7.1 Adjustment of Levelling Networks

Functional model:

$$\Delta h_{100,A} = H_A - H_{100}$$
$$\Delta h_{A,200} = H_{200} - H_A$$
$$\Delta h_{200,C} = H_C - H_{200}$$
$$\Delta h_{C,100} = H_{100} - H_C$$
$$\Delta h_{A,B} = H_B - H_A$$
$$\Delta h_{200,B} = H_B - H_{200}$$
$$\Delta h_{B,C} = H_C - H_B$$

We insert the fixed values for H_{100} and H_{200} and bring them to the left-hand side of the equations …

… Functional model

$$\begin{aligned}
\varphi_1&: \Delta h_{100,A} + 100.000 = H_A \\
\varphi_2&: \Delta h_{A,200} - 107.500 = -H_A \\
\varphi_3&: \Delta h_{200,C} + 107.500 = H_C \\
\varphi_4&: \Delta h_{C,100} - 100.000 = -H_C \\
\varphi_5&: \Delta h_{A,B} \qquad\qquad\quad = H_B - H_A \\
\varphi_6&: \Delta h_{200,B} + 107.500 = H_B \\
\varphi_7&: \underbrace{\Delta h_{B,C} \qquad\qquad\quad}_{\text{observation vector } \mathbf{L}'} = H_C - H_B
\end{aligned}$$

Linear or nonlinear? → Linear!

7.1 Adjustment of Levelling Networks

Observation equations:

$$
\begin{aligned}
\Delta h_{100,A} + 100.000 + v_1 &= \widehat{H}_A \\
\Delta h_{A,200} - 107.500 + v_2 &= -\widehat{H}_A \\
\Delta h_{200,C} + 107.500 + v_3 &= \widehat{H}_C \\
\Delta h_{C,100} - 100.000 + v_4 &= -\widehat{H}_C \\
\Delta h_{A,B} \qquad\qquad\quad + v_5 &= \widehat{H}_B - \widehat{H}_A \\
\Delta h_{200,B} + 107.500 + v_6 &= \widehat{H}_B \\
\Delta h_{B,C} \qquad\qquad\quad + v_7 &= \widehat{H}_C - \widehat{H}_B
\end{aligned}
$$

7.1 Adjustment of Levelling Networks

Observation vector:

$$\mathbf{L}' = \begin{bmatrix} \Delta h_{100,A} + 100.000 \\ \Delta h_{A,200} - 107.500 \\ \Delta h_{200,C} + 107.500 \\ \Delta h_{C,100} - 100.000 \\ \Delta h_{A,B} \\ \Delta h_{200,B} + 107.500 \\ \Delta h_{B,C} \end{bmatrix}$$

Vector of residuals:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

7.1 Adjustment of Levelling Networks

Stochastic Model of the observations:

$$p_1 = 1, p_2 = 1, \dots, p_7 = 1 \;\rightarrow \mathbf{P} = \mathbf{I}$$

Vector of unknowns:

$$\hat{\mathbf{X}} = \begin{bmatrix} \hat{H}_A \\ \hat{H}_B \\ \hat{H}_C \end{bmatrix}$$

Design Matrix (Matrix with coefficients of the linear functional model):

$$\mathbf{A} = \begin{array}{c} \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{array} \begin{array}{c} \begin{array}{ccc} H_A & H_B & H_C \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

7.1 Adjustment of Levelling Networks

Normal equations:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\widehat{\mathbf{X}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}' \qquad \text{with} \qquad \mathbf{P} = \mathbf{I}$$

$$\widehat{\mathbf{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}' \qquad \text{with} \qquad \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1} = \mathbf{Q}_{\hat{X}\hat{X}}$$

Residuals:

$$\mathbf{v} = \mathbf{A}\widehat{\mathbf{X}} - \mathbf{L}'$$

Adjusted observations:

$$\hat{\mathbf{L}}' = \mathbf{L}' + \mathbf{v}$$

7.1 Adjustment of Levelling Networks

Final check:

$$\hat{\mathbf{L}}' - \boldsymbol{\Phi}(\hat{\mathbf{X}}) \overset{!}{=} \mathbf{0}$$ → zero within computing precision

Computer: $\hat{\mathbf{L}}' - \boldsymbol{\Phi}(\hat{\mathbf{X}}) \leq \delta$ → e.g. 10^{-12}

Empirical reference standard deviation:

$$s_0 = \sqrt{\frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$$

VCM of adjusted unknowns:

$$\boldsymbol{\Sigma}_{\hat{X}\hat{X}} = \mathrm{s}_0^2 \cdot \mathbf{Q}_{\hat{X}\hat{X}}$$

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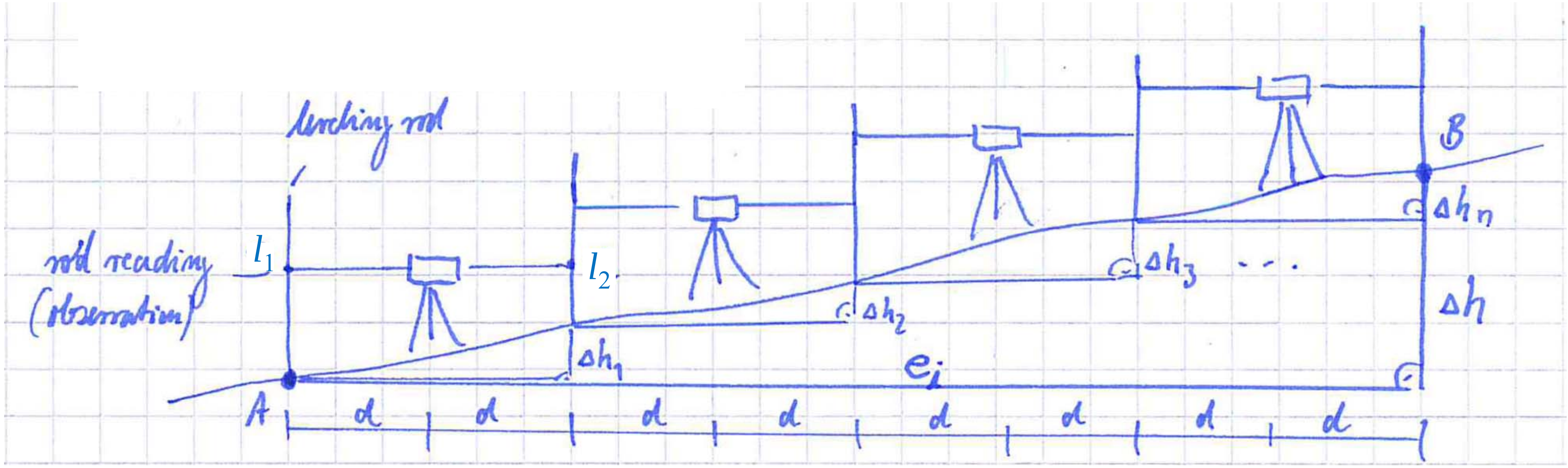
Weights in Differential Levelling



d: Length of sight distance

e_i: Length of the course between surveyed points, here points A and B

7.1 Adjustment of Levelling Networks

- Standard deviation in rod reading is usually expressed as ratio of the estimated standard error in rod reading per unit sight distance length
  → Standard deviation for rod reading

$$\sigma_{l_i} = d_i \cdot \sigma_{l/d}$$

d_i: length of sight distance

$$\Delta h_1 = l_1 - l_2$$

with equal length of sight distance

$$\sigma_{l_1} = \sigma_{l_2} = d \cdot \sigma_{l/d}$$

$$\rightarrow \sigma_{\Delta h_1}^2 = \left(d \cdot \sigma_{l/d}\right)^2 + \left(d \cdot \sigma_{l/d}\right)^2$$

$$= 2 \cdot d^2 \cdot \sigma_{l/d}^2$$

7.1 Adjustment of Levelling Networks

▶ Height difference Δh:

$$\Delta h = \Delta h_1 + \Delta h_2 + \cdots + \Delta h_n$$

$$\rightarrow \quad \sigma_{\Delta h}^2 = 2 \cdot n \cdot d^2 \cdot \sigma_{l/d}^2$$

▶ n as a function of length of sight distance d and length of the course e_i

$$n = \frac{e_i}{2d}$$

Example: $e_i = 200\ \mathrm{m},\ \ d = 25\ \mathrm{m} \quad \rightarrow \quad n = \frac{200\ \mathrm{m}}{2 \cdot 25\ \mathrm{m}} = 4$

$$\rightarrow \quad \sigma_{\Delta h}^2 = 2 \cdot \frac{e_i}{2d} \cdot d^2 \cdot \sigma_{l/d}^2 = e_i \cdot d \cdot \sigma_{l/d}^2$$

with $d, \sigma_{l/d}$ constant values, we introduce $k = d \cdot \sigma_{l/d}^2$

$$\boxed{\sigma_{\Delta h}^2 = e_i \cdot k}$$

7.1 Adjustment of Levelling Networks

▶ We know: Weights are the inverse values of the variances

$$p_i = \frac{1}{e_i \cdot k}$$

Since k is a constant and weights are relative, equation can be simplified to

$$\boxed{p_i = \frac{1}{e_i}}$$

Weights of different levelling lines are inversely proportional to their length. And since any course length is proportional to its number of instrument setups, weights are also inversely proportional to the number of instrument setups.

7.1 Adjustment of Levelling Networks

Example 1:

Line	Length e [km]	Weight p	Rel. weights p
1	4	0.25	3
2	3	$0.\overline{3}$	4
3	2	0.5	6
4	3	$\vdots$	$\vdots$
5	2		
6	2		
7	2		

Remember: Weights are relative

→ We can introduce relative weights

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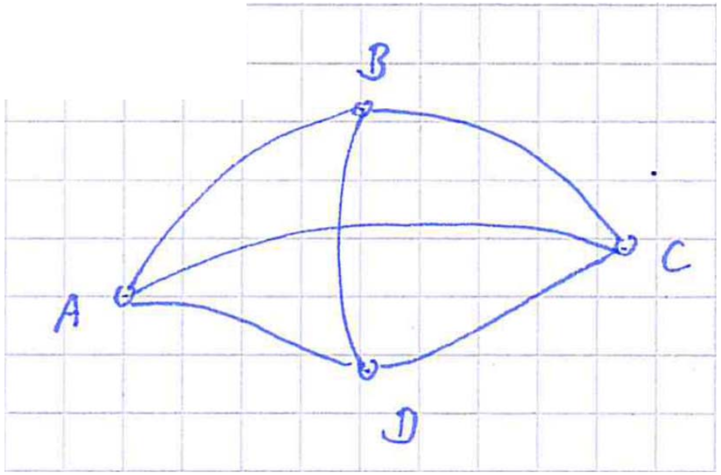
Example 2:

Given: Observed elevation differences and their standard deviations

From	To	Δh [m]	σ [m]
A	B	10.509	0.006
B	C	5.360	0.004
C	D	−8.235	0.005
D	A	−7.348	0.003
B	D	−3.167	0.004
A	C	15.881	0.012



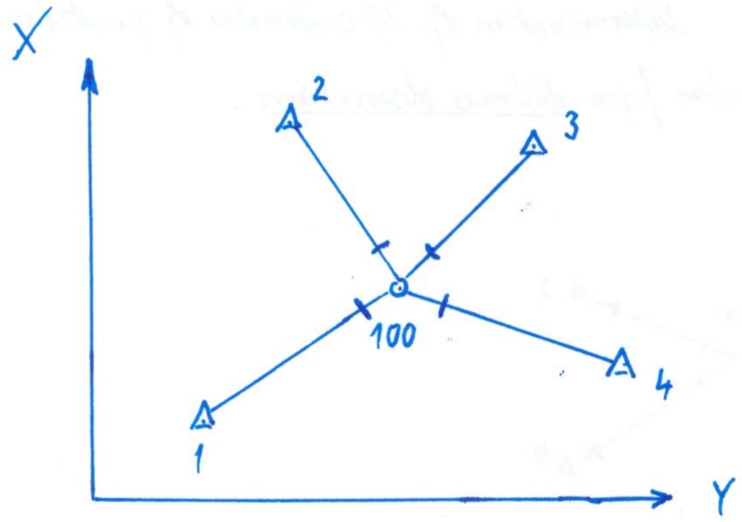
Determine the adjusted observations and their standard deviation

Problem: No height(s) of benchmark point(s) given $\rightarrow$ Solution?

7.2 Adjustment of Horizontal Surveys: Trilateration

Basic idea of trilateration:

Determination of 2D coordinates of points in a plane Cartesian coordinate system from <u>distance observations</u>



Cartesian coordinates in 2D

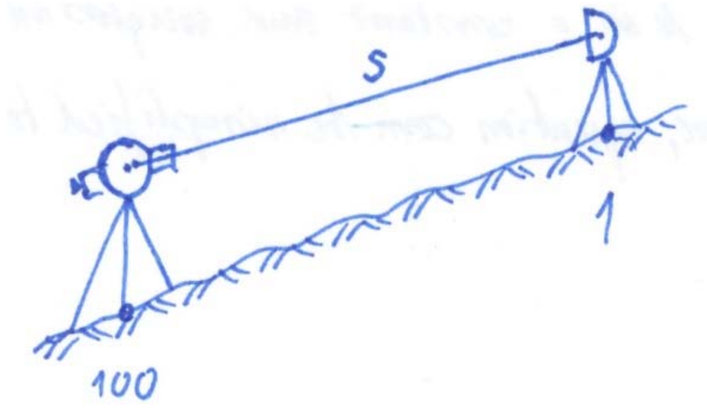
- Gauss-Krueger coordinates
- UTM coordinates

→ Projected coordinates into a plane

find points based on dist

7.2 Adjustment of Horizontal Surveys: Trilateration

Distance measurement between point 100 and 1



Question: What does our distance measurement between point 100 and point 1 have to do with distance between points in Gauss-Krueger coordinates?

Answer: Nothing!

7.2 Adjustment of Horizontal Surveys: Trilateration

Measurement with EDM: s

↓

Atmospheric correction: s_a

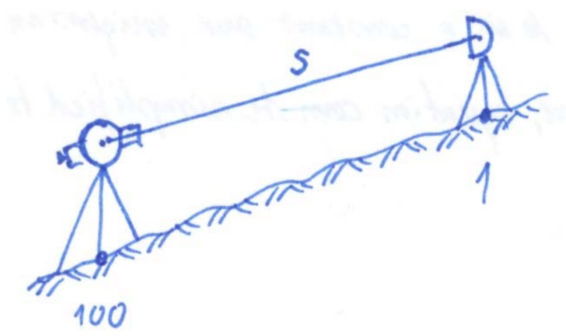
↓

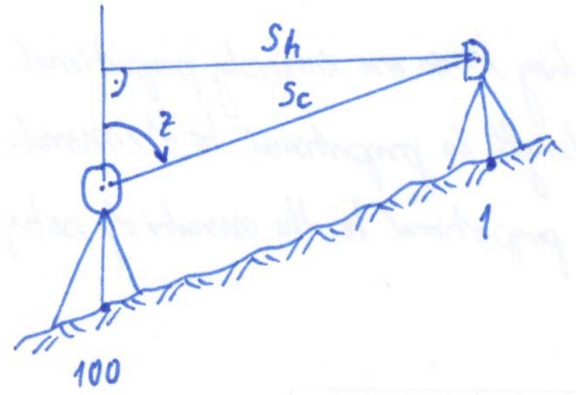
Correction for calibration parameters (zero correction, scale factor): s_c

↓

Reduction into the local horizontal plane: s_h

↓



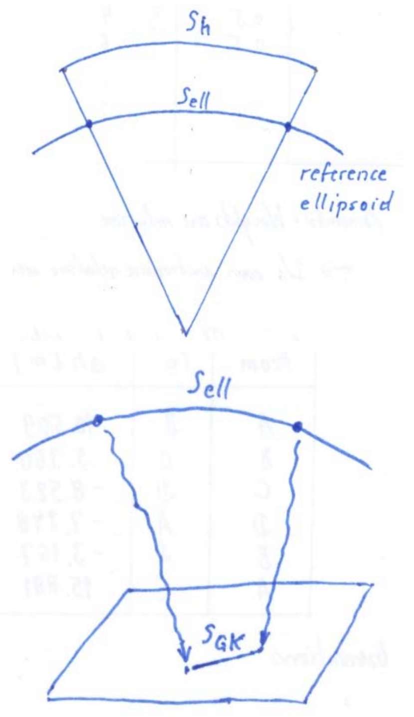


7.2 Adjustment of Horizontal Surveys: Trilateration

Height reduction
(projection onto the surface
of the reference ellipsoid): s_{ell}

↓

Projection into the GK- or
UTM-plane (by formulas): s_{GK}



$\rightarrow$ Finally s_{GK} corresponds with the given Gauss-Krueger coordinates

Attention: Pre-processing of distance measurements <u>must</u> be performed

7.2 Adjustment of Horizontal Surveys: Trilateration

In practice:

- s_{GK} or s_{UTM} is regarded as observation with corresponding precision/weight

  or

- s_h is regarded as observation and reduction and projection is performed within the application of the adjustment software

  $\rightarrow$ Check the pre-settings!

For an adjustment within the Gauss-Markov Model (parametric adjustment)

$$\mathbf{L} = \mathbf{\Phi}(\mathbf{X})$$

we have to introduce appropriate unknowns to express our observations as functions of the unknowns

$\rightarrow$ We introduce 2D coordinates as unknowns

7.2 Adjustment of Horizontal Surveys: Trilateration

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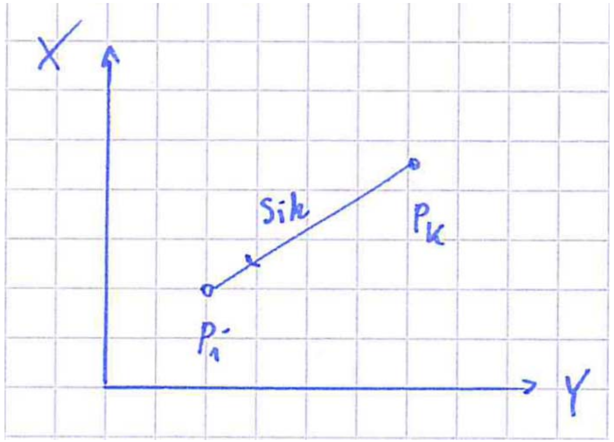
Functional model:

Measured from "i" to "k"

$$s_{ik} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}$$



Observation equations:

$$s_{ik} + v_{s_{ik}} = \sqrt{(\hat{x}_k - \hat{x}_i)^2 + (\hat{y}_k - \hat{y}_i)^2}$$

Nonlinear functional model

$\rightarrow$ for least squares adjustment we need a linearised functional model

$\rightarrow$ Jacobian matrix with partial derivatives

7.2 Adjustment of Horizontal Surveys: Trilateration

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Partial derivatives:

$$\frac{\partial s_{ik}}{\partial x_k} = \frac{1}{2\sqrt{}} \cdot 2(x_k - x_i) = \frac{x_k - x_i}{s_{ik}} = \frac{\Delta x_{ik}}{s_{ik}}$$

$$\frac{\partial s_{ik}}{\partial x_i} = \frac{1}{2\sqrt{}} \cdot 2(x_k - x_i) \cdot (-1) \qquad = \frac{-\Delta x_{ik}}{s_{ik}}$$

$$\frac{\partial s_{ik}}{\partial y_k} = \frac{\Delta y_{ik}}{s_{ik}}$$

$$\frac{\partial s_{ik}}{\partial y_i} = \frac{-\Delta y_{ik}}{s_{ik}}$$

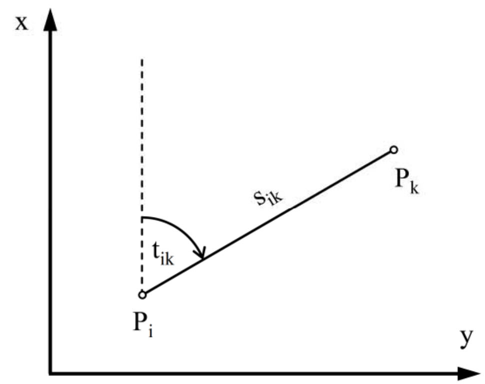
→ See handout (Partial derivatives of geodetic observation equations)

7.2 Adjustment of Horizontal Surveys: Trilateration

Partial derivatives of geodetic observation equations

Given: Observation Equations
Searched: Partial derivatives with respect to the unknowns for the linearization of the observation equations

1. Distances $s_{ik} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}$



Notation information:

Index i: Station
Index k: Target

measured from ‚i' to ‚k'

Partial derivatives can be further simplified by using the equations for the grid bearing t_{ik} and distance s_{ik} .

Partial Derivatives:

$$\frac{\partial s_{ik}}{\partial x_k} = \frac{x_k - x_i}{s_{ik}} = \frac{\Delta x_{ik}}{s_{ik}} = \cos t_{ik} \qquad \frac{\partial s_{ik}}{\partial x_i} = -\frac{x_k - x_i}{s_{ik}} = \frac{-\Delta x}{s_{ik}} = -\cos t_{ik}$$

$$\frac{\partial s_{ik}}{\partial y_k} = \frac{y_k - y_i}{s_{ik}} = \frac{\Delta y_{ik}}{s_{ik}} = \sin t_{ik} \qquad \frac{\partial s_{ik}}{\partial y_i} = -\frac{y_k - y_i}{s_{ik}} = -\frac{\Delta y_{ik}}{s_{ik}} = -\sin t_{ik}$$

Adjustment_Theory_I_Derivatives.pdf

7.2 Adjustment of Horizontal Surveys: Trilateration

Weights in trilateration networks

- Precision for distances from electronic distance measurement given in

$$\sigma_{s_i} = a_1 + a_2 \cdot d_i$$

a_1: constant part of precision
a_2: standard error per unit sight distance length
d_i: length of sight distance

7.2 Adjustment of Horizontal Surveys: Trilateration

- Typical values for the precision of an EDM:

$$3\ \mathrm{mm} + 2\ \mathrm{ppm}$$

ppm: parts per million $\widehat{=}$ $\mathrm{mm\ per\ km}$

$\rightarrow$ Standard deviation for a distance of

$$\begin{aligned} 500\ \mathrm{m}&: 4\ \mathrm{mm} \\ 1000\ \mathrm{m}&: 5\ \mathrm{mm} \\ 2000\ \mathrm{m}&: 7\ \mathrm{mm} \end{aligned}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

- Variance matrix of the observations

$$\mathbf{\Sigma}_{LL} = \begin{bmatrix} \sigma_{s_1}^2 & & & 0 \\ & \sigma_{s_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{s_n}^2 \end{bmatrix}$$

- With reference variance σ_0^2
- Cofactor matrix of observations: $\mathbf{Q}_{LL} = \frac{1}{\sigma_0^2}\mathbf{\Sigma}_{LL}$
- Weight matrix of observations: $\mathbf{P} = \mathbf{Q}_{LL}^{-1}$

Example

The measurements of the trilateration network depicted in Figure 1 are listed in Table 2. The points 1, 2 and 3 are control points (error-free) and their Gauss-Krueger coordinates are given in Table 1. Calculate the adjusted Gauss-Krueger coordinates of point 100 using least squares adjustment.

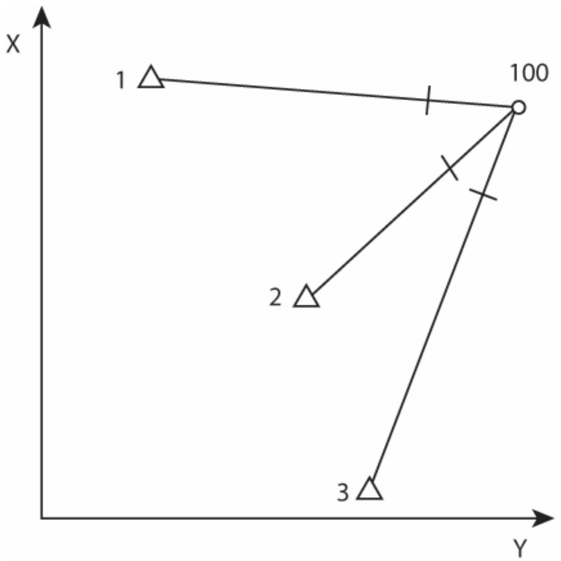


Figure 1: Trilateration network

Table 1: Gauss-Krueger coordinates of control points

Point No.	y [m]	x [m]
1	865.400	4527.150
2	2432.550	2047.250
3	2865.220	27.150

Approximate values for the coordinates of point 100:

$$y_{100}^{0} : 6861.3 \; ; \; x_{100}^{0} : 3727.8$$

(graphical coordinates from a map)

7.2 Adjustment of Horizontal Surveys: Trilateration

Table 2: Observed reduced distances

From	To	s [m]
100	1	6049.000
100	2	4736.830
100	3	5446.490

- The distance measurements have been performed with a precision of 1 mm + 2 ppm
- The distances are uncorrelated and already reduced into the Gauss-Krueger projection
- Set up an appropriate functional model as well as the observation equations
- Set up the stochastic model
- Choose appropriate values for the break-off conditions ε and δ and justify your decision
- Solve the normal equation system and determine the Gauss-Krueger coordinates of point 100 as well as their standard deviations
- Calculate the residuals and the adjusted observations as well as their standard deviations

7.2 Adjustment of Horizontal Surveys: Trilateration

General considerations:

- What are our unknowns?

  → Coordinates of point 100

  → We introduce y_{100}, x_{100}

- What are our observations?

  → Distances

  → $s_{100,1}, s_{100,2}, s_{100,3}$

- Observations reduced into projection?

  → Yes!

- What are our fixed values?

  → $y_1, x_1; y_2, x_2; y_3, x_3$

- Redundancy? → $r = n - u \;\rightarrow r = 3 - 2 \;\rightarrow r = 1$

7.2 Adjustment of Horizontal Surveys: Trilateration

Functional model:

$$s_{100,1} = \sqrt{(x_1 - x_{100})^2 + (y_1 - y_{100})^2}$$

$$s_{100,2} = \sqrt{(x_2 - x_{100})^2 + (y_2 - y_{100})^2}$$

$$s_{100,3} = \sqrt{(x_3 - x_{100})^2 + (y_3 - y_{100})^2}$$

Observation equations:

$$s_{100,1} + v_1 = \sqrt{(x_1 - \hat{x}_{100})^2 + (y_1 - \hat{y}_{100})^2}$$

$$s_{100,2} + v_2 = \sqrt{(x_2 - \hat{x}_{100})^2 + (y_2 - \hat{y}_{100})^2}$$

$$s_{100,3} + v_3 = \sqrt{(x_3 - \hat{x}_{100})^2 + (y_3 - \hat{y}_{100})^2}$$

Observation vector:

$$\mathbf{L} = \begin{bmatrix} 6049.000 \\ 4736.830 \\ 5446.490 \end{bmatrix}$$

Stochastic model of the observations:

$$\sigma_1 = 1\ \mathrm{mm} + 2\,\frac{\mathrm{mm}}{\mathrm{km}} \cdot 6.049\ \mathrm{km}$$

$$\sigma_2 = 1\ \mathrm{mm} + 2\,\frac{\mathrm{mm}}{\mathrm{km}} \cdot 4.73683\ \mathrm{km}$$

$$\sigma_3 = 1\ \mathrm{mm} + 2\,\frac{\mathrm{mm}}{\mathrm{km}} \cdot 5.44649\ \mathrm{km}$$

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$$\mathbf{\Sigma}_{LL} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

$$\mathbf{Q}_{LL} = \frac{1}{\sigma_0^2}\mathbf{\Sigma}_{LL}$$

with

$$\sigma_0^2 = 1$$

$$\mathbf{Q}_{LL} = \mathbf{\Sigma}_{LL}$$

$$\rightarrow \qquad \mathbf{P} = \mathbf{Q}_{LL}^{-1}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

Vector of adjusted unknowns:

$$\hat{\mathbf{X}} = \begin{bmatrix} \hat{x}_{100} \\ \hat{y}_{100} \end{bmatrix}$$

Nonlinear functional model

→ Solution from iterative computing with linearised functional model

→ Introduction of approximate values x_{100}^0, y_{100}^0

7.2 Adjustment of Horizontal Surveys: Trilateration

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Vector of starting values:

$$\mathbf{X}^0 = \begin{bmatrix} x_{100}^0 \\ y_{100}^0 \end{bmatrix}$$

Vector of adjusted reduced unknowns:

$$\hat{\mathbf{x}} = \hat{\mathbf{X}} - \mathbf{X}^0 = \begin{bmatrix} d\hat{x}_{100} \\ d\hat{y}_{100} \end{bmatrix} = \begin{bmatrix} \hat{x}_{100} - x_{100}^0 \\ \hat{y}_{100} - y_{100}^0 \end{bmatrix}$$

Vector of reduced observations:

$$\mathbf{l} = \begin{bmatrix} 6049.000 - \sqrt{\left(x_1 - x_{100}^0\right)^2 + \left(y_1 - y_{100}^0\right)^2} \\ 4736.830 - \sqrt{\left(x_2 - x_{100}^0\right)^2 + \left(y_2 - y_{100}^0\right)^2} \\ 5446.490 - \sqrt{\left(x_3 - x_{100}^0\right)^2 + \left(y_3 - y_{100}^0\right)^2} \end{bmatrix}$$

Jacobian matrix:

$$\mathbf{J} = \begin{array}{c} \\ s_{100,1} \\ s_{100,2} \\ s_{100,3} \end{array} \begin{array}{c} \begin{array}{cc} x_{100}^0 & y_{100}^0 \end{array} \\ \begin{bmatrix} \dfrac{\partial s_{100,1}^0}{\partial x_{100}^0} & \dfrac{\partial s_{100,1}^0}{\partial y_{100}^0} \\ \dfrac{\partial s_{100,2}^0}{\partial x_{100}^0} & \dfrac{\partial s_{100,2}^0}{\partial y_{100}^0} \\ \dfrac{\partial s_{100,3}^0}{\partial x_{100}^0} & \dfrac{\partial s_{100,3}^0}{\partial y_{100}^0} \end{bmatrix} \end{array}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

with

$$\frac{\partial s_{100,1}^0}{\partial x_{100}^0} = \frac{1}{2\sqrt{\left(x_1 - x_{100}^0\right)^2 + \left(y_1 - y_{100}^0\right)^2}} \cdot 2\left(x_1 - x_{100}^0\right) \cdot (-1)$$

$$= \frac{-(x_1 - x_{100}^0)}{s_{100,1}^0}$$

$$\vdots$$

$$\frac{\partial s_{100,3}^0}{\partial y_{100}^0} = \frac{1}{2\sqrt{\left(x_3 - x_{100}^0\right)^2 + \left(y_3 - y_{100}^0\right)^2}} \cdot 2\left(y_3 - y_{100}^0\right) \cdot (-1)$$

$$= \frac{-(y_3 - y_{100}^0)}{s_{100,3}^0}$$

Design matrix:

$$\mathbf{A} = \mathbf{J}$$

Normal equations:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Solution of normal equations:

$$\hat{\mathbf{x}} = \big(\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}}_{\mathbf{N}}\big)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Adjusted unknowns:

$$\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$$

Residuals:

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}$$

Adjusted observations:

$$\hat{\mathbf{L}} = \mathbf{L} + \mathbf{v}$$

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Check:

$$\max |\hat{x}_i| < \varepsilon$$

and

$$\max \left|\hat{L}_i - \Phi_i(\widehat{\mathbf{X}})\right| < \delta$$

no

Use $\widehat{\mathbf{X}}$ as new $\mathbf{X}^0$

yes

$\boxed{\widehat{\mathbf{X}} \text{ is final result}}$

Empirical reference standard deviation:

$$s_0 = \sqrt{\frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$$

VCM of adjusted unknowns:

$$\mathbf{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \cdot \mathbf{Q}_{\hat{X}\hat{X}} \qquad \text{with} \qquad \mathbf{Q}_{\hat{X}\hat{X}} = \mathbf{N}^{-1}$$

Standard deviation of unknowns:

$$\mathbf{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \cdot \begin{bmatrix} q_{\hat{x}\hat{x}} & q_{\hat{x}\hat{y}} \\ q_{\hat{y}\hat{x}} & q_{\hat{y}\hat{y}} \end{bmatrix}$$

$q_{\hat{x}\hat{x}}$: Cofactor of unknown value x_{100}

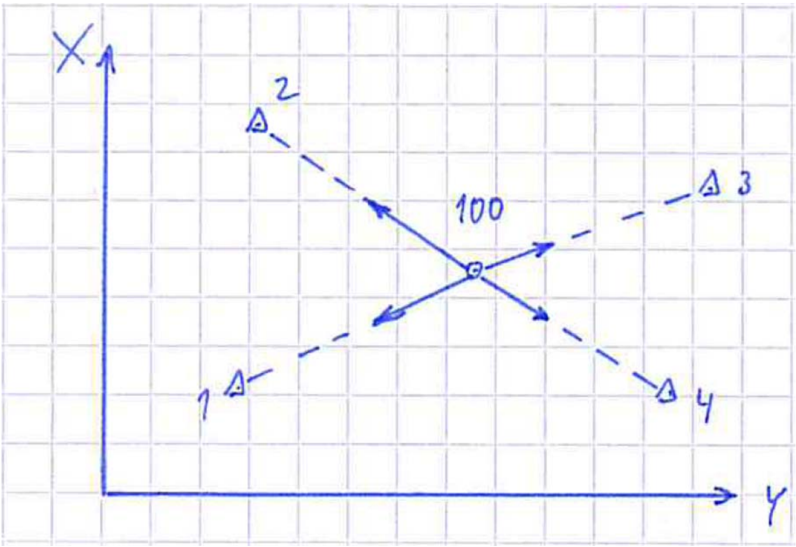
$q_{\hat{y}\hat{y}}$: Cofactor of unknown value y_{100}

$$s_{\hat{x}_{100}} = s_0 \cdot \sqrt{q_{\hat{x}\hat{x}}}$$
$$s_{\hat{y}_{100}} = s_0 \cdot \sqrt{q_{\hat{y}\hat{y}}}$$

7.3 Adjustment of Horizontal Surveys: Triangulation

Basic idea of triangulation:

Determination of 2D coordinates of points in a plane Cartesian coordinate system from observed directions



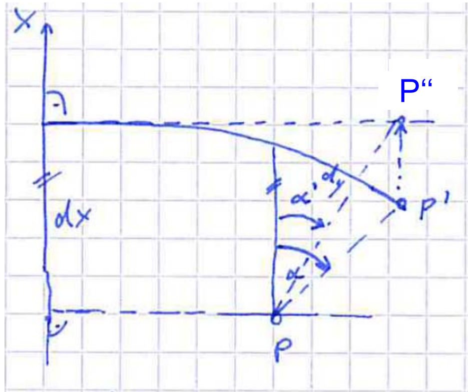
Cartesian coordinates in 2D

- Gauss-Krueger coordinates
- UTM coordinates

→ Projected coordinates into a plane

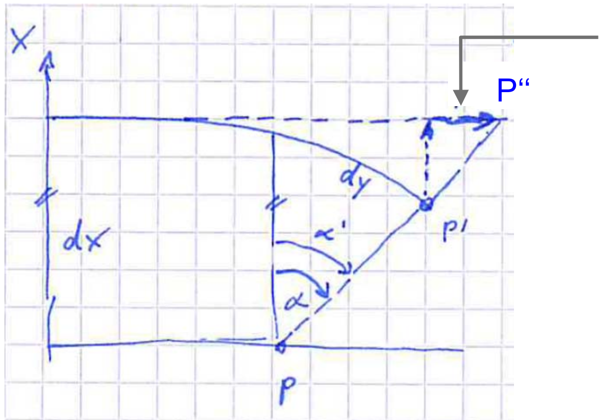
→ Conformal mapping

Non-conformal and conformal mapping



$$\alpha' \neq \alpha$$

e.g. Soldner coordinates



Ordinate difference is elongated
$\rightarrow$ elimination of angular distortion

$$\alpha' = \alpha$$

e.g. Gauss-Krueger coordinates, UTM coordinates

7.3 Adjustment of Horizontal Surveys: Triangulation

- A conformal mapping keeps a differential similarity between the original elliptic situation and the maps image

  $\rightarrow$ We can use our measured directions without corrections and reductions and combine them with GK or UTM coordinates

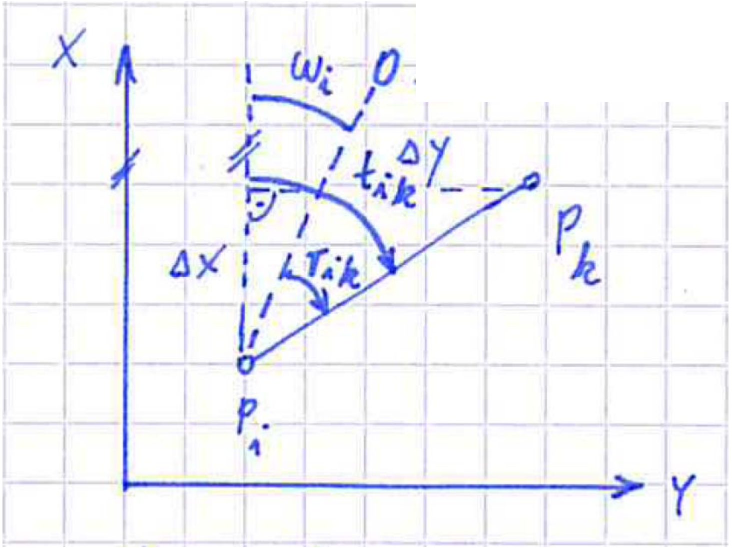
- Directions r_{ik} are our observations with corresponding precision / weight

- For an adjustment within the Gauss-Markov Model (parametric adjustment)

$$\mathbf{L} = \boldsymbol{\Phi}(\mathbf{X})$$

  we have to introduce appropriate unknowns to express our observations as functions of the unknowns

  $\rightarrow$ We introduce 2D coordinates as unknowns



0: zero direction of our instrument (tacheometer, theodolite)

i: instrument station

k: foresight station

r_{ik}: measured direction from i to k

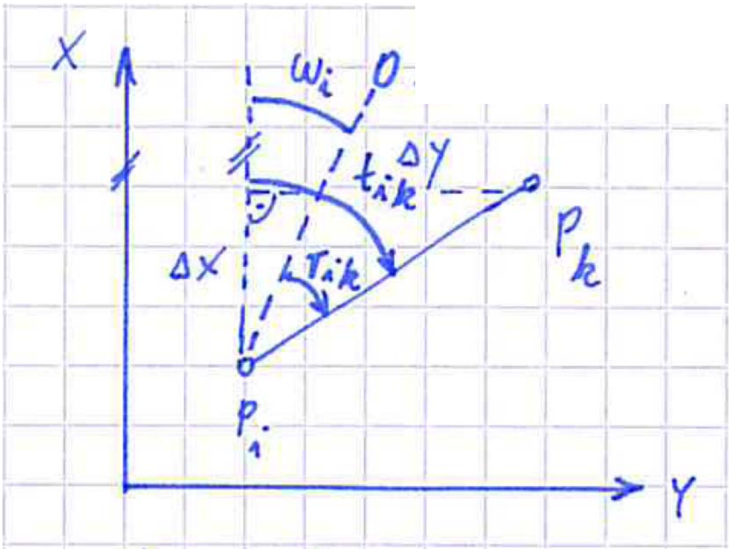
t_{ik}: azimuth from i to k

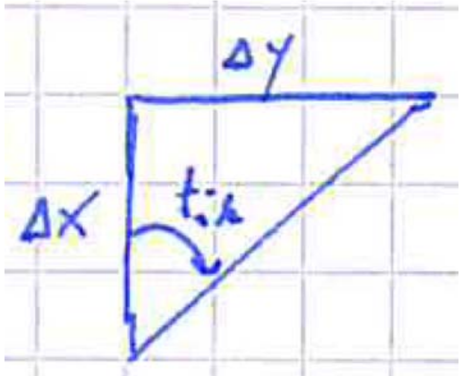
ω_i: orientation unknown

Functional model:

$$r_{ik} = t_{ik} - \omega_i$$

How to obtain t_{ik}?





$$\tan t_{ik} = \frac{\Delta y}{\Delta x} \quad \rightarrow \quad t_{ik} = \arctan \frac{\Delta y}{\Delta x}$$

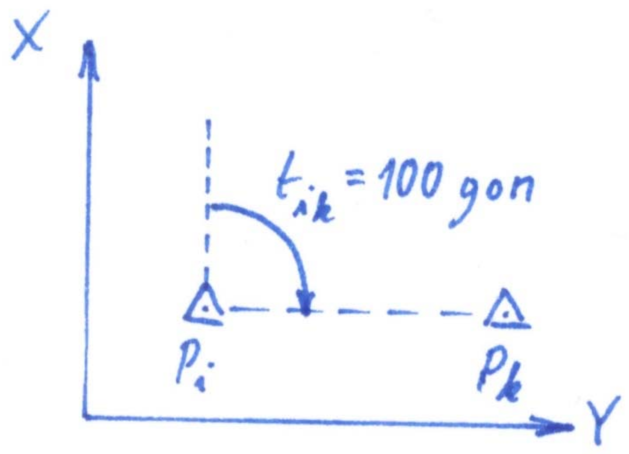
$$\rightarrow \quad t_{ik} = \arctan \frac{y_k - y_i}{x_k - x_i}$$

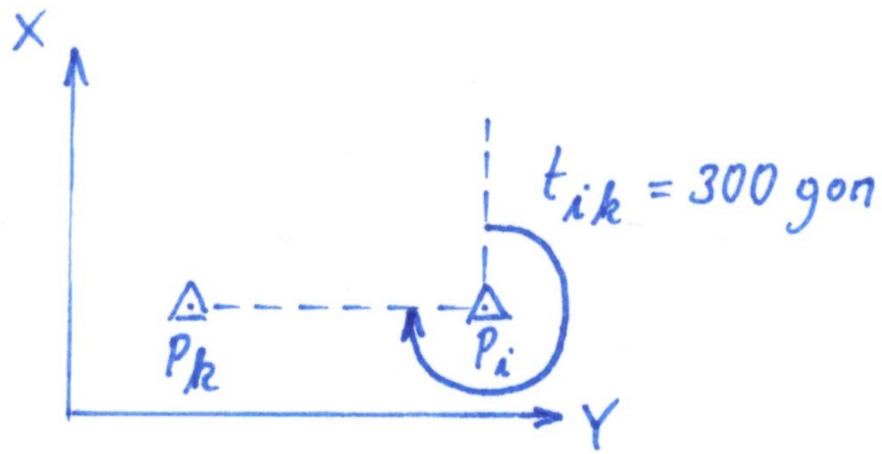
7.3 Adjustment of Horizontal Surveys: Triangulation

- Attention 1: What happens if $\Delta x = 0$?

  → Cannot use formula

  → two cases possible

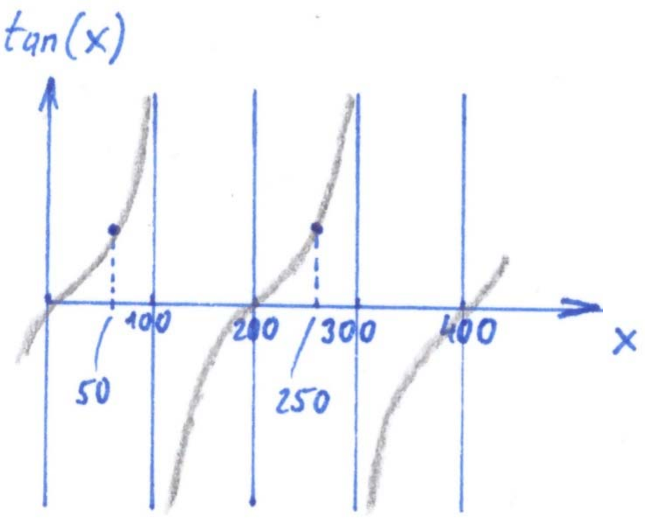




▶ Attention 2:

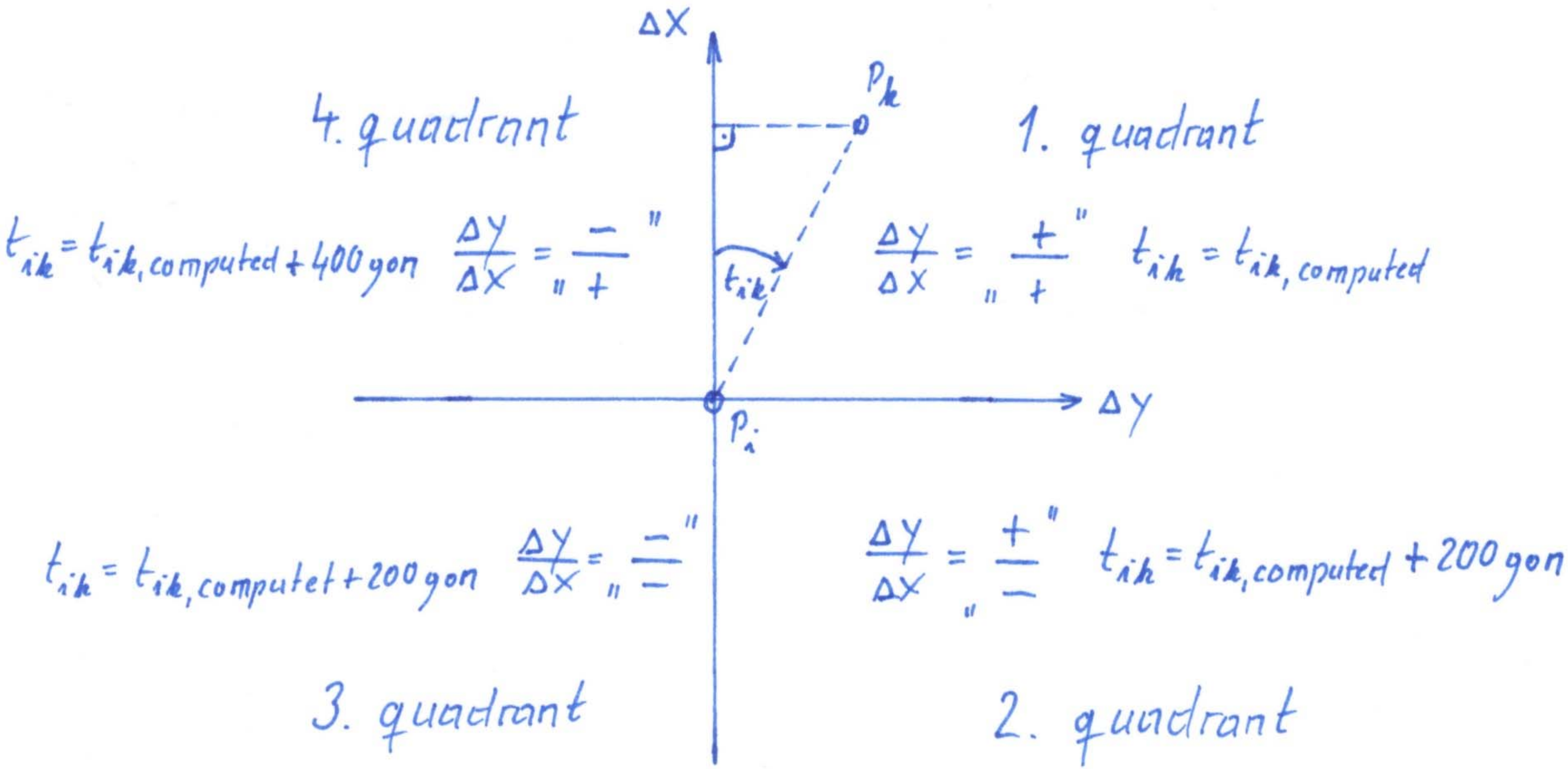
e.g. $\tan 50 \text{ gon} = 1$

$\tan 250 \text{ gon} = 1$



Problem: Which value is the desired one?

Solution: Analysis of the quadrants, $t_{ik,\mathrm{computed}} = \arctan\frac{\Delta y}{\Delta x}$



Remark: “atan2”

7.3 Adjustment of Horizontal Surveys: Triangulation

Functional model:

Measured from "i" to "k"

$$\boxed{r_{ik} = \arctan \frac{y_k - y_i}{x_k - x_i} - \omega_i}$$

Attention:
Quadrants!

Observation equations:

$$\boxed{r_{ik} + v_{ik} = \arctan \frac{\hat{y}_k - \hat{y}_i}{\hat{x}_k - \hat{x}_i} - \widehat{\omega}_i}$$

Nonlinear functional model

$\rightarrow$ for least squares adjustment we need a linearised functional model

$\rightarrow$ Jacobian matrix with partial derivatives

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Partial derivatives:

we know $(\arctan x)' = \frac{1}{1+x^2}$

$$\frac{\partial r_{ik}}{\partial y_k} = \frac{\partial\left(\arctan\frac{y_k-y_i}{x_k-x_i} - \omega_i\right)}{\partial y_k} = \frac{1}{1+\left(\frac{y_k-y_i}{x_k-x_i}\right)^2} \cdot \frac{1}{x_k-x_i}$$

$$= \frac{1}{1+\frac{(y_k-y_i)^2}{(x_k-x_i)^2}} \cdot \frac{1}{x_k-x_i} = \frac{1}{\frac{(x_k-x_i)^2}{(x_k-x_i)^2}+\frac{(y_k-y_i)^2}{(x_k-x_i)^2}} \cdot \frac{1}{x_k-x_i}$$

$$= \frac{(x_k-x_i)^2}{(x_k-x_i)^2+(y_k-y_i)^2} \cdot \frac{1}{x_k-x_i} = \frac{x_k-x_i}{s_{ik}^2} = \frac{\Delta x_{ik}}{s_{ik}^2}$$

7.3 Adjustment of Horizontal Surveys: Triangulation

$$\left.\frac{\partial r_{ik}}{\partial y_i}, \frac{\partial r_{ik}}{\partial x_k}, \frac{\partial r_{ik}}{\partial x_i}\right\}$$ See handout! $$\frac{\partial r_{ik}}{\partial \omega_i} = -1$$

2. Directions $$r_{ik} = \arctan\left(\frac{y_k - y_i}{x_k - x_i}\right) - \omega_i$$

Partial Derivatives:

$$\frac{\partial r_{ik}}{\partial y_k} = \frac{x_k - x_i}{s_{ik}^{\;2}} = \frac{\Delta x_{ik}}{s_{ik}^{\;2}} = \frac{\cos t_{ik}}{s_{ik}} \qquad \frac{\partial r_{ik}}{\partial y_i} = -\frac{x_k - x_i}{s_{ik}^{\;2}} = -\frac{\Delta x_{ik}}{s_{ik}^{\;2}} = -\frac{\cos t_{ik}}{s_{ik}}$$

$$\frac{\partial r_{ik}}{\partial x_k} = -\frac{y_k - y_i}{s_{ik}^{\;2}} = -\frac{\Delta y_{ik}}{s_{ik}^{\;2}} = -\frac{\sin t_{ik}}{s_{ik}} \qquad \frac{\partial r_{ik}}{\partial x_i} = \frac{y_k - y_i}{s_{ik}^{\;2}} = \frac{\Delta y_{ik}}{s_{ik}^{\;2}} = \frac{\sin t_{ik}}{s_{ik}} \qquad \frac{\partial r_{ik}}{\partial \omega_i} = -1$$

Adjustment_Theory_I_Derivatives.pdf