6.3 Gauss‘ Arguments for Least Squares Adjustment

Technische Universität Berlin

6.3.1 Gauss‘ First Argument

▶ We assume that our measurements

- Contain <u>no</u> systematic deviations
- Contain <u>no</u> blunders
- Density function of observations is known → Gaussian or Normal distribution

‖ Under the assumption of normal distribution we want to obtain the most probable solution ‖

▶ If we assume normally distributed measurements
→ random deviations ε_i are also normally distributed with density function:

$$f(\varepsilon_i) = \frac{1}{\sigma_i \cdot \sqrt{2\pi}} \mathrm{e}^{\left(-\frac{1}{2} \cdot \frac{\varepsilon_i^2}{\sigma_i^2}\right)}$$

6.3.1 Gauss' First Argument

▶ The same distribution can be applied for the case that we consider the empirical residuals v_i

$$f(v_i) = \frac{1}{\sigma_i \cdot \sqrt{2\pi}} \mathrm{e}^{\left(-\frac{1}{2} \cdot \frac{v_i^2}{\sigma_i^2}\right)}$$

▶ Now, if we consider the joint occurrence of all residuals we obtain the overall probability density from:

$$\Omega = \frac{1}{\sigma_1 \cdot \sqrt{2\pi}} \mathrm{e}^{\left(-\frac{1}{2} \cdot \frac{v_1^2}{\sigma_1^2}\right)} \cdot \frac{1}{\sigma_2 \cdot \sqrt{2\pi}} \mathrm{e}^{\left(-\frac{1}{2} \cdot \frac{v_2^2}{\sigma_2^2}\right)} \cdot \ldots \cdot \frac{1}{\sigma_n \cdot \sqrt{2\pi}} \mathrm{e}^{\left(-\frac{1}{2} \cdot \frac{v_n^2}{\sigma_n^2}\right)}$$

Probability density of

$$\Omega = \left(\prod_{i=1}^{n} \frac{1}{\sigma_i \cdot \sqrt{2\pi}}\right) \mathrm{e}^{(-K)} \qquad \text{with } K = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\sigma_i^2} v_i^2$$

$$2^1 \cdot 2^2 = 2^{(1+2)} = 8$$

$$2 \cdot 4 = 8$$

$$- = \frac{1}{K}$$

6.3.1 Gauss‘ First Argument

$\frac{1}{\sqrt{2}}$

Wanted: Maximum of density function K

We search for values v_i that yield maximum Ω → That is the case if K obtains a minimum value

$$\sum_{i=1}^{n} \frac{1}{\sigma_i^2} v_i^2 \rightarrow \min \quad \text{with} \quad p_i = \frac{1}{\sigma_i^2} \quad \text{“weight of observations } l_i\text{”}$$

Probability density of every point x* then

Weight matrix $\mathbf{P}$, here diagonal matrix $\mathbf{P} = \begin{bmatrix} p_{11} & & 0 \\ & \ddots & \\ 0 & & p_{nn} \end{bmatrix}$

$$\boxed{\sum_{i=1}^{n} p_i \, v_i^2 \rightarrow \min}$$

In matrix notation:

$$\boxed{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \rightarrow \min}$$

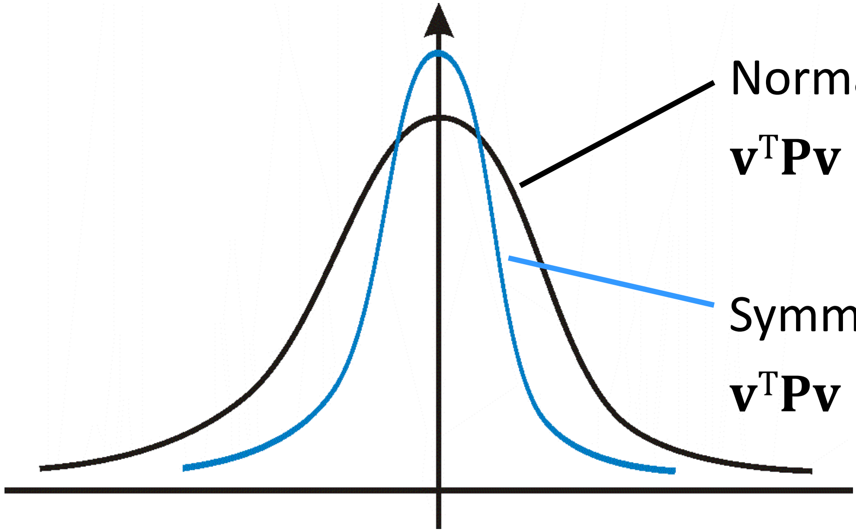
We want to Maximize the PDF

6.3.2 Gauss‘ Second Argument

Gauss was not satisfied with his first argument

- In his “Theoria Combinationes” he did <u>not</u> apply the <u>normal distribution</u>
- He has shown that the method of least squares yields the smallest standard deviations if the frequency function $f(\varepsilon)$ is only symmetric to $\varepsilon = 0$.

  But: In this case, the results are no longer the most probable solution!
  Results are referred to instead as the “most plausible” or “most appropriate”



Normal distribution:

$\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}$ → min yields “most probable” solution

Symmetric frequency function ≠ normal dist. :

$\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}$ → min yields “most plausible” solution

6.3.3 Extension to correlated observations

Technische Universität Berlin

Until now we have considered independent observations, precision represented by a diagonal matrix

$\sigma_1 = 1\,mm$

$\sigma_2 = 2\,mm$

$\Sigma_{ll} = \begin{bmatrix} (1)^2 & \\ & (2)^2 \end{bmatrix}$ with $\sigma_0 = 1$

$$\mathbf{P}_{n\times n} = \begin{bmatrix} p_{11} & & & 0 \\ & p_{22} & & \\ & & \ddots & \\ 0 & & & p_{nn} \end{bmatrix}$$

$\Sigma_{ll} = \begin{bmatrix} \sigma_T & 0 & 0 \\ 0 & \sigma_T & 0 \\ & & \sigma_0 \\ 0 & & \sigma_0 \\ 0 & & \end{bmatrix}$

$P = Q_{ll}^{-1} = \begin{bmatrix} \left(\frac{1}{(1)^2}\right) 1 & \\ & \left(\frac{1}{(2)^2}\right) \end{bmatrix} \rightarrow \frac{1}{4} = 0.25$

Gauss' derivations have been extended by e.g. Helmert, Tienstra and others:

1. Instead of original observations (e.g. horizontal directions) we can introduce derived observations (angles) if we consider the correlations.

2. We want to consider observations with known mathematical or physical correlations

$P = Q_{ll}^{-1}$ = weight Matrix

$= Q_{ll} = \frac{1}{\sigma_0^2} \cdot \Sigma_{ll}$

6.3.3 Extension to correlated observations

Generalisation:

P is a full Mattrix

$$\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{v} \rightarrow \min$$

with $$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

6.3.3 Extension to correlated observations

For full Matrix

We know: $\mathbf{Q}$ is symmetric and non-singular → $\mathbf{P}$ is symmetric and non-singular

→ we can apply Cholesky decomposition: $\mathbf{P} = \mathbf{C}^{\mathrm{T}}\mathbf{C}$

$\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}$ with $\mathbf{v} = \hat{\mathbf{l}} - \mathbf{l}$

$$
\begin{aligned}
\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} &= (\hat{\mathbf{l}} - \mathbf{l})^{\mathrm{T}}\mathbf{P}\,(\hat{\mathbf{l}} - \mathbf{l}) && \text{with} \quad \mathbf{P} = \mathbf{C}^{\mathrm{T}}\mathbf{C} \\
&= (\hat{\mathbf{l}} - \mathbf{l})^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\,(\hat{\mathbf{l}} - \mathbf{l}) \\
&= (\hat{\mathbf{l}}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} - \mathbf{l}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}) \cdot (\mathbf{C}\hat{\mathbf{l}} - \mathbf{C}\mathbf{l}) && \text{with} \quad \mathbf{l}' = \mathbf{C}\mathbf{l} \text{ and } \quad \hat{\mathbf{l}}' = \mathbf{C}\hat{\mathbf{l}} \\
&= (\hat{\mathbf{l}}' - \mathbf{l}')^{\mathrm{T}} \cdot (\hat{\mathbf{l}}' - \mathbf{l}') && \text{with} \quad \mathbf{v}' = \hat{\mathbf{l}}' - \mathbf{l}' \\
&= (\mathbf{v}')^{\mathrm{T}} \cdot (\mathbf{v}') && \text{equally weighted}
\end{aligned}
$$

$\boxed{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} = (\mathbf{v}')^{\mathrm{T}} \cdot (\mathbf{v}')}$ $\quad (\mathbf{v}')^{\mathrm{T}} \cdot (\mathbf{v}') \rightarrow \min$ could be solved via Gauss

6.3.3 Extension to correlated observations

→ Adjustment with $\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \rightarrow \min$ yields same result as $(\mathbf{v}')^{\mathrm{T}} \cdot (\mathbf{v}')$

→ We can in general introduce correlated observations

by introducing $\mathbf{P} = \mathbf{Q}^{-1}$ as weight matrix

$$\underset{n\times n}{\mathbf{P}} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

6.4 Functional Model and Stochastic Model

Technische Universität Berlin

6.4.1 Functional model

- Functional model in adjustment computation is a set of equations that represents an adjustment condition
- If the functional model represents the geometrical or physical situation adequately, observation errors can be expected to conform to the normal distribution
- Example: Functional model $\alpha + \beta + \gamma = 200 \text{ gon}$ in a triangle.
  But: This model is only adequate if the survey is limited to a small region.
  Large areas: Spherical excess has to be considered.

→ If the functional model does not fit the geometrical or physical situation, an incorrect adjustment will result!

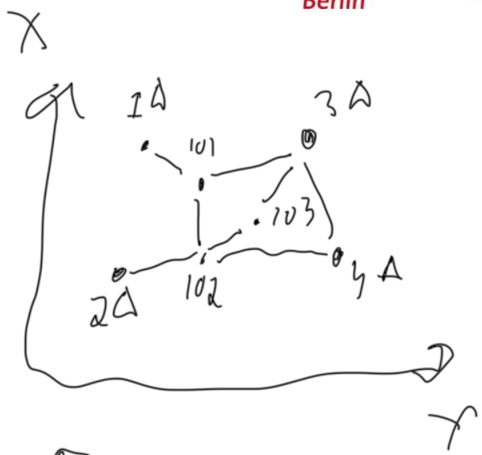
6.4.1 Functional Model

There are two basic forms for functional models:

▶ <u>Conditional adjustment</u> $\mathbf{\Phi}(\mathbf{L}) = \mathbf{0}$

▶ <u>Parametric adjustment</u> $\mathbf{L} = \mathbf{\Phi}(\mathbf{X})$

Gauss-Markov-Model



when its so hard for computer

▶ Conditional adjustment:

Geometric conditions are enforced on the observations and their residuals, e.g. $\alpha + v_\alpha + \beta + v_\beta + \gamma + v_\gamma = 200$ gon

- Advantage: Small equation systems
- Disadvantage: Often difficult and time consuming to find conditions, e.g. in complicated networks

→ Not well suited to computers

6.4.1 Functional Model

▶ Parametric adjustment:

Observations are expressed as functions of unknown parameters, e.g.

$$s_{ij} + v_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

- Disadvantage: large equation systems
- Advantages:
  - "Standard" formulas can be applied
  - Well suited to computers

→ Of course, conditional and parametric adjustment yield same results.

This semester: parametric adjustment

Next semester: conditional adjustment

6.4.2 Stochastic Model

- ▶ Determination of variances and, subsequently, the weights of the observations is know as stochastic model in least squares adjustment

- ▶ Weight of an observation controls the amount of correction it receives during the adjustment → It is very important to select a proper stochastic (weight) model!

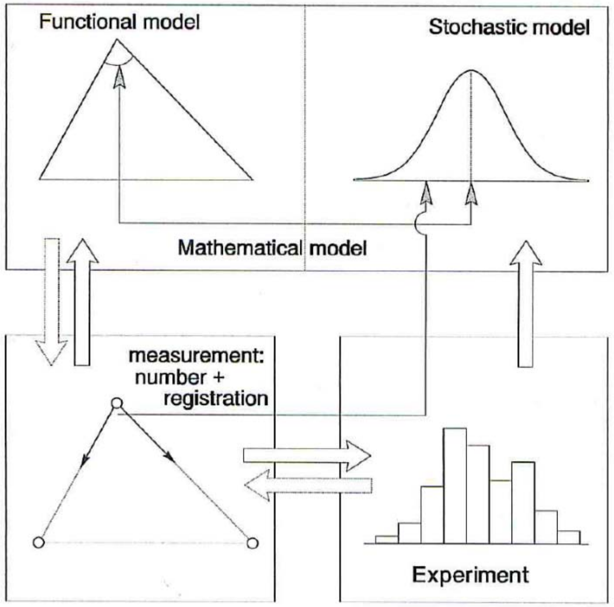
- ▶ Attention: When doing an “unweighted” adjustment (= all obs. have the same weight) then the stochastic model is created implicitly

- ▶ Failures to select the stochastic model properly will
  - Influence the adjusted parameters
  - Affect the ability to isolate blunders

→ Combination of functional and stochastic models is called the mathematical model

6.4.2 Stochastic Model

Diagram of the fundamental relations in adjustment theory



Both, stochastic and functional model must be correct if the adjustment is to yield the most probable values!

6.5 Observation Equations

- ▶ Equations that relate observed quantities (measurements) to both observational residuals and unknown parameters are called observation equations

- ▶ Functional model, e.g.

For adjust we have to introduce residuals to observations, then we get observation equation

x, y fixed

$$\underbrace{L_1}_{} = x + y \qquad (1)$$

$$L_2 = 2x - y \qquad (2)$$

$$\underbrace{L_3}_{n \text{ obs.}} = \underbrace{x - y}_{u \text{ unknowns}} \qquad (3)$$

$n = 3 \quad u = 2 \quad \rightarrow \quad n > u$ = over determined config

6.5 Observation Equations

▶ <u>If $n > u$</u> functional model can only be fulfilled by “true values”

$$"\sim" = \text{true value}$$

$$\tilde{L}_1 = \tilde{x} + \tilde{y}$$

$$\tilde{L}_2 = 2\tilde{x} - \tilde{y}$$

$$\tilde{L}_3 = \tilde{x} - \tilde{y}$$

6.5 Observation Equations

- Problem:

  We don't know the true values

  → Functional model has usually <u>no</u> solution

- Solution:

  We introduce residuals for the observations
  →Resulting set of equations is called

  ^ = adjusted param

  <u>Observation equations</u> (residual equations)

$$L_1 + v_1 = \hat{x} + \hat{y}$$

$$L_2 + v_2 = 2\hat{x} - \hat{y}$$

$$L_3 + v_3 = \hat{x} - \hat{y}$$

with $\hat{x}, \hat{y}$ = adjusted parameters

6.6 Least squares adjustment without matrix notation

- **Problem:**

  This equation system has no unique solution

- **Solution:**

  We introduce a target function for the residuals

$$\sum_{i=1}^{n} p_i v_i^2$$

  and we search for a solution with

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

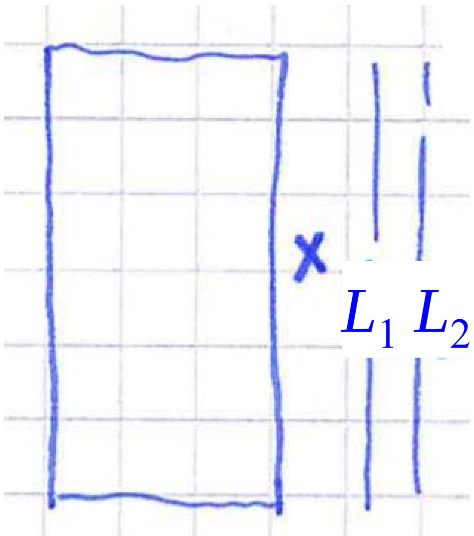
  → Least squares adjustment

Example:

Length of the classroom has been measured n-times with same precision

▶ Given: obs. $L_1, L_2, \ldots, L_n$, equally weighted

▶ Wanted: Adjusted unknown $\hat{x}$



Functional model:

$$\begin{aligned} L_1 &= x \\ L_2 &= x \\ &\vdots \\ L_n &= x \end{aligned}$$

mieght

Stochastic model:

$$p_1 = p_2 = \cdots = p_n = 1$$

6.6 Least squares adjustment without matrix notation

Observation equations:

$$
\begin{aligned}
L_1 + v_1 &= \hat{x} \\
L_2 + v_2 &= \hat{x} \\
&\vdots \\
L_n + v_n &= \hat{x}
\end{aligned}
$$

Rearranging:

$$
\begin{aligned}
v_1 &= \hat{x} - L_1 \\
v_2 &= \hat{x} - L_2 \\
&\vdots \\
v_n &= \hat{x} - L_n
\end{aligned}
$$

$$\sum_{i=1}^{n} p_i v_i^2 \to \min$$

$$\rightarrow \underbrace{1}_{p_1} \cdot \underbrace{(\hat{x} - L_1)^2}_{v_1^2} + \underbrace{1}_{p_2} \cdot \underbrace{(\hat{x} - L_2)^2}_{v_2^2} + \cdots + \underbrace{1}_{p_n} \cdot \underbrace{(\hat{x} - L_n)^2}_{v_n^2} \to \min$$

6.6 Least squares adjustment without matrix notation

How to obtain the min?

▶ Minimum value of a function can be found by taking its first derivative

▶ Equate the resulting function with zero

→ Taking first derivative with respect to x and setting the resulting function equal to zero yields

$$\Sigma p_i v_i^2 = 1 \cdot (\hat{x} - L_1)^2 + 1 \cdot (\hat{x} - L_2)^2 + \cdots + 1 \cdot (\hat{x} - L_n)^2$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial \hat{x}} = \underbrace{2 \cdot (\hat{x} - L_1) + 2 \cdot (\hat{x} - L_2) + \cdots + 2 \cdot (\hat{x} - L_n) = 0}_{\text{normal equation}}$$

6.6 Least squares adjustment without matrix notation

Technische Universität Berlin

Solution of normal equation:

$$(\hat{x} - L_1) + (\hat{x} - L_2) + \cdots + (\hat{x} - L_n) = 0$$

$$n \cdot \hat{x} = L_1 + L_2 + \cdots + L_n$$

Adjusted unknown:

arithmetic mean

$$\boxed{\hat{x} = \frac{L_1 + L_2 + \cdots + L_n}{n}}$$

$$\boxed{\hat{x} = \frac{\Sigma L_i}{n}}$$

6.6 Least squares adjustment without matrix notation

Example:

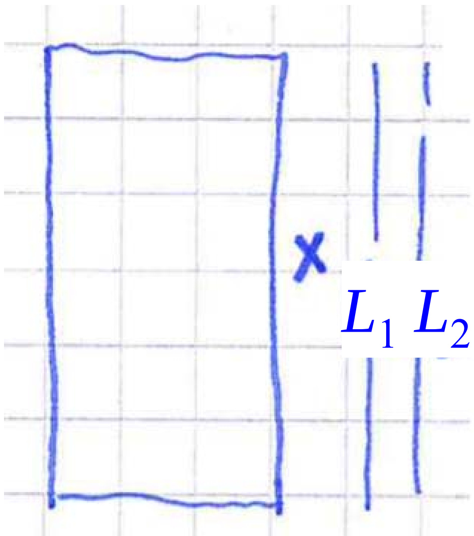
Length of the classroom has been measured n-times with different precision (standard deviation) σ_i

▶ Given: L_1, σ_1

L_2, σ_2

$\vdots$

L_n, σ_n



▶ Wanted: Adjusted unknown $\hat{x}$

6.6 Least squares adjustment without matrix notation

Technische Universität Berlin

Functional model:

$$
\begin{aligned}
L_1 &= x \\
L_2 &= x \\
&\vdots \\
L_n &= x
\end{aligned}
$$

Stochastic model:

$$
p_1 = \left(\frac{1}{\sigma_1}\right)^2, p_2 = \left(\frac{1}{\sigma_2}\right)^2, \ldots, p_n = \left(\frac{1}{\sigma_n}\right)^2
$$

weights STD

6.6 Least squares adjustment without matrix notation

Technische Universität Berlin

Observation equations:

$$
\begin{aligned}
L_1 + v_1 &= \hat{x} \\
L_2 + v_2 &= \hat{x} \\
&\vdots \\
L_n + v_n &= \hat{x}
\end{aligned}
$$

Rearranging:

$$
\begin{aligned}
v_1 &= \hat{x} - L_1 \\
v_2 &= \hat{x} - L_2 \\
&\vdots \\
v_n &= \hat{x} - L_n
\end{aligned}
$$

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

$$\rightarrow \quad p_1(\hat{x} - L_1)^2 + p_2(\hat{x} - L_2)^2 + \cdots + p_n(\hat{x} - L_n)^2 \rightarrow \min$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial \hat{x}} = \underbrace{2p_1(\hat{x} - L_1) + 2p_2(\hat{x} - L_2) + \cdots + 2p_n(\hat{x} - L_n) = 0}_{\text{normal equation}} \underbrace{= 0 \quad | \cdot 2}_{?}$$

Technische Universität Berlin

Solution of normal equation:

$$p_1(\hat{x} - L_1) + p_2(\hat{x} - L_2) + \cdots + p_n(\hat{x} - L_n) = 0$$

$$p_1\left(\hat{x} - p_1 L_1\right) + p_2\left(\hat{x} - p_2 L_2\right) + \cdots + p_n\left(\hat{x} - p_n L_n\right) = 0$$

$$\hat{x} \cdot \Sigma p_i = \Sigma p_i L_i$$

Modle
Atg. II
geodätisch

60% 6E
80% 3E

weight mean $= \dfrac{6 \cdot 60\% \quad 3 \cdot 80\%}{6 - 3}$

Adjusted unknown:

– weight arithmetic mean

$$\boxed{\hat{x} = \frac{\Sigma p_i L_i}{\Sigma p_i}}$$

6.6 Least squares adjustment without matrix notation

Example:

Functional model:

$$\begin{aligned} 3.0 &= x + y \\ 1.5 &= 2x - y \\ 0.2 &= x - y \end{aligned}$$

Values	$3.0, 1.5, 0.2$	are observations
Parameters	x, y	are unknowns

Stochastic model for the observations:

$$p_1 = 1, \qquad p_2 = 1, \qquad p_3 = 1$$

6.6 Least squares adjustment without matrix notation

Observation equations:

$$3.0 + v_1 = \hat{x} + \hat{y}$$
$$1.5 + v_2 = 2\hat{x} - \hat{y}$$
$$0.2 + v_3 = \hat{x} - \hat{y}$$

Rearranging:

$$v_1 = \hat{x} + \hat{y} - 3.0$$
$$v_2 = 2\hat{x} - \hat{y} - 1.5$$
$$v_3 = \hat{x} - \hat{y} - 0.2$$

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

→ $1 \cdot (\hat{x} + \hat{y} - 3.0)^2 + 1 \cdot (2\hat{x} - \hat{y} - 1.5)^2 + 1 \cdot (\hat{x} - \hat{y} - 0.2)^2 \rightarrow \min$

Technische Universität Berlin

Normal equations:

$$\Sigma p_i v_i^2 = 1 \cdot (\hat{x} + \hat{y} - 3.0)^2 + 1 \cdot (2\hat{x} - \hat{y} - 1.5)^2 + 1 \cdot (\hat{x} - \hat{y} - 0.2)^2$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial \hat{x}} = 2(\hat{x} + \hat{y} - 3.0) + 2 \cdot (2\hat{x} - \hat{y} - 1.5) \cdot 2 + 2 \cdot (\hat{x} - \hat{y} - 0.2) = 0$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial \hat{y}} = 2(\hat{x} + \hat{y} - 3.0) + 2 \cdot (2\hat{x} - \hat{y} - 1.5) \cdot (-1) + 2 \cdot (\hat{x} - \hat{y} - 0.2) \cdot (-1) = 0$$

<u>normal equations</u>

$$\hat{x} + \hat{y} - 3.0 + 4\hat{x} - 2\hat{y} - 3.0 + \hat{x} - \hat{y} - 0.2 = 0$$

$$\hat{x} + \hat{y} - 3.0 - 2\hat{x} + \hat{y} + 1.5 - \hat{x} + \hat{y} + 0.2 = 0$$

$$6\hat{x} - 2\hat{y} = 6.2 \qquad (1)$$

$$-2\hat{x} + 3\hat{y} = 1.3 \qquad (2)$$

6.6 Least squares adjustment without matrix notation

Solution of normal equations:

$(1) + 3 \cdot (2)$: $\quad 7\hat{y} = 10.1 \quad \Rightarrow \quad \hat{y} = 1.443$

$\hat{y}$ in (1): $\quad \hat{x} = \frac{6.2 + 2\hat{y}}{6} \quad \Rightarrow \quad \hat{x} = 1.514$

Residuals:

Can be computed from observation equations

$$\begin{aligned} v_1 &= -0.044 \\ v_2 &= 0.085 \\ v_3 &= -0.128 \end{aligned}$$

6.6 Least squares adjustment without matrix notation

Same example, but now we know the precision (standard deviation) s_i of the measured values

$$3.0, \qquad s_1 = 4\ \text{cm}$$
$$1.5, \qquad s_2 = 2\ \text{cm}$$
$$0.2, \qquad s_3 = 1\ \text{cm}$$

Stochastic model:

How to obtain weights p_1, p_2, p_3?

$$p_1 = \frac{1}{(s_1)^2} \qquad p_2 = \frac{1}{(s_2)^2} \qquad p_3 = \frac{1}{(s_3)^2}$$

$$p_1 = \frac{1}{16} \qquad p_2 = \frac{1}{4} \qquad p_3 = 1$$

6.6 Least squares adjustment without matrix notation

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

$$\rightarrow \quad \frac{1}{16} \cdot (\hat{x} + \hat{y} - 3.0)^2 + \frac{1}{4} \cdot (2\hat{x} - \hat{y} - 1.5)^2 + 1 \cdot (\hat{x} - \hat{y} - 0.2)^2 \rightarrow \min$$

$$\frac{\Sigma p_i v_i^2}{\partial \hat{x}} = 2 \cdot \frac{1}{16}(\hat{x} + \hat{y} - 3.0) + 2 \cdot \frac{1}{4}(2\hat{x} - \hat{y} - 1.5) \cdot 2 + 2 \cdot (\hat{x} - \hat{y} - 0.2) = 0$$

$$\frac{\Sigma p_i v_i^2}{\partial \hat{y}} = 2 \cdot \frac{1}{16}(\hat{x} + \hat{y} - 3.0) + 2 \cdot \frac{1}{4}(2\hat{x} - \hat{y} - 1.5)(-1) + 2 \cdot (\hat{x} - \hat{y} - 0.2)(-1) = 0$$

6.6 Least squares adjustment without matrix notation

$$\hat{x} + \hat{y} - 3.0 \; + 16\hat{x} - 8\hat{y} - 12 + 16\hat{x} - 16\hat{y} - 3.2 = 0$$

$$\hat{x} + \hat{y} - 3.0 \; + 8\hat{x} + 4\hat{y} - 6 - 16\hat{x} + 16\hat{y} + 3.2 = 0$$

$$33\hat{x} - 23\hat{y} = 18.2 \qquad (1)$$

$$-23\hat{x} + 21\hat{y} = -6.2 \qquad (2)$$

Solution of normal equations:

$21 \cdot (1) + 23 \cdot (2)$: $\quad (21 \cdot 33 - 23 \cdot 23)\hat{x} = (21 \cdot 18.2) - (23 \cdot 6.2)$

$$164\,\hat{x} = 239.6$$

$$\hat{x} = 1.4610$$

$\hat{x}$ in (1): $\quad \hat{y} = 1.3049$

Residuals:

Solution $\hat{x}, \hat{y}$ in observation equations → v_i

$$\hat{x} + \hat{y} - 3.0 \; + 16\hat{x} - 8\hat{y} - 12 + 16\hat{x} - 16\hat{y} - 3.2 = 0$$

$$\hat{x} + \hat{y} - 3.0 \; + 8\hat{x} + 4\hat{y} - 6 - 16\hat{x} + 16\hat{y} + 3.2 = 0$$

$$33\hat{x} - 23\hat{y} = 18.2 \qquad (1)$$

$$-23\hat{x} + 21\hat{y} = -6.2 \qquad (2)$$

Solution of normal equations:

$$\underbrace{\begin{bmatrix} 33 & -23 \\ -23 & 21 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}}_{\hat{\mathbf{X}}} = \underbrace{\begin{bmatrix} 18.2 \\ -6.2 \end{bmatrix}}_{\mathbf{n}}$$

$$\mathbf{N}\hat{\mathbf{X}} = \mathbf{n}$$

$$\hat{\mathbf{X}} = \mathbf{N}^{-1}\mathbf{n}$$