MAT 128A, Fall 2016

Programming Project 2

(due before class on Wednesday, November 9)

General Instructions

- The above deadline is hard. I will not accept any submissions after the lecture has started or after class.
- Write a report that includes all required numerical results and plots, a discussion of your results, and printouts of your Matlab programs.
- When you are asked to print out numerical results, print numbers in 16-digit floating-point format. You can use the Matlab command "`format long e`" to switch to that format from Matlab's default. For example, the number 10π would be printed out as `3.141592653589793e+01` in 16-digit floating-point format.
- Upload all your Matlab programs into your drop box on the MAT 128A site on SmartSite by the due date of the project.

We consider the problem of computing approximations to the first derivative $f'(x)$, $x \in [a, b]$, from given function values

$$y_i = f(x_i), \quad i = 0, 1, \ldots, n,$$

where

$$\big(a :=\big)\ x_0 < x_1 < x_2 < \cdots < x_n\ \big(=: b\big)$$

and $n \geq 1$.

1. The simplest solution to this problem is as follows:

   - For given $x \in [a, b]$, find the smallest $j \in \{1, 2, \ldots, n\}$ such that $x \in [x_{j-1}, x_j]$;
   - Use
     $$c(x) := \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$
     as an approximation to $f'(x)$.

   Write a Matlab program that implements this approach. The inputs for your program should be:

   - The value $x \in [a, b]$ at which $f'(x)$ is to be approximated;
   - A vector of length $n + 1$ containing $x_0, x_1, \ldots, x_n$;
   - A vector of length $n + 1$ containing $y_0, y_1, \ldots, y_n$.

The output for your program should be:

- The approximation to $f'(x)$.

2. Write a Matlab program that computes the approximation $p_n'(x) \approx f'(x)$, using the formula for $p_n'(x)$ presented in class. Here, $p_n = p_n(x)$ denotes the unique polynomial of degree at most n that interpolates the data points (x_i, y_i), $i = 0, 1, \ldots, n$.

   As discussed in class, a numerically stable way to compute $p_n'(x)$ is as follows:

   - For given $x \in [a, b]$, find the smallest $j \in \{ 0, 1, 2, \ldots, n \}$ such that
   $$|x - x_j| = \min_{0 \le i \le n} |x - x_i| \, ;$$
   - Set
   $$p_n'(x) = \frac{\hat{w}_j \left(q_j - y_j r_j + d_j (y_j t_j - s_j) \right) + (d_j)^2 (q_j t_j - r_j s_j)}{\left(\hat{w}_j + d_j r_j \right)^2},$$
   where
   $$d_j = d_j(x) := x - x_j,$$
   $$q_j = q_j(x) := \sum_{\substack{i=0 \\ i \ne j}}^{n} \frac{\hat{w}_i y_i}{x - x_i}, \qquad r_j = r_j(x) := \sum_{\substack{i=0 \\ i \ne j}}^{n} \frac{\hat{w}_i}{x - x_i},$$
   $$s_j = s_j(x) := \sum_{\substack{i=0 \\ i \ne j}}^{n} \frac{\hat{w}_i y_i}{(x - x_i)^2}, \quad t_j = t_j(x) := \sum_{\substack{i=0 \\ i \ne j}}^{n} \frac{\hat{w}_i}{(x - x_i)^2}.$$

   The inputs for your program should be:

   - The value $x \in [a, b]$ at which $f'(x)$ is to be approximated;
   - A vector of length $n + 1$ containing $x_0, x_1, \ldots, x_n$;
   - A vector of length $n + 1$ containing $y_0, y_1, \ldots, y_n$.
   - A vector of length $n + 1$ containing the scaled barycentric weights $\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n$.

   The output for your program should be:

   - The approximation $p_n'(x)$ to $f'(x)$.

3. Write a Matlab program that computes the approximation $s'(x) \approx f'(x)$. Here, $s = s(x)$ denotes the not-a-knot cubic spline that interpolates the data points (x_i, y_i), $i = 0, 1, \ldots, n$. We assume that $n \ge 3$; this guarantees the existence of s. In class, we discussed how to compute values of $s'(x)$ using the Matlab functions `spline`, `mkpp`, and `ppval`. Your Matlab program should implement this approach.

   The inputs for your program should be:

   - The value $x \in [a, b]$ at which $f'(x)$ is to be approximated;

- A vector of length $n+1$ containing $x_0, x_1, \ldots, x_n$;
- A vector of length $n+1$ containing $y_0, y_1, \ldots, y_n$.

The output for your program should be:

- The approximation $s'(x)$ to $f'(x)$.

4. Test your Matlab programs for the following problem:

- $a = -1, \quad b = 2;$
- $x_i = \dfrac{1}{2}(a+b) + \dfrac{1}{2}(b-a) \cos \dfrac{\pi(n-i)}{n}, \quad i = 0, 1, \ldots, n;$
- $y_i = f(x_i), \quad i = 0, 1, \ldots, n, \quad \text{where} \quad f(x) = e^x \sin(5x);$
- $n = 5$, $n = 10$, $n = 20$, $n = 40$, and $n = 80$.

For each of the 5 runs (with $n = 5, 10, 20, 40, 80$), print out

$$f'(x), \quad c(x), \quad p_n'(x), \quad s'(x)$$

and

$$\big|f'(x) - c(x)\big|, \quad \big|f'(x) - p_n'(x)\big|, \quad \big|f'(x) - s'(x)\big|$$

at the following 5 points:

$$x = -1, \quad x = -0.5, \quad x = 0, \quad x = 1, \quad \text{and} \quad x = 2.$$

Also, use Matlab's plotting functions to produce for each n a single plot showing the functions

$$f'(x), \quad c(x), \quad p_n'(x), \quad \text{and} \quad s'(x) \quad \text{for all} \quad a \le x \le b.$$

Note that you will need to use sufficiently many values of x to obtain realistic plots.