

[1] Michael Ching and Paolo Salvatore. Koszul duality for topological E_n operads. *Proceedings of the London Mathematical Society*, 125(1):1–60, 2022.

[2] Benoit Fresse. Koszul duality of E_n operads. *Selecta Mathematica*, 17(2):363–434. 2010.

[3] Jacob Lurie. Lie algebras in stable homotopy theory, 2023. Homotopy theory in honor of Paul Goerss.

[4] Connor Malin. Koszul self duality of manifolds, 2023.