Simplex method

Online lecture

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Linear program

maximize

$$y = c^T x$$

subject to

$$Ax \leq b$$
$$x \geq 0$$

where $x, c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m \quad b \geq 0$

Example

maximize

$$y = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} 4x_1 + 2x_2 &\leq 8 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, c = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

Standard form

maximize

$$y = c^T x + 0s$$

subject to

$$Ax + s = b$$
$$x, s \geq 0$$

where $x, c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

$s \in \mathbb{R}^m$ slack variables

Simplex method

maximize

$$y$$

subject to

$$\begin{aligned} y - c^T x + 0s &= 0 \\ 0y + Ax + s &= b \end{aligned}$$

$$x, s \geq 0$$

$$x = 0 \Rightarrow s = b, y = 0$$

x nonbasic variables

s basic variables

$$s_i \leftrightarrow x_j$$

x_j entering variable

s_i leaving variable

Simplex tableau

Basic variables	y	x	s	r.h.s
	1	$-c^T$	0	0
s	0	A	I	b

Simplex tableau

Entering : $\min -c_j^T$ Leaving : $a_{ij} > 0, \min \frac{b_i}{a_{ij}}$

Basic variables	y	$\cdots$	x_j	$\cdots$	$\cdots$	s_i	$\cdots$	r.h.s
	1	$\cdots$	$-c_j^T$	$\cdots$	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$			$\vdots$		$\vdots$
s_i	0	$\cdots$	a_{ij}	$\cdots$	$\cdots$	1	$\cdots$	b_i
$\vdots$	$\vdots$		$\vdots$			$\vdots$		$\vdots$

Simplex tableau

Basic variables	y	$\cdots$	x_j	$\cdots$	$\cdots$	s_i	$\cdots$	r.h.s
	1	$\cdots$	0	$\cdots$	$\cdots$	$\frac{c_j^T}{a_{ij}}$	$\cdots$	$\frac{b_i c_j^T}{a_{ij}}$
$\vdots$	$\vdots$		$\vdots$			$\vdots$		$\vdots$
x_j	0	$\cdots$	1	$\cdots$	$\cdots$	$\frac{1}{a_{ij}}$	$\cdots$	$\frac{b_i}{a_{ij}}$
$\vdots$	$\vdots$		$\vdots$			$\vdots$		$\vdots$

Gaussian elimination

Simplex tableau

Basic variables	y	$\cdots$	x_j	$\cdots$	$\cdots$	s_i	$\cdots$	r.h.s
	1	$\cdots$	0	$\cdots$	$\cdots$	$\dfrac{c_j^T}{a_{ij}}$	$\cdots$	$\dfrac{b_i c_j^T}{a_{ij}}$
$\vdots$	$\vdots$		$\vdots$			$\vdots$		$\vdots$
x_j	0	$\cdots$	1	$\cdots$	$\cdots$	$\dfrac{1}{a_{ij}}$	$\cdots$	$\dfrac{b_i}{a_{ij}}$
$\vdots$	$\vdots$		$\vdots$			$\vdots$		$\vdots$

Stop if the circled coefficients are non-negative

Example

maximize

$$y = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} 4x_1 + 2x_2 &\leq 8 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

maximize

$$y$$

subject to

$$\begin{aligned} y - 4x_1 - 3x_2 - 0s_1 - 0s_2 &= 0 \\ 0y + 4x_1 + 2x_2 + s_1 + 0s_2 &= 8 \\ 0y + x_1 + x_2 + 0s_1 + s_2 &= 3 \end{aligned}$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Example

Basic variables	y	x_1	x_2	s_1	s_2	r.h.s
	1	-4	-3	0	0	0
s_1	0	4	2	1	0	8
s_2	0	1	1	0	1	3

-4 is the smallest

Example

Basic variables	y	x_1	x_2	s_1	s_2	r.h.s
	1	-4	-3	0	0	0
s_1	0	4	2	1	0	8
s_2	0	1	1	0	1	3

The smallest ratio is 8/4=2

Example

Basic variables	y	x_1	x_2	s_1	s_2	r.h.s
	1	0	-1	1	0	8
x_1	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	2
s_2	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	1	1

-1 is a negative number

Example

Basic variables	y	x_1	x_2	s_1	s_2	r.h.s
	1	0	0	$\frac{1}{2}$	2	10
x_1	0	1	0	$\frac{1}{2}$	-1	1
x_2	0	0	1	$-\frac{1}{2}$	2	2

$$(x_1, x_2) = (1, 2)$$