Algorithms PART II: Partitioning and Divide & Conquer

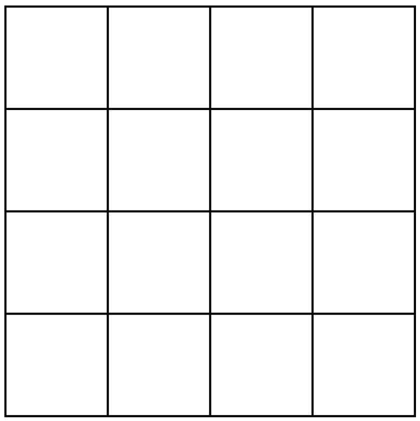
HPC Fall 2012

Prof. Robert van Engelen

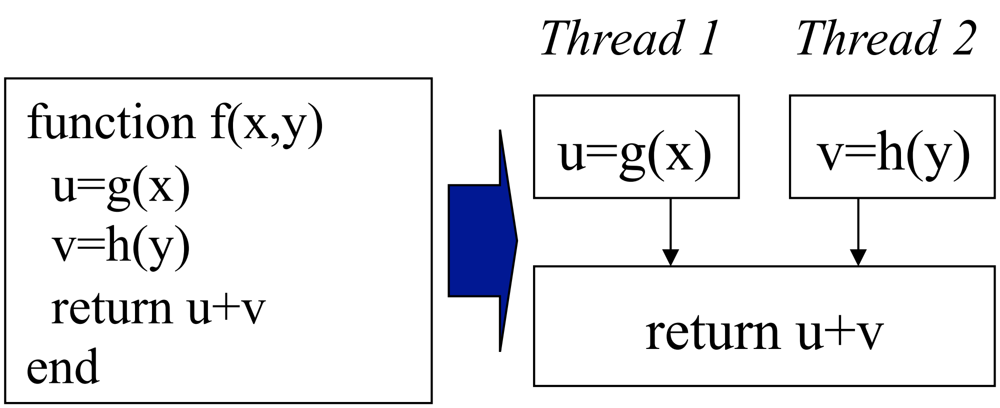
Overview

- Partitioning strategies
- Divide and conquer strategies
- Further reading

Partitioning Strategies



Block partitioning of a 2D domain



- *Data partitioning*
  - Perform *domain decomposition* to run parallel tasks on subdomains
  - "Scatter-compute-gather" where local computation may require communication and scatter/gather may involve computations
- *Task partitioning*
  - Decompose functions into independent subfunctions and execute the subfunctions in parallel

Partitioning Strategies

- *Partitioning strategy* (data partitioning):
  1. Break up a given problem into *P* subproblems
  2. Solve the *P* subproblems concurrently
  3. Collect and combine the *P* solutions

- Embarrassingly parallel
  - Is a simple form of data partitioning into independent subproblems without initial work and no communication between tasks (workers)

Partitioning Example 1: Summation

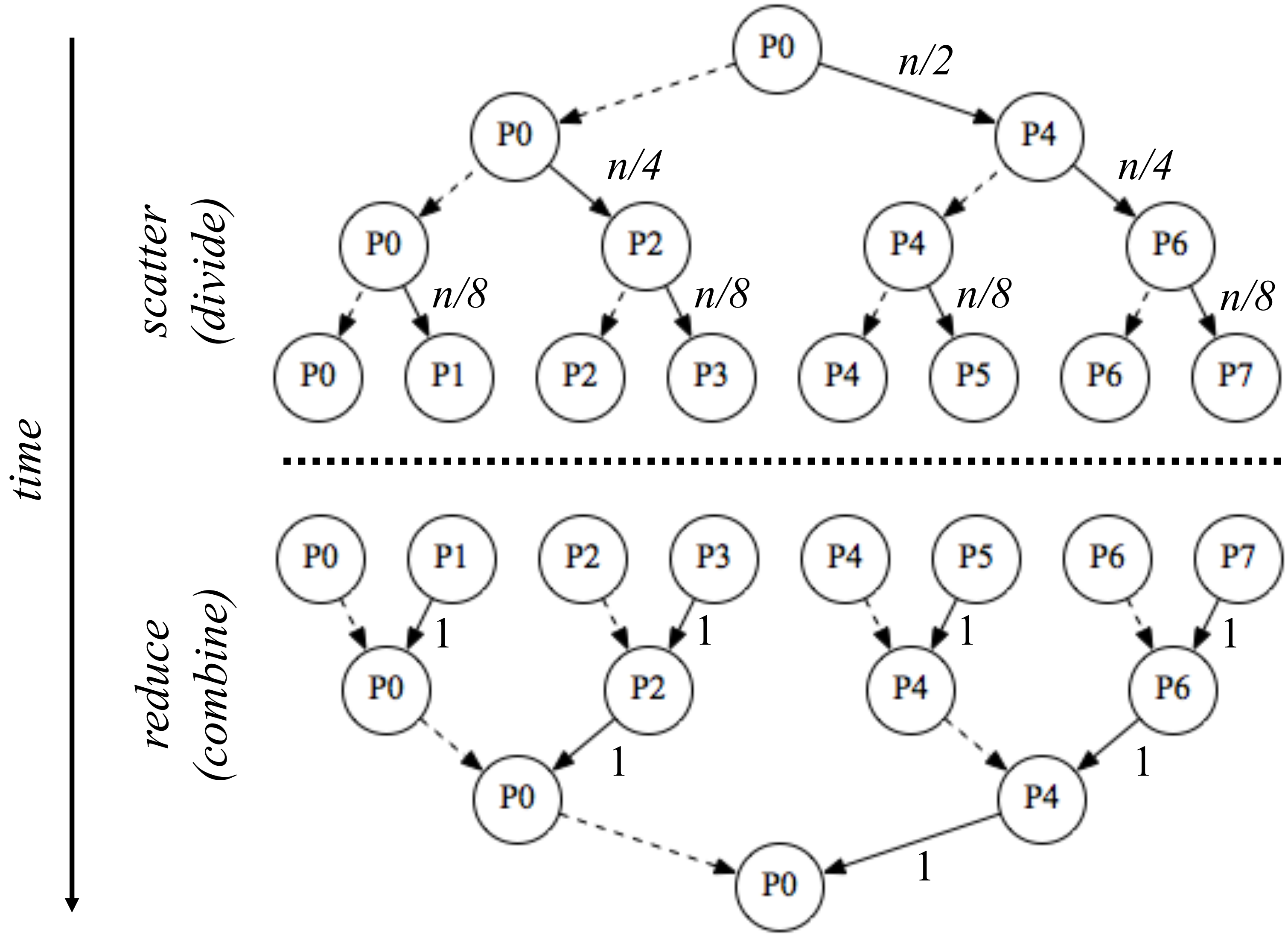
- Summation of n values $X = [x_1,\ldots,x_n]$

1. Divide X into P equally-sized sublists X_p, $p = 0,\ldots,P\text{-}1$ and distribute the X_p sublists to the P processors
2. The processors sum the local parts $s_p = \sum X_p$
3. Combine the local sums $s = \sum s_p$

- Algorithms:

1. Scatter list X using a scatter-tree
2. Serial summation of parts
3. Reduce local sums

Partitioning Example 1: Summation



$Log_2(P)$ *steps*

Total amount of data transferred: $n/2 \log_2(P)$

Local summations: n/P *steps*

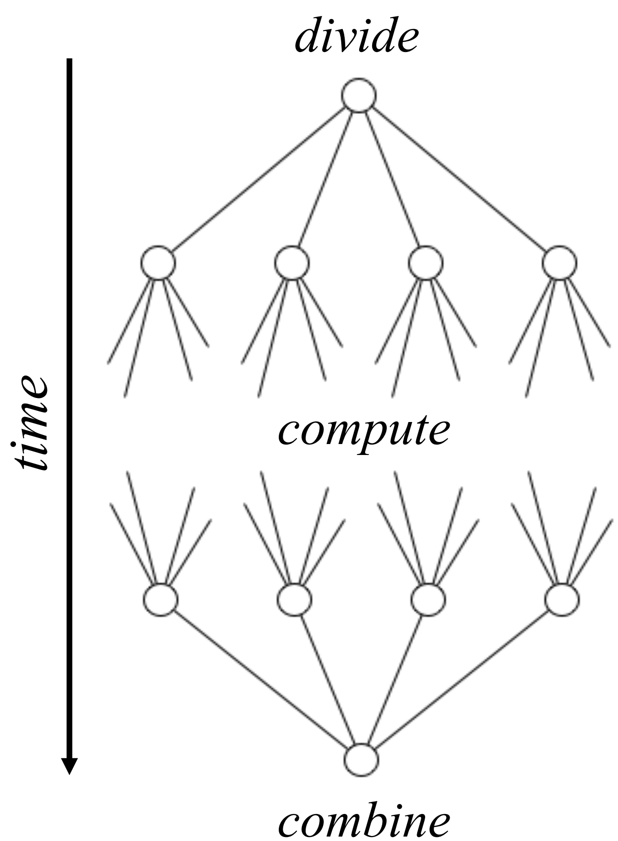
$Log_2(P)$ *steps*

Total amount of data transferred: P-1

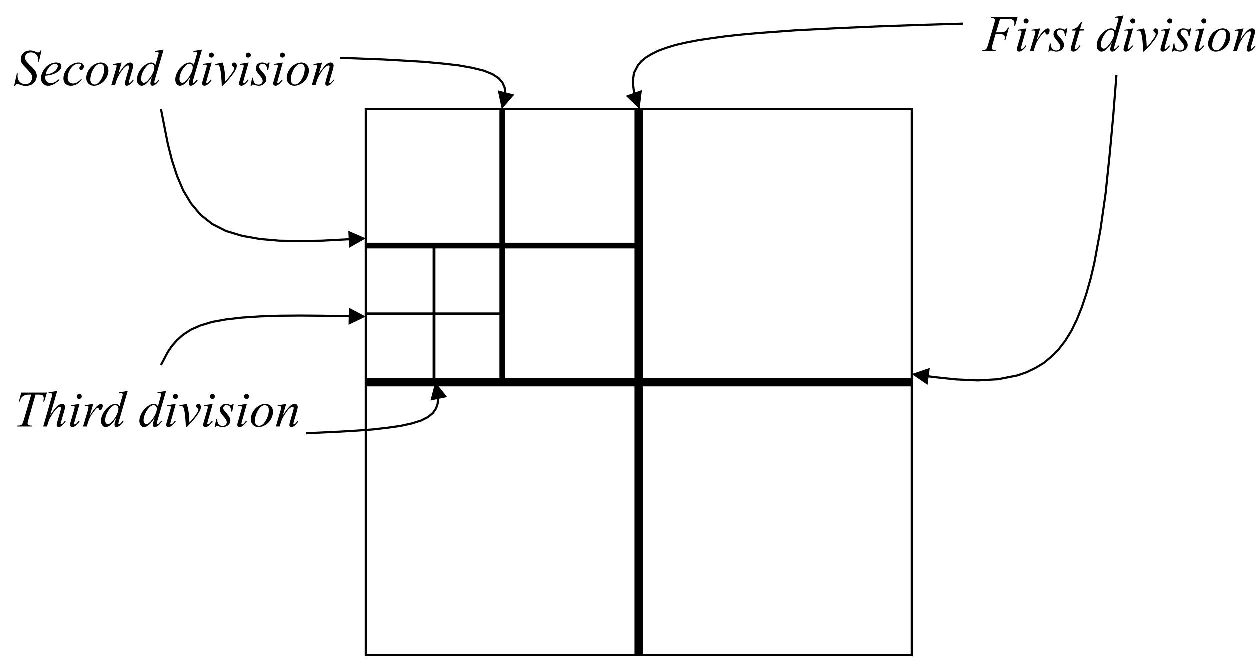
Partitioning Example 1: Summation

- Communication time
  - Scatter: $t_{comm1} = \sum_{k=1..\log2(P)} (t_{startup} + 2^{-k}n\ t_{data})$
    $= \log_2(P)t_{start} + n(P\text{-}1)/P\ t_{data}$
  - Reduce: $t_{comm2} = \log_2(P)\ (t_{start} + t_{data})$
  - Total: $t_{comm} = 2\ \log_2(P)t_{start} + (\ n(P\text{-}1)/P + \log_2(P)\ )\ t_{data}$
- Computation time
  - Local sum: $t_{comp1} = n/P$
  - Global sum: $t_{comp2} = \log_2(P)$
  - Total: $t_{comp} = n/P + \log_2(P)$
- Speedup, assuming $t_{startup} = 0$
  - Sequential time: $t_s = n\text{-}1$
  - Parallel time: $t_P = (\ n(P\text{-}1)/P + \log_2(P)\ )\ t_{data} + n/P + \log_2(P)$
  - Speedup: $S_P = t_s/t_P = \mathrm{O}(n\ /\ (n + \log(P)))$
  - Best speedup w/o communication: $S_P = \mathrm{O}(P/\log(P))$

General M-Ary Partitioning



Example: partitioning an image, e.g. to compute histogram by parallel reductions (summations to count color pixels)



3-level 4-ary partitioning for $4^3 = 64$ processors

Partitioning Example 2: Parallel Bucket Sort

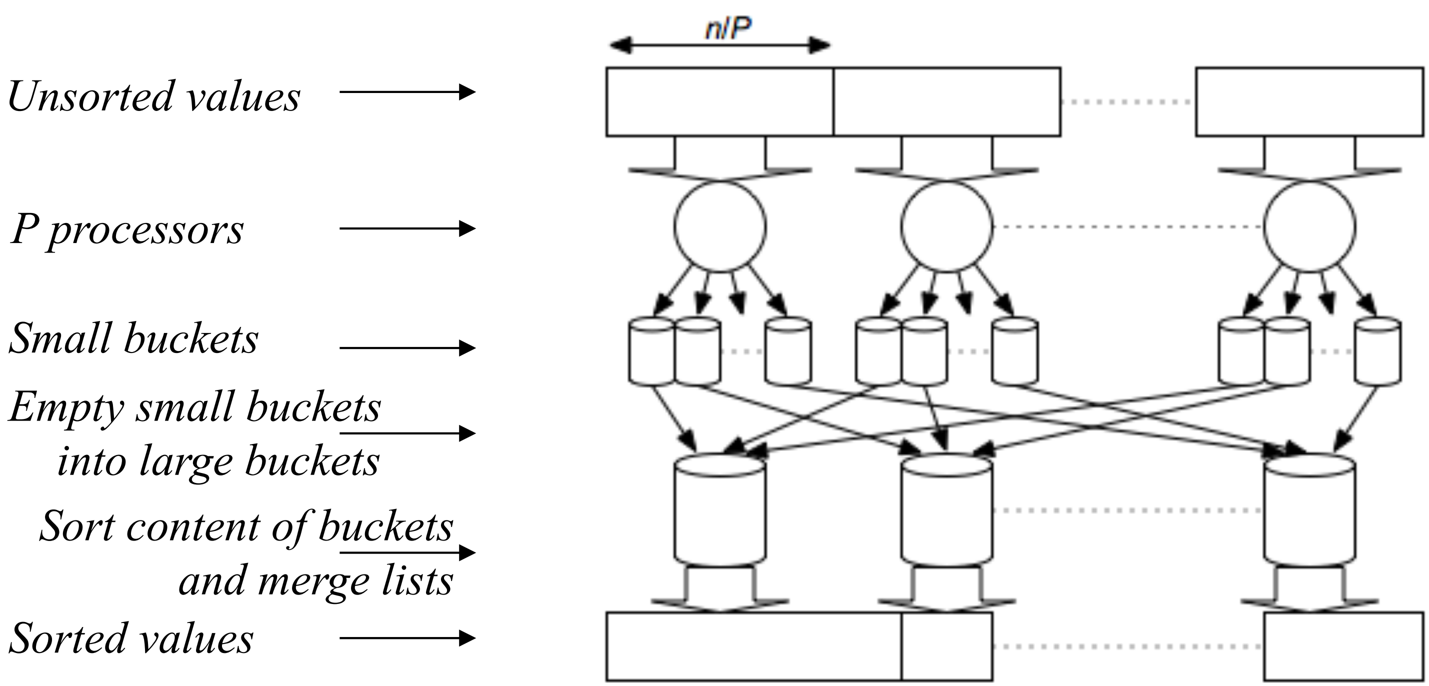
- Bucket sort of values $[x_1,\ldots, x_n]$ bounded within a range $x_i \in [lo \ldots hi]$

1. Partition the n values in n/P segments

2a. Sort each segment into P small buckets (local computation)

2b. Send content of small buckets to P large buckets

3. Sort P large buckets and merge lists



Partitioning Example 2: Parallel Bucket Sort

```
Input:  list X of length n with minimum value L and maximum U
Output: sorted list X

def function bucket(x) = P*(x-L)/(U-L);

scatter list X to local Xp lists each of size n/P
forall processors p = 0,…,P-1
  for i = 0,…,n/P-1
    x = Xp[i]
    put x into small bucket bp[bucket(x)]
all-to-all of small buckets bp into large buckets Bp
sort values in Bp[0,…,P-1] using a sequential sort algorithm
gather X from Bp into a merged sorted list
```

Partitioning Example 2: Parallel Bucket Sort

- Communication time (assuming uniform distribution in X)
  - Scatter: $t_{comm1} = \log_2(P) t_{startup} + n(P\text{-}1)/P\ t_{data}$
  - All-to-all: $t_{comm2} = (P\text{-}1)(t_{startup} + n/P^2\ t_{data})$
  - Gather: $t_{comm3} = \log_2(P) t_{startup} + n(P\text{-}1)/P\ t_{data}$
- Computation time (assuming uniform distribution in X)
  - Small bucket sort: $t_{comp1} = n/P$
  - Large bucket sort: $t_{comp2} = n/P \log_2(n/P)$
- Speedup
  - Sequential time: $t_s = n \log_2(n/P)$ (with P buckets)
  - Parallel time: $t_P = 2 \log_2(P) t_{startup} + 2\ n(P\text{-}1)/P\ t_{data}$
    $+ (P\text{-}1)(t_{startup} + n/P^2\ t_{data})$
    $+ n/P\ (1 + \log_2(n/P))$
  - Speedup w/o communication: $S_P = \mathrm{O}(P)$

Partitioning Example 3: Barnes Hut Algorithm

$$F = \frac{Gm_1m_2}{r^2}$$

Direction of the force between two bodies at points p and q

$$\vec{F} = \frac{Gm_pm_q}{r^2}\left(\frac{\vec{p}-\vec{q}}{r}\right)$$

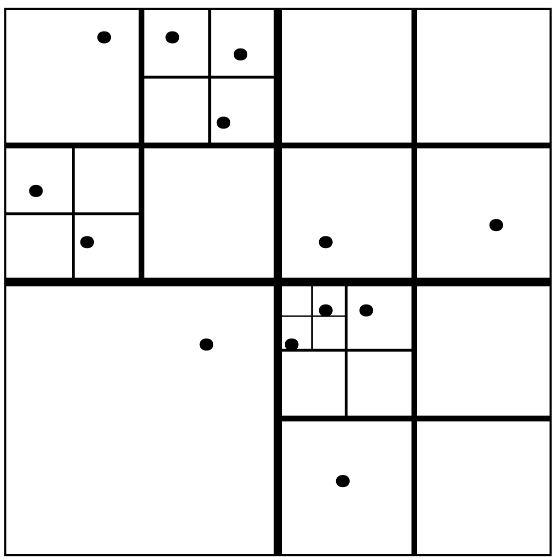
$$F = ma$$

$$\vec{F}^t = \frac{m(\vec{v}^{t+\frac{1}{2}} - \vec{v}^{t-\frac{1}{2}})}{\Delta t}$$

$$\vec{v}^{t+\frac{1}{2}} = \vec{v}^{t-\frac{1}{2}} + \frac{\vec{F}^t \Delta t}{m}$$

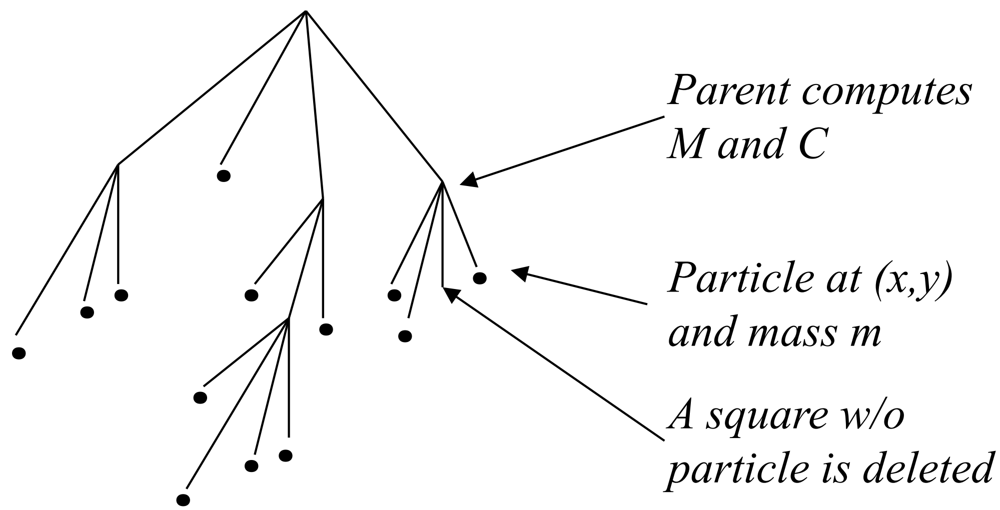
$$\vec{x}^{t+1} = \vec{x}^t + \vec{v}^{t+\frac{1}{2}}\Delta t$$

Partitioning Example 3: Barnes Hut Algorithm



Particles in 2D space

Quadtree



Mass of parent is sum of masses of children $M = \sum_{i=0}^{3} m_i$

Center of mass $C = \frac{1}{M} \sum_{i=0}^{3} m_i c_i$

Partitioning Example 3: Barnes Hut Algorithm

```
for (t = 0; t < tmax; t++)
{
  Build_tree();
  Compute_Total_Mass_Center();
  Compute_Force();
  Update_Positions();
}
```

Sequential time is O(n log n)

Assuming $P = n$ then $t_P = O(\log P)$

$$C = \frac{1}{M}\sum_{i=0}^{3} m_i c_i \qquad (*)$$

$$\vec{F} = \frac{G m_p m_q}{r^2}\left(\frac{\vec{p}-\vec{q}}{r}\right) \qquad (**)$$

```
Compute_Force()
{
  for (i = 0; i < n; i++)
    Compute_Tree_Force(i,root)
}
Compute_Tree_Force(i,node)
{
  if (box at node contains one particle)
    F = force using eq (**)
  else
  {
    r = distance from i to C (*) of box
    D = size of box at node
    if (D/r < theta)
      F = force using eq (**) with total M
    else
      for (all children c of box)
         F = F + Compute_Tree_Force(i,c);
  }
  return F;
}
```

Divide and Conquer

- Divide and conquer strategy (definition by JáJá 1992)
  1. Break up a given problem into independent subproblems
  2. Solve the subproblems ***recursively*** and concurrently
  3. Collect and combine the solutions into the overall solution

- In contrast to the partitioning strategy, divide and conquer uses *recursive partitioning* with concurrent execution to divide the problem down into independent subproblems
- In deeper levels of recursion the number of active processors may increase or decrease

Divide & Conquer Example 1: Parallel Recursive Matmul

- Block matrix multiplication in recursion by decomposing matrix in 2×2 submatrices and computing the submatrices recursively

```
Mat matmul(Mat A, Mat B, int s)
{ if (s == 1)
    C = A * B;
  else
  { s = s/2;
    P0 = matmul(A_{p,p}, B_{p,p}, s);
    P1 = matmul(A_{p,q}, B_{q,p}, s);
    P2 = matmul(A_{p,p}, B_{p,q}, s);
    P3 = matmul(A_{p,q}, B_{q,q}, s);
    P4 = matmul(A_{q,p}, B_{p,p}, s);
    P5 = matmul(A_{q,q}, B_{q,p}, s);
    P6 = matmul(A_{q,p}, B_{p,q}, s);
    P7 = matmul(A_{q,q}, B_{q,q}, s);
    C_{p,p} = P0 + P1;
    C_{p,q} = P2 + P3;
    C_{q,p} = P4 + P5;
    C_{q,q} = P6 + P7;
  }
  return C;
}
```

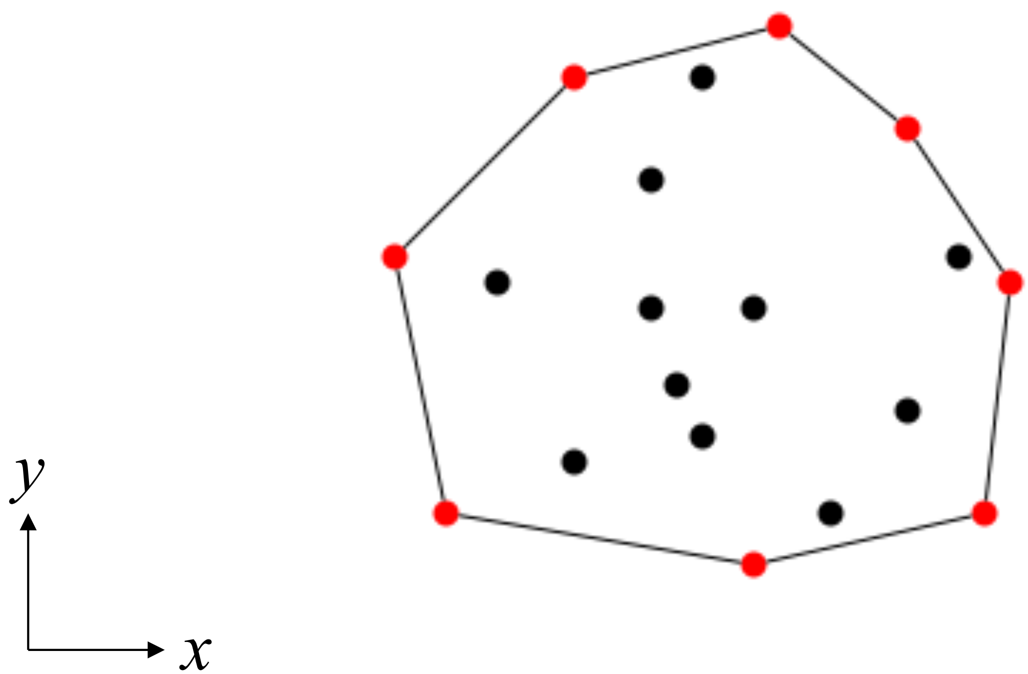
P0…**P7** *computed in parallel*

Can be computed in parallel

- Level of parallelism increases with deepening recursion
- Suitable for shared memory systems

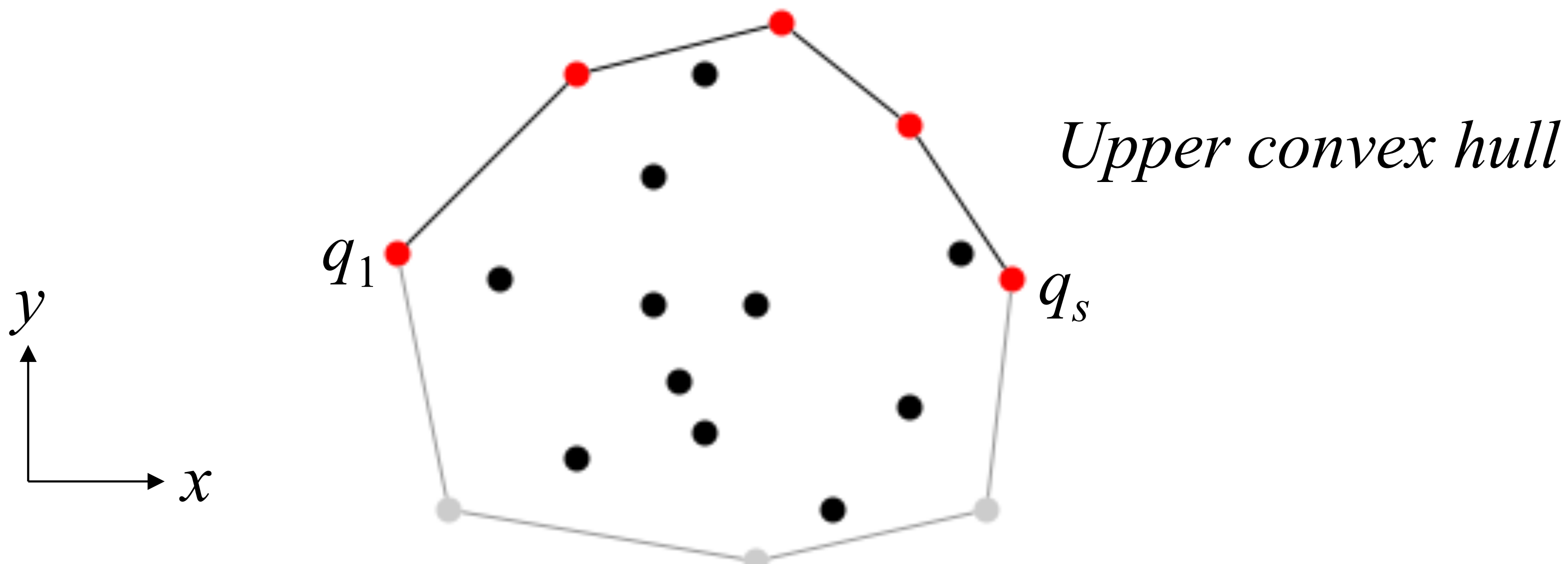
Divide and Conquer Example 2: Parallel Convex Hull Algorithm

- The *planar convex hull* of a set of points $S=\{p_1,p_2,\ldots,p_n\}$ of $p_i=(x,y)$ coordinates is the smallest convex polygon that encompasses all points S on the x-y plane



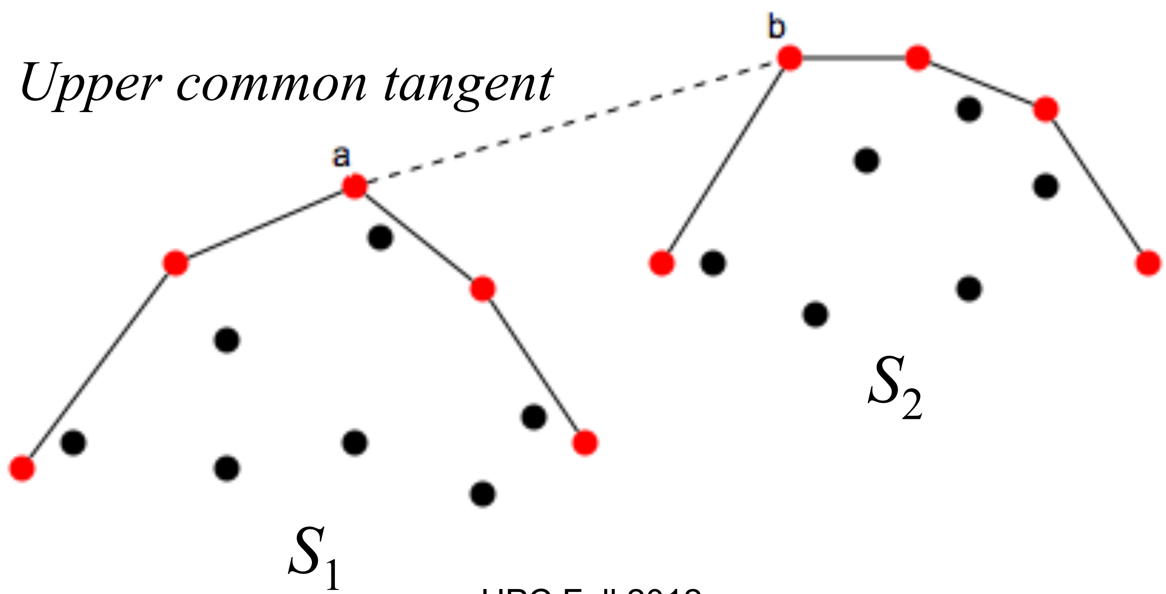
Divide and Conquer Example 2: Parallel Convex Hull Algorithm

- The *upper convex hull* spans points $\{q_1,\ldots,q_s\} \subseteq S$ from point q_1 with minimum x to q_s with maximum x
- The *convex hull* = *upper convex hull* + *lower convex hull*
- Problem:
  - Given points $S = \{p_1,\ldots,p_n\}$ such that $x(p_1) < x(p_2) < \ldots < x(p_n)$, construct the upper convex hull in parallel



Divide and Conquer Example 2: Parallel Convex Hull Algorithm

- Parallel convex hull:
  1. Divide the x-sorted points S into sets S_1 and S_2 of equal size
  2. Compute upper convex hull recursively on S_1 and S_2
  3. Combine $\mathrm{UCH}(S_1)$ and $\mathrm{UCH}(S_2)$ by computing the upper common tangent a to b to form $\mathrm{UCH}(S)$



Divide and Conquer Example 2: Parallel Convex Hull Algorithm

- Base case of recursion: two points, which are returned as $\mathrm{UCH}(S)$
- The line segment (a,b) can be computed sequentially in $\mathrm{O}(\log n)$ time with $n = |\mathrm{UCH}(S_1) + \mathrm{UCH}(S_2)|$ using a binary search method
- Line segments can be implemented as linked list of points, thus $\mathrm{UCH}(S_1)$ and $\mathrm{UCH}(S_2)$ can be connected using one pointer change of a to point to b in $\mathrm{O}(1)$ time

- Parallel convex hull time complexity recurrence relation:
  $$T(n) \leq T(n/2) + a \log n$$
  with solution:
  $$T(n) = \mathrm{O}(\log^2 n)$$
- Parallel convex hull operations recurrence relation:
  $$W(n) \leq 2W(n/2) + b\, n$$
  with solution:
  $$W(n) = \mathrm{O}(n \log n)$$
  which is cost optimal, since sequential algorithm is $\mathrm{O}(n \log n)$

Divide and Conquer Example 3: First-Order Linear Recurrences

- First-order linear recurrence

$$y_1 = b_1$$
$$y_i = a_i\, y_{i\text{-}1} + b_i \qquad\qquad 2 \leq i \leq n$$

- Example applications:
  - Prefix sum $y_i = \sum_{j=1..i} b_j$ is a special case of a first-order linear recurrence with $a_i = \mathbf{1}$ (the multiplicative unit element)
  - n-th order polynomial evaluation using Horner's rule
    $$p(x) = (((b_1\, x + b_2)\, x + b_3)\, x + \ldots + b_{n\text{-}1})\, x + b_n$$
    is a special case of a first-order linear recurrence with $a_i = x$
  - Solving a bi-diagonal system $\mathbf{By} = \mathbf{c}$,
    let $a_i = -\, l_i/d_i$
    $b_i = c_i/d_i$
    then solve linear recurrence
    to obtain solution $\boldsymbol{y}$

$$\begin{pmatrix} d_1 & & & & \\ l_2 & d_2 & & & \\ & l_3 & d_3 & & \\ & & \cdots & \cdots & \\ & & & l_n & d_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_n \end{pmatrix}$$

Divide and Conquer Example 3: First-Order Linear Recurrences

- Rewrite $y_i = a_i\, y_{i\text{-}1} + b_i$ into $y_i = a_i\,(a_{i\text{-}1}\, y_{i\text{-}2} + b_{i\text{-}1}) + b_i$
- This equation defines a linear recurrence of size $n/2$ for even index i

$$z_1 = b_1'$$
$$z_i = a_i'\, z_{i\text{-}1} + b_i' \qquad 2 \leq i \leq n/2$$

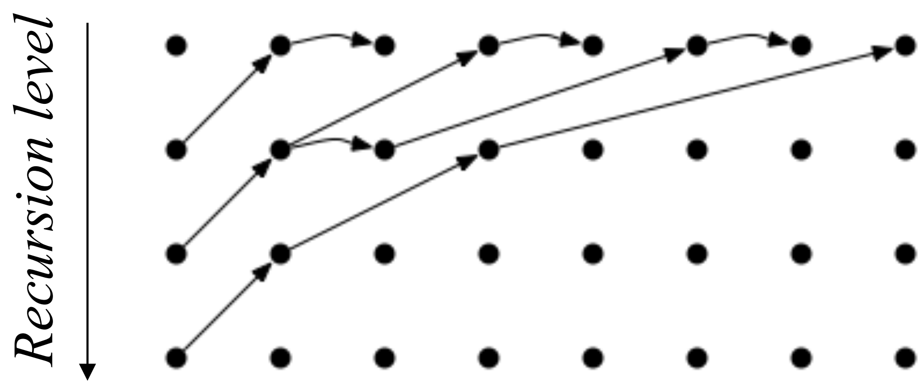
1. Let

$$a_i' = a_{2i}\, a_{2i\text{-}1}$$
$$b_i' = a_{2i}\, b_{2i\text{-}1} + b_{2i}$$

2. Solve z_i recursively
3. For $1 \leq i \leq n$ set

$$y_i = z_{i/2} \qquad \text{if } i \text{ is even}$$
$$y_i = a_i\, z_{(i\text{-}1)/2} + b_i \qquad \text{if } i \text{ is odd} > 1$$
$$y_i = b_1 \qquad \text{if } i = 1$$

Divide and Conquer Example 3: First-Order Linear Recurrences



$\log_2 n$ recursive steps

b_1

$b_1' = a_2\, b_1 + b_2$

$b_1'' = a_2'\, b_1' + b_2'$

$= ((a_2\, b_1 + b_2)\, a_3 + b_3)\, a_4 + b_4$

$b_1''' = a_2''\, b_1'' + b_2''$

$= ((a_2'\, b_1' + b_2')\, a_3' + b_3')\, a_4' + b_4'$

$= ((((a_2\, b_1 + b_2)\, a_3 + b_3)\, a_4 + b_4)\, a_5 + b_5)\, a_6 + b_6)\, a_7 + b_7)\, a_8 + b_8$

- Parallel algorithm:

```
linrecsolve(a[], b[], y[], n)
{
  if (n==1)
  { y[1] = b[1];
    return;
  }
  forall (i = 1 to n/2)
  { a_new[i] = a[2*i]*a[2*i-1];
    b_new[i] = a[2*i]*b[2*i-1]+b[2*i];
  }
  linrecsolve(a_new, b_new, z, n/2);
  forall (i = 1 to n)
  { if (i == 1)
      y[1] = b[1];
    else if (even(i))
      y[i] = z[i/2];
    else
      y[i] = a[i]*z[(i-1)/2]+b[i];
  }
}
```

Divide and Conquer Example 4: Triangular Matrix Inversion

- Consider $\mathbf{Ax} = \mathbf{b}$ with $n \times n$ triangular matrix $\mathbf{A}$

$$\begin{bmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ a_{31} & a_{32} & a_{33} & & \\ \dots & \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{bmatrix}$$

- Partition $\mathbf{A}$ into $(n/2) \times (n/2)$ blocks

$$\begin{bmatrix} \mathbf{A}_1 & \\ \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}$$

- Then $\mathbf{A}^{-1}$ is given by

$$\begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{0} \\ -\mathbf{A}_3^{-1}\mathbf{A}_2\mathbf{A}_1^{-1} & \mathbf{A}_3^{-1} \end{bmatrix}$$

Divide and Conquer Example 4: Triangular Matrix Inversion

- Parallel algorithm:
  1. Divide $\mathbf{A}$ into $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$
  2. Recursively compute inverses of $\mathbf{A}_1$ and $\mathbf{A}_3$ in parallel
  3. Multiply $-\mathbf{A}_3^{-1}\mathbf{A}_2\mathbf{A}_1^{-1}$ and combine with $\mathbf{A}_1^{-1}$ and $\mathbf{A}_3^{-1}$ to get $\mathbf{A}^{-1}$

- Time complexity is given by the recurrence relation

  $$T(n) = T(n/2) + c\, n$$

  with $P=n^2$ processors to compute $-\mathbf{A}_3^{-1}\mathbf{A}_2\mathbf{A}_1^{-1}$ in $\mathrm{O}(n)$ operations in parallel, thus $T(n) = \mathrm{O}(n)$ time

Divide and Conquer Example 5: Banded Triangular Systems

- Consider $\mathbf{Ax} = \mathbf{b}$ with banded matrix $\mathbf{A}$ with m=3

$$
\begin{pmatrix}
a_{11} & & & & & & & & \\
a_{21} & a_{22} & & & & & & & \\
a_{31} & a_{32} & a_{33} & & & & & & \\
 & a_{42} & a_{43} & a_{44} & & & & & \\
 & & a_{53} & a_{54} & a_{55} & & & & \\
 & & & a_{64} & a_{65} & a_{66} & & & \\
 & & & & a_{75} & a_{76} & a_{77} & & \\
 & & & & & a_{86} & a_{87} & a_{88} & \\
 & & & & & & a_{97} & a_{98} & a_{99}
\end{pmatrix}
\Rightarrow
\left(\begin{array}{|ccc|ccc|ccc|}
\hline
a_{11} & & & & & & & & \\
a_{21} & a_{22} & & & & & & & \\
a_{31} & a_{32} & a_{33} & & & & & & \\
\hline
 & a_{42} & a_{43} & a_{44} & & & & & \\
 & & a_{53} & a_{54} & a_{55} & & & & \\
 & & & a_{64} & a_{65} & a_{66} & & & \\
\hline
 & & & & a_{75} & a_{76} & a_{77} & & \\
 & & & & & a_{86} & a_{87} & a_{88} & \\
 & & & & & & a_{97} & a_{98} & a_{99} \\
\hline
\end{array}\right)
$$

- Define block diagonal $\mathbf{D}$ and inverse $\mathbf{D}^{-1}$

$$
\mathbf{D} = \begin{pmatrix}
\mathbf{A}_{11} & & & & \\
 & \mathbf{A}_{22} & & & \\
 & & \cdots & & \\
 & & & \cdots & \\
 & & & & \mathbf{A}_{n/m,n/m}
\end{pmatrix}
\qquad
\mathbf{D}^{-1} = \begin{pmatrix}
\mathbf{A}_{11}^{-1} & & & & \\
 & \mathbf{A}_{22}^{-1} & & & \\
 & & \cdots & & \\
 & & & \cdots & \\
 & & & & \mathbf{A}_{n/m,n/m}^{-1}
\end{pmatrix}
$$

Divide and Conquer Example 5: Banded Triangular Systems

- Compute $\mathbf{d} = \mathbf{D}^{-1}\mathbf{b}$ and $\mathbf{B} = \mathbf{D}^{-1}\mathbf{A}$ where $\mathbf{B}_{i,i\text{-}1} = \mathbf{A}_{ii}^{-1}\mathbf{A}_{i,i\text{-}1}$

$$\mathbf{d} = \mathbf{D}^{-1}\mathbf{b} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \cdots \\ \cdots \\ \mathbf{d}_{n/m} \end{pmatrix} \qquad \mathbf{B} = \mathbf{D}^{-1}\mathbf{A} = \begin{pmatrix} \mathbf{I}_\mathrm{m} & & & & \\ \mathbf{B}_{21} & \mathbf{I}_m & & & \\ & \mathbf{B}_{32} & \mathbf{I}_m & & \\ & & \cdots & \cdots & \\ & & & \mathbf{B}_{n/m,n/m\text{-}1} & \mathbf{I}_m \end{pmatrix}$$

- Solve first-order linear recurrence on $m \times m$ matrices $\mathbf{B}_{i,i\text{-}1}$

$$\mathbf{x}_1 = \mathbf{d}_1$$
$$\mathbf{x}_i = -\mathbf{B}_{i,i\text{-}1}\,\mathbf{x}_{i\text{-}1} + \mathbf{d}_i \qquad 2 \leq i \leq n/m$$

- Parallel time $\mathrm{O}(m + m\log(n/m))$ with $P{=}nm$ processors
  - Compute all $\mathbf{A}_{ii}^{-1}$ (each requiring $\mathrm{O}(m)$ operations) in parallel with parallel matrix inversion algorithm
  - Compute all $\mathbf{B}_{i,i\text{-}1} = \mathbf{A}_{ii}^{-1}\mathbf{A}_{i,i\text{-}1}$ in $\mathrm{O}(m)$ operations in parallel
  - Recurrence depth is $\log_2(n/m)$, each step has $\mathrm{O}(m)$ operations

Divide and Conquer Example 6: LU of Tridiagonal Matrix

- Consider tridiagonal matrix LU decomposition

$$\begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & b_3 & a_3 & c_3 & \\ & & \dots & \dots & \dots \\ & & & b_n & a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \dots & \dots & \\ & & & l_n & 1 \end{pmatrix} \begin{pmatrix} d_1 & u_1 & & & \\ & d_2 & u_2 & & \\ & & d_3 & u_3 & \\ & & & \dots & \dots \\ & & & & d_n \end{pmatrix}$$

- The LU decomposition $\mathbf{A} = \mathbf{L}\ \mathbf{U}$ satisfies

$$a_1 = d_1$$
$$c_i = u_i$$
$$a_i = d_i + l_i u_{i-1}$$
$$b_i = l_i d_{i-1}$$

thus

$$d_1 = a_1$$
$$d_i = a_i - l_i u_{i-1} = a_i - u_{i-1} b_i / d_{i-1} = [\ a_i\ d_{i-1} - b_i c_{i-1}\ ]\ /\ d_{i-1}$$

Divide and Conquer Example 6: LU of Tridiagonal Matrix

- Let

$$\mathbf{R}_1 = \begin{bmatrix} a_1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{R}_i = \begin{bmatrix} a_i & -b_i c_{i\text{-}1} \\ 1 & 0 \end{bmatrix} \qquad \mathbf{T}_i = \mathbf{R}_i\, \mathbf{R}_{i\text{-}1} \ldots \mathbf{R}_1$$

- From the Möbius transformation we have

$$d_i = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \mathbf{T}_i \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \mathbf{T}_i \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

- Algorithm:
  - Set up matrices $\mathbf{R}$
  - Solve first-order linear recurrence (prefix sum) of $\mathbf{T}$
  - Compute d_i
  - From the solution of d_i compute $l_i = b_i/d_{\text{i-1}}$

Further Reading

- [PP2] pages 106-131
- [PSC] pages 321-337