1) $P_{x_j|w_i} = \frac{1}{\Gamma(p_i)} \lambda_j^{p_i} x_j^{p_i - 1} e^{-\lambda_j x_j} \quad p_i, \lambda_j > 0$

a.

$$P(w_i | x_1 = x_1, \ldots, x_k = x_k) = \frac{P(x_1 = x_1, \ldots, x_k = x_k | w_i)}{P(x_1, x_2, w_j)}$$

$$\propto P(x_1 | w_i) P(x_2 | w_i) \cdots P(x_k | w_i) P(w_i)$$

$$= \prod_{j=1}^{k} \frac{1}{\Gamma(p_i)} \lambda_j^{p_i} x_j^{p_i - 1} e^{-\lambda_j x_j} P(w_i)$$

$$= \frac{P(w_i)}{\Gamma(p_i)} \prod_{j=1}^{k} (\lambda_j)^{p_i} (x_j)^{p_i - 1} e^{-\lambda_j x_j}$$

$$\frac{P(w_1)}{\Gamma(p_1)} \prod_{j=1}^{k} (\lambda_j)^{p_1} (x_j)^{p_1 - 1} \cancel{e^{-\lambda_j x_j}} > \frac{P(w_2)}{\Gamma(p_2)} \prod_{j=1}^{k} (\lambda_j)^{p_2} (x_j)^{p_2 - 1} \cancel{e^{-\lambda_j x_j}}$$

$$\boxed{\frac{P(w_1)}{\Gamma(p_1)} \prod_{j=1}^{k} (\lambda_j)^{p_1} (x_j)^{p_1 - 1} \overset{w_1}{>} \frac{P(w_2)}{\Gamma(p_2)} \prod_{j=1}^{k} (\lambda_j)^{p_2} (x_j)^{p_2 - 1}}$$

Bayes Optimal Classifier

b. Linear when $|p_2 - p_1| = 1$

c. $p_1 = 4$ $c = 2$ $\lambda_1 = \lambda_3 = 1$ $\underline{x} = (0.1, 0.2, 0.3, 4)$

$p_2 = 2$ $k = 4$ $\lambda_2 = \lambda_4 = 2$

$$\frac{\cancel{P(\omega_1)}}{\Gamma(p_1)} \prod_{j=1}^{k} (\lambda_j)^{p_1} (x_j)^{p_1 - 1} \overset{\omega_1}{>} \frac{\cancel{P(\omega_2)}}{\Gamma(p_2)} \prod_{j=1}^{k} (\lambda_j)^{p_2} (x_j)^{p_2 - 1}$$

$$\frac{1}{6} (4)^{\overset{2}{\cancel{4}}} \left[(0.1)(0.2)(0.3)(4)\right]^{\overset{2}{\cancel{3}}} \overset{\omega_1}{>} \frac{1}{1} \cancel{(4)^2} \cancel{\left[(0.1)(0.2)(0.3)(4)\right]^1}$$

$$\frac{16}{6} (0.024)^2 \underset{\underset{\omega_2}{<}}{\overset{\omega_1}{>}} 1 \rightarrow 0.0015 < 1 \rightarrow \omega_2$$

$$\boxed{C(\underline{x}) = 2}$$

d. $p_1 = 3.2$ $c = 2$ $\lambda_1 = 1$

$p_2 = 8$ $k = 1$

$$\frac{\cancel{P(\omega_1)}}{\Gamma(p_1)} \prod_{j=1}^{k} (\lambda_j)^{p_1} (x_j)^{p_1 - 1} \overset{\omega_1}{>} \frac{\cancel{P(\omega_2)}}{\Gamma(p_2)} \prod_{j=1}^{k} (\lambda_j)^{p_2} (x_j)^{p_2 - 1}$$

$$\frac{1}{\Gamma(3.2)} \cancel{(1)^{3.2}} x^{2.2} \overset{\omega_1}{>} \frac{1}{\Gamma(8)} \cancel{(1)^8} (x)^7$$

$$\frac{\Gamma(8)}{\Gamma(3.2)} = x^{4.8} \qquad x = \sqrt[4.8]{\frac{5040}{2.423}} = 4.912$$

$$\boxed{x^* = 4.912}$$

d. Type-1 Error (False Positive)

		GUESS	
		w_1	w_2
TRUE	w_1	correct	T 1
	w_2	T2	correct

$$T_1 = 1 - P(w_1 | x)$$ using zero-one loss function

$$= 1 - \frac{P(x_1|w_i) \cdots P(x_k|w_1)\, P(w_1)}{P(x}$$

$$= 1 - \frac{\frac{\cancel{P(w_1)}}{\Gamma(3.2)} x^{2.2}}{\frac{\cancel{P(w_1)}}{\Gamma(3.2)} x^{2.2} + \frac{\cancel{P(w_2)}}{\Gamma(8)} x^{7}} = 1 - \frac{1}{1 + \frac{\Gamma(3.2)}{\Gamma(8)} x^{4.8}}$$

$$= \frac{\Gamma(8)}{\Gamma(3.2)}$$

$$= 1 - \frac{1}{1+1} = \frac{1}{2}$$

$$\boxed{\text{T-1 Error} = \frac{1}{2}}$$

$$T_2 = 1 - P(w_2 | x)$$

$$= 1 - \frac{\frac{P(w_2)}{\Gamma(8)} x^{7}}{\frac{\cancel{P(w_1)}}{\Gamma(3.2)} x^{2.2} + \frac{\cancel{P(w_2)}}{\Gamma(8)} x^{7}} = 1 - \frac{1}{\frac{\Gamma(8)}{\Gamma(3.2)} x^{-4.8} + 1} = 1 - \frac{1}{2}$$

$$\boxed{\text{T-2 Error} = \frac{1}{2}}$$

e. $p_1 = p_2 = 4$ $\quad k = 2$

$c = 2$ $\quad \lambda_1 = 8$ $\quad \lambda_2 = 0.3$

$P(\omega_1) = 1/4$ $\quad P(\omega_2) = 3/4$

$$\frac{P(\omega_1)}{\cancel{\Gamma(p_1)}} \prod_{j=1}^{k} (\lambda_j)^{p_1} (x_j)^{p_1-1} = \frac{P(\omega_2)}{\cancel{\Gamma(p_2)}} \prod_{j=1}^{k} (\lambda_j)^{p_2} (x_j)^{p_2-1}$$

$$\left(\frac{1}{4}\right)\cancel{[(8)(0.3)]^4} (x_1 x_2)^3 = \frac{3}{4}\cancel{[(8)(0.3)]^4} (x_1 x_2)^3$$

$$2(x_1 x_2)^3 = 0$$

$$\boxed{f(x_1, x_2) = 0 \quad \text{when} \quad x_1 x_2 = 0}$$

2) $x_i | w_j \sim Lap(m_{ij}, \lambda_i)$

$$p(x_i|w_j) = \frac{\lambda_i}{2} e^{-\lambda_i |x_i - m_{ij}|}, \quad \lambda_i > 0 \qquad \begin{array}{l} i \in \{1 \dots k\} \\ j \in \{1 \dots c\} \end{array}$$

$$p(w_1) = p(w_2) = \dots = p(w_c)$$

$$p(w_i|x) \overset{w_i}{>} p(w_j|x) \quad \forall j \neq i$$

$$\frac{p(\underline{x}|w_i)p(w_i)}{\cancel{p(\underline{x})}} \overset{w_i}{>} \frac{p(\underline{x}|w_j)p(w_j)}{\cancel{p(\underline{x})}} \quad \forall j \neq i$$

$$p(w_i) \prod_{j=1}^{k} p(x_j|w_i) \overset{w_i}{>} p(w_\eta) \prod_{j=1}^{k} p(x_j|w_\eta) \quad \forall \eta \neq i$$

$$\cancel{p(w_i)} \prod_{j=1}^{k} \left[\frac{\lambda_j}{\cancel{2}} e^{-\lambda_j |x_j - m_{ji}|} \right] \overset{w_i}{>} \cancel{p(w_\eta)} \prod_{j=1}^{k} \left[\frac{\lambda_j}{\cancel{2}} e^{-\lambda_j |x_j - m_{j\eta}|} \right]$$

$$\sum_{j=1}^{k} -\lambda_j |x_j - m_{ji}| \cancelto{1}{\ln(e)} \overset{w_i}{>} \sum_{j=1}^{k} -\lambda_j |x_j - m_{j\eta}|$$

$$\boxed{\sum_{j=1}^{k} \left[\lambda_j \left(|x_j - m_{ji}| - |x_j - m_{j\eta}| \right) \right] \overset{w_i}{>} 0 \quad \forall \eta \neq i}$$

weighted Manhattan Distance classifier

= min Manhattan Distance classifier when $\boxed{m_{ji} = m_{j\eta}}$

3) a.

$$R(\alpha_i|x) = \sum_{j=1}^{4} \lambda(\alpha_i|\omega_j)p(\omega_j|x)$$

$$p(x_1) = p(x_1|\omega_1)p(\omega_1) + \cdots + p(x_1|\omega_4)p(\omega_4)$$

$$= \frac{1}{3}\left(\frac{1}{10}\right) + \frac{1}{2}\left(\frac{1}{5}\right) + \frac{1}{6}\left(\frac{1}{2}\right) + \frac{2}{5}\left(\frac{1}{5}\right)$$

$$p(x_1) = 0.2967$$

$$p(x_2) = \frac{1}{3}\left(\frac{1}{10}\right) + \frac{1}{4}\left(\frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{2}{5}\left(\frac{1}{5}\right)$$

$$p(x_2) = 0.33$$

$$p(x_3) = \frac{1}{3}\left(\frac{1}{10}\right) + \frac{1}{4}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{5}\left(\frac{1}{5}\right)$$

$$p(x_3) = 0.3733$$

$$R(\alpha_1|x_1) = \frac{1}{p(x_1)} \sum_{j=1}^{4} \left[\lambda(\alpha_i|\omega_j)p(x_i|\omega_j)p(\omega_j)\right]$$

$$= \frac{1}{0.2967} \; 0\cancel{\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)} + 2\left(\frac{1}{2}\right)\left(\frac{1}{5}\right) + 3\left(\frac{1}{6}\right)\left(\frac{1}{2}\right) + 4\left(\frac{2}{5}\right)\left(\frac{1}{5}\right)$$

$$= {}^{0.77}/_{0.2967} = 2.5952 = R(\alpha_1|x_1)$$

$R(\alpha_2|x_1) = 2.5506$

$R(\alpha_3|x_1) = 1.5506$ ← min for x_1

$R(\alpha_4|x_1) = 1.8539$

$R(\alpha_1|x_2) = 2.7879$

$R(\alpha_2|x_2) = 2.5455$

$R(\alpha_3|x_2) = \boxed{1.0909}$

$R(\alpha_4|x_2) = 1.4646$

$R(\alpha_1|x_3) = 2.7054$

$R(\alpha_2|x_3) = 1.6161$

$R(\alpha_3|x_3) = \boxed{0.7500}$

$R(\alpha_4|x_3) = 1.5179$

$R(\alpha_1|x_1) = 2.5955$

$R(\alpha_2|x_1) = 2.5506$

$R(\alpha_3|x_1) = \boxed{1.5506}$

$R(\alpha_4|x_1) = 1.8539$

b. $R = R(\alpha(x_1)|x_1)p(x_1) + \cdots + R(\alpha(x_3)|x_3)p(x_3)$

$= 1.5506(0.2967) + 1.0909(0.33) + 0.75(0.3733$

$\boxed{R = 1.1}$

4) ω_1 = evades taxes

ω_2 = doesn't evade taxes

a. $\boxed{p(\omega_1) = 3/10 \qquad p(\omega_2) = 7/10}$

b. Conditional Gaussianality

$$p(x|\omega_i) = N(\mu_i, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_i}{\sigma_i}\right)^2\right] \qquad x \in \mathbb{R}$$

$$\mu_1 = \frac{88+90+85}{3} = 87.67$$

$$\sigma_1^2 = \frac{\sum (x-\mu_i)^2}{n-1} = 6.33 \qquad \sigma_1 = 2.5166$$

$$\mu_2 = \frac{122+77+106+210+72+117+60}{7} = 109.143$$

$$\sigma_2^2 = 2539.5 \qquad \sigma_2 = 50.3932$$

$$p(x|\omega_1) = \frac{1}{6.3082} \exp\left[-\frac{1}{2}\left(\frac{x-87.67}{2.5166}\right)^2\right]$$

$$p(x|\omega_2) = \frac{1}{126.317} \exp\left[-\frac{1}{2}\left(\frac{x-109.143}{50.3932}\right)^2\right]$$

$x \rightarrow$ income

c. $x_1 \rightarrow$ {refund, no refund}

$P(x_1 = \text{refund} \mid w_1) = 0$ $\quad$ $P(x_1 = R \mid w_2) = 3/7$

$P(x_1 = NR \mid w_1) = 1$ $\quad$ $P(x_1 = NR \mid w_2) = 4/7$

$x_2 \rightarrow$ {Single, Married, Divorced}

$P(x_2 = S \mid w_1) = 2/3$ $\quad$ $P(x_2 = S \mid w_2) = 2/7$

$P(x_2 = M \mid w_1) = 0$ $\quad$ $P(x_2 = M \mid w_2) = 4/7$

$P(x_2 = D \mid w_1) = 1/3$ $\quad$ $P(x_2 = D \mid w_2) = 1/7$

The SLLN says that a certain amount of samples (n) is required to get an estimated distribution w/ confidence $1-\alpha$.

However, this is the most accurate estimate we can make w/o upscaling the 'evades taxes' group & downscaling the 'doesn't evade taxes' group.

d. Assuming these features are conditionally independent

$P(\underline{x} \mid w_i) = P(x_1 \mid w_i) P(x_2 \mid w_i) \cdots P(x_k \mid w_i) P(w_i)$

If one of these conditional probs $= 0$, then the entire expression is 0. Laplace correction corrects for this by not allowing each conditional prob to $= 0$.

e. Minimum Error Rate

$$P(\omega_1|x) \overset{\omega_1}{>} P(\omega_2|x)$$

$$\frac{\left[\prod_{i=1}^{k} P(x_i|\omega_1)\right]P(\omega_1)}{\cancel{P(x)}} \overset{\omega_1}{>} \frac{\left[\prod_{i=1}^{k} P(x_i|\omega_2)\right]P(\omega_2)}{\cancel{P(x)}}$$

$$P(\omega_1)\prod_{i=1}^{k} P(x_i|\omega_1) - P(\omega_2)\prod_{i=1}^{k} P(x_i|\omega_2) \overset{\omega_1}{>} 0$$

$$P(\omega_1)P(x_1|\omega_1)P(x_2|\omega_1)P(x_3|\omega_1) \overset{\omega_1}{>}$$

$$P(\omega_2)P(x_1|\omega_2)P(x_2|\omega_2)P(x_3|\omega_2)$$

where x_1 = refund, x_2 = Marital Status, x_3 = Income

$P(x_1 = R|\omega_1) = \frac{0+1}{3+2} = \frac{1}{5}$ $\quad P(x_1 = R|\omega_2) = \frac{3+1}{7+2} = \frac{4}{9}$

$P(x_1 = NR|\omega_1) = \frac{3+1}{3+2} = \frac{4}{5}$ $\quad P(x_1 = NR|\omega_2) = \frac{4+1}{7+2} = \frac{5}{9}$

$P(x_2 = S|\omega_1) = \frac{2+1}{3+3} = \frac{1}{2}$ $\quad P(x_2 = S|\omega_2) = \frac{2+1}{7+3} = \frac{3}{10}$

$P(x_2 = M|\omega_1) = \frac{0+1}{3+3} = \frac{1}{6}$ $\quad P(x_2 = M|\omega_2) = \frac{4+1}{7+3} = \frac{1}{2}$

$P(x_2 = D|\omega_1) = \frac{1+1}{3+3} = \frac{1}{3}$ $\quad P(x_2 = D|\omega_2) = \frac{1+1}{7+3} = \frac{1}{5}$

$$P(x_3|\omega_1) = \frac{1}{6.3082}\exp\left[-\frac{1}{2}\left(\frac{x-87.67}{2.5166}\right)^2\right]$$

$$P(x_3|\omega_2) = \frac{1}{126.317}\exp\left[-\frac{1}{2}\left(\frac{x-109.143}{50.3932}\right)^2\right]$$

5) Notes:

Gaussian class condition

$$N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$