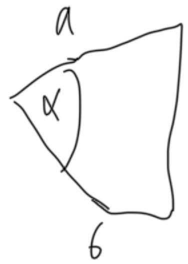
4.4.2 Special case: n uncorrelated observations, 1 unknown value

$$\mathbf{\Sigma}_{LL_{n\times n}} = \begin{bmatrix} \sigma_{l_1}^2 & & & 0 \\ & \sigma_{l_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{l_n}^2 \end{bmatrix}$$

$$x = \Phi(\mathbf{l}) = \varphi(l_1, l_2, \dots, l_n)$$



$$\underbrace{c = \sqrt{a^2 + b^2 - 2ab\cos\alpha}}_{\varphi(a,b,c,\alpha)}$$

$$\Sigma_{ll} =$$

- Step 1:

  Set up the functional model for one functional relation $\Phi(\mathbf{l})$ with n observations (measurements) l_i

$$x = \Phi(\mathbf{l}) = \varphi(l_1, l_2, \dots, l_n)$$

4.4.2 Special case: n uncorrelated observations, 1 unknown value

▶ Step 2: Set up the stochastic model for n uncorrelated observations

$$\mathbf{\Sigma}_{LL} = \begin{bmatrix} \sigma_{l_1}^2 & & & 0 \\ & \sigma_{l_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{l_n}^2 \end{bmatrix}$$

<u>Question</u>: Functional model linear or nonlinear?

- if functional model linear, design matrix $\mathbf{F}$ directly given
- if functional model nonlinear, proceed with step 3

▶ Step 3: Linearisation of the functional model

→ computation of partial derivatives $\frac{\partial \varphi}{\partial l_i}$ $\quad i = 1, 2, \dots, n$

$$\mathbf{J}_{1 \times n} = \begin{bmatrix} \frac{\partial \varphi}{\partial l_1} & \frac{\partial \varphi}{\partial l_2} & \frac{\partial \varphi}{\partial l_3} & \cdots & \frac{\partial \varphi}{\partial l_n} \end{bmatrix}$$

→ Insert elements of $\mathbf{J}$ into matrix $\mathbf{F}$

4.4.2 Special case: n uncorrelated observations, 1 unknown value

▶ Step 4: Computation of „VCM“ for the unknowns

F $\quad\quad\quad\quad$ Σ_{ll}

$$\boldsymbol{\Sigma}_{xx_{1\times 1}} = \begin{bmatrix} \dfrac{\partial\varphi}{\partial l_1} & \dfrac{\partial\varphi}{\partial l_2} & \dfrac{\partial\varphi}{\partial l_3} & \cdots & \dfrac{\partial\varphi}{\partial l_n} \end{bmatrix}_{1\times n} \begin{bmatrix} \sigma_{l_1}^2 & & & 0 \\ & \sigma_{l_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{l_n}^2 \end{bmatrix}_{n\times n} \begin{bmatrix} \dfrac{\partial\varphi}{\partial l_1} \\ \dfrac{\partial\varphi}{\partial l_1} \\ \vdots \\ \dfrac{\partial\varphi}{\partial l_n} \end{bmatrix}_{n\times 1}$$

F^T

- Theoretical variance:

$$\boxed{\sigma_x^2 = \left(\frac{\partial\varphi}{\partial l_1}\right)^2 \cdot \sigma_{l_1}^2 + \left(\frac{\partial\varphi}{\partial l_2}\right)^2 \cdot \sigma_{l_2}^2 + \ldots + \left(\frac{\partial\varphi}{\partial l_n}\right)^2 \cdot \sigma_{l_n}^2}$$

- Empirical variance:

$$\boxed{s_x^2 = \left(\frac{\partial\varphi}{\partial l_1}\right)^2 \cdot s_{l_1}^2 + \left(\frac{\partial\varphi}{\partial l_2}\right)^2 \cdot s_{l_2}^2 + \ldots + \left(\frac{\partial\varphi}{\partial l_n}\right)^2 \cdot s_{l_n}^2}$$

Variance propagation for uncorrelated observations

4.4.2 Special case: n uncorrelated observations, 1 unknown value

- Theoretical standard deviation:

$$\sigma_x = \sqrt{\sigma_x^2}$$

- Empirical standard deviation:

$$s_x = \sqrt{s_x^2}$$

→ σ_x respective s_x can easily be computed without applying matrix calculus

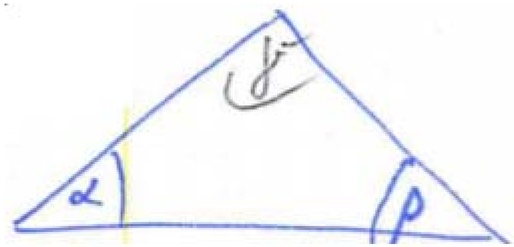
Please note: General case includes the special case, but the special case does not include the general case.

4.4.3 Variance – Covariance propagation: Some examples

Example 1:

▶ Angles α and β have been measured with a standard deviation of

$$s_\alpha = s_\beta = 1 \text{ mgon}$$



▶ Compute the standard deviation s_γ

▶ General or special case?

Special

▶ Step 1: Functional model $\gamma = 200 \text{ gon} - \alpha - \beta$

$$\underbrace{\gamma}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \underset{f_{11}}{\overset{\alpha}{-1}} & \underset{f_{12}}{\overset{\beta}{-1}} \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\mathbf{l}} + 200 \text{ gon}$$

4.4.3 Variance – Covariance propagation: Some examples

▶ Step 2: Stochastic model

$$\mathbf{S}_{LL} = \begin{bmatrix} (1\ \text{mgon})^2 & 0 \\ 0 & (1\ \text{mgon})^2 \end{bmatrix}$$

▶ Steps 3 and 4:

$$s_\gamma^2 = (f_{11})^2 \cdot s_\alpha^2 + (f_{12})^2 \cdot s_\beta^2$$

with $f_{11} = -1$ and $f_{12} = -1$

$$\Rightarrow s_\gamma^2 = 1 \cdot (1\ \text{mgon})^2 + 1 \cdot (1\ \text{mgon})^2$$

$$s_\gamma = \sqrt{2}\ \text{mgon}$$

$$s_\gamma = 1.41\ \text{mgon}$$

4.4.3 Variance – Covariance propagation: Some examples

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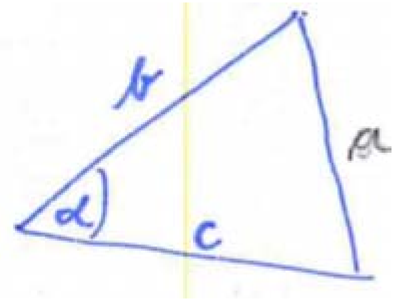
Example 2:

▶ Given: $b = 20.29\ \text{m}, \quad s_b = 1.0\ \text{cm}$

$c = 75.75\ \text{m}, \quad s_c = 2.0\ \text{cm}$

$\alpha = 27.292\ \text{gon}, \ s_\alpha = 0.015\ \text{gon}$



▶ Wanted: a, s_a

Special case

1 unkown

STD

▶ General or special case?

▶ Step 1: Functional model

$$a = \sqrt{b^2 + c^2 - 2bc \cdot \cos\alpha} \quad \rightarrow \quad \text{non-linear}$$

4.4.3 Variance – Covariance propagation: Some examples

▶ Step 2: Stochastic model

$a = \sqrt{b^2 + c^2 - 2bc \cdot \cos\alpha}$

$s_a^2 = \left(\frac{\partial a}{\partial b}\right)^2 \cdot s_b^2 + \left(\frac{\partial a}{\partial c}\right)^2 \cdot s_c^2 + \left(\frac{\partial a}{\partial \alpha}\right)^2 \cdot s_\alpha^2$

$\frac{\partial a}{\partial b} = \frac{1}{\cancel{2}\sqrt{\ }} \cdot (\cancel{2}b - \cancel{2}c \cdot \cos) = \frac{b - c \cdot \cos\alpha}{\sqrt{\ }} = \frac{b - c \cdot \cos\alpha}{a}$

$\frac{\partial a}{\partial c} = \frac{1}{\cancel{2}\sqrt{\ }} \cdot (\cancel{2}c - \cancel{2}b \cdot \cos\alpha) = \frac{c - b \cdot \cos\alpha}{\sqrt{\ }} = \frac{c - b\cos\alpha}{a}$

$\frac{\partial a}{\partial \alpha} = \frac{1}{\cancel{2}\sqrt{\ }} \cdot (\cancel{2}bc \cdot \sin\alpha) = \frac{bc \sin\alpha}{\sqrt{\ }} = \frac{bc \cdot \sin\alpha}{a}$

$$\mathbf{S}_{LL} = \begin{bmatrix} (s_b)^2 & & 0 \\ & (s_c)^2 & \\ 0 & & (s_\alpha)^2 \end{bmatrix} = \begin{bmatrix} (0.01)^2 & & 0 \\ & (0.02)^2 & \\ 0 & & \left(\frac{0.015}{\rho}\right)^2 \end{bmatrix}$$

with $\rho = \frac{200}{\pi}$

- Attention: Units must coincide!

radian = 2π = 400 gon = ○ = full circle = 360°

$s_a^2 = \left(\frac{b - c \cdot \cos\alpha}{a}\right)^2 \cdot s_b^2 + \left(\frac{c - b \cdot \cos\alpha}{a}\right)^2 \cdot s_c^2 + \left(\frac{bc \cdot \sin\alpha}{a}\right)^2 \cdot s_\alpha^2 \Rightarrow s_a = 0.022\,m$

- Solution: Choose [m] and [rad] in $\mathbf{S}_{LL}$

— Convert everything to m & gon

4.4.3 Variance – Covariance propagation: Some examples

$$a = \sqrt{b^2 + c^2 - 2bc \cdot \cos\alpha}$$

- Steps 3 and 4:

$$s_a^2 = \left(\frac{\partial a}{\partial b}\right)^2 \cdot (s_b)^2 + \left(\frac{\partial a}{\partial c}\right)^2 \cdot (s_c)^2 + \left(\frac{\partial a}{\partial \alpha}\right)^2 \cdot (s_\alpha)^2$$

with:

$$\frac{\partial a}{\partial b} = \frac{1}{2\sqrt{}} \cdot (2b - 2c \cdot \cos\alpha) = \frac{b - c \cdot \cos\alpha}{\sqrt{}} = \frac{b - c \cdot \cos\alpha}{a}$$

$$\frac{\partial a}{\partial c} = \frac{1}{2\sqrt{}} \cdot (2c - 2b \cdot \cos\alpha) = \frac{c - b \cdot \cos\alpha}{a}$$

$$\frac{\partial a}{\partial \alpha} = \frac{1}{2\sqrt{}} \cdot (2bc \cdot \sin\alpha) = \frac{bc \cdot \sin\alpha}{\sqrt{}} = \frac{bc \cdot \sin\alpha}{a}$$

$$s_a^2 = \left(\frac{b - c \cdot \cos\alpha}{a}\right)^2 \cdot (s_b)^2 + \left(\frac{c - b \cdot \cos\alpha}{a}\right)^2 \cdot (s_c)^2 + \left(\frac{bc \cdot \sin\alpha}{a}\right)^2 \cdot (s_\alpha)^2$$

$$\Rightarrow s_a = 0.022 \text{ m}$$

4.4.3 Variance – Covariance propagation: Some examples

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$Y_P = Y_S + d \cdot \sin t$
$X_P = X_S + d \cdot \cos t$

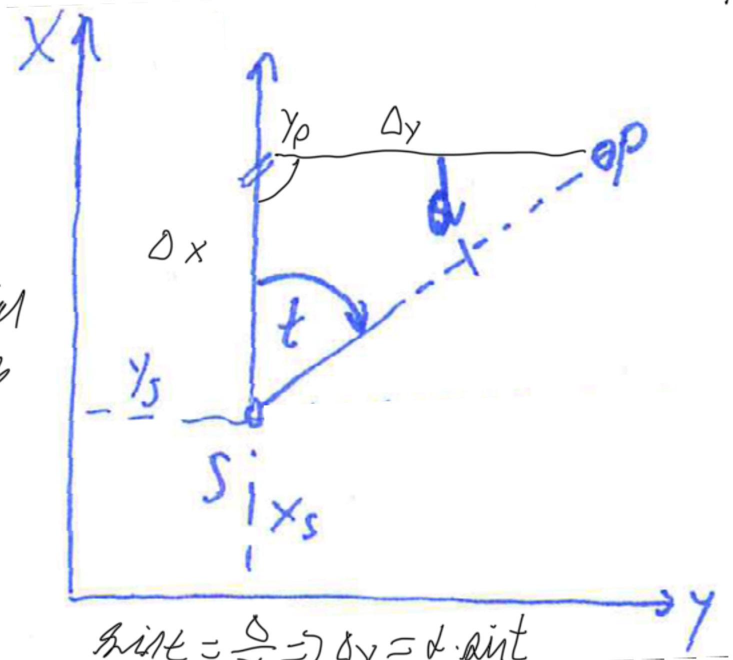
Example 3: Computation of Cartesian Coordinates

► Given:

$Y_S = 1000.000$ m, $X_S = 1000.000$ m
fixed values (error-free)

$t = 77.1234$ gon, $\sigma_t = 3.0$ mgon = radian

$d = 987.654$ m, $\sigma_d = 20.0$ mm = gon



$\sin t = \frac{\Delta}{y} \Rightarrow \Delta y = d \cdot \sin t$

$\cos t = \frac{\Delta}{x} \Rightarrow \Delta x = d \cdot \cos t$

► Wanted:

Y_P, X_P and $\sigma_{Y_P}, \sigma_{X_P}$ and ρ_{Y_P, X_P}

► General or special case?

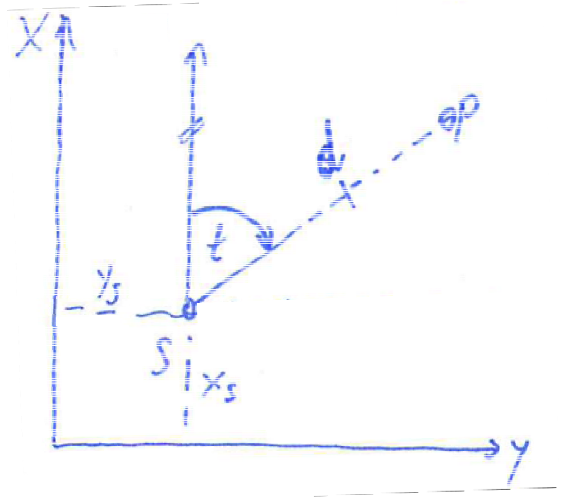
more than 1 unknown = general case

4.4.3 Variance – Covariance propagation: Some examples

▶ Step 1: Functional model

$$\begin{bmatrix} Y_p \\ X_p \end{bmatrix} = \begin{bmatrix} \Phi_1(\mathbf{l}) \\ \Phi_2(\mathbf{l}) \end{bmatrix} = \begin{bmatrix} \varphi_1(d,t) \\ \varphi_2(d,t) \end{bmatrix}$$

$$\begin{bmatrix} Y_P \\ X_P \end{bmatrix} = \begin{bmatrix} Y_S + d \cdot \sin t \\ X_S + d \cdot \cos t \end{bmatrix} = \begin{bmatrix} 1924.5700\ \mathrm{m} \\ 1347.3193\ \mathrm{m} \end{bmatrix}$$



$$J = \begin{array}{c|cc} & d & t \\ \hline Y_P & \frac{\partial y_P}{\partial \alpha} & \frac{\partial y_P}{\partial t} \\ X_P & \frac{\partial x_P}{\partial d} & \frac{\partial x_P}{\partial t} \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} \sin t & d \cdot \cos t \\ \cos t & -d \cdot \sin t \end{bmatrix}$$

$$\downarrow$$

$$F$$

▶ Step 2:

- Observation vector

$$\mathbf{l} = \begin{bmatrix} d \\ t \end{bmatrix} = \begin{bmatrix} 987.654\ \mathrm{m} \\ 77.1234\ \mathrm{gon} \end{bmatrix}$$

- Stochastic model

$$\mathbf{\Sigma}_{LL} = \begin{bmatrix} (\sigma_\alpha)^2 & 0 \\ 0 & (\sigma_t)^2 \end{bmatrix} = \begin{bmatrix} (0.02)^2 & 0 \\ 0 & \left(\frac{0.003}{\rho}\right)^2 \end{bmatrix}$$

4.4.3 Variance – Covariance propagation: Some examples

▶ Step 3: Linearisation, computation of partial derivatives

$$\frac{\partial Y_P}{\partial d} = \sin t\,; \quad \frac{\partial Y_P}{\partial t} = d \cdot \cos t$$

$$\begin{bmatrix} Y_P \\ X_P \end{bmatrix} = \begin{bmatrix} Y_S + d \cdot \sin t \\ X_S + d \cdot \cos t \end{bmatrix}$$

$$\frac{\partial X_P}{\partial d} = \cos t; \quad \frac{\partial X_P}{\partial t} = -d \cdot \sin t$$

Design matrix $\mathbf{F} = \mathbf{J}$:

$$\mathbf{J} = \begin{array}{c} \\ Y_P \\ X_P \end{array} \begin{array}{c} \begin{array}{cc} \quad d \quad & \quad t \quad \end{array} \\ \begin{bmatrix} \frac{\partial Y_P}{\partial d} & \frac{\partial Y_P}{\partial t} \\ \frac{\partial X_P}{\partial d} & \frac{\partial X_P}{\partial t} \end{bmatrix} \end{array} = \begin{bmatrix} \sin t & d \cdot \cos t \\ \cos t & -d \cdot \sin t \end{bmatrix}$$

4.4.3 Variance – Covariance propagation: Some examples

▶ Step 4: VCM of the unknowns

$$\boldsymbol{\Sigma}_{xx} = \mathbf{F} \cdot \boldsymbol{\Sigma}_{LL} \cdot \mathbf{F}^{\mathrm{T}}$$

$$\dots$$

$$\boldsymbol{\Sigma}_{xx} = \begin{bmatrix} 618.4138 & -581.4212 \\ -581.4212 & 1947.7531 \end{bmatrix}_{[\mathrm{mm}^2]}$$

$$\sigma_{Y_P} = \sqrt{618.4138} = 24.8679 \text{ mm}$$

$$\sigma_{X_P} = \sqrt{1947.7531} = 44.1334 \text{ mm}$$

$$\rho_{Y_P,X_P} = \frac{-581.4212}{24.8679 \cdot 44.1334} = -0.53$$ ← correlation between Y & X

4.4.4 Variance – Covariance propagation for complex functional relationships

$$Y_N = Y_M + s \cdot \sin t_{M,N}$$
$$X_N = X_M + s \cdot \cos t_{M,N}$$
general approach

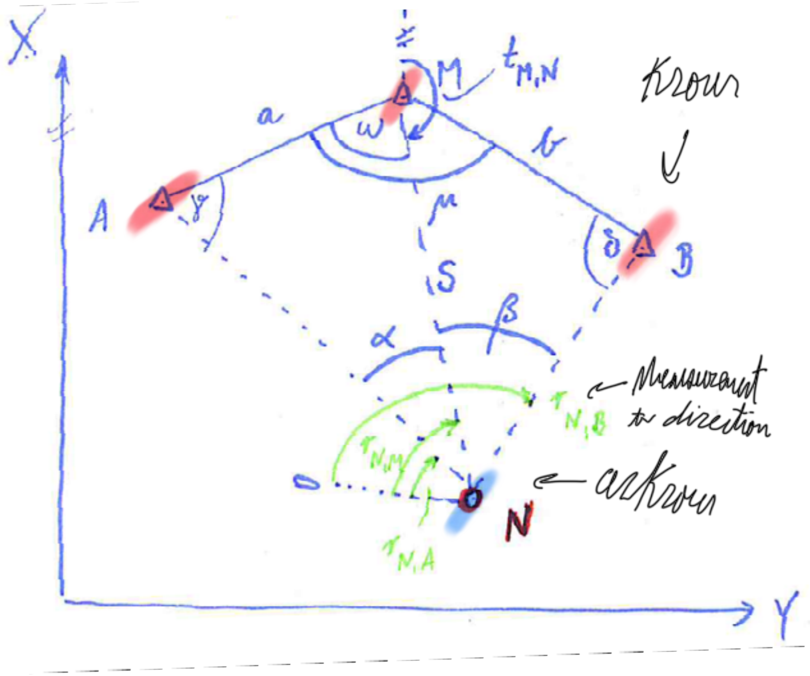
<u>Example</u>: Resection

▶ Given:

- Coordinates of points A, M, B as fixed values (error-free)
- Measurements: Horizontal directions $r_{N,A}, r_{N,M}, r_{N,B}$ with their standard deviations $\sigma_{r_{N,A}}, \sigma_{r_{N,M}}, \sigma_{r_{N,B}}$

▶ Wanted:

- Coordinates Y_N, X_N
- Standard deviations $\sigma_{Y_N}, \sigma_{X_N}$
- Correlation ρ_{Y_N,X_N}



4.4.4 Variance – Covariance propagation for complex functional relationships

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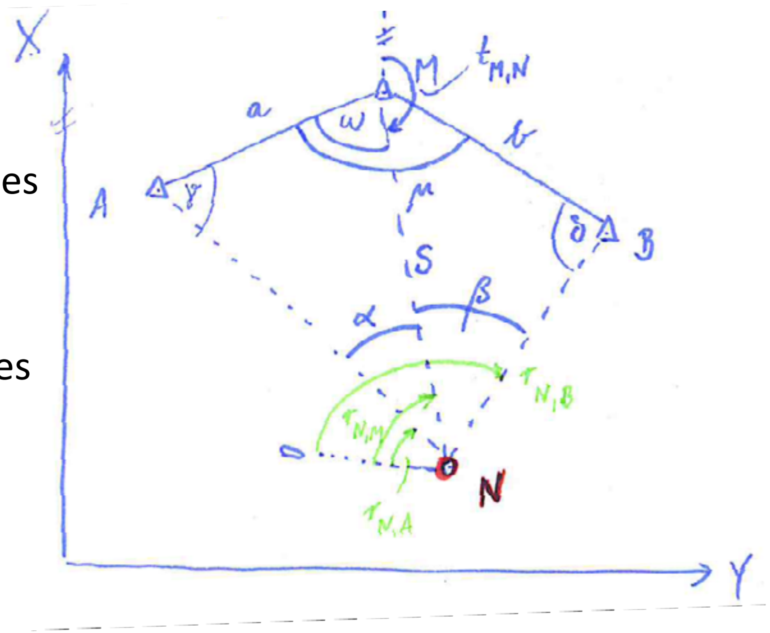
- Functional relationships:

1. $\alpha = r_{N,M} - r_{N,A}$
2. $\beta = r_{N,B} - r_{N,M}$
3. $\varepsilon = 2\pi - \alpha - \beta - \mu$ — computed from coordinates $(\varepsilon = \gamma + \delta)$
4. $a_1 = \frac{\sin \alpha}{a}$
5. $a_2 = \frac{\sin \beta}{b}$ — distances computed from coordinates
6. $\gamma = \arctan \frac{a_1 \cdot \sin \varepsilon}{a_2 + a_1 \cdot \cos \varepsilon}$
7. $s = \frac{\sin \gamma}{a_1}$
8. $t_{M,N} = t_{M,A} - \pi + \alpha + \gamma$ — computed from coordinates $\quad (t_{M,N} = t_{M,A} - \omega)$
9. $\boxed{Y_N = Y_M + s \cdot \sin t_{M,N}}$
10. $\boxed{X_N = X_M + s \cdot \cos t_{M,N}}$

Computation of Cartesian Coordinates



4.4.4 Variance – Covariance propagation for complex functional relationships

- Question: How can we compute σ_{Y_N}, σ_{X_N} ?

- First idea: Insert formulas into one another to obtain a formula for Y_N resp. X_N that depends only on the original measurements $r_{N,A}$, $r_{N,M}$, $r_{N,B}$

- We obtain:

$$Y_N = \varphi_1(r_{N,A}, r_{N,M}, r_{N,B})$$
$$X_N = \varphi_2(r_{N,A}, r_{N,M}, r_{N,B})$$

- Problem 1: We obtain very complicated equations, very complex part. derivatives

- Problem 2: Y_N, X_N are correlated! If we want to use Y_N, X_N for further computation, we need the full VCM → We have to apply the <u>general case</u> of VC propagation

- Solution: We keep the decomposition in partial steps and apply <u>chain rule of matrix multiplication</u>

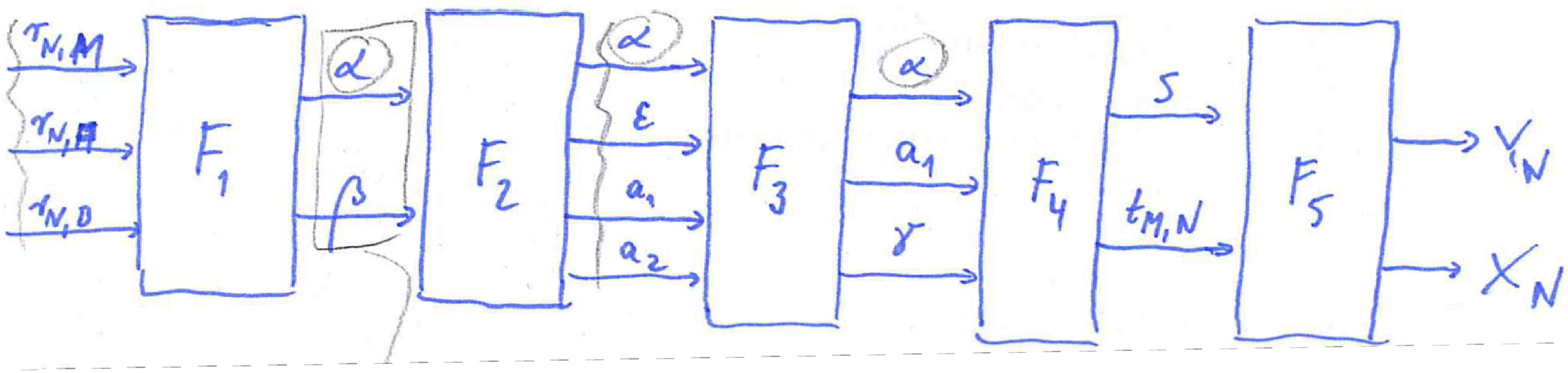
<u>Remark</u>: If (and only if) we are only interested in $\sigma_{Y_N}, \sigma_{X_N}$ and <u>not</u> in the correlation ρ_{Y_N,X_N} we can compute $\sigma_{Y_N}, \sigma_{X_N}$ separately by applying the <u>special case</u> two times

4.4.4 Variance – Covariance propagation for complex functional relationships

in exam

We have to consider ten functional relationships

<u>Flowchart</u>



<u>VC propagation</u>: $\boldsymbol{\Sigma}_{xx} = \mathbf{F} \cdot \boldsymbol{\Sigma}_{LL} \cdot \mathbf{F}^{\mathrm{T}}$

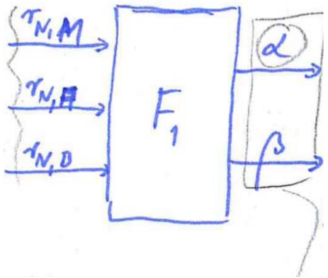
with $\mathbf{F} = \mathbf{F}_k \cdot \mathbf{F}_{k-1} \cdot \mathbf{F}_{k-2} \cdot \dots \cdot \mathbf{F}_1$

in our example $\mathbf{F} = \mathbf{F}_5 \cdot \mathbf{F}_4 \cdot \mathbf{F}_3 \cdot \mathbf{F}_2 \cdot \mathbf{F}_1$

4.4.4 Variance – Covariance propagation for complex functional relationships

$\mathbf{F}_1$:

- Functional model:



$$\begin{aligned} \alpha &= r_{N,M} - r_{N,A} \\ \beta &= r_{N,B} - r_{N,M} \end{aligned}$$

hidden 1st coefficient

- Design matrix:

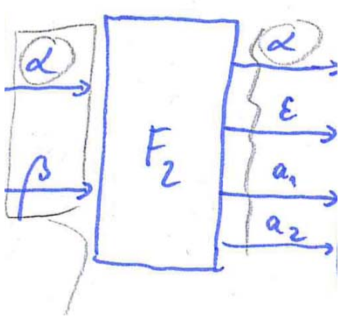
$$\mathbf{F}_1 = \begin{array}{c c} & \begin{array}{ccc} r_{N,M} & r_{N,A} & r_{N,B} \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}$$

New observations

4.4.4 Variance – Covariance propagation for complex functional relationships

$\mathbf{F}_2$:

- Functional model:



$$\alpha = \alpha$$

(α needed for further computation
→ we have to introduce "identity equation")

$$\varepsilon = 2\pi - \alpha - \beta - \mu$$

$$a_1 = \frac{\sin\alpha}{a}$$

$$a_2 = \frac{\sin\beta}{b}$$

- Design matrix:

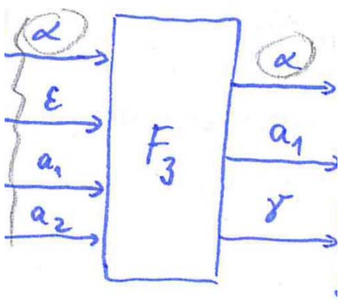
$$\mathbf{F}_2 = \begin{array}{c c} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \varepsilon \\ a_1 \\ a_2 \end{array} & \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ \dfrac{\cos\alpha}{a} & 0 \\ 0 & \dfrac{\cos\beta}{b} \end{bmatrix} \end{array}$$

New observations

4.4.4 Variance – Covariance propagation for complex functional relationships

$\mathbf{F}_3$:

- Functional model:



$$\alpha = \alpha$$

$$a_1 = a_1$$

$$\gamma = \arctan \frac{a_1 \cdot \sin\varepsilon}{a_2 + a_1 \cdot \cos\varepsilon}$$ = partial der

- Design matrix:

$$\mathbf{F}_3 = \begin{array}{c} \\ \alpha \\ a_1 \\ \gamma \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \varepsilon & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dfrac{\partial\gamma}{\partial\varepsilon} & \dfrac{\partial\gamma}{\partial a_1} & \dfrac{\partial\gamma}{\partial a_2} \end{bmatrix} \end{array}$$

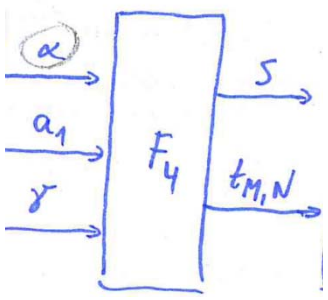
New obser

Propation XOXO

4.4.4 Variance – Covariance propagation for complex functional relationships

$\mathbf{F}_4$:

- Functional model:



$$s = \frac{\sin\gamma}{a_1}$$

$$t_{M,N} = t_{M,A} - \pi + \alpha + \gamma$$

- Design matrix:

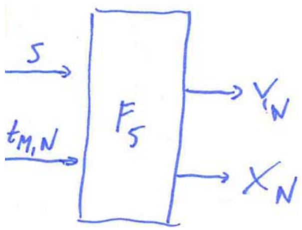
$$\mathbf{F}_4 = \begin{array}{c} \\ s \\ t_{M,N} \end{array} \begin{array}{c} \begin{array}{ccc} \alpha & a_1 & \gamma \end{array} \\ \begin{bmatrix} 0 & -\dfrac{\sin\gamma}{(a_1)^2} & \dfrac{\cos\gamma}{a_1} \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

New order

4.4.4 Variance – Covariance propagation for complex functional relationships

$\mathbf{F}_5$:

- Functional model:



$$Y_N = Y_M + s \cdot \sin t_{M,N}$$

$$X_N = X_M + s \cdot \cos t_{M,N}$$

- Design matrix:

$$\mathbf{F}_5 = \begin{array}{c} \\ Y_N \\ X_N \end{array} \begin{array}{cc} s & t_{M,N} \\ \multicolumn{2}{c}{\begin{bmatrix} \sin t_{M,N} & s \cdot \cos t_{M,N} \\ \cos t_{M,N} & -s \cdot \sin t_{M,N} \end{bmatrix}} \end{array}$$

4.4.4 Variance – Covariance propagation for complex functional relationships

VC propagation:

$$\mathbf{F} = \mathbf{F}_5 \cdot \mathbf{F}_4 \cdot \mathbf{F}_3 \cdot \mathbf{F}_2 \cdot \mathbf{F}_1$$

$$\boldsymbol{\Sigma}_{xx} = \mathbf{F} \cdot \boldsymbol{\Sigma}_{LL} \cdot \mathbf{F}^{\mathrm{T}}$$

with $$\boldsymbol{\Sigma}_{LL} = \begin{bmatrix} \left(\sigma_{r_{N,M}}\right)^2 & 0 & 0 \\ 0 & \left(\sigma_{r_{N,A}}\right)^2 & 0 \\ 0 & 0 & \left(\sigma_{r_{N,B}}\right)^2 \end{bmatrix}$$

→ Attention: Same order as columns in $\mathbf{F}_1$

Result:

$$\boldsymbol{\Sigma}_{xx\,2\times2} = \begin{bmatrix} \left(\sigma_{Y_N}\right)^2 & \mathrm{Cov}(Y_N, X_N) \\ \mathrm{Cov}(X_N, Y_N) & \left(\sigma_{X_N}\right)^2 \end{bmatrix}$$

Correlation: $$\rho_{Y_N,X_N} = \frac{\mathrm{Cov}(Y_N,X_N)}{\sigma_{Y_N} \cdot \sigma_{X_N}}$$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

Scaling of an observation with a constant factor

- Given: measurement l, standard deviation s_l
- Functional model:

factor?

$$f(l) = a \cdot l \qquad \text{with} \qquad a = \text{const.}$$

- Coefficient:

$$f_l = a$$

- VC propagation:

$$s_f^2 = f_l^2 \cdot s_l^2 \quad \Rightarrow \quad s_f^2 = a^2 \cdot s_l^2$$

Standard deviation $\quad s_f = \sqrt{s_f^2} \quad \Rightarrow \quad \boxed{s_f = a \cdot s_l}$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

Addition or subtraction of observations

- Given:

n uncorrelated measurements $l_1, l_2, \dots, l_n$ with their standard deviations $s_{l_1}, s_{l_2}, \dots, s_{l_n}$

- Functional model:

$$f(l_1, l_2, \dots, l_n) = l_1 \pm l_2 \pm \cdots \pm l_n$$

- Coefficients: Factor are ^2

$$f_{l_1} = +1, \qquad f_{l_2} = \pm 1, \qquad \dots, \qquad f_{l_n} = \pm 1$$

- VC propagation:

$$s_f^2 = s_{l_1}^2 + s_{l_2}^2 + \cdots + s_{l_n}^2 \qquad \text{(for addition and subtraction!)}$$

Standard deviation $\boxed{s_f = \sqrt{s_{l_1}^2 + s_{l_2}^2 + \cdots + s_{l_n}^2}}$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

▶ In the case that all observations have the same precision

→ Same standard deviation s_l for all observations l_i

$$s_l = s_{l_1} = s_{l_2} = \cdots = s_{l_n}$$

→ Standard deviation

$$s_f = \sqrt{s_l^2 + s_l^2 + \cdots + s_l^2}$$

$$s_f = \sqrt{n \cdot s_l^2}$$

$$\boxed{s_f = \sqrt{n} \cdot s_l}$$

<u>Rule</u>: Standard deviation grows with the square root if the number of summands.

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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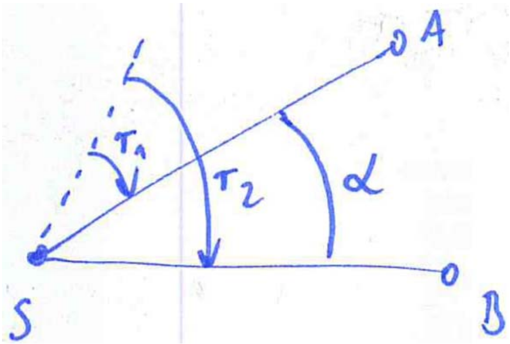
Example: Practical application



- Given:
  - Horizontal directions r_1, r_2 (observations)
  - Standard deviations $s_{r_1} = 1\ \mathrm{mgon}, s_{r_2} = 1\ \mathrm{mgon}$
- Wanted:
  - Horizontal angle α
  - Standard deviation s_α
- Solution:

$$s_\alpha = \sqrt{2} \cdot 1\ \mathrm{mgon}$$
$$s_\alpha = 1.41\ \mathrm{mgon}$$

∅ written exam

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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Arithmetic mean of observations

▶ Given:

- n uncorrelated measurements $l_1, l_2, \dots, l_n$
- Standard deviations $s_{l_1}, s_{l_2}, \dots, s_{l_n}$

▶ Functional model:

$$\bar{x} = f(l_1, l_2, \dots, l_n) = \frac{l_1 + l_2 + \cdots + l_n}{n} = \frac{1}{n} l_1 + \frac{1}{n} l_2 + \cdots + \frac{1}{n} l_n$$

▶ Coefficients:

$$f_{l_1} = \frac{1}{n} \;,\; f_{l_2} = \frac{1}{n} \;,\; \dots, \; f_{l_n} = \frac{1}{n}$$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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▶ VC propagation:

$$s_{\bar{x}}^2 = \left(\frac{1}{n}\right)^2 s_{l_1}^2 + \left(\frac{1}{n}\right)^2 s_{l_2}^2 + \cdots + \left(\frac{1}{n}\right)^2 s_{l_n}^2 = \frac{1}{n^2}(s_{l_1}^2 + s_{l_2}^2 + \cdots + s_{l_n}^2)$$

In the case that all observations have the same precision

→ same standard deviations s_l for all observations l_i

$$s_l = s_{l_1} = s_{l_2} = \cdots = s_{l_n}$$

$$s_{\bar{x}}^2 = \frac{1}{n^2}(s_l^2 + s_l^2 + \cdots + s_l^2)$$

$$s_{\bar{x}}^2 = \frac{1}{n^2} \cdot n \cdot s_l^2$$

Standard deviation $\boxed{s_{\bar{x}} = \frac{s_l}{\sqrt{n}}}$

<u>Rule</u>: If we perform measurements for the same random variable n times, the standard deviation for the arithmetic mean decreases by the factor $1/\sqrt{n}$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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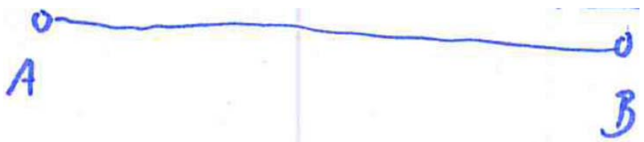
<u>Example</u>:



▶ Given:

- Distance $A - B$ has been measured 10 times
- Standard deviation $s_l = 1$ cm

▶ Wanted:

- Standard deviation of the arithmetic mean

$$s_{\bar{x}} = 1\text{ cm} \cdot \frac{1}{\sqrt{10}}$$

$$s_{\bar{x}} = 3.16\text{ mm}$$

How often do we have to measure to get

$s_{\bar{x}} = 2mm$

$s_l = 10mm = 1cm$

$s_{\bar{x}} = s_l \cdot \frac{1}{\sqrt{n}}$

$2mm = 10mm \cdot \frac{1}{\sqrt{n}} \quad | \cdot \sqrt{n}$

≙ 3xan

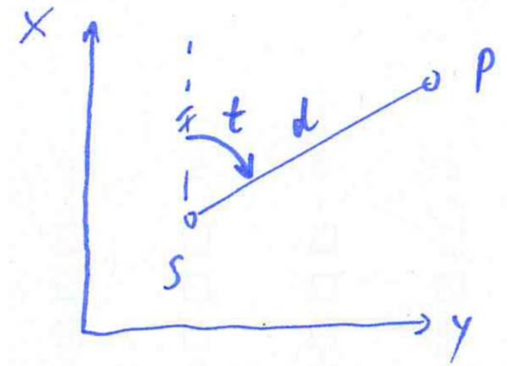
$\sqrt{n} \cdot 2mm = 10mm \quad | : 2mm$

$\sqrt{n} = \frac{10mm}{2mm} \; 5 = 25 = n$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

Some more remarks:

► $$\left.\begin{aligned} Y_P &= Y_s + d \cdot \sin t \\ X_P &= X_s + d \cdot \cos t \end{aligned}\right\}$$ General case, because we have two correlated unknowns



► If (and only if) we are interested in s_{Y_P}, s_{X_P} and not in the correlation r_{Y_P,X_P}, we can compute s_{Y_P}, s_{X_P} separately by applying the special case two times:

$$s_{Y_P}^2 = \left(\frac{\partial Y_P}{\partial d}\right)^2 \cdot s_d^2 + \left(\frac{\partial Y_P}{\partial t}\right)^2 \cdot s_t^2$$

$$s_{X_P}^2 = \left(\frac{\partial X_P}{\partial d}\right)^2 \cdot s_d^2 + \left(\frac{\partial X_P}{\partial t}\right)^2 \cdot s_t^2$$

► But: If we use our results (here Y_P, X_P) for further computations, we must consider the covariances and therefore apply the general case of VC propagation

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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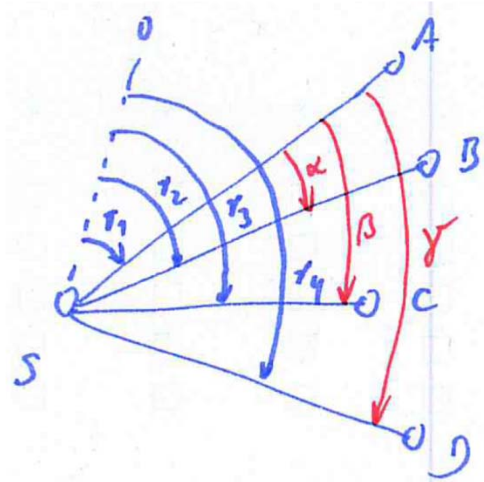
Example:



- Given:
  - Horizontal directions r_1, r_2, r_3, r_4 (observations)
  - Standard deviations $\sigma_{r_1} = \sigma_{r_2} = \sigma_{r_3} = \sigma_{r_4} = 0.5 \text{ mgon}$

- Wanted:
  - Horizontal angles α, β, γ
  - Standard deviations $\sigma_\alpha, \sigma_\beta, \sigma_\gamma$

- If we are only interested in $\sigma_\alpha, \sigma_\beta, \sigma_\gamma$ as a final result, we can apply the special case of VC propagation:

$$\alpha = r_2 - r_1 \;\Rightarrow\; \sigma_\alpha = \sqrt{2} \cdot 0.5 \text{ mgon} = 0.71 \text{ mgon}$$

$$\beta = r_3 - r_1 \;\Rightarrow\; \sigma_\beta = 0.71 \text{ mgon}$$

$$\gamma = r_4 - r_1 \;\Rightarrow\; \sigma_\gamma = 0.71 \text{ mgon}$$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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▶ If we want to use α, β, γ for further computation (which usually is the case), we have to apply the general case of VC propagation:

- Step 1: Functional model

$$\mathbf{X} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \Phi_1(\mathbf{l}) \\ \Phi_2(\mathbf{l}) \\ \Phi_3(\mathbf{l}) \end{bmatrix} = \begin{bmatrix} r_2 - r_1 \\ r_3 - r_1 \\ r_4 - r_1 \end{bmatrix}$$

- Step 2: Observation vector and stochastic model

$$\mathbf{l} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}, \qquad \mathbf{\Sigma}_{LL} = \begin{bmatrix} 0.5^2 & & 0 & \\ & 0.5^2 & & \\ & 0 & 0.5^2 & \\ & & & 0.5^2 \end{bmatrix}$$

4.4.5 Variance – Covariance propagation: Standard cases that often occur in practice

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- Step 3: Design matrix (coefficients of the linear functional model)

$$\mathbf{F} = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \end{array} \begin{array}{c} \begin{array}{cccc} r_1 & r_2 & r_3 & r_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

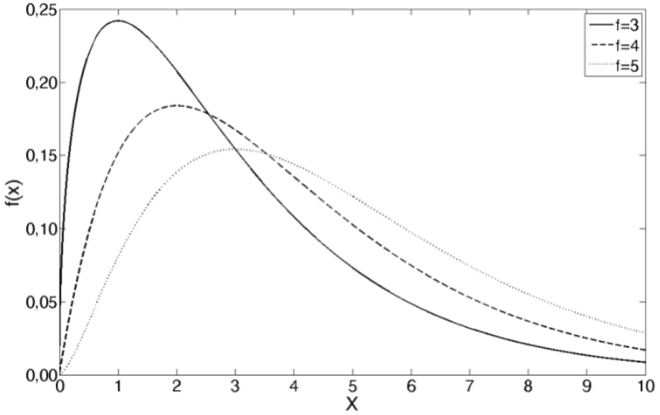
- Step 4: VCM of the unknowns

$$\mathbf{\Sigma}_{xx} = \mathbf{F} \cdot \mathbf{\Sigma}_{LL} \cdot \mathbf{F}^{\mathrm{T}} = \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

← Co-variances are correlated

Complete matrix $\mathbf{\Sigma}_{xx}$ must be considered for further computations!

Standard deviations: $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = \sqrt{0.5} = 0.71\ \mathrm{mgon}$



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Adjustment Theory I

Chapter 4 – Propagation of Observation Errors

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