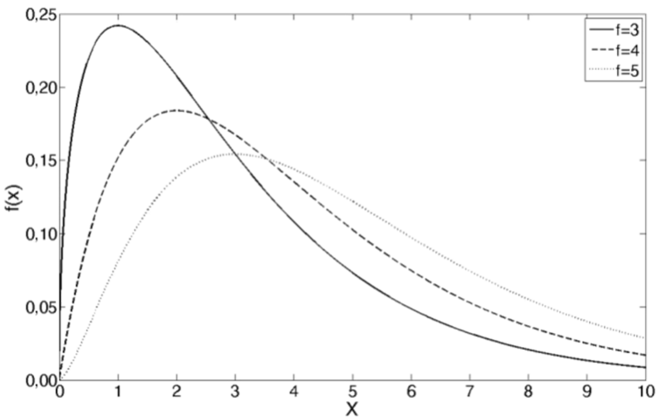

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Adjustment Theory I

Chapter 3 - The Random Vector

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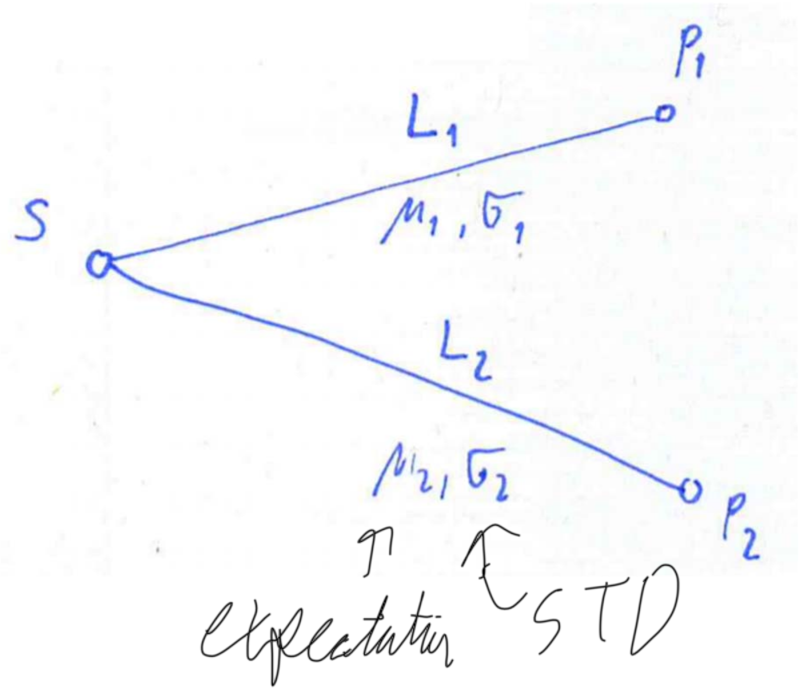
3.1.1 Theoretical variance and theoretical correlation coefficient

► Given: 2D random vector $\mathbf{L}_{2\times 1} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ with the random variables L_1 and L_2

Example: Measurement of two distances from the same point



3.1.1 Theoretical Variance and Theoretical Correlation Coefficient

- Vector of expectations

$$\boldsymbol{\mu}_{2\times1} = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} = E(\mathbf{L}_{2\times1}) = \begin{bmatrix}E(L_1)\\ E(L_2)\end{bmatrix}$$

- Vector of random deviations

$$\boldsymbol{\varepsilon}_{L\,2\times1} = \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} = \mathbf{L}_{2\times1} - \boldsymbol{\mu}_{2\times1} = \begin{bmatrix}L_1 - \mu_1\\ L_2 - \mu_2\end{bmatrix}$$

Vector = 1 · 1 = ein Vector

- as in 2.4.1

$$E\left(\boldsymbol{\varepsilon}_{L\,2\times1}\right) = E\{\mathbf{L}_{2\times1} - \mathbf{e}_{2\times1}\cdot\mu_L\} = \underbrace{E(\mathbf{L}_{2\times1})}_{\boldsymbol{\mu}_L} - \boldsymbol{\mu}_L = \mathbf{0}$$

allways

3x expectation

$$E\{\boldsymbol{\varepsilon}_{L\,2\times1}\cdot\boldsymbol{\varepsilon}^{\mathrm{T}}_{L\,1\times2}\} = E\left\{\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}_{2\times1}\cdot\begin{bmatrix}\varepsilon_1 & \varepsilon_2\end{bmatrix}_{1\times2}\right\} = E\left\{\begin{bmatrix}\varepsilon_1^2 & \varepsilon_1\cdot\varepsilon_2\\ \varepsilon_2\cdot\varepsilon_1 & \varepsilon_2^2\end{bmatrix}\right\}$$

$$= \begin{bmatrix}E(\varepsilon_1^2) & E(\varepsilon_1\cdot\varepsilon_2)\\ E(\varepsilon_2\cdot\epsilon_1) & E(\varepsilon_2^2)\end{bmatrix}$$

single E

- Theoretical Variance of $\mathbf{L}$

$$E\left(\varepsilon_i^2\right) = \sigma_i^2 \qquad \text{for } i = 1, 2$$

co variance

3.1.1 Theoretical Variance and Theoretical Correlation Coefficient

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▶ Definitions

- Theoretical Covariances between L_1 and L_2

they are the same

$$E(\varepsilon_1 \cdot \varepsilon_2) = \lim_{n\to\infty} \left\{ \frac{1}{n} \cdot \sum_{j=1}^{n} \varepsilon_{1j} \cdot \varepsilon_{2j} \right\} = \sigma_{12}$$

$$\sigma_{12} = \sigma_{21}$$

- Theoretical Variance-Covariance Matrix (VCM) of $\mathbf{L}$

we can learn about dependency

$$\mathbf{\Sigma}_{LL_{2\times 2}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = E\{\boldsymbol{\varepsilon}_{L_{2\times 1}} \cdot \boldsymbol{\varepsilon}_{L_{1\times 2}}^{\mathrm{T}}\}$$

- Variances are always positive ("+")
- Covariances are a "measure of the dependency" between L_1 and L_2

→ For stochastic independent values: $\sigma_{12} = 0$

3.1.1 Theoretical Variance and Theoretical Correlation Coefficient

▶ Definitions

- Theoretical Correlation Coefficient between L_1 and L_2

rho

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2}$$

$$\rho_{12} = \rho_{21}$$

- Fixed limits $-1 \leq \rho_{12} \leq +1$!
- Stochastic independent (no correlation): $\rho_{12} = 0$
- Maximum correlation in the "same direction": $\rho_{12} = +1$
- Maximum correlation in "opposite direction": $\rho_{12} = -1$

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2} \Rightarrow \sigma_{12} = \rho_{12} \cdot \sigma_1 \cdot \sigma_2$$

- Another possible representation for $\mathbf{\Sigma}_{LL}$

$$\mathbf{\Sigma}_{LL_{2\times2}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{12} \cdot \sigma_1 \cdot \sigma_2 \\ \rho_{12} \cdot \sigma_2 \cdot \sigma_1 & \sigma_2^2 \end{bmatrix}$$

3.1.2 Empirical Variance and Empirical Correlation Coefficient

▶ Given: 2D random vector $\mathbf{L}_{2\times1} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ with random variables L_1, L_2 and its realisations in the observation matrix $\mathbf{l}$

$$\mathbf{l}_{n\times2} = \begin{bmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \\ \vdots & \vdots \\ l_{1n} & l_{2n} \end{bmatrix}$$

n realisations of two random variables

Corresponding observations l_{1j} and l_{2j} are obtained "pairwise"

→ Measurements are performed in a small time interval
under almost the same conditions.

3.1.2 Empirical Variance and Empirical Correlation Coefficient

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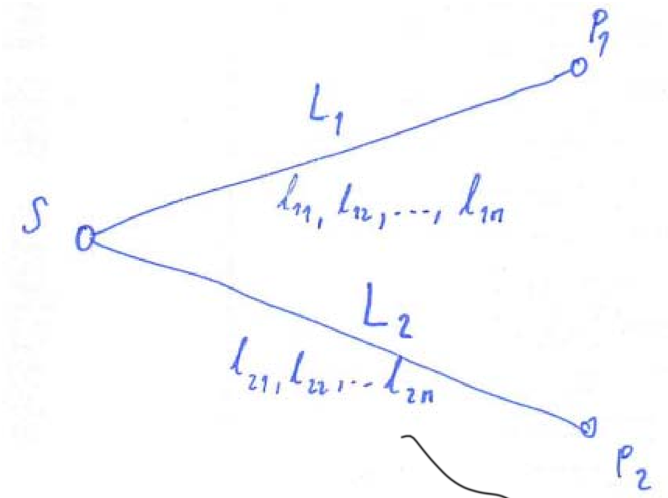
▶ <u>Example:</u> Measurement of two distances from the same point at the <u>same time</u>.



▶ Wanted: Empirical estimation r_{12} for the theoretical correlation coefficient ρ_{12}

→ We have to consider two cases, see 2.4.2

<u>CASE A</u>	<u>CASE B</u>
Expectation μ_L is known	Expectation μ_L is unknown

3.1.2.1 Empirical Variance-Covariance for known expectation μ_L (CASE A)

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- Vector of Expectations

$$\boldsymbol{\mu}_{2\times 1} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = E(\mathbf{L}_{2\times 1}) = \begin{bmatrix} E(L_1) \\ E(L_2) \end{bmatrix}$$

- Matrix of Random Deviations

$$\boldsymbol{\varepsilon}_{L_{n\times 2}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{21} \\ \varepsilon_{12} & \varepsilon_{22} \\ \vdots & \vdots \\ \varepsilon_{1n} & \varepsilon_{2n} \end{bmatrix} = \mathbf{L}_{n\times 2} - \mathbf{e}_{n\times 1} \cdot \boldsymbol{\mu}_{1\times 2}^{\mathrm{T}} = \begin{bmatrix} l_{11} - \mu_1 & l_{21} - \mu_2 \\ l_{12} - \mu_1 & l_{22} - \mu_2 \\ \vdots & \vdots \\ l_{1n} - \mu_1 & l_{2n} - \mu_2 \end{bmatrix}$$

3.1.2.1 Empirical Variance-Covariance for known expectation μ_L (CASE A)

▶ Definitions

$$\mathbf{S}_{LL_{2\times 2}} = \frac{1}{n} \cdot \boldsymbol{\varepsilon}^{\mathrm{T}}_{L_{2\times n}} \cdot \boldsymbol{\varepsilon}_{L_{n\times 2}} = \frac{1}{n} \cdot \begin{bmatrix} \sum_{j=1}^{n} \varepsilon_{1j}^2 & \sum_{j=1}^{n} \varepsilon_{1j} \cdot \varepsilon_{2j} \\ \sum_{j=1}^{n} \varepsilon_{2j} \cdot \varepsilon_{1j} & \sum_{j=1}^{n} \varepsilon_{2j}^2 \end{bmatrix}$$

- Empirical Variances of L_1 and L_2

$$s_i^2 = \frac{1}{n} \cdot \sum_{j=1}^{n} \varepsilon_{ij}^2 \qquad \text{for } i = 1, 2$$

- Empirical Covariances between L_1 and L_2

$$s_{12} = \frac{1}{n} \cdot \sum_{j=1}^{n} \varepsilon_{1j} \cdot \varepsilon_{2j} \qquad s_{12} = s_{21}$$

- Empirical Variance-Covariance Matrix VCM of $\mathbf{L}$

$$\mathbf{S}_{LL_{2\times 2}} = \frac{1}{n} \cdot \boldsymbol{\varepsilon}^{\mathrm{T}}_{L_{2\times n}} \cdot \boldsymbol{\varepsilon}_{L_{n\times 2}} = \begin{bmatrix} s_1^2 & s_{12} \\ s_{21} & s_2^2 \end{bmatrix}$$

3.1.2.2 Empirical Variance-Covariance for unknown expectation μ_L (CASE B)

▶ Vector of expectations $\boldsymbol{\mu}_L$ is not known → has to be replaced by

- Vector of mean values

$$\bar{\mathbf{l}}_{2\times 1} = \begin{bmatrix} \bar{l}_1 \\ \bar{l}_2 \end{bmatrix} = \frac{1}{n} \cdot \mathbf{l}_{2\times n}^{\mathrm{T}} \cdot \mathbf{e}_{n\times 1} = \frac{1}{n} \cdot \begin{bmatrix} l_{11} + l_{12} + \cdots + l_{1n} \\ l_{21} + l_{22} + \cdots + l_{2n} \end{bmatrix}$$

→ Matrix of residuals

$$\mathbf{v}_{n\times 2} = \mathbf{e}_{n\times 1} \cdot \bar{\mathbf{l}}_{1\times 2}^{\mathrm{T}} - \mathbf{l}_{n\times 2} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \\ \vdots & \vdots \\ v_{1n} & v_{2n} \end{bmatrix} = \begin{bmatrix} \bar{l}_1 - l_{11} & \bar{l}_2 - l_{21} \\ \bar{l}_1 - l_{12} & \bar{l}_2 - l_{22} \\ \vdots & \vdots \\ \bar{l}_1 - l_{1n} & \bar{l}_2 - l_{2n} \end{bmatrix}$$

3.1.2.2 Empirical Variance-Covariance for unknown expectation μ_L (CASE B)

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▶ Definitions

- Empirical Variances of L_1 and L_2

$$s_i^2 = \frac{1}{(n-1)} \cdot \sum_{j=1}^{n} v_{ij}^2 \qquad \text{for } i = 1, 2$$

- Empirical Covariances between L_1 and L_2

$$s_{12} = \frac{1}{(n-1)} \cdot \sum_{j=1}^{n} v_{1j} \cdot v_{2j}$$

$$s_{12} = s_{21}$$

- Empirical Variance-Covariance Matrix VCM of $\mathbf{L}$

$$\mathbf{S}_{LL_{2\times 2}} = \frac{1}{(n-1)} \cdot \mathbf{v}_{2\times n}^{\mathrm{T}} \cdot \mathbf{v}_{n\times 2} = \begin{bmatrix} s_1^2 & s_{12} \\ s_{21} & s_2^2 \end{bmatrix}$$

3.1.2.3 Fusion of CASE A and CASE B yields Empirical Correlation

▶ Definitions

- Empirical Correlation Coefficient between L_1 and L_2

$$r_{12} = \frac{s_{12}}{s_1 \cdot s_2}$$

$$s_{12} = s_{21}$$

- Fixed limits $-1 \leq r_{12} \leq +1$ **!**
- stochastic independent (no correlation): $r_{12} = 0$
- Maximum correlation in the “same direction”: $r_{12} = +1$
- Maximum correlation in “opposite direction”: $r_{12} = -1$

$$r_{12} = \frac{s_{12}}{s_1 \cdot s_2} \Rightarrow s_{12} = r_{12} \cdot s_1 \cdot s_2$$

- Another possible representation for $\mathbf{S}_{LL}$

$$\mathbf{S}_{LL_{2\times 2}} = \begin{bmatrix} s_1^2 & s_{12} \\ s_{21} & s_2^2 \end{bmatrix} = \begin{bmatrix} s_1^2 & r_{12} \cdot s_1 \cdot s_2 \\ r_{12} \cdot s_2 \cdot s_1 & s_2^2 \end{bmatrix}$$

Summary

- Theoretical Variance-Covariance Matrix

$$\boldsymbol{\Sigma}_{ll} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$$

If l_1, l_2 stochastic independent: Covariances $\sigma_{12} = \sigma_{21} = 0$

- Empirical Variance Covariance Matrix

$$\mathbf{S}_{ll} = \begin{bmatrix} s_1^2 & s_{12} \\ s_{21} & s_2^2 \end{bmatrix}$$

Summary

▶ Computation of empirical VCM for known expectation μ

$$\mathbf{S}_{ll} = \frac{\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}}{n} = \frac{1}{n} \cdot \begin{bmatrix} \sum_{j=1}^{n} \varepsilon_{1j}^{2} & \sum_{j=1}^{n} \varepsilon_{1j} \cdot \varepsilon_{2j} \\ \sum_{j=1}^{n} \varepsilon_{1j} \cdot \varepsilon_{2j} & \sum_{j=1}^{n} \varepsilon_{2j}^{2} \end{bmatrix}$$

▶ Computation of empirical VCM for unknown expectation μ

$$\mathbf{S}_{ll} = \frac{\mathbf{v}^{\mathrm{T}}\mathbf{v}}{n-1} = \frac{1}{n-1} \cdot \begin{bmatrix} \sum_{j=1}^{n} v_{1j}^{2} & \sum_{j=1}^{n} v_{1j} \cdot v_{2j} \\ \sum_{j=1}^{n} v_{1j} \cdot v_{2j} & \sum_{j=1}^{n} v_{2j}^{2} \end{bmatrix}$$

Summary

▶ Theoretical Correlation Coefficient

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \;, \quad -1 \leq \rho \leq +1$$

▶ Empirical Correlation Coefficient

$$r_{12} = \frac{s_{12}}{s_1 s_2} \;, \quad -1 \leq r \leq +1$$

Correlation can originate from

1. Mathematical correlation (functional correlation, algebraic correlation)

   If we apply a functional relationship between two or more realisations of a random variable, we obtain a correlation between the resulting estimations.

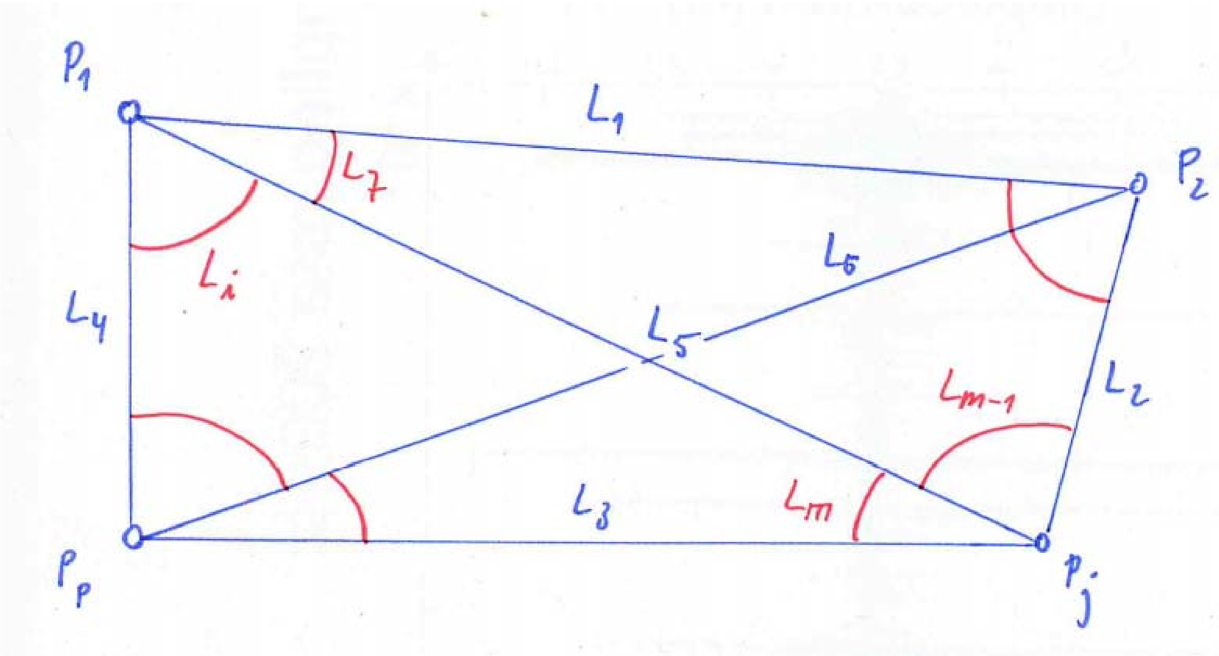
   → Originates on a purely mathematical basis

2. Physical correlation

   Correlation between two realisations of a random variable (→ originates from the measurement) due to small systematic deviations, that are not (or not sufficiently) considered in the functional model.

3.2.1 Theoretical expectation and theoretical covariance matrix

Example: Measurement of directions and distances in a geodetic network



3.2.1 Theoretical expectation and theoretical covariance matrix

- Given: m random variables $L_1, L_2, \dots, L_m$

- Random vector

$$\mathbf{L}_{m\times 1} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix}$$

- Vector of expectations

$$\boldsymbol{\mu}_{m\times 1} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix} = E\{\mathbf{L}_{m\times 1}\} = \begin{bmatrix} E(L_1) \\ E(L_2) \\ \vdots \\ E(L_m) \end{bmatrix}$$

3.2.1 Theoretical expectation and theoretical covariance matrix

- Vector of random deviations

$$\boldsymbol{\varepsilon}_{m\times1} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix} = \mathbf{L}_{m\times1} - \boldsymbol{\mu}_{m\times1} = \begin{bmatrix} L_1 - \mu_1 \\ L_2 - \mu_2 \\ \vdots \\ L_m - \mu_m \end{bmatrix}$$

- as in 2.4.1

$$E(\boldsymbol{\varepsilon}_{m\times1}) = E\{\mathbf{L}_{m\times1} - \boldsymbol{\mu}_{L_{m\times1}}\} = \underbrace{E(\mathbf{L}_{m\times1})}_{\boldsymbol{\mu}_L} - \boldsymbol{\mu}_{L_{m\times1}} = \mathbf{0}$$

$$E\{\boldsymbol{\varepsilon}_{L_{m\times1}} \cdot \boldsymbol{\varepsilon}^{\mathrm{T}}_{L_{1\times m}}\} = \begin{bmatrix} E(\varepsilon_1^2) & E(\varepsilon_1 \cdot \varepsilon_2) & \cdots & E(\varepsilon_1 \cdot \varepsilon_m) \\ E(\varepsilon_2 \cdot \varepsilon_1) & E(\varepsilon_2^2) & \cdots & E(\varepsilon_2 \cdot \varepsilon_m) \\ \vdots & \vdots & \ddots & \vdots \\ E(\varepsilon_m \cdot \varepsilon_1) & E(\varepsilon_m \cdot \varepsilon_2) & \cdots & E(\varepsilon_m \cdot \varepsilon_m) \end{bmatrix} = \boldsymbol{\Sigma}_{LL_{m\times m}}$$

3.2.1 Theoretical expectation and theoretical covariance matrix

▶ Theoretical VCM of L

$$\boldsymbol{\Sigma}_{LL_{m\times m}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\cdot\sigma_1\cdot\sigma_2 & \cdots & \sigma_{1m} \\ \rho_{21}\cdot\sigma_2\cdot\sigma_1 & \sigma_2^2 & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1}\cdot\sigma_m\cdot\sigma_1 & \rho_{m2}\cdot\sigma_m\cdot\sigma_2 & \cdots & \sigma_m^2 \end{bmatrix}$$

▶ Theoretical Variance of L_i

$$E\left(\varepsilon_i^2\right) = \sigma_i^2$$

3.2.2 Empirical expectation and empirical covariance matrix

- Given: m random variables $\mathbf{L}_{m\times 1} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix}$

- $m \times n$-dimensional observation matrix

$$\mathbf{l}_{m\times n} = \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{m1} \\ l_{12} & l_{22} & \cdots & l_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{mn} \end{bmatrix}$$

m different random variables were measured n times

- m-dimensional vector of mean values respective expectations

$$\bar{\mathbf{l}} = \begin{bmatrix} \bar{l}_1 \\ \bar{l}_2 \\ \vdots \\ \bar{l}_m \end{bmatrix} \qquad \text{resp.} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix}$$

3.2.2 Empirical expectation and empirical covariance matrix

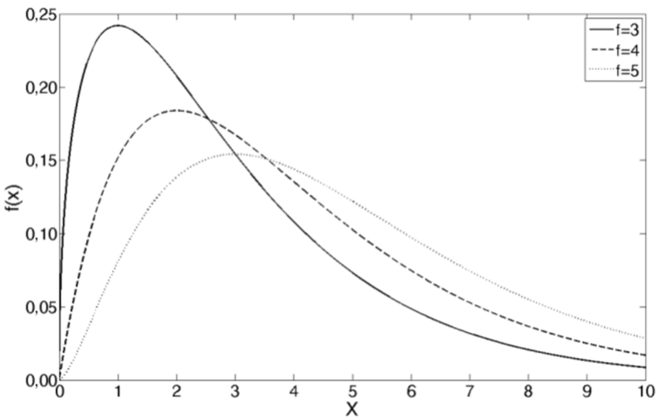
- m-dimensional VCM

$$\mathbf{S}_{ll_{m\times m}} = \begin{bmatrix} s_1^2 & s_{12} & \cdots & s_{1m} \\ s_{21} & s_2^2 & \cdots & s_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \cdots & s_m^2 \end{bmatrix}$$

- Correlation coefficient for the m-dimensional case

$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \cdot \sigma_j}$ respective $r_{ij} = \frac{s_{ij}}{s_i \cdot s_j}$ for $i, j = 1,2, \ldots, m$ and $i \neq j$

→ All computations (e.g. residuals, empirical variances and covariances and correlation coefficients as in 2-dimensional case)



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