Least-squares Adjustment for Linear Adjustment Problems

Linear functional model for the unknowns:

$$\begin{aligned} L_1 &= a_{11}X_1 + a_{12}X_2 + \cdots + a_{1u}X_u \\ L_2 &= a_{21}X_1 + a_{22}X_2 + \cdots + a_{2u}X_u \\ &\vdots \\ L_n &= a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nu}X_u \end{aligned}$$

Vector of observations:

$$\underset{n,1}{\mathbf{L}} = \begin{bmatrix} L_1 & L_2 & \cdots & L_n \end{bmatrix}^{\mathrm{T}}$$

Variance covariance matrix of the observations:

$$\underset{n,n}{\mathbf{\Sigma_{LL}}} = \begin{bmatrix} \sigma_{L_1}^2 & \sigma_{L_1L_2} & \cdots & \sigma_{L_1L_n} \\ \sigma_{L_2L_1} & \sigma_{L_2}^2 & \cdots & \sigma_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_nL_1} & \sigma_{L_nL_2} & \cdots & \sigma_{L_n}^2 \end{bmatrix}$$ with theoretical values σ_i

$$\underset{n,n}{\mathbf{S_{LL}}} = \begin{bmatrix} s_{L_1}^2 & s_{L_1L_2} & \cdots & s_{L_1L_n} \\ s_{L_2L_1} & s_{L_2}^2 & \cdots & s_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{L_nL_1} & s_{L_nL_2} & \cdots & s_{L_n}^2 \end{bmatrix}$$ with empirical values s_i

Theoretical reference standard deviation: σ_0 (or theoretical reference variance σ_0^2)

Cofactor matrix of the observations:

$$\underset{n,n}{\mathbf{Q_{LL}}} = \frac{1}{\sigma_0^2}\underset{n,n}{\mathbf{\Sigma_{LL}}}$$ respectively $$\underset{n,n}{\mathbf{Q_{LL}}} = \frac{1}{\sigma_0^2}\underset{n,n}{\mathbf{S_{LL}}}$$

Weight matrix of the observations:

$$\underset{n,n}{\mathbf{P}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}^{-1}}$$

Vector of adjusted unknowns:

$$\underset{u,1}{\hat{\mathbf{X}}} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \cdots & \hat{X}_u \end{bmatrix}^{\mathrm{T}}$$

Matrix of coefficients of the linear functional model:

$$\underset{n,u}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1u} \\ a_{21} & a_{22} & \cdots & a_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nu} \end{bmatrix}$$ "Design Matrix"

Vector of residuals:

$$\underset{n,1}{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{\mathrm{T}}$$

Observation equations:

$$\underset{n,1}{\mathbf{L}} + \underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{X}}}$$

Normal equations:

$$\underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}\,\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{L}}$$

Normal matrix:

$$\underset{u,u}{\mathbf{N}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,u}{\mathbf{A}}$$

Right hand side of normal equations:

$$\underset{u,1}{\mathbf{n}} = \underset{u,n}{\mathbf{A}^{\mathrm{T}}}\,\underset{n,n}{\mathbf{P}}\,\underset{n,1}{\mathbf{L}}$$

Normal equations:

$$\underset{u,u}{\mathbf{N}}\,\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,1}{\mathbf{n}}$$

Inversion of normal matrix:	$\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = \underset{u,u}{\mathbf{N}^{-1}}$
Solution for the unknowns:	$\underset{u,1}{\hat{\mathbf{X}}} = \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}\ \underset{u,1}{\mathbf{n}}$
Vector of residuals:	$\underset{n,1}{\mathbf{v}} = \underset{n,u}{\mathbf{A}}\ \underset{u,1}{\hat{\mathbf{X}}} - \underset{n,1}{\mathbf{L}}$
Vector of adjusted observations:	$\underset{n,1}{\hat{\mathbf{L}}} = \underset{n,1}{\mathbf{L}} + \underset{n,1}{\mathbf{v}}$
Final check:	$\underset{n,1}{\hat{\mathbf{L}}} \overset{!}{=} \underset{n,1}{\mathbf{\Phi}}\left(\underset{u,1}{\hat{\mathbf{X}}}\right)$
Empirical reference standard deviation:	$s_0 = \sqrt{\dfrac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$ (or empirical reference variance s_0^2)
Cofactor matrix of adjusted unknowns:	$\underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}$
VCM of adjusted unknowns:	$\underset{u,u}{\mathbf{\Sigma}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = s_0^2\ \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}$
Cofactor matrix of adjusted observations:	$\underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = \underset{n,u}{\mathbf{A}}\ \underset{u,u}{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}}\ \underset{u,n}{\mathbf{A}^{\mathrm{T}}}$
VCM of adjusted observations:	$\underset{n,n}{\mathbf{\Sigma}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}} = s_0^2\ \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$
Cofactor matrix of the residuals:	$\underset{n,n}{\mathbf{Q}_{\mathbf{vv}}} = \underset{n,n}{\mathbf{Q}_{\mathbf{LL}}} - \underset{n,n}{\mathbf{Q}_{\hat{\mathbf{L}}\hat{\mathbf{L}}}}$
VCM of the residuals:	$\underset{n,n}{\mathbf{\Sigma}_{\mathbf{vv}}} = s_0^2\ \underset{n,n}{\mathbf{Q}_{\mathbf{vv}}}$