MA1521 CALCULUS FOR COMPUTING

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Chapter 7: Ordinary Differential Equations **2**

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Introduction

- Recall that the derivative of a (differentiable) function determines the change of the function. More precisely, suppose $\dfrac{dy}{dx} = f(x)$ for all x.
  - Then $y = \displaystyle\int f(x)\,dx + C$.

  So if $\dfrac{dy}{dx}$ is known, we can determine y up to a constant.
- In general, if there is a relation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0,$$

  known as the **ordinary differential equation** (**ODE**), we want to determine the relation of of x and y explicitly.

The Simplest Ordinary Differential Equations

- $\dfrac{dy}{dx} = f(x) \Rightarrow y = \displaystyle\int f(x)\,dx + C.$
  - This is exactly the problem of integration.
- **Examples**.
  - $\dfrac{dy}{dx} = 1 - \sqrt{x}.$
    - $y = \displaystyle\int (1 - \sqrt{x})\,dx = x - \frac{2}{3}x^{3/2} + C.$
  - $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 - 1}}.$
    - $y = \displaystyle\int \frac{x}{\sqrt{x^2 - 1}}\,dx = \sqrt{x^2 - 1} + C.$
  - $\dfrac{d^2 y}{dx^2} = 0 \Rightarrow \dfrac{dy}{dx} = C \Rightarrow y = Cx + D.$

The Simplest Ordinary Differential Equations

- $\dfrac{dy}{dx} = g(y) \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{g(y)} \Rightarrow x = \displaystyle\int \frac{1}{g(y)}\,dy.$
  - $\dfrac{dy}{dx} = 1 + y^2 \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{1+y^2} \quad \therefore y = \tan(x - C).$
    - $x = \displaystyle\int \frac{1}{1+y^2}\,dy = \tan^{-1} y + C.$
  - $\dfrac{dy}{dx} = e^y \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{e^y} \quad \therefore y = -\ln(C - x).$
    - $x = \displaystyle\int e^{-y}\,dy = -e^{-y} + C.$
  - $\dfrac{dy}{dx} = \sec y \Rightarrow \dfrac{dx}{dy} = \cos y.$
    - $x = \displaystyle\int \cos y\,dy = \sin y + C.$

The Simplest Ordinary Differential Equations

- Suppose $\dfrac{dy}{dx} = y$. Find y in terms of x.
  - $\dfrac{dx}{dy} = \dfrac{1}{y} \Rightarrow x = \displaystyle\int \frac{1}{y}\,dy = \ln|y| + c.$ ←-- Problem!
    - $|y| = e^{x-c}$. Then $y = \pm e^{-c}e^{x} = Ce$

    However, y may be zero somewhere.
  - Define $z = ye^{-x}$. It is well-defined on $\mathbb{R}$.
    - $\dfrac{dz}{dx} = \dfrac{dy}{dx}e^{-x} + y(-e^{-x}) = e^{-x}\left(\dfrac{dy}{dx} - y\right) = 0.$

    So $z = ye^{-x} = C$ is constant on $\mathbb{R}$, i.e., $y = Ce^{x}$.
- For computation purpose, we still use the non-rigorous method by ignoring the zeros of y.
  We omit the detailed explanation of the existence and uniqueness of the solution in our course.

Separation of Variables

- Consider a general problem: $\dfrac{dy}{dx} = f(x)g(y)$.

  $f(x)g(y)$ is a product of a function in x and function in y.
  The variables x and y in $f(x)g(y)$ are **separable**.

  - In **differential forms**: $\dfrac{1}{g(y)}\, dy = f(x)\, dx$.
    - $\displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$
  - To be rigorous, $\dfrac{1}{g(y)} \dfrac{dy}{dx} = f(x)$.
    - $\displaystyle\int f(x)\, dx = \int \frac{1}{g(y)} \frac{dy}{dx}\, dx = \int \frac{1}{g(y)}\, dy.$
- This method is called **separation of variables**.

Examples

- $2\sqrt{xy}\,\dfrac{dy}{dx} = 1 \quad (x, y > 0) \qquad \therefore \quad y = \left(\frac{3}{2}\sqrt{x} + \frac{3}{4}C\right)^{2/3}$.
  - $\displaystyle\int 2\sqrt{y}\, dy = \int \frac{1}{\sqrt{x}}\, dx \Rightarrow \frac{4}{3}y^{3/2} = 2\sqrt{x} + C.$
- $\dfrac{dy}{dx} \sec x = e^{y + \sin x} \qquad \therefore \quad y = -\ln(-C - e^{\sin x})$.
  - $\displaystyle\int e^{-y}\, dy = \int e^{\sin x} \cos x\, dx \Rightarrow -e^{-y} = e^{\sin x} + C.$
- $\dfrac{dy}{dx} \ln x = \dfrac{y}{x} \quad \therefore \quad y = \pm e^{c} \ln x = C \ln x$.
  - $\displaystyle\int \frac{1}{y}\, dy = \int \frac{1}{x \ln x}\, dx \Rightarrow \ln|y| = \ln|\ln x| + c.$

Singular Solutions

- **Example**. $\dfrac{dy}{dx} = \sqrt[3]{xy} = \sqrt[3]{x} \cdot \sqrt[3]{y}$.
  - $\displaystyle\int \frac{dy}{\sqrt[3]{y}} = \int \sqrt[3]{x}\, dx \Rightarrow \frac{3}{2} y^{2/3} = \frac{3}{4} x^{4/3} + C$.
  - Note that $\sqrt[3]{y} = 0 \Rightarrow y = 0$.
    - $y = 0$ is also a solution to the equation.
- Suppose $\dfrac{dy}{dx} = f(x)g(y)$.
  - If $y = C$ is a solution to $g(y) = 0$,

    then it is a **singular solution** to $\dfrac{dy}{dx} = f(x)g(y)$.
  - The singular solution disappears if the equation is
    - $\dfrac{1}{g(y)} \dfrac{dy}{dx} = f(x)$.
  - We IGNORE the singular solutions in our course.

Example

- $\dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1 + \frac{y}{x}}{1 - \frac{y}{x}}$. It is NOT separable.
  - Let $z = \dfrac{y}{x}$. Then $y = zx$.
    - $\dfrac{dy}{dx} = x \dfrac{dz}{dx} + z = \dfrac{1+z}{1-z}$.
  - $x \dfrac{dz}{dx} = \dfrac{1+z}{1-z} - z = \dfrac{1+z^2}{1-z}$.
    - $\displaystyle\int \frac{1-z}{1+z^2}\, dz = \int \frac{1}{x}\, dx$.
    - $\tan^{-1} z - \dfrac{1}{2} \ln(1+z^2) = \ln|x| + C$.

$\therefore \tan^{-1} \dfrac{y}{x} = \dfrac{1}{2} \ln(x^2 + y^2) + C$.

Homogeneous Equations

- Consider $\frac{dy}{dx} = F(x,y)$.
  - Suppose $F(x,y)$ is **homogeneous of degree zero**.
    - i.e., $F(tx,ty) = F(x,y)$ for all $t \in \mathbb{R}\backslash\{0\}$.

    For example: $\frac{x+y}{x-y}, \frac{xy+y^2}{x^2+xy}, \frac{\sqrt{x^2+y^2}}{|x|}, \ldots$.
  - Let $z = \frac{y}{x}$. Then
    - $y = xz$ and $\frac{dy}{dx} = x\frac{dz}{dx} + z$.
    - $F(x,y) = F(\frac{x}{x}, \frac{y}{x}) = F(1,z)$.
  - The equation becomes
    - $x\frac{dz}{dx} + z = F(1,z)$, which is separable.

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Examples

- $x\frac{dy}{dx} = y + 2xe^{-y/x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + 2e^{-y/x}$.
  - Let $z = \frac{y}{x}$. Then $y = xz$ and $\frac{dy}{dx} = x\frac{dz}{dx} + z$.
    - $x\frac{dz}{dx} + z = z + 2e^{-z} \Rightarrow x\frac{dz}{dx} = 2e^{-z}$.
    - $\int e^z\, dz = \int \frac{2}{x}\, dx \Rightarrow e^z = 2\ln|x| + C$.

$\therefore \quad y = x(\ln|2\ln|x| + C|)$.

- $x\frac{dy}{dx} = y^2 + 2xy \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\,\frac{y}{x}$.
  - Let $z = \frac{y}{x}$. We have $x\frac{dz}{dx} + z = z^2 + 2z$.
    - $\int \frac{dz}{z(z+1)} = \int \frac{dx}{x} \Rightarrow \overset{\text{Exercise}}{\cdots} \Rightarrow y = \frac{x^2}{C-x}$.

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First Order Linear Equations

- The most important type of differential equation is the **linear equation**. For example,
  - $\dfrac{dy}{dx} = f(x)y + g(x)$.
  - $\dfrac{d^2y}{dx^2} + a(x)\dfrac{dy}{dx} = f(x)y + g(x)$.
- How to solve the **first order liner differential equation**?
  - $\dfrac{dy}{dx} + p(x)y = q(x)$.
    - If $p(x) = 0$: $\dfrac{dy}{dx} = q(x) \Rightarrow y = \displaystyle\int q(x)\,dx$.
    - If $q(x) = 0$: $\dfrac{dy}{dx} + p(x)y = 0$.
      $\displaystyle\int \frac{dy}{-y} = \int p(x)\,dx$, $y = \pm\exp\left(-\displaystyle\int p(x)\,dx\right)$.

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First Order Linear Equations

- We can solve $\dfrac{dy}{dx} + p(x)y = 0$ rigorously.
  - Take $P(x)$ such that $P'(x) = p(x)$. It is expected:
    - $y = \pm\exp\left(-\displaystyle\int p(x)\,dx\right) = C\exp(-P(x))$.
  - Let $z = ye^{P(x)}$. Then
    - $$\frac{dz}{dx} = \frac{d}{dx}\left(ye^{P(x)}\right) = \frac{dy}{dx}e^{P(x)} + yp(x)e^{P(x)} = \left(\frac{dy}{dx} + p(x)y\right)e^{P(x)} = 0.$$

    $\therefore\ ye^{P(x)} = C$, i.e., $y = Ce^{-P(x)}$.
- $e^{P(x)}$ plays an important role in this integration.
  It is called the **integrating factor**. We can use it to solve the general first order linear equations.

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First Order Linear Equations

- Consider the general equation $\dfrac{dy}{dx} + p(x)y = q(x)$.
  - Evaluate $P(x) = \int p(x)\,dx$.
  - Multiply an **integrating factor** $v(x) = e^{P(x)}$.
    - $e^{P(x)}\dfrac{dy}{dx} + e^{P(x)}p(x)y = e^{P(x)}q(x)$.
    - $\dfrac{d}{dx}\left(e^{P(x)}y\right) = e^{P(x)}q(x)$.
  - Integrate with respect to x:
    - $e^{P(x)}y = \int e^{P(x)}q(x)\,dx$.

$$\therefore \quad y = \frac{1}{e^{P(x)}}\int e^{P(x)}q(x)\,dx = \frac{1}{v(x)}\int v(x)q(x)\,dx.$$

Examples

- $x\dfrac{dy}{dx} = x^2 + 3y, \quad x > 0.$
  1. Convert the equation to the standard form:
     - $\dfrac{dy}{dx} - \dfrac{3}{x}\cdot y = x$
  2. Find an integrating factor $v(x)$:
     - $\displaystyle\int \frac{-3}{x}\,dx = -3\ln x + c.$
     - Take $v(x) = e^{-3\ln x} = x^{-3}$.
  3. Solve the equation:
     - $\displaystyle y = \frac{1}{v(x)}\int v(x)q(x)\,dx = \frac{1}{x^{-3}}\int x^{-3}\cdot x\,dx$
       $\displaystyle = x^3\int \frac{1}{x^2}\,dx = x^3\left(\frac{-1}{x} + C\right) = Cx^3 - x^2.$

Examples

- $\dfrac{dy}{dx} + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$
  1. The equation is already in the standard form.
  2. Find an integrating factor $v(x)$:
     - $\displaystyle\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\ln(\cos x) + c.$
     - $v(x) = e^{-\ln(\cos x)} = (\cos x)^{-1} = \sec x.$
  3. Solve the equation:
     - $\displaystyle y = \frac{1}{\sec x}\int \sec x \cdot \cos^2 x\, dx$
       $\displaystyle = \cos x \int \cos x\, dx = \cos x\,(\sin x + C)$
       $\displaystyle = \frac{1}{2}\sin 2x + C\cos x.$

Examples

- $\displaystyle (e^y - 2xy)\,\frac{dy}{dx} = y^2.$
  - It is not linear in y, but it is linear in x.
    - $\dfrac{dx}{dy} = \dfrac{e^y - 2xy}{y^2} = \dfrac{e^y}{y^2} - \dfrac{2x}{y}.$
    - $\dfrac{dx}{dy} + \dfrac{2}{y}\cdot x = \dfrac{e^y}{y^2}.$
  - Find an integrating factor $v(y)$:
    - $\displaystyle\int \frac{2}{y}\, dy = 2\ln|y| + c. \qquad v(y) = e^{2\ln|y|} = y^2.$
  - Solve the equation:
    - $\displaystyle x = \frac{1}{y^2}\int y^2 \cdot \frac{e^y}{y^2}\, dy = \frac{1}{y^2}\int e^y\, dy$
      $\displaystyle = \frac{1}{y^2}(e^y + C) = y^{-2}e^y + Cy^{-2}.$

Bernoulli's Equation

- Consider $\dfrac{dy}{dx} + p(x)y = q(x)y^n$.
  - If $n = 0$, $\dfrac{dy}{dx} + p(x)y = q(x)$;
  - If $n = 1$, $\dfrac{dy}{dx} + p(x)y = q(x)y$.

  The equation is linear if $n = 0$ or 1. Suppose $n \neq 0, 1$.
  - Let $z = y^{1-n}$. Then $\dfrac{dz}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$.
  - Multiply $(1-n)y^{-n}$ to the equation:
    - $(1-n)y^{-n}\dfrac{dy}{dx} + (1-n)p(x)y^{1-n} = (1-n)q(x)$.
  - The equation is reduced to a linear equation:
    - $\dfrac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$.

Examples

- $x\dfrac{dy}{dx} + y = x^4y^3$.
  - $\dfrac{dy}{dx} + \dfrac{1}{x} \cdot y = x^3y^3$.

  Let $z = y^{1-3} = y^{-2}$. The equation becomes
  - $\dfrac{dz}{dx} + (-2)\dfrac{1}{x} \cdot z = (-2)x^3$.

  $\displaystyle\int \frac{-2}{x}\,dx = -2\ln|x| + c \Rightarrow v(x) = e^{-2\ln|x|} = x^{-2}$.
  - $\displaystyle z = x^2\int x^{-2} \cdot (-2)x^3\,dx = x^2\int(-2x)\,dx$

    $= x^2(-x^2 + C)$.

  $\therefore\ y^{-2} = x^2(-x^2 + C)$.

Examples

- $\dfrac{dy}{dx} + \dfrac{y}{x} = \sqrt{y}$, $(x > 0, y > 0)$.
  - $\dfrac{dy}{dx} + \dfrac{1}{x} \cdot y = y^{1/2}$.

  Let $z = y^{1-1/2} = y^{1/2}$. The equation becomes
  - $\dfrac{dz}{dx} + \dfrac{1}{2}\dfrac{1}{x} \cdot z = \dfrac{1}{2}$.

    $\displaystyle\int \frac{1}{2x}\, dx = \frac{1}{2}\ln x + c \Rightarrow v(x) = e^{\frac{1}{2}\ln x} = x^{1/2}$.
  - $\displaystyle z = x^{-1/2} \int x^{1/2} \cdot \frac{1}{2}\, dx = x^{-1/2}(\frac{x^{3/2}}{3} + C)$

    $\displaystyle = \frac{x}{3} + \frac{C}{\sqrt{x}}$.

  $\displaystyle\therefore\ y = z^2 = \left(\frac{x}{3} + \frac{C}{\sqrt{x}}\right)^2$.

Initial Value Problem

- An **initial value problem** is an ordinary differential equation with specified values at given points.
  - In particular, a **first order differential equation** has one indeterminate, we need only one **initial condition**.
- **Example.** $\dfrac{dy}{dx} + (\tan x)y = \cos^2 x, \quad y(\pi/6) = \sqrt{3}$.
  - General solution: $y = \dfrac{1}{2}\sin 2x + C\cos x$.
  - Let $x = \pi/6$ and $y = \sqrt{3}$:
    - $\sqrt{3} = \dfrac{1}{2}\sin\dfrac{\pi}{3} + C\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{4} + \dfrac{C\sqrt{3}}{2}$.

      $\therefore\ C = 3/2$.

  The **particular solution** is $y = \dfrac{1}{2}\sin 2x + \dfrac{3}{2}\cos x$.

Example

- $\dfrac{dy}{dx}\sin 2x = 2y + 2\cos x, \quad y$ is bounded as $x \to \pi/2$.
  - Convert the equation into the standard form:
    - $\dfrac{dy}{dx} + \left(-\dfrac{1}{\sin x \cos x}\right) y = \dfrac{1}{\sin x}$.
  - Find an integrating factor:
    - $\displaystyle\int \frac{-dx}{\sin x \cos x} = -\int \frac{\sec^2 x}{\tan x}\, dx = -\ln|\tan x| + C$.
    - $e^{-\ln|\tan x|} = \dfrac{1}{|\tan x|}$.
    - Use $v(x) = \dfrac{1}{\tan x} = \cot x$.
  - Find the general solution:
    - $\displaystyle y = \frac{1}{v(x)} \int v(x) q(x)\, dx$

Example

- $\dfrac{dy}{dx}\sin 2x = 2y + 2\cos x, \quad y$ is bounded as $x \to \pi/2$.
  - Find the general solution:
    - $\displaystyle y = \tan x \int \cot x \csc x\, dx$
      $= \tan x (C - \csc x) = C \tan x - \sec x$.
  - Find the particular solution:
    - $y = (C \sin x - 1)/\cos x$.
    - $\displaystyle\lim_{x \to \pi/2} (C \sin x - 1) = \lim_{x \to \pi/2} (y \cdot \cos x) = 0$.
    - $C - 1 = 0$, i.e., $C = 1$.
  - Verification:
    - $\displaystyle\lim_{x \to \pi/2} (\tan x - \sec x) = \cdots = 0$. (Exercise!)

$\therefore \quad y = \tan x - \sec x$.

Exponential Growth and Decay

- **Continuously Compounded Interest.**
  - $r \cdot \Delta t = \dfrac{\Delta \$}{\$} \Rightarrow r \cdot \$ = \dfrac{\Delta \$}{\Delta t}$, where r is a constant.

  Suppose one deposits $\$\,621$ in a bank account that pays 6% compounded continuously.
  - How much money will he have 8 years later?

  Let $A(t)$ be the amount of money at time t (in year).
  - ODE: $\dfrac{dA}{dt} = 0.06A$; IC: $A(0) = 621$.
  - Solve the equation: $A(t) = 621e^{0.06t}$.
  - Answer: $A(8) = 621e^{0.06\times 8} \approx 1003.58$.

  Why in the real lift the interest is credited monthly or yearly but not continuously? Answer: $e^x > 1 + x$ for all $x > 0$.

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Exponential Growth and Decay

- **Radiocarbon Dating.**

  The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. The ratio of radiocarbon, Carbon-14, is often used to determine the age of carbonaceous materials.

  The half-life of Carbon-14 is about 5730 years.
  - Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

  Let $C(t)$ be the Carbon-14 left at time t (in year).
  - ODE: $\dfrac{dC}{dt} = kC$; IC: $C(0) = 1$.

    $C(t) = e^{kt}$. $C(5730) = 1/2 \Rightarrow k = -\dfrac{\ln 2}{5730}$.
  - Solve $(1 - 0.1) = e^{kt}$. Then $t = \dfrac{\ln 0.9}{k} \approx 871$ years.

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Logistic Growth

- **Population Growth**.
  - $r \cdot \Delta t = \frac{\Delta}{} \Rightarrow r \cdot = \frac{\Delta}{\Delta t}$. Is r a constant?

  The resource is limited! Only a maximum population M can be accommodated, called the **limiting population**.
  - If $> M$, $r < 0$;
  - If $< M$, $r > 0$; as increases, r decreases.

  It is reasonable to use $r(M -)$ as the rate.
  - $\dfrac{dP}{dt} = r(M - P)P$.

  This can also be applied to marking; It is known as the **logistic growth**, and M is called the **carrying capacity**.
  - The real-life problem is very complicated. Here we only estimate using a simple model.

Logistic Growth

- **Example**. A national park is known to be capable of supporting 100 grizzly bears, but no more. 10 bears are in the park at present.
  - Model the population in logistic growth with $r = 0.001$.
    When will be the bear population reach 50?

  Let $P(t)$ be the population of bear at time t (in year).
  - ODE: $\dfrac{dP}{dt} = 0.001P(100 - P)$; IC: $P(0) = 10$.
  - Solve the equation: $P(t) = \dfrac{100}{1 + 9e^{-0.1t}}$.
  - Let $P(t) = 50$. Then $t = 20 \ln 3 \approx 22$.
- **Remark**. The logistic growth model may not give reliable results for very small population levels.
  - As $t \to \infty$, $P(t) \to M$.

Heat Transfer

- **Second Law of Thermodynamics (Clausius Statement)**:
  - Heat transfer always occurs from a higher-temperature object to a cooler temperature.
- **Newton's Law of Cooling (1701)**:
  - The rate of heat loss is proportional to the difference of temperature. $(r > 0)$
  - $\dfrac{dT}{dt} = -r \cdot (T - T_S)$, $T_S =$ surrounding temperature.
    - $T > T_S \Rightarrow \dfrac{dT}{dt} < 0; \quad T < T_S \Rightarrow \dfrac{dT}{dt} > 0.$
  - The equation can be solved using separation of variable or integrating factor:
    - $T(t) - T_S = Ce^{-rt} = (T_0 - T_S)e^{-rt}$.
      As $t \to \infty$, $T(t) \to T_S$.

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Heat Transfer

- **Example**. A boiled egg at $98°\mathrm{C}$ is put in water of $18°\mathrm{C}$.
  - After 5 min, the temperature of egg becomes $38°\mathrm{C}$.
    Assume that the water is not warmed appreciably.
  - How much longer will it take the egg to reach $20°\mathrm{C}$?

  ODE: $\dfrac{dT}{dt} = -r(T - 18);$ IC: $T(0) = 98.$
  - $\displaystyle\int \frac{dt}{T - 18} = \int (-r)\, dt \Rightarrow \ln|T - 18| = -rt + c.$

  Solve the equation:
  - $T(t) = 18 + 80e^{-rt}$.
  - $T(5) = 38 \Rightarrow r = \frac{1}{5}\ln 4$.

  Solve for t when $T(t) = 20 = 18 + 80e^{-rt}$.
  - $t = \dfrac{\ln 40}{\frac{1}{5}\ln 4} \approx 13$ min.

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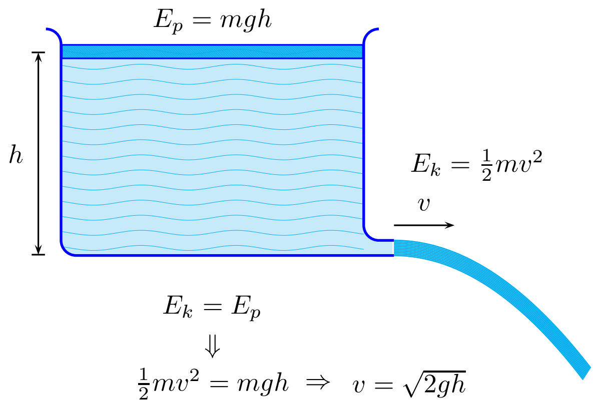
Draining Tank Problem

- Consider a tank with water:



$$E_k = E_p$$
$$\Downarrow$$
$$\tfrac{1}{2}mv^2 = mgh \;\Rightarrow\; v = \sqrt{2gh}$$

  - **Torricelli's Law.**
    - The rate of water runs out is proportional to the square root of the water's depth.

Draining Tank Problem

- A right circular cylindrical tank with radius 5 ft and height 16 ft is being drained at $0.5\sqrt{h}$ ft^3/min.
  - How long to empty the tank?

  At height h, $\quad V = \pi r^2 h = 25\pi h.$

  - $25\pi \dfrac{dh}{dt} = \dfrac{dV}{dt} = -0.5\sqrt{h}.$

  ODE: $\quad 25\pi \dfrac{dh}{dt} = -0.5\sqrt{h}; \qquad$ IC: $\quad h(0) = 16.$

  - $h(t) = \left(4 - \dfrac{t}{100\pi}\right)^2$

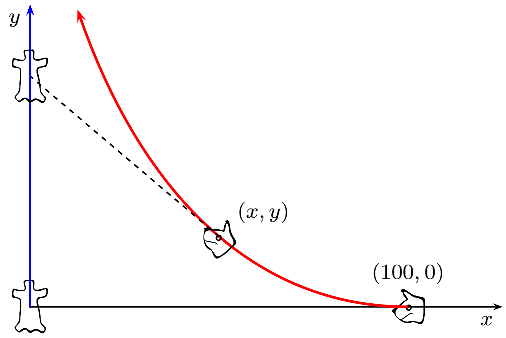
  Solve $h(t) = 0.$

  - $t = 400\pi$ min ≈ 21 hrs.
- **Exercise.** How about if the tank is a right circular cone?

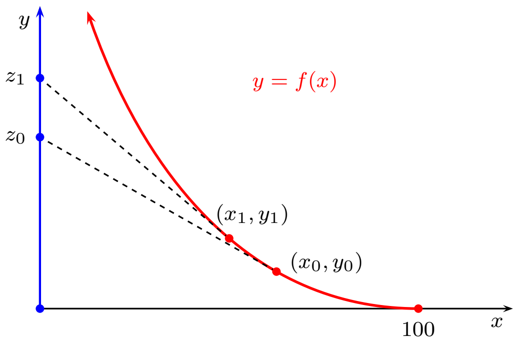
Dog and Rabbit

- **Example**. A dog sees a rabbit running in a straight line across an open field and gives chase. Assume
  - Rabbit is at $(0,0)$; dog is at $(100,0)$ (in meter).
  - Rabbit runs up the y-axis; dog runs straight for rabbit.
  - Speed of rabbit is 5 m/s; speed of dog is 6 m/s.

  How long can the dog catch the rabbit?



Dog and Rabbit



- Suppose at time t_0, dog is at (x_0, y_0), rabbit is at $(0, z_0)$.
  - Tangent line of $y = f(x)$ at (x_0, y_0):
    - $y - y_0 = f'(x_0)(x - x_0)$.
  - Let $x = 0$ in the tangent line:
    - $z_0 = y_0 - x_0 f'(x_0)$.
- Rabbit: $r(x) = f(x) - xf'(x)$.
  - Speed of rabbit: $r'(x) = -xf''(x)$.

  Dog: $d(x) = \displaystyle\int_x^{100} \sqrt{1 + (f'(t))^2}\, dt$.
  - Speed of dog: $d'(x) = -\sqrt{1 + (f'(x))^2}$.
- $r'(x) : d'(x) = 5 : 6$.

Dog and Rabbit

- $\frac{1}{5}xf''(x) = \frac{1}{6}\sqrt{1+(f'(x))^2}; \quad f'(100) = f(100) = 0.$

  Let $u = f'(x)$. It reduces to a first order equation:
  - $\frac{1}{5}xu' = \frac{1}{6}\sqrt{1+u^2}; \quad u(100) = 0.$

    Solution: $\quad u(x) = \sqrt[3]{10}\left(\frac{x^{5/6}}{200} - \frac{5\sqrt[3]{10}}{x^{5/6}}\right).$

  Solve $f'(x) = \sqrt[3]{10}\left(\frac{x^{5/6}}{200} - \frac{5\sqrt[3]{10}}{x^{5/6}}\right); \quad f(100) = 0.$
  - $f(x) = \frac{20\sqrt[3]{10}x^{11/6}}{1100} - 30\sqrt[3]{100}x^{1/6} + \frac{3000}{11}.$

  Therefore, $T = \frac{f(0)}{5} = \frac{600}{11} \approx 54.5$ seconds.

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Second Order Equations

- A **second order linear differential equation** has the form
  - $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x).$ (1)

  It is called **homogeneous** if $r(x)$ is the zero function:
  - $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$ (2)

  Theorem.
  - If y_1 and y_2 are solutions to (2) such that y_1/y_2 is non-constant. Then the **general solution** to (2) is
    - $y = C_1y_1 + C_2y_2, \quad C_1, C_2$ are constant.
  - If further y_p is a solution to (1), then the **general solution** to (1) is given by
    - $y = C_1y_1 + C_2y_2 + y_p, \quad C_1, C_2$ are constant.

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Second Order Equations

- In MA1521, we only consider the special case when $p(x)$ and $q(x)$ are constant functions.
  - $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + qy = r(x)$. (3)
- We first consider the **homogeneous** case.
  - $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + qy = 0$, or simply $y'' + py' + qy = 0$.

  Note that $(e^{\lambda x})' = \lambda e^{\lambda x}$. Let us try $y = e^{\lambda x}$:
  - $\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q\, e^{\lambda} = 0$.
  - $(\lambda^2 + p\lambda + q)e^{\lambda x} = 0$.

  Definition. The equation $\lambda^2 + p\lambda + q = 0$ is called the **characteristic equation** of the equation (3).

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Second Order Equations

- Given $\lambda^2 + p\lambda + q = 0$, its roots are given by
  - $\lambda_1, \lambda_2 = \dfrac{-p \pm \sqrt{\Delta}}{2}$, where $\Delta = p^2 - 4q$.
- **Theorem.** The general solution to $y'' + py' + qy = 0$ is given by $y = C_1 y_1 + C_2 y_2$,
  - $\Delta > 0 \Rightarrow \lambda_1 \neq \lambda_2$ are distinct real numbers.
    - $y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$.
  - $\Delta = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda$ is real.
    - $y_1 = e^{\lambda x}, \quad y_2 = xe^{\lambda x}$.
  - $\Delta < 0 \Rightarrow \lambda_1, \lambda_2 = a \pm bi, \quad a, b \in \mathbb{R}, b \neq 0$.
    - $y_1 = e^{ax}\cos bx, \quad y_2 = e^{ax}\sin bx$.

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Examples

- Find the general solutions of the following equations.
  - $y'' + y' - 6y = 0.$
    - $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = -3,\ 2.$

    Therefore, $y = C_1 e^{-3x} + C_2 e^{2x}$.
  - $y'' + y' = 0.$
    - $\lambda^2 + \lambda = 0 \Rightarrow \lambda = -1,\ 0.$

    Therefore, $y = C_1 e^{-1x} + C_2 e^{0x} = C_1 e^{-x} + C_2$.
  - $y'' - 9y' + 20y = 0.$
    - $\lambda^2 - 9\lambda + 20 = 0 \Rightarrow \lambda = 4,\ 5.$

    Therefore, $y = C_1 e^{4x} + C_2 e^{5x}$.

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Examples

- Find the general solutions of the following equations.
  - $y'' + 2y' + y = 0.$
    - $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -1.$

    Therefore, $y = C_1 e^{-x} + C_2 x e^{-x}$.
  - $y'' - 4y' + 4y = 0.$
    - $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = 2.$

    Therefore, $y = C_1 e^{2x} + C_2 x e^{2x}$.
  - $y'' = 0.$
    - $\lambda^2 = 0 \Rightarrow \lambda_{1,2} = 0.$

    Therefore, $y = C_1 e^{0x} + C_2 x e^{0x} = C_1 + C_2 x$.

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Examples

- Find the general solutions of the following equations.
  - $y'' - 6y' + 25y = 0.$
    - $\lambda^2 - 6\lambda + 25 = 0 \Rightarrow \lambda_{1,2} = 3 \pm 4i.$

    Therefore, $y = C_1 e^{3x} \cos 4x + C_2 e^{3x} \sin 4x.$
  - $y'' + 8y = 0.$
    - $\lambda^2 + 8 = 0 \Rightarrow \lambda_{1,2} = \pm 2\sqrt{2}\, i.$

    $$\begin{aligned} y &= C_1 e^{0x} \cos(2\sqrt{2}\, x) + C_2 e^{0x} \sin(2\sqrt{2}\, x) \\ &= C_1 \cos(2\sqrt{2}\, x) + C_2 \sin(2\sqrt{2}\, x). \end{aligned}$$
  - $y'' + 2y' + 3y = 0.$
    - $\lambda^2 + 2\lambda + 3 = 0 \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{2}\, i.$

    Therefore, $y = C_1 e^{-x} \cos(\sqrt{2}\, x) + C_2 e^{-x} \sin(\sqrt{2}\, x).$

Variation of Parameters

- We now discuss the general solution to
  - $y'' + py' + qy = r(x).$

  It is given by $y(x) = y_h(x) + y_p(x).$
  - $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$ is the general solution of the homogeneous equation $y'' + py' + qy = 0.$
  - $y_p(x)$ is a **particular solution** to $y'' + py' + qy = r(x).$
- We will use the method of **variation of parameters** to find a particular solution to $y'' + py' + qy = r(x).$
  - This method was invented by Joseph-Louis Lagrange (1736 – 1813), French mathematician and astronomer.
  - The method can be applied to any second order linear equation $y'' + p(x)y' + q(x) = r(x).$

Variation of Parameters

- Find a particular solution y_p to $y'' + py' + qy = r(x)$.
  - Suppose the general solution to $y'' + py' + qy = 0$ is
    - $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$.
  - It is suggested to try
    - $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$.

    Then $y_p' = (v_1' y_1 + v_1 y_1') + (v_2' y_2 + v_2 y_2')$.
  - Further assume that $v_1' y_1 + v_2' y_2 = 0$.
    - $y_p' = v_1 y_1' + v_2 y_2'$.
    - $y_p'' = (v_1' y_1' + v_1 y_1'') + (v_2' y_2' + v_2 y_2'')$.
  - $$\begin{aligned} r(x) &= y_p'' + py_p' + qy_p \\ &= (v_1' y_1' + v_1 y_1'') + (v_2' y_2' + v_2 y_2'') \\ &\quad + p(v_1 y_1' + v_2 y_2') + q(v_1 y_1 + v_2 y_2). \end{aligned}$$

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Variation of Parameters

- Find a particular solution y_p to $y'' + py' + qy = r(x)$.
  - Suppose the general solution to $y'' + py' + qy = 0$ is
    - $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$.
  - Assume that $y_p = v_1 y_1 + v_2 y_2$ and $v_1' y_1 + v_2' y_2 = 0$.
  - $$\begin{aligned} r(x) &= y_p'' + py_p' + qy_p \\ &= (v_1' y_1' + v_1 y_1'') + (v_2' y_2' + v_2 y_2'') \\ &\quad + p(v_1 y_1' + v_2 y_2') + q(v_1 y_1 + v_2 y_2) \\ &= v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_1) \\ &\quad + v_1' y_1' + v_2' y_2' \\ &= v_1' y_1' + v_2' y_2'. \end{aligned}$$
  - Solve the system in v_1' and v_2':
    - $v_1' y_1 + v_2' y_2 = 0$ and $v_1' y_1' + v_2' y_2' = r(x)$.

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Variation of Parameters

- Find a particular solution y_p to $y'' + py' + qy = r(x)$.
  - Suppose the general solution to $y'' + py' + qy = 0$ is
    - $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$.
  - Assume that $y_p = v_1 y_1 + v_2 y_2$ and $v_1' y_1 + v_2' y_2 = 0$.
  - Solve the linear system in v_1' and v_2':
    - $v_1' y_1 + v_2' y_2 = 0$ and $v_1' y_1' + v_2' y_2' = r(x)$.

    $\therefore \quad v_1' = \dfrac{-y_2 r(x)}{W(y_1, y_2)}$ and $v_2' = \dfrac{y_1 r(x)}{W(y_1, y_2)}$,

    where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ is the **Wronskian** of y_1 and y_2.
  - $v_1 = \displaystyle\int \frac{-y_2 r(x)}{W(y_1, y_2)}\, dx$ and $v_2 = \displaystyle\int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx$.

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Variation of Parameters

- **Variation of Parameters**. $y'' + py' + qy = r(x)$.
  1. Find the solution to the homogeneous equation
     - $y'' + py' + qy = 0, \quad$ say $y_h = C_1 y_1 + C_2 y_2$.
  2. Evaluate the Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
  3. Evaluate the parameters
     - $v_1 = \displaystyle\int \frac{-y_2 r(x)}{W(y_1, y_2)}\, dx, \; v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)}\, dx.$
  4. A particular solution is given by $y_p = v_1 y_1 + v_2 y_2$.
  5. The general solution is given by
     - $y = y_h + y_p = C_1 y_1 + C_2 y_2 + (v_1 y_1 + v_2 y_2)$.

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Examples

- $y'' - y' - 6y = e^{-x}$.
  1. $\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = -2,\ 3.$
     - $y_h = C_1 e^{-2x} + C_2 e^{3x};\quad y_1 = e^{-2x}, y_2 = e^{3x}$.
  2. $W(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix} = 5e^x.$
  3. $v_1' = \dfrac{-y_2 r(x)}{W(y_1, y_2)} = \dfrac{-e^{3x}e^{-x}}{5e^x} = -\dfrac{1}{5}e^x.$

     $v_2' = \dfrac{y_1 r(x)}{W(y_1, y_2)} = \dfrac{e^{-2x}e^{-x}}{5e^x} = \dfrac{1}{5}e^{-4x}.$
     - $v_1 = -\frac{1}{5}e^x, \quad v_2 = -\frac{1}{20}e^{-4x}.$
  4. $y_p = v_1 y_1 + v_2 y_2 = -\frac{1}{5}e^{-x} - \frac{1}{20}e^{-x} = -\frac{1}{4}e^{-x}.$

  $\therefore\ y = y_h + y_p = C_1 e^{-2x} + C_2 e^{3x} - \frac{1}{4}e^{-x}.$

Examples

- $y'' - 2y' + y = 2x$.
  1. $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1.$
     - $y_h = C_1 y_1 + C_2 y_2;\quad y_1 = e^x, y_2 = xe^x$.
  2. $W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}.$
  3. $v_1' = \dfrac{-y_2 r(x)}{W(y_1, y_2)} = \dfrac{-xe^x \cdot 2x}{e^{2x}} = -2x^2 e^{-x}.$

     $v_2' = \dfrac{y_1 r(x)}{W(y_1, y_2)} = \dfrac{e^x \cdot 2x}{e^{2x}} = 2xe^{-x}.$
     - $v_1 = 2(2 + 2x + x^2)e^{-x}, \quad v_2 = -2(1+x)e^{-x}.$
  4. $y_p = v_1 y_1 + v_2 y_2 = \cdots = 4 + 2x.$

  $\therefore\ y = y_h + y_p = C_1 e^x + C_2 xe^x + (4 + 2x).$

Examples

- $y'' + y = x$.
  1. $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$.
     - $y_h = C_1 y_1 + C_2 y_2; \quad y_1 = \cos x,\, y_2 = \sin x$.
  2. $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$.
  3. $v_1' = \dfrac{-y_2 r(x)}{W(y_1, y_2)} = \dfrac{-\sin x \cdot x}{1} = -x \sin x$.

     $v_2' = \dfrac{y_1 r(x)}{W(y_1, y_2)} = \dfrac{\cos x \cdot x}{1} = x \cos x$.
     - $v_1 = -\sin x + x \cos x, \quad v_2 = \cos x + x \sin x$.
  4. $y_p = v_1 y_1 + v_2 y_2 = \cdots = x$.

  $\therefore \quad y = y_h + y_p = C_1 \cos x + C_2 \sin x + x$.

Operator Methods

- Consider the first order differential equation
  - $\dfrac{dy}{dx} - ky = r(x)$.
    - $\displaystyle\int (-k)\, dx = -kx + c, \quad v(x) = e^{-kx}$.
    - $y = e^{kx} \displaystyle\int e^{-kx} r(x)\, dx$.
- Let $D = \dfrac{d}{dx}$. The equation has the form
  - $Dy - ky = r(x)$, or simply $(D - k)y = r(x)$.
    - Then $y = \dfrac{1}{D - k} r(x)$.
- Therefore, define $\dfrac{1}{D-k} r(x) = e^{kx} \displaystyle\int e^{-kx} r(x)\, dx$.

Operator Methods

- Let $D = \dfrac{d}{dx}$ in $\dfrac{d^2y}{dx^2} + p\,\dfrac{dy}{dx} + q\,y = r(x)$.
  - $D^2y + pDy + qy = r(x)$, or simply
  - $(D^2 + pD + q)y = r(x)$.

  Factorize $\lambda^2 + p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2)$.
  - $D^2 + pD + q = (D - \lambda_1)(D - \lambda_2)$.

  The equation becomes
  - $(D - \lambda_1)(D - \lambda_2)y = r(x)$.

  Therefore, $y = \dfrac{1}{D - \lambda_1}\,\dfrac{1}{D - \lambda_2}\,r(x)$.
- This is called the **operator method**, introduced by Oliver Heaviside (1850 – 1925), a self-taught English electrical engineer, mathematician, and physicist.

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Operator Methods

- Suppose $y'' + py' + qy = 0$.

  Then $y = \dfrac{1}{D - \lambda_1}\,\dfrac{1}{D - \lambda_2}\,0$,

  where λ_1, λ_2 are roots to $\lambda^2 + p\lambda + q = 0$.
  - $\dfrac{1}{D - \lambda_2}\,0 = e^{\lambda_2 x} \displaystyle\int e^{-\lambda_2 x} \cdot 0\,dx = Ce^{\lambda_2 x}$.
  - $y = \dfrac{1}{D - \lambda_1} Ce^{\lambda_2 x}$

    $= e^{\lambda_1 x} \displaystyle\int e^{-\lambda_1 x} \cdot Ce^{\lambda_2} x\,dx$

    $= Ce^{\lambda_1 x} \displaystyle\int e^{(\lambda_2 - \lambda_1)x}\,dx$.

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Operator Methods

- Suppose $y'' + py' + qy = 0$.
  Then $y = Ce^{\lambda_1 x} \int e^{(\lambda_2 - \lambda_1)x}\, dx$.
  - If $\lambda_1 = \lambda_2$, then
    $y = Ce^{\lambda_1 x}(x + D) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$.
  - If $\lambda_1 \neq \lambda_2$, then
    $y = Ce^{\lambda_1 x}\left(\dfrac{e^{(\lambda_2-\lambda_1)x}}{\lambda_2 - \lambda_1} + D\right) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

  In the 2nd case, suppose $\lambda_{1,2} = a \pm bi$, $a, b \in \mathbb{R}$, $b \neq 0$.
- $$\begin{aligned} y &= C_1 e^{ax+bxi} + C_2 e^{ax-bxi} \\ &= C_1 e^{ax}(\cos bx + i \sin bx) + C_2 e^{ax}(\cos bx - i \sin bx) \\ &= (C_1 + C_2)e^{ax}\cos bx + i(C_1 - C_2)e^{ax}\sin bx \\ &= C_1^* e^{ax}\cos bx + C_2^* e^{ax}\sin bx. \end{aligned}$$

Examples

- Find a particular solution of $y'' - y = e^{-x}$.
  - $e^{-x} = D^2 y - y = (D^2 - 1)y = (D-1)(D+1)y$.
  - $y = \dfrac{1}{D-1}\,\dfrac{1}{D+1}\, e^{-x}$.
    - $\dfrac{1}{D+1}\, e^{-x} = e^{-x}\int e^{x}\, e^{-x}\, dx = e^{-x} x$.
    - $$\begin{aligned} y &= \frac{1}{D-1}\, e^{-x} x = e^{x}\int e^{-x}\cdot e^{-x} x\, dx \\ &= e^{x}\int x e^{-2x}\, dx = -\frac{e^{x}}{2}\int x\, d(e^{-2x}) \\ &= -\frac{e^{x}}{2}\left(x e^{-2x} - \int e^{-2x}\, dx\right) \\ &= -\frac{e^{x}}{2}\left(x e^{-2x} + \frac{e^{-2x}}{2}\right) = -\frac{e^{-x}}{4}(2x+1). \end{aligned}$$

Examples

- Find a particular solution of $y'' - y = e^{-x}$.
  - $$\begin{aligned} y &= \frac{1}{(D-1)(D+1)}\, e^{-x} \\ &= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right] e^{-x} \\ &= \frac{1}{2}\frac{1}{D-1}\, e^{-x} - \frac{1}{2}\frac{1}{D+1}\, e^{-x} \\ &= \frac{1}{2}e^{x} \int e^{-x}e^{-x}\, dx - \frac{1}{2}e^{-x} \int e^{x}e^{-x}\, dx \\ &= \frac{1}{2}e^{x} \int e^{-2x}\, dx - \frac{1}{2}e^{-x} \int 1\, dx \\ &= \frac{1}{2}e^{x}\, \frac{-1}{2}e^{-2x} - \frac{1}{2}e^{-x}x \\ &= -\frac{1}{4}e^{-x} - \frac{1}{2}xe^{-x}. \end{aligned}$$

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Examples

- Find a particular solution of $y'' + 4y' + 4y = 20x^3e^{-2x}$.
  - $20x^3e^{-2x} = (D^2 + 4D + 4)y = (D+2)^2 y$.
    - $$\begin{aligned} \frac{1}{D+2}\, 20x^3e^{-2x} &= e^{-2x} \int x^{2x}\, 20x^3e^{-2x}\, dx \\ &= e^{-2x} \int 20x^3\, dx = 5e^{-2x}x^4. \end{aligned}$$
    - $$\begin{aligned} y &= \frac{1}{D+2}\, 5e^{-2x}x^4 \\ &= e^{-2x} \int e^{2x}\, 5e^{-2x}x^4\, dx \\ &= e^{-2x} \int 5x^4\, dx \\ &= e^{-2x}x^5. \end{aligned}$$

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Examples

- If $r(x)$ is a polynomial, we may try **series expansion**.
- Find a particular solution of $y'' + y = x^3 - 3x^2 + 1$.
  - $(D^2 + 1)y = x^3 - 3x^2 + 1$.
  - $\dfrac{1}{D^2+1} = \dfrac{1}{1-(-D^2)} = 1 - D^2 + D^4 - D^6 + \cdots$.
  - $$\begin{aligned} y &= \frac{1}{D^2+1}\left(x^3 - 3x^2 + 1\right) \\ &= \left(1 - D^2 + D^4 - D^6 + \cdots\right)\left(x^3 - 3x^2 + 1\right) \\ &= (x^3 - 3x^2 + 1) - D^2(x^3 - 3x^2 + 1) \\ &\quad + D^4(x^3 - 3x^2 + 1) - D^6(x^3 - 3x^2 + 1) + \cdots \\ &= (x^3 - 3x^2 + 1) - (6x - 6) + 0 - 0 + \cdots \\ &= x^3 - 3x^2 - 6x + 7. \end{aligned}$$
- Note that $D^n(x^k) = 0$ if $n > k$.

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Initial Value Problem

- Recall that an **initial value problem** is an ordinary differential equation with specified values at given points.
  - The solution to a **second order differential equation**
    - $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + qy = r(x)$

    is of the form $y = C_1y_1 + C_2y_2 + y_p$, which has two indeterminates. In order to uniquely determine the solution, we need **two initial conditions**.

- Usually, the initial conditions are given as following:

  (i) $y = A$ and $\dfrac{dy}{dx} = B$ at $x = x_0$;

  (ii) $y = A$ at $x = x_0$ and $y = B$ at $x = x_1$.

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Examples

- Suppose the general solution is $y = C_1 y_1 + C_2 y_2 + y_p$.
  - Initial conditions: $y(x_0) = A$ and $y(x_1) = B$.
    - $A = y(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) + y_p(x_0)$
    - $B = y(x_1) = C_1 y_1(x_1) + C_2 y_2(x_1) + y_p(x_1)$.

    Solve the above linear system to obtain C_1 and C_2.
- **Example**. $y'' + y = x$; $y(-\pi/4) = 0$ and $y(\pi/4) = 0$.
  - General solution: $y = C_1 \cos x + C_2 \sin x + x$.

$$\begin{aligned} 0 &= y(-\tfrac{\pi}{4}) = \frac{C_1}{\sqrt{2}} - \frac{C_2}{\sqrt{2}} - \frac{\pi}{4} \\ 0 &= y(\pi/4) = \frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} + \frac{\pi}{4} \end{aligned} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = -\dfrac{\sqrt{2}}{4} \end{cases}$$

  - Solution: $y = -\dfrac{\sqrt{2}}{4} \sin x + x$.

Examples

- Suppose the general solution is $y = C_1 y_1 + C_2 y_2 + y_p$.
  - Initial conditions: $y(x_0) = A$ and $y'(x_0) = B$.
    - $A = y(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) + y_p(x_0)$
    - $B = y'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) + y_p'(x_0)$.

    Solve the above linear system to obtain C_1 and C_2.
- **Example**. $y'' + y = x$; $y(\pi/4) = 0$ and $y'(\pi/4) = 0$.
  - General solution: $y = C_1 \cos x + C_2 \sin x + x$.

$$\begin{aligned} 0 &= y(\tfrac{\pi}{4}) = \frac{C_1}{\sqrt{2}} - \frac{C_2}{\sqrt{2}} + \frac{\pi}{4} \\ 1 &= y'(\tfrac{\pi}{4}) = -\frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} + 1 \end{aligned} \Rightarrow \begin{cases} C_1 = -\dfrac{\sqrt{2}\pi}{8} \\ C_2 = -\dfrac{\sqrt{2}\pi}{8} \end{cases}$$

  - Solution: $y = -\dfrac{\sqrt{2}\pi}{8} \cos x - \dfrac{\sqrt{2}\pi}{8} \sin x + x$.