Chapter 6
Dynamical Systems

Discrete Mathematics II/Mathematical Modelling

Nguyen An Khuong, Huynh Tuong Nguyen
Faculty of Computer Science and Engineering
University of Technology, VNU-HCM

Contents

Change?!

- Dynamical systems: Tools for constructing and manipulating models
- So we often have to model dynamic systems.
  - Discrete $\longrightarrow$ difference equations (“linear" vs “nonlinear", “single variable" vs “multivariate")
  - Continuous $\longrightarrow$ differential equations (“ordinary" vs “partial"; “linear" vs “nonlinear")
- We will formulate the equations, analyze their properties and learn how to solve them.
- To start there are many good references on this subject, including:
  - F.R. Giordano, W.P. Fox & S.B. Horton, *A First Course in Mathematical Modeling*, 5th ed., Cengage, 2014.
  - A Iserles, *A First Course in the Numerical Analysis of Differential Equations*, 2nd. Cambridge University Press, 2008

Single Species Equations: Growth

- Basic concept that individuals divide to increase a population can be modeled mathematically using a differential equation
- Can loosely be applied to populations that don't divide to populate
- Attributed to Malthus, who in 1798 found small group of organisms obeyed growth law
- The solution to the equation concerned him greatly

Exponential Growth

- As an example, consider the classic example of bacteria on a petri dish.
- Let's say the number on the plate grows by 10% each hour and the initial population is $x_0 = 1000$.
- After the first hour: $x_1 = 1000 + (0.1) * 1000 = 1000 * (1.1)$
- After the second hour: $x_2 = 1000 * (1.1) + (0.1) * (1000 * 1.1) = 1000 * (1.1) * (1.1) = 1000 * (1.1)^2$
- After the third hour: $x_3 = 1000 * (1.1)^3$ etc.
- In general, $x_t = x_o(1 + r)^t$ which can be written as $x_t = x_0 a^t$ where $a = (1 + r)$ The solution for the Bacteria is thus an exponential function of **base** $1 + r$.
- The (discrete) dynamical system: $x_t = ax_{t-1}$. (recurrent relation/difference equation)

Instantaneous Exponential Growth

- In the last slide we examined growth at some kind of finite increment using an average growth rate r over that increment.
- What happens if the growth process is continuous?
- Divide the growth process over the time increment into nt stages with the growth rate for each stage being $\frac{r}{n}$ where r is the average growth over the increment.

$$x(t) = x_o(1 + \frac{r}{n})^{nt}. \tag{1}$$

- Look at the limit as we divide our interval into ∞ pieces.

$$\lim_{n\to\infty} x_0[(1 + \frac{r}{n})^{n/r}]^{rt}. \tag{2}$$

- We can pull x_0 out of the limit. In brackets we have the irrational number *e*:

$$x(t) = x_0 e^{rt}. \tag{3}$$

Instantaneous Exponential Growth (cont'd)

- Our time-dependent population equation satisfies the fundamental differential equation:

$$\dot{x} = \frac{dx}{dt} = x' = rx \tag{4}$$

where r is the (in this situation) constant growth rate. (See how to derive the equation above in page 462, [Giordano et al.], eqn(11.5).)

- The solution to these equation can be found by separating variables and integrating.

$$ln|x(t)| = rt + c \tag{5}$$

and applying the initial condition that $x(0) = x_o$ at $t = 0$ to get

$$\ln|x(t)| = rt + \ln|x_o| \tag{6}$$

- Take the exponential of both sides to give the exact solution to the instantaneous growth equation

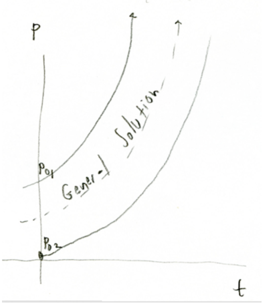
$$x(t) = x_o e^{rt} \tag{7}$$

Exponential Growth: Solution properties

- We can calculate the time to double the population (known as the "rule of 70" in financial circles)

$$t_{double} = \frac{\ln(2)}{r} \sim \frac{.70}{r}. \tag{8}$$

- What does the solution look like? On a log plot, it is a straight line of slope r.

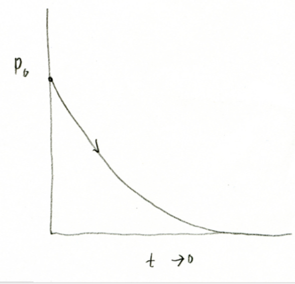


- Is this realistic?

What if our growth rate was negative?: Mortality

$$\dot{x} = -mx \tag{9}$$

- We have exponential decay. Solution will tend toward zero, no matter what the *initial condition* was



- Most systems have both growth and decay terms.
- Example, our phytoplankton equation will have terms for growth as a function of light, temperature, and nutrients, and decay (mortality).

More realistic model: Finite Resources

- We now know that growth cannot continue forever because of finite resources and in fact in simplified scenarios will reach a given constant level K know as the carrying capacity of the system.
- Verhulst noticed that simple populations appear to be capped and added an additional term to remove excess capacity.

$$\dot{x} = rx(1 - \frac{x}{K}) \tag{10}$$

- We can always nondimensionalize this by the carrying capacity to give

$$\dot{x} = rx(1 - x) \tag{11}$$

- This is commonly known as *logistic growth*

Logistic Growth: Solution

- This equation has solutions that tend toward K for all initial conditions except for $x = x_o$ for which there is no growth (assuming there is no such thing as a negative population).
- So, this is mathematically equivalent to a density dependent growth rate $r' = r(1 - x)$.
- Solution to this equation is

$$x(t) = \frac{x_o K}{x_o + (K - x_o)e^{-rt}} \tag{12}$$

Arbitrary Growth: Polynomial

- These equations are of the general form

$$\dot{x} = g(x) \tag{13}$$

where $g(x)$ is a polynomial: $g(x) = a_o + a_1 x + a_2 x^2 \ldots$.
  - Malthus, all coefficients are zero except for $a_1 = r$. Growth rate constant
  - Verhults (logistic), growth rate decreases monotonically
  - $a_0 = 0$ by argument that a population can't spontaneously exist
  - One can fit any population, but will not elucidate any dynamics
  - Allee effect: Growth rate might be maximal somewhere in between $x = 0$ and the carrying capacity. Low numbers, can't find mates, high numbers, competition/resources.

$$g(x) = a_0 + a_1 x + a_2 x^2 \tag{14}$$

- If $a_1 > 0$ and $a_2 < 0$ we have the Allee effect

Critical Points (Fixed Points)

- Fixed points are steady state solutions of ordinary differential equations.
- For example, consider the general form of our single species population equation $\dot{x} = f(x)$:

$$\dot{x} = 0 \Rightarrow f(x) = 0 \tag{15}$$

- In ecosystem dynamics, these fixed points are also known as *critical points*. For the Malthus system (exponential growth), we have:

$$\dot{x} = \alpha x = 0 \Rightarrow \hat{x} = 0 \tag{16}$$

- There is only one critical point $(x = 0)$. How could we reach this solution?

Stability of Critical Points

What is the nature of the solution near the critical point? Consider, two possibilities for a frog pond starting from a steady state:

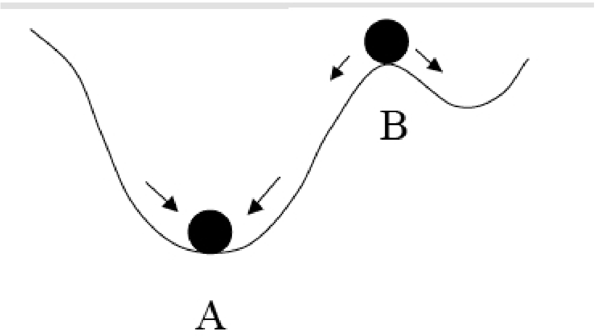
1. Five frogs are killed in a freak accident. The frog population returns to the steady state. *stable*
2. The frog population crashes: *unstable*

- Let's look at the stability of our exponential growth $\dot{x} = \alpha x$ critical point $x = 0$.
- If we perturb *x* to $x = 5$, what will happen?

$$x(t) = x_o e^{\alpha t} = 5e^{\alpha t} \tag{17}$$

- System will grow exponentially away from the critical point: *critical point is unstable*

Stability of Critical Points: cont'd



- B: Unstable
- A: Stable

This example relates to states of potential energy, but the analogy holds for energy in an ecosystem.

Stability of Critical Points: Logistic Equation

Logistic equation:

$$\dot{x} = \alpha x(1 - \frac{x}{K}) \tag{18}$$

Solve for fixed points with $\dot{x} = f(x) = 0$. Two fixed points:

1. $\hat{x_1} = 0$ Nothing exists
2. $\hat{x_2} = K$ Population is at carrying capacity

Though experiment: Are these stable:

1. $x = 0$: Perturb system from nothing, what will happen: *growth!* (Unstable)
2. $x = K$: Perturb system from carrying capacity, what will happen *return to K* (Stable)

We will analyze this more rigorously

Stability of Linear Systems

For linear ODEs of the form: $\dot{x} = \alpha x$, the stability can be determined simply from the sign of α:

- $\alpha > 0$: Unstable (growth)
- $\alpha < 0$: Stable (decay)

This is obvious from the solution: $x(t) = x_o e^{\alpha t}$.What about nonlinear ODEs:

$$\dot{x} = \alpha x(1 - \frac{x}{K})? \tag{19}$$

Trick is to *linearize about the critical point*. What is linearization: We approximate a nonlinear function with a linear function that is quite accurate at a given point.

Linearization

- We linearize a function near a given point using a Taylor Series.
- For example, take a nonlinear function $g(x)$.
- With approximation theory, we can show that the following series $f(x)$ converges to the exact function $g(x)$ at $x = a$

$$f(x) = g(a) + g^{'}(a)(x - a) + H.O.T. \tag{20}$$

- If we retain only the first two terms (zeroth order and linear) we have the linearization $L(x)$ of the function $g(x)$ about the point a.
- $$L(x) = g(a) + g^{'}(a)(x - a) \sim g(x) \tag{21}$$
- What do we need to compute this linearization?
  1. The first derivative of the function $g(x)$ evaluated at the point $x = a$
  2. The value of the function at a $(g(a))$

Linearization example: Sin

- The $\sin$ function is a complex, irrational function.
- However, if we are working around small angles, we can often replace it with a more compact function using the linearization.
- For example, the governing equation for a pendulum is:

$$I\ddot{\Theta} + mgl\sin\Theta = 0 \tag{22}$$

where Θ is the angle of perturbation from the vertical.

- We can simplify the solution if we linearize $\sin$ about $\Theta = 0$:

$$L(\sin\Theta) = 0 + \cos(\Theta)|_{Theta=0}(\Theta - 0) = \Theta \tag{23}$$

- Using the so-called small angle approximation: we can simplify the D.E. to be:

$$I\ddot{\Theta} + mgl\Theta = 0 \tag{24}$$

Stability of the Logistic Equation

- Let's analyze the stability of the critical points of the Logistic Equation.
- First, let's evaluate the first derivative:

$$f'(x) = \alpha - \frac{2\alpha x}{K} \quad (25)$$

- At the point $x = 0$, we have, for the Linearization:

$$L(x) = 0 + \alpha(x - 0) = \alpha x \quad (26)$$

- So, near the critical point $x = 0$, our logistic equation behaves as $L(x) = \alpha x$ (why is this not surprising).
- Thus it is an *unstable* critical point.

Stability of the Logistic Equation, cont'd

- At the point $x = K$, we have, for the Linearization:

$$L(x) = 0 - \frac{1}{2}\alpha(x - K) \tag{27}$$

- Thus, the linearized differential equation is:

$$\dot{x} \sim -\frac{1}{2}\alpha x + \frac{1}{2}K \tag{28}$$

- This is a stable differential equation with a single steady state value: $x = K$.
- So, the fixed point $x = K$ is stable. This is not surprising, given an understanding of the solution $x(t)$ of the differential equation near $x = K$.

System of ODEs: Species Interaction

- We will now expand our analysis to more realistic system which include multiple, interacting species.
- We will study two fundamental models, the *predator-prey* and the *competing species*.
- Later, when we begin studying marine ecosystems, you should see the analogy with these fundamental models.
- For example, a classic NP model is a predator-prey model where the nutrient is the prey and phytoplankton is the predator.
- What do we hope to gain from the analysis:
    - What are the steady states of my system
    - Are my steady states stable or unstable
    - How do the interaction parameterizations influence the system stability
    - How does the system parameterization influence the dynamics of the system response (time scales, oscillation rates, relative magnitude of populations, etc)

System of ODEs: Predator-Prey Models

We will start with a fundamental model of interaction in an ecosystem: The *predator-prey* model. History:

- Fur traders noted remarkable cycles in numbers of lynx and hare furs in the 1800s
- Later, Volterra noted similar fluctuations in fish populations and derived a fundamental set of equations to describe them
- Lotka derived simultaneously a similar set of equations

Predator Prey (Lotka-Volterra) Equations

$$\frac{dx}{dt} = (ax - bxy) \tag{29}$$

$$\frac{dy}{dt} = (-cy + dxy) \tag{30}$$

System of ODEs: Predator-Prey Models

Parameters:

- a: growth rate of prey
- b: consumption of x by y
- c: natural mortality of y
- d: consumption of x by y

Note: b and d are different because there is an efficiency to consumption

In this model, they were looking to answer two questions:

- Can we explain the cycles in a typical predator-prey system?
- Why is it that only in some systems the predation limits the prey density?
- What happens if the state or parameters are changed. For example, disease could alter the mortality rate, or a change in habitat could modify the ability of the prey to hid and affect the rate of predation.

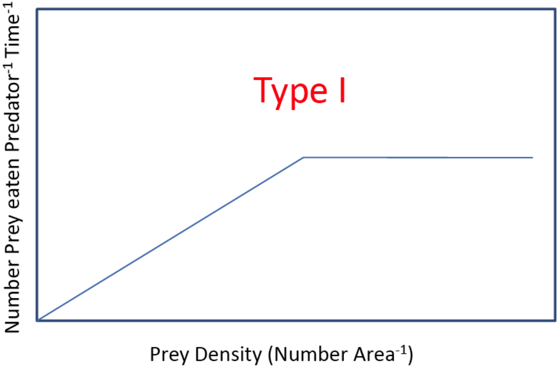
System of ODEs: Predator-Prey Models

The model relates prey x and predator y and these variables can represent biomass or population densities.Here we have used the following assumptions:

- There are no time lags, predators respond instantaneously to a change in the prey and vice versa
- Prey grow exponentially without predators to control their population. x
- Predators depend on prey to survive, otherwise, natural mortality will wipe them out c
- Predation rate depends on the likelihood that a predator comes across prey, thus is proportional to prey population
- growth rate of the predator is proportional to food intake

System of ODEs: Consumption Responses

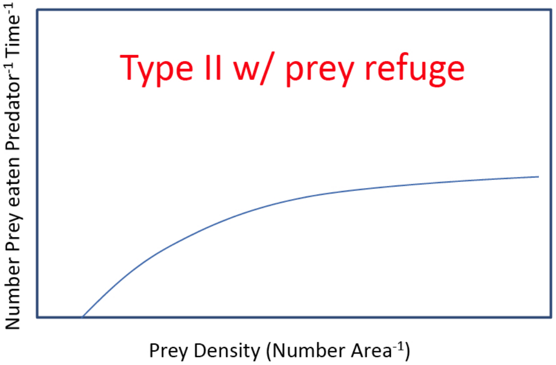
- Is a linear dependence of predation rate on prey realistic?
- No, realistically, should consider additional ecosystem dynamics:
- *Saturation* Above a certain density of prey, the consumption will level out (can only eat so much).
- We will use this when we parameterize grazing of Z on P



Hình: Type 1 Predation with Saturation (From UIdaho WLF 448)

System of ODEs: Consumption Responses, cont'd

Refuge : below a density, prey have places to hide:



Hình: Type 2 Predation with Refuge Effects (From UIdaho WLF 448)

Analysis of L-V model

- identify critical points in the system and think about their physical meaning
- plot the solution in phase (x,y) space using directional vectors
- examine Nullclines
- solve the full nonlinear system using Matlab and explore the results of phase plane trajectories by specifying a whole slew of initial conditions.

But first, we need to take a step backward and work on some quantitative tools. First let's look at the generic representation of a coupled set of two ordinary differential equations.Let's look at the generic representation first.

$$\frac{dx}{dt} = f(x, y) \tag{31}$$

$$\frac{dy}{dt} = g(x, y) \tag{32}$$

Phase Space

Phase space is the space of state variables. If we plot, for example, the time-dependent solution of two state variables together, we are working in the *phase plane*. For example:

Directional Gradients

Nguyen An Khuong,
Huynh Tuong Nguyen

Consider the governing equations in generic form:

$$\dot{x} = f(x, y) \tag{33}$$
$$\dot{y} = g(x, y) \tag{34}$$

We may not have an explicit solution for $x(t)$ or $y(t)$. But we do have direct information on the rate of change of our two species. For example, at the position in the phase plane x_1, y_1, we have:

$$\dot{x}|_{x_1,y_1} = f(x_1, y_1) \tag{35}$$
$$\dot{y}|_{x_1,y_1} = g(x_1, y_1) \tag{36}$$
$$\tag{37}$$

Directional Gradients, cont'd

We may not know the solution $x(t), y(t)$, but the differential equations tell as at any point in the phase plane (x, y) what the rate of change of the solution with time is. We can approximate the derivative using a forward Euler step:

$$\dot{x} = \frac{dx}{dt} \sim \frac{x_1 - x_0}{t_1 - t_0} \sim f(x_0, y_0) \tag{38}$$

or given a solution (x_0, y_0) at t_0, we can determine the approximate solution at t_1:

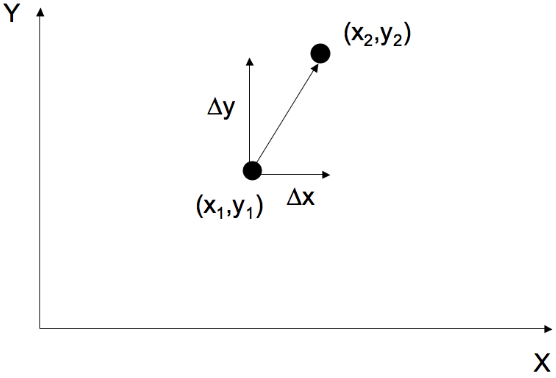
$$\Delta x = x_2 - x_1 \sim (t_2 - t_1) f(x_1, y_1) \tag{39}$$

and

$$\Delta y = y_2 - y_1 \sim (t_2 - t_1) g(x_1, y_1) \tag{40}$$

Directional Gradients, cont'd

So we can compute the vector at each point. If we only wish to piece together the gradients, then all we care about is the relative size of g and f and their signs. We can normalize the vectors so they are all the same length (magnitude) but give different directions. If we evaluate the gradients at enough points, we can get a sense of the behavior of the system.



Hình: Phase plane. Model state moves from x_1, y_1 to x_2, y_2

Directional Gradients, cont'd

The general procedure is as follows:

1. We plot our phase plane with species x on the x-axis and species y on the y-axis
2. We turn this into a regular grid of points, that is we divide up x into segments and y into segments so that we have a mesh of locations x, y. At each of these locations we evaluate our functions as above to find out what $\dot{x}, \dot{y}$ are at each point.
3. We then draw an arrow with the tail located at the point x, y. The head of the vector is located at the point $x + \delta x, y + \delta y$ where $\delta x = \frac{f(x,y)}{\sqrt{f^2+g^2}}$ and $\delta x = \frac{g(x,y)}{\sqrt{f^2+g^2}}$ are related the rates of changes of prey x and predator y. This arrow is pointing in the direction that ecosystem is going at the moment.

Nullclines (a.k.a. isoclines)

- *Nullcline*: Curve along which one of the variables is a steady state
- Along the x *isocline*, we have $f = 0$
- Along the x *isocline* the directional gradients must be parallel to the $y - axis$, but can be oriented up or down
- How do we determine the direction? We look at the equation for $\dot{y}$ and check the sign.
- If $\dot{y} = g(x, y) > 0$, y is increasing.
- We can also look at the $y - nullcline$ along which $\dot{y} = 0$ and the directional gradients are parallel to the $x - axis$.

Work at Example: Pred-Prey

$$\frac{dx}{dt} = (ax - bxy) \tag{41}$$

$$\frac{dy}{dt} = (-cy + dxy) \tag{42}$$

Identify the critical points:

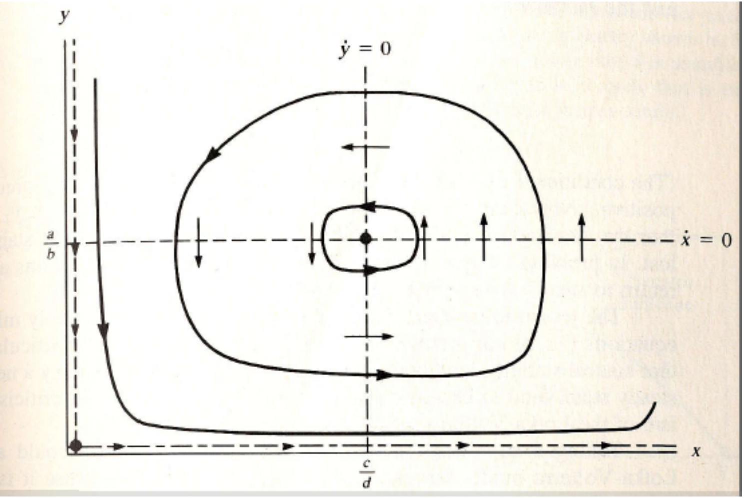
1. $\hat{x_1}, \hat{y_1} = 0, 0$: Extinction of both species
2. $\hat{x_2}, \hat{y_2} = \frac{c}{d}, \frac{a}{b}$: Growth equals mortality, clearly, the equations are coupled.

Nullclines of Pred-Prey

- *x nullcline* If we set $\dot{x} = 0$ we find that we have two straight lines, $y = \frac{a}{b}$ and $x = 0$. For this system, to the right of $x = \frac{c}{d}$ we have that $\dot{y} > 0$ and to the left we have that $\dot{y} < 0$.
- *y nullcline* If we set $\dot{y} = 0$ we find that we two straight lines given by $x = \frac{c}{d}$ and $y = 0$. Now, if we examine what happens to $\dot{x}$ along the y nullcline, we see that when $y > \frac{a}{b}$, we have that $\dot{x} < 0$ and below that line, it is positive.

Note: Where the nullclines intersect is, by definition, a critical point.

Sketch of pred-prey analysis



What is happening in the ecosystem?

Analysis of Critical Points

In our one-dimensional logistic equation we were able to analyze the stability of the critical points by linearizing the system at the critical point. Essentially, we transformed $\dot{x} = f(x) \Rightarrow \dot{x} \sim \alpha x$
For a coupled set of ODEs

$$\frac{dx}{dt} = f(x, y) \tag{43}$$

$$\frac{dy}{dt} = g(x, y) \tag{44}$$

The linear system is:

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y & (45) \\ \dot{y} &= a_{21}x + a_{22}y & (46) \end{aligned}$$

How do we linearize?

Solving nonlinear ODEs

- There are a few analytical techniques for solving nonlinear ODEs (e.g. separation of variables).
- However, for the complex form typical of dynamical systems, these do not generally work.
- We will have to use numerical techiques there (and Matlab will help greatly.)
- A well known solver we should re in detail is called *Runge-Kutta*.

Homeworks and next week plan

- Do **ALL** exercises in both:

  Section 1.4 (pages 52-56) **and**

  Section 11.1 (pages 468-470) in:

  F.R. Giordano, W.P. Fox & S.B. Horton, *A First Course in Mathematical Modeling*, 5th ed., Cengage, 2014.
- Due: a week from today lecture.
- Next week plan: Exercise Session