6.7.4 Nonlinear functional models

Example:

Find least squares solution for the following system of nonlinear equations

Functional model:

$$
\begin{aligned}
x + y - 2y^2 &= -4 \\
x^2 + y^2 &= 8 \\
3x^2 - y^2 &= 7.7
\end{aligned}
$$

Stochastic model for the observation:

$$p_1 = 1, \qquad p_2 = 1, \qquad p_3 = 1$$

6.7.4 Nonlinear functional models

Observation equations:

$$-4 + v_1 = \hat{x} + \hat{y} - 2\hat{y}^2$$

$$8 + v_2 = \hat{x}^2 + \hat{y}^2$$

$$7.7 + v_3 = 3\hat{x}^2 - \hat{y}^2$$

Rearranging:

$$v_1 = \hat{x} + \hat{y} - 2\hat{y}^2 + 4$$

$$v_2 = \hat{x}^2 + \hat{y}^2 - 8$$

$$v_3 = 3\hat{x}^2 - \hat{y}^2 - 7.7$$

$$\sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

$$\rightarrow \underbrace{1}_{p_1} \cdot \underbrace{(\hat{x} + \hat{y} - 2\hat{y}^2 + 4)^2}_{v_1^2} + 1 \cdot (\hat{x}^2 + \hat{y}^2 - 8)^2 + 1 \cdot (3\hat{x}^2 - \hat{y}^2 - 7.7)^2 \rightarrow \min$$

6.7.4 Nonlinear functional models

$$\frac{\partial \Sigma p_i v_i^2}{\partial x} =$$
$$2(\hat{x} + \hat{y} - 2\hat{y}^2 + 4) + 2(\hat{x}^2 + \hat{y}^2 - 8)(2\hat{x}) + 2(3\hat{x}^2 - \hat{y}^2 - 7.7)(6\hat{x}) = 0$$

$$\frac{\partial \Sigma p_i v_i^2}{\partial y} =$$
$$2(\hat{x} + \hat{y} - 2\hat{y}^2 + 4)(1 - 4\hat{y}) + 2(\hat{x}^2 + \hat{y}^2 - 8)(2\hat{y}) + 2(3\hat{x}^2 - \hat{y}^2 - 7.7)(-2\hat{y}) = 0$$

Nonlinear normal equations:

$$\boxed{\begin{gathered}\hat{x} + \hat{y} - 2\hat{y}^2 + 4 + 2\hat{x}^3 + 2\hat{x}\hat{y}^2 - 16\hat{x} + 18\hat{x}^3 - 6\hat{x}\hat{y}^2 - 46.2\hat{x} = 0 \\ \hat{x} - 6\hat{y}^2 + 4 - 4\hat{x}\hat{y} + 8\hat{y}^3 - 15\hat{y} + 2\hat{x}^2\hat{y} + 2\hat{y}^3 - 16\hat{y} - 6\hat{x}^2\hat{y} + 2\hat{y}^3 + 15.4\hat{y} = 0\end{gathered}}$$

<u>nonlinear normal equations</u>

6.7.4 Nonlinear functional models

▶ Nonlinear functional model → nonlinear normal equations

→ Question: How to solve nonlinear normal equations?

a) Direct solution (if possible, e.g. for quadratic equations), or
b) Linearisation (Taylor-Approximation) of normal equations and iterative solution, or
c) Heuristic Approaches, or
d) Replace the original functional model by a linearised functional model and iterative solution

6.7.4 Nonlinear functional models

Solution from linearisation of the original functional model

→ We replace our nonlinear functional model by an approximation (linear functional model)

Attention: We have to compute an iterative solution!

6.7.4 Nonlinear functional models

Nonlinear functional model:

$$
\begin{aligned}
-4.0 &= x + y - 2y^2 \\
8.0 &= x^2 + y^2 \\
7.7 &= 3x^2 - y^2
\end{aligned}
$$

$$
\mathbf{L}_{n\times1} = \mathbf{\Phi}_{n\times1}(\mathbf{X}_{u\times1}) = \begin{bmatrix} \varphi_1(\mathbf{X}_{u\times1}) \\ \varphi_2(\mathbf{X}_{u\times1}) \\ \vdots \\ \varphi_n(\mathbf{X}_{u\times1}) \end{bmatrix}
$$

Question: How can we find a linear approximation?

→ Taylor series approximation for each equation

6.7.4 Nonlinear functional models

Taylor series approximation:

$$\mathbf{L}_{n\times1} = \mathbf{\Phi}_{n\times1}\left(\mathbf{X}_{u\times1}\right) = \underbrace{\mathbf{\Phi}_{n\times1}(\mathbf{X}^0_{u\times1}) + \left(\frac{\partial\mathbf{\Phi}(\mathbf{X})}{\partial\mathbf{X}}\right)_{0\,n\times u} (\mathbf{X}-\mathbf{X}^0)_{u\times1}}_{\text{First-order Taylor series approximation at the place of the initial values } \mathbf{X}^0} + \cdots$$

Start of iterative computing

$\mathbf{X}^0$: Initial values (starting vales, approximations) for the unknowns

Initial values have to be close to the final solution, otherwise

- computation will not converge, or
- computation can converge to a local min for $\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \to \min$ → worst case!

6.7.4 Nonlinear functional models

We introduce:

$\mathbf{L}^0_{n\times 1} = \mathbf{\Phi}_{n\times 1}(\mathbf{X}^0_{u\times 1})$	observations as functions of the unknowns
$\mathbf{l}_{n\times 1} = \mathbf{L}_{n\times 1} - \mathbf{L}^0_{n\times 1}$	vector of reduced observations
$\mathbf{\Sigma}_{ll} = \mathbf{\Sigma}_{LL}$	from variance covariance propagation with functional model $\mathbf{l} = \mathbf{L} - \mathbf{L}^0$
$\mathbf{x}_{u\times 1} = \mathbf{X} - \mathbf{X}^0$	vector of reduced unknowns vector of corrections for the unknowns
$\mathbf{J}_{n\times u} = \left(\frac{\partial \mathbf{\Phi}(\mathbf{X})}{\partial \mathbf{X}}\right)_{0_{n\times u}}$	Jacobian matrix, contains the partial derivatives of each equation of the functional model

6.7.4 Nonlinear functional models

Linearised functional model:

$$\underbrace{\mathbf{L} - \underbrace{\boldsymbol{\Phi}(\mathbf{X}^0)}_{\mathbf{L}^0}}_{\mathbf{l}} = \underbrace{\left(\frac{\partial \boldsymbol{\Phi}(\mathbf{X})}{\partial \mathbf{X}}\right)_0}_{\mathbf{J}} \underbrace{(\mathbf{X} - \mathbf{X}^0)}_{\mathbf{x}}$$

$$\boxed{\mathbf{l}_{n\times 1} = \mathbf{J}_{n\times u}\mathbf{x}_{u\times 1}}$$

with

$$\mathbf{l}_{n\times 1} = \begin{bmatrix} L_1 - \varphi_1(\mathbf{X}^0) \\ L_2 - \varphi_2(\mathbf{X}^0) \\ \vdots \\ L_n - \varphi_n(\mathbf{X}^0) \end{bmatrix}$$

6.7.4 Nonlinear functional models

and with

$$\mathbf{J}_{n\times u} = \begin{bmatrix} \frac{\partial \varphi_1(\mathbf{X}^0)}{\partial X_1^0} & \frac{\partial \varphi_1(\mathbf{X}^0)}{\partial X_2^0} & \cdots & \frac{\partial \varphi_1(\mathbf{X}^0)}{\partial X_u^0} \\ \frac{\partial \varphi_2(\mathbf{X}^0)}{\partial X_1^0} & \frac{\partial \varphi_2(\mathbf{X}^0)}{\partial X_2^0} & \cdots & \frac{\partial \varphi_2(\mathbf{X}^0)}{\partial X_u^0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_n(\mathbf{X}^0)}{\partial X_1^0} & \frac{\partial \varphi_n(\mathbf{X}^0)}{\partial X_2^0} & \cdots & \frac{\partial \varphi_n(\mathbf{X}^0)}{\partial X_u^0} \end{bmatrix} = \begin{bmatrix} j_{11} & j_{12} & \cdots & j_{1u} \\ j_{21} & j_{22} & \cdots & j_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ j_{n1} & j_{n2} & \cdots & j_{nu} \end{bmatrix}$$

6.7.4 Nonlinear functional models

Now: We take elements of matrix $\mathbf{J}$ and insert them into matrix $\mathbf{A}$

$$\mathbf{A}_{n\times u} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1u} \\ a_{21} & a_{22} & \cdots & a_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nu} \end{bmatrix}$$

with

$$\begin{array}{cccc} a_{11} = j_{11}, & a_{12} = j_{12}, & \dots\,, & a_{1u} = j_{1u} \\ & \vdots & & \\ a_{n1} = j_{n1}, & a_{n2} = j_{n2}, & \dots\,, & a_{nu} = j_{nu} \end{array}$$

6.7.4 Nonlinear functional models

We obtain:

$$\boxed{\mathbf{l}_{n\times 1} = \mathbf{A}_{n\times u}\,\mathbf{x}_{u\times 1}}$$

Advantage of linearisation of the functional model:

We can apply the simple algorithm for the linear case!

Observation equations:

$$\boxed{\mathbf{l}_{n\times 1} + \mathbf{v}_{n\times 1} = \mathbf{A}_{n\times u}\,\hat{\mathbf{x}}_{u\times 1}}$$

with

6.7.4 Nonlinear functional models

Vector of residuals:

$$\mathbf{v}_{n\times 1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{here} \quad \mathbf{v}_{3\times 1} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Vector of adjusted reduced unknowns:

$$\hat{\mathbf{x}}_{u\times 1} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_u \end{bmatrix} \quad \text{here} \quad \hat{\mathbf{x}}_{2\times 1} = \begin{bmatrix} d\hat{x} \\ d\hat{y} \end{bmatrix}$$

6.7.4 Nonlinear functional models

$$\Omega = \sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$

In matrix notation: $\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \rightarrow \min$ with $\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}$

$$\Omega = (\mathbf{A}\hat{\mathbf{x}} - \mathbf{l})^{\mathrm{T}}\mathbf{P}(\mathbf{A}\hat{\mathbf{x}} - \mathbf{l})$$

$$\vdots$$

$$= \hat{\mathbf{x}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} - 2 \cdot \hat{\mathbf{x}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l} + \mathbf{l}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Minimum:

$$\frac{\partial\Omega}{\partial\hat{\mathbf{x}}^{\mathrm{T}}} = 2 \cdot \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} - 2 \cdot \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l} = 0$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

6.7.4 Nonlinear functional models

Normal Equations:

$$\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}}_{\substack{\mathbf{N}\\ u\times u}}\,\hat{\mathbf{x}} = \underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}}_{\substack{\mathbf{n}\\ u\times 1}}$$

- $\mathbf{N}_{u\times u} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}$: Normal matrix
- $\mathbf{n}_{u\times 1} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$: Right hand side of normal equations

If $\mathbf{N}$ regular → we can compute $\mathbf{N}^{-1}$

Solution of normal equations:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

$$\hat{\mathbf{x}} = \mathbf{N}^{-1}\mathbf{n}$$

What is $\hat{\mathbf{x}}$? $\rightarrow$ $\hat{\mathbf{x}}$ is a correction for the initial values $\mathbf{X}^0$

6.7.4 Nonlinear functional models

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Adjusted unknowns:

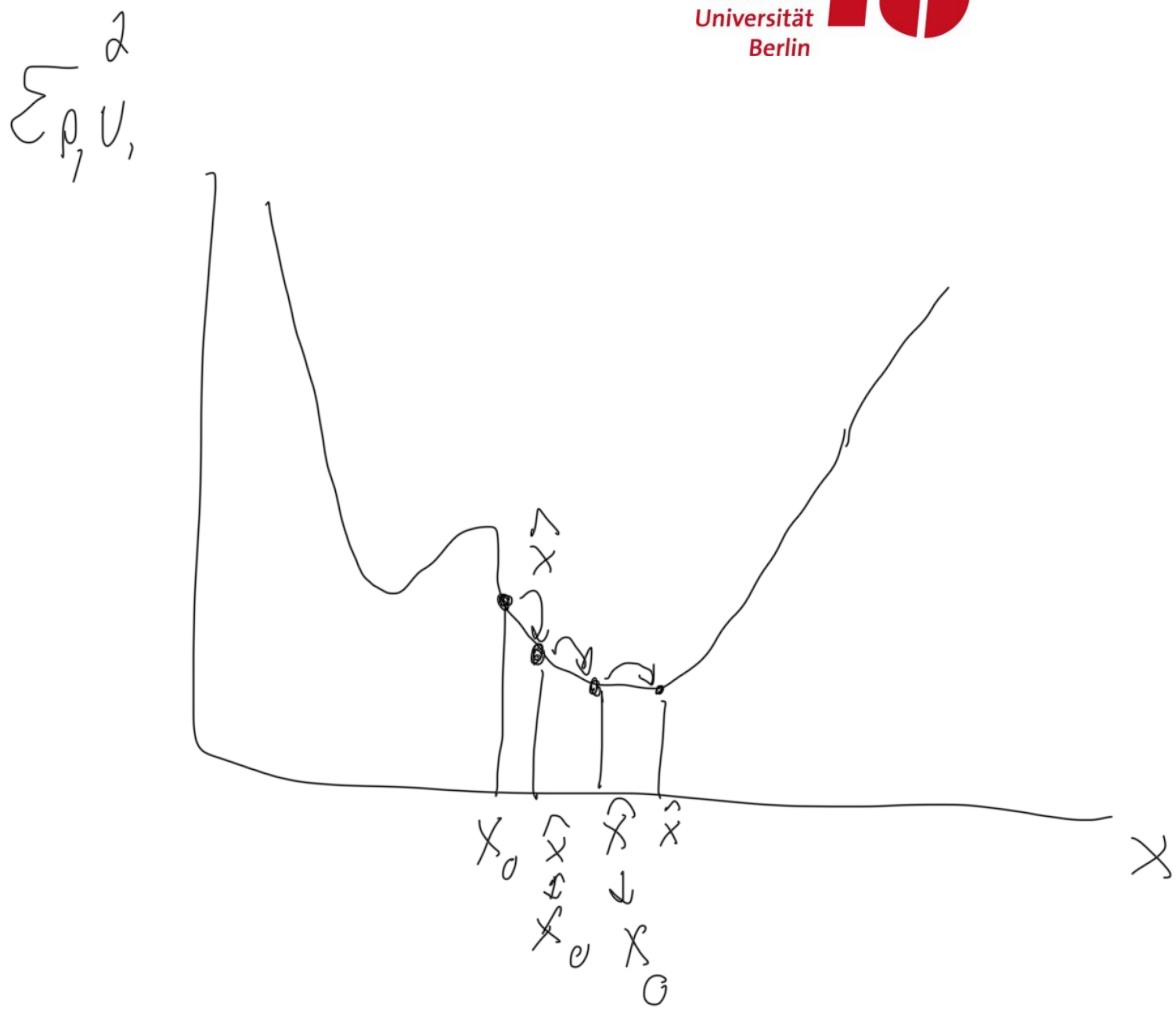
$$\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$$

Residuals:

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}$$

Adjusted observations:

$$\hat{\mathbf{L}} = \mathbf{L} + \mathbf{v}$$



6.7.4 Nonlinear functional models

Attention: $\widehat{\mathbf{X}}$ is solution for the linearised problem (approximation)

→ We have to perform iterative computation to reach final solution

→ We have to introduce $\widehat{\mathbf{X}}$ as new approximation $\mathbf{X}^0$ and perform the adjustment again and again!

Question: How often do we have to repeat the computation?

Answer: Until

a) the corrections for the unknowns $\hat{\mathbf{x}}$ become <u>sufficiently small</u>

b) final check $\hat{\mathbf{L}} - \mathbf{\Phi}(\widehat{\mathbf{X}}) \overset{!}{\approx} \mathbf{0}$ ← remaining linearization

6.7.4 Nonlinear functional models

a) „sufficiently small“

If our unknowns are coordinates and we want to obtain results with $\mathrm{mm}\ (10^{-3}\mathrm{m}) \rightarrow \max|\hat{x}_i|$ for $i = 1, \ldots, u \leq 10^{-5}$

Rule: $\max|\hat{x}_i|$ should be two orders of magnitude smaller than the resolution of the result

b) $\hat{\mathbf{L}} - \boldsymbol{\Phi}(\hat{\mathbf{X}}) \overset{!}{\approx} \mathbf{0}$ $\boldsymbol{\Phi}(\hat{\mathbf{X}})$: Original nonlinear functional model

Result contains:

- Computation error
- Linearisation error

→ “How far away is my solution for the linearized problem from the solution for the nonlinear problem”

6.7.4 Nonlinear functional models

Rule:

$\max \left| \hat{L}_i - \Phi_i(\hat{\mathbf{X}}) \right|$ should be smaller than 10^{-8}

→ If these break-off conditions are not met, continue with iterative computation

→ **Attention:** Introduce $\hat{\mathbf{X}}$ as new starting values $\mathbf{X}^0$ and start iterative computing at step "⇨"

6.7.5 Variance-Covariance Matrices (VCM) for the results

Precision measures (see also Section 6.7.2):

- Empirical reference standard deviation: $s_0 = \sqrt{\frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$
- $\mathbf{Q}_{\hat{X}\hat{X}} = \mathbf{Q}_{\hat{x}\hat{x}}$ from variance covariance propagation with functional model $\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$
- VCM of adjusted unknowns: $\boldsymbol{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \mathbf{Q}_{\hat{X}\hat{X}}$ with $\mathbf{Q}_{\hat{X}\hat{X}} = \mathbf{N}^{-1}$
- Cofactor matrix of adjusted observations: $\mathbf{Q}_{\hat{L}\hat{L}} = \mathbf{A}\mathbf{Q}_{\hat{X}\hat{X}}\mathbf{A}^{\mathrm{T}}$
- VCM of adjusted observations: $\boldsymbol{\Sigma}_{\hat{L}\hat{L}} = s_0^2 \mathbf{Q}_{\hat{L}\hat{L}}$
- Cofactor matrix of residuals: $\mathbf{Q}_{vv} = \mathbf{Q}_{LL} - \mathbf{Q}_{\hat{L}\hat{L}}$
- VCM of residuals: $\boldsymbol{\Sigma}_{vv} = s_0^2 \mathbf{Q}_{vv}$

6.7.6 Nonlinear functional models, Example

Example: Find least squares solution for the unknown parameters and compute their standard deviation

Functional model:

$$\begin{aligned}
\varphi_1:&\quad 4.0 = x + y - 2y^2 && (1)\\
\varphi_2:&\quad 8.0 = x^2 + y^2 && (2)\\
\varphi_3:&\quad 7.7 = 3x^2 - y^2 && (3)
\end{aligned}$$

Values $4.0, 8.0, 7.7$ are equally weighted and uncorrelated observations

Parameters x, y are unknowns

Observation vector:

$$\mathbf{L} = \begin{bmatrix} 4.0 \\ 8.0 \\ 7.7 \end{bmatrix}$$

6.7.6 Nonlinear functional models, Example

Stochastic model of the observations:

$$p_1 = 1, \quad p_2 = 1, \quad p_3 = 1$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Vector of adjusted unknowns:

$$\hat{\mathbf{X}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

Nonlinear functional model → solution from iterative computing with linearised functional model → introduction of approximate values x^0, y^0

6.7.6 Nonlinear functional models, Example

Vector of starting values:

$$\mathbf{X}^0 = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$$

Vector of adjusted reduced unknowns:

$$\hat{\mathbf{x}} = \begin{bmatrix} d\hat{x} \\ d\hat{y} \end{bmatrix} = \hat{\mathbf{X}} - \mathbf{X}^0 = \begin{bmatrix} \hat{x} - x^0 \\ \hat{y} - y^0 \end{bmatrix}$$

Vector of reduced observations:

$$\mathbf{l} = \begin{bmatrix} 4.0 - (x^0 + y^0 - 2(y^0)^2) \\ 8.0 - ((x^0)^2 + (y^0)^2) \\ 7.7 - (3(x^0)^2 - (y^0)^2) \end{bmatrix} = L - \Phi(X_0)$$

L $\qquad$ $\Phi(X_0)$

6.7.6 Nonlinear functional models, Example

Jacobian matrix:

$$
\mathbf{J} = \begin{array}{c} \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{array}
\begin{array}{c}
\begin{array}{cc} x^0 & y^0 \end{array} \\
\begin{bmatrix} \dfrac{\partial \varphi_1}{\partial x^0} & \dfrac{\partial \varphi_1}{\partial y^0} \\ \dfrac{\partial \varphi_2}{\partial x^0} & \dfrac{\partial \varphi_2}{\partial y^0} \\ \dfrac{\partial \varphi_3}{\partial x^0} & \dfrac{\partial \varphi_3}{\partial y^0} \end{bmatrix}
\end{array}
= \begin{array}{c} \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{array}
\begin{array}{c}
\begin{array}{cc} x^0 & y^0 \end{array} \\
\begin{bmatrix} 1 & (1-4y^0) \\ 2x^0 & 2y^0 \\ 6x^0 & -2y^0 \end{bmatrix}
\end{array}
$$

Design matrix:

$$
\mathbf{A} = \begin{bmatrix} 1 & (1-4y^0) \\ 2x^0 & 2y^0 \\ 6x^0 & -2y^0 \end{bmatrix}
$$

6.7.6 Nonlinear functional models, Example

Normal equations:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Solutions of normal equations:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Adjusted unknowns:

$$\widehat{\mathbf{X}} = \mathbf{X}^{0} + \hat{\mathbf{x}}$$

Residuals:

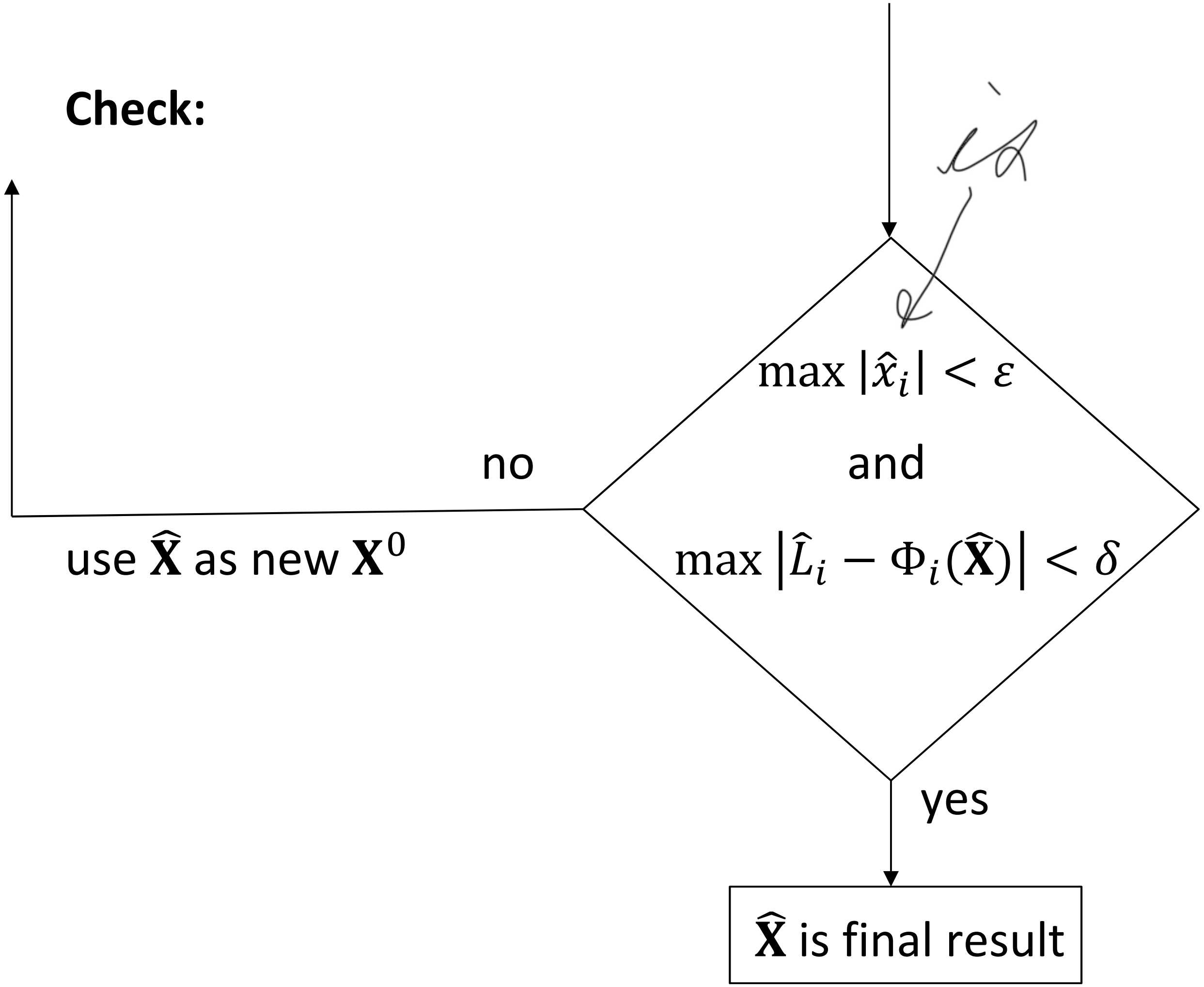
$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}$$

Adjusted observations:

$$\hat{\mathbf{L}} = \mathbf{L} + \mathbf{v}$$

6.7.6 Nonlinear functional models, Example

Check:



6.7.6 Nonlinear functional models, Example

Empirical reference standard deviation:

$$s_0 = \sqrt{\frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$$

VCM of adjusted unknowns:

$$\mathbf{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \cdot \mathbf{Q}_{\hat{X}\hat{X}} \qquad \text{with} \qquad \mathbf{Q}_{\hat{X}\hat{X}} = \mathbf{N}^{-1}$$

6.7.6 Nonlinear functional models, Example

cofactor

Standard deviation of unknowns:

$$\mathbf{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \cdot \begin{bmatrix} q_{\hat{x}\hat{x}} & q_{\hat{x}\hat{y}} \\ q_{\hat{y}\hat{x}} & q_{\hat{y}\hat{y}} \end{bmatrix}$$

- Cofactor for unknown value x: $q_{\hat{x}\hat{x}}$
- Cofactor for unknown value y: $q_{\hat{y}\hat{y}}$

$$s_{\hat{x}} = \sqrt{s_0^2 \cdot q_{\hat{x}\hat{x}}} = s_0 \cdot \sqrt{q_{\hat{x}\hat{x}}}$$

$$s_{\hat{y}} = \sqrt{s_0^2 \cdot q_{\hat{y}\hat{y}}} = s_0 \cdot \sqrt{q_{\hat{y}\hat{y}}}$$

6.7.7 Nonlinear functional models, Summary

Least-squares Adjustment for Nonlinear Adjustment Problems

- Iterative solution with linearized functional model -

Nonlinear functional model for the unknowns:

$$\begin{aligned} L_1 &= \varphi_1(X_1, X_2, \cdots, X_u) \\ L_2 &= \varphi_2(X_1, X_2, \cdots, X_u) \\ \vdots & \qquad \vdots \\ L_n &= \varphi_n(X_1, X_2, \cdots, X_u) \end{aligned}$$

Nonlinear functional model in matrix notation:

$$\underset{n,1}{\mathbf{L}} = \underset{n,1}{\mathbf{\Phi}}(\underset{u,1}{\mathbf{X}}) = \begin{bmatrix} \varphi_1(\underset{u,1}{\mathbf{X}}) \\ \varphi_2(\underset{u,1}{\mathbf{X}}) \\ \vdots \\ \varphi_n(\underset{u,1}{\mathbf{X}}) \end{bmatrix}$$

Vector of observations:

$$\underset{n,1}{\mathbf{L}} = \begin{bmatrix} L_1 & L_2 & \cdots & L_n \end{bmatrix}^{\mathrm{T}}$$

Variance covariance matrix of the observations:

$$\underset{n,n}{\mathbf{\Sigma}_{\mathbf{LL}}} = \begin{bmatrix} \sigma_{L_1}^2 & \sigma_{L_1L_2} & \cdots & \sigma_{L_1L_n} \\ \sigma_{L_2L_1} & \sigma_{L_2}^2 & \cdots & \sigma_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_nL_1} & \sigma_{L_nL_2} & \cdots & \sigma_{L_n}^2 \end{bmatrix}$$ with theoretical values σ_i

$$\underset{n,n}{\mathbf{S}_{\mathbf{LL}}} = \begin{bmatrix} s_{L_1}^2 & s_{L_1L_2} & \cdots & s_{L_1L_n} \\ s_{L_2L_1} & s_{L_2}^2 & \cdots & s_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{L_nL_1} & s_{L_nL_2} & \cdots & s_{L_n}^2 \end{bmatrix}$$ with empirical values s_i

6.7.7 Nonlinear functional models, Summary

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VCM of the reduced observations from variance covariance propagation with the functional model $\mathbf{l} = \mathbf{L} - \mathbf{L}^0$	$\underset{n,n}{\boldsymbol{\Sigma}_{\mathbf{ll}}} = \underset{n,n}{\boldsymbol{\Sigma}_{\mathbf{LL}}}$	with theoretical values σ_i
	$\underset{n,n}{\mathbf{S}_{\mathbf{ll}}} = \underset{n,n}{\mathbf{S}_{\mathbf{LL}}}$	with empirical values s_i
Theoretical reference standard deviation:	σ_0	(or theoretical reference variance σ_0^2)
Cofactor matrix of the observations and reduced observations:	$\underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} = \frac{1}{\sigma_0^2} \underset{n,n}{\boldsymbol{\Sigma}_{\mathbf{LL}}}$ respectively	$\underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} = \frac{1}{\sigma_0^2} \underset{n,n}{\mathbf{S}_{\mathbf{LL}}}$
Weight matrix of the observations and reduced observations:	$\underset{n,n}{\mathbf{P}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}^{-1}}$	
Vector of adjusted unknowns:	$\underset{u,1}{\hat{\mathbf{X}}} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \cdots & \hat{X}_u \end{bmatrix}^{\mathrm{T}}$	
Vector of $\begin{cases} \text{initial values} \\ \text{starting values} \\ \text{approximations} \end{cases}$ for the unknowns:	$\underset{u,1}{\mathbf{X}^0} = \begin{bmatrix} X_1^0 & X_2^0 & \cdots & X_u^0 \end{bmatrix}^{\mathrm{T}}$	
Vector of adjusted reduced unknowns:	$\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,1}{\hat{\mathbf{X}}} - \underset{u,1}{\mathbf{X}^0}$	

6.7.7 Nonlinear functional models, Summary

→ Observations as functions of the approximations for the unknowns:
$$\underset{n,1}{\mathbf{L}^0} = \underset{n,1}{\mathbf{\Phi}}\left(\underset{u,1}{\mathbf{X}^0}\right)$$

Vector of reduced observations:
$$\underset{n,1}{\mathbf{l}} = \underset{n,1}{\mathbf{L}} - \underset{n,1}{\mathbf{L}^0}$$

Jacobian matrix:
$$\underset{n,u}{\mathbf{J}} = \left(\frac{\partial \mathbf{\Phi}(\mathbf{X})}{\partial \mathbf{X}}\right)_{\mathbf{X}=\mathbf{X}^0} = \begin{bmatrix} \frac{\partial \varphi_1(\mathbf{X})}{\partial X_1} & \frac{\partial \varphi_1(\mathbf{X})}{\partial X_2} & \cdots & \frac{\partial \varphi_1(\mathbf{X})}{\partial X_u} \\ \frac{\partial \varphi_2(\mathbf{X})}{\partial X_1} & \frac{\partial \varphi_2(\mathbf{X})}{\partial X_2} & \cdots & \frac{\partial \varphi_2(\mathbf{X})}{\partial X_u} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_n(\mathbf{X})}{\partial X_1} & \frac{\partial \varphi_n(\mathbf{X})}{\partial X_2} & \cdots & \frac{\partial \varphi_n(\mathbf{X})}{\partial X_u} \end{bmatrix}_{\mathbf{X}=\mathbf{X}^0}$$

Coefficient matrix of the linearized functional model: $\underset{n,u}{\mathbf{A}} = \underset{n,u}{\mathbf{J}}$ “Design Matrix”

Vector of residuals:
$$\underset{n,1}{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{\mathrm{T}}$$

Observation equations:
$$\underset{n,1}{\mathbf{l}} + \underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{x}}}$$

Normal equations:
$$\underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{l}}$$

Normal matrix:
$$\underset{u,u}{\mathbf{N}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}$$

6.7.7 Nonlinear functional models, Summary

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Right hand side of normal equations:	$\underset{u,1}{\mathbf{n}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\ \underset{n,n}{\mathbf{P}}\ \underset{n,1}{\mathbf{l}}$
Normal equations:	$\underset{u,u}{\mathbf{N}}\ \underset{u,1}{\hat{\mathbf{x}}} = \underset{u,1}{\mathbf{n}}$
Inversion of normal matrix:	$\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}} = \underset{u,u}{\mathbf{N}^{-1}}$
Solution of normal equations:	$\underset{u,1}{\hat{\mathbf{x}}} = \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}}\ \underset{u,1}{\mathbf{n}}$
Solution for the unknowns:	$\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,1}{\hat{\mathbf{X}}^{0}} + \underset{u,1}{\hat{\mathbf{x}}}$
Vector of residuals:	$\underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\ \underset{u,1}{\hat{\mathbf{x}}} - \underset{n,1}{\mathbf{l}}$
Vector of adjusted observations:	$\underset{n,1}{\hat{\mathbf{L}}} = \underset{n,1}{\mathbf{L}} + \underset{n,1}{\mathbf{v}}$
Check 1:	$\max\lvert\hat{x}_i\rvert \le \varepsilon \quad \forall\, i = 1,\ldots,u$ MatLab: `max(abs(x_hat))`
Check 2:	$\max\left\lvert\hat{L}_i - \varphi_i\left(\hat{\mathbf{X}}\right)\right\rvert \le \delta \quad \forall\, i = 1,\ldots,n$

If $\left\{ \left(\max\lvert\hat{x}_i\rvert \le \varepsilon \quad \forall\, i = 1,\ldots,u\right) \wedge \left(\max\left\lvert\hat{L}_i - \varphi_i\left(\hat{\mathbf{X}}\right)\right\rvert \le \delta \quad \forall\, i = 1,\ldots,n\right) \right\}$

$\hat{\mathbf{X}}$ is the solution for the nonlinear adjustment problem

Else

Use $\hat{\mathbf{X}}$ as new approximation for the unknowns $\mathbf{X}^0$ and continue with step "→"

6.7.7 Nonlinear functional models, Summary

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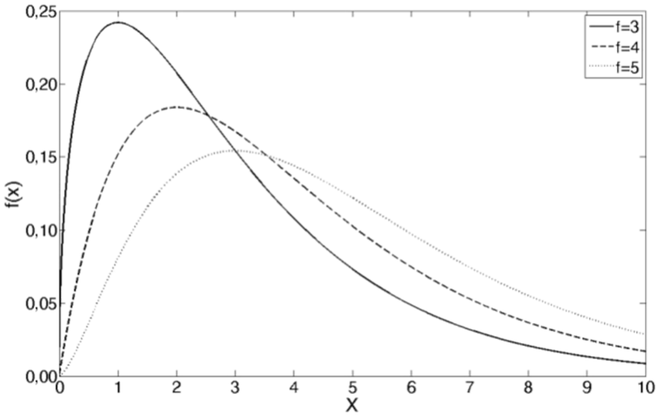
Empirical reference standard deviation:	$s_0 = \sqrt{\dfrac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$	(or empirical reference variance s_0^2)
Cofactor matrix of adjusted unknowns from variance covariance propagation with the functional model $\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$	$\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}}$	
VCM of adjusted unknowns:	$\underset{u,u}{\mathbf{\Sigma}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = s_0^2\,\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}$	
Cofactor matrix of adjusted observations:	$\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = \underset{n,u}{\mathbf{A}}\,\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}\,\underset{u,n}{\mathbf{A}^{\mathrm{T}}}$	
VCM of adjusted observations:	$\underset{n,n}{\mathbf{\Sigma}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = s_0^2\,\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$	
Cofactor matrix of the residuals:	$\underset{n,n}{\mathbf{Q}_{\mathbf{vv}}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} - \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$	
VCM of the residuals:	$\underset{n,n}{\mathbf{\Sigma}_{\mathbf{vv}}} = s_0^2\,\underset{n,n}{\mathbf{Q}_{\mathbf{vv}}}$	

Adjustment_Theory_I_LSA_Nonlinear.pdf



Adjustment Theory I

Chapter 6 – Introduction to Least Squares Adjustment

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