Partitioning and Divide-and-Conquer

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Introduction

Partitioning

Partitioning simply divides the problem into parts.

Divide and Conquer

Characterized by dividing problem into sub-problems of same form as larger problem. Further divisions into still smaller sub-problems, usually done by recursion.

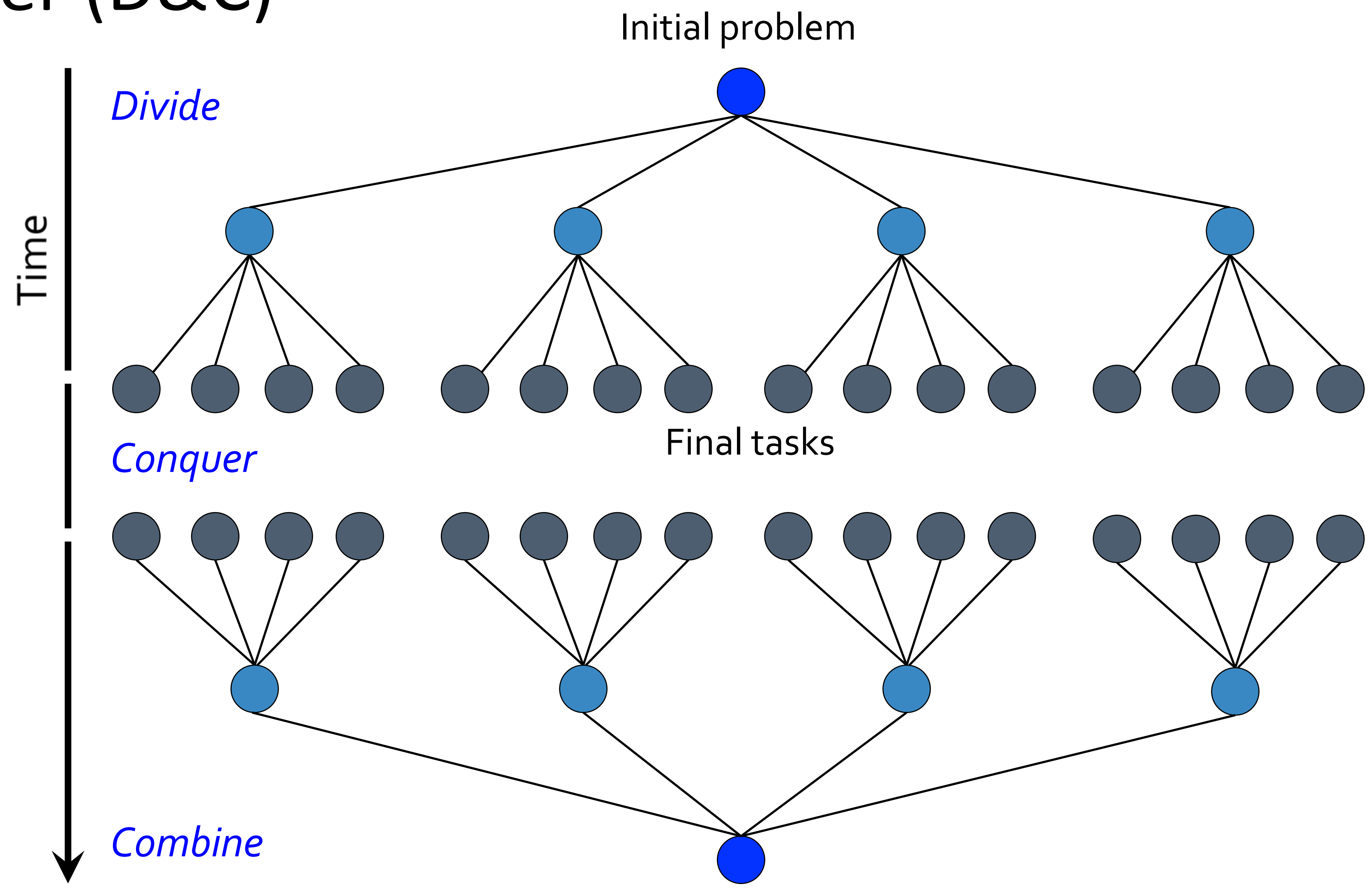
- ✓ Recursive divide and conquer amenable to parallelization because separate processes can be used for divided parts
- ✓ Also usually data is naturally localized.

Partitioning

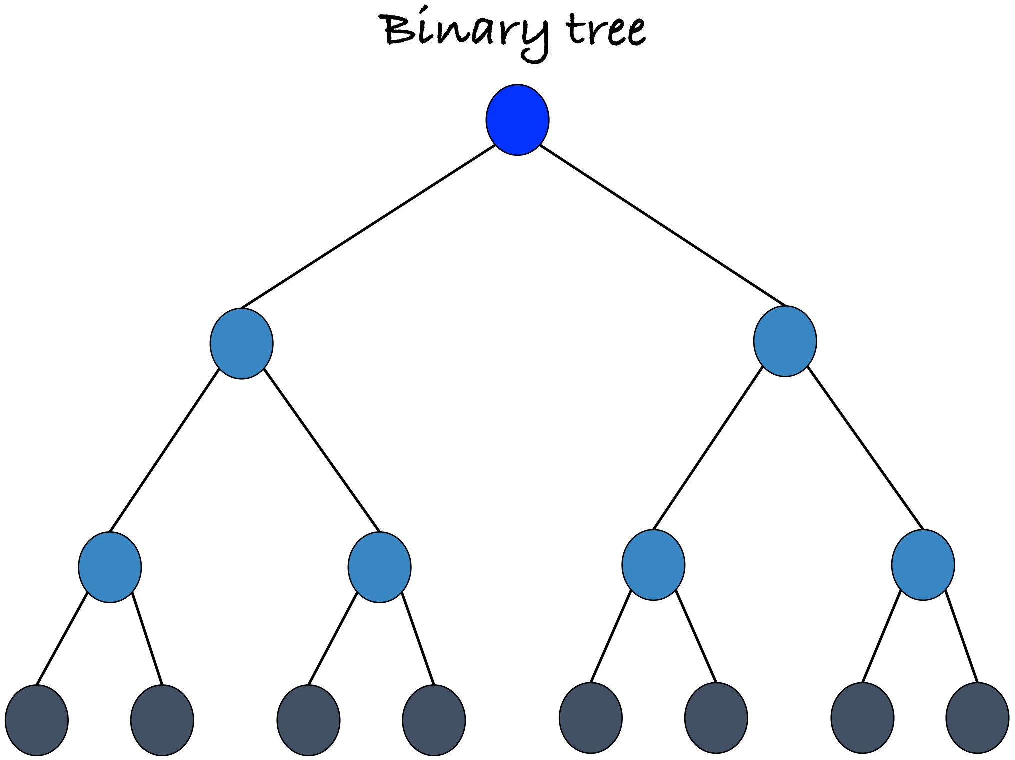
- Data partitioning or domain decomposition
    - Used in EPC?
- Functional decomposition

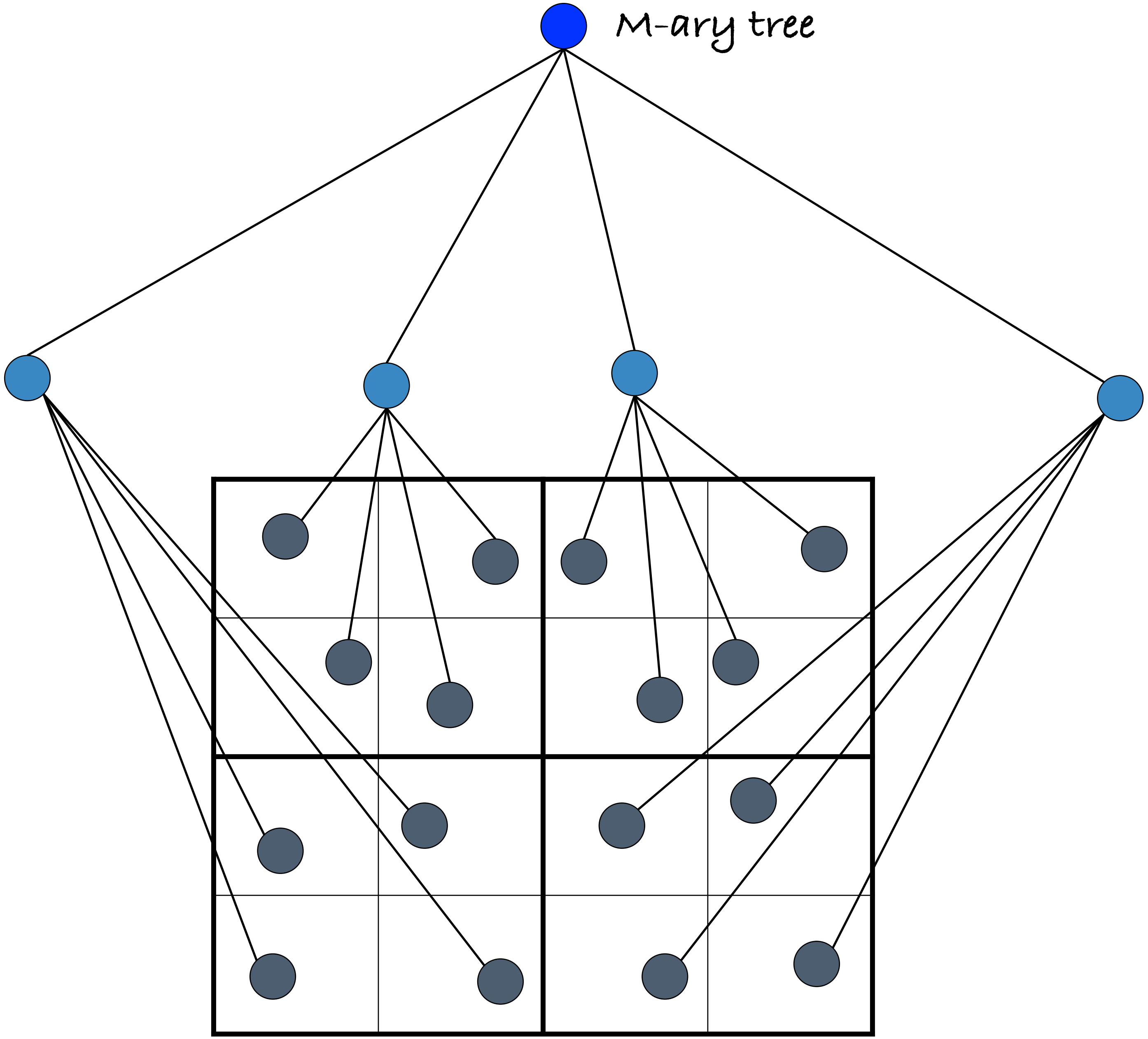
Divide-and-Conquer (D&C)

- Divide
  - Divide a problem into sub-problems that are of the same form as the large problem
  - Recursion: smallest sub-problems called final tasks
- Conquer
  - Run final tasks
- Combine
  - Aggregation

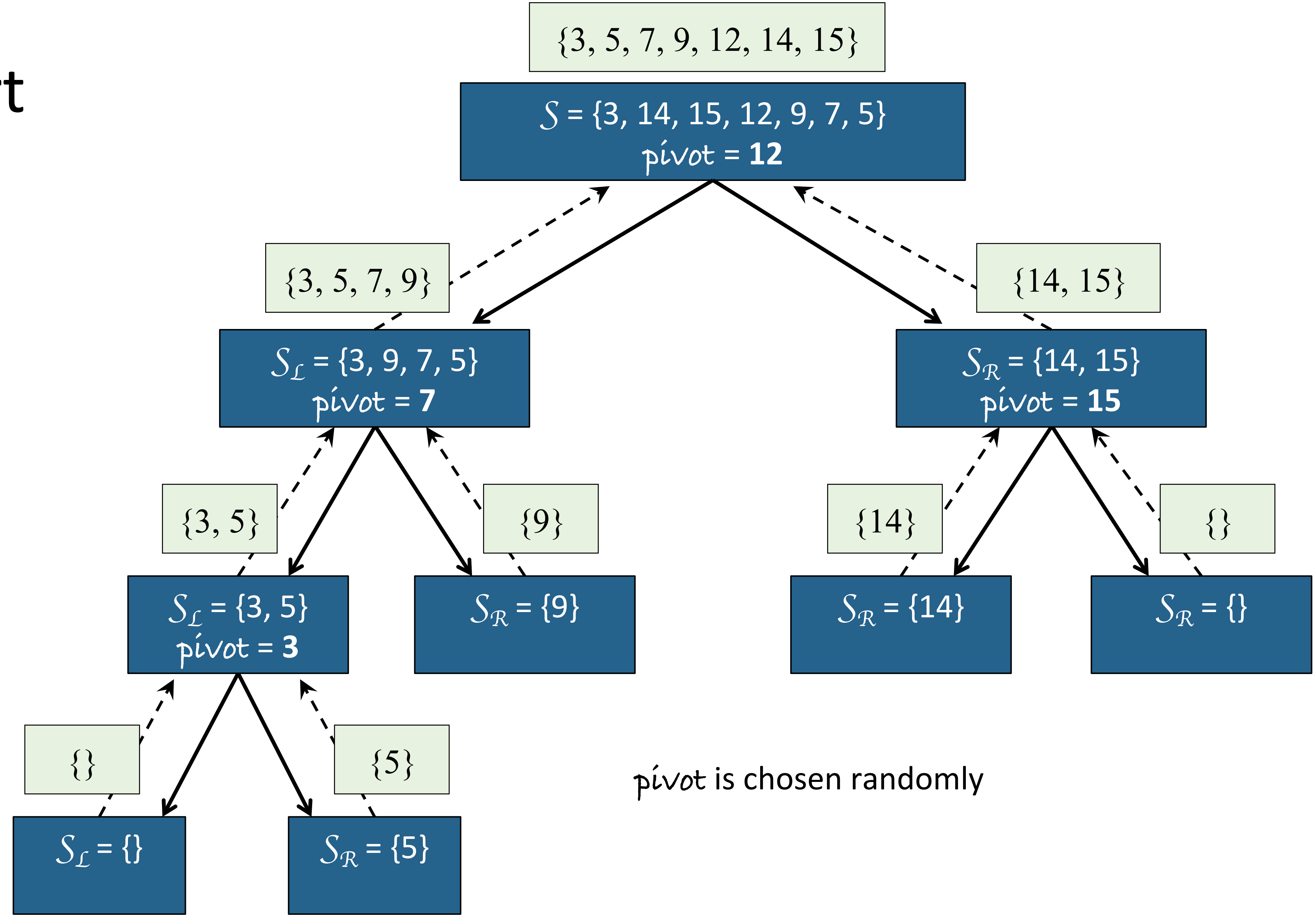


D&C trees





D&C Quicksort



pivot is chosen randomly

D&C matrix multiplication

- Divide
  - Divide A, B sizeof $n \times n$ into 8 matrix sizeof $n/2 \times n/2$:

    $A^{11}, A^{12}, A^{21}, A^{22}, B^{11}, B^{12}, B^{21}, B^{22}$
  - Matrix multiplication

$$X^{11} = A^{11}.B^{11} \qquad Y^{11} = A^{12}.B^{21}$$
$$X^{12} = A^{11}.B^{12} \qquad Y^{12} = A^{12}.B^{22}$$
$$X^{21} = A^{21}.B^{11} \qquad Y^{21} = A^{22}.B^{21}$$
$$X^{22} = A^{21}.B^{12} \qquad Y^{22} = A^{22}.B^{22}$$

- Combine

$$C^{11} = X^{11} + Y^{11} \qquad C^{12} = X^{12} + Y^{12}$$
$$C^{21} = X^{21} + Y^{21} \qquad C^{22} = X^{22} + Y^{22}$$

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix},\ B = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix},\ C = A\,.\,B = \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix}$$

$$C^{11} = A^{11}.B^{11} + A^{12}.B^{21} \qquad C^{12} = A^{11}.B^{12} + A^{12}.B^{22}$$
$$C^{21} = A^{21}.B^{11} + A^{22}.B^{21} \qquad C^{22} = A^{21}.B^{12} + A^{22}.B^{22}$$

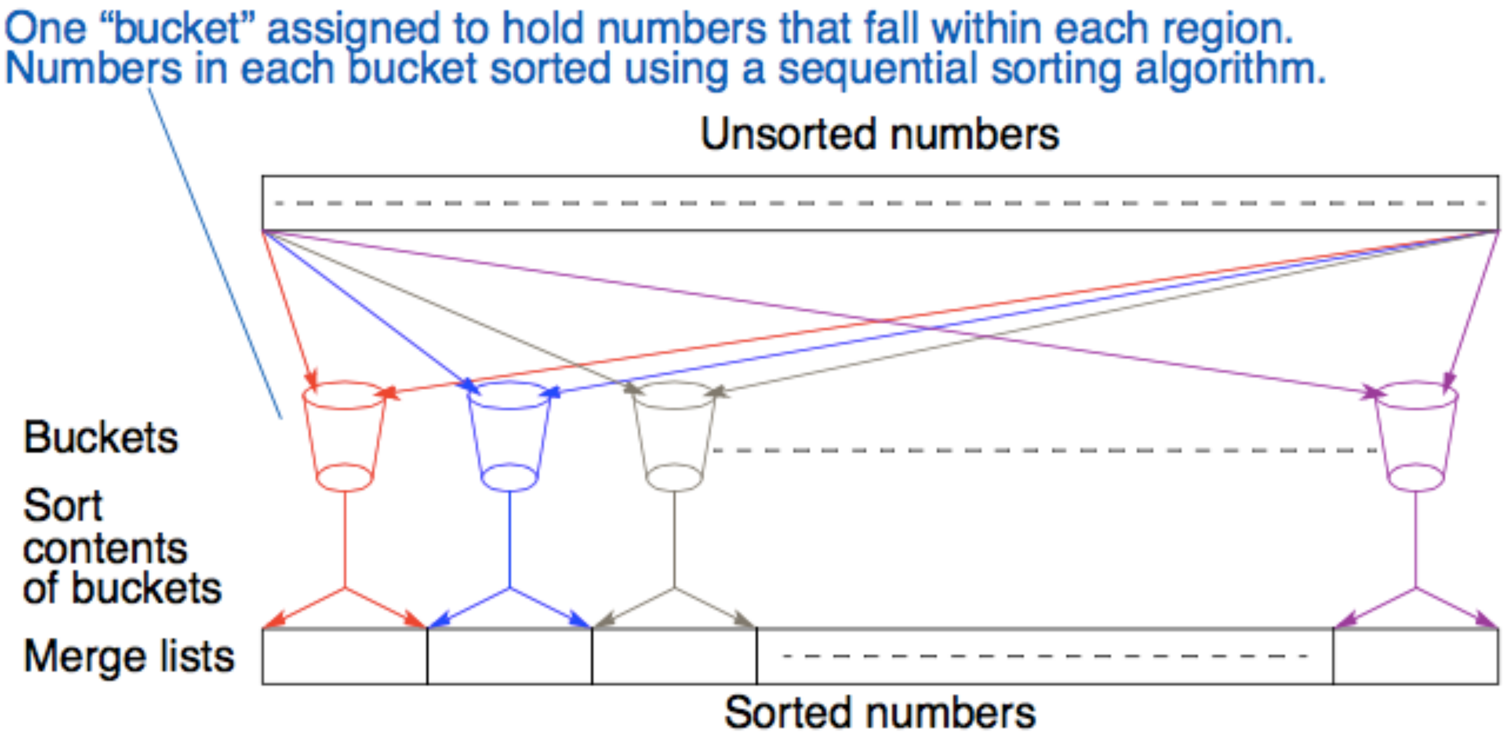
- Conquer
  - If matrix size is small enough, then C = A.B
  - Others, recursion & parallel computation

$$X^{11} = A^{11}.B^{11} \qquad Y^{11} = A^{12}.B^{21}$$
$$X^{12} = A^{11}.B^{12} \qquad Y^{12} = A^{12}.B^{22}$$
$$X^{21} = A^{21}.B^{11} \qquad Y^{21} = A^{22}.B^{21}$$
$$X^{22} = A^{21}.B^{12} \qquad Y^{22} = A^{22}.B^{22}$$

D&C matrix multiplication algorithm

```
Matrix_Multiplication (A, B, C, n) {    // n là kích thước
        if (n ≤ n0) { // n0 là kích thước bài toán nhỏ nhất
            C = A.B;  // Giải thuật tuần tự
            return;
        }
    // Bước (1)
        Phân rả A, B thành A11,A12,A21,A22,B11,B12,B21,B22 với
                kích thước n/2xn/2;
    // Bước (2)
        #pragma parallel for  // Song song vòng lặp for
        for (i=0; i<8; i++) {
            Matrix_Multiplication(A11, B11, X11, n/2);
            Matrix_Multiplication(A11, B12, X12, n/2);
            Matrix_Multiplication(A21, B11, X21, n/2);
            Matrix_Multiplication(A21, B12, X22, n/2);
            Matrix_Multiplication(A12, B21, Y11, n/2);
            Matrix_Multiplication(A12, B22, Y12, n/2);
            Matrix_Multiplication(A21, B22, Y21, n/2);
            Matrix_Multiplication(A22, B22, Y22, n/2);
        }
    // Bước (3)
        C11 = X11 + Y11;
        C12 = X12 + Y12;
        C21 = X21 + Y21;
        C22 = X22 + Y22;
        return;
}
```

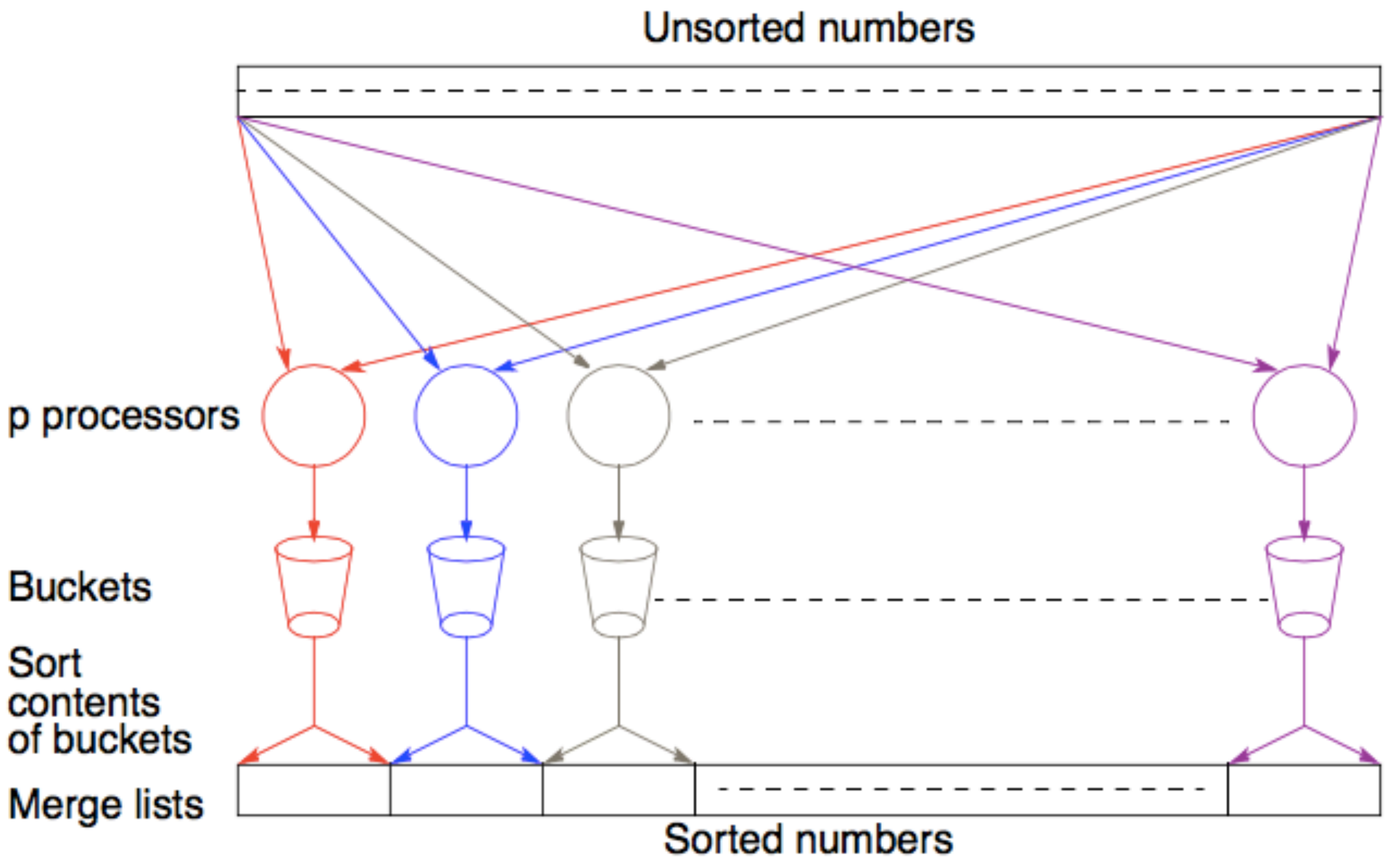
Bucket sort



- Sequential sorting time complexity: $O(n\log(n/m))$
- Works well if the original numbers uniformly distributed across a known interval, say 0 to a−1

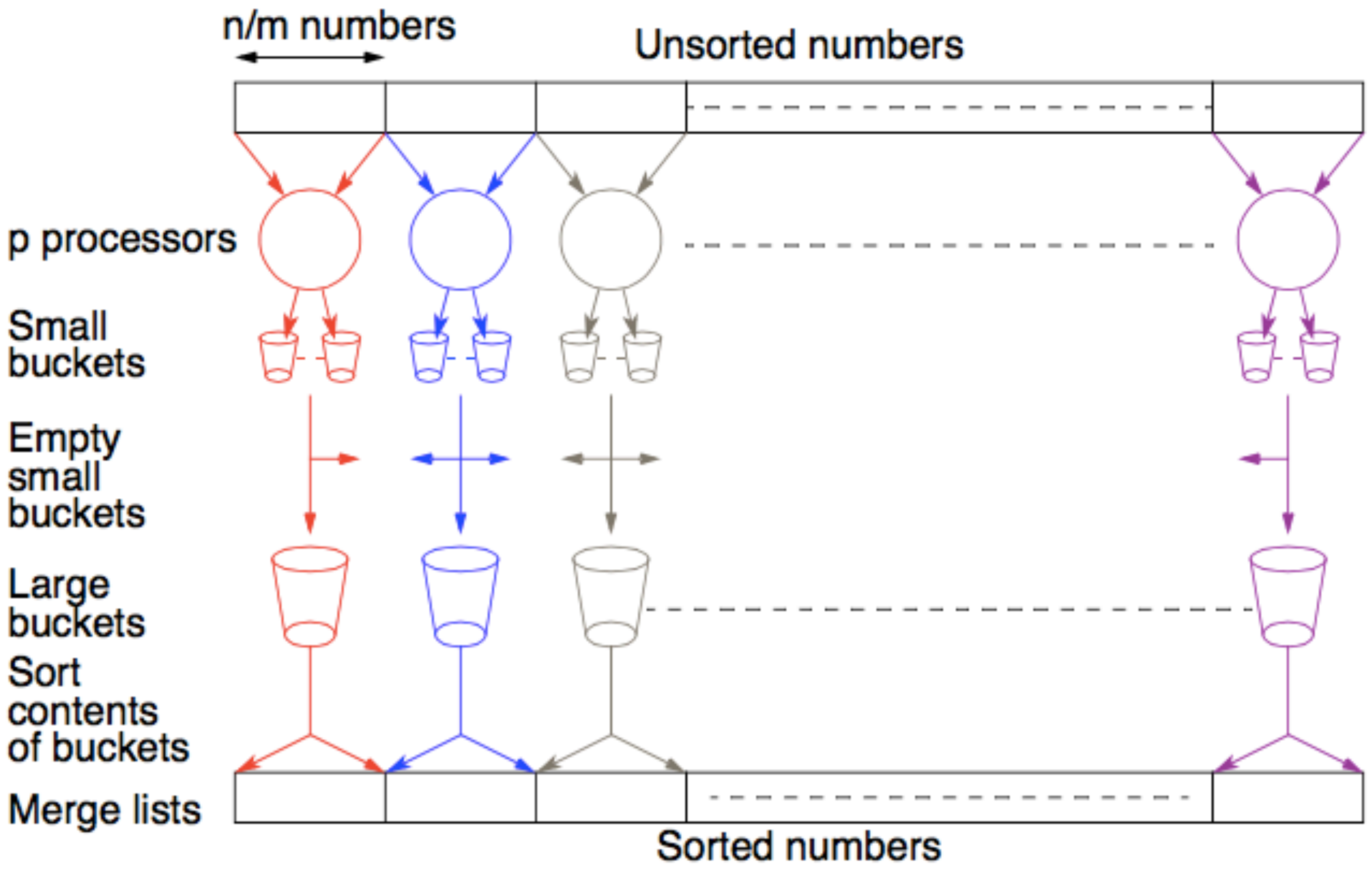
Parallel Bucket sort: version 1

- Assign one processor for each bucket



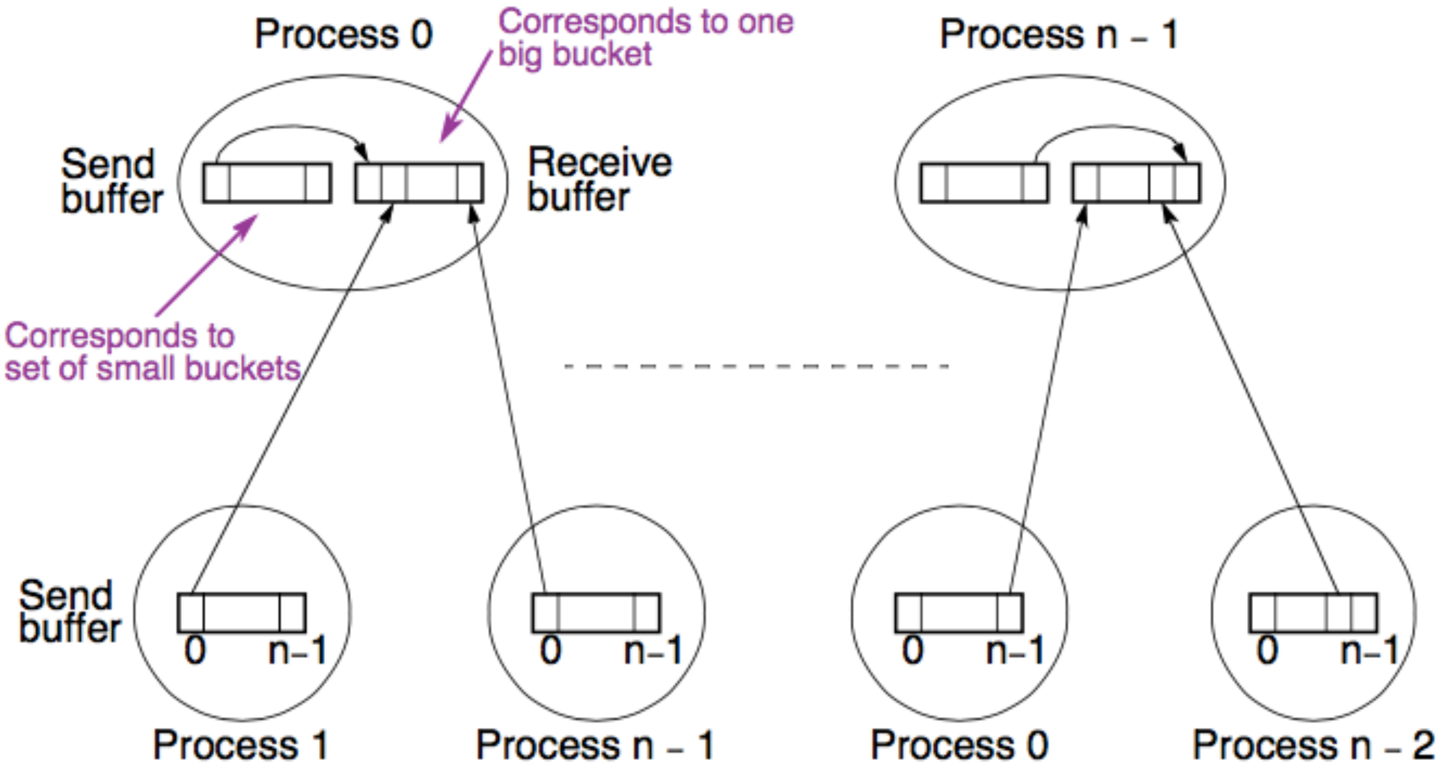
Parallel Bucket sort: version 2

- Partition sequence into m regions, one region for each processor
- Each processor maintains p "small" buckets and separates the numbers in its region into its own small buckets
- Small buckets then emptied into p final buckets for sorting, which requires each processor to send one small bucket to each of the other processors (bucket i to processor i)



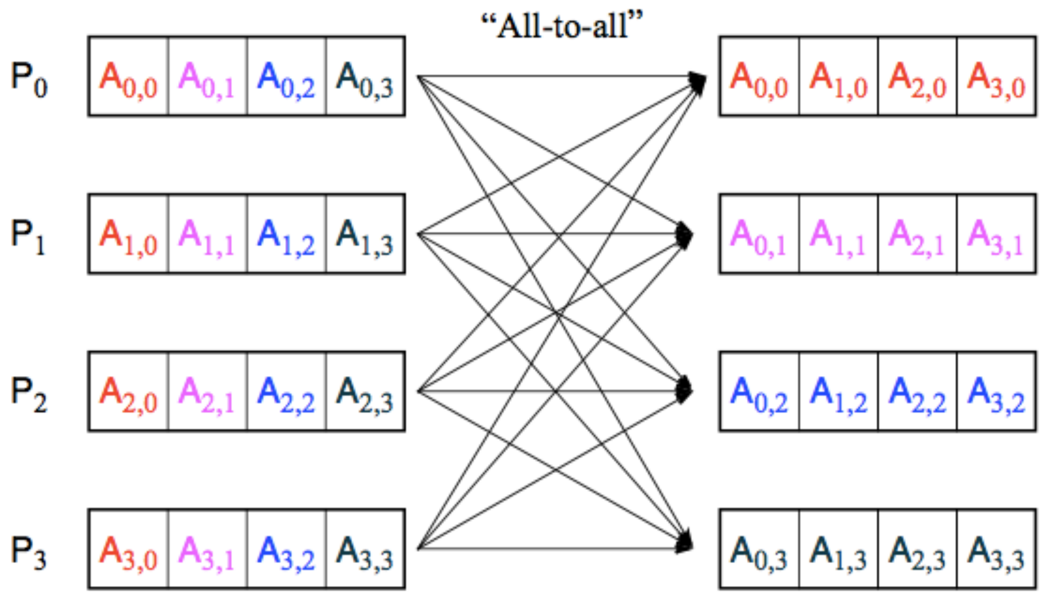
- Message-passing operation
  all-to-all broadcast

All-to-all broadcast



"all-to-all" routine actually transfers rows of an array to columns

Transpose a matrix



Numerical integration using rectangles

$$\text{Area} = \int_{x_1}^{x_2} f(x)\,dx = \lim_{n\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_r)\,(x_2 - x_1)$$

- Each region calculated using an approximation given by rectangles
- Aligning the rectangles: (a) & (b)

Figure (a)

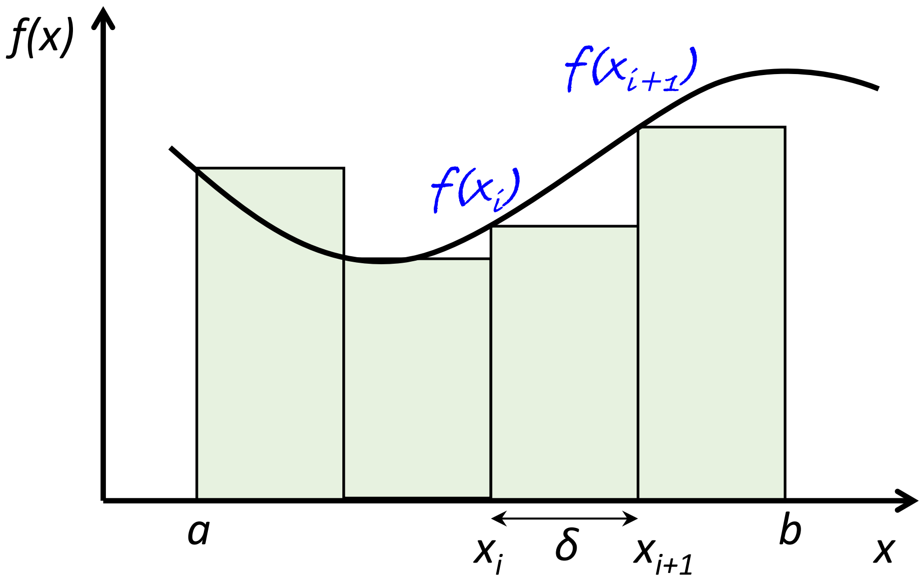
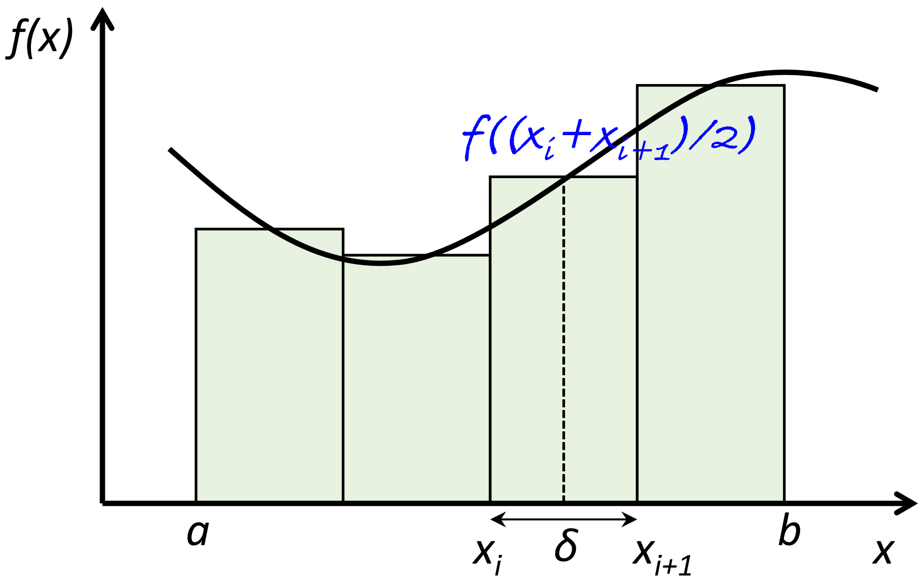


Figure (b)



Numerical integration using rectangles (1)

- Master/slave
- SPMD (Single-Program Multiple-Data)
- Static task assignment

- A better solution

```
11.        area = 0;
       // Tính diện tích trong vùng con tại mỗi P_i
12.        for (x=start; x<end; x=x+δ)
13.1.          area = area + f(x) + f(x+δ);
13.2.      area = 0.5 * area * δ;
```

```
// Tính $\int_a^b f(x)dx$ với chu kỳ lấy mẫu (b-a)/N_0
// P_group là nhóm p tiến trình/bộ xử lý (P_i)
1.  Integration(a, b) {
2.          i = Get_Rank(); // P_i có Rank=i
3.          if (i == 0) { // Master có Rank=0
4.              n = N_0;    // Số mẫu cầu lấy
5.          }
      // p bộ xử lý cùng thực hiện, Master cũng là một Slave
6.          bcast(&n, P_group); // Master truyền, tất cả P_i nhận
7.          δ = (b-a)/n;     // Chu kỳ lấy mẫu
8.          region = (b-a)/p;        // độ dài vùng con
9.          start = a + region * i; // điểm bắt đầu
10.         end = start + region;   // điểm kết thúc
11.         area = 0;               // diện tích vùng con
       // Tính diện tích trong vùng con tại mỗi P_i
12.         for (x=start; x<end; x=x+δ)
13.             area = area + 0.5 * (f(x) + f(x+δ)) * δ;
       // Tính tổng gộp diện tích các vùng con tại tất cả P_i
14.         Reduce_Add(&S, &area, P_group);
15.         Return(S);
16. }
```

Numerical integration using rectangles (2)

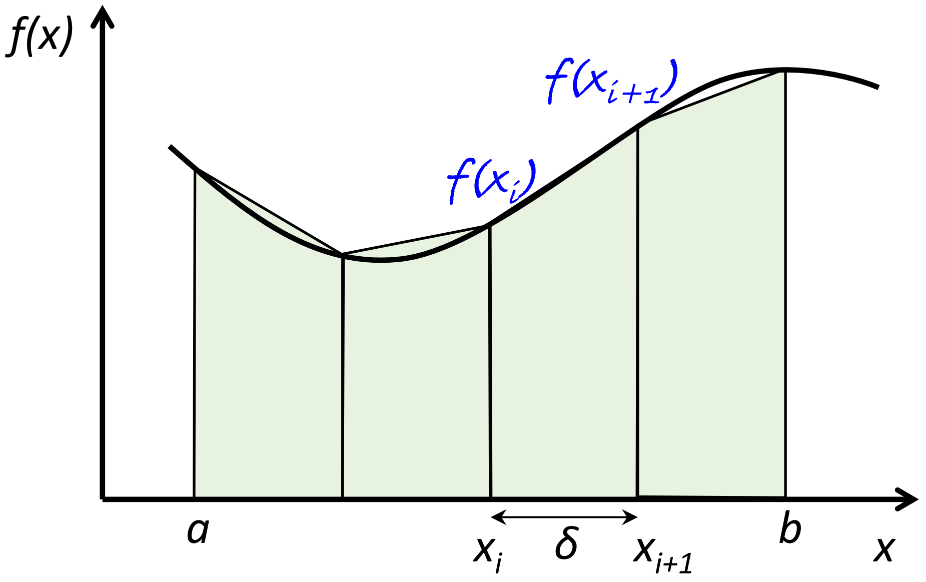
$$area = \frac{\delta(f(a)+f(a+\delta))}{2} + \frac{\delta(f(a+\delta)+f(a+2\delta))}{2} + \ldots + \frac{\delta(f(a+(n-1)\delta)+f(b))}{2}$$

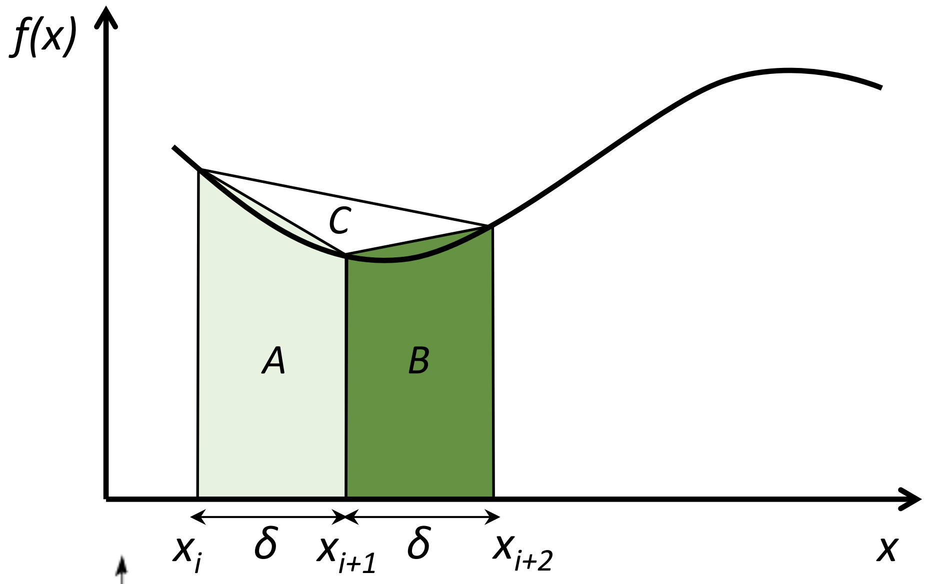
$$\Leftrightarrow area = \delta(\frac{f(a)}{2} + f(a+\delta) + f(a+2\delta) + \ldots + f(a+(n-1)\delta) + \frac{f(b)}{2}).$$

```
11.         area = 0;
        // Tính diện tích trong vùng con tại mỗi Pi
12.         for (x=start; x<end; x=x+δ)
13.1.           area = area + f(x) + f(x+δ);
13.2.       area = 0.5 * area * δ;
```

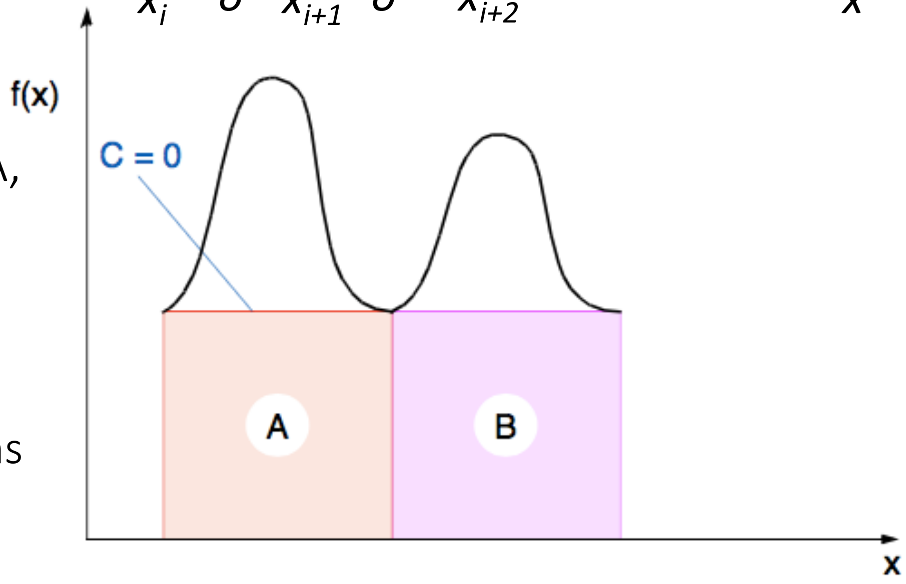
```
11.         area = 0.5 * (f(start) + f(end));
        // Tính diện tích trong vùng con tại mỗi Pi
12.         for (x=start+δ; x<end; x=x+δ)
13.1.           area = area + f(x);
13.2.       area = area * δ;
```

Numerical integration using trapezoidal method





- Adaptive Quadrature
  - Solution adapts to shape of curve. Use three areas, A, B, and C. Computation terminated when largest of A and B sufficiently close to sum of remain two areas
  - Adaptive quadrature with false termination
    - Might cause us to terminate early, as two large regions are the same (i.e., C = 0)



Numerical integration using trapezoidal method

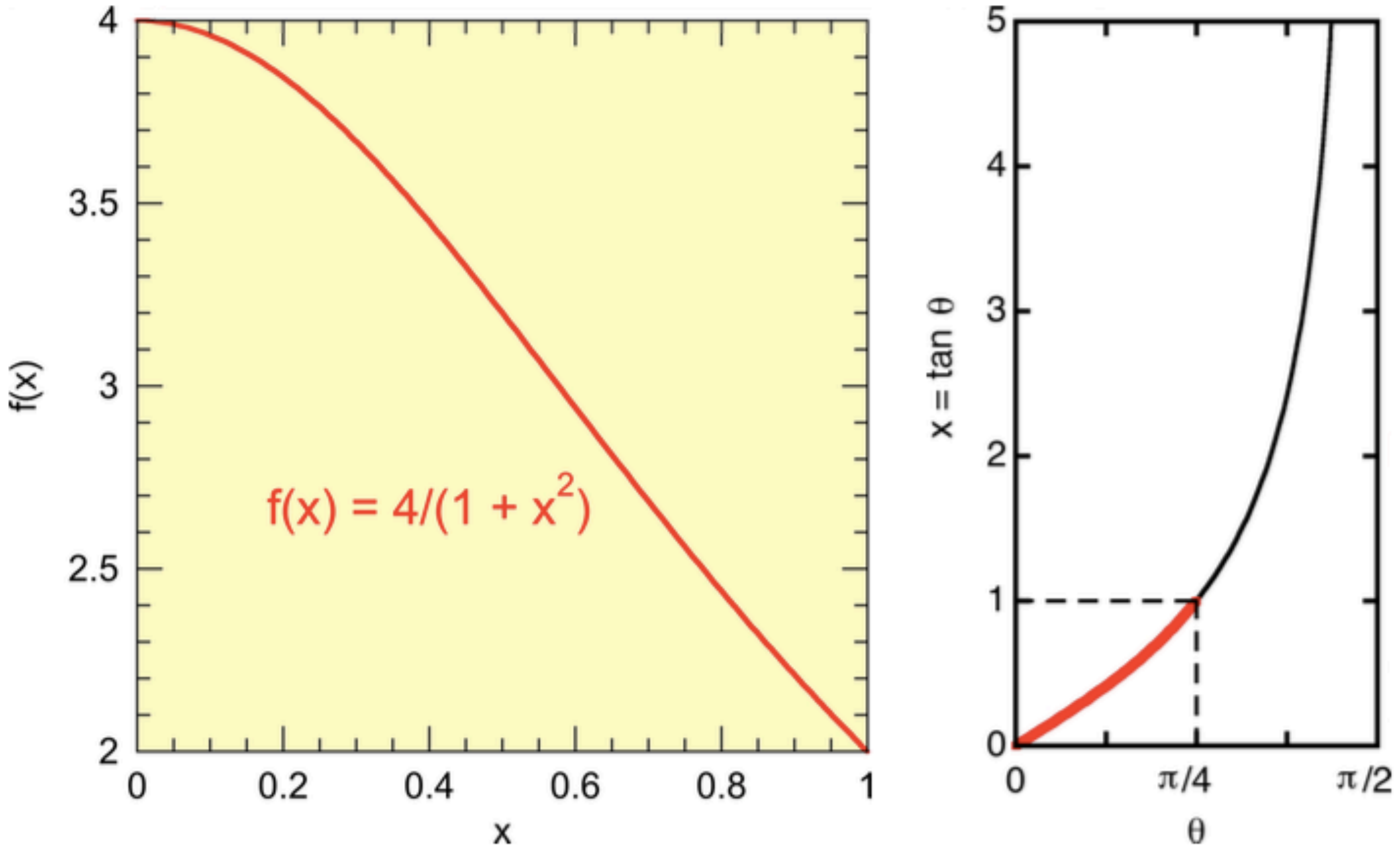
$F(L, \delta)$ with $L=[a, b]$ is an integral of $f(.)$ from a to b with the sampling frequency $1/\delta$.

- Divide
  - Divide $L=[a, b]$ into p parts $L_1, L_2, .., L_p$ with the same length $(b-a)/p$;
  - Concurrent computation $F(L_k, \delta)$ with all k from 1 to p
- Combine
  - Return $\sum Area_k$
- Conquer
  - Input: L_k and δ_k;
  - Compute $F(L_k, \delta_k)$
    - $Area_k$ = total area of sampling trapezoids in L_k;
  - If the accuracy Δ_k is hold, then
    - Return $Area_k$

    else
    - Divide L_k into $L_{k(L)} = [a_k, (b_k-a_k)/2]$ & $L_{k(R)} = [(b_k-a_k)/2 + 1, b_k]$
    - Compute $F(L_{k(L)}, \delta_k/2)$ & $F(L_{k(R)}, \delta_k/2)$

Integral representation of π

$$\int_0^1 dx \frac{4}{1+x^2} = \int_0^{\pi/4} \frac{d\theta}{\cos^2\theta} \frac{4}{1+\tan^2\theta} = \int_0^{\pi/4} 4 d\theta = \pi$$



Integral representation of π

```
/* Calculate value of π and compares with actual value (to 25 digits) of pi to give error.
Integrates function f(x)=4/(1+x^2)*/

#include <math.h>   //include files
#include <iostream.h>
#include "mpi.h"

void printit(); //function prototypes

int main(int argc, char *argv[])
{
    double actual_pi = 3.141592653589793238462643;
    int n, rank, num_proc, i;
    double temp_pi, calc_pi,
    int_size, part_sum, x;
    char response = 'y', resp1 = 'y’;
```

```
MPI::Init(argc, argv); //initiate MPI
num_proc = MPI::COMM_WORLD.Get_size();
rank = MPI::COMM_WORLD.Get_rank();
if (rank == 0)
   printit(); /* I am root node, print out welcome */
   while (response == 'y') {
         if (resp1 == 'y') {
              if (rank == 0) {   /*I am root node*/
              cout <______________________” <<endl;
              cout <<"\nEnter the number of intervals: (0 will exit)" << endl;
              cin >> n;
         }  else n = 0;
   MPI::COMM_WORLD.Bcast(&n, 1, MPI::INT, 0); //broadcast n
   if (n==0) break; //check for quit condition
   else {   int_size = 1.0 / (double) n; //calcs interval size
             part_sum = 0.0;
             for (i = rank + 1; i <= n; i += num_proc) { //calcs partial sums
                  x = int_size * ((double)i - 0.5);
                  part_sum += (4.0 / (1.0 + x*x));
         }
```

```
        temp_pi = int_size * part_sum; //collects all partial sums computes pi
        MPI::COMM_WORLD.Reduce(&temp_pi, &calc_pi, 1, MPI::DOUBLE, MPI::SUM, 0);
        if (rank == 0) { /*I am server*/
           cout << "pi is approximately " << calc_pi
           << ". Error is "
           << fabs(calc_pi - actual_pi)
           << endl
           <<"_______________________" << endl;
        }
     } //end else
     if (rank == 0) { /*I am root node*/
           cout << "\nCompute with new intervals? (y/n)" << endl;
           cin >> resp1;
     }
  } //end while
  MPI::Finalize(); //terminate MPI
  return 0;
} //end main
```

```
//functions
void printit() {
    cout << "\n_______________________" << endl
         << "Welcome to the pi calculator!" << endl
         << "Programmer: K. Spry" << endl
         << "You set the number of divisions \nfor estimating the integral: \n\tf(x)=4/(1+x^2)"
         << endl
         << "_______________________"
         << endl;
}  //end printit
```