MAT 128A, Fall 2016

Programming Project 1

(due before class on Wednesday, October 26)

General Instructions

- The above deadline is hard. I will not accept any submissions after the lecture has started or after class.
- Write a report that includes all required numerical results and plots, a discussion of your results, answers to all questions, and printouts of your Matlab programs.
- When you are asked to print out numerical results, print numbers in 16-digit floating-point format. You can use the Matlab command "`format long e`" to switch to that format from Matlab's default. For example, the number 10π would be printed out as `3.141592653589793e+01` in 16-digit floating-point format.
- Upload all your Matlab programs into your drop box on the MAT 128A site on SmartSite by the due date of the project.

We consider the problem of computing the interpolating polynomial p_n for $n+1$ given data points

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)$$

with

$$\big(a :=\big)\ x_0 < x_1 < x_2 < \cdots < x_n\ \big(=: b\big).$$

1. In class, we discussed an algorithm for generating divided differences. Write a Matlab program that employs this algorithm to compute the coefficients $a_0, a_1, \ldots, a_n$ of the Newton form

$$p_n(x) = a_0 + (x - x_0)\Bigg(a_1 + (x - x_1)\bigg(a_2 + (x - x_2)\Big(\cdots\big(a_{n-1} + (x - x_{n-1})a_n\big)\cdots\Big)\bigg)\Bigg) \quad (1)$$

of p_n. The inputs for your program should be:

- The integer n;
- A vector of length $n+1$ containing $x_0, x_1, \ldots, x_n$;
- A vector of length $n+1$ containing $y_0, y_1, \ldots, y_n$.

The output for your program should be:

- A vector of length $n+1$ containing $a_0, a_1, \ldots, a_n$.

2. Write a Matlab program that implements Horner's rule (as presented in class) to compute the value $y = p_n(x)$ for any given $x \in \mathbb{R}$.

   The inputs for your program should be:

   - The integer n;
   - The point x at which p_n is to be evaluated;
   - A vector of length n containing $x_0, x_1, \ldots, x_{n-1}$;
   - A vector of length $n+1$ containing the coefficients $a_0, a_1, \ldots, a_n$ of the representation (1) of p_n.

   The output for your program should be:

   - The value $y = p_n(x)$.

3. Consider the barycentric interpolation formula for p_n, as presented in class.

   (a) Formally, the barycentric interpolation formula is only defined for $x \neq x_j$, due to division by zero for $x = x_j$. Show that for $p_n(x)$ given by the barycentric interpolation formula, we have

   $$y_j = \lim_{x \to x_j} p_n(x), \quad j = 0, 1, \ldots, n.$$

   (b) For an actual implementation of the barycentric interpolation formula in floating-point arithmetic, consider the following remedy to deal with the division by zero for $x = x_j$:

   `If` $x - x_j \neq 0$,

   `compute` $p_n(x)$ `via the barycentric interpolation formula`,

   `else`,

   `set` $p_n(x) = y_j$.

   Is this a viable remedy? Give an explanation for your answer.

   (c) For some standard choices of the x_i's, the barycentric weights are known and thus do not need to be computed. For example, for equally-spaced x_i's in $[a, b]$, we have

   $$w_i = c(n, a, b)\, \hat{w}_i, \quad \text{where} \quad \hat{w}_i := (-1)^i \binom{n}{i} = (-1)^i \frac{n!}{i!\,(n-i)!}, \quad i = 0, 1, \ldots, n. \tag{2}$$

   For the Chebyshev interpolation points

   $$x_i = \frac{1}{2}\left(a + b + (b - a)\cos\frac{\pi(n-i)}{n}\right), \quad i = 0, 1, \ldots, n,$$

   in $[a, b]$, we have

   $$w_i = d(n, a, b)\, \hat{w}_i, \quad \text{where} \quad \hat{w}_i = (-1)^i \cdot \begin{cases} \frac{1}{2} & \text{for } i = 0 \text{ and } i = n, \\ 1 & \text{for } i = 1, 2, \ldots, n-1. \end{cases} \tag{3}$$

In (2) and (3), $c(n, a, b)$ and $d(n, a, b)$ are nonzero constants that are independent of i. Show that for the cases (2) and (3), the barycentric interpolation formula can be written in the form

$$p_n(x) = \frac{\displaystyle\sum_{i=0}^{n} \frac{\hat{w}_i}{x - x_i} y_i}{\displaystyle\sum_{i=0}^{n} \frac{\hat{w}_i}{x - x_i}}. \tag{4}$$

4. Write a Matlab program that employs the barycentric interpolation formula (4), together with the remedy from part 3(b), to compute $y = p_n(x)$ for any given $x \in \mathbb{R}$.

   The inputs for your program should be:

   - The integer n;
   - The point x at which p_n is to be evaluated;
   - A vector of length $n+1$ containing $x_0, x_1, \ldots, x_n$;
   - A vector of length $n+1$ containing $y_0, y_1, \ldots, y_n$.
   - A vector of length $n+1$ containing $\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n$.

   The output for your program should be:

   - The value $y = p_n(x)$.

5. Test your Matlab programs for the following interpolation problems:

   (a)
   - $x_i = -\pi + \frac{2\pi}{n} i, \quad i = 0, 1, \ldots, n;$
   - $y_i = f(x_i), \quad i = 0, 1, \ldots, n, \quad \text{where} \quad f(x) = \sin x;$
   - $n = 5$, $n = 10$, $n = 20$, and $n = 40$.

   For each of the four runs (with $n = 5, 10, 20, 40$), print out

   $$f(x), \quad p_n^{(1)}(x), \quad \left|f(x) - p_n^{(1)}(x)\right|, \quad p_n^{(2)}(x), \quad \text{and} \quad \left|f(x) - p_n^{(2)}(x)\right|$$

   at the following 6 points:

   $$x = -\pi, \quad x = \pi, \quad \text{and} \quad x = -\pi + \frac{\pi}{n}(2k-1), \quad k = 1, 2, n-2, n-1.$$

   Here, $p_n^{(1)}(x)$ and $p_n^{(2)}(x)$ denote the values of $p_n(x)$ computed with your programs for Horner's rule and for the barycentric interpolation formula, respectively.
   Also, use Matlab's plotting functions to produce for each n a single plot showing the three functions

   $$f(x), \quad p_n^{(1)}(x), \quad \text{and} \quad p_n^{(2)}(x) \quad \text{for all} \quad -\pi \le x \le \pi.$$

   Note that you will need to use sufficiently many values of x to obtain realistic plots.

(b)
- $x_i = -5 + \frac{10}{n}\, i, \quad i = 0, 1, \ldots, n;$
- $y_i = f(x_i), \quad i = 0, 1, \ldots, n, \quad$ where $\quad f(x) = \frac{1}{1+x^2};$
- $n = 5$, $n = 10$, $n = 20$, and $n = 40$.

For each of the four runs (with $n = 5, 10, 20, 40$), print out

$$f(x), \quad p_n^{(1)}(x), \quad \left|f(x) - p_n^{(1)}(x)\right|, \quad p_n^{(2)}(x), \quad \text{and} \quad \left|f(x) - p_n^{(2)}(x)\right|$$

at the following 6 points:

$$x = -5, \quad x = 5, \quad \text{and} \quad x = -5 + \frac{5}{n}\,(2k-1), \quad k = 1, 2, n-2, n-1.$$

Here, $p_n^{(1)}(x)$ and $p_n^{(2)}(x)$ denote the values of $p_n(x)$ computed with your programs for Horner's rule and for the barycentric interpolation formula, respectively.

Also, use Matlab's plotting functions to produce for each n a single plot showing the three functions

$$f(x), \quad p_n^{(1)}(x), \quad \text{and} \quad p_n^{(2)}(x) \quad \text{for all} \quad -5 \le x \le 5.$$

(c) Run problem (b) again, but now with the Chebyshev interpolation points

$$x_i = 5\cos\frac{\pi(n-i)}{n}, \quad i = 0, 1, \ldots, n.$$