Homework 1

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1 Online Passenger Allocation Problem

General Passenger Allocation Problem Given m buses, their routes, the capacity k_j for the j-th bus and n orders, the general passenger allocation (GPA) problem is to determine which order should be accepted to which bus such that the total number of the accepted orders is maximized and the capacity constraints of each bus cannot be violated.

Let x_{ij} represent that the passenger in i-th order is allocated to the j-th bus. Denote by d_i the destination station of the passenger in the i-th order and S_j the station set that the j-th bus passes. Therefore, we can use $d_i \in S_j$ to represent that the j-th bus can take the i-th passenger to his destination. Based on the above notations, we can formulate the GPA problem as a linear integer program (1). Constraint (1a) ensures that each passenger can allocated to at most one bus while Constraint (1b) guarantees that the total number of the passengers in each bus is bounded by the bus capacity.

Primal programming:

$$\max \quad \sum_{i=1}^{n} \sum_{j:d_i \in S_j} x_{ij}, \tag{1}$$

$$\text{s.t.} \quad \sum_{j:d_i \in S_j} x_{ij} \leq 1, \ \forall 1 \leq i \leq n, \tag{1a}$$

$$\sum_{i:d_i \in S_j} x_{ij} \leq k_j, \ \forall 1 \leq j \leq m, \tag{1b}$$

$$x_{ij} \in \{0, 1\}, \ \forall 1 \leq i \leq n, 1 \leq j \leq m \tag{1c}$$

Before the formal analysis, we relax the above integer program (1) to a linear program by replacing $x_{ij} \in \{0, 1\}$ with $x_{ij} \geq 0$ and take the dual as follows.

Dual programming:

$$\min \quad \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} k_j z_j, \tag{2}$$

$$\text{s.t.} \quad y_i + z_j \geq 1, \ \forall 1 \leq i \leq n, j : d_i \in S_j \tag{2a}$$

$$y_i \geq 0, \ \forall 1 \leq i \leq n \tag{2b}$$

$$z_j \geq 0, \ \forall 1 \leq j \leq m \tag{2c}$$

Algorithm 1 Most Remaining Seats First (MRSF) Algorithm

Input: $m, (S_1, \dots, S_m), (k_1, \dots, k_m)$.
Output: The passenger set in each bus $B_1, \dots, B_m$.

for $j \leftarrow 1$ **to** m **do**
  $B_j \leftarrow \emptyset$.
  $R_j \leftarrow k_j$.
end for
while order (i, d_i) arrives **do**
  $j^* \leftarrow \arg\max_{j:d_i \in S_j} R_j$.
  if $R_{j^*} \leq 0$ **then**
    reject order (i, d_i).
  else
    $B_{j^*} \leftarrow B_{j^*} \cup \{i\}$.
    $R_{j^*} \leftarrow R_{j^*} - 1$.
  end if
end while
return $B_1, \dots, B_m$.

Theorem 1. *Algorithm MRSF is a $\frac{1}{2}$-competitive algorithm for the online GPA problem.*

Proof.

$Add\ 2\ operations\ in\ MRSF:$

$Every\ time\ order(i, d_i)\ is\ accepted$

$y_i \leftarrow 1$

$z_{j^*} \leftarrow z_{j^*} + 1/k_{j*}$

$Let\ P\ and\ D\ be\ the\ values\ of\ the\ objective\ functions\ of\ the\ primal\ and\ dual\ solutions\ produced$

$Initially,\ P = D = 0 (x_{ij} = y_i = z_j = 0)$

$Let\ \varDelta P\ and\ \varDelta D\ be\ the\ changes\ in\ a\ single\ iteration$

$Let\ \tilde{x}\ and\ (\tilde{y}, \tilde{z})\ be\ the\ solution$

$Prove\ 3\ claims:$

$(1) \tilde{x}\ is\ feasible\ for\ the\ primal\ problem$

$(2) (\tilde{y}, \tilde{z})\ is\ feasible\ for\ the\ dual\ problem$

$(3) \varDelta D \leq 2 \varDelta P$

$For\ claim(1)$

$\because each\ i\ is\ at\ most\ in\ one\ of\ B_1, ..., B_m$

$\therefore \forall 1 \leq i \leq n,\ \sum_{j:d_i \in S_j} x_{ij} \leq 1\ is\ true$

$\because order(i, d_i)\ is\ rejected \iff \max_{j:d_i \in S_j} R_j \leq 0 (R_j = k_j - |B_j|)$

$\therefore \forall 1 \leq j \leq m,\ \sum_{i:d_i \in S_j} x_{ij} = |B_j| \leq k_j\ is\ true$

$\therefore claim(1)\ is\ proved$

$For\ claim(2)$

$if\ (i, d_i)\ is\ accepted$

$y_i = 1, z_j \geq 0, y_i + z_j \geq 1 (\forall j : d_i \in S_j)$

$else,\ \max_{j:d_i \in S_j} R_j \leq 0$
$so\ z_j \geq k_j \times \frac{1}{k_j} = 1,\ y_i + z_j \geq 1 (\forall j : d_i \in S_j)$
$\therefore \forall 1 \leq i \leq n, j : d_i \in S_j,\ y_i + z_j \geq 1$
$\therefore claim(2)\ is\ true$
$For\ claim(3)$
$if\ (i, d_i)\ is\ rejected$
$\Delta P = \Delta D = 0$
$else,\ \Delta P = 1$
$\Delta D = 1 + k_{j^*}/k_{j^*} = 2$
$\therefore in\ single\ iteration,\ \Delta D \leq 2\Delta P$
$\therefore claim(3)\ is\ true$
$Harkback$
$\because Initially,\ P = D = 0$
$\therefore At\ last,\ D \leq 2P$
$Let\ OPT\ be\ the\ optimal\ solution,\ we\ have\ P \leq OPT \leq D \leq 2P$
$\therefore P \geq \frac{1}{2} OPT$
$\therefore MRSF\ is\ a\ \frac{1}{2} - competitive\ algorithm\ for\ the\ online\ GPA\ program$