6.6 Least squares adjustment without matrix notation

Example:

Functional model:

$$
\begin{aligned}
3.0 &= x + y \\
1.5 &= 2x - y \\
0.2 &= x - y
\end{aligned}
$$

Values	$3.0, 1.5, 0.2$	are observations
Parameters	x, y	are unknowns

Stochastic model for the observations:

$$p_1 = 1, \qquad p_2 = 1, \qquad p_3 = 1$$

6.6 Least squares adjustment without matrix notation

Observation equations:

$$3.0 + v_1 = \hat{x} + \hat{y}$$
$$1.5 + v_2 = 2\hat{x} - \hat{y}$$
$$0.2 + v_3 = \hat{x} - \hat{y}$$

Rearranging:

$$v_1 = \hat{x} + \hat{y} - 3.0$$
$$v_2 = 2\hat{x} - \hat{y} - 1.5$$
$$v_3 = \hat{x} - \hat{y} - 0.2$$

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

→ $1 \cdot (\hat{x} + \hat{y} - 3.0)^2 + 1 \cdot (2\hat{x} - \hat{y} - 1.5)^2 + 1 \cdot (\hat{x} - \hat{y} - 0.2)^2 \rightarrow \min$

Normal equations:

$$\Sigma p_i v_i^2 = 1 \cdot (\hat{x} + \hat{y} - 3.0)^2 + 1 \cdot (2\hat{x} - \hat{y} - 1.5)^2 + 1 \cdot (\hat{x} - \hat{y} - 0.2)^2$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial \hat{x}} = \boxed{2(\hat{x} + \hat{y} - 3.0) + 2 \cdot (2\hat{x} - \hat{y} - 1.5) \cdot 2 + 2 \cdot (\hat{x} - \hat{y} - 0.2) = 0}$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial \hat{y}} = \boxed{2(\hat{x} + \hat{y} - 3.0) + 2 \cdot (2\hat{x} - \hat{y} - 1.5) \cdot (-1) + 2 \cdot (\hat{x} - \hat{y} - 0.2) \cdot (-1) = 0}$$

<u>normal equations</u>

$$\hat{x} + \hat{y} - 3.0 + 4\hat{x} - 2\hat{y} - 3.0 + \hat{x} - \hat{y} - 0.2 = 0$$

$$\hat{x} + \hat{y} - 3.0 - 2\hat{x} + \hat{y} + 1.5 - \hat{x} + \hat{y} + 0.2 = 0$$

$$6\hat{x} - 2\hat{y} = 6.2 \qquad (1)$$

$$-2\hat{x} + 3\hat{y} = 1.3 \qquad (2)$$

6.6 Least squares adjustment without matrix notation

Solution of normal equations:

$$(1) + 3 \cdot (2): \qquad 7\hat{y} = 10.1 \qquad \Rightarrow \qquad \hat{y} = 1.443$$

$$\hat{y} \text{ in } (1): \qquad \hat{x} = \frac{6.2 + 2\hat{y}}{6} \qquad \Rightarrow \qquad \hat{x} = 1.514$$

Residuals:

Can be computed from observation equations

$$\begin{aligned} v_1 &= -0.044 \\ v_2 &= 0.085 \\ v_3 &= -0.128 \end{aligned}$$

6.6 Least squares adjustment without matrix notation

Same example, but now we know the precision (standard deviation) s_i of the measured values

$$
\begin{aligned}
3.0, \quad & s_1 = 4\ \text{cm} \\
1.5, \quad & s_2 = 2\ \text{cm} \\
0.2, \quad & s_3 = 1\ \text{cm}
\end{aligned}
$$

Stochastic model:

How to obtain weights p_1, p_2, p_3?

$$p_1 = \frac{1}{(s_1)^2} \qquad p_2 = \frac{1}{(s_2)^2} \qquad p_3 = \frac{1}{(s_3)^2}$$

$$p_1 = \frac{1}{16} \qquad p_2 = \frac{1}{4} \qquad p_3 = 1$$

6.6 Least squares adjustment without matrix notation

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

$$\rightarrow \quad \frac{1}{16} \cdot (\hat{x} + \hat{y} - 3.0)^2 + \frac{1}{4} \cdot (2\hat{x} - \hat{y} - 1.5)^2 + 1 \cdot (\hat{x} - \hat{y} - 0.2)^2 \rightarrow \min$$

$$\frac{\Sigma p_i v_i^2}{\partial \hat{x}} = 2 \cdot \frac{1}{16}(\hat{x} + \hat{y} - 3.0) + 2 \cdot \frac{1}{4}(2\hat{x} - \hat{y} - 1.5) \cdot 2 + 2 \cdot (\hat{x} - \hat{y} - 0.2) = 0$$

$$\frac{\Sigma p_i v_i^2}{\partial \hat{y}} = 2 \cdot \frac{1}{16}(\hat{x} + \hat{y} - 3.0) + 2 \cdot \frac{1}{4}(2\hat{x} - \hat{y} - 1.5)(-1) + 2 \cdot (\hat{x} - \hat{y} - 0.2)(-1) = 0$$

6.6 Least squares adjustment without matrix notation

$$\hat{x} + \hat{y} - 3.0 \ + 16\hat{x} - 8\hat{y} - 12 + 16\hat{x} - 16\hat{y} - 3.2 = 0$$

$$\hat{x} + \hat{y} - 3.0 \ + 8\hat{x} + 4\hat{y} - 6 - 16\hat{x} + 16\hat{y} + 3.2 = 0$$

$$33\hat{x} - 23\hat{y} = 18.2 \qquad (1)$$

$$-23\hat{x} + 21\hat{y} = -6.2 \qquad (2)$$

Solution of normal equations:

$21 \cdot (1) + 23 \cdot (2)$: $\quad (21 \cdot 33 - 23 \cdot 23)\hat{x} = (21 \cdot 18.2) - (23 \cdot 6.2)$

$$164\,\hat{x} = 239.6$$

$$\hat{x} = 1.4610$$

$\hat{x}$ in (1): $\quad \hat{y} = 1.3049$

Residuals:

Solution $\hat{x}, \hat{y}$ in observation equations → v_i

6.6 Least squares adjustment without matrix notation

$$\hat{x} + \hat{y} - 3.0 \ + 16\hat{x} - 8\hat{y} - 12 + 16\hat{x} - 16\hat{y} - 3.2 = 0$$

$$\hat{x} + \hat{y} - 3.0 \ + 8\hat{x} + 4\hat{y} - 6 - 16\hat{x} + 16\hat{y} + 3.2 = 0$$

$$33\hat{x} - 23\hat{y} = 18.2 \qquad (1)$$

$$-23\hat{x} + 21\hat{y} = -6.2 \qquad (2)$$

Solution of normal equations:

$$\underbrace{\begin{bmatrix} 33 & -23 \\ -23 & 21 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}}_{\hat{\mathbf{X}}} = \underbrace{\begin{bmatrix} 18.2 \\ -6.2 \end{bmatrix}}_{\mathbf{n}}$$

$$\mathbf{N}\hat{\mathbf{X}} = \mathbf{n}$$

$$\hat{\mathbf{X}} = \mathbf{N}^{-1}\mathbf{n}$$

6.7 Least squares adjustment in matrix notation

Now: Application of matrices to build normal equations

6.7.1 Linear functional models

▶ Observations:

$$
\begin{aligned}
&3.0, \qquad s_1 = 4\ \text{cm}\\
&1.5, \qquad s_2 = 2\ \text{cm}\\
&0.2, \qquad s_3 = 1\ \text{cm}
\end{aligned}
$$

Observation vector:

$$\mathbf{L}_{n\times 1} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{bmatrix} \qquad \text{here} \qquad \mathbf{L}_{3\times 1} = \begin{bmatrix} 3.0 \\ 1.5 \\ 0.2 \end{bmatrix}$$

6.7.1 Linear functional models

Stochastic model of $\mathbf{L}$:

- For theoretical standard deviations σ_i: ($\mathbf{\Sigma}_{LL}$: VCM of $\mathbf{L}$)

$$\mathbf{\Sigma}_{LL_{n\times n}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

- For empirical standard deviations s_i:

$$\mathbf{S}_{LL_{n\times n}} = \begin{bmatrix} s_1^2 & s_{12} & \cdots & s_{1n} \\ s_{21} & s_2^2 & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_n^2 \end{bmatrix}$$

6.7.1 Linear functional models

▶ We choose arbitrary value σ_0

$$\boldsymbol{\Sigma}_{LL} = \sigma_0^2 \cdot \mathbf{Q}_{LL_{n\times n}} \qquad \Rightarrow \qquad \mathbf{Q}_{LL_{n\times n}} = \frac{1}{\sigma_0^2} \cdot \boldsymbol{\Sigma}_{LL_{n\times n}}$$

- σ_0: theoretical reference standard deviation or
  reference standard deviation à priori
- $\mathbf{Q}_{LL}$: Cofactor matrix of $\mathbf{L}$
- $\mathbf{P}$: Weight matrix of $\mathbf{L}$

$$\boxed{\mathbf{P}_{n\times n} = \mathbf{Q}_{LL}^{-1}{}_{n\times n}}$$

$\mathbf{Q}_{LL}$ regular

6.7.1 Linear functional models

- In our example:

$$\sigma_0 = 1 \qquad \text{(usually used in practice)}$$

$$\mathbf{S}_{LL} = \mathbf{Q}_{LL} = \begin{bmatrix} (4\ \text{cm})^2 & 0 & 0 \\ 0 & (2\ \text{cm})^2 & 0 \\ 0 & 0 & (1\ \text{cm})^2 \end{bmatrix}$$

$$\Rightarrow \mathbf{P} = \mathbf{Q}_{LL}^{-1} = \begin{bmatrix} \frac{1}{16\ \text{cm}^2} & 0 & 0 \\ 0 & \frac{1}{4\ \text{cm}^2} & 0 \\ 0 & 0 & \frac{1}{1\ \text{cm}^2} \end{bmatrix}$$

6.7.1 Linear functional models

Functional model:

$$
\begin{aligned}
3.0 &= 1x + 1y \\
1.5 &= 2x - 1y \\
0.2 &= 1x - 1y
\end{aligned}
\qquad
\underbrace{\begin{bmatrix} 3.0 \\ 1.5 \\ 0.2 \end{bmatrix}}_{\mathbf{L}_{n\times 1}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{A}_{n\times u}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{X}_{u\times 1}}
$$

- Observation vector $\mathbf{L}_{n\times 1}$
- Matrix with coefficients of the linear functional model
  → coefficient matrix or design matrix $\mathbf{A}_{n\times u}$
- Vector of unknowns $\mathbf{X}_{u\times 1}$

$$\boxed{\mathbf{L}_{n\times 1} = \mathbf{A}_{n\times u}\mathbf{X}_{u\times 1}}$$

6.7.1 Linear functional models

Observation equations:

$$\begin{aligned} 3.0 + v_1 &= \hat{x} + \hat{y} \\ 1.5 + v_2 &= 2\hat{x} - \hat{y} \\ 0.2 + v_3 &= \hat{x} - \hat{y} \end{aligned}$$

Vector of residuals:

$$\mathbf{v}_{n\times1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{here} \quad \mathbf{v}_{3\times1} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Vector of adjusted unknowns:

$$\hat{\mathbf{X}}_{u\times1} = \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \vdots \\ \hat{X}_u \end{bmatrix} \quad \text{here} \quad \hat{\mathbf{X}}_{2\times1} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

6.7.1 Linear functional models

$$\boxed{\mathbf{L}_{n\times 1} + \mathbf{v}_{n\times 1} = \mathbf{A}_{n\times m}\widehat{\mathbf{X}}_{u\times 1}}$$

$\widehat{\ }$: adjusted value

$$\mathbf{v} = \mathbf{A}\widehat{\mathbf{X}} - \mathbf{L}$$

$$\Omega = \sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

in matrix notation $\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \rightarrow \min$

with $\mathbf{v} = \mathbf{A}\widehat{\mathbf{X}} - \mathbf{L}$

6.7.1 Linear functional models

$$\Omega = \left(\mathbf{A}\hat{\mathbf{X}} - \mathbf{L}\right)^{\mathrm{T}}\mathbf{P}\left(\mathbf{A}\hat{\mathbf{X}} - \mathbf{L}\right)$$

$$= \left(\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} - \mathbf{L}^{\mathrm{T}}\right)\mathbf{P}\left(\mathbf{A}\hat{\mathbf{X}} - \mathbf{L}\right)$$

$$= (\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{L}^{\mathrm{T}}\mathbf{P})(\mathbf{A}\hat{\mathbf{X}} - \mathbf{L})$$

$$= \hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{X}} - \hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} - \mathbf{L}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{X}} + \mathbf{L}^{\mathrm{T}}\mathbf{P}\mathbf{L}$$

$1 \times u \quad u \times n \quad n \times n \quad n \times 1$

$$= \hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{X}} - \underbrace{\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}}_{\text{scalar}} - \underbrace{\left(\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}\right)^{\mathrm{T}}}_{\text{scalar}} + \mathbf{L}^{\mathrm{T}}\mathbf{P}\mathbf{L}$$

$$= \hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{X}} - 2 \cdot \hat{\mathbf{X}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{P}\mathbf{L}$$

Minimum:

$$\frac{\partial\Omega}{\partial\hat{\mathbf{X}}^{\mathrm{T}}} = 2 \cdot \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{X}} - 2 \cdot \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} = 0$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{X}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}$$

6.7.1 Linear functional models

Normal Equations:

$$\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}}_{\substack{\mathbf{N}\\u\times u}}\hat{\mathbf{X}} = \underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}}_{\substack{\mathbf{n}\\u\times 1}}$$

- $\mathbf{N}_{u\times u} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}$: Normal matrix
- $\mathbf{n}_{u\times 1} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}$: Right hand side of normal equations

If $\mathbf{N}$ regular → we can compute $\mathbf{N}^{-1}$

Solution of normal equations:

$$\hat{\mathbf{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}$$

$$\hat{\mathbf{X}} = \mathbf{N}^{-1}\mathbf{n}$$

Residuals:

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{X}} - \mathbf{L}$$

6.7.1 Linear functional models

Adjusted observations:

$$\boxed{\hat{\mathbf{L}}_{n\times 1} = \mathbf{L}_{n\times 1} + \mathbf{v}_{n\times 1}}$$

Final check:

$$\boxed{\hat{\mathbf{L}} = \mathbf{\Phi}(\hat{\mathbf{X}})}$$

Original functional model

$$\hat{\mathbf{L}} - \mathbf{\Phi}(\hat{\mathbf{X}}) \overset{!}{=} \mathbf{0}$$

zero within computing precision

6.7.1 Linear functional models

▶ Linear functional model

→ we obtain linear normal equations

→ very easy to solve (linear algebra)

▶ Empirical reference standard deviation (or empirical reference variance s_0^2)

- $n =$ number of observations
- $u =$ number of unknowns

$$s_0 = \sqrt{\frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$$

VCM for the vector of unknowns $\hat{\mathbf{X}}$:

- We know

$$\hat{\mathbf{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} \qquad (1)$$

→ For $\mathbf{L}$ we have given $\mathbf{\Sigma}_{LL}$ resp. $\mathbf{S}_{LL}$ → $\mathbf{Q}_{LL}$

- Question: What is the VCM ($\mathbf{\Sigma}_{\hat{X}\hat{X}}$) of $\hat{\mathbf{X}}$

- From Variance-Covariance Propagation (VCP), see Section 4.4, we know

$$\mathbf{x} = \mathbf{F}\mathbf{L}, \qquad \mathbf{Q}_{LL}$$

$$\mathbf{Q}_{xx} = \mathbf{F}\mathbf{Q}_{LL}\mathbf{F}^{\mathrm{T}}$$

6.7.2 Variance-Covariance Matrices (VCM) for the results

- Now we apply VCP to (1) $\hat{\mathbf{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}$:

F

$$\mathbf{Q}_{\hat{X}\hat{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\underbrace{\mathbf{P}\mathbf{Q}_{LL}}_{\mathbf{I}}\mathbf{P}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}$$

$$= \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}}_{\mathbf{I}}$$

$$\boxed{\mathbf{Q}_{\hat{X}\hat{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}}$$ Cofactor Matrix of the unknowns

$$\mathbf{Q}_{\hat{X}\hat{X}} = \mathbf{N}^{-1}$$ Inverse of normal matrix

$$\boldsymbol{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \cdot \mathbf{Q}_{\hat{X}\hat{X}}$$ VCM of the unknowns

VCM for functions of the parameters $\hat{\mathbf{X}}$:

For any linear function of the parameters

$$\mathbf{f} = \mathbf{F}\hat{\mathbf{X}}$$

we can apply VCP to obtain the cofactor matrix of $\mathbf{f}$ as

$$\mathbf{Q}_{ff} = \mathbf{F}\mathbf{Q}_{XX}\mathbf{F}^{\mathrm{T}} \qquad (2)$$

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1. Cofactor matrix $\mathbf{Q}_{\hat{L}\hat{L}}$ for the adjusted observations:

We know

$$\hat{\mathbf{L}} = \mathbf{L} + \mathbf{v} = \mathbf{A}\hat{\mathbf{X}}$$

Application of (2) yields

- Cofactor matrix of the adjusted observations:

$$\boxed{\mathbf{Q}_{\hat{L}\hat{L}} = \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}}$$

- VCM of the adjusted observations:

$$\mathbf{\Sigma}_{\hat{L}\hat{L}} = s_0^2 \cdot \mathbf{Q}_{\hat{L}\hat{L}}$$

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1. Cofactor matrix $\mathbf{Q}_{vv}$ for the residuals:

Residuals are obtained from

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{X}} - \mathbf{L}$$

with

$$\hat{\mathbf{X}} = \underbrace{\mathbf{N}^{-1}}_{\mathbf{Q}_{\hat{X}\hat{X}}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L}$$

$$\mathbf{v} = \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} - \mathbf{L}$$

$$\mathbf{v} = \left(\mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{I}\right)\mathbf{L}$$

→ Residuals as function of the observations

→ Application of VCP

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$$\mathbf{Q}_{vv} = \left(\mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{I}\right)\mathbf{Q}_{LL}\left(\mathbf{P}\mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}} - \mathbf{I}\right)$$

$$= \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\underbrace{\mathbf{A}^{\mathrm{T}}\underbrace{\mathbf{P}\mathbf{Q}_{LL}}_{\mathbf{I}}\mathbf{P}\mathbf{A}}_{\mathbf{Q}_{\hat{X}\hat{X}}^{-1}}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}} - \underbrace{\mathbf{Q}_{LL}\mathbf{P}}_{\mathbf{I}}\mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}\underbrace{\mathbf{P}\mathbf{Q}_{LL}}_{\mathbf{I}} + \mathbf{Q}_{LL}$$

$$= \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}} + \mathbf{Q}_{LL}$$

$$= \mathbf{Q}_{LL} - \underbrace{\mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}}_{\mathbf{Q}_{\hat{L}\hat{L}}}$$

- Cofactor matrix of the residuals:

$$\boxed{\mathbf{Q}_{vv} = \mathbf{Q}_{LL} - \mathbf{Q}_{\hat{L}\hat{L}}}$$

- VCM of the residuals:

$$\mathbf{\Sigma}_{vv} = s_0^2 \cdot \mathbf{Q}_{vv}$$

6.7.3 Linear functional models, Summary

Least-squares Adjustment for Linear Adjustment Problems

Linear functional model for the unknowns:

$$\begin{aligned} L_1 &= a_{11}X_1 + a_{12}X_2 + \cdots + a_{1u}X_u \\ L_2 &= a_{21}X_1 + a_{22}X_2 + \cdots + a_{2u}X_u \\ &\vdots \\ L_n &= a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nu}X_u \end{aligned}$$

Vector of observations:

$$\underset{n,1}{\mathbf{L}} = \begin{bmatrix} L_1 & L_2 & \cdots & L_n \end{bmatrix}^{\mathrm{T}}$$

Variance covariance matrix of the observations:

$$\underset{n,n}{\mathbf{\Sigma_{LL}}} = \begin{bmatrix} \sigma_{L_1}^2 & \sigma_{L_1L_2} & \cdots & \sigma_{L_1L_n} \\ \sigma_{L_2L_1} & \sigma_{L_2}^2 & \cdots & \sigma_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_nL_1} & \sigma_{L_nL_2} & \cdots & \sigma_{L_n}^2 \end{bmatrix}$$ with theoretical values σ_i

$$\underset{n,n}{\mathbf{S_{LL}}} = \begin{bmatrix} s_{L_1}^2 & s_{L_1L_2} & \cdots & s_{L_1L_n} \\ s_{L_2L_1} & s_{L_2}^2 & \cdots & s_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{L_nL_1} & s_{L_nL_2} & \cdots & s_{L_n}^2 \end{bmatrix}$$ with empirical values s_i

6.7.3 Linear functional models, Summary

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Theoretical reference standard deviation:	σ_0 (or theoretical reference variance σ_0^2)
Cofactor matrix of the observations:	$\underset{n,n}{\mathbf{Q}_{\mathrm{LL}}} = \frac{1}{\sigma_0^2}\underset{n,n}{\boldsymbol{\Sigma}_{\mathrm{LL}}}$ respectively $\underset{n,n}{\mathbf{Q}_{\mathrm{LL}}} = \frac{1}{\sigma_0^2}\underset{n,n}{\mathbf{S}_{\mathrm{LL}}}$
Weight matrix of the observations:	$\underset{n,n}{\mathbf{P}} = \underset{n,n}{\mathbf{Q}_{\mathrm{LL}}^{-1}}$
Vector of adjusted unknowns:	$\underset{u,1}{\hat{\mathbf{X}}} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \cdots & \hat{X}_u \end{bmatrix}^{\mathrm{T}}$
Matrix of coefficients of the linear functional model:	$\underset{n,u}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1u} \\ a_{21} & a_{22} & \cdots & a_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nu} \end{bmatrix}$ "Design Matrix"
Vector of residuals:	$\underset{n,1}{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{\mathrm{T}}$
Observation equations:	$\underset{n,1}{\mathbf{L}} + \underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{X}}}$
Normal equations:	$\underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{L}}$
Normal matrix:	$\underset{u,u}{\mathbf{N}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}$
Right hand side of normal equations:	$\underset{u,1}{\mathbf{n}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{L}}$
Normal equations:	$\underset{u,u}{\mathbf{N}}\,\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,1}{\mathbf{n}}$

6.7.3 Linear functional models, Summary

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Inversion of normal matrix:	$\underset{u,u}{\mathbf{Q}_{\hat{X}\hat{X}}} = \underset{u,u}{\mathbf{N}^{-1}}$
Solution for the unknowns:	$\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,u}{\mathbf{Q}_{\hat{X}\hat{X}}}\ \underset{u,1}{\mathbf{n}}$
Vector of residuals:	$\underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\ \underset{u,1}{\hat{\mathbf{X}}} - \underset{n,1}{\mathbf{L}}$
Vector of adjusted observations:	$\underset{n,1}{\hat{\mathbf{L}}} = \underset{n,1}{\mathbf{L}} + \underset{n,1}{\mathbf{v}}$
Final check:	$\underset{n,1}{\hat{\mathbf{L}}} \overset{!}{=} \underset{n,1}{\boldsymbol{\Phi}}\left(\underset{u,1}{\hat{\mathbf{X}}}\right)$
Empirical reference standard deviation:	$s_0 = \sqrt{\dfrac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$ (or empirical reference variance s_0^2)

6.7.3 Linear functional models, Summary

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Cofactor matrix of adjusted unknowns:	$\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}$
VCM of adjusted unknowns:	$\underset{u,u}{\mathbf{\Sigma}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = s_0^2 \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}$
Cofactor matrix of adjusted observations:	$\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = \underset{n,u}{\mathbf{A}} \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} \underset{u,n}{\mathbf{A}^{\mathrm{T}}}$
VCM of adjusted observations:	$\underset{n,n}{\mathbf{\Sigma}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = s_0^2 \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$
Cofactor matrix of the residuals:	$\underset{n,n}{\mathbf{Q}_{\mathbf{vv}}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} - \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$
VCM of the residuals:	$\underset{n,n}{\mathbf{\Sigma}_{\mathbf{vv}}} = s_0^2 \underset{n,n}{\mathbf{Q}_{\mathbf{vv}}}$

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