Chapter 1b

Propositional Logic Review II

(SAT Solving and Application)

Mathematics Modeling

(Materials drawn from **Chapter 1** in:
"Michael Huth and Mark Ryan. *Logic in Computer Science: Modelling and Reasoning about Systems*, 2nd Ed., Cambridge University Press, 2006"
and some other sources)

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2 2-SAT is in P

3 SAT Solvers

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Motivated Example – A Logic Puzzle

- If the unicorn is mythical, then it is immortal;and
- If the unicorn is not mythical, then it is a mortal mammal;and
- If the unicorn is either immortal or a mammal, then it is horned;and
- The unicorn is magical if it is horned.

- **Q:** Is the unicorn mythical? Is it magical? Is it horned?

CNF

- Boolean formula ϕ is defined over a set of propositional variables $p_1, ..., p_n$, using the standard propositional connectives $\neg, \wedge, \vee, \longrightarrow, \longleftrightarrow$, and parenthesis
  - The domain of propositional variables is $\{0, 1\}$.
  - Example: $\phi(p_1, p_2, p_3) = ((\neg p_1 \wedge p_2) \vee p_3) \wedge (\neg p_2 \vee p_3)$.
- A formula ϕ in conjunctive normal form (CNF) is a conjunction of disjunctions (clauses) of literals, where a literal is a variable or its complement.
  - Example: $\phi(p_1, p_2, p_3) = (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee p_3)$.

Proposition (see [2, Subsection 1.5.1])

There is an algorithm to translate *any* Boolean formula into CNF.

Proposition 1.45, p. 57

ϕ-satisfiable iff $\neg\phi$-not tautology.

SAT

Problem

Find an assignment to the variables $p_1, ..., p_n$ such that $\phi(p_1, ..., p_n) = 1$, or prove that no such assignment exists.

Facts: SAT is an NP-complete decision problem [Cook'71]

- SAT was the first problem to be shown NP-complete.
- There are no known polynomial time algorithms for SAT.
- More-than-35-year old conjecture:
  "Any algorithm that solves SAT is exponential in the number of variables, in the worst-case."

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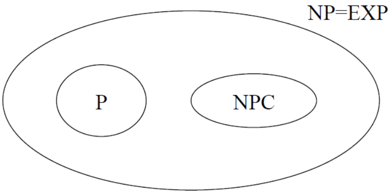
Polynomial time reductions and NP-Completeness

- Denote
  - EXP = {Decision problems solvable in exponential time}
  - P = {Decision problems solvable in polynomial time}
  - NP = {Decision problems where Yes solution can verified in polynomial time}
- A major open question in theoretical computer science is **if** $P = NP$ **or not**.
- Introduce the notion of **polynomial time reductions** $X \leq_P Y$:

  A problem X is polynomial time reducible to a problem Y ($X \leq_P Y$) if we can solve X in a polynomial number of calls to an algorithm for Y (and the instance of problem Y we solve can be computed in polynomial time from the instance of problem X).
- The class of NP-**complete** problems NPC: A problem Y is in NPC if
  a) $Y \in NP$, and
  b) $X \leq_P Y$ for all $X \in NP$.

P=NP question

- The problems in NPC are the hardest problems in NP and the key to resolving the $P = NP$ question.
- If one problem $Y \in NPC$ is in P then $P = NP$.
- If one problem $Y \in NP$ is not in P then $NPC \cap P = \emptyset$.
- By now a lot of problems have been proved NP-complete
- We think the world looks like this—but we really do not know:



- If someone found a polynomial time solution to a problem in NPC our world would "collapse" *and* a lot of smart people have tried really hard to solve NPC problems efficiently

$$\Downarrow$$

We regard $Y \in NPC$ a strong evidence for Y being hard!

NP-Complete Problems

- The following lemma helps us to prove a problem NP-complete using another NP-complete problem.

  Lemma: If $Y \in NP$ and $X \leq_P Y$ for some $X \in NPC$ then $Y \in NPC$

  Proof: To prove $Y \in NPC$ we just need to prove $Y \in NP$ (often easy) and reduce problem in NPC to Y (no lower bound proof needed!).
- Finding the first problem in NPC is somewhat difficult and require quite a lot of formalism
- It seems to be a easier problem 3Sat: Given a formula in 3-CNF, is it satisfiable?
  - A formula is in 3-CNF (conjunctive normal form) if it consists of an And of 'clauses' each of which is the Or of 3 'literals'
  - Example: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$
- We prove that 3SAT is in NPC, meaning that it is as hard as general SAT.
  - 3SAT $\in NP$
  - SAT $\leq_P$ 3SAT (we can show that transforming general formula into 3-CNF is in polynomial time.)

Example

- Consider the following 2-CNF formula consisting of the following clauses:

$$\bar{x}_1 \vee x_2, \qquad x_1 \vee x_2, \qquad \bar{x}_2 \vee x_3, \qquad x_3 \vee \bar{x}_4, \qquad x_1 \vee \bar{x}_2.$$

- Let's try to set $x_1 = 0$. Then the formula simplifies to:

$$T, \qquad x_2, \qquad \bar{x}_2 \vee x_3, \qquad x_3 \vee \bar{x}_4, \qquad \bar{x}_2.$$

  where T denotes the value "Truth".
- We are now *forced* to assign $x_2 = 1$ (as there is a unit-clause), and the formula simplifies to

$$T, \qquad T, \qquad x_3, \qquad x_3 \vee \bar{x}_4, \qquad \emptyset,$$

  where $\emptyset$ is the empty clause which denotes contradiction.
- So we have to backtrack to the last *free step*.
- Let's try $x_1 = 1$:

$$x_2, \qquad T, \qquad \bar{x}_2 \vee x_3, \qquad x_3 \vee \bar{x}_4, \qquad T.$$

- We are now forced to set $x_2 = 1$:

$$T, \qquad T, \qquad x_3, \qquad x_3 \vee \bar{x}_4, \qquad T.$$

- We are now forced to set $x_3 = 1$:

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Algorithm(ϕ)

Abstracting the above example, we present an algorithm that attempts to satisfy a 2-CNF formula ϕ as follows.

Algorithm(ϕ)

(0) Initialize empty assignment $\sigma = *^n$.

(1) If all variables are assigned return σ.

(2) Choose an unassigned variable x_i.

  (a) (Try $x_i = 1$)

   - Set $\sigma_i = 1$, $\phi' \leftarrow$ Simplify(ϕ, x_i).
   - $\phi' \leftarrow$ Unit Clause Propagation(ϕ').
   - If ϕ' does not contain $\emptyset$ goto (1).

  (b) (Try $x_i = 0$)

   - Unassign variables from step (a).
   - Set $\sigma_i = 0$, $\phi' \leftarrow$ Simplify($\phi, \bar{x}_i$).
   - $\phi' \leftarrow$ Unit Clause Propagation(ϕ').
   - If ϕ' does not contain $\emptyset$ goto (1).

(3) Halt with "UNSAT".

Simplify(ϕ, ℓ_i)

Simplify(ϕ, ℓ_i)

- $\forall$ clause $C \in \phi$:
  - If $\ell_i \in C$, remove C.
  - If $\bar{\ell}_i \in C$, $C \leftarrow C \setminus \bar{\ell}_i$.
  - Otherwise, copy C as is.
- Output the modified formula.

Unit Clause Propagation(ϕ)

- While $\exists$ unit clause ℓ_i:
  - Update σ: if $\ell_i = x_i$ set $\sigma_i = 1$, else $(\ell_i = \bar{x}_i)$ set $\sigma_i = 0$.
  - $\phi \leftarrow$Simplify(ϕ, ℓ_i).

Complexity: Let n denote the number of variables and let m denote the number of clauses. It is not hard to verify that there are at most n outer iterations and that each call to UCP takes at most $O(m)$ time, therefore the running time of Algorithm is $O(m \cdot n)$. (HW: Find an implementation in $O(n + m)$ complexity.)

Correctness of the Algorithm

Lemma

If the algorithm outputs an assignment σ, then σ satisfies ϕ.

We will need the following definition: A partial assignment $\sigma \in \{0, 1, *\}^n$ *violate* a clause $C = \ell_i \vee \ell_j$ if: σ_i and σ_j are assigned (i.e.,$\sigma_i, \sigma_j \neq *$) and σ_i doesn't satisfy ℓ_i and σ_j doesn't satisfy ℓ_j. The lemma follows from the following invariance.

Lemma

At the beginning of each iteration, the current partial assignment $\sigma^{(i)}$ does not violate any of the clauses of C.

Chứng minh.

Invariance 2 By induction on i. The basis is trivial as in the first iteration $\sigma = *^n$ and so none of the clauses are violated. Step: we'll prove that none of the clauses C are violated by $\sigma^{(i+1)}$. If both variables of C were assigned before the last iteration, then, by the induction hypothesis, $\sigma^{(i)}$ doesn't violate C, and therefore, so is $\sigma^{(i+1)}$. If both variables of C were assigned in the last iteration, then C must be satisfied by $\sigma^{(i+1)}$, otherwise, the algorithm finds a contradiction. □

Correctness of the Algorithm (cont.)

Lemma

If the algorithm outputs UNSAT, then ϕ is unsatisfiable.

Chứng minh.

- Let ϕ' be the formula at the beginning of the iteration in which A halts, and let x_i be the variable chosen at step (2) of this last iteration.
- Note that ϕ' is a 2-CNF formula and $\phi' \subseteq \phi$ (i.e., all the clauses of ϕ' appear as clauses in ϕ).
- Hence, it suffices to show that ϕ' is unsatisfiable.
- Let ϕ_0=Simplify($\phi', x_i = 0$) and ϕ_1=Simplify($\phi', x_i = 1$). It suffices to show that both ϕ_0 and ϕ_1 are unsatisfiable.
- Recall that the formula UCP(ϕ_0) and the formula UCP(ϕ_1) contain a contradiction.
- The proof now follows by noting that if UCP(ψ) contains a contradiction, then ψ is UNSAT.

□

Therefore, we have an efficient algorithm for SAT of 2-CNF.

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Graphical View of 2-SAT

- For a 2-CNF formula ϕ, define the implication graph $G = G_\phi$ as follows:
  - nodes $x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n$
  - for a clause $\ell_i \vee \ell_j$ define the edges:

    $\bar{\ell}_i \to \ell_j$

    $\bar{\ell}_j \to \ell_i$

  Main property: Let σ be a satisfying assignment.

  If σ satisfies a node v, then σ satisfies all nodes u achievable from v.

  The property can be proven by induction on the length of the path.

Theorem

ϕ is satisfiable iff the graph G does not contain a "contradiction path" of the form:

$$\ell_i \to \cdots \to \bar{\ell}_i \to \cdots \to \ell_i.$$

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Proof for the previous theorem

1. ($\exists$ contradiction path $\Rightarrow \phi$ is UNSAT):
   - Take a potential assignment σ.
   - If σ satisfies ℓ_i, then by Property it must satisfy $\bar{\ell}_i$. Contradiction.
   - If σ satisfies $\bar{\ell}_i$, then by Property it must satisfy ℓ_i. Contradiction.
2. (ϕ is UNSAT $\Rightarrow \exists$ contradiction path):
   If ϕ is UNSAT $\Rightarrow$ Algorithm Halts.
   $\Rightarrow$ for some x_i we have:
   (a) $\ell_j \leftarrow \cdots \leftarrow x_i \rightarrow \cdots \rightarrow \bar{\ell}_j$
   (b) $\ell_k \leftarrow \cdots \leftarrow \bar{x}_i \rightarrow \cdots \rightarrow \bar{\ell}_k$

   In our graph, if $\ell_i \rightarrow \ell_j$ is an edge, then $\bar{\ell}_j \rightarrow \bar{\ell}_i$ is also an edge.

   By reversing edges and negating:
   (a) $\Rightarrow x_i \rightarrow \cdots \rightarrow \bar{\ell}_j \rightarrow \cdots \rightarrow \bar{x}_i$
   (b) $\Rightarrow \bar{x}_i \rightarrow \cdots \rightarrow \bar{\ell}_k \rightarrow \cdots \rightarrow x_i$

   Therefore, there exists a contradiction path.

1 Introduction

2 2-SAT is in P

3 SAT Solvers

- WalkSAT: Idea
- DPLL: Idea
- A Linear Solver
- A Cubic Solver

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WalkSAT: An Incomplete Solver

- **Idea:** Start with a random truth assignment, and then iteratively improve the assignment until model is found.
- **Details:** In each step, choose an unsatisfied clause (clause selection), and "flip" one of its variables (variable selection).

WalkSAT: Details

- **Termination criterion:** No unsatisfied clauses are left.
- **Clause selection:** Choose a random unsatisfied clause.
- **Variable selection:**
  - If there are variables that when flipped make no currently satisfied clause unsatisfied, flip one which makes the most unsatisfied clauses satisfied.
  - Otherwise, make a choice with a certain probability between:
    - picking a random variable, and
    - picking a variable that when flipped minimizes the number of unsatisfied clauses.

DPLL: Idea

- Simplify formula based on pure literal elimination and unit propagation
- If not done, pick an atom p and split: $\phi \land p$ or $\phi \land \neg p$

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A Linear Solver: Idea

- Transform formula to tree of conjunctions and negations.
- Transform tree into graph.
- Mark the top of the tree as `T`.
- Propagate constraints using obvious rules.
- If all leaves are marked, check that corresponding assignment makes the formula true.

Transformation

$$
\begin{aligned}
T(p) &= p \\
T(\phi_1 \wedge \phi_2) &= T(\phi_1) \wedge T(\phi_2) \\
T(\neg\phi) &= \neg\phi(T) \\
T(\phi_1 \to \phi_2) &= \neg(T(\phi_1) \wedge \neg T(\phi_2)) \\
T(\phi_1 \vee \phi_2) &= \neg(\neg T(\phi_1) \wedge \neg T(\phi_2))
\end{aligned}
$$

Example

$$\phi = p \wedge \neg(q \vee \neg p)$$

$$T(\phi) = p \wedge \neg\neg(\neg q \wedge \neg\neg p)$$

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Binary Decision Tree: Example

See Example 1.48 and Figure 1.12 on page 70.

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Problem

What happens to formulas of the kind $\neg(\phi_1 \wedge \phi_2)$?

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A Cubic Solver: Idea

Improve the linear solver as follows:

- Run linear solver
- For every node n that is still unmarked:
  - Mark n with `T` and run linear solver, possibly resulting in temporary marks.
  - Mark n with `F` and run linear solver, possibly resulting in temporary marks.
  - Combine temporary marks, resulting in possibly new permanent marks

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An application of SAT solving: Solve Sudoku Boolean Formula

At the end of Chapter 0, we saw that

$$\phi = I \wedge R \wedge C \wedge B$$

- Note that ϕ is in CNF.
- ϕ can be altered so that it contains exactly 3 literals per clause (can be fed to 3-SAT solver).
- Problem: Solve this 3-SAT problem with a suitable solver?

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Homeworks and Next Week Plan?

Homeworks

- Read carefully all proofs in this note.
- Try to solve the Sudoku in the Introduction note
- Show that $kSAT \in NPC$ for all $k \geq 3$.
- Do ALL marked questions of Exercises 1.6 in [2].
- Read carefully Subsections 1.6.1 and 1.6.2 in [2].

Next Week?

Predicate Logic

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