Important Formulas for Adjustment Theory

Version 13
2023-12-19 17:57

1 Matlab / Octave basics

```
% clear command window, clear all variables, close all windows
clc
clear
close all

% square t element-wise
t .^ 2

% define matrix A
A = [1 2; 3 4]

% get value in second row and first column of matrix A
A(2,1)

% transpose matrix A (assuming A contains real numbers)
A'

% calculate inverse matrix
inv(A)

% define two symbolic variables x and y
syms x y

% define a SYMBOLIC function f that uses symbol x
f = x .^ 2

% calculate f(20) for symbolic function f(x)
subs(f, x, 20)

% assume that variable x is positive
assume(x, 'positive')

% find an x such that f becomes 1000
solve(f == 1000, x)
```

2 Matrices and vectors

$\boldsymbol{AB}$ = matrix multiplication (according to matrix multiplication rules)

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{(column vector)}$$

$\boldsymbol{x}^{\mathrm{T}}$ = transposed vector (or matrix), with rows and columns switched

$\boldsymbol{x} \cdot \boldsymbol{y}$ = scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$

$$= \sum_{i=1}^{n} x_i y_i = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 + \ldots x_n \cdot y_n$$

$$= \boldsymbol{x}^{\mathrm{T}} \boldsymbol{y}$$

$||\boldsymbol{x}||$ = norm ("length") of vector $\boldsymbol{x}$

$$= \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i x_i} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$= \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2}$$

$\boldsymbol{A}^{-1}$ = matrix inverse (only defined for square matrices)

3 Differentials and Integrals

3.1 Symbols

$$\begin{aligned}
\Delta x &= \text{change (difference) in } x \text{ (e. g. } \Delta x = x_2 - x_1) \\
\mathrm{d}x &= \text{infinitesimal small change (differential) in } x \\
\partial x &= \text{differential in } x \text{ while all other variables are kept constant} \\
\frac{\partial f}{\partial x} &= \text{derivative of } f \text{ with respect to } x \\
\int f(x)\,\mathrm{d}x &= \text{indefinite integral (without limits) of } f(x) \text{ with respect to } x \\
&= \text{“antiderivative” or “primitive” of } f(x) \\
&= F(x) \\
\int_a^b f(x)\,\mathrm{d}x &= \text{definite integral (from } a \text{ to } b) \text{ of } f(x) \text{ with respect to } x
\end{aligned}$$

3.2 Using the antiderivative / primitive

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$ such that $\frac{\partial F(x)}{\partial x} = f(x)$.

3.3 Symbolic differentials and integrals in Matlab or Octave

Assuming that `f` has been defined using a symbol `x` as variable, e. g. `f = x.^2;`

$\frac{\partial f}{\partial x} =$ `diff(f, x)`	result uses `x` as variable
$\int f(x)\,\mathrm{d}x =$ `int(f, x)`	result uses `x` as variable
$\int_a^b f(x)\,\mathrm{d}x =$ `int(f, x, a, b)`	result depends on `a` and `b`
$\int_a^x f(x')\,\mathrm{d}x' =$ `int(f, x, a, x)`	result uses `x` as variable

4 Mean, (co)variance, and standard deviation

4.1 Symbols

n = number of observations

i (or j) = index variable, e. g. for i^{th} (or j^{th}) observation

x_1 = first element in vector $\boldsymbol{x}$

x_2 = second element in vector $\boldsymbol{x}$

x_i = i^{th} element in vector $\boldsymbol{x}$

$\overline{x}$ = empirical mean of observations in vector $\boldsymbol{x}$

μ_X = theoretical mean of random variable X (expectation value)

$\widetilde{X}$ = true value (without systematic or random errors)

σ_X = theoretical standard deviation of random variable X

$\sigma_X^2 = \sigma_{XX}$ = theoretical variance of random variable X

σ_{XY} = theoretical covariance of random variables X and Y

s_x = standard deviation of (single) observations in vector $\boldsymbol{x}$

$s_x^2 = s_{xx}$ = variance of (single) observations in vector $\boldsymbol{x}$

s_{xy} = covariance of observations regarding $\boldsymbol{x}$ and $\boldsymbol{y}$

$s_{\overline{x}}$ = standard deviation of the *mean* $\overline{x}$, calculated from n observations

4.2 Important formulas

$$\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$s_x^2 = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{j=1}^{n}(x_j - \mu_X)^2 & \text{if theoretical mean } \mu_X \text{ is known} \\ \dfrac{1}{n-1}\displaystyle\sum_{j=1}^{n}(x_j - \overline{x})^2 & \text{if empirical mean } \overline{x} \text{ is used} \end{cases}$$

$$s_x = \sqrt{s_x^2}$$

$$s_{\overline{x}}^2 = \frac{s_x^2}{n}$$

$$s_{\overline{x}} = \frac{s_x}{\sqrt{n}} = \sqrt{\frac{s_x^2}{n}}$$

$$s_{xy} = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{j=1}^{n}(x_j - \mu_X)(y_j - \mu_Y) & \text{if theoretical means are known} \\ \dfrac{1}{n-1}\displaystyle\sum_{j=1}^{n}(x_j - \overline{x})(y_j - \overline{y}) & \text{if empirical means are used} \end{cases}$$

5 Distributions

5.1 Histograms

For histograms, we assign each of our n observations to one of m bins.

$$
\begin{aligned}
n &= \text{number of observations} \\
&= \sum_{i=1}^{m} k_i \\
m &= \text{number of bins} \approx \sqrt{n} \\
a &= \text{start value of a range or bin} \\
b &= \text{end value of a range or bin} \\
\Delta x &= \text{bin size} = \frac{b-a}{m} \\
i &= \text{index variable from 1 to } m\text{, denoting a certain bin } i \\
j &= \text{index variable from 1 to } m\text{, denoting a certain bin } j \\
k_i &= \text{(absolute) number of observations that fall into bin } i \\
h_i &= \text{relative number of observations that fall into bin } i \\
&= \frac{k_i}{n} \\
H_i &= \text{relative number of observations that fall into bins } 1, \ldots, i \\
&= \sum_{j=1}^{i} h_j
\end{aligned}
$$

All h_i form the frequency function, and all H_i form the cumulative frequency function.

5.2 Probability density function and distribution function

When $n \to \infty$, the frequency function (using h_i) becomes the probability density function $f(x)$, and the cumulative frequency function (using H_i) becomes the distribution function $F(x)$.

$$\begin{aligned} f(x) &= \text{probability density function} \\ &= P(\text{observation falls between } x \text{ and } x + \mathrm{d}x)/\mathrm{d}x \\ F(x) &= \text{distribution function} \\ &= P(\text{observation is equal to or less than } x) \end{aligned}$$

$f(x)$ and $F(x)$ relate to each other as follows:

$$\begin{aligned} f(x) &= \frac{\partial F}{\partial x} \\ F(x) &= \int_{-\infty}^{x} f(x')\,\mathrm{d}x' \end{aligned}$$

The following equations are always true:

$$0 \leq F(x) \leq 1$$

$$P(\text{observation falls between } a \text{ and } b) \;=\; \int_a^b f(x)\,\mathrm{d}x \;\;=\;\; F(b) - F(a)$$

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

When the probability density function $f(x)$ of a random variable X is known:

$$\begin{aligned} \mu_X &= \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x \\ \sigma_X^2 &= \int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f(x)\,\mathrm{d}x \end{aligned}$$

6 Propagation of random deviations

The following formulas hold for σ as well as for s, accordingly.

6.1 Special (easy) case of uncorrelated observations

$$X = \text{some variable calculated from observed variables } L_1,\ L_2,\ \ldots$$

$$\sigma_X^2 = \left(\frac{\partial X}{\partial L_1}\right)^2 \cdot \sigma_{L_1}^2 \ + \ \left(\frac{\partial X}{\partial L_2}\right)^2 \cdot \sigma_{L_2}^2 \ + \ \ldots$$

6.2 General case (correlations and/or multiple steps)

$$L_1 \ldots L_n = \text{observed variables (inputs)}$$
$$X_1 \ldots X_u = \text{calculated variables (outputs)}$$

$$\mathbf{J} = \text{Jacobian matrix, which contains derivatives of } X_i \text{ with respect to } L_j$$

$$= \begin{bmatrix} \frac{\partial X_1}{\partial L_1} & \frac{\partial X_1}{\partial L_2} & \frac{\partial X_1}{\partial L_3} & \cdots & \frac{\partial X_1}{\partial L_n} \\ \frac{\partial X_2}{\partial L_1} & \frac{\partial X_2}{\partial L_2} & \frac{\partial X_2}{\partial L_3} & \cdots & \frac{\partial X_2}{\partial L_n} \\ \frac{\partial X_3}{\partial L_1} & \frac{\partial X_3}{\partial L_2} & \frac{\partial X_3}{\partial L_3} & \cdots & \frac{\partial X_3}{\partial L_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_u}{\partial L_1} & \frac{\partial X_u}{\partial L_2} & \frac{\partial X_u}{\partial L_3} & \cdots & \frac{\partial X_u}{\partial L_n} \end{bmatrix}$$

$$\boldsymbol{\Sigma_{LL}} = \text{covariance matrix (variances } \mathbf{bold}\text{) for observed variables } L_1,\ L_2,\ \ldots$$

$$= \begin{bmatrix} \boldsymbol{\sigma_{L_1}^2} & \sigma_{L_1L_2} & \sigma_{L_1L_3} & \cdots & \sigma_{L_1L_n} \\ \sigma_{L_2L_1} & \boldsymbol{\sigma_{L_2}^2} & \sigma_{L_2L_3} & \cdots & \sigma_{L_2L_n} \\ \sigma_{L_3L_1} & \sigma_{L_3L_2} & \boldsymbol{\sigma_{L_3}^2} & \cdots & \sigma_{L_3L_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_nL_1} & \sigma_{L_nL_2} & \sigma_{L_nL_3} & \cdots & \boldsymbol{\sigma_{L_n}^2} \end{bmatrix}$$

$$\boldsymbol{\Sigma_{XX}} = \text{covariance matrix for calculated variables } X_1,\ X_2,\ \ldots,\ X_u$$
$$= \mathbf{J}\ \boldsymbol{\Sigma_{LL}}\ \mathbf{J}^{\mathrm{T}}$$

In case of multiple steps:

$$\boldsymbol{F}_i = \text{Jacobian matrix } (\mathbf{J}) \text{ for step } i$$
$$\boldsymbol{F} = \boldsymbol{F}_k\ \ldots\ \boldsymbol{F}_3\ \boldsymbol{F}_2\ \boldsymbol{F}_1$$
$$\boldsymbol{\Sigma_{XX}} = \boldsymbol{F}\ \boldsymbol{\Sigma_{LL}}\ \boldsymbol{F}^{\mathrm{T}}$$

6.3 Standard cases in practice

6.3.1 Scaling with constant factor

Let L be an observed variable, a be a constant factor, and X be the scaled variable.

$$\begin{aligned} X &= a \cdot L \\ \sigma_X &= a \cdot \sigma_L \end{aligned}$$

6.3.2 Addition or substraction of uncorrelated observations

Let L_1, L_2, ... be different observed variables and X be a sum or difference of those.

$$\begin{aligned} X &= L_1 \pm L_2 \pm \dots \\ \sigma_X^2 &= \sigma_{L_1}^2 \pm \sigma_{L_2}^2 \pm \dots \\ \sigma_X &= \sqrt{\sigma_{L_1}^2 \pm \sigma_{L_2}^2 \pm \dots} \end{aligned}$$

If all $L_1, \dots, L_n$ have the same precision σ_L:

$$\begin{aligned} X &= L_1 \pm L_2 \pm \dots \pm L_n \\ \sigma_X^2 &= n \cdot \sigma_L^2 \\ \sigma_X &= \sqrt{n} \cdot \sigma_L \end{aligned}$$

7 Normal distribution

$$\mathrm{N}_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}\right)$$

8 Adjustment

8.1 Linear function model without matrices

A linear function ($y = ax + b$) has 2 unknowns: factor a and addend b.
The functional model is:

$$y = ax + b$$

If number (n) of observations is bigger than number of unknowns ($u = 2$), we need to adjust the observations (y_i) by adding a residual v_i to each observation.
Observation equations:

$$\begin{aligned} y_1 + v_1 &= ax_1 + b \\ y_2 + v_2 &= ax_2 + b \\ &\vdots \\ y_n + v_n &= ax_n + b \end{aligned}$$

We aim to find our unknowns a and b such that the sum of the squared residuals, weighted by the observation's precision p_i, become as minimal as possible:

$$\Omega = \text{target function} = \sum_{i=1}^{n} p_i v_i^2 \rightarrow \min$$
$$\Omega = p_1 v_1^2 + p_2 v_2^2 + \ldots + p_n v_n^2$$

where

$$v_i = ax_i + b - y_i \quad \text{(residuals)}$$
$$p_i = \frac{1}{\sigma_i^2} \quad \text{(precision)}$$

The unknowns a and b such that $\Omega \rightarrow \min$ are found by solving:

$$\frac{\partial \Omega}{\partial a} = 0, \quad \frac{\partial \Omega}{\partial b} = 0 \quad \text{(normal equations)}$$

8.2 Linear function model with matrices

n = number of observations
u = number of unknowns
$\boldsymbol{L}$ = column vector with n observations
$\boldsymbol{\Sigma_{LL}}$ or $\boldsymbol{S_{LL}}$ = covariance matrix of observations
$\boldsymbol{X}$ = column vector with u unknowns
$\hat{\boldsymbol{X}}$ = column vector with u adjusted unknowns
$\boldsymbol{A}$ = "Design Matrix" ($n \times u$) needed for functional model

Functional model:

$$\boldsymbol{L} = \boldsymbol{A}\boldsymbol{X} \quad \text{(only works in theory, as } \boldsymbol{L} \text{ will have errors)}$$

Observation equations:

$$\boldsymbol{L} + \boldsymbol{v} = \boldsymbol{A}\hat{\boldsymbol{X}}$$

Calculating the weighting $\boldsymbol{P}$:

$$\begin{aligned} \sigma_0^2 &= \text{theoretical reference variance, usually set to 1} \\ \boldsymbol{Q}_{\boldsymbol{LL}} &= \frac{1}{\sigma_0^2}\boldsymbol{\Sigma}_{\boldsymbol{LL}} \quad \left(\text{or } = \frac{1}{\sigma_0^2}\boldsymbol{S}_{\boldsymbol{LL}}\right) \quad \text{where } \boldsymbol{\Sigma}_{\boldsymbol{LL}} \text{ (or } \boldsymbol{S}_{\boldsymbol{LL}}\text{) is covariance matrix} \\ \boldsymbol{P} &= {\boldsymbol{Q}_{\boldsymbol{LL}}}^{-1} \end{aligned}$$

Normal equations to find $\boldsymbol{X}$:

$$\boldsymbol{N}\hat{\boldsymbol{X}} = \boldsymbol{n} \quad \left(\text{solution: } \hat{\boldsymbol{X}} = \boldsymbol{N}^{-1}\boldsymbol{n}\right)$$

With:

$$\begin{aligned} \boldsymbol{N} &= \text{normal matrix} \\ &= \boldsymbol{A}^T\boldsymbol{P}\,\boldsymbol{A} \\ \boldsymbol{n} &= \text{right hand side of normal equations} \\ &= \boldsymbol{A}^T\boldsymbol{P}\,\boldsymbol{L} \end{aligned}$$

Residuals:

$$\boldsymbol{v} = \boldsymbol{A}\hat{\boldsymbol{X}} - \boldsymbol{L}$$

Adjusted observations and final check:

$$\begin{aligned} \hat{\boldsymbol{L}} &= \boldsymbol{L} + \boldsymbol{v} \\ \hat{\boldsymbol{L}} &= \boldsymbol{\Phi}(\hat{\boldsymbol{X}}) \left(= \boldsymbol{A}\hat{\boldsymbol{X}}\right) \quad \text{(i.e. } \boldsymbol{A}\hat{\boldsymbol{X}} \text{ should give the same values for } \hat{\boldsymbol{L}}\text{)} \end{aligned}$$

Calculation of covariances:

$$\begin{aligned} s_0 &= \sqrt{\frac{\boldsymbol{v}^T\boldsymbol{P}\,\boldsymbol{v}}{n-u}} = \sqrt{\frac{\text{objective function}}{\text{redundancy}}} = \text{empirical reference std. dev.} \\ \boldsymbol{Q}_{\hat{\boldsymbol{X}}\hat{\boldsymbol{X}}} &= \boldsymbol{N}^{-1} = \text{cofactor matrix of adjusted unknowns} \\ \boldsymbol{S}_{\hat{\boldsymbol{X}}\hat{\boldsymbol{X}}} &= s_0^2\,\boldsymbol{Q}_{\hat{\boldsymbol{X}}\hat{\boldsymbol{X}}} = \text{covariance matrix (VCM) of adjusted unknowns} \\ \boldsymbol{Q}_{\hat{\boldsymbol{L}}\hat{\boldsymbol{L}}} &= \boldsymbol{A}\,\boldsymbol{Q}_{\hat{\boldsymbol{X}}\hat{\boldsymbol{X}}}\,\boldsymbol{A}^T = \text{cofactor matrix of adjusted observations} \\ \boldsymbol{S}_{\hat{\boldsymbol{L}}\hat{\boldsymbol{L}}} &= s_0^2\,\boldsymbol{Q}_{\hat{\boldsymbol{L}}\hat{\boldsymbol{L}}} = \text{covariance matrix (VCM) of adjusted observations} \\ \boldsymbol{Q}_{\boldsymbol{vv}} &= \boldsymbol{Q}_{\boldsymbol{LL}} - \boldsymbol{Q}_{\hat{\boldsymbol{L}}\hat{\boldsymbol{L}}} = \text{cofactor matrix of the residuals} \\ \boldsymbol{S}_{\boldsymbol{vv}} &= s_0^2\,\boldsymbol{Q}_{\boldsymbol{vv}} = \text{covariance matrix (VCM) of the residuals} \end{aligned}$$

8.3 Non-linear function model with iterative solution

Calculate $\boldsymbol{P}$ as in the linear case.

$$\begin{aligned}
\boldsymbol{\Phi}(\boldsymbol{X}) &= \text{non-linear function with } u \text{ inputs } (\boldsymbol{X}) \text{ and } n \text{ outputs } (\boldsymbol{\Phi}(\boldsymbol{X})) \\
\boldsymbol{X}^0 &= \text{vector } (u \times 1) \text{ with initial approximations for unknowns} \\
\boldsymbol{L}^0 &= \boldsymbol{\Phi}(\boldsymbol{X}^0) = \text{vector } (n \times 1) \text{ of observations as functions of the unknowns} \\
\boldsymbol{l} &= \boldsymbol{L} - \boldsymbol{L}^0 = \text{vector } (n \times 1) \text{ of reduced observations} \\
\boldsymbol{\Sigma}_{ll} &= \boldsymbol{\Sigma}_{\boldsymbol{LL}} = \text{covariances of (reduced) observations} \\
\boldsymbol{A} &= \mathbf{J} = \left(\frac{\partial \boldsymbol{\Phi}(\boldsymbol{X})}{\partial \boldsymbol{X}} \right) \Bigg|_{\boldsymbol{X}=\boldsymbol{X}^0} = \text{Jacobian matrix } (n \times u) \text{ of function } \boldsymbol{\Phi}(\boldsymbol{X}) \text{ at } \boldsymbol{X}^0 \\
\boldsymbol{N} &= \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{A} = \text{normal matrix} \\
\boldsymbol{n} &= \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{l} = \text{right hand side of normal equations} \\
\hat{\boldsymbol{x}} &= \boldsymbol{N}^{-1} \boldsymbol{n} = \text{corrections for initial approximations} \\
\hat{\boldsymbol{X}} &= \boldsymbol{X}^0 + \hat{\boldsymbol{x}} = \text{adjusted unknowns} \\
\boldsymbol{v} &= \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{l} = \text{residuals} \\
\hat{\boldsymbol{L}} &= \boldsymbol{L} + \boldsymbol{v} = \text{adjusted observations}
\end{aligned}$$

Repeat with new $\boldsymbol{X}^0 = \hat{\boldsymbol{X}}$ of previous run until $\hat{\boldsymbol{x}}$ becomes sufficiently small.

Final check is $\hat{\boldsymbol{L}} \approx \boldsymbol{\Phi}(\hat{\boldsymbol{X}})$.

For calculation of covariances, proceed as in linear case.