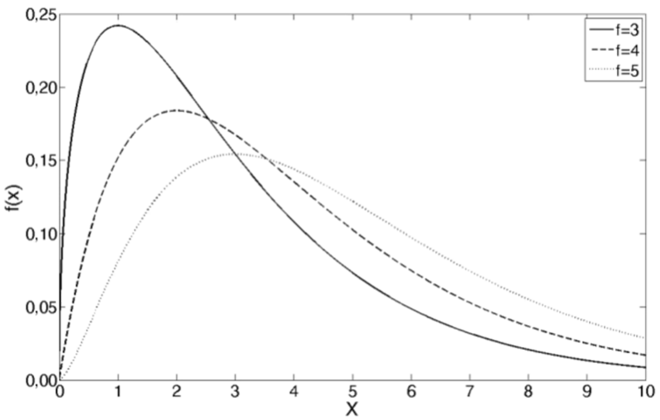


Adjustment Theory I

Chapter 6 – Introduction to Least Squares Adjustment

Prof. Dr.-Ing Frank Neitzel | Institute of Geodesy and Geoinformation Science

Version: 17 November 2024

Contents

Technische Universität Berlin

A Core of the semester

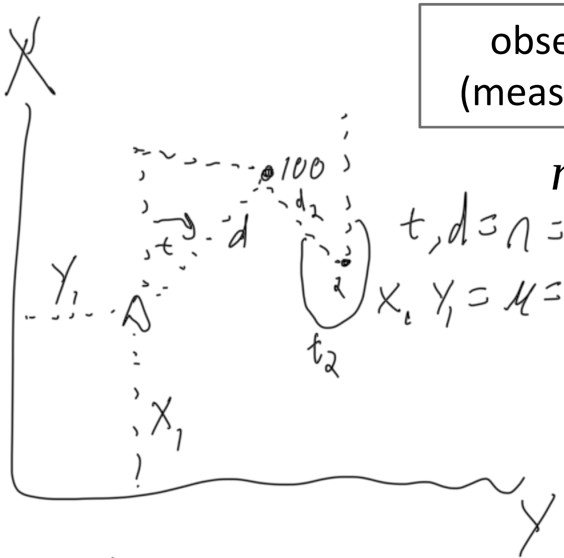
6. Introduction to Least Squares Adjustment

Technische Universität Berlin

6.1 Introduction

we can't measure them directly
i.e. cartesian coord

observations (measurements)	→	functional model	→	unknowns
n				u



$t, d = n =$ known
$X_1\ Y_1 = u =$ unknowns

Until now we have considered **minimal configurations**

$$n = u$$

redundant Measurements = = Reliability

$\Rightarrow$ Resulting equation system has a **unique solution**, but **the reliability of the solution is 0%**
$\Rightarrow$ no chance to detect blunders

6.1 Introduction

Now: We consider
overdetermined configurations

$$\boxed{n > u}$$

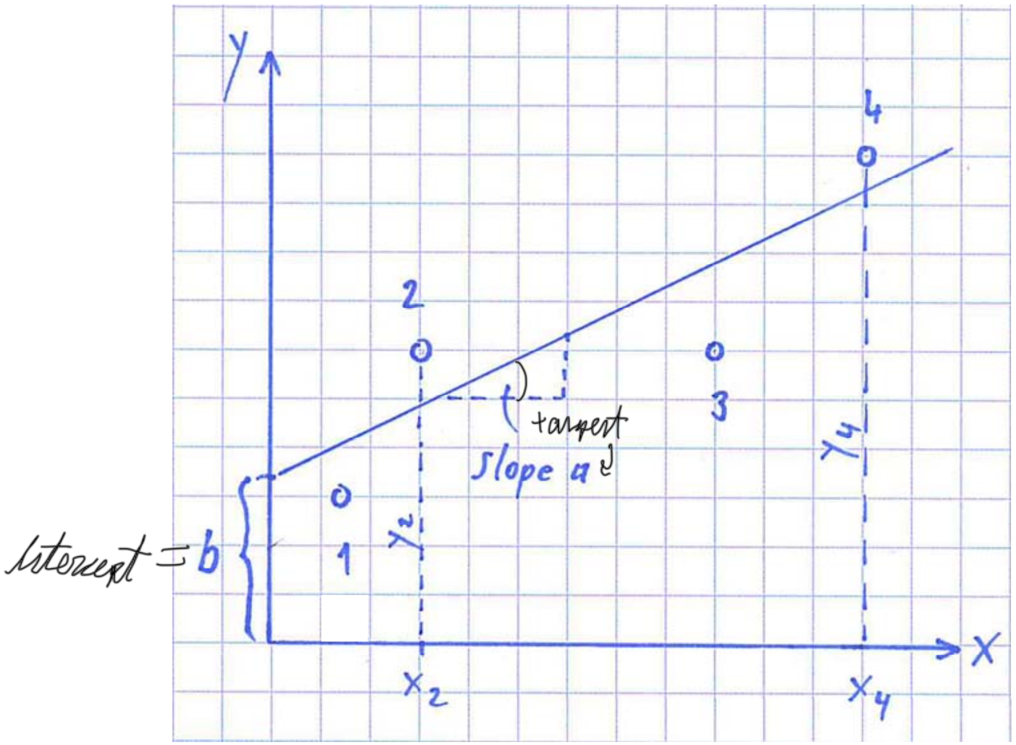
Fundamental principle in Geodesy!

$\Rightarrow$ **Reliable** solution
$\Rightarrow$ Increase of **precision** of the solution

6.1 Introduction

arithmatic men = you can compute residuals

Example: Computation of slope a and intercept b of a straight line



Functional model

$$y = ax + b$$

$$y_1 = ax_1 + b$$
$$y_2 = ax_2 + b$$
$$y_3 = ax_3 + b$$
$$y_4 = ax_4 + b$$

obervations, unknowns, fixed values

$n = 4, \quad u = 2$

$\Rightarrow$ Overdetermined configuration

6.1 Introduction

<u>We know</u>:

Our measurements are affected by random errors

→ Functional model can **not** be satisfied

→ To overcome this inconsistency, we introduce residuals v_i and obtain …

cut the roots

… Observation equations

$$y + vi = ax + b$$

$$\begin{aligned} y_1 + v_1 &= a x_1 + b \\ y_2 + v_2 &= a x_2 + b \\ y_3 + v_3 &= a x_3 + b \\ y_4 + v_4 &= a x_4 + b \end{aligned}$$

infinite solu

Functional model

$$y = ax + b$$

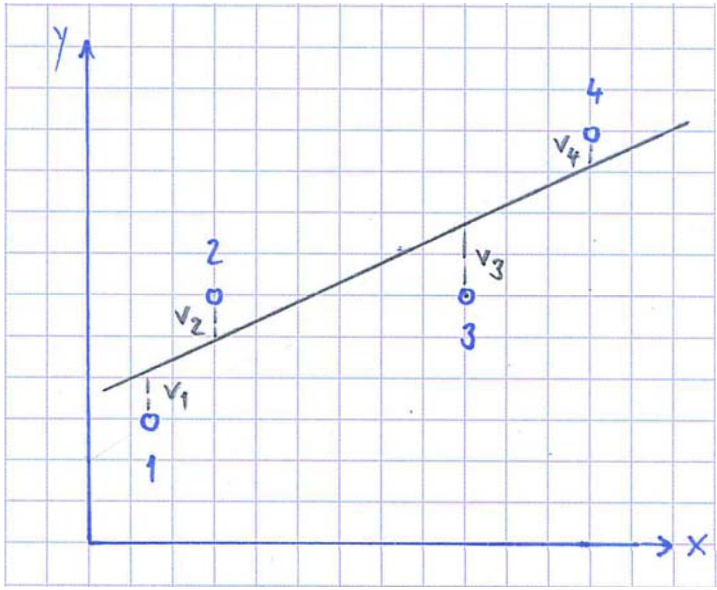
$$\begin{aligned} y_1 &= a x_1 + b \\ y_2 &= a x_2 + b \\ y_3 &= a x_3 + b \\ y_4 &= a x_4 + b \end{aligned}$$

<u>Problem</u>:

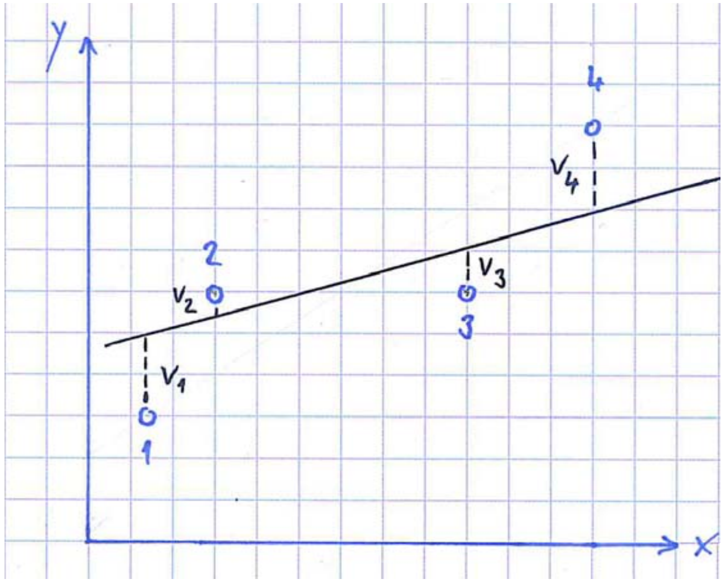
No unique solution, we can chose arbitrary straight lines as solution

6.1 Introduction

Possible solution 1



Possible solution 2



How to solve the problem?

⇒ We have to introduce a suitable constraint (target function) for the residuals to obtain **one specific solution**

⇒ Solution of overdetermined equation systems under consideration of a target function for the residuals is called **Adjustment Calculation**

⇒ Many different target functions for the residuals possible

⇒ We want to introduce specific target function that yields **Least Squares Adjustment**

6.1 Introduction

$v_1^2 + v_2^2 + v_n^2 \rightarrow Min$ = sum of squared residuals

$\sum_{i=1}^{n} v_i^2 \rightarrow Min$

$V = v^T v$

$$\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \rightarrow \min$$

Least Squares Adjustment

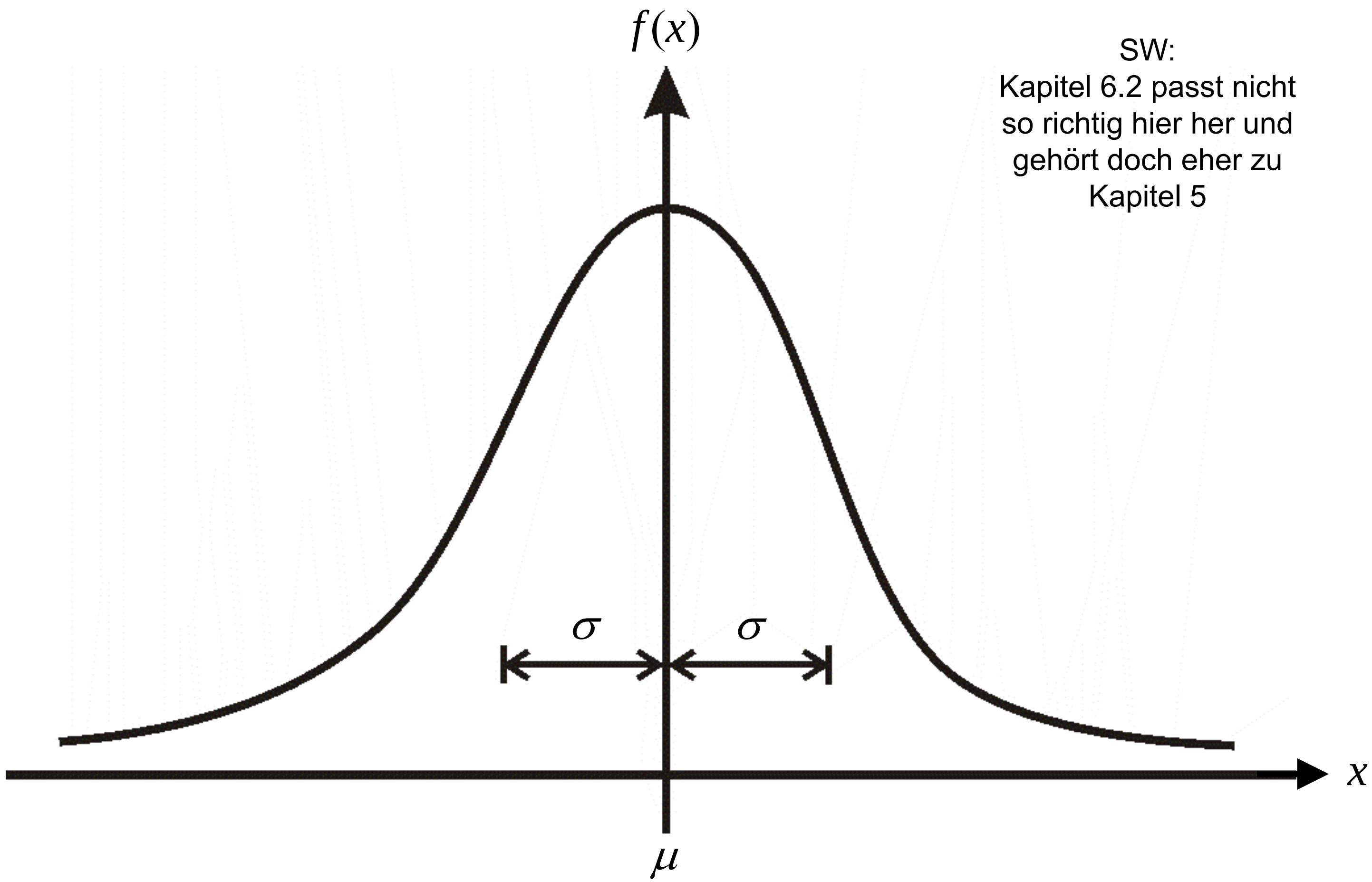
or

Method of Least Squares

$\Sigma_{ll} \rightarrow Q \rightarrow P = Q^{-1}$

6. Introduction to Least Squares Adjustment

6.2 Normal Distribution Density Function

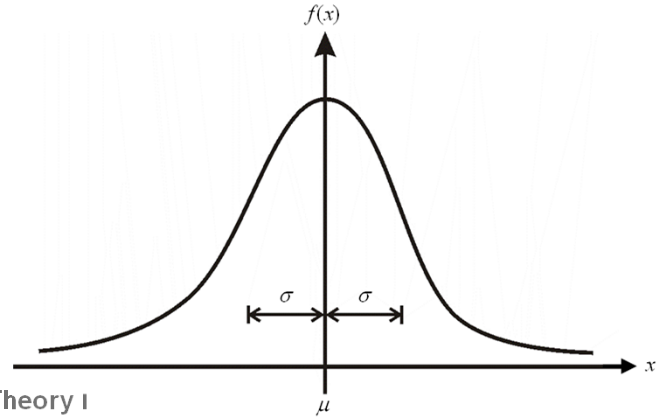


SW:
Kapitel 6.2 passt nicht so richtig hier her und gehört doch eher zu Kapitel 5

6.2 Normal Distribution Density Function

A few remarks describe the essential features of the normal distribution:

1. The normal density function $f(x)$ is symmetric about the expectation μ. Therefore all odd central moments are zero. Also the median and mode, which are two parameters of location sometimes used in practice, are equal to the expectation μ.
2. The maximum density value for the standardised variable is 0.399.
3. The density function approaches zero asymptotically as x goes to $\pm\infty$.
4. The density function has two points of inflection at $x = \mu \pm \sigma$



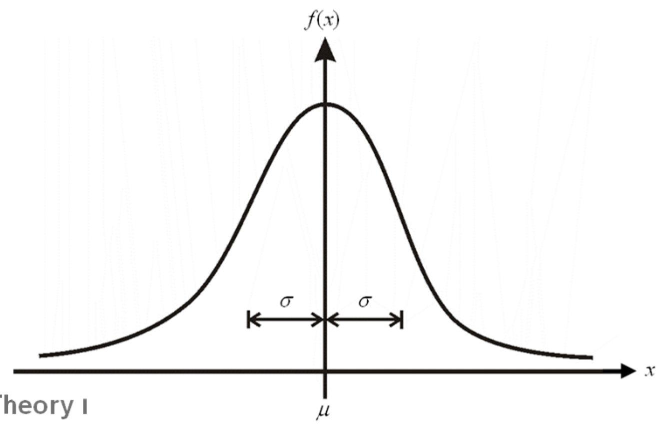
6.2 Normal Distribution Density Function

A few remarks describe the essential features of the normal distribution:

5. The probability from x taking values within x_1 and x_2 is given by the area between the x axis, the density function curve, and the boundaries of the interval $x = x_1, x = x_2$.

   In particular the probabilities for the deviation from the mean within some multiples of σ are as follows:

$$P[-\ \sigma < x - \mu < +\ \sigma] = 0.6827$$
$$P[-2\sigma < x - \mu < +2\sigma] = 0.9545$$
$$P[-3\sigma < x - \mu < +3\sigma] = 0.9973$$



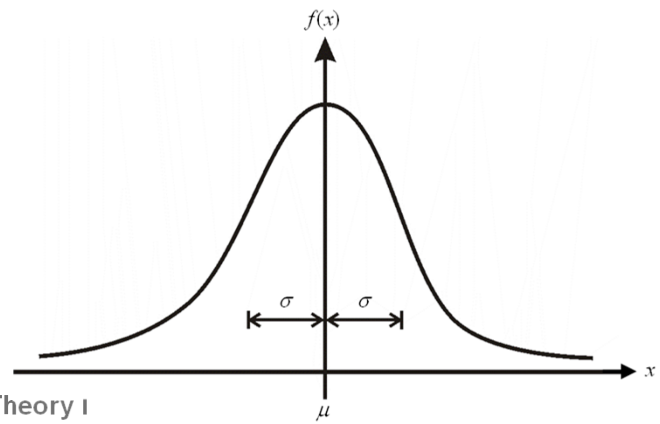
6.2 Normal Distribution Density Function

A few remarks describe the essential features of the normal distribution:

6. The abscissae associated with intervals covering probabilities of 0.90, 0.95 and 0.99 are

$$P[-1.645\sigma < x - \mu < +1.645\sigma] = 0.90$$
$$P[-1.960\sigma < x - \mu < +1.960\sigma] = 0.95$$
$$P[-2.576\sigma < x - \mu < +2.576\sigma] = 0.99$$

7. The probability that x takes on values on either side of μ (that is, either larger or smaller than μ) is equal 0.5.



6.2 Normal Distribution Density Function

Technische Universität Berlin

A few remarks describe the essential features of the normal distribution:

The theoretical and practical importance of the normal distribution is due to the "central limit theorem" which states that the sum $\sum_{i=1}^{n} \Delta_i$ of n independent variables $\Delta_1 + \cdots + \Delta_q$ will be asymptotically normally distributed as $n \rightarrow \infty$.

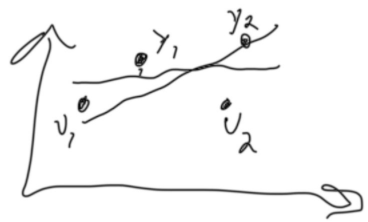
In practical applications, normal distributions are encountered very often. In particular, random variables that represent measurements in photogrammetry, geodesy or surveying are often nearly normally distributed.

6.3 Historical Development

Historical development concerning the optimal combination of redundant measurements

Method of selected points (before 1750)

- ▶ Select only as many observations ("points") as there are unknowns
- ▶ Remaining unused observations can be used to validate the estimated result
- ▶ Suppose that we use n observations → we obtain $\binom{n}{u}$ choices



Minimal config = 0% reliability

overdetermined config = $n > 4$

$$\left.\begin{aligned} y_1 + v_1 &= a x_1 + b \\ &\vdots \\ y_4 + v_4 &= a x_4 + b \end{aligned}\right\}$$ — We have to find a unique solution or target function for computing residuals

Historical development concerning the optimal combination of redundant measurements

Method of averages (ca. 1750)

- In 1714: Longitude Prize for determination of a ships longitude offered by British government
- Thomas Mayer (1723-1762, German mathematician): Determination of longitude (rather time) from motion of the moon. He obtained overdetermined equation System:

$$\mathbf{L}_{27\times1} = \mathbf{A}_{27\times3} \cdot \mathbf{x}_{3\times1}$$

27 observations, 3 unknowns

6.3 Historical Development

▶ Mayer's adjustment strategy:

- Distribute observations in 3 groups
- Sum up the equations within each group
- Solve the 3×3 system

<u>Euler's attempt (1749)</u>

▶ Leonhard Euler (1707-1783, Swiss mathematician and physicist)

▶ Orbital motion of Saturn under influence of Jupiter

▶ Prize (1748) of the Academy of Science, Paris

▶ 75 observations from the years 1582-1745, 6 unknowns

→ Euler could not solve the problem

6.3 Historical Development

Technische Universität Berlin

<u>Laplace's attempt (ca. 1787)</u>

- Pierre-Simon Laplace (1749-1827, French mathematician and astronomer)
- Motion of Saturn
- Best data: 24 observations, reformulated: 4 unknowns
- Approach: Like Mayer but other combinations

6.3 Historical Development

Method of least deviations (1760)

- Roger Boscovich (1711-1787, Croatian Jesuit, mathematician and physicist)
- Ellipticity of the earth
- 5 observations (Quito, Cape Town, Rome, Paris, Lapland), 2 unknowns
- First attempt:
  - All $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 10$ combinations with 2 observations each

6.3 Historical Development

Technische Universität Berlin

▶ First attempt [cont.]:

- 10 systems of equations (2×2) → 10 solutions, comparison of results
- His result shows gross variations of ellipticity → reject ellipsoidal hypothesis

▶ Second attempt:

- Mean deviation (or sum of deviations) should be zero

$$\sum_{i=1}^{n} v_i = 0$$

and sum of absolute deviations should be minimum

$$\sum_{i=1}^{n} |v_i| \rightarrow \min$$

→ This is an objective adjustment criterion, known (today) as L_1-norm estimation: Detection of outliers (blunders)

6.3 Historical Development

Technische Universität Berlin

Method of least squares (ca. 1805)

- ▶ Adrien-Marie Legendre (1752-1833)
- ▶ He published his method of least squares (in French “moindres carrés”)
- ▶ Application: Determination of orbits of comets
- ▶ After Legendre’s publication C.F. Gauss (1777-1855, German mathematician, astronomer, geodesist and physicist) states that he has already developed and used the method of least squares in 1794
- ▶ Today: It is acknowledged that Gauss’ claim of priority is very likely valid