Semiadditive geometry and classifications of Goodwillie towers

Operads and Calculus Workshop, Queen's University, Belfast

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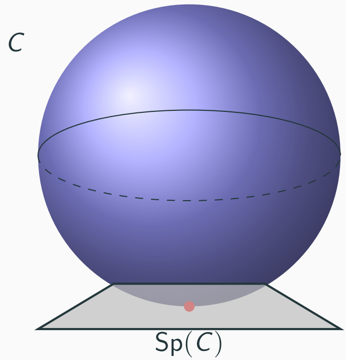
Calculus and geometry



Goodwillie calculus

We deal with differentiable ∞-categories C with

$$\mathrm{Sp}(C) = \mathrm{Sp}_E.$$

Goodwillie calculus associates to $F : C \to \mathrm{Sp}_E$ a tower:

$$\begin{tikzcd}
 & P_\infty(F) \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & P_2(F) \arrow[d] & (\partial_2(F) \wedge \Sigma^\infty_C(-)^2)_{h\Sigma_2} \arrow[l] \\
 & P_1(F) \arrow[d] & \partial_1 F \wedge \Sigma^\infty_C(-) \arrow[l] \\
F \arrow[r] \arrow[ur] \arrow[uuur] \arrow[uuuur] & P_0(F) &
\end{tikzcd}$$

Another notion of geometry

Definition

The category of polynomial functors $\mathrm{Poly}(C, \mathsf{Sp}_E)$ is the full subcategory of $\mathrm{Fun}(C, \mathsf{Sp}_E)$ for which $F \to P_n(F)$ is an equivalence, for some $n < \infty$.

Definition

A polynomial equivalence $C \simeq_P D$ is a symmetric monoidal equivalence

$$(\mathrm{Poly}(C, \mathsf{Sp}_E), \wedge_E) \simeq (\mathrm{Poly}(D, \mathsf{Sp}_{E'}), \wedge_{E'}).$$

Example

The categories of spaces S_* and n-connected spaces ${\mathsf{S}_*}^{\geqslant n+1}$ are polynomially equivalent even though they aren't equivalent.

Example: infinitesimal neighborhoods of $* \in C$

Example

The tangent ∞-category of C, $\mathrm{Sp}(C)$, is invariant under polynomial equivalence.

Proof.

A polynomial functor F is linear, if and only if there is no G such that F can be written as a sequence of extensions by $(X \wedge G^{\wedge n})_{h\Sigma_n}$ where $X \in \mathrm{Sp}_E^{B\Sigma_n}$ and $n \geqslant 2$. This condition is preserved under polynomial equivalence. From this one can deduce the ∞-category $\mathrm{Homog}_1(C)$ is preserved under equivalence. However this ∞-category of 1-homogeneous functors is equivalent to $\mathrm{Sp}(C)$. $\square$

Review: operads and their right modules

Recall there is a monoidal ∞-category

$$(\mathrm{SymSeq}(\mathsf{Sp}_E), \circ).$$

This ∞-category consists of sequences of objects of Sp_E with Σ_n-actions, $n > 0$. The composition product of symmetric sequences is

$$(A \circ B)(n) = \bigvee_{\pi \in \mathcal{P}(n)} A(|\pi|) \wedge \bigwedge_{\gamma \in \pi} B(|\gamma|)$$

Definition

Monoids for $\circ$ are called operads.

Definition

Given an operad O, a right O-module R is a symmetric sequence with a right action of O.

Review: right modules in Goodwillie calculus

For a functor $F : C \to \mathrm{Sp}_E$ we have a right action

$$F \circ \mathrm{Id}_C \to F.$$

Arone–Ching made a monumental observation that the process of taking derivatives turns the uninteresting action of functor composition with the identity into a very interesting operadic right action.

Theorem (Blans-Blom, cf. Arone-Ching)

Goodwillie derivatives can be made lax symmetric monoidal with respect to functor composition and composition product, respectively. Thus $\partial_* \mathrm{Id}_C$ is an operad. Consequently, for $F : C \to \mathrm{Sp}_E$ the derivatives $\partial_* F$ form a right $\partial_* \mathrm{Id}_C$-module.

The general structure of polynomial functors

Theorem (Arone-Ching)

Let Φ denote the right adjoint of $\partial_* : \mathrm{Fun}(C, \mathsf{Sp}_E) \to \mathrm{SymSeq}(\mathsf{Sp}_E)$. The adjunction is comonadic when restricted to polynomial functors and bounded symmetric sequences. Hence,

$$\mathrm{Poly}(C) \simeq \mathrm{CoAlg}^{<\infty}(\partial_* \Phi).$$

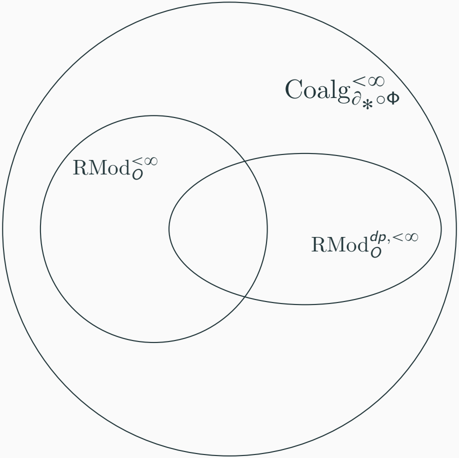
Theorem (Arone-Ching)

In the case of pointed spaces S_*, this comonad encodes divided power right lie *module structures on the derivatives, in the sense that it takes the form*

$$R(r) \to \prod_n \left[\prod_{n \twoheadrightarrow r} \mathrm{Map}(\mathrm{lie}(n_1) \wedge \cdots \wedge \mathrm{lie}(n_r), R(n)) \right]_{h\Sigma_n},$$

and a right lie *module structures can be obtained by composing with the Σ_n-norm.*

A naive map of some geometries



∞-categories with semiadditive stabilization

Recall that a stable ∞-category is (1-)semiadditive if the norm map

$$X_{hG} \to X^{hG}$$

is an equivalence for all finite group actions on all X. Equivalently, all Tate constructions

$$X^{tG} := \mathrm{cofiber}(X_{hG} \to X^{hG})$$

vanish.

Definition

We say C is infinitesimally semiadditive if $\mathrm{Sp}(C)$ is semiadditive.

Geometric characterization of infinitesimal semiadditivity

Proposition

The following are equivalent:

1. *C is infinitesimally semiadditive.*
2. *There is a symmetric monoidal equivalence* $(\mathrm{Poly}(C), \wedge) \simeq (\mathrm{RMod}_{O}^{<\infty}, \circledast)$ *for some operad O.*
3. *The derivative map* $\partial_* : \mathrm{Poly}(C) \to \mathrm{RMod}_{\partial_* \mathrm{Id}_C}^{<\infty}$ *is an equivalence.*

Proof:

$1 \implies 3$: Arone-Ching demonstrate that the obstruction to the (restricted) derivative map being an equivalence is precisely the Tate construction of the homogenous layers, and by our assumption the norm is an equivalence.

$3 \implies 1$: By precomposition with Σ^∞_C one can demonstrate the hypothesis implies all the Goodwillie towers in Sp_E split. This is known to be equivalent to semiadditivity of $\mathrm{Sp}(C) = \mathrm{Sp}_E$ by work of McCarthy and Kuhn.

$3 \implies 2$: This follows from the product rule for ∂_* (work in progress with Niall Taggart, Thomas Blom).

$2 \implies 3$: Omitted.

Divided power right modules

Definition

The ∞-category of right comodules RComod_O over an operad O in Sp_E is

$$\mathrm{Fun}_{\mathsf{Sp}_E}(\mathrm{Env}(O), \mathsf{Sp}_E).$$

Using the augmentation of a reduced operad O, we have an adjunction

$$\mathrm{Indecom}_O : \mathrm{RComod}_O \rightleftarrows \mathrm{SymSeq}(\mathsf{Sp}_E) : \mathrm{Triv}_O$$

Definition

The ∞-category $\mathrm{RMod}^{dp}_{K(O)}$ of divided power right $K(O)$ modules is the ∞-category of coalgebras for the comonad $\mathrm{Indecom}_O \circ \mathrm{Triv}_O$ on $\mathrm{SymSeq}(\mathsf{Sp}_E)$.

Geometry of S_*

Theorem

For $F \in \mathrm{Poly}(\mathsf{S}_*, \mathsf{Sp})$ there is a natural divided power right lie*-module structure on $\partial_* F$ which induces an equivalence*

$$\mathrm{Poly}(\mathsf{S}_*, \mathsf{Sp}) \simeq \mathrm{RMod}_{\mathrm{lie}}^{dp, <\infty}.$$

Explicitly, this divided power structure is obtained via an equivalence

$$\partial_* F \simeq \mathrm{Indecom}_{\mathrm{com}}(\epsilon(F|_{\mathrm{FinSet}_*})).$$

The notation ϵ denotes the augmentation coideal of an augmented commutative comodule. We let ϵ^{-1} denote the inverse construction.

Proof

For X a finite cell complex, Arone–Ching show that

$$P_n\Sigma^\infty \mathrm{Map}(X,-)(Y) \simeq \mathrm{RMod}_{\mathrm{lie}}(\partial_*\Sigma^\infty\mathrm{Map}(Y,-)^{\leqslant n}, \partial_*\Sigma^\infty\mathrm{Map}(X,-)^{\leqslant n}).$$

When $n \geqslant i$, a computation of $\partial_*\Sigma^\infty(-)^{\times i}$ implies

$$P_n(\Sigma^\infty\mathrm{Map}(X,-))([i]_+) = \epsilon^{-1}\mathrm{Prim}_{\mathrm{lie}}(\partial_*\Sigma^\infty\mathrm{Map}(X,-))(i).$$

For a Σ-finite right lie-module R, the counit $\mathrm{Indecom}_{\mathrm{com}}\mathrm{Prim}_{\mathrm{lie}}(R) \to R$ is an equivalence (a result due to Arone–Ching). From this, we deduce the claim for polynomial approximations to representable functors. The general result follows from the (co)Yoneda formula along with the Dwyer-Rezk classification in terms of right com-comodules.

Differentially semiadditive ∞-categories

Definition

An ∞-category C is differentially semiadditive if for all κ-compact $c \in C$ the Tate construction

$$(\partial_n \Sigma^\infty_E \mathrm{Map}(c, -) \wedge \Sigma^\infty_C(-)^{\wedge n})^{t\Sigma_n}$$

vanishes.

Example

If C is infinitesimally semiadditive, i.e. $\mathrm{Sp}(C)$ is semiadditive, then C is differentially semiadditive.

Example

The ∞-category S_* is differentially semiadditive.

Example

The ∞-category Sp is *not* differentially semiadditive.

Geometry of differentially semiadditive ∞-categories

Theorem

The following are equivalent:

1. *C is differentially semiadditive.*
2. *There is a symmetric monoidal equivalence* $(\mathrm{Poly}(C), \wedge) \simeq (\mathrm{RMod}_{O}^{<\infty, dp}, \circledast)$ *for some operad O.*
3. *The derivatives have a lift*

$$\partial_* : \mathrm{Poly}(C) \to \mathrm{RMod}_{\partial_* \mathrm{Id}_C}^{dp, <\infty}$$

which is an equivalence. The lift is also unique.

4. *For any n-polynomial functor F, one has the associated functor*

$$C \to \mathrm{Sp}_E$$

$$c \mapsto \mathrm{Nat}(P_{n-1}(\Sigma_E^\infty \mathrm{Map}(c, -)), F)$$

is $(n-1)$-polynomial.

A dichotomy for ∞-categories with semiadditive geometry

Definition

We say an operad O has significant Tate vanishing if the forgetful functor

$$\mathrm{RMod}_O^{dp} \to \mathrm{RMod}_O$$

is an equivalence.

Example

If each $O(n)$ is a finite Σ_n-spectrum, then O has significant Tate vanishing.

Corollary

If C is differentially semiadditive, then the following are equivalent:

- $\partial_* \mathrm{Id}_C$ *has significant Tate vanishing.*
- C *is infinitesimally semiadditive.*

Geometry of algebra categories

Proposition

$\mathrm{Alg}_{O}(\mathsf{Sp}_E)$ is differentially semiadditive, if and only if Sp_E is semiadditive.

Proof.

The forwards implication is the interesting one. By our characterization of differential semiadditivity, $\partial_* F$ has a contractible space of natural divided power structures. However, for any right O-module R, $\partial_* B(R, O, -) \simeq R$. Hence, R has a contractible space of divided power structures on it. As a consequence, one finds that $\mathrm{RMod}_{O}^{\mathrm{dp}} \simeq \mathrm{RMod}_{O}$, and thus O has significant Tate vanishing. By the previous corollary, this implies that $\mathrm{Alg}_{O}(\mathsf{Sp}_E)$ is infinitesimally semiadditive. $\square$

Example

If $O \in \mathrm{Operad}(\mathsf{Sp})$ has significant Tate vanishing, derivatives in $\mathrm{Alg}_{O}(\mathsf{Sp}_E)$ have divided powers, but these don't classify polynomials.

Snaith type localizations

Definition

A Snaith type localization ${\mathsf{S}_*}^{\geqslant n}[W^{-1}]$ requires a commutative square

$$\begin{tikzcd}
{\mathsf{S}_*}^{\geqslant n} \arrow[r, "\Sigma^\infty"] \arrow[d] & \mathsf{Sp} \arrow[d] \\
{\mathsf{S}_*}^{\geqslant n}[W^{-1}] \arrow[r, "\Sigma^\infty_E"'] & \mathsf{Sp}_E
\end{tikzcd}$$

and also that ${\mathsf{S}_*}^{\geqslant n}[W^{-1}]$ has a set of compact generators which are the localizations of compact objects of ${\mathsf{S}_*}^{\geqslant n}$.

Example

The categories of rational and v_h-periodic spaces ${\mathsf{S}_*}^{\geq 2}_{\mathbb{Q}}, {\mathsf{S}_*}^{\geq 1}_{v_h}$ are Snaith type localizations, where E is $H\mathbb{Q}$, T_h, respectively.

Geometry of Snaith type localizations

Proposition

If $\mathsf{S}_*{}^{\geqslant n}[W^{-1}]$ is Snaith type, then $\partial_* \mathrm{Id}_{\mathsf{S}_*[W^{-1}]}$ is lie_E and $\mathsf{S}_*{}^{\geqslant n}[W^{-1}]$ is differentially semiadditive.

Corollary

Suppose we have a polynomial equivalence

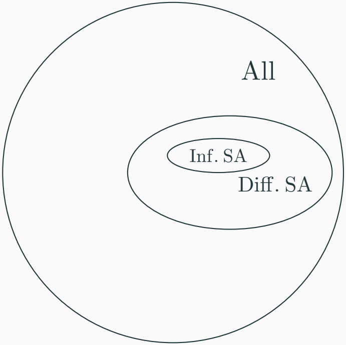
$$\mathsf{S}_*{}^{\geqslant n}[W^{-1}] \simeq_P \mathrm{Alg}_O(\mathsf{Sp}_E)$$

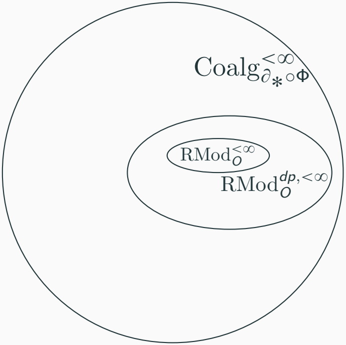
where $\mathsf{S}_*{}^{\geqslant n}[W^{-1}]$ is Snaith type. Then $O = \mathrm{lie}_E$ and Sp_E is semiadditive.

Remark

We expect Mathew-Clausen's results on Tate vanishing in the presence of Bousfield-Kuhn functors can be used to prove the statement when polynomial equivalence is replaced by equivalence, though the presence of even 0-connectivity hypotheses on S_* makes this rather delicate.

A summary in a picture





Some Questions

Question

For what operads O does $\mathrm{RMod}_O^{dp,<\infty}$ model the polynomial functors for an ∞-category C?

Question

Given a comonad C on $\mathrm{SymSeq}(\mathsf{Sp}_E)$, under what conditions do its bounded coalgebras model polynomial functors for some ∞-category C?

Question

What is the relation between polynomial equivalence of ∞-categories and other notions of equivalence, e.g. Goodwillie approximations of ∞-categories?

Thank you!