Least-squares Adjustment for Nonlinear Adjustment Problems

- Iterative solution with linearized functional model -

Nonlinear functional model for the unknowns:

$$\begin{aligned} L_1 &= \varphi_1(X_1, X_2, \cdots, X_u) \\ L_2 &= \varphi_2(X_1, X_2, \cdots, X_u) \\ \vdots \quad & \quad \vdots \\ L_n &= \varphi_n(X_1, X_2, \cdots, X_u) \end{aligned}$$

Nonlinear functional model in matrix notation:

$$\underset{n,1}{\mathbf{L}} = \underset{n,1}{\mathbf{\Phi}}\left(\underset{u,1}{\mathbf{X}}\right) = \begin{bmatrix} \varphi_1\left(\underset{u,1}{\mathbf{X}}\right) \\ \varphi_2\left(\underset{u,1}{\mathbf{X}}\right) \\ \vdots \\ \varphi_n\left(\underset{u,1}{\mathbf{X}}\right) \end{bmatrix}$$

Vector of observations:

$$\underset{n,1}{\mathbf{L}} = \begin{bmatrix} L_1 & L_2 & \cdots & L_n \end{bmatrix}^{\mathrm{T}}$$

Variance covariance matrix of the observations:

$$\underset{n,n}{\mathbf{\Sigma}_{\mathbf{LL}}} = \begin{bmatrix} \sigma_{L_1}^2 & \sigma_{L_1L_2} & \cdots & \sigma_{L_1L_n} \\ \sigma_{L_2L_1} & \sigma_{L_2}^2 & \cdots & \sigma_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_nL_1} & \sigma_{L_nL_2} & \cdots & \sigma_{L_n}^2 \end{bmatrix}$$ with theoretical values σ_i

$$\underset{n,n}{\mathbf{S}_{\mathbf{LL}}} = \begin{bmatrix} s_{L_1}^2 & s_{L_1L_2} & \cdots & s_{L_1L_n} \\ s_{L_2L_1} & s_{L_2}^2 & \cdots & s_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{L_nL_1} & s_{L_nL_2} & \cdots & s_{L_n}^2 \end{bmatrix}$$ with empirical values s_i

VCM of the reduced observations from variance covariance propagation with the functional model $\mathbf{l} = \mathbf{L} - \mathbf{L}^0$

$$\underset{n,n}{\mathbf{\Sigma}_{\mathbf{ll}}} = \underset{n,n}{\mathbf{\Sigma}_{\mathbf{LL}}}$$ with theoretical values σ_i

$$\underset{n,n}{\mathbf{S}_{\mathbf{ll}}} = \underset{n,n}{\mathbf{S}_{\mathbf{LL}}}$$ with empirical values s_i

Theoretical reference standard deviation: σ_0 (or theoretical reference variance σ_0^2)

Cofactor matrix of the observations and reduced observations:

$$\underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} = \frac{1}{\sigma_0^2}\underset{n,n}{\mathbf{\Sigma}_{\mathbf{LL}}}$$ respectively $$\underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} = \frac{1}{\sigma_0^2}\underset{n,n}{\mathbf{S}_{\mathbf{LL}}}$$

Weight matrix of the observations and reduced observations:

$$\underset{n,n}{\mathbf{P}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}^{-1}}$$

Vector of adjusted unknowns:

$$\underset{u,1}{\hat{\mathbf{X}}} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \cdots & \hat{X}_u \end{bmatrix}^{\mathrm{T}}$$

Vector of $\left\{\begin{array}{l}\text{initial values}\\ \text{starting values}\\ \text{approximations}\end{array}\right.$ for the unknowns:

$$\underset{u,1}{\mathbf{X}^0} = \begin{bmatrix} X_1^0 & X_2^0 & \cdots & X_u^0 \end{bmatrix}^{\mathrm{T}}$$

Vector of adjusted reduced unknowns:

$$\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,1}{\hat{\mathbf{X}}} - \underset{u,1}{\mathbf{X}^0}$$

→ Observations as functions of the approximations for the unknowns: $\underset{n,1}{\mathbf{L}^0} = \underset{n,1}{\mathbf{\Phi}}\left(\underset{u,1}{\mathbf{X}^0}\right)$

Vector of reduced observations: $\underset{n,1}{\mathbf{l}} = \underset{n,1}{\mathbf{L}} - \underset{n,1}{\mathbf{L}^0}$

Jacobian matrix:

$$\underset{n,u}{\mathbf{J}} = \left(\frac{\partial \mathbf{\Phi}(\mathbf{X})}{\partial \mathbf{X}}\right)_{\mathbf{X}=\mathbf{X}^0} = \begin{bmatrix} \frac{\partial \varphi_1(\mathbf{X})}{\partial X_1} & \frac{\partial \varphi_1(\mathbf{X})}{\partial X_2} & \cdots & \frac{\partial \varphi_1(\mathbf{X})}{\partial X_u} \\ \frac{\partial \varphi_2(\mathbf{X})}{\partial X_1} & \frac{\partial \varphi_2(\mathbf{X})}{\partial X_2} & \cdots & \frac{\partial \varphi_2(\mathbf{X})}{\partial X_u} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_n(\mathbf{X})}{\partial X_1} & \frac{\partial \varphi_n(\mathbf{X})}{\partial X_2} & \cdots & \frac{\partial \varphi_n(\mathbf{X})}{\partial X_u} \end{bmatrix}_{\mathbf{X}=\mathbf{X}^0}$$

Coefficient matrix of the linearized functional model: $\underset{n,u}{\mathbf{A}} = \underset{n,u}{\mathbf{J}}$ "Design Matrix"

Vector of residuals: $\underset{n,1}{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{\mathrm{T}}$

Observation equations: $\underset{n,1}{\mathbf{l}} + \underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{x}}}$

Normal equations: $\underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{l}}$

Normal matrix: $\underset{u,u}{\mathbf{N}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}$

Right hand side of normal equations: $\underset{u,1}{\mathbf{n}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{l}}$

Normal equations: $\underset{u,u}{\mathbf{N}}\,\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,1}{\mathbf{n}}$

Inversion of normal matrix: $\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}} = \underset{u,u}{\mathbf{N}^{-1}}$

Solution of normal equations: $\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}}\,\underset{u,1}{\mathbf{n}}$

Solution for the unknowns: $\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,1}{\hat{\mathbf{X}}^0} + \underset{u,1}{\hat{\mathbf{x}}}$

Vector of residuals: $\underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{x}}} - \underset{n,1}{\mathbf{l}}$

Vector of adjusted observations: $\underset{n,1}{\hat{\mathbf{L}}} = \underset{n,1}{\mathbf{L}} + \underset{n,1}{\mathbf{v}}$

Check 1: $\max\left|\hat{x}_i\right| \le \varepsilon \quad \forall\, i = 1,\ldots,u$ MatLab: `max(abs(x_hat))`

Check 2: $\max\left|\hat{L}_i - \varphi_i\left(\hat{\mathbf{X}}\right)\right| \le \delta \quad \forall\, i = 1,\ldots,n$

If $\left\{ \left(\max\left|\hat{x}_i\right| \le \varepsilon \quad \forall\, i=1,\ldots,u\right) \wedge \left(\max\left|\hat{L}_i - \varphi_i\left(\hat{\mathbf{X}}\right)\right| \le \delta \quad \forall\, i=1,\ldots,n\right) \right\}$

$\hat{\mathbf{X}}$ is the solution for the nonlinear adjustment problem

Else

Use $\hat{\mathbf{X}}$ as new approximation for the unknowns $\mathbf{X}^0$ and continue with step "→"

Empirical reference standard deviation: $s_0 = \sqrt{\dfrac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$ (or empirical reference variance s_0^2)

Cofactor matrix of adjusted unknowns from variance covariance propagation with the functional model $\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$: $\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}}$

VCM of adjusted unknowns: $\underset{u,u}{\mathbf{\Sigma}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = s_0^2 \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}$

Cofactor matrix of adjusted observations: $\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = \underset{n,u}{\mathbf{A}} \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} \underset{u,n}{\mathbf{A}^{\mathrm{T}}}$

VCM of adjusted observations: $\underset{n,n}{\mathbf{\Sigma}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = s_0^2 \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$

Cofactor matrix of the residuals: $\underset{n,n}{\mathbf{Q}_{\mathbf{vv}}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} - \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$

VCM of the residuals: $\underset{n,n}{\mathbf{\Sigma}_{\mathbf{vv}}} = s_0^2 \underset{n,n}{\mathbf{Q}_{\mathbf{vv}}}$