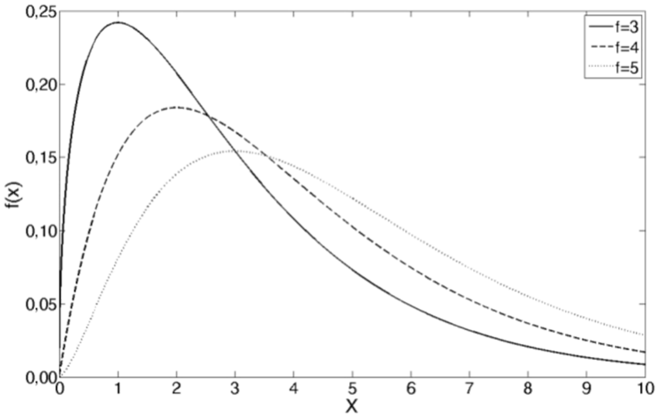

Adjustment Theory I

Chapter 4 – Propagation of Observation Errors

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4. Propagation of Observation Errors

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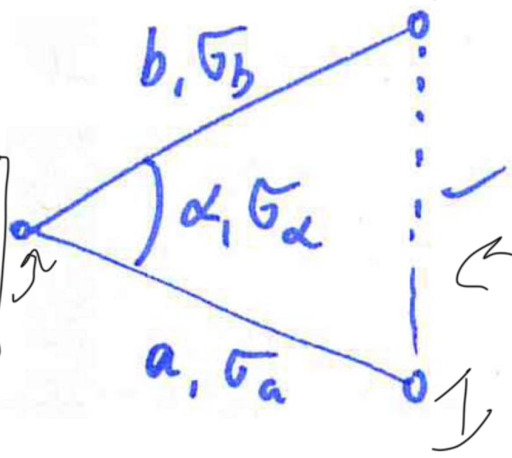
<u>Example</u>:

$c^2 = a^2 + b^2 - 2ab \cdot \cos\alpha$

$c = \sqrt{a^2 + b^2 - 2ab \cdot \cos\alpha}$



STD

c, σ_c ?

we want the length

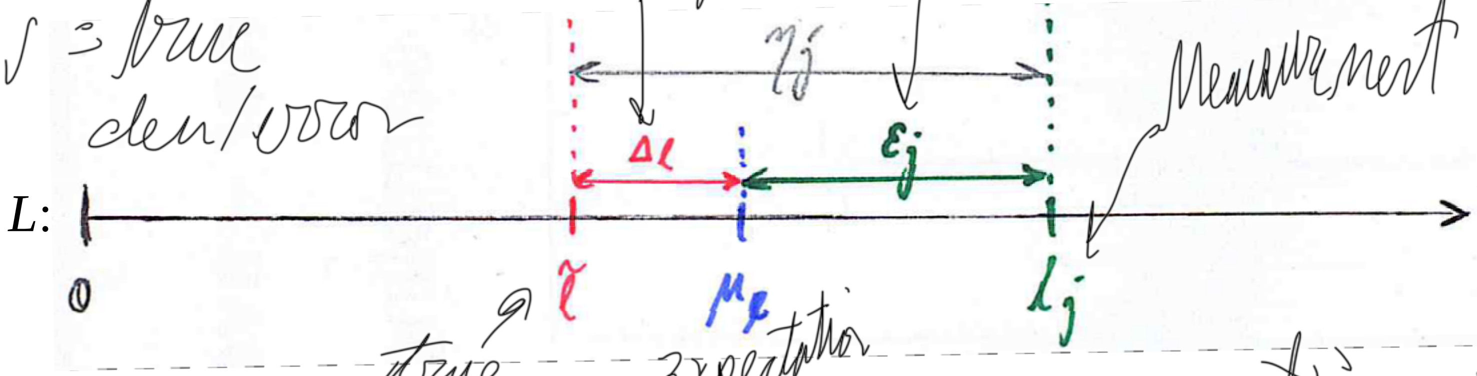
4.1 True, systematic and random deviations

η_j = true deviation/error

systematic

random

Measurement



true

expectation

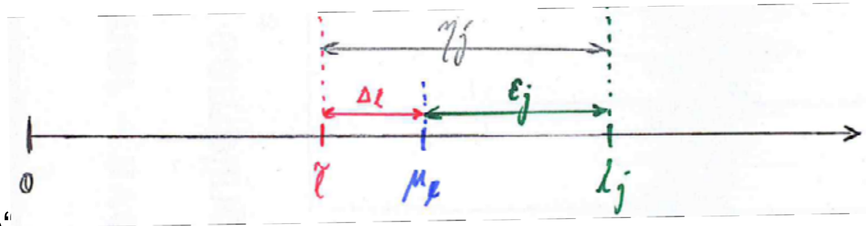
error propagation $\sigma_x \rightarrow \sigma_l$

4.1 Introduction

- For a single random variable L
  - Observation l_j
  - True value of the random variable $\tilde{l}$
  - Expectation $\mu_L = E(L) = \lim\limits_{n\to\infty} \bar{l} = \lim\limits_{n\to\infty} (\frac{1}{n}\sum_{j=1}^{n} l_j) = \lim\limits_{n\to\infty} (\frac{1}{n} \cdot \mathbf{e}_{1\times n}^{\mathrm{T}} \cdot \mathbf{l}_{n\times 1})$
  - Random deviations: $\varepsilon_j = l_j - \mu_L \quad \forall j$

    („random errors“) „actual – nominal“
  - Systematic deviations: $\Delta_l = \mu_L - \tilde{l}$

    („systematic errors“) „expectation – true value“



  - True deviations: $\eta_j = l_j - \tilde{l} = \Delta_l + \varepsilon_j \quad \forall j$

    („true errors“)

$$\boxed{l_j = \tilde{l} + \eta_j = \tilde{l} + \Delta_l + \varepsilon_j = \mu_L + \varepsilon_j} \qquad \forall j$$

4.1 Introduction

▶ For random vectors $\mathbf{L}$

- Random vector:

$$\mathbf{L}_{1\times n}^{\mathrm{T}} = [L_1 \quad L_2 \quad \cdots \quad L_n]$$

- Observation vector:

$$\mathbf{l}_{1\times n}^{\mathrm{T}} = [l_1 \quad l_2 \quad \cdots \quad l_n]$$

- Vector of true values:

$$\tilde{\mathbf{l}}_{1\times n}^{\mathrm{T}} = [\tilde{l}_1 \quad \tilde{l}_2 \quad \cdots \quad \tilde{l}_n]$$

- Expectation vector:

$$\boldsymbol{\mu}_{L\,1\times n}^{\ \mathrm{T}} = [\mu_1 \quad \mu_2 \quad \cdots \quad \mu_n] = E(\mathbf{L}_{1\times n}^{\mathrm{T}})$$

4.1 Introduction

▶ For random vectors $\mathbf{L}$

- Random deviation:

$$\boldsymbol{\varepsilon}_{l_{n\times1}} = \mathbf{l}_{n\times1} - \boldsymbol{\mu}_{L_{n\times1}}$$

- Systematic deviations:

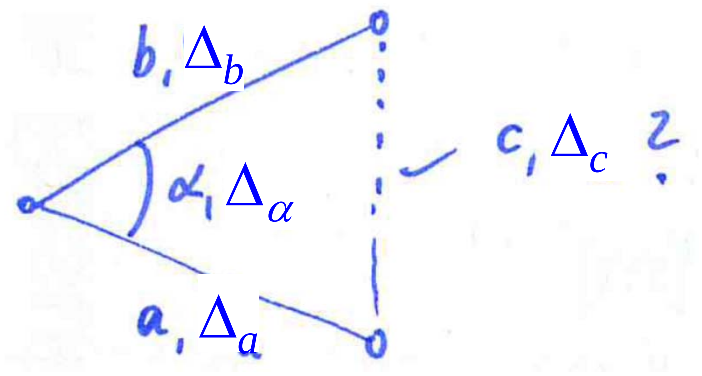
$$\Delta_{l_{n\times1}} = \boldsymbol{\mu}_{L_{n\times1}} - \tilde{\mathbf{l}}_{n\times1}$$

- True deviations: $\boldsymbol{\eta}_{l_{n\times1}} = \mathbf{l}_{n\times1} - \tilde{\mathbf{l}}_{n\times1} = \Delta_{l_{n\times1}} + \boldsymbol{\varepsilon}_{l_{n\times1}}$

$$\mathbf{l}_{n\times1} = \tilde{\mathbf{l}}_{n\times1} + \boldsymbol{\eta}_{l_{n\times1}} = \tilde{\mathbf{l}}_{n\times1} + \Delta_{l_{n\times1}} + \boldsymbol{\varepsilon}_{L_{n\times1}} = \boldsymbol{\mu}_{l_{n\times1}} + \boldsymbol{\varepsilon}_{l_{n\times1}}$$

4.2 Propagation of systematic deviations

Example:



- Given: Observation vector

$$\mathbf{l}_{n\times 1} = \tilde{\mathbf{l}}_{n\times 1} + \Delta_{l_{n\times 1}} + \boldsymbol{\varepsilon}_{l_{n\times 1}}$$

- Wanted: Quantities $\mathbf{f}_{u\times 1}$ that cannot be directly measured, but can be calculated from $\mathbf{l}_{n\times 1}$ as $\mathbf{f}_{u\times 1} = \boldsymbol{\Phi}_{u\times 1}(\mathbf{l}_{n\times 1})$.

"$\mathbf{f}$ can be derived from $\mathbf{l}$ via a functional model $\boldsymbol{\Phi}$"

Law of error propagation

4.2 Propagation of systematic deviations

- Question: What is the influence $\Delta\mathbf{f}_{u\times1}$ of systematic errors Δl_j in $\mathbf{l}_{n\times1}$ on the derived quantities $\mathbf{f}_{u\times1}$?

- Relation between $\mathbf{f}$ and $\mathbf{l}$ $\rightarrow$ functional model

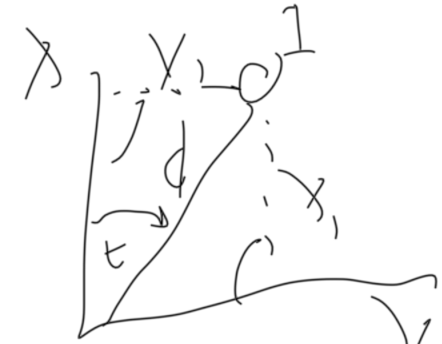
$$\mathbf{f}_{u\times1} = \boldsymbol{\Phi}_{u\times1}(\mathbf{l}_{n\times1}) = \begin{bmatrix} \varphi_1(\mathbf{l}_{n\times1}) \\ \varphi_2(\mathbf{l}_{n\times1}) \\ \vdots \\ \varphi_u(\mathbf{l}_{n\times1}) \end{bmatrix}$$

derived quantities $\rightarrow$ unknowns

functional relation

contains arbitrary (non-linear) but differentiable functions

Propagation of sys errors



$y_1 = \sin t \cdot \frac{y_1}{d} = d \cdot \sin t \quad \varphi_1\left[\begin{smallmatrix} d \\ t \end{smallmatrix}\right]$

$x_1 = d \cdot \cos t \quad \varphi_2\left[\begin{smallmatrix} d \\ t \end{smallmatrix}\right] \rightarrow \Phi(l)$

f_u

4.2.1 Linear Functional Models

If relation between $\mathbf{f}$ and $\mathbf{l}$ linear $\rightarrow$ linear functional model

$$\boxed{\mathbf{f}_{u\times 1} = \mathbf{F}_{u\times n} \cdot \mathbf{l}_{n\times 1}}$$

"Design Matrix" observation vector

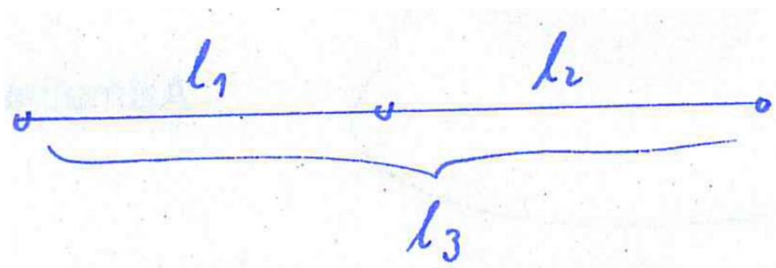
contains the coefficients

of the linear functional model

4.2.1 Linear Functional Models

Coefficients in front of measurements

<u>Example</u>: Zero correction of EDM



$$a = \underbrace{\overset{1\cdot}{l_1} + \overset{1\cdot}{l_2} - \overset{1\cdot}{l_3}}$$

Observation vector: $\mathbf{l}_{3\times1} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$, design matrix $\mathbf{F}_{1\times3} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

Propagation of true deviations („Propagation law for true deviations“)

$$\boxed{\boldsymbol{\eta}_{f_{u\times1}} = \mathbf{F}_{u\times n} \cdot \boldsymbol{\eta}_{l_{n\times1}}}$$

design Matrix   of true deviations

4.2.1 Linear Functional Models

▶ Now we consider systematic deviations

- True deviation of $\mathbf{l}_{n\times 1}$:

$$\boldsymbol{\eta}_{l_{n\times 1}} = \Delta_{l_{n\times 1}} + \boldsymbol{\varepsilon}_{l_{n\times 1}}$$

- True deviation of $\mathbf{f}_{u\times 1}$:

$$\boldsymbol{\eta}_{f_{u\times 1}} = \Delta_{f_{u\times 1}} + \boldsymbol{\varepsilon}_{f_{u\times 1}}$$

$$\boldsymbol{\eta}_f = \Delta_f + \boldsymbol{\varepsilon}_f = \mathbf{F}\cdot(\Delta_l + \boldsymbol{\varepsilon}_l) = \mathbf{F}\cdot\Delta_l + \mathbf{F}\cdot\boldsymbol{\varepsilon}_l$$

- Expectations:

$$E(\boldsymbol{\eta}_f) = \Delta_f + \underbrace{E(\boldsymbol{\varepsilon}_f)}_{=0} = \mathbf{F}\cdot\Delta_l + \mathbf{F}\cdot\underbrace{E(\boldsymbol{\varepsilon}_l)}_{=0}$$

- Propagation law for systematic deviations:

$$\Rightarrow \boxed{\Delta_{f_{u\times 1}} = \mathbf{F}_{u\times n}\cdot\Delta_{l_{n\times 1}}}$$

← systematic

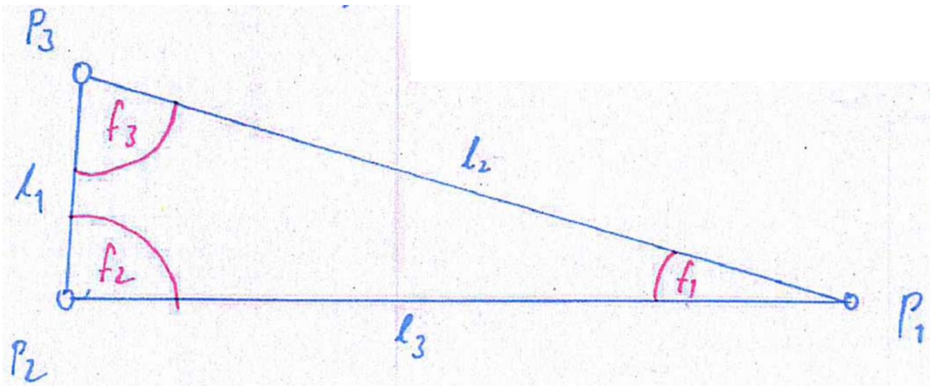
design Matrix - dev

4.2.2 Non-linear Functional Models

▶ Problem: Function of the observations $\mathbf{f}_{u\times 1} = \mathbf{\Phi}_{u\times 1}(\mathbf{l}_{n\times 1})$ oftentimes not linear

▶ Example:

- 3 measured distances in a triangle
- compute the three angles $\mathbf{f}_{u\times 1} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$
- and the corresponding systematic deviations $\Delta\mathbf{f}_{u\times 1} = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta f_3 \end{bmatrix}$



4.2.2 Non-linear Functional Models

▶ Solution: Linearisation of the function

$$\mathbf{f}_{u\times1} = \tilde{\mathbf{f}}_{u\times1} + \boldsymbol{\eta}_{f_{u\times1}} = \boldsymbol{\Phi}_{u\times1}(\mathbf{l}_{n\times1}) = \boldsymbol{\Phi}_{u\times1}(\tilde{\mathbf{l}}_{n\times1} + \boldsymbol{\eta}_{l_{n\times1}})$$

- $\boldsymbol{\eta}_{l_{n\times1}}$ true deviation of $\mathbf{l}_{n\times1}$
- $\tilde{\mathbf{f}}_{u\times1}$ true value of $\mathbf{f}_{u\times1}$
- $\boldsymbol{\eta}_{f_{u\times1}}$ true deviation of $\mathbf{f}_{u\times1}$

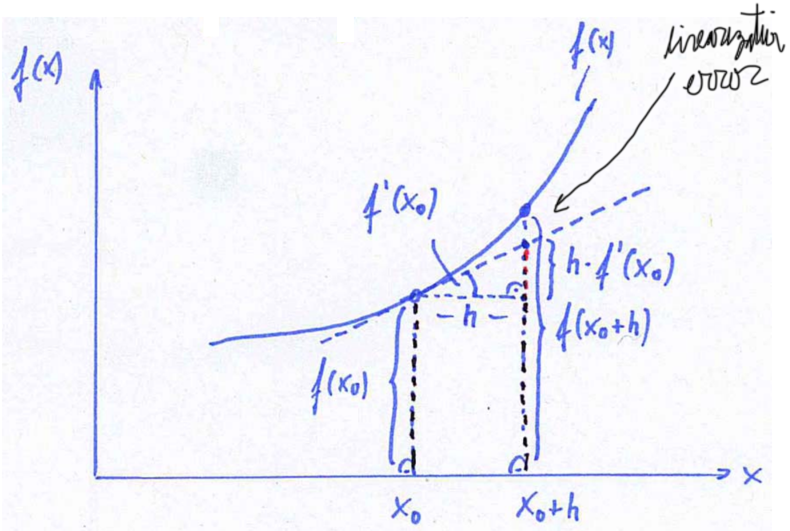
▶ Taylor Series introduced by Brook Taylor in 1715

▶ We know from mathematics: Taylor series is a representation of a function as an infinite sum of terms ($n \rightarrow \infty$) that are calculated from the values of the function's derivatives at a single point

▶ In practice: Approximation of a function by using a finite number n of terms of its Taylor Series

4.2.2 Non-linear Functional Models

linearization error gets larger the further away from x_0



linearization error

h = distance

$f'(x_0)$ = partial derivative = tangent slope

$\tan\alpha = \frac{y}{h}$

$y = h \cdot \tan\alpha \Rightarrow y = h \cdot f'(x_0)$

- Function at the point x_0: $f(x_0)$ with $x_0 \hat{=}$ “starting value” or “approximate value”

- Function at the point $x_0 + h$:

“factorial 3” $= 1 \cdot 2 \cdot 3 = 6$

$$f(x_0 + h) = f(x_0) + \frac{h}{1!} \cdot f'(x_0) \underbrace{+ \frac{h^2}{2!} \cdot f''(x_0) + \frac{h^3}{3!} \cdot f'''(x_0) + \cdots + \frac{h^n}{n!} \cdot f^{(n)}(x_0)}$$

Terms of higher order can be neglected if $h \ll x_0$!

$$\boxed{f(x_0 + h) = f(x_0) + h \cdot f'(x_0)}$$ Linearisation of a function

4.2.2 Non-linear Functional Models

Example 1:

$$f(x) = x^2,\ x_0 = 19,\ h = 1$$

$$f(x_0 + h) = x_0^2 + 1 \cdot 2x_0 = 19^2 + 2 \cdot 19 = 399$$

Direct solution: $f(x_0 + h) = (19 + 1)^2 = 400$

→ Difference, because we have assumed that terms of higher order $= 0$, but $h \ll x_0$ is not the case in this example

→ Difference is called "linearisation error"

▶ Question: How can we avoid a large linearisation error?

▶ Answer: h must be a small value

→ Starting value must be "close" to the solution

4.2.2 Non-linear Functional Models

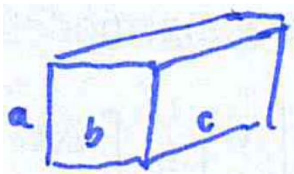
Example 2:

$$f(x) = x^2,\ x_0 = 19.9,\ h = 0.1$$

$$f(x_0 + h) = 19.9^2 + 0.1 \cdot 2 \cdot 19.9 = 396.01 + 3.98 = 399.99$$

- Function of more variables

  e.g. computation of volume $V = a \cdot b \cdot c$



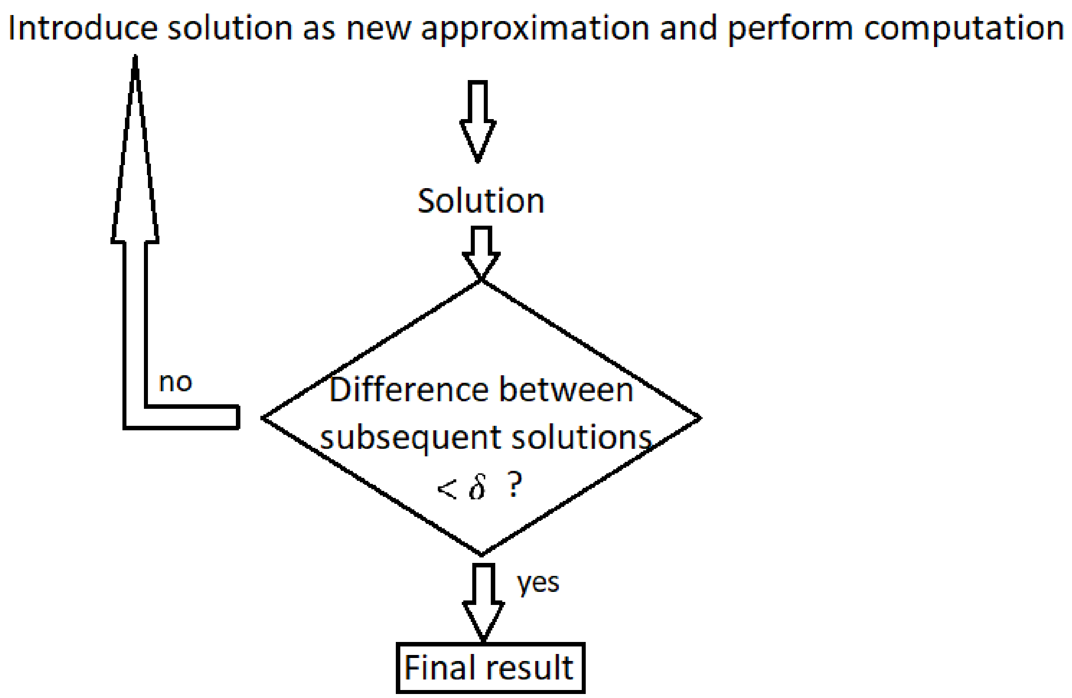
$$f(a_0 + h, b_0 + k, c_0 + m) = f(a_0, b_0, c_0) + h \cdot \left(\frac{\partial f}{\partial a}\right)_0 + k \cdot \left(\frac{\partial f}{\partial b}\right)_0 + m \cdot \left(\frac{\partial f}{\partial c}\right)_0$$

Partial derivative at the place of starting values

4.2.2 Non-linear Functional Models

▶ General problem: How can we be sure that we have reached an appropriate solution?

→ Iterative computing!



4.2.2 Non-linear Functional Models

$$\tilde{\mathbf{f}}_{u\times 1} + \boldsymbol{\eta}_{f\,u\times 1} = \boldsymbol{\Phi}_{u\times 1}(\tilde{\mathbf{l}}_{n\times 1} + \boldsymbol{\eta}_{l\,n\times 1})$$

contains arbitrary (non-linear)
differentiable functions

$$\tilde{\mathbf{f}} + \boldsymbol{\eta}_f = \boldsymbol{\Phi}_{u\times 1}(\tilde{\mathbf{l}}_{n\times 1}) + \frac{\partial \boldsymbol{\Phi}(\mathbf{l})}{\partial \mathbf{l}}_{u\times n} \cdot \boldsymbol{\eta}_{l\,n\times 1} + \frac{\partial^2 \boldsymbol{\Phi}(\mathbf{l})}{2\cdot \partial \mathbf{l}^2}_{u\times n} \cdot \boldsymbol{\eta}^2_{l\,n\times 1} + \cdots$$

Taylor Series

4.2.2 Non-linear Functional Models

Under the assumption that

much smaller

$$\|\boldsymbol{\eta}_l\|_{n\times1} \ll \|\mathbf{l}\|_{n\times1}$$

Euclidean norm

$$\sqrt{\sum_{i=1}^{n} {\eta_{l_i}}^2}$$

$$\sqrt{\sum_{i=1}^{n} {l_i}^2}$$

length of vector

we can truncate the Taylor Series after the first term

$$\tilde{\mathbf{f}} + \boldsymbol{\eta}_f = \boldsymbol{\Phi}(\tilde{\mathbf{l}}) + \frac{\partial \boldsymbol{\Phi}(\mathbf{l})}{\partial \boldsymbol{l}} \cdot \boldsymbol{\eta}_l$$

$$\text{with } \tilde{\mathbf{f}} = \boldsymbol{\Phi}(\tilde{\mathbf{l}})$$

true obs

$$\Rightarrow \boldsymbol{\eta}_{f_{u\times1}} = \underbrace{\frac{\partial \boldsymbol{\Phi}(\mathbf{l})}{\partial \mathbf{l}}}_{\mathbf{J}_{u\times n}} \cdot \boldsymbol{\eta}_{l_{n\times1}}$$

Jacobian Matrix

4.2.2 Non-linear Functional Models

Jacobian Matrix

$$\mathbf{J}_{u\times n}=\begin{bmatrix}\frac{\partial\varphi_1(\mathbf{l})}{\partial l_1} & \frac{\partial\varphi_1(\mathbf{l})}{\partial l_2} & \cdots & \frac{\partial\varphi_1(\mathbf{l})}{\partial l_n}\\ \frac{\partial\varphi_2(\mathbf{l})}{\partial l_1} & \frac{\partial\varphi_2(\mathbf{l})}{\partial l_2} & \cdots & \frac{\partial\varphi_2(\mathbf{l})}{\partial l_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial\varphi_u(\mathbf{l})}{\partial l_1} & \frac{\partial\varphi_u(\mathbf{l})}{\partial l_2} & \cdots & \frac{\partial\varphi_u(\mathbf{l})}{\partial l_n}\end{bmatrix}=\begin{bmatrix}j_{11} & j_{12} & \cdots & j_{1n}\\ j_{21} & j_{22} & \cdots & j_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ j_{u1} & j_{u2} & \cdots & j_{un}\end{bmatrix}$$

$y_1 = \underbrace{d \cdot \sin t}_{\varphi_1(d,t)}$

$x_1 = \underbrace{d \cdot \cos t}_{\varphi_2(d,t)}$

$$\begin{array}{c c} & \begin{array}{cc}\partial d & \partial t\end{array}\\ \begin{array}{c}\varphi_1\\ \varphi_2\end{array} & \begin{bmatrix}\sin t & d\cdot\cos t\\ \cos t & -d\cdot\sin t\end{bmatrix}\end{array}$$

Now: We take elements of matrix $\mathbf{J}$ and insert them into matrix $\mathbf{F}$

$$\mathbf{F}_{u\times n}=\begin{bmatrix}f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ f_{u1} & f_{u2} & \cdots & f_{un}\end{bmatrix}=\begin{matrix}f_{11}=j_{11}, f_{12}=j_{12}\ \ldots f_{1n}=j_{1n}\\ \vdots\\ f_{u1}=j_{u1}, f_{u2}=j_{u2}\ \ldots f_{un}=j_{un}\end{matrix}$$

4.2.2 Non-linear Functional Models

- Advantage of linearisation of the functional model:
  We can apply the simple formulas for the linear case!

- Propagation law for true deviations:

$$\boxed{\boldsymbol{\eta}_{f_{u\times 1}} = \mathbf{F}_{u\times n} \cdot \boldsymbol{\eta}_{l_{n\times 1}}}$$

- Propagation law for systematic deviations:

$$\boxed{\Delta_{f_{u\times 1}} = \mathbf{F}_{u\times 1} \cdot \Delta_{l_{n\times 1}}}$$