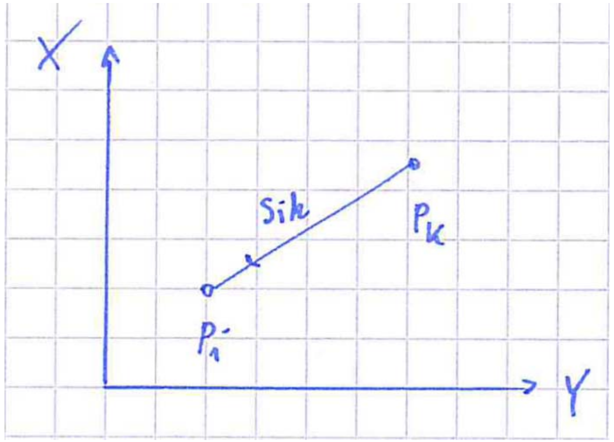
7.2 Adjustment of Horizontal Surveys: Trilateration

Functional model:

Measured from "i" to "k"

$$s_{ik} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}$$



Observation equations:

$$s_{ik} + v_{s_{ik}} = \sqrt{(\hat{x}_k - \hat{x}_i)^2 + (\hat{y}_k - \hat{y}_i)^2}$$

Nonlinear functional model

$\rightarrow$ for least squares adjustment we need a linearised functional model

$\rightarrow$ Jacobian matrix with partial derivatives

7.2 Adjustment of Horizontal Surveys: Trilateration

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Partial derivatives:

$$\frac{\partial s_{ik}}{\partial x_k} = \frac{1}{2\sqrt{}} \cdot 2(x_k - x_i) = \frac{x_k - x_i}{s_{ik}} = \frac{\Delta x_{ik}}{s_{ik}}$$

$$\frac{\partial s_{ik}}{\partial x_i} = \frac{1}{2\sqrt{}} \cdot 2(x_k - x_i) \cdot (-1) \qquad = \frac{-\Delta x_{ik}}{s_{ik}}$$

$$\frac{\partial s_{ik}}{\partial y_k} = \frac{\Delta y_{ik}}{s_{ik}}$$

$$\frac{\partial s_{ik}}{\partial y_i} = \frac{-\Delta y_{ik}}{s_{ik}}$$

$\rightarrow$ See handout (Partial derivatives of geodetic observation equations)

7.2 Adjustment of Horizontal Surveys: Trilateration

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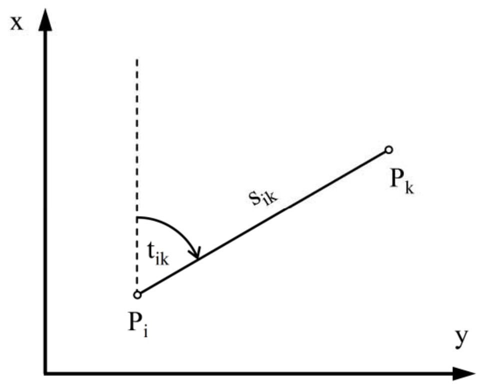
Partial derivatives of geodetic observation equations

Given: Observation Equations
Searched: Partial derivatives with respect to the unknowns for the linearization of the observation equations

1. Distances $s_{ik} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}$



Notation information:

Index i: Station
Index k: Target

measured from ‚i' to ‚k'

Partial derivatives can be further simplified by using the equations for the grid bearing t_{ik} and distance s_{ik} .

Partial Derivatives:

$$\frac{\partial s_{ik}}{\partial x_k} = \frac{x_k - x_i}{s_{ik}} = \frac{\Delta x_{ik}}{s_{ik}} = \cos t_{ik} \qquad \frac{\partial s_{ik}}{\partial x_i} = -\frac{x_k - x_i}{s_{ik}} = \frac{-\Delta x}{s_{ik}} = -\cos t_{ik}$$

$$\frac{\partial s_{ik}}{\partial y_k} = \frac{y_k - y_i}{s_{ik}} = \frac{\Delta y_{ik}}{s_{ik}} = \sin t_{ik} \qquad \frac{\partial s_{ik}}{\partial y_i} = -\frac{y_k - y_i}{s_{ik}} = -\frac{\Delta y_{ik}}{s_{ik}} = -\sin t_{ik}$$

Adjustment_Theory_I_Derivatives.pdf

7.2 Adjustment of Horizontal Surveys: Trilateration

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Weights in trilateration networks

- Precision for distances from electronic distance measurement given in

$$\sigma_{s_i} = a_1 + a_2 \cdot d_i$$

a_1: constant part of precision

a_2: standard error per unit sight distance length

d_i: length of sight distance

7.2 Adjustment of Horizontal Surveys: Trilateration

- Typical values for the precision of an EDM:

$$3\ \mathrm{mm} + 2\ \mathrm{ppm}$$

ppm: parts per million $\widehat{=}$ mm per km

$\rightarrow$ Standard deviation for a distance of

$$\begin{aligned} 500\ \mathrm{m}&: 4\ \mathrm{mm} \\ 1000\ \mathrm{m}&: 5\ \mathrm{mm} \\ 2000\ \mathrm{m}&: 7\ \mathrm{mm} \end{aligned}$$

- Variance matrix of the observations

$$\mathbf{\Sigma}_{LL} = \begin{bmatrix} \sigma_{s_1}^2 & & & 0 \\ & \sigma_{s_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{s_n}^2 \end{bmatrix}$$

- With reference variance σ_0^2
- Cofactor matrix of observations: $\mathbf{Q}_{LL} = \frac{1}{\sigma_0^2}\mathbf{\Sigma}_{LL}$
- Weight matrix of observations: $\mathbf{P} = \mathbf{Q}_{LL}^{-1}$

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Example

The measurements of the trilateration network depicted in Figure 1 are listed in Table 2. The points 1, 2 and 3 are control points (error-free) and their Gauss-Krueger coordinates are given in Table 1. Calculate the adjusted Gauss-Krueger coordinates of point 100 using least squares adjustment.

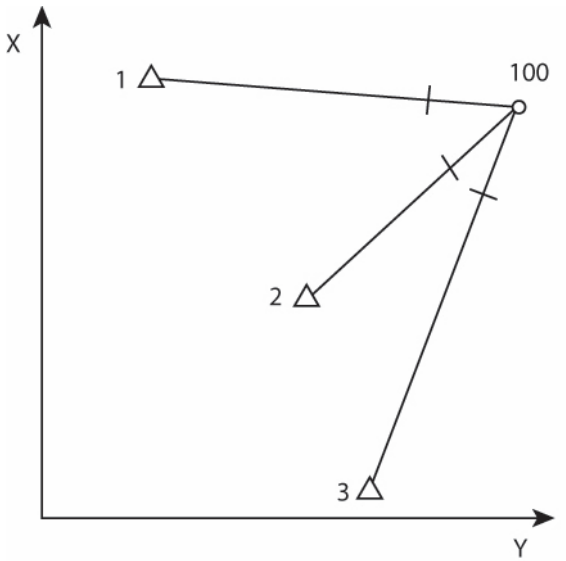

Figure 1: Trilateration network

Table 1: Gauss-Krueger coordinates of control points

Point No.	y [m]	x [m]
1	865.400	4527.150
2	2432.550	2047.250
3	2865.220	27.150

Approximate values for the coordinates of point 100:

$$y_{100}^{0} : 6861.3 \; ; \; x_{100}^{0} : 3727.8$$

(graphical coordinates from a map)

7.2 Adjustment of Horizontal Surveys: Trilateration

Table 2: Observed reduced distances

From	To	s [m]
100	1	6049.000
100	2	4736.830
100	3	5446.490

- The distance measurements have been performed with a precision of 1 mm + 2 ppm
- The distances are uncorrelated and already reduced into the Gauss-Krueger projection
- Set up an appropriate functional model as well as the observation equations
- Set up the stochastic model
- Choose appropriate values for the break-off conditions ε and δ and justify your decision
- Solve the normal equation system and determine the Gauss-Krueger coordinates of point 100 as well as their standard deviations
- Calculate the residuals and the adjusted observations as well as their standard deviations

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General considerations:

- What are our unknowns?

  → Coordinates of point 100

  → We introduce y_{100}, x_{100}

- What are our observations?

  → Distances

  → $s_{100,1}, s_{100,2}, s_{100,3}$

- Observations reduced into projection?

  → Yes!

- What are our fixed values?

  → $y_1, x_1; y_2, x_2; y_3, x_3$

- Redundancy? → $r = n - u \;\rightarrow r = 3 - 2 \;\rightarrow r = 1$

7.2 Adjustment of Horizontal Surveys: Trilateration

Functional model:

$$s_{100,1} = \sqrt{(x_1 - x_{100})^2 + (y_1 - y_{100})^2}$$

$$s_{100,2} = \sqrt{(x_2 - x_{100})^2 + (y_2 - y_{100})^2}$$

$$s_{100,3} = \sqrt{(x_3 - x_{100})^2 + (y_3 - y_{100})^2}$$

Observation equations:

$$s_{100,1} + v_1 = \sqrt{(x_1 - \hat{x}_{100})^2 + (y_1 - \hat{y}_{100})^2}$$

$$s_{100,2} + v_2 = \sqrt{(x_2 - \hat{x}_{100})^2 + (y_2 - \hat{y}_{100})^2}$$

$$s_{100,3} + v_3 = \sqrt{(x_3 - \hat{x}_{100})^2 + (y_3 - \hat{y}_{100})^2}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

Observation vector:

$$\mathbf{L} = \begin{bmatrix} 6049.000 \\ 4736.830 \\ 5446.490 \end{bmatrix}$$

Stochastic model of the observations:

$$\sigma_1 = 1\ \mathrm{mm} + 2\,\frac{\mathrm{mm}}{\mathrm{km}} \cdot 6.049\ \mathrm{km}$$

$$\sigma_2 = 1\ \mathrm{mm} + 2\,\frac{\mathrm{mm}}{\mathrm{km}} \cdot 4.73683\ \mathrm{km}$$

$$\sigma_3 = 1\ \mathrm{mm} + 2\,\frac{\mathrm{mm}}{\mathrm{km}} \cdot 5.44649\ \mathrm{km}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

$$\boldsymbol{\Sigma}_{LL} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

$$\mathbf{Q}_{LL} = \frac{1}{\sigma_0^2} \boldsymbol{\Sigma}_{LL}$$

with

$$\sigma_0^2 = 1$$

$$\mathbf{Q}_{LL} = \boldsymbol{\Sigma}_{LL}$$

$$\rightarrow \qquad \mathbf{P} = \mathbf{Q}_{LL}^{-1}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

Vector of adjusted unknowns:

$$\hat{\mathbf{X}} = \begin{bmatrix} \hat{x}_{100} \\ \hat{y}_{100} \end{bmatrix}$$

Nonlinear functional model

→ Solution from iterative computing with linearised functional model

→ Introduction of approximate values x_{100}^0, y_{100}^0

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Vector of starting values:

$$\mathbf{X}^0 = \begin{bmatrix} x_{100}^0 \\ y_{100}^0 \end{bmatrix}$$

Vector of adjusted reduced unknowns:

$$\hat{\mathbf{x}} = \hat{\mathbf{X}} - \mathbf{X}^0 = \begin{bmatrix} d\hat{x}_{100} \\ d\hat{y}_{100} \end{bmatrix} = \begin{bmatrix} \hat{x}_{100} - x_{100}^0 \\ \hat{y}_{100} - y_{100}^0 \end{bmatrix}$$

Vector of reduced observations:

$$\mathbf{l} = \begin{bmatrix} 6049.000 - \sqrt{\left(x_1 - x_{100}^0\right)^2 + \left(y_1 - y_{100}^0\right)^2} \\ 4736.830 - \sqrt{\left(x_2 - x_{100}^0\right)^2 + \left(y_2 - y_{100}^0\right)^2} \\ 5446.490 - \sqrt{\left(x_3 - x_{100}^0\right)^2 + \left(y_3 - y_{100}^0\right)^2} \end{bmatrix}$$

Jacobian matrix:

$$\mathbf{J} = \begin{array}{c} \\ s_{100,1} \\ s_{100,2} \\ s_{100,3} \end{array} \begin{array}{c} \begin{array}{cc} \quad x_{100}^{0} \quad & \quad y_{100}^{0} \quad \end{array} \\ \begin{bmatrix} \dfrac{\partial s_{100,1}^{0}}{\partial x_{100}^{0}} & \dfrac{\partial s_{100,1}^{0}}{\partial y_{100}^{0}} \\ \dfrac{\partial s_{100,2}^{0}}{\partial x_{100}^{0}} & \dfrac{\partial s_{100,2}^{0}}{\partial y_{100}^{0}} \\ \dfrac{\partial s_{100,3}^{0}}{\partial x_{100}^{0}} & \dfrac{\partial s_{100,3}^{0}}{\partial y_{100}^{0}} \end{bmatrix} \end{array}$$

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with

$$\frac{\partial s^0_{100,1}}{\partial x^0_{100}} = \frac{1}{2\sqrt{\left(x_1 - x^0_{100}\right)^2 + \left(y_1 - y^0_{100}\right)^2}} \cdot 2\left(x_1 - x^0_{100}\right) \cdot (-1)$$

$$= \frac{-\left(x_1 - x^0_{100}\right)}{s^0_{100,1}}$$

$$\vdots$$

$$\frac{\partial s^0_{100,3}}{\partial y^0_{100}} = \frac{1}{2\sqrt{\left(x_3 - x^0_{100}\right)^2 + \left(y_3 - y^0_{100}\right)^2}} \cdot 2\left(y_3 - y^0_{100}\right) \cdot (-1)$$

$$= \frac{-\left(y_3 - y^0_{100}\right)}{s^0_{100,3}}$$

7.2 Adjustment of Horizontal Surveys: Trilateration

Design matrix:

$$\mathbf{A} = \mathbf{J}$$

Normal equations:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Solution of normal equations:

$$\hat{\mathbf{x}} = (\underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}}_{\mathbf{N}})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l}$$

Adjusted unknowns:

$$\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$$

Residuals:

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}$$

Adjusted observations:

$$\hat{\mathbf{L}} = \mathbf{L} + \mathbf{v}$$

Check:

$$\max |\hat{x}_i| < \varepsilon$$

and

$$\max \left|\hat{L}_i - \Phi_i(\widehat{\mathbf{X}})\right| < \delta$$

no

Use $\widehat{\mathbf{X}}$ as new $\mathbf{X}^0$

yes

$\widehat{\mathbf{X}}$ is final result

7.2 Adjustment of Horizontal Surveys: Trilateration

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Empirical reference standard deviation:

$$s_0 = \sqrt{\frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{n-u}}$$

VCM of adjusted unknowns:

$$\mathbf{\Sigma}_{\hat{X}\hat{X}} = \mathrm{s}_0^2 \cdot \mathbf{Q}_{\hat{X}\hat{X}} \qquad \text{with} \qquad \mathbf{Q}_{\hat{X}\hat{X}} = \mathbf{N}^{-1}$$

Standard deviation of unknowns:

$$\mathbf{\Sigma}_{\hat{X}\hat{X}} = s_0^2 \cdot \begin{bmatrix} q_{\hat{x}\hat{x}} & q_{\hat{x}\hat{y}} \\ q_{\hat{y}\hat{x}} & q_{\hat{y}\hat{y}} \end{bmatrix}$$

$q_{\hat{x}\hat{x}}$: Cofactor of unknown value x_{100}

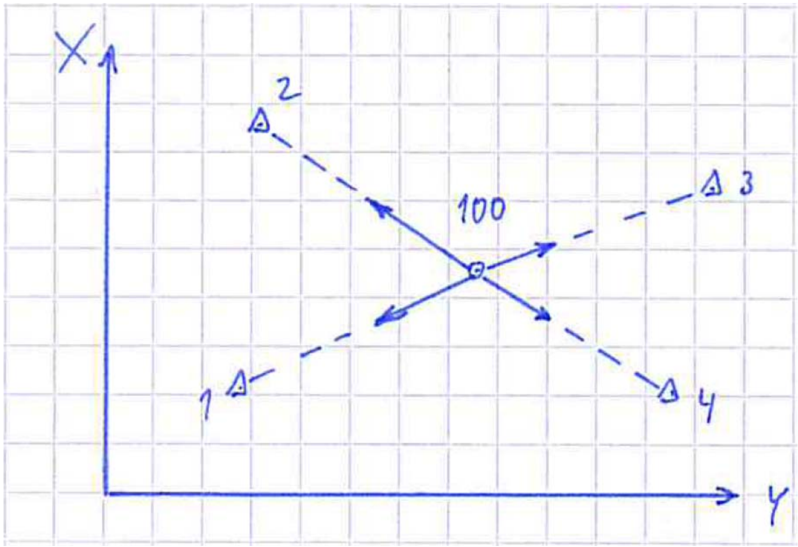
$q_{\hat{y}\hat{y}}$: Cofactor of unknown value y_{100}

$$s_{\hat{x}_{100}} = s_0 \cdot \sqrt{q_{\hat{x}\hat{x}}}$$
$$s_{\hat{y}_{100}} = s_0 \cdot \sqrt{q_{\hat{y}\hat{y}}}$$

7.3 Adjustment of Horizontal Surveys: Triangulation

Basic idea of triangulation:

Determination of 2D coordinates of points in a plane Cartesian coordinate system from <u>observed directions</u>



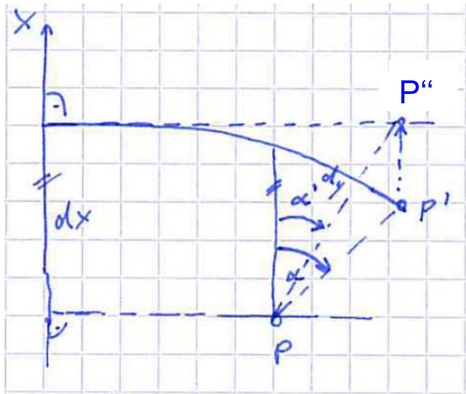
Cartesian coordinates in 2D

- Gauss-Krueger coordinates
- UTM coordinates

→ Projected coordinates into a plane

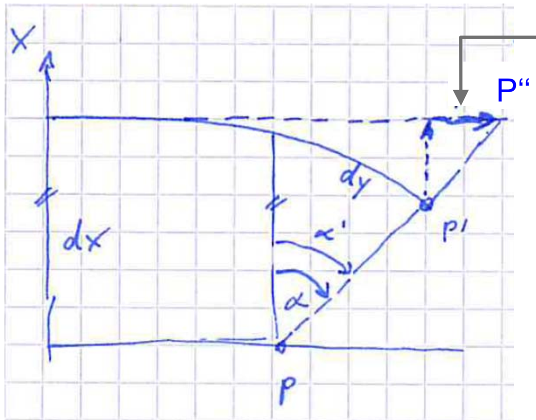
→ Conformal mapping

Non-conformal and conformal mapping



$$\alpha' \neq \alpha$$

e.g. Soldner coordinates



Ordinate difference is elongated
$\rightarrow$ elimination of angular distortion

$$\alpha' = \alpha$$

e.g. Gauss-Krueger coordinates, UTM coordinates

7.3 Adjustment of Horizontal Surveys: Triangulation

- A conformal mapping keeps a differential similarity between the original elliptic situation and the maps image

  $\rightarrow$ We can use our measured directions without corrections and reductions and combine them with GK or UTM coordinates

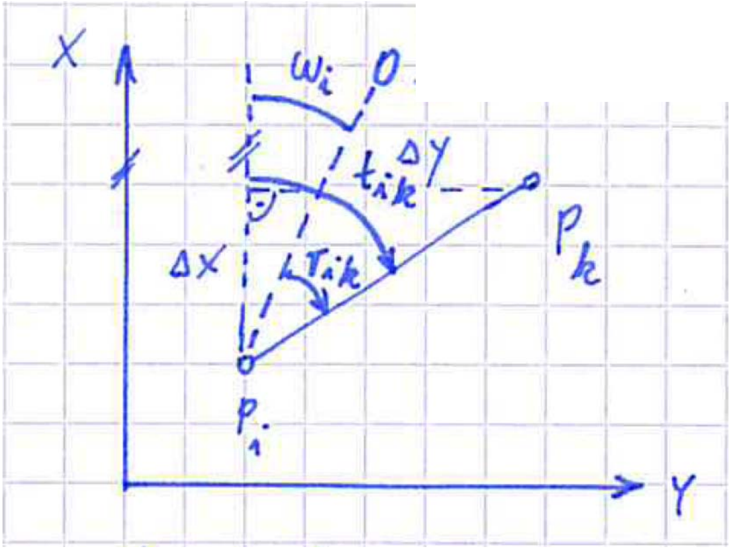
- Directions r_{ik} are our observations with corresponding precision / weight

- For an adjustment within the Gauss-Markov Model (parametric adjustment)

$$\mathbf{L} = \mathbf{\Phi}(\mathbf{X})$$

  we have to introduce appropriate unknowns to express our observations as functions of the unknowns

  $\rightarrow$ We introduce 2D coordinates as unknowns



0: zero direction of our instrument
(tacheometer, theodolite)

i: instrument station

k: foresight station

r_{ik}: measured direction from i to k

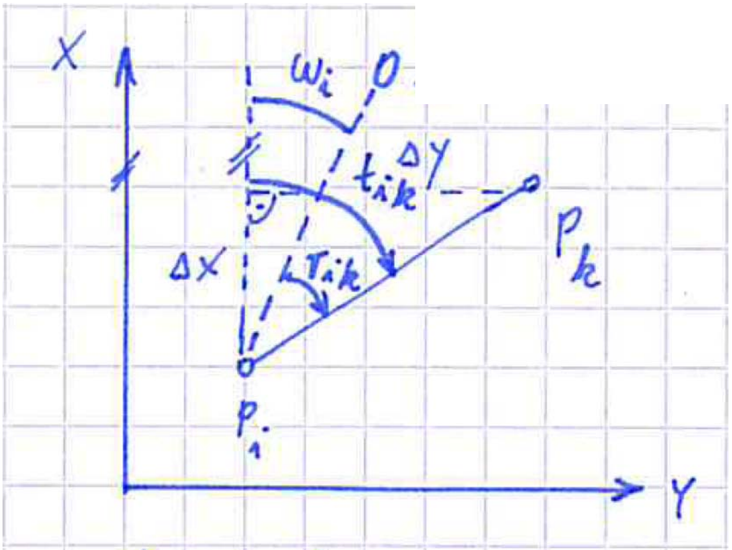
t_{ik}: azimuth from i to k

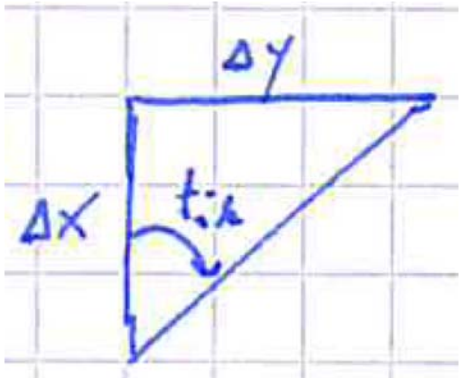
ω_i: orientation unknown

Functional model:

$$r_{ik} = t_{ik} - \omega_i$$

How to obtain t_{ik}?





$$\tan t_{ik} = \frac{\Delta y}{\Delta x} \rightarrow \qquad t_{ik} = \arctan\frac{\Delta y}{\Delta x}$$

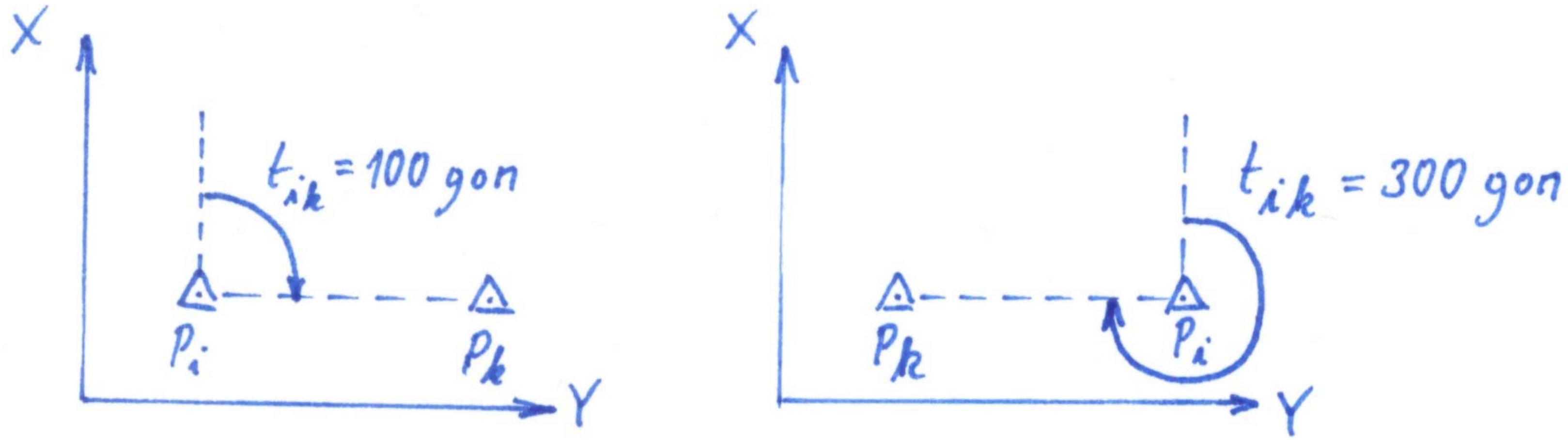
$$\rightarrow \qquad t_{ik} = \arctan\frac{y_k - y_i}{x_k - x_i}$$

7.3 Adjustment of Horizontal Surveys: Triangulation

▶ Attention 1: What happens if $\Delta x = 0$?

→ Cannot use formula

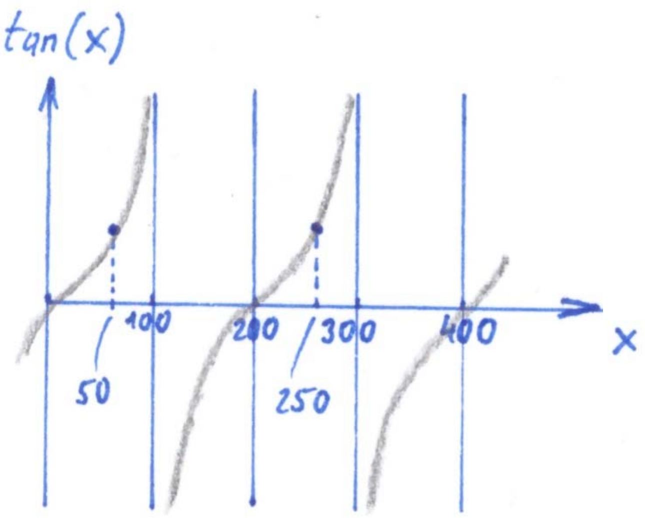
→ two cases possible



▶ Attention 2:

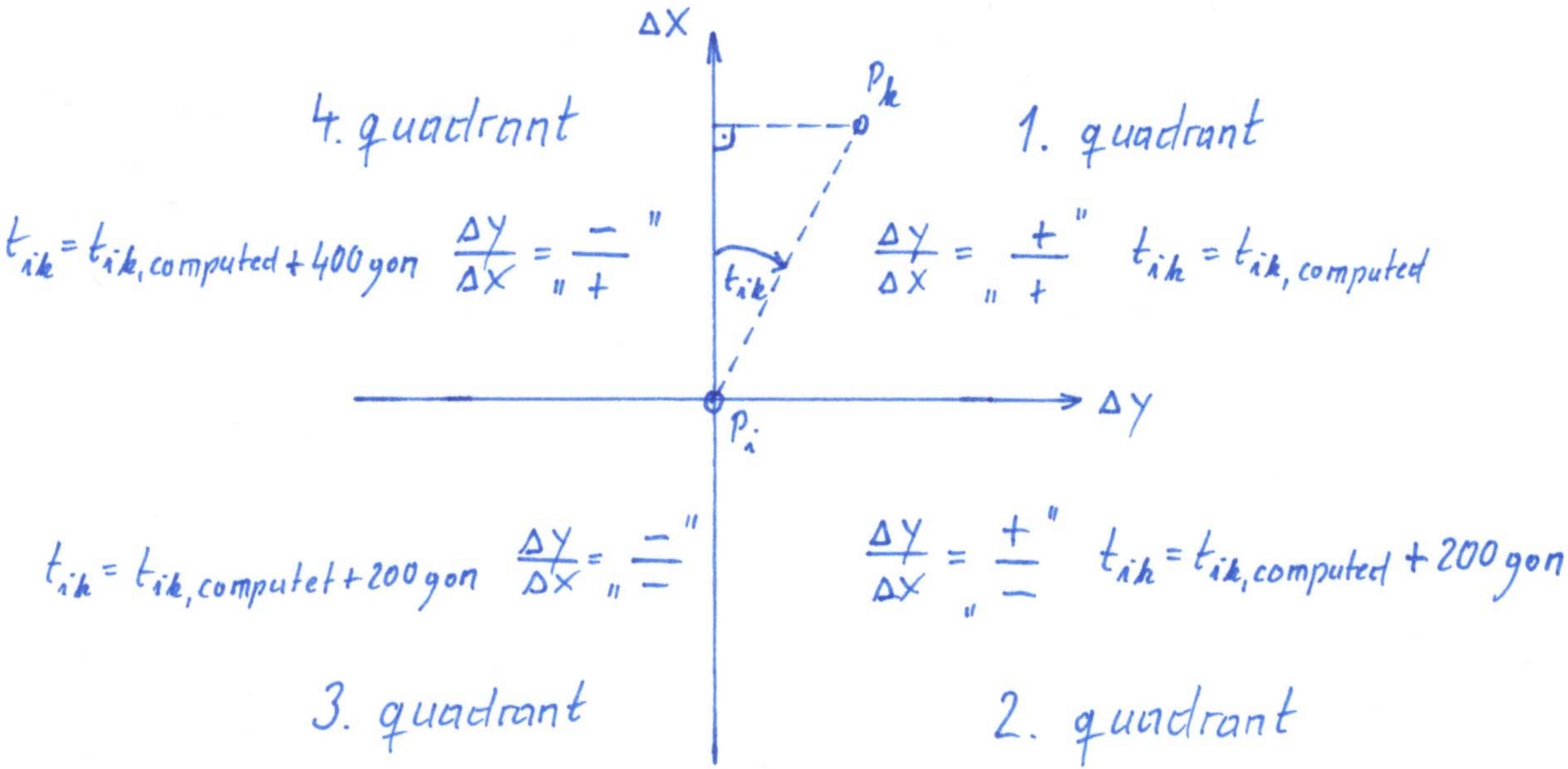
e.g. $\tan 50 \;\text{gon} = 1$

$\tan 250 \;\text{gon} = 1$



Problem: Which value is the desired one?

Solution: Analysis of the quadrants, $t_{ik,\mathrm{computed}} = \arctan\frac{\Delta y}{\Delta x}$



Remark: “atan2”

Functional model:

Measured from "i" to "k"

$$\boxed{r_{ik} = \arctan\frac{y_k - y_i}{x_k - x_i} - \omega_i}$$

Attention:
Quadrants!

Observation equations:

$$\boxed{r_{ik} + v_{ik} = \arctan\frac{\hat{y}_k - \hat{y}_i}{\hat{x}_k - \hat{x}_i} - \widehat{\omega}_i}$$

Nonlinear functional model

$\rightarrow$ for least squares adjustment we need a linearised functional model

$\rightarrow$ Jacobian matrix with partial derivatives